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Zhuchao Ji

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THÈSE DE DOCTORAT

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Zhuchao JI

Fatou-Julia dichotomy and non-uniform hyperbolicity for holomorphic endomorphisms on $\mathbb{P}^2(\mathbb{C})$

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Après avis des rapporteurs : WEIXIAO SHEN (Fudan University, Shanghai Math Center)
GABRIEL VIGNY (Université de Picardie Jules Verne, LAMFA)

Jury de soutenance :

WEIXIAO SHEN	(Fudan University, Shanghai Math Center) Rapporteur
GABRIEL VIGNY	(Université de Picardie Jules Verne, LAMFA) Rapporteur
VIVIANE BALADI	(LPSM et CNRS) Examinateur
XAVIER BUFF	(Université Paul Sabatier, IMT) Examinateur
HENRY DE THÉLIN	(Université Sorbonne Paris Nord, LAGA) Examinateur
ROMAIN DUJARDIN	(Sorbonne Université, LPSM) Directeur de thèse

Preface

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Paris, August 2020, Zhuchao Ji

Résumé

Cette thèse traite de deux aspects différents (produits semi-directs polynomiaux et endomorphismes post- critiquement fini) de la dynamique holomorphe sur le plan projectif $\mathbb{P}^2$. Elle contient les trois articles suivants:

I. Non-wandering Fatou components for strongly attracting polynomial skew products. (Publié dans *The Journal of Geometric Analysis.*) Nous prouvons une généralisation du théorème de non-errance de Sullivan pour les produits semi-directs polynomiaux de $\mathbb{C}^2$. Plus précisément, nous montrons que si f est un produits semi-direct polynomial avec une droite verticale invariante L attractive, et que de plus le multiplicateur correspondant est suffisamment petit, alors il n'y a pas de composante Fatou errante dans le bassin d'attraction de L.

II. Non-uniform hyperbolicity in polynomial skew products. (Soumis pour publication.) Soit f un produit semi-directs polynomial avec une droite verticale invariante attractive L. Supposons que f restreinte à L satisfait l'une des conditions non uniformément hyperboliques suivantes: 1. $f|_L$ est topologiquement Collet-Eckmann et Faiblement Régulière, 2. l'exposant de Lyapunov à chaque valeur critique se trouvant dans l'ensemble de Julia de $f|_L$ existe et est positif, et il n'y a pas de cycle parabolique. Alors l'ensemble de Fatou dans le bassin attractif de L est l'union des bassins des cycles d'attraction, et l'ensemble de Julia dans le bassin attractif de L est de mesure de Lebesgue nulle. En particulier il n'y a pas de composants Fatou errant dans le bassin d'attraction de L.

III. Structure of Julia sets for post-critically finite endomorphisms on $\mathbb{P}^2$. De Thélin a prouvé que pour l'endomorphisme post-critiquement fini sur $\mathbb{P}^2$, le courant de Green T est laminaire dans $J_1 \setminus J_2$, où J_1 est l'ensemble de Julia et J_2 est le support de la mesure de l'entropie maximale. Dans ce contexte nous donnons une description plus explicite de la dynamique sur $J_1 \setminus J_2$: ou bien x est contenu dans le bassin d'attraction d'un cycle de composantes critiques, ou bien il y a un disque Fatou passant par x. Nous montrons également que pour un endomorphisme post-critiquement fini de $\mathbb{P}^2$ tel que toutes les branches de $PC(f)$ sont lisses et se coupent transversalement, $J_2 = \mathbb{P}^2$ si et seulement si f est strictement post-critiquement fini. Cela est une réciproque partielle d'un résultat de Jonsson. Comme étape intermédiaire de la preuve, nous montrons que J_2 est l'adherence de l'ensemble des cycles répulsifs.

Abstract

This thesis deals with two different aspects (polynomial skew products and postctitically finite endomorphisms) of holomorphic dynamics on projective plane $\mathbb{P}^2$. It contains the following three papers:

I. Non-wandering Fatou components for strongly attracting polynomial skew products. (Published in *The Journal of Geometric Analysis.*) We prove a generalization of Sullivan's non-wandering domain theorem for polynomial skew products on $\mathbb{C}^2$. More precisely, we show that if f is a polynomial skew product with an invariant vertical line L, assume L is attracting and moreover the corresponding multiplier is sufficiently small, then there is no wandering Fatou component in the attracting basin of L.

II. Non-uniform hyperbolicity in polynomial skew products. (Submitted for publication.) We show that if f is a polynomial skew product with an attracting invariant vertical line L, assume the restriction of f on L satisfies one of the following non-uniformly hyperbolic condition: 1. $f|_L$ is topological Collet-Eckmann and Weakly Regular, 2. the Lyapunov exponent at every critical value point lying in the Julia set of $f|_L$ exist and is positive, and there is no parabolic cycle. Then the Fatou set in the attracting basin of L is union of basins of attracting cycles, and the Julia set in the attracting basin of L has Lebesgue measure zero. As a corollary, there are no wandering Fatou components in the attracting basin of L.

III. Structure of Julia sets for post-critically finite endomorphisms on $\mathbb{P}^2$. (Preprint.) De Thélin proved that for post-critically finite endomorphism on $\mathbb{P}^2$, the Green current T is laminar in $J_1 \setminus J_2$, where J_1 denotes the Julia set, and J_2 denotes the support of the measure of maximal entropy. We give a more explicit description of the dynamics on $J_1 \setminus J_2$ for post-critically finite endomorphism on $\mathbb{P}^2$: either x is contained in an attracting basin of a critical component cycle, or there is a Fatou disk passing through x. We also prove that for post-critically finite endomorphism on $\mathbb{P}^2$ such that all branches of $PC(f)$ are smooth and intersect transversally, $J_2 = \mathbb{P}^2$ if and only if f is strictly post-critically finite. This gives a partial converse of a result of Jonsson. As an intermediate step of the proof, we show that for post-critically finite endomorphism on $\mathbb{P}^2$, J_2 is the closure of the set of repelling cycles.

Contents

Introduction

This thesis deals with some aspects of holomorphic dynamics on the projective plane $\mathbb{P}^2(\mathbb{C})$. Let us first recall some basic history of this research domain. The object of the holomorphic dynamics is the iteration of a holomorphic self-map f on a complex manifold X.

When X is the Riemann sphere and f is a rational function, this is a classical subject, introduced by Fatou and Julia at the beginning of the last century. The study of complex dynamics made a spectacular rise in the early 1980s, thanks to works by well-known mathematicians such as Douady, Hubbard, Sullivan, Milnor etc. and the popularity of fractal images such as the Mandelbrot set and the Julia set. A celebrated result from this periods is Sullivan's non-wandering domain theorem [41], which gives a complete description of the dynamics on a dense open set of the phase space (the Fatou set). This will be an important theme in this thesis and we will come back on detail on this in the next section.

In higher dimension, the subject rise in the early 1990s under the impulse of mathematicians such as Bedford, Hubbard, Fornaess, Sibony and Smillie. A key input was the introduction of modern methods from higher dimensional complex analysis (pluripotential theory, currents, etc) to the study of holomorphic dynamics. Holomorphic dynamics in higher dimension is now a well established research topic, as a confluence of dynamical systems, complex analysis and geometry.

Many classes of higher dimensional holomorphic maps were studied so far. Among them holomorphic endomorphisms on $\mathbb{P}^k$ play a prominent role as they are the natural analogues of one-dimensional rational maps on $\mathbb{P}^1$. They belong to the more general class of meromorphic maps of algebraic varieties, which also includes for instance polynomial automorphisms on $\mathbb{C}^2$ (also known as complex Hénon maps). In this thesis we will mostly be concerned by polynomial endomorphisms on $\mathbb{C}^2$ and holomorphic endomorphisms on $\mathbb{P}^2$.

For such mappings we can define Fatou-Julia decomposition as in dimension 1. Let f be a holomorphic endomorphism on $\mathbb{P}^2$ or a polynomial endomorphisms on $\mathbb{C}^2$ (of degree ≥ 2). The Fatou set is defined as the locus where the iterates $\{f^n\}_{n\geq 1}$ locally form a normal family. The (first) Julia set J_1 is the complement of the Fatou set. Unlike the one-dimensional case, the classification of Fatou components is not known for holomorphic endomorphisms on $\mathbb{P}^2$. In particular, contrary to Sullivan's non-wandering domain theorem in dimension 1, there may exist wandering Fatou components in dimension 2, as showed recently by Astorg-Buff-Dujardin-Peters-Raissy [2]. One main question that we address in this thesis is the study of the Fatou-Julia decomposition in two dimensions, and in particular the question of existence of wandering Fatou components.

Post-critically finite rational functions play an important role in one-dimensional complex dynamics, because they serve as a kind of combinatorial skeleton of moduli space of rational functions of fixed degree. The study of post-critically finite endomorphisms on $\mathbb{P}^2$ start with the works of Fornaess-Sibony [18], [19]. Fatou components for post-critically finite endomorphisms on $\mathbb{P}^2$ were classified by Fornaess-Sibony, Ueda and Rong: the only possibility is super-attracting basins. We will discuss the structure of the Julia sets for post-critically finite endomorphisms on $\mathbb{P}^2$. More background on holomorphic dynamics (both one-dimensional and higher dimensional) will be given in Chapter 1.

There are three different projects in this thesis. In Chapter 2 we give a generalization of Sullivan's non-wandering domain theorem for some polynomial skew products. Polynomial skew products can be seen as a subclass of holomorphic endomorphisms on $\mathbb{P}^2$, which are intermediate between one-dimensional and two-dimensional maps. We classify the dynamics on the Fatou set for strongly attracting polynomial skew products.

In Chapter 3 we introduce some non-uniform hyperbolicity techniques for attracting polynomial skew products. We give a description of the dynamics for a.e. point in the Lebesgue sense, under the so called Topological Collet-Eckmann condition and the Weakly Regular condition for the dynamics on an invariant vertical line. As a corollary, we also show there are no wandering Fatous component in this case.

In Chapter 4 we discuss the structure of the Julia sets for post-critically finite endomorphisms on $\mathbb{P}^2$. Post-critically finite endomorphisms on $\mathbb{P}^2$ are holomorphic endomorphisms on $\mathbb{P}^2$ which have simple critical behavior, so their dynamics are expected to be better understood than general holomorphic endomorphisms.

1. Non-wandering Fatou components for polynomial skew products

In one-dimensional complex dynamics, Sullivan's no wandering domain theorem [41] asserts that every Fatou component of a rational function is pre-periodic. Together with the earlier results of classification of invariant Fatou component by Fatou, Siegel and Herman, the dynamics on the Fatou set of a rational function is completely clear: a Fatou component is pre-periodic to either an attracting basin, a parabolic basin, or a rotation domain.

The same question naturally arise in higher dimension, i.e. to classify the Fatou components for holomorphic endomorphisms on $\mathbb{P}^k$, $k \geq 1$. An invariant Fatou component Ω is called *recurrent* if there is a $x \in \Omega$ such that the ω-limit set $\omega(x)$ satisfies $\omega(x) \cap \Omega \neq \emptyset$. The classification of invariant recurrent Fatou components was proved by Fornaess-Sibony [20] when $k = 2$ and by Forneass-Rong [17] for general k. However the non-recurrent invariant case still remains open, some conditional results were proved by Lyubich-Peters [33]. On the other hand, one may expect that Sullivan's no wandering domain theorem holds in higher dimension, but it turns out that it is not the case. The following result was recently proved by Astorg-Buff-Dujardn-Peters-Raissy [2]:

THEOREM 1.1. *There exist a holomorphic endomorphism f on $\mathbb{P}^2$ of degree ≥ 2 induced by a parabolic polynomial skew product, possessing a wandering Fatou component.*

Note that from this theorem, wandering Fatou components can be constructed for holomorphic endomorphisms on $\mathbb{P}^k$ for any $k \geq 2$ by taking product. In the next few lines we explain the definition of parabolic polynomial skew product .

A *polynomial skew product* on $\mathbb{C}^2$ is a map of the following form:

$$f(z, w) = (p(z), q(z, w)),$$

where p is a one variable polynomial of degree ≥ 2 and q is a two variables polynomial of degree ≥ 2. When $\deg p = \deg q = d$ and $q(z, w) = a_d w^d + O(w^{d-1})$, f can be extended holomorphically to $\mathbb{P}^2$. Such a polynomial skew product is called a *regular polynomial skew product.*

To investigate the Fatou set of polynomial skew product f, let π_1 be the projection to the z-coordinate, i.e.

$$\pi_1 : \mathbb{C}^2 \to \mathbb{C}, \;\; \pi_1(z, w) = z.$$

We first notice that $\pi_1(F(f)) \subset F(p)$, and passing to some iterate of f, by Sullivan's non-wandering domain theorem, we may assume that the points in $F(p)$ will eventually land into an immediate basin or a Siegel disk (no Herman rings for polynomials), thus we only need to study the following semi-local case:

$$f = (p, q) : \Delta \times \mathbb{C} \to \Delta \times \mathbb{C},$$

where $0 \in \Delta$, $p(0) = 0$, which means the line $L : \{z = 0\}$ is invariant, and Δ is an immediate attracting or a parabolic basin or a Siegel disk of p. The map f is called *attracting, parabolic or elliptic* respectively according to the cases where the fixed point is attracting, parabolic or elliptic.

As we have mentioned, the wandering Fatou component constructed in Theorem 1.1 is for a parabolic polynomial skew product. At this stage, it is interesting to investigate the existence of wandering Fatou component for attracting polynomial skew products and for elliptic polynomial skew products. We mention several results in the next subsection.

1.1. Earlier results on wandering domain problem for polynomial skew products. Prior to this theorem, the investigation of wandering Fatou components for holomorphic endomorphisms on $\mathbb{P}^2$ was indeed mostly restricted to the class of polynomial skew products.

The first result of non-wandering Fatou components for polynomial skew products is due to Lilov [32]. Let

$$f = (p, q) : \Delta \times \mathbb{C} \to \Delta \times \mathbb{C},$$

be a polynomial skew product such that $p(0) = 0$. We begin with a definition.

DEFINITION 1.2. *We say a disk $D \subset \mathbb{C}^2$ is* vertical *if $D \subset \{z\} \times \mathbb{C}$ for some $z \in \mathbb{C}$. A vertical Fatou disk D is called a* vertical Fatou disk *if $\{f^n|_D\}$ is a normal family.*

Lilov's result is the following, we note that his result stress no conditions on $f|_L$:

THEOREM 1.3. *Let $f = (p, q)$ be a super-attracting polynomial skew product, i.e. $p'(0) = 0$, then there is no wandering Fatou component in the attracting basin of $L = \{z = 0\}$. Moreover, every Fatou component of $f|_L$ extend to a two-dimensional Fatou component of f (called a bulging Fatou component), and every vertical Fatou disk is contained in such a bulging Fatou component.*

We will see that the bulging property of Fatou components of $f|_L$ holds for every attracting polynomial skew product as well.

After Lilov, a result due to Peters-Vivas [38] implies that the problems of wandering Fatou components for attracting polynomial skew products is more difficult than expected, and the method used by Lilov can not easily be generalized to attracting but not super-attracting polynomial skew products.

THEOREM 1.4. *There exist an attracting polynomial skew product f and a vertical Fatou disk D such that D is not contained in a bulging Fatou component. In fact the $\omega-$limit set of D is contained in the Julia set of $f|_L$.*

We note that Peters-Vivas did not answer the question of existence of wandering Fatou components for attracting polynomial skew products, as they also proved in the same paper that the vertical Fatou disk they constructed is contained in $J(f)$.

On the other hand, it seems that putting some conditions on $f|_L$ makes the non-wandering domain theorem for attracting polynomial skew products easier to prove. This is indeed the case when $f|_L$ is hyperbolic. We say a rational function g is hyperbolic if g expands uniformly a Riemannian metric in a neighborhood of $J(g)$.

THEOREM 1.5. *Let f be an attracting polynomial skew product such that $f|_L$ is hyperbolic. Then there is no wandering Fatou component in the attracting basin of L.*

PROOF. Assume by contradiction that Ω is a wandering Fatou component, since Ω is not a bulging Fatou component, it is easy to see that the ω-limit set of any point in Ω should be contained in $J(f|_L)$, the Julia set of $f|_L$. Since $J(f|_L)$ is a hyperbolic set, by the shadowing lemma ([29] Theorem 18.1.3), any point $x \in \Omega$ is contained in a stable manifold of some $y \in J(f|_L)$. In particular the Lyapunov exponent of x in the vertical direction is positive (since x and y have same value of Lyapunov exponent in the vertical direction). Thus $x \in J(f)$, a contradiction. $\square$

The next result due to Peters-Smit relax the hyperbolic condition on $f|_L$ by sub-hyperbolic condition. By using different method than Lilov's, Peters-Smit [37] proved no wandering Fatou components for attracting polynomial skew products, under the sub-hyperbolic (weaker than hyperbolic) condition of $f|_L$.

THEOREM 1.6. *Let f be an attracting polynomial skew product such that $f|_L$ is sub-hyperbolic, then there is no wandering Fatou component in the attracting basin of $L = \{z = 0\}$.*

We note that for example in the Mandelbrot set, there are countably many sub-hyperbolic parameters which are not (uniformly) hyperbolic.

The elliptic case has been studied by Peters-Raissy [36].

THEOREM 1.7. *Let $f = (p, q)$ be an elliptic polynomial skew product such that $f|_L$ admits no critical point in the Julia set of $f|_L$, and $p'(0) = e^{i\pi\alpha}$ is such that α is a Brjuno number, then in a small neighborhood of $L = \{z = 0\}$ there is no wandering Fatou component.*

Finally we mention that recently Astorg-Boc Thaler-Peters [3] gave a new construction (slightly different than [2]) of wandering Fatou components for parabolic polynomial skew products. We also note that in the context of complex Hénon maps, there are some positive and negative results for the wandering domain problem. There are no wandering Fatou components for hyperbolic Hénon maps (Bedford-Smillie [5]) and for substantially dissipative partially hyperbolic Hénon maps (Lyubich-Peters [34]). However, recently Berger-Biebler proved that complex Hénon maps can have wandering Fatou components [6].

1.2. New results. In this subsection we summarize the main results in Chapters 2 and 3. The reference papers are [26] and [25].

Let f be an attracting polynomial skew product, that is, f is a polynomial skew product on $\mathbb{C}^2$ such that

$$f(z, w) = (p(z), q(z, w)), \tag{1.1}$$

and p satisfies

$$p(0) = 0, |p'(0)| < 1.$$

Let $L = \{z = 0\}$ be the invariant vertical line.

In Chapter 2 we first prove the following result.

PROPOSITION A. *Let f be an attracting polynomial skew product, let $g = f|_L$ which can be seen as a one variable polynomial map. Then every Fatou component of g in L is contained in a two-dimension Fatou component of f. Such Fatou components of f are called bulging Fatou components, and they are non-wandering.*

Then we prove the following result, which is a generalization of Lilov's theorem [32], and is the main result of the paper [26].

THEOREM B. *Let $f = (p, q)$ be an attracting polynomial skew product, there is a constant $\delta(q) > 0$ depending on q such that if $|p'(0)| < \delta$, there is no wandering Fatou component in the attracting basin of L.*

We note that the above result will be proved in Chapter 2, Theorem 6.4, while the "main theorem" in Chapter 2 is a local version of the above result.

We actually show a more precise version of Theorem B, i,e, we prove the non-existence of wandering vertical Fatou disk (see Definition 1.2) for strongly attracting polynomial skew products:

THEOREM C. *Let $f = (p, q)$ be an attracting polynomial skew product, there is a constant $\delta(q) > 0$ depending on q such that if $|p'(0)| < \delta$, then every vertical Fatou disk belongs to the attracting basin of L is contained in a bulging Fatou component. In particular in this situation, every Fatou component of f in the attracting basin of L is a bulging Fatou component, hence non-wandering.*

Let us now discuss the results in Chapter 3. First we introduce some *non-uniformly hyperbolic conditions* in one variable complex dynamics. Non-uniformly hyperbolic theory, also known as Pesin theory, is a generalization of uniformly hyperbolic theory. In Pesin theory we only require an invariant hyperbolic measure rather than the presence of invariant expanding and contracting directions.

In the one-dimensional complex setting, non-uniformly hyperbolic theory here is slightly different than the general setting. There are various non-uniformly hyperbolic conditions for rational functions such as sub-hyperbolicty, semi-hyperbolicity, Collet-Eckmann condition (CE for short), and Topological Collet Eckmann condition (TCE for short). These conditions are quantitative refinements of general Pesin theory. Among these non-uniformly hyperbolic condition, the weakest one is the TCE condition. We begin with its definition

DEFINITION 1.8 (Przytycki-Rivera Letelier-Smirnov [39]). *A rational function f on $\mathbb{P}^1$ of degree ≥ 2 is called TCE if there exist $\mu > 1$ and $r > 0$ such that for every $x \in J(f)$, $n \geq 0$ and every connected component W of $f^{-n}(B(x, r))$ we have*

$$\mathit{diam}\, W \leq \mu^{-n},$$

where $B(x, r)$ denotes the ball centered at x with radius r.

There are various equivalent definitions of TCE condition, see Przytycki-Rivera Letelier-Smirnov [39], which makes it in a sense the most natural non-uniformly hyperbolic condition for rational functions. Note that by [39] TCE condition is preserved under topological conjugacy.

The following condition is about the slow recurrence of critical points lying in Julia set.

DEFINITION 1.9. *A rational map f is called Weakly Regular (WR for short) if for all critical values $v \in CV(f)$ whose forward orbit does not meet critical points we have*

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \frac{1}{n} \sum_{\substack{j=0 \\ d(f^j(v), C(f) \cap J(f)) \leq \eta}}^{n-1} -\log |f'(f^j(v))| = 0.$$

Here $C(f)$ is the critical set and $CV(f)$ is the critical value set.

This condition means that for every $v \in CV(f) \cap J(f)$: the orbit of v does not come either too close nor too often to $C(f) \cap J(f)$.

The following condition Positive Lyapunov is stronger than CE.

DEFINITION 1.10. *A rational map f is called Positive Lyapunov if for every point $c \in C(f) \cap J(f)$ whose forward orbit does not meet other critical points the following limit exists and is positive*

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(f(c))| > 0.$$

In addition we ask that there are no parabolic cycles.

We are now in position to state the main theorem in Chapter 3.

THEOREM D. *Let f be an attracting polynomial skew product, let $g = f|_L$. Assume g satisfies one of the following conditions.*

(1) g is TCE and WR,

(2) g is Positive Lyapunov,

Then the Fatou set $F(f)$ in the attracting basin of L is the union of basins of attracting cycles, and the Julia set $J(f)$ in the attracting basin of L has zero Lebesgue measure.

As a corollary of the above theorem, we have the following generalization of Peters-Smit's result (Theorem 1.6).

COROLLARY E. *Let f be an attracting polynomial skew product, let $g = f|_L$. Assume g satisfies one of the following conditions.*

(1) g is TCE and WR,

(2) g is Positive Lyapunov,

Then there is no wandering Fatou component in the attracting basin of L.

To the best of our knowledge, Theorem D is the first time where the zero measure of Julia set is shown for a non-hyperbolic g (note that this is not expected to be true when no conditions of p are assumed, as even in one dimension Julia set can has positive Lebesgue measure, cf. Buff-Chéritat [8] and Avila-Lyubich[4]). The previous results we have mentioned only consider the dynamics on the Fatou set. However in the case g is uniformly hyperbolic, it is well-known that the Julia set has zero Lebesgue measure. In fact by the proof of Theorem 1.5, $J(f)$ in the attracting basin of L equals to the stable set of $W^s(J(g))$, where

$$W^s(J(g)) = \left\{x \in \mathbb{C}^2 : \text{dist}\,(f^n(x), J(g)) \to 0 \text{ when } n \to +\infty\right\}.$$

By the standard theory of hyperbolic set, $W^s(J(g))$ is foliated by stable manifolds and this foliation is absolutely continuous (cf. Young [44] Definition 6.2.5 and Theorem 6.2.6). Since $J(g)$ has zero area in L , $W^s(J(g))$ has Lebesgue measure zero in $\mathbb{C}^2$.

Based on the results of Graczyk-Swiatek [22] and [23], we also show that the one-dimensional conditions in Theorem 3.6 are satisfied by generic parameters in the unicritical family.

THEOREM F. *In the uni-critical family* $\{f_c(z) = z^d + c\}$, $d \geq 2$, *a.e.* $x \in \partial\mathcal{M}_d$ *in the sense of harmonic measure satisfies WR and TCE condition, or the Positive Lyapunov condition.*

Here $\mathcal{M}_d$ is the connectedness locus, when $d = 2$ this is the well-known Mandelbrot set.

1.3. Further discussion. There are several possible directions to generalize the non-wandering domain theorem proved in Theorem B and Theorem D. These two theorems both require f to be an attracting polynomial skew product, and also some additional condition such as the attracting rate being sufficiently large or f satisfying some non-uniformly hyperbolic condition. It is natural to ask whether such assumptions are necessary.

Problem 1: Let f be an attracting polynomial skew product of the form (1.1), can f have a wandering Fatou component in the basin of L without a smallness assumption on Theorem B. The argument in [26] does not apply, since in this general case a wandering Fatou disk can exist [37]. So new ideas need to be developed.

On the other hand, one may ask that whether Theorem B and Theorem D still holds, when we do not have a skew product structure, but still have an invariant projective line. This situation happens for an important subclass of holomorphic endomorphisms on $\mathbb{P}^2$: when f is a *regular polynomial endomorphism on* $\mathbb{C}^2$ (that is, a polynomial endomorphism on $\mathbb{C}^2$ extend holomorphically to $\mathbb{P}^2$), the line at infinity L_∞ is an invariant super-attracting projective line. In this case however we do not have any skew product structure.

Problem 2: Let f be a regular polynomial endomorphism on $\mathbb{C}^2$. In this case f has an attracting set L_∞, which is isomorphic to $\mathbb{P}^1$. Can f have a wandering Fatou component in the attracting basin of L_∞? If in addition $f|_{L_\infty}$ satisfies the TCE condition, is it true that the Fatou set in the attracting basin of L_∞ is the union of basins of attracting cycles, and Fatou set is either empty or has full Lebesgue measure in the attracting basin of L_∞?

2. Julia sets of post-critically finite endomorphisms on $\mathbb{P}^2$

A holomorphic endomorphism f on $\mathbb{P}^k$ is called post-critically finite (PCF for short) if the post-critical set

$$PC(f) := \bigcup_{n\geq 1} f^n(C(f))$$

is an algebraic curve of $\mathbb{P}^k$. Here

$$C(f) := \left\{x \in \mathbb{P}^k : Df(x) \text{ is not invertible }\right\}$$

is the critical set.

In dimension 1 , this coincides with the usual definition of PCF maps on the Riemann sphere $\mathbb{P}^1$. These play an important role in one-dimensional complex dynamics, mainly because the remarkable topological classification theorem of Thurston [14]. Let $\mathcal{M}_d$ be the moduli space of degree d rational functions. PCF maps are also important because they are very regularly distributed in $\mathcal{M}_d$: except the well-known flexible Lettès families, the set of PCF rational functions is a countable union of 0-dimensional varieties in $\mathcal{M}_d$ (a corollary of Thurston's result [14]). Moreover, every hyperbolic component in $\mathcal{M}_d$ with connected Julia set contains exactly one PCF map (McMullen [35]).

One may expect the similar properties of PCF endomorphisms still hold in higher dimension. The dynamics of PCF endomorphisms in higher dimension have been investigated by many authors. Let us start by discussing theorems related to Fatou and Julia sets for PCF endomorphisms on $\mathbb{P}^2$.

2.1. Earlier results on Fatou and Julia sets for post-critically finite endomorphisms on $\mathbb{P}^2$. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic endomorphism of degree ≥ 2, where $\mathbb{P}^2$ is the complex projective plane. The first *Julia set* J_1 is defined as the locus where the iterates $(f^n)_{n\geq 0}$ do not locally form a normal family, i.e. the complement of the *Fatou set*. Let T be the *dynamical Green current* of f, defined by $T = \lim_{n\to+\infty} d^{-n}(f^n)^*\omega$, where ω is the Fubini-Study (1,1) form on $\mathbb{P}^2$. The Julia set J_1 coincides with Supp (T), and the self intersection measure $\mu = T \wedge T$ is the unique measure of maximal entropy of f. See Dinh-Sibony [13] for background on holomorphic dynamics on projective spaces and for precise definitions.

We define the second Julia set to be $J_2 = \text{Supp}\ \mu$. From the definitions we know that $J_2 \subset J_1$. By Briend-Duval [7], J_2 is contained in the closure of the set of repelling periodic points. However contrary to dimension one there may exists repelling periodic point outside J_2 (see [24] and [21]).

The Fatou set of PCF endomorphisms on $\mathbb{P}^2$ have been studied by Fornaess-Sibony [19], Ueda [43] and Rong [40]. We have the following classification of Fatou components for PCF endomorphisms on $\mathbb{P}^2$ by Rong [40], which is not known for general holomorphic endomorphisms on $\mathbb{P}^2$.

THEOREM 2.1. *Let f be a PCF endomorphism on $\mathbb{P}^2$, then $F(f)$ is the union of attracting basins of super-attracting cycles.*

Next we study the dynamics on the Julia set for PCF endomorphisms on $\mathbb{P}^2$. The following definition was introduced by Ueda [43].

DEFINITION 2.2. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. A point q is said to be a point of bounded ramification if the following conditions are satisfied:*

(1) There exists a neighborhood W of q such that $PC(f) \cap W$ is an analytic subset of W.

(2) There exists an integer m such that for every $j > 0$ and every $p \in f^{-j}(q)$, we have that $\text{ord}\left(f^j, p\right) \leq m$.

In the case $k = 2$, the following characterization of points of bounded ramification for PCF endomorphism on $\mathbb{P}^2$ is due to Ueda [43].

LEMMA 2.3. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Then the points with unbounded ramification are the union of critical component cycles and critical point cycles.*

The above definition and theorem lead to the following definition introduced by Ueda [42].

DEFINITION 2.4. *Let f be a PCF endomorphism on $\mathbb{P}^k$ of degree ≥ 2. f is called strictly PCF is every point in $\mathbb{P}^k$ is of bounded ramification.*

In dimension 1, this definition is equivalent to PCF maps with $J(f) = \mathbb{P}^1$, also known as expanding Thurston maps.

The Julia set J_2 for strictly PCF endomorphisms on $\mathbb{P}^2$ was studied by Jonsson [28].

THEOREM 2.5. *Let f be a strictly PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Then $J_2 = \mathbb{P}^2$.*

We note that this result was generalized by Ueda [42] to arbitrary dimensions $k \geq 2$, i.e. $J_k = \mathbb{P}^k$ for strictly PCF endomorphism on $\mathbb{P}^k$, where J_k is the support of the unique measure of maximal entropy.

The dynamics of $J_1 \setminus J_2$ for PCF endomorphism on $\mathbb{P}^2$ was studied by de Thélin [12]. We start with a definition

DEFINITION 2.6. *Let f be a holomorphic endomorphism on $\mathbb{P}^2$ of degree ≥ 2. The dynamical Green current T is called laminar in an open set Ω if it expresses as an integral of integration currents over a measurable family of compatible holomorphic disks in Ω.*

Here compatible means these disks have no isolated intersections. De Thélin [11] gave a criterion for a current expressed as a limit of curves in a ball to be laminar. He proved the following result [12] by using his criterion of laminarity.

THEOREM 2.7. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. The the dynamical Green current T is laminar in $J_1 \setminus J_2$.*

We note that the above result (laminarity of dynamical Green current on $J_1 \setminus J_2$) does not hold for general holomorphic endomorphism on $\mathbb{P}^2$, as showed by Dujardin [16], but a related weaker result hold [15] (existence of Fatou direction).

The eigenvalues of fixed points of PCF endomorphism on $\mathbb{P}^2$ was recently studied by Le [31].

DEFINITION 2.8. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. A fixed point x_0 is called repelling if all eigenvalues of Df at x_0 have modules larger than* 1. *A fixed point x_0 is called super-saddle if Df at x_0 has one 0 eigenvalue and one eigenvalue with modulus larger than 1. A fixed point x_0 is called super-attracting if Df at x_0 has only 0 as eigenvalues.*

THEOREM 2.9. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Then every fixed point of f is either repelling, super-saddle or super-attracting.*

It is an important problem to find examples of PCF endomorphisms in higher dimensions. In dimension 1 this is solved by Thurston [14]. In higher dimension non-trivial examples of PCF endomorphisms were constructed by Crass [9], Fornaess-Sibony [18] and Koch [30].

Finally we mention that the the dynamics of PCF endomorphisms on $\mathbb{P}^k$, $k \geq 2$ was studied by Astorg [1] and Ueda [42].

2.2. New results. In this subsection we summarize the main result in chapter 4, the reference paper is [27]. The purpose of this paper is towards understanding the structure of the Julia set of PCF endomorphisms on $\mathbb{P}^2$. We begin with a definition.

DEFINITION 2.10. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2.. We call an irreducible component Λ of $C(f)$ periodic if there exist an integer $n \geq 1$ such that $f^n(\Lambda) = \Lambda$. Such an irreducible component will be called* periodic critical component. *The set $\{\Lambda, f(\Lambda), \ldots, f^{n-1}(\Lambda)\}$ is called a* critical component cycle. *Similarly, a critical point x satisfying $f^n(x) = x$ for some $n \geq 1$ is called a* periodic critical point. *The set $\{x, f(x), \ldots, f^{n-1}(x)\}$ is called a* critical point cycle.

The critical component cycle is an attracting set as proved by Fornaess-Sibony in [19]. Daurat [10] proved that the dynamical Green current T is laminar in the attracting basin of a critical component cycle.

Now we propose the following conjecture.

CONJECTURE G. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Then $J_1 \setminus J_2$ is contained in the attracting basins of critical component cycles.*

We will establish some intermediate results or some conditional results towards understanding the above conjecture. These results are the main theorems in Chapter 4.

The first result is about repelling cycles and J_2.

THEOREM H. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, then J_2 is the closure of the set of repelling periodic points. Moreover if all branches of $PC(f)$ are smooth and intersect transversally, then any periodic point in J_2 is repelling.*

Here are some comments about Theorem H. First, we note that repelling periodic point may not be contained in J_2 for general holomorphic endomorphisms on $\mathbb{P}^2$. Indeed there are examples with isolated repelling points outside J_2, see Fornaess-Sibony [21] and Hubbard-Papadopol [24]. Second, the assumption that all branches of $PC(f)$ are smooth and intersect transversally are satisfied by examples constructed by Crass [9], Fornaess-Sibony [19] and Koch [30]. Third, we note that the first part of Theorem H fits the picture of Conjecture G, since repelling periodic points does not belongs to the attracting basin of a critical component cycle.

Next we state a structure theorem of $J_1 \setminus J_2$ for PCF endomorphism on $\mathbb{P}^2$. Let f be a PCF endomorphism on $\mathbb{P}^2$, for point in $J_1 \setminus J_2$ which is not contained in an attracting basin of a critical component cycle, for a hypothetical point we show

THEOREM I. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree $d \geq 2$. Let $x \in J_1 \setminus J_2$ which is not contained in an attracting basin of a critical component cycle, then there is a Fatou disk D passing through x, i.e. the family $\{f^n|_D\}_{n\geq 1}$ is normal.*

We note that by the result of de Thélin [12], for σ_T a.e. point in $J_1 \setminus J_2$, there is a Fatou disk D passing through x. The above theorem together with Daurat's result about laminarity of Green current in the attracting basin gives a new proof of this fact.

We also prove a partial converse of the result of Theorem 2.5.

THEOREM J. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2 such that branches of $PC(f)$ are smooth and intersect transversally. Then $J_2 = \mathbb{P}^2$ if and only if f is strictly PCF.*

Finally we show that Conjecture G is true, when $PC(f) \subset C(f)$.

THEOREM K. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Assume that $PC(f) \subset C(f)$. Let B be the union of attracting basins of critical component cycles, then $J_2 = \mathbb{P}^2 \setminus B$, and J_2 is a repeller, i.e. there exists $k \geq 1$ and $\lambda > 1$ such that for every $x \in J_2$, for every $v \in T_x\mathbb{P}^2$ we have $\left|Df^k(v)\right| \geq \lambda|v|$.*

We show Conjecture G is true under some additional condition (which is conjectured to be always true):

PROPOSITION L. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Assume every super-saddle cycle is contained in a critical component cycle and f satisfies the "backward contracting property". Then $J_1 \setminus J_2$ is contained in the attracting basins of critical component cycles.*

We refer to chapter 4 for the definition of backward contracting property.

2.3. Further discussion. In Theorem H we proved that the repelling cycles of a PCF endomorphism on $\mathbb{P}^2$ are contained in J_2. On the other hand, Le [31] proved that every periodic point of a PCF endomorphism on $\mathbb{P}^2$ is either repelling, super-saddle or super-attracting. As a corollary, every periodic point in J_2 which is not in the critical set is repelling. If we consider invariant ergodic measure instead of periodic points, A natural question would be:

Problem 3: Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, let μ be an invariant ergodic measure such that μ does not have $-\infty$ Lyapunov exponent. Is that true that Supp $\mu \subset J_2$ and the Lyaponov exponents of μ are positive?

For one-dimensional PCF maps, the measurable dynamics are well known. Let f be a PCF map on $\mathbb{P}^1$, then either $J(f) = \mathbb{P}^1$ and f has an invariant ergodic measure equivalent to Lebesgue, or $J(f) \neq \mathbb{P}^1$ and $J(f)$ has Lebesgue measure zero. A natural question in dimension 2 would be:

Problem 4: Let f be a PCF endomorphisms on $\mathbb{P}^2$ of degree ≥ 2. If $J_2 = \mathbb{P}^2$, is there an invariant ergodic measure μ that is equivalent to Lebesgue measure on $\mathbb{P}^2$? If $J_2 \neq \mathbb{P}^2$, does J_2 have Lebesgue measure zero?

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CHAPTER 1

Background in holomorphic dynamics

1. Dynamics in one complex variable

In this subsection we studies the dynamics of rational functions on Riemann sphere $\mathbb{P}^1$. The study of local holomorphic dynamics around a fixed point was developed in the late 19th century. However the study of global dynamics of rational function on Riemann sphere dates back to Fatou and Julia in the beginning of 20th century. Begin in the early 1980s, the recommend mathematicians such as Douady, Hubbard, Sullivan, Milnor etc made great progress in this classical subject.

1.1. Fatou-Julia dichotomy. The Riemann sphere naturally divides into two parts: the *Fatou set*, where the dynamics is stable, and the *Julia set*, where the dynamics is chaotic.

DEFINITION 1.1. *Let f be a rational function on $\mathbb{P}^1$ of degree ≥ 2. The Fatou set $F(f)$ is the largest open set where the family of iterates $\{f^n\}_{n\geq 1}$ is a normal family. The Julia set $J(f)$ is the completement of $F(f)$.*

We have the following fundamental property of Fatou and Julia set, see Milnor [19] for the general background on dynamics in one complex variable.

PROPOSITION 1.2. *Let f be a rational function on $\mathbb{P}^1$ of degree $d \geq 2$. Then:*

(1) $F(f)$ is an open set and $J(f)$ is a non-empty closed set. $F(f)$ is a dense open set provided $F(f) \neq \emptyset$.

(2) $F(f)$ and $J(f)$ are totally invariant, i.e. $f^{-1}(F(f)) = F(f)$ and $f^{-1}(J(f)) = J(f)$.

(3) For any integer $k \geq 0$, $F(f^k) = F(f)$ and $J(f^k) = J(f)$.

(4) $J(f)$ is the closure of the set of repelling cycles, and there are at most $2d - 2$ non-repelling cycles.

(5) There is a unique measure of maximal entropy $\mu(f)$ (with entropy $\log d$). Moreover μ is mixing, has Hölder continuous local potential, has positive Lyapunov exponent, and Supp $\mu = J(f)$.

1.2. Classification of Fatou components. Since $F(f)$ is an open set, it leads to the following definition.

DEFINITION 1.3. *A Fatou component is a connected component of $F(f)$. A Fatou component Ω is called non-wandering if it is pre-periodic, i.e. there exist $k \geq 0, p \geq 0$ such that $f^{p+k}(\Omega) = f^k(\Omega)$. A Fatou component Ω is called wandering if it is not pre-periodic.*

To understand the dynamics on a non-wandering Fatou component, it is enough to study the dynamics on an invariant Fatou component, i.e. Fatou component Ω such that $f(\Omega) = \Omega$. We have the following classical result due to Fatou, Siegel and Herman.

THEOREM 1.4. *Let f be a rational function on $\mathbb{P}^1$ of degree $d \geq 2$. Let Ω be an invariant Fatou component. Then one of the following holds.*

(1) Ω is a super-attracting basin, i.e. there is a fixed point $p \in \Omega$ such that $Df(p) = 0$, and f is locally conjugate to $z \mapsto z^m$ for some integer $m \geq 2$.

(2) Ω is an attracting basin, i.e. there is a fixed point $p \in \Omega$ such that $Df(p) = \lambda$ satisfies $0 < |\lambda| < 1$, and f is locally conjugate to $z \mapsto \lambda z$.

(3) Ω is a parabolic basin, i.e. there is a fixed point $p \in \partial\Omega$ such that $Df(p) = \lambda$ satisfies $\lambda = e^{i\pi p/q}$, where p, q are integers, and f is semi-conjugate to $z \mapsto z + 1$ on $\mathbb{C}$ (the Fatou coordinate).

(4) Ω is a Siegel disk, i.e. Ω is biholomorphic to a disk, and there is a fixed point $p \in \Omega$ such that $Df(p) = \lambda$ satisfies $\lambda = e^{i\pi\alpha}$, where α is irrational, and f is conjugate to $z \mapsto \lambda z$.

(5) Ω is a Herman ring, i.e. Ω is biholomorphic to an annulus, and there exists $\lambda = e^{i\pi\alpha}$, where α is irrational, and f is conjugate to $z \mapsto \lambda z$.

We note that the Fatou coordinate in (3) was contructed by Fatou. The first examples of Siegel disks were constructed by Siegel and the examples of Herman rings were constructed by Herman. See [19] for a self-contained proof of the theorem.

At this stage the question of existence of wandering Fatou component was considerably very important, and the answer was unknown until the 1980's. In the seminal paper [26], Sullivan proved such wandering Fatou component can not exist for rational functions.

THEOREM 1.5. *Let f be a rational function on $\mathbb{P}^1$ of degree ≥ 2. Then there is no wandering Fatou component.*

Combine Theorem 1.4 and Theorem 1.5 we get a complete description of dynamics on $F(f)$ for rational function.

Note that for other classes of mappings (meromorphic functions or transcendental entire functions), there are classes with or without wandering Fatou components. The study of Fatou components is still an active topic for these classes of maps.

1.3. Hyperbolic rational functions. In general dynamical systems theory of smooth maps, a hyperbolic set with respect to some smooth map f is an invariant compact set such that its tangent bundle may be split into two invariant subbundles, one of which is contracting and the other is expanding under f. Hyperbolic set is interesting because one can understand the dynamics on it completely.

In the one dimensional complex setting, one of the most important problem in one-dimensional complex dynamics, usually refers to the Fatou conjecture, is whether the *hyperbolic rational functions* form a dense subset in the parameter space of given degree. We begin with the following definition

DEFINITION 1.6. *A rational function f on $\mathbb{P}^1$ of degree ≥ 2 is called hyperbolic if one of the following equivalent conditions holds:*

(1) The post critical set satisfies $\overline{PC(f)} \cap J(f) = \emptyset$, where $PC(f) = \bigcup_{n\geq 1}^{\infty} f^n(C(f))$ and $C(f)$ is the critical set of f.

(2) f has only attracting cycles and repelling cycles, and every critical point $c \in C(f)$ lies in an attracting basin of an attracting cycle.

(3) $J(f)$ is a hyperbolic set.

From the definition we know that a small pertubation of a hyperbolic rational function is still hyperbolic. See Milnor [19] for more details.

The measurable dynamics of hyperbolic rational functions is well understood.

THEOREM 1.7. *Let f be a hyperbolic rational function on $\mathbb{P}^1$. Then*

(1) The Hausdorff dimension of $J(f)$ is equal to the box dimension of $J(f)$ and is smaller than 2.

(2) There exist a unique invariant probability measure ν which is equivalent to the $\delta-$dimensional Hausdorff meausre, where δ is the Hausdorff dimension of $J(f)$. Moreover ν is mixing and has positive Lyapunov exponent.

For the proof of Theorem 1.7 see Przytycki-Urbański [22]. The non-wandering set for a hyperbolic rational function f is the union of $J(f)$ and the attracting cycles. It turns out that f satisfies Smale's axiom A. Let Rat_d denote the parameter space of degree d rational functions .As a corollary we have

THEOREM 1.8. *Let f be a hyperbolic rational function of degree d, then f is structurally stable, i.e. let $g \in Rat_d$ be a small perturbation of f, then f and g are topologically conjugate on their respective non-wandering set.*

The concept of structurally stable rational functions is closely related to the concept of J- *stable* rational functions.

DEFINITION 1.9. *A rational function f on $\mathbb{P}^1$ of degree $d \geq 2$ is called J-stable if one of the following equivalent conditions holds:*

(1) f is structurally stable.

(2) There exist a holomorphic motion of the Julia sets on a neighborhood U of f in Rat_d.

(3) The number of attracting cycles is constant on a neighborhood U of f in Rat_d.

There are various of equivalent definition of structurally stable rational functions, see Mané-Sad-Sullivan [18] and Lyubich [16] for the background and the definition of holomorphic motion. We have the following results due to Mané-Sad-Sullivan [18] and Lyubich [16].

THEOREM 1.10. *The set of J-stable rational functions is an open and dense subset in Rat_d, for every $d \geq 2$.*

It is conjectured that every $J-$stable rational function is actually hyperbolic. This conjecture is equivalent to the Fatou conjecture.

1.4. Non-uniformly hyperbolic rational functions. Now we introduce another important class of rational functions, which are called non-uniformly hyperbolic rational functions. They are rational functions carrying some hyperbolicity, but are not hyperbolic. Non-uniformly hyperbolic rational functions are useful, for example when we want to construct an invariant measure which is absolutely continuous with respect to Lebesgue measure, or when we want to study the parameters on the boundary of the Mandelbrot set.

There are various non-uniformly hyperbolic conditions for rational functions, such as sub-hyperbolicity, semi-hyperbolicity, the Collet-Eckmann condition (CE for short) and the Topological Collet-Eckmann condition (TCE for short). The weakest notion among these conditions is the TCE condition.

DEFINITION 1.11. *A rational function f on $\mathbb{P}^1$ of degree ≥ 2 is called TCE if there exist $\mu > 1$ and $r > 0$ such that for every $x \in J(f)$, $n \geq 0$ and every connected component W of $f^{-n}(B(x,r))$ we have*

$$diam\, W \leq \mu^{-n},$$

where $B(x,r)$ denotes the ball centered at x with radius r.

The TCE condition was introduced by Przytycki-Rohde in [21]. There are various equivalent characterization of the TCE condition, see Przytycki-Smirnov-Rivera-Letelier [23].

The measurable dynamics of TCE rational functions is well understood.

THEOREM 1.12. *Let f be a TCE rational function, then*

(1) The Fatou set $F(f)$ is a union of attracting basins.

(2) Either $J(f) = \mathbb{P}^1$ or the Hausdorff dimension δ of $J(f)$ is equal to the box dimension of $J(f)$ and it is smaller than 2.

(3) There is a unique invariant probability measure ν such that Supp $\nu \subset J(f)$ and ν is equivalent to the unique conformal measure with exponent δ (the conformal measure with exponent 2 is the Lebesgue measure). Moreover ν is mixing and has positive Lyapunov exponent.

For the proof of Theorem 1.12 see Przytycki-Smirnov-Rivera Letelier [23] and Przytycki-Rivera Letelier [20].

We note that the CE condition is stronger than the TCE condition (see [21]). It was proved by Aspenberg [1] and Rees [24] that non-hyperbolic TCE rational functions are abundant, in the Lebesgue sense.

THEOREM 1.13. *The set of CE rational functions such that $J(f) = \mathbb{P}^1$ is a positive Lebesgue measure subset in Rat_d, for every $d \geq 2$.*

The CE condition is also generic in the sense of harmonic measure by Graczyk-Swiatek for unicritical family [13].

THEOREM 1.14. *In the uni-critical family $\{f_c(z) = z^d + c\}$, $d \geq 2$, a.e. $x \in \partial\mathcal{M}_d$ in the sense of harmonic measure satisfies CE condition, where $\mathcal{M}_d$ is the connectedness locus.*

The above result was recently generalized by de Thélin-Gauthier-Vigny [14] to arbitrary algebraic families of rational functions.

THEOREM 1.15. *Let Λ be a quasi-projective sub-variety in Rat_d. Let $c(\lambda)$ be a marked critical value which is not stably pre-critical. Let $T_{\Lambda,c}$ be the bifurcation current of the pair (Λ, c). Then a.e. parameter in Λ in the sense of trace measure of $T_{\Lambda,c}$ satisfies that c is CE, i.e. $|(f_\lambda^n)'(c_\lambda)| \geq C\mu^n$ for some $C > 0$, $\mu > 1$.*

We note that in the uni-critical family $\{f_c(z) = z^d + c\}$, $d \geq 2$, $T_{\Lambda,c}$ (= the trace measure) in the above theorem is same as the harmonic measure on $\mathcal{M}_d$.

For more about non-uniform hyperbolic rational functions and for the measurable dynamics on $J(f)$, we refer the reader to Przytycki and Rivera-Letelier [20], Graczyk and Smirnov [12], and Rivera-Letelier and Shen [25].

2. Dynamics in several complex variables

2.1. Generalities. In this section we concentrate on study the dynamics of holomorphic endomorphisms on $\mathbb{P}^k$. Holomorphic endomorphisms on $\mathbb{P}^k$ can be seen as a direct generalization of rational maps on $\mathbb{P}^1$, and it is an important subclass of higher dimensional holomorphic maps for the dynamical studies. Some other higher dimensional holomorphic dynamical systems include Hénon-like maps on $\mathbb{C}^k$ and regular automorphisms on projective surfaces, etc.

For a holomorphic endomorphisms on $\mathbb{P}^k$, we can define the Fatou set and the Julia set for holomorphic endomorphisms on $\mathbb{P}^k$ similarly as in Definition 1.1. For $k \geq 2$, pluripotential theory has been a key idea to the understanding of the dynamics of holomorphic endomorphisms on $\mathbb{P}^k$ (the idea of using pluripotential theory in complex dynamics dates back to Sibony). An important object in the theory is a dynamically defined positive closed current T of bidegree (1,1) on $\mathbb{P}^k$.

Definition 2.1. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree $d \geq 2$. Let ω be the Fubini-Study (1,1) form. Then the following limit exist (in the sense of (1,1) currents) and is called the dynamical Green current:*

$$T := d^{-n} \lim_{n\to+\infty} (f^n)^*(\omega).$$

We refer the reader to Dinh-Sibony [6] for general background on the dynamics of holomorphic endomorphisms on $\mathbb{P}^k$. We have the following important result, for more details see [6].

Theorem 2.2. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree $d \geq 2$ and let T be the dynamical Green current. Then:*

(1) T has Hölder continuous local potential.

(2) for $1 \leq p \leq k$, the power $T^p := T \wedge T ... \wedge T$, p factors are well defined, and its support J_p is called the Julia set of order p.

(3) T is supported on the Julia set of f, i.e. $\mathrm{Supp}\, T := J_1(f) = J(f)$.

(4) $\mu := T^k$ is the unique invariant probability measure with maximal entropy $k \log d$. Moreover μ is mixing and has positive Lyapunov exponents.

2.2. Equidistribution problems in $\mathbb{P}^k$. In this paragraph, we will see that the dynamical Green current T and the measure of maximal entropy μ have many equidistribution properties: the pull back of a generic positive closed (1,1) current by f^n are equidistributed with respect to T, the pull back of a generic point by f^n are equidistributed with respect to μ, and the set of repelling periodic points are equidistributed with respect to μ.

The following result is due to Briend-Duval [4] and Dinh-Sibony [6].

Theorem 2.3 (Equidistribution of preimage of points). *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree $d \geq 2$ and let μ be the measure of maximal entropy. Then there exists a proper algebraic set $\mathcal{E}$, possibly empty, such that*

$$\lim_{n\to+\infty} d^{-kn}(f^n)^*(\delta_a) = \mu,$$

if and only if $a \notin \mathcal{E}$, here δ_a is the Dirac mass. Moreover $\mathcal{E}$ is totally invariant, $f^{-1}(\mathcal{E}) = \mathcal{E}$ and $\mathcal{E}$ is maximal in the sense that if a proper algebraic set E satisfies $f^{-n}(E) = E$ for some $n \geq 1$, then $E \subset \mathcal{E}$.

The following result is due to Briend-Duval [3].

THEOREM 2.4 (Equidistribution of repelling periodic points). *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree $d \geq 2$ and let μ be the measure of maximal entropy. Let P_n denote the set of repelling periodic points of period n. Then we have*

$$\lim_{n\to+\infty} d^{-kn} \sum_{a\in P_n} \delta_a = \mu.$$

The following result was proved by Favre-Jonsson [8] for $k = 2$, and by Dinh-Sibony [5] for general k.

THEOREM 2.5 (Equidistribution of preimage of positive closed (1,1) currents). *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree $d \geq 2$ and let T be the dynamical Greem current. Let $\mathcal{E}_m$ denote the union of the totally invariant proper algebraic sets in $\mathbb{P}^k$ which are minimal, i.e. do not contain smaller ones. Let S be a positive closed (1,1) current of mass 1 on $\mathbb{P}^k$ whose local potential are not identically $-\infty$ on any component of $\mathcal{E}_m$. Then we have*

$$\lim_{n\to+\infty} d^{-n}(f^n)^*(S) = T.$$

As a corollary we have

COROLLARY 2.6 (Equidistribution of preimage of hypersurface). *Let f, T, $\mathcal{E}_m$ be as above. Let H be a hypersurface of degree s in $\mathbb{P}^k$ such that H does not contain any component of $\mathcal{E}_m$, then*

$$\lim_{n\to+\infty} s^{-1}d^{-n}(f^n)^*[H] = T.$$

It is important to understand the properties of the exceptional set in the above equidistribution theorems. In dimension 1, such exceptional set is the union of at most two points. For $k > 1$, it is conjectured that an totally invariant proper irreducible algebraic set is a projective subspace, and there are also conjectures about the number of irreducible components of the exceptional set, see the survey paper Dinh-Sibony [7]. We have the following partial result, which is useful for our purpose.

THEOREM 2.7. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. Let E be a proper algebraic set such that $f^{-1}(E) = E$. Then $E \subset C(f)$, where $C(f)$ is the critical set. When $k = 2$, a totally invariant algebraic curve is a union of at most three projective lines.*

For the proof see Briend-Duval [4] and Fornaess-Sibony [10].

2.3. Fatou components in $\mathbb{P}^k$. Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2, unlike the one dimension case, the classification of Fatou components is not completely known yet when $k \geq 2$. The classification of invariant *recurrent* Fatou components is known, but the classification of invariant *non-recurrent* Fatou components remain mysterious. The existence of wandering Fatou components was unknown until recently.

DEFINITION 2.8. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. An invariant Fatou component Ω is called recurrent, if there exist a point $x \in \Omega$ such that the ω-limit set satisfies $\omega(x) \cap \Omega \neq \emptyset$.*

The following classification of invariant recurrent Fatou components was proved by Fornaess-Sibony [11] for $k = 2$, and by Fornaess-Rong [9] for general k.

THEOREM 2.9. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. Let Ω be an invariant recurrent Fatou component. Then one of the following happens:*

(1) Ω is an attracting basin, i.e. there is an attracting fixed point $p \in \Omega$.

(2) There exist a closed m-dimensional complex sub-manifold M of Ω ($1 \leq m \leq k-1$), and a holomorphic retraction $\rho : \Omega \to M$ such that any limit h of a convergent sub-sequence of $\{f^n\}$ is of the form $h = \phi \circ \rho$, where $\phi \in Aut\,(M)$.

(3) Ω is a Siegel domain. Any limit of a convergent subsequence of $\{f^n\}$ is an automorphism of Ω.

Here a Siegel domain means a invariant Fatou component such that there exists a sub-sequence $\{f^{n_j}\}$ converging uniformly on compact subsets of Ω to the identity. If $M \subset \Omega$ is a sub-variety, then a retraction $\rho : \Omega \to M$ is a holomorphic map such that $\rho|_M = \mathrm{Id}$.

For $m = 1$ in the above theorem, by a result of Ueda [27], M is biholomorphic to either a disc or an annulus, and $f|_M$ is conjugate to an irrational rotation.

As we have mentioned, the non-recurrent case is more delicate, and the classification of invariant non-recurrent Fatou components for $k \geq 2$ is not known. We have the following conditional theorem due to Lyubich-Peters [17].

THEOREM 2.10. *Let f be a holomorphic endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let Ω be an invariant non-recurrent Fatou component. Suppose the limit set $h(\Omega)$ is unique, then $h(\Omega)$ either consist of one point p, or $h(\Omega)$ is a injective immersed Riemann surface, conformally equivalent to either a disk, a punctured disk or an annulus, and $f|_{h(\Omega)}$ is conjugate to an irrational rotation.*

Here $h(\Omega)$ contains all images of limit maps of $\{f^{n_j}\}$ for all sub-sequence $\{n_j\}$.

Perhaps the main difference between one dimensional and higher dimensional Fatou theory is that when $k \geq 2$, a holomorphic endomorphism on $\mathbb{P}^k$ can have a wandering Fatou component. The following result is due to Astorg-Buff-Dujardin-Peters-Raissy [2].

THEOREM 2.11. *There exist a holomorphic endomorphism f on $\mathbb{P}^2$ of degree ≥ 2 induced by a parabolic polynomial skew product, possessing a wandering Fatou component.*

As already explained, wandering Fatou components can be constructed for holomorphic endomorphism on $\mathbb{P}^k$ by taking products for $k \geq 2$.

2.4. Dynamics of polynomial skew products. A *polynomial skew product* on $\mathbb{C}^2$ is a map of the following form:

$$f(z,w) = (p(z), q(z,w)),$$

where p is a one variable polynomial of degree ≥ 2 and q is a two variables polynomial of degree ≥ 2. When $\deg p = \deg q = d$ and $q(z,w) = a_d w^d + O(w^{d-1})$, f can be extended holomorphically to $\mathbb{P}^2$. Such a polynomial skew product is called a *regular polynomial skew product.*

Definition 2.12. *Let $f = (p, q)$ be a polynomial skew product. For every $z_0 \in \mathbb{C}$, the Julia set $J_{z_0} \subset \{z = z_0\} \times \mathbb{C}$ is defined as the locus where the sequence of iterates $\left\{q_{p^n(z)} \circ \cdots \circ q_z\right\}_{n\geq 1}$ do not locally form a normal family in $\{z = z_0\} \times \mathbb{C}$. Here q_z is a one variable polynomial defined by $q_z(w) := q(z,w)$.*

We have the following result due to Jonsson [15].

Theorem 2.13. *Let $f = (p, q)$ be a polynomial skew product. Then*

$$J_2 = \overline{\bigcup_{z\in J_p} \{z\} \times J_z},$$

where J_p is the Julia set of p.

Note that the above theorem implies that repelling periodic points belongs to J_2.

Jonsson proved the following topological characterization for uniformly expansion on J_2, which is analogous to the one dimensional case (see Definition 1.6).

Theorem 2.14. *Let f be a polynomial skew product. Then f is expanding on J_2 if and only if $J_2 \cap \overline{PC(f)} = \emptyset$, here $PC(f)$ is the post-critical set.*

For the proof see Jonsson [15]. We note that the result is unknown for general holomorphic endomorphisms on $\mathbb{P}^2$. As a corollary, Jonsson obtains [15]:

Corollary 2.15. *Let f be a polynomial skew product such that f is expanding on J_2. Then*

$$J_2 = \bigcup_{z\in J_p} \{z\} \times J_z.$$

Jonsson [15] also studied the Axiom A polynomial skew products. Again the following characterization is similar to the one dimensional case.

Theorem 2.16. *Let f be a polynomial skew product. Then f is Axiom A on $\mathbb{C}^2$ if and only if following conditions are satisfied:*

(1) f is expanding on J_2.

(2) p is hyperbolic.

(3) Let $z_0 \in \mathbb{C}^2$ be an attracting periodic point of p of periods n, then $f^n|_L$ is a hyperbolic map, where $L = \{z = z_0\}$.

Moreover, a regular polynomial skew product f is Axiom A on $\mathbb{P}^2$ if and only if f is Axiom A on $\mathbb{C}^2$ and $f|_{L_\infty}$ is hyperbolic, where L_∞ is the line at infinity.

Jonsson [15] also showed that Axiom A polynomial skew products are stable under perturbation.

THEOREM 2.17. *Let f be an Axiom A polynomial skew product. Then the chain recurrent set satisfies $R(f) = \overline{Per(f)}$. In particular, any holomorphic endomorphism on $\mathbb{P}^2$ which is sufficiently close (in C^1 topology) to f is also Axiom A.*

Note that in general Axiom A endomorphisms may not be stable under perturbation, but an Axiom A endomorphism satisfying $R(f) = \Omega(f)$ is stable under perturbation, where $\Omega(f)$ is the non-wandering set, see [15].

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CHAPTER 2

Non-wandering Fatou components for strongly attracting polynomial skew products

1. Introduction

Complex dynamics, also known as Fatou-Julia theory, is naturally subdivided according to these two terms. One is focused on the Julia set. This is the set where chaotic dynamics occurs. The other direction of investigation is concerned with the dynamically stable part - the Fatou set. In this paper we will concentrate on the Fatou theory.

In a general setting, let M be a complex manifold, and let $f : M \to M$ be a holomorphic self map. We consider f as a dynamical system, that is, we study the long-time behavior of the sequence of iterates $\{f^n\}_{n\geq 0}$. The **Fatou set** $F(f)$ is classically defined as the largest open subset of M in which the sequence of iterates is normal. Its complement is the **Julia set** $J(f)$. A **Fatou component** is a connected component of $F(f)$.

In one-dimensional case, we study the dynamics of iterated holomorphic self map on a Riemann surface. The classical case of rational functions on Riemann sphere $\mathbb{P}^1$ occupies an important place and produces a fruitful theory. The non-wandering domain theorem due to Sullivan [11] asserts that every Fatou component of a rational map is eventually periodic. This result is fundamental in the Fatou theory since it leads to a complete classification of the dynamics in the Fatou set: the orbit of any point in the Fatou set eventually lands in an attracting basin, a parabolic basin, or a rotation domain.

The same question arises in higher dimensions, i.e. to investigate the non-wandering domain theorem for higher dimensional holomorphic endomorphisms on $\mathbb{P}^k$. A good test class is that of polynomial skew products hence one-dimensional tools can be used.

A polynomial skew product is a map $P : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ of the form

$$P(t, z) = (g(t), f(t, z)),$$

where g is an one variable polynomial and f is a two variable polynomial. See Jonsson [4] for a systematic study of such polynomial skew products, see also Dujardin [3] for an application of polynomial skew products.

To investigate the Fatou set of P, let π_1 be the projection to the t-coordinate, i.e.

$$\pi_1 : \mathbb{C}^2 \to \mathbb{C}, \ \ \pi_1(t, z) = t.$$

We first notice that $\pi_1(F(P)) \subset F(g)$, pass to some iterates of P, we may assume that the points in $F(g)$ will eventually land into an immediate basin or a Siegel disk (no Herman rings for polynomials), thus we only need to study the following semi-local case:

$$P = (g, f) : \Delta \times \mathbb{C} \to \Delta \times \mathbb{C},$$

where $g(0) = 0$ which means the line $\{t = 0\}$ is invariant and Δ is the immediate basin or the Siegel disk of g. P is called an attracting, parabolic or elliptic local polynomial skew product when $g'(0)$ is attracting, parabolic or elliptic respectively.

The first positive result is due to Lilov. Under the assumption that $0 \leq |g'(0)| < 1$, Koenigs' Theorem and Böttcher's Theorem tell us that the dynamical system is locally conjugated to

$$P(t, z) = (\lambda t, f(t, z)),$$

when $g'(0) = \lambda \neq 0$, or

$$P(t, z) = (t^m, f(t, z)),\ m \geq 2,$$

when $g'(0) = 0$. In the first case the invariant fiber is called attracting and in the second case the invariant fiber is called super-attracting. Now f is no longer a polynomial, and f can be written as a polynomial in z,

$$f(t, z) = a_0(t) + a_1(t)z + \cdots + a_d(t)z^d,$$

with coefficients $a_i(t)$ holomorphic in t in a neighborhood of 0, we further assume that $a_d(0) \neq 0$ (and we make this assumption in the rest of the paper). In this case the dynamics in $\{t = 0\}$ is given by the polynomial

$$p(z) = f(0, z)$$

and is very well understood. In his unpublished PhD thesis [5], Lilov first showed that every Fatou component of p in the super-attracting invariant fiber is contained in a two-dimensional Fatou component, which is called a bulging Fatou component. We will show that this bulging property of Fatou component of p also holds in attracting case.

Lilov's main result is the non-existence of wandering Fatou components for local polynomial skew products in the basin of a super-attracting invariant fiber. Since this is a local result, it can be stated as follows.

Theorem (**Lilov**). *For a local polynomial skew product P with a super-attracting invariant fiber,*

$$P(t, z) = (t^m, f(t, z)),\ m \geq 2,$$

every forward orbit of a vertical Fatou disk intersects a bulging Fatou component. This implies that every Fatou component iterate to a bulging Fatou component. In particular there are no wandering Fatou components.

See Definition 2.1 for the definition of the vertical Fatou disk.

On the other hand, recently Astorg, Buff, Dujardin, Peters and Raissy [1] constructed a holomorphic endomorphism $h : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$, induced by a polynomial skew product $P = (g(t), f(t, z)) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ with parabolic invariant fiber, processing a wandering Fatou component, thus the non-wandering domain theorem does not hold for general polynomial skew products.

At this stage it remains an interesting problem to investigate the existence of wandering Fatou components for local polynomial skew products with attracting but not super-attracting invariant fiber. As it is clear from Lilov's theorem, Lilov actually showed a stronger result, namely that every forward orbit of a vertical Fatou disk intersects a bulging Fatou component. Peters and Vivas showed in [8] that there is an attracting local polynomial skew product with a wandering vertical Fatou disk, which shows that Lilov's proof breaks down in the general attracting case. Note that this result does not answer the existence question of wandering Fatou components, but shows that the question is considerably more complicated than in the super-attracting case. On the other hand, by using a different strategy from Lilov's, Peters and Smit in [7] showed that the non-wandering domain theorem holds in the attracting case, under the assumption that the dynamics on the invariant fiber is sub-hyperbolic. The elliptic case was studied by Peters and Raissy in [6]. See also Raissy [9] for a survey of the history of the investigation of wandering domains for polynomial skew products.

In this paper we prove a non-wandering domain theorem in the attracting local polynomial skew product case without any assumption of the dynamics on the invariant fiber. Actually we show that Lilov's stronger result holds in the attracting case when the multiplier λ is small.

THEOREM (**Main Theorem**). *For a local polynomial skew product P with an attracting invariant fiber,*

$$P(t,z) = (\lambda t, f(t,z)),$$

for any fixed f, there is a constant $\lambda_0 = \lambda_0(f) > 0$ such that if λ satisfies $0 < |\lambda| < \lambda_0$, every Fatou component iterates to a bulging Fatou component. In particular there are no wandering Fatou components.

We can also apply this local result to globally defined polynomial skew products, see Theorem 6.4 for the precise statement.

The proof of the main theorem basically follows Lilov's strategy. The difficulty is that Lilov's argument highly depends on the super-attracting condition and breaks down in the attracting case by [8]. The main idea of this paper are to use and adapt an one-dimensional lemma due to Denker-Przytycki-Urbanski(the DPU Lemma for short) to our case. This will give estimates of the horizontal size of bulging Fatou components and of the size of forward images of a wandering vertical Fatou disk (these concepts will be explained later). We note that some results in our paper already appear in Lilov's thesis [5] (Theorem 3.4, Lemma 4.1, Lemma 5.1 and Lemma 5.2). Since his paper is not easily available, we choose to present the whole proof with all details. On the other hand we believe that the introduction of the DPU Lemma makes the argument conceptually simpler even in the super-attracting case.

The outline of the paper is as follows. In section 2 we start with some definitions, then we present the DPU Lemma and some corollaries. In section 3 we show that every Fatou component of p in the invariant fiber bulges, i.e. is contained in a two-dimensional bulging Fatou component. This result follows classical ideas from normal hyperbolicity theory.

In section 4 we give an estimate of the horizontal size of the bulging Fatou components by applying the one-dimensional DPU Lemma. Let $z \in F(p)$ be a point in a Fatou component of the invariant fiber and denote by $r(z)$ the supremum radius of a horizontal holomorphic disk (see Definition 2.1 for the precise definition) centered at z that is contained in the bulging Fatou component. We have the following key estimate.

THEOREM 4.3. *If λ is chosen sufficiently small, then there are constants $k > 0$, $l > 0, R > 0$ such that for any point $z \in F(p) \cap \{|z| < R\}$,*

$$r(z) \geq k\, d(z, J(p))^l,$$

where $J(p)$ is the Julia set in the invariant fiber.

In section 5 we adapt the DPU Lemma to the attracting local polynomial skew product case, to show that the size of forward images of a wandering vertical Fatou disk shrinks slowly, which is also important in the proof of the main theorem.

PROPOSITION 5.5. *Let $\Delta_0 \subset \{t = t_0\}$ be a wandering vertical Fatou disk centered at $x_0 = (t_0, z_0)$. Let $x_n = (t_n, z_n) = P^n(x_0)$. Define a function ρ as follows: for a domain $U \subset \mathbb{C}$, for every $z \in U \subset \mathbb{C}$, define*

$$\rho(z, U) = \sup \{r > 0 |\ D(z, r) \subset U\}.$$

Set $\Delta_n = P^n(\Delta_0)$ for every $n \geq 1$ and let $\rho_n = \rho(z_n, \pi_2(\Delta_n))$, here π_2 is the projection $\pi_2 : (t, z) \mapsto z$. If λ is chosen sufficiently small, we have

$$\lim_{n\to\infty} \frac{|\lambda|^n}{\rho_n} = 0.$$

The proof of the main theorem is given in section 6. The main point are to combine Theorem 4.3 and Proposition 5.5 to show wandering vertical Fatou disk can not exist. We finish section 6 with some remarks around the main theorem. We also show how our main theorem can be applied to globally defined polynomial skew products in theorem 6.4.

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2. Preliminaries

2.1. Horizontal holomorphic disk and vertical Fatou disk. In this subsection we make the precise definitions appearing in the statement of Theorem 4.3 and Proposition 5.5.

Recall that after a local coordinate change our map has the form $P : \Delta \times \mathbb{C} \to \Delta \times \mathbb{C}$, here $\Delta \subset \mathbb{C}$ is a disk centered at 0, such that

$$P(t, z) = (\lambda t, f(t, z)),$$

here f is a polynomial in z with coefficients $a_i(t)$ holomorphic in Δ, and $a_d(0) \neq 0$, λ satisfies $0 < |\lambda| < 1$.

DEFINITION 2.1. • *A **horizontal holomorphic disk** is a subset of the form*

$$\{(t, z) \in \Delta \times \mathbb{C} \mid z = \phi(t),\ |t| < \delta\}$$

*where $\phi(t)$ is holomorphic in $\{|t| < \delta\}$ for some $\delta > 0$. δ is called the **size** of the horizontal holomorphic disk.*

- *Let π_2 denote the projection to the z-axis, that is*

$$\pi_2 : \Delta \times \mathbb{C} \longrightarrow \mathbb{C},$$
$$(t, z) \longmapsto z.$$

*A subset Δ_0 lying in some $\{t = t_0\}$ is called a **vertical disk** if $\pi_2(\Delta_0)$ is a disk in the complex plane. A vertical disk centered at x_0 with radius r is denoted by $\Delta(x_0, r)$. Δ_0 is called a **vertical Fatou disk** if the restriction of $\{P^n\}_{n\geq 0}$ to Δ_0 is a normal family.*

In the rest of the paper, for a disk on the complex plane centered at z with radius r, we denote it by $D(z, r)$ to distinguish.

REMARK 2.2. *A vertical disk contained in a Fatou component of P is a vertical Fatou disk.*

We define a positive real valued function $r(z)$, which measures the horizontal size of the bulging Fatou components.

DEFINITION 2.3. *Let $R > 0$ be a constant. For $z \in \mathbb{C}$ satisfying $|z| < R$ and z lying in the Fatou set of p, we define $r(z)$ to be the supremum of all positive real numbers r such that there exist a horizontal holomorphic disk passing through z with size $2r$, contained in $F(P) \cap \{|z| \leq R\}$.*

2.2. Denker-Przytycki-Urbanski's Lemma.

In this subsection we introduce the work of Denker-Przytycki-Urbanski in [2], and give some corollaries. Denker, Przytycki and Urbanski consider rational maps on $\mathbb{P}^1$, and study the local dynamical behavior of some neighborhood of a critical point lying in Julia set. As a consequence they deduce an upper bound of the size of the pre-images of a ball centered at a point in Julia set.

In the following let f be a rational map on $\mathbb{P}^1$, denote by $C(f)$ the set of critical points of f lying in Julia set. Assume that $\#C(f) = q$. We begin with a definition,

DEFINITION 2.4. *For a critical point $c \in C(f)$, define a positive valued function $k_c(x)$ by*

$$k_c(x) = \begin{cases} -\log d(x, c), & \textit{if } x \neq c \\ \infty, & \textit{if } x = c. \end{cases}$$

Define a function $k(x)$ by

$$k(x) = \max_{c \in C(f)} k_c(x).$$

Here the distance is relative to the spherical metric on $\mathbb{P}^1$.

Let x_0 be arbitrary and consider the forward orbit $\{x_0, x_1, \cdots, x_n, \cdots\}$, where $x_n = f^n(x_0)$. We let the function $k(x)$ acts on this orbit and the following DPU Lemma gives an asymptotic description of the sum of $k(x)$ on this orbit. Recall that q denotes the number of critical points lying in J.

LEMMA 2.5 (**Denker, Przytycki, Urbanski**). *There exist a constant $Q > 0$ such that for every $x \in \mathbb{P}^1$, and $n \geq 0$, there exists a subset $\{j_1, \cdots, j_{q'}\} \subset \{0, 1, \cdots, n\}$, such that*

$$\sum_{j=0}^{n} k(x_j) - \sum_{\alpha=1}^{q'} k(x_{j_\alpha}) \leq Qn,$$

here $q' \leq q$ is an integer.

Lemma 2.5 implies that in a sense the orbit of a point can not come close to $C(f)$ very frequently. As a consequence Denker, Przytycki, Urbanski deduce an upper bound of the size of the pre-images of a ball centered at a point in $J(f)$.

COROLLARY 2.6 (**Denker, Przytycki, Urbanski**). *There exist $s \geq 1$ and $\rho > 0$ such that for every $x \in J(f)$, for every $\epsilon > 0$, $n \geq 0$, and for every connected component V of $f^{-n}(B(x, \epsilon))$, one has diam $V \leq s^n \epsilon^\rho$.*

COROLLARY 2.7. *Let f be a polynomial map in $\mathbb{C}$. For fixed $R > 0$, there exist $s \geq 1$ and $\rho > 0$ such that for any $n \geq 0$ and any $z \in \mathbb{C}$ satisfying $f^n(z) \in \{|z| < R\}$, we have*

$$d(z, J(f)) \leq s^n d(f^n(z), J(f))^\rho,$$

where the diameter is relative to the Euclidean metric.

PROOF. Since the Euclidean metric and the spherical metric are equivalent on a compact subset of $\mathbb{C}$, by Corollary 2.6 for fixed $R > 0$, there exist $s \geq 1$ and $\rho > 0$ such that for every z satisfying $z \in J(f)$, $0 < \epsilon \leq R$, $n \geq 0$, and for every connected component V of $f^{-n}(D(z, \epsilon))$, one has diam $V \leq s^n \epsilon^\rho$.

For any z and n satisfy $f^n(z) \in \{|z| < R\}$, let $y \in J$ satisfy $d(f^n(z), J(f)) = d(y, f^n(z)) = \epsilon$. For every connected component V of $f^{-n}(D(y, 2\epsilon))$, one has diam $V \leq s_1^n (2\epsilon)^\rho$, so that

$$d(z, J(f)) \leq d(z, f^{-n}(y)) \leq \text{diam } V \leq s_1^n 2^\rho \epsilon^\rho.$$

Set $s = 2^\rho s_1$ and the proof is complete. □

REMARK 2.8. *The existence of such a result is intuitive since the Julia set is expected to be repelling in some sense - however the presence of critical points on J makes it non-trivial.*

3. Structure of bulging Fatou components

In this section we show that every Fatou component of p in the invariant fiber is actually contained in a Fatou component of P, which is called a **bulging Fatou component**, and in this case we call the Fatou component of p **bulges**. By Sullivan's theorem every Fatou component of p is pre-periodic, it is sufficient to show that every periodic Fatou component of p is contained in a Fatou component of P. There are three kinds of periodic Fatou components of p, i.e. attracting basin, parabolic basin and Siegel disk. For all these three kinds we study the structure of the associated bulging Fatou components.

We may iterate P many times to ensure that all periodic Fatou components of p are actually fixed, and all parabolic fixed points have multiplier equals to 1. In the following of this paper the metric referred to is the Euclidean metric.

3.1. Attracting basin case. In the attracting basin case, assume that we have an attracting basin B of p in the invariant fiber. Without loss generality we may assume 0 is the fixed point in B, so that $(0,0)$ becomes a fixed point of P, and $p'(0) = \lambda'$ with $|\lambda'| < 1$. We have the following well-known theorem [10].

THEOREM 3.1. *If $P : \Omega \to \Omega$ is a holomorphic self map, where Ω is an open set of $\mathbb{C}^2$ and $(0,0) \in \Omega$ is a fixed point. If all eigenvalues of the derivative $DP(0,0)$ are less than 1 in absolute value then P has an open attracting basin at the origin.*

In our case we have

$$DP(0,0) = \begin{pmatrix} \lambda & 0 \\ \frac{\partial f}{\partial t}(0,0) & \lambda' \end{pmatrix},$$

so that all the all eigenvalues of the derivative $DP(0,0)$ are less than 1 in absolute value. As a consequence B is contained in a two dimensional attracting basin of $(0,0)$, say U, so that B bulges.

3.2. Parabolic basin case. In the parabolic basin case suppose 0 is a parabolic fixed point of p. Assume that p is locally conjugated to $z \mapsto z + az^s + O(z^{s+1})$ for some $s \geq 2$, $a \neq 0$. We first prove that near the fixed point $(0,0)$, P is locally conjugated to

$$(t,z) \mapsto (\lambda t, z + az^s + O(z^{s+1})).$$

where $O(z^{s+1})$ means there are constant C such that the error term $\leq C|z|^{s+1}$, for all (t,z) in a neighborhood of the origin. Then we prove in this coordinate every parabolic basin of p bulges.

LEMMA 3.2. *Assume $(0,0)$ is a fixed point of P, and $|p'(0)| = 1$, then there exist a stable manifold through the origin in the horizontal direction. More precisely, there is a holomorphic function $z = \phi(t)$ defined on a small disk $\{|t| < \delta\}$ such that*

$$\phi(0) = 0, \text{ and } f(t, \phi(t)) = \phi(\lambda t).$$

PROOF. This is related to the two dimensional Poincaré's theorem. See [12] Theorem 3.1 for the proof. $\square$

We have the following theorem which is a special case of [12, §7.2].

THEOREM 3.3. *We assume that the local skew product is given by*

$$P(t,z) = (\lambda t, z + az^s + O(z^{s+1})),$$

then there exist a constant $\delta > 0$ *and s-1 pairwise disjoint simply connected open sets* $U_j \subset \{|t| < \delta\} \times \mathbb{C}$*, referred to as two dimensional attracting petals, with the following properties:*

(1) $P(U_j) \subset U_j$*, points in* U_j *converge to* $(0,0)$ *locally uniformly.*

(2) For any point $x_0 = (t_0, z_0)$ *such that* $P^n(x_0) \to (0,0)$*, there exist integer* N *and* j *such that for all* $n \geq N$ *either* $P^n(x_0) \in U_j$ *or* $z_n = 0$*.*

(3) $U_j = \{|t| < \delta\} \times (U_j \cap \{t = 0\})$*.*

Thus by Theorem 3.3, for fixed j, all the points x_0 whose orbit finally lands on U_j form an open subset Ω_j, which is contained in the Fatou set of P. It is obvious that every parabolic basin of p is contained in one of such Ω_j, this implies all parabolic basins of p bulge.

3.3. Siegel disk case. In the Siegel disk case, we assume that 0 is a Siegel point with a Siegel disk $D \subset \{t = 0\}$. We are going to prove that D is contained in a two dimensional Fatou component.

THEOREM 3.4. *Assume that* p *is locally conjugated to* $z \mapsto e^{i\theta} z$ *with* θ *an irrational multiple of* π_2*, then there is a neighborhood* Ω *of* D *such that* $D \subset \Omega \subset \mathbb{C}^2$*, and there exists a biholomorphic map* ψ *defined on* Ω *such that*

$$\psi \circ P \circ \psi^{-1}(t,z) = (\lambda t, e^{i\theta} z).$$

PROOF. We may assume that p is conjugated to $z \mapsto e^{i\theta} z$, then by Lemma 3.2 there is a stable manifold $z = \phi(t)$. A change of variables $z \mapsto z + \phi(t)$ straightens the stable manifold so that P is conjugated to

$$(t,z) \to (\lambda t, e^{i\theta} z + t g(t,z)),$$

where $g(t,z)$ is a holomorphic function. By an abuse of notation we rename this map by P.

Let U be a relatively compact neighborhood of $\overline{D}$ in $\mathbb{C}^2$. Set $C = \sup |g(t,z)|$ on U. Let δ be so small that $\frac{C\delta}{1-\delta} < \mathrm{dist}(D, \partial U)$, and then $\Omega = \{|t| < \delta\} \times D$ is an open subset of U. Let (t_0, z_0) be an arbitrary initial point in Ω, and denote $P^n(t_0, z_0)$ by (t_n, z_n), then

$$||z_{n+1}| - |z_n|| \leq |t_n g(t_n, z_n)| \leq C|\lambda|^n \delta.$$

Then we have

$$||z_n| - |z_0|| \leq \frac{C\delta}{1-\delta} + |z_0| \leq \mathrm{dist}(z_0, \partial U) + |z_0|,$$

so that (t_n, z_n) still lies in U. Thus $\{P^n\}$ is a normal family on Ω, for the reason that $P^n(\Omega)$ is uniformly bounded.

Thus we can select a sub-sequence $\{n_j\}$ for which the sequence

$$\phi_{t_0}(z_0) = \lim_{j\to\infty} e^{-in_j\theta} f_{t_{n_j}} \circ f_{t_{n_j}-1} \circ \cdots \circ f_{t_0}(z_0)$$

uniformly converges on compact subset of Ω. Thus $\phi_t(z)$ is a holomorphic function on Ω, and we have

$$\phi_{\lambda t_0} \circ f_{t_0}(z_0) = e^{i\theta}\phi_{t_0}(z_0)$$

for every $(t_0, z_0) \in \Omega$. Thus if we let $\psi(t,z) = (t, \phi_t(z))$, since $\phi_0(z) = z$ we can shrink Ω if necessary to make sure that ψ is invertible on Ω, and we have

$$\psi \circ P \circ \psi^{-1}(t,z) = (\lambda t, e^{i\theta} z).$$

For every $(t,z) \in \Omega$. □

It is obvious that Ω is contained in the Fatou set of P. Since $D \subset \Omega$, this implies that every Siegel disk of p bulges.

3.4. Wandering vertical Fatou disks. We finish section 3 with a definition.

DEFINITION 3.5. *A vertical Fatou disk Δ is called* ***wandering*** *if the forward images of Δ do not intersect any bulging Fatou component.*

We note that "wandering" has special meaning in our definition. The definition of wandering vertical Fatou disk we made here is not equivalent to vertical Fatou disks containing wandering points.

REMARK 3.6. *The forward orbit of a wandering vertical Fatou disk clusters only on $J(p)$.*

PROOF. This is simply because for every $x = (t,z) \in \Delta$, if $P^n(x)$ tends to $(0, z_0) \in F(p)$ then eventually $P^n((t,z))$ lands in the bulging Fatou component that contains $(0, z_0)$. This contradicts the fact x lying in a wandering Fatou disk. □

4. Estimate of horizontal size of bulging Fatou components

In this section we deduce an estimate of the horizontal size of the bulging Fatou components, by applying the one-dimensional DPU Lemma.

In the following we choose $R > 0$ such that if (t,z) satisfies $t \in \Delta$, $|z| > R$, then $|f(t,z)| \geq 2|z|$. This follows that the line at infinity is super-attracting. Thus for any holomorphic function $\phi(t)$ defined on $\{|t| < r\}$ such that $|\phi(t)| \leq R$, we have for all $|t| < r$,

$$|\phi(t) - \phi(0)| \leq 2R\frac{|t|}{r}, \tag{4.1}$$

this follows from the classical Schwarz Lemma.

We begin with a lemma.

LEMMA 4.1. *Let Crit*$(P) = \left\{(t,z) | \frac{\partial f}{\partial z}(t,z) = 0\right\}$, *then there exist constants* $0 < \delta_1 < 1$ *and* $K > K_1 > 0$ *such that any connected component* C_k *of Crit*$(P) \cap \{|t| < \delta_1\}$ *intersects the line* $\{t = 0\}$ *in a unique point, say* c_k, *and for any point* $x = (t,z) \in$ *Crit*(P), *say* $x \in C_k$, *we have*

$$|z - c_k| \leq K_1 |t|^{\frac{1}{d_1}}. \tag{4.2}$$

and

$$|f(t,z) - p(c_k)| \leq K |t|^{\frac{1}{d_1}}, \tag{4.3}$$

where $c_k = C_k \cap \{t = 0\}$, *and* d_1 *is the maximal multiplicity of critical points of* p.

PROOF. Since Crit(P) is an analytic variety, by Weirstrass preparation theorem we can let $\delta_1 < 1$ small enough so that Crit$(P) \cap \{|t| < \delta_1\} = \cup_{k=1}^{l} C_k$ where C_k, $1 \leq k \leq l$ are local connected analytic sets, $C_k \cap \{t = 0\} = \{c_k\}$. For each fixed component C intersect $\{t = 0\}$ at c, C is given by the zero set of a Weirstrass polynomial,

$$C = \{(t,z) \in \{|t| < \delta_1\} \times \mathbb{C}, g(t,z) = 0\},$$

where $g(t,z) = (z-c)^m + a_{m-1}(t)(z-c)^{m-1} + \cdots + a_0(t)$ is a Weirstrass polynomial, $m \leq d_1$ is an integer, $a_i(t)$ are holomorphic functions in t satisfying $|a_i(t)| \leq M|t|$ for some constant $M > 0$.

We show that

$$|z - c| \leq mM|t|^{\frac{1}{m}}.$$

We argue by contradiction. Suppose there exist a point $(t_0, z_0) \in C$ such that $\frac{|z_0 - c|}{|t_0|^{\frac{1}{m}}} = a > mM$, then we have

$$|z_0 - c|^m = a^m |t_0|,$$

and

$$|a_{m-1}(t_0)(z_0 - c)^{m-1} + \cdots + a_0(t_0)| \leq mMa^{m-1}|t_0|. \tag{4.4}$$

Thus we have $|z_0 - c|^m > |a_{m-1}(t_0)(z_0 - c)^{m-1} + \cdots + a_0(t_0)|$, which contradicts to $(t_0, z_0) \in C$. Setting $K_1 = 2d_1 M$ we get (4.2).

Let Ω be a relatively compact open set that contains Crit$(P) \cap \{|t| < \delta_1\}$. Let

$$M' = \max\left\{\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial t}\right| : (t,z) \in \Omega\right\}.$$

Then for $(t,z) \in C_k$ we have

$$\begin{aligned} |f(t,z) - p(c_k)| &\leq M'|t| + M'|z - c_k| \\ &\leq M'(1 + K_1)|t|^{\frac{1}{d_1}}. \end{aligned}$$

To get (4.3) we set $K = 2\max\{M'(1 + K_1), 2R\}$. Thus the proof is complete. ☐

REMARK 4.2. *We note that* K_1 *and* K *are invariant under a local coordinate change of the form* $t \mapsto \phi(t)$ *with* $\phi(0) = 0$ *and* $\phi'(0) = 1$. *To see this let* $a_i(t)$ *be the coefficients of the Weirstrass polynomial, the coordinate change* $t \mapsto \phi(t)$ *with* $\phi'(0) = 1$ *takes* $a_i(t)$ *become* $a_i(\phi(t))$. *We have* $|a_i(\phi(t)| \leq 2M|t|$ *by shrinking* $\delta_1(\phi)$ *if necessary , then we get*

(4.2) with the same constant K_1 (this is the reason for the constant 2 in definition of K_1). By shrinking $\delta_1(\phi)$ we see that Ω and R are invariant, and

$$\max\left\{\left|\frac{\partial f(\phi(t),z)}{\partial z}\right|,\left|\frac{\partial f(\phi(t),z)}{\partial t}\right|:\ (t,z)\in\Omega\right\}\le 2M'.$$

By the same reason we get (4.3) with the same constant K.

We are going to prove the following estimate of $r(z)$ under the assumption that the multiplier λ is sufficiently small.

THEOREM 4.3. *There exist a constant $\lambda_1=\lambda_1(f)>0$ such that for fixed $|\lambda|<\lambda_1$, there are constants $k>0$, $l>0$ such that for any point $z\in F(p)\cap\{|z|<R\}$,*

$$r(z)\ge k\,d(z,J(p))^l,$$

here $J(p)$ is the Julia set of p in the invariant fiber. Furthermore l depends only on p.

We would like to give an outline of the proof of Theorem 4.3 first. Since there are only finitely many invariant Fatou components of p, and every Fatou component is pre-periodic to one of them, it is enough to prove Theorem 4.3 holds for z in the basin of an invariant Fatou component. To do this, we first fix an invariant Fatou component U, and we prove Theorem 4.3 holds for a subset W satisfying $\cup_{i=0}^{\infty}p^{-i}(W)=$ the basin of U, this is the first step. In step 2, we use the following Pull Back Lemma to get the relation between $r(z)$ and $r(p(z))$, together with the DPU Lemma we are able to give the estimate for the points in $p^{-i}(W)$, for every i. We start with the Pull Back Lemma.

LEMMA 4.4 (**Pull Back lemma**). *There exist a constant $0<\epsilon<1$, such that if we let*

$$V=\{z\in F(p),d(z,J(p))<\epsilon\},$$

then for any $z_0\in F(p)\cap\{|z|<R\}$ such that $p(z_0)\in V$, at least one of the following holds:

$$r(z_0)\ge\frac{\alpha}{|\lambda|}r(p(z_0))d(z_0,C(p))^{d_1(d_1+1)};\tag{4.5}$$

or

$$r(z_0)\ge\beta\,d(z_0,J(p))^{d_1(d_1+1)}.\tag{4.6}$$

Here α,β are positive constants only depending on p and the constant K from Lemma 4.1, and $C(p)$ is the set of critical points lying in $J(p)$.

PROOF. Let $\mathrm{Crit}(p)$ be the set of critical points of p, We choose ϵ small such that $p(z)\in V$ implies $d(z,p(C(p)))=d(z,p(\mathrm{Crit}(p))$. Let ϕ be the associated holomorphic function with respect to $p(z_0)$ with size $r(p(z_0))$. We are going to show that the critical value set of P does not intersect the graph of ϕ when the domain of ϕ is small .

Suppose $x'=(t',z')$ lies in $\mathrm{Crit}(P)$ satisfying $t'<r(z_0)$ and $P(x')$ lying in the graph of ϕ. then by Lemma 4.1 the connected component of $\mathrm{Crit}(P)$ containing x' intersects

$\{t = 0\}$ at a unique point c. Then we have

$$\begin{aligned} d(p(z_0), p(C(p)) &\leq |p(z_0) - p(c)| \\ &= |\phi(0) - \phi(\lambda t') + f(t', z') - p(c)| \\ &\leq |\phi(0) - \phi(\lambda t')| + |f(t', z') - p(c)| \\ &\leq K\frac{|\lambda t'|}{r(p(z_0))} + K|t'|^{1/d_1}. \qquad (4.7) \end{aligned}$$

(4.7) holds by applying Lemma 4.1 and inequality (4.1).

Now there are two possibilities,

(a) If $\frac{|\lambda t'|}{r(p(z_0))} \geq 1$, then

$$|t'| \geq \frac{r(p(z_0))}{|\lambda|}.$$

(b) If $\frac{|\lambda t'|}{r(p(z_0))} < 1$, then

$$\frac{|\lambda t'|}{r(p(z_0))} \leq \frac{|\lambda t'|^{1/d_1}}{r(p(z_0))^{1/d_1}},$$

so that by (4.7) we have

$$d(p(z_0), p(C(p)) \leq K\frac{|\lambda t'|^{1/d_1}}{r(p(z_0))^{1/d_1}} + K|t'|^{1/d_1}.$$

For case (b), there are two subcases,

(b1) If $r(p(z_0)) \leq |\lambda|$, then

$$d(p(z_0), p(C(p)) \leq 2K\frac{|\lambda t'|^{1/d_1}}{r(p(z_0))^{1/d_1}},$$

by applying the fact that there is a constant $c = c(p) > 0$ such that $d(p(z_0), p(C(p)) \geq c\, d(z_0, C(p))^{d_1+1}$ we have

$$|t'| \geq \frac{\alpha}{|\lambda|} r(p(z_0)) d(z_0, C(p))^{d_1(d_1+1)},$$

where $\alpha = \left(\frac{c}{2K}\right)^{d_1}$.

(b2) If $r(p(z_0)) > |\lambda|$, then

$$d(p(z_0), p(C(p)) < 2K|t'|^{1/d_1}.$$

Thus we have

$$|t'| > \left(\frac{1}{2K}\right)^{d_1} d(p(z_0), J(p))^{d_1}.$$

By applying the fact that there is a constant $c = c(p) > 0$ such that $d(p(z_0), J(p)) \geq cd(z_0, J(p)^{d_1+1}$, we get

$$|t'| \geq \beta\, d(z_0, J(p))^{d_1(d_1+1)}.$$

where $\beta = c\left(\frac{c}{2K}\right)^{d_1}$.

We can let α small enough such that actually $\alpha d(z_0, C(p))^{d_1(d_1+1)} < 1$, thus for case (b1) we have

$$\frac{\alpha}{|\lambda|} r(p(z_0)) d(z_0, C(p))^{d_1(d_1+1)} \leq \frac{r(p(z_0))}{|\lambda|},$$

thus case (a) is actually contained in case (b1).

In either case (b1) or (b2) we get a lower bound on t'. Thus for any t which does not exceed that lower bound, $\phi(\lambda t)$ is not a critical value of f_t and so all branches of f_t^{-1} are well defined and holomorphic in a neighborhood of the graph of ϕ. Therefore, choose g_t to be the branch of f_t^{-1} for which $g_0(f_0(z)) = z$, then the function $\psi(t) = g_t(\phi(\lambda t))$ is well defined from $t = 0$ up to $|t| < \eta$ satisfying $\psi(0) = z_0$ and the graph of ψ containing in the Fatou set, where η is the lower bound from (4.5) and (4.6). We know that ψ is also bounded by R, since otherwise ϕ would not be bounded by R. To avoid the case $|t'| \geq \delta_1$, we can shrink β such that $\beta\ d(z_0, J(p))^{d_1(d_1+1)} < \delta_1$ for all z_0. Thus $|t'| \geq \delta_1$ implies $|t'| \geq \beta\ d(z_0, J(p))^{d_1(d_1+1)}$. Thus at least one of (4.5) and (4.6) holds.

□

Proof of Theorem 4.3. In the following we fix an invariant Fatou component U of p, denote the basin of U by B (If B is the basin of infinity we let B be contained in $\{|z| < R\}$). We can shrink ϵ to ensure that the set $\{z \in B, d(z, J(p)) < 2\epsilon\}$ is contained in $\{|z| < R\}$. In either case we first construct a subset W of B, satisfies the following conditions,

(1) W eventually traps the forward orbit of any point in B.

(2) W contains the compact subset $\{z \in B, d(z, J(p)) \geq \epsilon\}$.

(3) Theorem 4.3 holds for $z \in W$.

Finally we use the Pull Back Lemma to prove Theorem 4.3 holds for $z \in B$.

Step 1: Construction of W. We split the argument in several cases.

• **U is an immediate attracting basin.** Let $\omega \subset U$ be a compact neighborhood of the attracting fixed point. We set $W = \{z \in B, d(z, J(p)) \geq \epsilon\} \cup \omega$, then W automatically satisfies (1) and (2). Since W is also compact and contained in $F(P)$, there is a lower bound $a > 0$ such that $r(z) \geq a$ for every $z \in W$. So there exist $k > 0$ such that $r(z) \geq k\ d(z, J(p))$ for $z \in W$.

• **U is the attracting basin of ∞.** We set $W = \{z \in B, d(z, J(p)) \geq \epsilon\}$, then W automatically satisfies (1) and (2). There is a lower bound $a > 0$ such that $r(z) \geq a$ for every $z \in W$. So there exist $k > 0$ such that $r(z) \geq k\ d(z, J(p))$.

• **U is an immediate parabolic basin.** Let Q be the associated attracting petal of Theorem 3.3. We set $W = \{z \in B, d(z, J(p)) \geq \epsilon\} \cup Q$, then W automatically satisfies (1) and (2). By Theorem 3.3 there is a lower bound $a > 0$ such that $r(z) \geq a$ for every $z \in P$. Thus there is a lower bound $b > 0$ such that $r(z) \geq b$ for every $z \in W$. So there exist $k > 0$ such that $r(z) \geq k\ d(z, J(p))$.

• **U is a Siegel disk.** We set $W = U \cup \{z \in B, d(z, J(p)) \geq \epsilon\}$, then W automatically satisfies (1) and (2). To prove (3), it is enough to prove (3) for $z \in U$.

Lemma 4.5. *Let U be a Siegel disk, then there are constants $k > 0$, $l > 0$ such that for any point $z \in U$,*

$$r(z) \geq k\, d(z, J(p))^l.$$

Further more l only depends on p.

Proof. Since the technique of the proof is similar to that of Theorem 4.3, we postpone the proof to the end of this subsection. □

Step 2: Pull back argument.

We already have the estimate for $z \in W$. For every $z_0 \in B \backslash W$, let $\{z_i\}_{i \geq 0}$ be its forward orbit, and let n be the smallest integer such that z_n lies in W. Let m be the smallest integer such that case (4.5) does not happen, if this m dose not exist, let $m = n$, in either case we have

$$r(z_m) \geq k\, d(z_m, J(p))^l,$$

for some $k, l > 0$, and for all z_i, $0 \leq i \leq m-1$, we have

$$r(z_i) \geq \frac{\alpha}{|\lambda|} r(z_{i+1})) d(z_i, C(p))^{d_1(d_1+1)}. \tag{4.8}$$

By (4.8) we have

$$\begin{aligned} \log r(z_i) &\geq \log r(z_{i+1}) + \log d(z_i, C(p))^{d_1(d_1+1)} + \log \frac{\alpha}{|\lambda|} \\ &= \log r(z_{i+1}) - d_1(d_1+1) k(z_i) + \log \frac{\alpha}{|\lambda|}, \end{aligned}$$

for all $0 \leq i \leq m-1$, where $k(z_i)$ is as in Lemma 2.5.

Thus we have

$$\log r(z_0) \geq \log r(z_m) - d_1(d_1+1) \sum_{i=0}^{m-1} k(z_i) + m \log \frac{\alpha}{|\lambda|}.$$

By Lemma 2.5 there exist a subset $\left\{i_1, \cdots, i_{q'}\right\} \subset \{0, 1, \cdots, m-1\}$ such that

$$\sum_{i=0}^{m-1} k(z_i) - \sum_{\alpha=1}^{q'} k(z_{i_\alpha}) \leq Qm.$$

Therefore we have

$$\begin{aligned} \log r(z_0) &\geq \log r(z_m) - d_1(d_1+1) \sum_{\alpha=1}^{q'} k(z_{i_\alpha}) - d_1(d_1+1)Qm + m \log \frac{\alpha}{|\lambda|} \\ &\geq \log r(z_m) + d_1(d_1+1) \sum_{\alpha=1}^{q'} \log d(z_{i_\alpha}, J(p)) - d_1(d_1+1)Qm + m \log \frac{\alpha}{|\lambda|}. \end{aligned} \tag{4.9}$$

By Corollary 2.7 we have for each i_α,

$$\begin{aligned}\log d(z_{i_\alpha}, J(p)) &\geq \frac{1}{\rho}\log d(z_0, J(p)) - \frac{1}{\rho} i_\alpha \log s\\ &\geq \frac{1}{\rho}\log d(z_0, J(p)) - \frac{1}{\rho} m \log s.\end{aligned}$$

Likewise we have,

$$\begin{aligned}\log r(z_m) &\geq \log k + l \log d(z_m, J(p))\\ &\geq \log k + \frac{l}{\rho}\log d(z_0, J(p)) - \frac{l}{\rho} m \log s.\end{aligned}$$

Thus applying the estimates of $\log d(z_{i_\alpha}, J(p))$ and $\log r(z_m)$ to (4.9) gives

$$\begin{aligned}\log r(z_0) &\geq \log r(z_m) + d_1(d_1+1)\sum_{\alpha=1}^{q'} \log d(z_{i_\alpha}, J(p)) - d_1(d_1+1)Qm + m\log\frac{\alpha}{|\lambda|}\\ &\geq \log k + \frac{l + q d_1(d_1+1)}{\rho}\log d(z_0, J(p)) - \frac{l + q d_1(d_1+1)}{\rho} m \log s - d_1(d_1+1)Qm + m \log \frac{\alpha}{|\lambda|}.\end{aligned}$$

Let us now fix λ_1 so small such that

$$\log\frac{\alpha}{\lambda_1} \geq \frac{l + q d_1(d_1+1)}{\rho}\log s + d_1(d_1+1)Q, \tag{4.10}$$

then for every $|\lambda| < \lambda_1$ we have

$$\log r(z_0) \geq \log k + \frac{l + q d_1(d_1+1)}{\rho}\log d(z_0, J(p)),$$

which is equivalent to

$$r(z_0) \geq k\ d(z_0, J(p))^{l'},$$

where $l' = \frac{l+qd_1(d_1+1)}{\rho}$.

We have shown that there are constants $k > 0$, $l' > 0$ such that $r(z) \geq k\ d(z, J(p))^{l'}$ for $z \in B$, and l' only depends on p, this finishes the proof of Theorem 4.3. □

Proof of Lemma 4.5 It is enough to prove the estimate for an invariant subset $U_\epsilon \subset U \setminus \{z \in B, d(z, J(p)) \geq \epsilon\}$. First note that the conclusion of Lemma 4.4 holds for all $z_0 \in U_\epsilon$, since for all $z_0 \in U_\epsilon$ the condition $p(z_0) \in V$ holds. Since U is a Siegel disk, the forward orbit $\{z_n\}_{n\geq 0}$ lies in a compact subset S of U, where $z_n = p^n(z_0)$. Thus there is a lower bound $a > 0$ such that $r(z) \geq a$ for $z \in S$, a depending on S. By Lemma 4.4 there are two cases,

(1) There is no such integer n that $r(z_n) \geq \beta\ d(z_n, J(p))^{(d_1+1)d_1}$, thus all z_n satisfy $r(z_n) \geq \frac{\alpha}{|\lambda|} r(z_{n+1}) d(z_n, C(p))^{d_1(d_1+1)}$.

(2) There is an integer n such that $r(z_n) \geq \beta\ d(z_n, J(p))^{d_1(d_1+1)}$.

In case (1) for every $i \geq 0$

$$\begin{aligned}\log r(z_i) &\geq \log r(z_{i+1}) + \log d(z_i, C(p))^{d_1(d_1+1)} + \log \frac{\alpha}{|\lambda|} \\ &= \log r(z_{i+1}) - d_1(d_1+1)k(z_i) + \log \frac{\alpha}{|\lambda|}.\end{aligned}$$

Thus we have for every $n \geq 0$,

$$\begin{aligned}\log r(z_0) &\geq \log r(z_n) - d_1(d_1+1)\sum_{i=0}^{n-1} k(z_i) + n \log \frac{\alpha}{|\lambda|} \\ &\geq \log a + \frac{qd_1(d_1+1)}{\rho} \log d(z_0, J(p)) - \frac{qd_1(d_1+1)}{\rho} n \log s - d_1(d_1+1)Qn + n \log \frac{\alpha}{|\lambda|}.\end{aligned}$$

Let us now fix $|\lambda_1|$ so small such that

$$\log \frac{\alpha}{\lambda_1} \geq \frac{l + qd_1(d_1+1)}{\rho} \log s + d_1(d_1+1)Q + 1, \tag{4.11}$$

thus for every $|\lambda| < \lambda_1$ we have

$$\log r(z_0) \geq \log a + \frac{qd_1(d_1+1)}{\rho} \log d(z_0, J(p)) + n.$$

Let $n \to \infty$ then $r(z_0)$ can be arbitrary large, which is a contradiction, thus actually case (1) can not happen.

For the case (2), the proof is same as the proof of Theorem 4.3, thus the proof is complete. □

Remark 4.6. *The constant λ_1 appearing in Theorem 4.3 is invariant under local coordinate change $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$. To see this from Lemma 4.4 and Remark 4.2 we know that α is invariant since it only depends on p and K. By (4.10) and (4.11) λ_1 only depends on α and p, hence $\lambda_1(f)$ is invariant.*

5. Estimate of size of forward images of vertical Fatou disks

In this section we adapt the DPU Lemma to the attracting local polynomial skew product case, to show that the size of forward images of a wandering vertical Fatou disk shrinks slowly. We begin with two classical lemmas. We follow Lilov's presentation.

Lemma 5.1. *There exist $c_0 > 0$ depending only on p and $\delta_2 > 0$ such that when $|t_0| < \delta_2$, let $\Delta(x, r) \subset \{t = t_0\}$ be an arbitrary vertical disk, then $P(\Delta(x, r))$ contains a disk $\Delta(P(x), r') \subset \{t = \lambda t_0\}$ of radius $\geq c_0 r^d$.*

Proof. For fixed $x = (t, z)$ satisfying $|t| < \delta_2, z \in \mathbb{C}$, and for fixed $r > 0$, define a function

$$f_{t,z,r}(w) = \frac{1}{rM_{t,z.r}} (f_t(z - rw) - f_t(z))$$

which is a polynomial defined on the closed unit disk $\overline{D}(0, 1)$. The positive number $M_{t,z,r}$ is defined by

$$M_{t,z,r} = \sup_{w \in \pi_2(\overline{\Delta}(x,r))} |f_t'(w)|.$$

Let A be the finite dimensional normed space containing all polynomials with degree $\leq d$ on $\overline{D}(0,1)$, equipped with the uniform norm. Since $|f'_{t,z,r}(w)| \leq 1$ on $\overline{D}(0,1)$, the family $\{f'_{t,z,r}\}$ is bounded in A. Notice that $f_{t,z,r}(0) = 0$, so that $\{f_{t,z,r}\}$ is also bounded in A. The closure of $\{f_{t,z,r}\}$ contains no constant map since the derivative of constant map vanishes. but $\sup_{\overline{D}(0,1)} |f'_{t,z,r}(w)| = 1$.

Now suppose that there is a sequence $\{f_{t_n,z_n,r_n}\}$ such that $f_{t_n,z_n,r_n}(D(0,1))$ does not contains $D(0,\delta_n)$, with $\delta_n \to 0$. We can take a sub-sequence $f_{t_n,z_n,r_n} \to g$, where g is a non-constant polynomial map with $g(0) = 0$. Therefore by open mapping Theorem $g(D(0,\frac{1}{2}))$ contains $D(0,\delta)$ for some $\delta > 0$. Then for n large enough $f_{t_n,z_n,r_n}(D(0,1))$ also contains $D(0,\delta)$, which is a contradiction. Therefore for all parameter $\{t,z,r\}$, $f_{t,z,r}(D(0,1))$ contain a ball $D(0,\delta)$, which is equivalent to say that

$$\Delta(P(x), \delta r M_{t,z,r}) \subset P(\Delta(x,r)). \tag{5.1}$$

Next we estimate $M_{t,z,r}$ from below. Let $z_1(t), z_2(t), ..., z_{d-1}(t)$ be all zeroes of $f'_t(z)$. Then $f'_t(z) = da_d(t)(z - z_1(t)) \cdots (z - z_{d-1}(t))$. We choose δ_2 small such that $c_0 = \inf_{|t|\leq\delta_2} |da_d(t)| > 0$. Then we have

$$\begin{aligned} M_{t,z,r} = \sup_{w\in\pi_2(\overline{\Delta}(x,r))} |f'_t(w)| &= \sup_{w\in\pi_2(\overline{\Delta}(x,r))} |da_d(t)(z - z_1(t)) \cdots (z - z_{d-1}(t))| \\ &\geq c_0 r^{d-1} \sin^{d-1} \frac{\pi_2}{d-1}, \end{aligned}$$

this with (5.1) finishes the proof. $\square$

Lemma 5.2. *There exist $0 < c < c_0$, $\delta_2 > 0$ such that if a vertical disk $\Delta(x,r) \subset \{t = t_0\}$ satisfies $\Delta(x,r) \subset \{|z| < R\}$, $|t_0| < \delta_2$ and $\eta = d(\Delta(x,r), \{t = t_0\} \cap Crit(P)) > 0$, then $P(\Delta(z,r))$ contains a disk $\Delta(P(x), r') \subset \{t = \lambda t_0\}$ of radius $\geq c\eta^{2d-2}r$.*

Proof. Let $V = \{x_0 = (t_0, z_0) : |t_0| < \delta_2, |z_0| < R, d(x_0, \{t = t_0\} \cap \mathrm{Crit}(P)) > \eta\}$, and set

$$M_1 = \inf_V \left|\frac{\partial f}{\partial z}\right| > 0,\ M_2 = \sup_V \left|\frac{\partial^2 f}{\partial z^2}\right| < \infty,$$

here M_1 depends on η but M_2 does not.

Thus for $\Delta(x_0,r) \subset \{t = t_0\}$ satisfying $\Delta(x_0,r) \subset \{|z| < R\}$ and $\eta = d(\Delta(x_0,r), \{t = t_0\} \cap \mathrm{Crit}(P)) > 0$, we have $\Delta = \Delta(x_0,r) \subset V \cap \{t = t_0\}$. Pick an arbitrary a in the interior of $\pi_2(\Delta)$. Then for all $z \in \partial\pi_2(\Delta)$, we let

$$h(z) = f_{t_0}(z) - f_{t_0}(a) = f'_{t_0}(z)(z-a) + \frac{1}{2}(z-a)^2 g(z).$$

We know $g(z)$ satisfies $|g(z)| \leq M_2$, so that

$$|f'_{t_0}(z)(z-a)| \geq M_1|z-a| \geq M_1 \frac{|z-a|^2}{2r}.$$

In the case $r \leq \frac{M_1}{2M_2}$ we have

$$|f'_{t_0}(z)(z-a)| \geq M_2|z-a|^2 > \frac{1}{2}|(z-a)^2 g(z)|.$$

Thus by Rouché's Theorem the function $h(z)$ has the same number of zero points as $f'_{t_0}(z)(z-a)$, thus $h(z)$ has exactly one zero point $\{z=a\}$. Since $a \in \pi_2(\Delta)$ is arbitrary we have f_{t_0} is injective on Δ. The classical Koebe's one-quarter Theorem shows that $P(\Delta(x_0, r))$ contains a disk with radius at least

$$\frac{1}{4}\left|\frac{\partial f_{t_0}}{\partial z}(z_0)\right| r. \tag{5.2}$$

Now we estimate $\left|\frac{\partial f_{t_0}}{\partial z}(z_0)\right|$ from below. Let $z_1(t), z_2(t), ..., z_{d-1}(t)$ be all zeroes of $f'_t(z)$. Then $f'_t(z) = da_d(t)(z-z_1(t))\cdots(z-z_{d-1}(t))$. We choose δ_2 such that $c_0 = \inf_{|t|\leq\delta_2}|da_d(t)| > 0$. We have for every $1 \leq i \leq d-1$, $|z_0 - a_i(t_0)| \geq \eta$. Thus we have

$$\left|\frac{\partial f_{t_0}}{\partial z}(z_0)\right| = |da_d(t_0)(z_0 - z_1(t_0))\cdots(z_0 - z_{d-1}(t_0))| \geq c_0\eta^{d-1},$$

this with (5.2) gives

$$r' \geq \frac{1}{4}c_0\eta^{d-1}r.$$

In the case $r \geq \frac{M_1}{2M_2}$, by the same argument we have

$$r' \geq \frac{1}{4}c_0\eta^{d-1}\frac{M_1}{2M_2} \geq \frac{1}{8M_2}c_0^2\eta^{2d-2}. \tag{5.3}$$

Setting $c = \frac{1}{2}\min\left\{\frac{c_0}{4R^{d-1}}, \frac{1}{8RM_2}c_0^2\right\}$ we get the conclusion.

REMARK 5.3. *We note that c is invariant under a local coordinate change of the form $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$. To see this, we know c_0 and R are invariant under a local coordinate change of the form $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$, and by shrinking $\delta_2(\phi)$ we have*

$$\sup_V \left|\frac{\partial^2 f(\phi(t), z)}{\partial z^2}\right| \leq 2M_2,$$

thus from (5.3) and $c = \frac{1}{2}\min\left\{\frac{c_0}{4R^{d-1}}, \frac{1}{8RM_2}c_0^2\right\}$ we get that c is invariant.

□

Now we show that the size of forward images of a wandering vertical Fatou disk shrinks slowly. We begin with a definition.

DEFINITION 5.4. *Define the **inradius** ρ as follows: for a domain $U \subset \mathbb{C}$, for every $z \in U \subset \mathbb{C}$, define*

$$\rho(z, U) = \sup\{r > 0 | \ D(z, r) \subset U\},$$

here $D(z, r)$ is a disk centered at z with radius r.

PROPOSITION 5.5. *Let $\Delta_0 \subset \{t = t_0\}$ be a wandering vertical Fatou disk centered at $x_0 = (t_0, z_0)$. Let $x_n = (t_n, z_n) = P^n(x_0)$. Set $\Delta_n = P^n(\Delta_0)$ for every $n \geq 1$ and let $\rho_n = \rho(z_n, \pi_2(\Delta_n))$. There is a constant $\lambda_2(f)$ such that for fixed $|\lambda| < \lambda_2$, we have*

$$\lim_{n\to\infty}\frac{|\lambda|^n}{\rho_n} = 0.$$

Proof. Let λ_3 be a positive constant to be determined. It is sufficient to prove the result by replacing Δ_n by $\Delta_n \cap \Delta(x_n, \lambda_3^{n+1})$. In the following we let Δ_n always be contained in $\Delta(x_n, \lambda_3^{n+1})$.

Without loss generality we can assume that $|t_0| < \min\{\delta_1, \delta_2, \lambda_3\}$, where δ_1 is the constant in Lemma 4.1 and δ_2 is the constant in Lemma 5.1 and Lemma 5.2 . Let N be a fixed integer such that $N > d^q + 1$, where q is the number of critical points lying in $J(p)$. Let $K = \{|t| < \min\{\delta_1, \delta_2, \lambda_3\}\} \times \{|z| < R\}$ be a relatively compact subset of $\mathbb{C}^2$ such that for $(t, z) \notin K$, $|f(t, z)| \geq 2|z|$. Since the orbits of points in Δ_0 cluster only on $J(p)$,we have $\Delta_n \subset K$ for every n . We need the following lemma:

Lemma 5.6. *There is a constant $M > 0$ that if $|\lambda| < \lambda_3$, for every n and for every $x' = (t_n, w_n) \in \Delta_n$, for every integer m, letting $(t_{n+m}, w_{n+m}) = P^m(x')$ we have,*

$$|w_{n+m} - p^m(z_n)| \leq M^m \lambda_3^{n+1}. \tag{5.4}$$

Proof. We prove it by induction. Let M satisfying for $(t, z) \in K$, $\left|\frac{\partial f(t,z)}{\partial t}\right| \leq \frac{M}{2}$ and $\left|\frac{\partial f(t,z)}{\partial z}\right| \leq \frac{M}{2}$. We can also assume M is larger than the constant K in Lemma 4.1. Thus For $m = 0$ it is obviously true. Assume that when $m = k - 1$ is true, we have

$$\begin{aligned}
|w_{n+k} - p^k(z_n)| &= |f(t_{n+k-1}, w_{n+k-1}) - f(0, p^{k-1}(z_n))| \\
&\leq \frac{M}{2}|t_{n+k-1}| + \frac{M}{2}|w_{n+k-1} - p^{k-1}(z_n)| \\
&\leq \frac{M}{2}|\lambda|^{n+1} + \frac{M^k}{2}\lambda_3^{n+1} \\
&\leq M^k \lambda_3^{n+1}.
\end{aligned}$$

Thus for every m, (5.4) holds. □

Remark that when $w_n = z_n$, the same argument gives

$$|z_{n+m} - p^m(z_n)| \leq M^m |\lambda|^{n+1}.$$

Let $C(P)$ be the union of components of $\mathrm{Crit}(P)$ such that meet $C(p) = \mathrm{Crit}(p) \cap J(p)$ in the invariant fiber. For every point $x \in \Delta_n$, we define $k(x) = -\log d(x, C(P) \cap \{t = t_n\})$, and $k_n = \sup_{x \in \Delta_n} k(x)$. (This definition allows $k_n = +\infty$.) Recall that N is a fixed integer such that $N > d^q + 1$. We are going to prove a two dimensional DPU Lemma for attracting polynomial skew products:

Lemma 5.7 (**Two Dimensional DPU Lemma**). *Let $|\lambda| < \lambda_3$, then for every $N^k \leq n < N^{k+1}$, there is a subset*

$$\left\{\alpha_1, \cdots, \alpha_{q'}\right\} \subset \left\{N^k - 1, N^k, \cdots, n - 1\right\}$$

and a constant $Q > 0$ such that

$$\sum_{i=N^k-1}^{n-1} k_i - \sum_{i=1}^{q'} k_{\alpha_i} \leq Q(n - N^k + 1), \tag{5.5}$$

here k is an arbitrary integer, $q' \le q$ is an integer. Recall that q is the number of critical points lying in $J(p)$.

Proof. Recall that the DPU Lemma implies that there is a subset

$$\{\alpha_1, \cdots, \alpha_{q'}\} \subset \left\{N^k - 1, N^k, \cdots, n-1\right\}$$

and a constant $Q > 0$ such that

$$\sum_{i=N^k-1}^{n-1} k(p^{i-N^k+1}(z_{N^k-1})) - \sum_{j=1}^{q'} k(p^{\alpha_j - N^k+1}(z_{N^k-1})) \le \frac{Q}{2}(n - N^k + 1). \tag{5.6}$$

So it is sufficient to prove $k_i \le 2k(p^{i-N^k+1}(z_{N^k-1}))$ for every i not appearing in $\{\alpha_1, \ldots, \alpha_{q'}\}$. This is equivalent to

$$d(\Delta_i, C(P) \cap \{t = t_i\}) \ge d(p^{i-N^k+1}(z_{N^k-1}), C(p))^2. \tag{5.7}$$

To prove (5.7), assume that $d(p^{i-N^k+1}(z_{N^k-1}), C(p)) = d(p^{i-N^k+1}(z_{N^k-1}), c_k)$ for some point $c_k \in C(p)$, let C_k be the component of $C(P)$ which meats c_k at invariant fiber, by (5.4) and Lemma 4.1 we have

$$\begin{aligned} d(\Delta_i, C(P) \cap \{t = t_i\}) &\ge d(p^{i-N^k+1}(z_{N^k-1}), C(p)) - \sup_{x' \in \Delta_i} |\pi_2(x') - p^{i-N^k+1}(z_{N^k-1})| - |w_i - c_k| \\ &\ge d(p^{i-N^k+1}(z_{N^k-1}), C(p)) - M^{i-N^k+1}\lambda_3^{N^k} - M|\lambda|^{\frac{N^k}{d_1}}, \end{aligned}$$

where w_i is $\pi_2(C_k \cap \{t = t_i\})$.

By $|\lambda| < \lambda_3$ we have

$$M^{i-N^k+1}\lambda_3^{N^k} + M|\lambda|^{\frac{N^k}{d_1}} = \left(M^{i-N^k+1} + M\right)\lambda_3^{\frac{N^k}{d_1}}.$$

Thus we have

$$d(\Delta_i, C(P) \cap \{t = t_i\}) \ge d(p^{i-N^k+1}(z_{N^k-1}), C(p)) - \left(M^{i-N^k+1} + M\right)\lambda_3^{\frac{N^k}{d_1}}.$$

To prove (5.7) it is sufficient to prove

$$\left(M^{i-N^k+1} + M\right)\lambda_3^{\frac{N^k}{d_1}} \le d(p^{i-N^k+1}(z_{N^k-1}), C(p)) - d(p^{i-N^k+1}(z_{N^k-1}), C(p))^2. \tag{5.8}$$

By (5.6) we have

$$d(p^{i-N^k+1}z_{N^k-1}, C(p)) \ge e^{-\frac{Q}{2}(n-N^k+1)},$$

thus it is sufficient to prove

$$(M^{n-N^k+1} + M)\lambda_3^{\frac{N^k}{d_1}} \le e^{-\frac{Q}{2}(n-N^k+1)} - e^{-Q(n-N^k+1)}. \tag{5.9}$$

We can always choose λ_3 sufficiently small to make (5.9) holds for all $k \ge 0$. This ends the proof of the two dimensional DPU Lemma (5.5). □

By Lemma 5.1 and Lemma 5.2 there is a constant $c > 0$ such that

$$\rho_{n+1} \geq c e^{-(2d-2)k_n} \rho_n. \tag{5.10}$$

and

$$\rho_{n+1} \geq c\rho_n^d. \tag{5.11}$$

From the above we can now give some estimates of ρ_n. Recall that ρ_n is assumed smaller than $|\lambda_3|^{n+1}$ otherwise we replace it by $\min\left\{\rho_n, \lambda_3^{n+1}\right\}$.

Lemma 5.8. *There is a constant $c_1 > 0$ such that for $N^k \leq n < N^{k+1}$, we have*

$$\rho_n \geq c_1^{N^k} \rho_{N^k-1}^{d^q}.$$

Proof. For $N^k \leq i \leq n$, if $i - 1 \in \{\alpha_1, \cdots, \alpha_q\}$ we apply inequality (5.10), if $i \notin \left\{\alpha_1, \cdots, \alpha_{q'}\right\}$ we apply inequality (5.11). Thus we have

$$\begin{aligned}
\rho_n &\geq c^{n-\alpha_{q'}+1} \exp\left(-(2d-2)\sum_{j=\alpha_{q'}+1}^{n} k_j\right)\left(\cdots\left(c^{\alpha_1-N^k+1}\exp\left(-(2d-2)\sum_{j=N^k-1}^{\alpha_1-1} k_j\right)\rho_{N^k-1}\right)^d\cdots\right)^d \\
&\geq c^{(n-N^k+1)d^q}\exp\left(-d^q\sum_{j=N^k-1}^{n-1} k_i + d^q\sum_{j=1}^{q'} k_{\alpha_j}\right)\rho_{N^k-1}^{d^q} \quad \text{(because } q' \leq q\text{)} \\
&\geq c^{(n-N^k+1)d^q}\exp\left(-Qd^q\left(n - N^k + 1\right)\right)\rho_{N^k-1}^{d^q} \qquad \text{(by Lemma 5.7)} \\
&\geq c^{N^{k+1}d^q}\exp\left(-Qd^q N^{k+1}\right)\rho_{N^k-1}^{d^q}.
\end{aligned}$$

Setting $c_1 = \min\left\{c^{Nd^q}e^{-QNd^q}, \lambda_3\right\}$ we get the desired conclusion. □

Lemma 5.9. *For $N^k \leq n < N^{k+1}$, $\rho_0 \leq \lambda_3$ we have*

$$\rho_n \geq c_1^{N^{k+1}} \rho_0^{d^{q(k+1)}}.$$

Proof. By iterating Lemma 5.8 we get that

$$\rho_{N^k-1} \geq c_1^{\frac{N^k-d^{qk}}{N-d^q}} \rho_0^{d^{qk}} \geq c_1^{N^k} \rho_0^{d^{qk}},$$

so that

$$\rho_n \geq c_1^{N^k}\rho_{N^k-1}^{d^q} \geq c_1^{N^{k+1}}\rho_0^{d^{q(k+1)}},$$

this finishes the proof. □

Now we can conclude the proof of Proposition 5.5. For $N^k \leq n < N^{k+1}$ we get

$$\frac{|\lambda|^n}{\rho_n} \leq \frac{|\lambda|^{N^k}}{c_1^{N^{k+1}}\rho_0^{d^{q(k+1)}}}.$$

Choosing λ_2 small such that

$$\lambda_2 < c_1^N, \tag{5.12}$$

since $N > d^q + 1$ we deduce that for every $|\lambda| < \lambda_0$,

$$\lim_{k\to\infty} \frac{|\lambda|^{N^k}}{c_1^{N^{k+1}} \rho_0^{d^{q(k+1)}}} = 0,$$

finally $\lim_{n\to\infty} \frac{|\lambda|^n}{\rho_n} = 0$, which finishes the proof.

□

REMARK 5.10. *The constant λ_2 appearing in Proposition 5.5 is invariant under a local coordinate change of the form $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$. To see this we know that by (5.9) λ_3 depends only on M and p, M can be dealt with by replacing it everywhere by $2M$ (see Remark 4.2), so that λ_3 is invariant. By putting $c_1 = \min\left\{c^{Nd^q} e^{-QNd^q}, \lambda_3\right\}$ we get that c_1 is invariant. Then by (5.12) we get that λ_2 is invariant.*

COROLLARY 5.11. *In the same setting as Proposition 5.5, for every $l > 0$, if λ is chosen sufficiently small, we have*

$$\lim_{n\to\infty} \frac{|\lambda|^n}{\rho_n^l} = 0.$$

PROOF. By Proposition 5.5 if $|\lambda| < \lambda_2$, then $\lim_{n\to\infty} \frac{|\lambda|^n}{\rho_n} = 0$ holds. For any $l > 0$, we then let $|\lambda|$ smaller than λ_2^l to make the conclusion holds. □

6. Proof of the non-wandering domain theorem

In this section we prove the non-existence of wandering Fatou components. Let us recall the statement

THEOREM 6.1 (**No wandering Fatou components**). *Let P be a local polynomial skew product with an attracting invariant fiber,*

$$P(t, z) = (\lambda t, f(t, z)).$$

Then for any fixed f, there is a constant $\lambda_0(f) > 0$ such that if λ satisfies $0 < |\lambda| < \lambda_0$, every forward orbit of a vertical Fatou disk intersects a bulging Fatou component. In particular every Fatou component iterates to a bulging Fatou component, and there are no wandering Fatou components.

PROOF. We argue by contradiction. Suppose $\Delta_0 \subset \{t = t_0\}$ is a vertical disk lying in a Fatou component which does not iterate to a bulging Fatou component. Without loss generality we may assume $|t_0| < \min\{1, \delta_1, \delta_2, \lambda_3\}$. By Remark 2.2, Δ_0 is a vertical Fatou disk. Let $x_0 = (t_0, z_0) \in \Delta_0$ be the center of Δ_0 and set $x_n = (t_n, z_n) = P^n(t_0, z_0)$ and $\Delta_n = P^n(\Delta_0)$. We divide the proof into several steps, We set $\rho_n = \rho(z_n, \pi_2(\Delta_n))$ as before and assume that $\rho_0 \le \lambda_3$. Notice that Δ_0 can not be contained in the basin of infinity, thus Δ_n is uniformly bounded. Let $\lambda_0 < \min\left\{\lambda_1, \lambda_2^l\right\}$, where λ_1 and λ_2 come from Theorem 4.3 and Proposition 5.5. In the course of the proof we will have to shrink λ_0 one more time.

• **Step 1.** By Remark 3.6, the orbits of points in Δ_0 cluster only on $J(p)$.

• **Step 2.** We show that there exist $N_0 > 0$ such that when $n \geq N_0$, the projection $\pi_2\left(\Delta(x_n, \frac{\rho_n}{4})\right)$ intersects $J(p)$. We determine N_0 in the following. Suppose $\pi_2\left(\Delta(x_n, \frac{\rho_n}{4})\right)$ does not intersect $J(p)$. Thus $z_n \in F(p)$ and Theorem 4.3 implies $r(z_n) \geq k\ d(z_n, J(p))^l$, then we have

$$\frac{|t_n|}{r(z_n)} \leq \frac{|t_n|}{k\ d(z_n, J(p))^l} \leq \frac{4^l |t_n|}{k\ \rho_n^l}.$$

By Corollary 5.11 we can let N_0 large enough so that for all $n \geq N_0$, $\frac{4^l|t_n|}{k\ \rho_n^l} < 1$. From the definition of $r(z_n)$ we get a horizontal holomorphic disk defined by $\phi(t)$, $|t| < r(z_n)$ contained in the bulging Fatou components, with $\phi(0) = z_n$, and t_n is in the domain of ϕ. Then we have

$$|\phi(t_n) - z_n| = |\phi(t_n) - \phi(0)| \leq 2R\frac{|t_n|}{r(z_n)} \leq 2R\frac{|t_n|}{k\ d(z_n, J(p))^l} \leq 2R\frac{4^l|t_n|}{k\ \rho_n^l}.$$

Again by Corollary 5.11, we can let N_0 large enough that for all $n \geq N_0$, $2R\frac{4^l|t_n|}{k\ \rho_n^l} < \frac{\rho_n}{4}$. Thus $\phi(t_n) \in \Delta(x_n, \frac{\rho_n}{4}) \subset \Delta_n$. Since $\phi(t_n)$ is contained in the bulging Fatou components that contains z_n, this implies Δ_n intersects the bulging Fatou component so it can not be wandering. This contradiction shows that $\pi_2\left(\Delta(x_n, \frac{\rho_n}{4})\right)$ intersects $J(p)$.

Let $y_n \in \Delta_n$ satisfies $\pi_2(y_n) \in \pi_2\left(\Delta(x_n, \frac{\rho_n}{4})\right) \cap J(p)$, then for all $x \in \Delta(y_n, \frac{\rho_n}{4})$ we have $\rho(\pi_2(x), \pi_2(\Delta_n)) \geq \frac{\rho_n}{2}$.

• **Step 3.** We show that there is an integer $N_1 > N_0$ such that for every $x \in \Delta(y_{N_1}, \frac{\rho_{N_1}}{4})$, for every $m \geq 0$, $p^m(\pi_2(x)) \in \pi_2(\Delta_{m+N_1})$, here $\pi_2(y_{N_1}) \in \pi_2\left(\Delta(x_{N_1}, \frac{\rho_{N_1}}{4})\right) \cap J(p)$. This means that the orbit of $\pi_2(x)$ is always shadowed by the orbit of Δ_{N_1}, which will contradict the fact that $\pi_2(\Delta_{m+N_1})$ intersects $J(p)$. To show this, we inductively prove the more precise statement that for fixed $N > d^q + 1$, there exist a large $N_1 = N^{k_0} - 1 > N_0$, such that for every $k \geq k_0$, $N^k \leq n < N^{k+1}$, we have

$$p^{n-N_1}(\pi_2(x)) \in \pi_2(\Delta_n) \tag{6.1}$$

and

$$\rho_n' \geq c_2^{N^{k+1}} \rho_0^{d^{q(k+1)}}, \tag{6.2}$$

where $\rho_n' = \rho(p^{n-N_1}(\pi_2(x)), \pi_2(\Delta_n)$, $c_2 = \frac{c_1}{2}$ comes from Lemma 5.8 and Lemma 5.9. We will determine k_0 in the following.

From Lemma 5.9 we know that (6.1) and (6.2) hold for $n = N_1$. Assume that for some $k \geq k_0$, for all $n \leq N^k - 1$, (6.1) and (6.2) holds. Then for $N^k \leq n < N^{k+1}$, let $y = P^{n-N^k+1}(t_{N^k-1}, p^{N^k-1-N_1}(\pi_2(x)))$, by Lemma 5.8 we have

$$\rho(y, \Delta_n) \geq c_1^{N^k} \left(\rho'_{N^k-1}\right)^{d^q}. \tag{6.3}$$

To estimate the distance between $\pi_2(y)$ and $p^{n-N_1}(\pi_2(x))$, by Lemma 5.6 we have

$$|\pi_2(y) - p^{n-N_1}(\pi_2(x))| \leq M^{n-N^k+1} |\lambda|^{N^k}. \tag{6.4}$$

From (6.3) and (6.4) we have

$$\begin{aligned}\rho'_n &\geq c_1^{N^k}\left({\rho'}_{N^k-1}\right)^{d^q} - M^{n-N^k+1}|\lambda|^{N^k}\\ &\geq c_1^{N^k}(c_2^{N^k}\rho_0^{d^{qk}})^{d^q} - M^{n-N^k+1}|\lambda|^{N^k} \qquad \text{(By the induction hypothesis (6.2))}\\ &\geq c_1^{N^k}c_2^{N^k d^q}\rho_0^{d^{q(k+1)}} - M^{n-N^k+1}|\lambda|^{N^k}.\end{aligned}$$

By the choice $c_2 = \frac{c_1}{2}$ we have

$$\rho'_n \geq 2c_2^{N^{k+1}}\rho_0^{d^{q(k+1)}} - M^{n-N^k+1}|\lambda|^{N^k}.$$

To get (6.2) it is sufficient to prove

$$c_2^{N^{k+1}}\rho_0^{d^{q(K+1)}} \geq M^{n-N^k+1}|\lambda|^{N^k}.$$

We take λ_0 sufficiently small such that

$$\lambda_0 \leq (\frac{c_2}{M})^{2N}. \tag{6.5}$$

Thus to prove (6.2) it is sufficient to prove that when $|\lambda| < \lambda_0$,

$$\rho_0^{d^{q(k+1)}} \geq |\lambda|^{\frac{N^k}{2}}. \tag{6.6}$$

Since $N > d^q + 1$, we can choose k_0 large enough such that for every $k > k_0$ (6.6) holds. This finishes the induction.

This shows that (6.1) and (6.2) are true for all $n \geq N_1$.

• **Step 4.** Since for every $x \in \Delta(y_{N_1}, \frac{\rho_{N_1}}{4})$, for every $m \geq 0$, $p^m(\pi_2(x)) \in \pi_2(\Delta_{m+N_1})$, and Δ_n is uniformly bounded, the family $\{p^m\}_{m\geq 0}$ restricts on $D(\pi_2(y_{N_1}), \frac{\rho_{N_1}}{4}))$ is a normal family. Thus $\pi_2(y_{N_1})$ belongs to the Fatou set $F(p)$, this contradicts to $\pi_2(y_{N_1}) \in J(p)$. Thus the proof is complete.

□

REMARK 6.2. *The constant λ_0 appearing in Theorem 6.1 is invariant under a local coordinate change of the form $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$. To see this we know that the constants $c_2 = \frac{c_1}{2}$, M and N are invariant under a local coordinate change of the form $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$ (M can be dealt with by replacing it everywhere by $2M$, see Remark 4.2). Then by (6.5) λ_0 only depends on c_2, M and N, thus λ_0 is invariant.*

REMARK 6.3. *Lilov's Theorem can be seen as a consequence of Theorem 6.1. In fact, for the super-attracting case, the Fatou components of p bulge for a similar reason. Since when $|t|$ is very small, the contraction to the invariant fiber is stronger than any geometric contraction $t \mapsto \lambda t$, Theorem 4.3 and Proposition 5.5 follows easily. Thus following the argument of Theorem 6.1 gives the result.*

In the following theorem we show how the main theorem can be applied to globally defined polynomial skew products.

THEOREM 6.4. *Let*

$$P(t,z) = (g(t), f(t,z)) : \mathbb{C}^2 \to \mathbb{C}^2$$

be a globally defined polynomial skew product, where g, f are polynomials. Assume deg $f = d$ and the coefficient of the term z^d of f is non-vanishing, then there exist a constant $\lambda_0(t_0, f) > 0$ depending only on f and t_0 such that if $g(t_0) = t_0$ and $|g'(t_0)| < \lambda_0$ then there are no wandering Fatou components in $B(t_0)\times\mathbb{C}$, where $B(t_0)$ is the attracting basin of g at t_0 in the t-coordinate.

PROOF. First by a coordinate change $\phi_0 : t \mapsto t + t_0$, P is conjugated to

$$P_0 : (t,z) \mapsto (g_0(t), f_0(t,z)),$$

where $g_0(t) = g(t+t_0) - t_0$, and $f_0(t,z) = f(t+t_0, z)$. It is clear that $\{t = 0\}$ becomes an invariant fiber.

By Koenig's Theorem we can introduce a local coordinate change $\phi : t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$ such that P_0 is locally conjugated to

$$(t,z) \mapsto (\lambda t, f_0(\phi(t), z)), \tag{6.7}$$

where $\lambda = g'(t_0)$.

We have seen in Remark 6.2 that the constant $\lambda_0(f)$ is invariant under a local coordinate change of the form $t \mapsto \phi(t)$ with $\phi(0) = 0$ and $\phi'(0) = 1$. This means that for fixed f, for every such ϕ,

$$P_\phi : (t,z) \mapsto (\lambda t, f(\phi(t), z))$$

has no wandering Fatou components when $|\lambda| < \lambda_0(f)$. Thus applying this to (6.7) when $|\lambda| = |g'(t_0)| < \lambda_0(f_0)$ we get the local skew product $(t,z) \mapsto (\lambda t, f_0(\phi(t), z))$ has no wandering Fatou components. Thus by conjugation P has no wandering Fatou components in a neighborhood of $\{t = t_0\}$, thus actually P has no wandering Fatou components in $B(t_0)\times\mathbb{C}$, where $B(t_0)$ is the attracting basin of g at t_0 in the t-coordinate. □

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CHAPTER 3

Non-uniform hyperbolicity in polynomial skew products

1. Introduction

1.1. Background. In one-dimensional complex dynamics, i.e. in the theory of dynamics of rational maps on Riemann sphere $\mathbb{P}^1$, the classical Fatou-Julia dichotomy partitions the Riemann sphere into the Fatou set and the Julia set. Let f be a rational map on $\mathbb{P}^1$, the *Fatou set* $F(f)$ is defined as the largest open subset of $\mathbb{P}^1$ in which the sequence of iterates $(f^n)_{n\geq 0}$ is normal. Its complement is the *Julia set* $J(f)$. A *Fatou component* is a connected component of $F(f)$. A Fatou component is called *wandering* if it is not pre-periodic. One can show that the Fatou set is either empty or an open and dense subset. The dynamics on the Fatou set is completely understood, due to the work of Fatou, Julia, Siegel and Herman, supplemented with Sullivan's non-wandering domain theorem [46]: the orbit of any point in the Fatou set eventually lands in an attracting basin, a parabolic basin, a Siegel disk or a Herman ring. See Milnor [29] for a self-contained proof.

If in addition f satisfies some *non-uniformly hyperbolic* conditions, the measurable dynamics of f can also be understood. There are various hyperbolic conditions, such as uniform hyperbolicity, sub-hyperbolicity, the *Collet-Eckmann* condition (*CE* for short), the *Topological Collet-Eckmann* condition (*TCE* for short), the condition that Lyapunov exponent at all critical values exist and is positive, and f has no parabolic cycles (*Positive Lyapunov* for short), the *Weak regularity condition* (*WR* for short), etc. We say that f satisfies TCE if there is an "Exponential shrinking of components" on the Julia set, see the precise definition in Definition 2.6.

THEOREM. **(Przytycki, Rivera-Letelier, Smirnov [39, Theorem 4.3])** *Let f be a TCE rational map on $\mathbb{P}^1$ with degree at least 2 and such that $J(f) \neq \mathbb{P}^1$. Then the Fatou set $F(f)$ is equal to the union of a finite number of attracting basins, and the Julia set $J(f)$ has Hausdorff dimension strictly smaller than 2 (hence it has area zero).*

In higher dimensional complex dynamics, one of the major problems is to study the dynamics of *holomorphic endomorphisms* of $\mathbb{P}^k$, $k \geq 2$. The Fatou and Julia sets can be defined similarly. Unlike the one-dimensional case, little is known about the dynamics on the Fatou set in higher dimension. It is known that Sullivan's non-wandering domain theorem does not hold in general. Indeed Astorg, Buff, Dujardin, Peters and Raissy [3] constructed a holomorphic endomorphism $h : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ induced by a *polynomial skew product*, possessing a wandering Fatou component.

A polynomial skew product f is a polynomial map from $\mathbb{C}^2$ to $\mathbb{C}^2$, of the following form:

$$f(t,z) = (g(t), h(t,z)),$$

where g is a one variable polynomial and h is a two variables polynomial. We assume that g and h have degree at least 2. In the rest of the paper a polynomial map or a rational map is asked to have degree at least 2. See Jonsson [24] for a systematic study of such polynomial skew products, see also Dujardin [17], Astorg and Bianchi [2], Boc-Thaler, Fornaess and Peters [10] for related studies. As the definition suggests, the polynomial skew product leaves invariant a foliation by vertical lines, hence one-dimensional tools can be used. Our first purpose is to study the dynamics of a polynomial skew product on its Fatou set.

We assume h has the expression

$$h(t,z) = \sum_{i+j\leq n} a_{i,j} t^i z^j.$$

If in addition we assume the polynomial skew product f satisfies $\deg\ g = \deg\ h = n$, and $a_{0,n} \neq 0$, then f extends to $\mathbb{P}^2$ holomorphically. In this case the polynomial skew product is called *regular*. The regular polynomial skew products form a sub-class of holomorphic endomorphisms on $\mathbb{P}^2$.

To investigate the Fatou set of f, let π_1 be the projection to the t-coordinate, i.e.

$$\pi_1 : \mathbb{C}^2 \to \mathbb{C}, \ \ \pi_1(t,z) = t.$$

We first notice that $\pi_1(F(f)) \subset F(g)$, and passing to some iterate of f, we may assume that the points in $F(g)$ will eventually land into an immediate basin or a Siegel disk (no Herman rings for polynomials), thus we only need to study the following semi-local case:

$$f = (g,h) : \Delta \times \mathbb{C} \to \Delta \times \mathbb{C}, \tag{1.1}$$

where $g(0) = 0$ which means the line $L : \{t = 0\}$ is invariant and Δ is an immediate attracting or a parabolic basin or a Siegel disk of g. The map f is called attracting, parabolic or elliptic respectively when $g'(0)$ is attracting, parabolic, elliptic. The examples of wandering domains constructed in [3] are parabolic polynomial skew products. At this stage it remains an interesting problem to investigate the existence of wandering domains for attracting polynomial skew products, one part of our main theorem answer this question in the negative way under the non-uniformly hyperbolic condition.

In the geometrically attracting case, by Koenigs' Theorem, (1.1) is locally conjugated to

$$f(t,z) = (\lambda t, h(t,z)), \tag{1.2}$$

where $\lambda = g'(0)$. Beware that h is no longer a polynomial in t.

In the super-attracting case by Böttcher's Theorem, (1.1) is locally conjugated to

$$f(t,z) = (t^m, h(t,z)), \ m \geq 2.$$

In both attracting or super-attracting cases

$$h(t,z) = a_0(t) + a_1(t)z + \cdots + a_d(t)z^d$$

is a polynomial in z with coefficients $a_i(t)$ holomorphic in t in a neighborhood of 0. We furthermore assume that $a_d(0) \neq 0$, which means the degree of $h(t,z)$ in z is constant for $t \in \Delta$. This condition is needed in the proof of the main theorem, and it is satisfied for regular polynomial skew products. In the rest of the paper, an attracting polynomial skew product is assumed to have the normal form (1.2), and Δ denotes a small disk centered at 0.

1.2. Main theorem and outline of the proof. In this paper we show that under the non-uniformly hyperbolic hypothesis, we can exclude the existence of wandering domain, and give a classification of the dynamics on the Fatou set, and show that the Julia set has Lebesgue measure zero.

THEOREM. **(Main Theorem)** *Let f be an attracting polynomial skew product in the form of (1.1), let $p = f|_L$ be the restriction of f on the invariant fiber L. Assume that p satisfies one of the following conditions: 1. p satisfies TCE and WR. 2. p satisfies Positive Lyapunov. Then the Fatou set of f coincides with the union of the basins of attracting cycles, and the Julia set of f has Lebesgue measure zero.*

We let ∞ be the point at infinity of L. Since ∞ can be seen as an attracting fixed point, the Fatou set of f is never empty. The definitions of TCE condition, WR condition and Positive Lyapunov condition are given in section 2. The basins of attracting cycles are clearly non-wandering, as a consequence there are no wandering domains in the basin of L. In the rest of the paper we shall prove the main theorem for f geometrically attracting. In the super-attracting case the proof is completely similar, and left to the reader. Note that in that case the non existence wandering Fatou components was established by Lilov [28] (see also [23]).

The proof of the main theorem is divided into several steps. In section 2 we recall some preliminaries, and introduce some one-dimensional techniques such as the Koebe distortion lemma, Przytycki's lemma and Denker-Przytycki-Urbanski's lemma (DPU lemma for short). The one-dimensional non-uniformly hyperbolic theory is also introduced. Then proof of the main theorem goes as follows.

Step 1: We start with some definitions. Let $f : \Delta \times \mathbb{C} \to \Delta \times \mathbb{C}$ be an attracting polynomial skew product, the *critical set* of f is defined by $\mathrm{Crit} := \left\{(t,z) \in \Delta \times \mathbb{C} \mid \frac{\partial h}{\partial z}(t,z) = 0\right\}$. We let the radius of Δ be small enough so that each connected component of Crit intersect L at exactly one point. We define

$$\text{Crit'} := \{\text{the union of the connected components of Crit that intersect } J(p)\}.$$

DEFINITION 1.1. *Let $x \in \Delta \times \mathbb{C}$ be a point in the immediate basin of L, we say that x slowly approach Crit' if for every $\alpha > 0$, $\mathrm{dist}_v(f^n(x), \text{Crit'}) \geq e^{-\alpha n}$ for every sufficiently large n.*

Here dist_v denote the vertical distance which means that $\mathrm{dist}_v(x,y) = |\pi_2(x) - \pi_2(y)|$, where π_2 is the projection to the z-coordinate, i.e.

$$\pi_2 : \mathbb{C}^2 \to \mathbb{C}, \quad \pi_2(t,z) = z.$$

Let also $D_v(x,r)$ denote the vertical disk, $D_v(x,r) = \{y \in \Delta \times \mathbb{C} : \pi_1(y) = \pi_1(x),\ \mathrm{dist}_v(x,y) < r\}$.

This notion of slow approach was introduced by Levin, Przytycki and Shen in one-dimensional complex dynamics in [26]. Next we show that

THEOREM 1.2. *Lebesgue a.e. $x \in \Delta \times \mathbb{C}$ slowly approach Crit'.*

This is proved in sections 3 and 4. In section 3 the existence of a stable manifold at each critical value in $J(p)$ and the properties of renormalization maps associated with a critical value variety are studied. In section 4 we use the techniques developed in section 3 to prove Theorem 1.2. Step 1 is where we need the TCE and WR conditions or the Positive Lyapunov condition, in the remaining steps the TCE condition alone is sufficient for the proof.

Step 2: We define vertical Lyapunov exponent at one point as follows:

DEFINITION 1.3. *Let $x \in \Delta \times \mathbb{C}$ be a point in the immediate basin of L, the* lower vertical Lyapunov exponent *is defined by*

$$\chi_-(x) := \liminf_{n\to\infty} \frac{1}{n} \log |Df^n|_x(v)|.$$

Where $v = (0,1)$ is the unit vertical tangent vector.

It is well-known that the one-dimensional attracting basins of p extend to two-dimensional attracting basins, for example see [28] or [23, Section 3]. These two-dimensional attracting basins correspond to non-wandering Fatou components. We let $W^s(J(p))$ denote the *stable set* of $J(p)$,

$$W^s(J(p)) := \left\{x \in \Delta \times \mathbb{C} : \lim_{n\to\infty} \mathrm{dist}(f^n(x), J(p)) = 0\right\}.$$

It is easy to see that assuming p satisfies TCE, $W^s(J(p))$ is the union of the wandering Fatou components together with the Julia set $J(f)$. We show that

THEOREM 1.4. *If $x \in W^s(J(p))$ slowly approach Crit', then $\chi_-(x) \geq \log \mu_{Exp}$, where $\mu_{Exp} > 1$ is the constant appearing in the definition of the TCE condition.*

This is proved in section 5. Then by Theorem 1.2 Lebesgue a.e. point $x \in W^s(J(p))$ satisfies $\chi_-(x) \geq \log \mu_{Exp}$. This already implies the non-existence of wandering Fatou component thanks to the fact that points in Fatou set can not have positive Lyapunov exponent. Thus $W^s(J(p))$ coincide with the Julia set $J(f)$.

Step 3: Finally by using an adaption of a so called "telescope argument" in [26, Theorem 1.5], we show that

THEOREM 1.5. *The Julia set $J(f)$ has Lebesgue measure zero.*

This is proved in section 6, and the proof is complete.

In Appendix A we study the relations between TCE condition, CE condition, WR condition and Positive Lyapunov condition. In Appendix B we exhibit some families of polynomial maps satisfying the conditions in the main theorem.

1.3. Previous results. The first result of non-wandering domain theorem for polynomial skew products goes back to Lilov [28]. In his PhD thesis Lilov proved that super-attracting polynomial skew products do not have wandering Fatou components. He actually showed a stronger result, namely that there can not have vertical wandering Fatou disks.

In the geometrically attracting case, there are many works trying to understand the dynamics in the atrracting basin of the invariant line. Peters and Vivas showed in [32] that there is an attracting polynomial skew product with a wandering vertical Fatou disk. This result does not answer the existence question of wandering Fatou components, but showed that the question is considerably more complicated than in the super-attracting case. On the other hand, by using a different strategy from Lilov's, Peters and Smit in [31] showed that the non-wandering domain theorem holds in the attracting case, under the assumption that the dynamics on the invariant fiber is sub-hyperbolic. The author showed that the non-wandering domain theorem holds in the attracting case, under the assumption that the multiplier is sufficiently small, following Lilov's strategy, see [23].

In the parabolic case, the examples of wandering domains are constructed in [3], as we have mentioned. See also the recent paper [4].

The elliptic case was studied by Peters and Raissy in [30]. See Raissy [40] for a survey of the history of the investigation of wandering domains for polynomial skew products.

To the best of our knowledge, Theorem 1.5 is the first time where the zero measure of Julia set is shown for non-hyperbolic p (it is in general not true when no conditions of p are assumed, as even in one dimension Julia set can has positive Lebesgue measure, cf. [12] and [6]). The previous results we have mentioned only consider the dynamics on the Fatou set. However when p is uniformly hyperbolic, it is well-known that the Julia set has zero measure and the Fatou set coincide with the union of the basins of attracting cycles. In fact the stable set of $W^s(J(p))$ is foliated by stable manifolds, and this foliation is absolutely continuous hence $W^s(J(p))$ has Lebesgue measure zero.

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2. Preliminaries

In this section we introduce some one-dimensional tools used in the proof of the main theorem. Let $f:\ \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map , let Crit denote the set of critical points, and let Crit' denote the set of critical points lie in the Julia set. Let $CV(f)$ be the critical value set. We fix a Riemannian metric on $\mathbb{P}^1$, and $D(x, \varepsilon)$ denotes a small disk centered at x with radius ε.

2.1. Some technical lemmas. In [34, Lemma 1], Przytycki introduced a fundamental lemma which concerns the recurrence properties of small neighborhood of Crit'.

LEMMA 2.1 (**Przytycki**). *Let $c \in$ Crit'. There exist a constant $C > 0$ such that for every $\varepsilon > 0$ and $n > 0$, if $f^n(D(c,\varepsilon)) \cap D(c,\varepsilon) \neq \emptyset$, then $n \geq C \log \frac{1}{\varepsilon}$.*

We define a positive valued function on $\mathbb{P}^1$ as follows

DEFINITION 2.2. *Let $c \in Crit'$, define a positive valued function $\phi_c(x)$ by*

$$\phi_c(x) := \begin{cases} -\log \operatorname{dist}(x,c), & \text{if } x \neq c \\ \infty, & \text{if } x = c. \end{cases}$$

In terms of using the function $\phi_c(x)$, the Lemma 2.1 can be reformulated as: there exists a constant $Q > 0$, such that for every $x \in \mathbb{P}^1$, $c \in$ Crit', $n \geq 1$, we have

$$\min\left(\phi_c(x),\ \phi_c(f^n(x))\right) \leq Qn.$$

In a later paper by Denker, Przytycki, Urbanski [15, Lemma 2.3], Lemma 2.1 was generalized as the following DPU lemma. We let $\phi(x) := \max_{c \in \text{Crit'}} \phi_c(x)$.

LEMMA 2.3 (**Denker-Przytycki-Urbanski**). *There exist a constant $Q > 0$ such that for all $n \geq 0$ we have*

$$\sum_{\substack{j=0 \\ \text{except } M \text{ terms}}}^{n-1} \phi(f^j(x)) \leq Qn,$$

where the summation over all but at most $M = \#Crit'$ indices.

Note that the original statement differs slightly. This formulation also appeared in [23, Lemma 2.5]. Lemma 2.1 will be used in section 4 and Lemma 2.3 will be used in section 6.

Next we introduce a version of the Koebe distortion lemma for multivalent maps. We refer to [35, Lemma 1.4] and [37, Lemma 2.1] for more details. Consider a disk $D(x,\delta)$ of radius δ centered at x, let W be a connected component of $f^{-n}(D(x,\delta))$, assume that f^n restricted to W is D-critical, that is f^n has at most D critical points counted with multiplicity. Then $f^n|_W$ has distortion properties similar to univalent maps. In the following we assume δ smaller than $\operatorname{diam} \mathbb{P}^1/2$.

LEMMA 2.4. *For each $\varepsilon > 0$ and $D < \infty$ there are constants $C_1(\varepsilon, D) > 0$ and $C_2(\varepsilon, D) > 0$ such that the following holds.*

Let $D(x,\delta)$ denotes the ball in $\mathbb{P}^1$ centered at x with radius δ. Assume that W is a simply connected domain in $\mathbb{P}^1$ and $F : W \to D(x,\delta)$ is a proper holomorphic map. Let $W' \subset W$ be a connected component of $F^{-1}(D(x,\delta/2))$. Assume further that $\mathbb{P}^1 \setminus W$ contains a disk of radius ε and that F is D-critical on W. Then for every $y \in W'$,

$$|F'(y)| \operatorname{diam}(W') \leq C_1\delta. \tag{2.1}$$

In addition W' contains a disk B of radius r around every pre-image of $F^{-1}(x)$ contained in W', with

$$r \geq C_2 \operatorname{diam}(W'). \tag{2.2}$$

Assume further that W'' is a connected component of $F^{-1}(B')$, where $B' \subset D(x,\delta/2)$ is a disk, then there exist a constant $C_3(\varepsilon, D) > 0$ such that

$$\frac{\text{diam } W''}{\text{diam } W'} \le C_3 \left(\frac{\text{diam}(B')}{\delta}\right)^{2^{-D}}. \tag{2.3}$$

Finally if R is a measurable subset of $D(x,\delta/2)$, there exist a constant $C_4(\varepsilon, D) > 0$ such that

$$\frac{\text{meas}\left(F^{-1}(R) \cap W'\right)}{\text{meas } W'} \le C_4 \left(\frac{\text{meas}(R)}{\delta^2}\right)^{2^{-D}}. \tag{2.4}$$

where meas *denotes the Lebesgue measure induced by the Riemannian metric on $\mathbb{P}^1$.*

Note that we will use this lemma only for F being a polynomial, so the assumption that W is simply connected and $\mathbb{P}^1 \setminus W$ contains a disk are automatically satisfied.

Proof. The inequalities (2.1) and (2.2) were proved in [37, Lemma 2.1] and the inequality (2.3) was proved in [35, Lemma 1.4]. Here we prove inequality (2.4).

By the Riemann mapping theorem there is a surjective univalent map $\psi : D(0,1) \to W$. We consider the composition $F \circ \psi : D(0,1) \to D(x,\delta)$. By the classical Koebe distortion lemma for univalent maps, (2.4) is true for $D = 0$. To prove (2.4), it is sufficient to prove the following: Let $G : D(0,1) \to D(0,1)$ be a degree D Blaschke product on the unit disk, there exist a constant $C_4(D) > 0$ such that if R is a measurable set of $D(0,1/2)$, then

$$\text{meas}\, G^{-1}(R) \le C_4\, \text{meas}(R)^{2^{-D}}. \tag{2.5}$$

We let $a_i \in D(0,1)$ be the critical points of G, $1 \le i \le n$. Thus we have $n \le D$. We denote $A = \text{meas}(R)$. It is sufficient to prove (2.5) for A small. For small ε we cover R by ε-disks such that $N\pi\varepsilon^2 \le 2A$, where N is the number of disks in the covering. For $\delta > 0$ small there exist a uniform constant M such that $\text{dist}(G(x), G(a_i)) \ge \delta$ for every i and $G(x) \in D(0,3/4)$ imply $|G'(x)| \ge M\delta^{1/2}$. If an ε-disk D_ε is disjoint from the union $\bigcup_{i=1}^n D(G(a_i), A^{1/2})$, then by the change of variable formula we have

$$\text{meas}\, G^{-1}(D_\varepsilon) \min_{x \in G^{-1}(D_\varepsilon)} |G'(x)|^2 \le D\pi\varepsilon^2.$$

From $\min_{x\in G^{-1}(D_\varepsilon)} |G'(x)|^2 \ge M^2 A^{1/2}$ we get

$$\text{meas}\, G^{-1}(D_\varepsilon) \le \frac{D\pi\varepsilon^2}{M^2 A^{1/2}}.$$

Let B_1 be the union of all ε-disks disjoint from $\bigcup_{i=1}^n D(G(a_i), A^{1/2})$. Then we have

$$\text{meas}\, G^{-1}(B_1) \le \sum_{D_\varepsilon \cap \left(\bigcup_{i=1}^n D(G(a_i), A^{1/2})\right) = \emptyset} \text{meas}\, G^{-1}(D_\varepsilon) \le \frac{ND\pi\varepsilon^2}{M^2 A^{1/2}} \le \frac{2DA^{1/2}}{M^2}.$$

Let $B_{2,i}$ be the union of all ε-disks not disjoint from $D(G(a_i), A^{1/2})$, for $1 \le i \le n$. Let $B_2 = \bigcup_{i=1}^n B_{2,i}$. By (2.3) for small ε ($\varepsilon \le A^{1/2}$, say), there is a constant $C_3(D) > 0$ such that

$$\operatorname{meas} G^{-1}(D(G(a_i), A^{1/2} + \varepsilon)) \leq C_3 A^{2^{-D}}.$$

Thus we have

$$\operatorname{meas} G^{-1}(B_2) \leq \sum_{i=1}^{n} \operatorname{meas} G^{-1}(B_{2,i}) \leq \sum_{i=1}^{n} \operatorname{meas} G^{-1}(D(G(a_i), A^{1/2} + \varepsilon)) \leq DC_3 A^{2^{-D}}.$$

The last inequality holds since $n \leq D$. Finally we have

$$\operatorname{meas} G^{-1}(R) \leq \operatorname{meas} G^{-1}(B_1) + \operatorname{meas} G^{-1}(B_2) \leq \frac{2DA^{1/2}}{M^2} + DC_3 A^{2^{-D}}.$$

Setting $C_4 := DC_3 + 2D/M^2$ the proof is complete. □

Lemma 2.4 will be used frequently in the rest of the paper.

2.2. One-dimensional non-uniformly hyperbolic theory. A rational map f is uniformly hyperbolic if f expands a Riemannian metric on a neighborhood of $J(f)$. This is equivalent to Smale's Axiom A, and is equivalent to the condition that the closure of the post critical set $\overline{PC(f)}$ is disjoint from $J(f)$. The measurable dynamics of f is well-understood: the Fatou set is the union of finitely attracting basins, the Hausdorff dimension of $J(f)$ is equal to the Minkowski dimension of $J(f)$ and is smaller than 2, and there is a unique invariant probability measure μ such that $\operatorname{supp}(\mu) = J(f)$ which is absolutely continuous with respect to the δ-dimensional Hausdorff measure (δ is the Hausdorff dimension of $J(f)$). It can be shown that μ is mixing (hence ergodic) and has positive entropy. It is widely conjectured that uniformly hyperbolic maps are dense in the parameter space of fixed degree. This is known as Fatou conjecture and is a central problem in one-dimensional complex dynamics. Many weaker notions such as sub-hyperbolicity, semi-hyperbolicity have been defined. See [29, Section 19], [13] for more details.

Non-uniformly hyperbolic theory, also known as Pesin theory, is a generalization of uniformly hyperbolic theory. In Pesin theory we only require an invariant hyperbolic measure rather than the presence of invariant expanding and contracting directions. In this subsection we introduce some strong notions of non-uniform hyperbolicity in one-dimensional complex dynamics.

Definition 2.5. *A rational map f satisfies CE if there exists $\mu_{CE} > 1$ and $C > 0$ such that for every point $c \in$ Crit' whose forward orbit does not meet other critical points, and every $n \geq 0$ we have*

$$|(f^n)'(f(c))| \geq C\mu_{CE}^n.$$

In addition we ask that there are no parabolic cycles.

The CE condition was first introduced by Collet and Eckmann in [14] for S-unimodal maps of an interval. The CE condition was introduced in complex dynamics by Przytycki in [35]. The TCE condition was first introduced by Przytycki and Rohde in [37], as a generalization of the CE condition.

DEFINITION 2.6. *A rational map f satisfies TCE if there exist $\mu_{Exp} > 1$ and $r > 0$ such that for every $x \in J(f)$, every $n \geq 0$ and every connected component W of $f^{-n}(D(x, r))$ we have that*

$$\text{diam}\,(W) \leq \mu_{Exp}^{-n}.$$

There are various equivalent characterization of the TCE condition, see [39]. The following inclusions are strict:

$$\text{uniform hyperbolicity} \subsetneqq \text{sub-hyperbolicity} \subsetneqq \text{CE} \subsetneqq \text{TCE}.$$

It was proved by Aspenberg [1] that the set of non-hyperbolic CE maps has positive measure in the parameter space of rational maps of fixed degree, see also Rees [41]. In the family of unicritical polynomials, it was shown by Graczyk-Swiatek [20] and Smirnov [44] that for a.e. $c \in \partial\mathcal{M}_d$ in the sense of harmonic measure $f_c = z^d + c$ satisfies the CE condition, where $\mathcal{M}_d$ is the connectedness locus, and $\partial\mathcal{M}_d$ is the bifurcation locus. We list some useful property of TCE maps.

PROPOSITION 2.7. *Let f be a TCE map such that $J(f) \neq \mathbb{P}^1$ then*

(1) The Fatou set $F(f)$ is the union of attracting basins.

(2) The Hausdorff dimension δ of $J(f)$ is equal to the Minkowski dimension of $J(f)$ and is smaller than 2.

(3) There is a unique invariant probability measure μ such that $\text{supp}(\mu) \subset J(f)$ *and μ is absolutely continuous with respect to the conformal measure with exponent δ. Moreover μ is exponentially mixing (hence ergodic) and has positive Lyapunov exponent.*

For the proof see [39] and [36]. For more about measurable dynamics on $J(f)$, we refer the reader to Przytycki and Rivera-Letelier [36], Graczyk and Smirnov [19], and Rivera-Letelier and Shen [42].

In our presentation of the main theorem, we also ask that p satisfies WR or Positive Lyapunov. These additional conditions are used to construct stable manifold at $v \in CV(p) \cap J(p)$ in section 3.

DEFINITION 2.8. *A rational map f satisfies WR(η, ι) if there exists $\eta, \iota > 0$ and $C_0 > 0$ such that for all $v \in CV(f)$ whose forward orbit does not meet any critical point and for every integer $n \geq 0$, it holds*

$$\sum_{\substack{j=0 \\ d(f^j(v), \text{Crit'}) \leq \eta}}^{n-1} - \log |f'(f^j(v))| < n\iota + C_0.$$

This condition means that for every $v \in CV(f) \cap J(f)$: the orbit of v does not come either too close nor too often to Crit'. The following WR condition is stronger than WR(η, ι) with fixed η and ι.

Definition 2.9. *A rational map f satisfies WR if for all $v \in CV(f)$ whose forward orbit does not meet critical points we have*

$$\lim_{\eta\to 0}\limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{j=0\\ d(f^j(v),\text{Crit'})\le\eta}}^{n-1} -\log|f'(f^j(v))| = 0.$$

The condition Positive Lyapunov is stronger than CE.

Definition 2.10. *A rational map f satisfies Positive Lyapunov if for every point $c \in$ Crit' whose forward orbit does not meet other critical points the following limit exists and is positive*

$$\lim_{n\to\infty}\frac{1}{n}\log|(f^n)'(f(c))| > 0.$$

In addition we ask that there are no parabolic cycles.

The condition WR, as well as the condition Positive Lyapunov is satisfied for many polynomial maps, see Appendix B.

3. Stable manifolds and renormalization maps

In this section we work under the assumption that p satisfies TCE and WR(η,ι) with small ι (ι will be determined later) or p satisfies Positive Lyapunov. We construct stable manifolds at each $v \in CV(p)\cap J(p)$ under the above assumptions, we also study the renormalization maps associated to each critical value variety.

3.1. Stable manifold of a critical value. A (local) stable manifold $W^s_{\text{loc}}(v)$ of f is an embedded complex disk of $\mathbb{C}^2$ passing through v such that there exist $\delta > 0$ that for every $x \in W^s_{\text{loc}}(v)$ and $n \geq 0$, $\text{dist}(f^n(x), f^n(v))$ decreases exponentially fast. The construction of stable manifold is classical for hyperbolic periodic points and for uniformly hyperbolic invariant sets, cf. [25]. In Pesin's theory we can also construct stable manifold for a.e. point with respect to a hyperbolic invariant probability measure, cf. [9]. Since we deal with a single non-uniformly hyperbolic orbit, we construct a stable manifold at v by using Hubbard and Oberste-Vorth's graph transform associated to a sequence of crossed mappings, cf. [22]. In the first three subsections we prove the following theorem:

Theorem 3.1. *Let f be an attracting polynomial skew product and let $p = f|_L$ be the restriction of f to the invariant fiber. Assume p satisfies either TCE and WR, or p satisfies Positive Lyapunov. Then every $v \in J(p)\cap CV(p)$ admits a local stable manifold transverse to the invariant fiber L.*

We begin with some definitions.

Let $B_1 = U_1 \times V_1$, $B_2 = U_2 \times V_2$ be two bi-disks. Let Ω be a neighborhood of $\overline{B_1}$, let $f : \Omega \to \mathbb{C}^2$ be a holomorphic map such that $f(B_1)\cap B_2 \neq \emptyset$. Let π_1 (resp. π_2) be the projection map to the first (resp. second) coordinate.

Definition 3.2 (Hubbard and Oberste-Vorth). *The map f is called a* crossed mapping *of degree d from B_1 to B_2 if there exists $W_1 \subset U_1 \times V_1'$, where $V_1' \subset V_1$ is a relatively compact open subset and $W_2 \subset U_2' \times V_2$, where $U_2' \subset U_2$ is a relatively compact open subset, such that $f : W_1 \to W_2$ is bi-holomorphic, and for every $x \in U_1$, the mapping*

$$\pi_2 \circ f|_{W_1 \cap ((\{x\} \times V_1)} : W_1 \cap (\{x\} \times V_1) \to V_2$$

is proper of degree d, and for every $y \in V_2$ the mapping

$$\pi_1 \circ f^{-1}|_{W_2 \cap (U_2 \times \{y\})} : W_2 \cap (U_2 \times \{y\}) \to U_1$$

is proper of degree d.

Beware that coordinate are switched as compared to [22]: here the horizontal direction is contracted and the vertical direction is expanded.

Let B be the bidisk $D(0,1) \times D(0,1)$. We define the horizontal boundary as $\partial_h(B) := \{|x| < 1, |y| = 1\}$. The vertical boundary $\partial_v(B)$ can be defined similarly.

Definition 3.3. *An analytic curve X is called* horizontal (resp. vertical) *in B_1 if X is defined in a neighborhood of B_1, $X \cap B_1 \neq \emptyset$ and $X \cap \overline{\partial_h(B_1)} = \emptyset$ (resp. $X \cap \overline{\partial_v(B_1)} = \emptyset$).*

It follows that $\pi_1 : X \to U_1$ (resp. $\pi_2 : X \to V_1$) is proper of degree d, for some integer $d > 0$. We call this d the degree of the analytic curve.

Proposition 3.4. *If $f : B_1 \to B_2$ is a degree 1 crossed mapping and X is a degree d vertical curve in B_1, then $\pi_2 \circ f : X \cap W_1 \to V_2$ is proper of degree d, or equivalent to say, $f(X)$ is a degree d vertical curve in B_2 (defined in a neighborhood of B_2).*

For the proof see [22, Proposition 3.4].

Definition 3.5 (Dujardin). *f is called* Hénon-like *from B_1 to B_2 if the following three conditions are satisfied*

(1) f restricted to Ω is injective, where Ω is a neighborhood of $\overline{B_1}$.

(2) $f(\partial_h B_1) \cap \overline{B_2} = \emptyset$,

(3) $f(\overline{B_1}) \cap \partial B_2 \subset \partial_h B_2$.

Again, note that horizontal and vertical directions are switched as compared to [16].

Proposition 3.6. *If f is Hénon-like from B_1 to B_2, then f is a crossed mapping.*

For the proof see [16, Proposition 2.3].

The following theorem summarizes our approach to construct stable manifolds.

THEOREM 3.7. *Let $B_0 = U_0 \times V_0$, $B_1 = U_1 \times V_1$,... be an infinite sequence of bi-disks, and $f_i : B_i \to B_{i+1}$ be of degree 1 crossed mapping, with V_i' simply connected (with notation as in Definition 3.2) such that the modulus* $\mathrm{Mod}(V_i \setminus V_i')$ *is uniformly bounded from below. Then the set*

$$W^s_{(f_n)} = \{(x,y) \in B_0 \mid f_n \circ \cdots \circ f_0\,(x,y) \in B_n \ \text{for all}\, n \geq 0\}$$

is a degree 1 horizontal curve in B_0. $W^s_{(f_n)}$ is called the stable manifold for the sequence of crossed mappings.

For the proof see [22, Corollary 3.12].

We also need some one-dimensional preparations. The following lemma is due to Przytycki and Rivera-Letelier ([36, Lemma 3.3]).

THEOREM 3.8. *Let f be a rational function satisfying TCE with constants $\mu_{Exp} > 1$ and $r_0 > 0$. Then the following assertions hold.*

There are constants $C_0 > 0$ and $\theta_0 \in (0,1)$ such that for every $r \in (0, r_0)$, every integer $n \geq 1$, every $x \in J(f)$ and every connected component W of $f^{-n}(B(x,r))$, we have

$$diam\,(W) \leq C_0 \mu_{Exp}^{-n} r^{\theta_0}.$$

We now use the above property to show that p is *hyperbolic away from Crit'* in the following sense.

LEMMA 3.9. *Assume p satisfies TCE. Let $\eta > 0$ be a constant. Let $\{x_0, x_1, \ldots, x_{N-1}\} \subset J(p)$ be a segment of orbit such that for every $0 \leq i \leq N-1$, $\mathrm{dist}(x_i, \mathrm{Crit'}) > \eta$, then there exist C_1 and α uniform constants such that $|(p^N)'(x_0)| \geq C_1 \eta^\alpha \mu_{Exp}^N$.*

PROOF. We use the same notations C_0 and θ_0 as in Theorem 3.8. It is sufficient to prove the lemma for η small. Let r be a constant satisfying $\eta = C_0 r^{\theta_0}$ such that $r < r_0$. Let W_i be the pull back of $D(x_N, r)$ by p^i at x_{N-i} for $0 \leq i \leq N$. We show that $W_i \cap \mathrm{Crit'} = \emptyset$ for every i. Assume that $W_i \cap \mathrm{Crit'} \neq \emptyset$ for some i. By Theorem 3.8 we have $\mathrm{diam}(W_i) \leq C_0 \mu_{Exp}^{-i} r^{\theta_0} \leq \eta$. On the other hand since $\mathrm{dist}(x_i, \mathrm{Crit'}) > \eta$ for every i, we get $\mathrm{diam}(W_i) > \eta$, which is a contradiction. Thus $W_i \cap \mathrm{Crit'} = \emptyset$ for every i and p^N restricted to W_N is univalent.

By TCE we have $\mathrm{diam}\, W_N \leq \mu_{Exp}^{-N}$, by the classical Koebe distortion theorem we have

$$|(p^N)'(x_0)|\ \mathrm{diam}\, W_N \geq r/4 = 1/4 C_0^{-1/\theta_0} \eta^{1/\theta_0}.$$

Taking $C_1 = 1/4 C_0^{-1/\theta_0}$ and $\alpha = 1/\theta_0$ we get that $|(p^N)'(x_0)| \geq C_1 \eta^\alpha \mu_{Exp}^N$. ☐

3.2. The TCE+WR case. In this subsection, for an attracting skew product f such that p satisfies TCE and WR(η,ι) with small ι, we construct a sequence of bi-disks $\{B_i\}$ with low exponential size such that for every integer i, f is a degree 1 crossed mapping from B_i to B_{i+1} with B_i centered at $p^i(v)$. We use the same notations as in Lemma 3.9. We fix a constant $0 < \varepsilon_0 \ll \min\left\{|\lambda|^{-1/3} - 1,\ \mu_{Exp} - 1\right\}$. In the following we choose an integer N such that $C_1\eta^\alpha\mu_{Exp}^N \geq (1+\varepsilon_0)^N$, where η is given by WR(η,ι), α and C_1 are as in Lemma 3.9. We subdivide the integers into blocks of the form $[iN,(i+1)N)$. We say that a block of this subdivision is of *first type* if

$$\prod_{j=iN}^{(i+1)N-1} |p'(p^j(v))| \geq (1+\varepsilon_0)^N,$$

and we call this subdivision is of *second type* if the above inequality does not hold. By Lemma 3.9 if $\mathrm{dist}(p^j(v),\mathrm{Crit'}) > \eta$ for $iN \leq j < (i+1)N$, then $[iN,(i+1)N)$ is of first type.

Let $m \in [iN,(i+1)N)$ be a positive integer, if m is in a block of first type, we define

$$\mu_m := (1+\varepsilon_0)\left(\prod_{j=iN}^{(i+1)N-1} |p'(p^j(v))|\right)^{1/N}.$$

When m is in a block of second type, we define

$$\mu_m := (1+\varepsilon_0)^2.$$

Note that in both cases we have $\mu_m \geq (1+\varepsilon_0)^2 \geq 1+\varepsilon_0$.

We define

$$r_n := r_0 \prod_{m=0}^{n-1} \frac{a_m}{\mu_m},$$

where $r_0 > 0$ is a constant to be determined and $a_m := |p'(p^m(v))|$.

LEMMA 3.10. *There are constants $C_2, C_3 > 0$ such that the following estimates of r_n hold for $n \geq 0$:*

(1) $r_n \leq C_2 r_0 (1+\varepsilon_0)^{-n}$.

(2) $r_n \geq C_3 r_0 e^{-(\alpha+2)n\iota}(1+\varepsilon_0)^{-2n}$, where α is as in Lemma 3.9.

PROOF. To prove the first inequality, notice that for every $i \geq 0$, by the definition of μ_m we have

$$\prod_{j=iN}^{(i+1)N-1} \mu_j \geq (1+\varepsilon_0)^N \prod_{j=iN}^{(i+1)N-1} a_j. \tag{3.1}$$

Notice also that for every $iN \leq m < (i+1)N$, we have

$$\prod_{j=iN}^{m} a_j \leq \|Dp\|^N, \tag{3.2}$$

where $\|Dp\|$ is the uniform norm of Dp on the Julia set $J(p)$.

Combining (3.1) and (3.2), for $kN \le n < k(N+1)$ we have

$$r_n = r_0 \left(\prod_{i=0}^{k-1} \prod_{j=iN}^{(i+1)N-1} \frac{a_j}{\mu_j} \right) \prod_{j=kN}^{n} \frac{a_j}{\mu_j} \le r_0(1+\varepsilon_0)^{-kN} \frac{\|Dp\|^N}{(1+\varepsilon_0)^{n-kN}} = \|Dp\|^N r_0(1+\varepsilon_0)^{-n}.$$

Taking $C_2 = \|Dp\|^N$ the first inequality is proved.

To prove the second inequality, notice that if the block $[iN, (i+1)N)$ is of first type, then

$$\prod_{j=iN}^{(i+1)N-1} \mu_j = (1+\varepsilon_0)^N \prod_{j=iN}^{(i+1)N-1} a_j.$$

Assume that $\{i_0, i_2, \dots, i_{l-1}\} \subset \{0, 1, \dots, k-1\}$ are all the integers such that the block $[i_h N, (i_h+1)N)$ is of second type, $0 \le h \le l$, then we have

$$\begin{aligned} r_n &= r_0 \left(\prod_{i=0}^{k-1} \prod_{j=iN}^{(i+1)N-1} \frac{a_j}{\mu_j} \right) \prod_{j=kN}^{n} \frac{a_j}{\mu_j} \\ &= r_0(1+\varepsilon_0)^{(l-k)N} \left(\prod_{h=0}^{l-1} \prod_{j=i_hN}^{(i_h+1)N-1} \frac{a_j}{\mu_j} \right) \prod_{j=kN}^{n} \frac{a_j}{\mu_j} \\ &\ge r_0(1+\varepsilon_0)^{(l-k)N} \left(\prod_{h=0}^{l-1} \prod_{j=i_hN}^{(i_h+1)N-1} \frac{a_j}{(1+\varepsilon_0)^2} \right) \prod_{j=kN}^{n} \frac{a_j}{C_2(1+\varepsilon_0)^2} \\ &\ge \frac{r_0}{C_2}(1+\varepsilon_0)^{-2n} \left(\prod_{h=0}^{l-1} \prod_{j=i_hN}^{(i_h+1)N-1} a_j \right) \prod_{j=kN}^{n} a_j. \end{aligned} \tag{3.3}$$

Since the block $[i_hN, (i_h+1)N)$ is of second type, then necessarily there is an integer j satisfies $i_hN \le j < (i_h+1)N$ and $\mathrm{dist}(p^j(v), \mathrm{Crit'}) \le \eta$. By Lemma 3.9, the product of derivative between two points x_{n_1}, x_{n_2} such that $\mathrm{dist}(x_{n_i}, \mathrm{Crit'}) > \eta$, $n_1 < i < n_2$ satisfies

$$\prod_{i=n_1+1}^{n_2-1} a_j \ge C_1 \eta^\alpha. \tag{3.4}$$

Notice that the number of such maximal blocks $[x_{n_1}, x_{n_2}]$ in $[i_hN, (i_h+1)N)$ is equal to the cardinality $\#\left\{ j \in [i_hN, (i_h+1)N) : \mathrm{dist}(p^j(v), \mathrm{Crit'}) \le \eta \right\} + 1$.

There is also a constant $C_4 > 0$ such that $\mathrm{dist}(p^j(v), \mathrm{Crit'}) \le \eta$ implies $a_j \le C_4\eta$ for η small. Thus by choosing η small, for j satisfying $\mathrm{dist}(p^j(v), \mathrm{Crit'}) \le \eta$ we have

$$C_1\eta^\alpha \ge C_1 \left(\frac{a_j}{C_4} \right)^\alpha \ge C a_j^\alpha \ge a_j^{1+\alpha}. \tag{3.5}$$

Combining (3.5) and (3.4) we get

$$\left(\prod_{h=0}^{l-1}\prod_{j=i_hN}^{(i_h+1)N-1} a_j\right)\prod_{j=kN}^{n} a_j \geq \prod_{\substack{j=0\\ d(p^j(v),\mathrm{Crit'})\leq\eta}}^{n} C_1\eta^\alpha a_j \geq \prod_{j=0}^{n} a_j^{2+\alpha} \geq e^{(-n\iota - C_0)(\alpha+2)}. \tag{3.6}$$

Combining (3.3) and (3.6) we get

$$r_n \geq \frac{r_0}{C_2}(1+\varepsilon_0)^{-2n}e^{(-n\iota-C_0)(\alpha+2)}.$$

Setting $C_3 := e^{-(\alpha+2)C_0}/C_2$ the conclusion follows. □

The following proposition clearly implies Theorem 3.1 in the case p satisfying TCE and WR.

We let $U_i := D(p^i(v), r_0(1+\varepsilon_0)^{-3i})$, $V_i := D(p^i(v), r_i)$ and let $B_i = U_i \times V_i$ for every positive integer i.

PROPOSITION 3.11. *Assume p satisfies TCE and WR(η,ι) with small ι (to be determined in the proof). Then there exist $r_0 > 0$ such that for arbitrary $v \in J(p)\cap CV(p)$ and every i, the map $f : B_i \to B_{i+1}$ is a degree 1 crossed mapping and satisfies the condition of Theorem 3.7. As a consequence there is a stable manifold at v.*

PROOF. We first show that for carefully chosen ι and for r_0 sufficiently small, f restricted to a neighborhood of $\overline{B_n}$ is injective for every n. The WR condition implies the following Slow Recurrence property: there exist a small $\alpha(\iota) > 0$, such that $\mathrm{dist}(p^n(v), \mathrm{Crit'}) > e^{-n\alpha}$ for all large n, see Lemma A.2 for the proof. We let ι sufficiently small so that $\alpha << \log(1+\varepsilon_0)$. By Lemma 3.10 (1) we can let r_0 small such that $r_n << e^{-\alpha n}$ and also $r_0(1+\varepsilon_0)^{-3n} << e^{-\alpha n}$, for every n .

We need the following general fact: if $f : W \to f(W)$ is a proper holomorphic map satisfying no critical points and $f(W)$ is simply connected, then f is injective. Thus in our case, to show f restricted to a small neighborhood of $\overline{B_n}$ is injective, it is sufficient to show $f(\overline{B_n})$ is contained in a simply connected domain which is disjoint with the critical value set of f. Let $M = \sup_{x\in\Omega}\left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial t}\right|\right)$, where Ω is some compact subset such that $\Delta\times\mathbb{C}\setminus\Omega$ is in the basin of ∞. Then $f(\overline{B_n}) \subset U_{n+1}\times D(p^{n+1}(v), 2Mr_n)$. Let l be the maximal order of the critical points in $J(p)$, then there is a constant $C > 0$ such that $\mathrm{dist}(p^{n+1}(v), p(Crit')) \geq Ce^{l\alpha n}$. By choosing sufficiently small r_0 and ι we get that $U_{n+1}\times D(p^{n+1}(v), 2Mr_n)$ is disjoint with the critical value set of f. Thus f restricted to a small neighborhood of $\overline{B_n}$ is injective.

Next we prove $f : B_n \to B_{n+1}$ is a degree 1 Hénon-like map for every n. By $|\lambda| < (1+\varepsilon_0)^{-3}$ we get $\pi_1(f(\overline{B_n})) \subset \pi_1(B_{n+1})$, thus $f(\overline{B_n})\cap\partial_v B_{n+1} = \emptyset$, thus it is easy to verify $f(\overline{B_n})\cap\partial B_{n+1}\subset\partial_h B_{n+1}$.

To prove $f(\partial_h B_n)\cap\overline{B_{n+1}} = \emptyset$, we first show that if r_0 is sufficiently small, and if we let $\hat{V}_n := D(p^{n+1}(v), (1+\varepsilon_0)^{\frac{1}{2}}r_{n+1})$, then $\hat{V}_n \subset p(V_n)$ for every n. To see this, First by the definition of r_n we have

$$a_n r_n = \mu_n r_{n+1} \geq (1+\varepsilon_0) r_{n+1},$$

where $a_n = |p'(p^n(v))|$. We choose sufficiently small r_0 such that $r_n \ll \operatorname{dist}(p^n(v), \text{Crit'})$. Then the Koebe distortion theorem gives us $D(p^{n+1}(v), (1+\varepsilon_0)^{\frac{1}{2}} r_{n+1}) \subset p(V_n)$ as desired.

For every point $x \in \partial_h B_n$, let $y = \pi_2(x)$, by the above result we have $\operatorname{dist}(p(y), V_{n+1}) \geq \left((1+\varepsilon_0)^{1/2} - 1\right) r_{n+1}$, thus we get

$$\begin{aligned}
\operatorname{dist}_v(f(x), B_{n+1}) &\geq \left((1+\varepsilon_0)^{1/2} - 1\right) r_{n+1} - \operatorname{dist}_v(f(x), p(y)) \\
&\geq \left((1+\varepsilon_0)^{1/2} - 1\right) r_{n+1} - M r_0 (1+\varepsilon_0)^{-3n} \text{ (by mean value theorem)} \\
&\geq \left((1+\varepsilon_0)^{1/2} - 1\right) C_3 r_0 e^{-(\alpha+2)n\iota} (1+\varepsilon_0)^{-2n} - M r_0 (1+\varepsilon_0)^{-3n},
\end{aligned}$$

where $M = \sup_{x\in\Omega} \left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial t}\right| \right)$ as before. By choosing $\iota \ll \varepsilon_0$ we get $\operatorname{dist}_v(f(x), B_{n+1}) > 0$. Thus $f(\partial_h B_n) \cap \overline{B_{n+1}} = \emptyset$.

It is easy to show $f : B_n \to B_{n+1}$ has degree 1. The reason is that the forward image of a vertical disk is again a vertical disk, and $\pi_2 \circ f$ is of degree 1 when restricted to this vertical disk. Since f keep the degree of the curve fix, f must have degree 1.

Finally we set $V_n' := p^{-1} V_{n+1}$ and show that the modulus of the annulus $V_n - \overline{V_n'}$ is uniformly bounded from below. Since the modulus is invariant under univalent maps, we have that

$$\operatorname{Mod}(V_n \setminus V_n') = \operatorname{Mod}\left(p(V_n) \setminus V_{n+1}\right) \geq \operatorname{Mod}\left(\hat{V}_n \setminus V_{n+1}\right) = \frac{1}{4\pi} \log(1+\varepsilon_0).$$

Now all the conditions in Theorem 3.7 are checked, we conclude that there is a stable manifold in the sense of Theorem 3.7. Since the dynamics contracts exponentially transverse to L, this is a stable manifold in the usual sense. □

3.3. The Positive Lyapunov case. Next we assume p satisfies Positive Lyapunov instead of TCE+WR(η, ι). We can then construct the stable manifold by arguing as before. Indeed, let χ_v be the following vertical Lyapunov exponent

$$\chi_v := \lim_{n\to\infty} \frac{1}{n} \log |(p^n)'(v)| > 0. \tag{3.7}$$

We need to construct μ_n such that an estimate of r_n similar to that of Lemma 3.10 holds, and also show the Slow Recurrence property: for every $\alpha > 0$, $\operatorname{dist}(p^n(v), \text{Crit'}) > e^{-n\alpha}$ for all large n. That Positive Lyapunov implies Slow Recurrence is shown in Lemma A.3.

To show an estimate of r_n similar to Lemma 3.10, we define $\mu_n := (1+\varepsilon_0) e^{\chi_v}$ for every n, and let $r_n := r_0 \prod_{i=1}^{n-1} \frac{a_i}{\mu_i}$, where $r_0 > 0$ is a constant, $a_i := |p'(p^i(v))|$.

LEMMA 3.12. *There are constants $C_2, C_3 > 0$ such that the following estimates of r_n hold for $n \geq 0$:*

(1) $r_n \leq C_2 r_0 (1 + \frac{\varepsilon_0}{2})^{-n}$.

(2) $r_n \geq C_3 r_0 (1 + 2\varepsilon_0)^{-n}$.

PROOF. Since (3.7) holds, for every $\varepsilon > 0$ small, there exist constants $C_2 > 0$ and $C_3 > 0$ such that $\prod_{i=1}^{n-1} a_i \leq C_2(1+\varepsilon)^n e^{n\chi_v}$ and $\prod_{i=1}^{n-1} a_i \geq C_3(1-\varepsilon)^n e^{n\chi_v}$ for every $n \geq 0$. Now it is sufficient to choose $\varepsilon = \frac{\varepsilon_0}{2+\varepsilon_0}$. □

REMARK 3.13. *The above proof actually only use that the ratio of limsup and liminf of (3.7) is sufficiently close to 1, thus the condition Positive Lyapunov can be replaced by a weaker condition (ratio of upper and lower Lyapunov exponent is sufficient close to 1) to make the main theorem hold.*

Finally the same argument as in Proposition 3.11 gives the existence of a local stable manifold at v, in case that p satisfies Positive Lyapunov. Thus the proof of Theorem 3.1 is complete.

3.4. Renormalization map. Next we introduce the renormalization map of a critical value variety $\mathcal{V}$. We let $\mathcal{V}$ be a component of the critical value variety that defined in a neighborhood of B_0, where B_0 is as in Proposition 3.11. We assume that the germ of $\mathcal{V}$ at $(0, v)$ does not coincide with $W^s_{\text{loc}}(v)$ (the converse hypothesis that $\mathcal{V} = W^s_{\text{loc}}(v)$ makes the main theorem even easier to prove, we will see this later). Then by the definition of the stable manifold in Theorem 3.7, for N large $f^N(\mathcal{V}) \not\subset B_N$, thus $f^N(\mathcal{V}) \cap \partial B_N \neq \emptyset$. By the definition of a Hénon-like map we must have $f^N(\mathcal{V}) \cap \partial B_N \subset \partial_h B_N$. Thus for N large, $f^N(\mathcal{V})$ is a degree d vertical curve. Note that d is constant because our Hénon-like maps have degree 1. Without loss of generality, we may assume that $\mathcal{V}$ is a degree d vertical curve (in B_0), otherwise we may replace $\mathcal{V}$ by some $f^N(\mathcal{V})$. By the definition of degree 1 crossed mapping, for every $n \geq 0$, $f^n(\mathcal{V}) \cap B_n$ is also a degree d vertical curve (in B_n).

We assume $\mathcal{V}$ has the parametrization $\mathcal{V} = \{\gamma(t) : t \in D(0, r_0)\}$ of the form $\gamma(t) = (t^l, \psi(t))$, where r_0 is the radius of U_0 in Proposition 3.11, l is a positive integer and ψ is holomorphic. Since $\mathcal{V}$ is a vertical curve, we can further assume that $\psi'(0) \neq 0$, for otherwise $\mathcal{V}$ can not be a vertical curve in B_0 when r_0 is sufficiently small.

For $n \geq 0$ we let $W_n := f^{-n}(B_n) \cap B_0$. The map

$$\psi_n := \pi_2 \circ f^n \circ \gamma : \gamma^{-1}(W_n) \to V_n$$

is well defined.

DEFINITION 3.14. *For every integer $n \geq 0$, let ρ_n be the maximal positive real number such that $\psi_n^{-1}(\frac{1}{2}V_n)$ contains a disk $D(0, \rho_n)$, where $\frac{1}{2}V_n$ denotes a disk centered at the same point of V_n but with one-half radius.*

Concretely, ρ_n is the typical size of the piece of $\mathcal{V}$ which remains in B_j under iteration up to time n. We can then define the renormalization map as follows

DEFINITION 3.15. *For every integer $n \geq 0$, the n-th renormalization map ϕ_n is the holomorphic map from D_n to $\mathbb{C}$, defined by $\phi_n(z) = \psi_n \circ L_{\rho_n}(z)$ for $z \in D_n$, where $D_n := L_{\rho_n}^{-1}(\psi_n^{-1}(\frac{1}{2}V_n))$ is a domain in $\mathbb{C}$, and for $r \in \mathbb{C}$, $L_r : \mathbb{C} \to \mathbb{C}$ denotes the linear map $L_r(z) = rz$.*

By the definition of D_n we know that $D(0,1) \subset D_n$. The following proposition is crucial.

PROPOSITION 3.16. *The renormalization map ϕ_n has uniformly bounded (topological) degree. Moreover there exist a constant $C_0 > 0$ such that* $\operatorname{diam} D_n \leq C_0$*, and ρ_n is exponentially small, namely $\rho_n \leq C\mu_{CE}^{-n} r_n$, where $\mu_{CE} > 1$ and $C > 0$ are constants.*

PROOF. The constant μ_{CE} corresponds to the constant appeared in the CE condition (Definition 2.5). We will see in the proof that TCE+WR(η, ι) with ι small implies CE. By our construction, $f^n \circ \gamma\left(\gamma^{-1}(W_n)\right)$ is a degree d vertical curve (in B_n) for every n. Thus ψ_n is of uniformly bounded degree and so is ϕ_n.

To show $\operatorname{diam} D_n \leq C_0$, we observe that by Theorem 2.4 (2.2) there exist a uniform constant $C_0 > 0$ such that $C_0\rho_n \geq \operatorname{diam}(\psi_n^{-1}(\frac{1}{2}V_n))$, thus

$$\operatorname{diam} D_n = \frac{1}{\rho_n} \operatorname{diam}\left(\psi_n^{-1}\left(\frac{1}{2}V_n\right)\right) \leq C_0.$$

To show that ρ_n is exponentially small, first notice that TCE + WR(η, ι) with ι small implies CE (see Lemma A.4), and also Positive Lyapunov implies CE. Thus $|(p^n)'(v)|$ is exponentially large, that is there exist $C > 0$ such that $|(p^n)'(v)| \geq C\mu_{CE}^n$ with μ_{CE} slightly smaller than μ_{Exp}. Thus

$$|\psi_n'(0)| = |\psi'(0)||(p^n)'(v)| \geq C\mu_{CE}^n|\psi'(0)|,$$

which is exponentially large. By Theorem 2.4 (2.1) we get $\rho_n \leq C\mu_{CE}^{-n} r_n$, which is exponentially small. □

4. Slow approach to Crit'

In this section our aim is to prove Theorem 1.2. First we remark that it is not true that for every vertical fiber $\{t = t_0\}$ Lebesgue a.e. (in the sense of one-dimensional Lebesgue measure) point in this fiber slowly approach Crit', as pointed out in [32]. Indeed in [32] the authors construct a vertical Fatou disk which comes exponentially close to Crit'. Instead, we need to select a full measure family of vertical fibers such that Lebesgue a.e. point in the fiber slowly approach Crit'. This will be proved by studying the renormalization maps along critical varieties constructed in the previous section. Together with Przytycki's lemma we can actually track the orbits of points in Crit (Lemma 4.2 and Lemma 4.3). Thus we need the non-uniform hyperbolic conditions in section 3 to make sure that the stable manifolds at each $v \in CV(p) \cap J(p)$ exist.

Let W_0 be a forward invariant open subset of $F(p)$ satisfying $\overline{p(W_0)} \subset W_0$. Such a W_0 exist since $F(p)$ is a union of attracting basins. Let $W_m = p^{-m}(W_0)$, and let K_m be the complement $K_m = \mathbb{C} \setminus W_m$.

LEMMA 4.1. *If p satisfies TCE, then the Lebesgue measure of K_m decreases exponentially fast with m.*

PROOF. We may assume that K_0 is sufficiently close to $J(p)$, that is,

$$K_0 \subset \{x : \operatorname{dist}(x, J(p)) \leq r\},$$

where r is the constant appearing in the definition of the TCE condition. Thus by definition of the TCE condition we have

$$K_m \subset \left\{x : \operatorname{dist}(x, J(p)) \leq \mu_{Exp}^{-m}\right\}$$

for $m \geq 0$.

Denote N_ε the number of ε-disks needed to cover $J(p)$, by Proposition 2.7 (2), the Minkowski dimension of $J(p)$ is $h < 2$. Hence for $\varepsilon > 0$ sufficiently small we have $N_\varepsilon < \varepsilon^{-h}$. Choosing $\varepsilon = \mu_{Exp}^{-m}$, for m sufficiently large, K_m is covered by at most μ_{Exp}^{mh} disks of radius μ_{Exp}^{-m}. Thus the measure of K_m is at most $\pi\mu_{Exp}^{m(h-2)}$, and the conclusion follows.

□

In the following we assume p satisfies TCE+WR or Positive Lyapunov. Let $\Delta = D(0, r_0)$. The set $W = \Delta \times W_0$ is contained in $F(f)$, and $\overline{f(W)} \subset W$, let $K = (\Delta \times \mathbb{C}) \setminus W$. Let W' be the $\varepsilon-$neighborhood of W,

$$W' := \{x \in \Delta \times \mathbb{C} : \operatorname{dist}(x, W) < \varepsilon\}.$$

For ε sufficiently small W' is forward invariant.

Let $K' = (\Delta \times \mathbb{C}) \setminus W'$. By using Proposition 3.16, we show that for most vertical fibers, the critical points on the fiber move to the Fatou set fairly quickly. The argument is similar to Peters-Smit [31] who treated the sub-hyperbolic case.

Let us choose a critical value variety $\mathcal{V}$ passing through $v \in CV(p) \cap J(p)$, parametrized as before: $\mathcal{V} = \left\{(t^l, \psi(t)) : t \in \Delta\right\}$. Let ϕ_n be the renormalization map defined in subsection 3.4.

For every integer $s \geq 0$, we define $j(s)$ to be the maximal integer such that $|\lambda^s| \leq \rho_{j(s)}$, where ρ_n is as in Definition 3.14.

LEMMA 4.2. *For every critical value variety $\mathcal{V}$ passing through $v \in CV(p) \cap J(p)$ that does not coincide with the stable manifold at v, there is a full Lebesgue measure subset $E_v \subset \Delta$ such that for every $u \in E_v$ there exist an integer N_u and $\beta > 0$ independent of u such that for every $s \geq N_u$, we have*

$$f^{j(s)+\beta s}(\gamma(\lambda^s u)) \in W'. \text{ where } \gamma(t) = (t^l, \psi(t)).$$

We note that there is some abuse in notation. For the simplicity when we write a non-integer number s as an iteration number of a map f, we mean the iteration of $\lfloor s \rfloor$ times.

PROOF. Fix $\beta > 0$ arbitrary for the moment. For every integer $s \geq 0$, let A_s be the set

$$A_s = \left\{ u \in \Delta : f^{j(s)+\beta s}(\gamma(\lambda^s u)) \in K' \right\}.$$

By the definition of the renormalization map ϕ_n, we have

$$A_s = \left\{ u \in \Delta : f^{\beta s}\left((\lambda^s u)^l , \phi_{j(s)}\left(\frac{1}{\rho_{j(s)}}\lambda^s u\right)\right) \in K' \right\}.$$

Let $M = \sup_{x\in\Omega}\left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial t}\right|\right)$, where Ω is a compact subset such that $\Delta \times \mathbb{C} \setminus \Omega$ is in the basin of ∞. By a shadowing argument there exists $C > 0$ such that for every integer $m \geq 0$, if $f^m(x) \in K'$ and $|\pi_1(x)| < CM^{-m}$, then $\pi_2(x) \in K_m$. It is equivalent to say that if $x \in \Delta \times \mathbb{C}$ satisfies $|\pi_1(x)| < CM^{-m}$ and $\pi_2(x) \in W_m$, then $f^m(x) \in W'$.

We choose β sufficiently small, such that for large enough s we have

$$\left|(\lambda^s u)^l\right| < CM^{-\beta s} \ \left(\beta < \frac{-\log|\lambda|}{\log M} \text{ is enough}\right).$$

Thus we get

$$A_s \subset \left\{ u \in \Delta : \phi_{j(s)}\left(\frac{1}{\rho_{j(s)}}\lambda^s u\right) \in K_{\beta s} \right\}.$$

Next we estimate the measure of the slightly bigger set

$$\widetilde{A}_s := \left\{ u \in \Delta : \phi_{j(s)}\left(\frac{1}{\rho_{j(s)}}\lambda^s u\right) \in K_{\beta s} \right\} = \rho_{j(s)}\lambda^{-s}\phi_{j(s)}^{-1}(K_{\beta s}).$$

By Lemma 4.1 the Lebesgue measure of $K_{\beta s}$ decreases exponentially with s. Next we prove that we can choose ε_0 and ι in Lemma 3.10 (2) sufficiently small so that the ratio

$$\frac{\operatorname{meas}(K_{\beta s} \cap \frac{1}{2}V_{j(s)})}{r_{j(s)}^2}$$

is exponentially small. Since

$$\frac{\operatorname{meas}(K_{\beta s} \cap \frac{1}{2}V_{j(s)})}{r_{j(s)}^2} \leq \frac{\operatorname{meas}(K_{\beta s})}{r_{j(s)}^2},$$

it is sufficient to show $\operatorname{meas}(K_{\beta s})/r_{j(s)}^2$ is exponentially small.

If we choose $\beta = \frac{-\log|\lambda|}{2\log M}$, then by Lemma 4.1 (again for large enough s)

$$\operatorname{meas}(K_{\beta s}) \leq \pi \mu_{Exp}^{\frac{s\log|\lambda|(2-h)}{2\log M}}. \tag{4.1}$$

On the other hand, by Proposition 3.16 we have

$$\rho_{j(s)} \leq C\mu_{CE}^{-j(s)} r_{j(s)} \leq \mu_{CE}^{-j(s)} \ \text{ (since } r_0 \text{ small).}$$

Then by the definition of $j(s)$ we have

$$j(s) \le \frac{-s \log|\lambda|}{\log \mu_{CE}}.$$

Then by Lemma 3.10 (2) we have

$$r_{j(s)} \ge C_3 r_0 e^{-(\alpha+2)j(s)\iota}(1+\varepsilon_0)^{-2j(s)} \ge C_3 r_0 e^{\frac{(\alpha+2)\log|\lambda|}{\log \mu_{CE}}\iota s}(1+\varepsilon_0)^{\frac{2\log|\lambda| s}{\log \mu_{CE}}}. \tag{4.2}$$

By (4.1) and (4.2) we can choose ε_0 and ι sufficiently small such that meas$(K_{\beta s} \cap \frac{1}{2}V_{j(s)})/r_{j(s)}^2$ is exponentially small.

It is proved in Proposition 3.16 that the map ϕ_n has uniformly bounded degree, so by Lemma 2.4 (2.4) the measure of $\phi_{j(s)}^{-1}(K_{\beta s} \cap \frac{1}{2}V_{j(s)})$ also decreases exponentially with s. Finally since $\rho_{j(s)}\lambda^{-s}$ is uniformly bounded with s, the Lebesgue measure of $\widetilde{A}_s$ also decreases exponentially with s. Thus $\sum_{s=1}^{\infty} \operatorname{meas}(A_s) \le \sum_{s=1}^{\infty} \operatorname{meas}(\widetilde{A}_s) < \infty$. By the Borel-Cantelli lemma, there is a full measure subset $E_v \in \Delta$ such that for every $u \in E_v$, there exist an integer N_u such that when $s \ge N_u$, $u \notin A_s$. In other words, $f^{j(s)+\beta s}(\gamma(\lambda^s u)) \in W'$ and the conclusion follows. $\square$

In the case where the critical value variety $\mathcal{V}$ passing through $v \in CV(p) \cap J(p)$ coincides with the stable manifold of v, every $y \in \mathcal{V}$ will shadow v forever. Thus we get an estimate of the returning time to Crit' of y, simply by Przytycki's lemma (Lemma 2.1)

LEMMA 4.3. *Assume that the critical value variety $\mathcal{V}$ passing through $v \in CV(p) \cap J(p)$ coincides with the stable manifold of v. Let $\gamma : \Delta \to \mathbb{C}^2$ be the parametrization of this stable manifold such that $\gamma(0) = v$, $\gamma(t) = (t, \psi(t))$ where ψ is holomorphic.*

Let $\mathcal{C}$ be the critical variety of f such that $f(\mathcal{C}) = \mathcal{V}$. Then for every fixed $\alpha > 0$ there exists a constant $K(\alpha) > 0$ such that for every $n \ge 0$, $0 \le s \le n$ and $u \in \Delta$, if $\operatorname{dist}_v(f^n(\gamma(u)), \mathcal{C}) \le e^{-\alpha n}$ *and* $\operatorname{dist}_v(f^{n-s}(\gamma(\lambda^s u)), \mathcal{C}) \le e^{-\alpha n}$*, then $s \ge Kn$.*

PROOF. We let L_u to be the vertical line $\{t = u\}$. By the construction of bi-disks in Proposition 3.11 , there exist constants $C_0 > 0$, $\lambda_1 < 1$ such that for every $n \ge 0$ and $u \in \Delta$ we have $\operatorname{dist}_v(f^n(\gamma(u)), p^n(v)) \le C_0\lambda_1^n$. Together with $\operatorname{dist}_v(f^n(\gamma(u)), \mathcal{C}) \le e^{-\alpha n}$ we get $\operatorname{dist}_v(p^n(v), \mathcal{C} \cap L_{\lambda^n u}) \le e^{-\alpha n} + C_0\lambda_1^n$.

For similar reasons we have $\operatorname{dist}_v(f^{n-s}(\gamma(\lambda^s u)), p^{n-s}(v)) \le C_0\lambda_1^n$. Together with the inequality $\operatorname{dist}_v(f^{n-s}(\gamma(\lambda^s u)), \mathcal{C}) \le e^{-\alpha n}$ we get $\operatorname{dist}_v(p^{n-s}(v), \mathcal{C} \cap L_{\lambda^n u}) \le e^{-\alpha n} + C_0\lambda_1^n$.

On the other hand there exist $C_1 > 0$, $l' > 0$ such that $\operatorname{dist}_v(\mathcal{C} \cap L_{\lambda^n u}, c_0) \le C_1|\lambda|^{n/l'}$ (l' is related to the multiplicity of $\mathcal{C}$ at c_0), where $c_0 = \mathcal{C} \cap L$ is the unique intersection point of $\mathcal{C}$ and the invariant line L. Then by the triangle inequality we have $\operatorname{dist}(p^n(v), c_0) \le e^{-\alpha n} + C_0\lambda_1^n + C_1|\lambda|^{n/l'}$ and also $\operatorname{dist}(p^{n-s}(v), c_0) \le e^{-\alpha n} + C_0\lambda_1^n + C_1|\lambda|^{n/l'}$. Thus s is a return time of $p^{n-s}(v)$ into the small neighborhood $D(c_0, e^{-\alpha n} + C_0\lambda_1^n + C_1|\lambda|^{n/l'})$ of c_0. By Przytycki's lemma (Lemma 2.1) we get

$$s \ge -C \log(e^{-\alpha n} + C_0\lambda_1^n + C_1|\lambda|^{n/l'}) := Kn,$$

the conclusion follows.

□

The main result of this section is the following equivalent form of Theorem 1.2,

THEOREM 4.4. *There is a full Lebesgue measure subset $E \subset \Delta$ such that for every $u \in E$, for Lebesgue a.e. x in the fiber $L_u : \{t = u\}$, x slowly approach Crit'.*

PROOF. It is enough to prove that for each fixed $\alpha > 0$ and $u \in E$, the set of points in L_u satisfing $\mathrm{dist}_v(f^n(x), \mathrm{Crit'}) \geq e^{-\alpha n}$ for all large n has full Lebesgue measure in L_u. We let E be the intersection of all E_v, where E_v is in Lemma 4.2, and v ranges on the set of critical values. Thus E has full Lebesgue measure in Δ. For every $u \in E$ we consider the sets

$$E_n := \bigcup_{c \in \mathrm{Crit'} \cap L_{\lambda^n u}} f^{-n}\left(D_v(c, e^{-\alpha n})\right), \text{ and } E'_n := \bigcup_{c \in \mathrm{Crit'} \cap L_{\lambda^n u}} f^{-n}\left(D_v(c, e^{-2\alpha n})\right).$$

(Recall that D_v stands for vertical disk). For an arbitrary critical point $c \in \mathrm{Crit'} \cap L_{\lambda^n u}$, we let Γ be an arbitrary connected component of $f^{-n}\left(D_v(c, e^{-\alpha n})\right)$.

Step 1, we show that the cardinality $\#\{0 \leq s \leq n : \ f^s(\Gamma) \cap \mathrm{Crit} \neq \emptyset\}$ is uniformly bounded with respect to n.

For n large enough $f^s(\Gamma)$ has no intersection with any critical variety $\mathcal{C}$ such that $\mathcal{C} \not\subset$ Crit'. The reason is the following. Take the radius of Δ sufficiently small to make sure that $\mathcal{C} \subset\subset F(f)$. Thus if $c' \in \mathcal{C} \cap f^s(\Gamma)$ for some $0 \leq s \leq n$, then $\mathrm{dist}_v(f^{n-s}(c'), J(p)) > \delta$ for some uniform constant δ. On the other hand $\mathrm{dist}_v(c, J(p)) \leq C|\lambda|^{\frac{n}{l}}$, where c is as in the definition of E_n and E'_n. Thus $\mathrm{dist}_v(f^{n-s}(c'), c) > \delta'$ for some uniform constant δ', this is impossible when n large since $f^{n-s}(c') \in D_v(c, e^{-\alpha n})$. Thus it is sufficient to show that the cardinality $\#\{0 \leq s \leq n : \ f^s(\Gamma) \cap \mathrm{Crit'} \neq \emptyset\}$ is uniformly bounded with respect to n. For this it is sufficient to show that $\#\{0 \leq s \leq n : \ f^s(\Gamma) \cap \mathcal{C} \neq \emptyset\}$ is uniformly bounded with respect to n for every local component of critical variety $\mathcal{C} \subset$ Crit'.

Now there are two cases. Let $\mathcal{V} = f(\mathcal{C})$ be a critical value variety, and let v be the unique intersection point of $\mathcal{V}$ and L, $v \in CV(p) \cap J(p)$. In the first case we assume that $\mathcal{V}$ does not coincide with the stable manifold at v as in Lemma 4.2. We claim that if n is large, s satisfies $s + 1 + j(s+1) + \beta(s+1) \leq n$ and $s \geq N_u$, then we have $f^s(\Gamma) \cap \mathcal{C} = \emptyset$. For otherwise if $c' \in f^s(\Gamma) \cap \mathcal{C}$ then $v' := f(c') \in f^{s+1}(\Gamma) \cap \mathcal{V}$. Then by Lemma 4.2 we have $f^{j(s+1)+\beta(s+1)}(v') \in W'$. Since W' is forward invariant, when $n - s - 1 \geq j(s+1) + \beta(s+1)$ implies $f^{n-s-1}(v') \in W'$, thus $f^{n-s}(c') \in W'$. By the definition of Γ we also have $\mathrm{dist}_v(f^{n-s}(c'), \mathcal{C}) \leq e^{-\alpha n}$, which is a contradiction when n large. To summarize, there exist a uniform constant $0 < \theta < 1$ such that $f^s(\Gamma) \cap \mathcal{C} \neq \emptyset$ implies $s \geq \theta n$ or $s \leq N_u$.

We first show that the cardinality $\#\{(1-\kappa)n < s \leq n : \ f^s(\Gamma) \cap \mathcal{C} \neq \emptyset\}$ is uniformly bounded with respect to n, where κ is the constant defined by,

$$\kappa = \min\left(\frac{-\theta \log|\lambda|}{4l \log M}, \frac{1}{2}\right) \quad \text{with} \quad M = \sup_{x \in \Omega}\left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial t}\right|\right).$$

Note that by the definition of κ

$$M^\kappa \le |\lambda|^{\frac{-\theta}{4l}}. \tag{4.3}$$

Assume $s_1 < s_2$ satisfy $f^{s_i}(\Gamma)\cap\mathcal{C}\neq\emptyset$ and $(1-\kappa)n < s_i \le n$, $i=1,2$. Let $c_i\in f^{s_i}(\Gamma)\cap\mathcal{C}$, $i=1,2$. Let $c_0=\mathcal{C}\cap L$ be the unique intersection point. Then we have

(4.4)

$$\mathrm{dist}_v(f^{n-s_2}(c_2),c_0)\le \mathrm{dist}_v(f^{n-s_2}(c_2),\mathcal{C}\cap L_{\lambda^{n-s_2}u})+\mathrm{dist}_v(\mathcal{C}\cap L_{\lambda^{n-s_2}u},c_0)\le e^{-n\alpha}+C|\lambda|^{\frac{n}{l}}.$$

Similarly

$$\mathrm{dist}_v(f^{n-s_1}(c_1),c_0)\le \mathrm{dist}_v(f^{n-s_1}(c_1),\mathcal{C}\cap L_{\lambda^{n-s_1}u})+\mathrm{dist}_v(\mathcal{C}\cap L_{\lambda^{n-s_1}u},c_0)\le e^{-n\alpha}+C|\lambda|^{\frac{n}{l}}.$$

By the definition of θ we also have $\mathrm{dist}_v(c_1,c_2)\le C|\lambda|^{\frac{n\theta}{l}}$. Thus by (4.3) we have

$$\mathrm{dist}_v(f^{n-s_2}(c_1),f^{n-s_2}(c_2))\le M^{n-s_2}C|\lambda|^{\frac{n\theta}{l}}\le C|\lambda|^{\frac{3n\theta}{4l}}.\ \text{(By the choice of } s_2).$$

Let $y=\pi_2(f^{n-s_2}(c_2))$. Using (4.3) again we have

$$\mathrm{dist}_v(f^{n-s_1}(c_1),p^{s_2-s_1}(y))\le M^{s_2-s_1}\,\mathrm{dist}_v(f^{n-s_2}(c_1),f^{n-s_2}(c_2))\le C|\lambda|^{\frac{n\theta}{2l}}.$$

Thus we have

(4.5)

$$\mathrm{dist}(c_0,p^{s_2-s_1}(y))\le \mathrm{dist}_v(f^{n-s_1}(c_1),c_0)+\mathrm{dist}_v(f^{n-s_1}(c_1),p^{s_2-s_1}(y))\le e^{-n\alpha}+C|\lambda|^{\frac{n\theta}{2l}}.$$

Combining (4.4) and (4.5) we infer that s_2-s_1 is a return time of y in the small disk $D(c_0,e^{-n\alpha}+C|\lambda|^{\frac{n\theta}{2l}})$, by Przytycki's lemma (Lemma 2.1) there exist a constant $K(\alpha)$ such that $s_2-s_1\ge Kn$. Thus $\#\{(1-\kappa)n<s\le n:\ f^s(\Gamma)\cap\mathcal{C}\neq\emptyset\}\le\frac{\kappa}{K(\alpha)}+1$.

By Lemma 2.4 (2.3) there exist $\alpha_1>0$ such that $\mathrm{diam}\, f^{(1-\kappa)n}(\Gamma)\le e^{-\alpha_1 n}$.

Next if we consider the cardinality $\#\{(1-2\kappa)n<s\le(1-\kappa)n:\ f^s(\Gamma)\cap\mathcal{C}\neq\emptyset\}$, we replace $f^n(\Gamma)$ by $f^s(\Gamma)$ where s satisfies $(1-2\kappa)n<s\le(1-\kappa)n$, $f^s(\Gamma)\cap\mathcal{C}\neq\emptyset$ and is maximal. Repeating the same argument we know there is a constant $K(\alpha_1)>0$ such that $\#\{(1-2\kappa)n<s\le(1-\kappa)n:\ f^s(\Gamma)\cap\mathcal{C}\neq\emptyset\}\le\frac{\kappa}{K(\alpha_1)}+1$. After finitely many iteration of the argument we get $\#\{\theta n<s\le n:\ f^s(\Gamma)\cap\mathcal{C}\neq\emptyset\}$ is uniformly bounded with respect to n. We also have $\#\{0\le s\le\theta n:\ f^s(\Gamma)\cap\mathcal{C}\neq\emptyset\}\le N_u$. Thus $\#\{0\le s\le n:\ f^s(\Gamma)\cap\mathrm{Crit}\neq\emptyset\}$ is uniformly bounded with respect to n.

In the second case we assume that $\mathcal{V}=f(\mathcal{C})$ coincides with the stable manifold at v as in Lemma 4.3. We also want to show that $\#\{0\le s\le n:\ f^s(\Gamma)\cap\mathrm{Crit}\neq\emptyset\}$ is uniformly bounded with respect to n. Let as before γ be the parametrization of the stable manifold. Assume $0\le s_1<s_2\le n$ satisfy $f^{s_i}(\Gamma)\cap\mathcal{C}\neq\emptyset$, let $c_i\in f^{s_i}(\Gamma)\cap\mathcal{C}$, $i=1,2$. Let $f(c_1)=\gamma(u_1)$, then $f(c_2)=\gamma(\lambda^{s_2-s_1}u_1)$. By the definition of Γ we have $\mathrm{dist}_v(f^{n-s_1}(\gamma(u_1)),\mathcal{C}\cap L_{\lambda^{n-s_1}u_1})\le e^{-\alpha n}$, and $\mathrm{dist}_v(f^{n-s_2}(\gamma(\lambda^{s_2-s_1}u_1),\mathcal{C}\cap L_{\lambda^{n-s_1}u_1})\le e^{-\alpha n}$. By Lemma 4.3 we have $s_2-s_1\ge Kn$. Thus $\#\{0\le s\le n:\ f^s(\Gamma)\cap\mathrm{Crit}\neq\emptyset\}\le 1/K+1$ which is uniformly bounded.

Step 2, By Step 1 we already know that $f^n : \Gamma \to D_v(c, e^{-\alpha n})$ has uniformly bounded degree. Now we show that the conclusion of the theorem holds.

Let Γ' be the component of $f^{-n}(D_v(c, e^{-2\alpha n}))$ contained in Γ. By Lemma 2.4 (2.4) there exist a constant $\alpha' > 0$ such that meas $\Gamma'/$ meas $\Gamma \leq e^{-\alpha' n}$. Since ∞ is an attracting fixed point, the set E_n is uniformly bounded. Thus meas $E_n < A$ for some constant $A > 0$. Thus we have

$$\frac{\text{meas } E_n'}{\text{meas } E_n} = \frac{\sum \text{meas } \Gamma'}{\sum \text{meas } \Gamma} \leq e^{-\alpha' n},$$

where the sum ranges over all possible critical points and connected components.

Finally we have shown that meas $E_n' \leq A e^{-\alpha' n}$. Thus $\sum_{i=0}^{n} \text{meas } E_i' < \infty$, and by the Borel-Cantelli lemma for Lebesgue a.e. point in $x \in L_u$, $x \notin E_n'$ for large n, which means $\text{dist}_v(f^n(x), \text{Crit'}) \geq e^{-2\alpha n}$. In other words, x slowly approach Crit'. The conclusion follows. $\square$

5. Positive vertical Lyapunov exponent

In this section we prove Theorem 1.4: if x slowly approach Crit' and $\omega(x) \subset J(p)$ then $\chi_-(x) \geq \log \mu_{Exp}$, where $\mu_{Exp} > 1$ is the constant appearing in the definition of the TCE condition.

DEFINITION 5.1. *Let $x \in \Delta \times \mathbb{C}$ satisfies $\text{dist}_v(f^n(x), J(p)) \leq \frac{r}{4}$ for every $n \geq 0$, where r is as in the definition of TCE condition. We say a positive integer n is a* expanding time *of x if for every $0 \leq m \leq n$, the connected component Γ of $f^{-m}(D_v(f^n(x), \frac{r}{4}))$ containing $f^{n-m}(x)$, satisfies $\text{diam}(\Gamma) \leq \mu_{Exp}^{-m}$.*

LEMMA 5.2. *There exist a uniform constant $\theta > 0$ such that if $x \in \Delta \times \mathbb{C}$ satisfies $\text{dist}_v(f^n(x), J(p)) \leq \frac{r}{4}$, then every $n \leq -\theta \log |\pi_1(x)|$ is an expanding time, provided $|\pi_1(x)|$ is small enough.*

PROOF. Let n be an arbitrary integer, and for $0 \leq m \leq n$ let Γ be the connected component of $f^{-m}(D_v(f^n(x), \frac{r}{4}))$ containing $f^{n-m}(x)$. Let $M = \sup_{x \in \Omega} \left(\left| \frac{\partial f}{\partial z} \right|, \left| \frac{\partial f}{\partial t} \right| \right)$ as before. Then for arbitrary $x_1 \neq x_2 \in \Gamma$, let $y_1 = \pi_2(x_1)$, $y_2 = \pi_2(x_2)$. Then for $i = 1, 2$ we have

$$\text{dist}_v(p^m(y_i), f^m(x_i)) \leq M^m |\pi_1(x_i)| \leq M^m |\pi_1(x)|.$$

Let $z \in J(p)$ such that $\text{dist}_v(z, f^n(x)) \leq \frac{r}{4}$, If n satisfies $M^n |\pi_1(x)| \leq \frac{r}{4}$, then we have

$$\begin{aligned}
\text{dist}(p^m(y_i), z) &\leq \text{dist}_v(p^m(y_i), f^m(x_i)) + \text{dist}_v(f^m(x_i), z) \\
&\leq \text{dist}_v(p^m(y_i), f^m(x_i)) + \text{dist}_v(f^n(x), f^m(x_i)) + \text{dist}_v(z, f^n(x)) \\
&\leq \frac{r}{4} + \frac{r}{4} + \frac{r}{4} < r. \qquad \text{for } i = 1, 2.
\end{aligned}$$

Thus $p^m(y_i)$ is in the disk $D(z, r)$, $i = 1, 2$. By TCE we know for every connected component Γ' of $f^{-m} D(z, r)$, we have $\text{diam}(\Gamma') \leq \mu_{Exp}^{-m}$, thus $\text{dist}(y_1, y_2) \leq \mu_{Exp}^{-m}$. In other words, $\mathrm{dist}_v(x_1, x_2) \le \mu_{Exp}^{-m}$. Thus $\mathrm{diam}(\Gamma) \le \mu_{Exp}^{-m}$, which implies n is an exponential expanding time of x providing $M^n|\pi_1(x)| \le \frac{r}{4}$. Thus for any $0 < \theta < \frac{1}{\log M}$, the condition $n \le -\theta \log|\pi_1(x)|$ implies n is an expanding time. $\square$

The main result of this section is the following.

THEOREM 5.3. *If $x \in W^s(J(p))$ slowly approach Crit', then $\chi_-(x) \ge \log \mu_{Exp}$.*

PROOF. Without loss of generality we may assume that x satisfies $\mathrm{dist}_v(f^n(x), J(p)) \le \frac{r}{4}$ for every $n \ge 0$. For fixed sufficiently small $\alpha > 0$, there exist $N > 0$ such that for $n \ge N$, $\mathrm{dist}_v(f^n(x), \mathrm{Crit'}) \ge e^{-\alpha n}$ by slow approach. Since x is in $W^s(J(p))$ the orbit of x will stay away from any component of the critical variety which does not belongs to Crit'. Thus we have $\mathrm{dist}_v(f^n(x), \mathrm{Crit}) \ge e^{-\alpha n}$ as well. We may also assume that $|\pi_1(x)| < 1$.

Set $\delta := \frac{2\alpha \log M}{\log \mu_{Exp}}$, where $M = \sup_{x\in\Omega}\left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial t}\right|\right)$. Let $0 < \theta' < 1$ be a constant that will be determined later. For $(1-\theta')n < s \le n$, let Γ_s be the connected component of $f^{s-n}D_v(f^n(x), e^{-\delta n})$ containing $f^s(x)$. Set $\delta_0 = \frac{\delta}{2\log M}$, and note that for all large n, $\frac{r}{4}M^{-\delta_0 n} \ge e^{-\delta n}$. In particular

$$f^{\delta_0 n}(D_v(f^n(x), e^{-\delta n})) \subset D_v(f^{n+\delta_0 n}(x), r/4),$$

hence

$$D_v(f^n(x), e^{-\delta n}) \subset \mathrm{Comp}_{f^n(x)}\, f^{-\delta_0 n}(D_v(f^{n+\delta_0 n}(x), \frac{r}{4})).$$

Here Comp_y denotes the connected component containing y.

We claim that there exist $0 < \theta' < 1$ such that for large n, $\theta' n + \delta_0 n$ is an expanding time of $f^{n-\theta' n}(x)$. Indeed by Lemma 5.2, for every $m \le n$ large, $-m\theta \log|\lambda|$ is an expanding time of $f^m(x)$. Provided α is sufficiently small we have δ_0 is sufficiently small as well, thus $\theta' = \frac{\theta \log|\lambda| + \delta_0}{\theta \log|\lambda| - 1}$ is a positive number. We conclude that $\theta' n + \delta_0 n$ is an expanding time of $f^{n-\theta' n}(x)$. Thus for every $n - \theta' n < s \le n$, we have

$$\begin{aligned}\mathrm{diam}\ \mathrm{Comp}_{f^s(x)}\, f^{s-n}(D_v(f^n(x), e^{-\delta n})) &\le \mathrm{diam}\ \mathrm{Comp}_{f^s(x)}\, f^{s-(n+\delta_0 n)}(x)(D_v(f^{n+\delta_0 n}(x), \frac{r}{4})) \\ &\le \mu_{Exp}^{s-(n+\delta_0 n)} \le \mu_{Exp}^{-\delta_0 n} = e^{-\alpha n}. \text{ (By the choice of } \delta_0.)\end{aligned}$$

Thus from $\mathrm{dist}_v(f^s(x), \mathrm{Crit}) \ge e^{-\alpha s}$ we get that

$$\mathrm{Comp}_{f^s(x)}\, f^{s-n}(D_v(f^n(x), e^{-\delta n})) \cap \mathrm{Crit} = \emptyset.$$

This implies that $f^{\theta' n}$ restricted to $\mathrm{Comp}_{f^{n-\theta' n+1}(x)}\, f^{-\theta' n+1}(D_v(f^n(x), e^{-\delta n}))$ is univalent. Since $\theta' n$ is an expanding time of $f^{n-\theta' n}(x)$, we have

$$\mathrm{diam}\ \mathrm{Comp}_{f^{n-\theta' n+1}(x)}\, f^{-\theta' n+1}(D_v(f^n(x), e^{-\delta n})) \le \mu_{Exp}^{-\theta' n}. \tag{5.1}$$

Thus by (5.1) and Koebe distortion, there is a uniform constant $C > 0$ such that

$$\left|\left(Df^{\theta' n}\right)_{f^{n-\theta' n+1}(x)}(v)\right| \ge Ce^{-\delta n}\mu_{Exp}^{\theta' n},$$

where v is the unit vertical vector.

Next we replace $f^n(x)$ by $f^{n-\theta' n}(x)$, and repeat the argument above, we get an estimate

$$\left|\left(Df^{\theta' n_1}\right)_{f^{n_1-\theta' n_1+1}(x)}(v)\right| \geq Ce^{-\delta n_1}\mu_{Exp}^{\theta' n_1},$$

where $n_1 = n - \theta' n$.

We define $n_m := n_{m-1} - \theta' n_{m-1}$, $m \geq 1$, we set $n_0 = n$. We can repeat this procedure until for some k, $n - \sum_{i=0}^{k} n_i\theta' \leq N$. In this final time we can not define n_k as $n_k = n_{k-1} - \theta' n_{k-1}$, instead we choose the final n_k to satisfying $N < n - \sum_{i=0}^{k} n_i\theta' \leq 2N$. Combining these estimates, take the product of this derivatives, we have

$$\begin{aligned}|(Df^n)_x(v)| &\geq \varepsilon_1 \left|\left(Df^{\theta' n_k}\right)_{f^{n_k-\theta' n_k+1}(x)}(v)\right| \cdots \left|\left(Df^{\theta' n}\right)_{f^{n-\theta' n+1}(x)}(v)\right| \\ &\geq \varepsilon_1 Ce^{-\delta n_k}\mu_{Exp}^{\theta' n_k} \cdots Ce^{-\delta n}\mu_{Exp}^{\theta' n} \\ &\geq \varepsilon_1 C^{k+1}e^{-\delta(n-2N)/\theta'}\mu_{Exp}^{n-2N},\end{aligned}$$

where we take $\varepsilon_1 = \min_{0\leq j\leq 2N}|Df^j|_x(v)|$.

It is not hard to give an upper bound of k. Indeed we let $S_m := n - \sum_{i=0}^{m} n_i\theta'$, then S_m satisfies $S_m = (1-\theta')S_{m-1}$, for $1 \leq m \leq k$. Thus we get $S_k = (1-\theta')^{k+1}n$. Now $S_k > N$ implies $k < \frac{\log N - \log n}{\log(1-\theta')} - 1$. Thus C^{k+1} is a sub-exponentially large term with respect to n.

Taking the limit in the above inequality we get

$$\chi_-(x) = \liminf_{n\to\infty}\frac{1}{n}\log|Df^n|_x(v)| \geq \log\mu_{Exp} - \frac{\delta}{\theta'}.$$

Letting $\alpha \to 0$ then $\delta/\theta' \to 0$ as well, and we get $\chi_-(x) \geq \log\mu_{Exp}$. □

Corollary 5.4. *There are no wandering Fatou components in $\Delta\times\mathbb{C}$, $W^s(J(p)) = J(f)$, and Fatou set $F(f)$ is equal to the union of basins of attracting cycles. Moreover Lebesgue a.e. point $x \in J(f)$ slowly approach Crit' and $\chi_-(x) \geq \log\mu_{Exp}$.*

Proof. By Theorem 4.4 and Theorem 5.3, for Lebesgue a.e. point $x \in W^s(J(p))$, x slowly approach Crit' and $\chi_-(x) \geq \log\mu_{Exp}$. We also know $W^s(J(p))$ is the union of $J(f)$ and the wandering Fatou components.

It is clear that points in the Fatou set can not have a positive vertical Lyapunov exponent, thus there are no wandering Fatou component in $\Delta\times\mathbb{C}$, and $W^s(J(p)) = J(f)$. Since every attracting basin of p bulges to an attracting basin of f, for every point x such that $x \notin W^s(J(p))$ we get that x is in a basin of attracting cycle. Thus the Fatou set is the union of basins of attracting cycles. □

6. The Julia set $J(f)$ has Lebesgue measure zero

In this section we prove Theorem 1.5, thus finishing the proof of the main theorem. In the following we assume that $x \in J(f)$ is both slowly approaching Crit' and satisfies $\chi_-(x) \geq \log \mu_{Exp}$. We begin with a definition.

DEFINITION 6.1. *Let* $1 < \sigma < \log \mu_{Exp}$*, Let* m *be a positive integer. We say* m *is a* σ-hyperbolic time *for* x *if*

$$\left| \left(Df^{m-i}\right)_{f^i(x)}(v) \right| \geq \sigma^{m-i}$$

holds for each $0 \leq i \leq m-1$*, and* v *is the unit vertical vector.*

We fix once for all $1 < \sigma < \sigma' < \mu_{Exp}$.

Since $\chi_-(x) \geq \log \mu_{Exp}$, the hyperbolic times have positive density by Pliss's Lemma [33], in the following sense:

LEMMA 6.2. *There is a constant* $\theta > 0$ *such that if we consider the set*

$$H_n = \left\{ m \in \{1, \cdots, n\} : m \textit{ is a } \sigma'\textit{-hyperbolic time for } x \right\},$$

then for large n *we have*

$$\frac{\# H_n}{n} > \theta.$$

For the proof see [26, Theorem 3.1].

Next for a positive integer n we define $\phi(f^n(x)) := -\log \mathrm{dist}_v(f^n(x), \mathrm{Crit'})$. Multiplying the metric by a constant we can further assume ϕ is a positive function. We show that

LEMMA 6.3. *There exists a constant* $C = C(x) > 0$ *such that for every* $n \geq 0$*,*

$$\sum_{k=0}^{n-1} \phi(f^k(x)) \leq Cn.$$

PROOF. Fix $\alpha > 0$ small, by slow approach for large n we have $\mathrm{dist}_v(f^n(x), \mathrm{Crit'}) \geq e^{-\alpha n}$, or in other words $\phi(f^n(x)) \leq \alpha n$.

We claim that there exist constant $0 < \theta < 1$ and $C_1 > 0$ such that such for large n we have $\sum_{k=\theta n}^{n-1} \phi(f^k(x)) \leq C_1 n$. To show this, let $z = \pi_2(f^{\theta n}(x))$. By Lemma 2.3 we have

$$\sum_{\substack{j=0 \\ \text{except } M \text{ terms}}}^{(1-\theta)n-1} \phi(p^j(z)) \leq Q(1-\theta)n.$$

In particular $\phi(p^j(z)) \leq Q(1-\theta)n$ holds for j appearing in above sum. On the other hand there is a constant $K > 0$ so that $\mathrm{dist}_v(p^j(z), f^{\theta n + j}(x)) \leq K^j |\lambda|^{\theta n}$. We choose θ

sufficiently close to 1 so that $e^{Q(\theta-1)n} - K^{(1-\theta)n}|\lambda|^{\theta n} \geq e^{2Q(\theta-1)n}$. Thus we have

$$\begin{aligned}\operatorname{dist}_v(f^{\theta n+j}(x), \text{Crit'}) &\geq \operatorname{dist}(p^j(z), \text{Crit'}) - \operatorname{dist}_v(f^{\theta n+j}(x), p^j(z))\\ &\geq e^{-\phi(p^j(z))} - K^{(1-\theta)n}|\lambda|^{\theta n}\\ &\geq e^{-\phi(p^j(z))} + e^{2Q(\theta-1)n} - e^{Q(\theta-1)n} \geq e^{-2\phi(p^j(z))},\end{aligned}$$

which implies $\phi(f^{\theta n+j}(x)) \leq 2\phi(p^j(z))$. Then we get

$$\sum_{\substack{k=\theta n\\ \text{except } M \text{ terms}}}^{n-1} \phi(f^k(x)) \leq 2Q(1-\theta)n.$$

Together with slow approach $\phi(f^n(x)) \leq \alpha n$ we have

$$\sum_{k=\theta n}^{n-1} \phi(f^k(x)) \leq (2Q(1-\theta) + M\alpha)\, n.$$

Setting $C_1 := (2Q(1-\theta) + M\alpha)$ we get the conclusion.

Repeat the above argument we get the estimate in the time $\theta^2 n$ to θn we get

$$\sum_{k=\theta^2 n}^{\theta n-1} \phi(f^k(x)) \leq C_1\theta n.$$

Keep repeating the above argument in the time $\theta^j n$ to $\theta^{j-1}n$ until for some j, the slow approach property $\operatorname{dist}_v(f^{\theta^{j+1}n}(x), \text{Crit'}) \geq e^{\theta^{j+1}n}$ does not holds. The final step is a bounded time, and the sum of $\phi(f^k(x))$ in this bounded time is bounded by a constant depending on x. Summing up there is a constant $C = C(x) > 0$ such that

$$\sum_{k=0}^{n-1} \phi(f^k(x)) \leq Cn.$$

□

We now introduce some notions from [26, Theorem 3.1]. Given $K > 0$ we define the shadow $S(j, K)$ of a positive integer j to be the following interval of the real line:

$$S(j, K) := (j, j + K\phi(f^j(x))].$$

For a positive integer N, let $A(N, K)$ be the set of all positive integer n such that at most N integers j satisfy $n \in S(j, K)$. The following lemma and Theorem are proved in [26, Theorem 3.1], and both rely on the one-dimensional DPU lemma (Lemma 2.3). In our case we can replace the DPU lemma by Lemma 6.3, and get exactly the same statements.

Lemma 6.4. *For any N and K, for n sufficiently large we have*

$$\frac{\{A(N,K) \cap \{1, \cdots, n\}\}}{n} \geq 1 - \frac{CK}{N+1}.$$

THEOREM 6.5. *Suppose m is an σ'-hyperbolic time and $m \in A(N, \frac{1}{\log \sigma})$, then there exist a constant $\delta > 0$ such that if we let V_m be the connected component of $f^{-m} D_v(f^m(x), \delta)$ containing x, then $f^m : V_m \to D_v(f^m(x), \delta)$ has degree at most N.*

If we choose N sufficiently large, by Lemma 6.4 the density of $A(N, \frac{1}{\log \sigma})$ is close to 1. Together with Lemma 6.2, we get for m large, the intersection $H'_m := H_m \cap A(N, \frac{1}{\log \sigma})$ has uniform positive density when $m \to \infty$. Now by Theorem 6.5 we have

COROLLARY 6.6. *For large n there exist a subset $H'_n \subset \{1, \cdots, n\}$ and constants $\alpha > 0$, $\delta > 0$ such that $\# H'_n \geq \alpha n$ and for every $m \in H'_n$, $f^m : V_m \to D_v(f^m(x), \delta)$ has degree at most N.*

Now we are able to establish the main result of this section.

THEOREM 6.7. *The Julia set $J(f)$ in the basin of L has Lebesgue measure zero.*

PROOF. Let δ and N be as in Corollary 6.6. First we observe that the Fatou set $F(p)$ in the invariant fiber L has full Lebesgue measure, as a consequence of TCE. Let Ω be a relatively compact subset of $F(p)$, then there exist a constant $\varepsilon > 0$ such that for x satisfying $|\pi_1(x)| < \epsilon$, $\pi_2(x) \in \Omega$, we have $x \in F(f)$. Thus for $y \in \Delta \times \mathbb{C}$ with $\pi_1(y)$ sufficiently small we have

$$\frac{\operatorname{meas} D_v(y, \delta/2) \cap J(f)}{\operatorname{meas} D_v(y, \delta/2)} \leq \varepsilon,$$

here meas denote the one-dimensional Lebesgue measure, and ε is a constant to be determined in the next paragraph.

Now we argue by contradiction. Suppose $J(f)$ has positive Lebesgue measure, then by the Lebesgue density theorem and the Fubini theorem there exist $x \in J(f)$ such that x is a Lebesgue density point in the vertical line containing x. We may also assume that x has positive Lyapunov exponent and slowly approach Crit'. By Corollary 6.6 there is a sequence of positive integers $\{n_0, \cdots, n_k, \cdots\}$ such that $f^{n_k} : V_{n_k} \to D_v(f^{n_k}(x), \delta)$ has degree bounded by N for all $k \geq 0$. Let V'_{n_k} be the connected component of $f^{-n_k} D_v(f^{n_k}(x), \delta/2)$ containing x. Let ε sufficiently small such that $C_4 \varepsilon^{2^{-N}} << 1$, where C_4 is the constant in Lemma 2.4 (2.4). By Lemma 2.4 (2.4) we have

$$\frac{\operatorname{meas} V'_{n_k} \cap J(f)}{\operatorname{meas} V'_{n_k}} \leq C,$$

where $C < 1$ does not depend on k. Again by Lemma 2.4 (2.1) diam V'_{n_k} is exponentially small, and also by Lemma 2.4 (2.2), V'_{n_k} has uniformly good shape (the ratio of the diameter and the inradius of V'_{n_k} is uniformly bounded). This contradicts that x is a Lebesgue density point. Thus $J(f)$ must have Lebesgue measure zero. $\square$

REMARK 6.8. *The main theorem also holds in a slightly more general setting. Let Δ be a disk. Let $f : \Delta \times \mathbb{P}^1 \to \Delta \times \mathbb{P}^1$ be a skew product holomorphic map in the following form: $f(t, z) = (\lambda t, h(t, z))$, where $|\lambda| < 1$ and $h(t, z)$ is a rational map in z for fixed*

t. We assume moreover that the degree of $h(t,z)$ in z is a constant for $t \in \Delta$. Let $L = \{t = 0\}$ be the invariant fiber and let $p = f|_L$. Assume p has non-empty Fatou set and p satisfies either 1.TCE+WR or 2.Positive Lyapunov. Then the Fatou set of f is the union of the basins of attracting cycles and the Julia set of f has Lebesgue measure zero.

We notice that $\Delta \times \mathbb{P}^1$ can not be embedded into $\mathbb{P}^2$ since any two projective lines in $\mathbb{P}^2$ have non-trivial intersection. However $\Delta \times \mathbb{P}^1$ can be embedded into $\mathbb{P}^1 \times \mathbb{P}^1$, and the f above can be realized as a semi-local restriction of a globally defined meromorphic map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1 \times \mathbb{P}^1$. To construct such examples, we start with a skew product meromorphic self map $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, $f(t,z) = (g(t), h(t,z))$, where g is a one-variable rational function and h is a two-variable rational function. The function h has finite number of indeterminacy points, and f is holomorphic outside these indeterminacy points. We choose g such that g has an attracting fixed point t_0, and there are no indeterminacy points in the line $\{t_0\} \times \mathbb{P}^1$. Thus there exist a small neighborhood Ω of $\{t_0\} \times \mathbb{P}^1$ such that $f : \Omega \to \Omega$ is holomorphic, thus f is a skew product holomorphic map. We note that indeterminacy points are necessary since a globally holomorphic self map of $\mathbb{P}^1 \times \mathbb{P}^1$ must be a product map, see [18, Remark 1.6].

7. Appendix A: Relations between non-uniformly hyperbolic conditions

In this Appendix we study the relations between non-uniform hyperbolic conditions given in section 2. In the following we assume f is a rational map on $\mathbb{P}^1$ and distance are relative to the spherical metric.

DEFINITION 7.1. *A rational map f satisfies* Slow Recurrence condition with exponent α *(*SR(α) *for short) if for every critical point $c \in J(f)$, there exist an $\alpha > 0$ such that*

$$\mathrm{dist}(f^n(c), \mathrm{Crit'}) \geq e^{-n\alpha} \text{ for } n \text{ large.}$$

LEMMA 7.2. *WR(η,ι) implies SR(α) for some $\alpha(\iota) > 0$, and $\alpha \to 0$ when $\iota \to 0$.*

PROOF. By the definition of WR(η,ι) in particular we have $-\log|f'(f^n(c))| < n\iota + C_0$ for every $n \geq 0$, which is equivalent to say $|f'(f^n(c))| > e^{-n\iota - C_0}$, then it is straightforward that there exist an $\alpha > 0$ such that $\mathrm{dist}(f^n(c), \mathrm{Crit'}) \geq e^{-n\alpha}$ for large n, and $\alpha \to 0$ when $\iota \to 0$. □

LEMMA 7.3. *Positive Lyapunov implies SR(α) for every $\alpha > 0$.*

PROOF. By the definition of Positive Lyapunov in particular we have

$$\lim_{n \to 0} \frac{\log |f'(f^n(c))|}{n} = 0.$$

Thus for every $\beta > 0$ we have $|f'(f^n(c))| > e^{-\beta n}$ for large n . Similarly to Lemma 7.2, there exist an $\alpha > 0$ such that $\mathrm{dist}(f^n(c), \mathrm{Crit'}) \geq e^{-n\alpha}$ for large n, and $\alpha \to 0$ when $\beta \to 0$. Thus f satisfies SR(α) for every $\alpha > 0$. □

Lemma 7.4. *TCE+WR(η,ι) with η small or TCE+SR(α) with α small implies CE.*

Proof. This lemma was proved by Li in [27] for real maps, and Li's argument can also apply to rational maps. Here we give a simple proof for rational maps. This kind of argument has already appeared in [39]. By Lemma 7.2 it is sufficient to prove TCE+SR(α) with α small implies CE. Let $v = f(c)$. Let $M = \sup_{x\in\mathbb{P}^1} |f'(x)|$. Set $\alpha_1 = \frac{2\alpha \log M}{\log \mu_{Exp}}$ and $\varepsilon = \frac{\alpha_1}{2\log M}$. Note that for all large n, $rM^{-\varepsilon n} \geq e^{-\alpha_1 n}$, here r is the constant appearing in the definition of the TCE condition. We note that

$$D(f^n(v), e^{-\alpha_1 n}) \subset \mathrm{Comp}_{f^n(v)}\, f^{-\varepsilon n}(D(f^{n+\varepsilon n}(v), r)),$$

here Comp_y means the connected component containing y

For every $0 \leq s \leq n$ and n large we have

$$\mathrm{diam}\ \mathrm{Comp}_{f^s(v)}\, f^{s-(n+\varepsilon n)}(D(f^{n+\varepsilon n}(v), r)) \leq \mu_{Exp}^{s-(n+\varepsilon n)} \leq \mu_{Exp}^{-\varepsilon n}.$$

Since $D(f^n(v), e^{-\alpha_1 n}) \subset \mathrm{Comp}_{f^n(v)}\, f^{-\varepsilon n}(D(f^{n+\varepsilon n}(v), r))$, we have

$$\mathrm{diam}\ \mathrm{Comp}_{f^s(v)}\, f^{s-n}(D(f^n(v), e^{-\alpha_1 n})) \leq \mu_{Exp}^{-\varepsilon n} = e^{-\alpha n}$$

By SR(α), for all large n and all $0 \leq s \leq n$ we have

$$\mathrm{Comp}_{f^s(v)}\, f^{s-n}(D(f^n(v), e^{-\alpha_1 n})) \cap \mathrm{Crit} = \emptyset.$$

Hence f^n restricted to $\mathrm{Comp}_v\, f^{-n}(D(f^n(v), e^{-\alpha_1 n}))$ is univalent, by Koebe distortion lemma there exist a constant $C > 0$ such that $|(f^n)'(v)| \geq Ce^{-\alpha_1 n}/\mu_{Exp}^{-n}$. Since α is small we get f is CE. $\square$

8. Appendix B: Genericity of non-uniformly hyperbolic conditions

In this Appendix we give some families of polynomials satisfying the consitions in our main theorem (i.e. TCE+WR or Positive Lyapunov).

In real dynamics, the WR condition was first introduced in the Tsujii's paper [47]. Avila and Moreira proved that CE+WR condition is generic (has full Lebesgue measure) in every non-trivial analytic family of S-unimodal maps [7]. The condition CE+WR was also studied by Luzzatto and Wang [45], and also by Li [27] in relation to topological invariance. For the Positive Lyapunov condition, Avila and Moreira proved that this condition is generic (has full Lebesgue measure) in every non-trivial analytic family of quasi-quadratic maps [8]. The quadratic family $\left\{f_t(x) = t - x^2\right\}$ for $\frac{-1}{4} \leq t \leq 2$ is obviously a non-trivial analytic family of quasi-quadratic maps, and of S-unimodal maps. So our theorem also applies for these real polynomials (seen as complex dynamical systems).

In the rational map case it was shown by Astorg, Gauthier, Mihalache and Vigny that the CE and WR(η, ι) with arbitrarily small ι are robust [5, Lemma 5.5] (in the sense that there is a positive Lebesgue measure set in the parameter space satisfying both these two conditions).

Next we consider the family of uni-critical polynomials, i.e. the family $\{f_c(z) = z^d + c\}$, $c \in \mathbb{C}$ and $d \geq 2$ an integer. We let $\mathcal{M}_d$ be the connectedness locus, and let $\partial\mathcal{M}_d$ be the bifurcation locus. There is a harmonic measure (with pole at ∞) supported on $\partial\mathcal{M}_d$. It is shown by Graczyk and Swiatek in [21] that for a.e. $c \in \partial\mathcal{M}_d$ in the sense of harmonic measure the Lyapunov exponent at c exist and is equal to $\log d$.

For the WR condition, the author is told by Jacek Graczyk that WR condition is actually generic in the sense of harmonic measure in the family of uni-critical polynomials. We thank Jacek Graczyk for kindly let us write down his argument here.

THEOREM 8.1. *In the uni-critical family* $\{f_c(z) = z^d + c\}$, $d \geq 2$, *a.e.* $x \in \partial\mathcal{M}_d$ *in the sense of harmonic measure satisfies WR condition.*

PROOF. For $c \in \mathbb{C}$, let ω_c be the unique measure of maximal entropy of f_c. It is a result of Brolin [11] that its Lyapunov exponent $\int \log|f_c'(z)|\, d\omega_c$ is is equal to $\log d$. Since $|f_c'(z)| = d|z^{d-1}|$ we get $\int -\log|z|\, d\omega_c = 0$.

We define the truncation function H_δ on $J(f_c)$ for $\delta > 0$ as

$$H_\delta(z) = \begin{cases} -\log|z|, & \text{when } |z| > \delta, \\ -\log|\delta|, & \text{when } |z| \leq \delta. \end{cases}$$

Thus H_δ is a continuous function, and $H_\delta \to -\log|\cdot|$ when $\delta \to 0$ in $L^1(\omega_c)$. According to [21] section 1.1, for a.e. $c \in \partial\mathcal{M}_d$ in the sense of harmonic measure, the critical value c of f_c is typical with respect to ω_c. Here typical means for every continuous function H on $J(f_c)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H(f_c^i(c)) = \int H\, d\omega_c. \tag{8.1}$$

Applying (8.1) to H_δ, together with that fact that $H_\delta \to -\log|\cdot|$ in $L^1(\omega_c)$ as $\delta \to 0$ we get

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} H_\delta(f_c^i(c)) = \int -\log|z|\, d\omega_c = 0. \tag{8.2}$$

On the other hand, for every $\delta > 0$ let F_δ be a positive continuous function such that $\operatorname{supp} F_\delta \subset D(0, 2\delta)$, $\|F_\delta\|_\infty = 1$ and $F_\delta \geq \chi_{D(0,\delta)}$. Then for for a.e. $c \in \partial\mathcal{M}_d$ in the sense of harmonic measure we have

$$\lim_{n\to\infty} \frac{-\log\delta}{n} \sum_{i=0}^{n-1} F_\delta(f_c^i(c)) = -\log\delta \int F_\delta\, d\omega_c \leq -\log\delta\, \omega_c(D(0, 2\delta)). \tag{8.3}$$

By [38, Lemma 4] (or by the fact that the dynamical Green function is Hölder continuous, see [43] Theorem 1.7.3), for every $c \in \mathbb{C}$ there exist constants $C = C(c) > 0$, $\alpha = \alpha(c) > 0$ such that for every $r > 0$ we have $\omega_c(D(0, r)) \leq Cr^\alpha$.

Thus for c satisfying (8.3) we have

$$\lim_{n\to\infty} \frac{-\log\delta}{n} \sum_{i=0}^{n-1} F_\delta(f_c^i(c)) \le -\log\delta C(2\delta)^\alpha.$$

Thus we have

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{-\log\delta}{n} \sum_{i=0}^{n-1} \chi_{D(0,\delta)}(f_c^i(c)) &\le \lim_{\delta\to 0}\limsup_{n\to\infty} \frac{-\log\delta}{n} \sum_{i=0}^{n-1} F_\delta(f_c^i(c)) \\ &\le \lim_{\delta\to 0} -\log\delta C(2\delta)^\alpha = 0.\end{aligned}$$

We conclude that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{-\log\delta}{n} \sum_{i=0}^{n-1} \chi_{D(0,\delta)}(f_c^i(c)) = 0. \tag{8.4}$$

It is easy to check

$$\sum_{\substack{i=0 \\ f_c^i(c)\notin D(0,\delta)}}^{n-1} -\log|f_c^i(c)| = \sum_{i=0}^{n-1} H_\delta(f_c^i(c)) + \log\delta \sum_{i=0}^{n-1} \chi_{D(0,\delta)}(f_c^i(c))$$

Combining (8.2) and (8.4) we get for a.e. $c \in \partial\mathcal{M}_d$ in the sense of harmonic measure

$$\lim_{\delta\to 0}\liminf_{n\to\infty} \frac{1}{n} \sum_{\substack{i=0 \\ f_c^i(c)\notin D(0,\delta)}}^{n-1} -\log|f_c^i(c)| = 0. \tag{8.5}$$

Finally by the main theorem of [21], for a.e. $c \in \mathcal{M}_d$ in the sense of harmonic measure

$$\lim_{n\to\infty} \frac{1}{n} \log|(f_c^n)'(c)| = \log d,$$

which is equivalent to (since $|f_c'(z)| = d|z^{d-1}|$)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} -\log|f_c^i(c)| = 0. \tag{8.6}$$

Combining (8.5) and (8.6) we get for a.e. $c \in \mathcal{M}_d$ in the sense of harmonic measure

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{i=0 \\ d(f_c^i(c),0)\le\delta}}^{n-1} -\log|f_c^i(c)| = 0. \tag{8.7}$$

By Przytycki's lemma (Lemma 2.1) we have

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{D(0,\delta)}(f_c^i(c)) = 0. \tag{8.8}$$

Since $|f_c'(f_c^i(c))| = d|f_c^i(c)|^{d-1}$, combining (8.7) and (8.8) we get

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\sum_{\substack{i=0\\ d(f_c^i(c),0)\leq\delta}}^{n-1} -\log|f_c'(f_c^i(c))| = 0.$$

Thus the proof is complete. □

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CHAPTER 4

Structure of the Julia set for post-critically finite endomorphisms on $\mathbb{P}^2$

1. Introduction

1.1. Background. Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic endomorphism of degree ≥ 2, where $\mathbb{P}^2$ is the complex projective plane. The first *Julia set* J_1 is defined as the locus where the iterates $(f^n)_{n\geq 0}$ do not locally form a normal family, i.e. the complement of the *Fatou set*. Let T be the *dynamical Green current* of f, defined by $T = \lim_{n\to+\infty} d^{-n}(f^n)^*\omega$, where ω is the Fubini-Study (1,1) form on $\mathbb{P}^2$. The Julia set J_1 coincides with Supp (T), and the self intersection measure $\mu = T \wedge T$ is the unique measure of maximal entropy of f. See Dinh-Sibony [10] for background on holomorphic dynamics on projective spaces.

We define the second Julia set to be $J_2 = \text{Supp}\ \mu$. From the definitions we know that $J_2 \subset J_1$. By Briend-Duval [5], J_2 is contained in the closure of the set of repelling periodic points. However contrary to dimension one there may exists repelling periodic point outside J_2. A major problem in holomorphic dynamics is to investigate the structure of $J_1 \setminus J_2$. A promising picture is that $J_1 \setminus J_2$ is foliated (in some appropriate sense) by holomorphic disks D along which $(f^n|_D)_{n\geq 0}$ is a normal family. Such disks are called *Fatou disks*. The dynamical Green current T is called *laminar* in some open set Ω if it expresses as an integral of integration currents over a measurable family of compatible holomorphic disks (which means these disks have no isolated intersections) in Ω. These disks are automatically Fatou disks. Let $\sigma_T = T \wedge \omega$ be the trace measure of T, which is a natural reference measure on J_1. If T is laminar, then for σ_T a.e. $x \in J_1 \setminus J_2$, there exists a germ of holomorphic Fatou disk D containing x. De Thélin proved in [8] and [9] that T is laminar outside J_2 for *post-critically finite endomorphisms* on $\mathbb{P}^2$. We note that by the works of Dujardin in general T is not necessarily laminar [13], but a related weaker result holds [12].

A holomorphic endomorphism f on $\mathbb{P}^k, k \geq 1$ is called post-critically finite (PCF for short) if the post-critical set

$$PC(f) := \bigcup_{n\geq 1} f^n(C(f))$$

is an algebraic subset of $\mathbb{P}^k$, where

$$C(f) := \left\{ x \in \mathbb{P}^k : Df(x) \text{ is not invertible } \right\}$$

is the critical set.

In dimension 1 , this coincides with the usual definition of PCF maps on the Riemann sphere $\mathbb{P}^1$. These play an important role in one-dimensional complex dynamics, mainly because the remarkable topological classification theorem of Thurston [11].

PCF endomorphisms are still of interest in higher dimension, and their dynamics have been investigated by many authors. The dynamics on the Fatou set of PCF endomorphisms on $\mathbb{P}^2$ were studied by Fornaess-Sibony [15], Ueda [27] and Rong [23]. Jonsson [18] proved that $J_2=\mathbb{P}^2$ for strictly PCF endomorphism on $\mathbb{P}^2$. Le [21] proved that the eigenvalues of periodic points of PCF endomorphism on $\mathbb{P}^2$ are either 0 or larger than 1. The dynamics of PCF endomorphisms on $\mathbb{P}^k$, $k \geq 2$ were studied by Ueda [26] and Astorg [1]. Moreover, interesting examples of PCF endomorphisms were constructed by Crass [6], Fornaess-Sibony [14] and Koch [20].

1.2. Basins of critical component cycles. In this paper we investigate the dynamics on the Julia sets for PCF endomorphisms on $\mathbb{P}^2$. Let f be a PCF endomorphism on $\mathbb{P}^2$. Recall that the critical set $C(f)$ is an algebraic curve. We call an irreducible component Λ of $C(f)$ periodic if there exist an integer $n \geq 1$ such that $f^n(\Lambda)=\Lambda$. Such an irreducible component will be called *periodic critical component.* There are finitely many periodic critical components. The set $\left\{\Lambda, f(\Lambda), \ldots, f^{n-1}(\Lambda)\right\}$ is called a *critical component cycle*. Similarly, a critical point x satisfying $f^n(x)=x$ for some $n \geq 1$ is called a *periodic critical point.* The set $\left\{x, f(x), \ldots, f^{n-1}(x)\right\}$ is called a *critical point cycle.* Since f^n and f have the same Julia sets, to investigate the structure of $J_1 \setminus J_2$ we may assume all periodic critical components are invariant.

An important observation is that in $\mathbb{P}^2$, any invariant critical component Λ is an attracting set. By definition an attracting set Λ in $\mathbb{P}^2$ is an invariant compact subset such that there is a neighborhood U of Λ satisfying $f(U) \subset\subset U$, and $\Lambda=\bigcap_{n \geq 1} f^n(U)$. The open set U is called a trapping region of Λ. The attracting basin $\mathcal{B}(\Lambda)$ of an attracting set Λ is by definition the set $\bigcup_{n \geq 0} f^{-n}(U)$, where U is a trapping region of Λ. Equivalently $\mathcal{B}(\Lambda)$ is the set of points attracted by Λ, i.e.

$$B(\Lambda)=\left\{x \in \mathbb{P}^2: \operatorname{dist}\left(f^n(x), \Lambda\right) \rightarrow 0, \text{as } n \rightarrow+\infty\right\} .$$

An attracting basin in $\mathbb{P}^2$ is always disjoint from J_2, except when $\Lambda=\mathbb{P}^2$, see [25] Proposition 1.1 for a proof.

Fornaess and Sibony [15] proved that for any fixed Riemannian metric on $\mathbb{P}^2$, if Λ is an invariant critical component, then for $x \in \mathbb{P}^2$, when dist $(x, \Lambda) \rightarrow 0$, we have

$$\operatorname{dist}(f(x), \Lambda)=o(\operatorname{dist}(x, \Lambda))$$

It follows that for $\epsilon>0$ sufficiently small, the ϵ -neighborhood

$$U_\epsilon=\left\{x \in \mathbb{P}^2: \operatorname{dist}(x, \Lambda)<\epsilon\right\}$$

satisfies $f\left(U_\epsilon\right) \subset\subset U_\epsilon$, and $\Lambda=\bigcap_{n \geq 1} f^n\left(U_\epsilon\right)$. Then Λ is an attracting set with trapping region U_ϵ. Fornaess and Sibony [15] also showed that an invariant irreducible curve in $\mathbb{P}^2$ has genus 0 or 1. Bonifant, Dabija and Milnor showed that an elliptic curve can not be an attracting set [3], so the invariant critical component Λ must be a (possibly singular) rational curve.

Now let f be a PCF endomorphism on $\mathbb{P}^2$ with an invariant critical component Λ. Let $\pi : \widehat{\Lambda} \to \Lambda$ be the normalization of Λ, then f restricted to Λ lifts to a map $\hat{f}$ from $\widehat{\Lambda}$ to itself. By [15] Proposition 7.5, $\hat{f}$ is a rational map on $\mathbb{P}^1$ of degree ≥ 2. Let $J(f)$ be the Julia set of f, and let $\hat{\nu}$ be the unique measure of maximal entropy of $\hat{f}$. Then $\nu = \pi_*(\widehat{\nu})$ is an invariant measure on $\mathbb{P}^2$. It follows that Supp $\nu = \pi(J(f))$, and ν is a hyperbolic measure of saddle type. Daurat showed that the Green current T is laminar in the basin $\mathcal{B}(\Lambda)$ and subordinate to the stable manifolds $\bigcup_{x\in \text{ Supp } \nu} W^s(x)$. See [7] for the proof and for more details (it is easy to verify in our case the trapping region U_ϵ satisfies conditions (Tub) and (SJ) in Daurat's paper when ϵ sufficiently small). See also Bedford-Jonsson [2] for the case when Λ is totally invariant.

1.3. A conjectural picture of $J_1 \setminus J_2$. There are some examples of PCF endomorphisms on $\mathbb{P}^2$ for which we can compute J_1 and J_2 by hand. For example, for homogeneous PCF regular polynomials on $\mathbb{P}^2$, $J_1 \setminus J_2$ is contained in the attracting basins of critical component cycles. This is also true for the PCF endomorphism on $\mathbb{P}^2$ satisfying additional assumption, see Theorem 6.1. A conjectural picture of $J_1 \setminus J_2$ for PCF endomorphism on $\mathbb{P}^2$ is as follows:

CONJECTURE 1.1. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Then $J_1 \setminus J_2$ is contained in the attracting basins of critical component cycles.*

Although we can not prove the conjecture yet, we will prove several results towards Conjecture 1.1. In section 6 we will discuss a possible approach to prove Conjecture 1.1. A main difficulty in this approach is to prove the so called backward contracting property, see Question 2 in section 6 and Theorem 6.2 for details. Roughly speaking, it say that for a PCF endomorphism on $\mathbb{P}^2$, for every $x \in \mathbb{P}^2$ which is not contained in a critical component cycle nor a critical point cycle, there exist $r > 0$ such that for every component W_n of $f^{-n}(B(x,r))$, $\operatorname{diam} W_n \to 0$ when $n \to \infty$. We give possible strategy in section 6.

In Theorem 6.2 we show: Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Assume f satisfies the backward contracting property and every super-saddle cycle is contained in a critical component cycle, then $J_1 \setminus J_2$ is contained in the attracting basins of critical component cycles.

1.4. The main results. At this stage by Daurat's theorem we have a nice description of the part of J_1 contained in the attracting basin of critical component cycles for PCF endomorphisms on $\mathbb{P}^2$. Our first result is about the structure of J_2 for PCF endomorphisms on $\mathbb{P}^2$.

THEOREM 1.2. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, then J_2 is the closure of the set of repelling periodic points. Moreover if all branches of $PC(f)$ are smooth and intersect transversally, then any periodic point in J_2 is repelling.*

Here are some comments about Theorem 1.2. First, we note that for a general holomorphic endomorphism on $\mathbb{P}^2$, repelling periodic point may not contained in J_2. Indeed there exist examples possessing isolated repelling points outside J_2, see [16] and

[17]. Secondly, the assumption that all branches of $PC(f)$ are smooth and intersect transversally is satisfied by examples constructed by Crass [6], Fornaess-Sibony [14] and Koch [20] . Thirdly, we note that the first part of Theorem 1.2 fits the picture of Conjecture 1.1, since every repelling periodic point is not in an attracting basin of a critical component cycle.

Now, regarding the structure of $J_1 \setminus J_2$, assume that f is a PCF endomorphism on $\mathbb{P}^2$ and there exists points in $J_1 \setminus J_2$ which is not contained in an attracting basin of a critical component cycle. For such a hypothetical point we show:

THEOREM 1.3. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree $d \geq 2$. Let $x \in J_1 \setminus J_2$ which is not contained in an attracting basin of a critical component cycle, then there is a Fatou disk D passing through x, i.e. the family $\{f^n|_D\}_{n\geq 1}$ is normal.*

We note that by the result of de Thélin [9], for σ_T a.e. point in $J_1 \setminus J_2$, there is a Fatou disk D passing through x. The above theorem together with Daurat's result about laminarity of Green current in the attracting basin gives a new proof of this fact.

We also prove the following theorem, which gives a necessary and sufficient condition for PCF endomorphisms on $\mathbb{P}^2$ satisfying $J_2 = \mathbb{P}^2$, under smoothness and transversality condition on $PC(f)$.

THEOREM 1.4. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2 such that branches of $PC(f)$ are smooth and intersect transversally. Then $J_2 = \mathbb{P}^2$ if and only if f is strictly PCF.*

The definition of strictly PCF is given in section 5. We note that Jonsson [18] proved that if f is a strictly PCF endomorphisms on $\mathbb{P}^2$, then $J_2 = \mathbb{P}^2$. We will also give an alternative proof of Jonsson's theorem. We note that Theorem 1.4 fits the picture of Conjecture 1.1, see Remark 5.5.

The structure of this paper is as follows. Section 2 is devoted to some preliminaires. In particular we recall Ueda's results about Fatou maps and the normality of backward iteration of holomorphic endomorphisms on $\mathbb{P}^k$, $k \geq 1$. In section 3 we prove Theorem 1.3 as a combination of Theorem 3.3 and Corollary 3.8. The proof of Theorem 3.3, which is the first part of Theorem 1.2, actually follows rather simply by a result of Ueda [27]. We prove Theorem 1.3 in section 4 and we prove Theorem 1.4 in section 5. In section 6 we discuss some possible generalization of our results and list some open problems.

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2. Preliminaires

In this section we recall some results of Ueda that we will use later. We start with the following definitions of Ueda [27] Definition 4.5 and [26] Section 1.

DEFINITION 2.1. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. Let Z be a complex analytic space. A holomorphic map $h : Z \to \mathbb{P}^k$ is called a Fatou map if $\{f^n \circ h\}_{n\geq 1}$ is a normal family. A Fatou disk $D \subset \mathbb{P}^k$ is an image of a non-constant Fatou map $\phi : \mathbb{D} \to \mathbb{P}^k$, where $\mathbb{D}$ is the unit disk.*

Note that with this definition, a Fatou disk may be singular.

DEFINITION 2.2. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. A point q is said to be a point of bounded ramification if the following conditions are satisfied:*

(1) There exist a neighborhood W of q such that $PC(f) \cap W$ is an analytic subset of W.

(2) There exist an integer m such that for every $j > 0$ and every $p \in f^{-j}(q)$, we have that ord $\left(f^j, p\right) \leq m$.

In the case $k = 2$, we have the following characterization of points of bounded ramification for PCF endomorphism on $\mathbb{P}^2$. (See Ueda [27] Lemma 5.7.)

LEMMA 2.3. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Then the points with unbounded ramification are the union of critical component cycles and critical point cycles.*

Next we introduce the following abstract result of Ueda. (See [27] Lemma 3.7 and Lemma 3.8.)

LEMMA 2.4. *Let X be a complex manifold and D an analytic subset of X of codimension 1. For every point $x \in X$, for every neighborhood W of x such that W is a topological ball and the diameter of $W \cap D$ is sufficiently small, for every integer $m \geq 0$, there exist a complex manifold Z and $\eta : Z \to W$ holomorphic such that*

(1) η is m-universal, in the sense that for every $D\cap W$-branched holomorphic covering $h : Y \to W$ (i.e. the ramification locus of h is contained in $D \cap W$) with sheet number $\leq m$, there exist a holomorphic map $\gamma : Z \to Y$ such that $h \circ \gamma = \eta$.

(2) If $h : Y \to W$ is a $D\cap W$ branched holomorphic covering, then $h^{-1}(x)$ is a single point.

Specializing to PCF endomorphisms on $\mathbb{P}^k$, we get the following corollary:

COROLLARY 2.5. *Let f be a PCF endomorphism on $\mathbb{P}^k$ of degree ≥ 2. Let $x \in \mathbb{P}^k$ be a point with bounded ramification. Then for every neighborhood W of x such that W is a topological ball and the diameter of $PC(f) \cap W$ is sufficiently small, there exist a connected complex manifold Z, and $\eta : Z \to W$ a $PC(f) \cap W$ branched holomorphic covering map such that if W_n denotes a connected component of $f^{-n}(W)$, then there exist a holomorphic map $g_n : Z \to W_n$ such that $f^n \circ g_n = \eta$, i.e. the following diagram is commute.*

$$\begin{tikzcd} Z \arrow[r, "\eta"] \arrow[d, "g_n"'] & W \\ W_n \arrow[ur, "f^n"] & \end{tikzcd}$$

The map g_n constructed above can be seen as a kind of inverse branch of f^n.

PROOF. Since x has bounded ramification, there exists $m \geq 0$ such that ord $(f^n, x_n) \leq m$ for every $x_n \in f^{-n}(x)$ and $n \geq 0$. Take $X = \mathbb{P}^k$ and $D = PC(f)$ in Lemma 2.4. Then for W satisfying that W is a topological ball and the diameter of $PC(f) \cap W$ is sufficiently small, there exist a connected complex manifold Z, and $\eta : Z \to W$ a $PC(f) \cap W$ branched holomorphic covering map, satisfy the two conclusion of Lemma 2.4. Let W_n denote a connected component of $f^{-n}(W)$, then by Lemma 2.4 (2), $W_n \cap f^{-n}(x)$ contains a single point, thus $f^n : W_n \to W$ has sheet number $\leq m$. By Lemma 2.4 (1), there exists a holomorphic map $g_n : Z \to W_n$ such that $f^n \circ g_n = \eta$. Thus the proof is complete. $\square$

The sequence $\{g_v\}$ defined in Lemma 2.5 is in fact normal, and any limit map of $\{g_v\}$ is also a Fatou map, by the following result of Ueda ([26] Theorem 2.3).

LEMMA 2.6. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. Let Z be a complex analytic space and $h : Z \to \mathbb{P}^k$ a holomorphic map, for every integers n let $g_n : Z \to \mathbb{P}^k$ be a family of holomorphic maps such that $f^n \circ g_n = h$. Then $\{g_n\}$ is a normal family. Further more if ϕ is a limit map of a sub-sequence of $\{g_n\}$, then ϕ is a Fatou map.*

3. Location of periodic points.

Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. In this section we prove Theorem 1.2. By Briend-Duval [5], J_2 is contained in the closure of the set of repelling periodic points. Thus to prove J_2 is the closure of the set of repelling periodic points, we only need to prove repelling periodic points are contained in J_2. We also investigate the locations of super-saddle fixed point of PCF endomorphisms on $\mathbb{P}^2$ in Theorem 3.4. Theorem 1.2 will be a combination of Theorem 3.3 and Corollary 3.8. We start with a definition.

DEFINITION 3.1. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. A fixed point x_0 is called repelling if all eigenvalues of Df at x_0 have modulus larger than* 1. *A fixed point x_0 is called super-saddle if Df at x_0 has one 0 eigenvalue and one eigenvalue with modulus larger than 1. A fixed point x_0 is called super-attracting if Df at x_0 has only 0 eigenvalues.*

Note that by the result of Le [21], for PCF endomorphisms on $\mathbb{P}^2$ every periodic point is either repelling, super-saddle or super-attracting.

3.1. Repelling points. We first recall the following fundamental result, for the proof see Sibony [24] Corollaire 3.6.5.

PROPOSITION 3.2. *Let f be a holomorphic endomorphism on $\mathbb{P}^k$ of degree ≥ 2. Then $x \in J_k$ if and only if for every neighborhood U of x, $\mathbb{P}^k \setminus \bigcup_{n=0}^{\infty} f^n(U)$ is a pluri-polar set.*

Now we can prove the first part of Theorem 1.2.

THEOREM 3.3. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, then every repelling periodic point belongs to J_2.*

PROOF. Our argument concerns the backward iterates around a fixed point. This kind of argument has already appeared in Fornaess-Sibony [16] and Le [21]. Without loss of generality we may assume x_0 is fixed. Since $x_0 \notin C(f)$, by Lemma 2.3, x_0 is a point of bounded ramification. We are going to show that for every neighborhood U of x_0, we have $\mathbb{P}^2 \setminus PC(f) \subset \bigcup_{n=0}^{\infty} f^n(U)$. Thus since $PC(f)$ is algebraic, by Proposition 3.2 we will get $x_0 \in J_2$.

Let $y \in \mathbb{P}^2 \setminus PC(f)$ be an arbitrary point. Let W be a neighborhood of x_0 such that $y \in W$ and W satisfies the condition in Corollary 2.5. It can be achieved, by first joining x_0 and y by a smooth embedded curve, and let W be a sufficiently thin tubular neighborhood of this curve. Let $m \geq 0$ such that ord $(f^n, x_n) \leq m$ for every $x_n \in f^{-n}(x_0)$ and $n \geq 0$. Let $\eta : Z \to W$ a $PC(f) \cap W$ branched holomorphic covering map as in Corollary 2.5. For $n \geq 0$, let W_n denote the connected component of $f^{-n}(W)$ containing x_0. Then by Corollary 2.5 we can define holomorphic map $g_n : Z \to W_n$ such that $f^n \circ g_n = \eta$. By Lemma 2.6, $\{g_n\}$ is a normal family. We are going to show that actually g_n converges to the fixed point x_0. Since x_0 is repelling, there exists a small neighborhood $\Omega \subset W$ such that g_n converges to x_0 uniformly on $\eta^{-1}(\Omega)$. Now let ϕ be any limit map of some sub-sequence of $\{g_n\}$. Since ϕ is constant on an open set $\eta^{-1}(\Omega)$, ϕ is constant on Z. Thus any limit map of some sub-sequence of $\{g_n\}$ is the constant map $z \mapsto x_0$. This implies that g_n converges to the fixed point x_0. In particular if z_0 satisfies $\eta(z_0) = y$, we have $g_n(z_0) \to x_0$ when $n \to +\infty$.

Now let U be an arbitrary neighborhood of x_0. Since $g_n(z_0) \to x_0$, there exist $N > 0$ such that $g_N(z_0) \in U$. Since $f^N \circ g_N = \eta$ we get $y \in f^N(U)$. Since $y \in \mathbb{P}^2 \setminus PC(f)$ is arbitrary we get $\mathbb{P}^2 \setminus PC(f) \subset \bigcup_{n=0}^{\infty} f^n(U)$. By the arbitrariness of U, we have $x_0 \in J_2$, which completes the proof. $\square$

3.2. Super-saddle points. Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. We investigate the location of super-saddle cycles of f. Let x_0 be a super-saddle fixed point. Since Df has one zero eigenvalue, x_0 is contained in a critical component of f. If the critical component C containing x_0 is invariant, then there are infinitely many super-saddle cycles in C. For example, take an arbitrary homogeneous PCF polynomials f on $\mathbb{P}^2$, every repelling periodic point of $f|_{L_\infty}$ is a super-saddle periodic point of f, where L_∞ is the line at infinity. Actually for all known examples of PCF endomorphisms on $\mathbb{P}^2$, every super-saddle cycle is contained in a critical component cycle. We may conjecture that actually this is true for every PCF endomorphism on $\mathbb{P}^2$. For the moment we can only show that this is true under an additional assumption.

THEOREM 3.4. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, Let x_0 be a super-saddle fixed point in $\mathbb{P}^2$. If the branches of $PC(f)$ are smooth and intersect transversally at x_0, then x_0 is contained in a critical component cycle.*

PROOF. In the following we let $C_{x_0}(f)$ denote the union of critical components containing x_0 and $PC_{x_0}(f) = \bigcup_{n=1}^{\infty} f^n(C_{x_0}(f))$. It is easy to observe that $PC_{x_0}(f) = PC_{x_0}(f^n)$ for every $n \geq 1$. Then up to an iteration of f, we may assume that all periodic components of $PC_{x_0}(f)$ are fixed, and every component of $PC_{x_0}(f)$ is mapped to an invariant component by at most one iteration. It remains to show x_0 is contained in an invariant component of $C_{x_0}(f)$. We argue by contradiction. Assume x_0 is not contained in an invariant component of $C_{x_0}(f)$, then there exist V an invariant component of $PC_{x_0}(f)$ such that $V \not\subset C_{x_0}(f)$. We first show that $V \neq PC_{x_0}(f)$. We argue by contradiction, assuming $V = PC_{x_0}(f)$, then there exist a neighborhood U of x_0 such that $f : U \to f(U)$ is a V-branched covering. Then, since $x_0 \in C_{x_0}(f)$, $f^{-1}(V)$ contains a component of $C_{x_0}(f)$. Since $f^{-1}(V)$ also contains V, we deduce that $f^{-1}(V)$ is singular at x_0. We recall the following result of Ueda [27] Lemma 3.5.

LEMMA 3.5. *Let $f : U_1 \to U_2$ be a V-branched holomorphic covering, where U_1, U_2 are complex manifolds and V is a codimension 1 analytic subset of U_2. Suppose that $x_0 \in U_1$ is a singular point of $f^{-1}(V)$, then $f(x_0)$ is a singular point of V.*

Coming back to our situation, letting $U = U_1$ and $f(U) = U_2$ in above lemma, we know that x_0 is a singular point of V. This is impossible, since by our assumption V is smooth at x_0. Thus there must exist a component V_1 of $PC_{x_0}(f)$ such that $V_1 \neq V$.

Let V' be the invariant component of $PC_{x_0}(f)$ that is a forward image of V_1. We recall the following result of Le [21] Proposition 5.5.

LEMMA 3.6. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a proper holomorphic germ and let Σ_1, Σ_2 be two irreducible germs at 0 such that $\Sigma_1 \neq \Sigma_2$, $f(\Sigma_1) = \Sigma_2$ and $f(\Sigma_2) = \Sigma_2$. If Σ_2 is smooth at 0 then the eigenvalue of Df at 0 are 0 and λ where λ is the eigenvalue of $D_0 f|_{T_0\Sigma_2}$.*

Coming back to our situation, take $V = \Sigma_2$ (resp. $V' = \Sigma_2$) in above lemma, since x_0 is super-saddle, we get that the eigenvalue of f restricted to V (resp. V') at x_0 must be repelling. By our assumption V' and V intersect transversally, the only possible case that x_0 being super-saddle is when $V' = V$. Thus we must have $f(V_1) = V$ by our assumption that every component of $PC(f)$ is mapped to an invariant component by one iteration. To summarize, there exist $V_1 \subset PC_{x_0}(f)$ such that $V_1 \neq V$, and every component of $PC_{x_0}(f)$ is mapped to V by one iteration.

We do a local coordinate change such that $x_0 = (0,0), V = \{\{x = 0\} \cap U\}$, where U is a small neighborhood of $(0,0)$. By [21] Proposition 5.5, 0 is a repelling point of $f|_V$. Let λ be the eigenvalue of $f|_V$ at 0, $|\lambda| > 1$. Thus after a linear coordinate change fixing V, the expression of f in this coordinate has the following form

$$f(x, y) = (G(x, y), \lambda y + ax + H(x, y)), \tag{3.1}$$

Where $G(x,y) = O\left(\left|x^2\right|, \left|y^2\right|\right)$, x is a factor of G and $H(x,y) = O\left(|x|^2, |y|^2\right)$.

Since V_1 is a smooth curve that intersects $\{x = 0\}$ transversally, we may assume $V_1 = \{y = \phi(x)\}$ for some holomorphic function ϕ. We do a local coordinate change $x' \to x\ y' \to y - \phi(x)$. In this coordinate the expression of f has the same form as in (3.1), and $V_1 = \{y = 0\}$. In the following we work in this coordinate.

By our assumption that every component of $PC_{x_0}(f)$ is mapped to an invariant component by one iteration, there exists a critical component C such that $f(C) = V_1$. Thus C satisfies the equation

$$\lambda y + ax + H(x,y) = 0. \tag{3.2}$$

By the implicit function theorem, C is a smooth curve that intersects with $\{x = 0\}$ transversally. We let $C = \{y = \psi(x)\}$ for some holomorphic function ψ.

By direct calculation, the Jacobian of f is

$$\mathrm{Jac}(f) = \frac{\partial G}{\partial x}\left(\lambda + \frac{\partial H}{\partial y}\right) - \frac{\partial G}{\partial y}\left(a + \frac{\partial H}{\partial x}\right).$$

Since C is in the critical set of f, ψ satisfies the following equation

$$\frac{\partial G}{\partial x}(x,\psi(x))\left(\lambda + \frac{\partial H}{\partial y}(x,\psi(x))\right) - \frac{\partial G}{\partial y}(x,\psi(x))\left(a + \frac{\partial H}{\partial x}(x,\psi(x))\right) = 0. \tag{3.3}$$

Take differential of x in the both sides of (3.2) we get

$$\lambda\psi'(x) + a + \frac{\partial H}{\partial x}(x,\psi(x)) + \frac{\partial H}{\partial y}(x,\psi(x))\psi'(x) = 0. \tag{3.4}$$

Combining (3.3) and (3.4) we get

$$\left(\lambda + \frac{\partial H}{\partial y}(x,\psi(x))\right)\left(\frac{\partial G}{\partial x}(x,\psi(x)) + \frac{\partial G}{\partial y}(x,\psi(x))\psi'(x)\right) = 0.$$

Since $\left(\lambda + \frac{\partial H}{\partial y}(x,\psi(x))\right) \neq 0, \psi$ satisfies

$$\frac{\partial G}{\partial x}(x,\psi(x)) + \frac{\partial G}{\partial y}(x,\psi(x))\psi'(x) = 0.$$

This implies that ψ satisfies $G(x,\psi(x)) = 0$ for every x. Then by the expression (3.1), we must have $f(C) \subset V$. Since $f(C) = V_1$, we have $f(C) = (0,0)$, this is a contradiction since f is a locally finite to one map. Thus x_0 is contained in an invariant critical component, and the proof is complete. □

REMARK 3.7. *The post-critical set of examples of PCF endomorphisms constructed by Crass, Fornaess-Sibony and Koch [6, 14, 20] are the union of projective lines, so these examples satisfy the assumption in Theorem 3.4.*

Corollary 3.8. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2 such that all branches of $PC(f)$ are smooth and intersect transversally, then every periodic point in J_2 is repelling.*

Proof. Let $x_0 \in J_2$ be a periodic point. By Le [21], x_0 is either repelling, super-saddle or super-attracting. Since super-attracting periodic points belong to the Fatou set, x_0 is not super-attracting. By Theorem 3.4, super-saddle periodic points are contained in critical component cycles, in particular they are not in J_2. So x_0 must be repelling. $\square$

4. Structure of $J_1 \setminus J_2$.

In this section we prove Theorem 1.3. We will first prove several lemmas, which are true for PCF endomorphism on $\mathbb{P}^2$.

Lemma 4.1. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let $x_0 \in \mathbb{P}^k$ and let v be a sub-sequence of integers such that $x_v \to y$ and y is of bounded ramification, where $x_v = f^v(x_0)$. Let W be a neighborhood of y and $\eta : Z \to W$ a holomorphic covering as in Corollary 2.5. Let W_v denote the connected component of $f^{-v}(W)$ containing x_0. Let $g_v : Z \to W_v$ such that $f^v \circ g_v = \eta$. Assume g_v converges to a constant map, then $x_0 \in J_2$ and $y \in J_2$*

Proof. We first prove $y \in J_2$. We take $W = B(y, r)$ for sufficiently small r. Let $Z' = \eta^{-1}(B(y, r/2))$. Let N large enough such that $x_v \in B(y, r/2)$ when $v \geq N$. Since g_v converges to a constant map and $Z' \subset\subset Z$, we have $\operatorname{diam} g_v(Z') \to 0$. Let v large enough such that $W' := f^N(g_v(Z')) \subset\subset W$. Thus $f^{v-N} : W' \to W$ is a polynomial-like map, By [10] Theorem 2.22, there exist a fixed point of f^{v-N} in W. Letting $r \to 0$, we get that y is approximated by periodic points. By the result of Le [21], every periodic point of a PCF endomorphism f on $\mathbb{P}^2$ is repelling, super-saddle or super-attracting. Since y is not in a critical component cycle, and since there are only finitely many super-attracting periodic points and super-saddle periodic points outside critical component cycles, y is approximated by repelling periodic points. By Theorem 3.3, $y \in J_2$. Let $z = \eta^{-1}(y)$. The convergence of g_v to x_0 implies $g_v(z) \to x_0$, thus there is a sequence of pre-image $\{y_v\}$ of y such that y_v converges to x_0. By the backward invariance of J_2, we conclude that $x_0 \in J_2$. $\square$

Lemma 4.2. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let $x_0 \in \mathbb{P}^2$ and let v be a sub-sequence of integers such that $x_v \to y$ and y is of bounded ramification, where $x_v = f^v(x_0)$. Let W be a neighborhood of y and $\eta : Z \to W$ a holomorphic covering as in Corollary 2.5. Let W_v denotes the connected component of $f^{-v}(W)$ containing x_0. Let $g_v : Z \to W_v$ such that $f^v \circ g_v = \eta$. Assume g_v converges to a non-constant map ϕ, then there exist a Fatou disk passing through x_0.*

Proof. Let $W = B(y, r)$ for small r. Let $M = \phi(Z)$, we will show that M contains a Fatou disk passing through x_0. Let N large enough such that $x_v \in W$ when $v \geq N$. Then for $v \geq N$, there exist $z_v \in Z'$ such that $g_v(z_v) = x_0$. Let $z = \eta^{-1}(y)$, it is clear that $z_v \to z$ when $v \to +\infty$. Let $v \to +\infty$ in the equation $g_v(z_v) = x_0$ we get $\phi(z) = x_0$, then $x_0 \in M$. By Lemma 2.6, $\phi : Z \to \mathbb{P}^2$ is a Fatou map, by definition, this implies that $\{f^n|_M\}_{n\geq 1}$ is a normal family. Let $D \subset Z$ be a holomorphic disk passing through z such that ϕ is not a constant map when restricted to D, then $\phi(D) \subset M$ is a Fatou disk passing through x_0. □

LEMMA 4.3. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let $x_0 \in J_1$ such that x_0 is not contained in the attracting basin of a critical component cycle nor contained in the stable manifold of a super-saddle cycle, then there exist a sub-sequence v of positive integers such that $x_v = f^v(x_0) \to y$, where y is a point of bounded ramification.*

PROOF. There are at most finitely many critical point cycles, which are not contained in the critical component cycles. We denote this finite set by E. We show that if $x_0 \in J_1$ is such that x_0 is not contained in the attracting basins of critical component cycles and $\omega(x_0)$ contains only points of unbounded ramification, then x_0 is contained in the stable manifold of a super-saddle cycle. If x_0 satisfies the above assumption, by Lemma 2.3 we know that $\omega(x_0) \subset E$. We recall the following basic property of ω -limit set:

LEMMA 4.4. *Let X be a compact metric space, let $f : X \to X$ be a continuous map, and let $g : \omega(x_0) \to \omega(x_0)$ be the restriction of f on the ω-limit set of x_0, then there is no non-trivial open subset U of $\omega(x_0)$ such that $g(\overline{U}) \subset U$.*

PROOF. Bowen ([4] Theorem 1) proved the above lemma for homeomorphisms, but the proof also holds for non-invertible maps. For the completion we give a proof here. Assume by contradiction that such open subset U exist, let $Y := \omega(x_0)$. By our assumption $2\epsilon := \mathrm{dist}(Y \setminus U, g(\overline{U})) > 0$. Choose $0 < \delta < \epsilon$ such that $\mathrm{dist}(x_1, x_2) < \delta$ implies $\mathrm{dist}(f(x_1), f(x_2)) < \epsilon$ for every $x_1, x_2 \in X$. Now it is clear that there is $N > 0$ such that $\mathrm{dist}(f^n(x_0), Y) < \delta$ when $n > N$ (otherwise $\omega(x_0)$ will be strictly larger than Y). Pick $M \geq N$ such that $\mathrm{dist}(f^M(x_0), g(\overline{U})) < \epsilon$ and $\mathrm{dist}(f^M(x_0), y) < \delta$ for some $y \in Y$. Then $\mathrm{dist}(g(\overline{U}), y) < 2\epsilon$, which implies $y \in U$. Then

$$\mathrm{dist}(f^{M+1}(x_0), g(\overline{U})) \leq \mathrm{dist}(f^{M+1}(x_0), g(y)) < \epsilon.$$

Inductively for all $m \geq M$ we have $\mathrm{dist}(f^m(x_0), g(\overline{U})) < \epsilon$. This implies $Y \cap (Y \setminus U) = \emptyset$, which is a contradiction. □

Coming back to our situation, since $\omega(x_0)$ is a finite set, the only possibility is that $\omega(x_0)$ contains a single periodic cycle. Since $x_0 \in J_1$ and $\omega(x_0)$ contains no points of bounded ramification, this cycle must be a super-saddle cycle. Thus x_0 is contained in the stable manifold of this super-saddle cycle. It follows that if x_0 is not contained in the attracting basin of a critical component cycle nor contained in the stable manifold of a super-saddle cycle, $\omega(x_0)$ has non-empty intersection with the set of points of bounded ramification. This completes the proof. □

Now we are in position to prove Theorem 1.3. Recall the statement.

THEOREM 4.5. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let $x \in J_1 \setminus J_2$ which is not contained in an attracting basin of a critical component cycle, then there is a Fatou disk D passing through x.*

PROOF. Suppose first x_0 is contained in the stable manifold of a super-saddle cycle. Then there exist an embedded holomorphic disc D passing through x_0 such that D coincides with the local stable manifold at x_0, and it is clear that $\{f^n|D\}_{n\geq 1}$ is normal. Now suppose x_0 satisfies the assumptions of the theorem, and x_0 is not contained in the stable manifold of a super-saddle cycle. By Lemma 4.3, we can choose a sub-sequence v of positive integers such that $x_v = f^v(x_0) \to y$, and y is of bounded ramification. Consider a neighborhood W of y and $\eta : Z \to W$ the holomorphic covering as in Corollary 2.5. Let W_v denotes the connected component of $f^{-v}(W)$ containing x_0. Let $g_v : Z \to W_v$ such that $f^v \circ g_v = \eta$. By Lemma 2.6 $\{g_v\}$ is a normal family. By passing to some sub-sequence, we may assume g_v converges to a holomorphic map ϕ. ϕ can not be a constant map, since otherwise by Lemma 4.1, $x_0 \in J_2$, which is a contradiction. Thus ϕ is a non-constant map. By Lemma 4.2, there is a Fatou disk passing through x_0, which completes the proof. $\square$

5. PCF endomorphisms satisfying $J_2 = \mathbb{P}^2$

In this section we prove Theorem 1.4. We start with a definition of Ueda [26].

DEFINITION 5.1. *Let f be a PCF endomorphism on $\mathbb{P}^k$ of degree $\geq 2, k \geq 1$. f is called strictly PCF if all points in $\mathbb{P}^k$ have bounded ramification.*

We note that this definition coincide with the definition of 2-critically finite maps in Jonsson [18], when $k = 2$. In [18] Jonsson proved the following result, here we give an alternative proof.

THEOREM 5.2. *Let f be a strictly PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, then $J_2 = \mathbb{P}^2$.*

PROOF. We start with a lemma.

LEMMA 5.3. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let V be an invariant irreducible component of $PC(f)$ such that $V \not\subset C(f)$, let $\pi : \hat{V} \to V$ be the normalization of V, and let $\hat{f}$ be the lift of f on $\widehat{V}$. Let $J(\hat{f})$ be the Julia set of $\hat{f}$. Then $\pi(J(\hat{f})) \subset J_2$.*

PROOF. We have the following commutative diagram

$$\begin{array}{ccc} \hat{V} & \xrightarrow{\hat{f}} & \hat{V} \\ \pi\downarrow & & \downarrow\pi \\ V & \xrightarrow{f} & V \end{array}$$

It is clear that if p is a periodic point of $\hat{f}$, then $\pi(p)$ is a periodic point of f. Since periodic points of $\hat{f}$ are dense in $J(\hat{f})$, periodic points of f are dense in $\pi(J(\hat{f}))$. since $J(\hat{f})$ does not have isolated points, $\pi(J(\hat{f}))$ does not have isolated points. It is clear that a periodic point of f in $\pi(J(\hat{f}))$ is not super-attracting. Since $V \not\subset C(f)$, there are at most finitely many super-saddle periodic points in $\pi(J(\hat{f}))$. By the main theorem of [21], every periodic point is either repelling or super-attracting or super-saddle. Therefore periodic points in $\pi(J(f))$, except finitely many of them, are repelling. Thus repelling points are dense in $\pi(J(\hat{f}))$. By Theorem 3.3, $\pi(J(\hat{f})) \subset J_2$. □

We come back to the proof of Theorem 5.2. Suppose f is strictly PCF on $\mathbb{P}^2$. Then f does not have critical component cycles, nor critical point cycles. Thus if V is an invariant irreducible component of $PC(f)$, then $V \not\subset C(f)$. Let $\pi : \hat{V} \to V$ be the normalization of V, and let $\widehat{f}$ be the lift of f on $\hat{V}$. By [18] Lemma 2.6, $\widehat{f}$ is PCF on $\hat{V}$. If p is a super-attracting fixed point of $\hat{f}$, then $\pi(p)$ is a fixed critical point of f, since f is strictly PCF, such p can not exist. Thus $J(\hat{f}) = \hat{V}$ and $\pi(J(\hat{f})) = V$. This implies $V \subset J_2$ by Lemma 5.3.

We recall the theorem of Dinh-Sibony [10] Corollary 1.65:

For holomorphic endomorphism on $\mathbb{P}^k$ of degree $\geq 2, k \geq 1$, there exist an exceptional set E which is a totally invariant algebraic subset of $C(f)$ such that if H is a hypersurface that does not contain any component of E then $(\deg H)^{-1} d^{-n} f^{n*}[H]$ converges to the Green current T when $n \to +\infty$.

In our case $V \cap C(f)$ is finite number of points, and $V \cap C(f)$ is not invariant since f is strictly PCF. Thus V does not contain any component of E and $(\deg V)^{-1} d^{-n} f^{n*}[V]$ converges to the Green current T when $n \to +\infty$. This implies every point in J_1 is approximated by points in $f^{-n}(V)$, which are in J_2, so $J_1 = J_2$. Since f does not have super-attracting cycles, the Fatou set of f is empty by Rong [23], thus $J_2 = J_1 = \mathbb{P}^2$. The proof is complete.

□

Now we prove the following theorem, which together with Theorem 5.2 completes the proof of Theorem 1.4.

THEOREM 5.4. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2 such that all branches of $PC(f)$ are smooth and intersect transversally. Then $J_2 = \mathbb{P}^2$ implies that f is strictly PCF.*

PROOF. By Lemma 2.3, it is enough to prove that $J_2 = \mathbb{P}^2$ implies that f does not have critical component cycles nor critical point cycles. First, f does not have critical component cycles, since for any invariant critical component C of f, C is an attracting set thus $C \cap J_2 = \emptyset$. Next, let x be a fixed critical point of f. If $J_2 = \mathbb{P}^2$ then the Fatou set is empty, thus x can not be super-attracting. By [21], x must be a super-saddle fixed point. By Theorem 3.4, x is contained in an invariant critical component, this is impossible since we just showed f does not have critical component cycles. This completes the proof. □

REMARK 5.5. *If conjecture 1.1 holds, we can give a simple proof of Theorem 1.3 as follows: $J_2 = \mathbb{P}^2 \Leftrightarrow J_1 = J_2$ and Fatou set is empty $\Leftrightarrow$ No critical component cycle and super-attracting cycle (since Fatou set are super-attracting basins) $\Leftrightarrow$ f is strictly PCF (by Theorem 3.4 and Lemma 2.3).*

6. Further discussion

In Theorem 3.4 we proved that for PCF endomorphisms on $\mathbb{P}^2$, under some additional assumptions, every super-saddle cycle is contained in a critical component cycle. We do not know any counterexample when we remove the additional assumptions. Thus a natural question is:

Question 1: Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2, is every super-saddle cycle be contained in a critical component cycle?

In Lemma 4.1 we proved that if f is a PCF endomorphism on $\mathbb{P}^2$ with the "backward contracting property", then an orbit x_0 such that $\omega(x_0)$ contains a point of bounded ramification should be contained in J_2. We want to ask the following:

Question 2 (backward contracting property): Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Let $x_0 \in \mathbb{P}^2$ and let v be a sub-sequence of integers such that $x_v \to y$ and y is of bounded ramification. Then there exist a neighborhood W of y and $\eta : Z \to W$ the holomorphic covering as in Corollary 2.5. Let W_v denotes a connected component of $f^{-v}(W)$ containing x_0. Let $g_v : Z \to W_v$ such that $f^v \circ g_v = \eta$. Assume g_v converges to a holomorphic map ϕ, then must ϕ be a constant map?

We note that, the answer to Question 2 is yes for rational functions on $\mathbb{P}^1$. The reason is that if f is a rational function on $\mathbb{P}^1$, f expands a Thurston metric in a neighborhood of $J(f)$, which is a smooth metric blowing only at $PC(f)$ ([22] section 19). We can also prove that the answer to Question 2 is yes in two dimensions, under the assumption that $PC(f) \subset C(f)$, the reason is the following:

THEOREM 6.1. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Assume $PC(f) \subset C(f)$ Let B be the union of attracting basins of critical component cycles, then $J_2 = \mathbb{P}^2 \setminus B$, and J_2 is a repeller, i.e. there exists $k \geq 1$ and $\lambda > 1$ such that for every $x \in J_2$, for every $v \in T_x\mathbb{P}^2$ we have $\left|Df^k(v)\right| \geq \lambda |v|$.*

PROOF. Since $PC(f) \subset C(f)$, up to an iteration of f we may assume $PC(f)$ contains only invariant critical components. Let $J := \mathbb{P}^2 \setminus B$. Let Ω be a neighborhood of J such that $\Omega \subset\subset f(\Omega)$. Let C be an invariant critical component. Let W be a neighborhood of C such that $f(\Omega) \cap W = \emptyset$. It is clear that W contains five curves in general position, i.e. three curves can not intersect at one point. Then by [18] Proposition 3.8, $\mathbb{P}^2 \setminus W$ is Kobayashi hyperbolic. So $f(\Omega)$ and Ω are Kobayashi hyperbolic. Let $|.|_{K,\Omega}$ denotes the Kobayashi metric on Ω. (We refer to [19] for the background on Kobayashi metric). Since $\Omega \subset\subset f(\Omega)$, there exists a constant $\lambda > 1$ such that for every $x \in \Omega$, for every $v \in T_x\mathbb{P}^2$, we have

$$|v|_{K,\Omega} \geq \lambda |v|_{K,f(\Omega)}.$$

Since $f : \Omega \to f(\Omega)$ is a covering map, for every $x \in \Omega$, for every $v \in T_x\mathbb{P}^2$, we have

$$|v|_{K,\Omega} = |Df(v)|_{K,f(\Omega)}.$$

Thus we have for every $x \in \Omega$, for every $v \in T_x\mathbb{P}^2$

$$|Df(v)|_{K,f(\Omega)} \geq \lambda |v|_{K,f(\Omega)}.$$

This implies that for the usual Fubini-Study metric, there exists $k \geq 1$ and $\lambda > 1$ such that for every $x \in J_2$, for every $v \in T_x\mathbb{P}^2$, $\left|Df^k(v)\right| \geq \lambda|v|$. Thus for every $x \in J$, for $r > 0$ small, the diameter of the backward image of $B(x, r)$ by f^n converges to 0. By Lemma 4, 1, this implies $x \in J_2$. Thus $J = J_2$ and the proof is complete. $\square$

Now it is easy to see that Theorem 6.1 implies a positive answer to Question 2 , since in this case every point in $PC(f)$ is of unbounded ramification. Thus x_0 in Question 2 must be contained in J_2, and the backward contracting property follows. Finally, we show that the positive answers to Question 1 and Question 2 will confirm Conjecture 1.1.

THEOREM 6.2. *Let f be a PCF endomorphism on $\mathbb{P}^2$ of degree ≥ 2. Assume every super-saddle cycle is contained in a critical component cycle and f satisfies the backward contracting property of Question 2. Then $J_1 \backslash J_2$ is contained in the union of the attracting basins of critical component cycles.*

PROOF. Suppose f satisfies the assumptions in the theorem. Let B be the union of attracting basins of critical component cycles. It is equivalent to prove $J_1 \setminus B = J_2$. Our strategy is the same as in Section 4. Let $x_0 \in J_1 \setminus B$,. We show that $\omega(x_0)$ is contained in the set of points of bounded ramification. Suppose not, since $x_0 \notin B$, by Lemma 2.3 $\omega(x_0)$ must contain a critical point cycle. This critical point cycle can not be super-attracting, since x_0 is not in the Fatou set. By Le [21], this critical point cycle must be a super-saddle cycle. Then by our assumption, x_0 must be contained in a critical component cycle, which is impossible. Thus $\omega(x_0)$ is contained in the set of points of bounded ramification. Since f satisfies the backward contracting property with conditions as in Lemma 4.1 and 4.2. Since every limit map of g_v is a constan map, by Lemma 4.1 $x_0 \in J_2$, which completes the proof. $\square$

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