# Pancyclicity in hamiltonian graph theory

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*Pancyclicité dans la théorie des graphes hamiltonienne*

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# Abstract


In this thesis, we focus on the following topics in graph theory: hamiltonian problem, pancyclicity, chorded pancyclicity in the claw-free graphs, $k$-fan-connected graphs.

This thesis includes seven chapters. The first chapter introduces definitions and background. Then our main studies are presented in Chapters 2-6. Finally, in Chapter 7, we summarize the main results of this thesis and introduce the future research.

In Chapter 1, we give a short but relatively complete introduction. In the first part, some basic definitions and notations are given. In the second section, we introduce some background of hamiltonian graphs and generalizations of hamiltonian problem. And we reviewed the classic results on these topics. In the last section, we show the motivations and overview of our main topics.

The hamiltonian graph theory has been studied widely as one of the most important problems in graph theory. In fact, the hamiltonian problem includes also the generalization of hamiltonian cycles such as circumferences, dominating cycles, pancyclic, cyclability, etc. In this thesis, we will work on the generalizations of hamiltonian graph theory.

There are four fundamental results that deserve special attention here, both for their contribution to the overall theory and their effect on the area's development.

The first result is Dirac's theorem (in 1952), where the search for sufficient conditions for graphs to become hamiltonian graphs usually involves some kind of edge density condition. Enough edges are provided for the existence of a hamiltonian cycles. Dirac's theorem is the first sufficient condition for a graph to be hamiltonian. It is shown that if the degree of each vertex is at least half of the order of the graph, then the graph is hamiltonian.

The second result is Ore's theorem (in 1960), which relaxes Dirac's condition and extends the methods for controlling the degrees of the vertices in the graph. This is the first important generalization of Dirac's theorem. Ore's theorem is that if for any two nonadjacent vertices, their degree sum is greater than or equal to $n$, then the graph of order $n$ is hamiltonian.

The $k$-closure $Cl_k(G)$ is obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $k$, until no such pair remains. The $k$-closure is independent of the order of the addition of the edges.

The third fundamental result is that a graph $G$ of order $n$ is hamiltonian if and only if $Cl_n(G)$ is hamiltonian.

The fourth fundamental result presents a sufficient condition of hamiltonian graphs on the relation between the independence number and the connectivity of the graphs. If $G$ is a graph with connectivity $k$ such that $\alpha(G) \leq k$, where $\alpha(G)$ is the independence number of $G$, then $G$ is hamiltonian.

Many achievements have been made in the research related to these four fundamental results, but many questions remain to be solved. In this thesis, we will focus on a few questions related to the four basic results.

A cycle containing all vertices of a graph $G$ is called a hamiltonian cycle and $G$ is called hamiltonian if it contains a hamiltonian cycle. A graph $G$ is called pancyclic if it contains cycles of all length $k$ for $3 \leq k \leq |V(G)|$. Analogously, a bipartite graph $G$ is called bipancyclic if it contains cycles of all even lengths from $4$ to $|V(G)|$.

In Chapters 2 and 3, we study the pancyclicity of a connected graph. Ore showed in 1960 that if the degree sum of any pair of nonadjacent vertices is at least $n$ in a graph $G$ of order $n$, then $G$ is hamiltonian. Bondy proved that under the same condition, $G$ is pancyclic or $G = K_{n/2,n/2}$. Thus, Bondy suggested the interesting "metaconjecture": almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs).

A vertex-cut of $G$ is a subset $V'$ of $V(G)$ such that $G - V'$ is disconnected. If the vertex-cut $V'$ has only one vertex $\{v\}$, then we call $v$ as a cut-vertex. A $k$-vertex-cut is a vertex-cut of $k$ elements. If $G$ has at least one pair of distinct nonadjacent vertices, the connectivity $\kappa(G)$ of $G$ is the minimum $k$ for which $G$ has a $k$-vertex-cut; otherwise, we define $\kappa(G)$ to be $|V(G)| - 1$. $G$ is said to be $k$-connected if $\kappa(G) \geq k$.

The hamiltonian problem also includes the generalization of hamiltonian cycles. Cyclable problem is one of the most important generalizations of hamiltonian cycles.

Let $S$ be a subset of $V(G)$. We say that $G$ is $S$-*cyclable* if $G$ has an $S$-*cycle*, i.e., a cycle containing all vertices of $S$. In 2005, Flandrin, Li, Marczyk and Wozniak showed the following theorem which is an Ore-type condition for graphs to be $S$-cyclable. Let $G = (V, E)$ be a $k$-connected graph of order $n$ with $k \geq 2$, and $X_1, X_2, \ldots, X_k$ be subsets of the vertex set $V$, $X = X_1 \cup X_2 \cup \ldots \cup X_k$. If for each $i = 1, 2, \ldots, k$, for any pair of nonadjacent vertices in $X_i$, their degree sum is at least $n$, then $G$ is $X$-cyclable.

From the above result and Bondy's "metaconjecture", we propose our conjecture: if $G = (V, E)$ is a $k$-connected graph ($k \geq 2$) of order $n$ with $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$, and for any pair of nonadjacent vertices $x, y$ in $X_i$ with $i = 1, 2, \ldots, k$, we have $d(x) + d(y) \geq n$, then $G$ is pancyclic or $G$ is a bipartite graph.

In Chapter 2, we prove that our conjecture is true for $k = 2$. We prove that if $G = (V, E)$ is a $2$-connected graph of order $n$ with $V(G) = X \cup Y$ such that for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$, $d(x_1) + d(x_2) \geq n$ and for any pair of nonadjacent vertices $y_1$ and $y_2$ in $Y$, $d(y_1) + d(y_2) \geq n$, then $G$ is pancyclic or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$. It is easy to see that our result is stronger than Bondy's result.

To prove our result, we present some lemmas.

The first lemma is that let $G = (V, E)$ be a 2-connected balanced bipartite graph of order $n$ and $V(G) = X \cup Y$, if for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$ ($y_1$ and $y_2$ in $Y$), $d(x_1) + d(x_2) \geq n$ ($d(y_1) + d(y_2) \geq n$), then $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.

The second lemma is as follows. Let $P = u_1u_2u_3 \cdots u_p$ be a path in a graph $G$. If for any vertices $x, y \in V(G) - V(P)$ such that $(N_P(x) - \{u_1\})^- \cap N_P(y) = \emptyset$, then $d_P(x) + d_P(y) \leq p + 1$. If $d_P(x) + d_P(y) = p + 1$, then (1) $V(P) = (N_P(x) - \{u_1\})^- \cup N_P(y)$; (2) $xu_1, yu_p \in E(G)$; (3) If $u_i \notin N_P(x)$ for some $i, 2 \leq i \leq p$, then $u_{i-1} \in N_P(y)$. And if $u_j \notin N_P(y)$ for some $j, 1 \leq j \leq p-1$, then $u_{j+1} \in N_P(x)$; (4) If $u_i, u_j \notin N_P(x) \cup N_P(y)$ with $2 \leq i < j \leq p-1$ such that $\{u_{i+1}, u_{i+2}, \ldots, u_{j-1}\} \subseteq N_P(x) \cup N_P(y)$, then there exists an exact one $k$ with $i+1 \leq k \leq j-1$, such that $\{u_{i+1}, u_{i+2}, \ldots, u_k\} \subseteq N_P(x)$ and $\{u_k, u_{k+1}, \ldots, u_{j-1}\} \subseteq N_P(y)$; (5) If $N_P(x)$ does not contain consecutive vertices on $P$ and $N_P(y)$ does not contain consecutive vertices on $P$, then $p$ is odd and $N_P(x) = N_P(y) = \{u_1, u_3, u_5, \ldots, u_{p-2}, u_p\}$.

In Chapter 3, we prove that our conjecture is true for $k = 3$. It is kind of a continuation of the work in Chapter 2. Our main result is to prove that a 3-connected graph $G = (V, E)$ of order $n$ and $V(G) = X_1 \cup X_2 \cup X_3$, and any pair of nonadjacent vertices $v_1$ and $v_2$ in $X_i$, $d(v_1) + d(v_2) \geq n$ with $i = 1, 2, 3$, then $G$ is pancyclic or $G$ is a bipartite graph.

The main idea and the main tools of the proof of Theorem in Chapter 3 and Theorem in Chapter 2 are similar, but there are also some differences. To make this chapter complete, we will give the whole proof of the Theorem in Chapter 3.

In the results of the Chapter 3 of the proof, we give the following lemma. Let $G = (V, E)$ be a 3-connected graph of order $n$ and $V(G) = X_1 \cup X_2 \cup X_3$. If for each $i$, $i = 1, 2, 3$, $G[X_i]$ is a clique, then $G = K_{3,3}$ or $G$ is pancyclic.

A digraph $D$ is strongly connected if there exists a path from $x$ to $y$ and a path from $y$ to $x$ for every pair of distinct vertices $x, y$. A digraph $D$ is $k$-strongly ($k \geq 1$) connected (or $k$-strong), if $|V(D)| \geq k+1$ and $D(V(D) \setminus A)$ is strongly connected for any subset $A \subseteq V(D)$ of at most $k-1$ vertices. A digraph $D$ is bipartite if there exists a partition $X, Y$ of $V(D)$ into two partite sets such that every arc of $D$ has its end-vertices in different partite sets. It is called balanced if $|X| = |Y|$.

For two distinct vertices $x, y$ in $D$, $\{x, y\}$ dominates a vertex $z$ if $x \to z$ and $y \to z$; in this case, we call the pair $\{x, y\}$ dominating.

A digraph $D$ is called non-hamiltonian if it is not hamiltonian. A balanced bipartite digraph of order $2m$ is even pancyclic (or bipancyclic) if it contains a cycle of length $2k$ for any $k, 2 \leq k \leq m$.

In Chapter 4, we consider pancyclic and hamiltonian problems in digraph or bipartite digraph. In Section 1, we present a list of hamiltonian results of digraph or bipartite digraph. In Section 2, we give a sufficient condition for a balanced bipartite digraph to be hamiltonian. We prove that for each dominating pair of vertices when their degree sum is at least $3a$, the strongly connected balanced bipartite directed graph $D$ of order $2a \geq 10$ is hamiltonian. In

Section 3, we show some new sufficient conditions for bipancyclic and cyclability of digraphs.

Chorded pancyclic is one of the generalizations of the hamiltonian problem.

In Chapter 5, we consider chorded pancyclic problems on $K_{1,3}$-free graph. A non-induced cycle is called a chorded cycle. A graph $G$ of order $n$ is chorded pancyclic if $G$ contains a chorded cycle of each length from $4$ to $n$. A graph is called $K_{1,3}$-free if it has no induced $K_{1,3}$ subgraph. If a cycle has at least two chords, then the cycle is called a doubly chorded cycle. A graph $G$ of order $n$ is doubly chorded pancyclic if $G$ contains a doubly chorded cycle of each length from $4$ to $n$.

Bondy's metaconjecture was extended as follows. Almost any condition that implies a graph is hamiltonian will also imply it is chorded pancyclic, possibly with some class of well-defined exceptional graphs and some small order exceptional graphs.

We study a minimum degree condition for $K_{1,3}$-free graphs to be chorded pancyclic. In 1986, Flandrin, Fournier and Germa gave a condition of minimum degree for $K_{1,3}$-free graphs to be pancyclic, i.e., a $2$-connected $K_{1,3}$-free graph $G$ of the order $n \geq 35$, if $\delta(G) \geq \frac{n-2}{3}$, then $G$ is pancyclic.

In Chapter 5, from the above result and the extension of Bondy's metaconjecture, we obtain the results of the extension of pancyclic to the chorded pancyclic. We prove the following result: every $2$-connected $K_{1,3}$-free graph $G$ with $|V(G)| \geq 35$ is chorded pancyclic if the minimum degree is at least $\frac{n-2}{3}$. This result supports for the extension of Bondy's metaconjecture. Furthermore, we show the number of chords in the chorded cycle of length $m$ $(4 \leq m \leq n)$. Let $CH_m$ be the maximum number of chords in cycle $C_m \subseteq G$ with $4 \leq m \leq n$, and $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then we obtain the size of $CH_m$: if $4 \leq m \leq 5$, then $CH_m \geq \frac{m(m-1)}{2} - m$; if $6 \leq m \leq \frac{n+1}{3}$, $CH_m \geq m$; if $\frac{n+4}{3} \leq m \leq \frac{2n+8}{3}$, $CH_m \geq [\frac{m}{6}]$; if $\frac{2n+11}{3} \leq m \leq n$, $CH_m \geq \frac{m(\delta-(n-m))}{2} - m$.

Moreover, we prove $CH_m \geq 2$. So, we can obtain $G$ is doubly chorded pancyclic.

A hamiltonian path of a graph $G$ is a path that contains all vertices of $V(G)$. A graph G is Hamilton-connected if there is a hamiltonian path connecting every two distinct vertices.

In 1991, Flandrin, Jung and Li proved that if for any three independent vertices $x_1, x_2, x_3$ in a $2$-connected graph $G$ of order $n$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq n$, then $G$ is hamiltonian.

As a generalization of Hamilton-connected and hamiltonian path, Lin et $al.$ introduced the $k$-fan-connectivity of graphs: for any integer $t \geq 2$, let $v$ be a vertex of a graph $G$ and let $U = \{u_1, u_2, \ldots, u_t\}$ be a subset of $V(G) \setminus \{v\}$. A $(v, U)$-*fan* is a set of paths $P_1, P_2, \ldots, P_t$ such that $P_i$ is a path connecting $v$ and $u_i$ for $1 \leq i \leq t$ and $P_i \cap P_j = \{v\}$ for $1 \leq i < j \leq t$.

It follows from Menger theorem that there is a $(v, U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| \leq k$ if and only if $G$ is $k$-connected. If a $(v, U)$-fan spans $G$, then it is called a spanning $(v, U)$-fan of $G$. $G$ is $k$-fan-connected if $G$ has a spanning $(v, U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with

$|U| = k$. Clearly, the $k$-fan-connectivity generalizes the Hamilton-connectivity. Thus, if a graph $G$ has order at least three, it is easy to obtain that $G$ is Hamilton-connected is equivalent to $G$ is $2$-fan-connected.

In Chapter 6, we show the proposition: a graph $G$ is $k$-fan-connected with $k \geq 2$, then $G$ is $(k+1)$-connected.

In 2009, Lin, Cheng-Kuan, et al. proved that if for any two nonadjacent vertices $x, y$ in a graph $G$ with $k \geq 2$, $d(x) + d(y) \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected.

In Chapter 6, we improve the above Lin, Cheng-Kuan, et al.'s result by showing that the Flandrin-Jung-Li's degree sum condition is a new sufficient condition of $k$-fan-connected graphs. We prove that if for any three independent vertices $x_1, x_2, x_3$ in a graph $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected and the lower bound is sharp.

In Chapter 6, we also give an example that satisfies our main result's conditions but does not satisfy the degree sum condition of Lin, Cheng-Kuan, et al.'s theorem. And we show Lin, Cheng-Kuan, et al.'s theorem can be derived from our result.

From our result, we can obtain a corollary: if for any three independent vertices $x_1, x_2, x_3$ in a $3$-connected graph $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + 1$, then $G$ is Hamilton-connected.

This corollary is stronger than Ore's theorem (Let $G$ be a graph. If for any two nonadjacent vertices $x, y$ such that $d(x) + d(y) \geq |V(G)| + 1$, then $G$ is Hamilton-connected.) in the case of $3$-connected graphs.

We prove our result of Chapter 6 by contradiction and induction. In the first section, we will present Menger's Theorem and give some other related introductions. The lower bound of $\overline{\sigma_3}(G)$ in our result is sharp as shown in the second section. With some preliminaries introduced in the third section, we prove our result in the last section.

In Chapter 7, we briefly describe the obtained results. And, we would like to mention several new studies related to this thesis that is not included in the thesis. Moreover, Chapter 7 also covers other topics that I am interested in, such as hamiltonian line graphs, fault-tolerant hamiltonicity, graph coloring and so on. These topics are likely to become my further research fields.

**Keywords:** Pancyclicity, Hamiltonian cycle, Digraph, Bipartite digraph, Chorded pancyclicity, Claw-free graph, $k$-fan-connected.

# Résumé


Dans cette thèse, nous nous concentrons sur les sujets suivants en théorie des graphes : problème hamiltonien, panpsychisme, pancyclique à cordes dans les graphes sans griffes, graphes $k$-fan-connectés.

Cette thèse comprend sept chapitres. Le premier chapitre présente les définitions et le contexte. Ensuite, nos principales études sont présentées dans les Chapitres 2-6. Enfin, dans le Chapitre 7, nous résumons les principaux résultats de cette thèse et introduisons les recherches futures.

Au Chapitre 1, nous donnons une introduction courte mais relativement complète. Dans la première partie, quelques définitions et notations de base sont données. Dans la deuxième section, nous introduisons un aperçu des graphes hamiltoniens et des généralisations du problème hamiltonien. Et nous avons passé en revue les résultats classiques sur ces sujets. Dans la dernière section, nous montrons les motivations et un aperçu de nos principaux sujets.

La théorie des graphes hamiltonienne a été largement étudiée comme l'un des problèmes les plus importants de la théorie des graphes. En fait, le problème hamiltonien inclut également la généralisation des cycles hamiltoniens tels que les circonférences, les cycles dominants, pancyclique, cyclabilité, etc. Dans cette thèse, nous travaillerons sur les généralisations de la théorie des graphes hamiltonienne.

Il y a quatre résultats fondamentaux qui méritent une attention particulière ici, à la fois pour leur contribution à la théorie globale et leur effet sur le développement de la région.

Le premier résultat est le théorème de Dirac (en 1952), où la recherche de conditions suffisantes pour que les graphes deviennent des graphes hamiltoniens implique généralement une sorte de condition de densité d'arêtes. Suffisamment d'arêtes sont fournies pour l'existence d'un cycle hamiltonien. Le théorème de Dirac est la première condition suffisante pour qu'un graphe soit hamiltonien. On montre que si le degré de chaque sommet est au moins la moitié de l'ordre du graphe, alors le graphe est hamiltonien.

Le second résultat est le théorème d'Ore (en 1960), qui assouplit la condition de Dirac et étend les méthodes de contrôle des degrés des sommets du graphe. C'est la première généralisation importante du théorème de Dirac. Le théorème de Ore est que si pour deux sommets non adjacents, leur somme de degrés est supérieure ou égale à $n$, alors le graphe d'ordre $n$ est hamiltonien.

La $k$-clôture $Cl_k(G)$ est obtenue à partir de $G$ en joignant récursivement des paires de sommets non adja-

cents dont la somme des degrés est d'au moins $k$, jusqu'à ce qu'il ne reste plus une telle paire. La $k$-clôture est indépendante de l'ordre d'adjacent des arêtes.

Le troisième résultat fondamental est qu'un graphe $G$ d'ordre $n$ est hamiltonien si et seulement si $Cl_n(G)$ est hamiltonien.

Le quatrième résultat fondamental présente une condition suffisante des graphes hamiltoniens sur la relation entre le nombre d'indépendances et la connectivité des graphes. Si $G$ est un graphe de connectivité $k$ tel que $\alpha(G) \leq k$, où $\alpha(G)$ est le nombre d'indépendances de $G$, alors $G$ est hamiltonien.

De nombreuses réalisations ont été réalisées dans la recherche liée à ces quatre résultats fondamentaux, mais de nombreuses questions restent à résoudre. Dans cette thèse, nous nous concentrerons sur quelques questions liées aux quatre résultats de base.

Un cycle contenant tous les sommets d'un graphe $G$ est appelé cycle hamiltonien et $G$ est dit hamiltonien s'il contient un cycle hamiltonien. Un graphe $G$ est dit pancyclique s'il contient des cycles de toute longueur $k$ pour $3 \leq k \leq |V(G)|$. De manière analogue, un graphe bipartite $G$ est dit bipancyclique s'il contient des cycles de tous pairs longueurs de $4$ à $|V(G)|$.

Dans les Chapitres 2 et 3, nous étudions la pancyclicité d'un graphe connecté. Ore a montré en 1960 que si la somme des degrés d'une paire de sommets non adjacents est d'au moins $n$ dans un graphe $G$ d'ordre $n$, alors $G$ est hamiltonien. Bondy a prouvé que sous la même condition, $G$ est pancyclique ou $G = K_{n/2,n/2}$. Ainsi, Bondy a suggéré l'intéressante "métaconjecture" : presque toutes les conditions non triviales sur les graphes qui impliquent que le graphe soit hamiltonien implique aussi que le graphe est pancyclique (il peut y avoir une famille de graphes exceptionnels).

Un sommet-coupe de $G$ est un sous-ensemble $V'$ de $V(G)$ tel que $G - V'$ est déconnecté. Si le sommet-coupe $V'$ n'a qu'un seul sommet $\{v\}$, alors on appelle $v$ comme coupe-sommet. Un $k$-sommet-coupe est un sommet-coupe de $k$ éléments. Si $G$ a au moins une paire de sommets distincts non adjacents, la connectivité $\kappa(G)$ de $G$ est le $k$ minimum pour lequel $G$ a un $k$-sommet-coupe; sinon, nous définissons $\kappa(G)$ comme étant $|V(G)| - 1$. $G$ est dit $k$-connecté si $\kappa(G) \geq k$.

Le problème hamiltonien comprend également la généralisation des cycles hamiltoniens, le problème cyclable est l'une des généralisations les plus importantes des cycles hamiltoniens.

Soit $S$ un sous-ensemble de $V(G)$. On dit que $G$ est $S$-cyclable si $G$ a un $S$-cycle, c'est-à-dire un cycle contenant tous les sommets de $S$. En 2005, Flandrin, Li, Marczyk et Wozniak ont montré le théorème suivant qui est une condition de type Ore pour que les graphes soient $S$-cyclables. Soit $G = (V, E)$ un graphe $k$-connecté d'ordre $n$ avec $k \geq 2$, et $X_1$, $X_2$, $\ldots$, $X_k$ des sous-ensembles de l'ensemble de sommets $V$, $X = X_1 \cup X_2 \cup \ldots \cup X_k$. Si pour chaque $i = 1, 2, \ldots, k$, pour toute paire de sommets non adjacents dans $X_i$, leur somme de degrés est d'au moins $n$, alors $G$ est $X$-cyclable.

À partir du résultat ci-dessus et de la “metaconjecture” de Bondy, nous proposons notre conjecture : si $G = (V,E)$ est un graphe $k$-connecté ($k \geq 2$) d'ordre $n$ avec $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$, et pour toute paire de sommets non adjacents $x, y$ dans $X_i$ avec $i = 1, 2, \ldots, k$, on a $d(x) + d(y) \geq n$, alors $G$ est pancyclique ou $G$ est un graphe bipartite.

Au Chapitre 2, nous prouvons que notre conjecture est vraie pour $k = 2$. On montre que si $G = (V,E)$ est un graphe $2$-connecté d'ordre $n$ avec $V(G) = X \cup Y$ tel que pour toute paire de sommets non adjacents $x_1$ et $x_2$ dans $X$, $d(x_1) + d(x_2) \geq n$ et pour toute paire de sommets non adjacents $y_1$ et $y_2$ dans $Y$, $d(y_1) + d(y_2) \geq n$, alors $G$ est pancyclique ou $G = K_{n/2,n/2}$ ou $G = K_{n/2,n/2} - \{e\}$. Il est facile de voir que notre résultat est plus fort que celui de Bondy.

Pour prouver notre résultat, nous présentons quelques lemmes.

Le premier lemme est que soit $G = (V,E)$ un graphe biparti équilibré $2$-connecté d'ordre $n$ et $V(G) = X \cup Y$, si pour une paire de sommets non adjacents $x_1$ et $x_2$ dans $X$ (resp. $y_1$ et $y_2$ dans $Y$), $d(x_1)+d(x_2) \geq n$ ($d(y_1)+d(y_2) \geq n$, resp.), alors $G = K_{n/2,n/2}$ ou $G = K_{n/2,n/2} - \{e\}$.

Le deuxième lemme est le suivant. Soit $P = u_1u_2u_3 \cdots u_p$ un chemin dans un graphe $G$. Si pour tout sommet $x, y \in V(G) - V(P)$ tel que $(N_P(x) - \{u_1\})^- \cap N_P(y) = \emptyset$, alors $d_P(x) + d_P(y) \leq p + 1$. Si $d_P(x) + d_P(y) = p + 1$, alors (1) $V(P) = (N_P(x) - \{u_1\})^- \cup N_P(y)$ ; (2) $xu_1, yu_p \in E(G)$; (3) Si $u_i \notin N_P(x)$ pour quelque $i, 2 \leq i \leq p$, alors $u_{i-1} \in N_P(y)$. Et si $u_j \notin N_P(y)$ pour quelque $j, 1 \leq j \leq p-1$, alors $u_{j+1} \in N_P(x)$; (4) Si $u_i, u_j \notin N_P(x) \cup N_P(y)$ avec $2 \leq i < j \leq p-1$ tel que $\{u_{i+1}, u_{i+2}, \cdots, u_{j-1}\} \subseteq N_P(x) \cup N_P(y)$, alors il existe exactement $k$ avec $i+1 \leq k \leq j-1$, tel que $\{u_{i+1}, u_{i+2}, \cdots, u_k\} \subseteq N_P(x)$ et $\{u_k, u_{k+1}, \ cdots, u_{j-1}\} \subseteq N_P(y)$; (5) Si $N_P(x)$ ne contient pas de sommets consécutifs sur $P$ et $N_P(y)$ ne contient pas de sommets consécutifs sur $P$, alors $p$ est impair et $N_P(x) = N_P(y) = \{u_1, u_3, u_5, \cdots, u_{p-2}, u_p\}$.

Au Chapitre 3, nous prouvons que notre conjecture est vraie pour $k = 3$. C'est une sorte de continuation du travail du Chapitre 2. Notre résultat principal est de prouver qu'un graphe connecté à $3$ $G = (V,E)$ d'ordre $n$ et $V(G) = X_1 \cup X_2 \cup X_3$, et toute paire de sommets non adjacents $v_1$ et $v_2$ dans $X_i$, $d(v_1) + d(v_2) \geq n$ avec $i = 1, 2, 3$, alors $G$ est pancyclique ou $G$ est un graphe bipartite.

L'idée principale et les principaux outils de la preuve du théorème du Chapitre 3 et du théorème du Chapitre 2 sont similaires, mais il y a aussi quelques différences. Pour compléter ce chapitre, nous donnerons la preuve complète du théorème au Chapitre 3.

Dans les résultats du Chapitre 3 de la preuve, nous donnons le lemme suivant. Soit $G = (V,E)$ un graphe $3$-connecté d'ordre $n$ et $V(G) = X_1 \cup X_2 \cup X_3$. Si pour chaque $i$, $i = 1, 2, 3$, $G[X_i]$ est une clique, alors $G = K_{3,3}$ ou $G$ est pancyclique.

Un digraphe $D$ est fortement connecté s'il existe un chemin de $x$ à $y$ et un chemin de $y$ à $x$ pour chaque paire de sommets distincts $x, y$. Un digraphe $D$ est $k$-fortement ($k \geq 1$) connecté (ou $k$-fort), si $|V(D)| \geq k+1$ et $D(V(D) \setminus A)$

est fortement connecté pour tout sous-ensemble $A \subseteq V(D)$ d'au plus $k-1$ sommets. Un digraphe $D$ est biparti s'il existe une partition $X, Y$ de $V(D)$ en deux ensembles partites tels que chaque arc de $D$ a ses extrémités-sommets dans différents ensembles de partitions. Il est dit équilibré si $|X| = |Y|$.

Pour deux sommets distincts $x, y$ dans $D$, $\{x, y\}$ domine un sommet $z$ si $x \to z$ et $y \to z$; dans ce cas, nous appelons le couple $\{x, y\}$ dominant.

Un digraphe $D$ est dit non hamiltonien s'il n'est pas hamiltonien. Un digraphe bipartite équilibré d'ordre $2m$ est même pancyclique (ou bipancyclique) s'il contient un cycle de longueur $2k$ pour tout $k, 2 \leq k \leq m$.

Dans le Chapitre 4, nous considérons le problème pancyclique et hamiltonien en digraphe ou digraphe bipartite. Dans la section 1, nous présentons une liste de résultats hamiltoniens de digraphe ou de digraphe bipartite. Dans la section 2, nous donnons une condition suffisante pour qu'un digraphe bipartite équilibré soit hamiltonien. Nous montrons que pour chaque paire dominante de sommets lorsque leur somme de degrés est d'au moins $3a$, le graphe orienté bipartite équilibré fortement connecté $D$ d'ordre $2a \geq 10$ est hamiltonien. Dans la section 3, nous montrons quelques nouvelles conditions suffisantes pour la bipancyclique et la cyclabilité des digraphes.

Le pancyclique à cordes est l'une des généralisations du problème hamiltonien.

Dans le Chapitre 5, nous considérons des problèmes pancycliques à cordes sur un graphe $K_{1,3}$-libre. Un cycle non induit est appelé cycle à cordes. Un graphe $G$ d'ordre $n$ est pancyclique à cordes si $G$ contient un cycle à cordes de chaque longueur de $4$ à $n$. Un graphe est dit $K_{1,3}$-libre s'il n'a pas de sous-graphe $K_{1,3}$ induit. Si un cycle a au moins deux cordes, alors le cycle est appelé un cycle à double corde. Un graphe $G$ d'ordre $n$ est pancyclique à double corde si $G$ contient un cycle à double corde de chaque longueur de $4$ à $n$.

La métaconjecture de Bondy a été étendue comme suit. Presque toutes les conditions qui impliquent qu'un graphe est hamiltonien impliqueront également qu'il est pancyclique à cordes, peut-être avec une classe de graphes exceptionnels bien définis et des graphes exceptionnels de petit ordre.

Nous étudions une condition de degré minimum pour que les graphes $K_{1,3}$-libres soient pancycliques à cordes. En 1986, E. Flandrin, I. Fournier et A. Germa ont donné une condition de degré minimum pour que les graphes $K_{1,3}$-libres soient pancycliques, c'est-à-dire un graphe $G$ $K_{1,3}$-libre $2$-connecté d'ordre $n \geq 35$, si $\delta(G) \geq \frac{n-2}{3}$, alors $G$ est pancyclique.

Au Chapitre 5, à partir du résultat ci-dessus et de l'extension de la métaconjecture de Bondy, on obtient les résultats de l'extension du pancyclique au pancyclique à cordes. Nous prouvons le résultat suivant : tout graphe $G$ $K_{1,3}$-libre $2$-connecté avec $|V(G)| \geq 35$ est pancyclique à cordes si le degré minimum est au moins $\frac{n-2}{3}$. Ce résultat soutient l'extension de la métaconjecture de Bondy. De plus, nous montrons le nombre de cordes dans le cycle à cordes de longueur $m$ ($4 \leq m \leq n$). Soit $CH_m$ le nombre maximum de cordes dans le cycle $C_m \subseteq G$ avec $4 \leq m \leq n$, et $G$ un graphe $K_{1,3}$-libre $2$-connecté avec l'ordre $n \geq 35$. Si $\delta(G) \geq \frac{n-2}{3}$, alors on obtient la taille de

$CH_m$ : si $4 \leq m \leq 5$, alors $CH_m \geq \frac{m(m-1)}{2} - m$ ; si $6 \leq m \leq \frac{n+1}{3}$, $CH_m \geq m$ ; si $\frac{n+4}{3} \leq m \leq \frac{2n+8}{3}$, $CH_m \geq [\frac{m}{6}]$ ; if $\frac{2n+11}{3} \leq m \leq n$, $CH_m \geq \frac{m(\delta-(n-m))}{2} - m$.

De plus, nous prouvons $CH_m \geq 2$. Ainsi, nous pouvons obtenir que $G$ soit un pancyclique à double corde.

Un chemin hamiltonien d'un graphe $G$ est un chemin qui contient tous les sommets de $V(G)$. Un graphe G est connecté à Hamilton s'il existe un chemin hamiltonien reliant tous les deux sommets distincts.

En 1991, E. Flandrin, H.A. Jung et H.Li ont prouvé que si pour trois sommets indépendants $x_1, x_2, x_3$ dans un graphe $G$ 2-connecté d'ordre $n$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq n$, alors $G$ est hamiltonien.

Comme généralisation du chemin Hamilton-connecté et hamiltonien, Lin et $al.$ ont introduit la $k$-fan-connectivité des graphes : Pour tout entier $t \geq 2$, soit $v$ un sommet d'un graphe $G$ et soit $U = \{u_1, u_2, \ldots, u_t\}$ un sous-ensemble de $V(G) \setminus \{v\}$. Un $(v, U)$*-fan* est un ensemble de chemins $P_1, P_2, \ldots, P_t$ tel que $P_i$ est un chemin reliant $v$ et $u_i$ pour $1 \leq i \leq t$ et $P_i \cap P_j = \{v\}$ pour $1 \leq i < j \leq t$.

Il résulte du théorème de Menger qu'il existe un $(v, U)$-fan pour chaque sommet $v$ de $G$ et chaque sous-ensemble $U$ de $V(G) \setminus \{v\}$ avec $|U| \leq k$ si et seulement si $G$ est $k$-connecté. Si un $(v, U)$-fan couvre $G$, alors il est appelé $(v, U)$-fan couvrant de $G$. $G$ est $k$-fan-connecté si $G$ a un $(v, U)$-fan couvrant pour chaque sommet $v$ de $G$ et chaque sous-ensemble $U$ de $V(G) \setminus \{v\}$ avec $|U| = k$. Clairement, la $k$-fan-connectivité généralise la Hamilton-connectivité. Ainsi, si un graphe $G$ est d'ordre au moins trois, il est facile d'obtenir que $G$ est Hamilton-connecté équivaut à $G$ est $2$-fan-connecté.

Au Chapitre 6, nous montrons la proposition : un graphe $G$ est $k$-fan-connecté avec $k \geq 2$, alors $G$ est $(k+1)$-connecté.

En 2009, Lin, Cheng-Kuan et al. ont prouvé que si pour deux sommets non adjacents $x, y$ dans un graphe $G$ avec $k \geq 2$, $d(x) + d(y) \geq |V(G)| + k - 1$, alors $G$ est $k$-fan-connecté.

Au Chapitre 6, nous améliorons le résultat de Lin, Cheng-Kuan et al. ci-dessus en montrant que la condition de somme des degrés de Flandrin-Jung-Li est une nouvelle condition suffisante des graphes $k$-fan-connecté. Nous montrons que si pour trois sommets indépendants $x_1, x_2, x_3$ dans un graphe $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + k - 1$, alors $G$ est $k$-fan-connecté et la borne inférieure est tranchant.

Au Chapitre 6, nous donnons également un exemple qui satisfait les conditions de notre résultat principal, mais ne satisfait pas la condition de somme des degrés du théorème de Lin, Cheng-Kuan et al. Et nous montrons que le théorème de Lin, Cheng-Kuan et al. peut être dérivé de notre résultat.

De notre résultat, nous pouvons obtenir un corollaire : si pour trois sommets indépendants $x_1, x_2, x_3$ dans un graphe $G$ 3-connecté, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + 1$, alors $G$ est Hamilton-connecté.

Ce corollaire est plus fort que le théorème de Ore (Soit $G$ un graphe. Si pour deux sommets non adjacents $x, y$ tels que $d(x) + d(y) \geq |V(G)| + 1$ , alors $G$ est Hamilton-connecté.) dans le cas de graphes $3$-connectés.

Nous prouvons notre résultat du Chapitre 6 par contradiction et récurrence. Dans la première section, nous

présenterons le théorème de Menger et donnerons quelques autres introductions connexes. La borne inférieure de $\overline{\sigma_3}(G)$ dans notre résultat est tranchant comme indiqué dans la deuxième section. Avec quelques préliminaires introduits dans la troisième section, nous prouvons notre résultat dans la dernière section.

Au Chapitre 7, nous décrivons brièvement les résultats obtenus. Et, nous aimerions mentionner plusieurs nouvelles études liées à cette thèse qui n'est pas incluses dans la thèse. De plus, le Chapitre 7 couvre également d'autres sujets qui m'intéressent, tels que les graphes de ligne hamiltoniens, l'hamiltonicité tolérante aux pannes, la coloration de graphe, etc. Ces sujets sont susceptibles de devenir mes autres domaines de recherche.

**Mots clés :** Pancyclicité, Cycle hamiltonien, Digraphe, Digraphe bipartite, Pancyclicité à cordes, Graphe sans griffe, $k$-fan-connecté.

# Contents

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# Symbols

| | |
|---|---|
| $\alpha(G)$ | Independence Number of $G$ |
| $\Delta(G)$ | Maximum Degree of $G$ |
| $\delta(G)$ | Minimum Degree of $G$ |
| $\Delta^{+}(G)$ | Maximum Out-degrees of $G$ |
| $\delta^{+}(G)$ | Minimum Out-degrees of $G$ |
| $\Delta^{-}(G)$ | Maximum In-degrees of $G$ |
| $\delta^{-}(G)$ | Minimum In-degrees of $G$ |
| $\kappa(G)$ | Connectivity of $G$ |
| $c(G)$ | Circumference of $G$ |
| $diam(G)$ | The Diameter of $G$ |
| $g(G)$ | The Girth of $G$ |
| $G \to H$ | $G$ has a homomorphism to $H$ |

# Chapter 1

# Introduction

Graph theory originated from the well-known Seven Bridges of Königsberg problem. This problem was proposed by Leonhard Euler in 1736. Graph theory has experienced tremendous growth in recent decades. There are many well-known problems on graph theory, e.g., hamiltonian problem, four-color problem, Chinese postman problem, the optimal assignment problem, etc. Graph theory serves to analyze many concrete real-world problems successfully. Certain problems in physics, chemistry, communication science, computer technology, genetics, psychology, sociology, linguistics, etc. can be formulated as problems in graph theory.

In this thesis, we will focus on the following topics: hamiltonian graphs, pancyclicity, chorded pancyclic in claw-free graphs, $k$-fan-connected graphs.

In this chapter, we give a short but relatively complete introduction. In the first part, some basic definitions and notations are given. In the second section, we introduce some background of hamiltonian graphs and generalizations of hamiltonian problem. And we reviewed the classic results on these topics. In the last section, we show the motivations and overview of our main topics.

## 1.1 Basic definitions and notations

### 1.1.1 Definitions and notations of graph

A *graph* $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$, disjoint from $V(G)$, of *edges*, and an incidence function $\psi_G$ that associates with each edge of $G$ an unordered pair of (not necessarily distinct) vertices of $G$. If $e$ is an edge and $u$ and $v$ are vertices such that $\psi_G(e) = uv$, then $e$ is said to *join* $u$ and $v$; the vertices $u$ and $v$ are called the *ends* of $e$; the ends $u$ and $v$ are *incident* with an edge $e$. Two vertices $x, y$ are *adjacent*, if $xy$ is an edge of the graph; Two edges $e \neq f$ are *adjacent* if they are incident with a common vertex.

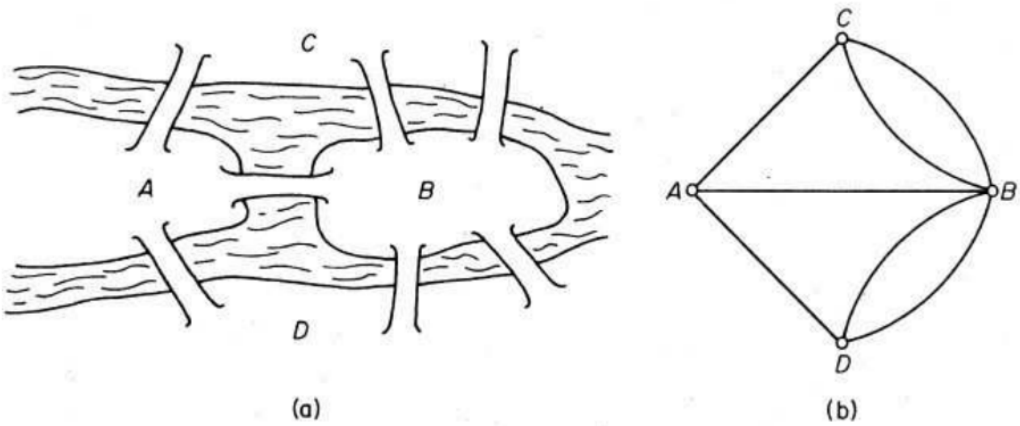



Figure 1.1: The seven bridges and the graph of the Königsberg bridge problem [24]

An edge with identical ends is called a *loop*. Two edges $e$ and $f$ (which are not loops) are said to be *parallel* if they have the same pair of ends. A graph is *simple* if it has neither loops nor parallel edges. A graph with parallel edges and without loops is called a *multigraph*. All graphs considered in this thesis are finite and without loops or multiple edges.

The number of vertices of a graph $G$ is its *order*, written as $|G|$ or $|V(G)|$; its number of edges is its *size*, denoted by $||G||$. Graphs are finite, infinite, countable and so on according to their order.

**Isomorphism**

Let $G$ and $H$ be two graphs. An *isomorphism* between $G$ and $H$ is a bijection $\varphi : V(G) \to V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ if and only if $uv \in E(G)$ for all $u, v \in V(G)$. Two graphs are *isomorphic* if there exists an isomorphism between them.

**Subgraph**

A graph $H$ is a *subgraph* of $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. We write $H \subseteq G$ if $H$ is a subgraph of $G$. When $H \subseteq G$ but $H \neq G$, we call $H$ a *proper subgraph* of $G$.

Suppose that $V'$ is a nonempty subset of $V(G)$. The subgraph of $G$ whose vertex set is $V'$ and whose edge set is the set of those edges of $G$ that have both ends in $V'$ is called the subgraph of $G$ *induced* by $V'$ and is denoted by $G[V']$; we say that $G[V']$ is an *induced subgraph* of $G$. The induced subgraph $G[V(G) \setminus V']$ is denoted by $G - V'$. If $V' = \{v\}$, we write $G - v$ for $G - \{v\}$. A *spanning subgraph* of $G$ is a subgraph of $H$ with $V(H) = V(G)$.

Suppose that $E'$ is a nonempty subset of $E(G)$. The subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and whose edge set is $E'$ is called the subgraph of $G$ *induced* by $E'$ and is denoted by $G[E']$; $G[E']$ is an *edge-induced subgraph* of $G$. The *spanning subgraph* of $G$ with edge set $E(G) \setminus E'$ is written simply as $G - E'$. The graph obtained from $G$ by adding a set of edges $E'$ is denoted by $G + E'$. If $E' = \{e\}$, we write $G - e$ and $G + e$ instead of $G - \{e\}$ and $G + \{e\}$.

**Disjoint union of graphs**

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *disjoint union* of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

**Complete join of graphs**

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *complete join* of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is the graph obtained by starting with $G_1 \cup G_2$ and adding edges joining every vertex of $G_1$ to every vertex of $G_2$.

**Neighbors and degree**

Let $G = (V(G), E(G))$ be a (non-empty) graph. The *set of neighbors* of a vertex $v$ in $G$ is the set of all vertices adjacent to $v$, denoted by $N_G(v)$. Put $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. More generally for $U \subseteq V(G)$, the neighbors in $V \setminus U$ of vertices in $U$ are called *neighbors* of $U$; their set is denoted by $N_G(U)$. If there is no ambiguity, we also write $N(v)$ for $N_G(v)$ and $N(U)$ for $N_G(U)$.

For any vertex $v$ of a simple graph $G = (V(G), E(G))$, the *degree* of $v$ is the number of vertices adjacent to $v$ in $G$, which is equal to the number of neighbors of $v$. We will use $d_G(v)$ to denote the degree of $v$, if there is no confusion arises, simplified as $d(v)$. So $d_G(v) = |N_G(v)|$. A vertex of degree $0$ is *isolated*. We denote $\delta(G)$ and $\Delta(G)$ the *minimum and maximum degrees*, respectively, of vertices of $G$, where $\delta(G) := \min\{d(v) | v \in V(G)\}$ and $\Delta(G) := \max\{d(v) | v \in V(G)\}$.

If all the vertices of $G$ have the same degree $k$, then $G$ is $k$*-regular*, or simply *regular*. A $3$-regular graph is called *cubic*.

**Walk, path and cycle**

A *walk* in a graph $G = (V(G), E(G))$ is a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$, whose terms are alternately vertices and edges, such that, for any $1 \leq i \leq k$, the ends of $e_i$ are $v_{i-1}$ and $v_i$. We say that $W$ is a *walk* from $v_0$ to $v_k$, or a $(v_0, v_k)$*-walk*. The vertices $v_0$ and $v_k$ are called the *initial vertex and terminal vertex* of $W$, respectively. And $v_1, \ldots, v_{k-1}$ are its internal vertices. The integer $k$ is the length of $W$, i.e., the *length of a walk* is the number of its edge. A walk of length $k$ is also called a $k$*-walk*.

If $W = v_0 e_1 v_1 \cdots e_k v_k$ and $W' = v_k e_{k+1} v_{k+1} \cdots e_l v_l$, are walks, the walk $v_k e_k v_{k-1} \cdots e_1 v_0$, obtained by *reversing* $W$, is denoted by $W^{-1}$ and the walk $v_0 e_1 v_1 \cdots e_l v_l$, obtained by concatenating $W$ and $W'$ at $v_k$, is denoted by $WW'$. A *section of a walk* $W = v_0 e_1 v_1 \cdots e_k v_k$ is a walk that is a subsequence $v_i e_{i+1} v_{i+1} \cdots e_j v_j$ of consecutive terms of $W$; we refer to this subsequence as the $(v_i, v_j)$-section of $W$.

In a simple graph, a walk $v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$ can be simply expressed as $v_0 v_1 \cdots v_k$. If the edges $e_1, e_2, \ldots, e_k$ of a walk $W$ are distinct, $W$ is called a *trail*.

If the vertices $v_0, v_1, \ldots, v_k$ of $W$ are distinct, then $W$ is called a *path* or $v_0 - v_k$*-path*. Usually, denote the section $v_i v_{i+1} \cdots v_j$ of the path $P = v_0 v_1 \cdots v_k$ by $P[v_i, v_j]$.

A walk is *closed* if it has positive length and its initial vertex and terminal vertex are the same. A closed trail whose terminal vertex and internal vertex are distinct is a *circuit*; and a closed path is a *cycle*.

The *length of a path or a cycle* is the number of its edges. A path or a cycle of length $k$ is called a $k$-path or $k$-cycle, respectively; the path or cycle is *odd or even* according to the parity of its length.

**Girth, circumference and chord**

The minimum length of a cycle (contained) in a graph $G$ is the *girth* of $G$, denoted by $g(G)$. The *odd-girth* of a graph is the length of the shortest odd-cycle contained in the graph.

The maximum length of a cycle (contained) in $G$ is its *circumference*, denoted by $c(G)$. If a graph does not contain any cycle, its girth and circumference are defined to be infinity.

An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle.

**Distance and diameter**

The *distance* $d_G(x, y)$ in $G$ of two vertices $x, y$ is the length of the shortest $x - y$ path in $G$; if no such path exists, we set $d_G(x, y) = \infty$. Whenever the underlying graph is clear from the context, we will write $d(x, y)$ instead of $d_G(x, y)$.

The greatest distance between any two vertices in a connected graph $G$ is the *diameter* of $G$, denoted by $diamG$.

**Acyclic graph and tree**

An *acyclic graph* is one that contains no cycle in the graph.

A *tree* is a connected acyclic graph. A *spanning tree* of $G$ is a spanning subgraph of $G$ that is a tree.

**Connected and component**

Two vertices $u$ and $v$ of $G = (V(G), E(G))$ are said to be *connected* if there is a $(u, v)$-path in G. A graph $G$ is called *connected* if any two of its vertices are linked by a path in $G$. If $U \subseteq V(G)$ and $G[U]$ is connected, we also call $U$ itself *connected* in $G$. Instead of not connected we usually say *disconnected*.

Let $G = (V, E)$ be a graph. A maximal connected subgraph of $G$ is a *component* of $G$. Clearly, the components are induced subgraphs, and their vertex sets partition $V$. Since connected graphs are non-empty, the empty graph has no components.

**Vertex-cut, connectivity $\kappa(G)$ and $k$-connected**

A *vertex-cut* of $G$ is a subset $V'$ of $V(G)$ such that $G - V'$ is disconnected. If the vertex-cut $V'$ has only one vertex $\{v\}$, then call $v$ as a *cut-vertex*. A $k$*-vertex-cut* is a vertex-cut of $k$ elements. If $G$ has at least one pair of distinct nonadjacent vertices, the *connectivity* $\kappa(G)$ of $G$ is the minimum $k$ for which $G$ has a $k$-vertex-cut; otherwise, we define $\kappa(G)$ to be $|V(G)| - 1$. $G$ is said to be $k$-connected if $\kappa(G) \geq k$.

**Edge-cut, edge-connectivity $\lambda(G)$ and $k$-edge-connected**

An *edge-cut* of $G$ is a subset $E'$ of $E(G)$ such that $G - E'$ is disconnected. If the edge-cut $E' = \{e\}$, then call $e$ as a *cut-edge* or *bridge*. A $k$*-edge-cut* is an edge-cut of $k$ elements. Define the *edge-connectivity* $\lambda(G)$ of $G$ to be the minimum $k$ for which $G$ has a $k$-edge-cut. $G$ is said to be $k$*-edge-connected* if $\lambda(G) \geq k$.

**Independent set and independence number $\alpha(G)$**

An *independent set* of a graph $G$ is a subset of the vertices such that no two vertices in the subset induce an edge of $G$. The cardinality of a maximum independent set in a graph $G$ is called the *independence number* of $G$, denoted by $\alpha(G)$.

**The definitions of $\sigma_m(G)$ and $\overline{\sigma_m}(G)$**

For any integer $m \geq 2$, if $\alpha(G) \geq m$, put

$$\sigma_m(G) = \min\left\{ \sum_{i=1}^{m} \deg_G(x_i) \,\middle|\, x_1, x_2, \ldots, x_m \text{ are pairwisely nonadjacent vertices in G} \right\}$$

$$\overline{\sigma_m}(G) = \min\left\{ \sum_{i=1}^{m} \deg_G(x_i) - |\bigcap_{i=1}^{m} N_G(x_i)| \,\middle|\, x_1, x_2, \ldots, x_m \text{ are pairwisely nonadjacent vertices in G} \right\}.$$

If $G$ does not have $m$ vertices that are independent, we define $\sigma_m(G) = \overline{\sigma_m}(G) = \infty$.

**Hamiltonian cycle and hamiltonian**

A cycle containing all vertices of $G$ is called a *hamiltonian cycle* and $G$ is called *hamiltonian* if it contains a hamiltonian cycle. For two vertices $u$ and $v$, a $(u, v)$*-path* is a path connecting $u$ and $v$. A path in $G$ containing every vertex of $G$ is a *hamiltonian path*. A *hamiltonian* $(u, v)$*-path* is a hamiltonian path connecting $u$ and $v$.

**Traceable, $1$-edge hamiltonian and $1$-hamiltonian**

A graph $G$ is *traceable* if it contains a spanning path (that is, the path containing all the vertices of $G$).

A graph $G = (V, E)$ is $1$*-edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$. Obviously, any $1$-edge hamiltonian graph is hamiltonian. The graph $G$ is $1$*-node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. A graph $G$ is $1$-*hamiltonian* if it is $1$-edge hamiltonian and $1$-node hamiltonian.

In this thesis, we mainly consider simple graphs. We conclude this section by introducing some special classes of graphs.

**Complete graphs and cliques**

A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. If there is just one complete graph on n vertices; it is denoted by $K_n$.

A *clique* of a graph $G$ is a complete graph contained in $G$ as a subgraph. The *clique number* $\omega(G)$ of a graph $G$ is the order of a maximum clique in $G$.

**Bipartite graphs and $k$-partite graphs**

A *bipartite graph* is one whose vertex set can be partitioned into two subsets $X$ and $Y$, so that each edge has one end in $X$ and one end in $Y$; such a partition $(X, Y)$ is called a *bipartition* of graph.

A *complete bipartite graph* is a simple bipartite graph with bipartition $(X, Y)$ in which each vertex of $X$ is joined to each vertex of $Y$; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

A $k$*-partite graph* is one whose vertex set can be partitioned into $k$ subsets so that no edges has both ends in any one subset; a *complete* $k$*-partite graph* is one that is simple and in which each vertex is joined to every vertex that is not in the same subset.

**Line graphs**

The *line graph* of a graph $G$, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a *vertex cut* of $L(G)$ if and only if $X$ is an essential edge-cut of $G$.

**Planar graphs**

A graph is *planar* if it can be drawn on the plan such that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of the graph. Given a planar embedding of a planar graph, it divides the plan into a set of connected regions, including an outer unbounded connected region. Each of these regions is called a *face* of the planar graph. The boundary of a face is the cycle of the graph containing it. A planar graph with a given planar embedding is called a *plane graph*.

**Pancyclic and bipancyclic graphs**

A graph $G$ is called *pancyclic* if it contains cycles of all length $k$ for $3 \leq k \leq |V(G)|$. Analogously, a bipartite graph $G$ is called *bipancyclic* if it contains cycles of all even lengths from $4$ to $|V(G)|$.

**Chorded pancyclic and doubly chorded pancyclic**

A *chord* of a cycle is an edge between two nonadjacent vertices of the cycle. We say that a cycle is *chorded* if the cycle has at least one chord, and we call such a cycle *chorded cycle*. If a cycle has at least two chords, then the cycle is called a *doubly chorded cycle*. A graph $G$ of order $n$ is *chorded pancyclic* (*doubly chorded pancyclic*) if $G$ contains a chorded cycle (doubly chorded cycle) of each length from $4$ to $n$.

In the following, we give some basic terminology and notations of digraphs.

### 1.1.2 Definitions and notations of digraph

A *directed graph* $D$ is an ordered triple $(V(D), A(D), \psi_D)$ consisting of a nonempty set $V(D)$ of *vertices*, a set $A(D)$, disjoint from $V(D)$, of *arcs*, and an *incidence function* $\psi_D$ that associates with each arc of $D$ an ordered pair of (not necessarily distinct) vertices of $D$. If $a$ is an arc and $u$ and $v$ are vertices such that $\psi_D(a) = (u, v)$, then $a$ is said to join $u$ to $v$; $u$ is the *tail* of $a$, and $v$ is its *head*. For convenience, we shall abbreviate directed graph to *digraph*. A digraph is *strict* if it has no loops and no two arcs with the same ends have the same orientation.

**Subdigraph**

A digraph $D'$ is a *subdigraph* of $D$ if $V(D') \subseteq V(D)$, $A(D') \subseteq A(D)$ and $\psi_{D'}$ is the restriction of $\psi_D$ to $A(D')$. The terminology and notation for subdigraphs is similar to that used for subgraphs.

**Directed walks, directed trails, directed paths and directed cycles**

A *directed walk* in $D$ is a finite non-null sequence $W = (v_0, a_1, v_1, \ldots, a_k, v_k)$, whose terms are alternately vertices and arcs, such that, for $i = 1, 2, \ldots, k$, the arc $a_i$ has head $v_i$ and tail $V_{i-1}$. As with walks in graphs, a directed walk $(v_0, a_1, v_1, \ldots, a_k, v_k)$ is often represented simply by its vertex sequence $(v_0, v_1, \ldots, v_k)$. A *directed trail* is a directed walk that is a trail, i.e., a directed trail is a directed walk in which all edges are distinct.

A *directed path* is a directed trail in which all vertices are distinct.

A *directed circuit* is a non-empty directed trail in which the first vertex is equal to the last vertex.

A *directed cycle* is a directed circuit in which the only repeated vertex is the first / last vertex.

**Reachable and diconnected**

If there is a directed $(u, v)$-path in $D$, vertex $v$ is said to be *reachable* from vertex $u$ in $D$.

Two vertices are *diconnected* in $D$ if each is reachable from the other.

The subdigraphs $D[V_1], D[V_2], \cdots, D[V_m]$ induced by the resulting partition $(V_1, V_2, \ldots, V_m)$ of $V(D)$ are called the *dicomponents* of $D$. A digraph $D$ is *diconnected* if it has exactly one dicomponent.

**In-degree, out-degree and degree**

The *in-degree* $d_D^-(v)$ of a vertex $v$ in $D$ is the number of arcs with head $v$; the *out-degree* $d_D^+(v)$ of $v$ is the number of arcs with tail $v$. The *degree* $d_D(v)$ of the vertex $v$ in $D$ is defined as $d_D(v) = d_D^+(v) + d_D^-(v)$.

The number $\min\{d_D^+(x) : x \in V(D)\}$ is called the *minimum out-degree* of $D$ and is denoted by $\delta^+(D)$. *Minimum out-degrees, maximum in-degrees and out-degrees* are similarly defined. We denote the *minimum and maximum in-degrees and out-degrees* in $D$ by $\delta^-(D)$, $\Delta^-(D)$, $\delta^+(D)$ and $\Delta^+(D)$, respectively.

The number $\min\{d^+(x) + d^-(x) : x \in V(D)\}$ is called the *minimum degree* of $D$.

**Out-neighborhood and in-neigborhood**

The *out-neighborhood* of a vertex $x$ is the set $N^+(x) = \{y \in V(D) | xy \in A(D)\}$ and $N^-(x) = \{y \in V(D) | yx \in A(D)\}$ is the *in-neighborhood* of $x$. Similarly, if $A \subseteq V(D)$, then $N^+(x, A) = \{y \in A | xy \in A(D)\}$ and $N^-(x, A) = \{y \in A | yx \in A(D)\}$. The *out-degree* of $x$ is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the *in-degree* of $x$. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$.

**Tournament**

A *tournament* is a digraph, where there is precisely one arc between every pair of distinct vertices.

**Bipartite digraph**

A *bipartite digraph* $D = (X, Y; A)$ has the vertex set partitioned into two partite sets $X$ and $Y$ of cardinalities $a$ and $b$, respectively, where $A$ denotes the set of arcs; each arc has one vertex in $X$ and the other in $Y$. If $a = b$ then $D$ is called *balanced*. $K^*_{a,b}$ denotes a *complete bipartite digraph* with partite sets of cardinalities $a$ and $b$.

**Matching**

A *matching* $M$ from $X$ to $Y$ is a set of arcs such that any vertex in $X \cup Y$ is incident with at most one arc in $A$ and moreover each arc in $M$ has its tail in $X$ and a head in $Y$; $M$ is *perfect* if each vertex has incident arc in $M$.

**Hamiltonian, pancyclic and cyclable**

A cycle (path) is called *hamiltonian* if it includes all the vertices of $D$. A digraph $D$ is *hamiltonian* if it contains a hamiltonian cycle and is *pancyclic* if it contains a cycle of length $k$ for any $3 \leq k \leq n$, where $n$ is the order of $D$. A balanced bipartite digraph of order $2m$ is *even pancyclic* if it contains a cycle of length $2k$ for any $k, 2 \leq k \leq m$. A set $S$ of vertices in a directive graph $D$ is said to be *cyclable* (*pathable*) in $D$ if $D$ contains a directed cycle (path) through all vertices of $S$.

## 1.2 Some background

In 1857, the Irish mathematician Sir William Hamilton (1805-1865) invented a game (Icosian Game, now also known as Hamilton's puzzle) of traveling around the edges of a graph from vertex to vertex. Hamilton described the game, in a letter to his friend Graves, as a mathematical game on the dodecahedron. Each vertex of the dodecahedron is labeled with the name of a city and the game's object is finding a (hamiltonian) cycle along the edges of the dodecahedron such that every vertex is visited a single time, and the ending point is the same as the starting point (see Figure 1.2). Since then, the hamiltonian problem, determining when a graph contains a hamiltonian cycle, has been fundamental in graph theory. For a long time, there was no elegant characterization of hamiltonian graphs, although several necessary and sufficient conditions were known.

Today, however, the constant stream of results in this area continues to supply us with new and interesting theorems and still further questions. The hamiltonian problem came out to be a fruitful branch of graph theory.

The hamiltonian graph theory has been studied widely as one of the most important problems in graph theory. In fact, the hamiltonian problem also includes the generalization of hamiltonian cycles such as circumferences, dominating cycles, pancyclic, cyclability, etc. In this thesis, we will work on the generalizations of hamiltonian graph theory.

### 1.2.1 Some background of hamiltonian problem

Hamiltonian problem is one of the most significant problems in graph theory. Finding its proof has greatly promoted the development of graph theory.

Determining whether hamiltonian cycles exist in graphs is NP-complete. Therefore, it is natural and interesting to study sufficient conditions for hamiltonian problems. On the hamiltonian problems, one may find many well-known theorems in graph theory. Thus, it is not necessary and also impossible to give a detailed survey in this thesis.

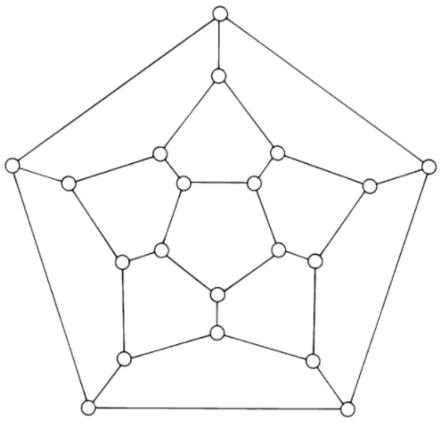


Figure 1.2: The Hamilton's puzzle: the graph of the dodecahedron

There are four fundamental results that I feel deserve special attention here-both for their contribution to the overall theory and their effect on the area's development.

The first result is Dirac's theorem [41] (in 1952), where the search for sufficient conditions for graphs to become hamiltonian graphs usually involves some kind of edge density condition. Enough edges are provided for the existence of a hamiltonian cycles. Dirac's theorem is the first sufficient condition for a graph to be hamiltonian. It is shown that if the degree of each vertex is at least half of the order of the graph, the graph is hamiltonian. More precisely see the following,

**Theorem 1.2.1 (Dirac's theorem, [41])** *If $G$ is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then $G$ is hamiltonian.*

This original result started a new approach to develop sufficient conditions on degrees for a graph to be hamiltonian. A lot of effort has been made by various people in the generalization of Dirac's theorem, and this area is one of the core subjects in hamiltonian graph theory and extremely graph theory.

The second result is Ore's theorem [109] (in 1960), which relaxes Dirac's condition and extends the methods for controlling the degrees of the vertices in the graph. This is the first important generalization of Dirac's theorem.

**Theorem 1.2.2 (Ore's theorem, [109])** *Let $G$ be a graph of order $n$. If $d(x) + d(y) \geq n$ for any pair of nonadjacent vertices $x$ and $y$ in $G$, then $G$ is hamiltonian.*

Any path or cycle problem is really a part of a hamiltonian problem. The founding results of Dirac [41] and Ore [109] established interest in hamiltonian graphs. Degree condition is the classic method to solve the hamiltonian problem, and a neighborhood union is an important form of generalized degree condition.

Let

$$\sigma_k(G) = \min\{d(x_1) + \cdots + d(x_k) | x_1, \ldots, x_k \; are \; independent \; in \; G\}.$$

Graphs satisfying lower bounds on $\sigma_k$ with $k = 2$ will often be called Ore-type graphs, while if $k = 1$, they will be called Dirac-type graphs.

The number of components of a graph $G$ is denoted by $\omega(G)$. The graph $G$ is $t$-tough ($t \geq 0$) if $|S| \geq t \cdot \omega(G - S)$ for every subset $S$ of the vertex set $V(G)$ with $\omega(G - S) > 1$. The toughness of $G$, denoted by $\tau(G)$, is the maximum $t$ for which $G$ is $t$-tough. Thus, a graph $G$ is called $1$-tough if for any subset $S$ of vertices the number of components in $G - S$ is at most $|S|$.

The case where the degree sum is less than Ore's theorem (Theorem 1.2.2) has also been extensively studied. In 1978, Jung [79] showed that a $1$-tough graph $G$ of order $n \geq 11$ with $\sigma_2(G) \geq n - 4$ is hamiltonian. Ainouche and Christofides [5] showed that all $2$-connected maximal non-hamiltonian graphs of order $n$ such that $\sigma_2(G) \geq n - 2$ are isomorphic to one of the following graphs: $K_{(n-1)/2} + \overline{K}_{(n+1)/2}$, $K_{(n-2)/2} + \overline{K}_{(n+2)/2}$, $K_{(n-2)/2} + (\overline{K}_{(n+2)/2} \cup K_2)$, $K_2 + (2K_2 \cup K_1)$ and $K_2 + 3K_2$.

However, degree sum conditions that apply to very few graphs have a major shortcoming. To be more applicable, it is natural to consider changes in such conditions.

In 1980, Bondy [20] also gave a sufficient condition for $G$ to contain a cycle $C$ with $G - V(C)$ contains no clique $K_k$.

Häggkvist and Nicoghossian [68] in 1981 further improved Dirac's theorem by incorporating the connectivity ($k$) of the graph into the degree bound, such as minimum degree $\delta \geq (n + k)/3$, $\sigma_3(G) \geq n + k$ and so on.

In 1984, Fan [45] considered a condition on a particular subset of non-adjacent vertices. Fan's theorem [45] combines local conditions and density conditions. This raises the question, is it possible to use a sparser set of vertices? This idea can be used with other adjacency conditions and structures outside the vertex's neighborhood.

In 1987, Bondy and Fan [22] provided an Ore-type result for finding a dominating cycle, where a *dominating cycle* $C$ is such that every edge of the graph has at least one adjacent vertex on the cycle $C$. Harary and Nash-Williams [72] showed that the existence of a dominating cycle in $G$ is essentially equivalent to the line graph of $G$ is hamiltonian.

Dirac's theorem concerns a degree condition on every vertex. Ore's theorem concerns a degree sum condition on any pair of nonadjacent vertices. It is natural to generalize them into degree and neighborhood conditions on more independent vertices. The results [56] obtained in 1991 use degrees and neighborhood intersection of any set of three independent vertices.

**Theorem 1.2.3 ([56])** *If $G$ is a 2-connected graph of order $n$ such that $\overline{\sigma_3}(G) \geq n$, then $G$ is hamiltonian.*

Li in 2000 extended into conditions on degree sum and neighborhood intersection of four independent vertices in 3-connected graphs.

**Theorem 1.2.4 ([83])** *Let $G$ be a 3-connected graph of order $n$. If $\overline{\sigma_4}(G) \geq n+3$, $G$ has a dominating maximum cycle.*

Bondy [20] gave a sufficient hamiltonian condition that relates the degree sum of any $k+1$ independent vertices.

**Theorem 1.2.5 ([20])** *Let $G$ be a $k$-connected graph of order $n \geq 3$. If $\sigma_{k+1}(G) > 1/2(k+1)(n-1)$, then $G$ is hamiltonian.*

In 2010, Li, Zhou and Wang [90] developed Theorem 1.2.4 to the degree sum of $k+3$ independent vertices.

The Dirac-type condition requires that every vertex has a large degree. However, for some vertices that may have a smaller degree, we hope to use some large degree vertices to replace the small degree vertices in the correct position considered in the proof to constructing a longer cycle. This idea leads to the definitions of implicit degrees given by Zhu, Li, and Deng in 1989.

For any vertex $u$ in a graph $G$, define $N_1(u) = N(u)$ and $N_2(u) = \{x \in V(G) : d(x,u) = 2\}$, where $d(u,v)$ is the distance between $x$ and $u$, i.e., the number of edges in the shortest path between $x$ and $u$.

**Definition 1.2.6** *Let $d(u) = k+1$, and put*

$$M_2 = \max\{d(v) : v \in N_2(u)\} \text{ and } m_2 = \min\{d(v) : v \in N_2(u)\}.$$

*Let*

$$d_1 \leq d_2 \leq \cdots \leq d_k \leq d_{k+1} \leq \cdots$$

*be the degree sequence of the vertices of $N_1(u) \cup N_2(u)$. If $N_2(u) \neq \emptyset$, then we define two kinds of implicit-degrees of $u$, denoted by $d_1(u)$ and $d_2(u)$, as follows:*

$$d_1(u) = \begin{cases} \max\{d_{k+1}, k+1\} & \text{if } d_{k+1} > M_2, \\ \max\{d_k, k+1\} & \text{otherwise.} \end{cases}$$

*and*

$$d_2(u) = \begin{cases} \max\{m_2, k+1\} & \text{if } m_2 > d_k, \\ d_1(u) & \text{otherwise.} \end{cases}$$

*If $N_2(u) = \emptyset$, then define $d_1(u) = d_2(u) = d(u)$.*

*It is clear from the definition that $d_2(u) \geq d_1(u) \geq d(u)$ for every vertex $u$. Let $\delta_i = \min\{d_i(u) : \forall u \in V(G)\}$ and for $i = 1, 2$*

$$\sigma_{i,k}(G) = \min\left\{\sum_{j=1}^{k} d_i(x_j) \,\middle|\, x_1, x_2, \ldots, x_k \text{ are k independent vertices of G}\right\}.$$

In 2012, Li, Ning, Cai extended Theorem 1.2.5 into condition with implicit degrees.

**Theorem 1.2.7 ([92])** *Let $G$ be a $k$-connected graph of order $n \geq 3$. If $\sigma_{(2,k+1)}(G) > (k+1)(n-1)/2$, then $G$ is hamiltonian.*

In 1976, Bondy and Chvátal [21] introduced classical results on stability and closure.

The $k$-closure $Cl_k(G)$ is obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $k$, until no such pair remains. The $k$-closure is independent of the order of the addition of the edges. Obviously, any graph of order $n$ satisfies $G = Cl_{2n-3}(G) \subseteq Cl_{2n-4}(G) \subseteq \cdots \subseteq Cl_1(G) \subseteq Cl_0(G) = K_n$.

The third fundamental result is that a graph $G$ of order $n$ is hamiltonian if and only if $Cl_n(G)$ is hamiltonian.

The following theorem motivated Bondy and Chvátal to the definition of closure. This developed a powerful tool that is very useful in the proofs of many results.

**Theorem 1.2.8 ([21])** *Let $u$ and $v$ be distinct nonadjacent vertices of a graph $G$ of order $n \geq 3$ such that $d_G(u) + d_G(v) \geq n$. Then $G$ is hamiltonian if and only if $G + uv$ is hamiltonian.*

Zhu, Li, and Deng [127] obtained the following result on hamiltonian graphs under the condition of implicit degree.

**Theorem 1.2.9 ([127])** *Let $G$ be a simple graph of order $n$. If $u$ and $v$ are nonadjacent vertices with $d_1(u)+d_1(v) \geq n$, then $G$ is hamiltonian if and only if $G + uv$ is hamiltonian.*

The fourth fundamental result due to Chvátal and Erdös [34] gives a sufficient condition of hamiltonian graphs on the relation between the independence number and the connectivity of the graphs. If $G$ is a graph with connectivity $k$ such that $\alpha(G) \leq k$, where $\alpha(G)$ is the independence number of $G$, then $G$ is hamiltonian.

A graph $G = (V, E)$ is $1$-*edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$. Obviously, any $1$-edge hamiltonian graph is hamiltonian. The graph $G$ is $1$-*node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. A graph $G$ is $1$-*hamiltonian* if it is $1$-edge hamiltonian and $1$-node hamiltonian.

**Theorem 1.2.10 ([34])** *A $k$-connected graph $G$ is*

*(1) Traceable if $\alpha(G) \leq \kappa(G) + 1$.*

*(2) Hamiltonian if $\alpha(G) \leq \kappa(G)$.*

*(3) $1$-hamiltonian, $1$-edge hamiltonian and hamiltonian connected if $\alpha(G) < \kappa(G)$.*

This result also produced many new results.

If $G$ contains no induced subgraph isomorphic to any graph in the set $F = \{H_1, H_2, \ldots, H_k\}$, we say $G$ is $F$-free. If $F = \{H_1\}$, we say $G$ is $H_1$-free.

In 1990, Ainouche et al. [6] showed that $K_{1,3}$-free graph $G$ can reduce the condition of Theorem 1.2.10. The square $G^2$ of $G$ is the graph $(V(G), \{uv|u, v \in V(G); d(u, v) \leq 2\})$, where $d(u, v)$ is the distance in $G$ from $u$ to $v$.

**Theorem 1.2.11 ([6])** *A $k$-connected $K_{1,3}$-free graph $G$ ($k \geq 2$) is hamiltonian if $\alpha(G^2) \leq k$.*

Many achievements have been made in the research related to these four fundamental results, but many questions remain to be solved. In this thesis, we will focus on a few questions related to the four basic results.

### 1.2.2 Some background of generalization of hamiltonian problem

Many results generalize or reinforce Dirac's theorems. Some results generalize hamiltonian cycles to the circumference of graphs, and some results look for more edge-disjoint hamiltonian cycles. In addition, some results attempt to construct cycles of all lengths from 3 to the order of the graph, i.e., to prove that the graph is pancyclic, which is one of the main topics of this thesis.

We will introduce some results which generalize hamiltonian cycles and Dirac's theorems. In addition to the results I introduced, there are many results regarding the generalization of the hamiltonian problem. For some results concerning independence number and connectivity conditions, please refer to [27, 34, 73]; for some results on pancyclic, please refer to [47, 52, 75]. For more details, we refer to the survey paper by Li [84].

A generalization of Dirac's theorem is from the parameter of circumferences of graphs.

**Circumference**

If a graph satisfies the Dirac-type condition or Ore-type condition, then it is hamiltonian. Thus, the circumference of the graph is its order. Bermond, Bondy and Linial show the following result.

**Theorem 1.2.12 ([15], [18] and [98])** *Let $G$ be a $2$-connected graph of order $n$. Then the circumference $c(G) \geq \min\{n, \sigma_2(G)\}$.*

One of the necessary conditions for the hamiltonian graph is $1$-tough, and the $1$-tough graph must be $2$-connected. Therefore, it is natural to want to know the lower bound of the circumference in Dirac-type or Ore-type conditions. Let $G$ be a $1$-tough graph. In 1986, Bauer and Schmeichel [11] proved that $c(G) \geq \min\{n, \sigma_2(G) + 2\}$.

In 1997, Wei [123] generalized Theorem 1.2.3 into circumference in the case that the graph is $3$-connected.

**Theorem 1.2.13 ([123])** *If $G$ is $3$-connected graph, then the circumference $c(G) \geq \min\{n, \overline{\sigma_3}(G)\}$.*

Let $diff(G) = p(G) - c(G)$, where $p(G)$ and $c(G)$ are the orders of the longest path and the longest cycle, respectively. There are many studies on $diff(G)$. In 1995, Enomoto, Van Den Heuvel, Kaneko, and Saito [43] showed that for a $2$-connected graph $G$ of order $n$, if $\sigma_3(G) \geq n+2$, then $diff(G) \leq 1$. And in 2009, Ozeki, Tsugaki, and Yamashita [113] proved that for a $3$-connected graph $G$ of order $n$ with $\sigma_4(G) \geq n+6$, $diff(G) \leq 2$.

For the condition of implicit degree, in [127], Zhu, Li, and Deng obtain results about the circumference. See Definition 1.2.6 for the definition of $\sigma_{(2,2)}(G)$.

**Theorem 1.2.14 ([127])** *Let $G$ be a $2$-connected graph of order $n$. Then the circumference $c(G) \geq \min\{n, \sigma_{(2,2)}(G)\}$.*

When constructing hamiltonian graphs, the transformation of non-hamiltonian graphs into hamiltonian graphs often produces many spanning cycles. Therefore, sometimes it is in nature to count the number of disjoint cycles that exist and prove the existence of several edge-disjoint cycles. One of the generalizations of the hamiltonian problem is edge-disjoint hamiltonian cycles.

**Edge-disjoint hamiltonian cycles**

Edge-disjoint hamiltonian cycles are important in telecommunication networks. Using the hamiltonian cycle, we can design a simple protocol for network communications. If a network has $k$ edge-disjoint hamiltonian cycles, then $k$ different messages can circulate independently in the network. And when less than $k$ edges do not work, the network can still work with some hamiltonian cycles. One of the fundamental results about edge-disjoint hamiltonian cycles in graphs under Dirac-type condition is due to Nash-Williams who showed in [106] that a graph of order $n$ satisfying Dirac-type condition admits at least $\lfloor \frac{5(n+10)}{224} \rfloor$ edge-disjoint hamiltonian cycles. Nash Williams asked if that number could be improved, and it has been a matter of interest ever since. Nash-Williams [106] gave an example of a graph on $n = 4m$ vertices with minimum degree $2m$ having at most $\lfloor (n+4)/8 \rfloor$ edge disjoint hamiltonian cycles.

Nash-Williams [106] noted that the construction given above depends on the graph being non-regular. He conjectured [106] the following, which is the best possible, and was also conjectured independently by Jackson [76].

**Conjecture 1.2.15** *Let $G$ be a $d$-regular graph on at most $2d$ vertices. Then $G$ contains $\lfloor n/2 \rfloor$ edge-disjoint hamiltonian cycles.*

In 1985, Faudree, Rousseau, and Schelp obtained the first results about edge-disjoint hamiltonian cycles in graphs under the Ore-type condition. But they required $n + 2k - 2$ instead of $n$ in Ore-type condition. In 1986, Faudree and Schelp conjectured that if $n$ is sufficiently larger than $\delta$ and $\sigma_2(G) \geq n$, then the graph of order $n$ has $\lfloor \frac{\delta-1}{2} \rfloor$ edge-disjoint hamiltonian cycles. Their conjecture was confirmed in 1989 by Li. In regular graphs, Nash-Williams' result [106] has been extended by Jackson and Li, independently.

Therefore, it is interesting to see if the Ore-type condition $\sigma_2(G) \geq n$ may ensure more edge-disjoint hamiltonian cycles. We have the following,

**Theorem 1.2.16 ([88])** *Let $G$ be a graph of order $n \geq 20$. If $\delta \geq 5$ and $\sigma_2(G) \geq n$, then $G$ has at least two edge-disjoint hamiltonian cycles.*

In regular graphs, the Nash-Williams result [106] has been extended independently by Jackson and Li. A $k$-regular graph is a graph in which every vertex has degree $k$.

**Theorem 1.2.17 (Jackson, [76])** *Let $G$ be a $k$-regular graph of order $n \geq 14$. If $k \geq \frac{n-1}{2}$, then $G$ has at least $\lfloor \frac{3k-n+1}{6} \rfloor$ edge-disjoint hamiltonian cycles.*

**Theorem 1.2.18 (Li, [82])** *Let $G$ be a $k$-regular graph of order at most $3k-2$. If $k \geq 16$ and $G - \{e', e''\}$ is 2-connected for any two edges $e'$ and $e''$, then $G$ admits two edge-disjoint hamiltonian cycles.*

Pancyclicity is one of the most important generalizations of the hamiltonian problem. And pancyclicity is one of the main topics of this thesis.

**Pancyclic, vertex pancyclic and edge pancyclic**

A graph $G$ of order $n$ is said to be vertex pancyclic if, for any vertex $x$, there is a cycle in G of length $l$ containing $x$, for each $l$, $3 \leq l \leq n$. In 1971, Bondy [19] initiated the study of pancyclic and vertex pancyclic graphs, and he showed that if $\delta(G) \geq (n+1)/2$, then $G$ is vertex pancyclic. Many results concerning pancyclic graphs are based upon edge density conditions.

For several sufficient conditions, Bondy's metaconjecture has been verified. This is motivation to examine these sufficient conditions even for vertex pancyclicity since vertex pancyclicity implies pancyclicity, and pancyclicity implies hamiltonian.

Obviously, when $k \geq 3$, we cannot place $k$ vertices on the 3-cycle. Therefore, two methods have recently appeared to adjust the concept of pancyclic meaning. The first method is due to Goddard [62]. For $k \geq 2$, we say $G$ is $k$-vertex pancyclic if every set $S$ of $k$ vertices is in a cycle of every possible length. Further, $G$ is set-pancyclic if $G$ is $k$-vertex pancyclic for all $k \geq 2$.

Now by "possible length", Goddard means at least $k+$ the path cover number of $G[S]$, where the path cover number of $G[S]$ is the least number of paths that cover all the vertices of $G[S]$. This is easily seen to be a reasonable range, since if $G[S]$ has path cover number $t$, then at least $t$ new vertices will be needed to link the paths (containing our $k$ vertices) into a cycle. Goddard [62] showed: If $G$ has order $n$ and $\delta(G) \geq (n+1)/2$, then $G$ is set pancyclic.

In [51] a second approach is proposed. Let $k \geq 0$, $s \geq 0$, and $t \geq 1$ be fixed integers with $s \leq t$ and $G$ be a graph of order $n$. For an integer $m$ with $k+t \leq m \leq n$, a graph $G$ is $(k,t,s,m)$-pancyclic if for each $(k,t,s)$-linear forest $F$, there is a cycle $C_r$ of length $r$ in $G$ containing $F$ for each $m \leq r \leq n$.

We now switch from the Ore-type condition to a condition on the minimum degree. We investigate the edge pancyclicity of graphs by considering the vertex pancyclicity of a related digraph.

**Theorem 1.2.19 ([114])** *Let $G$ be a graph of order $n$ such that $\delta(G) \geq (n+2)/2$. Then $G$ is edge pancyclic.*

There are several new strong hamiltonian properties and generalizations of old properties. Brandt [25] proposed one such generalization as weak pancyclic.

**Weakly pancyclic**

If a graph contains cycles of all lengths between its girth and circumference, it is called a weak pancyclic. In 1997, Brandt showed the following.

**Theorem 1.2.20 ([25])** *If $G$ is a nonbipartite graph of order $n$ and size $q > \lfloor (n-1)^2/4 + 1 \rfloor$, then $G$ is weakly pancyclic.*

**Conjecture 1.2.21 ([25])** *Every nonbipartite graph of order $n$ and size at least $(n-1)(n-3)/4+4$ is weakly pancyclic.*

In 1999, Bollobás and Thomason [16] were very close to solving this conjecture. In 2013, Brandt [26] also considered other degree conditions for weakly pancyclic graphs.

**Theorem 1.2.22 ([26])** *Let $G \neq C_5$ be a nonbipartite triangle-free graph of order $n$. If $\delta(G) > n/3$, then $G$ is weakly pancyclic with girth $4$ and circumference $\min\{2, n - \alpha(G)\}$, (where $\alpha(G)$ is the independence number of $G$).*

Let $S$ be a subset of vertices. We ask if we may get some properties on cycles under conditions on the subset $S$ of vertices. Two questions arise: is there a path/cycle containing a maximum number of vertices in $S$? Does the graph admit a path/cycle of large length? Another generalization of hamiltonian graphs is the idea of cyclable sets.

**Cyclable**

A subset $S$ of $V(G)$ is called cyclable in $G$ if all the vertices of $S$ belong to a common cycle in $G$. If $V(G)$ is cyclable, then $G$ is hamiltonian. Several set restricted density results imply cyclability. The first extends the well-known Chvátal-Erdös Theorem. The following result is due independently to Bollobás and Brightwell [17] and Shi [115]. It uses the classic Dirac-type density condition for the subset $S$ of $V(G)$. Let $\delta(S)$ be the minimum degree in $G$ of a vertex of $S$.

**Theorem 1.2.23 ([17], [115])** *Let $G$ be a $2$-connected graph and $S$ a subset of $V(G)$. If $\delta(S) \geq n/2$, then $S$ is cyclable in $G$.*

In 1995, Ota [111] made the natural extension to degree sums of pairs of nonadjacent vertices in $S$, denoted by $\sigma_2(S)$.

**Theorem 1.2.24 ([111])** *Let $G$ be a $2$-connected graph and $S$ a subset of $V(G)$. If $\sigma_2(S) \geq n/2$, then $S$ is cyclable in $G$.*

**Theorem 1.2.25 ([58])** *Let $G = (V, E)$ be $k$-connected graph, $k \geq 2$, of order $n$. Denote by $X_1$, $X_2$, ..., $X_k$ subsets of the vertex set $V$ and let $X = X_1 \cup X_2 \cup \ldots \cup X_k$. If for each $i$, $i = 1, 2, \ldots, k$, and for any pair of nonadjacent vertices $x, y \in X_i$, we have $d(x) + d(y) \geq n$, then $G$ is $X$-cyclable.*

The following result generalizes Theorem 1.2.25 into the implicit degree condition. [91] give examples that do not satisfy the condition of Theorem 1.2.25, and verify the implicit degree condition in the following theorem.

**Theorem 1.2.26** *Let $G$ be a $k$-connected graph on $n$ vertices with $k \geq 2$. Denote by $X_1$, $X_2$, ..., $X_k$ subsets of the vertex set $V(G)$ and let $X = X_1 \cup X_2 \cup \ldots \cup X_k$ If $\sigma_{(1,2)}(X_j) \geq n$ for each $j, 1 \leq j \leq k$, then $X$ is cyclable in $G$.*

An extension of the idea of cyclable sets is the following. A graph $G$ is said to be $S$-pancyclable if for every integer $l, 3 \leq l \leq |S|$, there is a cycle in $G$ that contains exactly $l$ vertices of $S$. An Ore-type result in this direction is the following:

**Theorem 1.2.27 ([52])** *If $G$ is a graph of order $n$ and $\sigma_2(G) \geq n$, then either $G$ is $S$-pancyclable or else $n$ is even, $S = V(G)$ and $G = K_{n/2,n/2}$, or $|S| = 4$, $G[S] = K_{2,2}$ and the structure of $G$ is well characterized.*

[1] also, consider bipartite graphs.

**Theorem 1.2.28** *Let $G$ be a $2$-connected balanced bipartite graph of order $2n$ and bipartition $(X, Y)$. Let $S$ be a subset of $X$ of cardinality at least $3$. Then if the degree sum of every pair of nonadjacent vertices $x \in S$ and $y \in Y$ is at least $n + 3$, then $G$ is $S$-pancyclable.*

Most of this thesis will focus on the generalization of the hamiltonian problem.

## 1.3 Motivations and overview

### 1.3.1 Motivations and overview of pancyclicity

A graph of order $n$ is said to be pancyclic if it contains cycles of all lengths from $3$ to $n$.

“The study of pancyclic graphs arose from the conviction that existing sufficient conditions for a graph to be hamiltonian are satisfied only by graphs with a much more specific structure.”-J.A. Bondy, 1971.

In 1971, Bondy [118] suggested the following interesting “metaconjecture”: almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs).

Pancyclicity is one of the main topics of this thesis. It is NP-complete to test whether a graph is pancyclic.

Let’s recall some results that support the “metaconjecture”.

**Theorem 1.3.1 (Bondy's theorem, [19])** *Let $G$ be a graph of order $n$. If $d(x)+d(y) \geq n$ for any pair of nonadjacent vertices $x$ and $y$ in $G$, then $G$ is pancyclic or isomorphic to $K_{n/2,n/2}$.*

In 1981, Amar, Flandrin Fournier, and Germa [9] showed the following:

**Theorem 1.3.2 ([9])** *Let $G$ be a hamiltonian, nonbipartite graph of order $n \geq 162$. If $\delta(G) \geq (2n+1)/5$, then $G$ is pancyclic.*

In 1982, Mitchem and Schmeichel [104] proposed that the degree bound in theorems that guarantee pancyclicity or bipancyclicity can be reduced if the assumption is hamiltonian. This is clearly a strengthening over simply assuming $G$ is $2$-connected. As it turns out, Faudree, Häggkvist, and Schelp [70] had already asked a question of this type.

**Theorem 1.3.3** *If $G$ is a hamiltonian graph on $n$ vertices with $q > \lfloor (n-1)^2/4 \rfloor + 1$ edges, then $G$ is either pancyclic or bipartite.*

**Theorem 1.3.4 ([14])** *Let $G$ be a $2$-connected graph on $n$ vertices. If for all vertices $x$ and $y$, $dis(x,y)=2$ implies max $\{d(x),d(y)\} \geq \frac{n}{2}$, then $G$ is either pancyclic, $K_{\frac{n}{2},\frac{n}{2}}, K_{\frac{n}{2},\frac{n}{2}}-e$, or the graph shown in the following figure.*

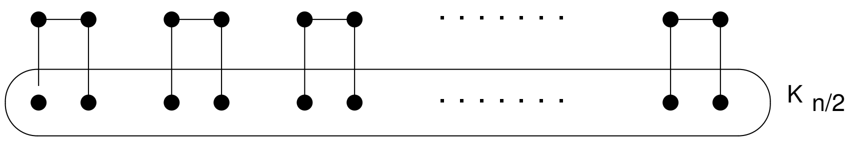



Figure of Theorem 1.3.4

**Theorem 1.3.5 ([117])** *Let $G$ be a $2$-connected graph on $n$ vertices. If for all independent vertices $x$, $y$ and $z$, we have $d(x)+d(y)+d(z) \geq \frac{3n}{2}-1$, then $G$ is either pancyclic, $K_{\frac{n}{2},\frac{n}{2}}, K_{\frac{n}{2},\frac{n}{2}}-e$, or $C_5$.*

If only a pair of consecutive vertices on the hamiltonian cycle is considered, then the edge density can be reduced. In 1988, Hakimi and Schmeichel [117] showed the following theorem:

**Theorem 1.3.6 ( [117])** *If $G$ is a hamiltonian graph of order $n$ with hamiltonian cycle $C = x_1x_2...x_nx_1$ such that $d(x_1)+d(x_n) \geq n$, with say $d(x_1) \leq d(x_n)$, then $G$ is either*

*(1) pancyclic,*

*(2) bipartite, or*

*(3) missing only an $(n-1)$-cycle.*

*Moreover, if (3) holds, then $d(x_{n-2}), d(x_{n-1}), d(x_2), d(x_3) < n/2$, and $G$ has one of two possible adjacency structures near $x_1$ and $x_n$. In the first structure, vertices $x_{n-2}, x_{n-1}, x_n, x_1, x_2, x_3$ are independent except for edges of $C$, and $x_nx_{n-3}, x_nx_{n-4}, x_1x_4, x_1x_5 \in E(G)$. The second structure (which can occur only if $d(x_1) < d(x_n)$) is identical to the first except that $x_nx_3 \in G$ and $x_1x_5 \notin G$.*

In 1996, this idea was generalized by Faudree, Favaron, Flandrin, and Li in the case that the graph admits a hamiltonian path.

**Theorem 1.3.7 ( [47])** *Let $G$ be a graph of order $n$. If $G$ has a hamiltonian $(u, v)$-path for a pair of nonadjacent vertices $u$ and $v$ such that $d(u) + d(v) \geq n$, then $G$ is pancyclic. Moreover, if $u$ (or $v$) has degree at least $\frac{n}{2}$, it is contained in a triangle and for any $m$, $4 \leq m \leq n$, there exists some $C_m$ in $G$ that contains both $u$ and $v$.*

For the bipartite graph, in 1988, Entringer and Schmeichel [44] gave the following theorem.

**Theorem 1.3.8 ([44])** *Let $G$ be a hamiltonian bipartite graph on $2n$ vertices and $q > n^2/2$ edges. Then $G$ is bipancyclic.*

This result is also the best possible that can be seen by taking five $k$-sets of independent vertices and cyclically joining all vertices in one set to all vertices in the next set. This graph has a degree sum of $4n/5$ but lacks triangles.

In 1989, Tian and Zang [120] got the following result.

**Theorem 1.3.9 ([120])** *If $G$ is a hamiltonian bipartite graph on $2n$ vertices where $n \geq 60$ and $\delta(G) \geq 5n/2 + 2$, then $G$ is bipancyclic.*

In [46] and [64], they asked the following more general problem.

**Problem 1.3.10** *Given a result, assuming that $G$ is $2$-connected and has properties $P_1, \ldots, P_k$ to obtain property $P$, when does the hamiltonian hypothesis instead of $2$-connectivity allow us to reduce the other hypotheses and obtain the same result?*

Then, we have the theorem: a graph with order $n$ and vertex degree sequence $d_l < d_2 < \cdots < d_n$, such that $d_k < k < n/2$ implies $d_{n-k} > n - k$ is either pancyclic or bipartite.

In 2004, combining Ramsey number conditions gave new results. $R(a, b)$ stands for the standard graph Ramsey number.

**Theorem 1.3.11 ([57])** *Let $G$ be a $k$-connected graph with independence number $\alpha$ such that*

$$k > \alpha + (\alpha + 1)R(\alpha + 1, \alpha + 1).$$

*Then $G$ is pancyclic.*

In 2009, Hu and Li [75] were able to show pancyclic in a graph obtained from a graph with Ore-type condition by deleting some edges.

We must mention that other important conditions for pancyclic and weakly pancyclic are about the number of edges. Bondy [19] obtained that every hamiltonian graph of order n and size at least $n^2/4$ is pancyclic. A result of Häggkvist, Faudree, and Schelp [70] states that a hamiltonian nonbipartite graph of order $n$ and size at least $\lfloor\frac{(n-1)^2}{4}\rfloor + 2$ is pancyclic. From this, Brandt [25] deduced that every nonbipartite graph of order n and size at least $\lfloor\frac{(n-1)^2}{4}\rfloor+2$ is weakly pancyclic. He conjectured that it suffices to have the size at least $\lceil\frac{n^2}{4}\rceil-n+5$ . This conjecture is proved by Bollobás and Thomason [16]. They showed that every graph of order $n$ and size at least $\lceil\frac{n^2}{4}\rceil - n + 59$ is weakly pancyclic or bipartite.

In [91] and [92], Li, Ning, and Cai get results about cyclable. There are also some results on pancyclicity that use implicit degrees.

From Bondy's metaconjecture, we propose the following conjecture.

**Conjecture 1.3.12 ([85])** *Let $G = (V, E)$ be a $k$-connected graph ($k \geq 2$) of order $n$. Suppose that $V(G) = \cup_{i=1}^{k} X_i$. If for any pair of nonadjacent vertices $x, y \in X_i$ with $i = 1, 2, \dots, k$, $d(x) + d(y) \geq n$, then $G$ is pancyclic or $G$ is bipartite graph.*

In Chapter 2, we prove Conjecture 1.3.12 is true for $k = 2$. Our main result is the following.

**Theorem 1.3.13 ([85])** *Let $G = (V, E)$ be a 2-connected graph of order $n$ and $V(G) = X \cup Y$. If for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$, $d(x_1) + d(x_2) \geq n$ and for any pair of nonadjacent vertices $y_1$ and $y_2$ in $Y$, $d(y_1) + d(y_2) \geq n$. Then $G$ is pancyclic or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.*

It is easy to see that Theorem 1.3.13 is stronger than Bondy's theorem (Theorem 1.3.1).

In Chapter 3, we prove that the conjecture 1.3.12 is true for $k = 3$. The following is our main result.

**Theorem 1.3.14 ([86])** *Let $G = (V, E)$ be a 3-connected graph of order $n$ and $V(G) = X_1 \cup X_2 \cup X_3$. For any pair of nonadjacent vertices $v_1$ and $v_2$ in $X_i$, $d(v_1) + d(v_2) \geq n$ with $i = 1, 2, 3$. Then $G$ is pancyclic or $G$ is bipartite.*

### 1.3.2 Motivations and overview on forbidden graphs

Given a family of graphs $\mathcal{F}$, we say a graph $G$ is $\mathcal{F}$-free if $G$ contains no induced subgraph isomorphic to a graph in $\mathcal{F}$. The graphs of $\mathcal{F}$ are called forbidden subgraphs. If $G$ contains no induced subgraph isomorphic to any graph in the set $F = \{H_1, H_2 \dots, H_k\}$, we say $G$ is $F$-free. If $F = \{H_1\}$, we say $G$ is $H_1$-free. Forbidden subgraphs are a method to the hamiltonian problem, which started with an observation by Goodman and Hedetniemi [63]. The forbidden subgraph's problem has been studied for $G$ being traceable, hamiltonian, pancyclic, Hamilton-connected, and so on.

| | $\delta \geq \frac{n-2}{3}$ | $\sigma_3 \geq n-2$ | $U_2 > \frac{2n-5}{3}$ |
|---|---|---|---|
| Traceability | [99]$(S)$ | [125, 28] $(S)$ | [12]$(S)$ |

Table 1.1: 1-connected claw-free graphs

| | $\delta$ | $\sigma_2$ | $\sigma_3$ | $U_2$ |
|---|---|---|---|---|
| Traceability | | | | $\geq \frac{n-2}{2}$ [49] |
| Hamiltonicity | $\geq \frac{n-2}{3}$ [99]$(S)$ | $\geq \frac{2n-5}{3}$ [55] | $\geq n-2$ [125, 28]$(S)$ | $\geq \frac{2n-5}{3}$ [12]$(S)$ |
| Pancyclicity | $\geq \frac{n-2}{3}$ [54] | | | $\geq \frac{2n-2}{3}$ [49] |

Table 1.2: 2-connected claw-free graphs

The complete bipartite graph $K_{1,n}$ is called a star, and the $K_{1,3}$ is called a claw. A graph is claw-free if it contains no claw as its induced subgraph.

Many of the results mentioned in this thesis are also included in the survey by Gould [65].

The circumference of $2$-connected claw-free graphs was investigated by Broersma et al. [30].

So, first, let's introduce some of the notation that we're going to use.

For $1 \leq k \leq n$ we denote by $U_k(G)$ the minimum of the neighborhood union $|N(x_1) \cup \cdots \cup N(x_k)|$, where the minimum is taken over all subsets $\{x_1, x_2, \ldots, x_k\}$ of $k$ independent vertices of $V(G)$.

For the sake of clarity and ease of reference, the results concerning traceability, hamiltonicity and pancyclicity in claw-free graphs as a function of $\delta$, $\sigma_k$ and $U_k$ have been placed in Tables 1.1,1.2 (depending on the connectivity of the graph). As $S$ (for sharp) in Table 1.1 indicates that the bound cannot be improved.

The following result gives a minimum degree condition for $K_{1,3}$-free graphs to be pancyclic.

**Theorem 1.3.15 ([54])** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then $G$ is pancyclic.*

The lower bound of Theorem 1.3.15 is sharp because there is a graph of order $34$, which satisfies the degree sum condition in Theorem 1.3.15 but is not pancyclic.

For non-hamiltonian $3$-connected claw-free graphs, in Table 1.3, we gave some results regarding traceability, hamiltonicity and Hamilton-connected. Li Mingchu [100] verified $4\delta$ as a lower bound for the circumference.

In the 1980s, some results showed that a $2$-connected graph is a hamiltonian graph when specific induced subgraph pairs are prohibited. Notable among these were the following results (see Figure 1.3 for graphs and note that $Z_2$ is obtained from $Z_3$ by removing the vertex of degree one).

**Theorem 1.3.16** *(1) [42] If $G$ is a $2$-connected $\{K_{1,3}, N\}$-free graph, then $G$ is hamiltonian.*

*(2) [29] If $G$ is a $2$-connected $\{K_{1,3}, P_6\}$-free graph, then $G$ is hamiltonian.*

*(3) [66] If $G$ is a $2$-connected $\{K_{1,3}, Z_2\}$-free graph, then $G$ is hamiltonian.*

| | $\delta$ | $\sigma_3$ | $U_2$ |
|---|---|---|---|
| Traceability | | $\geq n+1$ [71] | |
| Hamiltonicity | $\geq \frac{n+7}{6}$ [81] | | $\geq \frac{11(n-7)}{21}$ [71] |
| Hamilton-connected | | $\geq n+1$ [53] | |

Table 1.3: 3-connected claw-free graphs

*(4) [13] If $G$ is a $2$-connected $\{K_{1,3}, W\}$-free graph, then $G$ is hamiltonian.*

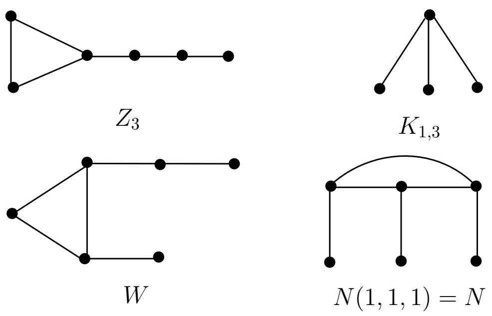



Figure 1.3: The forbidden graphs

The fundamental conjecture of Matthews and Sumner [99] is still open.

In 1979, Oberly and Sumner [107] obtained the following results by associating forbidden subgraphs with local connectivity: a connected, locally connected, $K_{1,3}$-free graph of order $n \geq 3$ is hamiltonian. A graph $G$ is locally connected if, for each vertex $x$, the subgraph $G[N(x)]$ is a connected graph.

In 1988, Zhang [128] considered degree sums in $K_{1,3}$-free graphs. He showed that if $G$ is a $k$-connected, $K_{1,3}$-free graph of order $n$ such that $\sigma_{k+1}(G) \geq n - k$, then $G$ is hamiltonian.

**Conjecture 1.3.17 ( Matthews-Sumner conjecture )** *Every 4-connected claw-free graph is hamiltonian.*

In 2001, Broersma, Kriesell, and Ryjácek [31] showed that the above conjecture is true for some graphs.

For the hamiltonian problem, there are still some special problems. Such as alternating hamiltonian cycles, making weighted graphs hamiltonian, and so on.

**Theorem 1.3.18 ([80])** *Every $5$-connected line graph with minimum degree at least $6$ is hamiltonian.*

To solve the problems of the Matthews-Sumner conjecture and the completeness of the general theory, the $3$-connected case is generally considered. There are a lot of new results here.

**Theorem 1.3.19 ([81])** *Every $3$-connected claw-free graph with minimum degree $\delta$ and order at most $6\delta - 7$ is hamiltonian.*

**Theorem 1.3.20 ([95])** *Every $3$-connected claw-free graph with minimum degree $\delta$ and order $n \leq 5\delta - 8$ is Hamilton-connected.*

In [67], it described the pancyclicity of $3$-connected graphs with forbidden pairs.

**Theorem 1.3.21 ([67])** *If $X$ and $Y$ are connected graphs of order at least $3$ with $X, Y \neq P_3$ and $Y \neq K_{1,3}$, then a $3$-connected $XY$-free graph G is pancyclic if and only if $X = K_{1,3}$ and $Y$ is a subgraph of a member of the family $\{P_7, L_1, N(4,0,0), N(3,1,0), N(2,2,0), N(2,1,1)\}$.*

In 2011, Ryjáček and Vrána [116] proposed the following conjecture.

**Conjecture 1.3.22 ([116])** *Every $4$-connected claw-free graph is Hamilton-connected.*

For more results of claw-free graphs, we refer to the survey paper by Faudree et al. [48].

Chorded pancyclic on claw-free graphs is one of the main topics of this thesis. We study a minimum degree condition for $K_{1,3}$-free graphs to be chorded pancyclic in this thesis.

A *chord* of a cycle is an edge between two nonadjacent vertices of the cycle. We say that a cycle is *chorded* if the cycle has at least one chord, and we call such a cycle *chorded cycle*. If a cycle has at least two chords, then the cycle is called a *doubly chorded cycle*. A graph $G$ of order $n$ is *chorded pancyclic* (*doubly chorded pancyclic*) if $G$ contains a chorded cycle (doubly chorded cycle) of each length from $4$ to $n$.

Bondy's metaconjecture was extended into almost any condition that implies a graph is hamiltonian will imply it is chorded pancyclic, possibly with some class of well-defined exceptional graphs and some small order exceptional graphs. As support for the extension of Bondy's metaconjecture, there are the following results. For graphs $G$ and $H$, let $G \square H$ denote the Cartesian product of $G$ and $H$.

**Theorem 1.3.23 ([35])** *Let $G$ be a graph of order $n \geq 4$. If $d(x) + d(y) \geq n$ for any two nonadjacent vertices in $G$, then $G$ is chorded pancyclic, or $G = K_{\frac{n}{2},\frac{n}{2}}$, or $G = K_3 \square K_2$*

**Theorem 1.3.24 ([60])** *A hamiltonian graph G of order $n \geq 4$ with $|E(G)| \geq \frac{1}{4}n^2$ is chorded pancyclic unless $G = K_{\frac{n}{2},\frac{n}{2}}$, or $G = K_3 \square K_2$.*

**Theorem 1.3.25 ([36])** *Let $G$ be a $2$-connected graph of order $n \geq 10$. If $G$ is $\{K_{1,3}, Z_2\}$-free then $G = C_n$ or $G$ is chorded pancyclic, where $C_n$ be a cycle with $n$ vertices.*

**Theorem 1.3.26 ([36])** *Let $G$ be a $2$-connected graph of order $n \geq 13$. If $G$ is $\{K_{1,3}, P_6\}$-free then $G$ is chorded pancyclic.*

In Chapter 5, we obtain the results which the extension of the pancyclicity to the corded pancyclicity from Theorem 1.3.15. Our main results are as follows:

**Theorem 1.3.27 ([93])** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then $G$ is chorded pancyclic.*

Let $CH_m$ be the maximum number of chords in cycle $C_m \subseteq G$ with $4 \leq m \leq n$. We obtain the following theorem.

**Theorem 1.3.28 ([93])** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then*

$$CH_m \geq \begin{cases} \frac{m(m-1)}{2} - m & \text{if } 4 \leq m \leq 5, \\ m & \text{if } 6 \leq m \leq \frac{n+1}{3}, \\ [\frac{m}{6}] & \text{if } \frac{n+4}{3} \leq m \leq \frac{2n+8}{3}, \\ \frac{m(\delta-(n-m))}{2} - m & \text{if } \frac{2n+11}{3} \leq m \leq n. \end{cases}$$

Moreover, by Theorem 1.3.28, $CH_m \geq 2$. Therefore, we can obtain that $G$ is doubly chorded pancyclic.

**Corollary 1.3.29 ([93])** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then $G$ is doubly chorded pancyclic.*

In the previous part of this section, we gave several theorems for forbidden graphs, from which we can generalize the conditions of Theorem 1.3.27 to obtain chorded pancyclic.

### 1.3.3 Motivation and overview of hamiltonicity in digraphs

Let $D$ be a digraph. A cycle (path) is called *hamiltonian* if it includes all the vertices of $D$. A digraph $D$ is *hamiltonian* if it contains a hamiltonian cycle and is *pancyclic* if it contains a cycle of length $k$ for any $3 \leq k \leq n$, where $n$ is the order of $D$. A balanced bipartite digraph of order $2m$ is *even pancyclic* if it contains a cycle of length $2k$ for any $k, 2 \leq k \leq m.$

In [77], Jackson pointed out that for undirected regular graphs, the degree condition of Dirac's theorem can be greatly reduced by adding the connectivity condition. He got the result that every $2$-connected $d$-regular graph on $n$ vertices with $d \geq n/3$ contains a hamiltonian cycle. In addition to the Petersen graph, Hilbig [74] and Zhu et al. [126] raised the degree condition to $n/3 - 1$. There is an example to prove that the degree condition cannot be reduced further and that the connectivity condition is necessary. For directed graphs, the following conjecture is obtained.

**Conjecture 1.3.30** *Every strongly $2$-connected $d$-regular digraph on $n$ vertices with $d \geq n/3$ contains a hamiltonian cycle.*

The conjecture of Bang-Jensen et al. [10] would strengthen Meyniel's theorem (A strongly connected directed graph of order $n$ whose degree sum of any pair of nonadjacent vertices is at least $2n-1$ is hamiltonian.) by requiring the degree condition only for dominated pairs of vertices (a pair of vertices is dominated if there is a vertex which sends an edge to both of them). Nash-Williams [105] proposes a conjecture about degree sequence conditions in directed graphs similar to Chvátal's theorem.

Another sufficient hamiltonian condition in undirected graphs is the Chvátal–Erdös theorem [34]. The connectivity $\kappa(G)$ of a digraph is defined to be the size of the smallest set of vertices $S$ so that $G - S$ is either not strongly connected or consists of a single vertex. Let $\alpha_2(G)$ be the size of the largest set $S$ so that $S$ induces no cycle of length $2$. Jackson and Ordaz [78] got the following conjecture.

**Conjecture 1.3.31 ([78])** *If $G$ is a digraph with $\kappa(G) \geq \alpha_2(G) + 1$, then $G$ contains a hamiltonian cycle.*

In 1960, Ore [109] generalized Dirac's [41] well-known theorem about hamiltonian cycles in graphs. Bondy [19] extended this result and proved that a graph satisfying the Ore-type condition is not only hamiltonian but even pancyclic, unless the graph is regular, completes bipartite. Ghouila-Houri [61] and Woodall [124] generalized Dirac's theorem and Ore's theorem to digraphs, respectively.

One can use Ghouila-Houri's theorem [61] to deduce that every digraph on $n$ vertices with a minimum semidegree greater than $n/2$ is pancyclic.

We say that a digraph with $n$ vertices satisfies the condition $(c_i)$ if, for each pair of nonadjacent vertices, the degree sum is at least $2n - 2 + i$.

In 1973, Meyniel [103] generalized the results of Ghouila-Houri and Woodall ([61] and [124]) by showing that a strongly connected digraph satisfying $c_i$ is hamiltonian. Overbeck-Larisch [112] and Bondy and Thomassen [119] gave a short proof of Meyniel's theorem. In 1976, Häggkvist and Thomassen [69] generalized Ghouila-Houri's theorem by showing that a strongly connected digraph $D$ with $n$ vertices and minimum degree at least $n$ is pancyclic unless $n$ is even and $G = K_{n/2,n/2}$.

**Theorem 1.3.32 ([69])** *If a strongly connected digraph $D$ with $n$ vertices has minimum degree at least $n$, then $D$ is pancyclic, or $n$ is even and $G = K_{n/2,n/2}$.*

In 1971, Bondy [19] proved that the number of edges in an undirected hamiltonian nonpancyclic graph with $n$ vertices is less than or equal to $n^2/4$ and conjectured that the number of edges in a hamiltonian nonpancyclic digraph with $n$ vertices is less than or equal to $n^2/2$.

Every hamiltonian digraph with $n$ vertices and $n/2(n+1) - 1$ or more edges is pancyclic.

Another natural way to generalize Dirac's theorem is to require finding a certain set of vertex-disjoint cycles in G that together cover all vertices of $G$. For directed and oriented graphs, factors with specified cycles length and $k$-ordered hamiltonian cycles are also taken into account.

A graph $G$ is $k$-ordered if for every sequence $s_1, s_2, \ldots, s_k$ of distinct vertices of $G$ there is a cycle which encounters $s_1, s_2, \ldots, s_k$ in this order. $G$ is a $k$-ordered hamiltonian if it contains a hamiltonian cycle with this property.

In 1977, Thomassen [119] proved that the Ore-type condition implies that every digraph with minimum in-degree and minimum out-degree $> n/2$ is pancyclic. In 1997, Alon and Gutin [7] observed that one can use Ghouila-Houri's theorem [61] to show that every digraph $G$ with minimum in-degree and minimum out-degree $> n/2$ is even vertex-pancyclic.

A digraph $D$ is *strongly connected* (or, just, strong) if there exists a path from $x$ to $y$ and a path from $y$ to $x$ for every pair of distinct vertices $x, y$. A digraph $D$ is $k$-strongly ($k \geq 1$) connected (or $k$-strong), if $|V(D)| \geq k+1$ and $D(V(D) \setminus A)$ is strongly connected for any subset $A \subseteq V(D)$ of at most $k-1$ vertices.

Recently, there has been a renewed interest in various Meyniel-type hamiltonian conditions in bipartite digraphs. Let us recall the following well-known degree conditions that guarantee that a balance bipartite digraph is hamiltonian.

We begin with the following theorem due to Adamus Janusz.

**Theorem 1.3.33 ([2])** *Let $D$ be a strong connected balanced bipartite digraph of order $2a \geq 6$. Suppose that $d(x)+d(y) \geq 3a$ for each pair of distinct vertices $x, y$ with a common out-neighbor or a common in-neighbor, then $D$ is hamiltonian.*

The following theorems are generalizations of Theorem 1.3.33.

**Theorem 1.3.34 ([121])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 4$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$ and $d(y) \geq a+1$ or $d(y) \geq 2a-1$ and $d(x) \geq a+1$. Then $D$ is hamiltonian.*

Before starting the following theorems, we need to introduce additional notation.

Let $D(8)$ be the bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, $A(D(8))$ contains exactly the arcs $y_0x_1, y_1x_0, x_2y_3, x_3y_2$ and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i, i \in [0,3], y_0 \leftrightarrow x_2, y_0 \leftrightarrow x_3, y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$, and it contains no other arcs.

**Theorem 1.3.35 ([39])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 4$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a-1$ or $d(y) \geq 2a-1$ ($\max\{d(x), d(y)\} \geq 2a-1$). Then $D$ is hamiltonian or isomorphic to the digraph $D(8)$.*

**Theorem 1.3.36 ([39])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 8$. Suppose that $d(x)+d(y) \geq 4a-3$ for every pair of vertices $x, y$ with a common out-neighbour. Then $D$ is hamiltonian.*

In 1971, Bondy suggested [19] "metaconjecture". There are many results that support this "metaconjecture" in digraph. Let us cite for examples the followings:

**Theorem 1.3.37 ([39])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 8$ with partite sets $X$ and $Y$. If $D$ is not a directed cycle and $max\{d(x), d(y)\} \geq 2a-1$ for every pair of distinct vertices $\{x, y\}$ with a common out-neighbor, then either $D$ contains cycles of all even lengths less than or equal to $2a$ or $D$ is isomorphic to the digraph $D(8)$.*

**Theorem 1.3.38 ([102])** *Let $D$ be a balanced bipartite digraph of order $2a \geq 4$ with partite sets $X$ and $Y$. Suppose that $d(x) + d(y) \geq 3a + 1$ for each two vertices $x, y$ either both in $X$ or both in $Y$. Then $D$ contains cycles of all even lengths $4, 6, \ldots, 2a$ (i.e., $D$ is bipancyclic).*

**Theorem 1.3.39 ([3])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 6$. Suppose that $d(x) + d(y) \geq 3a$ for every pair of vertices $x, y$ with a common in-neighbour or a common out-neighbour. Then $D$ is either bipancyclic or $D$ is a directed cycle of length $2a$.*

In view of the next theorem we need the following definition.

**Definition 1.3.40** *Let $D$ be a balanced bipartite digraph of order $2a \geq 10$, and let $k$ be an integer. We say that $D$ satisfies the condition $\aleph_k$ if for every dominating pair of vertices $\{x, y\}$, $d(x) + d(y) \geq 3a + k$.*

In Chapter 4, we prove the following theorem which improves the result of Theorem 1.3.33.

**Theorem 1.3.41 ([87])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$. Suppose that $D$ satisfies the condition $\aleph_0$, i.e., $d(x) + d(y) \geq 3a$ for every dominating pair of vertices $\{x, y\}$, $D$ is hamiltonian.*

We also proved some new sufficient conditions for bipancyclic of digraphs.

**Theorem 1.3.42 ([87])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 8$ with partite sets $X$ and $Y$. Suppose that $D$ contains a cycle of length $2a-2$ and $d(x) + d(y) \geq 4a - 4$ for every dominating pair of vertices $\{x, y\}$. Then $D$ is even pancyclic.*

**Theorem 1.3.43 ([87])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. If $D$ contains a cycle of length $2a-2$ and $D$ satisfies the condition $\aleph_1$, i.e., $d(x) + d(y) \geq 3a + 1$ for every dominating pair of vertices $\{x, y\}$, then $D$ contains a cycle of length $2k$ for all $k$, where $1 \leq k \leq a$ (i.e., $D$ is even pancyclic).*

Let $D$ be a digraph and let $S$ be a nonempty subset of vertices of $D$. We say that a digraph $D$ is $S$-strongly connected if, for any pair $x, y$ of distinct vertices of $S$, there exists a path from $x$ to $y$ and a path from $y$ to $x$.

A set $S$ of vertices in a directive graph $D$ is said to be *cyclable* (*pathable*) in $D$ if $D$ contains a directed cycle (path) through all vertices of $S$.

Many well-known conditions guarantee the cyclability of a set of vertices in an undirected graph. In 2007, Li, Flandrin and Shu [89] proved the following theorem which gives a sufficient condition for cyclability of digraphs.

**Theorem 1.3.44 ([89])** *Let $D$ be a digraph of order $n$ and $S \subseteq V(D)$. If $D$ is $S$-strong and if $d(x)+d(y) \geq 2n-1$ for any two nonadjacent vertices $x, y \in S$, then $S$ is cyclable in $D$.*

**Theorem 1.3.45 ([89])** *Let $D$ be a digraph of order $n$ and $S \subseteq V(D)$. If $D$ is $S$-strong and if $d(x)+d(y) \geq 2n-3$ for any two nonadjacent vertices $x, y \in S$, then $S$ is pathable in $D$.*

In this thesis, we show the following theorem.

**Theorem 1.3.46 ([87])** *Let $D$ be a 2-strong digraph of order $n$ and $S \subseteq V(D)$. If $D$ is $S$-strong and if $d(x)+d(y)+d(w)+d(z) \geq 4n-3$ for all distinct pairs of non-adjacent vertices $x, y$ and $w, z$ in $S$, then $S$ is cyclable in $D$ or $D$ contains a cycle through all the vertices of $S$ except one.*

The proof of Theorem 1.3.46 is in Chapter 4.

### 1.3.4 Motivation and overview of $k$-fan-connected graphs

To facilitate the reading, we state again the definitions and notations here.

A vertex cut is a set $S \subset V(G)$ such that $G-S$ has more components than $G$. A graph is $k$-connected if every vertex cut has at least $k$ vertices. The connectivity of $G$, $\kappa(G)$, is the minimum size of a vertex cut, i.e., $\kappa(G)$ is the maximum $k$ such that $G$ is $k$-connected.

One of these subclasses of hamiltonian graphs is the family of Hamilton-connected graphs introduced by Ore [110] in 1963. A graph $G$ is said to be Hamilton-connected if each pair $u, v$ of distinct vertices are joined by a $u, v$-path containing all the vertices of $G$.

If $G$ is a Hamilton-connected graph, then $G$ is hamiltonian. It is well known that the complete bipartite graph is not Hamilton-connected.

In 1963, Ore [110] gave a sufficient condition for a graph to be Hamilton-connected: a graph whose degree sum for each pair of nonadjacent vertices is at least its order plus one is Hamilton-connected. In 1969 and 1970, Chartrand, Kapoor, and Kronk [59] and Lick [32] found another sufficient condition for Hamilton-connected graphs, that is, $G$ is a graph of order $n \geq 3$ such that for every $j$ with $2 \leq j \leq n/2$, the number of vertices of degree not exceeding $j$ is less than $j-1$, then $G$ is Hamilton-connected. In 1970, Lick [96] proposed a sufficient condition about the degree sequence for hamiltonian connectivity. In 1972, Chvátal and Erdös [34] considered the relationship between the independent number and the connectivity as a condition to get the hamiltonian connectivity of graphs.

Faudree et al. [50] and Wei [122] studied sufficient degree and/or neighborhood union conditions for Hamilton-connected graphs.

In 1979, Chartrand, Gould, and Polimeni [33] proved that if a graph $G$ is connected, locally 3-connected, and contains no induced subgraph isomorphic to $K_{1,3}$, then $G$ is Hamilton-connected.

The following theorem is a well-known result due to Ore.

**Theorem 1.3.47 ([110])** *Let $G$ be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n+1$, then $G$ is Hamilton-connected.*

Theorem 1.3.47 is generalized into a sufficient condition on any three independent vertices. In 1991, Flandrin, Jung and Li proved the followings:

**Theorem 1.3.48 ([56])** *Let $G$ be a 2-connected graph of order $n$ such that $\overline{\sigma_3}(G) \geq n$, then $G$ is hamiltonian.*

When $\overline{\sigma_3}(G) \geq n-1$, we have the following theorem:

**Theorem 1.3.49 ([Flandrin, Jung and Li [56])** *Let $G$ be a connected graph of order $n$ such that $\overline{\sigma_3}(G) \geq n-1$, then $G$ has a hamiltonian path.*

As a generalization of Hamilton-connected and hamiltonian path, Lin et al. introduced the $k$-fan-connectivity of graphs in [97]. For any integer $t \geq 2$, let $v$ be a vertex of a graph $G$ and let $U = \{u_1, u_2, \ldots, u_t\}$ be a subset of $V(G) \setminus \{v\}$. A $(v,U)$*-fan* is a set of paths $P_1, P_2, \ldots, P_t$ such that $P_i$ is a path connecting $v$ and $u_i$ for $1 \leq i \leq t$ and $P_i \cap P_j = \{v\}$ for $1 \leq i < j \leq t$.

It follows from Menger Theorem [101] that there is a $(v,U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| \leq k$ if and only if $G$ is $k$-connected. If a $(v,U)$-fan spans $G$, then it is called a *spanning* $(v,U)$*-fan* of $G$. If $G$ has a spanning $(v,U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| = k$, then $G$ is $k$*-fan-connected.*

**Theorem 1.3.50 ([40])** *A graph $G$ is $k$-connected if and only if $|G| > k+1$ and for any $k$-set $U \subseteq V(G)$ and $x \in V(G) - U$, there is an $xU$-fan.*

Let $k$ be a positive integer. In 2009, Lin et al. [97] established some results about $k$-fan. A hamiltonian path $P$ is nothing but a spanning $1$-fan rooted at the endpoints of $P$. A graph $G$ is spanning $k$-fan-connected if it has at least $k+1$ vertices and contains a spanning $k$-$(x,U)$-fan for every choice of $x \in V(G)$ and $U \in \binom{V(G)/\{x\}}{k}$; In [97], it is an easy observation that a graph with at least three vertices is spanning $1$-fan-connected if and only if it is spanning $2$-fan-connected. More generally, if $G$ is spanning $(k+1)$-fan-connected, then it must be spanning $k$-fan-connected.

**Theorem 1.3.51 ([97])** *Assume that $k$ is a positive integer. Let $G$ be a graph with order $n$. If $u$ and $v$ be two non-adjacent vertices with $d(u)+d(v) \geq n+k-1$, then $G$ is $k$-fan-connected if and only if $G+uv$ is $k$-fan-connected.*

Lin et al., in [97], obtained an Ore-type condition for graphs to be $k$-fan-connected.

**Theorem 1.3.52 ([97])** *Let $k \geq 2$ be an integer and $G$ be a graph. If $\sigma_2(G) \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected.*

In Chapter 6, we studied the $k$-fan-connected graphs. Our main theorem is as follows:

**Theorem 1.3.53 ([94])** *Let $k \geq 2$ be an integer and $G$ be a $(k+1)$-connected graph. If $\overline{\sigma_3}(G) \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected.*

The lower bound of $\overline{\sigma_3}(G)$ in Theorem 1.3.53 is sharp as shown in Chapter 6.

# Chapter 2

# Pancyclicity in hamiltonian graphs

In this chapter, we will discuss the result related to Conjecture 1.3.12.

Let $S$ be a subset of $V(G)$. We say that $G$ is *$S$-cyclable* if $G$ has an *$S$-cycle*, i.e., a cycle containing all vertices of $S$. The following theorem is an Ore-type condition for a graph to be $S$-cyclable.

**Theorem 2.0.1 ([58])** *Let $G = (V, E)$ be a $k$-connected graph, $k \geq 2$, of order $n$. Denote by $X_1$, $X_2$, ..., $X_k$ subsets of the vertex set $V$ and let $X = X_1 \cup X_2 \cup \cdots \cup X_k$. If for each $i$, $i = 1, 2, \ldots, k$, and for any pair of nonadjacent vertices $x, y \in X_i$, we have $d(x) + d(y) \geq n$, then $G$ is $X$-cyclable.*

Bondy suggested the following interesting "metaconjecture": almost any nontrivial condition on graphs which implies that the graph is hamiltonian also implies that the graph is pancyclic (there may be a family of exceptional graphs).

From Bondy's "metaconjecture" and Theorem 2.0.1, we propose Conjecture 1.3.12. We recall Conjecture 1.3.12 here.

**Conjecture 2.0.2** *Let $G = (V, E)$ be a $k$-connected graph, $k \geq 2$, of order $n$. Suppose that $V(G) = \cup_{i=1}^{k} X_i$ such that for each $i$, $i = 1, 2, \ldots, k$, and for any pair of nonadjacent vertices $x, y \in X_i$, $d(x)+d(y) \geq n$. Then $G$ is pancyclic or G is bipartite graph.*

The main result of this chapter is to prove that the above conjecture is true for $k = 2$. Our main result is the following theorem.

**Theorem 2.0.3 ([85])** *Let $G = (V, E)$ be a $2$-connected graph of order $n$ and $V(G) = X \cup Y$. If for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$, $d(x_1) + d(x_2) \geq n$ and for any pair of nonadjacent vertices $y_1$ and $y_2$ in $Y$, $d(y_1) + d(y_2) \geq n$, then $G$ is pancyclic or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.*

It is easy to see that Theorem 2.0.3 is stronger than Bondy's result in Theorem 1.3.1. For ease of reading, we reiterate Theorem 1.3.1 here.

**Theorem 2.0.4 (Bondy's theorem, [19])** *If a graph $G$ satisfies the Ore-type condition that the degree sum of any pair of nonadjacent vertices is at least the order of $G$, then $G$ is pancyclic or isomorphic to $K_{n/2,n/2}$.*

We will prove Theorem 2.0.3 in Section 2.2. Section 2.1 contains two lemmas with their proofs.

## 2.1 Preliminaries

### 2.1.1 Some definitions, notations and theorems

Now, we introduce some definitions, notations and theorems which can be used in the proof of Theorem 2.0.3.

All graphs considered in this chapter are finite, undirected and without loops or multiple edges. Given a graph $G$, we write $\overline{G}$ as the complement of $G$. Let

$$\sigma_2(G) = min\{d(x) + d(y) \mid x, y \in V(G), x \neq y, xy \notin E(G)\}.$$

A cycle containing all vertices of $G$ is called a *hamiltonian cycle* and $G$ is called *hamiltonian* if it contains a hamiltonian cycle. For two vertices $u$ and $v$, a $(u, v)$*-path* is a path connecting $u$ and $v$. A *hamiltonian* $(u, v)$*-path* is a hamiltonian path connecting $u$ and $v$. For any integer $m$, denote by $C_m$ a cycle of length $m$. Other notations and terminology not defined in this chapter can be found in section 1.1 of Chapter 1.

For a cycle $C = c_1c_2 \cdots c_pc_1$ in $G$ with a given orientation, the order $1, 2, \ldots p$ following the orientation of $C$, we denote by $c_i^- = c_{i-1}$ the predecessor of $c_i$ and by $c_i^+ = c_{i+1}$ the successor of $c_i$. For a subset $X$ of $V(C)$, $X^+$ and $X^-$ denote the set of the successors and the predecessor of the vertices of $X$ in $C$, respectively. For any $x \in V(G)$, we put

$$N_C^-(x) = \{c_i^- \mid c_i \in C \cap N(x)\},\ N_C^+(x) = \{c_i^+ \mid c_i \in C \cap N(x)\}.$$

We define similarly for the predecessor and the successor of a vertex on a path $P[p_1, p_q] = p_1p_2 \cdots p_q$. We denote by $\overline{P}[p_q, p_1] = p_qp_{q-1} \cdots p_1$.

The following theorems play an important role in the proof of Theorem 2.0.3.

**Theorem 2.1.1 ([117])** *If $G$ is a hamiltonian graph of order $n$ with hamiltonian cycle $C = x_1x_2...x_nx_1$ such that $d(x_1) + d(x_n) \geq n$, with say $d(x_1) \leq d(x_n)$, then $G$ is either*

*(1) pancyclic,*

*(2) bipartite, or*

*(3) missing only an $(n-1)$-cycle.*

*Moreover, if (3) holds, then $d(x_{n-2}), d(x_{n-1}), d(x_2), d(x_3) < n/2$, and $G$ has one of two possible adjacency structures near $x_1$ and $x_n$. In the first structure, vertices $x_{n-2}, x_{n-1}, x_n, x_1, x_2, x_3$ are independent except for edges of $C$, and $x_n x_{n-3}, x_n x_{n-4}, x_1 x_4, x_1 x_5 \in E(G)$. The second structure (which can occur only if $d(x_1) < d(x_n)$) is identical to the first except that $x_n x_3 \in G$ and $x_1 x_5 \notin G$.*

**Theorem 2.1.2 ([47])** *Let $G$ be a graph of order $n$. If $G$ has a hamiltonian $(u, v)$-path for a pair of nonadjacent vertices $u$ and $v$ such that $d(u) + d(v) \geq n$, then $G$ is pancyclic. Moreover, if $u$ (or $v$) has degree at least $\frac{n}{2}$, it is contained in a triangle and for any $m$, $4 \leq m \leq n$, there exists some $C_m$ in $G$ that contains both $u$ and $v$.*

### 2.1.2 Lemmas

In this section, we present some lemmas which will be used in the proof of Theorem 2.0.3.

**Lemma 2.1.3** *Let $G = (V, E)$ be a $2$-connected balanced bipartite graph of order $n$ and $V(G) = X \cup Y$. If for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$ (resp., $y_1$ and $y_2$ in $Y$), $d(x_1) + d(x_2) \geq n$ (resp., $d(y_1) + d(y_2) \geq n$), then $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$ .*

**Proof of Lemma 2.1.3.** Suppose that $G \neq K_{n/2,n/2}$. Let $V_1$ and $V_2$ be the bipartitions of $G$. Clearly $n \geq 6$. Let $v_1 \in V_1$ and $v_2 \in V_2$ be a pair of non-adjacent vertices. Then $d(v_1) < n/2$ and $d(v_2) < n/2$. Without loss of generality, we assume $v_1 \in X$. Since the maximum degree of $G$ is $n/2$, $v_1$ must be adjacent to every vertex in $X$. Hence $(V_1 - \{v_1\}) \cup \{v_2\} \subseteq Y$. Similarly, $(V_2 - \{v_2\}) \cup \{v_1\} \subseteq X$. Since for any pair of vertices $x_1, x_2 \in V_1 - \{v_1\}$, $d(x_1) + d(x_2) \geq n$, then $N_G(x_1) = N_G(x_2) = V_2$. And for any pair of vertices $y_1, y_2 \in V_2 - \{v_2\}$, $N_G(y_1) = N_G(y_2) = V_1$. So, we deduce that $G = K_{n/2,n/2} - \{e\}$. ∎

**Lemma 2.1.4 ([85])** *Let $P = u_1 u_2 u_3 \cdots u_p$ be a path in $G$ and $x, y \in V(G) - V(P)$ such that $(N_P(x) - \{u_1\})^- \cap N_P(y) = \emptyset$. Then $d_P(x) + d_P(y) \leq p + 1$ and if $d_P(x) + d_P(y) = p + 1$,*

*(1) $V(P) = (N_P(x) - \{u_1\})^- \cup N_P(y)$;*

*(2) $xu_1, yu_p \in E(G)$;*

*(3) If $u_i \notin N_P(x)$ for some $i, 2 \leq i \leq p$, then $u_{i-1} \in N_P(y)$, and if $u_j \notin N_P(y)$ for some $j, 1 \leq j \leq p-1$, then $u_{j+1} \in N_P(x)$;*

*(4) If $u_i, u_j \notin N_P(x) \cup N_P(y)$ with $2 \leq i < j \leq p-1$ such that $\{u_{i+1}, u_{i+2}, \ldots, u_{j-1}\} \subseteq N_P(x) \cup N_P(y)$, then there exists exact one $k,\ i+1 \leq k \leq j-1$, such that $\{u_{i+1}, u_{i+2}, \ldots, u_k\} \subseteq N_P(x)$ and $\{u_k, u_{k+1}, \ldots, u_{j-1}\} \subseteq N_P(y)$;*

*(5) If $N_P(x)$ does not contain consecutive vertices on $P$ and $N_P(y)$ does not contain consecutive vertices on $P$, then $p$ is odd and $N_P(x) = N_P(y) = \{u_1, u_3, u_5, \dots, u_{p-2}, u_p\}$.*

**Proof of Lemma 2.1.4.** Since $(N_P(x) - \{u_1\})^- \cap N_P(y) = \emptyset$, we deduce that

$$\begin{aligned} d_P(x) + d_P(y) &= |N_P(x)| + |N_P(y)| \\ &\leq |(N_P(x) - \{u_1\})^-| + 1 + |N_P(y)| \\ &= |(N_P(x) - \{u_1\})^- \cup N_P(y)| + 1 \\ &\leq p + 1. \end{aligned} \tag{2.1}$$

It follows that if $d_P(x) + d_P(y) = p + 1$, $(N_P(x) - \{u_1\})^- \cup N_P(y) = V(P)$ ((1) is proved.) and $u_1 \in N_P(x)$. Since $u_p \in V(P) - N_P(x)^-$, then $u_p \in N_P(y)$. ((2) is proved.) If $u_i \notin N_P(x)$ for some $i$ with $2 \leq i \leq p$, then $u_{i-1} \notin N_P(x)^-$ and hence $u_{i-1} \in N_P(y)$. If $u_j \notin N_P(y)$ for some $j$ with $1 \leq j \leq p-1$, then $u_j \in N_P(x)^-$ and $u_{j+1} \in N_P(x)$. ((3) is proved.) Suppose $V(P) - (N_P(x) \cup N_P(y)) = \{u_{i_1}, u_{i_2}, \dots, u_{i_t}\}$. Let $P_0 = u_1 u_2 \cdots u_{i_1-1}$, $P_s = u_{i_s+1} u_{i_s+2} \cdots u_{i_{s+1}-1}$ with $1 \leq \ s \leq t-1$, $P_t = u_{i_t+1} u_{i_t+2} \cdots u_p$. By the same argument with (2.1) on every $P_k$, $0 \leq k \leq t$, it follows that $d_{P_k}(x) + d_{P_k}(y) \leq |P_k| + 1$ and

$$\begin{aligned} p+1 &= d_P(x) + d_P(y) \\ &\leq \sum_{k=0}^{t} (d_{P_k}(x) + d_{P_k}(y)) \\ &\leq \sum_{k=0}^{t} (|P_k| + 1) = |P| + 1. \end{aligned}$$

This implies that $d_{P_k}(x) + d_{P_k}(y) = |P_k| + 1$ with $0 \leq k \leq t$. Since $P_k \subseteq N_P(x) \cup N_P(y)$ and $(N_P(x) - \{u_1\})^- \cap N_P(y) = \emptyset$, then there exists a vertex $u_{j_k} \in P_k$ for any $k,\ 0 \leq k \leq t$, such that $N_{P_0}(x) = \{u_1, u_2, \dots, u_{j_0}\}$ and $N_{P_0}(y) = \{u_{j_0}, u_{j_0+1}, \dots, u_{i_1-1}\}$, $N_{P_k}(x) = \{u_{i_k+1}, u_{i_k+2}, \cdots, u_{j_k}\}$ and $N_{P_k}(y) = \{u_{j_k}, u_{i_k+1}, \dots, u_{i_{k+1}-1}\}$ with $1 \leq k \leq t-1$, $N_{P_t}(x) = \{u_{i_t+1}, u_{i_t+2}, \cdots, u_{j_t}\}$ and $N_{P_t}(y) = \{u_{j_t}, u_{i_t+1}, \dots, u_p\}$. ((4) is proved.)

If there are two consecutive vertices in $N_P(x) \cup N_P(y)$, by (4), either $x$ or $y$ must contain consecutive neighbors, a contradiction. By (2), we deduce that $p$ is odd and $N_P(x) \cup N_P(y) = \{u_1, u_3, u_5, \dots, u_{p-2}, u_p\}$.((5) is proved.) ∎

## 2.2 The proof of main result

Now we prove the Theorem 2.0.3.

To the contrary, we assume that $G$ is a counterexample, $i.e.$ $G$ is not pancyclic, $G \neq K_{n/2,n/2}$ and $G \neq K_{n/2,n/2} -$

$\{e\}$, such that $|V(G)|$ is minimum among all counterexamples. Without loss of generality, let $X \cap Y = \emptyset$ and $|X| \geq |Y|$.

### 2.2.1 The connectivity of G is at least 3

First, we get an important result.

**Claim 2.2.1** *The connectivity of $G$ is at least $3$.*

To prove Claim 2.2.1, we assume that the connectivity of $G$ is 2. Let $\{w', w''\}$ be a cut-set which cuts $G$ into $H_1$ and $H_2$. Let $|H_1| = n_1$ and $|H_2| = n_2$.

Suppose first that $H_1 \cap X \neq \emptyset$ and $H_2 \cap X \neq \emptyset$. For any $u \in H_1 \cap X$ and $v \in H_2 \cap X$, we have

$$n \leq d(u) + d(v) \leq |H_1| - 1 + 2 + |H_2| - 1 + 2 \leq n,$$

which implies $N(u) = (H_1 - \{u\}) \cup \{w', w''\}$ and $N(v) = (H_2 - \{v\}) \cup \{w', w''\}$. If moreover $H_1 \cap Y \neq \emptyset$ and $H_2 \cap Y \neq \emptyset$, by similar reason, we obtain that both $H_1$ and $H_2$ are cliques and clearly $G$ is pancyclic or $G = K_{2,2}$. Thus, without loss of generality, we may assume that $H_1 \cap Y = \emptyset$, hence $Y \subseteq H_2 \cup \{w', w''\}$ and $V(H_1) \subset X$ is a clique such that each vertex in $H_1$ is adjacent to both $w'$ and $w''$. By Theorem 2.0.1, $G$ has a hamiltonian cycle $C_n$. $\{w', w''\}$ is a 2-cut which cuts $C_n$ into two parts such that all vertices $H_2$ must lie on the same part of $C_n$ and that of $H_1$ on the other part. So it is easy to get all $C_m$, $n \geq m \geq n - n_1 + 1$.

Define a new graph $D$ as follows:

$$D := \begin{cases} G - H_1 & \text{if } w'w'' \in E(G), \\ (G - H_1) \cup \{w'w''\} & \text{if } w'w'' \notin E(G). \end{cases}$$

Let $X' = X \cap V(D)$ and $Y' = Y \cap V(D)$. Then $D$ is 2-connected, and $D(X')$ is a clique. Clearly any vertex $u \in X' - \{w'w''\}$ forms a triangle with $w'$ and $w''$ and hence $D$ is not bipartite. For any pair of nonadjacent vertices $v_1, v_2 \in Y'$, at least one of $v_1$ and $v_2$ is in $H_2$ and $d_D(v_1) + d_D(v_2) \geq d_G(v_1) + d_G(v_2) - |H_1| \geq n - |H_1| = |D|$. Since $G$ is a minimum counterexample and $D$ is not bipartite, there exists a cycle $C_k$ in $D$ for any $k$ ,$3 \leq k \leq |D|$. When $w'w'' \notin C_k$, $C_k \subseteq G$. When $w'w'' \in C_k$, let $x_1 \in H_2 \cap X \subseteq D(X')$ and $x_2 \in H_1$. For $k \geq 4$ and $x_1 \notin C_k$, since $x_1$ is adjacent to every vertex in $C_k$, it is easy to construct a path $P_{k-1}$ of $k-1$ vertices in $D$ connecting $w'$ and $w''$. Put $C'_k := x_2 w' P_{k-1} w'' x_2$ that is a cycle of length $k$ in $G$. For $k \geq 4$ and $x_1 \in C_k$, since $x$ is adjacent to every vertex in $C_k$, similarly it is easy to construct a path $P'_{k-1}$ of $k-1$ vertices in $D$ connecting $w'$ and $w''$, which gives a cycle of length $k$, $C''_k = x_2 w' P'_{k-1} w'' x_2$ in $G$. When $k = 3$, we may deduce directly that $w'w'' \notin E(G)$ and $|H_1| = 1$ since otherwise we have a $C_3$. Let $x \in X \cap H_2$. If $|H_2| \geq 2$, we have $u \in H_2 - \{x\}$ which is adjacent to $w'$ or $w''$. Now $xuw'x$ (or $xuw''x$) is a triangle in $G$. So $|H_2| = 1$ and $G = C_4 = K_{2,2}$.

Suppose, without loss of generality, that $H_1 \cap Y = \emptyset$ and $H_2 \cap X = \emptyset$. If there exist $u_1, v_1 \in H_1$ and $u_2, v_2 \in H_2$ such that $u_1v_1 \notin E(G)$ and $u_2v_2 \notin E(G)$, then

$$\begin{aligned} 2n &\leq d(u_1) + d(v_1) + d(u_2) + d(v_2) \\ &\leq 2(|H_1| - 2 + 2) + 2(|H_2| - 2 + 2) \\ &\leq 2(|H_1| + |H_2|), \end{aligned}$$

a contradiction. So, without loss of generality, we assume $H_2$ is a clique.

Since $H_2$ is clique and with the cycle $C_n$ define above, it is easy to get all $C_m$, $n - n_2 + 2 \leq m \leq n$. Let $P = x_0x_1x_2x_3 \cdots x_{n_1}x_{n_1+1}$, with $x_0 = w'$ and $x_{n_1+1} = w''$, be a hamiltonian path of $G(H_1 \cup \{w', w''\})$. We first prove the followings:

**Fact 2.2.2** *Either $G(H_1 \cup \{w', w''\})$ contains a path $P^*$ connecting $w'$ and $w''$ such that $|P^*| = n_1 + 1$, or $n_2 = 1$ and for any $i$, $1 \leq i \leq n_1 - 2$, such that $x_ix_{i+2} \notin E(G)$ and $x_0x_{i+2}, x_ix_{n_1+1} \in E(G)$.*

**Proof.** For some $i$, $1 \leq i \leq n_1 - 2$, if $x_ix_{i+2} \in E(G)$, then put $P^* = w'x_1x_2 \cdots x_ix_{i+2}x_{i+3} \cdots x_{n_1}w''$. Suppose for any $i$, $1 \leq i \leq n_1 - 2$, $x_ix_{i+2} \notin E(G)$. If there is a $j$, $0 \leq j \leq i - 2$, such that $x_jx_i \in E(G)$ and $x_{j+1}x_{i+2} \in E(G)$, then put $P^* = x_0x_1 \cdots x_jx_ix_{i-1} \cdots x_{j+1}x_{i+2}x_{i+3} \cdots x_{n_1}x_{n_1+1}$. It follows that $P[x_0, x_{i-1}] \cap N(x_i)^+ \cap N(x_{i+2}) = \emptyset$. By Lemma 2.1.4,

$$d_{P[x_0,x_{i-1}]}(x_i) + d_{P[x_0,x_{i-1}]}(x_{i+2}) \leq |P[x_0, x_{i-1}]| + 1$$

and the equality implies $x_0x_{i+2} \in E(G)$. Similarly, we have

$$d_{P[x_{i+3},x_{n_1+1}]}(x_i) + d_{P[x_{i+3},x_{n_1+1}]}(x_{i+2}) \leq |P[x_{i+3}, x_{n_1+1}]| + 1$$

and the equality implies $x_ix_{n_1+1} \in E(G)$. Thus, we obtain that

$$n \leq d_G(x_i) + d_G(x_{i+2}) \leq |P[x_0, x_{i-1}]| + 1 + |P[x_{i+3}, x_{n_1+1}]| + 1 + 2|\{x_{i+1}\}| = n_1 + 3,$$

which implies that $n_2 = 1$ and the equality implies $x_0x_{i+2}, x_ix_{n_1+1} \in E(G)$. The Fact is proved. ∎

When there is a $y \in H_2 \cap N(w') \cap N(w'')$, we have a cycle $yw'Pw''y$ of length $n_1+3$. When $H_2 \cap N(w') \cap N(w'') = \emptyset$, we get $y_1 \in H_2 \cap N(w')$ and $y_2 \in H_2 \cap N(w'')$ such that $y_1y_2 \in E(G)$. And by Fact 2.2.2 and since $|H_2| \geq 2$, we have a path $P^*$ in $G - H_2$ connecting $w'$ and $w''$ such that $|P^*| = n_1 + 1$. It follows that $y_1w'P^*w''y_2y_1$ is a cycle of length $n_1 + 3$. Therefore, we have obtained all cycles $C_m$, $n_1 + 3 \leq m \leq n$.

To prove that $G$ contains a $C_{n_1+2}$, we suppose first that there is a $y \in H_2 \cap N(w') \cap N(w'')$. If $G(H_1 \cup \{w', w''\})$

contains a path $P^*$ connecting $w'$ and $w''$ such that $|P^*| = n_1 + 1$, then the cycle $yw'P^*w''y$ is of length $n_1 + 2$. If no such path exists, by Fact 2.2.2, $w'x_{i+2}, x_iw'' \in E(G)$ for any $i$, $1 \leq i \leq n_1 - 2$. It follows that $w'x_3, w''x_2 \in E(G)$ when $n_1 \geq 4$. It gives a cycle $w'x_3x_4 \cdots x_{n_1}w''x_2x_1w'$ of length $n_1 + 2$.

We may directly deduce that when $n_1 \leq 3$, either there is $C_{n_1+2}$ or $G = K_{2,2}$ or $G = K_{3,3} - \{e\}$.

Suppose that $H_2 \cap N(w') \cap N(w'') = \emptyset$. Clearly we have a cycle of length $n_1 + 2$ if $w'w'' \in E(G)$. We assume $w'w'' \notin E(G)$. If $w', w'' \in Y$ (or $w', w'' \in X$), since $d_G(w') + d_G(w'') \geq n$, $d_{H_1}(w') + d_{H_1}(w'') \geq n_1 + 2$. By Lemma 2.1.4 and with the path $P$ define above, it exists an $i$, $1 \leq 1 \leq n_1 - 1$ such that $w'x_{i+1}, w''x_i \in E(G)$. Hence, we have a cycle $w'x_{i+1}x_{i+2} \cdots x_{n_1}w''x_ix_{i-1} \cdots x_1w'$ with length $n_1 + 2$. without loss of generality, we consider the case that $w' \in X$ and $w'' \in Y$ . Put $G_1 = G(H_1 \cup \{w', w''\})$ with $X_1 = V(H_1) \cup \{w'\}$ and $Y_1 = \{w''\}$. If $N(w') \cap H_1 = \{z\}$, then for any $z' \in V(H_1) - \{z\}$, $n \leq d(w') + d(z') \leq n_1 + n_2 + 1$, a contradiction. So $|N(w') \cap H_1| \geq 2$. If $|N(w'') \cap H_1| \geq 2$, we can see that $G_1$ is 2-connected, and it satisfies that condition of the theorem with a smaller order.

So, $G_1$ has a cycle of length $n_2 + 2$. If $N(w'') \cap H_1 = \{x\}$, then $\{w', x\}$ is a 2-cut. By the above argument, we may have that $G(H_2 \cup \{w''\})$ is a clique in $Y$ and hence $H_2 \cap N(w') \cap N(w'') \neq \emptyset$, a contradiction.

Therefore, we obtain a cycle $C_{n_1+2}$ in $G$.

We will show the existence of $C_m$, $3 \leq m \leq n_1 + 1$ or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.

When $|N(w') \cap H_1| \geq 2$, we define $G_2 = G(H_1 \cup \{w'\})$ with $X_2 = V(H_1)$ and $Y_2 = \{w'\}$. If $x'$ and $x''$ are nonadjacent vertices in $X_2$,

$$d_{G_2}(x') + d_{G_2}(x'') \geq d_G(x') - 1 + d_G(x'') - 1 \geq n - 2 \geq |V(G_2)|,$$

which implies that $G_2$ is 2-connected. Since $|V(G_2)| < V(G)|$, by the minimality assumption, $G_2$ is pancyclic or $G_2 = K_{(n_1+1)/2,(n_1+1)/2}$ or $G_2 = K_{(n_1+1)/2,(n_1+1)/2} - \{e\}$. In the last two cases, for any pair of nonadjacent vertices $x'$ and $x''$ in $G_2 - \{w'\}$, $d_{G_2}(x') + d_{G_2}(x'') \leq n_1 + 1$ and hence $n \leq d_G(x') + d_G(x'') \leq n_1 + 3$. It follows that $|H_2| = 1$, $n_1$ is odd and $x'w'', x''w'' \in E(G)$. When $n_1 \geq 5$, $V(H_1) \subset N(w'')$. It is easy to see now that $G(H_1 \cup \{w', w''\})$ contains all cycles $C_m$, for $3 \leq m \leq n_1 + 2$. When $n_1 = 3$, we deduce that $G = K_{3,3} - \{e\}$.

Without loss of generality, we assume that $N(w') \cap H_1 = \{x'\}$ and $N(w'') \cap H_1 = \{x''\}$. If $w'w'' \in E(G)$, let $G_1 = G(H_1 \cup \{w', w''\})$ with $X_1 = V(H_1)$ and $Y_1 = \{w', w''\}$. It is easy to verify that $G_1$ satisfies the condition of the theorem and $|G_1| < |G|$. By the minimality assumption of $G$, we have $G_1$ is pancyclic or $G_1 = K_{(n_1+2)/2,(n_1+2)/2}$ or $G_2 = K_{(n_1+2)/2,(n_1+2)/2} - \{e\}$. If $n_1 = 2$, by degree sum condition, then $G$ is pancyclic. If $n_1 \geq 3$, from $d_{G_1}(w') = d_{G_1}(w'') = 1$, we get that $G_1$ is pancyclic and hence $G$ has all cycles $C_m$, for $3 \leq m \leq n_1 + 2$. So we assume that $w'w'' \notin E(G)$.

Clearly $\{x', x''\}$ is a 2-cuts of $G$. By the above argument, either $H_2 \cup \{w', w''\} \subseteq Y$ is a clique (which is not possible because $w'w'' \notin E(G)$) or $H_1 - \{x', x''\} \subseteq X$ is a clique. If there are two nonadjacent vertices $x_a$ and $x_b$ in

$X$, we obtain

$$2n \leq d_G(x_a) + d_G(x_b) + d_G(w') + d_G(w'') \leq 2(n_1 - 1) + 2(n_2 + 1) = 2(|G| - 2),$$

a contradiction. So $H_1$ is a clique and there are all cycles $c_m$, for $3 \leq m \leq n_1$. Since

$$n_1 + n_2 + 2 = n \leq d_G(w') + d_G(w'') \leq 2 + n_2 + |H_2 \cap N(w') \cap N(w'')|,$$

it follows that $|H_2 \cap |N(w') \cap N(w'')| \geq n_1$. Clearly there is a cycle $C_{n_1+1}$ in $G$.

Claim 2.2.1 is proved. □

### 2.2.2 Constructing the desired hamiltonian cycle

By Claim 2.2.1 we assume that $G$ is 3-connected. If both $G[X]$ and $G[Y]$ are cliques, clearly $G$ is pancyclic or $G = K_{2,2}$. It follows that we may assume that there exists a pair of nonadjacent $x_1$ and $x_2$ in $X$ or $Y$.

Let $P = v_1v_2v_3 \cdots v_p$ be a path in $G$ such that

(1) $v_1v_p \notin E(G)$ and $v_1, v_p \in X$ or $v_1, v_p \in Y$, say $v_1, v_p \in X$;

(2) subject to (1), $p$ is as large as possible.

When $V(P) = V(G)$, by Theorem 2.1.2, $G$ is pancyclic. So there is a vertex $w^0 \in V(G) - V(P)$. Since $G$ is 3-connected, there are three internal disjoint paths $P^1[w^0, v_d]$, $P^2[w^0, v_l]$ and $P^3[w^0, v_m]$ connecting $w^0$ and three distinct vertices $\{v_d, v_l, v_m\} \subseteq V(P)$ with $d < l < m$. It follows that $w^0, v_{d+1}(= v_d^+), v_{l+1}(= v_l^+)$ are pairwisely nonadjacent (otherwise there would be a path longer than $P$ that connects $v_1$ and $v_p$, a contradiction). Then two of the three vertices $w^0, v_{d+1}(= v_d^+), v_{l+1}(= v_l^+)$ should be in the same part of $X$ and $Y$.

If these two vertices are $w^0$ and $v_{d+1}$,

put $P_1[v_1, w^0] = P[v_1, v_d]\overline{P^1}(v_d, w^0] = v_1v_2 \cdots v_d\overline{P^1}(v_d, w^0]$ and $P_2 = P[v_{d+1}, v_p] = v_{d+1}v_{d+2} \cdots v_p$;

If these two vertices are $w^0$ and $v_{l+1}$,

put $P_1[v_1, w^0] = P[v_1, v_l]\overline{P^2}(v_l, w^0] = v_1v_2 \cdots v_l\overline{P^2}(v_l, w^0]$ and $P_2 = P[v_{l+1}, v_p] = v_{l+1}v_{l+2} \cdots v_p$;

If these two vertices are $v_{d+1}$ and $v_{l+1}$,

put $P_1[v_1, v_{d+1}] = v_1v_2 \cdots v_d\overline{P^1}(v_d, w^0]\ P^2(w^0, v_l)v_lv_{l-1} \cdots v_{d+1}$ and $P_2 = v_{l+1}v_{l+2} \cdots v_p$.

In all the above cases, these two paths $P_1$ and $P_2$ satisfy $|P_1| + |P_2| \geq p+1$, one endpoint of $P_1$ and one endpoint of $P_2$ are not adjacent and both belong to $X$, the other endpoint of $P_1$ and the other endpoint of $P_2$ are not adjacent and both belong to $X$ or $Y$. We assume that $Q' = u_1u_2u_3 \cdots u_q$ and $Q'' = u_{q+1}u_{q+2} \cdots u_t$ are two disjoint paths such that $t$ ($t \geq p+1$) is maximum, subject to $u_1, u_t \in X$, $u_q, u_{q+1} \in X$ or $u_q, u_{q+1} \in Y$ and $u_1u_t, u_qu_{q+1} \notin E(G)$.

If there exists a vertex $w^* \in (G-(Q' \cup Q'')) \cap N(u_q) \cap N(u_{q+1})$, then there is a new path $P^* := Q'w^*Q'' = u_1u_2\cdots u_q w^* u_{q+1}u_{q+2}\cdots u_t$ which contradicts the maximality of $P$. So $(G-(Q' \cup Q'')) \cap N(u_q) \cap N(u_{q+1}) = \emptyset$. Similarly $(G-(Q' \cup Q'')) \cap N(u_1) \cap N(u_t) = \emptyset$.

For any $i,\ 2 \le i \le q-1$, if $u_iu_t, u_{i+1}u_1 \in E(G)$, then $Q = u_qu_{q-1}\cdots u_{i+1}u_1u_2u_3\cdots u_iu_tu_{t-1}\cdots u_{q+1}$ is a new path. Since $u_q, u_{q+1}$ are nonadjacent and both belong to $X$ or $Y$ and $t \ge p+1$, $Q$ contradicts with the choice of $P$. So $N_{Q'}(u_1)^- \cap N_{Q'}(u_t) = \emptyset$. Similarly, $N_{Q''}(u_t)^+ \cap N_{Q''}(u_1) = \emptyset$. It follows that

$$\begin{aligned}
n &\le d_G(u_1) + d_G(u_t) \\
&\le |G - V(Q' \cup Q'')| + d_{Q'-\{u_1\}}(u_1) + d_{Q'-\{u_1\}}(u_t) \\
&+ d_{Q''-\{u_t\}}(u_1) + d_{Q''-\{u_t\}}(u_t) \\
&\le |G - V(Q' \cup Q'')| + |Q' - \{u_1\}| + 1 + |Q'' - \{u_t\}| + 1 \\
&\le n - t + t = n.
\end{aligned} \tag{2.2}$$

It implies that $d_{Q'-\{u_1\}}(u_1) + d_{Q'-\{u_1\}}(u_t) = |Q' - \{u_1\}| + 1$ and $d_{Q''-\{u_t\}}(u_1) + d_{Q''-\{u_t\}}(u_t) = |Q'' - \{u_t\}| + 1$. Therefore $Q' - \{u_1\}$, $Q'' - \{u_t\}$, $u_1$ and $u_t$ satisfy Lemma 2.1.4. So $u_1u_{q+1}, u_qu_t \in E(G)$. Hence, we have a cycle $C := u_1u_2\cdots u_qu_tu_{t-1}\cdots u_{q+1}u_1$.

Now, we constructed a hamiltonian cycle $C$. Next, we will give the properties of the hamiltonian cycle $C$.

**Claim 2.2.3** $N_G(u_1) \subseteq V(C)$, $N_G(u_t) \subseteq V(C)$, $N_G(u_q) \subseteq V(C)$ *and* $N_G(u_{q+1}) \subseteq V(C)$.

**Proof.** Suppose that there is $w \in N_G(u_1) - V(C)$. It follows that $w \in Y$ since otherwise when $wu_t \in E(G)$, the path $u_qu_{q-1}\cdots u_1wu_tu_{t-1}\cdots u_{q+1}$, contradicts with the choice of $P$, and when $wu_t \notin E(G)$, $w, u_t \in X$, the two paths $wQ'[u_1, u_q] = wu_1u_2\cdots u_q$ and $Q''$ contradict with the property of $Q'$ and $Q''$.

Since $G$ is 3-connected, there are two internal disjoint paths $F_1[w, u_i]$ and $F_2[w, u_j]$ between $w$ and $u_i, u_j \in V(C) - \{u_1\}$. If $u_i = u_t$, then a path $u_qu_{q-1}\cdots u_1wF_1(w, u_t)u_tu_{t-1}\cdots u_{q+1}$ contradicts the choice of $P$. So $i \ne t$ and $j \ne t$.

Similarly, we may show that at least one of $u_i$ and $u_j$, say $u_i \notin \{u_q, u_{q+1}\}$. Hence, we may assume $u_i \notin \{u_1, u_t, u_q, u_{q+1}\}$. If $u_2 = u_i$, we put $Q_1' = u_1wF_1(w, u_2)u_2u_3\cdots u_q$ and $Q_1'' = Q''$, which contradict the definitions of $Q'$ and $Q''$. So $u_2 \ne u_i$ and $u_2 \ne u_j$, in particular, $wu_2 \notin E(G)$.

If $u_2 \in Y$, then a path $u_2u_3\cdots u_qu_tu_{t-1}\cdots u_{q+1}u_1w$ contradicts the maximality of $P$. So $u_2 \in X$. Suppose $q+2 \le i \le t-1$. If $u_tu_{i-1} \in E(G)$ (resp. $u_tu_{i-2} \in E(G)$ when $t-1 \ge i \ge q+3$), then

$$u_{q+1}u_{q+2}\cdots u_{i-2}u_{i-1}u_tu_{t-1}\cdots u_i\overline{F}(u_i, w)wu_1u_2\cdots u_q$$

$$(resp.\ u_{q+1}u_{q+2}\cdots u_{i-3}u_{i-2}u_tu_{t-1}\cdots u_i\overline{F}(u_i, w)wu_1u_2\cdots u_q)$$

is a path of length at least $t > p$, a contradiction. Hence, $u_tu_{i-1} \notin E(G)$ when $t-1 \geq i \geq q+2$ and $u_tu_{i-2} \notin E(G)$ when $t-1 \geq i \geq q+3$.

By (2.2) and Lemma 2.1.4 (3), $u_1u_i \in E(G)$ when $t-1 \geq i \geq q+2$ and $u_1u_{i-1} \in E(G)$ when $t-1 \geq i \geq q+3$. From $u_1u_{q+1} \in E(G)$, $i \neq q+1, q+2$. Therefore, we always obtain $u_1u_{i-1} \in E(G)$.

If $u_2u_t \in E(G)$, then there is a path $u_{q+1}u_{q+2}\cdots u_{i-1}u_1wF(w,u_i)u_iu_{i+1}\cdots u_tu_2u_3\cdots u_{q+1}$ whose length is at least $t+1 > p$, a contradiction.

If $u_2u_t \notin E(G)$, two paths $u_2u_3\cdots u_q$ and $u_{q+1}u_{q+2}\cdots u_{i-1}u_1wF(w,u_i)u_iu_{i+1}\cdots u_t$, contradict with the choice of $Q'$ and $Q''$.

Thus, we may assume $u_i \in Q'$ $(3 \leq i \leq q-1)$.

If $wu_{i+1} \in E(G)$ (resp. $u_2u_{i+1} \in E(G)$), two paths

$$u_1u_2\cdots u_i\overline{F}(u_i,w)wu_{i+1}u_{i+2}\cdots u_q(resp.u_1wF(w,u_i)u_iu_{i-1}\cdots u_2u_{i+1}u_{i+2}\cdots u_q)$$

and $Q''$ contradict the choice of $Q'$ and $Q''$. So $wu_{i+1} \notin E(G)$ and $u_2u_{i+1} \notin E(G)$. It follows that a path

$$Q = wF(w,u_i)u_iu_{i-1}\cdots u_1u_{q+1}u_{q+2}\cdots u_tu_qu_{q-1}\cdots u_{i+1} \quad \text{if } u_{i+1} \in Y \qquad or$$
$$Q = u_2u_3\cdots u_i\overline{F}(u_i,w)wu_1u_{q+1}u_{q+2}\cdots u_tu_qu_{q-1}\cdots u_{i+1} \quad \text{if } u_{i+1} \in X.$$

contradicts the maximality of $P$.

Thus, $N_G(u_1) \subseteq V(C)$. Similarly, $N_G(u_t) \subseteq V(C)$, $N_G(u_q) \subseteq V(C)$ and $N_G(u_{q+1}) \subseteq V(C)$.

The proof of Claim 2.2.3 is completed. ∎

**Claim 2.2.4** *$C$ is a hamiltonian cycle of $G$.*

**Proof.** In (2.2), by Claim 2.2.3, we have

$$\begin{aligned} n &\leq d_G(u_1) + d_G(u_t) \\ &\leq d_{Q'-\{u_1\}}(u_1) + d_{Q'-\{u_1\}}(u_t) \\ &\quad + d_{Q''-\{u_t\}}(u_1) + d_{Q''-\{u_t\}}(u_t) \\ &\leq |Q'-\{u_1\}| + 1 + |Q''-\{u_t\}| + 1 \leq t, \end{aligned}$$

which implies $t = n$ and hence $C$ is a hamiltonian cycle. ∎

### 2.2.3 The rest of the proof of Theorem 2.0.3

$C$ is a hamiltonian cycle, in which $u_1$ and $u_{q+1}$ are consecutive and $u_q$ and $u_t$ are consecutive. Since $d_G(u_1) + d_G(u_t) + d_G(u_q) + d_G(u_{q+1}) \geq 2n$, we have either $d_G(u_1) + d_G(u_{q+1}) \geq n$ or $d_G(u_t) + d_G(u_q) \geq n$. By Theorem 2.1.1, $G$ is either pancyclic or bipartite or missing only an $(n-1)$-cycle.

**Case 1** *$G$ is bipartite.*

Let $A$ and $B$ be the bipartitions of $G$. Without loss of generality, we assume $|A| \geq |B|$. If $|A| = 2$, $G = K_{2,2}$. If $|A| \geq 3$, every pair of vertices in $X \cap A$ (resp., $Y \cap A$) have degree sum at most $2|B|$. Hence, they must be adjacent to all vertices of $B$ and $|A| = |B| = \frac{n}{2}$.

By Lemma 2.1.3, it follows that $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.

**Case 2** *$G$ is missing only an $(n-1)$-cycle.*

If $d_G(u_1) + d_G(u_{q+1}) \geq n+1$, from the proof of Theorem 2.0.4, $G$ is pancyclic. So we assume $d_G(u_1) + d_G(u_{q+1}) = n$ and similarly $d_G(u_t) + d_G(u_q) = n$.

If $u_1u_3 \in E(G)$, then there is a $(n-1)$-cycle: $u_1u_3u_4 \cdots u_qu_tu_{t-1} \cdots u_{q+1}u_1$, a contradiction. So $u_1u_3 \notin E(G)$ and from Lemma 2.1.4, $u_2u_t \in E(G)$.

Without loss of generality, assume $q \geq t-q$. When $q = 2$ and $t = 4$, clearly $G = K_{2,2}$. When $q = 3$, $u_1u_2u_tu_{t-1} \cdots u_{q+1}u_1$ is a $(n-1)$-cycle. When $q = 4$, by Theorem 2.1.1, $u_qu_{q+1} \in E(G)$ which is a contradiction. So we assume that $q \geq 5$. Similarly, we may assume that $t - q \geq 5$.

From Theorem 2.1.1, we obtain $d(u_2) < n/2$, $d(u_3) < n/2$, $d(u_{q+2}) < n/2$, $d(u_{q+3}) < n/2$ and $u_2u_{q+2}, u_2u_{q+3}, u_3u_{q+2}, u_3u_{q+3} \notin E(G)$. It follows that $u_2, u_3$ belong to one of $X$ and $Y$, say $X$, and $u_{q+2}, u_{q+3}$ belong to $Y$.

Similarly, $d(u_{q-1}), d(u_{q-2}), d(u_{t-1}), d(u_{t-2}) < n/2$, $u_{q-1}, u_{q-2}$ belong to one of $X$ and $Y$ and $u_{t-1}, u_{t-2}$ belong to the other one of $X$ and $Y$. If $u_2u_{t-1} \in E(G)$, we get a $(n-1)$-cycle: $u_1u_4u_5 \cdots u_qu_tu_2u_{t-1}u_{t-2} \cdots u_{q+1}u_1$, a contradiction. Thus $u_2u_{t-1} \notin E(G)$, which implies $u_{t-1}, u_{t-2} \in Y$ and hence $u_{q-1}, u_{q-2} \in X$. We have $u_2 \in N(u_{q-1}) \cap N(u_{q-2})$. The $(n-1)$-cycle $C_{n-1} = u_1u_4u_5 \cdots u_{q-2}u_2u_{q-1}u_qu_tu_{t-1} \cdots u_{q+1}u_1$ is a contradiction.

The proof of Theorem 2.0.3 is complete. $\square$

## 2.3 Open problems

In 1960, Ore [109] showed that if the degree sum of any pair of nonadjacent vertices is at least $n$ in a graph $G$ of order $n$, then $G$ is hamiltonian (Theorem 1.2.2). Bondy proved that under the same condition, $G$ is pancyclic or $G = K_{n/2,n/2}$ (Theorem 1.3.1).

In this chapter, we prove that if $G = (V, E)$ is a $2$-connected graph of order $n$ with $V(G) = X \cup Y$ such that for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$, $d(x_1) + d(x_2) \geq n$ and for any pair of nonadjacent vertices $y_1$ and $y_2$ in $Y$, $d(y_1) + d(y_2) \geq n$, then $G$ is pancyclic or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.

Note that the main result of this chapter is to prove that the conjecture 2.0.2 is true for $k = 2$. For all other cases ($k \geq 3$) of Conjecture 2.0.2, we haven't given proof. In the next chapter (Chapter 3), we will prove that Conjecture 2.0.2 is true for $k = 3$.

We try to prove Conjecture 1.3.12 with $k \geq 4$, but unfortunately, we did not succeed yet. This will be one of our further works.

For Conjecture 1.3.12, it is natural to generalize them into degree and neighborhood conditions on more independent vertices. Therefore, this is our other further work.

# Chapter 3

# Pancyclicity in 3-connected graphs

In this chapter, we give the proof of Conjecture 1.3.12 for graphs of $k = 3$. It is kind of a continuation of the work in Chapter 2. To facilitate reading, we reiterate Conjecture 1.3.12 here.

**Conjecture 3.0.1** *Let $G = (V, E)$ be a $k$-connected graph, $k \geq 2$, of order $n$. Suppose that $V(G) = \cup_{i=1}^{k} X_i$ such that for each $i$, $i = 1, 2, \ldots, k$, and for any pair of nonadjacent vertices $x, y \in X_i$, $d(x)+d(y) \geq n$. Then $G$ is pancyclic or $G$ is a bipartite graph.*

The main result of this chapter is to prove that the above conjecture is true for $k = 3$.

**Theorem 3.0.2** *Let $G = (V, E)$ be a $3$-connected graph of order $n$ and $V(G) = X_1 \cup X_2 \cup X_3$. For any pair of nonadjacent vertices $v_1$ and $v_2$ in $X_i$, $d(v_1) + d(v_2) \geq n$ with $i = 1, 2, 3$. Then $G$ is pancyclic or $G$ is a bipartite graph.*

## 3.1 Introduction

In Chapter 2, we gave proof of Conjecture 1.3.12 for a $2$-connected graph, i.e., $k = 2$ in Conjecture 1.3.12.

**Theorem 3.1.1 (Theorem 2.0.3)** *Let $G = (V, E)$ be a $2$-connected graph of order $n$ and $V(G) = X \cup Y$. If for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$, $d(x_1) + d(x_2) \geq n$ and for any pair of nonadjacent vertices $y_1$ and $y_2$ in $Y$, $d(y_1) + d(y_2) \geq n$. Then $G$ is pancyclic or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.*

Here we will prove that Conjecture 1.3.12 is true for $k = 3$ by showing Theorem 3.0.2.

The main idea and the main tools of the proof of Theorem 3.0.2 and Theorem 2.0.3 are similar, but there are also some differences. To make this chapter complete, we will give the whole proof of Theorem 3.0.2. We will follow all notations, such as hamiltonian $(u, v)$-path, the predecessor and the successor of a vertex, $S$-cyclable etc., as in Chapter 2.

### 3.1.1 Well-known results

In our proof of Theorem 3.0.2, we will use some well-known results.

**Theorem 3.1.2 (Theorem 2.1.2)** *Let $G$ be a graph of order $n$. If $G$ has a hamiltonian $(u, v)$-path for a pair of nonadjacent vertices u and v such that $d(u) + d(v) \geq n$, then G is pancyclic. Moreover, if u (or v) has degree at least $\frac{n}{2}$, it is contained in a triangle and for any $m$, $4 \leq m \leq n$, there exists some $C_m$ in $G$ that contains both $u$ and $v$.*

**Theorem 3.1.3 ([47])** *Let $C = x_1x_2 \cdots x_nx_1$ be a hamiltonian cycle in a graph $G$. If $d(x_1) + d(x_n) \geq n + 1$, then $G$ is pancyclic.*

**Theorem 3.1.4 ([117])** *If $G$ is a hamiltonian graph of order $n$ with hamiltonian cycle $x_1, x_2, ..., x_n, x_1$ such that $d(x_1)+ d(x_n) \geq n$, then $G$ is either pancyclic or bipartite or missing only an $(n - 1)$-cycle. Moreover, if $G$ is missing only an $(n - 1)$-cycle, then $d(x_{n-2}), d(x_{n-1}), d(x_2), d(x_3) < n/2$, and $G$ has one of two possible adjacency structures near $x_1$ and $x_n$. In the first structure, vertices $x_{n-2}, x_{n-1}, x_n, x_1, x_2, x_3$ are independent except for edges of $C$, and $x_nx_{n-3}, x_nx_{n-4}, x_1x_4, x_1x_5 \in E(G)$. The second structure (which can occur only if $d(x_1) < d(x_n)$) is identical to the first except that $x_nx_3 \in G$ and $x_1x_5 \notin G$.*

### 3.1.2 Outline of the proof

In our proof for Theorem 3.0.2, we will use Menger's Theorem (see section 6.1 in Chapter 6).

In Theorem 3.0.2, let $V(G) = X_1 \cup X_2 \cup X_3$. We first consider the situation for each $i$, $i = 1, 2, 3$, $G[X_i]$ is a clique (Lemma 3.2.2).

Next, we can find a path $P$. There is a vertex $w^0 \in V(G) - V(P)$, and there are (at least) three internal disjoint paths $P^1[w^0, v_{d_1}]$, $P^2[w^0, v_{d_2}]$, and $P^3[w^0, v_{d_3}]$ connecting $w^0$ and three distinct vertices $\{v_{d_1}, v_{d_2}, v_{d_3}\} \subseteq V(P)$ with $d_1 < d_2 < d_3$. Then we talk about it in two cases: non-extremal case ($v_{d_1} \neq v_1$ or $v_{d_3} \neq v_p$) and extremal case ($v_{d_1} = v_1$ and $v_{d_3} = v_p$).

In section 3.3, we will talk about non-extremal case. First, we show the existence of a cycle $C := u_1u_2 \cdots u_qu_tu_{t-1} \cdots u_{q+1}u_1$. such that $|C| \geq |P| + 1$ and $|C| \neq n$. So, there exists a vertex $w \in V(G - C)$. And there are three disjoint paths $P'_1[w, u_{l_1}]$, $P'_2[w, u_{l_2}]$ and $P'_3[w, u_{l_3}]$ between $w$ and $u_{l_1}, u_{l_2}, u_{l_3} \in V(C)$. With that, according to the relationship between $\{u_{l_1}, u_{l_2}, u_{l_3}\}$ and $\{u_1, u_t, u_q, u_{q+1}\}$, it is proved that G is pancyclic or a bipartite graph in this non-extremal case.

Let the component where $w^0$ is located be $H$. In section 3.4, let's first show some properties of $H$. In the end, we have proved Theorem 3.0.2 with the extremal case based on the number of vertices in $H$.

## 3.2 Some lemmas

Some lemmas in our proof are the same as in Chapter 2. We will give these lemmas without proof here.

**Lemma 3.2.1 (Lemma 2.1.4)** *Let $P = u_1u_2u_3\cdots u_p$ be a path in $G$ and $x, y \in V(G) - V(P)$ such that $(N_P(x) - \{u_1\})^- \cap N_P(y) = \emptyset$. Then $d_P(x) + d_P(y) \leq p + 1$ and if $d_P(x) + d_P(y) = p + 1$,*

*(1) $V(P) = (N_P(x) - \{u_1\})^- \cup N_P(y)$;*

*(2) $xu_1, yu_p \in E(G)$;*

*(3) If $u_i \notin N_P(x)$ for some $i, 2 \leq i \leq p$, then $u_{i-1} \in N_P(y)$, and if $u_j \notin N_P(y)$ for some $j, 1 \leq j \leq p-1$, then $u_{j+1} \in N_P(x)$;*

*(4) If $u_i, u_j \notin N_P(x) \cup N_P(y)$ with $2 \leq i < j \leq p-1$ such that $\{u_{i+1}, u_{i+2}, \ldots, u_{j-1}\} \subseteq N_P(x) \cup N_P(y)$, then there exists exact one $k,\ i+1 \leq k \leq j-1$, such that $\{u_{i+1}, u_{i+2}, \ldots, u_k\} \subseteq N_P(x)$ and $\{u_k, u_{k+1}, \ldots, u_{j-1}\} \subseteq N_P(y)$;*

*(5) If $N_P(x)$ does not contain consecutive vertices on $P$ and $N_P(y)$ does not contain consecutive vertices on $P$, then $p$ is odd and $N_P(x) = N_P(y) = \{u_1, u_3, u_5, \ldots, u_{p-2}, u_p\}$.*

If $V(G) = X_1 \cup X_2 \cup X_3$ and for each $i$, $i = 1, 2, 3$, $G[X_i]$ is a clique, we have the following lemma:

**Lemma 3.2.2** *Let $G = (V, E)$ be a $3$-connected graph of order $n$ and $V(G) = X_1 \cup X_2 \cup X_3$. If for each $i$, $i = 1, 2, 3$, $G[X_i]$ is a clique. Then $G = K_{3,3}$ or $G$ is pancyclic.*

**Proof of Theorem 3.2.2:** Suppose, on the contrary, that $G$ is not pancyclic. By Theorem 2.0.1, $G$ is hamiltonian.

Suppose there exists $i \in \{1, 2, 3\}$ such that $|X_i| = 1$. Since $G$ is $3$-connected graph, then $G[V - X_i]$ is $2$-connected graph. By Theorem 2.0.3, $G[V - X_i]$ is pancyclic or isomorphic to $K_{2,2}$. Since $G$ is a $3$-connected graph, then $G$ is pancyclic. This is a contradiction.

Suppose $X_i = \{u_i, v_i\}$ for any $i$, $i = 1, 2, 3$. We obtain the following proposition:

**Proposition 3.2.3** $N(x) \cap X_j \neq \emptyset$ *for any $x \in \{u_i, v_i\}$ with each $i \neq j \in \{1, 2, 3\}$.*

**Proof.** Without loss of generality, let $N(v_1) \cap X_3 = \emptyset$. Since $G$ is $3$-connected graph, then $v_1v_2, v_1u_2 \in E$ and $G[V - X_1]$ is $2$-connected graph. So, $v_1v_2u_2v_1$ is a cycle of length $3$, and we have a cycle $C$ of length $4$ in $G[V - X_1]$ such that $u_2v_2 \in C$. Then $C' = (C - \{u_2v_2\}) \cup \{v_1v_2, v_1u_2\}$ is a cycle of length $5$ in $G$. It follows $G$ is pancyclic from $G$ is hamiltonian. This is a contradiction. By the symmetry of $G[X_i]$, we obtain this proposition. ∎

By the Proposition 3.2.3, then $G[V - X_3]$ is $2$-connected graph. It follows that $G$ is pancyclic or $G$ isomorphic to $K_{3,3}$ from Theorem 2.0.3 and Proposition 3.2.3.

Suppose there exists $i \in \{1,2,3\}$ such that $|X_i| \geq 3$. We assume $e_1 = u_1v_1 \in G[X_1]$ and $e_2 = u_2v_2 \in G[X_2]$ such that $u_1u_2, v_1v_2 \in E$. Let $e_3 = u_3v_3 \in G[X_3]$ and $u, v \in G[V - X_3]$ such that $u_3u, v_3v \in E$. Since $G[X_i]$ is a clique for any $i \in \{1,2,3\}$, for each $k$, $1 \leq k \leq |X_i| - 1$, there is a $(u_i, v_i)$-path $P_k^i$ in $G[X_i]$ of length $k$. So, we have cycles of all lengths from $4$ to $|X_1 \cup X_2|$. Since $G$ is $3$-connected, without loss of generality, we assume $u, v \in X_2$.

If $u \notin \{u_2, v_2\}$ or $v \notin \{u_2, v_2\}$, there is $(u, v)$-paths $Q$ in $G[V - X_3]$ of all lengths from 1 to $|X_1 \cup X_2| - 1$. When $|G[X_1]| \geq 3$ or $|G[X_2]| \geq 4$, since $G[X_1]$ and $G[X_2]$ are cliques, we can find a $(u, v)$-paths $Q$ such that $|V(Q)| = |X_1 \cup X_2| - 1$. Then $C' = Q \cup \{u_3v_3, u_3u, v_3v\}$ is a cycle of length $|X_1 \cup X_2| + 1$. Also, we can find a $(u, v)$-paths $Q$ such that $|V(Q)| = |X_1 \cup X_2|$, then $C_{k'} = P_k^3 \cup Q \cup \{u_3u, v_3v\}$ are cycles of all lengths from $|X_1 \cup X_2| + 2$ to $n$. Thus, $G$ is pancyclic, a contradiction.

When $|X_1| = 2$ and $|X_2| = 3$, if $|X_3| \geq 3$, we choose $(u, v)$-paths $Q$ such that $|V(Q)| = 3$, then $C' = Q \cup P_3^3$ is a cycle $|C'| = 6$. And We can find $(u, v)$-paths $Q$ such that $|V(Q)| = |X_1 \cup X_2|$, then $C_{k'} = P_k^3 \cup Q$ are cycles of all lengths from $7$ to $n$. Then $G$ is pancyclic, a contradiction. If $|X_3| = 2$, since $G$ is $3$-connected, it is easy to construct $G$ is pancyclic.

If $u = u_2, v = v_2$. If $|X_3| \geq 3$ and $|X_2| = 2$, since $G[X_i]$ is a clique for any $i = 1,2,3$, it is easy to construct cycles of all lengths from $3$ to $n$ in $G$. Then $G$ is pancyclic. This is a contradiction. So, $|X_3| = 2$ or $|X_2| \geq 3$. If $|X_2| \geq 3$, since $G$ is $3$-connected, there is a vertex $w \in X_2/\{u_2, v_2\}$ such that $N(w) \cap (X_3 \cup X_1) \neq \emptyset$. When $N(w) \cap X_3 \neq \emptyset$, from the same argument with $u \notin \{u_2, v_2\}$ or $v \notin \{u_2, v_2\}$, it follows that $G$ is pancyclic. When $N(w) \cap X_1 \neq \emptyset$, by the symmetry between $G[X_1]$ and $G[X_3]$, $G$ is pancyclic. So $|X_2| = 2$. Also, by the symmetry between $G[X_1]$ and $G[X_3]$, then $|X_1| = |X_2| = |X_3| = 2$. This is a contradiction.

The proof of this lemma is complete. ☐

**Lemma 3.2.4** *Let $G$ be a 1-connected graph with the order $n$ and $V(G) = X_1 \cup X_2$. Suppose that for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X_i$ with $i = 1,2$, $d(x_1) + d(x_2) \geq n$. If $w$ cuts $G$ into $G_1$ and $G_2$, then $V(G_1) \subseteq X_i$ and $V(G_2) \subseteq X_j$ with $i \neq j \in \{1,2\}$. Moreover, $G_1$ is a clique or $G_2$ is a clique.*

**Proof:** Suppose that $G_1 \cap X_i \neq \emptyset$ and $G_2 \cap X_i \neq \emptyset$ with $i = 1,2$, then

$$n \leq d(x) + d(y) \leq |G_1| - 1 + 1 + |G_2| - 1 + 1 < n$$

for any vertex $x \in X_i \cap G_1$ and $y \in X_i \cap G_2$, a contradiction. So, $V(G_1) \subseteq X_i$ and $V(G_2) \subseteq X_j$ with $i \neq j \in \{1,2\}$.

If there exist $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ such that $u_1v_1 \notin E(G)$ and $u_2v_2 \notin E(G)$, then

$$2n \leq d(u_1) + d(v_1) + d(u_2) + d(v_2) \leq 2(|G_1| - 2 + 1) + 2(|G_2| - 2 + 1) < 2n,$$

a contradiction. Thus, $G_1$ is a clique or $G_2$ is a clique. ☐

**Lemma 3.2.5** *Let $G$ be a 2-connected graph with the order $n$ and $V(G) = X_1 \cup X_2$. Suppose that for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X_i$ with $i = 1, 2$, $d(x_1) + d(x_2) \geq n$. If $\{w, w_1\}$ cuts $G$ into $G_1$ and $G_2$, $G_1 \cap X_i \neq \emptyset$ and $G_2 \cap X_i \neq \emptyset$ with $i = 1, 2$, then $G_1$ and $G_2$ are cliques. Moreover, $G$ is pancyclic.*

**Proof:** For any vertex $x \in X_i \cap G_1$ and $y \in X_i \cap G_2$ with $i = 1, 2$, $n \leq d(x) + d(y) \leq |G_1| + |G_2| + 2 \leq n$. So, $N(x) = G_1 \cup \{w, w_1\}$ and $N(y) = G_2 \cup \{w, w_1\}$. $G_1$ and $G_2$ are cliques. Thus, $G$ is pancyclic. $\square$

## 3.3 Non-extremal case

To the contrary, we suppose that $G$ is not pancyclic graph or a bipartite graph. And $|V(G)|$ is minimum among all counter example. By Lemma 3.2.2, there exists $i \in \{1, 2, 3\}$ such that $G[X_i]$ is not a clique. Therefore, we may assume that there exists a pair of nonadjacent vertices in $X_i$ for some $i \in \{1, 2, 3\}$ .

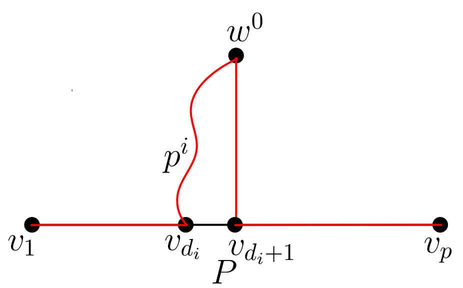
Let $P = v_1v_2v_3 \cdots v_p$ be a path in $G$ such that

(1) $v_1v_p \notin E(G)$ and $v_1, v_p \in X_i,\ i \in \{1, 2, 3\}$;

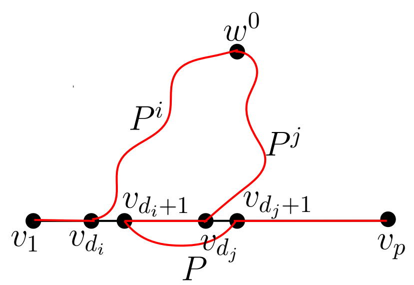
(2) subject to (1), $p$ is as large as possible.

If $V(P) = V(G)$, by Theorem 2.1.2, $G$ is pancyclic. So, there is a vertex $w^0 \in V(G) - V(P)$. Since $G$ is a 3-connected graph, there are (at least) three internal disjoint paths $P^1[w^0, v_{d_1}]$, $P^2[w^0, v_{d_2}]$, and $P^3[w^0, v_{d_3}]$ connecting $w^0$ and three distinct vertices $\{v_{d_1}, v_{d_2}, v_{d_3}\} \subseteq V(P)$ with $d_1 < d_2 < d_3$.

We will prove it in two cases: $v_{d_1} \neq v_1$ or $v_{d_3} \neq v_p$ (say Non-extremal case) and $v_{d_1} = v_1$ and $v_{d_3} = v_p$ (say extremal case). Let's start with the non-extremal case.



(a) when $w^0v_{d_i+1} \in E(G)$



(b) when $v_{d_i+1}v_{d_j+1} \in E(G)$

Figure 3.1: A path is longer than $P$ if $\{w^0, v_{d_1+1}, v_{d_2+1}, v_{d_3+1}\}$ is not independent vertex set

**Case 1** $v_{d_1} \neq v_1$ *or* $v_{d_3} \neq v_p$.

### 3.3.1 The existence of cycle longer than $|P|+1$

Without loss of generality, we assume $v_{d_3} \neq v_p$. It follows that $w^0, v_{d_1+1}, v_{d_2+1}, v_{d_3+1}$ are pairwisely nonadjacent otherwise there would be a path longer than $P$ that connects $v_1$ and $v_p$ (see Figure 3.1), a contradiction. Then two of these four vertices $w^0, v_{d_1+1}, v_{d_2+1}, v_{d_3+1}$ should be in the same part $X_i$ for $i \in \{1,2,3\}$. Thus,

- if these two vertices are $w^0$ and $v_{d_i+1}$ where $i \in \{1,2,3\}$ (see figure 3.2), put $P_1[v_1, w^0] = P[v_1, v_{d_i}]\overline{P^i}(v_{d_i}, w^0]$ and $P_2 = P[v_{d_i+1}, v_p]$;

- if these two vertices are $v_{d_i+1}$ and $v_{d_j+1}$ (see Figure 3.3), put $P_1[v_1, v_{d_i+1}] = P[v_1, v_{d_i}]\overline{P^i}(v_{d_i}, w^0]P^j(w^0, v_{d_j})$ $\overline{P}[v_{d_j}, v_{d_i+1}]$ and $P_2[v_{d_j+1}, v_p] = P[v_{d_j+1}, v_p]$, where $i, j \in \{1,2,3\}$

In all above cases, the two paths $P_1$ and $P_2$ satisfy $|P_1| + |P_2| \geq p+1$, one endpoint of $P_1$ and one endpoint of $P_2$ are not adjacent and both belong to $X_i$, the other endpoint of $P_1$ and the other endpoint of $P_2$ are not adjacent and both belong to $X_j$, where $i, j \in \{1,2,3\}$.

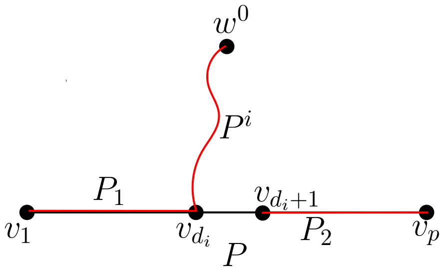



Figure 3.2: $w^0$ and $v_{d_i+1}$ are both belong to the same $X_j$

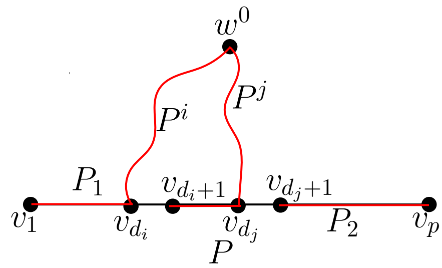



Figure 3.3: $v_{d_j+1}$ and $v_{d_i+1}$ are both belong to the same $X_k$

We assume that $Q' = u_1u_2u_3 \cdots u_q$ and $Q'' = u_{q+1}u_{q+2} \cdots u_t$ are two disjoint paths such that $t$ ($t \geq p+1$) is maximum, subject to $u_1, u_t \in X_i$, $u_q, u_{q+1} \in X_j$ with $i, j \in \{1,2,3\}$, and $u_1u_t \notin E$, $u_qu_{q+1} \notin E(G)$.

By the choice of $P$, then $(G - (Q' \cup Q'')) \cap N(u_q) \cap N(u_{q+1}) = \emptyset$, $(G - (Q' \cup Q'')) \cap N(u_1) \cap N(u_t) = \emptyset$, $N_{Q'}(u_1)^- \cap N_{Q'}(u_t) = \emptyset$ and $N_{Q''}(u_t)^+ \cap N_{Q''}(u_1) = \emptyset$. It follows from Lemma 3.2.1 that

$$\begin{aligned} n &\leq d_G(u_1) + d_G(u_t) \\ &\leq |G - V(Q' \cup Q'')| + d_{Q'-\{u_1\}}(u_1) + d_{Q'-\{u_1\}}(u_t) + d_{Q''-\{u_t\}}(u_1) + d_{Q''-\{u_t\}}(u_t) \\ &\leq |G - V(Q' \cup Q'')| + |Q' - \{u_1\}| + 1 + |Q'' - \{u_t\}| + 1 \\ &\leq n - t + t = n. \end{aligned} \tag{3.1}$$

This implies that $d_{Q'-\{u_1\}}(u_1) + d_{Q'-\{u_1\}}(u_t) = |Q' - \{u_1\}| + 1$ and $d_{Q''-\{u_t\}}(u_1) + d_{Q''-\{u_t\}}(u_t) = |Q'' - \{u_t\}| + 1$. By Lemma 3.2.1, $u_1u_{q+1}, u_qu_t \in E(G)$. Hence, we have a cycle $C := u_1u_2 \cdots u_qu_tu_{t-1} \cdots u_{q+1}u_1$.

When $|C| = n$, $C$ is a hamiltonian cycle, where $u_1$ and $u_{q+1}$ are consecutive vertices on $C$, and $u_q$ and $u_t$ are consecutive vertices on $C$. Since $d_G(u_1) + d_G(u_t) + d_G(u_q) + d_G(u_{q+1}) \geq 2n$, we have either $d_G(u_1) + d_G(u_{q+1}) \geq n$ or $d_G(u_t) + d_G(u_q) \geq n$. Then $d_G(u_1) + d_G(u_{q+1}) = n$ and $d_G(u_t) + d_G(u_q) = n$ otherwise by Theorem 3.1.3, $G$ is pancyclic, a contradiction. By Theorem 3.1.4, we assume $G$ is missing only an $(n-1)$-cycle.

Then $u_1u_3 \notin E(G)$ otherwise $u_1u_3u_4 \cdots u_qu_tu_{t-1} \cdots u_{q+1}u_1$ is a $(n-1)$-cycle, a contradiction. Similarly $u_{q-2}u_q \notin E(G)$, $u_{q+1}u_{q+3} \notin E(G)$ and $u_tu_{t-2} \notin E(G)$. By Lemma 3.2.1, it follows that $u_2u_t \in E(G)$, $u_{q-1}u_{q+1} \in E(G)$, $u_1u_{t-1} \in E(G)$ and $u_qu_{q+2} \in E(G)$.

Suppose that $u_1$ has two consecutive neighbor $u_i$ and $u_{i+1}$ in $Q'$. Then $u_2u_3 \cdots u_iu_1u_{i+1}u_{i+2} \cdots u_qu_{q+2} \cdots u_tu_2$ is a $(n-1)$-cycle, a contradiction. So, $u_1$ does not have two consecutive neighbors in $Q'$. Similarly, $u_1$ does not have two consecutive neighbors in $Q''$ and $u_q$ (resp., $u_{q+1}, u_t$) does not have two consecutive neighbors in $Q'$ and $Q''$.

By Lemma 3.2.1, we deduce that $q$ and $t-q$ are even, and suppose

$A_1 = N_{Q'}(u_1) = N_{Q'}(u_t) = \{u_2, u_4, u_6, \ldots, u_{q-2}, u_q\}$,

$A_2 = N_{Q''}(u_1) = N_{Q''}(u_t) = \{u_{q+1}, u_{q+3}, \ldots, u_{t-3}, u_{t-1}\}$ and $A = A_1 \cup A_2$.

$B_1 = V(Q') - A_1 = N_{Q'}(u_q) = N_{Q'}(u_{q+1}) = \{u_1, u_3, u_5, \ldots, u_{q-3}, u_{q-1}\}$,

$B_2 = V(Q'') - A_2 = N_{Q''}(u_q) = N_{Q''}(u_{q+1}) = \{u_{q+2}, u_{q+4}, \ldots, u_{t-2}, u_t\}$ and $B = B_1 \cup B_2$.

When there are $u_i, u_j \in A_1$ such that $u_iu_j \in E(G)$, if $j = q$, then $u_{i-1}, u_{i+1} \in N_{Q'}(u_q)$. It contradicts that $u_q$ has no two consecutive neighbors in $Q'$. So, we have $j \leq q-2$. Then $u_{i+1}, u_{j+1} \in N_{Q'}(u_q)$, and

$$u_2u_3 \cdots u_iu_ju_{j-1} \cdots u_{i+1}u_qu_{j+1}u_{j+2} \cdots u_{q-1}u_{q+1}u_{q+2} \cdots u_tu_2$$

is a $(n-1)$-cycle, a contradiction.

When there are $u_i \in A_1$ and $u_j \in A_2$ such that $u_iu_j \in E(G)$, then $u_{i-1} \in N_{Q'}(u_q)$, $u_{j+1} \in N_{Q''}(u_{q+1})$. It follows that

$$u_2u_3 \cdots u_{i-1}u_qu_{q-1} \cdots u_iu_ju_{j-1} \cdots u_{q+1}u_{j+1}u_{j+2} \cdots u_tu_2$$

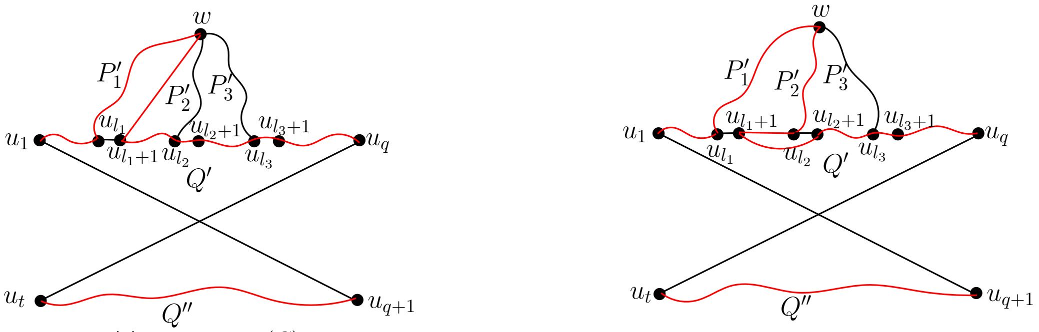
is a $(n-1)$-cycle, a contradiction. Thus, similarly, $A$ and $B$ are independent sets, independently. Hence, $G$ a is a bipartite graph.

When $|C| \neq n$, there exists a vertex $w \in V(G-C)$. Since $G$ is a 3-connected graph, there are three internal disjoint paths $P_1'[w, u_{l_1}]$, $P_2'[w, u_{l_2}]$ and $P_3'[w, u_{l_3}]$ between $w$ and $u_{l_1}, u_{l_2}, u_{l_3} \in V(C)$. By the maximality of $P$, then there does not exist two vertices $u_{l_i}, u_{l_j} \in \{u_{l_1}, u_{l_2}, u_{l_3}\}$ such that $u_{l_i} = u_1, u_{l_j} = u_t$ or $u_{l_i} = u_q, u_{l_j} = u_{q+1}$.

Thus, we have two cases: at most one vertex in $\{u_{l_1}, u_{l_2}, u_{l_3}\}$ belong to $\{u_1, u_t, u_q, u_{q+1}\}$. And there exists only two vertices of $\{u_{l_1}, u_{l_2}, u_{l_3}\}$ belong to $\{u_1, u_t, u_q, u_{q+1}\}$. First, we analyze the first case.

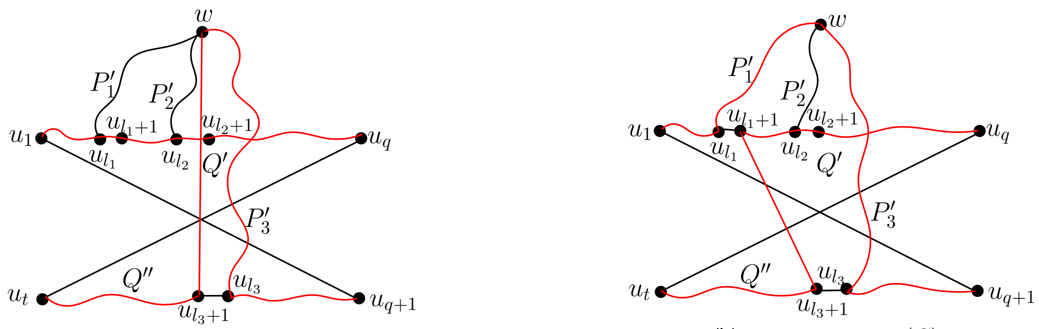
### 3.3.2 At most one vertex in $\{u_{l_1}, u_{l_2}, u_{l_3}\}$ belong to $\{u_1, u_t, u_q, u_{q+1}\}$

Without loss of generality, it follows that $w, u_{l_1+1}(= u_{l_1}^+), u_{l_2+1}(= u_{l_2}^+), u_{l_3+1}(= u_{l_3}^+)$ are pairwisely nonadjacent since otherwise there would be two paths which contradict with the choice of $Q'$ and $Q''$ (see Figures 3.4 and 3.5 ). Then two of four vertices $w, u_{l_1+1}, u_{l_2+1}, u_{l_3+1}$ should be in the same parity $X_i$ with $i \in \{1, 2, 3\}$.




(a) $wu_{l_1+1} \in E(G)$ (b) $u_{l_1+1}u_{l_2+1} \in E(G)$

Figure 3.4: $w, u_{l_1+1}(= u_{l_1}^+), u_{l_2+1}(= u_{l_2}^+), u_{l_3+1}(= u_{l_3}^+)$ are pairwisely nonadjacent with $u_{l_3} \in Q'$




(a) $wu_{l_3+1} \in E(G)$ (b) $u_{l_1+1}u_{l_3+1} \in E(G)$

Figure 3.5: $w, u_{l_1+1}(= u_{l_1}^+), u_{l_2+1}(= u_{l_2}^+), u_{l_3+1}(= u_{l_3}^+)$ are pairwisely nonadjacent with $u_{l_3} \in Q''$

If these two vertices are $w$ and $u_{l_i+1}$ where $i \in \{1, 2, 3\}$ (see Figure 3.6(a)), put $Q_1 = Q'[u_1, u_{l_i}]\overline{P_i'}(u_{l_i}, w]$, $Q_2 = Q'[u_{l_i+1}, u_q]$ and $Q_3 = Q''$; or put $Q_1 = Q''[u_{q+1}, u_{l_i}]\overline{P_i'}(u_{l_i}, w]$, $Q_2 = Q''[u_{l_i+1}, u_t]$ and $Q_3 = Q'$.

If these two vertices are $u_{l_i+1}$ and $u_{l_j+1}$, where $u_{l_i+1}$ and $u_{l_j+1}$ in the same path $Q'$( $Q''$) (see Figure 3.6(b)), put

$Q_1 = Q'[u_1, u_{l_i}]\overline{P_i'}(u_{l_i}, w]P_j'(w, u_{l_j})\overline{Q'}[u_{l_j}, u_{l_i+1}]$, $Q_2 = Q'[u_{l_j+1}, u_q]$ and $Q_3 = Q''$;

or put $Q_1 = Q''[u_{q+1}, u_{l_i}]\overline{P_i'}(u_{l_i}, w]P_j'(w, u_{l_j})\overline{Q''}[u_{l_j}, u_{l_i+1}]$, $Q_2 = Q''[u_{l_j+1}, u_t]$ and $Q_3 = Q'$.

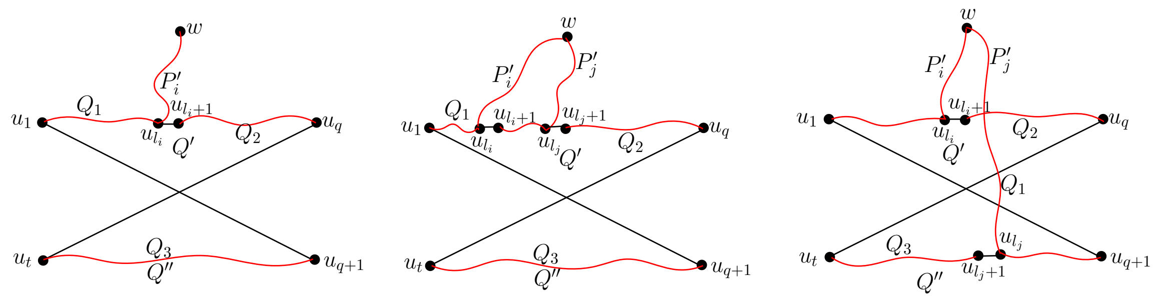
If these two vertices are $u_{l_i+1} \in Q'$ and $u_{l_j+1} \in Q''$ (see Figure 3.6(c)), put

$Q_1 = Q'[u_1, u_{l_i}]\overline{P_i'}(u_{l_i}, w]P_j'(w, u_{l_j}]\overline{Q''}(u_{l_j}, u_{q+1}]$, $Q_2 = Q'[u_{l_i+1}, u_q]$ and $Q_3 = Q''[u_{l_j+1}, u_t]$.

In all above cases, three paths $Q_1$, $Q_2$ and $Q_3$ satisfy $|Q_1| + |Q_2| + |Q_3| \geq t + 1$, one endpoint of $Q_1$ and one endpoint of $Q_2$ are not adjacent and both belong to $X_i$, the other endpoint of $Q_1$ and the endpoint of $Q_3$ are not adjacent and both belong to $X_j$ and the other endpoint of $Q_2$ and the other endpoint of $Q_3$ are not adjacent and both belong to $X_k$ with $i, j, k \in \{1, 2.3\}$.

We assume that $S_1 = w_1w_2w_3 \cdots w_q$, $S_2 = w_{q+1}w_{q+2} \cdots w_l$ and $S_3 = w_{l+1}w_{l+2} \cdots w_{t'}$ are three disjoint paths such that $t'$ ($t' \geq t+1$) is maximum, subject to $w_1, w_{t'} \in X_1$, $w_q, w_{q+1} \in X_2$, $w_l, w_{l+1} \in X_3$ and $w_1w_{t'}, w_qw_{q+1}, w_lw_{l+1} \notin E(G)$.

By the choice of $Q'$ and $Q''$, $(G-(S_1 \cup S_2 \cup S_3)) \cap N(w_q) \cap N(w_{q+1}) = \emptyset$, $(G-(S_1 \cup S_2 \cup S_3)) \cap N(w_1) \cap N(w_{t'}) = \emptyset$ and $(G-(S_1 \cup S_2 \cup S_3)) \cap N(w_l) \cap N(w_{l+1}) = \emptyset$.




(a) these two vertices are $w$ and $u_{l_i+1}$ (b) these two vertices are $u_{l_i+1}$ and $u_{l_j+1}$ in $Q'$) (c) these two vertices are $u_{l_i+1} \in Q'$ and $u_{l_j+1} \in Q''$

Figure 3.6: Two of four vertices $w, u_{l_1+1}, u_{l_2+1}, u_{l_3+1}$ should be in the same parity $X_i$ with $i \in \{1, 2, 3\}$

Suppose $2 \leq i \leq q-1$. If $w_iw_{q+1}, w_{i-1}w_q \in E$, two paths $S_1[w_1, w_{i-1}]w_q\overline{S_1}(w_q, w_i]w_{q+1}S_2$ and $S_3$, which contradict the choice of $Q'$ and $Q''$. So, by Lemma 3.2.1, then $d_{S_1}(w_q) + d_{S_1}(w_{q+1}) \leq |S_1|$. Similarly, $d_{S_2}(w_q) + d_{S_2}(w_{q+1}) \leq |S_2|$ and $d_{S_3}(w_q) + d_{S_3}(w_{q+1}) \leq |S_3| + 1$. It follows that:

$$\begin{aligned} n &\leq d_G(w_q) + d_G(w_{q+1}) \\ &\leq |G - V(S_1 \cup S_2 \cup S_3)| + d_{S_1}(w_q) + d_{S_1}(w_{q+1}) + d_{S_2}(w_q) \\ &+ d_{S_2}(w_{q+1}) + d_{S_3}(w_q) + d_{S_3}(w_{q+1}) \leq n+1 \end{aligned} \tag{3.2}$$

Suppose that $d(w_q) + d(w_{q+1}) = n+1$, it implies that $d_{S_1}(w_q) + d_{S_1}(w_{q+1}) = |S_1|$, $d_{S_2}(w_q) + d_{S_2}(w_{q+1}) = |S_2|$ and $d_{S_3}(w_q) + d_{S_3}(w_{q+1}) = |S_3| + 1$. By Lemma 3.2.1, $w_qw_l \in E$ and $w_{q+1}w_{l+1} \in E$. Hence, path $P' = S_1[w_1, w_q]w_l\overline{S_2}(w_l, w_{q+1}]w_{l+1}S_3(w_{l+1}, w_{t'}]$ contradicts the choice of $P$. So $d(w_q) + d(w_{q+1}) = n$.

If $d_{S_3}(w_q) + d_{S_3}(w_{q+1}) = |S_3| + 1$, then $d_{S_1}(w_q) + d_{S_1}(w_{q+1}) = |S_1|$ or $d_{S_2}(w_q) + d_{S_2}(w_{q+1}) = |S_2|$. We assume $d_{S_1}(w_q) + d_{S_1}(w_{q+1}) = |S_1|$. It follows that $w_1w_{q+1}, w_qw_{t'} \in E$ from Lemma 3.2.1. Then there is a path $\overline{S_2}[w_l, w_{q+1}]w_1S_1(w_1, w_q]w_{t'}\overline{S_3}(w_{t'}, w_{l+1}]$ which contradicts the choice of $P$. Thus, $d_{S_3}(w_q) + d_{S_3}(w_{q+1}) \leq |S_3|$.

It follows that $d_{S_1}(w_q) + d_{S_1}(w_{q+1}) = |S_1|$ and $d_{S_2}(w_q) + d_{S_2}(w_{q+1}) = |S_2|$. By Lemma 3.2.1, $w_1w_{q+1}, w_qw_l \in E$.

The same argument with $w_q, w_{q+1}$, it follows that $d_{S_1}(w_1) + d_{S_1}(w_{t'}) = |S_1|$ and $d_{S_3}(w_1) + d_{S_3}(w_{t'}) = |S_3|$.

When $d_{S_1}(w_1) + d_{S_1}(w_{t'}) = |S_1|$, by Lemma 3.2.1, $w_q w_{t'} \in E$. Then path $\overline{S_2}[w_l, w_{q+1}]w_1 S_1(w_1, w_q]w_{t'}\overline{S_3}(w_{t'}, w_{l+1}]$ contradicts the choice of $P$. So, $G$ is pancyclic or a bipartite graph.

### 3.3.3 There exists only two vertices of $\{u_{l_1}, u_{l_2}, u_{l_3}\}$ in $\{u_1, u_t, u_q, u_{q+1}\}$

Without loss of generality, we assume $u_{l_1} = u_1$, then there are four subcases:

**Subcase 1.1** $u_{l_3} = u_q$ *and* $u_{l_2} \in Q'$.

It follows that $w, u_2, u_{l_2+1}, u_{q+1}$ are pairwisely nonadjacent by the choice of $Q'$, $Q''$ and $P$. Then two of these four vertices $w, u_2, v_{l_2+1}, v_{q+1}$ should be in the same parity $X_i$, for some $i \in \{1,2,3\}$. Let $j \in \{2, l_2+1\}$,

$$P'_s = \begin{cases} P'_1 & j = 2, \\ P'_2 & j = l_2+1. \end{cases} \tag{3.3}$$

By the choice of $Q'$ and $Q''$, $wu_j \notin E$ and $u_2 u_{l_2+1} \notin E$. By the maximality of $P$, then $u_j u_{q+1} \notin E$ and $wu_{q+1} \notin E$. If $wu_j \in E$, then two paths $Q'[u_1, u_{j-1}]\overline{P'_s}[u_{j-1}, w]u_j Q'(u_j, u_q]$ and $Q''$ contradict with the choice of $Q'$ and $Q''$. If $u_j u_{q+1} \in E$, then there is a path $Q'[u_1, u_{j-1}]\overline{P'_s}[u_{j-1}, w]P'_3(w, u_q]\overline{Q'}[u_q, u_j]u_{q+1}Q''(u_{q+1}, u_t]$ whose length is at least $t+1 \geq |P|$, a contradiction. If $wu_{q+1} \in E$, then there is a path $Q'[u_1, u_q]\overline{P'_3}(u_q, w]u_{q+1}Q''(u_{q+1}, u_t]$ longer than $P$. If $u_2 u_{l_2+1} \in E$, two paths $\overline{P'_1}[u_1, w]P'_2(w, u_{l_2}]\overline{Q'}(u_{l_2}, u_2]u_{l_2+1}Q'(u_{l_2+1}, u_q]$ and $Q''$ contradict with the choice of $Q'$ and $Q''$. If $w, u_j \in X_i$, there is a $(w, u_j)$-path $C - \{u_{j-1}u_j\} \cup P'_s[w, u_{j-1}]$ which contradicts the choice of $P$. If

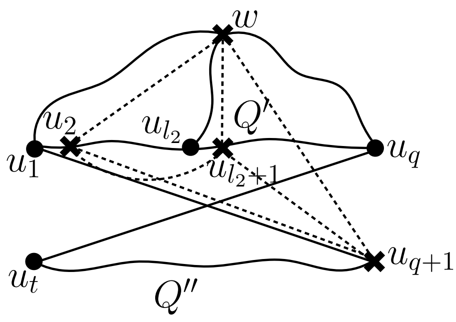



Figure 3.7: When $u_{l_3} = u_q$ and $u_{l_2} \in Q'$

$u_j, u_{q+1} \in X_i$, then two paths $Q'[u_1, u_{j-1}]\overline{P'_s}[u_{j-1}, w]P'_3[w, u_q]\overline{Q'}(u_q, u_j]$ and $Q''$ contradict the choice of $Q'$ and $Q''$. If $w, u_{q+1} \in X_i$, then two paths $Q'[u_1, u_q]\overline{P'_3}[u_q, w]$ and $Q''$ contradict the choice of $Q'$ and $Q''$. If $u_{l_2+1}, u_2 \in X_i$, then there is a $(u_2, u_{l_2+1})$-path $C - \{u_{l_2}u_{l_2+1}, u_1 u_2\} \cup P'_1 \cup P'_2$ which contradicts the choice of $P$.

**Subcase 1.2** $u_{l_2} = u_q$ *and* $u_{l_3} \in Q'' - \{u_{q+1}, u_t\}$.

It follows that $w, u_t, u_{l_3-1}, u_{q-1}$ are pairwisely nonadjacent. And two of these four vertices $w, u_t, u_{l_3-1}, u_{q-1}$ should be in the same parity $X_i$, for some $i \in \{1,2,3\}$.

The proof of Subcase 1.2 is similar to the proof of Subcase 1.1. If $w, u_t \in X_i$ or $u_{q-1}, u_t \in X_i$ or $u_t, u_{l_3-1} \in X_i$,

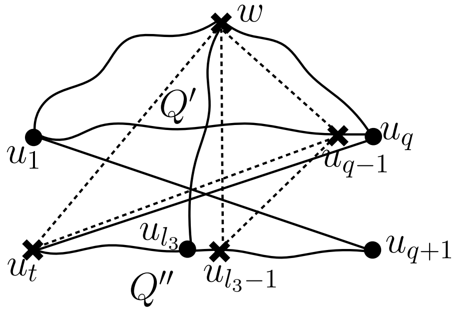



Figure 3.8: When $u_{l_2} = u_q$ and $u_{l_3} \in Q'' - \{u_{q+1}, u_t\}$

then there are two paths longer than $Q'$ and $Q''$, a contradiction. If $w, u_{l_3-1} \in X_i$ or $w, u_{q-1} \in X_i$, there is a $(w, u_j)$-path $C/\{u_j u_{j+1}\} \cup P'_s[w, u_{j+1}]$ longer than $P$, where $j \in \{l_3 - 1, q - 1\}$ and

$$P'_s = \begin{cases} P'_3 & j = l_3 - 1, \\ P'_2 & j = q - 1. \end{cases} \tag{3.4}$$

This is a contradiction. If $u_{q-1}, u_{l_3-1} \in X_i$, there are two paths $u_q u_t \overline{Q''}(u_t, u_{l_3}] \overline{P'_3}(u_{l_3}, w] P'_1[w, u_1] Q'(u_1, u_{q-1}]$ and $Q''[u_{l_3-1}, u_{q+1}]$ longer than $Q'$ and $Q''$, a contradiction.

**Subcase 1.3** $u_{l_3} = u_{q+1}$, $u_{l_2} \in Q' - \{u_1, u_q\}$.

It follows that $w, u_2, u_{l_2+1}, u_{q+2}$ are pairwisely nonadjacent by the choice of $Q'$ and $Q''$.

If $wu_2 \in E$ or $u_{q+2}w \in E$ or $wu_{l_2+1} \in E$ or $u_2 u_{l_2+1} \in E$, then there are two paths which contradict with the choice of $Q'$ and $Q''$. If $u_2 u_{q+2} \in E$, there are two paths $\overline{P'_1}[u_1, w] P'_3(w, u_{q+1}]$ and $\overline{Q'}[u_q, u_2] u_{q+2} Q''(u_{q+2}, u_t]$ which contradict with the choice of $Q'$ and $Q''$. If $u_{l_2+1} u_{q+2} \in E$, there are two paths $Q'[u_1, u_{l_2}] \overline{P'_2}[u_{l_2}, w] P'_3(w, u_{q+1}]$ and $\overline{Q'}[u_q, u_{l_2+1}] u_{q+2} Q''(u_{q+2}, u_t]$ which contradict with the choice of $Q'$ and $Q''$. Then two of these four vertices

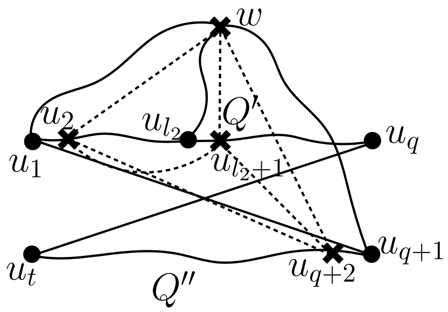



Figure 3.9: When $u_{l_3} = u_{q+1}$, $u_{l_2} \in Q' - \{u_1, u_q\}$

$w, u_2, u_{l_2+1}, u_{q+2}$ should be in the same parity $X_i$, for some $i \in \{1, 2, 3\}$.

If $w, u_2 \in X_i$ or $u_{q+2}, w \in X_i$ or $w, u_{l_2+1} \in X_i$, there is $(w, u_j)$-path $C - \{u_{j-1}u_j\} \cup P'_s[w, u_{j-1}]$ which contradicts the choice of $P$, where $j \in \{2, l_2+1, q+2\}$,

$$P'_s = \begin{cases} P'_1 & j = 2, \\ P'_2 & j = l_2 + 1, \\ P'_3 & j = q + 2. \end{cases} \tag{3.5}$$

If $u_2, w_{l_2+1} \in X_i$, there is $(u_2, u_{l_2+1})$-path $C - \{u_1u_2, u_{l_2+1}, u_{l_2}\} \cup P'_1[w, u_1] \cup \overline{P'_2}[w, u_{l_2}]$ which contradicts the choice of $P$. If $u_2, u_{q+2} \in X_i$, there are three paths $Q_1 = \overline{P'_1}[u_1, w]P'_3[w, u_{q+1}]$, $Q_2 = Q'[u_2, u_q]$ and $Q_3 = \overline{Q''}[u_t, u_{q+2}]$, by Section 3.3.2 , a contradiction. If $u_{l_2+1}, u_{q+2} \in X_i$, there are three paths $Q_1 = Q'[u_1, u_{l_2}]\overline{P'_2}(u_{l_2}, w]P'_3(w, u_{q+1}]$, $Q_2 = Q'[u_{l_2+1}, u_q]$ and $Q_3 = \overline{Q''}[u_t, u_{q+2}]$. It follows that $G$ is pancyclic from Section 3.3.2.

**Subcase 1.4** $u_{l_2} = u_{q+1}$, $u_{l_3} \in Q'' - \{u_{q+1}, u_t\}$.

It follows that $w, u_2, u_{q+2}, u_{l_3+1}$ are pairwisely nonadjacent by the choice of $Q'$ and $Q''$.

The proof of Subcase 1.4. is similar to the proof of Subcase 1.3. So again, let's skip the proof step. Thus, in

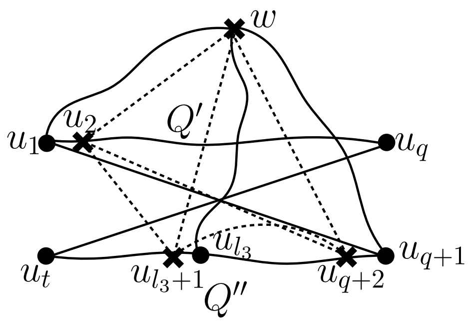



Figure 3.10: When $u_{l_2} = u_{q+1}$, $u_{l_3} \in Q'' - \{u_{q+1}, u_t\}$

Case 1 (in non-extremal case), $G$ is pancyclic or $G$ is a bipartite graph. Now let's talk about the extreme case, which is Case 2.

## 3.4 Extremal case

**Case 2** $v_{d_1} = v_1$ *and* $v_{d_3} = v_p$.

So, $\{v_1, v_p, v_{d_2}\}$ is cut-set of $G$ and let the component where $w^0$ is located be $H$.

Let's first show some properties of $H$.

### 3.4.1 Some properties of H

**Claim 3.4.1** *For any two vertices $x, y$ in $H$, then $x, y \in X_i$ for some $i \in \{1,2,3\}$. And there does not exist other components apart from $H$ and $P$.*

**Proof.** Without loss of generality, we assume $w^0 \in X_1$. Suppose that there is a vertex $u \in (H - w^0) \cap X_i$ with $i \in \{2,3\}$. It follows that $w^0, v_2, v_{d_2+1}$ are pairwisely nonadjacent by the choice of $P$. Similarly, $u, v_2, v_{d_2+1}$ are pairwisely nonadjacent. If there are at least two vertices of $w^0, v_2, v_{d_2+1}$ in the same parity $X_i$, by Case 1, we are done. So, $w^0, v_2, v_{d_2+1}$ should be in different parity $X_i$. Then there are two of $u, v_2, v_{d_2+1}$ should be in the same parity. By Case 1, we are done. So, for any two vertices of $H$ in the same $X_i$.

Suppose that there is another component $H'$ apart from $H$ and $P$, then $H$ and $H'$ are disconnected.

By the same argument with $H$, every vertex in $H'$ should be in the same parity of $X_i$ with $i \in \{1,2,3\}$. For $v \in H'$, there are three internal disjoint paths $P_i[w^0, v_{t_i}]$ connecting $v$ and three distinct vertices $v_{t_i} \in P$ with $i = 1,2,3$. If there are two vertices in $\{v_{t_1}, v_{t_2}, v_{t_3}\}$ that are not $\{v_1, v_p\}$, by Case 1, we are done. We assume $v_1 = v_{t_1}$ and $v_{t_3} = v_p$. Since $w^0, v_2$ and $v_{d_2+1}$ are in different parity $X_i$ for $i = 1,2,3$. Let $v_2 \in X_2$ and $v_{d_2+1} \in X_3$. Similarly, the vertices $v$, $v_2$ and $v_{t_2+1}$ should be in different parity $X_i$ with $i = 1,2,3$. If $v \in X_1$, then path $P^1[w^0, v_1]P[v_1, v_p]\overline{P_1}(v_p, v]$ contradicts the choice of $P$. So $v \in X_3$ and $v_{t_2+1} \in X_1$, then path $P^1[w^0, v_1]P(v_1, v_{t_2}]\overline{P_2}(v_{t_2}, v]P_3(v, v_p]\overline{P}(v_p, v_{t_2+1}]$ contradicts the choice of $P$ by $w^0 v_{t_2+1} \notin E$. So, there does not exist another component apart from $H$ and $P$. ∎

**Claim 3.4.2** *$H$ is a clique.*

**Proof.** Suppose $V(H) = \{u, v\}$, and $uv \notin E(G)$, by Claim 3.4.1 and the choice of $P$, a contradiction. Thus, suppose $|H| \geq 3$. Since $G$ is a 3-connected graph, then there are three vertices $x, y, z$ in $H$ such that $xv_1, v_p y, z v_{d_2+1} \in E$. Then $xy \in E$ otherwise there is a $(x, y)$-path which contradicts the choice of $P$ by Claim 3.4.1. Let $C_1 = P \cup \{xy, xv_1, yv_p\}$.

If there is a vertex $x' \in H$ such that $xx' \notin E$, then there are three internal disjoint paths $F_i[x', x_i]$ connecting $x'$ and three distinct vertices $x_i \in V(C_1)$ with $i = 1,2,3$. Since $\{v_1, v_{d_2}, v_p\}$ is cut-set of $G$, there is a vertex $x_i \in \{x_1, x_2, x_3\}$ such that $x_i \in \{y, v_1, v_p\}$. When $x_i = y$ or $x_i = v_p$, there is a $(x, x')$-path $xv_1 P v_p x_i \overline{F_i} x'$ which contradicts the choice of $P$. If $x_i = v_1$, there is a $(x, x')$-path $xyv_p\overline{P}v_1\overline{F_i}x'$, which contradicts the choice of $P$. By the symmetry between $x$ and $y$, so every vertex in $H$ connects with $x$ and $y$.

If there are two vertices $u', v' \in H$ such that $u'v' \notin E$, then $xu', yv' \in E$ and there is a $(u', v')$-path $u'xv_1Pv_pyv'$ which contradicts the choice of $P$. So, $H$ is a clique. ∎

By Claims 3.4.1 and 3.4.2, let $V(G) = V(H \cup P)$, $P = v_1 v_2 \cdots v_p$ and $N_P(V(H)) = \{v_1, v_d, v_p\}$.

**Claim 3.4.3** *If $V(H) \subseteq X_1$, then $V(P) \setminus \{v_1, v_d, v_p\} \subseteq X_2 \cup X_3$.*

**Proof.** Suppose there exists $v_i \in (V(P) \setminus \{v_1, v_d, v_p\}) \cap X_1$ , then $d(x) + d(v_i) \geq n$ for any $x \in V(H)$. So, there exists at most one vertex on $V(P) \setminus \{v_i\}$ which does not adjacent to $v_i$. If $v_i$ adjacent to every vertex in $V(P) \setminus \{v_i\}$, then it is easy to prove that $G$ is pancyclic. So, we assume $v_j \in V(P) \setminus \{v_i\}$ such that $v_iv_j \notin E(G)$.

Suppose $|V(H)| \geq 2$, let $u, v \in V(H)$ such that $uv_1, v_pv \in E$. By Claim 3.4.2, there are $(u, v)$-paths $P_{k'}$ of each length $k'$, $1 \leq k' \leq |V(H)| - 1$, in $H$.

If $i = 2$, then there is a path $v_2v_3 \cdots v_pvu$ which contradicts the choice of $P$.

If $3 \leq i \leq \frac{p+1}{2} - 1$ where $p$ is odd ($3 \leq i \leq \frac{p}{2} - 1$ where $p$ is even). Suppose that $i + 2 \leq j \leq p - 1$, then there are cycles $C_k$ with $3 \leq k \leq n$ in $G$: let $C_3 = v_iv_{i-1}v_{i-2}v_i$ and $C_4 = v_iv_{j-1}v_jv_{j+1}v_i$; for $1 \leq k' \leq |V(H)| - 1$,

$$C_k = \begin{cases} v_1v_2 \cdots v_{k-4}v_iv_pvuv_1 & \text{when } 5 \leq k \leq i+3, \\ v_1v_2 \cdots v_iv_{p-k+i+3}v_{p-k+i+4} \cdots v_pvuv_1 & \text{when } i+4 \leq k \leq p-j+i+2, \\ v_1v_2 \cdots v_{i-2}v_iv_{p-k+i+2}v_{p-k+i+3} \cdots v_pvuv_1 & \text{when } p-j+i+3 \leq k \leq p+1, \\ P \cup \{v_1u, v_pv\} \cup P_{k'} & \text{when } p+2 \leq k \leq n. \end{cases}$$

Suppose $j = p$, then there are cycles $C_k$ with $3 \leq k \leq n$ in $G$, for $1 \leq k' \leq |V(H)| - 1$.

$$C_k = \begin{cases} v_iv_{i+1} \cdots v_{k+i-1}v_i & \text{when } 3 \leq k \leq p-i, \\ v_1v_2 \cdots v_iv_{p-k+i+3} \cdots v_pvuv_1 & \text{when } p-i+1 \leq k \leq p+2, \\ P \cup \{v_1u, v_pv\} \cup P_{k'} & \text{when } p+2 \leq k \leq n. \end{cases}$$

Similarly, if $1 \leq j \leq i - 2$, then $G$ is pancyclic.

If $\frac{p+1}{2} + 1 \leq i \leq p - 1$ where $p$ is odd ($\frac{p}{2} + 1 \leq i \leq p - 1$ where $p$ is even), by the symmetry, $G$ is pancyclic.

If $i = \frac{p+1}{2}$, where $p$ is odd. Suppose that $2 \leq j \leq i - 2$, there are cycles $C_k$ with $3 \leq k \leq n$ in $G$, for $1 \leq k' \leq |V(H)| - 1$

$$C_k = \begin{cases} v_iv_{i+1} \cdots v_{i+k-1}v_i & \text{when } 3 \leq k \leq \frac{p+1}{2}, \\ uv_1v_iv_{p-k+5} \cdots v_pvu & \text{when } \frac{p+1}{2} + 1 \leq k \leq \frac{p+1}{2} + 3, \\ uv_1v_2 \cdots v_iv_{p-k+3+i} \cdots v_pvu & \text{when } \frac{p+1}{2} + 4 \leq k \leq p+2, \\ P \cup \{v_1u, v_pv\} \cup P_{k'} & \text{when } p+2 \leq k \leq n. \end{cases}$$

Suppose that $i + 2 \leq j \leq p - 1$, by the symmetry, $G$ is pancyclic.

Suppose that $j = 1$, there are cycles $C_k$ with $3 \leq k \leq n$ in $G$, for $1 \leq k' \leq |V(H)| - 1$

$$C_k = \begin{cases} v_i v_{i+1} \cdots v_{i+k-1} v_i & \text{when } 3 \leq k \leq \frac{p+1}{2}, \\ v_1 v_2 \cdots v_{k-4} v_i v_p v u v_1 & \text{when } \frac{p+1}{2} + 1 \leq k \leq \frac{p+1}{2} + 3, \\ u v_1 v_2 \cdots v_i v_{p-k+3+i} \cdots v_p v u & \text{when } \frac{p+1}{2} + 4 \leq k \leq p + 2, \\ P \cup \{v_1 u, v_p v\} \cup P_{k'} & \text{when } p + 2 \leq k \leq n. \end{cases}$$

Similarly, when $j = p$ and $i = \frac{p}{2}$ if $p$ is even, $G$ is pancyclic.

Suppose that $|V(H)| = 1$, let $u \in H$. By the choice of $Q'$ and $Q''$ in Case 1., $i \neq 2$ and $i \neq p - 1$. It is a similar argument with $|V(H)| \geq 2$, there are cycles $C_k$ with $3 \leq k \leq p - 1$ and $k = p + 1$. There is a cycle $C_p$ in $G$: if $j \neq i+2$, $C_p = v_1 v_2 \cdots v_i v_{i+2} v_{i+3} \cdots v_p u v_1$; if $j = i+2$, let $C_p = v_1 v_2 \cdots v_{i-2} v_i v_{i+1} \cdots v_p u v_1$, a contradiction. ∎

By Claim 3.4.3, let $V(H) \subseteq X_1$ and $V(P) \setminus \{v_1, v_d, v_p\} \subseteq X_2 \cup X_3$. By the choice of $P$ and Case 1, we have the following fact:

**Fact 3.4.4** *$v_2 v_{d+1}, v_{p-1} v_{d-1} \notin E$, $v_2, v_{d+1}$ are in different part $X_2, X_3$ and $v_{p-1}, v_{d-1}$ are in different part $X_2, X_3$.*

If $|V(P[v_2, v_{d-1}])| \leq 4$ and $|V(P[v_{d+1}, v_{p-1}])| \leq 4$, by the maximality of $P$, then $|H| \leq \min\{d - 2, p - d - 1\} \leq 4$. Then $n \leq 15$. And $d(v_1) + d(v_p) \geq n$, we can obtain $G$ is pancyclic or $G$ is a bipartite graph. In Appendix A, we will give a detailed proof of the following claim 3.4.5.

**Claim 3.4.5** *If $|V(P[v_2, v_{d-1}])| \leq 4$ and $|V(P[v_{d+1}, v_{p-1}])| \leq 4$, then $G$ is pancyclic or a bipartite graph.*

In the following, we prove that if two vertices with a distance of $2$ on $P[v_2, v_{d-3}]$ or a distance of $3$ on $P[v_2, v_{d-4}]$ are adjacent, and any two vertices on $P[v_{d+1}, v_{p-1}]$ are adjacent, then $G$ is pancyclic or a bipartite graph. So, we got the following result.

**Claim 3.4.6** *If for any $v_i \in V(P[v_2, v_{d-3}])$ and $v_j \in V(P[v_2, v_{d-4}])$ such that $v_i v_{i+2} \in E(G)$ and $v_j v_{j+3} \in E(G)$. And for any $v_k, v_l \in V(P[v_{d+1}, v_{p-1}])$, $v_k v_l \in E(G)$. Then $G$ is pancyclic or a bipartite graph.*

**Proof.** If $d \geq 7$ and $p - d \geq 3$. Then, we can construct all cycles $C_k$ with $3 \leq k \leq n$ in $G$.

Let $C_3 = v_2 v_3 v_4 v_2$ and $C_4 = v_2 v_3 v_4 v_5 v_2$.

When $5 \leq k \leq d - 2$, let $C_k = v_2 v_4 v_6 \cdots v_i v_{i+2} \cdots v_{k+1} v_k v_{k-2} \cdots v_j v_{j-2} \cdots v_2$ (if $k$ is odd) or $C_k = v_2 v_4 v_6 \cdots v_i v_{i+2} \cdots v_k v_{k+1} v_{k-1} \cdots v_j v_{j-2} \cdots v_2$ (if $k$ is even).

According to the number of vertices in $H$, we construct all cycles $C_k$ with $d - 1 \leq k \leq n$.

Suppose $|H| \geq 3$. we may assume $u, v, a \in V(H)$ such that $v_1 u, v_p v, v_d a \in E(G)$. By Claim 3.4.2, there are $(u, v)$-paths $P_l$ of each length $l$, $1 \leq l \leq |H| - 1$, in $H$.

When $k = d-1$, if $d \geq 8$, let $C_k = v_1v_2v_5v_7v_8\cdots v_dauv_1$; if $d = 7$, let $C_k = vv_pv_{p-1}v_{d+1}v_dav$. When $k = d$, let $C_k = v_1v_2v_5v_6\cdots v_dauv_1$. When $k = d+1$, let $C_k = v_1v_2v_4v_5\cdots v_dauv_1$. When $k = d+2$, let $C_k = v_1v_2v_3\cdots v_dauv_1$. When $k = d+3$, let $C_k = v_1v_2v_5v_6\cdots v_dv_{d+1}v_{p-1}v_pvuv_1$. When $k = d+4$, let $C_k = v_1v_2v_4v_5\cdots v_dv_{d+1}v_{p-1}v_pvuv_1$. When $k = d+5$, let $C_k = v_1v_2v_3\cdots v_dv_{d+1}v_{p-1}v_pvuv_1$. When $d+6 \leq k \leq p+2$, let $C_k = uv_1v_2v_3\cdots v_dv_{d+1}v_{p+d-k+4}v_{p+d-k+5}\cdots v_pvu$. When $p+3 \leq k \leq n$, let $C_k = P \cup \{uv_1, v_pv\} \cup P_l$ for $2 \leq l \leq |H|-1$.

If $|H| = 2$, since $G$ is a 3-connected graph, without loss of generality, we may assume $u, v \in V(H)$ such that $v_1u, v_pv, v_du \in E(G)$ and $vv_d \in E$ or $vv_1 \in E$. By Claim 3.4.2, the $uv \in E(G)$.

When $d+3 \leq k \leq n$, we can construct all cycles $C_k$, which are the same as when $|H| \geq 3$. When $k = d-1$, let $C_k = v_1v_2v_4v_6v_7\cdots v_duv_1$. When $k = d$, let $C_k = v_1v_2v_4v_5\cdots v_duv_1$. When $k = d+1$, let $C_k = v_1v_2v_3v_4\cdots v_duv_1$. When $k = d+2$, let $C_k = v_1v_2v_4v_5\cdots v_dvuv_1$ (if $vv_d \in E(G)$) or let $C_k = v_1v_2v_4v_5\cdots v_duvv_1$ (if $vv_1 \in E(G)$).

Suppose $V(H) = \{u\}$. Since $G$ is 3-connected graph, then $v_1u, v_pu, v_du \in E(G)$.

$$C_k = \begin{cases} v_1v_2v_5v_6\cdots v_duv_1 & \text{when } k = d-1, \\ v_1v_2v_4v_5\cdots v_duv_1 & \text{when } k = d, \\ v_1v_2v_3\cdots v_duv_1 & \text{when } k = d+1, \\ v_1v_2v_5v_6\cdots v_dv_{d+1}v_{p-1}v_puv_1 & \text{when } k = d+2, \\ v_1v_2v_4v_5\cdots v_dv_{d+1}v_{p-1}v_puv_1 & \text{when } k = d+3, \\ v_1v_2v_3\cdots v_dv_{d+1}v_{p-1}v_puv_1 & \text{when } k = d+4, \\ uv_1v_2v_3\cdots v_dv_{d+1}v_{p+d-k+3}v_{p+d-k+4}\cdots v_pvu & \text{when } d+5 \leq k \leq n. \end{cases}$$

If $d \geq 7$ and $p - d = 2$, by the maximality of $P$, then $|H| = 1$. The same argument with above, it is easy to construct $G$ is pancyclic.

If $d \leq 6$ and $p - d \geq 6$, then $p - (d-1) \geq 7$. Since for any $v_i \in V(P[v_2, v_{d-3}])$ and $v_j \in V(P[v_2, v_{d-4}])$ such that $v_iv_{i+2} \in E(G)$ and $v_jv_{j+3} \in E(G)$. It follows from $d \leq 6$ that for any $v_i, v_j \in V(P[v_2, v_{d-1}])$ such that $v_iv_j \in E(G)$. Because for any $v_k, v_l \in V(P[v_{d+1}, v_{p-1}])$ such that $v_kv_l \in E(G)$, so the same argument with $d \geq 7$. Thus, we can construct all cycles $C_k$, for $3 \leq k \leq n$, in $G$.

If $d \leq 6$ and $p - d \leq 5$, by Claim 3.4.5, then $G$ is pancyclic or $G$ is a bipartite graph. ∎

According to the number of vertices in $V(H)$, we go ahead and prove the rest of the proof.

### 3.4.2 H has at least three vertices

In this section, we will show that if $|V(H)| \geq 3$, then $G$ is pancyclic or $G$ is a bipartite graph.

Let $G' = G - (H \cup \{v_1, v_d, v_p\})$ be a subgraph of $G$. We may assume $u, v, w' \in V(H)$ such that $uv_1, vv_p, w'v_d \in E$. By Claim 3.4.3, then $V(G') \subseteq X_2 \cup X_3$. For any two nonadjacent vertices $x, y \in X_i$ with $i \in \{2, 3\}$, we get $d_{G'}(x) + d_{G'}(y) \geq d(x) + d(y) - 6 \geq |G'|$.

When $G'$ is a 2-connected graph, by Theorem 2.0.3, $G'$ is pancyclic or $G' = K_{|G'|/2,|G'|/2}$ or $G' = K_{|G'|/2,|G'|/2} - \{e\}$.

Suppose $G' = K_{|G'|/2,|G'|/2}$ or $G' = K_{|G'|/2,|G'|/2} - \{e\}$. Let $X$ and $Y$ be the bipartitions of $G'$. If $v_2, v_{d+1} \in X$, then $v_3v_{d+1} \in E$ or $v_2v_{d+2} \in E$. If $v_2 \in X$ and $v_{d+1} \in Y$, then $v_3v_{d+2} \in E$. In the both cases, there is a $(v_1, v_p)$-path which contradicts the choice of $P$. So, $G'$ is pancyclic, and we assume there are cycles $C_k$, $3 \leq k \leq p-3$, in $G$. Suppose there does not exist cycles $C_m$ with $p - 2 \leq m \leq n$ in $G$. By Claim 3.4.5, we can assume $|V(P[v_2, v_{d-1}])| \geq 5$.

If $v_2, v_{p-1} \in X_2$, by Fact 3.4.4, then $v_{d-1}, v_{d+1} \in X_3$. Since $|H| \geq 3$, then $v_{d-1}v_{d+1} \in E$ otherwise there is a $(v_{d-1}, v_{d+1})$-path $v_{d+1}v_{d+2}\cdots v_pvuv_1v_2\cdots v_{d-1}$ which contradicts the choice of $P$. By the maximality of $P$, then $|V(P[v_{d+1}, v_{p-1}])| \geq 4$.

Then $v_4v_{d+1} \notin E(G)$ otherwise path $P_1 = v_1uvwv_dv_{d-1}v_{d-2}\cdots v_4v_{d+1}v_{d+2}\cdots v_p$ contradicts with the choice of $P$. If $v_4 \in X_3$, then there are two paths $Q_1 = v_1uvw'v_dv_{d-1}\cdots v_4$ and $Q_2 = v_{d+1}v_{d+2}\cdots v_p$ such that $|Q_1| + |Q_2| \geq p + 1$. By Case 1, we have done. So, $v_4 \in X_2$, then $v_2v_4 \in E$ by the choice of $P$. Similarly, $v_{d-2} \in X_3$ and $v_{d-2}v_{d+1}, v_{p-1}v_{p-3} \in E$. Then let

$$C_{p-2} = v_1v_2v_4\cdots v_{d-2}v_{d+1}\cdots v_{p-3}v_{p-1}v_pvuv_1,\ C_{p-1} = v_1v_2v_4\cdots v_{d-2}v_{d+1}\cdots v_pvuv_1,$$

$$C_p = v_1v_2\cdots v_{d-2}v_{d+1}\cdots v_pvuv_1,\ C_{p+1} = P - \{v_d\} \cup \{v_{d-1}v_{d+1}, v_1u, vv_p, vu\}.$$

By Claim 3.4.2, then there are cycle $C_m$ with $n \geq m \geq p + 2$, a contradiction. So, we assume $v_2, v_{d-1} \in X_2$ and $v_{d+1}, v_{p-1} \in X_3$.

By the choice of $P$ and Case 1, then $v_4v_{d+1} \notin E$, $v_4 \in X_2$ and $v_2v_4 \in E$. Similarly, $v_{p-3}v_{p-1} \in E$, $v_{d-3}v_{d-1} \in E$ and $v_{d+3}v_{d+1} \in E$ ($v_{p-1}v_{d+1} \in E$). In the same argument with $v_2, v_{p-1} \in X_2$, we can construct all cycles $C_k$, with $n \geq m \geq p - 2$. Then $G$ is pancyclic, a contradiction. So, the connectivity of $G'$ is 1. Let $w_1$ cuts $G'$ into $G_1$ and $G_2$.

It follows that $|V(P[v_2, v_{d-1}])| \geq 5$ or $|V(P[v_{d+1}, v_{p-1}])| \geq 5$ from Claim 3.4.5. By Lemma 3.2.4 and Fact 3.4.4, we can assume $V(G_1) \subseteq X_2$, $V(G_2) \subseteq X_3$, $w_1 \in X_3$ and $G_1$ is a clique, and $v_2 \in X_2$ and $v_{d+1} \in X_3$. When $v_2v_i \in E$ ($i \leq d - 1$ and $i$ is as large as possible), then $v_{i-1}v_{d+1} \notin E$ otherwise path $v_1uw'v_dv_{d-1}\cdots v_iv_2v_3\cdots v_{i-1}v_{d+1}v_{d+2}\cdots v_p$ contradicts the choice of $P$. If $v_{i-1} \in X_3$, there are two paths $Q^1 = v_{i-1}v_{i-2}\cdots v_2v_iv_{i+1}\cdots v_dw'uv_1$ and $Q^2 = v_{d+1}v_{d+2}\cdots v_p$ such that $|Q^1| + |Q^2| \geq p + 2$, by the Case 1, we have done. So $v_{i-1} \in X_2$ and $G[P[v_2, v_{i-1}]]$ is a clique.

If $v_2v_j \in E$ ($d+2 \leq j \leq p-1$ and $j$ is as small as possible), similarly $G[P[v_{j+1}, v_{p-1}]]$ is a clique. Since $w_1$ is a cut-vertex of $G'$, then $G[P[v_{i+1}v_{d-1}]]$ and $G[P[v_{d+1}, v_{j-1}]]$ are disconnected. So, $v_{d-1}v_{d+1} \notin E$. By the choice of $P$, then $v_{d-1} \in X_2$. So, $G[P[v_2, v_{d-1}] \cup P[v_{j+1}, v_{p-1}]]$ is a clique. However, $v_{d-1}v_{p-1} \notin E$, then $v_{p-1} = v_j$. By the choice of $P$, for any vertex $v_l \in P[v_{d+1}, v_{j-3}]$ such that $v_lv_{l+2}, v_lv_{l+3} \in E$. By Claim 3.4.6, $G$ is pancyclic or a bipartite graph. So, $v_2v_j \notin E$ (for any $j \geq d+2$) and $P[v_{d+1}, v_{p-1}] \subseteq X_3$. And for any vertex $v_l \in P[v_{d+1}, v_{p-1}]$ such that $v_lv_{l+2}, v_lv_{l+3} \in E$.

If $P[v_{i+1}, v_{d-1}] \subseteq X_3$, by the choice of $P$, then $v_{d-1}v_{d+1}, v_{d-1}v_{d+2} \in E$ and $V(P[v_{i+1}, v_{d-1}]) \subseteq N(v_{d+1})$. For any vertex $v_l \in P[v_{i+1}, v_{d-1}] \cup P[v_{d+1}, v_{p-1}]$ such that $v_lv_{l+2}, v_lv_{l+3} \in E$, by the same argument with Claim 3.4.6, this is a contradiction. Then $V(P[v_2, v_{d-1}]) \subseteq X_2$ and $G[P[v_2, v_{d-1}]]$ is a clique. By Claim 3.4.6, then $G$ is pancyclic or a bipartite graph.

When $G'$ is disconnected, let $G_1 = G[P[v_2, v_{d-1}]]$ and $G_2 = G[P[v_{d+1}, v_{p-1}]]$. By the degree sum condition, we assume $V(G_1) \subseteq X_2$, $V(G_2) \subseteq X_3$ and $G_2$ is a clique. By the choice of $P$, then $v_iv_{i+2} \in E(G)$ and $v_iv_{i+3} \in E(G)$ for $v_i \in V(P[v_2, v_{d-3}])$. By Claim 3.4.6, then $G$ is pancyclic or a bipartite graph.

Thus, if $|V(H)| \geq 3$, then $G$ is pancyclic or $G$ is a bipartite graph.

### 3.4.3 H has two vertices

In this section, we will show that if $|V(H)| = 2$, then $G$ is pancyclic or $G$ is a bipartite graph.

In this case, let $V(H) = \{u, v\}$, $uv_1, vv_p \in E$ and $G' = G - (H \cup \{v_1, v_p\})$. Put $W_1 = \{v_d\}$, $W_2 = X_2 - \{v_d\}$ and $W_3 = X_3 - \{v_d\}$. For any two nonadjacent vertices $x, y \in W_i$ with $i = 1, 2, 3$, we can obtain

$$d_{G'}(x) + d_{G'}(y) \geq d(x) + d(y) - 4 \geq |G'|. \tag{3.6}$$

When $G'$ is a 3-connected graph, by the minimality of $G$, then there are cycles $C_k$ with $3 \leq k \leq n-4$ in $G'$ ( or $G$). By Theorem 2.0.1, there is a cycle $C_n$ in $G$.

Let $C' = u_1u_2 \cdots u_{p'}$ and $P' = v_2v_3 \cdots v_{p-1}$ be hamiltonian cycle and hamiltonian path of $G'$, respectively, where $u_i \in V(G')$ and $p' = p-2$. So, $u_i$ is a certain $v_j$ in $V(G')$. Next, we will show that there are cycles $C_k$ with $n-3 \leq k \leq n-1$ in $G$.

If $d_{P'}(v_1) + d_{P'}(v_p) \geq |P'| + 2$. Let $G^* = G - H$, then $P$ is hamiltonian $(v_1, v_p)$-path in $G^*$. By Theorem 2.1.2, there are cycles $C_{p-1}$ (i.e., $C_{n-3}$) and $C_p$ (i.e., $C_{n-2}$) in $G$.

Suppose there does not exist a cycle $C_{p+1}$. Then $uv_p, vv_1, v_{d-1}v_{d+1} \notin E$ and for any $v_i \in V(P[v_2, v_{p-2}])$, $v_iv_{i+2} \notin E$. Then $v_i$ and $v_{i+2}$ are in different part $W_j$ with $j \in \{1, 2, 3\}$, otherwise there is a path $v_iv_{i-1} \cdots v_1uvv_pv_{p-1} \cdots v_{i+2}$

which contradicts the choice of $P$. Without loss of generality, we may assume $v_{d-1} \in W_2$ and $v_{d+1} \in W_3$. So, by Fact 3.4.4, $v_2 \in W_2$ and $v_{p-1} \in W_3$. Since $G$ is 3-connected, then $uv_d, vv_d \in E$. By the choice of $P$ and Case 1, then $v_3 \in W_2$. By Claim 3.4.5, we can assume $|V(P[v_{d+1}, v_{p-1}])| \geq 5$.

If $d-1 \geq 8$, since $v_i$ and $v_{i+2}$ are in different part $W_j$ with $j \in \{1,2,3\}$, then $v_3, v_6 \in W_2$ and $v_3v_6 \in E$ otherwise $Q = v_6v_7 \cdots v_pvuv_1v_2v_3$ such that $|P| = |Q|$ and $V(H) = \{v_4, v_5\}$, this contradicts Claim 3.4.3. Similarly, $v_8v_5 \in E$. By the choice of $P$, then $v_7v_3 \in E$. So, $C_{p+1} = v_1v_2v_3v_7v_6v_5v_8v_9 \cdots v_pvuv_1$.

If $d-1 = 7$, the same argument with $v_3v_6 \in E$, $v_{d+1}v_5 \in E$. By the choice of $P$, $v_7v_3 \in E$. Then $C_{p+1} = v_1v_2v_3v_7v_6v_5v_{d+1}v_{d+2} \cdots v_pvv_duv_1$. If $d-1 = 6$, the same argument with $v_3v_6 \in E$, $v_5v_{d+1} \in E$. Then $C_{p+1} = v_1v_2v_3v_6v_5v_{d+1}v_{d+2} \cdots v_pvv_duv_1$. So, this contradicts that $G$ is not pancyclic.

So, $|P'| \leq d_{P'}(v_1) + d_{P'}(v_p) \leq |P'| + 1$. We can assume $uv_1, uv_p \in E$. Then there is cycle $C_{n-1} = P \cup \{uv_1, uv_p\}$ in $G$. Suppose there does not exist cycle $C_m$ with $m = n-2, n-3$.

Suppose that $m = n-3$. If $p'$ is odd, it follows that $u_iv_1, u_{i+1}v_1 \in E$ or $u_iv_p, u_{i+1}v_p \in E$ from $d_{C'}(v_1) + d_{C'}(v_p) \geq p'$, then it is easy to construct the cycle $C_{n-3}$ in $G$. So $p'$ is even. When $d_{C'}(v_1) \geq \frac{p'}{2} + 1$ or $d_{C'}(v_p) \geq \frac{p'}{2} + 1$, we also obtain $u_iv_1, u_{i+1}v_1 \in E$ or $u_iv_p, u_{i+1}v_p \in E$. So, $d_{C'}(v_1) = d_{C'}(v_p) = \frac{p'}{2}$, exactly one of the two edges $u_iv_1$ and $u_{i+1}v_1$ does exist. If $N_{C'}(v_1) = N_{C'}(v_p) = \{u_1, u_3, \ldots, u_{p'-1}\}$ or $N_{C'}(v_1) = N_{C'}(v_p) = \{u_2, u_4, \ldots, u_{p'}\}$, then $C_{n-3} = u_1u_2 \cdots u_iv_1uvv_pu_{i+4} \cdots u_{p'}u_1$. Without loss of generality, $N_{C'}(v_1) = \{u_1, u_3, \ldots, u_{p'-1}\}$ and $N_{C'}(v_p) = \{u_2, u_4, \ldots, u_{p'}\}$, then $C_{n-3} = u_1u_2 \cdots u_{p'-3}v_1uv_pu_{p'}u_1$.

So, $m = n-2$. Since $G$ is a 3-connected graph and Claim 3.4.5, we can assume $vv_d \in E$ and $|V(P[v_2, v_{d-1}])| \geq 5$. There does not exist cycle $C_{n-2}$, then for any $v_i \in V(P[v_2, v_{p-2}])$, $v_iv_{i+2} \notin E$. By the choice of $P$, $v_i$ and $v_{i+2}$ are in different part $W_j$ with $j \in \{1,2,3\}$, $|V(P[v_{d+1}, v_{p-1}])| \geq 2$. So, we can assume $v_{d-1}, v_2 \in W_2$ and $v_{d+1}, v_{p-1} \in W_3$. The same argument with Fact 3.4.4, then $v_3, v_6 \in W_2$ and $v_3v_6 \in E$ otherwise $Q = v_6v_7 \cdots v_pvuv_1v_2v_3$ such that $|P| = |Q|$ and $H = \{v_4, v_5\}$, this contradicts Claim 3.4.3. So, $C_m = v_1v_2v_3v_6v_7 \cdots v_puv_1$, this is a contradiction.

Suppose that the connectivity of $G'$ is 2 and $\{v_i, v_j\}$ is a cut-set that cuts $G'$ into $G_1$ and $G_2$. Let $P' = v_2v_3 \cdots v_{p-1}$ be a path of $G'$. Assume $|G_1| = n_1$ and $|G_2| = n_2$.

Suppose that $G_1 \cap W_i \neq \emptyset$ and $G_2 \cap W_i \neq \emptyset$ for any $i = 2, 3$. The similar with Lemma 3.2.5, $G'$ is pancyclic. The same argument with $G'$ is 3-connected, $G$ is pancyclic.

Suppose that $G_1 \cap W_2 \neq \emptyset$ and $G_1 \cap W_3 \neq \emptyset$, $G_2 \cap W_2 \neq \emptyset$ and $G_2 \cap W_3 = \emptyset$. By (3.6), then we have the following:

**Fact 3.4.7** *For any vertex $x \in W_2 \cap G_2$ and $y \in W_2 \cap G_1$, $N(x) = G_2 \cup \{v_i, v_j\}$ and $N(y) = G_1 \cup \{v_i, v_j\}$.*

Next, we will show if $|G_2| \geq 2$ and $G_1$ is pancyclic graph, then $G$ is pancyclic.

**Proposition 3.4.8** *If $|G_2| \geq 2$ and $G_1$ is pancyclic graph, then $G$ is pancyclic.*

**Proof.** Let $C = u_1u_2 \cdots u_{n_1}u_1$ be a hamiltonian cycle of $G_1$. Assume $u_1 \in W_2 \cap G_1$ and $u_jv_j \in E$. We will show that there exists a hamiltonian cycle $C''$ in $G_1$ such that $u_1u_j \in E(C'')$. Suppose there does not exist a hamiltonian cycle $C''$ in $G_1$ such that $u_1u_j \in E(C'')$. Then $u_2u_{n_1} \notin E(G)$ otherwise $C'' = u_1u_ju_{j-1}\cdots u_2u_{n_1}u_{n_1-1}\cdots u_{j+1}u_1$. So, by Fact 3.4.7, $u_2, u_{n_1} \in W_3$. If $u_2v_j \in E$ ($u_{n_1}v_j \in E(G)$), then $u_j = u_2$ ($u_{n_1} = u_j$). This is a contradiction. Thus, by (3.6), $d_{G_1}(u_2) + d_{G_1}(u_{n_1}) \geq |G_1| + 2$. Let $P_1 = C[u_3, u_j]$ and $P_2 = C[u_{j+1}, u_{n_1-1}]$. If $\exists u_i \in P_2$ such that $u_iu_{n_1}, u_{i+1}u_2 \in E$, then $C'' = u_1u_ju_{j-1}\cdots u_2u_{i+1}u_{i+2}\cdots u_{n_1}u_iu_{i-1}\cdots u_{j+1}u_1$. This is a contradiction. By Lemma 3.2.1, $d_{P_2}(u_2) + d_{P_2}(u_{n_1}) \leq |P_2| + 1$. Similarly, $d_{P_1}(u_2) + d_{P_1}(u_{n_1}) \leq |P_1| + 1$. So, $d_{G_1}(u_2) + d_{G_1}(u_{n_1}) \leq |G_1| + 1$, a contradiction. So, there exists a hamiltonian cycle $C''$ in $G_1$ such that $u_1u_j \in E(C'')$.

Then, by Fact 3.4.7, it is easy to construct cycles $C_k$ of length $3 \leq k \leq n$ in $G$. ∎

If $v_d \in G_2$, then $|G_2| \geq 2$ and $d_{G_1}(x) + d_{G_1}(y) \geq |G'| - 4 \geq |G_1|$ for any pair of nonadjacent vertices $x, y \in G_1$. By Theorem 2.0.4, Fact 3.4.7 and Proposition 3.4.8, $G$ is pancyclic.

If $v_d \in G_1$. When $W_2 \cap G_1 = \{x'\}$ is cut-set and cuts $G_1$ into $G_1^1$ and $G_1^2$. If $W_3 \cap G_1^a \neq \emptyset$ with $a = 1, 2$, by (3.6), then $|G_2| = 1$. For any $x \in W_3 \cap G_1^a$, $N(x) = V(G_1^a) \cup \{x', v_i, v_j\}$ with $a = 1, 2$, and $G_1^1$ and $G_1^2$ are cliques. Assume $G^* = G[V(G_1) \cup \{v_i\}]$, then $\{v_i, x'\}$ cuts $G^*$ into $G_1^1$ and $G_1^2$. So, $G^*$ is pancyclic. By (3.6), $G$ is pancyclic. Under the definition of $G_1$, $G_1^1$ and $G_1^2$, $x', W_2, W_3$, we obtain the following:

**Proposition 3.4.9** *If $W_3 \cap G_1^a \neq \emptyset$ with $a = 1, 2$, $G$ is pancyclic.*

If $V(G_1^1) = \{v_d\}$. When $v_{d-1} = v_i$ and $v_{d+1} = v_j$, by the choice of $P$ and Fact 3.4.7, $v_2, v_{p-1} \in W_3$, this contradicts the definition $P$. When $x' \in \{v_{d-1}, v_{d+1}\}$, this contradicts Fact 3.4.4.

When $G_1$ is a $2$-connected graph, let $M_1 = (W_2 \cap V(G_1)) \cup \{v_d\}$ and $M_2 = W_3$. By Fact 3.4.7 and Theorem 2.0.3, $G_1$ is pancyclic. When $|G_2| \geq 2$, by Proposition 3.4.8, we can obtain $G$ is pancyclic. Under the definition of $G_1, W_2, W_3, v_i, v_j$, we obtain the following:

**Proposition 3.4.10** *If $|V(G_2)| = 1$, let $V(G_2) = \{w_1\}$, then $G$ is pancyclic.*

**Proof.** Assume $i < j$ and $w \in \{u, v\}$ or $w = uv$.

When $v_{p-1} \in G_1$, suppose $v_2 \neq w_1$. We can assume $d \geq j + 1$. By Facts 3.4.4 and 3.4.7, then $v_2 = v_i$. Similarly, $v_j = v_{d-1} = v_4$, $w_1 = v_3$ and $v_{d+1}, v_{p-1} \in W_3$. By Fact 3.4.7, there exists a vertex $v_l \in P[v_{d+2}, v_{p-2}] \cap W_2 \cap G_1$ such that $v_lv_d, v_lv_i \in E$. If $v_{d+1}v_{l+1} \notin E$, then $v_{l+1} \in W_3$ and path $v_{d+1}v_{d+2}\cdots v_lv_dv_{d-1}\cdots v_1wv_pv_{p-1}\cdots v_{l+1}$ contradicts the choice of $P$. So, $v_{d+1}v_{l+1} \in E$. Then path $v_1wv_dv_jw_1v_iv_lv_{l-1}\cdots v_{d+1}v_{l+1}v_{l+2}\cdots v_p$ contradicts the choice of $P$.

So, $v_2 = w_1$. If $j \geq d + 1$, by Facts 3.4.4 and 3.4.7, $v_i = v_{d-1} \in W_2$ and $v_{d+1}, v_{j+1} \in W_3$. Then $v_{d+1}v_{j+1} \in E$ otherwise $R = v_{d+1}v_{d+2}\cdots v_jw_1v_iv_dwv_pv_{p-1}\cdots v_{j+1}$, when $|R| > |P|$, a contradiction. When $|R| = |P|$ and $v_1 \in V(H)$, since $G$ is $3$-connected, by Case 1, we are done. So, path $v_1wv_dv_iw_1v_jv_{j-1}\cdots v_{d+1}v_{j+1}\cdots v_p$ contradicts the choice of $P$. If $j \leq d - 1$, it follows that $v_j = v_{d-1}$ from Facts 3.4.4 and 3.4.7. If there exists a vertex $v_l \in P[v_{d+2}, v_{p-2}] \cap W_2 \cap G_1$, the same with above, then $v_{d+1}v_{l+1} \in E$. So, $R_1 = v_1wv_dv_j\cdots v_iv_lv_{l-1}\cdots v_{d+1}v_{l+1}v_{l+2}\cdots v_p$, similarly argument with $R$, a contradiction. So, by $G_1 \cap W_2 \neq \emptyset$, then there exists a vertex $v_{l'} \in P[v_4, v_{d-2}] \cap W_2 \cap G_1$

such that $v_{l'}v_{p-1}, v_{l'}v_{d+2} \in E$. If $v_{l'-1} = v_i$, then path $T = v_1w_1v_jv_{j-2}\cdots v_lv_{p-1}v_{p-2}\cdots v_dwv_p$, similarly argument with $R$, a contradiction. Then $v_{l'-1}v_{d+1} \in E$ otherwise $v_{l'-1} \in W_3$ and path $v_{l'-1}\cdots v_1wv_p\cdots v_{d+2}v_{l'}v_{l'+1}\cdots v_{d+1}$ contradicts the choice of $P$. So, there is a path $v_1v_2\cdots v_{l'-1}v_{d+1}\cdots v_{p-1}v_{l'}\cdots v_dwv_p$ which contradicts the choice of $P$.

When $v_{p-1} = w_1$ or $v_j = v_{p-1}$, the proof is similar to the proof of $v_{p-1} \in G_1$. So the proof of this proposition is complete. ∎

When $v_d = v_i$ or $v_d = v_j$, if $|G_2| \geq 2$, by (3.6), Theorem 2.0.4 and Proposition 3.4.8, $G$ is pancyclic. If $|G_2| = 1$, the same argument with Proposition 3.4.10, $G$ is pancyclic.

Suppose that $V(G_1) \subseteq W_2 \cup \{v_d\}$ and $V(G_2) \subseteq W_3 \cup \{v_d\}$. When $|G_1 \cap W_2| \geq 2$ and $|G_2 \cap W_3| \geq 2$, by (3.6), we can assume $G_2$ is a clique.

If $v_d \in G_1$. When $v_2, v_{p-1} \in G_1$, then $v_{d-1} \in W_2$ or $v_{d+1} \in W_2$. By Fact 3.4.4, a contradiction. When $v_2 \in G_1$ and $v_{p-1} = v_i$ or $v_{p-1} = v_j$, by Fact 3.4.4, then $P[v_2, v_{d-1}] \subseteq W_2$, $P[v_{d+1}, v_{p-2}] \subseteq W_3$ and $v_{d+1} \in \{v_i, v_j\}$. So, $G(V(P[v_{d+1}, v_{p-2}]))$ is a clique. By the choice of $P$, then $v_lv_{l+2}, v_lv_{l+3} \in E$ for any $2 \leq l \leq d-2$, and $yv_i, yv_j \in E(G)$ for any $y \in V(P[v_{d+2}, v_{p-2}])$. Since $G$ is $3$-connected graph, then there is a vertex $v_h \in P[v_{d+2}, v_{p-2}]$ such that $v_hv_p \in E(G)$ or $v_hv_1 \in E(G)$. We can assume $v_hv_p \in E(G)$. So, $v_{d+1}v_{p-1} \in E(G)$ otherwise $v_{p-1}v_{p-2}\cdots v_{h+1}v_{d+2}v_{d+3}\cdots v_hv_pvuv_1v_2\cdots v_{d+1}$ is a path which contradicts the maximality of $P$. Hence, $G[V(P[v_{d+1}, v_{p-1}])]$ is a clique. By Claim 3.4.6, $G$ is pancyclic. So, we can obtain the following fact:

**Fact 3.4.11** *If $v_{d+1} = v_i$, $v_{p-1} = v_j$, $V(P[v_{d+1}, v_{p-1}]) \subseteq W_3$ and $V(P[v_{d+2}, v_{p-2}]) = V(G_2)$, then $G[V(P[v_{d+1}, v_{p-1}])]$ is a clique.*

When $v_2 \in G_1$ and $v_{p-1} \in G_2$, we can assume there exists $v_a \in P[v_{d+1}, v_{p-1}]$ such that $P[v_2, v_{d-1}] \cup P[v_{d+2}, v_a] \subseteq W_2$ and $P[v_{a+1}, v_{p-1}] \subseteq W_3$. By the choice of $P$, for $v_l \in P[v_2, v_{d-3}] \cup P[v_{d+1}, v_{a-2}]$, then $v_lv_{l+2} \in E$ and $v_lv_{l+3} \in E$ otherwise a $(v_l, v_{l+3})$-path $P_1$ such that $|P_1| = |P|$ and $H = \{v_{l+1}, v_{l+2}\}$, by Claim 3.4.3 and $v_{l+1}, v_{l+2} \in W_2$, a contradiction. Similarly, for any $v_b \in P[v_{d+2}, v_a]$ and $v_c \in P[v_2, v_{d-1}]$ such that $v_bv_c \in E(G)$. The similar to Claim 3.4.6, $G$ is pancyclic.

Similarly, when $v_2 \in G_2$ and $v_{p-1} = v_i$ ($v_{p-1} = v_j$), or when $v_2, v_{p-1} \in G_2$, then $G$ is pancyclic.

The same argument with $v_d \in G_1$, if $v_d = v_j$, then $G$ is pancyclic. When $|G_1 \cap W_2| = 1$ or $|G_2 \cap W_3| = 1$, by Claim 3.4.6, $G$ is pancyclic.

When $z$ cuts $G'$ into $G_1$ and $G_2$. By Lemma 3.2.4, we assume $G_1 \subseteq W_2 \cup \{v_d\}$, $G_2 \subseteq W_3 \cup \{v_d\}$ and $G_2$ is a clique. Suppose that $v_2 \in G_1$, $v_{p-1} \in G_2$. When $z \neq v_d$, let $v_d \in G_1$. By Fact 3.4.4, $z = v_{d+1} \in W_3$. By the choice of $P$, $zv_{d+3} \in E$ and for any vertex $v_i$ with $2 \leq i \leq d-2$, $v_iv_{i+2}, v_iv_{i+3} \in E$. By Claim 3.4.6, $G$ is pancyclic. Similarly, if $v_d = z$, $G$ is pancyclic.

### 3.4.4 H has only one vertex

In this section, we will prove if $|H| = 1$, assume $V(H) = \{w\}$, then $G$ is pancyclic or $G$ is a bipartite graph.

First, we show that there is a cycle $C_p$ in $G$. Suppose there does not exist a cycle $C_p$, then $v_1v_3, v_pv_{p-2} \notin E$. Let $P' = v_2v_3 \cdots v_{p-1}$, then $(N_{P'}(v_1) - \{v_2\})^- \cap N_{P'}(v_p) = \emptyset$. Since $d(v_1) + d(v_p) \geq n$, by Lemma 3.2.1, $d_{P'}(v_1) + d_{P'}(v_p) = |P'| + 1$ and $v_1v_{p-1}, v_pv_2 \in E$.

If $v_1v_i, v_1v_{i+1} \in E$ with $3 \leq i \leq p-1$, then $C_p = v_2v_3 \cdots v_iv_1v_{i+1}v_{i+2} \cdots v_pv_2$, a contradiction. Similarly, $v_iv_p \notin E$ or $v_pv_{i+1} \notin E$. By lemma 3.2.1 (5), $|P'|$ is odd and $N_{P'}(v_1) = N_{P'}(v_p) = \{v_2, v_4, \ldots, v_{p-1}\}$. Let $B = \{v_3, v_5, \ldots, v_{p-2}\}$. If there exist $v_i, v_j \in B$ such that $v_iv_j \in E$, then $C_p = v_pv_{j+1} \cdots v_{p-1}v_1v_{i+1}v_{i+2} \cdots v_jv_iv_{i-1} \cdots v_2v_p$. So, $B$ is an independent set. By Claim 3.4.5, we can assume $|B| \geq 4$, So, there exist $v_l, v_j \in B$ such that $v_l, v_j \in X_i$ with $i = 1, 2, 3$. So, $n \leq d(v_l) + d(v_j) \leq \frac{p-1}{2} + \frac{p-1}{2} + 1 = n - 1$, this is a contradiction. Thus, there exist the cycle $C_p$.

Next, we suppose $v_d$ is adjacent to at least one of $v_1$ and $v_p$, then we will show $G$ is pancyclic or $G$ is a bipartite graph. Without loss of generality, we assume $v_1v_d \in E$.

Put $G' = G - \{w, v_p\}$ and $W_1 = \{v_1, v_d\}$, $W_2 = X_2 - \{v_1, v_d\}$, $W_3 = X_3 - \{v_1, v_d\}$. For any two nonadjacent vertices $x, y \in W_i$

$$d_{G'}(x) + d_{G'}(y) \geq d(x) + d(y) - 2 \geq |G'|. \tag{3.7}$$

When $G'$ is 3-connected, by the minimality of $G$, then $G$ is pancyclic. If $x$ is a cut-set of $G'$, by $v_1v_d \in E$, then $\{v_p, x\}$ is a 2-cutset of $G$. This contradicts $G$ is 3-connected. So, we assume the connectivity of $G'$ is 2 and $\{v_i, v_j\}$ cuts $G'$ into $G_1$ and $G_2$.

Suppose $G_1 \cap W_i \neq \emptyset$ and $G_2 \cap W_i \neq \emptyset$ with $i = 2, 3$, by Lemma 3.2.5, then $G$ is pancyclic.

Suppose $G_1 \cap W_2 \neq \emptyset$ and $G_1 \cap W_3 \neq \emptyset$ and $G_2 \cap W_2 \neq \emptyset$ and $G_2 \cap W_3 = \emptyset$.

If $v_1, v_d \in G_1$, when $G_1$ is 1-connected, let $\{x'\} = V(G_1) \cap W_2$ be a cut-set and cuts $G_1$ into $G_1^1$ and $G_1^2$. If $W_3 \cap G_1^a \neq \emptyset$ with $a = 1, 2$, by Proposition 3.4.9, then $G$ is pancyclic. If $G_1^1 = \{v_1, v_d\}$, then $v_{d-1}, v_{d+1}, v_2 \in \{x', v_i, v_j\}$. By Facts 3.4.4 and 3.4.7, $x' \notin \{v_2, v_{d+1}\}$, $x' = v_{d-1}$ and $v_{p-1} \in W_3$. By the definition of $P$, this is a contradiction. When $G_1$ is 2-connected, let $M_1 = V(G_1) - \{v_1, v_d\}$ and $M_2 = \{v_1, v_d\}$. When $|G_2| \geq 2$, by Fact 3.4.7, Theorem 2.0.3 and Proposition 3.4.8, $G$ is pancyclic. When $V(G_2) = \{w_1\}$, by the Proposition 3.4.10, $G$ is pancyclic.

If $v_1 \in G_1$ and $v_d = v_i$, when $G_1$ is 1-connected, let $x' = G_1 \cap W_2$ be a cut-set and cuts $G_1$ into $G_1^1$ and $G_1^2$. If $W_3 \cap G_1^a \neq \emptyset$ with $a = 1, 2$, by Proposition 3.4.9, $G$ is pancyclic. If $V(G_1^1) = \{v_1\}$, then $v_2 \in \{x', v_j\}$. By Facts 3.4.4 and 3.4.7, $v_2 = v_j$. Since $G'$ is a 2-connected graph, then there is $v_l \in V(P[v_{d+2}, v_{p-2}])$ such that $v_lv_j \in E$ and $v_{l+1}v_{d+1} \in E$ otherwise $v_{l+1}, v_{d+1} \in W_2$ or $v_{l+1}, v_{d+1} \in W_3$, then path $v_{d+1}v_{d+2} \cdots v_lv_2v_3 \cdots v_dv_1wv_pv_{p-1} \cdots v_{l+1}$ contradicts the choice of $P$. So, path $v_1wv_dv_{d-1} \cdots v_2v_lv_{l-1} \cdots v_{d+1}v_{l+1} \cdots v_p$ contradicts the choice of $P$. When $G_1$

is $2$-connected and $|G_2| \geq 2$, by Proposition 3.4.8, $G$ is pancyclic. When $V(G_2) = \{w_1\}$, it follows that $G$ is pancyclic from the similar proof to Proposition 3.4.10.

If $\{v_1, v_d\} = \{v_i, v_j\}$, when $|G_2| \geq 2$, by (3.7), Theorem 2.0.4 and Proposition 3.4.8, $G$ is pancyclic. Suppose $V(G_2) = \{w_1\}$. We may assume $v_{p-1} \in G_1$, let $P_1 = v_1 w v_d v_{d+1} \cdots v_p$ such that $|P_1| = |P|$ and $V(H) = \{w_1\} \subseteq W_2$, this contradicts Claim 3.4.3.

If $v_1, v_d \in G_2$, then $|G_2| \geq 2$, by (3.7), Theorem 2.0.4 and Proposition 3.4.8, $G$ is pancyclic.

Suppose $G_1 \subseteq W_2 \cup \{v_1, v_d\}$ and $G_2 \subseteq W_3 \cup \{v_1, v_d\}$, we can assume $G_2$ is a clique by (3.7).

When $v_1, v_{p-1} \in G_1$, if $v_2 \in G_1$, then $v_{p-1}, v_{d-1} \in W_2$ or $v_2, v_{d+1} \in W_2$ which contradicts Fact 3.4.4. So, $v_2 = v_i$ (or $v_2 = v_j$). If $v_d = v_j$ (or $v_d = v_i$), then $V(P[v_2, v_{d-1}]) \subseteq W_3$ and $V(P[v_{d+1}, v_{p-1}]) \subseteq W_2$. So, by Fact 3.4.11, $G[P[v_2, v_{d-1}]]$ is a clique.

By Claim 3.4.5, we assume $|V(P[v_{d+1}, v_{p-1}])| \geq 5$. Then $v_{d+1}v_{d+3} \in E$, otherwise there is a path $P_1 = v_{d+3}v_{d+4} \cdots v_p w v_1 v_2 \cdots v_{d+1}$ such that $|P_1| = |P|$ and $v_{d+2} \in H \cap W_2$, which contradicts Claim 3.4.3. Similarly, for any $v_k \in P[v_{d+1}, v_{p-3}]$ such that $v_k v_{k+2} \in E$ and $v_k v_{k+3} \in E(G)$. By Claim 3.4.6, $G$ is pancyclic. Similarly, if $v_d \in G_1$, this is a contradiction.

When $v_1, v_{p-1} \in G_2$ or when $v_1 \in G_1$, $v_{p-1} = v_i$ or when $v_1 \in G_1$ and $v_{p-1} \in G_2$, the same argument with $v_1, v_{p-1} \in G_1$, so, $G$ is pancyclic or a bipartite graph.

Last, suppose $v_1 v_d \notin E$ and $v_p v_d \notin E$. Put $G' = G - \{w, v_d\}$ and $W_1 = \{v_1, v_p\}$, $W_2 = X_2 - \{v_1, v_p\}$ and $W_3 = X_3 - \{v_1, v_p\}$. For any two nonadjacent vertices $x, y \in W_i$ with $i = 1, 2, 3$, then we can obtain (3.7).

If $G'$ is $3$-connected, by the minimality of $G$, then $G$ is pancyclic. If $x'$ cuts $G'$ into $G_1$ and $G_2$. When $v_1, v_p \in G_1$ or $v_1, v_p \in G_2$ or $v_1 \in G_1$, $v_p = x'$, then $\{v_d, x'\}$ is cutset of $G$, this contradicts that $G$ is $3$-connected. When $v_1 \in G_1$ and $v_p \in G_2$, then $|G'| \leq d_{G'}(v_1) + d_{G'}(v_p) \leq |G_1| + |G_2|$, a contradiction. So, we assume the connectivity of $G'$ is 2 and $\{v_i, v_j\}$ cuts $G'$ into $G_1$ and $G_2$.

Suppose that $G_1 \cap W_i \neq \emptyset$ and $G_2 \cap W_i \neq \emptyset$ with $i = 2, 3$. If $v_1, v_p \in V(G_i)$, by Lemma 3.2.5, $G_i - \{v_1, v_p\}$ is a clique and $G' - \{v_1, v_p\}$ is pancyclic. Since $V(G_i) - \{v_p\} \subseteq N_{G'}(v_1)$, $V(G_i) - \{v_1\} \subseteq N_{G'}(v_p)$ and (3.7), then $G'$ is pancyclic. If $v_1 \notin V(G_i)$ or $v_p \notin V(G_i)$ with $i = 2, 3$, by Lemma 3.2.5, $G'$ is pancyclic.

Suppose that $G_1 \cap W_2 \neq \emptyset$ and $G_1 \cap W_3 \neq \emptyset$, $G_2 \cap W_2 \neq \emptyset$, $G_2 \cap W_3 = \emptyset$.

When $v_1, v_p \in G_1$, we assume that $\{x'\} = V(G_1) \cap W_2$ cuts $G_1$ into $G_1^1$ and $G_1^2$. If $v_1 \in G_1^1$ and $v_p \in G_1^2$, by (3.7), it is easy to know that $G$ is pancyclic. If $v_1, v_p \in G_1^1$ ($v_1, v_p \in G_1^2$), by (3.7), then $G_1^1 \cap W_3 \neq \emptyset$, $|G_2| = 1$ and $G_1^2$ is a clique. And $(G_1^1 - \{v_1, v_p\}) \cup \{x'\} \subseteq N(v_1)$, $(G_1^1 - \{v_1, v_p\}) \cup \{x'\} \subseteq N(v_p)$, we can obtain that $G$ is pancyclic. So, $G_1$ is $2$-connected, when $|G_2| \geq 2$, by Theorem 2.0.3 and Proposition 3.4.8, $G$ is pancyclic.

Suppose that $V(G_2) = \{w_1\}$ and $i < j$.

If $v_{d-1}, v_{d+1} \in G_1$, by Facts 3.4.4, 3.4.7 and the definition of $P$, this is a contradiction. If $v_{d-1} = v_j$ and $v_{d+1} \in G_1$, by Facts 3.4.4 and 3.4.7, then $v_i = v_2$ and $v_{d+1}, v_{p-1} \in W_3$. There exists a vertex $v_l \in P[v_{d+2}, v_{p-2}] \cap W_2 \cap H_2$ such that $v_l v_i, v_l v_1 \in E$. Then $v_{d+1}v_{l+1} \in E$ otherwise $v_{d+1}, v_{l+1} \in X_3$ and a path $v_{d+1}v_{d+2}\cdots v_l v_1 v_2 \cdots v_d w v_p v_{p-1} \cdots v_{l+1}$ contradict with the choice of $P$. So, path $v_1 w v_d v_{d-1} v_{d-2} \cdots v_2 v_l v_{l-1} \cdots v_{d+1} v_{l+1} v_{l+2} \cdots v_p$ contradicts the choice of $P$. If $v_{d-1} = w_1$ and $v_{d+1} = v_j$, then $v_{d-1}v_{d+1} \in E$ and path $v_1 v_2 \cdots v_{d-1} v_{d+1} v_{d+2} \cdots v_p$ is a hamiltonian path of $G'$. By Theorem 2.1.2, it follows that $G$ is pancyclic from (3.7). Similarly, if $v_{d+1} = w_1$ and $v_{d-1} = v_i$ or if $v_{d-1} = w_1$ and $v_{d+1} \in G_1$ or if $v_{d+1} = w_1$ and $v_{d-1} \in G_1$, then $G$ is pancyclic.

When $v_1 \in G_1$ and $v_p = v_j$. When $x' \in W_2 \cap G_1$ cuts $G_1$ into $G_1^1$ and $G_1^2$, if $W_3 \cap G_1^a$ with $a = 1, 2$, by Proposition 3.4.9, $G$ is pancyclic. If $V(G_1^1) = \{v_1\}$, then $d_{G'}(v_1) = 2$. By (3.7), then $N(v_p) = V(G) - \{v_1, v_d, v_p\}$. So, $G$ is pancyclic. If $G_1$ is $2$-connected, when $|G_2| \geq 2$, by Proposition 3.4.8, $G$ is pancyclic. Suppose $\{w_1\} = V(G_2)$. The same argument with $v_1, v_p \in G_1$, $G$ is pancyclic.

When $v_1 \in G_1$ and $v_p \in G_2$, by Fact 3.4.7, (3.7), Theorem 2.0.4, $G_1$ is pancyclic. Since $|G_2| \geq 2$ and Proposition 3.4.8, $G$ is pancyclic. Similarly, when $v_1, v_p \in G_2$ or when $v_1 = v_i$ and $v_p = v_j$, by the choice of $P$ and (3.7), $G$ is pancyclic.

Suppose that $V(G_1) \subseteq W_1 \cup W_2$ and $V(G_2) \subseteq W_3 \cup W_1$. We assume $G[G_2 \cap W_3]$ is a clique.

When $v_1 \in G_1$ and $v_p \in G_2$, by (3.7), then $N_{G'}(v_1) = G_1 \setminus \{v_1\} \cup \{v_i, v_j\}$ and $N_{G'}(v_p) = G_2 \setminus \{v_p\} \cup \{v_i, v_j\}$. We assume $v_{d-1}, v_{d+1} \in G_1$. By Fact 3.4.4, then $v_2 \in W_3$, $v_2 = v_i$ and $V(P[v_3, v_{d-1}]) \in G_1$. And $v_1 v_3 \in E(G)$. If $v_{d-1}v_{d+1} \notin E$, so path $v_{d-1}v_{d-2}\cdots v_3 v_1 v_2 v_p v_{p-1} \cdots v_{d+1}$ is a hamiltonian path of $G'$. By (3.7) and Theorem 2.1.2, $G$ is pancyclic. So, $v_{d-1}v_{d+1} \in E$ and path $P - \{v_d\} \cup \{v_{d-1}v_{d+1}\}$ is a hamiltonian path of $G'$. By (3.7) and Theorem 2.1.2, $G$ is pancyclic. Then we can obtain the following:

**Fact 3.4.12** *If $v_{d-1}v_{d+1} \in E$, $G$ is pancyclic.*

We give the following result for the rest of proof of Theorem 3.0.2.

**Proposition 3.4.13** *If there exists a vertex $v_l \in V(P[v_3, v_{d-1}])$ such that $v_k v_l \in E(G)$ and $v_{l-1}, v_{d-1} \in W_i$ with $i = 2, 3$, where $v_k \in V(P[v_{d+1}, v_{p-2}])$ and $v_{k+1}v_{d+1} \in E(G)$, then $G$ is pancyclic.*

**Proof.** If $v_{l-1}v_{d-1} \notin E(G)$, then $P' = v_{d-1}v_{d-2}\cdots v_l v_k v_{k-1} \cdots v_{d+1} v_{k+1} \cdots v_p w v_1 v_2 \cdots v_{l-1}$ is a path such that $|P'| = |P|$ and $V(H) = \{v_d\}$, by case 1, a contradiction. So, $v_{l-1}v_{d-1} \in E(G)$.
Then $v_1 v_2 \cdots v_{l-1} v_{d-1} v_{d-2} \cdots v_l v_k v_{k-1} \cdots v_{d+1} v_{k+1} \cdots v_p$ is hamiltonian path of $G'$. By (3.7) and Theorem 2.1.2, $G$ is pancyclic. ∎

If $v_{d-1} \in G_1$ and $v_{d+1} \in G_2$ ($v_{d-1} \in G_2$ and $v_{d+1} \in G_1$), we may assume $P[v_2, v_{i-1}] \cup P[v_{i+1}, v_{j-1}] \cup P[v_{j+1}, v_{d-1}] \subseteq G_1$ and $G[P[v_{d+1}, v_p]]$ is a clique. Since $G'$ is $2$-connected, $v_j v_k \in E$ with $v_k \in P[v_{d+1}, v_{p-1}]$.

By Proposition 3.4.13, $G$ is pancyclic. If $v_{d-1} \in G_1$ and $v_{d+1} = v_i$ ($v_{d+1} = v_j$), or if $v_{d+1} \in G_2$ and $v_{d-1} = v_i$ ($v_{d-1} = v_j$), or if $v_{d-1} = v_i$ and $v_{d+1} = v_j$, the same argument with $v_{d-1} \in G_1$ and $v_{d+1} \in G_2$, $G$ is pancyclic.

When $v_1, v_p \in G_1$ ($v_1, v_p \in G_2$). If $v_{d-1}, v_{d+1} \in G_1$, by Fact 3.4.4, $v_2, v_{p-1} \in W_3$. By the definition of path $P$, this is a contradiction. If $v_{d-1}, v_{d+1} \in G_2$, since $G[G_2 \cap W_3]$ is a clique, by Fact 3.4.12, $G$ is pancyclic. If $v_{d-1} \in G_2$ and $v_{d+1} = v_i$, then $v_j \in V(P[v_2, v_{d-1}])$ and $V(P[v_{j+1}, v_{d-1}]) = V(G_2)$. So, there is $v_k \in P[v_{j+2}, v_{d-1}]$ such that $v_i v_k \in E$, by Proposition 3.4.13, $G$ is pancyclic. Similarly, if $v_{d-1} = v_i$ and $v_{d+1} \in G_2$, $G$ is pancyclic.

If $v_{d-1} = v_j$ and $v_{d+1} \in G_1$, by Facts 3.4.4 and 3.4.11, then $v_2 = v_i \in W_3$, $V(P[v_3, v_{d-2}]) = V(G_2)$ and $G[P[v_2, v_{d-1}]]$ is a clique. If there is $v_k \in P[v_{d+1}, v_{p-2}]$ such that $v_k v_2 \in E$, then $v_{d+1} v_{k+1} \in E$ otherwise $P_1 = v_{d+1} v_{d+2} \cdots v_k v_2 v_3 \cdots v_d w v_p v_{p-1} \cdots v_{k+1}$ such that $|P_1| = |P|$, by case 1 and $v_1 v_2, v_1 w \in E$, a contradiction. So, path $v_1 w v_d v_{d-1} \cdots v_2 v_k v_{k-1} \cdots v_{d+1} v_{k+1} v_{k+2} \cdots v_p$ longer than $P$, a contradiction. Thus, for any vertex $v_k \in P[v_{d+1}, v_{p-2}]$ such that $v_2 v_k \notin E$. Similarly, for any vertex $v_k \in P[v_{d+1}, v_{p-2}]$ such that $v_{d-1} v_k \notin E$.

If $v_{p-1} v_{p-3} \notin E$, then $S' = v_{p-1} v_p w v_1 v_2 \cdots v_{p-3}$ such that $|S'| = |P|$. If $|P[v_{d+1}, v_{p-3}]| \geq 3$, by Claim 3.4.3, a contradiction. If $|P[v_{d+1}, v_{p-3}]| = 2$, if $v_{d+1} v_{p-2} \notin E$, by Claim 3.4.3, a contradiction. So, $v_{d+1} v_{p-2} \in E$ and path $v_{p-1} v_p w v_1 v_2 \cdots v_{d+1} v_{p-2} v_{p-3}$ contradicts the choice of $P$. If $|P[v_{d+1}, v_{p-3}]| = 1$, since $G$ is 3-connected, if $v_1 v_{p-2} \in E$, then path $v_{p-3} v_{p-2} v_1 v_2 \cdots v_d w v_p v_{p-1}$ contradicts the choice of $P$. If $v_{p-2} v_p \in E$, then $v_{p-1} v_{p-2} v_p w v_1 v_2 \cdots v_{p-3}$ contradicts the choice of $P$. If $v_d v_{p-2} \in E$, then $v_{d+1} v_{p-2} v_d v_{d-1} \cdots v_1 w v_p v_{p-1}$ contradicts the choice of $P$. So, $v_{p-1} v_{p-3} \in E$.

Then $v_{p-2} v_{p-4} \in E$ otherwise path $v_{p-2} v_{p-3} v_{p-1} v_p w v_1 v_2 \cdots v_{p-4}$ contradicts the choice of $P$. Similarly, for any vertex $v_l \in P[v_{d+1}, v_{p-3}]$ such that $v_l v_{l+2} \in E$. Suppose $v_{d+1} v_{d+4} \notin E$, then $v_{d+4} = v_{p-1}$ otherwise path $v_{d+1} v_d \cdots v_1 w v_p v_{p-1} \cdots v_{d+5} v_{d+3} v_{d+2} v_{d+4}$ longer than $P$. Since $G$ is 3-connected, assume $N(v_{d+2}) \cap \{v_1, v_d, v_p\} \neq \emptyset$. If $v_{d+2} v_1 \in E$, then there is a path $v_{d+1} v_{d+3} v_{d+2} v_1 v_2 \cdots v_d w v_p v_{p-1} \cdots v_{d+4}$ longer than $P$, a contradiction. If $v_d v_{d+2} \in E$, path $v_{d+1} v_{d+3} v_{d+2} v_d v_{d-1} \cdots v_1 w v_p \cdots v_{d+4}$ longer than $P$. If $v_p v_{d+2} \in E$, then path $v_{d+4} v_{d+3} v_{d+2} v_p w v_1 v_2 \cdots v_{d+1}$ contradict with the choice of $P$. So, $v_{d+1} v_{d+4} \in E$. Similarly, for any vertex $v_l \in P[v_{d+1}, v_{p-4}]$ such that $v_l v_{l+3} \in E$. It follows that $G$ is pancyclic from Claim 3.4.6. Similarly, if $v_{d-1} \in G_2$ and $v_{d+1} \in G_1$, then $G$ is pancyclic.

If $v_{d+1} \in G_2$ and $v_{d-1} \in G_1$, when $V(P[v_2, v_{i-1}] \cup P[v_{i+1}, v_{d-1}]) \subseteq W_2$, the same argument with above, we can get a contradiction. When $V(P[v_2, v_{d-1}]) \subseteq G_1$ and $V(P[v_{d+1}, v_{i-1}] \cup P[v_{i+1}, v_{p-2}]) \subseteq G_2$, by Fact 3.4.4 and Proposition 3.4.13, then $v_{p-1} = v_j$ and there does not exist $v_l \in P[v_2, v_{d-1}]$ such that $v_l v_i \in E$ or $v_l v_j \in E$. Since $G'$ is 2-connected, so, we can assume $v_1 v_i \in E(G)$. If $v_i \in W_2$, then $v_i v_1 w v_p v_{p-1} \cdots v_{i+1} v_{i-1} v_{i-2} \cdots v_2$ is a path which contradicts the choice of $P$. So, $v_i \in W_3$. The similar proof to Fact 3.4.11, $G[v_{d+1}, v_{p-1}]$ is a clique. By Claim 3.4.6, $G$ is pancyclic or a bipartite graph.

The same argument with $v_1, v_p \in G_1$, when $v_1 \in G_1$ and $v_p = v_j$ or when $v_1 = v_i$ and $v_p = v_j$, $G$ is pancyclic or a bipartite graph.

Thus, $G$ is pancyclic or $G$ is a bipartite graph. The proof of the theorem 3.0.2 is complete.

## 3.5 Concluding remarks and further work

In this chapter, we prove that if $G = (V, E)$ is a $3$-connected graph of order $n$ with $V(G) = X_1 \cup X_2 \cup X_3$, for any pair of nonadjacent vertices $v_1$ and $v_2$ in $X_i$, $d(v_1) + d(v_2) \geq n$ with $i = 1, 2, 3$, then $G$ is pancyclic or a bipartite graph.

Note that the main result of this chapter is to prove that the conjecture 2.0.2 is true for $k = 3$. For all other cases ($k \geq 4$) of Conjecture 2.0.2, we haven't given proof. Thus, this is our other further work.

# Chapter 4

# Pancyclicity and hamiltonicity in digraphs or bipartite digraphs

In this chapter, we consider the hamiltonian properties of a digraph or bipartite digraph.

Let $D$ be a strongly connected balanced bipartite directed graph of order $2a \geq 10$. Let $x, y$ be distinct vertices in $D$, $\{x, y\}$ dominates a vertex $z$ if $x \rightarrow z$ and $y \rightarrow z$; in this case, we call the pair $\{x, y\}$ dominating.

In this chapter, we show that if for every dominating pair of vertices whose degree sum is at least $3a$ in a strongly connected balanced bipartite directed graph $D$, then $D$ is hamiltonian. More precisely, we prove the following. Before we go any further, we need the following definition.

**Definition 4.0.1** *Let $D$ be a balanced bipartite digraph of order $2a \geq 10$, and let $k$ be an integer. We say that $D$ satisfies the condition $\aleph_k$ if for every dominating pair of vertices $\{x, y\}$, $d(x) + d(y) \geq 3a + k$.*

**Theorem 4.0.2** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$. If $D$ satisfies the condition $\aleph_0$, i.e., $d(x) + d(y) \geq 3a$ for every dominating pair of vertices $\{x, y\}$, then $D$ is hamiltonian.*

We will prove this theorem by contradiction and König-Hall theorem. In Section 4.1, we will present a list of hamiltonian results of a digraph or bipartite digraph. In Section 4.2, we proposed some lemmas to prove Theorem 4.0.2. Also, we give the proof of Theorem 4.0.2. In Section 4.3, We show some new sufficient conditions for bipancyclic and cyclability of digraphs.

## 4.1 Introduction and notations

We start with some terminology and notations.

In this chapter, we consider finite digraphs without loops and multiple arcs. Terminology and notations not described below follow Section 1.1.

For a digraph $D$, we denote by $V(D)$ the vertex set of $D$ and by $A(D)$ the set of arcs in $D$. The order of $D$ is the number of its vertices. The arc of a digraph $D$ directed from $x$ to $y$ is denoted by $xy$ or $x \to y$ (we also say that $x$ dominates $y$ or $y$ is an out-neighbour of $x$ and $x$ is an in-neighbour of $y$), and $x \leftrightarrow y$ denotes that $x \to y$ and $y \to x$ ($x \leftrightarrow y$ is called 2-cycle). If $x \to y$ and $y \to z$ we write $x \to y \to z$. If there is no arc from $x$ to $y$, we shall use the notation $xy \notin A(D)$. For disjoint subsets $V_1$ and $V_2$ of $V(D)$, we define $A(V_1 \to V_2)$ as the set $\{xy \in A(D)|x \in V_1, y \in V_2\}$ and $A(V_1, V_2) = A(V_1 \to V_2) \cup A(V_2 \to V_1)$. If $x \in V(D)$ and $V_1 = \{x\}$, we sometimes write $x$ instead of $\{x\}$. If $V_1$ and $V_2$ are two disjoint subsets of $V(D)$ such that every vertex of $V_1$ dominates every vertex of $V_2$, then we say that $V_1$ dominates $V_2$, denoted by $V_1 \to V_2$. $V_1 \leftrightarrow V_2$ means that $V_1 \to V_2$ and $V_2 \to V_1$.

The *out-neighborhood* of a vertex $x$ is the set $N^+(x) = \{y \in V(D)|xy \in A(D)\}$ and $N^-(x) = \{y \in V(D)|yx \in A(D)\}$ is the *in-neighborhood* of $x$. Similarly, if $U \subseteq V(D)$, then $N^+(x, U) = \{y \in U|xy \in A(D)\}$ and $N^-(x, U) = \{y \in U|yx \in A(D)\}$. The *out-degree* of $x$ is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the *in-degree* of $x$. Similarly, $d^+(x, U) = |N^+(x, U)|$ and $d^-(x, U) = |N^-(x, U)|$. The degree of the vertex $x$ in $D$ is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, U) = d^+(x, U) + d^-(x, U)$). The subdigraph of $D$ induced by a subset $U$ of $V(D)$ is denoted by $D\langle U\rangle$ or $\langle U\rangle$ brevity.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ $(m \geq 2)$ and the arcs $x_ix_{i+1}, i \in [1, m-1]$ (respectively, $x_ix_{i+1}, i \in [1, m-1]$, and $x_mx_1$), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). The length of a cycle or a path is the number of its arcs. We say that $x_1x_2 \cdots x_m$ is a path from $x_1$ to $x_m$ or is a $(x_1, x_m)$-path. The length of a cycle or a path is the number of its arcs.

If $P$ is a path containing a subpath from $x$ to $y$, we let $P[x, y]$ denote that subpath. Similarly, if $C$ is a cycle containing vertices $x$ and $y$, $C[x, y]$ denotes the subpath of $C$ from $x$ to $y$. Given a vertex $x$ of a path $P$ or a cycle $C$, we denote by $x^+$ (respectively, by $x^-$) the successor (respectively, the predecessor) of $x$ (on $P$ or $C$), and in case of ambiguity, we use $P$ or $C$ as a subscript (that is $x^+_P \cdots$).

A digraph $D$ is *strongly connected* (or, just, strong) if there exists a path from $x$ to $y$ and a path from $y$ to $x$ for every pair of distinct vertices $x, y$. A digraph $D$ is $k$-strongly ($k \geq 1$) connected (or $k$-strong), if $|V(D)| \geq k+1$ and $D(V(D) \setminus A)$ is strongly connected for any subset $A \subseteq V(D)$ of at most $k-1$ vertices.

A digraph $D$ is bipartite if there exists a partition $X, Y$ of $V(D)$ into two partite sets such that every arc of $D$ has its end-vertices in different partite sets. It is called balanced if $|X| = |Y|$. The underlying graph of a digraph $D$ is denoted by $UG(D)$. It contains an edge $xy$ if $x \to y$ or $y \to x$ (or both).

A cycle (path) is called *hamiltonian* if it includes all the vertices of $D$. A digraph $D$ is *hamiltonian* if it contains a hamiltonian cycle and is *pancyclic* if it contains a cycle of length $k$ for any $3 \leq k \leq n$, where $n$ is the order of $D$. A digraph $D$ is called non-hamiltonian if it is not hamiltonian. A balanced bipartite digraph of order $2m$ is *even* *pancyclic* if it contains a cycle of length $2k$ for any $k, 2 \leq k \leq m$.

For general digraphs, there are not in the literature as many sufficient conditions as for undirected graphs that guarantee the existence of a hamiltonian cycle in a digraph. The more general and classical ones is the following theorem of M. Meyniel:

**Theorem 4.1.1 (M. Meyniel [103])** *If $D$ is a strongly connected digraph of order $n \geq 2$ and $d(x) + d(y) \geq 2n - 1$ for all pairs of nonadjacent vertices $x$ and $y$ of $D$, then $D$ is hamiltonian.*

Notice that Meyniel's theorem is a common generalization of well-known classical theorems of Ghouila-Houri [61] and Woodall [124]. A beautiful short proof Meyniel's theorem can be found in [23].

Recently, there has been a renewed interest in various Meyniel-type hamiltonian conditions in bipartite digraphs (see, e.g., [4, 2, 37, 121]). The following theorem due to Adamus Janusz.

**Theorem 4.1.2 ([2])** *Let $D$ be a strong connected balanced bipartite digraph of order $2a \geq 6$. Suppose that $d(x) + d(y) \geq 3a$ for each pair of distinct vertices $x, y$ with a common out-neighbor or a common in-neighbor, then $D$ is hamiltonian.*

The following theorems are the generalization of Theorem 4.1.2.

**Theorem 4.1.3 ([121])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 4$. Suppose that, for every dominating pair of vertices $\{x, y\}$, either $d(x) \geq 2a - 1$ and $d(y) \geq a + 1$ or $d(y) \geq 2a - 1$ and $d(x) \geq a + 1$. Then $D$ is hamiltonian.*

Before starting the following theorems, we need to introduce additional notation.

Let $D(8)$ be the bipartite digraph with partite sets $X = \{x_0, x_1, x_2, x_3\}$ and $Y = \{y_0, y_1, y_2, y_3\}$, $A(D(8))$ contains exactly the arcs $y_0x_1, y_1x_0, x_2y_3, x_3y_2$ and all the arcs of the following 2-cycles: $x_i \leftrightarrow y_i, i \in [0, 3], y_0 \leftrightarrow x_2, y_0 \leftrightarrow x_3, y_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow x_3$, and it contains no other arcs.

**Theorem 4.1.4 ([39])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 8$. Suppose that $d(x) + d(y) \geq 4a - 3$ for every pair of vertices $x, y$ with a common out-neighbour. Then $D$ is hamiltonian.*

There are many results that support Bondy's "metaconjecture" in digraph. Let us cite for example the following:

**Theorem 4.1.5 ([102])** *Let $D$ be a balanced bipartite digraph of order $2a \geq 4$ with partite sets $X$ and $Y$. Suppose that $d(x) + d(y) \geq 3a + 1$ for each two vertices $x, y$ either both in $X$ or both in $Y$. Then $D$ contains cycles of all even lengths $4, 6, \ldots, 2a$ (i.e., $D$ is bipancyclic);*

Next, we will give a sufficient condition for the existence of hamiltonian cycles in balance bipartite digraph.

## 4.2 The hamiltonicity of balance bipartite digraph

This section mainly presents the proof of Theorem 4.0.2. First, we propose some lemmas to prove Theorem 4.0.2.

### 4.2.1 Lemmas

Throughout this section, we assume that $D$ is a strongly connected balanced bipartite digraph with partite sets of cardinalities $a \geq 5$, which satisfies the condition $d(x) + d(y) \geq 3a$ for every dominating a pair of vertices $\{x, y\}$.

**Lemma 4.2.1** *Suppose that $D$ is non-hamiltonian. Then, for every vertex $u \in V(D)$, there exists a vertex $v \in V(D) \setminus \{u\}$ such that $u$ and $v$ have a common out-neighbour.*

**Proof of Lemma 4.2.1.** Suppose, on the contrary, that $D$ contains a vertex $x_0$ which has no common out-neighbor with any other vertex in $D$. Let $P = x_0x_1 \cdots y$ be the largest path in $D$. Then $d^-(x_1) = 1$ and $d(x_1) \leq a + 1$. If there exists a vertex $w \in V(D)$ such that $\{x_1, w\}$ is a dominating pair, then $d(w) \geq 2a - 1$. If $d(w) = 2a$, then $x_0$ would have $w$ as a common out-neighbor with some vertices, a contradiction. So $d(w) = 2a - 1$, $d(x_1) = a + 1$ and $x_0w \notin A(D)$.

By strong connectedness of $D$, for any $x \in V(D)$, $d^+(x) \geq 1$. Thus, $d^+(x_1) = a$ and $x_1$ would have a common out-neighbor with any vertex $v$ from its partite set. The same argument with $w$, $d(v) = 2a - 1$ and $x_0v \notin A(D)$. So. $D[V(D) - \{x_0, x_1\}]$ be a complete bipartite digraph. Since $D$ is a strongly connected digraph, then it is easy to construct a hamiltonian cycle of $D$. This contradicts $D$ is non-hamiltonian. It follows that $x_1$ has no common out-neighbor with any other vertex in $D$. Repeating the above argument for all vertices on $P$, so, $y$ has no common out-neighbor with any other vertex in $D$. Since $P$ be the largest path in $D$, it follows from the strong connectedness of $D$ that $D$ is a cycle of length $2a$, i.e., $D$ is hamiltonian, a contradiction. □

Similarly, we can obtain the following lemma:

**Lemma 4.2.2** *Suppose that $D$ is not a cycle of length $2a$. If $d(x)+d(y) \geq 3a+1$ for every dominating pair of vertices $\{x, y\}$, then, for every vertex $u \in V(D)$, there exists a vertex $v \in V(D) \setminus \{u\}$ such that $u$ and $v$ have a common out-neighbour.*

The next lemma is the key of the proof of Theorem 4.0.2.

**Lemma 4.2.3 ([4])** *Suppose that $D$ is non-hamiltonian, and let $\{C_1, C_2, \ldots, C_l\}$ be a cycle factor in $D$ with a minimal number of elements, and $|C_1| \leq |C_2| \cdots \leq |C_l|$. Then,*

$$|A(V(C_1), V(D) \setminus V(C_1))| \leq \frac{|C_1|(2a - |C_1|)}{2}.$$

Now, we are ready to prove Theorem 4.0.2.

### 4.2.2 The proof of Theorem 4.0.2

Now, let $D$ be a balanced bipartite satisfying the conditions of Theorem 4.0.2. Let $X$ and $Y$ denote its partite sets. For a proof by contradiction, suppose that $D$ is not hamiltonian.

By Lemma 4.2.1 and condition $\aleph_0$, for every vertex $x \in V(D)$, $d(x) \leq 2a$. Then, we have the follow claim:

**Claim 4.2.4** *For every vertex $u$ in $D$, $d(u) \geq a$.*

To complete the proof, we now will prove the following claim.

**Claim 4.2.5** *$D$ contains a cycle factor.*

**Proof.** $D$ contains a cycle factor if and only if there exist both a perfect matching from $X$ to $Y$ and a perfect matching from $Y$ to $X$. By the König-Hall theorem [108], it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset X$ and $|N^+(T)| \geq |T|$ for every $T \subset Y$.

Suppose, on the contrary, that a nonempty set $S \subset X$ such that $|N^+(S)| < |S|$.

By the strong connectedness of $D$, $d^+(x) \geq 1$ for every vertex $x$ in $D$. Then $|S| \geq 2$. It follows from $|N^+(S)| < |S|$ that there exist vertices $x_1, x_2 \in S$ such that $N^+(x_1) \cap N^+(x_2) \neq \emptyset$. Thus, $\{x_1, x_2\}$ be a dominating pair. By condition $\aleph_0$, we can obtain

$$3a \leq d(x_1) + d(x_2) = (d^+(x_1) + d^+(x_2)) + (d^-(x_1) + d^-(x_2)) \leq 2(|S| - 1) + 2a,$$

and so, $2|S| \geq a + 2$.

Since $S \subset X$ and $|N^+(S)| < |S|$, then $|S| \leq a$ and $|Y \setminus N^+(S)| \geq 1$.

If there exist $y_1, y_2 \in Y \setminus N^+(S)$ such that $\{y_1, y_2\}$ is a dominating pair, then

$$3a \leq d(y_1) + d(y_2) \leq 2(2a - |S|) \leq 4a - (a + 2),$$

a contradiction. So, no two vertices of $Y \setminus N^+(S)$ form a dominating pair. Thus, $|N^+(Y \setminus N^+(S) - \{y\})| \geq |Y \setminus N^+(S) - \{y\}|$. For every vertex $y \in Y \setminus N^+(S)$,

$$d^+(y) \leq a - (|Y \setminus N^+(S)| - 1) = a - |Y \setminus N^+(S)| + 1 = |N^+(S)| + 1 \leq |S|.$$

By Claim 4.2.4, $a \leq d(y) = d^+(y) + d^-(y) \leq |S| + (a - |S|) = a$. So, $d(y) = a$ and $d^+(y) = |S|$. If there are two vertices $y_1$, $y_2$ in $Y \setminus N^+(S)$, then $d^+(y_1) = d^+(y_2) = |S|$. Since $\{y_1, y_2\}$ is not a dominating pair, then $N^+(y_1) \cap N^+(y_2) = \emptyset$. Thus, $2|S| = d^+(y_1) + d^+(y_2) = |N^+(y_1) \cup N^+(y_2)| \leq a$, which contradicts $2|S| \geq a + 2$. Hence $S = X$. However, $|Y \setminus N^+(S)| \geq 1$, so $y'' \in Y \setminus N^+(S)$ such that $d^-(y'') = 0$, which contradicts the strong connectedness of $D$.

This Claim is proved. ∎

By Claim 4.2.5, $D$ contains a cycle factor $\{C_1, C_2, \ldots, C_l\}$. Now suppose $l$ is the minimum possible, since $D$ is not hamiltonian, so $l \geq 2$. We assume $|C_1| \leq |C_2| \leq \cdots \leq |C_l|$ and $|C_1| = 2t$, then $1 \leq t \leq \frac{a}{2}$. Now, we have the following claim:

**Claim 4.2.6** $t \geq 2$.

**Proof.** For a proof by contradiction, suppose $t = 1$. Then $C_1$ is a 2-cycle, and let $C_1 = x_1y_1x_1$. By Lemma 4.2.3, then $d_{C_1^c}(x_1) + d_{C_1^c}(y_1) \leq 2(a-1)$. And by Claim 4.2.4,

$$2a \leq d(x_1) + d(y_1) = d_{C_1}(x_1) + d_{C_1}(y_1) + d_{C_1^c}(x_1) + d_{C_1^c}(y_1) \leq 2a + 2.$$

Without loss of generality, assume $d(x_1) \leq d(y_1)$. We distinguish the following four cases.

**Case 1** $d(x_1) = d(y_1) = a$.

By Lemma 4.2.1, there exists a vertex $x' \in X \setminus \{x_1\}$ such that $\{x_1, x'\}$ is a dominating pair. It follows from condition $\aleph_0$ that $d(x') = 2a$. So $x'y_1 \in A(D)$ and $y_1x' \in A(D)$. Let $x' \in C_j$ for some $1 < j \leq l$ and $y'$ be the successor of $x'$ on the cycle $C_j$. Then $\{y_1, y'\}$ is a dominating pair and $d(y') = 2a$. So, $x_1y' \in A(D)$ and cycle $C_1$ can be merged into $C_j$. This contradicts the minimality of $l$.

**Case 2** $d(x_1) = a$ *and* $d(y_1) = a + 1$.

By Lemma 4.2.1, there exists a vertex $x' \in X \setminus \{x_1\}$ such that $\{x_1, x'\}$ is a dominating pair. By condition $\aleph_0$, $d(x') = 2a$. Let $x' \in C_i$ and $y'$ be the successor of $x'$ on the cycle $C_i$ with $2 \leq i \leq l$. Then $\{y_1, y'\}$ is a dominating pair and $d(y') \geq 2a - 1$. By the minimality of $l$, $d(y') = 2a - 1$, $x_1y' \notin A(D)$ and $y'x_1 \in A(D)$. If $|C_i| = 2$, then $C_1$ can be merged into $C_i$, a contradiction. So, $|C_i| \geq 4$. Let $x''y''x'y' \subset C_i$, by $d(x') = 2a$, then $\{y_1, y''\}$ is a dominating pair and $d(y'') \geq 2a - 1$. If $y''x_1 \in A(D)$, then $C_1$ can be merged into $C_i$, a contradiction. So, $y''x_1 \notin A(D)$. By $d(y'') \geq 2a - 1$, then $x_1y'' \in A(D)$ and $\{x_1, x''\}$ is a dominating pair. Hence, $d(x'') = 2a$ and $x''y_1 \in A(D)$. Then the cycle $C_1$ can be merged into $C_i$ by replacing the arc $x''y''$ on $C_i$ with the path $x''y_1x_1y''$. This contradicts the minimality of $l$.

**Case 3** $d(x_1) = a$ *and* $d(y_1) = a + 2$.

The same argument with Case 2, $\{x', x_1\}$ and $\{y_1, y'\}$ are both dominating pairs, and $x'y' \in A(C_i)$. By $\aleph_0$, $d(y') \geq 2a - 2$. It follows from the minimality of $l$ and $d(x') = 2a$ that $x_1y' \notin A(D)$. If $|C_i| = 2$, by the minimality of $l$, then $y'x_1 \notin A(D)$. Since $a \geq 5$ and $d(y') \geq 2a - 2$, then there is $C_k$ with $k \neq 1, i$. Let $uv \in A(C_k)$, then $x'v, uy' \in A(G)$. So, $C_i$ can be merged into $C_k$, a contradiction. Thus, $|C_i| \geq 4$. The definitions of $y''$ and $x''$ are the same as Case

2. By the minimality of $l$ and $d(x') = 2a$, $y''x_1 \notin A(D)$. If $x_1y'' \in A(D)$, then $\{x'', x_1\}$ is a dominating pair. So $x''y_1 \in A(D)$ by $\aleph_0$. $C_1$ can be merged into $C_i$, a contradiction. So $x_1y'' \notin A(D)$. By $d(x') = 2a$, then $\{y'', y_1\}$ is a dominating pair and $d(y'') \geq 2a - 2$.

If there exists $C_j$ with $j \neq 1, i$. Let $yx \in A(C_j)$, since $d(y'') \geq 2a-2$ and $y''x_1, x_1y'' \notin A(D)$, then $y''x, yx' \in A(D)$. So, $C_i$ can be merged into $C_j$. This contradicts the minimality of $l$.

It follows from $a \geq 5$ that $|C_i| \geq 6$. Let $x''y''x'y'x'''y''' \subseteq C_i$. Since $d(y'') \geq 2a - 2$ and $d(x') = 2a$, then $y''x''', x'y''', x'y'' \in A(D)$. Suppose $y'x_1 \in A(D)$, if $x'''y' \in A(D)$, then

$$C = C_i \setminus \{y''x', x'y', y'x''', x'''y'''\} \cup \{y''x''', x'''y', y'x_1, x_1y_1, y_1x', x'y'''\}$$

is a hamiltonian cycle, a contradiction. By $d(y') \geq 2a - 2$, then $x''y' \in A(D)$. Similarly, we can find a hamiltonian cycle

$$C = C_i \setminus \{x''y'', y''x', x'y', y'x'''\} \cup \{x''y', y'x_1, x_1y_1, y_1x', x'y'', y''x'''\},$$

a contradiction. So, $y'x_1 \notin A(D)$.

By $d(x_1) = a \geq 5$, there exists $y \in C_i$ such that $y$ connects with $x_1$. Let $x$ be the successor vertex of $y$ on cycle $C_i$, then $y'x \in A(D)$ by $d(y') \geq 2a - 2$. If $yx_1 \in A(D)$, then $C = C_i \setminus \{yx, y''x', y'x'''\} \cup \{yx_1, x_1y_1, y_1x', y'x, y''x'''\}$ is a hamiltonian cycle, a contradiction. So $x_1y \in A(D)$. Similarly, we can find a hamiltonian cycle, a contradiction.

**Case 4** $d(x_1) = d(y_1) = a + 1$.

By Lemma 4.2.1, there exists a vertex $x' \in X \setminus \{x_1\}$ such that $\{x_1, x'\}$ is a dominating pair. It follows from condition $\aleph_0$ that $d(x') \geq 2a - 1$. Let $x' \in C_i$ for some $1 < i \leq l$ and $y'$ be the successor of $x'$ on the cycle $C_i$.

If $\{y_1, y'\}$ is not a dominating pair, then $y_1x' \notin A(D)$ or $y'x' \notin A(D)$. By $d(x') \geq 2a - 1$, $y'x_1 \notin A(D)$. When $|C_i| = 2$, then $y'x' \in A(D)$, $y_1x' \notin A(D)$ and $x'y_1 \in A(D)$. By the minimality of $l$, $x_1y' \notin A(D)$. By Claim 4.2.4, then $d(y') \geq a \geq 5$, so there exists $x'' \in C_j$ with $j \neq 1, i$ such that $x''y' \in A(D)$ or $y'x'' \in A(D)$. If $x''y' \in A(D)$, let $y''$ be the successor vertex of $x''$ on $C_j$. Then $x'y'' \in A(D)$ since $d(x') \geq 2a - 1$. So, $C_i$ can be merged into cycle $C_j$, a contradiction. Similarly, if $y'x'' \in A(D)$, a contradiction. So $|C_i| \geq 4$. Without loss of generality, let $C_i = v_1u_1 \cdots v_su_sv_1$, where for any $1 \leq i \leq s$, $v_i \in Y$, $u_i \in X$ and $x' = u_1$.

When $y_1u_1 \notin A(D)$, by $d(u_1) \geq 2a - 1$, we have $u_1y_1 \in A(D)$. Then, we obtain the following fact:

**Fact 4.2.7** *If $x'y_1 \in A(D)$ ($u_1y_1 \in A(D)$), then $D$ would be hamiltonian.*

**Proof.** If there exists $u_k \in C_i$ such that $y_1u_k \in A(D)$, then $\{v_k, y_1\}$ is a dominating pair. So, $d(v_k) \geq 2a - 1$. By the minimality of $l$, $v_kx_1 \notin A(D)$. Since $d(v_k) \geq 2a - 1$, then $x_1v_k \in A(D)$, and $\{x_1, u_{k-1}\}$ is a dominating pair. Thus, $d(u_{k-1}) \geq 2a - 1$. By the minimality of $l$, then $u_{k-1}y_1 \notin A(D)$ and $y_1u_{k-1} \in A(D)$. Repeating the above argument for all the subsequent vertices on $C_i$, then $y_1u_1 \in A(D)$. So $C_1$ an be merged into $C_i$, a contradiction. Hence,

$N^+(y_1) \cap V(C_i) = \emptyset$. Similarly, $N^+(x_1) \cap V(C_i) = \emptyset$. By the strong connectedness of $D$, then there exists $C_j$ with $j \neq 1, i$. Without loss of generality, let $xy \in A(C_j)$ such that $x_1y \in A(D)$. Then $\{x_1, x\}$ is a dominating pair, and $d(x) \geq 2a-1$ by $\aleph_0$. It follows from the minimality of $l$ that $xy_1 \notin A(D)$. So $xy' \in A(D)$. Then

$$C_j \setminus \{xy\} \cup C_i \setminus \{x'y'\} \cup \{x_1y, xy', x'y_1, y_1x_1\}$$

is a cycle that contradicts the minimality of $l$. Thus, $D$ is hamiltonian. ∎

By Fact 4.2.7, since $D$ is not hamiltonian, then $x'y_1 \notin A(D)$. And by $d(x') \geq 2a-1$, $y_1x' \in A(D)$ ($y_1u_1 \in A(D)$). Then $\{v_1, y_1\}$ is dominating pair, $d(v_1) \geq 2a-1$. Since $y'x' \notin A(D)$ and $d(x') \geq 2a-1$, then $u_1y_1 \in A(D)$. By the minimality of $l$, $v_1x_1 \notin A(D)$. So, $x_1v_1 \in A(D)$. Similarly, $\{u_s, x_1\}$ is dominating pair and $d(u_s) \geq 2a-1$. By the minimality of $l$, then $u_sy_1 \notin A(D)$. So, $y_1u_s \in A(D)$ and $v_2u_s \in A(D)$ (i.e., $y'u_s \in A(D)$). Then $\{y_1, y'\}$ is a dominating pair, a contradiction.

Hence, $\{y_1, y'\}$ is a dominating pair, then $d(y') \geq 2a-1$.

If $|C_i| = 2$, assume $x'y_1 \in A(D)$ by $d(x') \geq 2a-1$, it follows that $x_1y', y_1x' \notin A(D)$ and $y'x_1 \in A(D)$ from the minimality of $l$. Since $a \geq 5$, there exists $C_j$ with $j \neq 1, i$ and $x''y'' \in A(C_j)$. So, $x''y', x'y'' \in A(D)$ and $C_i$ can be merged into $C_j$, a contradiction. Hence, $|C_i| \geq 4$. If $x'y_1 \in A(D)$, by Fact 4.2.7, $D$ is hamiltonian. This is a contradiction. So $x'y_1 \notin A(D)$ and $y_1x' \in A(D)$ by $d(x') \geq 2a-1$. Let $y_2$ be a predecessor vertex of $x'$ on $C_i$ and $x_2$ be a predecessor vertex of $y_2$ on $C_i$. Then $\{y_1, y_2\}$ is a dominating pair, and $d(y_2) \geq 2a-1$. By the minimality of $l$, $y_2x_1 \notin A(D)$. So, $x_1y_2 \in A(D)$. Repeating the above argument for all vertices on $C_i$, we can obtain $N^-(V(C_1)) \cap V(C_i) = \emptyset$. Since $D$ is strongly connected, then there exists $C_j$ with $j \neq 1, i$ and $xy \in A(C_j)$. By $d(x') \geq 2a-1$, $d(y') \geq 2a-1$ and $N^-(V(C_1)) \cap V(C_i) = \emptyset$, so $x'y, xy' \in A(D)$. Then $C_i$ can be merged into $C_j$, which contradicts the minimality of $l$.

Hence, $t \geq 2$. ∎

By Lemma 4.2.3, without loss of generality, assume

$$|A(V(C_1) \cap X, V(D) \setminus V(C_1))| \leq t(a-t). \qquad (*)$$

By Claim 4.2.6, assume $d_{C_1^c}(x_1) \leq \cdots \leq d_{C_1^c}(x_t)$ and $d_{C_1^c}(y_1) \leq \cdots \leq d_{C_1^c}(y_t)$, where $x_1, x_2, \ldots x_t \in V(C_1) \cap X$ and $y_1, y_2, \ldots y_t \in V(C_1) \cap Y$. By $(*)$, $d_{C_1^c}(x_1) \leq a-t$. Then, we have the following claim.

**Claim 4.2.8** *When $d_{C_1^c}(x_1) = a-t$, then $D$ would be hamiltonian.*

**Proof.** For all $1 \leq i \leq t$, by $(*)$, $d_{C_1^c}(x_i) = a-t$. If there exist $x_i, x_j \in X \cap V(C_1)$ such that $\{x_i, x_j\}$ is a dominating pair, then

$$3a \leq d(x_i) + d(x_j) = d_{C_1}(x_i) + d_{C_1}(x_j) + d_{C_1^c}(x_i) + d_{C_1^c}(x_j) \leq 4t + 2(a-t) \leq 3a$$

by $t \leq a/2$. So $d(x_i) + d(x_j) = 3a$, $t = a/2$, $d_{C_1}(x_i) = d_{C_1}(x_j) = a$, $l = 2$ and $d_{C_1^c}(x_k) = a/2$ for all $1 \leq k \leq t$. Let $C_1^c = C_2$. Then, every two vertices in $V(C_1) \cap X$ can form a dominating pair. By $\aleph_0$, then $D[V(C_1)]$ is a complete bipartite digraph.

If existing $x_d \in V(C_1)$ and $y \in V(C_2)$ such that $x_d y \in A(D)$, let $x$ be a predecessor of $y$ on $C_2$. Then $\{x, x_d\}$ is a dominating pair. So, $d(x) \geq a + a/2$ by $\aleph_0$ and $d_{C_1}(x) \geq a/2$.

We will show $N^+(x) \cap V(C_1) = \emptyset$. If existing $y_k \in V(C_1)$ such that $xy_k \in A(D)$, since $D[V(C_1)]$ is a complete bipartite digraph, $C_1$ can be merged into $C_2$, a contradiction. So, $N^+(x) \cap V(C_1) = \emptyset$. Let $x_d y_d \in A(C_1)$, then $y_d x \in A(D)$ by $d_{C_1}(x) \geq a/2$. Let $y''$ be the predecessor of $x$ on $C_2$, then $\{y_d, y''\}$ is a dominating pair.

If there is $x_b \in V(C_1)$ such that $y'' x_b \in A(D)$, the same argument with above, a contradiction. So $N^+(y'') \cap V(C_1) = \emptyset$. We can assume $x_c \in V(C_1)$ such that $x_c y'' \in A(D)$ by $d(y_d) + d(y'') \geq 3a$. Repeating the above argument for all the vertices on $C_2$, so $N^+(V(C_2)) \cap V(C_1) = \emptyset$. This contradicts the strong connectedness of $D$.

For all $x_k \in V(C_1) \cap X$ such that $N^+(x_k) \cap V(C_2) = \emptyset$ and for all $y \in V(C_2) \cap Y$ such that $y \in N^-(x_k)$. By $y$ and $x_k$ were arbitrary and the strong connectedness of $D$, there exist $y_f x_f \in A(C_1)$ and $y^1 x^1 \in A(C_2)$ such that $y_f x^1 \in A(D)$ and $y^1 x_f \in A(D)$. So, $C_1$ can be merged into $C_2$, a contradiction. Hence, no two vertices $x_i$ and $x_j$ in $V(C_1) \cap X$ form a dominating pair. So $d^-_{C_1}(y_i) = 1$ for all $1 \leq i \leq t$. In particular, $d^+_{C_1}(x_1) = 1$. Since $d(x_1) \geq a$ and $d_{C_1^c}(x_1) = a - t$, $d(x_1) = d^+_{C_1}(x_1) + d^-_{C_1}(x_1) + d_{C_1^c}(x_1)$, then $d^-_{C_1}(x_1) \geq t - 1$.

When $t \geq 3$, without loss of generality, assume $\{y_2, y_3\}$ is a dominating pair. By $(*)$ and Lemma 4.2.3, then $|A(V(C_1) \cap Y, V(D) \setminus V(C_1))| \leq t(a-t)$ and $d_{C_1^c}(y_1) + d_{C_1^c}(y_2) + d_{C_1^c}(y_3) \leq 3(a-t)$. So, $d_{C_1^c}(y_2) + d_{C_1^c}(y_3) \leq 3(a-t)$, and

$$3a \leq d(y_2) + d(y_3) = d_{C_1}(y_2) + d_{C_1}(y_3) + d_{C_1^c}(y_2) + d_{C_1^c}(y_3) \leq 2(t+1) + 3(a-t).$$

Then $t \leq 2$, a contradiction. So, $t = 2$.

If $\{y_1, y_2\}$ is a dominating pair, then $d_{C_1^c}(y_1) + d_{C_1^c}(y_2) \leq 2(a-2)$, and

$$3a \leq d(y_1) + d(y_2) \leq 2(2+1) + 2(a-2) = 2a + 2,$$

which contradicts $a \leq 3$. So $d_{C_1}(x_1) = d_{C_1}(x_2) = 2$ and $d(x_1) = d(x_2) = a$, $d(y_1) \leq a$.

If there is $y \in C_j$ with $j \neq 1$ such that $x_1 y \in A(D)$, let $x$ be a predecessor vertex of $y$ on $C_j$. So, $\{x_1, x\}$ is a dominating pair. By $\aleph_0$, $d(x) = 2a$, $C_1$ can be merged into $C_j$, a contradiction. Thus, $N^+(x_1) \cap V(C_1^c) = \emptyset$. Similarly, $N^+(V(C_1)) \cap V(C_1^c) = \emptyset$, which contradicts $D$ is strongly connected. Hence, $D$ is hamiltonian. ∎

By Claim 4.2.8, suppose that $d_{C_1^c}(x_1) = a - t - \alpha_1$ for some $\alpha_1 > 0$. Then

$$d^+_{C_1}(x_1) = d(x_1) - d^-_{C_1}(x_1) - d_{C_1^c}(x_1) \geq a - d^-_{C_1}(x_1) - a + t + \alpha_1 \geq \alpha_1 \qquad (M)$$

by $d(x_1) \geq a$. So $x_1$ dominates at least $\alpha_1$ vertices on $C_1$.

If $x_i, x_j$ satisfy $d_{C_1^c}(x_i) + d_{C_1^c}(x_j) \leq 2(a-t) - 2$ for $1 \leq i < j \leq t$, $\{x_i, x_j\}$ is a dominating pair. Then

$$3a \leq d(x_i) + d(x_j) \leq 4t + 2(a-t) - 2 \leq 3a - 2,$$

a contradiction. So, $\{x_i, x_j\}$ is not a dominating pair.

By Lemma 4.2.1, for all above $x_i, x_j$, if there exist $x', x'' \in C_1$ such that $\{x_i, x'\}$ and $\{x_j, x''\}$ are dominating pairs, then

$$\begin{aligned}
d(x') + d(x'') &\geq 6a - d(x_i) - d(x_j) \\
&= 6a - [(d^+_{C_1}(x_i) + d^+_{C_1}(x_j)) + (d^-_{C_1}(x_i) + d^-_{C_1}(x_j)) + (d_{C_1^c}(x_i) + d_{C_1^c}(x_j))] \\
&\geq 6a - t - 2t - 2(a-t) + 2 \\
&= 4a - t + 2.
\end{aligned}$$

So $d(x') \geq 4a - t + 2 - 2a = 2a - t + 2$ and

$$d_{C_1^c}(x') \geq a - t + 2. \qquad (M_1)$$

Let $s \geq 1$, for all $1 \leq i \leq s$, $d_{C_1^c}(x_i) = a - t - \alpha_i$ with $1 \leq \alpha_s \leq \cdots \leq \alpha_1$, and for all $s+1 \leq j \leq t$, $d_{C_1^c}(x_j) \geq a - t$.

In the same argument with $x_1$, by $(M)$, for each $1 \leq i \leq s$, $x_i$ dominates at least $\alpha_i$ vertices on $C_1$. Denote by $S_i$ the vertex set of the predecessors of $x_i$ which dominates at least $\alpha_i$ vertices and apart from $x_i$. For all $1 \leq i < j \leq s$, it follows from $d_{C_1^c}(x_i) + d_{C_1^c}(x_j) \leq 2(a-t) - 2$ that $\{x_i, x_j\}$ is not a dominating pair. So $S_i \cap S_j = \emptyset$. Let

$$R = \cup_{i=1}^{i=s} S_i$$

and

$$\overline{R} = V(C_1) \cap X \setminus (\cup_{i=1}^{i=s}\{x_i\} \cup R),$$

$I'$ denotes all $i$ that $x_i$ dominates at least $\alpha_i$ vertices apart from its own on $C_1$, and $I''$ denotes all $i$ that $x_i$ dominates exactly $\alpha_i - 1$ vertices apart from its own on $C_1$. Then $|\overline{R}| = (t - \sum_{i \in I'} \alpha_i - \sum_{j \in I''}(\alpha_j - 1) - s)$. By $(M_1)$, for any vertex $x_k \in R$, $d(x_k) \geq a - t + 2$. So, by $(*)$, we obtain:

$$\begin{aligned} t(a-t) &\geq \sum_{i=1}^{t} d_{C_1^c}(x_i) = \sum_{i=1}^{s} d_{C_1^c}(x_i) + \sum_{x_j \in R} d_{C_1^c}(x_j) + \sum_{x_k \in \overline{R}} d_{C_1^c}(x_k) \\ &\geq \sum_{i=1}^{s}(a-t-\alpha_i) + (\sum_{i \in I'} \alpha_i + \sum_{j \in I''} (\alpha_j - 1))(a-t+2) \\ &\quad +(t - \sum_{i \in I'} \alpha_i - \sum_{j \in I''} (\alpha_j - 1) - s)(a-t) \\ &= t(a-t) + \sum_{i=1}^{s} \alpha_i - 2|I''|. \qquad (**) \end{aligned}$$

So, $\sum_{i \in I'} \alpha_i + \sum_{j \in I''} \alpha_j = \sum_{k=1}^{s} \alpha_k \leq 2|I''|$.

If there is $i \in I''$ such that $\alpha_i = 1$, by the definition of $I''$, then

$$d^+_{C_1}(x_i) = 1. \qquad (M_2)$$

By Claim 4.2.4, then

$$a \leq d(x_i) = d^+_{C_1}(x_i) + d^-_{C_1}(x_i) + d_{C_1^c}(x_i) \leq 1 + t + a - t - 1 = a,$$

and $d_{C_1^c}(x_i) = a - t - 1$. So, $d^-_{C_1}(x_i) = t$.

Next, we will show $N^+(x_i) \cap V(C_1^c) = \emptyset$.

Suppose there exists $y \in A(C_j)$ with $j \neq 1$ such that $x_i y \in A(D)$, then $\{x, x_i\}$ is a dominating pair, where $x$ be a predecessor vertex of $y$ on $C_j$. By $d(x_i) = a$ and $\aleph_0$, we obtain $d(x) = 2a$. Let $y_i$ be a successor vertex of $x_i$ on $C_1$. So, $xy_i \in A(D)$ and $C_1$ can be merged into $C_j$. This contradicts the minimality of $l$. Hence,

$$N^+(x_i) \cap V(C_1^c) = \emptyset. \qquad (M_3)$$

Suppose there exists $x_j \in V(C_1) \cap X$ such that $\{x_j, x_i\}$ is a dominating pair, by $d(x_i) = a$ and $\aleph_0$, then $d(x_j) = 2a$. Since $t \geq 2$, let $y'$ and $y''$ are predecessor and successor of $x_j$ on $C_1$, respectively. If there exists $yx \in A(C_j)$ with $j \neq 1$ such that $y'x \in A(D)$, by $d(x_j) = 2a$, then $yx_j \in A(D)$. So, $C_1$ can be merged into $C_j$, a contradiction. Thus, $N^+(y') \cap V(C_1^c) = \emptyset$. Similarly, $N^-(y'') \cap V(C_1^c) = \emptyset$. By $(M_2)$, then $d_{C_1}(y') + d_{C_1}(y'') \leq 4t - 1$. It follows that $\{y', y''\}$ is a dominating pair from $d(x_j) = 2a$. So,

$$3a \leq d(y') + d(y'') = d_{C_1}(y') + d_{C_1}(y'') + d_{C_1^c}(y') + d_{C_1^c}(y'') \leq 4t - 1 + 2(a-t) = 2a + 2t - 1,$$

we obtain $t \geq \frac{a+1}{2}$, which contradicts $t \leq \frac{a}{2}$. Hence, there does not exist any vertex $x_j$ in $V(C_1) \cap X$ such that $x_j$ and $x_i$ have a common out-neighbour.

By Lemma 4.2.1, let $x' \in V(C_j)$ such that $\{x_i, x'\}$ is a dominating pair. Then $d(x') = 2a$ by $d(x_i) = a$. Let $y_i$ be a predecessor vertex of $x_i$ on $C_1$, $y'$ be a predecessor vertex of $x'$ on $C_j$.

If there exists $x \in V(C_1) \cap X$ such that $y'x \in A(D)$. By $d(x') = 2a$, then $C_1$ can be merged into $C_j$. This contradicts the minimality of $l$. So $d^+_{C_1}(y') = 0$. By $(M_3)$, then $x_i y' \notin A(D)$. So, $d^-_{C_1}(y') \le t-1$. And $d_{C_1^c}(y') \le 2(a-t)$. Thus, $d(y') \le 2(a-t) + t - 1$.

By $(M_2)$, then $d^-_{C_1}(y_i) \le t-1$. And $d^+_{C_1}(y_i) \le t$. If there exists $x'' \in V(C_j) \setminus \{x'\}$ such that $y_i x'' \in A(D)$, by $(M_3)$, $d^+_{C_1}(y') = 0$, and $d_{C_1^c}(x_i) = a - t - 1$, then $y'' x_i \in A(D)$, where $y''$ be a predecessor vertex of $x''$ on $C_j$. So, $C_1$ can be merged into $C_j$, a contradiction. Thus, $d^+_{C_j \setminus \{x'\}}(y_i) = 0$. Similarly, for any $k \ne 1, j$, $d^+_{C_k}(y_i) = 0$. And by $d(x') = 2a$, then $y_i x' \in A(D)$. Thus, $N^+_{C_1^c}(y_i) = \{x\}$, i.e., $d^+_{C_1^c}(y_i) = 1$. And $d^-_{C_1^c}(y_i) \le a - t$. So, $d(y_i) = d^+_{C_1}(y_i) + d^-_{C_1}(y_i) + d^+_{C_1^c}(y_i) + d^-_{C_1^c}(y_i) \le a + t$.

It follows that $\{y_i, y'\}$ is a dominating pair from $d(x') = 2a$. Thus,

$$d(y_i) + d(y') \le 2(a-t) + t - 1 + a + t = 3a - 1,$$

which contradicts $d(y_i) + d(y') \ge 3a$.

Hence, for all $i \in I''$, $\alpha_i \ge 2$ and the $(**)$ inequalities are equal. Then $|I'| = 0$ and $\alpha_i = 2$ with $i \in I''$. Let $x \in V(C_1) \cap X$ such that $\{x, x_i\}$ is a dominating pair. Since

$$d(x_i) = d^+_{C_1}(x_i) + d^-_{C_1}(x_i) + d_{C_1^c}(x_i) \le 2 + t + a - t - 2 = a,$$

so $d(x) = 2a$ and $d_{C_1^c}(x) = a - t + 2$ by $\aleph_0$. Then $2a = d(x) \le a - t + 2 + 2t = a + t + 2$ and $t = a - 2$. It follows from $t \le a/2$ and $t \ge 2$ that $t = 2$ and $a = 4$. This contradicts $a \ge 5$.

Hence, $D$ is hamiltonian.

The proof of Theorem 4.0.2 is completed.

## 4.3 The bipancyclicity and cyclability of digraph

In this section, first, we proved some new sufficient conditions for bipancyclic of digraphs.

From Theorem 4.1.4, we obtain the following theorem.

**Theorem 4.3.1** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \ge 8$ with partite sets $X$ and $Y$. Suppose that $D$ contains a cycle of length $2a-2$ and $d(x) + d(y) \ge 4a - 4$ for every dominating pair of vertices $\{x, y\}$. Then $D$ is even pancyclic.*

To prove Theorem 4.3.1, we use the following theorem:

**Theorem 4.3.2 ([38])** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$ which contains a pre-hamiltonian cycle (i.e., a cycle of length $2a-2$). Assume that $max\{d(x), d(y)\} \geq 2a-2$ for every dominating pair of vertices $\{x, y\}$. Then for any $k, 1 \leq k \leq a-1$, $D$ contains cycles of every length $2k$.*

**Proof of Theorem 4.3.1:** On the contrary, we suppose $D$ is not bipancyclic. By Theorem 4.0.2 and $a \geq 4$, let $C$ be a cycle of length $2a$ and for any $u \in V(D)$ such that $d^+(u) \leq a-1$ and $d^-(u) \leq a-1$, i.e., $d(u) \leq 2a-2$. By Lemma 4.2.2, for all $x \in V(D)$, $2a-2 \geq d(x) \geq 4a-4-(2a-2) = 2a-2$, i.e., $d(x) = 2a-2$. For any $u, v \in V(D)$ from the same partite set of $D$,

$$2(2a-2) \leq d(u) + d(v) = (d^+(u) + d^+(v)) + (d^-(u) + d^-(v)).$$

And $d^-(u) + d^-(v) \leq 2a-2$, then $d^+(u) + d^+(v) \geq a+1$. So $\{u, v\}$ is a dominating pair. By Theorem 4.3.2, for any $k, 1 \leq k \leq a$, $D$ contains cycles of every length $2k$. □

The next theorem is our second theorem which improves the result of Theorem 4.1.5.

**Theorem 4.3.3** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. If $D$ contains a cycle of length $2a-2$ and $D$ satisfies the condition $\aleph_1$, i.e., $d(x) + d(y) \geq 3a+1$ for every dominating pair of vertices $\{x, y\}$, then $D$ contains a cycle of length $2k$ for all $k$, where $1 \leq k \leq a$ (i.e., $D$ is even pancyclic).*

To prove Theorem 4.3.3, we need the following lemma.

**Lemma 4.3.4 ([8])** *Let $D$ be a bipartite digraph of order $n$ which contains a cycle $C$ of length $2b$, where $2 \leq 2b \leq n-1$. Let $x$ be a vertex not contained in $C$. If $d(x, V(C)) \geq b+1$, then $D$ contains cycles of every even length $m$, $2 \leq m \leq 2b$, through $x$.*

**Proof of Theorem 4.3.3:** By Theorem 4.0.2, $D$ contains a Hamilton cycle.

Without loss of generality, let $C = x_1y_1x_2y_2 \cdots x_{a-1}y_{a-1}x_1$ be a cycle of length $2a-a$, where $x_i \in X$ and $y_i \in Y$ for all $1 \leq i \leq a-1$.

Suppose $x$ and $y$ are not on $C$ with $x \in X$ and $y \in Y$. The remainder of the proof splits into two cases depending on the degrees of vertices $x$ and $y$.

**Case 1** $d(x) \geq a+2$ *or* $d(y) \geq a+2$.

Without loss of generality, we assume that $d(x) \geq a+2$. Since $d(x) = d_{\{y\}}(x) + d_C(x) \geq a+2$ and $d_{\{y\}}(x) \leq 2$, then $d_C(x) \geq a+2-2 = a > a-1$.

By Lemma 4.3.4, $D$ contains a cycle of all even lengths less than or equal to $2a-2$.

**Case 2** $d(x) \leq a+1$ *and* $d(y) \leq a+1$.

Since $D$ is a strongly connected balance bipartite digraph and by Lemma 4.2.1, we assume, without loss of generality, $xy_1 \in A(D)$. So $\{x, x_1\}$ is a dominating pair and $d(x) + d(x_1) \geq 3a + 1$.

Then $d(x_1) \geq 3a + 1 - a - 1 = 2a$. Hence, $x_1$ together with every vertex $y_i$ forms a $2$-cycle.

So, we can obtain that $D$ contains a cycle of all even lengths $2k$ with $1 \leq k \leq a$. The proof of this theorem is completed. □

Before proceeding further, we give more notations.

Let $D$ be a digraph and let $S$ be a nonempty subset of vertices of $D$. We say that a digraph $D$ is $S$-strongly connected if, for any pair $x, y$ of distinct vertices of $S$, there exists a path from $x$ to $y$ and a path from $y$ to $x$.

A set $S$ of vertices in a directive graph $D$ is said to be *cyclable* (*pathable*) in $D$ if $D$ contains a directed cycle (path) through all vertices of $S$.

There are many well-known conditions which guarantee the cyclability of a set of vertices in an undirected graph. H. Li, E. Flandrin and J. Shu [89] proved the following theorem which gives a sufficient condition for cyclability of digraphs.

**Theorem 4.3.5 ([89])** *Let $D$ be a digraph of order $n$ and $S \subseteq V(D)$. If $D$ is $S$-strong and if $d(x) + d(y) \geq 2n - 1$ for any two nonadjacent vertices $x, y \in S$, then $S$ is cyclable in $D$.*

In this section, we will show the following theorem.

**Theorem 4.3.6** *Let $D$ be a $2$-strong digraph of order $n$ and $S \subseteq V(D)$. If $D$ is $S$-strong and if $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$ for all distinct pairs of nonadjacent vertices $x, y$ and $w, z$ in $S$, then $S$ is cyclable in $D$ or $D$ contains a cycle through all the vertices of $S$ except one.*

**Proof of Theorem 4.3.6:** Since for all distinct pairs of nonadjacent vertices $x, y$ and $w, z$ in $S$, $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Then $S$ contains at most one pair of nonadjacent vertices $u, v$ such that $d(u) + d(v) \leq 2n - 2$.

If for any pair of nonadjacent vertices $x, y$ in $S$ such that $d(x) + d(y) \geq 2n - 1$, by Theorem 4.3.5, we obtain $S$ is cyclable in $D$. So, we assume that there is a pair of nonadjacent vertices $u, v$ in $S$ such that $d(u) + d(v) \leq 2n - 2$.

Let $S' = S - \{u\}$, then $D$ is clearly $S'$-strongly connected and for two nonadjacent vertices of $S'$ have degree sum in $D$ greater or equal to $2n - 1$. It follows that $S'$ is cyclable in $D$ from Theorem 4.3.5. Let $C$ be a cycle which contains all vertices of $S'$, i.e., $C$ contains a cycle through all the vertices of $S$ except one vertex $u$.

Theorem 4.3.6 has completed. □

## 4.4 Concluding remarks and further work

In this chapter, we gave sufficient conditions for a balanced bipartite digraph to be hamiltonian. And we show some sufficient conditions for a digraph to be even pancyclic and cyclable.

Note that our result show that a balance bipartite digraph with order $2a$, if $d(x) + d(y) \geq 3a$ for every dominating pair of vertices $\{x, y\}$, we can find a hamiltonian cycle. We also show that if a digraph $D$ of order $2a$ is not a directed cycle and $D$ contains a cycle of length $2a - 2$, if $d(x) + d(y) \geq 3a + 1$ for every dominating pair of vertices $\{x, y\}$, then $D$ contains a cycle of length $2k$ for all $k$, where $1 \leq k \leq a$.

We get the following question:

**Problem 4.4.1** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. If $D$ satisfies the condition $\aleph_1$, i.e., $d(x) + d(y) \geq 3a$ for every dominating pair of vertices $\{x, y\}$, then $D$ is even pancyclic?*

Also, we have a question to know if Theorem 4.0.2 (or the sufficient hamiltonian condition of digraphs) has a cyclable version. These will be our further works.

# Chapter 5

# Chorded pancyclicity in claw-free graphs

Chorded pancyclic is one of the generalizations of the hamiltonian problem. In this chapter, we study a new sufficient condition of chorded pancyclic graphs.

We study a minimum degree condition for $K_{1,3}$-free graphs to be chorded pancyclic. Theorem 1.3.15 gives a condition of minimum degree for $K_{1,3}$-free graphs to be pancyclic. We reaffirm this theorem here.

**Theorem 5.0.1 ([54])** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then $G$ is pancyclic.*

The lower bound of Theorem 5.0.1 is sharp because there is a graph of order $34$, which satisfies the degree sum condition in Theorem 5.0.1 but is not pancyclic.

From Theorems 5.0.1, we obtain the results of the extension of pancyclic to the chorded pancyclic. The following theorems are the main results of this chapter.

**Theorem 5.0.2** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then $G$ is chorded pancyclic.*

Let $CH_m$ be the maximum number of chords in cycle $C_m \subseteq G$ with $4 \leq m \leq n$.

**Theorem 5.0.3** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35$. If $\delta(G) \geq \frac{n-2}{3}$, then*

$$CH_m \geq \begin{cases} \frac{m(m-1)}{2} - m & \text{if } 4 \leq m \leq 5, \\ m & \text{if } 6 \leq m \leq \frac{n+1}{3}, \\ [\frac{m}{6}] & \text{if } \frac{n+4}{3} \leq m \leq \frac{2n+8}{3}, \\ \frac{m(\delta-(n-m))}{2} - m & \text{if } \frac{2n+11}{3} \leq m \leq n. \end{cases}$$

Moreover, by Theorem 5.0.3, $CH_m \geq 2$. So, we can obtain $G$ is doubly chorded pancyclic.

**Corollary 5.0.4** *Let $G$ be a $2$-connected $K_{1,3}$-free graph with the order $n \geq 35.$ If $\delta(G) \geq \frac{n-2}{3}$, then $G$ is doubly chorded pancyclic.*

## 5.1 Terminology and notations

A *chord* of a cycle is an edge between two nonadjacent vertices of the cycle. We say that a cycle is *chorded* if the cycle has at least one chord, and we call such a cycle *chorded cycle*. If a cycle has at least two chords, then the cycle is called a *doubly chorded cycle*. A graph $G$ of order $n$ is *chorded pancyclic* (*doubly chorded pancyclic*) if $G$ contains a chorded cycle (doubly chorded cycle) of each length from 4 to $n$.

Bondy's metaconjecture (see Chapter 1 or Chapter 2) was extended into almost any condition that implies a graph is hamiltonian will imply it is chorded pancyclic, possibly with some class of well-defined exceptional graphs and some small order exceptional graphs. As support for the extension of Bondy's metaconjecture, there are many results (see Section 1.3.2 in Chapter 1).

For a vertex set $S$ of $V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$.

Given a family $\pounds = \{H_1, H_2, \ldots, H_k\}$ of graphs, we say that a graph $G$ is *$\pounds$-free* if $G$ has no induced subgraph isomorphic to any $H_i$ with $i = 1, 2, \ldots, k$. In particular, if $\pounds = \{H\}$, we simply say $G$ is *$H$-free*.

From Theorem 5.0.1, we got our main result (Theorem 5.0.2). Theorem 5.0.2 supports for extension of Bondy's metaconjecture.

When $G$ is chorded pancyclic, it is in nature to consider how many chords in a cycle of length $l$, for any $1 \leq l \leq n$, where $n$ is the order of $G$. Thus, we obtain Theorem 5.0.3.

It is necessary to introduce the followings.

We say that a graph $G$ is *traceable* if it contains a spanning path (that is, the path containing all the vertices of $G$ ). For any integer $m$, denote by $C_m$ a cycle of length $m$.

## 5.2 The proof of main results

### 5.2.1 Preparation for the proof

To prove main results, we use the following theorem:

**Theorem 5.2.1 ([34])** *Let $G$ be a graph with at least three vertices. For some $s$, if $G$ is $s$-connected and contains no independent set of more than $s$ vertices, then $G$ has a hamiltonian cycle.*

From Theorem 5.2.1, we obtain the following lemma:

**Lemma 5.2.2** *Let $G$ be a $K_{1,3}$-free graph. For any $x \in V(G)$, then $G[N_G(x)]$ is either traceable, or two disjoint cliques.*

**Proof.** We assume that $x$ is any vertex in $V(G)$. Suppose that $G[N_G(x)]$ is disconnected, then there are only two components $G_1$ and $G_2$ in $G[N_G(x)]$ since $G$ is $K_{1,3}$-free.

For the sake of contradiction, suppose that there are two nonadjacent vertices $u$ and $v$ in $V(G_1)$. Let $z$ be a vertex in $V(G_2)$. Then $\{x, u, v, z\}$ induces a $K_{1,3}$ in $G$, which contradicts that $G$ is $K_{1,3}$-free. Hence, $G[N_G(x)]$ is two disjoint cliques.

If $G[N_G(x)]$ is $1$-connected, then let $u$ be a vertex-cut of $G[N_G(x)]$. Since $G$ is $K_{1,3}$-free, then let $u$ cuts $G[N_G(x)]$ into two components $G'$ and $G''$. The same argument as when $G[N_G(x)]$ is disconnected, then $G'$ and $G''$ are cliques. It follows that $G[N_G(x)]$ is traceable.

If $G[N_G(x)]$ is $2$-connected, since $G$ is $K_{1,3}$-free, it follows from Theorem 5.2.1 that $G[N_G(x)]$ is traceable.

The proof of this lemma is completed. ∎

### 5.2.2 Proof of Theorem 5.0.2

In this section we prove Theorem 5.0.2.

Note that $\delta(G) \geq \frac{n-2}{3} \geq 11$ since $n \geq 35$. For the sake of a contradiction, we suppose that $G$ is not chorded pancyclic. Let $m$ be the largest value with $4 \leq m \leq n$ such that $G$ has no chorded cycle of length $m$. By Theorem 5.0.1, there exists a chorded cycle of length $n$, and so $m \neq n$.

By Theorem 5.0.1, $G$ is pancyclic. We divide the proof into some cases according to the value of $m$.

**Case 1** $m \geq 9$.

Let $C = v_1v_2v_3 \cdots v_mv_1$ be such a cycle in $G$. For any two vertices $v, w \in V(C)$ with $vw \notin E(C)$, since $C$ is not a chorded cycle, then $vw \notin E(G)$. We will show that $N(v_1) \cap N(v_4) = \emptyset$.

Suppose that there exists a vertex $x \in N(v_1) \cap N(v_4)$. Since $\delta(G) \geq \frac{n-2}{3} \geq 11$, there is a vertex $y \in V(G-C)-\{x\}$ such that $v_6y \in E(G)$. As $G$ is $K_{1,3}$-free and $v_5v_7 \notin E(G)$, then $y$ is adjacent to either $v_5$ or $v_7$.

If $y$ is adjacent to $v_5$, then $v_1xv_4v_5yv_6v_7 \cdots v_mv_1$ is a cycle of length $m$ with the chord $v_5v_6$. Otherwise, $v_1xv_4v_5v_6yv_7 \cdots v_mv_1$ is a cycle of length $m$ with the chord $v_6v_7$. This is a contradiction.

Similarly, $N(v_4) \cap N(v_7) = \emptyset$. We show that $N(v_1) \cap N(v_7) = \emptyset$. If $v_{10} = v_1$, the similar to $N(v_1) \cap N(v_4) = \emptyset$, we are done.

We may assume that $v_{10} \neq v_1$. Suppose that there is a vertex $z \in N(v_1) \cap N(v_7)$. Since $\delta(G) \geq \frac{n-2}{3} \geq 11$ and $G$ is $K_{1,3}$-free, there must exist four vertices $x_1, x_2, x_3, x_4 \in N(v_9)$ such that $x_1x_2x_3x_4$ is a path in $G$. Since $G$ is $K_{1,3}$-free, then $x_4v_8 \in E(G)$ or $x_4v_{10} \in E(G)$.

Let

$$C' = \begin{cases} v_1zv_7v_8x_4x_3x_2x_1v_9v_{10}\cdots v_mv_1 & \text{if } x_4v_8 \in E(G), \\ v_1zv_7v_8v_9x_1x_2x_3x_4v_{10}\cdots v_mv_1 & \text{if } x_4v_{10} \in E(G) \end{cases}$$

Then $C'$ is a cycle of length $m$ with the chord $x_2v_9$, a contradiction.

Hence, $N(v_1) \cap N(v_7) = \emptyset$. Since $N(v_1) \cap N(v_4) = N(v_4) \cap N(v_7) = N(v_1) \cap N(v_7) = \emptyset$, we obtain that

$$\begin{aligned} n-2 &\leq d(v_1) + d(v_4) + d(v_7) \\ &\leq 6 + |V(G-C)| \\ &= n - m + 6. \end{aligned}$$

So, we obtain $m \leq 8$, which contradicts that $m \geq 9$.

**Case 2** $4 \leq m \leq 8$.

First, we give the following result.

**Claim 5.2.3** *If there exists a cycle $C_l = v_1v_2\cdots v_lv_1$ of length $l$ in $G$ for some $3 \leq l \leq 7$ and there does not exist a chorded cycle $C$ of length $l+1$ in $G$, then for any two vertices $v_i, v_j \in V(C_l)$, $v_i$ and $v_j$ has no common neighbor in $V(G) \setminus V(C_l)$.*

**Proof.** Without loss of generality, let $x \in N_{G-C_l}(v_1)$. Since there exists no chorded cycle of length $l+1$ in $G$, then $x$ is not adjacent to two consecutive vertices in $C_l$.

To the contrary, we assume $v_jx \in E(G)$ with $3 \leq j \leq \lceil \frac{l}{2} \rceil$. Note that $3 \leq j \leq 4$ since $3 \leq l \leq 7$. Since $G$ is $K_{1,3}$-free, $v_{j-1}v_{j+1} \in E(G)$. Let

$$C' = \begin{cases} C_l - \{v_1v_l, v_2v_3\} \cup \{v_1x, v_3x, v_2v_l\} & \text{if } v_3x \in E(G), \\ C_l - \{v_1v_l, v_2v_3, v_4v_5\} \cup \{v_1x, xv_4, v_2v_l, v_3v_5\} & \text{otherwise.} \end{cases}$$

Then $C'$ is a cycle of length $l+1$ with the chord $v_2v_3$. This is a contradiction.

By the symmetry, this claim is proved. ∎

Now, we have two subcases.

**Subcase 2.1** $m = 8$.

Let $C_7 = v_1v_2\cdots v_7v_1$ is a cycle of length 7 in $G$. By Claim 5.2.3, for any $v_i, v_j \in V(C_7)$, $N_{G-C_7}(v_i) \cap N_{G-C_7}(v_j) = \emptyset$. And $|N_{G-C_7}(v_i)| \geq \delta - 6$ for any $v_i \in V(C_7)$. Thus, since $\delta(G) \geq \frac{n-2}{3}$,

$$\begin{aligned} n-7 &\geq \sum_{1\leq i\leq 7} |N_{G-C_7}(v_i)| \\ &\geq 7(\delta - 6) \\ &\geq \frac{7n-14}{3} - 42. \end{aligned}$$

So, we obtain $n < 30$, which contradicts that $n \geq 35$.

**Subcase 2.2** $4 \leq m \leq 7$

The following property which is important for our work, is that:

**Claim 5.2.4** *If there exists a cycle $C_l$ of length $l$ in $G$ for some $3 \leq l \leq 6$, then there exists a chorded cycle $C$ of length $l+1$ in $G$.*

**Proof.** Let $C_l = v_1v_2\cdots v_lv_1$ is a cycle of length $l$ in $G$ with $3 \leq l \leq 6$. To be contrary, we assume that there does not exist a chorded cycle $C$ of length $l+1$ in $G$. Since $\delta(G) \geq \frac{n-2}{3} \geq 11$, then $|N_{G-C_l}(v_i)| \geq 6$ for each $1 \leq i \leq l$.

Since $G$ is $K_{1,3}$-free, it follows from Claim 5.2.3 and Lemma 5.2.2 that $G[N_{G-C_l}(v_i)]$ is a clique for each $1 \leq i \leq l$. When $3 \leq l \leq 6$, $|N_{G-C_l}(v_i)| \geq 6$ since $\delta(G) \geq \frac{n-2}{3} \geq 11$. Hence, there is a chorded cycle with length $l+1$ in $G[N_{G-C_l}(v_i) \cup \{v_i\}]$ for each $1 \leq i \leq l$. The proof of Claim 5.2.4 is completed. ∎

Since $G$ is pancyclic, it follows from Claim 5.2.4 that $G$ has a chorded cycle of length $m$ with $4 \leq m \leq 7$. This is a contradiction. Hence, this theorem holds. □

Next we will prove Theorem 1.3.28 (i.e., Theorem 5.0.3).

### 5.2.3 Proof of Theorem 5.0.3

By Theorem 5.0.2, $G$ is chorded pancyclic. Let $C_m$ be a chorded cycle in $G$ with $4 \leq m \leq n$. We have the following cases.

**Case 1** $4 \leq m \leq 5$.

When $m = 4$. For any vertex $x \in V(G)$, let $y \in N(x)$. If there are 3 vertices $u_1, u_2, u_3 \in N(x) - \{y\}$ such that $u_1, u_2, u_3 \notin N(y)$. Since $G$ is $K_{1,3}$-free, then $G[\{x, u_1, u_2, u_3\}]$ is clique, we are done. It follows from $\delta \geq \frac{n-2}{3} \geq 11$

that there exist 3 vertices $v_1, v_2, v_3 \in (N(x) - \{y\}) \cap N(y)$. Since $G$ is $K_{1,3}$-free, then we may assume $v_1v_2 \in E(G)$. Hence, $G[\{v_1, v_2, x, y\}]$ is clique, we are done.

When $m = 5$. We suppose that there does not exist chorded cycle $C_5$ in $G$ such that $CH_5 \geq \frac{m(m-1)}{2} - m = 5$. For any vertex $x \in V(G)$, let $y \in N(x)$.

**Subcase 1.1** $|N(y) \cap (N(x) - \{y\})| \leq d(x) - 5$.

There are 4 vertices $u_1, u_2, u_3, u_4 \in N(x) - \{y\}$ such that $u_1, u_2, u_3, u_4 \notin N(y)$. Since $G$ is $K_{1,3}$-free, then $G[\{u_1, u_2, u_3, u_4, x\}]$ is clique. We have done.

**Subcase 1.2** $|N(y) \cap (N(x) - \{y\})| \geq d(x) - 4$.

Since $\delta \geq \frac{n-2}{3} \geq 11$, then $|N(y) \cap (N(x) - \{y\})| \geq d(x) - 4 \leq 7$. By $R(3, 3) = 6$, since $G$ is $K_{1,3}$-free graph, then there are $v_1, v_2, v_3 \in (N(x) - \{y\}) \cap N(y)$ such that $v_1v_2v_3v_1$ is a cycle. Hence, $G[\{v_1, v_2, v_3, x, y\}]$ is clique. This is a contradiction.

**Case 2** $6 \leq m \leq \frac{n+1}{3}$.

We prove this case by induction on $m$.

When $m = 6$, by Case 1, let $C_5 = v_1v_2v_3v_4v_5v_1$ be a chorded cycle, and $G[\{v_1, v_2, v_3, v_4, v_5\}]$ be a clique. Suppose there exists $v_i \in V(C_5 \setminus \{v_1\})$ such that $N_{G-C_5}(v_1) \cap N_{G-C_5}(v_i) \neq \emptyset$. We assume $x \in N_{G-C_5}(v_1) \cap N_{G-C_5}(v_i)$, then $C = xv_1v_2 \cdots v_{i-1}v_5v_4 \cdots v_ix$ is a cycle of length 6 with $CH_6 \geq 6$ chords. Hence, for any $v_i, v_j \in V(C_5)$, $N_{G-C_5}(v_i) \cap N_{G-C_5}(v_j) = \emptyset$. And $|N_{G-C_5}(v_i)| \geq \delta - 4$ for any $v_i \in V(C_5)$. Thus, since $\delta(G) \geq \frac{n-2}{3}$,

$$\begin{aligned} n - 5 &\geq \sum_{1 \leq i \leq 5} |N_{G-C_5}(v_i)| \\ &\geq 5(\delta - 4) \\ &\geq \frac{5n - 10}{3} - 20. \end{aligned}$$

So, we obtain $n < 28$, which contradicts that $n \geq 35$.

Next, we suppose there is a cycle $C_m$ with $CH_m \geq m$ chords for any $m < \frac{n+1}{3}$. We will show there is a cycle $C_{m+1}$ with $CH_{m+1} \geq m + 1$ chords. Let $C_m = v_1v_2 \cdots v_mv_1$ be such cycle with $CH_m \geq m$ chords. For the sake of a contradiction, we suppose that $G$ does not exist a cycle $C_{m+1}$ with $CH_{m+1} \geq m + 1$ chords.

If $m = 6$, then $|N_{G-C_6}(v_i)| \geq \delta - 5 \geq 6$. Since $C_6$ is a chorded cycle with 6 chorded, and $G$ is $K_{1,3}$-free, then for any vertex $x \in V(N_{G-C_6}(v_i))$ such that $xv_j \notin E(G)$, where $v_j \in V(C_6 \setminus \{v_i\})$. By Lemma 5.2.2, then $G[N_{G-C_6}(v_i) \cup \{v_i\}]$ is clique. So, there is a cycle $C_7$ with chords $CH_7 \geq 7$, a contradiction. So, $m \geq 7$.

Suppose there exists $v_i \in V(C_m)$ such that $v_{i-1}v_{i+1} \notin E(G)$. Since $d(v_i) \geq \delta \geq \frac{n-2}{3}$, it follows from $G$ is $K_{1,3}$-free that there exists $x \in N_{G-C_m}(v_i)$ such that $xv_{i-1} \in E(G)$ or $xv_{i+1} \in E(G)$. Let

$$C = \begin{cases} xv_iv_{i+1}\cdots v_mv_1\cdots v_{i-1}x & \text{if } xv_{i-1} \in E(G), \\ xv_{i+1}\cdots v_mv_1\cdots v_ix & \text{if } xv_{i+1} \in E(G). \end{cases}$$

Then $C$ is a cycle of length $m+1$ with $CH_{m+1} \geq m+1$ chords, a contradiction.

So, for any $v_i \in V(C_m)$ such that $v_{i-1}v_{i+1} \in E(G)$ ($v_0 = v_m, v_{m+1} = v_1$). Suppose there exists $u \in N_{G-C_m}(v_i)$ such that $v_j \in N(u) \cap C_m$. Without loss of generality, assume $v_j = v_1$. Let

$$C' = \begin{cases} uv_iv_{i-2}v_{i-4}\cdots v_kv_{k-2}\cdots v_2v_3v_5v_7\cdots v_lv_{l+2}\cdots v_{i-1}v_{i+1}v_{i+2}\cdots v_1u & \text{if } i \text{ is even}, \\ uv_iv_{i-2}v_{i-4}\cdots v_kv_{k-2}\cdots v_3v_2v_4\cdots v_lv_{l+2}\cdots v_{i-1}v_{i+1}v_{i+2}\cdots v_1u & \text{if } i \text{ is odd}. \end{cases}$$

Then $C'$ is a cycle of length $m+1$ with $CH_{m+1} \geq m+1$ chords, a contradiction. So $N(u) \cap V(C_m \setminus \{v_i\}) = \emptyset$ for any $u \in N_{G-C_m}(v_i)$ and $v_i \in C_m$.

We will show $d_{C_m}(v_i) = m-1$ with any $v_i \in V(C_m)$. Since $m \leq \frac{n-2}{3} \leq \delta$, $N_{G-C_m}(v_j) \neq \emptyset$ for any $v_j \in C_m$. Assume $v \in N_{G-C_m}(v_{i-2})$, then $\{v_{i-2}, v, v_{i-3}, v_i\}$ induces $K_{1,3}$ in $G$ unless $v_{i-3}v_i \in E(G)$. Assume $v' \in N_{G-C_m}(v_{i-3})$, $\{v_{i-3}, v', v_{i-4}, v_i\}$ induces $K_{1,3}$ in $G$ unless $v_{i-4}v_i \in E(G)$. So $v_iv_j \in E(G)$ for any $v_j \in V(C_m - \{v_i\})$ and $G[V(C_m)]$ is clique.

Next, we will show that for any $x \in N_{G-C_m}(v_i)$ and $y \in N_{G-C_m}(v_{i+1})$, we have $xy \notin E(G)$.

To the contrary, suppose $x \in N_{G-C_m}(v_i)$ and $y \in N_{G-C_m}(v_{i+1})$ such that $xy \in E(G)$.

Let $C'' = xv_iv_{i-2}v_{i-3}v_{i-4}\cdots v_{i+1}yx$, then $C''$ is a cycle of length $m+1$ with the chords $CH_{m+1} \geq 2(m-4)+1+(m-5) \geq m+1$ with $m \geq 7$. This is a contradiction. So, for any $x \in N_{G-C_m}(v_i)$ and $y \in N_{G-C_m}(v_{i+1})$ such that $xy \notin E(G)$.

Further, we will prove that for any vertex $x_1 \in N_{G-C_m}(v_i)$ and $y_1 \in N_{G-C_m}(v_{i+1})$ such that $N(x_1) \cap N(y_1) = \emptyset$.

Suppose $x \in N_{G-C_m}(v_i)$ and $y \in N_{G-C_m}(v_{i+1})$ such that $z \in N(x) \cap N(y)$.

When $m \geq 8$. Since $d_{C_m}(v_i) = m-1$ with $v_i \in V(C_m)$, then $C^* = zxv_iv_{i-3}v_{i-4}\cdots v_{i+1}yz$ is a cycle of length $m+1$ with chords $CH_{m+1} \geq \frac{(m-2)(m-3)}{2} - (m-2) + 1 \geq m+1$, a contradiction.

When $m = 7$. If $|N_{G-C_7}(v_i)| \geq 7$, then $G[N_{G-C_7}(v_i) \cup \{v_i\}]$ is clique, we are done. So $|N_{G-C_7}(v_i)| \leq 6$. It follows from $\frac{n-2}{3} \leq \delta \leq d(v_i) \leq 12$ that $n \leq 38$. Since $\bigcap_{i=1}^{7} N_{G-C_7}(v_i) = \emptyset$ and $|N_{G-C_7}(v_i)| \geq \delta - 6 \geq 5$, then $n \geq \sum_{i=1}^{7} |N_{G-C_7}(v_i)| + 7 \geq 42$. This is a contradiction.

Thus, for any vertex $x_1 \in N_{G-C_m}(v_i)$ and $y_1 \in N_{G-C_m}(v_{i+1})$ such that $N(x_1) \cap N(y_1) = \emptyset$. Since $G[V(C_m)]$ is clique, then for any vertex $z_1 \in N_{G-C_m}(v_{i+2})$, $N(y_1) \cap N(z_1) = \emptyset$ and $N(x_1) \cap N(z_1) = \emptyset$. Then

$$\begin{aligned} n-2 &\leq d(x_1) + d(y_1) + d(z_1) \\ &\leq 3 + |V(G - C_m)| \\ &= n - m + 3. \end{aligned}$$

Thus, we obtain $m \leq 5$, which contradicts that $m \geq 7$.

**Case 3** $\frac{n+4}{3} \leq m \leq \frac{2n+8}{3}$.

For the sake of a contradiction, we suppose that $G$ does not exist a cycle $C_m$ with chords $CH_m \geq [\frac{m}{6}]$. By Theorem 5.0.2, let $C_m = v_1 v_2 \cdots v_m v_1$ be a chorded cycle with chords $CH_m \leq [\frac{m}{6}] - 1$.

Assume $S = \{v_i \in V(C_m) | d_{C_m}(v_i) = 2\}$, then $|S| \geq \frac{4m}{6} + 1$ otherwise $CH_m \geq \frac{2\times\frac{4m}{6}+3\times\frac{2m}{6}}{2} - m \geq [\frac{m}{6}]$, a contradiction.

Now we show $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{2+[\frac{m}{6}]}) = \emptyset$ with $[\frac{m}{6}] \geq 3$. Suppose $x \in N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{2+[\frac{m}{6}]})$.

Assume $S_1 = S \cap V(C_m(v_{2+[\frac{m}{6}]}, v_1))$, then $|S_1| \geq \frac{3m}{6} - 1$. If for any vertex $v_i \in S_1$ such that $v_{i-1}v_{i+1} \in E(G)$, then there are $CH_m \geq \frac{3m}{6} - 1 \geq [\frac{m}{6}]$ chords in $C_m$, a contradiction. So, there exists $v_i \in S_1$ such that $v_{i-1}v_{i+1} \notin E(G)$. Let $T = N_{G-C_m}(v_i) \cap N_{G-C_m}(v_{i+1})$. Without loss of generality, assume $|T| \geq \frac{\delta-2}{2}$. It follows from $m \leq 2\delta + 4$ and $\delta \geq 11$ that $|T| \geq [\frac{m}{6}] - 1$.

By Lemma 5.2.2, when $G[T]$ is traceable, let $P$ be a path in $G[T]$ such that $|P| = [\frac{m}{6}] - 1$, then $C' = v_1 x v_{2+[\frac{m}{6}]} v_{3+[\frac{m}{6}]} \cdots v_i P v_{i+1} \cdots v_m v_1$ is a cycle of length $m$ with $CH_m \geq [\frac{m}{6}]$ chords.

When $G[T]$ is two disjoint cliques. It follows from $G$ is $K_{1,3}$-free that there exists a vertex $v \in T$ such that $vv_{i-1} \in E(G)$. So, we can find two paths $P_1$ and $P_2$ in $G[T]$ such that $v$ is the endpoint of $P_2$ and $|P_1| + |P_2| = [\frac{m}{6}] - 1$. Then $C'' = v_1 x v_{2+[\frac{m}{6}]} v_{3+[\frac{m}{6}]} \cdots v_{i-1} P_2 v_i P_1 v_{i+1} \cdots v_m v_1$ is a cycle of length $m$ with $CH_m \geq [\frac{m}{6}]$ chords. This is a contradiction.

So, $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{2+[\frac{m}{6}]}) = \emptyset$. Similarly, $N_{G-C_m}(v_{3+[\frac{2m}{6}]}) \cap N_{G-C_m}(v_{2+[\frac{m}{6}]}) = \emptyset$, where $[\frac{m}{6}] \geq 3$.

Next, we will show $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{3+[\frac{2m}{6}]}) = \emptyset$. Suppose $x' \in N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{3+[\frac{2m}{6}]})$. Let $S_2 = S \cap V(C_m(v_{3+[\frac{2m}{6}]}, v_1))$, then $|S_2| \geq \frac{2m}{6} - 2$.

Suppose for any vertex $v_i \in S_2$ such that $v_{i-1}v_{i+1} \in E(G)$, then there are $CH_m \geq \frac{2m}{6} - 2 \geq [\frac{m}{6}]$ chords in $C_m$, a contradiction.

So, there exists $v_i \in S_2$ such that $v_{i-1}v_{i+1} \in E(G)$. Let $A_1 = \{x_j \in N_{G-C_m}(v_i) | v_{i-1}x_j \in E(G), v_{i+1}x_j \notin E(G)\}$ and $A_2 = N_{G-C_m}(v_i) - A_1$. Then $|A_1| + |A_2| \geq \delta - 2 \geq [\frac{2m}{6}]$. By Lemma 5.2.2, $G[A_1]$ is a clique or $A_1 = \emptyset$. So, there is a hamiltonian path $Q$ in $G[A_1]$.

By Lemma 5.2.2, suppose $G[A_2]$ is traceable, then there is a path $Q_1$ such that $|Q_1| + |Q| = [\frac{2m}{6}]$. Then $C^1 = v_1x'v_{3+[\frac{2m}{6}]}v_{4+[\frac{2m}{6}]} \cdots v_{i-1}Qv_iQ_1v_{i+1} \cdots v_mv_1$ is a cycle of length $m$ with $CH_m \geq [\frac{m}{6}]$ chords.

Suppose $G[A_2]$ is two disjoint cliques. If $A_1 \neq \emptyset$, since $G$ is $K_{1,3}$-free, there exist $v' \in A_2$ and $u \in A_1$ such that $uv' \in E(G)$. So, we can find two paths $Q_2$ and $Q_3$ in $G[A_2]$ such that $v'$ is the endpoint of $Q_3$ and $|Q_2|+|Q_3|+|Q| = [\frac{2m}{6}]$. Then $C^2 = v_1x'v_{3+[\frac{2m}{6}]}v_{4+[\frac{2m}{6}]} \cdots v_{i-1}Quv'Q_3v_iQ_2v_{i+1} \cdots v_mv_1$ is a cycle of length $m$ with $CH_m \geq [\frac{m}{6}]$ chords. This is a contradiction. If $A_1 = \emptyset$, since $G$ is $K_{1,3}$-free, then there exist $v'' \in A_2$ such that $v''v_{i-1} \in E(G)$. So, we can find two paths $Q_4$ and $Q_5$ in $G[A_2]$ such that $v''$ is the endpoint of $Q_5$ and $|Q_4| + |Q_5| = [\frac{2m}{6}]$. Then $C^2 = v_1x'v_{3+[\frac{2m}{6}]}v_{4+[\frac{2m}{6}]} \cdots v_{i-1}v''Q_5v_iQ_4v_{i+1} \cdots v_mv_1$ is a cycle of length $m$ with $CH_m \geq [\frac{m}{6}]$ chords. This is a contradiction.

So $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{3+[\frac{2m}{6}]}) = \emptyset$. Then

$$\begin{aligned} n-2 &\leq d(v_1) + d(v_{2+[\frac{m}{6}]}) + d(v_{3+[\frac{2m}{6}]}) \\ &\leq |V(G-C_m)| + 6 + 6 + [\frac{m}{6}] - 4 \\ &= n - m + [\frac{m}{6}] + 8. \end{aligned}$$

Thus, we obtain $m \leq 12$, which contradicts that $m \geq \frac{n+4}{3} \geq 13$, where $[\frac{m}{6}] \geq 3$.

Suppose $[\frac{m}{6}] = 2$, by Theorem 1.3.27, $C_m$ is a cycle with a chord. Since $G$ is $K_{1,3}$-free, without loss of generality, we assume $v_1v_3 \in E(G)$. Now we show $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_4) = \emptyset$. Suppose $u \in N_{G-C_m}(v_1) \cap N_{G-C_m}(v_4)$.

Since there does not exist 2 chords in $C_m$, we can assume $w \in N_{G-C_m}(v_i) \cap N_{G-C_m}(v_{i+1})$ with $v_i \in V(C_m[v_5, v_m])$. Let $C = v_1uv_4v_5 \cdots v_iwv_{i+1} \cdots v_1$. If $uv_m \in E(G)$, then $C$ is a cycle of length $m$ with the chords $uv_m$ and $v_iv_{i+1}$, a contradiction. It follows from $G$ is $K_{1,3}$-free that $uv_3 \in E(G)$. Then, $C^* = v_1uv_3v_4 \cdots v_mv_1$ is a cycle of length $m$ with the chords $v_1v_3$ and $uv_4$, a contradiction. So $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_4) = \emptyset$. Similarly, $N_{G-C_m}(v_4) \cap N_{G-C_m}(v_7) = \emptyset$.

It follows from $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_{3+[\frac{2m}{6}]}) = \emptyset$ that $N_{G-C_m}(v_1) \cap N_{G-C_m}(v_7) = \emptyset$. Hence, we obtain that

$$\begin{aligned} n-2 &\leq d(v_1) + d(v_4) + d(v_7) \\ &\leq 7 + |V(G-C_m)| \\ &= n - m + 7. \end{aligned}$$

So, we obtain $m \leq 9$, which contradicts that $m \geq \frac{n+4}{3} \geq 13$.

**Case 4** $\frac{2n+11}{3} \leq m \leq n$.

Assume $C_m = v_1v_2 \cdots v_mv_1$ be a cycle in $G$ with $CH_m$ chords. For any vertex $v_i \in V(C_m)$, $d_{G-C_m}(v_i) \leq n-m$ and $d_{C_m} \geq \delta - (n-m)$. So, $CH_m \geq \frac{m(\delta-(n-m))}{2} - m$.

Hence, the theorem holds. □

## 5.3 Open problems

A non-induced cycle is called a chorded cycle. A graph $G$ of order $n$ is chorded pancyclic if $G$ contains a chorded cycle of each length from 4 to $n$. A graph is called $K_{1,3}$-free if it has no induced $K_{1,3}$ subgraph.

In this chapter, we prove that the following result: every $2$-connected $K_{1,3}$-free graph $G$ with $|V(G)| \geq 35$ is chorded pancyclic if the minimum degree is at least $\frac{n-2}{3}$. We show the number of chords in the chorded cycle of length $l$ $(4 \leq l \leq n)$. Moreover, $G$ is doubly chorded pancyclic.

At present, there are not many types of research on chorded pancyclic. So, there's a lot of room for research. Can we find more necessary and sufficient conditions for a graph to be chorded pancyclic? That's what we're going to work on.

# Chapter 6

# $k$-fan-connected graphs

In this chapter, we will show the result of $k$-fan-connected graph by improving the degree sum condition of Theorem 3.1. We recall Theorem 3.1 by Lin, Tan, et al. here.

**Theorem 6.0.1 (Lin, Tan, et al. [97])** *Let $k \geq 2$ be an integer and $G$ be a graph. If $\sigma_2(G) \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected.*

Our main result is Theorem 1.3.53. We reaffirm this theorem here.

**Theorem 6.0.2** *Let $k \geq 2$ be an integer and $G$ be a $(k+1)$-connected graph. If $\overline{\sigma_3}(G) \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected.*

We can obtain the following corollary that is stronger than Theorem 6.1.7 in the case of $3$-connected graphs.

**Corollary 6.0.3** *Let $G$ be a $3$-connected graph. If $\overline{\sigma_3}(G) \geq |G| + 1$, then $G$ is Hamilton-connected.*

In this chapter, we use some new notations. Let $T$ be a tree and let $r \in V(T)$. The *outdirected tree* concerning $(T, r)$ is the directed tree obtained from $T$ in which all the edges are directed away from $r$. For $X \subset V(T)$ and $Y \subset V(T)$, $X^-_{T,r}$ and $Y^+_{T,r}$, denote the set of the predecessors and the successors of the vertices of $X$ and $Y$ in $(T, r)$, respectively. Similarly, for $x \in V(T)$, $x^-_{T,r}$ denote the predecessor of $x$ in $(T, r)$, respectively. If there is no ambiguity, we write $X^-_r$, $Y^+_r$, and $x^-_r$ for $X^-_{T,r}$, $Y^+_{T,r}$, and $x^-_{T,r}$, respectively.

We shall prove Theorem 1.3.53 (i.e., Theorem 6.0.2) by contradiction and induction. In section 6.1, we will present Menger's Theorem and give some other related introductions. The lower bound of $\overline{\sigma_3}(G)$ in Theorem 1.3.53 (i.e., Theorem 6.0.2) is sharp, as shown in Section 6.2. In section 6.3, to prove the theorem 1.3.53 (i.e., Theorem 6.0.2), we're going to introduce some preliminaries. In section 6.4, we will prove Theorem 1.3.53 (i.e., Theorem 6.0.2).

## 6.1 Menger's Theorem and introduction

### 6.1.1 Menger's Theorem

We start with Menger's Theorem which is one of the cornerstones of graph theory.

We first give some definitions about Menger's theorem.

Let $G = (V, E)$ be a graph and $A, B \subseteq V$, we call $P = x_0 \cdots x_k$ an $A - B$ *path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$. We write $a - B$ *path* rather than $\{a\} - B$ path. If $X \subseteq V \cup E$ are such that every $A - B$ path in $G$ contains a vertex or an edge from $X$, we say that $X$ *separates* the sets $A$ and $B$ in $G$.

Menger's theorem takes many versions. A simple, very general versions of Menger's Theorem is as follows:

**Theorem 6.1.1 (Menger 1927 [101])** *Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating $A$ from $B$ in $G$ is equal to the maximum number of disjoint $A - B$ paths in $G$.*

From this Theorem, we get the following Corollaries:

**Corollary 6.1.2** *For $B \subseteq V$ and $a \in V \setminus B$, the minimum number of vertices $\neq a$ separating a from $B$ in $G$ is equal to the maximum number of paths forming an $a - B$ fan in $G$.*

**Corollary 6.1.3** *Let $a$ and $b$ be two distinct vertices of $G$.*

1. *If $ab \notin E(G)$, then the minimum number of vertices $\neq a, b$ separating a from $b$ in $G$ is equal to the maximum number of independent $a - b$ paths in $G$.*

2. *The minimum number of edges separating $a$ from $b$ in $G$ is equal to the maximum number of edge-disjoint $a - b$ paths in $G$.*

The following is a global Version of Menger's Theorem.

**Theorem 6.1.4 (Global Version of Menger's Theorem)**

1. *A graph is $k$-connected if and only if it contains $k$ independent paths between any two vertices.*

2. *A graph is $k$-edge-connected if and only if it contains $k$ edge-disjoint paths between any two vertices*

This version of Menger's Theorem is the one we usually use the most. In section 6.4, our proof of Theorem 1.3.53 uses a global version of Menger's Theorem.

### 6.1.2 Introduction and notations

We will use standard notations and terminology of graph theory. To make it easier to read, in this section we again introduce some definitions and notations. For a vertex $x \in V(G)$, we denote the degree of $x$ in $G$ by $\deg_G(x)$ and the set of neighbors of the vertex $x$ in $G$ by $N_G(x)$, where $N_G(x) = \{v \in V(G) | xv \in E(G)\}$ and $d_G(x) = |N_G(x)|$.

A vertex cut is a set $S \subset V(G)$ such that $G - S$ has more than one component. A graph is $k$-connected if every vertex cut has at least $k$ vertices. The connectivity of $G$, $\kappa(G)$, is the minimum size of a vertex cut, i.e., $\kappa(G)$ is the maximum $k$ such that $G$ is $k$-connected. Let $\alpha(G)$ be the number of the vertices of a maximum independent set in $G$. For any integer $m \geq 2$, if $\alpha(G) \geq m$, put

$$\sigma_m(G) = \min\left\{ \sum_{i=1}^{m} \deg_G(x_i) \,\middle|\, x_1, x_2, \ldots, x_m \text{ are pairwisely nonadjacent vertices in G} \right\}$$

$$\overline{\sigma_m}(G) = \min\left\{ \sum_{i=1}^{m} \deg_G(x_i) - |\bigcap_{i=1}^{m} N_G(x_i)| \,\middle|\, x_1, x_2, \ldots, x_m \text{ are pairwisely nonadjacent vertices in G} \right\}$$

If $G$ does not have $m$ vertices that are independent, we define $\sigma_m(G) = \overline{\sigma_m}(G) = \infty$. By the definition of $\sigma_m(G)$ and $\overline{\sigma_m}(G)$, we obtain the following proposition.

**Proposition 6.1.5** *For a graph $G$, $\sigma_m(G) \leq \overline{\sigma_{m+1}}(G)$.*

The proof of Proposition 6.1.5 is easy. Now I will prove it briefly.

**Proof.** Let $\{x_1, x_2, \ldots, x_m\}$ be an independent set of vertices in $G$ such that $\sigma_m(G) = \sum_{i=1}^{m} \deg_G(x_i)$. And assume $\{y_1, y_2, \ldots, y_{m+1}\}$ be independent set of vertices in $G$ such that $\overline{\sigma_{m+1}}(G) = \sum_{i=1}^{m+1} \deg_G(y_i) - |\bigcap_{i=1}^{m+1} N_G(y_i)|$.

From the definition of $\sigma_m(G)$, we can obtain $\sigma_m(G) \leq \sum_{i=1}^{m} \deg_G(y_i)$. And it is easy to know that $\deg_G(y_i) \geq |\bigcap_{i=1}^{m+1} N_G(y_i)|$. It follows that $\deg_G(y_{m+1}) \geq |\bigcap_{i=1}^{m+1} N_G(y_i)|$. Thus $\sigma_m(G) \leq \overline{\sigma_{m+1}}(G)$. ∎

The related definition of hamiltonian was introduced in the section 1.1 of the chapter 1, here I will explain it again.

A *hamiltonian path* of a graph $G$ is a path that contains all vertices of $V(G)$. A graph $G$ is *Hamilton-connected* if there is a hamiltonian path between every two different vertices. A cycle containing all vertices of $G$ is called a *hamiltonian cycle* and $G$ is called *hamiltonian* if it contains a hamiltonian cycle. Let $K_m$ and $C_m$ denote the complete graph of $m$ vertices and the cycle of length $m$, respectively.

One of the core subjects in hamiltonian graph theory is to develop sufficient conditions for a graph to have a hamiltonian path/cycle (refer to [84] for a survey). Some further sufficient conditions related to degrees of vertices with distance exactly two for hamiltonian graphs can be found in Chapters 1 and 2.

We begin with a well-known result due to Ore.

**Theorem 6.1.6 (Ore [109])** *Let $G$ be a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$. Then $G$ is hamiltonian.*

The following result gives the degree sum condition for graphs to be Hamilton-connected by Ore [110] in 1963.

**Theorem 6.1.7 (Ore [110])** *Let $G$ be a graph. If $\sigma_2(G) \geq |V(G)| + 1$, then $G$ is Hamilton-connected.*

Theorem [109] is generalized into a sufficient condition on any three independent vertices. In 1991, Flandrin, Jung and Li proved the followings:

**Theorem 6.1.8 (Flandrin, Jung and Li [56])** *Let $G$ be a 2-connected graph of order $n$ such that $\overline{\sigma_3}(G) \geq n$, then $G$ is hamiltonian.*

When $\overline{\sigma_3}(G) \geq n-1$, we have the following theorem:

**Theorem 6.1.9 (Flandrin, Jung and Li [56])** *Let $G$ be a connected graph of order $n$ such that $\overline{\sigma_3}(G) \geq n-1$, then $G$ has a hamiltonian path.*

As a generalization of Hamilton-connected and hamiltonian path, Lin et $al.$ introduced the $k$-fan-connectivity of graphs in [97]. Now we again introduce the concept of $k$-fan-connected which was mentioned in section 1.3.4.

For any integer $t \geq 2$, let $v$ be a vertex of a graph $G$ and let $U = \{u_1, u_2, \ldots, u_t\}$ be a subset of $V(G) \setminus \{v\}$. A $(v, U)$*-fan* is a set of paths $P_1, P_2, \ldots, P_t$ such that $P_i$ is a path connecting $v$ and $u_i$ for $1 \leq i \leq t$ and $P_i \cap P_j = \{v\}$ for $1 \leq i < j \leq t$.

It follows from Menger Theorem [101] that there is a $(v, U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| \leq k$ if and only if $G$ is $k$-connected. If a $(v, U)$-fan spans $G$, then it is called a *spanning* $(v, U)$*-fan* of $G$. If $G$ has a spanning $(v, U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| = k$, then $G$ is $k$*-fan-connected*.

If a graph $G$ has order at least three, it is easy to obtain that "$G$ is Hamilton-connected" is equivalent to "$G$ is $2$-fan-connected".

We show the followings.

**Proposition 6.1.10** *Let $k \geq 2$ be an integer. If a graph $G$ is $k$-fan-connected, then $G$ is $(k+1)$-connected.*

**Proof.** Suppose that $G$ is not $(k+1)$-connected. There exists a cut-set $S$ with size at most $k$. Let $U$ be a subset of $V(G)$ with $S \subseteq U$ such that $|U| = k$. It follows that there exists no spanning $(v, U)$-fan in $G$ for any vertex $v$ of $V(G) \setminus U$, contrary to the $k$-fan-connectivity of $G$. ∎

In this chapter, we improve Theorem 6.0.1 by showing that the Flandrin-Jung-Li's condition in Theorem 6.1.8 is a new sufficient condition of $k$-fan-connected graphs. We get our main result Theorem 6.0.2.

## 6.2 Sharpness of the lower bound

The lower bound of $\overline{\sigma_3}(G)$ in Theorem 6.0.2 is sharp as shown in this section.

The following example gives many graphs which satisfy the conditions of Theorem 6.0.2, but does not satisfy the degree sum condition of Theorem 6.0.1.

Example: let $n$ be a large integer and a graph $G = (K_1 \cup C_{(n-k+3)/2}) + K_{(n+k-5)/2}$ (see Figure 6.1). Then $|V(G)| = n$, $G$ is $(k+1)$-connected, and $\overline{\sigma_3}(G) = n + k - 1$. The degree sum of $x \in V(K_1)$ and $y \in V(C_{(n-k+3)/2})$

is $n+k-3$. It follows that $G$ satisfies all conditions of Theorem 6.0.2, but does not satisfy the degree sum condition of Theorem 6.0.1.

If $\sigma_2(G) \geq |V(G)| + k - 1$ with $k \geq 2$, then it is easy to verify that $G$ is $k$-connected. By proposition 6.1.5, we got $\overline{\sigma_3}(G) \geq |V(G)| + k - 1$. It follows that $G$ is $k$-fan-connected from Theorem 6.0.2. Thus, the result of Theorem 6.0.1 can be derived from Theorem 6.0.2.

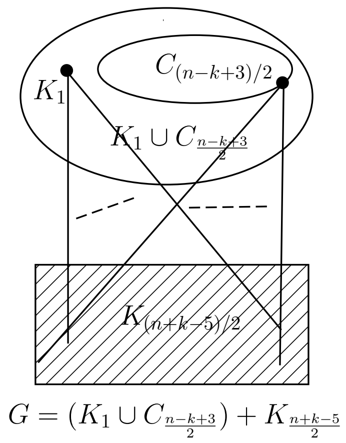



Figure 6.1: The graph of $G = (K_1 \cup C_{(n-k+3)/2}) + K_{(n+k-5)/2}$

Let us see the following example that shows the lower bound of $\overline{\sigma_3}(G)$ in Theorems 6.0.2 is sharp.

In the sense that we cannot replace the lower bound $|V(G)| + k - 1$ by $|V(G)| + k - 2$.

Let $n$ be a sufficiently large integer, and let $k \geq 2$ be an integer. Let $G := K_{(n+k-2)/2} + \overline{K_{(n-k+2)/2}}$ (see Figure 6.2). Then $\overline{\sigma_3}(G) = |V(G)| + k - 2$. Let $U$ be a subset of $V(K_{(n+k-2)/2})$ with size $k$ and $v \in V(\overline{K_{(n-k+2)/2}})$. We will show that $G$ has no spanning $(v, U)$-fan.

Suppose that $G$ has a spanning $(v, U)$-fan $T$. Then the number of the edges of $T$ having one end vertex in $V(K_{(n+k-2)/2})$ and the other in $V(\overline{K_{(n-k+2)/2}})$ is

$$k + 2 \times ((n-k+2)/2 - 1) = n$$

since $\deg_T(w) = 2$ for each $w \in V(\overline{K_{(n-k+2)/2}}) \setminus \{v\}$ and $\deg_T(v) = k$. On the other hand, the number of the edges of $T$ is

$$\sum_{w \in V(K_{(n+k-2)/2})} \deg_T(w) = k + 2 \times ((n+k-2)/2 - k) = n - 2.$$

This is a contradiction. So, the lower bound of $\overline{\sigma_3}(G)$ in Theorems 6.0.2 is sharp.

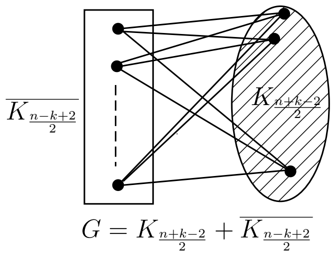



Figure 6.2: The graph of $G = K_{\frac{n+k-2}{2}} + \overline{K_{\frac{n-k+2}{2}}}$

## 6.3 Preliminaries

In this section, We introduce some lemmas which are used in the proof of Theorem 6.0.2.

The first lemma has already been introduced in Chapter 2, and now we reintroduce it under new notation.

**Lemma 6.3.1** *[85] Let $P = u_1u_2u_3\cdots u_p$ be a path in a graph $G$. Let $w_1$ and $w_2$ be two vertices in $V(G) - V(P)$ such that $(N_G(w_1) \cap (V(P) \setminus \{u_1\}))^-_{u_1} \cap N_G(w_2) = \emptyset$. Then $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| \le p+1$. Moreover, if $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| = p+1$, then*

(i) *$w_1u_1, w_2u_p \in E(G)$,*

(ii) *if $w_1$ is not adjacent to consecutive two vertices on $P$, then $w_2u_1 \in E(G)$, and*

(iii) *if $w_2$ is not adjacent to consecutive two vertices on $P$, then $w_1u_p \in E(G)$.*

Now, let's state this lemma briefly. When $|N_G(w_1)\cap V(P)|+|N_G(w_2)\cap V(P)| = p+1$, we have $(i)$ $w_1u_1, w_2u_p \in E(G)$. If $w_1$ is not adjacent to consecutive two vertices on $P$ and $w_2u_1 \notin E(G)$, then $|N_G(w_1) \cap (V(P) - \{u_1\})| + |N_G(w_2) \cap (V(P) - \{u_1\})| = p = |V(P) - \{u_1\}|$. By using the conclusion of (i) again, we can get $w_1u_2 \in E(G)$. Then $w_1$ is adjacent to consecutive two vertices on $P$, a contradiction. So (ii) holds. Similarly, if $w_2$ is not adjacent to consecutive two vertices on $P$, then $w_1u_p \in E(G)$.

**Lemma 6.3.2** *Let $P = u_1u_2u_3\cdots u_p$ be a path in graph $G$. Let $w_1$, $w_2$, and $w_3$ be three vertices in $V(G) - V(P)$ such that $(N_G(w_1) \cap (V(P) \setminus \{u_1\}))^-_{u_1} \cap N_G(w_2) = \emptyset$ and $N_G(w_3) \cap V(P) \subseteq \{u_p\}$. If $w_2$ is not adjacent to consecutive two vertices on $P$, then*

$$\sum_{1\le i\le 3} |N_G(w_i) \cap V(P)| - |\bigcap_{1\le i\le 3} (N_G(w_i) \cap V(P))| \le \begin{cases} p & \text{if } u_1w_1 \notin E(G), \\ p+1 & \text{otherwise.} \end{cases}$$

**Proof.** First, we consider the case $u_1w_1 \notin E(G)$. By Lemma 6.3.1, then $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| \le p$. If $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| \le p-1$, since $N_G(w_3) \cap V(P) \subseteq \{u_p\}$, so the lemma holds.

Hence, we may assume that $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| = p$. If $w_3$ is not adjacent to $u_p$, then the lemma holds. So, we assume $w_3$ is adjacent to $u_p$. If $u_1w_2 \notin E(G)$, by applying Lemma 6.3.1 to $P - \{u_1\}$, $w_1u_p, w_2u_p \in E(G)$ and so we obtain

$$\sum_{1\le i\le 3} |N_G(w_i) \cap V(P)| - |\bigcap_{1\le i\le 3} (N_G(w_i) \cap V(P))| = p + 1 - 1 = p.$$

We may assume that $u_1w_2 \in E(G)$. Since $(N_G(w_1) \cap (V(P) \setminus \{u_1\}))^-_{u_1} \cap N_G(w_2) = \emptyset$ and $w_2$ is not adjacent to consecutive two vertices on $P$, $w_1u_2, w_2u_2 \notin E(G)$. Let $P' := P - \{u_1, u_2\}$, then $|N_G(w_1) \cap V(P')| + |N_G(w_2) \cap V(P')| = p - 1$. By applying Lemma 6.3.1 to $P'$, $w_1u_p, w_2u_p \in E(G)$. So

$$\begin{aligned}\sum_{1\le i\le 3} |N_G(w_i) \cap V(P)| - &|\bigcap_{1\le i\le 3} (N_G(w_i) \cap V(P))| \\ &= \sum_{1\le i\le 3} |N_G(w_i) \cap V(P')| - |\bigcap_{1\le i\le 3} (N_G(w_i) \cap V(P'))| \\ &\le p + 1 - 1 = p.\end{aligned}$$

This completes the case $u_1w_1 \notin E(G)$.

Next, we consider the case $u_1w_1 \in E(G)$. If $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| \le p$, then we obtain the desired inequality since $N_G(w_3) \cap V(P) \subseteq \{u_p\}$. We may assume that $|N_G(w_1) \cap V(P)| + |N_G(w_2) \cap V(P)| = p + 1$ and $w_1u_p, w_2u_p \in E(G)$ by Lemma 6.3.1. If $w_3$ is not adjacent to $u_p$, then the lemma holds. If $w_3$ is adjacent to $u_p$, then we obtain

$$\sum_{1\le i\le 3} |N_G(w_i) \cap V(P)| - |\bigcap_{1\le i\le 3} (N_G(w_i) \cap V(P))| \le p + 2 - 1 = p + 1.$$

Hence, the lemma holds. ∎

## 6.4 Proof of Theorem 6.0.2

In this section, we will prove Theorem 6.0.2.

The sketch of the proof:

Firstly, to prove this theorem, we introduce the segment insertion operation. An important Claim 6.4.5 derived from this operation is also given. It will be shown in section 6.4.1.

Secondly, because Theorem 6.0.2 is based on $\overline{\sigma_3}(G)$, so in section 6.4.2 we're going to find three independent vertices $w_1$, $w_2$ and $w_3$. At the same time, we get some relationships among their neighborhood sets.

Thirdly, in Section 6.4.3, we divide the vertex set of the graph $G$ into several partitions. And then we find the degree sum of the three independent vertices $w_1$, $w_2$ and $w_3$ in each partition.

Lastly, according to whether $w_2$ belongs a segment to discuss, then we get contradiction. Thus, the theorem is further proved.

### 6.4.1 Segment insertion operation

On the contrary, suppose that $G$ is not $k$-fan-connected, then there exists a vertex $v$ and a subset $U = \{u_1, u_2, \ldots, u_k\}$ of $V(G)\setminus\{v\}$ such that $G$ has no spanning $(v, U)$-fan. Since $G$ is $(k+1)$-connected, it follows from Menger's Theorem that $G$ has a $(v, U)$-fan. Let $T$ be an order maximum $(v, U)$-fan of $G$ and $H$ be a component of $G - T$.

For two vertices $a$ and $b$ of $T$, $P[a, b]$ denotes the path in $T$ connecting $a$ and $b$. If $P$ is a path in $T$ connecting vertices $x$ and $y$ of $T$ such that $(N_G(V(H)) \cap V(P)) = \{x, y\}$ and $v \notin V(P) \setminus \{x, y\}$, then we call the path $P$ a *segment* of $T$. By the maximality of $T$, then $|V(P)| \geq 3$.

Let $Q$ be a segment of $T$ and $w$ be an internal vertex of $Q$. If there are two vertices $a, b \in N_G(w)$ such that $ab \in E(T) \setminus E(Q)$, then $w$ is called an *insertible vertex* of $Q$.

**Segment insertion operation:** Suppose that $w_1, w_2, \ldots, w_s$ are insertible vertices of $Q$ in order along $Q$. Let

$$h_1 := \max\{i\colon w_i \text{ can be inserted in an edge which } w_1 \text{ can be inserted in}\}$$

and suppose that $w_1$ and $w_{h_1}$ can be inserted in an edge $a_1b_1$. Let

$$h_2 := \max\{i\colon w_i \text{ can be inserted in an edge which } w_{h_1+1} \text{ can be inserted in}\}$$

and suppose that $w_{h_1+1}$ and $w_{h_2}$ can be inserted in an edge $a_2b_2$. Continuing in the same manner, we will have $h_t = s$ for some $t \geq 1$. Then we insert $Q[w_1, w_{h_1}]$ between $a_1$ and $b_1$, $Q[w_{h_1+1}, w_{h_2}]$ between $a_2$ and $b_2, \ldots, Q[w_{h_{t-1}+1}, w_{h_t}]$ between $a_t$ and $b_t$. We call such an operation a *segment insertion* and denote it by $SI[Q[w_1, w_s]]$.

It's easy to get the following claim, which plays an important role in the whole proof of Theorem 6.0.2.

**Claim 6.4.1** *Every segment of $T$ contains a non-insertible vertex.*

**Proof.** On the contrary, we assume that there exists a segment $P = w_1w_2 \ldots w_s$ not containing a non-insertible vertex. Let $Q$ be a path connecting $w_1$ and $w_s$ such that $V(Q) \setminus \{w_1, w_s\} \subseteq V(H)$. We use a segment insertion $SI[P[w_2, w_{s-1}]]$ and let $T'$ be the resulting graph. Then $T' \cup Q$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$. This contradicts the maximality of $T$. ∎

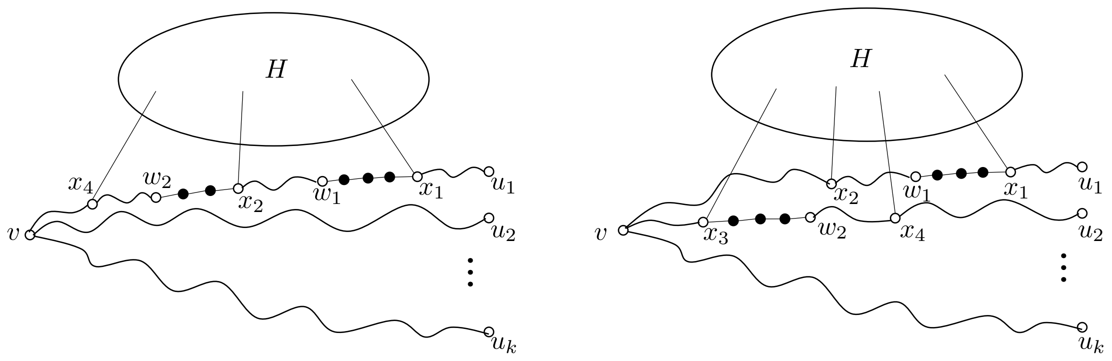



Figure 6.3: The definition of $w_1$ and $w_2$, where black vertices are insertible vertices.

### 6.4.2 The relationships among three independent vertices

Since $G$ is $(k+1)$-connected, $|N_G(V(H)) \cap V(T)| \geq k+1$. Then $|N_G(V(H)) \cap V(P[v, u_i])| \geq 2$ for some $1 \leq i \leq k$. Without loss of generality, we may assume that $V(P[v, u_1])$ has the most vertices in $N_G(V(H))$ among $V(P[v, u_i])$ for all $1 \leq i \leq k$. And assume that there is a segment of $T$ in $P[v, u_1]$. Let $x_1$ and $x_2$ be the end vertices of the segment of $T$ in $P[v, u_1]$ such that $V(P[x_1, u_1]) \cap N_G(V(H)) = \{x_1\}$. Let $w_1$ be the non-insertible vertex of $P[x_1, x_2]$ such that $|V(P[x_1, w_1])|$ is as small as possible. Write $P[x_1, w_1] = y_0 y_1 \ldots y_m$ where $y_0 = x_1$ and $y_m = w_1$.

If there is a segment $P[x_3, x_4]$ of $T$ other than $P[x_1, x_2]$, we choose the segment $P[x_3, x_4]$ so that if there is a segment of $T$ other than $P[x_1, x_2]$ in $P[v, u_1]$, then we assume $x_3 = x_2$ (see the graph in the left of Fig. 6.3) otherwise without loss of generality, we may assume that the segment $P[x_3, x_4]$ is in $P[v, u_2]$ such that $|V(P[v, x_3])|$ is as small as possible (see the graph in the right of Fig. 6.3). Now let $w_2$ be the non-insertible vertex of $P[x_3, x_4]$ such that $|V(P[x_3, w_2])|$ is as small as possible. Then $w_2$ is in a segment, and $w_2 \in V(P[v, u_1])$ or $w_2 \in V(P[v, u_2])$. Write $P[x_3, w_2] = y'_0 y'_1 \ldots y'_\ell$ where $y'_0 = x_3$ and $y'_\ell = w_2$.

If there is only one segment $P[x_1, x_2]$ in $T$, let $w_2 \in N_T(x_2) \setminus V(P[x_1, x_2])$. Now $w_2$ is not in a segment, and $w_2$ is in $V(P[v, u_1])$. In this case, let $y'_1 = w_2$ (see Fig. 6.4).

Let $w_3$ be an arbitrary vertex of $V(H)$. For two vertices $a$ and $b$, we denote $aHb$ a path connecting $a$ and $b$ through $H$ if such a path exists.

The relationship among three vertices $w_1$, $w_2$ and $w_3$ be as following claims.

**Claim 6.4.2** *The vertex $w_3$ is not adjacent to $w_1$ and $w_2$.*

**Proof.** Suppose that $w_1 w_3 \in E(G)$. We use a segment insertion $SI[P[y_1, y_{m-1}]]$ and let $T'$ be a resulted graph. Then $T' + w_1 w_3 \cup w_3 H x_1$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$. This is a contradiction.

Suppose that $w_2 w_3 \in E(G)$. From the maximality of $T$, $w_2 x_2 \notin E(T)$. Thus, $w_2$ is in a segment of $T$. Then we deduce a contradiction by the similar argument of the above one.

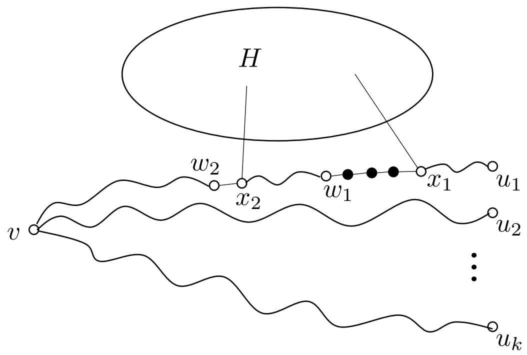



Figure 6.4: The definition of $w_1$ and $w_2$, where black vertices are insertible vertices.

Therefore, $w_3$ is not adjacent to $w_1$ and $w_2$. ∎

**Claim 6.4.3** *For any $1 \leq i \leq m$ and $1 \leq j \leq \ell$, $y_i$ and $y'_j$ are not adjacent.*

**Proof.** We prove this claim by induction on $i+j$ with $1 \leq i \leq m$ and $1 \leq j \leq \ell$. Suppose that $y'_1 y_1 \in E(G)$. Then $T + y'_1 y_1 - x_1 y_1 - y'_1 x_2 \cup x_1 H x_3$ is a $(v, U)$-fan with the order of at least $|V(T)|+1$, a contradiction. Suppose that this claim holds for $2 \leq i'+j' < i+j$ with $i+j \geq 3$. Suppose that $y_i y'_j \in E(G)$. We use segment insertions $SI[P[y_1, y_{i-1}]]$ (if $i \geq 2$) and $SI[P[y'_1, y'_{j-1}]]$ (if $j \geq 2$). Let $T'$ be a resulted graph. According to the induction hypothesis of this claim, for each $1 \leq i' \leq i-1$, $y_{i'}$ is not inserted into any edge of $P[x_2, y'_j]$, and for each $1 \leq j' \leq j-1$, $y'_{j'}$ is not inserted into any edge of $P[x_1, y_j]$. Then $T' + y'_j y_i \cup x_1 H x_3$ is a $(v, U)$-fan with the order of at least $|V(T)|+1$, a contradiction.

Hence, Claim 6.4.3 holds. ∎

By Claims 6.4.2 and 6.4.3, the set $\{w_1, w_2, w_3\}$ is an independent set of $G$.

**Claim 6.4.4** *The following statements hold for each $1 \leq i \leq m$ and $1 \leq j \leq \ell$.*

(i) $N_G(y_i) \cap (N_G(w_3) \cap V(T))^+_{u_1} = \emptyset$,

(ii) *if $w_2$ is in $V(P[v, u_1])$, then* $N_G(y'_j) \cap (N_G(w_3) \cap V(T))^+_{u_1} = \emptyset$,

(iii) *if $w_2$ is in $V(P[v, u_2])$, then* $N_G(y'_j) \cap (N_G(w_3) \cap V(T))^-_{u_2} = \emptyset$,

(iv) *if $w_2$ is in $V(P[v, u_1])$, then* $N_G(y_i) \cap (N_G(y'_j) \cap (V(T) \setminus V(P[w_1, x_2])))^-_{u_1} = \emptyset$,

(v) *if $w_2$ is in $V(P[v, u_2])$, then* $N_G(y'_j) \cap (N_G(y_i) \cap (V(T) \setminus V(P[v, w_1] \cup P[x_3, v])))^-_{u_2} = \emptyset$,

(vi) *if $w_2$ is in $V(P[v, u_2])$, then* $N_G(y'_j) \cap (N_G(y_i) \cap V(P[v, w_1] \cup P[x_3, v]))^-_{u_1} = \emptyset$. *And if $w_2$ is in $V(P[v, u_1])$, then* $N_G(y'_j) \cap (N_G(y_i) \cap V(P[w_1, x_2]))^-_{u_1} = \emptyset$.

**Proof.** (i) We show that $N_G(y_i) \cap (N_G(w_3) \cap V(T))^+_{u_1} = \emptyset$ for each $1 \le i \le m$ by induction on $i$ with $1 \le i \le m$. Suppose that there is a vertex $w \in N_G(w_3) \cap V(T)$ such that there is a vertex $w^+ \in N_G(y_i) \cap \{w\}^+_{u_1}$ for some $1 \le i \le m$. If $i = 1$, then $T + w^+y_1 - ww^+ - x_1y_1 \cup wHx_1$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction. We assume that this claim holds for $1 \le j < i$ with $i \ge 2$. We use a segment insertion $SI[P[y_1, y_{i-1}]]$ and let $T'$ be a resulted graph. It follows from the induction hypothesis of this claim that for each $1 \le j < i$, $y_j$ is not inserted in $ww^+$. Then $T' + y_iw^+ - ww^+ \cup x_1Hw$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction.

(ii) We show $N_G(y'_j) \cap (N_G(w_3) \cap V(T))^+_{u_1} = \emptyset$ for each $1 \le j \le \ell$. If $w_2$ is in a segment of $T$, then we can deduce a contradiction by the similar argument of the above one. If $w_2$ is not in any segments of $T$, then we can also deduce a contradiction by the similar argument of the above one in the case $i = 1$.

(iii) We can show $N_G(y'_j) \cap (N_G(w_3) \cap V(T))^-_{u_2} = \emptyset$ by induction on $j$ with $1 \le j \le \ell$. The proof is similar to the proof of (i).

(iv) We show this claim by induction on $i + j$ with $1 \le i \le m$ and $1 \le j \le \ell$. Suppose that there is a vertex $w \in N_G(y'_1) \cap V(T) \setminus V(P[w_1, x_2])$ such that $w^-_{u_1}y_1 \in E(G)$. Then $T + y_1w^-_{u_1} + y'_1w - x_1y_1 - y'_1x_2 - ww^-_{u_1} \cup x_1Hx_2$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction. We assume that this claim holds for $3 \le i' + j' < i + j$. Suppose that there is a vertex $w \in N_G(y'_j) \cap V(T) \setminus V(P[w_1, x_2])$ such that $w^-_{u_1}y_i \in E(G)$ for some $1 \le i \le m$ and $1 \le j \le \ell$. We use segment insertions $SI[P[y_1, y_{i-1}]]$ (if $i \ge 2$) and $SI[P[y'_1, y'_{j-1}]]$ (if $j \ge 2$). Let $T'$ be a resulted graph. It follows from Claim 6.4.3 and the induction hypothesis of this claim that $y_{i'}$ is not inserted into an edge in $P[x_2, w_2] \cup \{ww^-_{u_1}\}$ for each $1 \le i' < i$ and $y'_{j'}$ is not inserted into an edge in $P[x_1, w_1] \cup \{ww^-_{u_1}\}$ for each $1 \le j' < j$. Hence, $T' + y_iw^-_{u_1} + y'_jw - ww^-_{u_1} \cup x_1Hx_2$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction.

(v) We can show this claim by induction on $i + j$ with $1 \le i \le m$ and $1 \le j \le \ell$. The proof is similar to the proof of (iv).

(vi) We show this claim by induction on $i + j$ with $1 \le i \le m$ and $1 \le j \le \ell$. If $w_2$ is in $V(P[v, u_2])$. Suppose that there is a vertex $w \in (N_G(y_1) \cap V(P[v, w_1] \cup P[x_3, v]))_{u_1}$ such that $w^-_{u_1}y'_1 \in E(G)$. Then $T + y_1w + y'_1w^-_{u_1} - y_1x_1 - y'_1x_3 - ww^-_{u_1} \cup x_1Hx_3$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction. We assume that this claim holds for $3 \le i' + j' < i + j$. Suppose that there is a vertex $w \in (N_G(y_i) \cap V(P[v, w_1] \cup P[x_3, v]))_{u_1}$ such that $w^-_{u_1}y'_j \in E(G)$ for some $1 \le i \le m$ and $1 \le j \le \ell$. We use segment insertions $SI[P[y_1, y_{i-1}]]$ (if $i \ge 2$) and $SI[P[y'_1, y'_{j-1}]]$ (if $j \ge 2$). Let $T'$ be a resulted graph. It follows from Claim 6.4.3 and the induction hypothesis of this claim that for each $1 \le j' < j$, $y'_{j'}$ is not inserted in an edge into $P[x_1, w_1] \cup \{ww^-_{u_1}\}$. Then $T' + y_iw + y'_jw^-_{u_1} - ww^-_{u_1} \cup x_1Hx_3$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction. Similarly, if $w_2$ is in $V(P[v, u_1])$, then $N_G(y'_j) \cap (N_G(y_i) \cap V(P[w_1, x_2]))^-_{u_1} = \emptyset$. ∎

### 6.4.3 The rest of the proof of Theorem 6.0.2

Note that each vertex of $H$ satisfies the property of $w_3$ in Claims 6.4.2 and 6.4.4 since $w_3$ is an arbitrary vertex of $H$.

For the path $P$ contained in $T$, the first vertex of $P$ in order along $(T, r)$ is denoted by $s_r(P)$, where $r$ is a vertex of $T$. Let $v_i$ be the vertex in $N_T(v) \cap V(P[v, u_i])$ for each $1 \leq i \leq k$. If $V(P[v, u_i]) \cap N_G(V(H)) \neq \emptyset$ for $1 \leq i \leq k$, then let $s_i$ (resp. $t_i$) be the vertices of $V(P[v, u_i]) \cap N_G(V(H))$ such that $|V(P[v, s_i])|$ (resp. $|V(P[t_i, u_i])|$) is as small as possible.

So first, let's calculate $\overline{\sigma_3}(G)$ on a segment of $T - V(P[x_1, x_2])$ and path $P[t_i, u_i]$. Then we have the following claim.

**Claim 6.4.5** *Let $P$ be either a segment of $T - V(P[x_1, x_2])$ or $P[t_i, u_i]$ for $2 \leq i \leq k$. Then*

$$\sum_{1 \leq i \leq 3} |N_G(w_i) \cap V(P - s_{u_1}(P))| - |\bigcap_{1 \leq i \leq 3} (N_G(w_i) \cap V(P - s_{u_1}(P)))| \leq |V(P)| - 1.$$

**Proof.** Suppose $P = P[x_3, x_4]$, then $w_2$ is a non-insertible vertex. By Claim 6.4.3, then

$$|N_G(w_1) \cap V(P[x_3, w_2] - x_3)| + |N_G(w_2) \cap V(P[x_3, w_2] - x_3)| \leq |V(P[x_3, w_2] - x_3)| - 1.$$

By Claim 6.4.4 (iv), then $N_G(w_1) \cap (N_G(w_2) \cap V(P))^-_{u_1} = \emptyset$. By Lemma 6.3.2,

$$\begin{aligned} \sum_{1 \leq i \leq 3} |N_G(w_i) \cap V(P[w_2, x_4] - w_2)| - |\bigcap_{1 \leq i \leq 3} (N_G(w_i) \cap V(P[w_2, x_4] - w_2))| \\ \leq |V(P[w_2, x_4] - w_2)| + 1. \end{aligned}$$

Thus, we obtain the desired inequality.

Suppose $P \neq P[x_3, x_4]$. If $w_2$ is in $V(P[v, u_1])$, by Claim 6.4.4 (i), (ii) and (iv), then $w_1 s_{u_1}(P - s_{u_1}(P)), w_2 s_{u_1}(P - s_{u_1}(P)) \notin E(G)$ and $N_G(w_1) \cap (N_G(w_2) \cap V(P))^-_{u_1} = \emptyset$. Since $w_1$ is not adjacent to consecutive two vertices on $P$, it follows from Lemma 6.3.2 that we obtain the desired inequality.

If $w_2$ is in $V(P[v, u_2])$, then $w_2$ is a non-insertible vertex. When $P \subseteq P[v, u_2]$, by Claim 6.4.4 (v), then $N_G(w_1) \cap (N_G(w_2) \cap V(P))^-_{u_1} = \emptyset$. It follows from Lemma 6.3.2 that we obtain the desired inequality. When $P \not\subseteq P[v, u_2]$, by Claim 6.4.4 (v), then $N_G(w_2) \cap (N_G(w_1) \cap V(P))^-_{u_1} = \emptyset$. It follows from Lemma 6.3.2 and $w_2$ is non-insertible vertex that we obtain the desired inequality. ∎

Next, the following claim is to calculate $\overline{\sigma_3}(G)$ on path $P[t_1, u_1]$.

**Claim 6.4.6** *The following inequality holds.*

$$\sum_{1\le i\le 3} |N_G(w_i)\cap V(P[t_1,u_1])| - |\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[t_1,u_1]))| \le |V(P[t_1,u_1])|+1$$

**Proof.** By Claim 6.4.4 (iv), (v), then $N_G(w_1)\cap(N_G(w_2)\cap V(P[t_1,u_1]))^-_{u_1}=\emptyset$. We obtain the desired inequality from Lemma 6.3.2. ∎

The following claims calculate $\overline{\sigma_3}(G)$ on $V(P[v_j,s_j])$ with $2\le j\le k$, $V(P[v,s_1]-s_1)$ and $V(P[x_1,x_2]-x_1)$, respectively.

**Claim 6.4.7** *Suppose that $vw_3\notin E(G)$. For each $2\le j\le k$, the following inequality holds.*

$$\sum_{1\le i\le 3} |N_G(w_i)\cap V(P[v_j,s_j])| - |\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[v_j,s_j])|$$
$$\le \begin{cases} |V(P[v_j,s_j])|+1 & \textit{if } vw_1\notin E(G)\\ |V(P[v_j,s_j])| & \textit{otherewise.}\end{cases}$$

**Proof.** First, we consider the case $vw_1\notin E(G)$. If $w_2$ in $P[v,u_1]$, it follows that $N_G(w_1)\cap(N_G(w_2)\cap V(P[v_j,s_j]))^-_{u_1}=\emptyset$ from Claim 6.4.4 (iv). Since $w_1$ is a non-insertible vertex, by Lemma 6.3.2, we obtain the desired inequality in the case that $vw_1\notin E(G)$. If $w_2$ in $P[v,u_2]$, then $w_2$ is a non-insertible vertex. By Claim 6.4.4 (v) and (vi), then $N_G(w_2)\cap(N_G(w_1)\cap V(P[v_j,s_j]))^-_{u_1}=\emptyset$. We obtain the desired inequality from Lemma 6.3.2.

Next, we consider tha case $vw_1\in E(G)$. Since $w_1$ is a non-insertible vertex, $w_1$ is not adjacent to $v_j$ for each $2\le j\le k$. When $w_2\in V(P[v,u_1])$, by Claim 6.4.4 (iv), then for each $2\le j\le k$, $w_2v_j\notin E(G)$ and $N_G(w_1)\cap(N_G(w_2)\cap V(P[v_j,s_j]))^-_{u_1}=\emptyset$. It follows from Lemma 6.3.2 that we obtain the desired inequality. When $w_2\in V(P[v,u_2])$, then $w_2$ is a non-insertible vertex. By Claim 6.4.4(v) and (vi), then $N_G(w_2)\cap(N_G(w_1)\cap V(P[v_j,s_j]))^-_{u_1}=\emptyset$. We obtain the desired inequality from Lemma 6.3.2. ∎

**Claim 6.4.8** *Suppose that $vw_3\notin E(G)$. The following inequality holds.*

$$\sum_{1\le i\le 3} |N_G(w_i)\cap V(P[v,s_1]-s_1)| - |\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[v,s_1]-s_1)|$$
$$\le \begin{cases} |V(P[v,s_1]-s_1)|-1 & \textit{if } vw_1\notin E(G)\\ |V(P[v,s_1]-s_1)| & \textit{otherewise.}\end{cases}$$

**Proof.** It follows that $w_1$ and $w_2$ are not adjacent to $s_{u_1}(P[v,s_1]-s_1)$ from Claim 6.4.4 (i), (ii) and (iii). By Claim

6.4.4 (iv), (vi) and Lemma 6.3.2, we obtain the desired inequality in the case that $vw_1 \in E(G)$.

Suppose that $vw_1 \notin E(G)$. By Claim 6.4.4 (iv), (vi) and Lemma 6.3.1, we obtain

$$|N_G(w_1) \cap V(P[v, s_1] - s_1)| + |N_G(w_2) \cap V(P[v, s_1] - s_1)| \leq |V(P[v, s_1] - s_1)|.$$

If $w_2 \in V(P[v, u_1])$, by Claim 6.4.4 (iv) and Lemma 6.3.1 (i), then this claim holds in the case that $vw_1 \notin E(G)$. We may assume $w_2 \in V(P[v, u_2])$. Then $w_2$ is a non-insertible vertex. By Claim 6.4.4 (vi) and Lemma 6.3.1 (iii), hence, this claim holds in the case that $vw_1 \notin E(G)$. ∎

**Claim 6.4.9** *The following inequality holds.*

$$\sum_{1 \leq i \leq 3} |N_G(w_i) \cap V(P[x_1, x_2] - x_1)| - |\bigcap_{1 \leq i \leq 3} (N_G(w_i) \cap V(P[x_1, x_2] - x_1)|$$
$$\leq \begin{cases} |V(P[x_1, x_2] - x_1)| & \textit{if } w_2 \textit{ is in a segment} \\ |V(P[x_1, x_2] - x_1))| + 1 & \textit{otherwise.} \end{cases}$$

**Proof.** By Claim 6.4.3, then

$$|N_G(w_1) \cap V(P[x_1, w_1] - x_1)| + |N_G(w_2) \cap V(P[x_1, w_1] - x_1)|$$
$$\leq |V(P[x_1, w_1] - x_1)| - 1. \tag{6.1}$$

By Claim 6.4.4 (vi), then $N_G(w_2) \cap (N_G(w_1) \cap V(P[w_1, x_2]))^-_{u_1} = \emptyset$. By Lemma 6.3.1 and (6.1), we obtain

$$|N_G(w_2) \cap V(P[x_1, x_2] - x_1)| + |N_G(w_1) \cap V(P[x_1, x_2] - x_1)|$$
$$= |N_G(w_2) \cap V(P[x_1, x_2] - x_1)| + |(N_G(w_1) \cap V(P[x_1, x_2] - x_1))^-_{u_1}|$$
$$\leq |V(P[x_1, x_2] - x_1)|$$

Suppose that $w_2$ is in a segment. Then $w_2$ is a non-insertible vertex. By Lemma 6.3.1 (ii) and (iii), $w_1$ and $w_2$ are adjacent to $x_2$. Since $N_G(w_3) \cap V(P[x_1, x_2] - x_1) \subseteq \{x_2\}$, we obtain the desired inequality. Hence, we may assume that $w_2$ is not in a segment. By $N_G(w_3) \cap V(P[x_1, x_2] - x_1) \subseteq \{x_2\}$, we obtain the desired inequality. ∎

By Claim 6.4.2, $(N_G(w_1) \cup N_G(w_2)) \cap V(H) = \emptyset$ and so

$$|(N_G(w_1) \cup N_G(w_2) \cup N_G(w_3)) \cap V(H)| \leq |V(H)| - |\{w_3\}| \leq |V(G)| - |V(T)| - 1. \tag{6.2}$$

Let $\mathcal{P}$ be the set of segments of $T$ and paths $P[t_i, u_i]$ for $2 \leq i \leq k$.

The discussion is then classified according to whether $vw_3$ is an edge of $G$. So let's first look at the case where $vw_3$ is an edge.

Suppose that $vw_3 \in E(G)$, since $G$ is $k+1$ connected, then there are at least two segments. So $w_1$ and $w_2$ are non-insertible vertices. Then $V(T) = \cup_{P \in \mathcal{P}}(V(P) - s(P)) \cup V(P[t_1, u_1])$. By Claims 6.4.5, 6.4.6 and 6.4.9, we obtain

$$\begin{aligned}
\sum_{1\leq i\leq 3} |N_G(w_i) \cap V(T)| - |\bigcap_{1\leq i\leq 3} (N_G(w_i) \cap V(T)| \\
&= \sum_{P\in\mathcal{P}} \left( \sum_{1\leq i\leq 3} |N_G(w_i) \cap V(P - s_{u_1}(P))| - |\bigcap_{1\leq i\leq 3} (N_G(w_i) \cap V(P - s_{u_1}(P))| \right) \\
&+ \sum_{1\leq i\leq 3} |N_G(w_i) \cap V(P[t_1, u_1]))| - |\bigcap_{1\leq i\leq 3} (N_G(w_i) \cap V(P[t_1, u_1]))| \\
&\leq \sum_{P\in\mathcal{P}} (|V(P)| - 1) + |V(P[t_1, u_1])| + 1 \\
&= |V(T)| + 1. \qquad (6.3)
\end{aligned}$$

By (6.3) and (6.2), we obtain

$$\sum_{1\leq i\leq 3} |N_G(w_i)| - |\bigcap_{1\leq i\leq 3} N_G(w_i)| \leq |V(G)|.$$

Since $k \geq 2$, this contradicts to $\overline{\sigma_3}(G) \geq |V(G)| + k - 1$.

Let's talk about the case where $vw_3$ is not an edge in $G$.

Suppose that $vw_3 \notin E(G)$. Let $\mathcal{Q}$ be the set of paths $P[v, s_i]$ for $2 \leq i \leq k$. Then

$$V(T) = \bigcup_{P\in\mathcal{P}\cup\mathcal{Q}} (V(P) - s_{u_1}(P)) \cup V(P[v, s_1] - s_1) \cup V(P[t_1, u_1]).$$

By Claims 6.4.7 and 6.4.8, we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}} \left( \sum_{1\leq i\leq 3} |N_G(w_i) \cap V(Q - s_{u_1}(Q))| - |\bigcap_{1\leq i\leq 3} (N_G(w_i) \cap V(Q - s_{u_1}(Q))| \right) \\
&+ \sum_{1\leq i\leq 3} |N_G(w_i) \cap V(P[v, s_1] - s_1)| - |\bigcap_{1\leq i\leq 3} (N_G(w_i) \cap V(P[v, s_1] - s_1))| \\
&\leq \begin{cases} \sum_{Q\in\mathcal{Q}}(|V(Q)| - 1) + k - 1 + |V(P[v, s_1] - s_1)| - 1 & \text{if } vw_1 \notin E(G) \\ \sum_{Q\in\mathcal{Q}}(|V(Q)| - 1) + |V(P[v, s_1] - s_1)| & \text{otherwise} \end{cases} \\
&\leq \sum_{Q\in\mathcal{Q}} (|V(Q)| - 1) + |V(P[v, s_1] - s_1)| + k - 2. \qquad (6.4)
\end{aligned}$$

Under the condition that $vw_3$ is not an edge in $G$, we separately discuss and analyze whether $w_2$ is in a segment.

Suppose $w_2$ is in a segment, then by Claims 6.4.5, 6.4.6, 6.4.9 and (6.4), we obtain

$$\begin{aligned}
&\sum_{1\le i\le 3}|N_G(w_i)\cap V(T)|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(T)|\\
&=\sum_{P\in\mathcal{P}\cup\mathcal{Q}}\left(\sum_{1\le i\le 3}|N_G(w_i)\cap V(P-s_{u_1}(P))|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P-s_{u_1}(P))|\right)\\
&+\sum_{1\le i\le 3}|N_G(w_i)\cap V(P[v,s_1]-s_1)|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[v,s_1]-s_1))|\\
&+\sum_{1\le i\le 3}|N_G(w_i)\cap V(P[t_1,u_1]))|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[t_1,u_1]))|\\
&\le\sum_{P\in\mathcal{P}}(|V(P)|-1)+\sum_{Q\in\mathcal{Q}}(|V(Q)|-1)+|V(P[v,s_1]-s_1)|+k-2+|V(P[t_1,u_1]|+1\\
&\le|V(T)|+k-1.
\end{aligned}\tag{6.5}$$

By (6.2) and (6.5), we obtain

$$\sum_{1\le i\le 3}|N_G(w_i)|-|\bigcap_{1\le i\le 3}N_G(w_i)|\le|V(G)|+k-2.$$

This contradicts to $\overline{\sigma_3}(G)\ge|V(G)|+k-1$.

Suppose $w_2$ is not in a segment, since $G$ is $(k+1)$-connected, then for any $2\le i\le k$, $|N_G(V(H))\cap V(P[v,u_i])|=1$. By Claims 6.4.5, 6.4.6, 6.4.9 and (6.4), we obtain

$$\begin{aligned}
&\sum_{1\le i\le 3}|N_G(w_i)\cap V(T)|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(T)|\\
&=\sum_{P\in\mathcal{P}\cup\mathcal{Q}}\left(\sum_{1\le i\le 3}|N_G(w_i)\cap V(P-s_{u_1}(P))|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P-s_{u_1}(P))|\right)\\
&+\sum_{1\le i\le 3}|N_G(w_i)\cap V(P[v,s_1]-s_1)|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[v,s_1]-s_1))|\\
&+\sum_{1\le i\le 3}|N_G(w_i)\cap V(P[t_1,u_1]))|-|\bigcap_{1\le i\le 3}(N_G(w_i)\cap V(P[t_1,u_1]))|\\
&\le\sum_{P\in\mathcal{P}}(|V(P)|-1)+\sum_{Q\in\mathcal{Q}}(|V(Q)|-1)+|V(P[v,s_1]-s_1)|+k-1+|V(P[t_1,u_1]|+1\\
&\le|V(T)|+k.
\end{aligned}\tag{6.6}$$

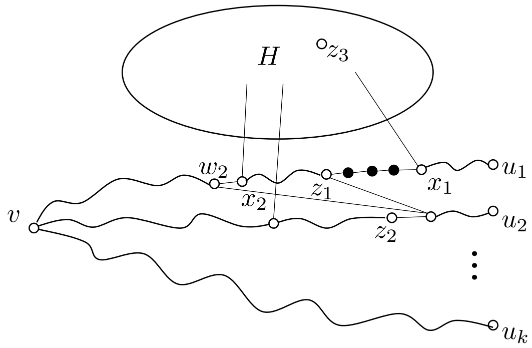



Figure 6.5: The definition of $z_1$, $z_2$, and $z_3$ where black vertices are insertible vertices.

By (6.2) and (6.6), we obtain

$$\sum_{1\le i\le 3} |N_G(w_i)| - |\bigcap_{1\le i\le 3} N_G(w_i)| \le |V(G)| + k - 1.$$

Since $\overline{\sigma_3}(G) \ge |V(G)| + k - 1$, the above inequalities are equal. By Claim 6.4.5,

$$\sum_{1\le i\le 3} |N_G(w_i) \cap V(P[t_2, u_2] - t_2)| - |\bigcap_{1\le i\le 3} (N_G(w_i) \cap V(P[t_2, u_2] - t_2))| = |V(P[t_2, u_2] - t_2)|.$$

Since $w_1 s_{u_1}(P[t_2, u_2] - t_2), w_2 s_{u_1}(P[t_2, u_2] - t_2) \notin E(G)$, and $w_1$ is a non-insertible vertex, it follows from Claim 6.4.4 (iv) and Lemma 6.3.1 that $w_2u_2, w_1u_2 \in E(G)$. This implies $N_G(w_1) \cap N_G(w_2) \cap V(P(t_2, u_2]) \neq \emptyset$. Let $z \in N_G(w_1) \cap N_G(w_2) \cap V(P[t_2, u_2] - t_2)$ such that $|V(P[t_2, z])|$ is as small as possible. By Claim 6.4.4 (i), then the set $\{w_1, w_3, z^-_{u_1}\}$ is an independent set of $G$ since $w_1$ is a non-insertible vertex.

For convenience, let $z_1 = w_1$, $z_2 = z^-_{u_1}$ and $z_3 = w_3$ (see Fig. 6.5). By Claim 6.4.4 (iv), for any $1 \le i \le m$, $y_i$ and $z_2$ are not adjacent, where $y_m = w_1 = z_1$. We consider the degree sum of $\{z_1, z_2, z_3\}$ to divide $T$ into some parts. Fig. 6.6 illustrates how to divide $T$ and when we consider the parts.

Now we will show that for $1 \le i \le m$,

$$N_G(z_2) \cap (N_G(y_i) \cap (V(T) \setminus (V(P[v_2, z_2]) \cup V(P[v, z_1]))))^+_{u_1} = \emptyset. \tag{6.7}$$

We prove this equation by induction on $i$ with $1 \le i \le m$. Suppose that there is a vertex $y \in V(T) \setminus (V(P[v_2, z_2]) \cup V(P[v, z_1])))$ such that $y_1y \in E(G)$ and $z_2y^+_{u_1} \in E(G)$. $T + w_2z + z_2y^+_{u_1} + y_1y - w_2x_2 - z_2z - yy^+_{u_1} \cup x_1Hx_2$ is a $(v, U)$-fan (see Figure 6.7) with the order of at least $|V(T)| + 1$, a contradiction. We assume that this equation (6.7) holds for $1 \le j < i$. Suppose that there is a vertex $w \in N_G(y_i) \cap (V(T) \setminus (V(P[v_2, z_2]) \cup V(P[v, z_1])))$ such that

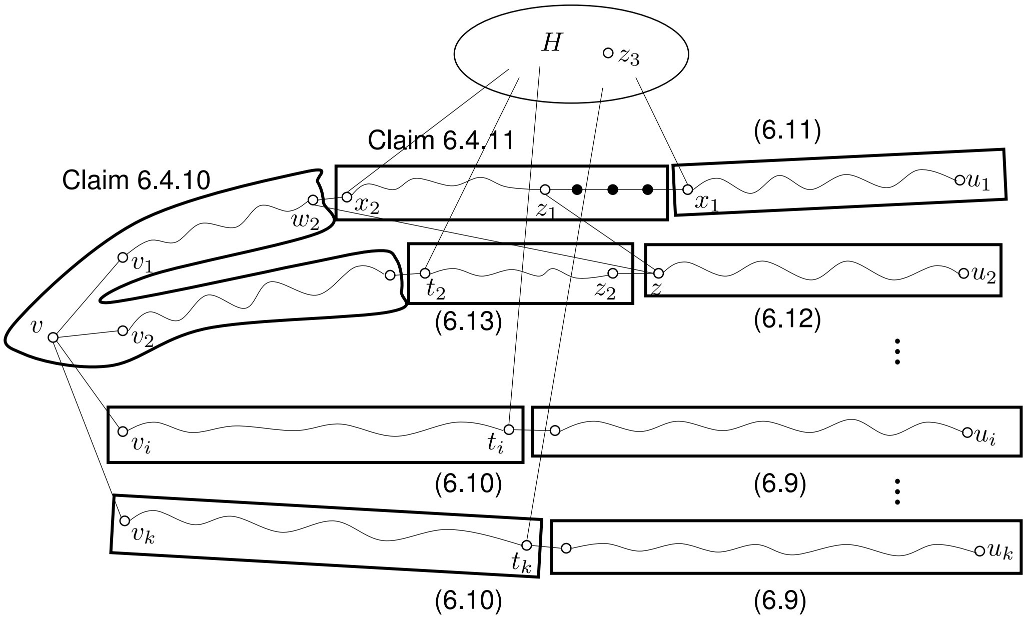



Figure 6.6: Summary of the following proofs.

$w^+_{u_1} z_2 \in E(G)$ for some $1 \leq i \leq m$. We use segment insertion $SI[P[y_1, y_{i-1}]]$ (if $i \geq 2$) and let $T'$ be a resulted graph. It follows from the induction hypothesis that $y_j$ is not inserted into an edge in $\{ww^+_{u_1}, z_2z, x_2w_2\}$ for each $1 \leq j < i$. Hence, $T' + z_2w^+_{u_1} + y_iw + w_2z - ww^+_{u_1} - zz_2 - w_2x_2 \cup x_1Hx_2$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction. Similarly, we obtain that for $1 \leq i \leq m$,

$$N_G(z_2) \cap (N_G(y_i) \cap (V(P[v_2, z_2]) \cup V(P[v, z_1])))^-_{u_1} = \emptyset. \tag{6.8}$$

For $3 \leq i \leq k$, then $s_{u_1}(P[t_i, u_i] \setminus \{t_i\})z_2 \notin E(G)$. Otherwise, there is a $(v, U)$-fan $T + w_2z + z_2s_{u_1}(P[t_i, u_i] \setminus \{t_i\}) - w_2x_2 - zz_2 - t_is_{u_1}(P[t_i, u_i] \setminus \{t_i\}) \cup t_iHx_2$ which contradicts the maximality of $T$. By Lemma 6.3.1 and (6.7), we obtain the following, for $3 \leq i \leq k$,

$$\sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[t_i, u_i] \setminus \{t_i\})| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[t_i, u_i] \setminus \{t_i\}))| \leq |V(P[t_i, u_i] \setminus \{t_i\})| \tag{6.9}$$

By (6.7), for $3 \leq i \leq k$, then $(N_G(z_2) \cap (V(P[v_i, t_i]) \setminus \{v_i\}))^-_{u_1} \cap N_G(z_1) = \emptyset$. Since $z_1$ is a non-insertible vertex and $N_G(V(H)) \cap V(P[v_i, t_i]) \subseteq \{t_i\}$, it follows from Lemma 6.3.2 that we obtain the following, for $3 \leq i \leq k$,

$$\sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[v_i, t_i])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[v_i, t_i]))| \leq |V(P[v_i, t_i])| + 1 \tag{6.10}$$

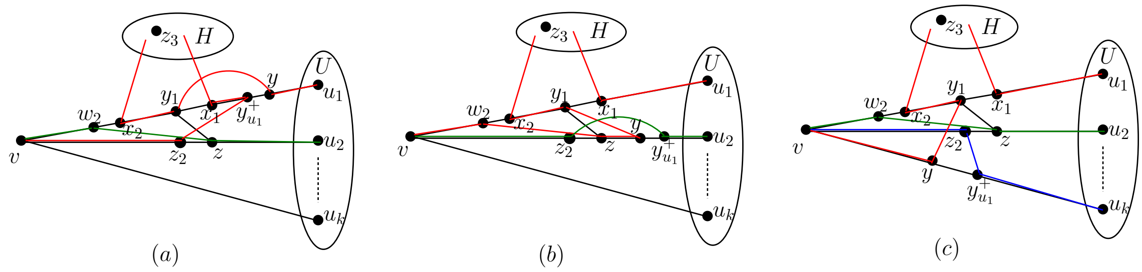


Figure 6.7: The construction of a larger $(v, U)$-fan

Similarly, we have

$$\sum_{1 \le i \le 3} |N_G(z_i) \cap V(P[t_1, u_1])| - |\bigcap_{1 \le i \le 3} (N_G(z_i) \cap V(P[t_1, u_1]))| \le |V(P[t_1, u_1])| + 1. \tag{6.11}$$

and

$$\sum_{1 \le i \le 3} |N_G(z_i) \cap V(P[z, u_2])| - |\bigcap_{1 \le i \le 3} (N_G(z_i) \cap V(P[z, u_2]))| \le |V(P[z, u_2])| + 1. \tag{6.12}$$

and

$$\sum_{1 \le i \le 3} |N_G(z_i) \cap V(P[t_2, z_2])| - |\bigcap_{1 \le i \le 3} (N_G(z_i) \cap V(P[t_2, z_2]))| \le |V(P[t_2, z_2])|. \tag{6.13}$$

**Claim 6.4.10** *The following inequality holds.*

$$\begin{aligned}\sum_{1 \le i \le 3} |N_G(z_i) \cap (V(P[v_2, t_2] \setminus \{t_2\} \cup P[v, w_2]))| - |\bigcap_{1 \le i \le 3} (N_G(z_i) \cap (V(P[v_2, t_2] \setminus \{t_2\} \cup P[v, w_2]))| \\ \le |V(P[v_2, t_2] - \{t_2\})| + |V(P[v, w_2])| - 1\end{aligned}$$

**Proof.** Let $x = s_{u_2}(P[v_2, t_2] - \{t_2\})$. If $z_2 x \in E(G)$, then $T + w_2 z + z_2 x - w_2 x_2 - x t_2 - z z_2 \cup t_2 H x_2$ is a $(v, U)$-fan (see Figure 6.8) with the order of at least $|V(T)| + 1$, a contradiction. So $z_2 x \notin E(G)$. By (6.8) and Lemma 6.3.1(i), we obtain

$$|N_G(z_1) \cap V(P[v_2, t_2] - \{t_2\})| + |N_G(z_2) \cap V(P[v_2, t_2] - \{t_2\})| \le |V(P[v_2, t_2] - \{t_2\})|. \tag{6.14}$$

Suppose $v z_1 \in E(G)$, since $z_1$ is a non-insertible vertex, then $v_2 z_1 \notin E(G)$. When

$$|N_G(z_1) \cap V(P[v_2, t_2] - \{t_2\})| + |N_G(z_2) \cap V(P[v_2, t_2] - \{t_2\})| = |V(P[v_2, t_2] - \{t_2\})|,$$

let $P_1 = P[v_2, t_2] - \{v_2, t_2\}$. If $z_2v_2 \notin E(G)$, then $|N_G(z_1) \cap V(P_1)| + |N_G(z_2) \cap V(P_1)| = |V(P_1)| + 1$. By (6.8) and Lemma 6.3.1(i), $z_2x \in E(G)$. This is a contradiction. So, $z_2v_2 \in E(G)$. By (6.8), $s_{u_1}(P_1)z_1 \notin E(G)$. The similar argument of the above, $z_2x \in E(G)$, a contradiction. Thus, we obtain the following inequality:

$$|N_G(z_1) \cap V(P[v_2, t_2] - \{t_2\})| + |N_G(z_2) \cap V(P[v_2, t_2] - \{t_2\})| \leq |V(P[v_2, t_2] - \{t_2\})| - 1.$$

By Claim 6.4.4(iv), $w_2z_1 \notin E(G)$. It follows from (6.8) and Lemma 6.3.1(i) that

$$|N_G(z_1) \cap V(P[v, w_2])| + |N_G(z_2) \cap V(P[v, w_2])| \leq |V(P[v, w_2])|. \tag{6.15}$$

Hence, we obtain the desired inequality and may assume that $vz_1 \notin E(G)$.

If either inequality (6.14) or inequality (6.15) is not equal, then we obtain the desired inequality. Therefore, we assume that the equal signs of inequalities (6.14) and (6.15) are both true.

Suppose that $z_2v \notin E(G)$. Then $|N_G(z_1) \cap V(P[v_1, w_2])| + |N_G(z_2) \cap V(P[v_1, w_2])| = |V(P[v, w_2])|$. By Lemma 6.3.1 (i), $z_1w_2 \in E(G)$, a contradiction. So, $z_2v \in E$.

When $z_1v_2 \notin E(G)$. Suppose $z_2v_2 \notin E(G)$. By (6.14), we obtain $|N_G(z_1) \cap V(P_1)| + |N_G(z_2) \cap V(P_1)| = |V(P_1)| + 1$. This together with Lemma 6.3.1 (i), $xz_2 \in E(G)$, a contradiction. So $z_2v_2 \in E(G)$. Then $s_{u_1}(P_1)z_1 \notin E(G)$ by (6.8), the similar argument of the above, $s_{u_1}(P_1)z_2 \in E(E)$. Repeating the above argument for all vertices on $P[v_2, t_2] - t_2$, we get $xz_2 \in E(G)$, a contradiction. So, $z_1v_2 \in E(G)$.

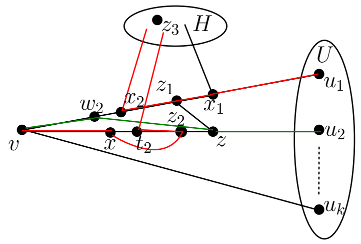



Figure 6.8: The construction of a larger $(v, U)$-fan with $xz_2 \in E$ in Claim 6.4.10

Thus, $z_2v \in E(G)$ and $z_1v_2 \in E(G)$. This contradicts to (6.8). Hence, the claim holds. ∎

**Claim 6.4.11** *The following inequality holds for.*

$$\sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[x_1, x_2]) \setminus \{x_1\}| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[x_1, x_2]) \setminus \{x_1\}| \leq |V(P[x_1, x_2]) \setminus \{x_1\}|$$

**Proof.** Since for $1 \leq i \leq m$, $z_2y_i \notin E(G)$, then $|N_G(z_1) \cap V(P[x_1, z_1]) \setminus \{x_1, z_1\}| + |N_G(z_2) \cap V(P[x_1, z_1]) \setminus \{x_1, z_1\}| \leq$

$|V(P[x_1, z_1]) \setminus \{x_1, z_1\}|$. By Lemma 6.3.1, (6.8) and $z_1 z_2 \notin E(G)$, then

$$|N_G(z_1) \cap V(P[z_1, x_2])| + |N_G(z_2) \cap V(P[z_1, x_2])| \leq |V(P[z_1, x_2])|. \tag{6.16}$$

If the inequality (6.16) is not equal, then we obtain the desired inequality since $N_G(z_3) \cap V(P[x_1, x_2]) \setminus \{x_1\} \subseteq \{x_2\}$. If $N_G(z_3) \cap V(P[x_1, x_2]) \setminus \{x_1\} = \emptyset$, then we also obtain the desired inequality by (6.16). Hence, we may assume that the equal sign of the inequality (6.16) holds and $N_G(z_3) \cap V(P[x_1, x_2]) \setminus \{x_1\} = \{x_2\}$.

Suppose that $z_1 x_2 \in E(G)$. Then $z_1$ and $z_2$ are adjacent to $x_2$ by Lemma 6.3.1 (i). This together with $x_2 z_3 \in E(G)$, we obtain the desired inequality. Hence, we may assume that $z_1 x_2 \notin E(G)$. Then $|N_G(z_1) \cap (V(P[z_1, x_2]) \setminus \{x_2\})| + |N_G(z_2) \cap (V(P[z_1, x_2]) \setminus \{x_2\})| = |V(P[z_1, x_2])| - 1$. Let $x' = s_{u_2}(P[x_2, z_1] - x_2)$. By Lemma 6.3.1 (i), $z_2 x' \in E(G)$. We use a segment insertion $SI[P[y_1, y_{m-1}]]$ and let $T'$ be a resulted graph. So, $T' + z_2 x' + z_1 z - z_2 z - x_2 x' \cup x_1 H x_2$ is a $(v, U)$-fan with the order of at least $|V(T)| + 1$, a contradiction. Hence, we obtain the desired inequality. ∎

By (6.9) and (6.10), we obtain

$$\begin{aligned}
&\sum_{3 \leq j \leq k} \left( \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[v_j, u_j])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[v_j, u_j])| \right) \\
&= |V(T) \setminus (V(P[v, u_1] \cup P[v, u_2]))| + k - 2
\end{aligned} \tag{6.17}$$

By Claims 3.4.12, 6.4.10, (6.11), (6.12), and (6.13), we obtain

$$\begin{aligned}
&\sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[v, u_1] \cup P[v, u_2])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[v, u_1] \cup P[v, u_2])| \\
&= \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[t_1, u_1])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[t_1, u_1]))| \\
&+ \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[x_1, x_2))| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P(x_1, x_2])| \\
&+ \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[v_2, t_2))| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[v_2, t_2))| \\
&+ \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[z, u_2])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[z, u_2]))| \\
&+ \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[t_2, z_2])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[t_2, z_2]))| \\
&+ \sum_{1 \leq i \leq 3} |N_G(z_i) \cap V(P[v, w_2])| - |\bigcap_{1 \leq i \leq 3} (N_G(z_i) \cap V(P[v, w_2]))| \\
&\leq V(P[v, u_1] \cup P[v, u_2]) + 1.
\end{aligned} \tag{6.18}$$

Therefore, by (6.2), (6.17), and (6.18), we obtain

$$\sum_{1 \leq i \leq 3} |N_G(z_i)| - |\bigcap_{1 \leq i \leq 3} N_G(z_i)| \leq |V(G)| + k - 2.$$

This contradicts to $\overline{\sigma_3}(G) \geq |V(G)| + k - 1$.

The proof of Theorem 6.0.2 (i.e., Theorem 1.3.53) is complete. □

## 6.5 Concluding remarks and further work

For any integer $t \geq 2$, let $v$ be a vertex of a graph $G$ and let $U = \{u_1, u_2, \ldots, u_t\}$ be a subset of $V(G) \setminus \{v\}$. A $(v, U)$*-fan* is a set of paths $P_1, P_2, \ldots, P_t$ such that $P_i$ is a path connecting $v$ and $u_i$ for $1 \leq i \leq t$ and $P_i \cap P_j = \{v\}$ for $1 \leq i < j \leq t$. If a $(v, U)$-fan spans $G$, then it is called a spanning $(v, U)$-fan of $G$. $G$ is $k$-fan-connected if $G$ has a spanning $(v, U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| = k$. Clearly, the $k$-fan-connectivity generalizes the Hamilton-connectivity.

In this chapter, we prove that if for any three independent vertices $x_1, x_2, x_3$ in a graph $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected and the lower bound is sharp.

Note that the conditions for our results are better than those previously obtained. Is there any other better condition for a graph to be $k$-fan-Connected? Such as Chvátal and Erdös condition ($\alpha(G) \leq \kappa(G) + 1$) and so on. This will be one of our further works.

# Chapter 7

# Conclusions and future research

In this thesis, we (mainly) studied hamiltonian graph theory. We briefly describe the obtained results here. In addition, we would like to mention several new studies that are relevant but not included in this thesis.

## 7.1 Results obtained and open questions

In Chapter 2, we proved that if $G = (V, E)$ is a $2$-connected graph of order $n$ with $V(G) = X \cup Y$ such that for any pair of nonadjacent vertices $x_1$ and $x_2$ in $X$, $d(x_1) + d(x_2) \geq n$ and for any pair of nonadjacent vertices $y_1$ and $y_2$ in $Y$, $d(y_1) + d(y_2) \geq n$, then $G$ is pancyclic or $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - \{e\}$.

Note that the main result of Chapter 2 is to prove that the conjecture 2.0.2 is true for $k = 2$.

In Chapter 3, we proved that Conjecture 2.0.2 is true for $k = 3$.

We showed that if $G = (V, E)$ is a $3$-connected graph of order $n$ with $V(G) = X_1 \cup X_2 \cup X_3$, for any pair of nonadjacent vertices $v_1$ and $v_2$ in $X_i$, $d(v_1) + d(v_2) \geq n$ with $i = 1, 2, 3$, then $G$ is pancyclic or a bipartite graph.

We haven't given a proof for Conjecture 1.3.12 with $k \geq 4$. That's what we're going to do next.

**Conjecture 7.1.1** *Let $G = (V, E)$ be a $k$-connected graph ($k \geq 4$) of order $n$. Suppose that $V(G) = \cup_{i=1}^{k} X_i$. If for any pair of nonadjacent vertices $x, y \in X_i$ with $i = 1, 2, \ldots, k$, $d(x) + d(y) \geq n$, then $G$ is pancyclic or $G$ is a bipartite graph.*

This Conjecture 7.1.1 is still open.

For Conjecture 1.3.12, it is natural to generalize them into degree and neighborhood conditions on more independent vertices. So, this is our other further work. When we consider the topic above, we posed the following problem:

**Question 7.1.1** *Let $G = (V, E)$ be a 2-connected graph of order $n$. Suppose that $V(G) = X \cup Y$. If $\overline{\sigma_3}(X) \geq n + c$ and $\overline{\sigma_3}(Y) \geq n + c$, where $c$ be an integer, then $G$ is hamiltonian.*

The symbols $\overline{\sigma_3}(X)$ and $\overline{\sigma_3}(Y)$ that appear in Question 7.1.1 can be found in section 1.3.4 of Chapter 1. From Bondy's "metaconjecture", we further ask the following questions:

**Question 7.1.2** *Let $G = (V, E)$ be a 2-connected graph of order $n$. Suppose that $V(G) = X \cup Y$. If $\overline{\sigma_3}(X) \geq n + c$ and $\overline{\sigma_3}(Y) \geq n + c$ where $c$ be an integer, then $G$ is pancyclic or a bipartite graph.*

**Question 7.1.3** *Let $G = (V, E)$ be a $k$-connected graph, $k \geq 2$, of order $n$. Suppose that $V(G) = \cup_{i=1}^{k} X_i$ such that for each $i$, $i = 1, 2, \ldots, k$, and $\overline{\sigma_3}(X_i) \geq n + c$ where $c$ be an integer, then $G$ is pancyclic or $G$ is bipartite graph.*

In Chapter 1, we defined implicit degree (Definition 1.2.6). For the condition of implicit degree, Li proposes the following conjecture:

**Conjecture 7.1.2** *Let $G = (V, E)$ be a 2-connected graph of order $n$. $S$ be a subset of $V(G)$. If $\sigma_{i,2} \geq n$, then $G$ is S-pancyclic or $G$ is exceptional graph.*

If we change the degree condition to the implicit degree condition in Conjecture 2.0.2, is there the same conclusion? What is the lower bound after changing to the implicit degree condition? Can it be characterized? These are the questions we will continue to study next.

In Chapter 4, we gave sufficient conditions for a balanced bipartite digraph to be hamiltonian. And we show some sufficient conditions for a digraph to be even pancyclic and cyclable.

We showed that in a balance bipartite digraph with order $2a$, if $d(x) + d(y) \geq 3a$ for every dominating pair of vertices $\{x, y\}$, we can find a hamiltonian cycle.

According to Bondy's metaconjecture, we got the following question.

**Problem 7.1.3** *Let $D$ be a strongly connected balanced bipartite digraph of order $2a \geq 10$ other than a directed cycle of length $2a$. If $D$ satisfies the condition $\aleph_1$, i.e., $d(x) + d(y) \geq 3a$ for every dominating pair of vertices $\{x, y\}$, then $D$ is even pancyclic?*

We also showed that if a digraph $D$ of order $2a$ is not a directed cycle and $D$ contains a cycle of length $2a - 2$, if $d(x) + d(y) \geq 3a + 1$ for every dominating pair of vertices $\{x, y\}$, then $D$ contains a cycle of length $2k$ for all $k$, where $1 \leq k \leq a$.

We want to know whether there is a cyclable version of Theorem 4.0.2 (or the sufficient hamiltonian condition for directed graphs). This will be our further works.

Similarly, can we get $D$ is hamiltonian by replacing the condition of degree with the condition of implicit degree? For example, starting with Theorem 4.1.1, we have the following problem:

**Question 7.1.4** *If $D$ is a strongly connected digraph of order $n \geq 2$ and $d_i(x) + d_i(y) \geq 2n - 1$ for all pairs of nonadjacent vertices $x$ and $y$ of $D$, then $D$ is hamiltonian.*

A non-induced cycle is called a chorded cycle. A graph $G$ of order $n$ is chorded pancyclic if $G$ contains a chorded cycle of each length from 4 to $n$. A graph is called $K_{1,3}$-free if it has no induced $K_{1,3}$ subgraph.

In Chapter 5, we prove that the following result: every 2-connected $K_{1,3}$-free graph $G$ with $|V(G)| \geq 35$ is chorded pancyclic if the minimum degree is at least $\frac{n-2}{3}$. We show the number of chords in the chord cycle of length $l$ $(4 \leq l \leq n)$. Moreover, $G$ is doubly chorded pancyclic.

At present, there are not many kinds of researches on chorded pancyclic. So, there's a lot of room for research. Can we find more necessary and sufficient conditions for a graph to be chorded pancyclic? That's what we're going to work on.

For any integer $t \geq 2$, let $v$ be a vertex of a graph $G$ and let $U = \{u_1, u_2, \ldots, u_t\}$ be a subset of $V(G) \setminus \{v\}$. A $(v, U)$*-fan* is a set of paths $P_1, P_2, \ldots, P_t$ such that $P_i$ is a path connecting $v$ and $u_i$ for $1 \leq i \leq t$ and $P_i \cap P_j = \{v\}$ for $1 \leq i < j \leq t$. If a $(v, U)$-fan spans $G$, then it is called a spanning $(v, U)$-fan of $G$. $G$ is $k$-fan-connected if $G$ has a spanning $(v, U)$-fan for every vertex $v$ of $G$ and every subset $U$ of $V(G) \setminus \{v\}$ with $|U| = k$. Clearly, the $k$-fan-connectivity generalizes the Hamilton-connectivity.

In Chapter 6, we prove that if for any three independent vertices $x_1, x_2, x_3$ in a graph $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected and the lower bound is sharp.

Note that the conditions for our results are better than those previously obtained. Is there any other better condition for a graph to be $k$-fan-connected? Such as Chvátal and Erdös condition ($\alpha(G) \leq \kappa(G) + 1$) and so on. This will be one of our further works.

If for any pair of vertices $x$ and $y$, and for $k$ distinct vertices $\{u_1, u_2, \ldots, u_k\}$ in $V - \{x, y\}$, there are $k$ internal disjoint paths $P_1, P_2, \ldots, P_k$ connecting $x$ and $y$, respectively, such that

$$u_i \in P_i - \{x, y\} \; for \; 1 \leq i \leq k; \; and \; \bigcup_{1 \leq i \leq k} V(P_i) = V(G)$$

Then $G$ is called *$k$-fan-Hamilton-connected.*

We will show the result about $k$-fan-Hamilton-connected of a graph for Dirac-type condition. Our main theorem is as follows:

**Theorem 7.1.4** *Let $k \geq 2$ be an integer and $G$ be a graph with order $n \geq 2$. If $\delta(G) \geq \frac{n+k}{2}$, then $G$ is $k$-fan-Hamilton-connected.*

Similarly, we will prove that the result about $k$-fan-Hamilton-connected of a graph for ore-type condition. We obtain the following theorem:

**Theorem 7.1.5** *Let $k \geq 2$ be an integer and $G$ be a graph with order $n \geq 2$. If $\sigma_2(G) \geq n+k$, then $G$ is $k$-fan-Hamilton-connected.*

For Theorems 7.1.4 and 7.1.5, We intend to prove in two steps. The first step is to prove that for any pair of vertices $x$ and $y$, and for $k$ distinct vertices $\{u_1, u_2, \ldots, u_k\}$ in $V - \{x, y\}$, there are $k$ internal disjoint paths $P_1, P_2, \ldots, P_k$ connecting $x$ and $y$, respectively, such that $u_i \in P_i - \{x, y\}$, for any $1 \leq i \leq k$. The second step to prove $\bigcup_{1 \leq i \leq k} V(P_i) = V(G)$. Now that we have completed the second part of the proof, we only have to prove the existence. This will be our future work.

## 7.2 Future research

Here, we would like to mention future research.

### 7.2.1 Hamiltonian line graphs

One of the topics in the hamiltonian graph is the hamiltonicity of claw-free graphs. As we all know, every line graph is claw-free.

The line graph transformation is probably the most interesting of all graph transformations, and it is certainly the most widely studied. The line graph concept is quite natural and has been introduced in several ways. We want to consider the hamiltonian line graphs next. Even we want to study pancyclicity on the line graphs. For example, we will consider the following problems:

**Question 7.2.1** *Let $G = (V, E)$ be a $k$-connected line graph, $k \geq 2$, of order $n$. Suppose that $V(G) = \cup_{i=1}^{k} X_i$ such that for each $i$, $i = 1, 2, \ldots, k$, and for any pair of nonadjacent vertices $x, y \in X_i$, $d(x) + d(y) \geq n$, then $G$ is pancyclic or $G$ is bipartite graph.*

**Question 7.2.2** *For 3-connected line graphs, can high essential connectivity guarantee chorded pancyclic? Or what are the sufficient conditions to determine the line graph to be chorded pancyclic?*

### 7.2.2 Fault-tolerant hamiltonicity

The consideration of fault-tolerance ability is a major factor in evaluating the performance of networks. A graph $G$ is called a $k$-vertex fault-tolerant hamiltonian, or simply $k$-hamiltonian, if it remains hamiltonian after removing no more than $k$ vertices from $G$. Hence, using the notion of fault-tolerance the $k$-hamiltonian-connected graphs, $k$-pancyclic graphs, and $k$-panconnected graphs can be defined similarly. Fault-tolerant hamiltonicity has been widely studied in many network topologies, such as hypercubes, de Bruijn networks, double loop networks, twisted cubes, bubble-sort graphs, and star graphs.

**Definition 7.2.1** *Let $\Gamma$ be a group, $S$ be a set of elements of $\Gamma$ not including the identity element. Suppose, furthermore, that the inverse of every element of S also belongs to $S$. The* Cayley graph *$C(\Gamma, S)$ is the graph with vertex set $\Gamma$ in which two vertices $x$ and $y$ are adjacent if and only if $xy^{-1} \in S$.*

Given a graph $G$, we assign a sign $+$ or $-$ to each edge of $G$. The edges labeled $+$ are called *positive edges* while the ones labeled $-$ are called *negative edges*. We can see this assignment as a mapping of the edges of $G$ to the set $\{+, -\}$. Such a mapping is called a *signature* of $G$. We normally denote the set of negative edges by $\Sigma$. Note that a signature of $G$ is given if and only if the set of negative edges is given, thus the set of edges $\Sigma$ will be referred to as the signature of $G$, and $(G, \Sigma)$ is called a *signed graph*.

Since edge faults can occur when a network is put into service, it is important to consider faulty networks. So, fault-tolerance ability is a very important factor of interconnection networks. Therefore, we want to consider edge fault-tolerant hamiltonicity and edge fault-tolerant pancyclicity (bipancyclicity) in many graphs, such as singed graphs and so on.

### 7.2.3 Graph coloring

Due to the four-color problem and the modeling of several applications, graph coloring is one of the most studied areas of graph theory. It consists of assigning colors to the vertices or edges of an input graph under various constraints.

Edge-colorings are interesting not only because of the mathematical point of view but also because of the many applications they have in real life, for example in scheduling problems and frequency assignment for fiber optic networks, etc. Therefore, many types of edge-colorings have been studied over the years.

An edge-colored graph is a graph whose edges have been colored in some way with $c$ different colors. There is a question: given an edge-colored graph, how can we find (if possible) or guarantee the existence of some subgraphs with certain properties? For example, how to find or guarantee the existence of a hamiltonian cycle that is properly colored. So, we want to study proper hamiltonian cycles, proper hamiltonian paths, proper trees, proper cycles, rainbow trees, rainbow paths, rainbow cliques, monochromatic cliques, monochromatic cycles, etc. on some conditions such as several edges, connectivity, rainbow degree, etc.

A graph is $k$*-proper connected* if any two vertices are connected by $k$-vertex disjoint paths whose adjacent edges have distinct colors. A *strong edge-coloring* of a graph $G$ is an edge-coloring such that any two vertices belonging to distinct edges with the same color are not adjacent.

We also want to study the proper connection of graphs and strong edge-colorings of graphs.

### 7.2.4 Other works

We can study graph structural properties with algorithmic aspects. We also consider the parameters for several classes of graphs like graphs without induced $P_4$ (path on $4$ vertices), bipartite graphs, grids, etc.

Furthermore, we study the hamiltonian properties of the graph that can be combined with the algorithm.

The vertex coloring problem: the vertices of the input graph are presented to a coloring algorithm one at a time in some arbitrary order. The algorithm must choose a color for each vertex, based only on the colors assigned to the already-processed vertices.

We also studied the graph coloring problem by the algorithm such as polynomial-time algorithms. The most popular on-line coloring algorithm is the greedy algorithm.

# Appendix A

# The supplement of Claim 3.4.5

In this chapter, we will give a detailed proof of Claim 3.4.5 in Chapter 3.

Since $|V(P[v_2, v_{d-1}])| \leq 4$ and $|V(P[v_{d+1}, v_{p-1}])| \leq 4$, by the maximality of $P$, then $|H| \leq \min\{d-2, p-d-1\} \leq 4$.

Suppose $V(H) = \{u\}$. If $|V(P[v_2, v_{d-1}])| = 1$ and $|V(P[v_{d+1}, v_{p-1}])| = 1$, since $d(v_1) + d(v_p) \geq n$ and $G$ is not pancyclic, it is easy to know $G = K_{3,3}$.

Suppose $2 \leq |V(P[v_2, v_{d-1}])| \leq 4$ or $2 \leq |V(P[v_{d+1}, v_{p-1}])| \leq 4$. By $d(v_1) + d(v_p) \geq n$, if $|V(P[v_2, v_{d-1}])| = 3$, $|V(P[v_{d+1}, v_{p-1}])| = 1$, $|V(P[v_2, v_{d-1}])| = 3$ and $|V(P[v_{d+1}, v_{p-1}])| = 3$, we obtain $G$ is a bipartite graph. Otherwise, we can construct all cycles $C_k$, $3 \leq k \leq n$.

Taking $d = 6$ and $p = 11$ as an example, we construct all the cycles $C_k$, for $3 \leq k \leq n$, in $G$. Since $n = 12$ and $G$ is hamiltonian, then we just construct all cycles $C_k$, $3 \leq k \leq 11$. And $d_P(v_1) + d_P(v_{11}) \geq 10$.

First, we construct the cycle $C_3$. Suppose there does not exist a cycle $C_3$. Then, for any $v_i \in V(P[v_2, v_{10}])$, $v_iv_1 \notin E(G)$ or $v_{i+1}v_1 \notin E(G)$. Since $d_P(v_1) + d_P(v_{11}) \geq 10$, then $N_P(v_1) = N_P(v_p) = \{v_2, v_4, v_6, v_8, v_{10}\}$. Thus, $C_3 = v_1v_6uv_1$, a contradiction.

If $C_4$ does not exist in $G$, then $v_1v_4 \notin E(G)$. And $v_1v_5 \notin E(G)$ otherwise let $C_4 = v_1v_5v_6uv_1$. Similarly, $v_1v_7, v_1v_{10} \notin E(G)$. So, $N_P(v_1) \subseteq \{v_2, v_3, v_6, v_8, v_9\}$. By the symmetry $v_1$ and $v_p$, then $N_P(v_{11}) \subseteq \{v_3, v_4, v_6, v_9, v_{10}\}$. Since $d_P(v_1) + d_P(v_{11}) \geq 10$, then $v_1v_6, v_1v_8 \in E(G)$. Let $C_4 = v_1v_6v_7v_8v_1$, a contradiction.

The same argument with $C_4$, if $C_5$ does not exist, then

$$N_P(v_1) \subseteq \{v_2, v_3, v_6, v_7, v_{10}\} \text{ and } N_P(v_{11}) \subseteq \{v_2, v_5, v_6, v_9, v_{10}\}$$

. Since $d_P(v_1) + d_P(v_{11}) \geq 10$, then $v_1v_3, v_1v_6 \in E(G)$. So, let $C_5 = v_1v_3v_4v_5v_6v_1$, a contradiction.

The same with above, we can construct the cycle $C_6$. And $C_7 = v_1v_2 \cdots v_6uv_1$.

If there does not exist cycle $C_8$ in $G$, then $v_1v_6, v_{11}v_6, v_1v_8, v_{11}v_4 \notin E(G)$. There is at most one edge between $v_1v_3$ and $v_1v_9$. And we have $v_1v_4 \notin E(G)$ or $v_1v_{10} \notin E(G)$. So $d_P(v_1) \leq 5$. Since $d_P(v_1) + d_P(v_{11}) \geq 10$, by the symmetry $v_1$ and $v_{11}$, then $v_1v_7, v_1v_5, v_{11}v_5, v_{11}v_7 \in E(G)$. And without loss of generality, let $v_1v_3 \in E(G)$. So, $C_8 = v_1v_3v_4v_5v_6uv_{11}v_7v_1$, a contradiction.

If there does not exist cycle $C_9$ in $G$, then $v_1v_9, v_1v_5 \notin E(G)$. If $v_1v_{10} \in E(G)$, then $C_9 = v_1v_2v_3v_4v_5v_6uv_{11}v_{10}v_1$, a contradiction. And $v_1v_4 \notin E(G)$ or $v_1v_8 \notin E(G)$. So

$$N_P(v_1) = \{v_2, v_3, v_4, v_6, v_7\} \text{ or } N_P(v_1) = \{v_2, v_3, v_6, v_7, v_8\}.$$

By the symmetry,

$$N_P(v_p) = \{v_4, v_5, v_6, v_9, v_{10}\} \text{ or } N_P(v_p) = \{v_5, v_6, v_8, v_9, v_{10}\}.$$

Since $d_P(v_1) + d_P(v_{11}) \geq 10$, then $v_1v_7, v_{11}v_5 \in E(G)$. So, let $C_9 = v_1v_7v_8v_9v_{10}v_{11}v_5v_6uv_1$, a contradiction.

Suppose that there does not exist cycle $C_{10}$ in $G$, then $v_1v_9, v_1v_{10}, v_1v_4 \notin E(G)$. And $v_1v_3 \notin E(G)$ or $v_1v_8 \notin E(G)$ otherwise $C_{10} = v_1v_3v_4v_5v_6uv_{11}v_{10}v_9v_8v_1$, a contradiction. So $N_P(v_1) = \{v_2, v_3, v_5, v_6, v_7\}$ or $N_P(v_1) = \{v_2, v_5, v_6, v_7, v_8\}$. By the symmetry, $N_P(v_p) = \{v_4, v_5, v_6, v_7, v_{10}\}$ or $N_P(v_p) = \{v_5, v_6, v_7, v_9, v_{10}\}$. Since $d_P(v_1) + d_P(v_{11}) \geq 10$,

$$C_{10} = \begin{cases} v_1v_5v_4v_{11}v_{10}v_9v_8v_7v_6uv_1 & \text{if } v_{11}v_4 \in E(G), \\ v_1v_2v_3v_4v_5v_{11}v_9v_8v_7v_6v_1 & \text{if } v_9v_{11} \in E(G). \end{cases}$$

This is a contradiction.

If $C_{11}$ does not exist in $G$, then $v_1v_3, v_1v_8 \notin E(G)$ and $(N_P(v_1))^- \cap N_P(v_{11}) = \emptyset$. since $d_P(v_1) + d_P(v_{11}) \geq 10$, by Lemma 6.3.1, then $v_2v_{11}, v_1v_{10} \in E(G)$ from $v_1v_3, v_9v_{11} \notin E(G)$. If $v_i \in V(P[v_4, v_6]) \cup \{v_9\}$ such that $v_1v_i, v_1v_{i+1} \in E(G)$, then $C_{11} = v_1v_iv_{i-1}\cdots v_2v_{11}v_{10}\cdots v_{i+1}v_1$, a contradiction. So $d_P(v_1) \leq 4$. Similarly, $d_P(v_{11}) \leq 4$. This contradicts to $d_P(v_1) + d_P(v_{11}) \geq 10$.

So, in this case, we can construct all cycles $C_k$, $3 \leq k \leq n$, in $G$.

Similarly, when $2 \leq |V(H)| \leq 4$, we can obtain $G$ is pancyclic or $G$ is a bipartite graph.

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# Appendix B

# Publications and manuscripts

1. H. LI and Z. Tian, On pancyclic 2-connected graphs, Graphs and Combinatorics. (Reference [85])

2. H. Li, S. Maezawa and Z. Tian, New sufficient condition for graphs to be k-fan-connected, will submitted. (Reference [94])

3. H. Li and Z. Tian, A new condition for Pancyclicity of 3-connected graph, manuscript. (Reference [86])

4. H. Li and Z. Tian, Sufficient condition for a balanced bipartite digraph to be hamiltonian and even pancyclic, manuscript.(Reference [87])

5. H. Li, S. Maezawa and Z. Tian, Chorded pancyclicity on $K1,3$-free graph, manuscript. (Reference [93])

ÉCOLE DOCTORALE
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**Titre:** Pancyclicité dans la théorie des graphes hamiltonienne

**Mots clés:** Pancyclicité, Cycle hamiltonien, Pancyclicité à cordes, Graphe sans griffe, $k$-fan-connecté.

**Résumé:** La théorie hamiltonienne des graphes a été largement étudiée comme l'un des problèmes les plus importants de la théorie des graphes. Dans cette thèse, nous travaillons sur des généralisations de la théorie hamiltonienne des graphes, et nous nous concentrons sur les sujets suivants : hamiltonien graphes, pancyclicité, pancyclicité à cordes dans les graphes sans griffes, graphes $k$-fan-connectés.

Pour le problème du pancyclic, on montre pour $k = 2, 3$, si $G = (V, E)$ est un graphe $k$-connecté d'ordre $n$ avec $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$, et pour toute paire de sommets non adjacents $x, y$ dans $X_i$ avec $i = 1, 2, \ldots, k$, on a $d(x) + d(y) \geq n$, alors $G$ est pancyclique ou $G$ est un graphe bipartite.

Pour le problème hamiltonien du digraphe biparti, soit $D$ un graphe orienté biparti équilibré fortement connecté d'ordre $2a \geq 10$. Soit $x, y$ des sommets distincts dans $D$, $\{x, y\}$ domine un sommet $z$ si $x \to z$ et $y \to z$; dans ce cas, nous appelons le couple $\{x, y\}$ dominant. Nous montrons que $D$ est hamiltonien pour chaque paire de sommets dominants si leur somme de degrés est d'au moins $3a$. En outre, nousmontrons quelques nouvelles conditions suffisantes pour la bi-pancyclique et la cyclabilit é des digraphes.

Pour le problème pancyclique à cordes dans les graphes sans griffes, nous prouvons que tout graphe $G$ sans griffes 2-connecté avec $|V(G)| \geq 35$ est pan-cyclique à cordes si le degré minimum est d'au moins $\frac{n-2}{3}$. De plus, nous montrons le nombre de cordes dans le cycle à cordes de longueur $l$ ($4 \leq l \leq n$). De plus, $G$ est un pancyclique à double corde.

Pour le problème $k$-fan-connecté, nous prouvons que si pour trois sommets indépendants $x_1, x_2, x_3$ dans un graphe $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + k - 1$, alors $G$ est $k$-fan-connecté et la borne inférieure est tranchant. Ce résultat principal en déduit qu'un graphe 3-connexe, sous les mêmes hypothèses, est un Hamilton-connexe.
Enfin, nous aimerions mentionner plusieurs nouvelles études liées à cette thèse qui n'est pas incluses dans la thèse. De plus, nous couvrons également d'autres sujets qui m'intéressent, tels que les graphes de ligne hamiltoniens, l'hamiltonicité tolérante aux pannes, la coloration de graphe, etc. Ces sujets sont susceptibles de devenir mes autres domaines de recherche.

**Title:** Pancyclicity in hamiltonian graph theory

**Keywords:** Pancyclicity, Hamiltonian cycle, Chorded pancyclicity, Claw-free graph, $k$-fan-connected.

**Abstract:** Hamiltonian graph theory has been widely studied as one of the most important problems in graph theory. In this thesis, we work on generalizations of hamiltonian graph theory, and focus on the following topics: hamiltonian graphs, pancyclicity, chorded pancyclic in the claw-free graphs, $k$-fan-connected graphs.

For pancyclic problem, we show for $k = 2, 3$, if $G = (V, E)$ is a $k$-connected graph of order $n$ with $V(G) = X_1 \cup X_2 \cup \cdots \cup X_k$, and for any pair of nonadjacent vertices $x, y$ in $X_i$ with $i = 1, 2, \ldots, k$, we have $d(x) + d(y) \geq n$, then $G$ is pancyclic or $G$ is a bipartite graph.

For hamiltonian problem in bipartite digraph, let $D$ be a strongly connected balanced bipartite directed graph of order $2a \geq 10$. Let $x, y$ be distinct vertices in $D$, $\{x, y\}$ dominates a vertex $z$ if $x \to z$ and $y \to z$; in this case, we call the pair $\{x, y\}$ dominating. We show that $D$ is hamiltonian for each dominating pair of vertices if their degree sum is at least $3a$. In addition, we show some new sufficient conditions for bipancyclic and cyclability of digraphs.

For chorded pancyclic problem in claw-free graphs, we prove that every 2-connected claw-free graph $G$ with $|V(G)| \geq 35$ is chorded pancyclic if the minimum degree is at least $\frac{n-2}{3}$. Furthermore, we show the number of chords in the chord cycle of length $l$ ($4 \leq l \leq n$). In addition, $G$ is doubly chorded pancyclic.

For $k$-fan-connected problem, we prove that if for any three independent vertices $x_1, x_2, x_3$ in a graph $G$, $\sum_{i=1}^{3} \deg_G(x_i) - |\bigcap_{i=1}^{3} N_G(x_i)| \geq |V(G)| + k - 1$, then $G$ is $k$-fan-connected and the lower bound is sharp. This main result deduces a 3-connected graph, under the same assumptions, is a Hamilton-connected.
Finally, we would like to mention several new studies related to this thesis that is not included in the thesis. Moreover, we also cover other topics that I am interested in, such as hamiltonian line graphs, fault-tolerant hamiltonicity, graph coloring and so on. These topics are likely to become my further research fields.