Long-Time Behavior of Kinetic Equations with Boundary Effects

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Abstract

This thesis studies the long-time behavior of several partial differential equations arising in kinetic theory. Those equations have an equilibrium towards which, roughly, some solution converges. We employ both probabilistic and deterministic methods to derive the rate of this convergence. This work is divided into three parts.

In the first one, we study the free-transport equation enclosed in a bounded domain with Maxwell boundary condition, as already considered by Aoki and Golse [1] and Kuo et al. [87, 88, 86]. We extend the (almost) optimal rate $\frac{1}{t^d}$, where d is the dimension of the problem, to the case of a general regular domain, without the symmetry assumption required in earlier works. We use two different methods: a probabilistic coupling in Chapter 2, allowing us to generalize the boundary condition, and a deterministic version of some subgeometric Harris' theorem in Chapter 4, with which we can consider the case where the temperature varies at the boundary. In Chapter 3 we provide some numerical evidences supporting our result that the polynomial rate of convergence observed in the case where the spatial domain is symmetric should extend to non-symmetric, regular domains.

In the second part of this thesis, we focus on the subgeometric convergence towards the invariant distribution of Markov processes. We exhibit a new set of conditions, close in spirit to the ones of Douc, Fort and Guillin [44] and Hairer [71], leading to the subgeometric convergence of a strong Markov process. Our conditions are chosen in order to be equivalent, as in the exponential theory of Meyn and Tweedie [98], whereas only one implication holds in the usual set of conditions.

In the last part, we study collisional kinetic models, namely the linearized Boltzmann and linearized Landau equations, enclosed in a regular, bounded domain. We prove constructive L^2 hypocoercivity estimates for the generalized Maxwell boundary condition, which includes the case of the specular reflection boundary condition. With those estimates, one concludes to the exponential relaxation towards equilibrium for those models.

Résumé

Cette thèse est dédiée à l'étude du comportement en temps long de plusieurs équations aux dérivées partielles, issues de la théorie cinétique, pour lesquelles, informellement, un équilibre vers lequel la solution converge existe. Nous utilisons des méthodes probabilistes et déterministes pour obtenir le taux de convergence associé. Ce travail est divisé en trois parties.

Dans la première, nous étudions un modèle de transport libre à l'intérieur d'un domaine borné, avec la condition au bord de Maxwell déjà considérée par Aoki et Golse [1] et Kuo et al. [87, 88, 86]. Nous étendons le taux (quasi) optimal $\frac{1}{t^d}$, où d est la dimension du problème, au cas de domaines réguliers généraux, sans l'hypothèse de symétrie nécessaire dans les travaux précédents. On utilise deux méthodes différentes: un couplage probabiliste dans le Chapitre 2, qui nous permet de traiter des versions généralisées de la condition de bord, et un théorème de Harris sous-géométrique déterministe dans le Chapitre 4, qui permet notamment de traiter le cas où la température varie au bord. Au Chapitre 3, nous présentons des simulations numériques qui viennent à l'appui de nos résultats, selon lesquels le taux polynomial observé quand le domaine spatial est symétrique doit aussi s'appliquer à des domaines non symétriques réguliers.

Dans la seconde partie de cette thèse, on s'intéresse à la convergence sous-géométrique vers la distribution invariante de processus de Markov. On présente un nouvel ensemble de conditions, proches de celles de Douc, Fort et Guillin [44] et de Hairer [71], à partir desquelles l'on peut déduire la convergence sous-exponentielle d'un processus de Markov fort. Un point particulièrement intéressant est que ces nouvelles conditions sont équivalentes, comme dans le cas de la théorie pour les taux exponentielles de Meyn et Tweedie [98], et contrairement aux conditions pré-existantes.

Dans la dernière partie, on étudie des modèles de cinétique collisionnelle, en particulier l'équation de Boltzmann linéarisée et l'équation de Landau linéarisée dans un domaine borné régulier. On prouve des estimées d'hypocoercivité constructives dans L^2 pour la condition de Maxwell généralisée au bord, qui inclut la cas de la réflexion spéculaire. Ces estimées permettent de conclure à la relaxation exponentielle vers l'équilibre de ces modèles.

Contents

Résumé détaillé

Cette thèse s'intéresse à la dynamique de systèmes physiques impliquant un grand nombre de particules. La théorie cinétique se concentre sur une description *mésoscopique* de tels systèmes, en les décrivant statistiquement au moyen de l'étude du comportement typique d'une particule. En particulier, ce cadre est adapté lorsque l'on s'intéresse à la convergence vers l'équilibre thermodynamique et plus largement à l'asymptotique du système physique.

Considérons un système de sphères dures (molécules de gaz par exemple) évoluant dans un volume. Chaque particule se déplace à vitesse constante jusqu'à en heurter une autre, ce qui modifie alors les vitesses des deux particules en jeu, ou jusqu'à ce qu'elle heurte le bord lorsqu'il existe. Pour décrire un tel système hors équilibre de sphères dures identiques voyageant à vitesse constante entre les collisions et rebondissant de façon élastique entre elles, on utilise la célèbre **équation de Boltzmann**

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \mathcal{Q}_B(t,x,v), \qquad (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \tag{0.0.1}$$

qui décrit l'évolution de la densité de probabilité f de particules en position $x \in \Omega$, avec vitesse $v \in \mathbb{R}^d$, au temps $t \geq 0$. Il y a plusieurs choix possibles pour le domaine spatial Ω :

1. l'espace entier $\mathbb{R}^d$;
2. le tore $\mathbb{T}^d$;
3. un domaine borné inclus dans $\mathbb{R}^d$.

Dans le dernier cas, il est nécessaire de compléter l'équation par des conditions décrivant comment la densité évolue au bord de Ω.

L'opérateur $\mathcal{Q}_B$ qui apparaît à droite dans (0.0.1) est l'opérateur de collision de Boltzmann et modélise l'effet des collisions entre molécules sur la dynamique de la densité. Pour un modèle de sphères dures, cet opérateur s'écrit, pour $(t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, en désignant par $|y|$ la norme euclidienne de $y \in \mathbb{R}^d$, pour deux fonctions f, g suffisamment régulières,

$$\mathcal{Q}_B(f,g)(t,x,v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \omega)(f' g'_* + f'_* g' - f g_* - f_* g) d\omega dv_*, \tag{0.0.2}$$

avec, pour $h \in \{f, g\}$,

$$h = h(t, x, v), \qquad h' = h(t, x, v'), \qquad h_* = h(t, x, v_*), \qquad h'_* = h(t, x, v'_*),$$

où les valeurs v', v'_* sont celles des vitesses post-collisionnelles (collision de la particule de vitesse v avec celle de vitesse v_* au point x) données par

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\omega, \qquad v'_* = \frac{v - v_*}{2} + \frac{|v - v_*|}{2}\omega.$$

La fonction B est le noyau de collision, qui permet d'adapter la forme des collisions au modèle considéré. Plusieurs choix sont possibles pour B.

Le célèbre théorème H de Boltzmann montre qu'une quantité macroscopique du système, appelée entropie est monotone en temps : cette quantité est donnée par

$$\mathcal{H}(t) := \int_{\Omega \times \mathbb{R}^d} f(t, x, v) \ln(f)(t, x, v)\, dvdx, \quad (t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \tag{0.0.3}$$

et le théorème H indique que

$$\frac{d}{dt}\mathcal{H}(t) \leq 0.$$

Ceci implique que l'équation de Boltzmann est **irréversible**, puisque la quantité $\mathcal{H}$ décroît au cours du temps (on ne peut pas regarder la dynamique « à l'envers »). En apparence, on voit émerger là un paradoxe : la dynamique des sphères dures que l'on considère pour obtenir l'équation de Boltzmann est réversible, mais l'équation obtenue à la limite ne l'est pas. En particulier, Zermelo a soulevé la contradiction apparente avec le théorème de récurrence de Poincaré, qui implique que le système reviendra à sa configuration initiale après un temps très long. La réponse à ce paradoxe est que le temps de la récurrence dans le théorème de Poincaré est extraordinairement long, et donc que le système étudié ne sera de toute façon plus décrit par l'équation de Boltzmann sur cette échelle. Cependant, cela pose le cadre d'une question plus générale, centrale en théorie cinétique : bien souvent, l'on s'intéresse à la convergence vers un état d'équilibre. Mais l'étude du temps nécessaire pour cette convergence est aussi de grande importance ! On souhaite donc obtenir des versions **quantitatives** du théorème H, qui permettent d'estimer finement le temps nécessaire pour converger vers l'équilibre. Idéalement, on voudrait aussi obtenir des taux **constructifs**, c'est à dire avoir des formes explicites pour les constantes qui apparaissent dans ces estimations quantitatives. Cette thèse s'articule autour de ces questions, et se concentre plus particulièrement sur l'influence sur la vitesse de convergence des conditions au bord que l'on introduit dans le cas où le domaine Ω est borné. On considèrera particulièrement deux familles de modèles: le transport libre, et les équations de cinétique collisionnelle (dont l'équation de Boltzmann) dans le régime proche de l'équilibre. On s'intéressera à la **condition de Maxwell** au bord, dans sa version générale, i.e. en considérant aussi les deux cas particuliers de la réflexion diffuse pure et de la réflexion spéculaire pure, que l'on introduit à présent.

Conditions de bord. On considère le cas d'un domaine $\Omega \subset \mathbb{R}^d$, $d \geq 2$, borné, régulier, admettant un champ de vecteur $x \to n_x$ qui coïncide, au bord $\partial\Omega$, avec le vecteur normal unitaire sortant. On introduit les notations

$$\Sigma := \partial\Omega \times \mathbb{R}^d, \qquad \Sigma_\pm := \{(x,v) \in \Sigma, \pm(v \cdot n_x) > 0\}, \qquad \Sigma_0 := \{(x,v) \in \Sigma, v \cdot n_x = 0\}.$$

On pose aussi, pour tout $x \in \partial\Omega$, $\Sigma_\pm^x := \{v \in \mathbb{R}^d, (x,v) \in \Sigma_\pm\}$. La condition de Maxwell s'écrit, pour $(t,x,v) \in \mathbb{R}_+ \times \Sigma_-$,

$$f(t,x,v) = \alpha(x) c M(v) \int_{\Sigma_+^x} f(t,x,v')|v' \cdot n_x| dv' + (1-\alpha(x)) f(t,x,\eta_x(v)), \tag{0.0.4}$$

où $\eta_\cdot(\cdot)$ est l'opérateur de réflexion spéculaire, donné pour tout $x \in \partial\Omega$ par

$$\eta_x(v) := v - 2(v \cdot n_x) n_x, \qquad v \in \mathbb{R}^d.$$

Le coefficient $\alpha(x) \in [0,1]$ est appelé **coefficient d'accommodation** au point $x \in \partial\Omega$. Le cas le plus pertinent pour la fonction M est celui de la maxwellienne du bord, donnée par

$$M(v) = \frac{1}{(2\pi T)^{d/2}} e^{-\frac{|v|^2}{2T}}, \qquad v \in \mathbb{R}^d,$$

où $T > 0$ est la température au bord. La constante $c > 0$ est choisie de façon à ce que $\int_{\Sigma_+^x} cM(v) dv = 1$ pour tout $x \in \partial\Omega$ (on prendra toujours M radialement symétrique). Pour simplifier la présentation, on prendra $T \equiv 1$ dans la suite de ce résumé, mais certains chapitres de cette thèse traitent la température en toute généralité, autorisant une dépendance en x de la fonction T.

Le cas $\alpha \equiv 0$ est appelé **réflexion spéculaire pure**, et correspond à une condition purement déterministe. Le cas $\alpha \equiv 1$ est appelé **réflexion diffuse pure**, et peut être vu comme une condition purement probabiliste. Ce choix de modélisation a été introduit par Maxwell lui-même, sur la base de considérations physiques.

Équation de transport libre avec conditions de bord. Le premier modèle auquel cette thèse s'intéresse est l'équation de transport libre avec la condition de Maxwell présentée ci-dessus. Ce modèle émerge comme description d'un système de sphères dures lorsque la densité de particules est très faible : dans ce cas, l'opérateur (0.0.2) décrivant les collisions dans l'équation de Boltzmann s'annule et l'on obtient l'équation plus simple

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = 0, \qquad (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d. \tag{0.0.5}$$

On considèrera ici cette équation avec condition de Maxwell au bord, avec l'hypothèse supplémentaire qu'il existe une constante $\alpha_0 > 0$ telle que $\alpha(x) \geq \alpha_0$ pour tout $x \in \partial\Omega$. Cette hypothèse est naturelle lorsque l'on s'intéresse à la convergence vers l'équilibre : lorsque $\alpha \equiv 0$, la dynamique est entièrement déterministe, il n'y a pas de mélange et donc pas de convergence vers un équilibre. Le cas où α peut s'annuler sur certains points du bord est également problématique. Cette hypothèse permet donc d'éviter ces situations pathologiques.

Partie 1. Taux de convergence à l'équilibre pour l'équation de transport libre

Dans la première partie de cette thèse, on présente deux approches pour l'étude de la convergence vers l'équilibre de (0.0.5), ainsi que des simulations basées sur le système de particules utilisé pour la première approche.

Approche par le couplage probabiliste

Le chapitre 2 présente une stratégie probabiliste, et correspond à un article soumis rédigé avec Nicolas Fournier. L'idée est d'exploiter le fait que la dynamique liée à (0.0.5) avec les conditions au bord de Maxwell peut naturellement être associée à un processus markovien, dans lequel une particule $(X_t, V_t)_{t\geq 0}$ évolue à vitesse constante jusqu'à toucher le bord. Lorsque la collision avec le bord se produit, on peut modéliser de façon probabiliste les deux issues possibles (réflexion spéculaire ou diffuse) et la nouvelle vitesse dans le cas de la réflexion diffuse. La composante spatiale reste identique au moment de cette collision.

Partant de cette description, on peut alors opérer un couplage stochastique, en tirant partie de plusieurs faits. Dans toute la suite, on note $X \sim \mu$ si X a pour distribution la mesure de probabilité μ.

1. Si $(X_0, V_0) \sim \mu_\infty$, la distribution d'équilibre, alors $(X_t, V_t) \sim \mu_\infty$ pour tout $t \geq 0$.

2. On peut mesurer le taux de convergence dans une distance entre mesures, la distance en variation totale, telle que $\|\mu - \nu\|_{TV} = \inf \mathbb{P}(X \neq Y)$, où l'infimum porte sur tous les couples (X, Y) tels que $X \sim \mu$, $Y \sim \nu$. Donc si $(X_t, V_t) \sim \mu_t$, avec μ_t la solution au sens des mesures au temps $t \geq 0$, et si $(\tilde{X}_t, \tilde{V}_t) \sim \mu_\infty$, on a

$$\|\mu_t - \mu_\infty\|_{TV} \leq \mathbb{P}\Big((X_t, V_t) \neq (\tilde{X}_t, \tilde{V}_t)\Big) = \mathbb{P}(\tau > t),$$

où $\tau = \inf\{t \geq 0, (X_{t+s}, V_{t+s})_{s\geq 0} = (\tilde{X}_{t+s}, \tilde{V}_{t+s})_{s\geq 0}\}$.

Le problème devient alors d'estimer des quantités liées à τ. En particulier pour le cas typique de la maxwellienne au bord, on souhaite utiliser l'inégalité de Markov pour prouver que, pour tout $\epsilon > 0$,

$$\mathbb{P}(\tau > t) \leq \frac{\mathbb{E}[\tau^{d-\epsilon}]}{t^{d-\epsilon}},$$

de sorte que le contrôle de l'espérance assure le résultat anticipé par la littérature - bien qu'établi seulement dans des cas restreints avec des hypothèses fortes de symétrie jusqu'ici - d'une convergence en $\frac{1}{t^{d-\epsilon}}$ pour tout $\epsilon > 0$, pour une condition initiale bornée. Le couplage lui-même est très technique, du fait de la nature particulière de l'aléatoire qui ne se manifeste qu'au bord. Nous ne rentrerons donc pas ici dans les détails. Notons seulement que cette stratégie permet aussi de considérer des fonctions M différentes de la maxwellienne de bord, pour lesquelles on obtient des taux différents.

Simulations du comportement asymptotique

Dans le chapitre 3, on utilise la description stochastique du chapitre 2 pour simuler la dynamique par le biais d'un système de particules. Cela permet d'abord d'obtenir des résultats qui semblent confirmer les propriétés qualitatives : dès lors que l'on a un peu de réflexion diffuse ($\alpha(x) \geq \alpha_0$ pour tout $x \in \partial\Omega$), on observe une convergence vers une loi qui semble bien uniforme en espace, comme attendu. On peut d'ailleurs montrer que, dans le cas spéculaire, on n'observe *a priori* pas de convergence. On cherche ensuite à confirmer les résultats établis au chapitre précédent, mais l'on se heurte sur ce point à une difficulté fondamentale : la distance en variation totale est très difficile à estimer numériquement. Pour tenter d'observer, malgré cela, les taux polynômiaux attendus dans le cas où M est la maxwellienne de bord par exemple, on utilise la description suivante de la variation totale entre deux mesures μ et ν sur un espace mesurable $(E, \mathcal{E})$,

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{\phi : E \to [-1,1]} \Big| \int \phi d\mu - \int \phi d\nu \Big|.$$

On tente donc d'estimer la variation totale à partir d'approximations de la différence des intégrales pour les choix $\phi(x,v) = |x|^4 + |v|^2$ et $\phi_2(x,v) = \sqrt{|x|} + \sqrt{|v|}$. Les résultats sont probants d'un point de vue qualitatif : outre le taux polynômial observé, on obtient des réactions qualitatives du taux de convergence conformes à la théorie lorsque l'on fait varier la fonction M. En revanche, l'erreur due à l'utilisation de ces fonctions tests ne permet pas de retrouver les valeurs exactes prédites pour les exposants de cette convergence polynômiale.

Étude par le biais d'un théorème de Harris sous-géométrique

Dans le chapitre 4, on utilise un pendant déterministe d'un outil classique en théorie des probabilités : le théorème de Harris (fortement lié au couplage). La stratégie est, sans rentrer dans les détails, la suivante : on considère un semi-groupe de générateur infinitésimal $\mathcal{L}$, et l'on suppose que les deux hypothèses suivantes sont satisfaites,

i) il existe une hiérarchie de poids $(m_i)_{i=1,\dots,N}$ telle que l'adjoint $\mathcal{L}^*$ de $\mathcal{L}$ satisfait

$$\mathcal{L}^* m_{i+1} \leq -C_i m_i + K, \qquad i \geq 1,$$

pour des constantes $C_i, K > 0$;

ii) une condition de Doeblin est valide : pour tout $R \geq R_0 > 0$, il existe un $T(R) > 0$ et une mesure non triviale ν tels que

$$e^{\mathcal{L}T} f \geq \nu \int_{\{|x| \leq R\}} f dx, \qquad \forall f \geq 0.$$

Il faut de plus une condition d'interpolation sur les poids $(m_i)_{i=1,\dots,N}$ que nous ne détaillerons pas ici. À partir de ce cadre, on peut obtenir un résultat sur le taux de convergence à l'équilibre du semi-groupe de Markov $(e^{\mathcal{L}t})_{t\geq 0}$.

L'adaptation de ce squelette de base au problème du transport libre n'est pas évidente. D'abord, le rôle particulier du bord dans la dynamique oblige à passer à une version intégrée en temps des inégalités de i) qui permet de contrôler le flux au bord et d'obtenir une condition presque équivalente. Ensuite, la condition de Doeblin est très technique et se prouve en utilisant fortement les spécificités de la dynamique du transport libre. Pour ces deux conditions, une quantité clé est

$$\sigma(x, v) = \inf\{t > 0, x + tv \in \partial\Omega\},$$

le temps mis par une particule partant de $(x, v) \in \Omega \times \mathbb{R}^d$ pour toucher le bord. Cette quantité vérifie

$$v \cdot \nabla_x \sigma(x, v) = -1$$

ce qui permet d'obtenir les inégalités de Lyapunov, et, pour tout R assez grand, il existe $T(R)$ tel que

$$e^{\mathcal{L}T} f \geq \nu \int_{\{\sigma(x,v) \leq R\}} f \, dxdv \qquad \forall f \geq 0,$$

pour une mesure non triviale ν, ce qui donne une condition de Doeblin. On conclut par des manipulations algébriques à partir de ces deux conditions. Ces manipulations sont en grande partie indépendantes du fait que l'on travaille avec la dynamique du transport libre. On retrouve, pour la norme L^1, le résultat de convergence polynômiale à taux $\frac{1}{t^{d-\epsilon}}$ vers l'équilibre, pour tout $\epsilon > 0$. En outre, cette approche déterministe permet facilement de considérer le cas où la température varie au bord. En revanche, les résultats présentés sont limités au cas de la maxwellienne au bord.

Partie 2: Convergence sous-géométrique vers l'équilibre de processus markoviens

Lorsque l'on considère des processus markoviens convergeant exponentiellement vite vers une distribution invariante, on peut identifier deux conditions équivalentes : une condition d'intégrabilité du temps de retour vers un ensemble satisfaisant de bonnes propriétés (ce qui, en un sens, est semblable aux outils utilisés dans le couplage), et une condition dite de Foster-Lyapunov sur le générateur infinitésimal. Dans le cas sous-géométrique, en particulier polynômial, deux conditions similaires ont été identifiées dans la littérature, mais il n'y a pas d'équivalence entre elles. Au chapitre 5, nous nous concentrons sur ce cas sous-géométrique et donnons deux nouvelles conditions qui sont équivalentes. La première est une intégrabilité d'un temps aléatoire qui correspond au temps nécessaire pour que le processus passe, dans un ensemble aux propriétés adaptées C, un temps exponentiel. La seconde est une nouvelle condition de Foster-Lyapunov qui fait intervenir un paramètre de temps. On montre ensuite que ces deux conditions sont impliquées par la condition de Foster-Lyapunov usuelle présente pour ce cas sous-géométrique dans la littérature, et qu'elles impliquent le résultat de convergence en variation totale connu dans ce cadre.

Partie 3: Hypocoercivité d'équations linéaires avec une condition de Maxwell générale

Cette partie est constituée du seul chapitre 6, qui est un travail en cours d'achèvement réalisé en collaboration avec Kléber Carrapatoso, Stéphane Mischler et Isabelle Tristani. On s'intéresse cette fois à une version linéarisée de l'équation de Boltzmann (0.0.1), avec ou sans cut-off angulaire. Le cas de l'équation de Landau linéarisée peut également être traité par la même méthode.

On cherche à appliquer la méthode dite hypocoercive à ces équations complétées par une condition de Maxwell générale, en particulier on autorise ici le cas $\alpha \equiv 0$ de la réflexion spéculaire pure. Pour ces équations linéarisées, on a un opérateur collisionnel, noté $\mathcal{S}$, qui induit naturellement une forme de dissipation dans un espace de Hilbert $\mathcal{H}$, mais l'opérateur de transport est conservatif et ne participe donc pas au mélange. Pour mieux comprendre, on peut diviser la solution f en une composante macroscopique πf et une composante microscopique $\pi^\perp f$, et l'on a alors

$$f = \pi f + \pi^\perp f,$$

avec un contrôle naturel sur $\pi^\perp f$ grâce à l'opérateur collisionnel $\mathcal{S}$, mais pas de contrôle sur la partie macroscopique πf. La stratégie est alors de construire un nouveau produit scalaire sur $\mathcal{H}$ pour obtenir ce contrôle. La partie macroscopique peut elle-même se diviser en trois composantes, chacune correspondant à une loi de conservation de l'opérateur $\mathcal{S}$: conservation de la masse, de la quantité de mouvement et de l'énergie. Pour contrôler ces quantités, on fait appel à une équation elliptique adaptée à chaque cas. Si les composantes associées à la masse et à l'énergie sont simplement liées à des équations de Poisson avec condition de Robin, le contrôle de la partie correspondant à la quantité de mouvement est beaucoup plus délicat, et repose sur l'utilisation d'un système de Lamé avec une condition au bord bien spécifique. On propose de surcroît une preuve pour les estimations elliptiques associées à ces problèmes, en utilisant en particulier l'inégalité de Korn et des variantes de cette dernière. En particulier, ces résultats sont les premiers qui prouvent la convergence exponentielle vers l'équilibre dans une norme hilbertienne, dans le cas de la réflexion spéculaire pure, pour l'équation de Boltzmann linéaire dans un domaine régulier quelconque.

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Chapter 1

Introduction

1.1 Kinetic theory

1.1.1 An overview of kinetic theory

This thesis studies the dynamics of physical systems involving a large number N of particles. Typically, $N \sim 10^{24}$, the order of the Avogadro number. Several levels of descriptions can be adopted for such situations. The *microscopic* description is the study of each trajectory, using Newton's laws, giving rise to an Hamiltonian system in the phase space of positions and velocities of particles $\Omega^N \times (\mathbb{R}^d)^N$, where Ω is the spatial domain where the particles evolve and $d \geq 1$ is the space dimension. This framework has two major drawbacks: first, the very large number of particles in the system makes it very hard to study, both analytically and numerically. Second, physical quantities of interest, also called observables, such as mass, average velocity and temperature can not be accessed at this level of description. A different approach consists in focusing on the dynamics of those observables. This is the *macroscopic description* of the system, where one studies the Euler equations for inviscid fluids and the Navier-Stokes equations for viscid fluids.

Between those two scales, the kinetic theory provides an intermediate level: the *mesoscopic description*. The idea is to describe statistically the system by focusing on the "typical" behavior of a particle. This level of description is fundamental for several reasons. The statistical point of view reduces substantially the number of degrees of freedom compared to the Hamiltonian system. On the other hand, it allows one to take into account physical properties which can not be captured at the macroscopic level. Another, very important, feature of the mesoscopic description is the possible study of systems which are not at the *local thermodynamic equilibrium*. When one uses the macroscopic description of the system, it is always assumed to be at equilibrium: in each infinitesimal volume of the fluid, particle velocities have the equilibrium distribution, hence motivating the definition of observables. The mesoscopic level thus gives a framework to study the *convergence towards equilibrium*, the goal being to understand the long-time behavior of the system. Finally, the kinetic theory plays an important role in Hilbert's sixth problem, the axiomatization of physics, in particular in the derivation of the macroscopic equations from the microscopic models. The suggestion of Hilbert was to use the Boltzmann equations (which lie at the level of the mesoscopic description) as an intermediate state.

A fourth level of description exists, corresponding to the situation where the macroscopic fluid becomes turbulent, and where some averaging is required to obtain quantitative or qualitative results, but we will not discuss it here. The interested reader can find more details on the corresponding models in Bardos [5].

Ultimately we can sum up the situation as follows.

Microscopic description

Hamiltonian system describing the trajectory of all particles

Newton's laws of motion

Mesoscopic description

Evolution equation on the density of particles

Boltzmann equations, Landau equations, Fokker-Planck equations, Vlasov equations...

Macroscopic description

Evolution equation on observables (mass, macroscopic velocity, temperature)

Euler equations, Navier-Stokes equations

Larger scales description

Diffuse models, turbulence models...

This thesis focuses on the mesoscopic description. The state of the system is described by a density $f = f(t, x, v)$ of particles (that we assume here, for simplicity, to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$), with $t \geq 0$ the time, $x \in \Omega$ the position and $v \in \mathbb{R}^d$ the velocity. In this context, $f(t, x, v)dxdv$ is thus the quantity of particles in the volume element $dxdv$ centered in $(x, v) \in \Omega \times \mathbb{R}^d$ at time t. We will sometimes consider the case where Ω is the whole space $\Omega = \mathbb{R}^d$ or the torus $\Omega = \mathbb{T}^d$, but in the core of this thesis, Ω is a bounded domain (open, connected set) with boundary $\partial\Omega$, and the dynamics considered will involve boundary conditions describing the behavior of f at $\partial\Omega$. For a wider view on collisional kinetic theory, we refer to Villani [121] and the more recent review of Mouhot and Villani [104], see also applications and context from the physical point of view in Krall and Trivelpiece [84] and Binney and Tremaine [11].

1.1.2 Informal derivation of the models and of the boundary conditions

We study partial differential equations modeling the evolution of the particles density in the phase space. In this section, we present an informal derivation from the particle system of a central partial differential equation in kinetic theory, the Boltzmann equation, in order to provide the reader some general context. From this derivation we will also obtain the free-transport equation, and detail the physical motivations of the boundary conditions.

We mention here that the rigorous derivation of the Boltzmann equation from the system of particles is an open problem. On this matter a crucial reference is the work of Lanford [90], see also the recent advances by Gallagher, Saint-Raymond and Texier [62] and Bodineau, Gallagher and Saint-Raymond [13]. The informal material that we present here is largely inspired by Cercignani, Illner and Pulvirenti [27].

We consider N hard, elastic spheres (particles, for instance gas molecules), identical, with diameter $\lambda > 0$, in a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. When $\Omega \notin \{\mathbb{R}^d, \mathbb{T}^d\}$, we will assume that $\partial\Omega$ is smooth, with a unit normal outward vector n_x at all $x \in \partial\Omega$. The state of the system is given by a phase point

$$z = (x_1, v_1, \ldots, x_N, v_N,) \in (\Omega \times \mathbb{R}^d)^N,$$

and the phase space is

$$\Gamma = \Big\{ z \in (\Omega \times \mathbb{R}^d)^N ; |x_i - x_j| \geq \lambda \text{ if } i \neq j \Big\},$$

where $|\cdot|$ is the Euclidian norm in $\mathbb{R}^d$, the idea being that since we consider hard spheres, those should not overlap. When two particles collide, i.e. when $x_j = x_i + n\lambda$ for some $i \neq j$ with $n \in \mathbb{S}^{d-1}$ the unit sphere in $\mathbb{R}^d$, a collision occurs. This collision turns the two velocities v_i and v_j into v_i', v_j'. Before the collision we should have $n \cdot (v_i - v_j) > 0$ (ingoing collision configuration) and after, we should have $n \cdot (v_i' - v_j') < 0$ (outgoing collision configuration).

1.1.2.1 A brief study of the collisions between particles

How do we obtain v_i', v_j' from (x_i, v_i) and (x_j, v_j) ? Without loss of generality (wlog) we assume $j = 1$, $i = 2$. We consider elastic collisions, hence collisions should preserve the momentum and the energy, which gives us two equations

$$v_1 + v_2 = v_1' + v_2' \tag{1.1.1}$$

$$|v_1|^2 + |v_2|^2 = |v_1'|^2 + |v_2'|^2. \tag{1.1.2}$$

We parametrize the representation as follows: we introduce $n = \frac{v_1' - v_1}{\|v_1' - v_1\|} = \frac{1}{C}(v_1' - v_1)$, with $C > 0$ constant. By definition of n, we have $v_1 = v_1' - nC$, and, from (1.1.1), $v_2 = v_2' + nC$.

We then plug those equations into (1.1.2) to obtain

$$|v_1'|^2 - 2C(v_1' \cdot n) + C^2 + |v_2'|^2 + 2C(v_2' \cdot n) + C^2 = |v_1'|^2 + |v_2'|^2,$$

where $a \cdot b$ denotes the scalar product of $a, b \in \mathbb{R}^d$, thus $C = n \cdot (v_1' - v_2')$. Hence,

$$v_1 = v_1' - n[(v_1' - v_2') \cdot n], \qquad v_2 = v_2' + n[(v_1' - v_2') \cdot n], \tag{1.1.3}$$

and

$$v_1 - v_2 = v_1' - v_2' - 2n[(v_1' - v_2') \cdot n].$$

Taking the dot product with n on both sides, we have

$$n \cdot (v_1 - v_2) = -n \cdot (v_1' - v_2'),$$

which allows us to revert (1.1.3) to get

$$v_1' = v_1 - n[n \cdot (v_1 - v_2)], \qquad v_2' = v_2 + n[n \cdot (v_1 - v_2)]. \tag{1.1.4}$$

We call J the collision transformation $J : (v_1, n, v_2) \to (v_1', -n, v_2')$ and from (1.1.4) we have that J is an involution, i.e. $J^2 = I_d$. More importantly, it preserves the Lebesgue measure on $\mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{R}^d$. Let us compute the Jacobian to prove this. We introduce the relative velocities

$$V := v_1 - v_2, \qquad V' := v_1' - v_2',$$

and the velocities of the center of mass $\bar{V} := \frac{1}{2}(v_1 + v_2)$, $\bar{V}' := \frac{1}{2}(v_1' + v_2')$. It follows from (1.1.1) that $\bar{V}' = \bar{V}$ and, from (1.1.4), that

$$V' = V - 2n(n \cdot V).$$

Clearly the map $(v_1, v_2) \to (V, \bar{V})$ has unit Jacobian. Moreover, we may decompose V into a normal component V_n directed along n and a tangential component $V_\perp$ in the plane normal to n. Since

$$V' \cdot n = -V \cdot n, \quad V' \cdot m = V \cdot m,$$

for all m with $n \cdot m = 0$, we find

$$V_n' := V' \cdot n = -V_n, \qquad V_\perp' = V_\perp,$$

where $V_\perp'$ is the projection of V' on $n^\perp$. Hence the transformation from $(V_n, V_\perp)$ to $(V_n', V_\perp')$ has Jacobian -1. Moreover, we clearly have $\bar{V}_n' = \bar{V}_n$ and $\bar{V}_\perp' = \bar{V}_\perp$, with the obvious definitions for

$\bar{V}_n, \bar{V}'_n$ and $\bar{V}_\perp, \bar{V}'_\perp$. Since $n \to -n$ also has Jacobian -1, we conclude that the overall Jacobian of J is 1.

1.1.2.2 Dynamics and collisions against the wall

We assume that the spheres evolve without any additional forces in the background. As a result, between collisions, they move on straight lines with unchanged velocities, according to the classical Newton's laws of motion. For the particle 1, writing $x_1(t), v_1(t)$ for the position and velocity at time $t \geq 0$, and assuming that no collision occurs between 0 and t, we have

$$x_1(t) = x_1(0) + tv_1(0), \qquad v_1(t) = v_1(0).$$

When a collision occurs (with the boundary wall $\partial\Omega$ or another hard sphere) at some time τ, the post-collisional velocity $v'_1(\tau)$ is deduced from the quantities involved in the collision

- as detailed above in the case of a collision against another hard sphere;
- according to the boundary condition otherwise,

and $x_1(\cdot)$ is continuous in time. After the collision the particle goes again in straight line with velocity $v'_1(\tau)$ until its next collision. Regarding the collisions with the boundary, we consider for now (and will discuss other possible conditions later) the so-called specular reflection boundary condition: setting for all $(x, v) \in \partial\Omega \times \mathbb{R}^d$,

$$\eta_x(v) := v - 2(v \cdot n_x)n_x; \tag{1.1.5}$$

we have

$$v'_1(\tau) = \eta_{x_1(\tau)}(v_1(\tau)) = v_1(\tau) - 2n_{x_1(\tau)}(n_{x_1(\tau)} \cdot v_1(\tau)), \tag{1.1.6}$$

where $x_1(\tau) \in \partial\Omega$, $n_{x_1(\tau)}$ is the unit outward normal vector of $\partial\Omega$ at $x_1(\tau)$, and where $v_1(\tau) \cdot n_{x_1(\tau)} > 0$ (outgoing velocity). Physically this condition is easy to understand from our previous computations: it corresponds to the case where the hard sphere (x_1, v_1) hits a wall made of hard-spheres at rest with no space between them. Of course this condition preserves the energy $|v_1(\tau)|^2$ of the particle.

The free flow between collisions and the two conditions (1.1.4) and (1.1.6) completely determine the time evolution of any phase point z, both forward and backward, as long as there is no high-order collisions (collisions involving three or more particles at the same time) and no multiple collisions at the boundary (situations where two particles collide together and with $\partial\Omega$ at the same time). In those cases, the flow of the phase space point is not uniquely determined. However, we have the following theorem.

Theorem 1.1.1 ([27, Theorem 4.2.1]). *The following sets are of Lebesgue measure zero in the phase space:*

i. the set of all phase points which are led to a high-order collision or a multiple collision under forward or backward evolution,

ii. the set of all phase points such that there is a cluster point of collision instants under forward or backward evolution,

iii. the set of all phase points such that there is a cluster point of collision instants with the boundary under forward or backward evolution.

We let Γ_0 be the phase space Γ minus those zero Lebesgue measure sets. On this set, the time evolution of any phase point z is globally defined, both backward and forward. We may thus introduce a group $(T^t)_{t\in\mathbb{R}}$ of operators such that, if $z \in \Gamma_0$ is the state of the system at time 0, $T^t z$ is the state of the system at time t. As a group, $(T^t)_{t\geq 0}$ satisfies $T^0 = I_d$ and $T^t \circ T^s = T^{t+s}$ for all $s, t \in \mathbb{R}$.

1.1.2.3 The Liouville equation

As mentioned in §1.1.1, we shall adopt a mesoscopic description of the system. For this we introduce the probability density $P(\cdot,\cdot)$ on $\mathbb{R}_+ \times \Gamma$, and $P(t, z)$ is the probability density of the system at the phase point $z \in \Gamma$, at time $t \geq 0$. We consider an initial probability density $P_0 \in L^1_+(\Gamma)$ for the initial configuration of the system, where $L^1_+(\Gamma)$ is the Lebesgue space of positive functions integrable with respect to the Lebesgue measure on Γ (recall that $\Gamma \setminus \Gamma_0$ has Lebesgue measure 0).

Assuming that $P(t,\cdot) \in L^1_+(\Gamma)$ for all $t \geq 0$, for A a Borel set of Γ, we then have

$$\int_{T^t A} P(t, z)dz = \int_A P_0(z)dz.$$

The Lebesgue measure is invariant under T^t on Γ_0. Informally this follows from three facts:

i. as we proved, the inter-particle collision transformation is an involution preserving this measure;

ii. the Lebesgue measure is invariant under the free motion;

iii. the boundary condition (1.1.6) implies an involution preserving the measure, following a similar proof to the one for the map J already presented.

The flow on Γ_0 can be split into those three behaviors, hence it preserves the Lebesgue measure.

Therefore,

$$\int_A P(t, T^t z)dz = \int_A P_0(z)dz,$$

and since A is arbitrary, we conclude that for all $t \geq 0$, for almost all z, $P(t, T^t z) = P_0(z)$, and thus

$$\frac{d}{dt}\Big(P(t, T^t z)\Big) = 0. \tag{1.1.7}$$

Equation (1.1.7) is the Liouville equation in mild formulation, and is completed by a boundary condition at the boundary $\partial\Gamma$ of Γ: for $i, j \in \{1, \dots, N\}$, $i \neq j$,

$$\begin{aligned} &P(t, x_1, v_1, \dots, x_i, v_i, \dots, x_j, v_j, \dots, x_N, v_N) \\ &\quad = P(t, x_1, v_1, \dots, x_i, v_i - n_{ij}(n_{ij} \cdot V_{ij}), \dots, x_j, v_j + n_{ij}(n_{ij} \cdot V_{ij}), \dots, x_N, v_N), \end{aligned} \tag{1.1.8}$$

if $|x_i - x_j| = \lambda$, where $V_{ij} = v_i - v_j$ and $n_{ij} = \frac{x_i - x_j}{|x_i - x_j|}$, and by the specular reflection boundary condition if $x_k \in \partial\Omega$ for some k. This condition is necessary to have that P is constant along the flow even though the velocity variables encounter a discontinuity when a collision (between two particles or with the boundary wall) occurs.

1.1.2.4 The BBGKY hierarchy

For all $s \in \{1, \dots, N\}$, $z^s \in \Gamma_0^{(s)}$, we introduce the s-th marginal of the system

$$P^{(s)}(t, z^s) = \int_{\Gamma_0^{(N-s)}} P(t, z^s, z^{N-s}) dz^{N-s},$$

where $z^s = (x_1, v_1, \dots, x_s, v_s)$, with the notation $z^{N-s} = (x_{s+1}, v_{s+1}, \dots, x_N, v_N)$, and where $\Gamma_0^{(k)}$ is the phase space for a system with k particles, excluding the zero Lebesgue measure set of configurations leading to pathological behaviors, as defined in Theorem 1.1.1. In the limit where the number N of particles tends to infinity, we want an "average" description of the system. For this we will focus on the evolution of one typical particle, whose probability density is given by the first marginal of the system, $P^{(1)}$.

Consider a permutation $\gamma \in S_N$, where S_N is the permutation group of $\{1, \dots, N\}$. Write, for $z \in \Gamma_0$, $\gamma z := (x_{\gamma(1)}, v_{\gamma(1)}, \dots, x_{\gamma(N)}, v_{\gamma(N)})$. Because all particles are identical, we assume that, for all $\gamma \in S_N$, $P_0(z) = P_0(\gamma z)$. A consequence of the fact that the flow is well-defined on Γ_0 is then that $P(t, \gamma z) = P(t, z)$ for all $z \in \Gamma_0$.

We make the additional hypothesis that $t \to P_0(T^t z)$ is continuous for almost all $z \in \Gamma$. This can be interpreted as the fact that P_0 does not distinguish between precollisional and postcollisional configurations.

One can then derive the Bogolioubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy ([14, 15, 17, 83, 124]) from the Liouville equation. The idea is to integrate the Liouville equation against some of the variables to derive an equation on the s-th marginal. For the sake of conciseness, we will not expand on this process, but the interested reader can refer to [27] or the detailed treatment of Simonella [112]. Ultimately, we find, for $s \leq N-1$, $t \geq 0$, $z \in \Gamma_0^{(s)}$,

$$\begin{aligned}
&\partial_t P^{(s)}(t,z) + \sum_{i=1}^{s} v_i \cdot \partial_{x_i} P^{(s)}(t,z) \qquad (1.1.9)\\
&= (N-s)\lambda^2 \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{\{n\in\mathbb{S}^{d-1}: V_i\cdot n>0\}} |V_i\cdot n| \Big(P^{(s+1)}(t, z_s', x_i + n\lambda, v_*') \\
&\qquad\qquad - P^{(s+1)}(t, z_s, x_i - n\lambda, v_*)\Big) dn dv_*,
\end{aligned}$$

where $z_s' = (x_1, v_1, \ldots, x_i, v_i', \ldots, x_s, v_s)$, $V_i = v_i - v_*$ and

$$v_i' = v_i - n(V_i \cdot n), \qquad v_*' = v_* + n(V_i\cdot n).$$

The physical interpretation of this equation is quite straightforward: the s-th marginal function evolves in time according to the s-particles dynamics, corrected by the interaction with the remaining $N-s$ particles, whose effect is detailed by the right-hand side of (1.1.9). Note that we can distinguish between a gain term (post-collisional velocities) and a loss term (pre-collisional velocities) on the right-hand side. Equation (1.1.9) is completed by the specular reflection boundary condition

$$P^{(s)}(t, x_1, v_1, \ldots, x_i, v_i, \ldots, x_s, v_s) = P^{(s)}(t, x_1, v_1, \ldots, x_i, \eta_{x_i}(v_i), \ldots, x_s, v_s), \qquad (1.1.10)$$

where $(x_i, v_i) \in \partial\Omega \times \mathbb{R}^d$ with $v_i \cdot n_{x_i} < 0$, $i \in \{1, \ldots, s\}$.

1.1.2.5 Informal derivation of the equations from the hierarchy

To derive the Boltzmann equation from the BBGKY hierarchy, we will consider the Boltzmann-Grad limit, which corresponds to the case where $\lambda \simeq \frac{1}{N^{1/2}}$, so that $N\lambda^2 \to l \in (0,\infty)$ when $N \to \infty$.

Before focusing on this case, we point out that in the case where $\lambda \ll \frac{1}{N^{\frac{1}{2}}}$, $N\lambda^2 \to 0$ as $N \to \infty$, and assuming that a limit for the first marginal exists, we obtain at the limit the much simpler equation

$$\partial_t P^{(1)}(t,x,v) + v \cdot \nabla P^{(1)}(t,x,v) = 0, \qquad \text{for almost all } (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d.$$

This equation is nothing but the free-transport equation that we will heavily study in this thesis. In this sense the free-transport equation is the equation describing the evolution of a gas with a very low density, see also the notions of mean free paths and Knudsen number in [26, Chapter 5]. More details on the physical insights are given in [116, Chapters 1 and 2]. Of course the equation has to be completed with a suitable boundary condition at the boundary $\partial\Omega$ of the domain, for instance of the form (1.1.10).

From the BBGKY hierarchy, in the Boltzmann-Grad limit, one derives informally the Boltzmann equation by first considering the limit of (1.1.9) as $N \to \infty$ (and so $\lambda \to 0$) to obtain the *Boltzmann hierarchy*

$$\partial_t P^{(s)}(t,z) + \sum_{i=1}^{s} v_i \cdot \partial_{x_i} P^{(s)}(t,z) = l \sum_{i=1}^{s} \int_{\mathbb{R}^d} \int_{\{n \in \mathbb{S}^{d-1} : V_i \cdot n > 0\}} |V_i \cdot n| \Big(P^{(s+1)}(t, z_s', x_i, v_*') - P^{(s+1)}(t, z_s, x_i, v_*) \Big) dn dv_*, \tag{1.1.11}$$

where again $z_s' = (x_1, v_1, \ldots, x_i, v_i', \ldots, x_s, v_s)$, $V_i = v_i - v_*$ and

$$v_i' = v_i - n(V_i \cdot n), \qquad v_*' = v_* + n(V_i \cdot n).$$

Note that, in comparison with (1.1.9), we have x_i rather than $x_i + n\lambda$ and $x_i - n\lambda$ inside the integral on the right-hand side. The next key hypothesis is the *Boltzmann chaos assumption*, which can be decomposed into two parts.

1. The initial chaos assumption simply states that particles are independent and identically distributed at time $t = 0$, hence, for all $s \geq 1$, $(x_1, v_1, \ldots, x_s, v_s) \in \Gamma_0^{(s)}$,

$$P^{(s)}(0, x_1, v_1, \ldots, x_s, v_s) = \prod_{j=1}^{s} P^{(1)}(0, x_j, v_j).$$

2. The propagation of chaos assumption states that this property remains valid at all time, i.e., for all $t \geq 0$, $s \geq 1$, $(x_1, v_1, \ldots, x_s, v_s) \in \Gamma_0^{(s)}$,

$$P^{(s)}(t, x_1, v_1, \ldots, x_s, v_s) = \prod_{j=1}^{s} P^{(1)}(t, x_j, v_j).$$

The second assumption is very strong: since particles collide with each other, one can argue that the independence between them disappears very quickly. However, one might expect this property to hold almost everywhere. A thorough discussion of this hypothesis can be found in [27, Section 2.3]. Under the chaos assumption, considering the first marginal of the system, we find, for all $t \geq 0$, almost all $(x_1, v_1) \in \Omega \times \mathbb{R}^d$,

$$\begin{aligned} &\partial_t P^{(1)}(t, x_1, v_1) + v_1 \cdot \partial_{x_1} P^{(1)}(t, x_1, v_1) \\ &\quad = l \int_{\mathbb{R}^d} \int_{\{n \in \mathbb{S}^{d-1} : V \cdot n > 0\}} |V \cdot n| \Big(P^{(1)}(t, x_1, v_1') P^{(1)}(t, x_1, v_*') \\ &\qquad - P^{(1)}(t, x_1, v_1) P^{(1)}(t, x_1, v_*) \Big) dn dv_*, \end{aligned} \tag{1.1.12}$$

where $V = v_1 - v_*$, $v_1' = v_1 - n(V \cdot n)$ and $v_*' = v_* + n(V \cdot n)$. This is a first form of Boltzmann equation. Note that the factor $|V \cdot n|$ in the integral is heavily dependent of the choice of collisions for the model considered, here a gas of hard spheres. Letting f play the role of $P^{(1)}$, we will consider in what follows a slightly more general form of the Boltzmann equation, given by

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q_B(f,f)(t,x,v), \quad (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \tag{1.1.13}$$

where $Q_B(\cdot,\cdot)$ is a quadratic operator describing, as before, the effect of the interaction between particles, and is given, for all $(t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, by

$$Q_B(f,f)(t,x,v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|,\xi)\big(f(v')f(v'_*) - f(v)f(v_*)\big) d\omega dv_*, \tag{1.1.14}$$

where we do not write the dependency of f in (t,x) in the integral and

$$\xi = \frac{|\omega \cdot (v - v_*)|}{|v-v_*|}.$$

This is a slightly different parameterization of the binary collisions, in which $\omega \in \mathbb{S}^{d-1}$ and

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\omega, \quad v'_* = \frac{v-v_*}{2} + \frac{|v-v_*|}{2}\omega.$$

The function B generalises the previous factor $|V \cdot n|$ in (1.1.12), and is called the *Boltzmann collision kernel*. In the next paragraph, we discuss several possible values for B.

1.1.2.6 Collision kernel and Landau equation

Based on physical insight, it is natural that B only depends on the relative speed $v - v_*$ and of the cosine of the deviation angle ω, and is non-negative. The interaction may be short-range or long-range.

When the interparticle forces are proportional to r^{-s}, $s > 2$, where r is the distance between the two particles under consideration, we have

$$B(|q|,\xi) = |q|^{\frac{s-5}{s-1}} \beta_s(\xi),$$

$$\beta_s(\xi) \sim_{\xi \to 0} |\xi|^{-\frac{s+1}{s-1}},$$

so that Q_B is defined only as a singular integral operator. *Hard potentials* correspond to $s \geq 5$, *Maxwellian potential* correspond to $s = 5$ and *soft potentials* correspond to $s \in (2,5)$. The *Grad's angular cutoff assumption* consists in removing the singularity of β_s, so that $B \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{S}^{d-1})$, in which case the collision integral behaves (roughly) as a bounded operator on functions of the velocity variable.

In the physically important case of the Coulomb interaction in dimension 3, the force scales like $|v - v_*|^{-2}$, and we can not give meaning to the Boltzmann operator anymore. For this reason, Landau [89] introduced a "diffusive" version of the Boltzmann collision operator, today known as the *Landau collision operator*, defined by

$$\mathcal{Q}_L(g,f) = \nabla_v \cdot \Big(\int_{\mathbb{R}^d} \phi(v-v')[f(v')\nabla_v g(v) - g(v)\nabla_v f(v')]\,dv' \Big),$$

where ϕ is the collision kernel for the Coulombic particle interactions

$$\phi(z) = \Big(I - \frac{z}{|z|} \otimes \frac{z}{|z|} \Big) \cdot |z|^{\gamma+2},$$

which is a symmetric and non-negative matrix such that $\phi_{ij}(z) z_i z_j = 0$. In dimension $d = 3$, when $\gamma = -d$, we recover the Coulomb case. In general, the case $\gamma \in [-d, 0)$ is known as *soft potentials*, $\gamma = 0$ is known as *Maxwell molecules* and $\gamma > 0$ is known as *hard potentials*. *Moderately soft potentials* correspond to the case $\gamma \in (-2, 0)$.

The Landau equation is a limit case of the Boltzmann equation in the sense that solutions to the Boltzmann equation tend to solutions of the Landau equation if all collisions tend to be grazing, in other words if β_s tends to be more singular. For detailed studies of the connection between the two equations, see Arsenev and Buryak [4], Bobylev [12], Degond and Lucquin-Desreux [35] and Desvillettes [37].

1.1.2.7 Boundary conditions

For both the *free-transport equation*, that we rewrite

$$\partial_t f + v \cdot \nabla_x f = 0, \qquad (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \tag{1.1.15}$$

and the Boltzmann equation (1.1.13), as well as for the other models that we will consider, we need to complete the problem by prescribing the value of the density f at time $t = 0$

$$f(0,x,v) = f_0(x,v) \qquad (x,v) \in \Omega \times \mathbb{R}^d,$$

for some function f_0, and its value at the boundary of the domain $\partial\Omega$.

The boundary condition describes the interactions of the molecules with the solid wall $\partial\Omega$. In particular those interactions encode the heat transfer between the gas and the solid boundary. There are essentially three situations:

A. if $\Omega = \mathbb{R}^d$, there is no boundary condition for say, but we require some integrability of f, which plays in some sense a similar role;

B. if $\Omega = \mathbb{T}^d$ the d-dimensional torus, the boundary condition is in fact a periodicity condition;

C. if $\Omega \subset \mathbb{R}^d$ is a domain (open, connected) bounded with some regularity at the boundary, let us introduce the notations

$$\Sigma_\pm = \{(x,v) \in \partial\Omega \times \mathbb{R}^d, \pm(v \cdot n_x) > 0\} \tag{1.1.16}$$
$$\text{and } \forall x \in \partial\Omega, \quad \Sigma_\pm^x = \{v \in \mathbb{R}^d, (x,v) \in \Sigma_\pm\},$$

where we recall that n_x is the unit outward normal vector at $x \in \partial\Omega$. We will consider mainly four boundary conditions:

i. the **bounce-back** boundary condition, see Figure 1.1: for all $t \geq 0$, $(x,v) \in \Sigma_-$,

$$f(t,x,v) = f(t,x,-v); \tag{1.1.17}$$

ii. the **specular reflection** boundary condition, see Figure 1.2, corresponding to (1.1.6): for all $t \geq 0$, $(x,v) \in \Sigma_-$, recalling the definition of $\eta_x(\cdot)$ from (1.1.5),

$$f(t,x,v) = f(t,x,\eta_x(v)). \tag{1.1.18}$$

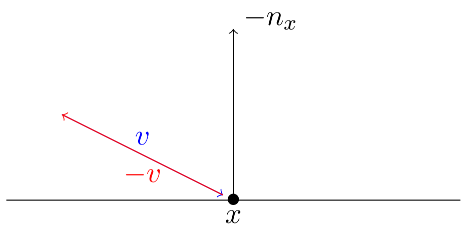

Fig. 1.1 The bounce-back boundary condition.

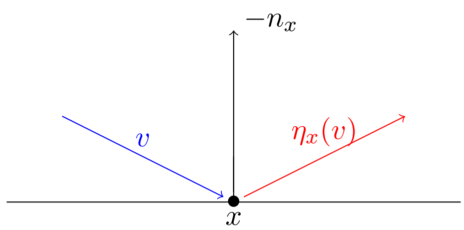

Fig. 1.2 The specular reflection boundary condition.

iii. the **pure diffusive** boundary condition, see Figure 1.3: for all $t \geq 0$, $(x,v) \in \Sigma_-$,

$$f(t,x,v) = \Big(\int_{\{v \cdot n_x > 0\}} f(t,x,v)(v \cdot n_x)dv\Big) M(x,v), \tag{1.1.19}$$

where M is a kernel satisfying some hypotheses detailed below;

iv. the **Maxwell** boundary condition, which is a mix between the specular reflection boundary condition and the pure diffusive boundary condition, see Figure 1.4: for all $t \geq 0$, $(x,v) \in \Sigma_-$,

$$f(t,x,v) = \alpha(x)\Big(\int_{\Sigma_+^x} f(t,x,v)(v \cdot n_x)dv\Big) M(x,v) + (1-\alpha(x)) f(t,x,\eta_x(v)), \tag{1.1.20}$$

where $\alpha : \partial\Omega \to [0,1]$ is a function. The value $\alpha(x)$ is called the *accommodation coefficient* at $x \in \partial\Omega$.

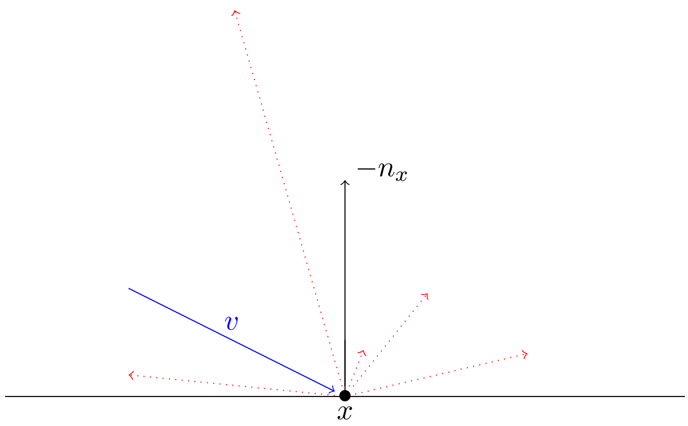

Fig. 1.3 The pure diffusive boundary condition. Dotted red vectors are possible outcoming velocities.

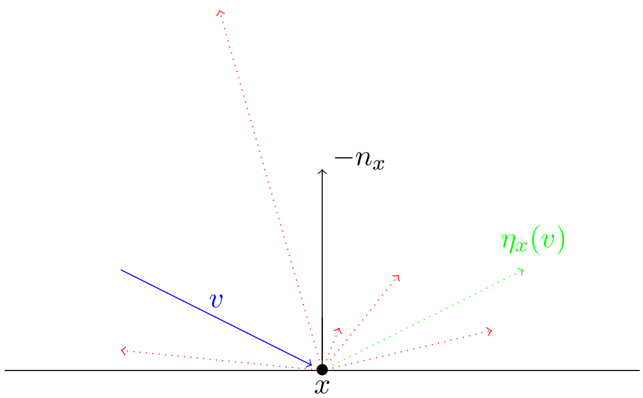

Fig. 1.4 The Maxwell boundary condition. The situation is the same as in the case of the pure diffusive reflection, except that the outcoming velocity vector is $\eta_x(v)$ with probability $1-\alpha(x)$.

The specular reflection, the pure diffusive and the Maxwell boundary conditions were derived by Maxwell [95, Appendix]. Under the assumption that the wall is a perfectly elastic smooth fixed surface without any asperities, a molecule striking the surface will have the normal component (with respect to the boundary) of its velocity reversed, while the other components will not be altered (specular reflection). The point of Maxwell is then that, with this condition, the gas can not exert stress on the surface, except in the direction of the normal, which goes against the experimentations. Physically it makes more sense to allow several outgoing velocity directions. In particular, Maxwell considered two cases.

1. A stratum of the boundary is made of elastic spheres so far apart from one another that no sphere is "protected" from the impact of molecules, and the stratum is deep enough so that no molecule can pass through it without striking one or more of the spheres. Then every molecule coming out from the wall must have struck one or more spheres, making all

directions of its velocity equally probable. This is the physical idea behind the pure diffusive boundary condition.

2. The spheres at the boundary are so near together that there is some protection of some wall spheres by others. If we call the pole of the sphere the point lying furthest from the solid, a greater proportion of molecules will strike any one of the outer layer of spheres near its pole than near its equator. In this case the directions given by the specular reflection are more likely to occur than any others. The Maxwell boundary condition is then the most relevant description of the behavior of the system.

The hypotheses on M should be the following: first, we need a normalization condition. For all $x \in \partial\Omega$,

$$\int_{\{v \cdot n_x < 0\}} |v \cdot n_x| M(x,v) dv = 1. \tag{1.1.21}$$

We also need a positivity constraint

$$\forall (x,v) \in \partial\Omega \times \mathbb{R}^d, \quad M(x,v) \geq 0.$$

One possibility for the value of M is to consider the *wall Maxwellian*, which is radial in velocity, and given by

$$M(x,v) = M_{T(x)}(v) = e^{-\frac{|v|^2}{2T(x)}} \frac{c(x)}{(2\pi T(x))^{\frac{d}{2}}}, \tag{1.1.22}$$

with $c(x)$ a normalizing constant ensuring that (1.1.21) is satisfied, and $T(x)$ the temperature of the wall at the point $x \in \partial\Omega$, but other kernels are possible and can be physically relevant, see for instance the case of a photon gas [111, Section 4.5] and the corresponding mathematical studies in [1]. A thorough study of the interactions with the wall is given in [26, Chapter 3]. When T is independent of x, so that $M_{T(\cdot)}(\cdot)$ only depends on v, we will write $M_T : \mathbb{R}^d \to \mathbb{R}_+$. When $T \equiv 1$, we will write M rather than M_1.

1.1.3 Properties of the Boltzmann equation and the free-transport equation

1.1.3.1 Conservation laws for the Boltzmann operator

We now focus on some physical features of the Boltzmann equation (1.1.13), and in particular of the Boltzmann operator (1.1.14). The conservation of momentum (1.1.1) and energy (1.1.2) at the microscopic level have consequences at the macroscopic level, as we shall see now.

By applying several changes of variable, one can prove informally that, if ϕ is a test function and f has enough regularity for the integrals to make sense,

$$\int_{\mathbb{R}^d} Q_B(f,f)\phi(v)dv = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B(|v-v_*|,\omega)ff_*(\phi'+\phi'_*-\phi-\phi_*)dvdv_*d\omega$$

where $\phi=\phi(v)$, $\phi'=\phi(v')$, $\phi_*=\phi(v_*)$, $\phi'_*=\phi(v'_*)$, $f=f(v)$, $f'=f(v')$ and the usual notations for v', v'_*, see (1.1.14). From this formulation, one can prove that

$$\Big[\int_{\mathbb{R}^d} Q_B(f,f)\phi(v)dv = 0\Big] \Longleftrightarrow \Big[\phi(v)\in \mathrm{Span}\{1, v_1, \ldots, v_d, |v|^2\}\Big].$$

With those invariant functions, one can define the macroscopic quantities at time $t\geq 0$

$$\int_{\Omega\times\mathbb{R}^d} f(t,x,v)\phi(v)dvdx,$$

which corresponds to

i. the mass, with the choice $\phi(v)=1$ for all $v\in\mathbb{R}^d$,

ii. the total momentum, with the choice $\phi(v)=\sum_{i=1}^d v_i$ for all $v\in\mathbb{R}^d$,

iii. the energy, with the choice $\phi(v)=|v|^2$ for all $v\in\mathbb{R}^d$.

We then have, for all $t\geq 0$,

$$\begin{aligned}\frac{d}{dt}\Big(\int_{\Omega\times\mathbb{R}^d} f(t,x,v)\phi(v)dvdx\Big) &= \int_{\Omega\times\mathbb{R}^d}\big(-v\cdot\nabla_x f(t,x,v)\big)\phi(v)dvdx\\ &\quad+\int_\Omega\Big(\int_{\mathbb{R}^d} Q_B(f,f)(t,x,v)\phi(v)dv\Big)dx\\ &=\int_{\partial\Omega\times\mathbb{R}^d} f_{|\partial\Omega}(t,x,v)(v\cdot n_x)\phi(v)dvd\sigma_x+0\end{aligned}$$

for all the previous choices of ϕ, where $d\sigma_x$ is the Lebesgue surface measure of $\partial\Omega$. Informally we can here conclude that all three macroscopic quantities are conserved in the case where $\Omega=\mathbb{R}^d$ or $\Omega=\mathbb{T}^d$, and that both mass and energy are conserved when Ω is a bounded domain with bounce-back or specular reflection boundary condition.

1.1.3.2 Entropy dissipation and the H-Theorem

We focus here on the case of the specular reflection boundary condition, but the discussion applies also, with minor adaptations, to the case of the bounce-back boundary condition, and to the situation where $\Omega=\mathbb{R}^d$ or $\Omega=\mathbb{T}^d$. We assume again that f is sufficiently regular for all the integrals (and boundary integrals) used below to make sense. We define the quantity $D[f]$ given, for all $(t,x)\in\mathbb{R}_+\times\Omega$, by

$$D[f](t,x) := -\int_{\mathbb{R}^d} Q_B(f,f)(t,x,v)\ln(f)(t,x,v)dv.$$

We can again apply several changes of variables to obtain

$$D[f] = -\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B(|v-v_*|,\omega)(f'f'_* - ff_*)\ln\Big(\frac{ff_*}{f'f'_*}\Big)dv_*dvd\omega,$$

with the usual notations. Since $(z-y)\ln(\frac{y}{z}) \leq 0$ in $\mathbb{R}^*_+\times\mathbb{R}^*_+$, we get the *Boltzmann inequality*

$$D[f] \geq 0. \tag{1.1.23}$$

We introduce the entropy of the solution f, defined, for all $t\geq 0$, by

$$\mathcal{H}(f)(t) := \int_{\Omega\times\mathbb{R}^d} f(t,x,v)\ln(f)(t,x,v)dvdx.$$

We now compute the evolution of $\mathcal{H}$. We have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t) &= \frac{d}{dt}\Big(\int_{\Omega\times\mathbb{R}^d} f\ln(f)dvdx\Big)\\ &= -\int_{\Omega\times\mathbb{R}^d}(v\cdot\nabla_x f)\ln(f)dvdx + \int_{\Omega\times\mathbb{R}^d} Q_B(f,f)\ln(f)dvdx\\ &\quad -\int_{\Omega\times\mathbb{R}^d}(v\cdot\nabla_x f)dvdx + \int_{\Omega\times\mathbb{R}^d} Q_B(f,f)dvdx.\end{aligned}$$

Note that the last term on the right-hand side is worth 0 since $\phi\equiv 1$ is a collision invariant, see §1.1.3.1. Moreover, a simple application of the Green's Theorem gives

$$\int_{\Omega\times\mathbb{R}^d}(v\cdot\nabla_x f)\ln(f)dvdx = -\int_{\Omega\times\mathbb{R}^d}(v\cdot\nabla_x f)dvdx + \int_{\partial\Omega\times\mathbb{R}^d} f_{|\partial\Omega}\ln(f_{|\partial\Omega})(v\cdot n_x)dvd\sigma_x.$$

Using the specular reflection boundary condition and the notation $\Sigma_\pm$ from (1.1.16), we have

$$\begin{aligned}&\int_{\partial\Omega\times\mathbb{R}^d} f_{|\partial\Omega}\ln(f_{|\partial\Omega})(v\cdot n_x)dvd\sigma_x\\ &\qquad = \int_{\partial\Omega\times\mathbb{R}^d}|v\cdot n_x|\Big(f_{|\Sigma_+}\ln(f_{|\Sigma_+})(v)\mathbf{1}_{\Sigma_+}(v) - f_{|\Sigma_+}\ln(f_{|\Sigma_+})(\eta_x(v))\mathbf{1}_{\Sigma_-}(v)\Big)dvd\sigma_x.\end{aligned}$$

Applying the change of variable $v\to\eta_x(v)$ of Jacobian 1 in the second term on the right-hand side, we obtain

$$\int_{\partial\Omega\times\mathbb{R}^d} f_{|\partial\Omega}\ln(f_{|\partial\Omega})(v\cdot n_x)dvd\sigma_x = 0.$$

Coming back to the evolution of $\mathcal{H}(f)$, we conclude that

$$\frac{d}{dt}\mathcal{H}(f)(t) = \int_{\Omega\times\mathbb{R}^d} Q_B(f,f)\ln(f)dvdx = -\int_\Omega D[f](t,x)dx \leq 0.$$

This result is known as the **H-Theorem** of Boltzmann, see [16]. Moreover, if we consider a distribution f such that, for some $(t,x) \in \mathbb{R}_+ \times \Omega$,

$$D[f](t,x) = 0,$$

we have that $\ln(f) \in \mathrm{Span}\{1, v_1, \dots, v_d, |v|^2\}$ according to §1.1.3.1. Hence, f is of the form

$$\forall (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \qquad f(t,x,v) = A(t,x) e^{-\frac{|v-\xi|^2}{2T(x)}} =: \mu_{(t,x)}(v),$$

for some constants (in v) $A(t,x), T(x) \geq 0$, $\xi \in \mathbb{R}^d$. The constant $T(x)$ is the temperature of the gas at $x \in \Omega$, and can be defined as a macroscopic quantity from f. The constant A also has an explicit form related to macroscopic quantities. In its full version, the theorem actually states that there is an equivalence, see [27, 121]:

$$\Big[D[f] = 0\Big] \Longleftrightarrow \Big[f(t,x,v) = \mu_{(t,x)}(v)\Big].$$

The function $\mu_{(t,x)}$ is known in the literature as the *Maxwellian* (notice the analogy with the wall Maxwellian (1.1.22) when μ is independent of t), and is said to be a local thermodynamic equilibrium of the system at the point (t,x). To find global equilibrium, we want to find μ such that

$$\forall x \in \Omega, v \in \mathbb{R}^d, v \cdot \nabla_x \mu_{(t,x)}(v) = 0,$$

which leads to Maxwellian distributions depending only on v. A natural expectation is that the density function will converge, as time goes to infinity, towards some global Maxwellian. A key problem is to establish a *quantitative* version of the H-theorem, i.e., to derive the rate at which this convergence occurs. We will come back to this point, as this is the main focus of Chapter 6.

1.1.3.3 Linearized Boltzmann equation

We present an heuristic of the derivation of the linearized Boltzmann equation from the standard one. We also explain why this equation, which will be studied in this thesis, is simpler yet still relevant for the study of the full Boltzmann equation. Let us suppose here that $\Omega = \mathbb{R}^d$. Starting from the Boltzmann equation (1.1.13), we may focus on the fluctuations around the global equilibrium, homogeneous in space, and given by

$$\mu(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}.$$

In the *perturbative regime*, we study $h = f - \mu$, where f is a solution to (1.1.13). Recall the definition of $Q_B(\cdot,\cdot)$ from (1.1.14). We introduce the bilinear operator Q given by

$$Q(f,g)(v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|,\omega)(f' g'_* + f'_* g' - f g_* - f_* g) dv_* d\omega, \qquad v \in \mathbb{R}^d,$$

so that $Q(f,f) = Q_B(f,f)$. Assuming that f is smooth for the sake of the argument, we have, using that μ is an equilibrium,

$$(\partial_t + v \cdot \nabla_x)h = (\partial_t + v \cdot \nabla_x)f = Q(f,f) = Q(h,h) + Q(h,\mu) + Q(\mu,h) + Q(\mu,\mu).$$

We note that $Q(\mu,\mu) = Q_B(\mu,\mu) = 0$ using the conservation of energy (1.1.2). Moreover, the symmetry of Q leads to the following equation on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$:

$$(\partial_t + v \cdot \nabla_x)h = Q(h,h) + 2Q(\mu,h). \tag{1.1.24}$$

Since $Q(\cdot,\cdot)$ is a bilinear operator, the term $Q(h,h)$ appearing in (1.1.24) is quadratic. A starting point for the study of (1.1.24) is to consider only the linear part of the equation:

$$(\partial_t + v \cdot \nabla_x)h(t,x,v) = 2Q(\mu,h)(t,x,v), \qquad (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d. \tag{1.1.25}$$

This equation is the *linearized Boltzmann equation*, and is of interest in two situations:

A. When the physical model dictates that the term $Q(h,h)$ above should be small with respect to the other terms in the equation, for example in the case where a gas is a mixture of two components, of species P_1 and P_2, assuming that the specie P_1 has a very small density with respect to the one of P_2. If we also suppose that the particles P_2 are at a global equilibrium (Maxwellian distribution), the relevant term for the evolution of the density of particles P_1 is the one taking into account the interactions between the particles P_1 and the particles P_2 which are in thermal equilibrium. Equation (1.1.25) models this problem. Other physical contexts with this structure are the transport in ionized gases and the radiative transfer through a stellar or planetary atmosphere in (local) thermal equilibrium, see also [26] on those physical insights, as well as the recent survey [6] for medical applications.

B. The linearized Boltzmann equation still retains some key aspects of the full non-linear Boltzmann equation. For some features on which the non-linear nature of the collision operator is expected to have little impact, the study of the linearized equation can be of great interest. Moreover a whole range of techniques has been developed in recent years to obtain conclusions on the full non-linear Boltzmann equation in the perturbative regime from the linearized equation, see Guo [68], Briant and Guo [19], as well as the review paper of Duan [46] and the references within.

Let us point out that a similar procedure can be applied to the Landau equation, leading to the *linearized Landau equation*.

1.1.3.4 The case of the free-transport with Maxwell or pure diffusive boundary condition

Going back to the free-transport equation (1.1.15), one can wonder whether a result playing the role of the H-theorem can be derived. The situation is however quite different, since there is no collision operator to “mix” the particles. If a Lyapunov functional exists, playing the role of the entropy in the previous case, it is due solely to the boundary condition. For this reason, purely “deterministic” boundary conditions, such as the bounce-back (1.1.17) and the specular reflection (1.1.18) boundary conditions, do not imply any form of convergence towards an equilibrium. In fact, in those cases, as well as in the case where $\Omega = \mathbb{T}^d$ or $\Omega = \mathbb{R}^d$, one can derive exact formulas for the solution by using the characteristics of the free-transport (and of the possibly associated boundary conditions).

A very elementary example is given by the case $\Omega = \mathbb{R}^d$, where, if $f_0 \in C^1(\Omega \times \mathbb{R}^d)$, a strong solution $f \in C^1(\mathbb{R}_+ \times \Omega \times \mathbb{R}^d)$ is given, for all $(t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d$, by

$$f(t, x, v) = f_0(x - tv, v).$$

Similar formulas can be given in the torus and in a bounded domain with bounce-back boundary condition. Even in the case of the specular reflection, an explicit formula can be derived under suitable hypothesis, see Briant [18, Appendix A.3].

On the other hand, when one focuses on the pure diffusive (1.1.19) or on the Maxwell (1.1.20) boundary condition, a mixing effect is induced by the wall, and one can expect a result similar to the H-theorem in the case of the Boltzmann equation.

Indeed, consider the case where Ω is a bounded domain with pure diffusive boundary condition, and assume for simplicity that we are in the case of a wall Maxwellian whose temperature is constant, i.e. we focus on the problem (recall the notation Σ_+^x from (1.1.16))

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t, x, v) = 0, & (t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d, \\ f_{|\Sigma_-}(t, x, v) = M(v) \Big(\int_{\Sigma_+^x} |v' \cdot n_x| f_{|\Sigma_+}(t, x, v') dv' \Big), & (t, x, v) \in \mathbb{R}_+ \times \Sigma_- \\ f(0, x, v) = f_0(x, v), & (x, v) \in \Omega \times \mathbb{R}^d, \end{cases} \tag{1.1.26}$$

with M given by (1.1.22) with $T \equiv 1$. Assume that f is non-negative and sufficiently regular, so that all the integrals considered below are well-defined.

We will first need an inequality which will play a role similar to the Boltzmann inequality for the Boltzmann equation, established by Darrozès and Guiraud [33] and generalized by Cercignani [24, 26].

Theorem 1.1.2. *Let $C(\cdot)$ be a strictly convex function on $\mathbb{R}_+$. Then, for all $t \geq 0$, $x \in \partial\Omega$,*

$$\int_{\mathbb{R}^d} M(v)(v \cdot n_x) C\Big(\frac{f(t, x, v)}{M(v)}\Big) dv \geq 0,$$

with equality if, for all $t \geq 0$, $x \in \partial\Omega$, $v \in \mathbb{R}^d$, $f(t,x,v) = \kappa M(v)$ for some constant $\kappa > 0$.

Proof. Recall that, for all $x \in \partial\Omega$,

$$\int_{\Sigma_\pm^x} M(v)|v \cdot n_x| dv = 1. \tag{1.1.27}$$

Hence, by Jensen's inequality,

$$C\Big(\int_{\Sigma_\pm^x} M(v)|v \cdot n_x|\Big(\frac{f(t,x,v)}{M(v)}\Big)dv\Big) \leq \int_{\Sigma_\pm^x} M(v)|v \cdot n_x| C\Big(\frac{f(t,x,v)}{M(v)}\Big)dv, \tag{1.1.28}$$

with equality if $\frac{f(t,x,v)}{M(v)} = \kappa$ constant. Using the pure diffusive boundary condition (1.1.19), and omitting the dependency in t and x of f for the sake of simplicity, we find

$$\begin{aligned}
\int_{\Sigma_-^x} M(v)|v \cdot n_x| C\Big(\frac{f(v)}{M(v)}\Big)dv &= \int_{\Sigma_-^x} M(v)|v \cdot n_x| C\Big(\int_{\Sigma_+^x} M(v')|v' \cdot n_x|\frac{f(v')}{M(v')}dv'\Big)dv \\
&\leq \int_{\Sigma_-^x} M(v)|v \cdot n_x| \Big(\int_{\Sigma_+^x} M(v')|v' \cdot n_x| C\Big(\frac{f(v')}{M(v')}\Big)dv'\Big)dv \\
&= \int_{\Sigma_+^x} M(v')|v' \cdot n_x| C\Big(\frac{f(v')}{M(v')}\Big)dv',
\end{aligned}$$

where we used (1.1.27) twice. Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^d} M(v)(v \cdot n_x) C\Big(\frac{f(v)}{M(v)}\Big)dv &= \int_{\Sigma_+^x} M(v)|v \cdot n_x| C\Big(\frac{f(v)}{M(v)}\Big)dv - \int_{\Sigma_-^x} M(v)|v \cdot n_x| C\Big(\frac{f(v)}{M(v)}\Big)dv \\
&\geq 0.
\end{aligned}$$

The equality statement is straightforward in view of (1.1.28). $\square$

With this at hand, we introduce the relative entropy function defined for all $t \geq 0$ by

$$W(t) := \int_{\Omega \times \mathbb{R}^d} f(t,x,v) \ln\Big(\frac{f(t,x,v)}{M(v)}\Big)dvdx.$$

Proposition 1.1.1. *We have $\frac{d}{dt}W(t) \leq 0$, and W is constant if $f(t,x,v) = \kappa M(v)$ on the set $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$ for some constant $\kappa > 0$.*

Proof. We have, for all $t \geq 0$,

$$\frac{d}{dt}W(t) = -\int_{\Omega \times \mathbb{R}^d} (v \cdot \nabla_x f) \ln\Big(\frac{f}{M}\Big)dvdx - \int_{\Omega \times \mathbb{R}^d} M(v \cdot \nabla_x f)dvdx.$$

Using Green's Theorem, we obtain

$$\begin{aligned}\frac{d}{dt}W(t) &= +\int_{\Omega\times\mathbb{R}^d} M(v\cdot\nabla_x f)dvdx - \int_{\partial\Omega\times\mathbb{R}^d}(v\cdot n_x)f\ln\Big(\frac{f}{M}\Big)dvd\sigma_x \\ &\qquad\qquad\qquad\qquad\qquad\qquad - \int_{\Omega\times\mathbb{R}^d} M(v\cdot\nabla_x f)dvdx \\ &= -\int_{\partial\Omega}\Big(\int_{\mathbb{R}^d} M(v)(v\cdot n_x)C\Big(\frac{f}{M}\Big)dv\Big)d\sigma_x \\ &\le 0,\end{aligned}$$

with C defined on $\mathbb{R}_+^*$ by $C(x) = x\ln(x)$ strictly convex. From Theorem 1.1.2, the last inequality is an equality in the case where $f(t,x,v) = \kappa M(v)$. □

Under some appropriate conditions on α, this heuristic can be extended to the Maxwell boundary condition (1.1.20). Those conditions should prevent pathological cases, for instance the situation where a particle gets "trapped" between two points where $\alpha = 0$.

1.2 Mathematical problems and methods

The main focus of this thesis is **the understanding of boundary effects in the long-term behavior of several kinetic equations**. An underlying subject is the study of the connections between the deterministic techniques and the probabilistic methods which are used to tackle those problems. In particular, the manuscript is organized around three topics and the corresponding contributions.

1. **The quantitative study of the decay towards equilibrium of the free-transport equation enclosed in a domain, with boundary conditions**. As shown informally above, one expects a decay towards the equilibrium when the boundary condition involves some mixing. This decay is actually difficult to investigate quantitatively, as most of the usual techniques do not apply. Previous approaches to the problem were restricted to the case of spherically symmetric domains and used heavily this property. We will see two methods to tackle this problem, one based on a deterministic subgeometric Harris' theorem, and a probabilistic one called the coupling method. A connected question is whether one can obtain explicit constants associated to the rate of convergence.

2. **The derivation of hypocoercivity estimates for linear collisional kinetic equations enclosed in a domain**. We focus on some collisional kinetic equations, for which a whole spectrum of methods, based on the notion of hypocoercivity, have been developed over the last two decades. The impact of the boundary operator for more involved boundary conditions, i.e. when Ω is bounded with specular reflection, pure diffusive or Maxwell boundary condition, has gained interest in the last ten years and we will see how new

estimates can be obtained for a whole range of linear equations, whose collision operators share key properties, including some linearized Boltzmann and Landau equations. For those studies we would also like to derive constructive constants.

3. **The subgeometric convergence towards equilibrium of Markov processes.** The deterministic version of the Harris' theorem mentioned above is an adaptation from Harris' theorem (all references are given below). On the other hand, the Meyn-Tweedie theory developed in the 1990's provides a systematic framework for the study of the convergence rate of Markov processes in the exponential case. In particular, they give an equivalence result which might be interpreted as a link between the coupling method and Harris' theory, based on a Foster-Lyapunov inequality. While the Meyn-Tweedie theory was adapted in the last fifteen years to the subgeometric case, such an equivalence result is specific to exponential rates. We present intermediate conditions of slightly different nature which allow one to recover some equivalence result, in the subgeometric framework.

1.2.1 Convergence rate towards equilibrium of the free-transport equation enclosed in a domain

1.2.1.1 Hypotheses and main problems

We focus here on the free-transport equation (1.1.15), in the case where $\Omega \subset \mathbb{R}^d$, $d \geq 2$ is a bounded domain with the Maxwell (1.1.20) or pure diffusive (1.1.19) boundary condition. For the kernel M of the diffusive component, we make the following hypothesis.

Hypothesis 1.2.1. The function $M : \partial\Omega \times \mathbb{R}^d \to \mathbb{R}_+$ is radially symmetric in its second variable, and for all $x \in \partial\Omega$, $\int_{\mathbb{R}^d} M(x,v)|v|dv < \infty$.

This radial symmetry is critical for the results that we will derive. Of course the wall Maxwellian (1.1.22) satisfies this hypothesis. Finally, we assume that there exists $\underline{\alpha} > 0$ such that $\alpha(x) \geq \underline{\alpha}$ for all $x \in \partial\Omega$.

Let us introduce some notations. For a measure space $(S, \mathcal{S}, \mu)$ we define the corresponding Lebesgue space of exponent $p \in [1, \infty)$ by

$$L^p(S) = \{f : S \to \mathbb{R}, f \text{ measurable }, \|f\|_p < \infty\},$$

where $\|f\|_p = \Big(\int_S |f(x)|^p d\mu(x)\Big)^{\frac{1}{p}}$. We set $G := \Omega \times \mathbb{R}^d$, that we endow with its Borel σ-algebra $\mathcal{G}$ and we consider the Lebesgue measure on $(G, \mathcal{G})$.

As mentioned above, at least under assumptions of regularity on f, we have an entropy decay which provides the intuition of a convergence towards equilibrium. Our goal here is to study rigorously this decay in a well-chosen norm. Several obstacles must be overcome.

(P1) For the boundary condition to make sense, we first need to give meaning to the **traces** of f, in particular when we work in the Lebesgue space $L^1(G)$, in which case the restrictions of f on $\partial\Omega$ are not a priori well-defined.

(P2) We need to rigorously prove that an equilibrium actually exists, and provide a **qualitative result** of convergence towards it.

(P3) The model has no spectral gap in the L^2 norm, and as such many heavily used tools to investigate **quantitatively the convergence towards equilibrium** of a kinetic equation do not apply. We need different techniques. Ideally we would like to obtain a rate of convergence with constructive (explicit) constants.

Arkeryd and Cercignani [2, Sections 2 and 3] investigated (P1) in depth, see also Cannone and Cercignani [23], based on the earlier work of Ukai [120]. The problem is quite subtle, and we will not expand on it. Overall, Arkeryd and Cercignani proved that, if the initial data f_0 belongs to $L^1(G)$, the traces are well-defined.

1.2.1.2 Some results about (P2)

Regarding (P2), consider first the situation where $M(\cdot,\cdot)$ is independent of x. In the wall Maxwellian case, see (1.1.22), this corresponds to the case where the temperature $T(\cdot)$ at the boundary is constant. We assume that the initial data f_0 has global mass 1. Letting $f(t,\cdot,\cdot)$ be the solution at time $t \geq 0$, $f(t,\cdot,\cdot)$ also has mass one. Indeed

$$\begin{aligned}\frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} f(t,x,v)dvdx &= -\int_{\Omega\times\mathbb{R}^d} v\cdot\nabla_x f(t,x,v)dvdx\\ &= -\int_{\partial\Omega\times\mathbb{R}^d} |v\cdot n_x|\Big(f_{|\Sigma_+}(v)\mathbf{1}_{\Sigma_+}(v) - (1-\alpha(x))f_{|\Sigma_+}(\eta_x(v))\mathbf{1}_{\Sigma_-}(v)\\ &\qquad\qquad - \alpha(x)M(v)\widetilde{f_{|\Sigma_+}}(x)\mathbf{1}_{\Sigma_-}(v)\Big)dvd\sigma_x,\end{aligned}$$

and using twice the change of variables $v \to \eta_x(v)$ from Σ_- to Σ_+, along with the facts that $|v\cdot n_x| = |\eta_x(v)\cdot n_x|$ for all $(x,v) \in \partial\Omega\times\mathbb{R}^d$ and that M is radial, we have

$$\begin{aligned}\frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} f(t,x,v)dvdx &= \int_{\partial\Omega}\alpha(x)\int_{\Sigma_+^x}|v\cdot n_x|\Big(f_{|\Sigma_+}(v) - M(v)\widetilde{f_{|\Sigma_+}}(x)\Big)dvd\sigma_x\\ &= \int_{\partial\Omega}\alpha(x)\Big(\widetilde{f_{|\Sigma_+}}(x) - \widetilde{f_{|\Sigma_+}}(x)\int_{\Sigma_+^x}M(v)|v\cdot n_x|dv\Big)d\sigma_x = 0,\end{aligned}$$

since $M(v)|v\cdot n_x|$ is a probability distribution on Σ_+^x.

Since the kernel M only depends on v, it is an obvious steady state of (1.1.15), satisfying the boundary condition as a consequence of its radial symmetry (for the specular component) and of the condition itself (1.1.20). Because of the mass conservation, our equilibrium candidate is thus:

$$f_\infty(x,v) = \frac{\beta M(v)}{|\Omega|}, \tag{1.2.1}$$

where $|\Omega|$ denotes the volume of Ω and $\beta = (\int_{\mathbb{R}^d} M(v)dv)^{-1}$.

In the case where M depends on x, the situation is more complicated, yet one can still obtain an explicit formula for the equilibrium f_∞ in the form of an infinite series. This derivation was done by Sone [115, Chapter 2, Section 2.5], see also [113, 114], in the most important case where $M(\cdot,\cdot)$ is a wall Maxwellian. Here, we extend this result to any kernel satisfying Hypothesis 1.2.1. For clarity we rewrite the Maxwell boundary condition in a slightly different form

$$f_{|\Sigma_-}(t,x,v) = (1-\alpha(x))f_{|\Sigma_+}(t,x,\eta_x(v)) + \alpha(x)c(x)\underline{M}(x,v)\widetilde{f_{|\Sigma_+}}(t,x), \qquad (t,x,v) \in \mathbb{R}_+ \times \Sigma_-,$$

where, for all $x \in \partial\Omega$, $\underline{M}(x,\cdot)$ is a probability distribution on $\mathbb{R}^d$ and the normalization function c appears explicitely in the condition: it is given by

$$c(x) = \Big(\int_{\{v\cdot n_x>0\}} \underline{M}(x,v)|v\cdot n_x|dv\Big)^{-1}.$$

Recall that the Jacobian of the hyperspherical change of variables $v \to (r,\theta_1,\dots\theta_{d-1})$ from $\mathbb{R}^d$ to $[0,\infty)\times[-\pi,\pi)\times[0,\pi)^{d-2}$ is given by $r^{d-1}\prod_{j=1}^{d-2}\sin(\theta_j)^{d-1-j}$. Then, setting

$$\tilde{c}(x) := \int_0^\infty r^d \underline{M}(x,v)dv,$$

using that $\underline{M}$ is radial in v, we have that $c(x) = (\kappa\tilde{c}(x))^{-1}$, where κ is a constant independent of x, given by

$$\kappa := \int_{\mathcal{A}} |u(\theta_1,\dots,\theta_d)\cdot n_x| \prod_{j=1}^{d-2}\sin(\theta_j)^{d-1-j}d\theta_1\dots d\theta_d,$$

where $\mathcal{A} := (-\frac{\pi}{2},\frac{\pi}{2})\times[0,\pi)^{d-2}$ and $u(\theta_1,\dots,\theta_d)$ is the unit vector directed according to those angles.

On $\bar{\Omega}\times\mathbb{R}^d$, using the notation $\Sigma_\pm$ from (1.1.16) and setting

$$\Sigma_0 := \{(x,v)\in\partial\Omega\times\mathbb{R}^d, v\cdot n_x = 0\},$$

we define the map τ by:

$$\tau(x,v) = \begin{cases} \inf\{t>0, x+tv\in\partial\Omega\}, & (x,v)\in\Sigma_-\cup G, \\ 0, & (x,v)\in\Sigma_+\cup\Sigma_0. \end{cases} \tag{1.2.2}$$

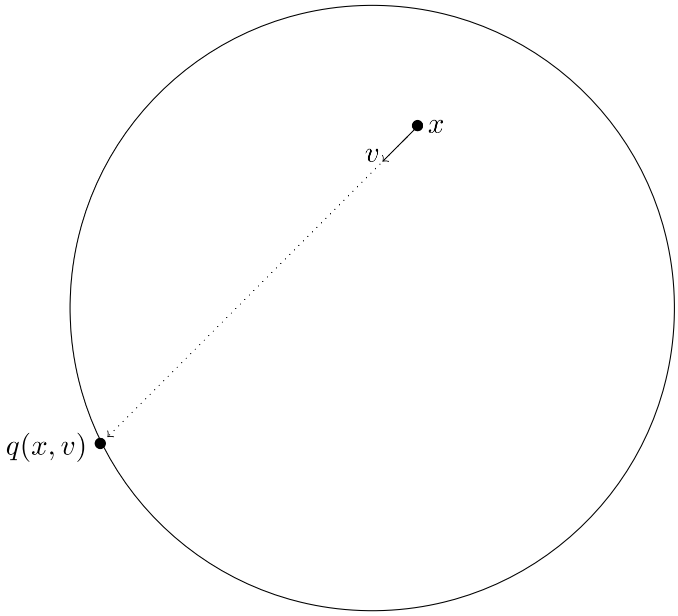

Fig. 1.5 An example for the function q

This function gives, for a particle in position x with velocity v at time $t = 0$, its first instant of collision with the boundary. Set, for all $(x, v) \in \bar{\Omega} \times \mathbb{R}^d$,

$$q(x, v) = x + \tau(x, v)v, \tag{1.2.3}$$

which is the point of $\partial\Omega$ where this collision occurs, see also Figure 1.5.

Recall the definition of η from (1.1.5). For $(x, v) \in \bar{\Omega} \times \mathbb{R}^d$, we introduce the sequence of points and velocities from the specular backward trajectory, that is

$$\begin{aligned} v^1 &= v, & x^1 &= q(x, -v^1), \\ v^{n+1} &= \eta_{x^n}(v^n), & x^{n+1} &= q(x^n, -v^{n+1}), \qquad n \geq 1. \end{aligned}$$

By following the characteristics of the free-transport equation, we have

$$f(t, x, v) = f(t - \tau(x, -v), q(x, -v), v)\mathbf{1}_{\{t > \tau(x,-v)\}} + f_0(x - tv, v)\mathbf{1}_{\{t \leq \tau(x,-v)\}},$$

and by using the boundary condition, this leads to

$$\begin{aligned} f(t, x, v) = \mathbf{1}_{\{t>\tau(x,-v)\}} \Big\{ &(1 - \alpha(x^1)) f(t - \tau(x, -v), x^1, v^2) \\ &+ \alpha(x^1) c(x^1) \underline{M}(x^1, v^1) \Big(\int_{\Sigma_+^{x^1}} f(t - \tau(x, -v), x^1, v') |v' \cdot n_{x^1}| dv' \Big) \Big\} \\ &+ f_0(x - tv, v) \mathbf{1}_{\{t \leq \tau(x,-v)\}}. \end{aligned}$$

Assume for now that there exists an equilibrium f_∞ on $\bar\Omega \times \mathbb{R}^d$, and suppose that f_∞ has enough regularity so that its trace is well-defined. Since f_∞ satisfies $(\partial_t + v\cdot\nabla_x)f_\infty = 0$ and f_∞ does not depend on t, taking the limit as $t\to\infty$, we have from the previous equation,

$$f_\infty(x,v) = \alpha(x^1)c(x^1)\underline{M}(x^1,v^1)\widetilde{(f_\infty)_{|\Sigma_+}}(x^1) + (1-\alpha(x^1))(f_\infty)_{|\Sigma_+}(x^1,v^2).$$

We may iterate this formula to find

$$f_\infty(x,v) = \sum_{m=1}^{\infty}\Big(\prod_{h=1}^{m-1}(1-\alpha(x^h))\Big)\alpha(x^m)\widetilde{(f_\infty)_{|\Sigma_+}}(x^m)c(x^m)\mathcal{M}(m),$$

where $\mathcal{M}(m) = \underline{M}(x^m,v^m)$. The previous series satisfies the boundary condition and the free-transport equation. Yet, the flux values $\widetilde{(f_\infty)_{|\Sigma_+}}(\cdot)$ are unknown. On the other hand, if we plug this series into the definition of the flux $\widetilde{(f_\infty)_{|\Sigma_+}}(\cdot)$, we obtain an integral equation for the latter. We prove that setting $\widetilde{(f_\infty)_{|\Sigma_+}} \equiv \varpi_0$ for some constant $\varpi_0 > 0$ solves this integral equation. Indeed, denoting $\alpha^{(i)} := \alpha(x^i)$ for all $i \geq 1$, we have

$$\begin{aligned}
\widetilde{(f_\infty)_{|\Sigma_+}}(x) &= \int_{\Sigma_+^x}(f_\infty)_{|\Sigma_+}(x,v)|v\cdot n_x|dv\\
&= \varpi_0\int_{\Sigma_+^x}|v\cdot n_x|(\alpha^{(1)}c(x^1)\mathcal{M}(1) + (1-\alpha^{(1)})\alpha^{(2)}c(x^2)\mathcal{M}(2)+\dots)dv\\
&= \varpi_0\int_{\mathcal{A}}|u(\theta_1,\dots,\theta_{d-1})\cdot n_x|\Big(\prod_{j=1}^{d-2}\sin(\theta_j)^{d-1-j}\Big)\Big(\alpha^{(1)}c(x^1)\Big\{\int_0^\infty \underline{M}(x^1,r)r^d dr\Big\}\\
&\quad + (1-\alpha^{(1)})\alpha^{(2)}c(x^2)\Big\{\int_0^\infty \underline{M}(x^2,r)r^d dr\Big\}+\dots\Big)d\theta_1\dots d\theta_{d-1}.
\end{aligned}$$

Since, for all $x\in\partial\Omega$,

$$\int_0^\infty r^d\underline{M}(x,r)dr = \tilde c(x), \qquad \text{and} \quad c(x) = (\kappa\tilde c(x))^{-1},$$

we find

$$\begin{aligned}
\widetilde{(f_\infty)_{|\Sigma_+}}(x) &= \varpi_0\kappa^{-1}\int_{\mathcal{A}}|u(\theta_1,\dots,\theta_{d-1})\cdot n_x|\Big(\prod_{j=1}^{d-2}\sin(\theta_j)^{d-1-j}\Big)\\
&\qquad\qquad \times\Big(\alpha^{(1)} + (1-\alpha^{(1)})\alpha^{(2)}+\dots\Big)d\theta_1\dots d\theta_{d-1}\\
&= \varpi_0,
\end{aligned}$$

where we used the definition of κ and that the infinite series $\alpha^{(1)} + (1-\alpha^{(1)})\alpha^{(2)}+\dots$ converges to unity. Indeed, the series is uniformly convergent with respect to x since $\alpha \geq \underline{\alpha}$ by assumption.

Moreover, we can rewrite it as

$$\alpha^{(1)} + (1-\alpha^{(1)})(1-(1-\alpha^{(2)}) + \dots,$$

hence for all $N > 1$

$$\sum_{m=1}^{N} \Big(\prod_{h=1}^{m-1} (1-\alpha^{(h)}) \Big) \alpha^{(m)} = 1 - \prod_{m=1}^{N} (1-\alpha^{(m)}),$$

from which the convergence to unity is obvious. Ultimately, we proved that the series

$$f_\infty(x,v) = \varpi_0 \sum_{m=1}^{\infty} \Big(\prod_{h=1}^{m-1} (1-\alpha^{(h)}) \Big) \alpha^{(m)} c(x^m) \mathcal{M}(m)$$

is a steady state of (1.1.15) with Maxwell boundary condition. The constant ϖ_0 is determined by the total mass of the system (since this quantity is conserved). The series converges uniformly by hypothesis on α.

1.2.1.3 Bibliographical review

Although some qualitative convergence results were given in restricted contexts, see for instance [3, Theorem 1.4], we will follow a different strategy, where we find a quantitative decay towards the candidate equilibrium, thus solving (P3) above and deduce the qualitative convergence from this result.

The first study of (P3) was the numerical investigation of Tsuji, Aoki and Golse [119]. Using a method based on the decay of the relative entropy W as presented in §1.1.3.4, they obtained numerical evidences supporting the idea that the decay is polynomial in this case, as opposed to the expected exponential convergence of the Boltzmann equation [68, 54, 19]. Considering a spherical vessel of dimension $d \in \{1,2,3\}$, they exhibit a decay of order $\frac{1}{t^d}$ in a kind of L^1 norm in the position and the molecular velocity.

Aoki and Golse [1] later pursued this work, aiming at deriving this rate of convergence. They first prove some negative results and lower bounds on the rate of convergence towards equilibrium, in an appropriate L^1 norm. Then, in the particular case of a spherically symmetric domain in $\mathbb{R}^3$, considering as the initial distribution a radial function of both the space and velocity variables and in the case of the pure diffusive boundary condition, they prove that the decay rate in L^p norm is better than $\frac{1}{t^{\min(1,\frac{3}{p})}}$. We emphasize that, in the L^1 norm, this gives an upper bound in $\frac{1}{t}$ for the decay rate, while the numerical results of [119] suggest a rate of order $\frac{1}{t^3}$. The main tool of Aoki and Golse is the renewal theorem of Feller.

A second method was developed by Kuo, Liu and Tsai [88, 87] and Kuo [86]. The starting point is to build a stochastic process $(X_t, V_t)_{t\geq 0}$ with the appropriate dynamic, i.e. such that its density is a solution to (1.1.15) with pure diffusive or Maxwell boundary condition. Again, the study only holds for the case of spherically symmetric domain, but this time no symmetry assumption is made on the initial data. The case where the temperature varies at the boundary is also investigated in [88]. The idea behind the method is that, because of the symmetry of the domain, the intervals of time between two collisions of the process with the boundary are i.i.d. (independent and identically distributed) random variables in $\mathbb{R}_+$. Then, one can derive a law of large numbers for those time intervals and use it to obtain some control on the L^∞ norm of the flux at the boundary, from which one concludes on the rate of convergence in the L^p norm, for all $p \in [1, \infty)$. Interestingly, this approach also allows the authors to tackle the linearized Boltzmann equation.

At last, let us mention other recent advances in collisionless kinetic theory with boundary conditions. Mokhtar-Kharroubi and Seifert [102] studied the case of monoenergetic free-transport equations in slab geometry with azimuthal symmetry and abstract boundary operators, and provided a quantitative result of convergence by means of Ingham's tauberian theorem. This strategy was also used very recently in a multi-dimensional case, for a kernel which goes beyond the cases studied here, by Lods and Mokhtar-Kharroubi [91]. On the other hand, a drawback from this method is that it seems to provide suboptimal rates, at least in some situations.

1.2.2 Hypocoercivity for linear kinetic equations

In general, the hypocoercivity technique is adapted for the study of evolution equations involving

1. a degenerate dissipative operator (the collision operator in the context of kinetic theory),

2. a conservative operator (the transport operator in our context),

and such that the combination of both operators implies the convergence towards a uniquely determined equilibrium state. There are two main methods: the H^1 one and the L^2 one. The H^1 approach was introduced first for hypoelliptic operators (see the books of Hörmander [75]) by Hérau and Nier [77] and Eckmann and Hairer [53]. This was later extended by Helffer and Nier [74], and Villani [123] and Mouhot and Neumann [103] generalized the method to more kinetic operators. In particular, Mouhot and Neumann obtained the exponential convergence towards equilibrium for two linear kinetic models with a more simple structure than the linearized Boltzmann equation: the BGK equation and the kinetic Fokker-Planck equation, in the torus. Roughly, the strategy is to endow the H^1 space with a new scalar product which makes coercive the considered operator and whose associated norm is equivalent to the usual H^1 one. Those ideas were adapted in the L^2 setting in order to deal with more general operators and geometries by Hérau [76]. We will focus on the case of linear kinetic equations preserving mass, for which a more simple setting was introduced by Dolbeault, Mouhot and Schmeiser [42, 43] in the whole space.

In this thesis, the conservative operator will always be the free-transport operator from kinetic theory: for a particle evolving in the phase space $\Omega \times \mathbb{R}^d$, this operator, denoted $\mathcal{T}$, is given by

$$\mathcal{T} = -v \cdot \nabla_x,$$

and is completed by an operator $\mathcal{K}$ acting only on the boundary. The techniques are based on elementary algebraic tricks, and the introduction of modified Dirichlet forms. For this, one can in general find a natural scalar product in which a degenerated coercivity estimate for the collisional operator holds. The problem is then to add extra terms in a way that allows to recover a fully coercive estimate, while handling the fact that, in this new setting, the transport and the collisional operator may not be negative anymore, creating ill-directed terms which must be absorbed.

Let us be more precise on this L^2 coercivity method, as we will adapt it in Chapter 6. We consider a general space inhomogeneous kinetic equation

$$\partial_t f = \mathcal{L} f = \mathcal{S} f + \mathcal{T} f, \quad f(0) = f_0, \tag{1.2.4}$$

for a function $f = f(t, x, v)$ with $t \geq 0$, $x \in \Omega$ for Ω a bounded, regular domain in $\mathbb{R}^d$, $v \in \mathbb{R}^d$ with $d \geq 2$. The collisional operator $\mathcal{S}$ acts only on the velocity variable v and is dissipative and symmetric in some Hilbert space $\mathcal{H}$ detailed below, while $\mathcal{T}$ given above acts on both the space and the velocity variables (x, v) and is skew-symmetric in $\mathcal{H}$, i.e. we have

$$\mathcal{L} = \mathcal{S} + \mathcal{T}, \quad \mathcal{S}^* = \mathcal{S} \leq 0 \quad \text{and } \mathcal{T}^* = -\mathcal{T}. \tag{1.2.5}$$

If μ is the equilibrium of the evolution equation, the Hilbert spaces of interest are

$$\mathcal{H} = L^2_{\Omega\times\mathbb{R}^d}(\mu^{-1}) := \Big\{ f : G \to \mathbb{R} \text{ measurable with } \int_G |f|^2(x, v)\mu^{-1}(v)dvdx < \infty \Big\},$$
$$\mathcal{H}_v = L^2_{\mathbb{R}^d}(\mu^{-1}) := \Big\{ g : \mathbb{R}^d \to \mathbb{R} \text{ measurable with } \int_{\mathbb{R}^d} |g|^2(v)\mu^{-1}(v)dv < \infty \Big\}.$$

We write $(\cdot, \cdot)_{\mathcal{H}}$, $(\cdot, \cdot)_{\mathcal{H}_v}$ for the scalar products on $\mathcal{H}$ and $\mathcal{H}_v$, respectively, and $\| \cdot \|_{\mathcal{H}}$, $\| \cdot \|_{\mathcal{H}_v}$ for the corresponding Hilbert norms. The collisional operator $\mathcal{S}$ has a nullspace in $\mathcal{H}_v$ that we write $\mathcal{N}(\mathcal{S})$. We furthermore introduce the operator π of projection towards this null set. Aside from technical hypotheses, the main ingredient that we require is the existence of some degenerated estimate on $\mathcal{S}$: we assume that there exists $\kappa^\perp > 0$ such that

$$(\mathcal{S} f, f)_{\mathcal{H}_v} \leq -\kappa^\perp \|\pi^\perp f\|_{\mathcal{H}_v}, \quad \forall f \in D(\mathcal{S}) \cap \mathcal{H}_v, \tag{1.2.6}$$

with $\pi^\perp = I - \pi$.

Splitting the solution as $f = \pi^\perp f + \pi f$, we thus already have a control on the **microscopic part** πf. The problem is to define a new scalar product on $\mathcal{H}$ in which one recovers a control of the so-called **macroscopic part** πf. For this, we start with the usual inner product of $\mathcal{H}$ and add, step by step, new terms in order to control the missing terms appearing on the macroscopic part πf.

We thus introduce a new norm $|||.|||$ associated to a new scalar product

$$((f,g)) = (f,g)_{\mathcal{H}} + a[f,g] + a[g,f],$$

for some non-symmetric bilinear form a. The new Dirichlet form is then given by

$$D[f] := ((-\mathcal{L}f, f)) = (-\mathcal{L}f, f) + a[-\mathcal{L}f, f] + a[f, -\mathcal{L}f],$$

and the form a is chosen so that $|||\cdot|||$ is equivalent to $\|\cdot\|_{\mathcal{H}}$ and there exists some constructive constant $\lambda > 0$ such that

$$D[f] \geq \lambda |||f|||^2, \quad \forall f \in \mathcal{H}_0,$$

where $\mathcal{H}_0$ is a subset of $\mathcal{H}$ containing $g - \mu$ if g is a solution to (1.2.5).

Hypocoercivity and quantitative H-Theorem. In Chapter 6, we use those hypocoercivity techniques to prove the exponential convergence towards equilibrium, in some weighted L^2 norm, of various forms of Boltzmann and Landau linearized equations in a bounded domain with general Maxwell boundary conditions.

The question of deriving a quantitative version of the H-Theorem presented in §1.1.3.2 is a key problem in kinetic theory, which goes back to Boltzmann. In the context of the theorem, proving (exponential) convergence towards equilibrium with constructive arguments is fundamental, as the validity of the Boltzmann equation breaks for very large time, see Villani [121, Chapter 1]. It is then crucial to show that the time scale of the convergence is much smaller than the time scale of the validity of the model. This issue is strongly linked to the famous *Cercignani's conjecture*, or more accurately, *conjectures*, since various forms have been stated. On this subject, and on the related topic of the Cercignani's conjecture for the Landau equation, we refer to Cercignani [25], Villani [122], Desvillettes, Mouhot and Villani [36] and the references therein. As mentioned by Grad [64], this exponential convergence towards equilibrium for the Boltzmann equation should also hold in the presence of compatible boundary conditions in a general domain. Many progresses have been during the last decade for the *perturbative regime*, corresponding to the linearized equation, see §1.3.5 for references. In Chapter 6, we extend those results using an L^2 hypocoercivity method, and derive estimates with constructive constants.

1.2.3 Convergence towards equilibrium of Markov processes

1.2.3.1 Context and historical aspects

In the theory of probabilities and statistics, a Markov process is a stochastic process satisfying the Markov property, that is, the future of the process only depends on its past through its present, and not through its whole history. Another way to say this is that, conditionally on the present of the process, its past and its future are independent. Mathematically, we have the following definition.

Definition 1.2.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, $(X_t)_{t\geq 0}$ an adapted process with state space $(E, \mathcal{E})$ is a homogeneous Markov process with transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ if, for all $f : E \to \mathbb{R}_+$ measurable, for all $0 \leq s < t$,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathcal{P}_{t-s}(X_s) \quad a. - s..$$

The homogeneity property says that the transition laws are constant in time, hence the family of transition operators form indeed a semigroup.

Markov processes are vastly spread in our environment: the dynamic of the queue length at the airport, the nuclear fission, the evolution of the exchange rate between two currencies, the growth of populations and a large amount of human endeavors (word usage by famous authors, record levels in sport...) are some examples of phenomena in which a Markov process is playing a key role. Markov himself investigated the first chapter of Pushkin's novel *Eugene Onegin* and focused on the distribution of vowels and consonants [93]. For a more in-depth list of examples, we refer to Meyn and Tweedie [98].

Historically, Markov processes were introduced by Markov [92] in order to show that a central limit theorem could also hold for dependent variables. The study of stability issues for Markov processes can be traced back to Markov itself, while the question of deriving quantitative bounds for the convergence was studied by Doeblin [40, 41], Harris [73] and Chung [28, 29], focusing mostly on the exponential case. Decades later, Nummelin [105] and Meyn and Tweedie [98] provided a more systematic treatment.

1.2.3.2 Recurrence in the countable and general state space setting

For this section we refer mainly to the work of Meyn and Tweedie, in both the discrete time case [99], including of course the extensive treatment given in [98] and the continous-time case. A first simple family of Markov processes are Markov chains on countable state space. In this case, the process $(X_n)_{n\geq 0}$ is now indexed by integers, and evolves on a countable state space E. Two complexifications, allowing one to model more situations, can be considered: the continuous-time case where the process is indexed by $\mathbb{R}_+$ or even $\mathbb{R}$, and the case of a general state space with a structure more flexible than the countable case.

One of the key question in the study of Markov processes is the one of stability: what is the behavior of the system asymptotically ? Under suitable assumptions, the Markov process has an invariant measure, i.e. there exists a positive measure μ on $(E, \mathcal{E})$ such that $\mathcal{P}_t\mu = \mu$ for all $t \geq 0$, where, for all $A \in \mathcal{E}$,

$$(\mathcal{P}_t\mu)(A) = \int_E \mathcal{P}_t(x, A)\mu(dx).$$

Whether an invariant measure exists is a question strongly related to the notion of recurrence. To illustrate this concept and the one of transience, consider a Markov chain $(X_n)_{n\geq 0}$ with state space $\mathbb{Z}$, and write $P_{ij} = \mathbb{P}(X_1 = j|X_0 = i)$ for all $(i, j) \in \mathbb{Z}^2$ for the transition matrix. For i in $\mathbb{Z}$, let τ_i be the first return time of the process to $\{i\}$:

$$\tau_i = \inf\{k \geq 1 : X_k = i\},$$

and denote $\mathbb{P}_i$ the probability measure defined by

$$\mathbb{P}_i(A) = \mathbb{P}(X_1 \in A|X_0 = i), \qquad \forall A \in \mathcal{P}(\mathbb{Z}).$$

We then say that the state i is

1. transient if $\mathbb{P}_i(\tau_i = \infty) > 0$;
2. recurrent if $\mathbb{P}_i(\tau_i < \infty) = 1$;
3. positive recurrent if i is recurrent and $\mathbb{E}_i[\tau_i] := \mathbb{E}[\tau_i|X_0 = i] < \infty$.

Introducing the number of visits to the state $i \in \mathbb{Z}$ starting from i defined by

$$N_i := \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n=i|X_0=i\}},$$

and letting $f_i = \mathbb{P}_i(\tau_i < \infty)$, we note that, by Markov property,

$$\mathbb{P}(N_i = n) = f_i^{n-1}(1 - f_i), \qquad n \geq 1,$$

so that $N_i \sim \mathcal{G}(1 - f_i)$, the geometric distribution of parameter $1 - f_i$. Hence,

$$\mathbb{E}[N_i] = \frac{1}{1 - f_i},$$

with the convention $\mathbb{E}[N_i] = \infty$ if $f_i = 1$, and we have the following result.

Proposition 1.2.1.

1. *The state $i \in \mathbb{Z}$ is recurrent if and only if $\mathbb{E}[N_i] = \infty$.*
2. *The state $i \in \mathbb{Z}$ is transient if and only if $\mathbb{E}[N_i] < \infty$.*

If the Markov chain is transient, the notion of invariant measure is useless: every state is visited finitely often. Once the recurrence is established, a useful result to study the positive recurrence is the following theorem from Durrett's book [51, Theorem 5.6.1].

Theorem 1.2.1. *Suppose that $(X_n)_{n\geq 0}$ is recurrent (i.e. all $i \in \mathbb{Z}$ are recurrent), then*

$$\forall i \in \mathbb{Z}, \qquad \frac{1}{n}\sum_{m=0}^{n} \mathbf{1}_{\{X_m=i|X_0=i\}} \xrightarrow{n\to\infty} \frac{\mathbf{1}_{\{\tau_i<\infty\}}}{\mathbb{E}_i[\tau_i]} \quad \mathbb{P}_i - a.s.$$

where, by convention $\frac{1}{\infty} = 0$.

We focus from now on on irreducible chains, i.e. chains such that for all $(i,j) \in \mathbb{Z}^2$, there exists $n \geq 1$ satisfying $P^n_{ij} = \mathbb{P}(X_n = j|X_0 = i) > 0$. Irreducibility implies that all the states are of the same type, see for instance [71, 98].

Example 1.2.1 (A recurrent process). An example of a recurrent process is the random walk on $\mathbb{Z}$, with $P_{i,i+1} = P_{i,i-1} = \frac{1}{2}$ for all $i \in \mathbb{Z}$ and $P_{i,j} = 0$ if $|i-j| \neq 1$. This chain is clearly irreducible. Let us assume that $X_0 = 0$. Obviously $P^n_{0,0} = 0$ if n is odd. Moreover, to go back to 0 in $2n$ steps, we need n jumps right and n jumps left, each having probability $\frac{1}{2}$ to occur, hence

$$\mathbb{P}_0(X_{2n} = 0) = P^{2n}_{0,0} = C^n_{2n}\frac{1}{2^{2n}}, \qquad \text{where } C^k_n = \frac{n!}{k!(n-k)!}, 0 \leq k \leq n.$$

Using Stirling's formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$, as $n \to \infty$, we obtain that

$$P^{2n}_{0,0} \sim \sqrt{\frac{1}{\pi n}}.$$

We then have

$$\mathbb{E}[N_0] = 1 + \sum_{n=1}^{\infty} \mathbb{P}_0(X_{2n} = 0) = 1 + \sum_{n=1}^{\infty} P^{2n}_{0,0} = \infty.$$

Using Proposition 1.2.1, we conclude that the chain is recurrent. Moreover, since $\mathbb{P}_0(X_n = 0)$ converges to 0 as $n \to \infty$, we can apply Cesàro's theorem and conclude that

$$\frac{1}{n}\sum_{m=0}^{n} \mathbb{P}_0(X_m = 0) \to 0.$$

On the other hand, by Theorem 1.2.1 and the dominated convergence theorem

$$\frac{1}{n}\sum_{m=0}^{n} \mathbb{P}_0(X_m = 0) \to \frac{\mathbb{P}_0(\tau_0 < \infty)}{\mathbb{E}_0[\tau_0]}.$$

Since $\mathbb{P}_0(\tau_0 < \infty) > 0$, we conclude that $\mathbb{E}_0[\tau_0] = \infty$. Hence the random walk in one dimension is not positive recurrent. Such a process is sometimes called null recurrent. A related fact is that any measure μ such that there exists $c > 0$ with, for all $i \in \mathbb{Z}$, $\mu(\{i\}) = c$ is an invariant measure, which can not be normalized to 1 to give an invariant probability measure.

Example 1.2.2. The random walk in $\mathbb{Z}^3$ is a transient process. The proof relies on a similar argument: one can show that

$$P_{0,0}^{2n} \sim \frac{C}{n^{\frac{3}{2}}}$$

for some constant $C > 0$, and conclude with the help of Proposition 1.2.1. See for instance Durrett [51, Theorem 5.4.4].

Example 1.2.3 (The finite state space case). Any irreducible Markov chain on a finite state space is positive recurrent.

Those three notions of transience, recurrence and positive recurrence can be adapted to the general state space setting, under some assumptions. We will not detail the definitions of transience in this case. Let us consider the continuous-time setting, as most of the results in this thesis will focus on this situation.

We consider, in this whole subsection, a state space $(E, \mathcal{E})$, where $\mathcal{E}$ is the Borelian σ-algebra of E, which is assumed to be a locally compact, separable metric space. We make the hypothesis that $(X_t)_{t\geq 0}$ is a Borel right process, hence it is strongly Markovian with right-continuous sample paths. We introduce the filtration $(\mathcal{F}_t)_{t\geq 0}$ given, for all $t \geq 0$, by $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$. For all $t \geq 0$, $\mathcal{P}_t$ is an operator acting on $\mathcal{B}_b(E)$ the space of bounded measurable functions $f : E \to \mathbb{R}$ and on σ-finite measures μ on $(E, \mathcal{E})$ in the following way

$$\mathcal{P}_t f(x) = \int_E f(y)\mathcal{P}_t(x, dy), \qquad \mu\mathcal{P}_t(A) = \int_E \mathcal{P}_t(x, A)\mu(dx),$$

for all $A \in \mathcal{E}$. We assume the strong Markov property: for any stopping time τ, any $f \in \mathcal{B}_b(E)$,

$$\mathbb{E}_x[f(X_{t+\tau})|\mathcal{F}_\tau] = \mathcal{P}_t f(X_\tau), \qquad 0 \leq t < \infty.$$

For any $A \in \mathcal{E}$, we set

$$T_A = \inf\{t \geq 0 : X_t \in A\}, \qquad S_A = \int_0^\infty \mathbf{1}_{\{X_t \in A\}} dt.$$

To generalize the previous notion we say that $(X_t)_{t\geq 0}$ is ν-irreducible (or, simply, irreducible) if, for a σ-finite measure ν on $(E, \mathcal{E})$,

$$\Big[\nu(B) > 0\Big] \implies \Big[\forall x \in E, \quad \mathbb{E}_x[S_B] > 0\Big],$$

in which case we say that ν is an irreducibility measure for the process. A set A such that $\nu(A) > 0$ for some irreducibility measure ν is said to be *accessible*. The notion of *Harris recurrence* generalizes the previous concept of recurrence.

Definition 1.2.2 (Harris recurrent process)**.** The process $(X_t)_{t\geq 0}$ is called Harris recurrent if either

1. there exists a σ-finite measure ν such that $\mathbb{P}_x(S_A = \infty) = 1$ for all A such that $\nu(A) > 0$; or

2. there exists a σ-finite measure μ such that $\mathbb{P}_x(T_A < \infty) = 1$ for all A such that $\mu(A) > 0$.

It is often much easier to verify the second condition, although both are equivalent. In some cases, the two measures μ and ν do not coincide. An Harris recurrent process is clearly irreducible.

A key result of the theory is that the Harris recurrence property implies the existence of a unique (up to constant multiples) invariant measure, and in some cases we can thus normalize this measure to obtain a unique invariant probability measure, leading to a notion similar to the positive recurrence in the countable state space setting. We take the pragmatic view of defining the positive recurrence in this new setting based on this property.

Definition 1.2.3. The process $(X_t)_{t\geq 0}$ is called positive Harris recurrent if it is Harris recurrent with a finite invariant measure π.

Most of the theory is exposed in the book of Nummelin [105], see also the paper of Meyn and Tweedie [100].

To sum up, we have the following analogies.

Countable state space	**General state space**
Transient process	Transient process
Recurrent process	Harris recurrent process
Positive recurrent process	Positive Harris recurrent process

Let us recall some more definitions. We will heavily use the notion of petite set.

Definition 1.2.4. A non-empty measurable set C is said to be petite if there exist a probability measure a on $\mathcal{B}(\mathbb{R}_+)$ (the Borel σ-algebra) and a non-trivial σ-finite measure ν on $\mathcal{E}$ such that

$$\forall x \in C, \int_0^\infty \mathcal{P}_t(x, \cdot)a(dt) \geq \nu(\cdot).$$

Definition 1.2.5. A continuous-time Markov process $(X_t)_{t\geq 0}$ with values in E is non-explosive if there exists a family of pre-compact open sets $(O_n)_{n\geq 0}$ such that $O_n \to E$ as $n \to \infty$ (i.e. $\cup_n O_n = E$ and $O_n \subset O_{n+1}$ for all $n \geq 0$), and such that, setting for all $m \geq 0$, $T_m = \inf\{t > 0, X_t \notin O_m\}$, for all $x \in E$,

$$\mathbb{P}_x\Big(\lim_{m\to\infty} T_m = \infty\Big) = 1.$$

Definition 1.2.6. We say that a process $(X_t)_{t\geq 0}$ with associated semigroup $(\mathcal{P}_t)_{t\geq 0}$ is aperiodic if there exists an $m > 0$ such that, denoting by δ_m the Dirac mass at m, there exists an accessible δ_m-petite set C (i.e. petite with measure $a = \delta_m$ on $\mathbb{R}_+$) and some $t_0 \geq 0$ such that for all $x \in C$, $t \geq t_0$, $\mathcal{P}_t(x, C) > 0$.

Definition 1.2.7. We say that the process $(X_t)_{t\geq 0}$ is strong Feller if the transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ is such that, for all $\phi \in \mathcal{B}_b(E)$, for all $t \geq 0$, $\mathcal{P}_t\phi \in \mathcal{C}_b(E)$ (the space of continuous bounded functions on E). We say that it is (weak) Feller if, for all $\phi \in \mathcal{C}_b(E)$, for all $t \geq 0$, $\mathcal{P}_t\phi \in \mathcal{C}(E)$ (the space of continuous functions on E).

1.2.3.3 Exponential convergence towards equilibrium: three methods

In this thesis, we focus on essentially three (more or less equivalent) ways to obtain, for a positive Harris recurrent process, the exponential convergence towards its unique probability measure.

A. The study of the probabilistic generator of the process, and the existence of a corresponding Foster-Lyapunov inequality. This is the key ingredient of the Meyn-Tweedie theory.

B. The study of the deterministic generator of the process. This is essentially equivalent to the previous method. We distinguish them in order to emphasize that the same Harris' theorem can be obtained by means of purely deterministic method.

C. The construction of an appropriate coupling of the process of interest. This probabilistic technique requires a more *ad hoc* adaptation, but is quite simple in its principle.

In the following part of this subsection we will briefly present the corresponding results and illustrate them with the example of the Ornstein-Uhlenbeck process $(Y_t)_{t\geq 0}$ on $\mathbb{R}$, solution to the SDE

$$dY_t = \sqrt{2}dB_t - Y_t dt, \tag{1.2.7}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. The stochastic generator of this process is given by

$$\mathcal{L}f = \partial^2_{xx}f - x\partial_x f,$$

for all $f \in C^2(\mathbb{R})$, and equivalently the law of the process is solution to the following (deterministic) Fokker-Planck equation

$$\partial_t f = \partial^2_{xx}f + \partial_x(xf), \qquad x \in \mathbb{R}. \tag{1.2.8}$$

This equation has a steady state given by $\mu_\infty(x) = \exp(-\frac{x^2}{2})$ on $\mathbb{R}$, and we will prove the exponential convergence towards this steady state using the three methods mentioned above.

As an Itô diffusion with Lispschitz drift, the Ornstein-Uhlenbeck process exists (i.e. there exists a solution to (1.2.7) in some appropriate sense), is Feller and irreducible, see Peszat and Zabczyk [107, Corollary 1.1 and Theorem 1.3].

We will work essentially with the L^1 norm and the first Wasserstein distance. For this we make the further assumption that E can be equipped with a distance d so that (E,d) is a Polish (i.e. separable, completely metrizable) metric space, and we endow it with the corresponding Borel σ-algebra denoted $\mathcal{E}$. For a random variable X, we write $X \sim \mu$ if X has law μ.

Definition 1.2.8 (Coupling)**.** A coupling of two probability measures μ, ν on $(E, \mathcal{E})$ is the realisation of a pair of random variables $(Z, \tilde{Z})$ defined on the same probability space and such that $Z \sim \mu$, $\tilde{Z} \sim \nu$.

Of course, the simplest possible coupling is the one where Z and $\tilde{Z}$ are independent, but the interest of the notion relies precisely in the possibility of taking Z and $\tilde{Z}$ appropriately correlated.

Definition 1.2.9 (p-th Wasserstein distance)**.** For $p \geq 1$, the p-th Wasserstein distance between two probability measures μ, ν is defined on (E,d) by

$$W_p(\mu,\nu) = \Big(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{E \times E} d(x,y)^p d\gamma(x,y) \Big)^{\frac{1}{p}},$$

where $\Gamma(\mu,\nu)$ is the set of measures on $E \times E$ having marginals μ and ν for, respectively, their first and second factors. Equivalently,

$$W_p(\mu,\nu) = \Big(\inf_{\substack{(X,Y) \\ X \sim \mu, Y \sim \nu}} \mathbb{E}[d(X,Y)^p] \Big)^{\frac{1}{p}}.$$

When $d(x,y) = \mathbf{1}_{\{x \neq y\}}$, the 1st Wasserstein distance corresponds to the **total variation distance**, defined between two measures μ and ν on E by

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{E}} |\mu(A) - \nu(A)|. \tag{1.2.9}$$

The connection between the total variation distance of (μ,ν) and the corresponding couplings is given by

$$\|\mu - \nu\|_{TV} = \inf_{\substack{(X,Y) \\ X \sim \mu, Y \sim \nu}} \mathbb{P}(X \neq Y).$$

A. The Meyn-Tweedie theory. The general result of Meyn and Tweedie [97] and Down, Meyn and Tweedie [45] regarding the exponential convergence towards equilibrium is the following one. A set C is called absorbing if $C \neq \emptyset$ and $\mathcal{P}_t(x, C) = 1$ for all $x \in C$, $t \geq 0$.

Theorem 1.2.2 (Exponential case, [97, 45])**.** *Assume that $(X_t)_{t\geq 0}$ is non-explosive, irreducible, and aperiodic. Then the following conditions are equivalent.*

1. *There exist a closed petite set $C \in \mathcal{E}$ and some constants $\delta > 0$ and $\kappa > 1$ such that, setting*
$$\tau_C(\delta) := \inf\{t > \delta, X_t \in C\}, \qquad \text{we have} \qquad \sup_{x\in C} \mathbb{E}_x[\kappa^{\tau_C(\delta)}] < \infty.$$
2. *There exist a closed petite set $C \in \mathcal{E}$, some constants $b > 0$, $\beta > 0$ and a function $V : E \to [1, \infty]$ finite at some $x_0 \in E$ such that*
$$\mathcal{L}V \leq -\beta V + b\mathbf{1}_C. \tag{1.2.10}$$

Any of those conditions implies that the set $S_V = \{x : V(x) < \infty\}$ is absorbing for any V solution of (1.2.10), that the process is positive Harris recurrent with an invariant probability measure π, and that there exist $\rho < 1$ and $d > 0$ such that, for all $x \in E$, for all $t \geq 0$,

$$\|\mathcal{P}_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq dV(x)\rho^t.$$

Loosely speaking, the idea is that, under the assumptions of condition 2, one can identify a set at the "center" of the space in which the process behaves well in the sense of the stability (a petite set), and for which a Lyapunov inequality (sometimes called Foster-Lyapunov inequality) telling us that the process comes back to this set sufficiently fast exists. Note that, for $x \notin C$ in condition 2, $\mathcal{L}V(x) \leq -\beta V(x)$ with $\beta > 0$, hence the process comes back "exponentially fast" to small values of V by Grönwall's lemma. The idea is similar for the first condition, which even gives an explicit control of some moment of the time of the excursion outside of C.

Remark 1.2.1. Rigorously, the condition 2 should be understand in the sense of the weak generator, see Davis [34], that is, the condition is equivalent to the fact that for all $x \in E$, the process given for all $t \geq 0$ by

$$M_t = V(X_t) - V(x) + \beta \int_0^t V(X_s)ds - b\int_0^t \mathbf{1}_C(X_s)ds,$$

is a $\mathbb{P}_x$-local supermartingale. The domain of the corresponding weak generator contains the canonical one $\mathcal{D}(\mathcal{L})$. Both generators agree on $\mathcal{D}(\mathcal{L})$.

Example 1.2.4. Regarding the Ornstein-Uhlenbeck process $(Y_t)_{t\geq 0}$ solution to (1.2.7), we consider $V : \mathbb{R} \to [1, \infty)$ defined by $V(x) = \exp(a|x|)$ for some $a > 0$ (e.g. $a = 1$). For $x > 0$,

$$\begin{aligned}\mathcal{L}V(x) = a^2e^{ax} - xae^{ax} &\leq (a^2 - xa)e^{ax}(\mathbf{1}_{\{x\in(0,2a]\}} + \mathbf{1}_{\{x\in(2a,\infty)\}})\\ &\leq -a^2e^{ax} + a^2e^{ax}\mathbf{1}_{\{x\leq 2a\}}\\ &\leq -a^2V(x) + a^2e^{2a^2}\mathbf{1}_{\{|x|\leq 2a\}}.\end{aligned}$$

A similar computation shows that the result also holds for $x \leq 0$. Moreover, V is continuous and $\{x : |x| \leq 2a\} = \{x : V(x) \leq e^{2a^2}\}$ is thus compact. Since $(X_t)_{t\geq 0}$ is Feller and irreducible with an associated measure with non-empty interior [107], all compact sets of $\mathbb{R}$ are petite sets by [99, Theorem 3.4].

B. The deterministic Harris' theorem. In 2011, Hairer and Mattingly [72] gave a different proof of the result of Harris presented above in the framework of the Meyn-Tweedie theory (condition 2). The proof relies on the introduction of a modified norm and the extensive use of its properties, with a purely deterministic approach. While the result is essentially the same as the one above, we distinguish them because the subexponential counterpart will be different, see subsection 1.2.3.4. The material exposed here is almost entirely taken from Cañizo and Mischler [21]. A slightly different formulation is given in Cañizo, Cao, Evans and Yoldas [22] in the context of linear kinetic equations, but the method can be applied independently from this framework.

We first introduce two notions of semigroups of linear operators on a Banach lattice (roughly, a Banach space ordered by some order relation) A, for instance $A := L^p(E, \mathcal{E}, \mu)$ with μ a positive, σ-finite measure. Let $C_0(E)$ be the space of continuous functions which tend to 0 at infinity, endowed with the uniform norm $\|\cdot\|_\infty$ and let $\mathcal{M}(E)$ be the space of Radon measures on E, defined as the dual space of $C_0(E)$. We write $|\cdot|$ for the norm on E.

Definition 1.2.10. On a Banach lattice $B \supset C_0(E)$, $(\mathcal{P}_t)_{t\geq 0}$ is a Markov semigroup if

1. $(\mathcal{P}_t)_{t\geq 0}$ is a (strongly) continuous semigroup in B,
2. $(\mathcal{P}_t)_{t\geq 0}$ is positive, i.e. $\mathcal{P}_t \geq 0$ for any $t \geq 0$,
3. $(\mathcal{P}_t)_{t\geq 0}$ is constant conservative, i.e. $\mathbf{1} \in B$ and $\mathcal{P}_t\mathbf{1} = \mathbf{1}$ for any $t \geq 0$.

Definition 1.2.11. We say that $(S_t)_{t\geq 0}$ is a stochastic semigroup on a Banach lattice B if

1. $(S_t)_{t\geq 0}$ is a (strongly) continuous semigroup in B,
2. $(S_t)_{t\geq 0}$ is positive, i.e. $S_t \geq 0$ for all $t \geq 0$,
3. $(S_t)_{t\geq 0}$ is mass conservative, i.e. $\langle S_t f\rangle = \langle f\rangle$ for all $t \geq 0$, all $f \in B$, where $\langle f\rangle = \int_E f dx$.

We point out that if $(S_t)_{t\geq 0}$ is a strongly continuous Markov semigroup on B, then the associated Kato's inequality holds: denoting $\mathcal{L}$ the corresponding operator

$$\operatorname{sign}(f)\mathcal{L}f \leq \mathcal{L}|f|, \qquad \forall f \in \mathcal{D}(\mathcal{L}).$$

The converse is also true: a strongly continuous semigroup satisfying Kato's inequality is Markov. Another result that we will use is the following.

Proposition 1.2.2. *Let $B \subset L^1(E, \mathcal{E}, \mu)$ a Banach lattice. A stochastic semigroup on B is a semigroup of contraction for the L^1 norm, i.e. if $(S_t)_{t\geq 0}$ is a stochastic semigroup on B,*

$$\forall t \geq 0, \quad \forall f \in B, \qquad \|S_t f\|_{L^1} \leq \|f\|_{L^1}. \tag{1.2.11}$$

We consider a stochastic semigroup $(S_t)_{t\geq 0}$ on $L^1(E)$ and denote $\mathcal{L}$ the associated generator. For $m : E \to [1, \infty)$ a weight, we set

$$L^1_m(E) = \{f \in L^1(E), \|fm\|_{L^1} < \infty\}, \tag{1.2.12}$$

and denote $\|f\|_m = \|fm\|_{L^1}$ for all $f \in L^1_m(E)$. We make two hypotheses, reminiscent of the condition 2. of Theorem 1.2.2.

Hypothesis 1.2.2. There exist $m : E \to [1, \infty)$, $\alpha, b > 0$ two constants such that

$$\mathcal{L}^* m \leq -\alpha m + b.$$

Hypothesis 1.2.3. For any $\rho > 0$, there exist a constant $T > 0$ and $\nu \in \mathcal{M}(E)_+$ the space of non-negative σ-finite measure on E such that $\nu \not\equiv 0$ and

$$S_T f \geq \nu \int_{\{x : |x| \leq \rho\}} f dx, \qquad \forall f \in L^1(E)_+ = \{g \in L^1(E), g \geq 0\}.$$

Remark 1.2.2. Hypothesis 1.2.3 is sometimes referred to as the Doeblin minorisation condition, and indicates that a petite set exists in the previous terminology.

Theorem 1.2.3 (Harris' Theorem)**.** *Let $(S_t)_{t\geq 0}$ be a stochastic semigroup on $L^1(E)$ with generator $\mathcal{L}$, satisfying Hypotheses 1.2.2 and 1.2.3. There holds*

$$\forall t \geq 0, \, \forall f \in L^1(E), \, \langle f \rangle = 0, \qquad \|S_t f\|_m \leq C e^{at} \|f\|_m,$$

for some constructive constants $C \geq 1$ and $a < 0$.

Let us expand on the link between this theorem and the usual PDE formulation for evolution problems.

Coming back to the situation where we have a Markov process $(X_t)_{t\geq 0}$, the transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ is a Markov semigroup. We write $\mathcal{L}_2$ for the associated generator. Suppose that the law of the process is a solution to the PDE

$$\partial_t f = \mathcal{L} f,$$

for some operator $\mathcal{L} : L^1(E) \to L^1(E)$. Assume that this evolution equation is well-posed, in the sense that it admits a unique solution in L^1 for all initial data $f_0 \in L^1(E)$. In this case we can associate a stochastic semigroup $(S_t)_{t\geq 0}$ of linear operators such that for all $t \geq 0$, $x \in E$,

$S_t f_0(x) = f(t,x)$ is the unique solution at time $t \geq 0$ of the evolution equation satisfying the initial condition. It turns out that, for all $t \geq 0$,

$$\mathcal{P}_t = S_t^*, \qquad \mathcal{L}_2 = \mathcal{L}^*,$$

i.e., the transition semigroup of the stochastic operator is the adjoint semigroup of $(S_t)_{t\geq 0}$, and the associated operator is also the adjoint of the operator of the evolution equation. Hence, Hypothesis 1.2.2 and condition 2 in Theorem 1.2.2 both give an inequality on the generator of the Markov semigroup associated to the process.

The starting point of the proof of Theorem 1.2.3 is the following. We have, for all $t \geq 0$, for m given by Hypothesis 1.2.2, and using Kato's inequality

$$\frac{d}{dt}\int_E |S_t f|(x) m(x) dx = \int_E \mathcal{L}|S_t f|(x) m(x) dx = \int_E |S_t f| \mathcal{L}^* m(x) dx,$$

so that, using Hypothesis 1.2.2,

$$\frac{d}{dt}\int_E |S_t f|(x) m(x) dx \leq -\alpha \int_E |S_t f|(x) m(x) dx + b \int_E |S_t f|(x) dx.$$

By Proposition (1.2.2), we thus have

$$\frac{d}{dt}\|S_t f\|_m \leq -\alpha \|S_t f\|_m + b\|f\|_{L^1},$$

and we conclude that for all $T > 0$,

$$\|S_T f\|_m + \alpha \int_0^T \|S_s f\|_m \leq \|f\|_m + bT\|f\|_{L^1}. \tag{1.2.13}$$

The idea is then to combine (1.2.13) and Hypothesis 1.2.3 to find a strict contraction in some norm derived from the norm $\|\cdot\|_m$, and to conclude from there.

Example 1.2.5. We can use the function V of example 1.2.4 as a weight m. The Lyapunov condition of this case gives directly Hypothesis 1.2.2. Since the set $\{x : |x| < 2a\}$ is compact, it is a petite set, hence Hypothesis 1.2.3 is also satisfied. We obtain the exponential convergence towards the equilibrium μ_∞, in the appropriate norm $\|\cdot\|_V$, of the solution to the Fokker-Planck equation (1.2.8), as a direct result of Theorem 1.2.3.

C. The coupling method. The coupling method was first used to derive quantitative bounds of convergence towards equilibrium by Pitman [108] in the context of Markov chains. Thorisson [118] wrote an in-depth presentation of the theory, see also [117].

When one studies the convergence of a Markov, positive Harris recurrent process $(X_t)_{t\geq 0}$ towards its invariant measure π, starting from some $x_0 \in E$, one can obtain an upper bound on the convergence rate in the following way:

1. one builds a coupling $(X_t, X'_t)_{t\geq 0}$ with $X_0 = x_0$, $X'_0 \sim \pi$, in such a way that for all $t \geq 0$, $X_t \sim \mathcal{P}_t(x_0, \cdot)$, $X'_t \sim \pi$ since π is an invariant measure. The coupling is chosen in order to control the coupling time

$$T_C = \inf\{t \geq 0 : X_t = X'_t\}.$$

By the strong Markov property, the innovation which determines $(X_{T_C+s})_{s\geq 0}$ from $\mathcal{F}_{T_C}$, where $\mathcal{F}_s = \sigma((X_u, X'_u), 0 \leq u \leq s)$ only depends on the past trajectory through X_{T_C}. The same property holds for $(X'_{T_C+s})_{s\geq 0}$. Hence, conditionally on $\mathcal{F}_{T_C}$, we can build a trajectory $(Y_s)_{s\geq 0}$ with transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ and $Y_0 = X_{T_C}$ and set, in our construction of the coupling $X_{T_C+s} = X'_{T_C+s} = Y_s$ for all $s \geq 0$.

2. We then have, for all $t \geq 0$, by Markov's inequality

$$\|\mathcal{P}_t(x_0, \cdot) - \pi\|_{TV} \leq \mathbb{P}(X_t \neq X'_t) \leq \mathbb{P}(t > T_C) \leq \frac{\mathbb{E}[r(T_C)]}{r(t)}, \tag{1.2.14}$$

for some rate function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mathbb{E}[r(T_C)] < \infty$. The optimal choice of r thus gives an upper bound on the rate of convergence towards equilibrium. In the exponential setting $r(t) = e^{at}$ for some $a > 0$ and for all $t \geq 0$.

3. Alternatively, one can focus on the first Wasserstein distance (or another such distance with a different order) with the distance d on E and build a coupling $(X_t, X'_t)_{t\geq 0}$ such that $\mathbb{E}[d(X_t, X'_t)] \leq \frac{C}{r(t)}$ for some constant $C > 0$ independent of t and some rate r. The control of the first Wasserstein distance then follows immediatly from the definition.

Example 1.2.6. Coming back to the Ornstein-Uhlenbeck process $(Y_t)_{t\geq 0}$ solution to (1.2.7), the solution starting from $x \in \mathbb{R}$ has the following explicit form [107]

$$\forall t \geq 0, \qquad Y_t = Y_0 e^{-t} + \int_0^t \sqrt{2} e^{-(t-s)} dB_s.$$

Consider now the process Y'_t with initial distribution μ_∞, which we recall is the invariant distribution, and solution of (1.2.7). Then

$$\forall t \geq 0, \qquad Y'_t = Y'_0 e^{-t} + \int_0^t \sqrt{2} e^{-(t-s)} dB'_s,$$

for a Brownian motion $(B'_t)_{t\geq 0}$. We will choose to correlate $(Y_t)_{t\geq 0}$ and $(Y'_t)_{t\geq 0}$ by taking $B' = B$. This gives a coupling $(Y_t, Y'_t)_{t\geq 0}$ such that for all $t \geq 0$, $Y_t \sim \mathcal{P}_t(x, \cdot)$, $Y'_t \sim \mu_\infty$. Such a choice provides an upper bound on the convergence towards equilibrium in the first Wasserstein distance on $\mathbb{R}$ equipped with the distance $d(x, y) = |x - y|$, indeed

$$W_1(\mathcal{P}_t(x, \cdot), \mu_\infty) \leq \mathbb{E}[d(Y_t, Y'_t)] = e^{-t}\mathbb{E}[|Y_0 - Y'_0|] = e^{-t} W_1(\delta_x(\cdot), \mu_\infty),$$

where we used that since x is deterministic, all the couplings of $(\delta_x(\cdot), \mu_\infty)$ lead to the same expectation. This shows the exponential convergence towards equilibrium in the W_1 distance. Of course this argument does not provide directly an exponential rate for the total variation distance. In fact, the previous choice of coupling is not adapted at all to the study of the total variation distance, since $T_C = +\infty$.

1.2.3.4 Subexponential convergence towards equilibrium

We now turn to the subexponential case, roughly (see [44] for a precise definition), we want to establish the convergence towards equilibrium at some rate r of the form

$$r(t) = t^\alpha \ln(t)^\beta \exp(\gamma t^\delta), \qquad \delta \in (0,1) \text{ and } \begin{cases} \gamma > 0, \alpha, \beta \in \mathbb{R} \text{ or}, \\ \gamma = 0, \alpha > 0, \beta \in \mathbb{R} \text{ or}, \\ \gamma = \alpha = 0, \beta > 0. \end{cases}$$

We can again identify three methods allowing one to obtain quantitative bounds. Let us first point out that of course, such models, for which the geometric ergodicity fails, exist. The main such object in this thesis will be the free-transport equation with pure diffusive or Maxwell boundary condition, in which the occurence of small velocities prevents the possibility of an exponential ergodicity, see Aoki and Golse [1, Proposition 3.3]. Another example is the compound Poisson-process driven Ornstein-Uhlenbeck process (i.e., the previous Ornstein-Uhlenbeck process (1.2.7) where the Brownian motion is replaced by a compound Poisson-process), see Fort and Roberts [59, Section 3.3 and Lemma 17].

The three methods presented here for the study of the subgeometric case are essentially counterparts of those in the geometric case.

A. The study of the probabilistic generator of the process, and the existence of a Foster-Lyapunov inequality. This corresponds to the extension of the Meyn-Tweedie theory.

B. The study of the deterministic generator of the evolution equation. In constrast with the exponential case, the results obtained with deterministic techniques are slightly different from the probabilistic ones, and can be more flexible in some cases.

C. The construction of an appropriate coupling of the process of interest. The method is exactly the same, but some interesting properties can be acknowledged that differ from the exponential setting.

Our typical example for this subsection will be the gradient dynamic on $\mathbb{R}$, i.e. the process $(Y_t)_{t\geq 0}$ solution to the SDE

$$dY_t = -\partial_x V(Y_t)dt + \sqrt{2}dB_t, \tag{1.2.15}$$

with $(B_t)_{t\geq 0}$ is a standard Brownian motion, with the choice

$$V(x) = 2(1+|x|^2)^{\frac{1}{4}}, \qquad x \in \mathbb{R}.$$

The stochastic generator is given by

$$\mathcal{L} = \partial_{xx}^2 - \partial_x V \partial_x,$$

and equivalently, the law of the process is solution to the evolution equation given by

$$\partial_t f = \partial_{xx}^2 f + \partial_x (f \partial_x V) \qquad \text{in } \mathbb{R}.$$

The equilibrium distribution is given by $\mu_\infty(x) = e^{-V(x)}$.

A. Methods based on the probabilistic generator. The first systematic extension of the Meyn-Tweedie theory to the subexponential case is the above-mentioned paper of Fort and Roberts [59] on the polynomial case. Later, Douc, Fort and Guillin [44] provided a more systematic study of a whole class of subgeometric rates. Finally Hairer [71] gave an enlightning proof in a slightly stronger case. Let us state a result.

Theorem 1.2.4 ([44, 71])**.** *Assume that $(X_t)_{t\geq 0}$ is non-explosive, irreducible and aperiodic. Let $\phi : [1,\infty) \to \mathbb{R}_+^*$ be C^1, strictly increasing, strictly concave with $\phi(x) \leq x$ for all $x \geq 1$ with $\frac{\phi(x)}{x} \downarrow 0$, $\phi(x) - x\phi'(x) \uparrow \infty$ when $x \to \infty$. Define the function $H_\phi(\cdot)$ on $[1,\infty)$ by*

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)},$$

and let $H_\phi^{-1} : [0,\infty) \to [1,\infty)$ be its inverse function. Consider the two following conditions.

1. *There exist a compact petite subset C of E and some $\delta > 0$ such that, for $\tau_C(\delta)$ defined by*

$$\tau_C(\delta) = \inf\{t \geq \delta, X_t \in C\}, \tag{1.2.16}$$

   we have

$$\mathbb{E}_x[H_\phi^{-1}(\tau_C(\delta))] < \infty \quad \text{for all } x \in E \quad \text{and} \quad \sup_{x\in C} \mathbb{E}_x[H_\phi^{-1}(\tau_C(\delta))] < \infty. \tag{1.2.17}$$

2. *There exist a compact petite subset C of E, a constant $K > 0$ and $V : E \to [1,\infty)$ continuous with precompact sublevel sets such that for all $x \in E$,*

$$\mathcal{L}V(x) \leq -\phi(V(x)) + K\mathbf{1}_C(x). \tag{1.2.18}$$

In those two cases, there exists an invariant probability measure π for $(\mathcal{P}_t)_{t\geq 0}$ on E and for all $x \in E$,

$$\lim_{t\to\infty} \phi(H_\phi^{-1}(t))\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

Let us comment on this theorem.

Remark 1.2.3. There is *a priori* no equivalence between the two conditions. In fact, condition 2 implies condition 1, but the converse does not seem to hold. We will come back to this issue when presenting Chapter 5.

Remark 1.2.4. The innocuous properties of ϕ are not detailed in Hairer [71]. In Chapter 5, we carefully rewrite the proof from [71] that condition 2 implies the result, showing where those hypotheses are required.

Example 1.2.7. For the gradient dynamic, let $W(x) = e^{\alpha V(x)}$ with $\alpha \in (0,1)$ constant. A straightforward computation gives

$$\mathcal{L}W = \alpha W(1+x^2)^{-7/4}\Big(1 - \frac{1}{2}x^2 + (\alpha-1)x^2(1+x^2)^{\frac{1}{4}}\Big).$$

The factor in brackets is obviously negative and upper bounded by some negative constant outside a compact set of the form $C := \{x : W(x) \leq \overline{W}\}$ for some constant $\overline{W} > 0$. Hence for two constants $\beta, K > 0$, we have

$$\mathcal{L}W \leq -\beta \frac{W}{\ln(W)^7} + K\mathbf{1}_C.$$

As for the Ornstein-Uhlenbeck process, the process is Feller and irreducible with an associated measure with non-empty interior [107], so all compact sets are petite. To compute the rate of convergence, we have first, for all $u \geq 1$,

$$H_\phi(u) \propto \int_1^u \ln(x)^7 \frac{dx}{x} = \int_0^{\ln(u)} y^7 dy = \frac{\ln(y)^8}{8},$$

hence $H_\phi^{-1}(u) \propto e^{cu^{\frac{1}{8}}}$ for some constant $c > 0$. The upper bound on the rate of convergence towards equilibrium is thus the rate defined on $\mathbb{R}_+^*$ by

$$r(t) := \phi(H_\phi^{-1}(t)) \propto t^{-\frac{7}{8}} e^{ct^{\frac{1}{8}}}.$$

B. Deterministic Harris' subgeometric theorem. In the subgeometric case, the deterministic version of Harris' theorem takes a slightly different form. While the Doeblin condition remains unchanged, the Lyapunov condition is now established between different weights, and used in several forms. We state here the result corresponding to the strategy adopted for the free-transport equation with pure diffusive or Maxwell boundary condition in Chapter 4, i.e. in the L^1 setting, but the theorem holds with straightforward adaptations for higher-order L^p spaces, $p > 1$, and even for the space of Radon measures or the space of continuous functions. The results presented here are due to Cañizo and Mischler [21].

Theorem 1.2.5. *Let $m_i : E \to [1, \infty)$, $i \in \{0, \dots, 3\}$ be four weight functions such that*

$$m_0 \leq m_1, m_2 \leq m_3$$

and such that both m_0 and m_2 diverge to infinity as $x \to \infty$. Consider a stochastic semigroup $(S_t)_{t \geq 0}$ with generator $\mathcal{L}$ such that for all $t \geq 0$, $S_t : L^1(m_i) \to L^1(m_i)$, $i \in \{0, \dots, 3\}$ and assume the following.

(H1) Doeblin condition: for any $R \geq R_0 > 0$, there exist $T = T(R) \geq T_0 > 0$ and a measure $\nu \in \mathcal{M}(E)_+ \setminus \{0\}$ such that

$$S_T f \geq \nu \int_{\{x \in E, |x| \leq R\}} f dx, \quad \forall f \in L^1(E), f \geq 0.$$

(H2) Lyapunov conditions: there exists $b > 0$ such that

$$\mathcal{L}^* m_1 \leq -m_0 + b;$$

$$\mathcal{L}^* m_3 \leq -m_2 + b; \qquad \mathcal{L}^* m_2 \leq b.$$

(H3) Interpolation condition: there exists a family $(\epsilon_\lambda)_{\lambda > 0}$ of positive real numbers such that, for all $\lambda \geq 0$,

$$m_1 \leq \lambda m_0 + \epsilon_\lambda m_3, \quad \text{with } \epsilon_\lambda \to 0 \text{ as } \lambda \to \infty,$$

the weight function m_0 is an interpolation between 1 and m_3, and $\frac{m_2(R)}{T(R)} \to \infty$ as $R \to \infty$.

Then

$$\|S_t f\|_{L^1} \leq \Theta(t) \|f\|_{m_3}, \qquad \forall t \geq 0, \forall f \in L^1(m_3) \text{ with } \langle f \rangle = 0,$$

where the decay rate function is given, for some constant $\kappa > 0$, by

$$\Theta(t) := \inf_{\lambda > 0} \left(e^{-\kappa t / \lambda} + \epsilon_\lambda \right).$$

Remark 1.2.5. The Lyapunov condition can be replaced by inequalities of the form

$$\|S_t f\|_{m_3} + \int_0^t \|S_s f\|_{m_2} ds \leq \|f\|_{m_3} + b_3(1+t)\|f\|_{L^1},$$

which is similar to (1.2.13), for some constant $b_3 > 0$. From the condition on m_2 only, we have, for some constant $b_2 > 0$, $0 \leq s \leq t$,

$$\|S_t f\|_{m_2} \leq \|S_s f\|_{m_2} + b_2(1 + (t-s))\|f\|_{L^1}.$$

Ultimately we obtain the integrated Lyapunov condition

$$\|S_T f\|_{m_3} + T\|S_T f\|_{m_2} \le \|f\|_{m_3} + K\|f\|_{L^1}, \tag{1.2.19}$$

for $K = K(T) = b_3(1+T) + b_2(T + \frac{T^2}{2})$. In some cases, and in particular in the application of Chapter 4, the Lyapunov condition as written in Theorem 1.2.5 is difficult to obtain, but this integrated version of the condition can be proved and is enough to derive the result.

Remark 1.2.6. One advantage of this deterministic version is that constants are easily tractable, providing a constructive rate, a key point in applications.

Remark 1.2.7. If (m_1, m_0) is a couple of weight functions satisfying

$$\mathcal{L}^* m_1 \le -m_0 + b,$$

for some constant $b > 0$, then $\phi(m_1)$ satisfies a similar inequality for all $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ concave and increasing. In some sense we thus only need one such inequality.

Remark 1.2.8. The rates given in Theorem 1.2.4 and 1.2.5 are not necessarily equal in situations where both results apply.

In Chapter 4, see in particular Section 5, we give a proof for a particular choice of weights and starting from the integrated Lyapunov condition (1.2.19) which may be adapted to obtain Theorem 1.2.5. The original statement and the proof, as well as many refinements, can be found in Cañizo and Mischler [21].

Example 1.2.8. For the gradient dynamic, let us first write

$$\langle x \rangle = (1 + x^2)^{\frac{1}{2}},$$

and consider a weight $m(x) = e^{\kappa \langle x \rangle^s}$ for two constants $\kappa, s \in (0, \frac{1}{2})$. A computation shows that, for $\mathcal{L}^*$ given by $\mathcal{L}^* g = \partial^2_{xx} g - \partial_x V \partial_x g$ for all $g \in \mathcal{D}(\mathcal{L}^*)$,

$$\mathcal{L}^* m(x) \le -C e^{\kappa \langle x \rangle^s} \langle x \rangle^{s-4} + b,$$

for two constants $C, b > 0$ depending on s, κ. With $0 < \gamma < s < \frac{1}{2}$ and the choices

$$m_0(x) = e^{\kappa \langle x \rangle^\gamma} \langle x \rangle^{\gamma - 4}, \quad m_1(x) = e^{\kappa \langle x \rangle^\gamma}, \quad m_2(x) = e^{\kappa \langle x \rangle^s} \langle x \rangle^{s-4}, \quad m_3(x) = e^{\kappa \langle x \rangle^s},$$

we can apply the Theorem (a tedious computation shows that $\mathcal{L}^* m_2 \le b$) and find a (non-optimal) upper bound of $\Theta(t) \simeq e^{-\kappa t^{\frac{7s}{8(4-\gamma)}}}$. For the (illicit) limit case $s = \gamma = \frac{1}{2}$, this would correspond to $e^{-\kappa t^{\frac{1}{8}}}$. Instead, we can conclude that for all $\epsilon > 0$, there exists $0 < \gamma < s < \frac{1}{2}$ such that $\Theta(t) \lesssim e^{-\kappa t^{\frac{1}{8} - \epsilon}}$. Note that this result is very similar to the one of Example 1.2.7.

C. Coupling in the subgeometric case. The strategy remains the same, except that we now apply step (2) of the coupling method with subgeometric rates. We will not apply the method on the gradient dynamic, but rather refer to Chapter 2 which gives a more involved example of application, with the study of the free-transport equation.

Let us just point out an interesting feature, specific to the subgeometric framework. The rate of convergence between two solutions with equal mass and high regularity can be better than the one towards equilibrium. This was observed in particular by Douc, Fort and Guillin [44] and Hairer [71]. The best way to formulate this is probably to come back to the formulation of Theorem 1.2.4, although this can be expressed in the framework of the coupling method or of Theorem 1.2.5. Then, another result that one can obtain from condition 2 is that, for all $x, y \in E$,

$$H_\phi^{-1}(t)\|\mathcal{P}_t(x,\cdot) - \mathcal{P}_t(y,\cdot)\|_{TV} \to 0,$$

as $t \to \infty$. Since $\frac{\phi(x)}{x} \to 0$ at infinity, this provides a faster convergence than the one towards equilibrium. An example of this is provided in Chapter 4, see in particular Remark 3.

1.3 Contributions

1.3.1 Chapter 2: Study of the convergence of a collisionless gas towards its equilibrium with a coupling method.

In Bernou and Fournier, [10], we use a coupling method to tackle the problem of convergence towards equilibrium of the free-transport equation with Maxwell boundary condition. Let us rewrite the problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0, & (t,x,v) \in \mathbb{R}_+ \times G, \\ f(0,x,v) = f_0(x,v) & (x,v) \in G, \\ f(t,x,v) = (1-\alpha(x)) f(t,x,\eta_x(v)), & \\ \qquad\qquad +\alpha(x)\Big(\int_{\Sigma_+^x} f(t,x,v)(v\cdot n_x)dv\Big) M(x,v), & (t,x,v) \in \mathbb{R}_+ \times \Sigma_-, \end{cases} \tag{1.3.1}$$

where we recall the definition of η from (1.1.5) and that $G = \Omega \times \mathbb{R}^d$. The starting point of our study lies, much like in the papers of Kuo et al. [87, 88, 86] in the construction of a stochastic process $(X_t, V_t)_{t\geq 0}$ whose law is a solution, in an adapted sense, of (1.3.1). Once the process is built, however, the strategy differs widely. Indeed, we consider a general bounded domain Ω, so there is no symmetry on which one can rely to obtain a form of law of large numbers as in [87, 88, 86]. Instead, we use a coupling method to study the convergence towards equilibrium in the total variation distance.

In this work, we consider only the case where M is independent of x. On the other hand, we allow the density M of particles outgoing from the boundary to be quite general, as detailed in the hypothesis below: in particular we do not assume that M is a wall Maxwellian.

The main contributions of [10] are the following.

i. We recover the (almost) optimal rate $\frac{1+\ln^2(t+1)}{(t+1)^d}$ for the case where M is the wall Maxwellian and the initial data f_0 has a bounded density.

ii. We extend the result on the convergence rate to a general regular domain, without symmetry assumption, and detail the influence of both M and the initial data on this rate in a precise manner.

iii. The result is derived in the wider space of probability measures. In particular we introduce a notion of solutions in the sense of measures for (1.3.1), so that we do not need the existence of a probability density function for f_0.

Similar models from probability theory have been studied. First of all, the monokinetic version of our problem, that is the case where M has support in $\{v \in \mathbb{R}^d : \|v\| = 1\}$ is the well-known stochastic billard. Stochastic billards models are obtained when considering a billard taking a new random direction at every collision with the boundary. Several studies proved the exponential convergence towards equilibrium for such models, often with a coupling method, see for instance Evans [55], whose result on straight paths between boundary points will be widely used in this thesis, Comets, Popov, Schütz and Vachkovskaia [31] and the recent work of Fétique [61] in the convex setting. Such stochastic billiards are strongly related to the Markov chain appearing in the so-called *shake-and-bake* algorithms for simulating distributions on the boundary of a convex polytope. They can also be seen as models for a Knudsen gas enclosed in a wall, in the case where the reflection is random, yet energy conservative. Finally, the stochastic process that we will define is a pathological instance of a piecewise deterministic Markov process (PDMP) as introduced by Davis [34], since the jump instants are predictable. The rate of convergence towards equilibrium of PDMPs has been heavily studied recently, see in particular Fontbona, Guérin and Malrieu [56, 57] and the recent work of Durmus, Guillin and Monmarché [50].

1.3.1.1 Hypotheses and main result

We consider an open, connected, bounded set Ω in $\mathbb{R}^d$ with $d \geq 2$ with C^2 regularity at the boundary. We assume that $\alpha \geq \alpha_0$ on $\partial\Omega$ for some constant $\alpha_0 > 0$. Since the temperature is constant at the boundary, the equilibrium distribution is given, for all $(x, v) \in G$, by

$$f_\infty(x, v) = c_\infty \frac{M(v)}{|\Omega|},$$

where $c_\infty = (\int_{\mathbb{R}^d} M(v)dv)^{-1} > 0$ constant and $|\Omega|$ denotes the Lebesgue measure of Ω in $\mathbb{R}^d$, under the assumption that the initial data has mass 1. We assume Hypothesis 1.2.1 on M, and additionally that M is lower bounded by some continuous, radially symmetric function around 0.

We introduce a notion of weak solution in the sense of measures, on which the main result focuses.

Theorem 1.3.1. *Let the initial distribution, f_0 be a probability measure on G. Assume that $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing function such that there exists a constant $C > 0$ satisfying $r(x+y) \leq C(r(x)+r(y))$ for all $x, y \in \mathbb{R}_+$ and such that*

$$\int_{\mathbb{R}^d} r\Big(\frac{1}{\|v\|}\Big) M(v)dv + \int_G r\Big(\frac{1}{\|v\|}\Big) f_0(dx,dv) < \infty. \tag{1.3.2}$$

Then, there exist some constant $\kappa > 0$ and a weak solution $\rho(dt,dx,dv) = dt f_t(dx,dv)$ to (1.3.1) *such that for all $t \geq 0$,*

$$\|f_t - f_\infty\|_{TV} \leq \frac{\kappa}{r(t)}.$$

This weak solution is also unique provided that the initial distribution f_0 admits a density in $L^1(D \times \mathbb{R}^d)$.

In the case where M is the wall Maxwellian, we recover the (almost) optimal rate, as well as a result detailing the influence of the regularity of the initial data.

Corollary 1.3.1. *We take the same hypotheses and notations as in Theorem 1.3.1, and assume furthermore that M is bounded (for instance, M is the wall Maxwellian* (1.1.22) *with $T \equiv 1$).*

a) If f_0 has a bounded density, there exists a constant $\kappa > 0$ such that, for all $t \geq 0$,

$$\|f_t - f_\infty\|_{TV} \leq \frac{\kappa(1+\ln^2(t+1))}{(t+1)^d}.$$

b) If there exists $\beta \in (0,d)$ such that

$$\int_G \frac{1}{\|v\|^\beta} f_0(dx,dv) < \infty,$$

there exists a constant $\kappa > 0$ such that for all $t \geq 0$,

$$\|f_t - f_\infty\|_{TV} \leq \frac{\kappa}{(1+t)^\beta}.$$

The strategy of this study is the following:

i. We introduce a notion of weak solution in the sense of measures.

ii. We build a stochastic process whose law is a weak solution in the sense of measures, and coincide with the unique such solution when f_0 admits a density.

iii. We prove the result in the easier case where Ω is uniformly convex using a coupling method. The boundary condition, which is quite unusual in this stochastic context, leads to non-standard difficulties.

iv. Finally, we extend the result to general domains with C^2 boundary.

1.3.1.2 Stochastic process

In order to build the stochastic process whose law is a solution to (1.3.1), we first need to introduce a law $\mathcal{Q}$. With this distribution, independent of the space point considered, we will obtain the new velocity of a particle hitting the boundary in the case of a diffuse reflection (the stochastic component of the Maxwell boundary condition). We introduce the law Υ of a couple (R, Θ) where R has values in $\mathbb{R}_+$, Θ takes values in $\mathcal{A} := (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi)^{d-2}$ and both variables are independent, as well as a map $\vartheta : \partial\Omega \times \mathcal{A} \to \mathbb{R}^d$ such that, for any $x \in \partial\Omega$, $(R, \Theta) \sim \Upsilon$,

$$R\vartheta(x, \Theta) \sim c_0 M(v)|v \cdot n_x| \mathbf{1}_{\{v \cdot n_x < 0\}}.$$

The idea is the following.

1. The density $c_0 M(v)|v \cdot n_x| \mathbf{1}_{\{v \cdot n_x < 0\}}$ is the one of the outgoing velocity in the case of diffuse reflection at $x \in \partial\Omega$.
2. R is the norm of this outgoing velocity.
3. Θ is a random angle taking values in $\mathcal{A}$.
4. The map ϑ allows one to simulate (R, Θ) independently of the point $x \in \partial\Omega$ where the velocity changes, and to compute a new velocity outgoing from x in the case of the diffuse reflection and based on the values of R and Θ.

Let $\mathcal{Q}$ be the density of the triplet (U, R, Θ) where $U \sim \mathcal{U}_{[0,1]}$ and $(R, \Theta) \sim \Upsilon$. We define the map $w : \partial\Omega \times \mathbb{R}^d \times [0, 1] \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}^d$ by

$$w(x, v, u, r, \theta) = \eta_x(v) \mathbf{1}_{\{u > \alpha(x)\}} + r\vartheta(x, \theta) \mathbf{1}_{\{u \leq \alpha(x)\}}.$$

Recall the definition of τ from (1.2.2) and q from (1.2.3). Set, for any process $(Y_t)_{t \geq 0}$, any $t \geq 0$, $Y_{t-} = \lim_{s \to t, s < t} Y_s$. The stochastic process that we will use is defined in the following way.

Definition 1.3.1. Consider an initial distribution ρ_0 on $G \cup \Sigma_-$, a sequence of i.i.d. random vectors $(U_i, R_i, \Theta_i)_{i \geq 1}$ of law $\mathcal{Q}$. We define the stochastic process $(X_t, V_t)_{t \geq 0}$ as follows:

Step 0: $(X_0, V_0) \sim \rho_0$.

Step 1: Set $T_1 = \tau(X_0, V_0)$.
For $t \in [0, T_1)$, set $V_t = V_0$ and $X_t = X_0 + tV_0$.
Set $X_{T_1} = X_{T_1-}$ and $V_{T_1} = w(X_{T_1}, V_{T_1-}, U_1, R_1, \Theta_1)$.

Step $k+1$: Set $T_{k+1} = T_k + \tau(X_{T_k}, V_{T_k})$.
For $t \in (T_k, T_{k+1})$, set $X_t = X_{T_k} + (t - T_k)V_{T_k}$, $V_t = V_{T_k}$. Set $X_{T_{k+1}} = X_{T_{k+1}-} \in \partial\Omega$ and $V_{T_{k+1}} = w(X_{T_{k+1}}, V_{T_{k+1}-}, U_{k+1}, R_{k+1}, \Theta_{k+1})$.

etc.

We say that $(X_s, V_s)_{s\geq 0}$ is a free-transport process with initial distribution ρ_0.

Let $\rho_0 \in \mathbf{P}(G)$, where for all K, $\mathbf{P}(K)$ denotes the set of probability measures on K. We denote, for all $t \geq 0$, f_t the law of $(X_t, V_t)_{t\geq 0}$ from Definition 1.3.1 with initial distribution ρ_0. We then define the measure ρ on $\mathbb{R}_+ \times \bar{G}$ by

$$\rho(dt, dx, dv) = f_t(dx, dv)dt.$$

One can show that ρ is a weak solution to (1.1.15) in the sense of measures, and also that $t \to f_t(dx, dv)$ is right-continuous from $(0, \infty)$ to $\mathbf{P}(\bar{G})$.

1.3.1.3 Coupling in the convex case

For the sake of conciseness, we present mostly the convex case. In this subsection, Ω is strictly convex. We introduce some more notations.

Notation 1.3.1. We define four maps:

i. the map $\xi : \partial\Omega \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, such that

$$\xi(x, r, \theta) = \tau(x, r\vartheta(x, \theta)),$$

ii. the map $y : \partial\Omega \times \mathcal{A} \to \partial\Omega$, such that

$$y(x, \theta) = q(x, \vartheta(x, \theta)),$$

iii. the map $\tilde{\xi} : \bar{\Omega} \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, such that

$$\tilde{\xi}(x, v, r, \theta) = \tau(x, v) + \tau\Big(q(x, v), r\vartheta(q(x, v), \theta)\Big),$$

iv. the map $\tilde{y} : \bar{\Omega} \times \mathbb{R}^d \times \mathcal{A} \to \partial\Omega$, such that

$$\tilde{y}(x, v, \theta) = q\Big(q(x, v), \vartheta(q(x, v), \theta)\Big).$$

The first idea is to show that one can couple two free-transport processes "in one step" using the strict convexity, i.e. that starting from a configuration with two free-transport processes from Definition 1.3.1, $(X_t, V_t)_{t\geq 0}$ and $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ at time $t \geq 0$ with $(X_t, V_{t-}) \in \Sigma_+$, $\tilde{X}_t \in \Omega$ and $\|\tilde{V}_t\| \geq 1$, we can couple six random variables $(U, R, \Theta, \tilde{U}, \tilde{R}, \tilde{\Theta})$ in such a way that, letting $T = t + \tau(X_t, V_t)$, we have

$$\mathbb{P}(X_T = \tilde{X}_T) \geq c,$$

for some constant $c > 0$ independent of the configuration at time t. With the previous notations, this corresponds to the following result.

Proposition 1.3.1. *There exists a constant $c > 0$ such that for all $x_0 \in \partial\Omega$, $\tilde{x}_0 \in \Omega$, $\tilde{v}_0 \in \mathbb{R}^d$ with $\|\tilde{v}_0\| \geq 1$, there exists $\Lambda_{x_0,\tilde{x}_0,\tilde{v}_0} \in \mathbf{P}(((0,\infty) \times \mathcal{A})^2)$ such that, if $(R, \Theta, \tilde{R}, \tilde{\Theta})$ has law $\Lambda_{x_0,\tilde{x}_0,\tilde{v}_0}$, both (R, Θ) and $(\tilde{R}, \tilde{\Theta})$ have law Υ, and for*

$$E_{x_0,\tilde{x}_0,\tilde{v}_0} := \left\{(r, \theta, \tilde{r}, \tilde{\theta}) \in (\mathbb{R}_+ \times \mathcal{A})^2 : y(x_0, \theta) = \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\theta}), \xi(x_0, r, \theta) = \tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{r}, \tilde{\theta})\right\},$$

we have

$$\mathbb{P}((R, \Theta, \tilde{R}, \tilde{\Theta}) \in E_{x_0,\tilde{x}_0,\tilde{v}_0}) \geq c.$$

The final coupling is built with the help of this law Λ, an elementary coupling for the $(U, \tilde{U})$, and the addition of a control variable $(Z_t)_{t\geq 0}$ allowing us to keep track of the correlations. We omit here the tedious details of this construction. Ultimately, we obtain a process $(X_t, V_t, \tilde{X}_t, \tilde{V}_t, Z_t)_{t\geq 0}$ which is strong Markov, such that $(X_t, V_t)_{t\geq 0}$ and $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ are both free-transport processes with initial distribution ρ_0 and f_∞ respectively, and satisfying, for all $t \geq 0$,

$$\left\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\right\} \subset \left\{(X_{t+s}, V_{t+s})_{s\geq 0} = (\tilde{X}_{t+s}, \tilde{V}_{t+s})_{s\geq 0}\right\}.$$

Moreover, we can show that, introducing

$$T_C := \inf\{t \geq 0, (X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\},$$

and recalling the definition of r from Theorem 1.3.1,

$$\mathbb{E}[r(T_C)] < \infty.$$

The conclusion then follows from the usual inequality linking total variation and coupling (1.2.14).

Remark 1.3.1. Following our presentation of the various methods existing for studying the rate of convergence towards equilibrium of a Markov process, it is worth making the following point. This result on $\mathbb{E}[r(T_C)]$ relies heavily on the fact that a free-transport process $(X'_t, V'_t)_{t\geq 0}$ comes back to the situation with $\|V'_t\| \geq 1$ sufficiently often. In a very loose sense one might understand the set $\{(x,v) \in G : \|v\| \geq 1\}$ as a petite set, although we do not show any rigorous result towards this direction, nor use the structure of petite set.

1.3.1.4 Extension to the general case

The strategy for the general case where Ω is no longer strictly convex is similar, yet requires suitable modifications of Proposition 1.3.1. We prove the existence of a subset $F \subset \partial\Omega$ of the boundary having positive Hausdorff measure in $\partial\Omega$, such that a similar result holds.

Proposition 1.3.2. *There exists a constant $c > 0$ such that for all $x_0 \in F$, $\tilde{x}_0 \in \Omega$, $\tilde{v}_0 \in \mathbb{R}^d$ with $\|\tilde{v}_0\| \geq 1$ and $q(\tilde{x}_0, \tilde{v}_0) \in F$, there exists $\Lambda_{x_0,\tilde{x}_0,\tilde{v}_0} \in \mathbf{P}(((0,\infty) \times \mathcal{A})^2)$ such that if $(R, \Theta, \tilde{R}, \tilde{\Theta})$ has law $\Lambda_{x_0,\tilde{x}_0,\tilde{v}_0}$, $(R,\Theta) \sim \Upsilon$, $(\tilde{R}, \tilde{\Theta}) \sim \Upsilon$ and for $E_{x_0,\tilde{x}_0,\tilde{v}_0}$ defined by*

$$E_{x_0,\tilde{x}_0,\tilde{v}_0} := \left\{(r,\theta,\tilde{r},\tilde{\theta}) \in (\mathbb{R}_+ \times \mathcal{A})^2 : y(x_0,\theta) = \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\theta}), \xi(x_0, r, \theta) = \tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{r}, \tilde{\theta})\right\},$$

we have

$$\mathbb{P}((R, \Theta, \tilde{R}, \tilde{\Theta}) \in E_{x_0,\tilde{x}_0,\tilde{v}_0}) \geq c.$$

In this context, we need not only for a free-transport process $(X'_t, V'_t)_{t\geq 0}$ to come back sufficiently often to the space $\{(x,v) \in G, \|v\| \geq 1\}$ as mentioned in Remark 1.3.1, but rather some recurrence result on the set $\{(x,v) \in G,\, q(x,v) \in F, \|v\| \geq 1\}$. To obtain this, we use results established by Evans [55], which tell us that for all Ω with C^1 regularity, there exist $N \geq 1$ and a finite set Δ_0 of boundary points $x_1, \dots, x_N$ such that, for any $(y,z) \in (\partial\Omega)^2$, writing $a \leftrightarrow b$ if $ta + (1-t)b \in \Omega$ for all $t \in (0,1)$, we have, for some $r \in \{1,\dots,N\}$, for $(x_{i_j})_{j=1,\dots,r}$ a subset of Δ_0,

$$y \leftrightarrow x_{i_1}, x_{i_1} \leftrightarrow x_{i_2}, \dots, x_{i_r} \leftrightarrow z.$$

As a consequence, we show that there exists $c_2 > 0$ constant such that, for all $x \in \partial\Omega$, letting $(X'_t, V'_t)_{t\geq 0}$ be a free-transport process starting from x and setting $T_0 = 0$ and for $i \geq 0$, $T_{i+1} = \inf\{t > T_i, X'_t \in \partial\Omega\}$,

$$\mathbb{P}(X'_{T_{N+1}} \in F) \geq c_2.$$

From there, the strategy is a rather direct adaptation from the one in the convex setting.

1.3.2 Chapter 3: Numerical simulations of the free-transport equation

The construction of the stochastic process associated to the problem (1.3.1) gives a starting point for computing numerical simulations of the convergence of the model with a particle system, rather than with a study of the entropy function as in Tsuji, Aoki and Golse [119]. In Chapter 3, we compute such numerical simulations in order to check the behavior of the rate of convergence towards equilibrium. The main points of this chapter are the following.

1. We give strong evidences in favour of the qualitative result stating that the convergence towards equilibrium is polynomial rather than exponential.

2. We run some computations in a non-radial domain: a star-shaped domain.

3. We go slightly beyond the physical case of the wall Maxwellian by considering various distributions M putting less or more weights on the small (in terms of Euclidian norm) velocities. As predicted, this modifies the observed rate of convergence.

On the other hand, this method of simulation exhibits some drawbacks. The main one is the following: the results of Chapter 2 are given in terms of the total variation distance between two measures. However, this distance is difficult to compute in general, in particular when the size of the sample becomes large. For this reason, we have to use an observable of the system (i.e. the mean value of a certain function of the sample), but the rate might differ from the theoretical one for the total variation distance. As a result, we only give qualitative conclusions, and can not give a definitive answer regarding the exact value of the exponent of the polynomial rate of convergence.

Let us present briefly some of the results. First, in the unit disk, we can look at the distribution of the first velocity coordinate, Figure 1.6 to see some qualitative convergence towards the equilibrium distribution. In the following discussion we consider an initial data given by

$$f_0(x,v) = \mathcal{U}_{B(0,1)}(dx)\mathcal{N}_2(0_2, 0.01 I_2)(dv), \qquad x \in B(0,1), v \in \mathbb{R}^2, \tag{1.3.3}$$

where $\mathcal{U}_{B(0,1)}$ is the uniform law in the unit disk.

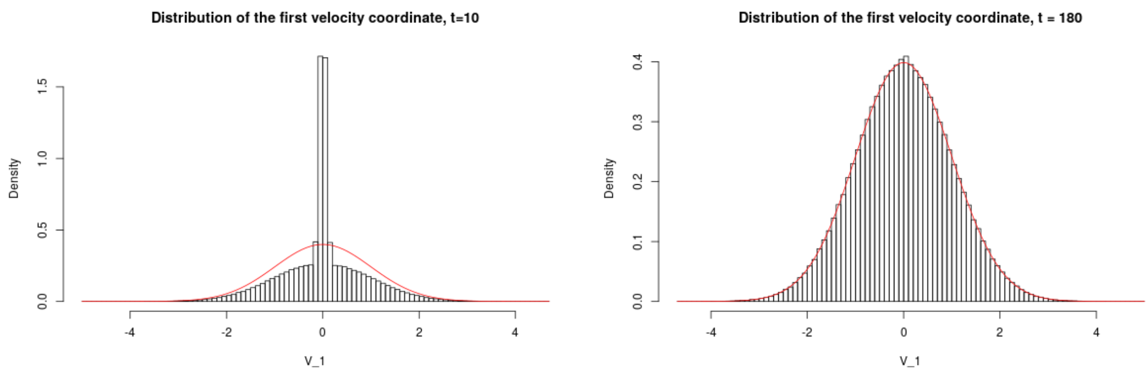

Fig. 1.6 Distribution of the first velocity coordinate at time $t = 10$ and $t = 180$ in the case where the initial data is given by (1.3.3).

Another way to write the total variation (1.2.9) between two measures μ and ν on a measurable space $(E, \mathcal{E})$ is given by

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{\phi : E \to [-1,1]} \Big| \int \phi d\mu - \int \phi d\nu \Big|.$$

The naive estimator of this quantity exhibits too much noise. Hence, to compute our estimator, we rather use a function $\phi : E \to [-1, 1]$ that we find empirically. We thus let, for all $(x, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\phi(x, v) = \|x\|^4 + \|v\|^2$, simulate $(X_i^t, V_i^t)_{1 \le i \le n}$ i.i.d. of law f_t, $(\tilde{X}_i, \tilde{V}_i)_{1 \le i \le n}$ of law f_∞ and use the following estimator

$$\hat{\epsilon}_n(f_t, f_\infty) = \frac{1}{2} \Big| \frac{1}{n} \sum_{i=1}^{n} \big(\phi(X_i^t, V_i^t) - \phi(\tilde{X}_i, \tilde{V}_i) \big) \Big|.$$

In what follows, we call this the ϕ-estimate. We also consider a different test function,

$$\phi_2(x, v) = \sqrt{\|x\|} + \sqrt{\|v\|}, \qquad x \in \mathbb{R}^2, v \in \mathbb{R}^2,$$

We call the corresponding estimate the ϕ_2-estimate, with the obvious definition.

We only present two results for the sake of conciseness. In both cases the simulations are performed with 10^6 particles, and with pure diffusive boundary condition, i.e. $\alpha \equiv 1$ in (1.3.1). First, in the most standard situation where M is given by (1.1.22) with temperature $T \equiv 1$ and f_0 is given by (1.3.3), we obtain a graph that strongly indicates the convergence at a polynomial rate, although the exponent is far from the theoretical one.

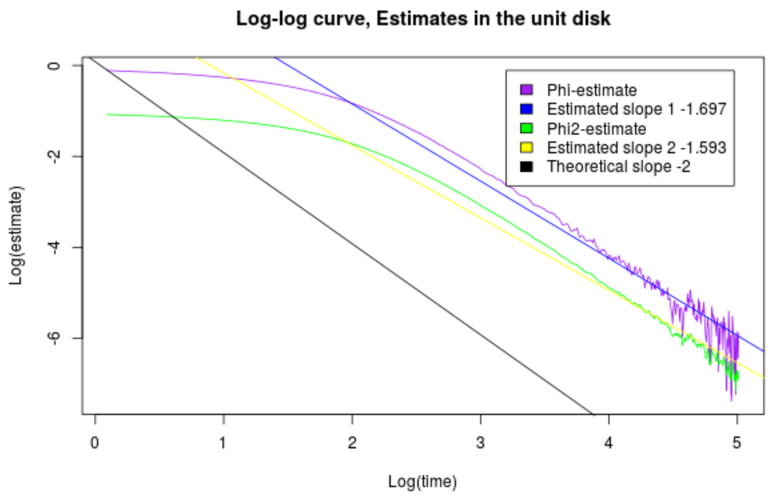

Fig. 1.7 Log-log convergence of the ϕ-estimate and the ϕ_2-estimate with f_0 given by (1.3.3) and M given by (1.1.22) with $T \equiv 1$.

For the next result, we introduce a generalized reflection law given, for $\theta > 0$, $a \in (0,3)$, by

$$M_{a,\theta}(v) = \frac{e^{-\frac{\|v\|^{\frac{2}{a}}}{2\theta}}}{a\pi(2\theta)^{\frac{3}{2}-\frac{a}{2}}} \frac{\|v\|^{\frac{3}{a}-3}}{\Gamma(\frac{3}{2}-\frac{a}{2})}, \qquad v \in \mathbb{R}^2. \tag{1.3.4}$$

This is a modification of the wall Maxwellian with more weight on small velocities, in the case $a > 1$, and a smaller weight when $a < 1$. When $a = 1$, we recover the usual wall Maxwellian with variance θ. The idea is that, considering $M_{a,1}$ instead of M in the boundary condition, we will be able to test a prediction of Theorem 1.3.1, which is that the rate of convergence, for a well-chosen initial data, should be different in this case. In particular, a computation shows that one could expect a rate of order $\frac{1}{t}$ in the case where $a = 1.5$ and $f_0 = \frac{M_{1.5,0.01}}{|\Omega|}$, where $|\Omega|$ denotes the volume of Ω. Testing this prediction in the unit disk, with the estimators mentioned above, we obtain Figure 1.8.

As one can see the rate is again polynomial, and has been largely deviated towards the expected value, although it remains quite different from it. Again we suffer from the fact that our estimators are poor proxies of the total variation distance.

In Chapter 3 we also present similar results for a star-shaped domain, for which there is no radial symmetry, in order to confirm our result of Chapter 2 that the absence of such symmetry should not modify the rate of convergence towards equilibrium.

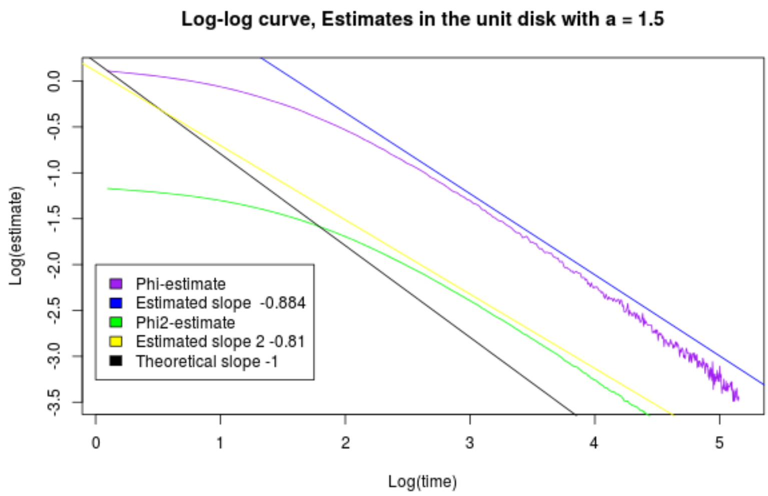

Fig. 1.8 Log-log convergence of the ϕ-estimate and the ϕ_2-estimate in the case $a = 1.5$, with $f_0 = \frac{M_{1.5,0.01}}{|\Omega|}$.

1.3.3 Chapter 4: Deterministic subgeometric Harris' theorem and application to the study of a collisionless gas

In [8], we investigate again (1.3.1), using this time a deterministic method. The result of this paper is also useful in collisional kinetic theory, since this provides new properties of the semigroup associated with the free-transport operator completed with the Maxwell or pure diffusive boundary condition.

The contributions of this chapter are as follows.

i. We obtain the (almost) optimal decay rate $\frac{\ln(1+t)^{d+1}}{(1+t)^d}$ in the L^1 norm for the convergence towards equilibrium of the solution to (1.3.1), with weak hypotheses on the accommodation coefficient α, and in a general C^2 domain. In particular we consider the case where $M = M(x,v)$ is a wall Maxwellian whose temperature varies at the boundary, see (1.1.22).

ii. We obtain a slightly better rate $\frac{\ln(1+t)^{d+2}}{(1+t)^{d+1}}$ for the convergence towards zero of a difference between two solutions with regular initial data of equal masses. This non-equality of the two rates of convergence is usual when one studies subexponential rates of convergence, see §1.2.3.4.

iii. We compare the model with the case of the free-transport equation with an absorbing boundary condition, i.e. with the problem given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f &= 0, & (t,x,v) \in \mathbb{R}_+^* \times G, \\ f(t,x,v) &= 0, & (t,x,v) \in \mathbb{R}_+^* \times \Sigma_-, \\ f_{|t=0}(x,v) &= f_0(x,v), & (x,v) \in G, \end{cases} \tag{1.3.5}$$

and obtain an exponential rate of convergence for very regular initial data, yet a similar rate for typical initial data, e.g. f_0 with a Maxwellian distribution in velocity.

iv. An important point is that the constants associated to the rates of convergence in those results may be constructive, in the sense that for some domains, it is possible to obtain explicit values.

Hypotheses and main results. We assume that $\alpha : \partial\Omega \to [0,1]$ is such that there exists $c_0 \in (0,1)$ with

$$\alpha(x) \geq c_0, \quad \forall x \in \partial\Omega.$$

This assumption is quite natural, as the specular reflection provides no mixing. Then, we introduce, for all $f \in L^1(G)$, the mean of f, defined by

$$\langle f \rangle = \int_G f(x,v)dvdx.$$

Recall that we denote $L^1_w(G)$ the weighted L^1 space with weight $w : G \to [1, \infty)$, $\| \cdot \|_w$ the corresponding norm.

Since the problem is well-posed, see Arkeryd and Cercignani [2], we may associate to the equation a strongly continuous semigroup $(S_t)_{t\geq 0}$ of linear operators, such that, for $f_0 \in L^1(G)$, for all $t \geq 0$, $S_t f_0 = f(t, \cdot)$ is the unique solution in $L^\infty([0, \infty); L^1(\bar{G}))$ to (1.3.1). This is a stochastic semigroup in the sense of Definition 1.2.11, and as such it preserves mass, is a contraction in the large sense, so that $\|S_t f\|_{L^1} \leq \|f\|_{L^1}$, and satisfies the positivity property. The results on the convergence will be presented at the level of this semigroup.

We introduce some weights ω_i, $i \geq 1$ on G such that, for some constant $\kappa > 0$

$$\omega_i(x, v) \sim \left(e^2 + \frac{\kappa}{\|v\|}\right)^i \ln\left(e^2 + \frac{\kappa}{\|v\|}\right)^{-1.6},$$

when $v \to 0$, see [8] for the precise definition, which uses crucially the existence of c_0. Our first result focuses on the difference between two solutions, both starting from initial data with enough regularity.

Theorem 1.3.2. *There exists a constant $C > 0$ such that for all $t \geq 0$, for all $f, g \in L^1_{\omega_{d+1}}(G)$ with $\langle f \rangle = \langle g \rangle$, there holds*

$$\|S_t(f - g)\|_{L^1} \leq C \frac{\ln(1+t)^{d+2}}{(1+t)^{d+1}} \|f - g\|_{\omega_{d+1}}.$$

The proof of Theorem 1.3.2 allows one to recover easily the result of existence of an equilibrium, even in the case where the temperature varies at the boundary. Let f_∞ be the equilibrium with mass 1. To obtain the result on the convergence rate towards f_∞, we use an interpolation argument and Theorem 1.3.2. To this end, we introduce the weights $(m_i)_{i\geq 1}$ satisfying, for some constant $\kappa > 0$,

$$m_i(x, v) \sim \left(e^2 + \frac{\kappa}{\|v\|}\right)^i \ln\left(e^2 + \frac{\kappa}{\|v\|}\right)^{-1.6\frac{d}{d+1}},$$

when $v \to 0$.

Corollary 1.3.2. *There exists a constant $C' > 0$ such that for all $t \geq 0$, for all $f \in L^1_{m_d}(G)$ with $\langle f \rangle = 1$,*

$$\|S_t(f - f_\infty)\|_{L^1} \leq \frac{C' \ln(1+t)^{d+1}}{(1+t)^d} \|f - f_\infty\|_{m_d}.$$

Recall the definition of τ, see (1.2.2). The result on the free-transport equation with absorbing boundary condition is as follows.

Theorem 1.3.3. *For any $f \in L^1_m(G)$, $t \geq 0$,*

$$\|S_t f\|_{L^1} \leq \Theta(t) \|f\|_m,$$

with the following choices

i. $m(x,v) = e^{\tau(x,v)}$ *in* $\bar{\Omega} \times \mathbb{R}^d$*, in which case* $\Theta(t) = e^{-t}$.

ii. $m(x,v) = (1+\tau(x,v))^\nu$ *in* $\bar{\Omega} \times \mathbb{R}^d$*, with* $\nu > 1$*, in which case* $\Theta(t) = \frac{1}{(t+1)^\nu}$.

Overview of the strategy. The strategy for the proof of Theorem 1.3.2 is adapted from the deterministic subgeometric Harris' theorem introduced by Cañizo and Mischler [21] and presented in §1.2.3.4. We rewrite first the problem as a Cauchy problem for some operator $\mathcal{L}$:

$$\begin{cases} \partial_t f &= \mathcal{L} f, \qquad \text{in } \bar{\Omega} \times \mathbb{R}^d, \\ f(0,\cdot) &= f_0(\cdot), \qquad \text{in } G. \end{cases}$$

a. Lyapunov and interpolation conditions. We clearly have some interpolation relations inside our two families of weights $(\omega_i)_{i \geq 1}$ and $(m_i)_{i\geq 1}$. Because the whole dissipative behavior of the model is due to the action of the boundary operator, it seems very difficult to obtain the usual inequality required in Theorem 1.2.5

$$\mathcal{L}^* \omega_{d+1} \leq -\omega_d + \delta, \tag{1.3.6}$$

for some constant $\delta > 0$, where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$, and the corresponding one for the other weights. On the other hand, a key property of the model is that

$$v \cdot \nabla_x \tau(x,v) = -1 \quad \text{ in } G,$$

which allows one to obtain an integrated version of (1.3.6) that captures this dissipative property: there exist $C_1, b_1 > 0$ constants such that for all $T > 0$, $f \in L^1_{\omega_{d+1}}(G)$,

$$\|S_T f\|_{\omega_{d+1}} + C_1 \int_0^T \|S_s f\|_{\omega_d} ds \leq \|f\|_{\omega_{d+1}} + b_1(1+T)\|f\|_{L^1}, \tag{1.3.7}$$

as well as similar inequalities for other couples of weights.

b. Doeblin-Harris (or Doeblin) condition. We prove, by following the characteristics of (1.3.1) backward, that there exists $R_0 > 0$ such that for all $R > R_0$, there exist $T(R) > 0$ and a non-negative measure ν on $\bar{\Omega} \times \mathbb{R}^d$ with $\nu \not\equiv 0$ such that for all $(x,v) \in G$,

$$S_{T(R)} f(x,v) \geq \nu(x,v) \int_{\{(y,w)\in\Omega\times\mathbb{R}^d : \tau(y,w) \leq R\}} f(y,w) dw dy. \tag{1.3.8}$$

The form of ν is the key point regarding whether the constants associated to the rates of convergence are constructive. For some domains Ω, explicit formulas for ν can be derived, from which the constructivity follows.

Idea of the proof. To derive the proof of Theorem 1.3.2 from the two previous conditions, we assume without loss of generality that $g = 0$ and that $f \in L^1_{\omega_{d+1}}(G)$ with $\langle f \rangle = 0$. We fix $T > 0$ large enough and introduce a modified norm

$$|||\cdot|||_{\omega_{d+1}} := \| \cdot \|_{L^1} + \beta \| \cdot \|_{\omega_{d+1}} + \alpha \| \cdot \|_{\omega_d},$$

for two well-chosen constants $\alpha, \beta > 0$ depending on T. Using the Lyapunov condition (1.3.7) and the Doeblin-Harris condition (1.3.8), we show that

$$|||S_T f|||_{\omega_{d+1}} \leq |||f|||_{\omega_{d+1}}. \tag{1.3.9}$$

We then introduce some further weights w_0, w_1 such that $1 \leq w_0 \leq w_1 \leq \omega_{d+1}$. With a similar argument, we find that, for some modified norm $|||\cdot|||_{w_1}$, for T as above and some $\tilde{\alpha} > 0$ well chosen,

$$|||S_T f|||_{w_1} + 2\tilde{\alpha} \|f\|_{w_0} \leq |||f|||_{w_1}. \tag{1.3.10}$$

We then combine the two inequalities (1.3.9) and (1.3.10) along with the inequalities satisfied by the weights to conclude.

The proof of Theorem 1.3.3 is a relatively straightforward application of Grönwall's lemma.

1.3.4 Chapter 5: A version of Theorem 1.2.4 with equivalent conditions

This chapter focuses on the subgeometric convergence towards equilibrium of Markov processes. As mentioned above, one of the key results of the exponential theory is the equivalence of two conditions, both implying an exponential convergence towards equilibrium: the control of the moment of the hitting time of a set with appropriate properties and the existence of some test function satisfying a Foster-Lyapunov inequality with respect to the generator, see Theorem 1.2.2 above, and in particular the equivalence between the two conditions.

On the other hand, the two conditions in Theorem 1.2.4 are not equivalent. An interesting problem, already pointed by Fort and Roberts [59] in their study of the polynomial case, is then to obtain some sort of equivalence between a Foster-Lyapunov type condition, as 2. in Theorem 1.2.4, and the hitting time of some petite set, as condition 1.

This chapter presents a new Foster-Lyapunov type condition, with the important difference that the functional built here also depends on time, as well as a new hitting time condition, which are equivalent and imply the result. There are two main contributions.

1. In a first part, we write, with extended details, the theorem and the proof of convergence at a subgeometric rate from a Foster-Lyapunov condition holding for a function V whose sublevel sets are petite, as presented by Hairer [71]. In particular we complete slightly the statement with necessary innocuous hypothesis and detail several points of the proof.

2. In a second part, we present our new conditions, logical relations between them and with the condition of [71], and we show that they indeed imply the final result.

On the first of those point, let us mention, in connection with §1.2.3.4, the interesting fact that the proof follows from a coupling argument. Let us present the theorem and the key ideas informally. Assume that, for some $V : E \to [1, \infty)$ with precompact sublevel sets,

$$\mathcal{L}V \leq -\phi(V) + K,$$

with $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing, strictly concave function (and additionnal innocuous properties), and $K > 0$ a constant. Assume also that, for all $\kappa > 0$ there exist $T > 0$ and $\alpha \in (0,1)$ such that for all x, y such that $V(x) + V(y) \leq \kappa$,

$$\|\mathcal{P}_T(x,\cdot) - \mathcal{P}_T(y,\cdot)\|_{TV} \geq 2(1-\alpha). \tag{1.3.11}$$

Firstly, we show that this set of hypotheses is enough to imply the existence of an invariant probability measure thanks to Bogolyoubov' criteria. The proof of convergence at a subgeometric rate follows by building a function ψ_2 on $[0,\infty) \times E \times E$ for which we can identify a compact, petite set

$$\mathcal{K} := \{(x,y) \in E^2, V(x) + V(y) \leq \kappa\} \subset E^2,$$

with $\kappa > 0$ constant, such that ψ_2 decreases outside of $\mathcal{K}$ and $\psi_2(0,\cdot,\cdot)$ is bounded on $\mathcal{K}$. Write $T > 0$ for the constant associated to $\mathcal{K}$ in (1.3.11). We use a coupling argument based on the following ideas: we introduce two processes $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$. We let $(T_n)_{n\geq 1}$ be (roughly) the hitting times of $\mathcal{K}$ for $(X_t, \tilde{X}_t)_{t\geq 0}$. For all $n \geq 1$, we try to couple $(X_{T_n+s})_{s\in(0,T]}$ and $(\tilde{X}_{T_n+s})_{s\in(0,T]}$ using (1.3.11) on $\mathcal{K}$. If this fails, the properties of $\psi_2(0,\cdot,\cdot)$ allow us to prove that $T_{n+1} \geq T_n + T$ is controlled, so that we will be able to use the petiteness assumption again. On the other hand, if $X_{T_n+T} = \tilde{X}_{T_n+T}$, the coupling is built so that the two processes will remain equal for ever.

For the second problem, there are two main ideas.

a) To modify the stopping time related to the petite set C considered in the first condition. Usually, the condition is given in terms of a delayed hitting time defined for $\delta > 0$ by

$$\tau_C(\delta) = \inf\{t > \delta, X_t \in C\},$$

but this stopping time interacts poorly with the generator.

b) Introducing a new Foster-Lyapunov condition which involves time, and interacts well with the stopping time considered to solve the first issue. Of course this condition has to be compatible with the previous one, in the sense that it should imply or be implied by it.

The final statement is the following.

Theorem 1.3.4. *Assume that $(X_t)_{t\geq 0}$ is strong Markov, non-explosive, irreducible and aperiodic. Let $\phi : [1,\infty) \to \mathbb{R}_+^*$ be a C^1 function, strictly increasing, strictly concave with $\phi(x) \leq x$ for all $x \geq 1$ and $\frac{\phi(x)}{x} \downarrow 0$, $\phi(x) - x\phi'(x) \uparrow \infty$ when $x \to \infty$. Define the function $H_\phi(\cdot)$ on $[1,\infty)$ by*

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)},$$

and let $H_\phi^{-1} : [0,\infty) \to [1,\infty)$ be its inverse function. Consider the three following conditions.

1. *There exist a compact petite subset C of E and some $r > 0$ such that, for $\tilde{\tau}_C^r$ defined by*

$$\tilde{\tau}_C^r = \inf\Big\{t > 0, \int_0^t \mathbf{1}_C(X_s)ds \geq \frac{T}{r}\Big\},$$

where T is an exponential random variable with parameter 1 independent of everything else, we have

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty \quad \text{for all } x \in E \quad \text{and} \quad \sup_{x\in C} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty. \tag{1.3.12}$$

2. *There exist a compact petite subset C of E, two constants $\kappa, \eta > 0$ and a function $\psi : \mathbb{R}_+ \times E \to [1,\infty)$, continuous and non-decreasing in its first argument, continuous in its second argument, such that for all $t \geq 0$, $x \in E$,*

$$H_\phi^{-1}(t) \leq \psi(t,x) \qquad \text{and} \quad (\partial_t + \mathcal{L})\psi(t,x) \leq \kappa H_\phi^{-1}(t)\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)),$$

with moreover $\psi(0,x) \leq \kappa$ for all $x \in C$ and for all $x \in E$, $\mathcal{L}\psi(0,x) \leq \kappa \mathbf{1}_C(x) - \eta$.

3. *There exist a compact petite subset C of E, a constant $K > 0$ and $V : E \to [1,\infty)$ continuous with precompact sublevel sets such that for all $x \in E$,*

$$\mathcal{L}V(x) \leq -\phi(V(x)) + K\mathbf{1}_C(x).$$

Conditions 1. and 2. are equivalent, and both are implied by Condition 3. Moreover, in those three cases, there exists an invariant probability measure π for $(\mathcal{P}_t)_{t\geq 0}$ on E and for all $x \in E$,

$$\lim_{t\to\infty} \phi(H_\phi^{-1}(t))\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

We see in this statement that, for the idea a) mentioned above, the solution is to consider a randomized stopping time $\tilde{\tau}_C^r$ rather than $\tau_C(\delta)$. Because of this additional randomness in the definition of $\tilde{\tau}_C^r$, and of the flexibility that the presence of r in the definition allows, we will be able to obtain some well-behaved interaction with the generator: condition 1 implies condition 2, while still retaining the implication of the final result.

For the idea b), there are several results.

i. The new condition 2. implies the control of $\tilde{\tau}_C^r$. This relies heavily on the flexibility offered by introducing the parameter r.

ii. The new condition 2. implies the existence of an invariant probability measure and the convergence towards it at the desired rate. This is based on a control of $\tau_C(\delta)$ for some $\delta > 0$ that is deduced quite easily from the inequalities.

iii. The third condition, the original Foster-Lyapunov inequality, implies the new one. We use an idea which is reminiscent of the proof that condition 3 implies the result by setting, for all $(t,x) \in \mathbb{R}_+ \times E$,
$$\psi(t,x) = 2H_\phi^{-1}(H_\phi(V(x)) + t) - H_\phi^{-1}(t),$$
and the result follows quite easily.

1.3.5 Chapter 6: Hypocoercivity with general Maxwell boundary conditions

In this chapter, which corresponds to an article in preparation in collaboration with Kléber Carrapotoso, Stéphane Mischler and Isabelle Tristani, we study linear kinetic equations with general Maxwell boundary conditions i.e. both cases $\alpha \equiv 0$ (specular reflection boundary condition) and $\alpha \equiv 1$ (pure diffusive boundary condition) are allowed.

1.3.5.1 Hypotheses and main result

We consider a linear kinetic equation of the form

$$\partial_t f = \mathcal{L} f = -v \cdot \nabla_x f + \mathcal{S} f, \quad f = f(t,x,v), \quad (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \tag{1.3.13}$$

with the Maxwell boundary condition (1.1.20). Here $\Omega \subset \mathbb{R}^d$ is a regular bounded domain (open, connected) of $\mathbb{R}^d$, $d \geq 2$. We assume that there exists a regular vector field n on $\bar{\Omega}$ such that for all $x \in \partial\Omega$, n_x coincides with the unit outward normal vector at x. We complete the equation with the Maxwell boundary condition (1.1.20) in the large sense where $\alpha : \partial\Omega \to [0,1]$ with $M \equiv M_1$ the wall Maxwellian of temperature 1, see (1.1.22), independent of x. For simplicity, we rather write the Maxwell boundary condition as

$$f_{|\Sigma_-}(t,x,v) = \mathcal{R}(f_{|\Sigma_+}(t,\cdot,\cdot))(x,v) \qquad \text{on } (0,\infty) \times \Sigma_-,$$

with

$$(\mathcal{R}g)(x,v) = (1-\alpha(x))g(x,\eta_x(v)) + \alpha(x) c_\mu \mu(v) \tilde{g}(x), \qquad (x,v) \in \Sigma_-,$$

where η_x is given by (1.1.5),

$$\tilde{g}(x) := \int_{\mathbb{R}^d} g(x,v)|v \cdot n_x| dv,$$

is the flux at $x \in \partial\Omega$, $g : \Sigma_+ \to \mathbb{R}$ is any function, $\mu(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{\frac{d}{2}}}$ and c_μ is the positive normalization constant so that $\widetilde{c_\mu \mu}(x) = 1$ with the previous notation (of course this quantity is actually independent of x).

We make several hypotheses on the collisional operator $\mathcal{S}$. Those assumptions cover the cases of the linearized operators associated to Boltzmann with angular cutoff operators for hard potentials and Maxwell molecules, Boltzmann without angular cutoff and Landau operators for hard potentials, Maxwell molecules and moderately soft potentials. We refer to §1.1.2.6 for the corresponding definitions. We assume that the operator acts locally in time and position, namely

$$(\mathcal{S}f)(t,x,v) = \mathcal{S}(f(t,x,\cdot))(v).$$

We introduce the Hilbert space

$$L^2_v(\mu^{-1}) := \left\{ f : \mathbb{R}^d \to \mathbb{R} \,\Big|\, \int_{\mathbb{R}^d} f^2 \mu^{-1} \, dv < +\infty \right\}$$

endowed with the scalar product

$$(f,g)_{L^2_v(\mu^{-1})} := \int_{\mathbb{R}^d} f g \mu^{-1} \, dv \qquad \text{and the norm} \qquad \|f\|^2_{L^2_v(\mu^{-1})} := \int_{\mathbb{R}^d} f^2 \mu^{-1} \, dv.$$

We assume that the operator $\mathcal{S}$ satisfies the following on $L^2_v(\mu^{-1})$.

(i) Its kernel is given by

$$\ker(\mathcal{S}) = \mathrm{Span}\{\mu, v_1\mu, \dots, v_d\mu, |v|^2\mu\},$$

and we denote by πf the projection onto $\ker(\mathcal{S})$ given by

$$\pi f = \left(\int_{\mathbb{R}^d} f \, dw \right) \mu + \left(\int_{\mathbb{R}^d} w f \, dw \right) \cdot v\mu + \left(\int_{\mathbb{R}^d} \frac{(|w|^2 - d)}{\sqrt{2d}} f \, dw \right) \frac{(|v|^2 - d)}{\sqrt{2d}} \mu. \quad (1.3.14)$$

(ii) The operator is self-adjoint and negative $(\mathcal{S}f, f)_{L^2_v(\mu^{-1})} \leq 0$, so that its spectrum is included in $\mathbb{R}_-$. We assume further that $\mathcal{S}$ satisfies a degenerated coercivity estimate: there is a positive constant $\lambda > 0$ such that for any $f \in \mathrm{Dom}(\mathcal{S}) \cap L^2_v(\mu^{-1})$ one has

$$(-\mathcal{S}f, f)_{L^2_v(\mu^{-1})} \geq \lambda \|f^\perp\|^2_{L^2_v(\mu^{-1})}, \quad (1.3.15)$$

where $f^\perp = f - \pi f$.

(iii) For any polynomial function $\phi = \phi(v) : \mathbb{R}^d \to \mathbb{R}$ of degree ≤ 4, one has

$$\left| \int_{\mathbb{R}^d} \phi(v) f^{\perp} \, dv \right| \leq C \|f^{\perp}\|_{L^2_v(\mu^{-1})}$$

and

$$(\mathcal{S} f^{\perp}, \phi\mu)_{L^2_v(\mu^{-1})} = (f^{\perp}, \mathcal{S}(\phi\mu))_{L^2_v(\mu^{-1})} \leq C \|f^{\perp}\|_{L^2_v(\mu^{-1})}.$$

We also introduce the Hilbert space

$$\mathcal{H} = L^2_{x,v}(\mu^{-1}) := \Big\{ f : G \to \mathbb{R} \Big| \int_G f^2 \mu^{-1} dv dx < \infty \Big\},$$

and write $(\cdot, \cdot)$ for the associated scalar product, $\|\cdot\|_{\mathcal{H}}$ for the associated norm. For all $f \in \mathcal{H}$, we decompose $f = \pi f + f^{\perp}$, where the macroscopic part πf is given by (1.3.14). The (normalized) macroscopic quantities of interests are the mass, momentum and energy defined respectively by

$$\rho(x) = \int_{\mathbb{R}^d} f(x,v) dv, \quad m(x) = \int_{\mathbb{R}^d} v f(x,v) dv, \quad \theta(x) = \int_{\mathbb{R}^d} \frac{|v|^2 - d}{\sqrt{2d}} f(x,v) dv.$$

Note that

$$\|f\|^2_{\mathcal{H}} = \|f^{\perp}\|^2_{\mathcal{H}} + \|\pi f\|^2_{L^2_x(\Omega)} = \|f^{\perp}\|^2_{\mathcal{H}} + \|\rho\|^2_{L^2_x(\Omega)} + \|m\|^2_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)}.$$

In this section, we write $\langle \cdot \rangle$ for the mean on Ω, i.e., for a in $L^1(\Omega)$,

$$\langle a \rangle := \frac{1}{|\Omega|} \int_{\Omega} a dx.$$

To take into account the symmetries of Ω we introduce the set

$$\mathscr{R}_{\Omega} := \{ \Omega \ni x \to Ax \in \mathbb{R}^d : A \in \mathcal{M}^a_d(\mathbb{R}),\ Ax \cdot n(x) = 0 \quad \forall x \in \partial\Omega \},$$

where $\mathcal{M}^a_d(\mathbb{R}^d)$ is the space of skew-symmetric $d \times d$ matrices.

Our boundary condition leads to the conservation of the mass for all values of α,

$$\frac{d}{dt} \int_{\mathbb{R}^d} f dv dx = 0.$$

In the case where $\alpha \equiv 0$, we also have the conservation of energy

$$\frac{d}{dt} \int_{\Omega \times \mathbb{R}^d} |v|^2 f dv dx = \int_{\Omega \times \mathbb{R}^d} |v|^2 (\mathcal{S}(f) - v \cdot \nabla_x f) dv dx = 0,$$

using that $|\eta_x(v)|^2 = |v|^2$ for all $(x, v) \in \Sigma$. In addition, we have the conservation of angular momentum: for all $R \in \mathscr{R}_\Omega$,

$$\frac{d}{dt} \int_{\Omega \times \mathbb{R}^d} R(x) \cdot v f \, dv dx = 0. \tag{1.3.16}$$

The main result of Chapter 6 is the following.

Theorem 1.3.5. *Let $f_0 \in L^2_{x,v}(\mu^{-1})$ satisfy*

1. *in the case of the specular reflection boundary condition $\alpha \equiv 0$,*

$$\int_G f_0 dv dx = 0, \qquad \int_G |v|^2 f_0 dv dx = 0, \qquad \int_G R \cdot v \, f_0 \, dv dx = 0,$$

   for all $R \in \mathscr{R}_\Omega$;

2. *otherwise*

$$\int_G f_0 dv dx = 0.$$

   There exist positive constants $\kappa, C > 0$ such that for any solution f to (1.3.13) with Maxwell boundary condition (1.1.20) associated to the initial data f_0, for any $t \geq 0$, there holds

$$\|f(t)\|_{L^2_{x,v}(\mu^{-1})} \leq C e^{-\kappa t} \|f_0\|_{L^2_{x,v}(\mu^{-1})}.$$

1.3.5.2 Previous results, contribution and strategy

The Cauchy theory, as well as the trend-to-equilibrium issue for the cutoff Boltzmann equation with hard potentials or hard-spheres in a perturbative regime, that is for the corresponding linearized equation as detailed in §1.1.3.3, has been developed by Guo [68], who proved exponential convergence towards equilibrium in a weighted $L^\infty_{x,v}$ space considering either the specular reflection boundary condition with Ω strictly convex and analytic, or the pure diffusive boundary condition with Ω smooth and convex. Briant [18] obtained similar results for more general weights. The $L^2 - L^\infty$ theory of [68] works as follows: the coercive property of the linearized collision operator is captured in $L^2_{x,v}$, and $L^\infty_{x,v}$ estimates rely on this non-constructive $L^2_{x,v}$ theory.

More recently, still for the Boltzmann equation with hard potentials or hard spheres, Briant and Guo [20], obtained constructive results in $L^2_{x,v}$ for positive constant accommodation coefficient $\alpha > 0$ with no convexity assumptions on Ω, from which they deduced exponential convergence in a weighted $L^\infty_{x,v}$ space. Furthermore, for the specular reflection boundary condition, well-posedness and stability results relying on non-constructive L^2 estimates were derived in restricted contexts (convex or periodical cylindrical domain with non-convex analytic cross-section) by Kim and Lee [81, 82].

The only results we are aware of in the case of long-range interaction, i.e. for non-cutoff Boltzmann and Landau collision operators in a bounded domain, are the very recent works of Guo-Hwang-Jang-Ouyang [69, 70] for the Landau equation with specular reflection boundary condition and of Duan-Liu-Sakamoto-Strain [49] for non cut-off Boltzmann and Landau equations in a finite channel with inflow or specular reflection boundary condition.

This chapter improves the existing result regarding the stability of the linearized Boltzmann and Landau equations in two regards.

1. We study a general, smooth enough, non-convex domain.
2. Our L^2 estimates are constructive, and our method encompasses the three boundary conditions (pure diffusive, specular reflection and Maxwell) in a single treatment.

Our method focuses on the decomposition $f = \pi f + f^{\perp}$ mentioned above. The estimate (1.3.15) gives immediatly

$$\frac{d}{dt}\|f_t\|_{\mathcal{H}}^2 = \langle f, \mathcal{L} f\rangle \le -\lambda \|f^{\perp}\|_{\mathcal{H}}^2$$

for some $\lambda > 0$. On the other hand the transport operator $-v \cdot \nabla_x$ does not provide any dissipative property. In order to control the missing macroscopic part πf, the idea is to construct a new inner product on $\mathcal{H}$, with an associated norm equivalent to the usual one, for which $\mathcal{L}$ is coercive. For this we consider the usual inner product of $\mathcal{H}$ and add new terms to control the missing terms appearing on the macroscopic part πf, see also the discussion on hypocoercivity §1.2.2.

Those new terms rely heavily on some existence and regularity results on elliptic equations and systems. To control the mass and energy terms in πf, we shall use an auxiliary Poisson equation with Robin or Neumann boundary conditions, for which Poincaré-type inequalities hold. The control of the momentum term is more subtle, and we shall introduce a tailored Lamé-type system with mixed Robin-type boundary condition to handle it. The regularity estimates for this system are based on Korn-type inequalities, see Duvaut and Lions [52], Desvillettes-Villani [38] and Ciarlet and Ciarlet [30].

1.3.5.3 Brief presentation of the elliptic regularity results

To derive the result, we use crucially the elliptic estimates mentioned above for variational solutions to the Poisson equation with Robin boundary condition, i.e.

$$\begin{aligned} \Delta u = \xi \quad &\text{in } \Omega, \\ (2-\alpha(x))\nabla_x u \cdot n(x) + \alpha(x) u = 0 \quad &\text{on } \quad \partial\Omega, \end{aligned}$$

We write $\Delta^{-1}\xi$ for the associated solution, which is uniquely defined in some variational sense as soon as $\xi \in (H^1(\Omega))'$. Note that we also need $\langle \xi \rangle = 0$ in the case $\alpha \equiv 0$, in which case the variational formulation is slightly different. If $u = \Delta^{-1}\xi$, the two ellipticity estimates are

$$\|u\|_{H^1(\Omega)} \lesssim \|\xi\|_{(H^1(\Omega))'},$$

and in the case where $\xi \in L^2(\Omega)$, $u \in H^2(\Omega)$ with

$$\|u\|_{H^2(\Omega)} \lesssim \|\xi\|_{L^2(\Omega)}.$$

We also use the following Poincaré-Korn inequality: there exists a positive constant $C_{PK} > 0$ such that for any vector field $U \in H^1(\Omega)$ satisfying $U \cdot n_x = 0$ on $\partial\Omega$,

$$\inf_{R_A \in \mathscr{R}_\Omega} \|U - R_A x\|^2_{L^2_x(\Omega)} \leq C_{PK} \|\nabla^{\mathrm{sym}}_x U\|^2_{L^2_x(\Omega)}. \tag{1.3.17}$$

This allows one to prove some H^2 estimates on the following Lamé-type system, which is used to control the momentum term: for $\Xi : \mathbb{R}^d \to \mathbb{R}^d$,

$$\begin{cases} -\operatorname{div}_x(\nabla^{sym}_x U) = \Xi & \text{in} \quad \Omega, \\ \nabla^{\mathrm{sym}}_x U \cdot n(x) = 0 & \text{on} \quad \partial\Omega, \\ (2-\alpha(x))\left[\nabla^{\mathrm{sym}}_x U n(x) - (\nabla^{\mathrm{sym}}_x U : n(x) \otimes n(x)) n(x)\right] + \alpha(x) U = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3.18}$$

To simplify the presentation we assume that $\mathscr{R}_\Omega = \{0\}$ from now on. We consider the Hilbert spaces $\mathcal{V}_1 = \{W : \mathbb{R}^d \to \mathbb{R}^d, W \in H^1(\Omega),\ W \cdot n_x = 0 \text{ on } \partial\Omega\}$, endowed with the $\|\cdot\|_{H^1(\Omega)}$-norm. As is classical for the Poisson equation, we can introduce a bilinear form which is coercive thanks to some Korn-type inequalities. We can then prove, for all Ξ in $(\mathcal{V}_1)'$, the existence of a unique variational solution U to (1.3.18) satisfying

$$\|U\|_{H^1(\Omega)} \lesssim \|\xi\|_{\mathcal{V}_1'}.$$

Furthermore, if $\Xi \in L^2(\Omega)$ then $U \in H^2(\Omega)$, U satisfies

$$\|U\|_{H^2(\Omega)} \lesssim \|\Xi\|_{L^2(\Omega)},$$

and is a solution to (1.3.18) almost everywhere.

1.3.5.4 Hypocoercivity estimates

The proof of Theorem 1.3.5 requires the introduction of the quantity M_ϕ for any polynomial function $\phi : \mathbb{R}^d \to \mathbb{R}$, defined by

$$M_\phi[f] := \int_{\mathbb{R}^d} \phi(v) f(x,v)\, dv.$$

Two families of polynomials will be repeatedly used: $p = (p_i)_{1\le i\le d}$ defined on $\mathbb{R}^d$ by

$$p_i(v) = v_i \frac{|v|^2 - d - 2}{\sqrt{2d}},$$

and $q_{ij} = (q_{ij})_{1\le i,j\le d}$ given by

$$q_{ij}(v) = v_i v_j - \delta_{ij}.$$

We introduce the macroscopic quantities defined for each g with enough regularity by

$$\rho[g] = \int_{\mathbb{R}^d} g dv, \quad m[g] = \int_{\mathbb{R}^d} v g dv, \quad \theta[g] = \int_{\mathbb{R}^d} \frac{|v|^2 - d}{\sqrt{2d}} g dv,$$

and set $\rho := \rho[f]$, $m := m[f]$, $\theta := \theta[f]$.

The result is obtained through four estimates. First, for the microscopic part $f^\perp$, we have, for some constant λ, from (1.3.15),

$$\langle -\mathcal{L} f, f\rangle_{\mathcal{H}} \ge \lambda \|f^\perp\|_{\mathcal{H}}. \tag{1.3.19}$$

Then, we turn to the estimate for the energy. We let $u_\theta = \Delta^{-1}\theta$ and $u_{\theta[\mathcal{L}f]} = \Delta^{-1}\theta[\mathcal{L}f]$. Such a solution is indeed well defined since computations show that $\theta[\mathcal{L}f]$ belongs to $(H^1_x(\Omega))'$ with $\langle \theta[\mathcal{L}f]\rangle = 0$. Using this and the ellipticity estimates, one can show that for some $\kappa_1, C > 0$,

$$\begin{aligned}\langle -\nabla_x u_\theta, M_p[\mathcal{L}f]\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \Big\langle -\nabla_x u_{\theta[\mathcal{L}f]}, M_p[f]\Big\rangle_{L^2_x(\Omega)} \\ \ge \kappa_1 \|\theta\|^2_{L^2_x(\Omega)} - C\|m\|^2_{L^2_x(\Omega)}\|f^\perp\|^2_{\mathcal{H}} - C\|f^\perp\|^2_{\mathcal{H}} - C\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+,\mu^{-1}(v\cdot n_x))},\end{aligned} \tag{1.3.20}$$

where $D^\perp g = g - c_\mu \mu \tilde{g}$ for all $g : \Sigma_+ \to \mathbb{R}$ measurable.

The next term is associated with mass, and is handled with a similar strategy: we let $u_\rho = \Delta^{-1}\rho$, so that $\|u_\rho\|_{H^2_x(\Omega)} \lesssim \|\rho\|_{L^2_x(\Omega)}$.

We can show that $\rho[\mathcal{L}f] \in (H^1(\Omega))'$ and $\langle \rho[\mathcal{L}f]\rangle = 0$. Therefore, we can also consider the solution $u_{\rho[\mathcal{L}f]} = \Delta^{-1}\rho[\mathcal{L}f]$. We then prove that there exist $\kappa_2, C > 0$ such that

$$\begin{aligned}\langle -\nabla_x u_\rho, \rho[\mathcal{L}f]\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \Big\langle -\nabla_x u_{\rho[\mathcal{L}f]}, \rho[f]\Big\rangle_{L^2_x(\Omega)} \\ \ge \kappa_2 \|\rho\|^2_{L^2_x(\Omega)} - C\Big(\|m\|^2_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)} + \|f^\perp\|^2_{\mathcal{H}}\Big).\end{aligned} \tag{1.3.21}$$

For the term corresponding to the momentum, we consider the solution U_m of (1.3.18) with $\Xi = m[f] \in L^2_x(\Omega)$. We then show that $m[\mathcal{L}f] \in \mathcal{V}_1'$, so that the solution $U_{m[\mathcal{L}f]}$ of (1.3.18) with $\Xi = m[\mathcal{L}f]$ is well-defined in $H^1_x(\Omega)$.

We can hence prove that for two constants $\kappa_3, C > 0$,

$$\begin{aligned}
&\langle -\nabla_x^{sym} U_m, M_q[\mathcal{L}f]\rangle_{\mathcal{V}_1,\mathcal{V}_1'} + \left\langle -\nabla_x^{sym} U_{m[\mathcal{L}f]}, M_q[f]\right\rangle_{L^2_x(\Omega)} \qquad (1.3.22)\\
&\geq \kappa_3 \|m\|^2_{L^2_x(\Omega)} - C\Big(\|\rho\|_{L^2_x(\Omega)}\|f^\perp\|_{\mathcal{H}} + \|\theta\|_{L^2_x(\Omega)}\|\rho\|_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)} \\
&\qquad + \|f^\perp\|^2_{\mathcal{H}} + \|\alpha D^\perp f_{|\Sigma_+}\|^2_{L^2(\Sigma_+,\mu^{-1}(v)(v\cdot n_x))}\Big).
\end{aligned}$$

We finally define the scalar product $\langle\!\langle\cdot\rangle\!\rangle$ by

$$\begin{aligned}
\langle\!\langle f, g\rangle\!\rangle := \langle f, g\rangle_{\mathcal{H}} &+ \epsilon_1 \left\langle \nabla_x u_{\theta[f]}, M_p[g]\right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \epsilon_1 \left\langle \nabla_x u_{\theta[g]}, M_p[f]\right\rangle_{L^2_x(\Omega)} \\
&+ \epsilon_2 \left\langle \nabla_x^{sym} U_{m[f]}, M_q[g]\right\rangle_{\mathcal{V}_1,\mathcal{V}_1'} + \epsilon_2 \left\langle \nabla_x^{sym} U_{m[g]}, M_q[f]\right\rangle_{L^2_x(\Omega)} \\
&+ \epsilon_3 \left\langle \nabla_x u_{\rho[f]}, m[g]\right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \epsilon_3 \left\langle -\nabla_x u_{\rho[g]}, m[f]\right\rangle_{L^2_x(\Omega)}.
\end{aligned}$$

The equivalence between the norm derived from $\langle\!\langle\cdot\rangle\!\rangle$ and $\|\cdot\|_{\mathcal{H}}$ is quite straightforward. For an appropriate choice of ϵ_1, ϵ_2 and $\epsilon_3 > 0$, we also obtain by multiple applications of Young's inequality, that, for some constant $\kappa' > 0$,

$$\langle\!\langle -\mathcal{L}f, f\rangle\!\rangle \geq \kappa' \|f\|_{\mathcal{H}},$$

from which the conclusion follows.

Part I

Rate of convergence towards equilibrium of the free-transport equation

Chapter 2

A coupling approach for the convergence to equilibrium for a collisionless gas

This chapter corresponds to the paper [10], submitted, which is a joint work with Nicolas Fournier[1].

Abstract: We use a probabilistic approach to study the rate of convergence to equilibrium for a collisionless (Knudsen) gas in dimension equal to or larger than 2. The use of a coupling between two stochastic processes allows us to extend and refine, in total variation distance, the polynomial rate of convergence given in [1] and [87]. This is, to our knowledge, the first quantitative result in collisionless kinetic theory in dimension equal to or larger than 2 that does not require any symmetry of the domain, nor a monokinetic regime. Our study is also more general in terms of reflection at the boundary: we allow for rather general diffusive reflections and for a specular reflection component.

Keywords: stochastic billiards, Markov process, collisionless gas, coupling, long-time behaviour, subexponential convergence to equilibrium.

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2.1 Introduction

We consider a Knudsen (collisionless) gas enclosed in a vessel and investigate the rate of convergence to equilibrium. We study a C^2 bounded domain (open, connected) D in $\mathbb{R}^n$, with $n \geq 2$. The boundary of this domain, ∂D, is considered at rest, and when a gas particle collides with the boundary, a reflection which is either diffuse or specular occurs. For a point x in ∂D, n_x denotes the unit inward normal at x.

The distribution function of the gas, $f(t,x,v)$, represents the density of particles with position $x \in \bar{D}$ and velocity $v \in \mathbb{R}^n$ at time $t \geq 0$. We assume that it satisfies the free-transport equation with both a boundary condition and an initial condition:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0, & (x,v) \in D \times \mathbb{R}^n, \\ f(t,x,v)(v \cdot n_x) = -\alpha(x) c_0 M(v)(v \cdot n_x) \displaystyle\int_{\{v' \cdot n_x < 0\}} f(t,x,v')(v' \cdot n_x) dv' & \\ \qquad\qquad + (1-\alpha(x)) f(t,x,v - 2(v \cdot n_x)n_x)(v \cdot n_x), & x \in \partial D, v \cdot n_x > 0, \\ f(0,x,v) = f_0(x,v), & (x,v) \in D \times \mathbb{R}^n, \end{cases} \tag{2.1.1}$$

where the constant $c_0 > 0$ is given by

$$c_0 = \int_{\{u \cdot n_x > 0\}} M(u)(u \cdot n_x) du, \tag{2.1.2}$$

for any choice of $x \in \partial D$. The independence of c_0 with respect to x is a consequence of the radial symmetry assumption made below on the density M.

This dynamic does not take into account collisions between particles that may occur inside D. This is legitimate for the study of Knudsen gases, which are dilute enough. This model represents particles moving in D following the free transport dynamic until they collide with the boundary. When a particle reaches the boundary at some point $x \in \partial D$, it is specularly reflected with probability $1 - \alpha(x)$, and diffusively reflected with probability $\alpha(x)$. In the latter case, its new velocity is chosen using M. See Definition 2.3.1 for the precise probabilistic interpretation of the model.

Here are our main assumptions.

Hypothesis 2.1.1.

- D is a C^2 open connected bounded set in $\mathbb{R}^n$, with $n \geq 2$.
- $\alpha : \partial D \to [0,1]$ is uniformly bounded from below, i.e. there exists $\alpha_0 > 0$ such that:
$$\alpha(x) \geq \alpha_0, \quad \forall x \in \partial D. \tag{2.1.3}$$
- $M : \mathbb{R}^n \to \mathbb{R}_+$ is radially symmetric with $\int_{\mathbb{R}^n} M(v)dv = 1$, $\int_{\mathbb{R}^n} \|v\| M(v) dv < \infty$, and there exist $\delta_1 > 0$ and some continuous, radially symmetric, $\bar{M} : \mathbb{R}^n \to \mathbb{R}_+$ such that $0 < \bar{M}(v) \leq M(v)$ for all $v \in \mathbb{R}^n$ such that $0 < \|v\| \leq \delta_1$.

The paradigmatic example (and most physically relevant one) of such M is the Maxwellian distribution with parameter (temperature) θ, that fits into this framework:

$$M(v) = \frac{1}{(2\pi\theta)^{\frac{n}{2}}} e^{-\frac{\|v\|^2}{2\theta}}, \qquad v \in \mathbb{R}^n. \tag{2.1.4}$$

Observe that informally, (2.1.1) preserves mass. Indeed, for a strong solution to (2.1.1), Green's formula gives:

$$\frac{d}{dt}\int_{D\times\mathbb{R}^n} f(t,x,v)dvdx = -\int_{D\times\mathbb{R}^n} \nabla_x(vf(t,x,v))dvdx = \int_{\partial D\times\mathbb{R}^n} f(t,x,v)(v\cdot n_x)dvdx = 0,$$

where the last equality is a consequence of the boundary condition in (2.1.1).

2.1.1 Main result

The stationary problem corresponding to (2.1.1) leads to an equilibrium in the phase space. Its distribution is given by (assuming the initial data to be of total mass 1)

$$\mu_\infty(x,v) = \frac{M(v)}{|D|}, \quad \forall (x,v) \in D\times\mathbb{R}^n,$$

where $|D|$ denotes the Lebesgue measure of D in $\mathbb{R}^n$. Note that (unsurprisingly) the equilibrium distribution is space-homogeneous in D.

It is known that there is convergence towards this equilibrium distribution in L^1 distance, see for instance Arkeryd and Nouri [3, Theorem 1.1] for a proof in the case where $\alpha \equiv 1$ and with slight restrictions on D. The goal of this paper is to characterize the rate of this convergence.

Recall that the total variation distance of a signed measure μ on a measurable space $(E,\mathcal{E})$ is given by

$$\|\mu\|_{TV} = \frac{1}{2}\sup\Big\{\int_E g d\mu, g : E\to\mathbb{R}, \|g\|_\infty \le 1\Big\}.$$

In the whole paper, we use the notation $f(t,x,v)$ when f is a L^1-function on $\mathbb{R}_+\times D\times\mathbb{R}^n$ and $f_t(dx,dv)$ when f is measure-valued. Our main result is the following, see Definition 2.2.1 and Theorem 2.2.3 for the precise meaning of weak solutions.

Theorem 2.1.1. *Assume that Hypothesis 2.1.1 is satisfied. Let the initial distribution, f_0, be a probability measure on $D\times\mathbb{R}^n$. Let $r:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous increasing function such that there exists a constant $C>0$ satisfying $r(x+y)\le C(r(x)+r(y))$ for all $x,y\in\mathbb{R}_+$ and such that*

$$\int_{\mathbb{R}^n} r\Big(\frac{1}{\|v\|}\Big)M(v)dv < \infty \quad \textit{and} \quad \int_{D\times\mathbb{R}^n} r\Big(\frac{1}{\|v\|}\Big)f_0(dx,dv) < \infty. \tag{2.1.5}$$

Then, there exist some constant $\kappa > 0$ and a weak solution $\rho(dt, dx, dv) = dt f_t(dx, dv)$ to (2.1.1) such that for all $t \geq 0$,

$$\|f_t - \mu_\infty\|_{TV} \leq \frac{\kappa}{r(t)}.$$

Moreover, in the case where f_0 admits a density in $L^1(D \times \mathbb{R}^n)$, the solution f is unique among "regular" solutions (see Theorem 2.2.2).

The typical example for the rate r is $r(t) = (t+1)^n$, or rather $r(t) = (t+1)^{n-}$, as exemplified by the following situation.

Corollary 2.1.1. *We take the same hypotheses and notations as in Theorem 2.1.1, and assume furthermore that M is bounded (for instance, M is a Maxwellian distribution of the form (2.1.4)).*

a) If f_0 has a bounded density, there exists a constant $\kappa > 0$ such that, for all $t \geq 0$,

$$\|f_t - \mu_\infty\|_{TV} \leq \frac{\kappa(1 + \log^2(t+1))}{(t+1)^n}.$$

b) If there exists $d \in (0, n)$ such that

$$\int_{D \times \mathbb{R}^n} \frac{1}{\|v\|^d} f_0(dx, dv) < \infty,$$

there exists a constant $\kappa > 0$ such that for all $t \geq 0$,

$$\|f_t - \mu_\infty\|_{TV} \leq \frac{\kappa}{(t+1)^d}.$$

Physically, the most interesting case is the following: consider a collisionless gas enclosed in a vessel represented by the domain D. The boundary of the domain is kept at temperature $\theta > 0$. A particle colliding with this boundary at $x \in \partial D$ is either specularly reflected, with probability $1 - \alpha(x)$, or exchanges energy with the boundary and is diffusively reflected with probability $\alpha(x)$, the distribution M being the Maxwellian with temperature θ.

2.1.2 Bibliography and discussion

Relaxation to equilibrium is a key aspect in statistical mechanics. In general, this relaxation, which is known since the H-theorem in the case of the Boltzmann equation, is the result of two main physical equilibrating effects: the collisions between gas molecules and their interactions with the boundary. In [39], Desvillettes and Villani find that the distance between the distribution function of the gas at time t and the final equilibrium state decays at a rate $\mathcal{O}(\frac{1}{t^m})$ for all $m > 0$, in the case of space inhomogeneous solutions to the Boltzmann equation satisfying strong conditions of regularity, positivity and decay at large velocities. The rate of [39] is completed by an exponential rate in the case where the initial data is close to equilibrium in Guo [68]. In these works, the authors assume that the spatial domain is either the flat torus or a smooth bounded domain with specular or bounce-back reflection at the boundary. Hence the focus is on the equilibrating effect of the collisions between gas molecules rather than the interaction with the boundary, and the equilibrium is entirely determined by the total mass and energy. Later, in [123], Villani works on the case of a diffuse or accomodation reflection at the wall of a bounded smooth domain, with a constant temperature at the boundary. The equilibrium is thus slightly changed, as the total mass is now the only conserved quantity. In this case, both collisions between gas molecules and interactions with the boundary play an important role in the relaxation to equilibrium, and give an example of the so-called "hypocoercivity" method.

Concerning the model studied in this paper, here are the main available results. In [119], Tsuji, Aoki and Golse find numerically a rate of convergence in t^{-n} for bounded initial data. An upper bound for the convergence rate in t^{-1} is obtained by Aoki and Golse in [1], assuming some spherical symmetry on the domain and on the initial condition and that $\alpha \equiv 1$. Using a stochastic approach, Kuo, Liu and Tsai in [87] obtain the (optimal) convergence rate of t^{-n} in a spherically symmetric domain for $n = 1, 2, 3$ with again $\alpha \equiv 1$ and with bounded initial data satisfying some technical conditions. Later, Kuo [86] extended this work, in dimensions 1 and 2, to the case of Maxwell boundary conditions (with additionally some specular reflections). All the above results assume that M is a Maxwellian distribution. We also refer to the connected paper by Mokhtar-Kharroubi and Seifert [102] who studied a similar problem in slab geometry (in dimension 1) using Ingham's tauberian theorem.

Our rate confirms, up to a logarithmic term, both the suggestions made in [119] in view of their numerical results, see Corollary 2.1.1, and the rate obtained by Kuo [86]. It also extends this result to higher dimensions, considers more complicated domains and allows more general initial conditions.

For the most interesting case where M is given by (2.1.4), we can sum up our conclusions as follows: if f_0 is bounded on $\{v \in \mathbb{R}^n, \|v\| \leq \epsilon\}$ for some $\epsilon > 0$, e.g. if $f_0(x, v) = g_0(x)\delta_{v_0}(v)$ for some density g_0 on D and some $v_0 \neq 0$, the convergence rate towards equilibrium is (up to a logarithmic factor) in $\frac{1}{t^n}$. On the other hand, if f_0 is unbounded around 0, e.g. $f_0(x, v) = \frac{c}{\|v\|^\alpha}\mathbf{1}_{\{\|v\|\leq 1\}}$ with $\alpha \in (0, n)$, the convergence rate towards equilibrium is $\frac{1}{t^{(n-\alpha)-}}$ using Theorem 2.1.1 with $r(t) = t^{(n-\alpha)-}$.

In Kuo et al. [87], the authors point out that $f_0 - \mu_\infty$ (with f_0 bounded) is the limiting factor that prevents from a better rate of convergence. We believe that, indeed, our method might allow one to prove the following extension: when one considers two solutions f_t and g_t with $f_0 = \delta_{(x_0,v_0)}$ and $g_0 = \delta_{(y_0,w_0)}$, $\|f_t - g_t\|_{TV} \lesssim t^{-n-1}$ as soon as $v_0 \neq 0$ and $w_0 \neq 0$.

Stochastic billards have also been studied in details, see the works of Evans [55], Comets, Popov, Schütz and Vachkovskaia [31] and the recent work of Fétique [61] in the convex setting. This corresponds to the monokinetic case of our model: the velocity of particles has a constant norm 1 (f_0 and the distribution M are carried by the unit sphere). They prove exponential convergence to equilibrium by coupling methods. Let us mention that we use a result from Evans on the geometry of C^1 domains.

The stochastic process studied in this paper is similar to the family of Piecewise Deterministic Markov Processes (PDMP) introduced by Davis [34]. However it does not entirely fit this framework, since the jumps are predictable in our case. In the past few years, several long time behaviours for models corresponding to PDMP have been studied, exhibiting a geometric convergence towards equilibrium. We refer to the study of the telegraph process by Fontbona, Guérin and Malrieu [56, 57], and on the recent work of Durmus, Guillin and Monmarché [50].

In conclusion, our result is, to the best of our knowledge, the first quantitative result for this problem for a non-symmetric domain in dimension $d \geq 2$, in a non-monokinetic regime. We also consider a more general law M for the reflection at the boundary, with a larger class of initial data f_0.

2.1.3 Strategy for the proof and plan of the paper

The next Section 2.2 is devoted to the rigorous introduction of our notion of weak solutions, and to the proof of uniqueness under a regularity assumption on f_0, in the spirit of [65] and [96].

In Section 2.3, we construct the stochastic process which we use in the proof of Theorem 2.1.1. We show that the law of this stochastic process is a weak solution in the sense of measures to (2.1.1), and that it is the unique weak solution under further regularity assumptions of f_0. The unusual boundary conditions leads to rather non-standard difficulties.

In Section 2.4, we derive the proof of our large time result in the context of a uniformly convex domain with C^2 boundary, following the strategy described below, and we extend in Section 2.5 the previous result to general domains. For the sake of clarity we start by proving the result in a uniformly convex domain, because the coupling is easier since from any point at the boundary of the domain, we can join any other point at the boundary in one step.

It is worth mentioning that the coupling method which we use is close, at least in spirit, to methods based on the study of the Feller nature of the corresponding semigroup. Those methods are known since the work of Meyn and Tweedie [98] for exponential rates of convergence, and have recently been extended by Douc, Fort and Guillin [44] for subgeometric convergence rates.

They involve the derivation of the modulated moments of the delayed hitting time of some "petite" set, a computation that is straightforward once the coupling time is estimated.

In a companion paper [8], we investigate the same problem by a purely analytic approach. Of course, the main issue is the absence of a spectral gap for the operator corresponding to (2.1.1), which is the key reason for the polynomial rate of convergence.

To prove Theorem 2.1.1, we introduce a coupling $(X_t, V_t)_{t\geq 0}, (\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ with (X_t, V_t) distributed according to f_t and $(\tilde{X}_t, \tilde{V}_t)$ distributed according to μ_∞, in such a way that the coupling time

$$\tau = \inf\{t \geq 0, (X_{t+s})_{s\geq 0} = (\tilde{X}_{t+s})_{s\geq 0}, (V_{t+s})_{s\geq 0} = (\tilde{V}_{t+s})_{s\geq 0}\},$$

is as small as possible. We show that it is possible to build a coupling such that the following occurs.

i) When one process collides with the boundary (Proposition 2.4.1), if the other one has a large enough speed, so that its next collision occurs sufficiently soon after the one of the first process, there is a positive probability that the two processes coincide for all times following the next collision with the boundary.

ii) We come back to the previous situation after a random number of collisions with the boundary for both processes, and this number of collisions is controlled by a geometric random variable.

The construction of such a coupling is quite subtle. Indeed, the random nature of $(X_t, V_t)_{t\geq 0}$ only appears when $X_t \in \partial D$. When one tries to couple two such processes, complex situations can occur, for instance one of the process can hit the boundary several times before the other one does so. To construct a global process satisfying the Markov property, we introduce an extra variable, $(Z_s)_{s\geq 0}$, in the process, see Definition 2.4.1, which allows us to memorize the randomness generated at some rebound of $(X_t)_{t\geq 0}$ until $(\tilde{X}_t)_{t\geq 0}$ hits the boundary.

We then show that $r(\tau)$ has finite expectation, roughly, as soon as

$$\int_{D\times\mathbb{R}^n} r\Big(\frac{1}{\|v\|}\Big) f_0(x,v)dvdx + \int_{\mathbb{R}^n} r\Big(\frac{1}{\|v\|}\Big) M(v)dv < \infty.$$

This assumption is crucial: the velocity of a particle has roughly for law either f_0 or M, the time needed to cross the domain is proportional to the inverse of this velocity, and the coupling can occur only at the boundary.

We then conclude using the fact that:

$$\|f_t - \mu_\infty\|_{TV} \leq \mathbb{P}\Big((X_t, V_t) \neq (\tilde{X}_t, \tilde{V}_t)\Big) \leq \mathbb{P}(\tau > t) = \mathbb{P}\Big(r(\tau) > r(t)\Big) \leq \frac{\mathbb{E}[r(\tau)]}{r(t)} \tag{2.1.6}$$

from Markov's inequality, leading us to the rate of convergence in Theorem 2.1.1.

2.2 Weak Solutions

In this section, we give a definition of weak solutions in the sense of measures for (2.1.1). Existence of this weak solution for any initial probability measure, without further assumption, will be obtained in Section 2.3 by a probabilistic method. We show uniqueness of sufficiently regular weak solutions. Let us mention that uniqueness for boundary value problems such as (2.1.1) cannot be derived in general. For a discussion on those well-posedness issues, we refer to Greenberg, van der Mee and Protopopescu [65, Chapter 11]

We recall that D is a C^2 domain (open, connected) in $\mathbb{R}^n$ and set $G = D \times \mathbb{R}^n$. We let $\Sigma = (0,\infty) \times G$. We write $\cdot$ for the scalar product in $\mathbb{R}^n$, $\|.\|$ for the Euclidian norm. We also define

$$\begin{aligned}
F_t &= \{(t,x,v), (x,v) \in G\}, \quad t \in \mathbb{R}_+, \\
\partial_\pm G &= \{(x,v), \pm v \cdot n_x < 0, x \in \partial D, v \in \mathbb{R}^n\}, \\
\partial_0 G &= \{(x,v) \in \partial D \times \mathbb{R}^n, v \cdot n_x = 0\},
\end{aligned}$$

where we recall that n_x is the unit normal vector at $x \in \partial D$ pointing towards D. In words, $\partial_+ G$ corresponds to points coming from D towards the boundary, while $\partial_- G$ is the set of points coming from the boundary towards D. For a topological space A, we write $\mathcal{M}(A)$ for the set of non-negative Radon measures on A, $\mathcal{P}(A)$ for the set of probability measures on A. We denote $\langle .,. \rangle$ the scalar product for the duality $\mathcal{M}(A)$, $\mathcal{M}(A)^*$. We write $\mathcal{B}(A)$ for the Borel sigma-algebra on A. For any set B, we denote $\bar{B}$ for the closure of B, and set $d(D)$ to be the diameter of D :

$$d(D) = \sup_{x,y \in \partial D} \|x - y\|.$$

For any space E, we write $\mathcal{D}(E) = C_c^\infty(E)$ for the space of test functions (smooth with compact support) on E. We set

$$L = \partial_t + v \cdot \nabla_x. \tag{2.2.1}$$

We deal with two reference measures:

- the n-dimensional Lebesgue measure (on D, $\bar{D}$ and $\mathbb{R}^n$).
- the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

To lighten the notations, the same symbols $dx, dv, dz, \ldots$ denote all of them. Possible ambiguity can be resolved by checking the space of integration. Similarly the volume of a set A, denoted $|A|$ in all cases, refer to the corresponding ambiant space endowed with the appropriate measure.

We let $K : \mathcal{M}((0,\infty) \times \partial_+ G) \to \mathcal{M}((0,\infty) \times \partial_- G)$, given, for any $\nu \in \mathcal{M}((0,\infty) \times \partial_+ G)$, any test function $\phi \in \mathcal{D}((0,\infty) \times \partial_- G)$, by

$$\begin{aligned}\langle K\nu, \phi\rangle_{(0,\infty)\times\partial_- G} = &\int_{(0,\infty)\times\partial_+ G} \Big(\int_{\{v'\cdot n_x>0\}} \alpha(x)\phi(t,x,v')c_0 M(v')|v'\cdot n_x| dv'\Big)\nu(dt,dv,dx)\\ &+ \int_{(0,\infty)\times\partial_+ G} (1-\alpha(x))\phi(t,x,\eta_x(v))\nu(dt,dv,dx),\end{aligned} \tag{2.2.2}$$

for c_0 defined by (2.1.2). The operator $\eta_x(.)$ is the one of specular reflection at $x \in \partial D$, given by

$$\eta_x(v) = v - 2(v\cdot n_x)n_x, \quad v \in \mathbb{R}^n. \tag{2.2.3}$$

Hence, if $(x,v) \in \partial_\pm G$, $(x, \eta_x(v)) \in \partial_\mp G$.

Whenever necessary, we extend the definition of K to an operator $\bar{K}$ from $\mathcal{M}(\partial_+ G)$ to $\mathcal{M}(\partial_- G)$ defined similarly. For any measure $\nu \in \mathcal{M}(\partial_+ G)$, any test function $\phi \in \mathcal{D}(\partial_- G)$, we set

$$\begin{aligned}\langle \bar{K}\nu, \phi\rangle_{\partial_- G} = &\int_{\partial_+ G} \Big(\int_{\{v'\cdot n_x>0\}} \alpha(x)\phi(x,v')c_0 M(v')|v'\cdot n_x| dv'\Big)\nu(dv,dx)\\ &+ \int_{\partial_+ G} (1-\alpha(x))\phi(x,\eta_x(v))\nu(dv,dx).\end{aligned} \tag{2.2.4}$$

With this at hand, we define our notion of weak solution in the sense of measures.

Definition 2.2.1. We say that a non-negative Radon measure $\rho \in \mathcal{M}(\bar{\Sigma})$ is a weak solution to (2.1.1) with non-negative initial datum $\rho_0 \in \mathcal{M}(G)$ if

i) for all $T > 0$, $\rho((0,T) \times G) < \infty$;

ii) there exists a couple of non-negative Radon measures $\rho_\pm$ on $(0,\infty) \times \partial_\pm G$ such that :

$$\rho_- = K\rho_+, \tag{2.2.5}$$

and for all $\phi \in \mathcal{D}(\bar{\Sigma})$ with $\phi = 0$ on $(0,\infty) \times \partial_0 G$,

$$\langle \rho, L\phi\rangle_\Sigma = -\langle \rho_0, \phi(0,\cdot)\rangle_G + \langle \rho_+, \phi\rangle_{(0,\infty)\times\partial_+ G} - \langle \rho_-, \phi\rangle_{(0,\infty)\times\partial_- G}. \tag{2.2.6}$$

As we will see in Section 2.3.3, such a solution always exists. If $f \in C^\infty([0,\infty) \times \bar{D} \times \mathbb{R}^n)$ is a strong solution to (2.1.1), then $\rho(dt,dx,dv) = f(t,x,v)dtdxdv$ on $(0,\infty) \times D \times \mathbb{R}^n$ is a weak solution with

$$\rho_+(dt,dx,dv) = f(t,x,v)|v \cdot n_x|dtdxdv, \qquad \text{in } (0,\infty) \times \partial_+ G,$$
$$\rho_-(dt,dx,dv) = f(t,x,v)|v \cdot n_x|dtdxdv, \qquad \text{in } (0,\infty) \times \partial_- G.$$

Indeed, this can be understood reading the proof of Theorem 2.2.2 and mainly relies on the following fact: using that $\partial_t f + v \cdot \nabla_x f = 0$ in $D \times \mathbb{R}^n$ and Green's formula, we find that

$$\begin{aligned}\langle \rho, L\phi \rangle_\Sigma &= \int_0^\infty \int_{D\times\mathbb{R}^n} f L\phi dvdxdt \\ &= -\int_{D\times\mathbb{R}^n} f_0 \phi(0,\cdot) dvdx - \int_0^\infty \int_{\partial D\times\mathbb{R}^n} \phi f (n_x \cdot v) dvdxdt \\ &= -\langle \rho_0, \phi(0,\cdot)\rangle_G + \langle \rho_+, \phi\rangle_{(0,\infty)\times\partial_+ G} - \langle \rho_-, \phi\rangle_{(0,\infty)\times\partial_- G}.\end{aligned}$$

The fact that $\rho_- = K\rho_+$ is explained by the boundary condition in (2.1.1), see Remark 2.2.1 below.

In [96, Proposition 1], Mellet and Mischler show uniqueness of the solution in an L^1 setting for a slightly harder case (namely the Vlasov equation rather than the free transport), with the additional hypothesis that the initial datum belongs to the space $L^1(D \times \mathbb{R}^n) \cap L^2(D \times \mathbb{R}^n)$. We adapt this proof in Theorem 2.2.2 below.

When a weak solution can be identified with a function having a few regularity, we can define its trace on ∂D in a precise manner. We recall here a result of Mischler [101].

Theorem 2.2.1. *[101, Theorem 1, $E \equiv 0$, $G \equiv 0$] If $f \in L^\infty_{loc}([0,\infty); L^1_{loc}(\bar{D} \times \mathbb{R}^n))$ satisfies*

$$Lf = 0 \qquad \textit{in } \mathcal{D}'((0,\infty) \times D \times \mathbb{R}^n),$$

then there holds that $f \in C([0,\infty), L^1_{loc}(\bar{D} \times \mathbb{R}^n))$ and the trace γf of f on $(0,\infty) \times \partial D \times \mathbb{R}^n$ is well defined, it is the unique function

$$\gamma f \in L^1_{loc}([0,\infty) \times \partial D \times \mathbb{R}^n, (n_x \cdot v)^2 dvdxdt)$$

satisfying the Green's formula: for all $0 \leq t_0 < t_1$, for all $\phi \in \mathcal{D}(\bar{\Sigma})$ such that $\phi = 0$ on $(0,\infty) \times \partial_0 G$,

$$\int_{t_0}^{t_1} \int_G f L\phi dvdxdt = \Big[\int_G f\phi dvdx\Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \int_{\partial D\times\mathbb{R}^n} (\gamma f)(n_x \cdot v)\phi dvdxdt. \tag{2.2.7}$$

Observe that all the terms are well-defined in (2.2.7). In particular, our test functions satisfy $\phi(t,x,v) \leq C|v \cdot n_x|$ for all $(t,x,v) \in (0,\infty) \times \partial D \times \mathbb{R}^n$.

Remark 2.2.1. For any $g \in L^1((0,\infty)\times\partial_+G, |v\cdot n_x|dvdxdt)$, it holds that $K(g|v\cdot n_x|)$ belongs to $L^1_{\text{loc}}((0,\infty)\times\partial_-G, |v\cdot n_x|dvdxdt)$ and we have

$$K(|v\cdot n_x|g)(t,x,v) = \alpha(x)c_0M(v)\int_{\{v'\cdot n_x<0\}} g(t,x,v')|v'\cdot n_x|dv' + (1-\alpha(x))g(t,x,\eta_x(v)), \quad (2.2.8)$$

for almost every $(t,x,v)\in(0,\infty)\times\partial_-G$.

Proof of (2.2.8). Set $\nu(dt,dx,dv) = g(t,x,v)|v\cdot n_x|dtdxdv$ on $(0,\infty)\times\partial_+G$ and consider a test function $\phi\in\mathcal{D}((0,\infty)\times\partial_-G)$. We have

$$\begin{aligned}
&\langle K(\nu),\phi\rangle_{(0,\infty)\times\partial_-G}\\
&\quad= \int_0^\infty\int_{\partial_+G}\alpha(x)\Big(\int_{\{v'\cdot n_x>0\}}\phi(t,x,v')c_0M(v')|v'\cdot n_x|dv'\Big)g(t,x,v)|v\cdot n_x|dvdxdt\\
&\qquad+\int_0^\infty\int_{\partial_+G}(1-\alpha(x))\phi(t,x,\eta_x(v))g(t,x,v)|v\cdot n_x|dvdxdt\\
&\quad= \int_0^\infty\int_{\partial_-G}\phi(t,x,v)\Big(\alpha(x)c_0M(v)\int_{\{v'\cdot n_x<0\}}g(t,x,v')|v'\cdot n_x|dv'\Big)|v\cdot n_x|dvdxdt\\
&\qquad+\int_0^\infty\int_{\partial_-G}\phi(t,x,v)(1-\alpha(x))g(t,x,\eta_x(v))|v\cdot n_x|dvdxdt.
\end{aligned}$$

In the first integral, we only exchanged the roles of v and v'. In the second one, we performed the involutive change of variables $v'=\eta_x(v)$ and used that $|\eta_x(v)\cdot n_x| = |v\cdot n_x|$ for all $(x,v)\in\partial_+G$. Since this holds for any $\phi\in\mathcal{D}((0,\infty)\times\partial_-G)$, (2.2.8) follows. $\square$

For f with the same regularity as in Theorem 2.2.1, $\gamma_\pm f$ denote the restrictions of γf to $(0,\infty)\times\partial_\pm G$. From (2.2.6) and (2.2.7) and the uniqueness of this trace function it is clear that if the measures $\rho_\pm$ in Definition 2.2.1 admit two densities $f_\pm$ with respect to the measure $|v\cdot n_x|dvdxdt$ on $(0,\infty)\times\partial_\pm G$, those densities can be identified with $\gamma_\pm f$.

We now adapt the uniqueness result in Proposition 1 in [96].

Theorem 2.2.2. *Consider $f\in C_w([0,\infty);L^1(\bar D\times\mathbb{R}^n))$ for all $T>0$ (i.e. f is weakly continuous in time in the sense of measures) admitting a trace function γf belonging to $L^1([0,T]\times\partial D\times\mathbb{R}^n, |v\cdot n_x|dvdxdt)$ (for all $T>0$) such that formula (2.2.7) holds. Assume that $\rho(dt,dx,dv) = f(t,x,v)dtdxdv$ is a weak solution to (2.1.1) with initial condition f_0 in $L^1(D\times\mathbb{R}^n)$. Then, we have*

$$\begin{cases}
Lf = (\partial_t + v\cdot\nabla_x)f = 0 & \text{in } \mathcal{D}'((0,\infty)\times D\times\mathbb{R}^n),\\
f(0,.) = f_0 & \text{a.e. in } D\times\mathbb{R}^n,\\
(v\cdot n_x)\gamma_-f = K\Big(|v\cdot n_x|\gamma_+f\Big) & \text{a.e. in } (0,\infty)\times\partial_-G.
\end{cases}\quad (2.2.9)$$

Moreover, f is the unique solution to (2.2.9) with this regularity.

As we will see in Theorem 2.2.3, such a solution always exists, assuming of course that f_0 is a probability density function.

Proof. **Step 1.** Here, we prove that f solves (2.2.9).

We first claim that we have the two equalities $\rho_+(dt, dx, dv) = \gamma_+ f(t, x, v)|v \cdot n_x|dtdxdv$ and $\rho_-(dt, dx, dv) = \gamma_- f(t, x, v)|v \cdot n_x|dtdxdv$. Indeed, consider a test function ϕ belonging to $\mathcal{D}((0, \infty) \times \bar{D} \times \mathbb{R}^n)$, with $\phi = 0$ on $(0, \infty) \times \partial_0 G$. Using (2.2.6), the definition of ρ and (2.2.7), we obtain

$$\begin{aligned}\langle \rho_+, \phi\rangle_{(0,\infty)\times\partial_+ G} - \langle \rho_-, \phi\rangle_{(0,\infty)\times\partial_- G} = \langle \rho, L\phi\rangle_\Sigma &= \int_0^\infty \int_{D\times\mathbb{R}^n} fL\phi dvdxdt \\ &= -\int_0^\infty \int_{\partial D\times\mathbb{R}^n} (v \cdot n_x)(\gamma f)\phi dvdxdt.\end{aligned}$$

from which we deduce that $\rho_+(dt, dx, dv) - \rho_-(dt, dx, dv) = \gamma f(t, x, v)(v \cdot n_x)dtdxdv$ whence the claim. With this at hand, the third equation of (2.2.9) follows immediatly from (2.2.5) and Remark 2.2.1.

The first equation of (2.2.9) follows from (2.2.6) and the definition of ρ, since for all $T > 0$, the right-hand side of (2.2.6) is 0 for $\phi \in \mathcal{D}((0, T) \times D \times \mathbb{R}^n)$.

For the second equation of (2.2.9), we want to prove that for any $\phi \in \mathcal{D}(D \times \mathbb{R}^n)$,

$$\int_{D\times\mathbb{R}^n} \phi(x, v)f(0, x, v)dvdx = \langle f_0, \phi\rangle_{D\times\mathbb{R}^n}. \tag{2.2.10}$$

Using the definition of ρ and the equation (2.2.6) we obtain immediatly

$$\int_0^\infty \int_{D\times\mathbb{R}^n} L\psi f dvdxdt = -\langle f_0, \psi(0, .)\rangle_{D\times\mathbb{R}^n}$$

for any $\psi \in \mathcal{D}([0, \infty) \times D \times \mathbb{R}^n)$. Let $\phi \in \mathcal{D}(D \times \mathbb{R}^n)$, $\epsilon \in (0, 1)$ and define the function β_ϵ by $\beta_\epsilon(t) = e^{-\frac{t}{\epsilon-t}}\mathbf{1}_{\{t\in[0,\epsilon)\}}$. Therefore β_ϵ is smooth with compact support in $[0, \infty)$ and we can apply the previous equation with $\psi(t, x, v) = \beta_\epsilon(t)\phi(x, v)$ to find

$$\int_0^\infty \int_{D\times\mathbb{R}^n} \Big(\beta'_\epsilon(t)\phi(x, v) + \beta_\epsilon(t)v \cdot \nabla_x\phi(x, v)\Big) f(t, x, v)dvdxdt = -\langle f_0, \phi\rangle_{D\times\mathbb{R}^n}. \tag{2.2.11}$$

We set

$$J_\epsilon = \int_0^\infty \int_{D\times\mathbb{R}^n} \beta_\epsilon(t)v \cdot \nabla_x\phi(x, v)f(t, x, v)dvdxdt,$$

and

$$I_\epsilon = \int_0^\infty \int_{D\times\mathbb{R}^n} \beta'_\epsilon(t)\phi(x, v)f(t, x, v)dvdxdt,$$

so that (2.2.11) writes

$$I_\epsilon + J_\epsilon = -\langle f_0, \phi\rangle_{D\times\mathbb{R}^n}.$$

Since $\phi \in \mathcal{D}(D\times\mathbb{R}^n)$, $\beta_\epsilon \le 1$ and $\beta_\epsilon(t) \to 0$ a.e. as ϵ converges to 0 the dominated convergence theorem gives immediatly $\lim\limits_{\epsilon\to 0} J_\epsilon = 0$. On the other hand, since $\int_0^\epsilon |\beta_\epsilon'(t)|dt = -\int_0^\epsilon \beta_\epsilon'(t)dt = 1$,

$$I_\epsilon = \Delta_\epsilon - \int_{D\times\mathbb{R}^n} \phi(x,v) f(0,x,v)dvdx,$$

with

$$\Delta_\epsilon = \int_0^\infty \int_{D\times\mathbb{R}^n} \beta_\epsilon'(t)\phi(x,v)\Big(f(t,x,v) - f(0,x,v)\Big)dvdxdt.$$

We have,

$$\begin{aligned}|\Delta_\epsilon| &\le \int_0^\epsilon |\beta_\epsilon'(t)|dt\Big|\int_{D\times\mathbb{R}^n} \phi(x,v)\Big(f(t,x,v) - f(0,x,v)\Big)dvdx\Big| \\ &\le \sup_{t\in[0,\epsilon]} \Big|\int_{D\times\mathbb{R}^n} \phi(x,v)\Big(f(t,x,v) - f(0,x,v)\Big)dvdx\Big|.\end{aligned}$$

The resulting supremum converges to 0 as ϵ goes to 0 using the weak continuity of f. Taking the limit as ϵ goes to 0 in (2.2.11) completes the proof of (2.2.10).

Step 2. We now show uniqueness of the solution through a contraction result in $L^1(D\times\mathbb{R}^n)$. Consider two solutions g_1, g_2 of (2.2.9) with the same initial datum g_0. By linearity, $f = g_1 - g_2$ is again a solution to (2.2.9) the problem with an initial datum $f_0 \equiv 0$ (the trace being $\gamma f = \gamma g_1 - \gamma g_2$ by linearity of the Green's formula (2.2.7)). Let $\beta \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ such that $|\beta(y)| \le C_\beta(1+|y|)$, for some constant $C_\beta > 0$ and for all $y \in \mathbb{R}$. From [101, Proposition 2] (note that our hypothesis on γf implies $\gamma f \in L^1_{\text{loc}}((0,\infty)\times\partial D\times\mathbb{R}^n, |v\cdot n_x|^2 dvdxdt)$), we know that

$$\begin{aligned}L\beta(f) = (\partial_t + v\cdot\nabla_x)\beta(f) &= 0, &&\text{in } \mathcal{D}'((0,\infty)\times D\times\mathbb{R}^n), \\ \gamma\beta(f) &= \beta(\gamma f), &&\text{in } (0,T)\times\partial D\times\mathbb{R}^n.\end{aligned}$$

We now choose $\beta(y) = |y|$, which satisfies the previous requirements. We set $0 < t_0 < t_1$ and for all $\epsilon \in (0,t_0)$, $\delta_\epsilon(t) = \mathbf{1}_{(t_0,t_1)}(t) + e^{-\frac{t-t_1}{\epsilon+t_1-t}}\mathbf{1}_{[t_1,t_1+\epsilon)} + e^{-\frac{t_0-t}{\epsilon+t-t_0}}\mathbf{1}_{(t_0-\epsilon,t_0)}$ and apply the Green's formula (2.2.7) to $|f|$ with the test function $\psi(t,x,v) = \delta_\epsilon(t)\phi(x,v)$ for all (t,x,v) in $[0,\infty)\times\bar{D}\times\mathbb{R}^n$, where $\phi \in \mathcal{D}(D\times\mathbb{R}^n)$, so that $\psi \in \mathcal{D}((0,\infty)\times D\times\mathbb{R}^n)$ using that δ_ϵ is smooth with support in $(t_0-\epsilon, t_1+\epsilon)$. We obtain

$$0 = \int_{t_0}^{t_1} \int_{D\times\mathbb{R}^n} |f| L\psi dvdxds = \Big[\int_{D\times\mathbb{R}^n} |f|\psi dvdx\Big]_{t_0}^{t_1}.$$

Since $\delta_\epsilon(t_1) = \delta_\epsilon(t_0) = 1$, we deduce

$$\int_{D\times\mathbb{R}^n} (|f(t_1)| - |f(t_0)|)\phi(x,v)dxdv = 0.$$

Since f is weakly continuous, we let $t_0 \to 0$, and, using $|f(0)| = |f_0| = 0$ almost everywhere in $D\times\mathbb{R}^n$, we conclude that for all $t_1 > 0$

$$\int_{D\times\mathbb{R}^n} |f(t_1,x,v)|\phi(x,v)dxdv = 0,$$

for all $\phi \in \mathcal{D}(D\times\mathbb{R}^n)$. This completes the proof. □

In the next subsection, we construct a stochastic process from which we obtain a weak solution to the problem. Ultimately, we show the following well-posedness result, which follows from Theorem 2.2.2, Propositions 2.3.2 and 2.3.3.

Theorem 2.2.3.

(i) Let $\rho_0 \in \mathcal{P}(D\times\mathbb{R}^n)$. There exists a weak solution ρ in the sense of Definition 2.2.1 to (2.1.1) with inital data ρ_0. This solution writes $\rho(dt,dx,dv) = dt f_t(dx,dv)$ on Σ, with $t \to f_t$ right-continuous from $[0,\infty)$ to $\mathcal{P}(D\times\mathbb{R}^n)$.

(ii) If moreover ρ_0 admits a density $f_0 \in L^1(D\times\mathbb{R}^n)$, then for all $t \geq 0$, f_t admits a density $f(t,.)$ with respect to the Lebesgue measure on $D\times\mathbb{R}^n$. We have, for all $T > 0$, $f \in C([0,T]; L^1(\bar{D}\times\mathbb{R}^n))$ and the trace measure of f, that γf belongs to the space $L^1([0,T]\times\partial D\times\mathbb{R}^n, |v\cdot n_x|dtdxdv)$. Hence f is the unique weak solution to (2.1.1) with such regularity.

2.3 Probabilistic setting

In this section, we build a stochastic process which corresponds to the evolution of a gas particle. Then we show that its law (roughly speaking) is a weak solution in the sense of Definition 2.2.1 of (2.1.1), and enjoys the regularity requirements of Theorem 2.2.2 when the initial condition admits a density.

2.3.1 Construction of the process

We start by setting some notations that will show useful in the construction of the stochastic process. We set $\mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0,\pi)^{n-2}$. We recall that the Jacobian of the hyperspherical change of variables $v \to (r, \theta_1, \dots, \theta_{n-1})$ from $\mathbb{R}^n$ to the space $(0,\infty)\times[-\pi,\pi)\times[0,\pi)^{n-2}$ is given by $r^{n-1}\prod_{j=1}^{n-2}\sin(\theta_j)^{n-1-j}$. For $r \in \mathbb{R}_+$, we abusively write $M(r) = M(v)$ with $v \in \mathbb{R}^n$, $\|v\| = r$.

Lemma 2.3.1. *We define the density* $h_R : \mathbb{R}_+ \to \mathbb{R}_+$ *by* $h_R(r) = c_R r^n M(r)\mathbf{1}_{\{r\geq 0\}}$*, where* c_R *is a normalizing constant. Let also* h_Θ *the density on* $\mathcal{A}$ *defined by*

$$h_\Theta(\theta_1, \dots, \theta_{n-1}) = c_\Theta \cos(\theta_1) \prod_{j=1}^{n-2} \sin(\theta_j)^{n-1-j}.$$

We write Υ *for the law of* (R, Θ)*,* R *having density* h_R*,* Θ *having density* h_Θ *independent of* R*.*

There exists a measurable function $\vartheta : \partial D \times \mathcal{A} \to \mathbb{R}^n$ *such that for any* $x \in \partial D$*, any* Υ*-distributed random variable* (R, Θ)*,*

$$R\vartheta(x, \Theta) \sim c_0 M(v)|v \cdot n_x|\mathbf{1}_{\{v\cdot n_x>0\}}, \tag{2.3.1}$$

and such that for all $\theta = (\theta_1, \dots, \theta_{n-1}) \in \mathcal{A}$*,* $x \in \partial D$

$$\vartheta(x, \theta) \cdot n_x = \cos(\theta_1). \tag{2.3.2}$$

Proof. For $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$, we define, $P : \mathbb{R}^n \to [0, \infty)\times[-\pi, \pi)\times[0, \pi]^{n-2}$, which, to a vector expressed in the $(e_1, \dots, e_n)$ coordinates, gives the associated hyperspherical coordinates (with polar axis e_1). For $x \in \partial D$, we fix an orthonormal basis $(n_x, f_2 \dots, f_n)$ of $\mathbb{R}^n$ and consider the isometry ξ_x that sends $(e_1, \dots, e_n)$ to $(n_x, f_2, \dots, f_n)$. We then set, for $\theta \in \mathcal{A}$,

$$\vartheta(x, \theta) = \Big(\xi_x \circ P^{-1}\Big)(1, \theta).$$

With this construction, ϑ is such that (2.3.1) holds. Finally, by definition of P and ξ_x, we have

$$\cos(\theta_1) = P^{-1}(1, \theta) \cdot e_1 = \xi_x\Big(P^{-1}(1, \theta)\Big) \cdot n_x,$$

as desired. □

Remark 2.3.1. Note that the fact that $\int_{\mathbb{R}_+} s^n M(s)ds < \infty$ follows from $\int_{\mathbb{R}^n} \|v\|M(v)dv < \infty$, see Hypothesis 2.1.1, using hyperspherical coordinates.

Notation 2.3.1. We introduce two important deterministic maps. Define $\zeta : \bar{D} \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$\zeta(x, v) = \begin{cases} \inf\{s > 0, x + sv \in \partial D\}, & \text{if } (x, v) \in G \cup \partial_- G, \\ 0, & \text{if } (x, v) \in \partial_+ G \cup \partial_0 G. \end{cases} \tag{2.3.3}$$

We also define $q : \bar{D} \times \mathbb{R}^n \to \partial D$ by

$$q(x, v) = \begin{cases} x + \zeta(x, v)v, & \text{if } (x, v) \in G \cup \partial_- G, \\ x, & \text{if } (x, v) \in \partial_+ G \cup \partial_0 G. \end{cases} \tag{2.3.4}$$

For a gas particle governed by the dynamics of (2.1.1), in position $(x, v) \in \bar{D} \times \mathbb{R}^n$ at time $t = 0$, $\zeta(x, v)$ is the time of its first collision with the boundary, while $q(x, v)$ is the point of ∂D where this collison occurs. The value attributed to those functions on $\partial_0 G$ has no consequences on our study, since our dynamic forbids the occurence of this situation.

Recall that $\eta_x(v) = v - 2(v \cdot n_x)n_x$ for all $(x, v) \in \partial D \times \mathbb{R}^n$.

Notation 2.3.2. We define the map $w : \partial D \times \mathbb{R}^n \times [0, 1] \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}^n$ by

$$w(x, v, u, r, \theta) = \eta_x(v)\mathbf{1}_{\{u>\alpha(x)\}} + r\vartheta(x, \theta)\mathbf{1}_{\{u\leq\alpha(x)\}}. \tag{2.3.5}$$

We write $\mathcal{U}$ for the uniform distribution over $[0, 1]$, and denote $\mathcal{Q}$ the measure $\mathcal{U} \otimes \Upsilon$.

Let us define, given an appropriate sequence of inputs, our process.

Definition 2.3.1. Consider an initial distribution ρ_0 on $(D \times \mathbb{R}^n) \cup \partial_- G$, a sequence of i.i.d. random vectors $(U_i, R_i, \Theta_i)_{i\geq 1}$ of law $\mathcal{Q}$. We define the stochastic process $(X_t, V_t)_{t\geq 0}$ as follows:

Step 0: Let (X_0, V_0) be distributed according to ρ_0.

Step 1: Set $T_1 = \zeta(X_0, V_0)$.
For $t \in [0, T_1)$, set $V_t = V_0$ and $X_t = X_0 + tV_0$.
Set $X_{T_1} = X_{T_1-}$ and $V_{T_1} = w(X_{T_1}, V_{T_1-}, U_1, R_1, \Theta_1)$.

Step k+1: Set $T_{k+1} = T_k + \zeta(X_{T_k}, V_{T_k})$.
For all $t \in (T_k, T_{k+1})$, set $X_t = X_{T_k} + (t - T_k)V_{T_k}$, $V_t = V_{T_k}$.
Set $X_{T_{k+1}} = X_{T_{k+1}-} \in \partial D$ and
$V_{T_{k+1}} = w(X_{T_{k+1}}, V_{T_{k+1}-}, U_{k+1}, R_{k+1}, \Theta_{k+1})$.

etc.

We say that $(X_s, V_s)_{s\geq 0}$ is a free-transport process with initial distribution ρ_0.

Remark 2.3.2. We extend the previous definition to the case where $(x, v) \in \partial_+ G$ and $\rho_0 = \delta_x \otimes \delta_v$, with, informally, $X_0 = x, V_{0-} = v$. In this case, we pick an extra triplet $(U_0, R_0, \Theta_0) \sim \mathcal{U} \otimes \Upsilon$ independent of everything else and we set

$$X_0 = x, \quad V_0 = w(x, v, U_0, R_0, \Theta_0).$$

Step 1 and further remain the same.

2.3.2 Non-explosion

In this section, we show that the process constructed in Definition 2.3.1 is almost surely well defined for all times $t > 0$. For $m \geq 1$, we write $\mathbb{S}^m = \{x \in \mathbb{R}^{m+1}, \|x\| = 1\}$ for the unit sphere in $\mathbb{R}^{m+1}$. Recall that any C^2 bounded domain satisfies the uniform interior ball condition and therefore the following interior cone condition, see for instance Fornaro [58, Proposition B.0.16 and its proof].

Definition 2.3.2. We say that a bounded set $D \subset \mathbb{R}^n$ satisfies the uniform cone condition if there exist $\beta \in (0,1)$, $h > 0$, such that for all $x \in \partial D$,

$$C_x = \{x + tu, t \in (0,h), u \cdot n_x > \beta, u \in \mathbb{S}^{n-1}\} \subset D.$$

Proposition 2.3.1. *Under Hypothesis 2.1.1, the sequence $(T_i)_{i\geq 1}$ of Definition 2.3.1 almost surely satisfies $T_i \to +\infty$ as $i \to +\infty$. More precisely, for any $T > 0$, $\mathbb{E}[\#\{i \geq 1 : T_i \leq T\}] < \infty$.*

Proof. Let h and β be the positive constants of the uniform cone condition corresponding to D. Recall that there exists a constant $\alpha_0 > 0$ such that for any $x \in \partial D$, $\alpha(x) \geq \alpha_0$. For N large enough, writing $\Theta_1 = (\Theta_1^1, \dots, \Theta_1^{n-1}) \in \mathbb{R}^{n-1}$, we have

$$p = \mathbb{P}\Big(U_1 \leq \alpha_0, \cos(\Theta_1^1) > \beta, R_1 \in [0,N]\Big) > 0.$$

Using Borel-Cantelli's lemma, one concludes that almost surely, an infinite number of elements of the sequence $\Omega_i = \{U_i \leq \alpha_0, \cos(\Theta_i^1) > \beta, R_i \in [0,N]\}$ is realized. For all $i \geq 1$, on Ω_i, $\vartheta(X_{T_i}, \Theta_i) \cdot n_{X_{T_i}} > \beta$, whence $X_{T_i+t} = X_{T_i} + tV_{T_i} \in C_{X_{T_i}} \subset D$ for all $t \in [0, \frac{h}{N}]$, because $V_{T_i} = R_i \vartheta(X_{T_i}, \Theta_i)$ has a norm smaller than N.

Set $T_0 = 0$ and $\tau_i = T_{i+1} - T_i$ for all $i \geq 1$. By the previous observation, we have, on Ω_i,

$$\tau_i = \zeta(X_{T_i}, V_{T_i}) = \frac{|q(X_{T_i}, V_{T_i}) - X_{T_i}|}{R_i} \geq \frac{h}{N} > 0,$$

To conclude, note first that

$$\lim_{i\to+\infty} T_i \geq \sum_{j\geq 1} \tau_j \mathbf{1}_{\Omega_j} \geq \frac{h}{N} \sum_{j\geq 1} \mathbf{1}_{\Omega_j} = +\infty \quad \text{a.s.}$$

For the second part of the propositon, we let $T > 0$ and $N_T := \sup\{i \geq 1, \tau_1 + \dots + \tau_i \leq T\}$. For all $i \geq 1$, we let $(\sigma_i)_{i\geq 1}$ be the i.i.d. sequence defined by $\sigma_i = \frac{h}{N}\mathbf{1}_{\Omega_i}$, and define the random variable M_T by $M_T := \sup\{i \geq 1, \sigma_1 + \dots + \sigma_i < T\}$. We have

$$\mathbb{E}[\#\{i \geq 1 : T_i \leq T\}] \leq \mathbb{E}[N_T] + 1 \leq \mathbb{E}[M_T] + 1,$$

since for all $i \geq 1, \tau_i \geq \sigma_i$ almost surely. Since the sequence $(\sigma_i)_{i\geq 1}$ is i.i.d., it follows from a classical result of renewal theory that $\mathbb{E}[M_T] < \infty$, which terminates the proof. $\square$

2.3.3 Law of the process

Proposition 2.3.2. *Let $\rho_0 \in \mathcal{P}(D \times \mathbb{R}^n)$ and consider the process $(X_t, V_t)_{t\geq 0}$ from Definition 2.3.1. Set, for all $t \geq 0$, f_t to be the law of (X_t, V_t), and define the measure ρ on $\bar{\Sigma}$ by*

$$\rho(dt, dx, dv) = f_t(dx, dv)dt.$$

Then ρ is a weak solution to (2.1.1) in the sense of Definition 2.2.1. Moreover $t \to f_t(dx, dv)$ is right-continuous from $(0, \infty)$ to $\mathcal{P}(\bar{D} \times \mathbb{R}^n)$ endowed with the weak convergence of measures.

Remark 2.3.3. The boundary measures corresponding to ρ in Definition 2.2.1 are given by

$$\rho_{\pm}(A) = \mathbb{E}\big[\sum_{i\geq 1} \mathbf{1}_{(T_i, X_{T_i}, V_{T_i})\in A}\big], \quad A \in \mathcal{B}((0,\infty) \times \partial_{\pm} G).$$

Proof of Proposition 2.3.2. From its definition, it is clear that ρ is a non-negative Borel measure on $\bar{\Sigma}$. For all $T > 0$,

$$\rho((0,T) \times G) = \int_0^T \mathbb{E}[\mathbf{1}_{(t,X_t,V_t)\in\Sigma}]dt \leq T,$$

so that ρ is also Radon.

For $i \geq 1$, we introduce two probability measures $\rho^i_{\pm}$ on $\mathbb{R}_+ \times \partial_{\pm} G$: ρ^i_+ is the law of the triple (T_i, X_{T_i}, V_{T_i-}) and ρ^i_- is the law of the triple (T_i, X_{T_i}, V_{T_i}).

We now prove that for all $i \geq 1$, $\rho^i_- = K\rho^i_+$. For $B \in \mathcal{B}(\mathbb{R}_+ \times \partial_- G)$, using the definition of $(V_t)_{t\geq 0}$, we have

$$\begin{aligned}
\rho^i_-(B) &= \mathbb{E}[\mathbf{1}_{(T_i,X_{T_i},V_{T_i})\in B}] \\
&= \mathbb{E}\Big[\alpha(X_{T_i})\mathbf{1}_{\left(T_i,X_{T_i},R_i\vartheta(X_{T_i},\Theta_i)\right)\in B}\Big] + \mathbb{E}\Big[(1-\alpha(X_{T_i}))\mathbf{1}_{\left(T_i,X_{T_i},\eta_{X_{T_i}}(V_{T_i-})\right)\in B}\Big]
\end{aligned}$$

Using (2.3.1), we deduce,

$$\begin{aligned}
\rho^i_-(B) &= \int_{(0,\infty)\times\partial_+ G}\int_{\{w\in\mathbb{R}^n, w\cdot n_x>0\}} \alpha(x)\mathbf{1}_{\{(t,x,w)\in B\}} c_0 M(w)|w\cdot n_x|dw\rho^i_+(dt,dx,dv) \\
&\quad + \int_{(0,\infty)\times\partial_+ G} \mathbf{1}_{\{(t,x,\eta_x(v))\in B\}}(1-\alpha(x))\rho^i_+(dt,dx,dv) \\
&= K\rho^i_+(B),
\end{aligned}$$

recall (2.2.2). Setting $\rho_+(A) = \sum_{i\geq 1}\rho^i_+(A)$ for all $A \in \mathcal{B}(\mathbb{R}_+ \times \partial_+ G)$, $\rho_-(B) = \sum_{i\geq 1}\rho^i_-(B)$ for all $B \in \mathcal{B}(\mathbb{R}_+ \times \partial_- G)$, we deduce that $\rho_- = K\rho_+$ on $\mathbb{R}_+ \times \partial_- G$.

We now prove (2.2.6). Let $\phi \in \mathcal{D}(\bar{\Sigma})$. We have, by definition of ρ and using Definition 2.3.1,

$$\begin{aligned}
\langle \rho, L\phi \rangle_\Sigma &= \int_0^\infty \mathbb{E}[L\phi(t, X_t, V_t)]dt \\
&= \int_0^\infty \mathbb{E}\Big[\sum_{i=0}^\infty \mathbf{1}_{\{T_i \le t < T_{i+1}\}} L\phi(t, X_{T_i} + (t - T_i)V_{T_i}, V_{T_i})\Big]dt \\
&= \sum_{i=0}^\infty \mathbb{E}\Big[\int_{T_i}^{T_{i+1}} (\partial_t + V_{T_i} \cdot \nabla_x)\phi(t, X_{T_i} + (t - T_i)V_{T_i}, V_{T_i})dt\Big] \\
&= \sum_{i=0}^\infty \mathbb{E}\Big[\int_{T_i}^{T_{i+1}} \frac{d}{dt}\Big(\phi(t, X_{T_i} + (t - T_i)V_{T_i}, V_{T_i})\Big)dt\Big].
\end{aligned}$$

As a conclusion,

$$\begin{aligned}
\langle \rho, L\phi \rangle_\Sigma &= \mathbb{E}\Big[\sum_{i=0}^\infty \phi(T_{i+1}, X_{T_i} + (T_{i+1} - T_i)V_{T_i}, V_{T_i})\Big] - \mathbb{E}\Big[\sum_{i=1}^\infty \phi(T_i, X_{T_i}, V_{T_i})\Big] \\
&\quad - \mathbb{E}[\phi(0, X_0, V_0)] \\
&= \mathbb{E}\Big[\sum_{i=0}^\infty \phi(T_{i+1}, X_{T_{i+1}}, V_{T_{i+1}-})\Big] - \langle \rho_-, \phi \rangle_{(0,\infty)\times\partial_- G} - \langle \rho_0, \phi(0, .) \rangle_{D\times\mathbb{R}^n} \\
&= \langle \rho_+, \phi \rangle_{(0,\infty)\times\partial_+ G} - \langle \rho_-, \phi \rangle_{(0,\infty)\times\partial_- G} - \langle \rho_0, \phi(0, .) \rangle_{D\times\mathbb{R}^n},
\end{aligned}$$

which concludes the proof that ρ is a weak solution. Observe that all the computations above can easily be justified because there exists some $T > 0$ such that $\mathrm{supp}(\phi) \subset [0, T] \times \bar{D} \times \mathbb{R}^n$.

The right-continuity of $t \to f_t$ on $(0, \infty)$ is a straightforward result given that $(X_t)_{t\ge0}$ is continuous and $(V_t)_{t\ge0}$ is càdlàg on $(0, \infty)$ according to Definition 2.3.1. $\square$

In the next proposition, we study the regularity of the solution given by Proposition 2.3.2 in the case where the initial data ρ_0 has a density in $D \times \mathbb{R}^n$.

Proposition 2.3.3. *For ρ_0 having a density $f_0 \in L^1(D \times \mathbb{R}^n)$, the Radon measure ρ defined in Proposition 2.3.2 admits a density f with respect to the Lebesgue measure in $\mathbb{R}_+ \times \bar{D} \times \mathbb{R}^n$. Moreover, $f \in C([0, \infty); L^1(\bar{D} \times \mathbb{R}^n))$. The non-negative measures $\rho_\pm$ satisfy*

$$\rho_\pm(dt, dx, dv) = \gamma_\pm f(t, x, v)|v \cdot n_x| dtdxdv \ \ on \ (0, \infty) \times \partial_\pm G,$$

where $\gamma f \in L^1([0, T] \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dvdxdt)$ for all $T > 0$ is the trace measure of f given by Theorem 2.2.1 and where we write $\gamma_\pm f$ for its restrictions to $(0, \infty) \times \partial_\pm G$.

Observe that we can indeed apply Theorem 2.2.1 because (i) we have the inclusion $L^1([0, T] \times \partial D \times \mathbb{R}^n, |v \cdot n_x| dvdxdt) \subset L^1_{\mathrm{loc}}([0, \infty) \times \partial D \times \mathbb{R}^n, (v \cdot n_x)^2 dvdxdt)$, and (ii) $Lf = 0$ in $\mathcal{D}'((0, \infty) \times D \times \mathbb{R}^n)$ since ρ is a weak solution to (2.1.1), see Step 7 below.

Proof. Recall that for $i \geq 1$, ρ_+^i denotes the law of (T_i, X_{T_i}, V_{T_i-}), with the sequence $(T_i)_{i\geq 1}$ and the process $(X_t, V_t)_{t\geq 0}$, of Definition 2.3.1. For all $i \geq 1$, we also write ρ_-^i for the law of (T_i, X_{T_i}, V_{T_i}).

Step 1. We show that ρ_+^1 has a density with respect to $|v\cdot n_x|dvdxdt$. For $A \in \mathcal{B}(\mathbb{R}_+\times\partial_+ G)$,

$$\begin{aligned}\rho_+^1(A) &= \mathbb{E}[\mathbf{1}_{\{(T_1, X_{T_1}, V_{T_1-})\in A\}}] = \mathbb{E}[\mathbf{1}_{\{(\zeta(X_0,V_0), q(X_0,V_0), V_0)\in A\}}] \\ &= \int_{D\times\mathbb{R}^n} \mathbf{1}_{\{(\zeta(x,v), q(x,v), v)\in A\}} f_0(x,v)dvdx.\end{aligned}$$

For any fixed $v \in \mathbb{R}^n$, the map $x \to \big(y = q(x,v), s = \zeta(x,v)\big)$ is a C^1 diffeomorphism from D to $\{(y,s) : y \in \partial D, v\cdot n_y < 0, s \in [0, \zeta(y,-v))\}$, and the Jacobian is given by $|v\cdot n_y|$, see Lemma 2.3 of [54] where $\tau_b(x,v) = \zeta(x,-v)$ with our notations. Applying this change of variables, we obtain

$$\rho_+^1(A) = \int_{\partial_+ G}\int_0^{\zeta(y,-v)} \mathbf{1}_{\{(s,y,v)\in A\}} f_0(y - sv, v)|v\cdot n_y| ds dv dy.$$

Hence ρ_+^1 has a density with respect to the measure $|v\cdot n_x|dvdxdt$ on $\mathbb{R}_+\times\partial_+ G$.

Step 2. We show that for all $i \geq 1$, assuming that ρ_+^i has a density g_+^i, ρ_-^i has a density g_-^i with respect to the measure $|v\cdot n_x|dvdxdt$ on $\mathbb{R}_+\times\partial_- G$. For $A \in \mathcal{B}(\mathbb{R}_+\times\partial_- G)$,

$$\begin{aligned}\rho_-^i(A) &= \mathbb{E}[\mathbf{1}_{\{(T_i, X_{T_i}, V_{T_i})\in A\}}] \\ &= \mathbb{E}\Big[\alpha(X_{T_i})\mathbf{1}_{\{(T_i, X_{T_i}, R_i\vartheta(X_{T_i},\Theta_i))\in A\}}\Big] + \mathbb{E}\Big[(1-\alpha(X_{T_i}))\mathbf{1}_{\{(T_i, X_{T_i}, \eta_{X_{T_i}}(V_{T_i-}))\in A\}}\Big],\end{aligned}$$

where we recall that $\eta_x(v) = v - 2(v\cdot n_x)n_x$. We obtain, recalling Lemma 2.3.1,

$$\begin{aligned}\rho_-^i(A) &= \int_{\partial_+ G\times\mathbb{R}_+}\alpha(x)\Big(\int_{\{v'\cdot n_x>0\}}\mathbf{1}_{\{(\tau,x,v')\in A\}} c_0 M(v')|v'\cdot n_x|dv'\Big) g_+^i(\tau,x,v)|v\cdot n_x|d\tau dv dx \\ &\quad + \int_{\partial_+ G}\int_{\mathbb{R}_+}(1-\alpha(x))\mathbf{1}_{\{(\tau,x,\eta_x(v))\in A\}} g_+^i(\tau,x,v)|v\cdot n_x|d\tau dv dx \\ &= \int_{\partial_- G\times\mathbb{R}_+}\mathbf{1}_{\{(\tau,x,v')\in A\}}\Big(\alpha(x)c_0M(v')\int_{\{v\cdot n_x<0\}} g_+^i(\tau,x,v)|v\cdot n_x|dv\Big)|v'\cdot n_x|d\tau dv' dx \\ &\quad + \int_{\partial_- G}\int_{\mathbb{R}_+}\mathbf{1}_{\{(\tau,x,v)\in A\}}\Big((1-\alpha(x))g_+^i(\tau,x,v-2(v\cdot n_x)n_x)\Big)|v\cdot n_x|d\tau dv dx,\end{aligned}$$

where we have used that the change of variable $v \to (w = \eta_x(v))$ is involutive for any $x \in \partial D$. We conclude that for $(t,x,v) \in (0,\infty)\times\partial_- G$,

$$g_-^i(t,x,v) = \alpha(x)c_0M(v)\int_{\{v'\cdot n_x<0\}} g_+^i(t,x,v')|v'\cdot n_x|dv' + (1-\alpha(x))g_+^i(t,x,v-2(v\cdot n_x)n_x),$$

and therefore for all $i \geq 1$, ρ_-^i has a density with respect to $|v\cdot n_x|dvdxdt$ on $\mathbb{R}_+\times\partial_- G$.

Step 3. We show that for all $i \geq 1$, for all $t \geq 0$, assuming that ρ_-^i has a density g_-^i, the law f_t^i of (X_t, V_t) restricted to (T_i, T_{i+1}) has a density on $D \times \mathbb{R}^n$ with respect to the Lebesgue measure. For $A \in \mathcal{B}(D \times \mathbb{R}^n)$,

$$\begin{aligned} f_t^i(A) &= \mathbb{E}[\mathbf{1}_{\{(X_t,V_t)\in A\}}\mathbf{1}_{\{T_i<t<T_{i+1}\}}] \\ &= \mathbb{E}[\mathbf{1}_{\{(X_{T_i}+(t-T_i)V_{T_i},V_{T_i})\in A\}}\mathbf{1}_{\{T_i<t<T_i+\zeta(X_{T_i},V_{T_i})\}}] \\ &= \int_{\partial_- G}\int_0^t \mathbf{1}_{\{(x+(t-\tau)v,v)\in A\}}\mathbf{1}_{\{\tau<t<\tau+\zeta(x,v)\}}|v\cdot n_x|g_-^i(\tau,x,v)d\tau dv dx. \end{aligned}$$

For any fixed $v \in \mathbb{R}^n$, $t \in (0,\infty)$, $(x,\tau) \to (y = x + (t-\tau)v)$, is a C^1-diffeomorphism from $\{(x,\tau) \in \partial D \times (0,\infty) : v \cdot n_x > 0,\ \tau < t < \tau + \zeta(x,v)\}$ to D such that $x = q(y,-v)$, $t - \tau = \zeta(y,-v)$ and is the inverse of the C^1-diffeomorphism of Step 1. Hence, its Jacobian is given by $\frac{1}{|v\cdot n_x|} \neq 0$, and we obtain,

$$f_t^i(A) = \int_{D\times\mathbb{R}^n} \mathbf{1}_{\{(y,v)\in A\}} g_-^i(t-\zeta(y,-v), q(y,-v), v)dydv,$$

and therefore f_t^i has a density g_t^i over $D \times \mathbb{R}^n$.

Step 4. One easily shows that for all $t \geq 0$, f_t^0, the law of (X_t, V_t) restricted to $[0, T_1)$ also has a density with respect to the Lebesgue measure. Indeed, it is enough to write, for any $A \in \mathcal{B}(D \times \mathbb{R}^n)$,

$$f_t^0(A) = \mathbb{E}[\mathbf{1}_{\{t<T_1,(X_0+tV_0,V_0)\in A\}}],$$

and to use that (X_0, V_0) has a density.

Step 5. We now prove that, for all $i \geq 0$, if f_t^i has a density g_t^i for all $t \geq 0$, then ρ_+^{i+1} has a density with respect to the measure $|v \cdot n_x|dtdvdx$ on $\mathbb{R}_+ \times \partial_+ G$. For $A \in \mathcal{B}(\mathbb{R}_+ \times \partial_+ G)$,

$$\begin{aligned} \rho_+^{i+1}(A) &= \mathbb{E}[\mathbf{1}_{\{(T_{i+1},X_{T_{i+1}},V_{T_{i+1}-})\in A\}}] \\ &= \mathbb{E}\Big[\int_{T_i}^{T_{i+1}} \mathbf{1}_{\{(T_{i+1},X_{T_{i+1}},V_{T_{i+1}-})\in A\}}\frac{1}{T_{i+1}-T_i}dt\Big] \\ &= \int_0^\infty \mathbb{E}\Big[\mathbf{1}_{\{(t+\zeta(X_t,V_t),q(X_t,V_t),V_t)\in A\}}\frac{\mathbf{1}_{\{T_i<t<T_{i+1}\}}}{t+\zeta(X_t,V_t)-(t-\zeta(X_t,-V_t))}\Big]dt \\ &= \int_0^\infty\int_{D\times\mathbb{R}^n} \mathbf{1}_{\{(t+\zeta(x,v),q(x,v),v)\in A\}}\frac{1}{t+\zeta(x,v)-(t-\zeta(x,-v))}g_t^i(x,v)dvdxdt. \end{aligned}$$

We used that for $t \in (T_i, T_{i+1}), T_i = t - \zeta(X_t, -V_t)$, $T_{i+1} = t + \zeta(X_t, V_t)$, $X_{T_{i+1}} = q(X_t, V_t)$ and $V_{T_{i+1}-} = V_t$. We use a slightly modified change of variables compared to Step 1: for a fixed $t \in \mathbb{R}^+$ and a fixed $v \in \mathbb{R}^n$, we consider $x \to \big(y = q(x,v), \tau = t + \zeta(x,v)\big)$. This diffeomorphism from D to $\{(y,\tau) \in \partial D \times (0,\infty) : v \cdot n_y < 0, t < \tau < t + \zeta(y,-v)\}$ has a Jacobian equal to

$|v \cdot n_y|$, as in Step 1. Therefore, since $\zeta(x,v)+\zeta(x,-v)=\zeta(y,-v)$ and $x=y-(\tau-t)v$,

$$\begin{aligned}\rho_+^{i+1}(A) &= \int_0^\infty \int_{\partial_+ G}\int_0^\infty \frac{\mathbf{1}_{\{(\tau,y,v)\in A\}}}{\zeta(y,-v)}\mathbf{1}_{\{\tau-\zeta(y,-v)<t<\tau\}} g_t^i(y-(\tau-t)v,v)|v\cdot n_y| d\tau dv dy dt \\ &= \int_{\partial_+ G}\int_0^\infty \mathbf{1}_{\{(\tau,y,v)\in A\}}|v\cdot n_y| \frac{1}{\zeta(y,-v)}\Big(\int_{\tau-\zeta(y,-v)}^{\tau} g_t^i(y-(\tau-t)v,v)dt\Big) d\tau dv dy,\end{aligned}$$

and this shows that ρ_+^{i+1} has a density with respect to the measure $|v\cdot n_x| dv dx dt$ on $(0,\infty)\times\partial_+ G$.

Step 6. From Steps 1 to 5, we conclude that for all $i\geq 1$, $\rho_\pm^i$ have a density $g_\pm^i$ with respect to the measure $|v\cdot n_x|dvdxdt$ on $(0,\infty)\times\partial_\pm G$. Thus, $\rho_\pm=\sum_{i\geq 1}\rho_\pm^i$ also have a density with respect to $|v\cdot n_x|dvdxdt$ on $(0,\infty)\times\partial_\pm G$ that we write $g_\pm$. The function defined by

$$g(t,x,v)=g_+(t,x,v)\mathbf{1}_{\{v\cdot n_x<0\}}+g_-(t,x,v)\mathbf{1}_{\{v\cdot n_x>0\}}, \quad (t,x,v)\in\mathbb{R}_+\times\partial D\times\mathbb{R}^n, \tag{2.3.6}$$

belongs to $L^1([0,T)\times\partial D\times\mathbb{R}^n, |v\cdot n_x|dtdxdv)$ for all $T>0$, because

$$\rho_\pm([0,T]\times\partial_\pm G)=\mathbb{E}[\#\{i: T_i\leq T\}]<\infty,$$

by Proposition 2.3.1. Consequently, g belongs to $L^1_{\text{loc}}([0,T)\times\partial D\times\mathbb{R}^n, |v\cdot n_x|^2 dtdxdv)$. A second conclusion from those steps is that the measure f_t has a density on $D\times\mathbb{R}^n$ for all $t\geq 0$. Hence ρ has a density f on $\mathbb{R}_+\times\bar{D}\times\mathbb{R}^n$.

Step 7. Note that, because $\rho(dt,dx,dv)=f(t,x,v)dtdxdv$ satisfies (2.2.6), we obviously have that f satisfies

$$Lf=0\in\mathcal{D}'((0,\infty)\times D\times\mathbb{R}^n).$$

Using Theorem 2.2.1, we conclude that $f\in C([0,\infty);L^1_{\text{loc}}(\bar{D}\times\mathbb{R}^n))$, and then that f belongs to $C([0,\infty);L^1(\bar{D}\times\mathbb{R}^n))$ since for all $t\geq 0$, $f(t,.)$ is a probability density.

Step 8. There only remains to prove that the function g defined by (2.3.6) is the trace of f in the sense of Theorem 2.2.1. We want to show that for any $0\leq t_0<t_1$, any $\phi\in\mathcal{D}((0,\infty)\times\bar{D}\times\mathbb{R}^n)$ such that $\phi=0$ on $(0,\infty)\times\partial_0 G$, we have

$$\int_{t_0}^{t_1}\int_G fL\phi dvdxdt=\Big[\int_G f\phi dvdx\Big]_{t_0}^{t_1}-\int_{t_0}^{t_1}\int_{\partial D\times\mathbb{R}^n} g(t,x,v)(n_x\cdot v)\phi dvdxdt.$$

By substraction, this can be reduced to proving that

$$\int_0^{t_1}\int_G fL\phi dvdxdt=\int_G f(t_1,x,v)\phi(t_1,x,v)dxdv-\int_0^{t_1}\int_{\partial D\times\mathbb{R}^n} g\,(n_x\cdot v)\phi dvdxdt, \tag{2.3.7}$$

for any $t_1>0$, any $\phi\in\mathcal{D}((0,\infty)\times\bar{D}\times\mathbb{R}^n)$, $\phi=0$ on $(0,\infty)\times\partial_0 G$.

For any $\epsilon \in (0,1)$, let $\beta_\epsilon(t) = \mathbf{1}_{(0,t_1)}(t) + e^{-\frac{t-t_1}{\epsilon+t_1-t}}\mathbf{1}_{[t_1,t_1+\epsilon)}(t)$. Applying (2.2.6) with the test function $\beta_\epsilon\phi$, recalling that $\rho(dt,dx,dv) = f(t,x,v)dtdxdv$, $\rho_\pm(dt,dx,dv) = g_\pm(t,x,v)|v\cdot n_x|$ so that $(\rho_+ - \rho_-)(dt,dx,dv) = -g(t,x,v)(v\cdot n_x)dtdxdv$, we find

$$\int_0^\infty \int_G \beta_\epsilon' f\phi dvdxdt + \int_0^\infty \int_G \beta_\epsilon f L\phi dvdxdt = -\int_0^\infty \int_{\partial D\times\mathbb{R}^n} g(v\cdot n_x)\beta_\epsilon\phi dvdxdt.$$

We rewrite this equation as

$$A_\epsilon + B + C_\epsilon = -D_\epsilon,$$

by setting

$$\begin{aligned} A_\epsilon &= \int_{t_1}^{t_1+\epsilon}\int_G \beta_\epsilon'(t)\Big(f(t,x,v)\phi(t,x,v) - f(t_1,x,v)\phi(t_1,x,v)\Big)dvdxdt,\\ B &= -\int_{D\times\mathbb{R}^n} f(t_1,x,v)\phi(t_1,x,v)dvdx,\\ C_\epsilon &= \int_0^\infty\int_G \beta_\epsilon f L\phi dvdxdt,\\ D_\epsilon &= \int_0^\infty\int_{\partial D\times\mathbb{R}^n} g(v\cdot n_x)\beta_\epsilon\phi dvdxdt, \end{aligned}$$

where we used that $\int_{t_1}^{t_1+\epsilon}\beta_\epsilon'(t)dt = -1$. We have

$$|A_\epsilon| \le \sup_{t\in[t_1,t_1+\epsilon]}\Big|\int_{D\times\mathbb{R}^n}(f(t,x,v)\phi(t,x,v) - f(t_1,x,v)\phi(t_1,x,v))dvdx\Big| \times \int_{t_1}^{t_1+\epsilon}|\beta_\epsilon'(t)|dt.$$

Hence $A_\epsilon \to 0$ as $\epsilon \to 0$, because $f \in C([0,\infty), L^1(\bar{D}\times\mathbb{R}^n))$ and by regularity of ϕ, see Step 7.

Since $\beta_\epsilon(t) \le 1$ for all $t \ge 0$, since $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+\times\bar{D}\times\mathbb{R}^n)$, by regularity of ϕ, and since $\beta_\epsilon(t) \to \mathbf{1}_{[0,t_1]}(t)$, a straightforward application of the dominated convergence theorem gives that $C_\epsilon \to \int_0^{t_1}\int_G fL\phi dvdxdt$ as $\epsilon \to 0$.

The same argument, along with the fact that $g \in L^1((0,T)\times\partial D\times\mathbb{R}^n, |v\cdot n_x|dtdvdx)$ allows us to conclude that

$$\lim_{\epsilon\to 0} D_\epsilon = \int_0^{t_1}\int_{\partial D\times\mathbb{R}^n} g\phi(v\cdot n_x)dvdxdt.$$

Overall, we obtain that g satisfies (2.3.7) for any $t_1 \ge 0$, any $\phi \in \mathcal{D}((0,\infty)\times\bar{D}\times\mathbb{R}^n)$ with $\phi = 0$ on $(0,\infty)\times\partial_0 G$, so that g is the trace of f in the sense of Theorem 2.2.1.

$\square$

2.4 The convex case

In this section, we prove Theorem 2.1.1 in the easier case where D is a C^2 uniformly convex bounded domain (open, connected) in $\mathbb{R}^n$.

The strategy is to build a coupling of two stochastic processes with the dynamic of Definition 2.3.1, $(X_t, V_t)_{t\geq 0}$ with initial distribution f_0, $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ with initial distribution μ_∞, where μ_∞ is the equilibrium distribution. For this couple of processes, two different regimes can be identified: a low-speed regime and a high-speed regime.

In a first step, we collect several results on the high-speed regime. In this situation, we find a coupling which is successful, in a sense to be defined, with a probability admitting a positive lower bound. In a second step, we detail the construction of the processes. Finally, we prove that

$$\tau = \inf\{t \geq 0 : (X_{t+s})_{s\geq 0} = (\tilde{X}_{t+s})_{s\geq 0}, (V_{t+s})_{s\geq 0} = (\tilde{V}_{t+s})_{s\geq 0}\}, \tag{2.4.1}$$

satisfies $\mathbb{E}[r(\tau)] < \infty$.

2.4.1 A coupling result.

Recall the notations h_R, Υ introduced in Lemma 2.3.1. Since M admits a density, there exists $a > 0$ such that,

$$\int_0^a h_R(x)dx > 0, \qquad \int_a^\infty h_R(x)dx > 0, \tag{2.4.2}$$

and we assume for simplicity that $a = 1$ in the sequel. Recall also that $\mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi]^{n-2}$, and $d(D) := \sup\limits_{(x,y)\in D^2} \|x - y\|$, which corresponds to the diameter of D. We introduce some more notations.

Notation 2.4.1. We define four maps:

i. the map $\xi : \partial D \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, such that

$$\xi(x, r, \theta) = \zeta(x, r\vartheta(x, \theta)),$$

ii. the map $y : \partial D \times \mathcal{A} \to \partial D$, such that

$$y(x, \theta) = q(x, \vartheta(x, \theta)),$$

iii. the map $\tilde{\xi} : \bar{D} \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{A} \to \mathbb{R}_+$, such that

$$\tilde{\xi}(x, v, r, \theta) = \zeta(x, v) + \zeta\Big(q(x, v), r\vartheta(q(x, v), \theta)\Big),$$

iv. the map $\tilde{y} : \bar{D} \times \mathbb{R}^n \times \mathcal{A} \to \partial D$, such that

$$\tilde{y}(x, v, \theta) = q\Big(q(x, v), \vartheta(q(x, v), \theta)\Big).$$

The main result in this section is the following proposition:

Proposition 2.4.1. *There exists a constant $c > 0$ such that for all $x_0 \in \partial D$, $\tilde{x}_0 \in D$, $\tilde{v}_0 \in \mathbb{R}^n$ with $\|\tilde{v}_0\| \geq 1$, there exists $\Lambda_{x_0,\tilde{x}_0,\tilde{v}_0} \in \mathcal{P}(((0,\infty) \times \mathcal{A})^2)$ such that, if $(R, \Theta, \tilde{R}, \tilde{\Theta})$ has law $\Lambda_{x_0,\tilde{x}_0,\tilde{v}_0}$, both (R, Θ) and $(\tilde{R}, \tilde{\Theta})$ have law Υ, and for*

$$E_{x_0,\tilde{x}_0,\tilde{v}_0} := \Big\{(r, \theta, \tilde{r}, \tilde{\theta}) \in (\mathbb{R}_+ \times \mathcal{A})^2 : y(x_0, \theta) = \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\theta}), \xi(x_0, r, \theta) = \tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{r}, \tilde{\theta})\Big\},$$

we have

$$\mathbb{P}\big((R, \Theta, \tilde{R}, \tilde{\Theta}) \in E_{x_0,\tilde{x}_0,\tilde{v}_0}\big) \geq c. \tag{2.4.3}$$

The rest of this subsection is devoted to the proof of this proposition.

Lemma 2.4.1. *There exist two constants $r_1 > 0$ and $c_1 > 0$ such that for all $(x, y) \in (\partial D)^2$,*

$$\int_{\{z \in \partial D, \|z-x\| \wedge \|z-y\| \geq r_1\}} \big(|(z-x) \cdot n_x||(z-x) \cdot n_z|\big) \wedge \big(|(z-y) \cdot n_y||(z-y) \cdot n_z|\big) dz \geq c_1. \tag{2.4.4}$$

Proof. Without loss of generality we assume that $0 \in D$. Recall that we write $\mathcal{H}$ for the $n-1$ dimensional Hausdorff measure.

We show first that there exists $c > 0$ such that for all $(x, y) \in (\partial D)^2$, $\mathcal{H}(A_{x,y}) \geq c$, where $A_{x,y} := \{z \in \partial D, \|z - x\| \wedge \|z - y\| \geq r_1\}$ for some $r_1 > 0$. Set $r_0 := \inf_{z \in \partial D} \|z\|$.

Note that for all $(x, y) \in (\partial D)^2$, for $\delta \in (0, 1)$, with $r_1 = r_0\sqrt{2 - 2\delta}$, we have the inclusion $A_{x,y} \subset A'_{x,y} := \{z \in \partial D, \frac{z}{\|z\|} \cdot \frac{x}{\|x\|} < \delta, \frac{z}{\|z\|} \cdot \frac{y}{\|y\|} < \delta\}$ since for all $z \in A'_{x,y}$,

$$\|x - z\|^2 \geq \|x\|^2 - 2\delta\|x\|\|z\| + \|z\|^2 \geq (\|x\| - \|z\|)^2 + (2 - 2\delta)\|z\|\|x\| \geq r_1^2, \tag{2.4.5}$$

and $\|y - z\| \geq r_1$ as well.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\phi(x) = \frac{x}{(2\|x\|) \vee r_0} r_0$ for any $x \in \mathbb{R}^n$. Note that ϕ is the projection on the closed ball $\bar{B}(0, \frac{r_0}{2}) := \{z \in \mathbb{R}^n, \|z\| \leq \frac{r_0}{2}\}$ and is thus 1-Lispschitz. By definition of r_0, setting $S := \{y \in \mathbb{R}^n, \|y\| = \frac{r_0}{2}\}$, we have $\phi(\partial D) = S$.

We apply the following statement: for $m \in \mathbb{N}^*$, for any Lipschitz map $f : \mathbb{R}^m \to \mathbb{R}^m$ with Lipschitz constant $L > 0$, for any $A \subset \mathbb{R}^m$,

$$\mathcal{H}(f(A)) \leq L^m \mathcal{H}(A), \tag{2.4.6}$$

see [94, Theorem 7.5]. We obtain that

$$\mathcal{H}\Big(\phi(A'_{x,y})\Big) \leq \mathcal{H}(A'_{x,y}).$$

Observe that $\phi(A'_{x,y}) = \{u \in S, \frac{u}{\|u\|} \cdot \frac{x}{\|x\|} < \delta, \frac{u}{\|u\|} \cdot \frac{y}{\|y\|} < \delta\}$ so that

$$\mathcal{H}\Big(\phi(A'_{x,y})\Big) \geq \mathcal{H}(S) - 2\mathcal{H}\Big(\Big\{u \in S, \frac{u \cdot e_1}{\|u\|} < \delta\Big\}\Big) \geq \frac{1}{2}\mathcal{H}(S),$$

if $\delta < \delta_0$ for some $\delta_0 > 0$ not depending on x and y, since $\mathcal{H}(\{u \in S, \frac{u \cdot e_1}{\|u\|} < \delta\})$ converges to 0 when δ goes to 0.

To conclude, it suffices to use that

$$\inf_{(a,b)\in(\partial D)^2, \|a-b\|\geq r_1} |(a-b)\cdot n_a| > 0,$$

which follows by compactness from the fact that D is C^1, bounded and uniformly convex. $\square$

Recall that the constant c_0 is defined by (2.1.2).

Lemma 2.4.2. *For $x \in \partial D$ and $V \sim c_0 M(v)|v \cdot n_x| \mathbf{1}_{\{v \cdot n_x > 0\}}$, the law of $(\zeta(x,V), q(x,V))$ admits a density μ_x on $\mathbb{R}_+^* \times (\partial D \setminus \{x\})$ given by*

$$\mu_x(\tau, z) = c_0 M\Big(\frac{z-x}{\tau}\Big)\frac{1}{\tau^{n+2}}|(z-x)\cdot n_x||(z-x)\cdot n_z|.$$

Proof. Let $A \in \mathcal{B}(\mathbb{R}_+ \times (\partial D \setminus \{x\}))$. We have

$$\mathbb{P}\Big((\zeta(x,V), q(x,V)) \in A\Big) = \int_{\{v\cdot n_x > 0\}} \mathbf{1}_{\{(\zeta(x,v),q(x,v))\in A\}} c_0 M(v)|v\cdot n_x| dv. \tag{2.4.7}$$

We show that this quantity is equal to

$$I := \int_0^\infty \int_{\partial D} \mathbf{1}_{\{(\tau,z)\in A\}} c_0 M\Big(\frac{z-x}{\tau}\Big)\frac{1}{\tau^{n+2}}|(z-x)\cdot n_x||(z-x)\cdot n_z| dz d\tau.$$

Consider the change of variable $(\tau, z) \to v$ given by $v = \frac{z-x}{\tau} =: \phi(\tau, z)$. Note that by uniform convexity, we have $v \cdot n_x > 0$ and $(\tau, z) = (\zeta(x,v), q(x,v))$. The map ϕ is a C^1 diffeomorphism between $\mathbb{R}_+ \times (\partial D \setminus \{x\})$ and $\{v \in \mathbb{R}^n, v \cdot n_x > 0\}$. Note that

1. the tangent space to $\mathbb{R}_+$ at $\tau \in \mathbb{R}_+$ is $\mathbb{R}$,
2. the tangent space to $\partial D \setminus \{x\}$ at $z \in \partial D \setminus \{x\}$ is $n_z^\perp \subset \mathbb{R}^n$,
3. the tangent space to $\{v \in \mathbb{R}^n, v\cdot n_x > 0\}$ at v is $\mathbb{R}^n$.

For $(\tau, z) \in \mathbb{R}_+ \times (\partial D \setminus \{z\})$, the differential of ϕ in the direction (s, y) with $s \in \mathbb{R}$, $y \in n_z^\perp$ is given by

$$D\phi_{(\tau,z)}(s,y) = \frac{y}{\tau} - \frac{(z-x)s}{\tau^2}.$$

Let $(f_1, \dots, f_{n-1})$ be an orthonormal basis of $n_z^\perp$, f_n such that $(f_1, \dots, f_{n-1}, f_n)$ is an orthonormal basis of $n_z^\perp \times \mathbb{R}$. The Jacobian matrix of ϕ in the bases $(f_1, \dots, f_n)$ for $n_z^\perp \times \mathbb{R}$ and $(f_1, \dots, f_{n-1}, n_z)$ for $\mathbb{R}^n$ is thus

$$J_\phi(\tau, z) = \begin{pmatrix} \frac{1}{\tau} & 0 & \dots & 0 & -\frac{(z-x)\cdot f_1}{\tau^2} \\ 0 & \frac{1}{\tau} & \dots & 0 & -\frac{(z-x)\cdot f_2}{\tau^2} \\ & \dots & \dots & & \\ 0 & \dots & \dots & \frac{1}{\tau} & -\frac{(z-x)\cdot f_{n-1}}{\tau^2} \\ 0 & \dots & 0 & 0 & -\frac{(z-x)\cdot n_z}{\tau^2} \end{pmatrix}.$$

The Jacobian at the point (τ, z) is therefore given by $\frac{|(z-x)\cdot n_z|}{\tau^{n+1}}$.

Recalling (2.4.7), using that $(\tau, z) = (\zeta(x,v), q(x,v))$, we find

$$I = \int_{\{v\cdot n_x>0\}} \mathbf{1}_{\{(\zeta(x,v),q(x,v))\in A\}} c_0 \frac{M(v)}{\zeta(x,v)} |\zeta(x,v)(v\cdot n_x)| dv = \mathbb{P}\Big((\zeta(x,V), q(x,V) \in A\Big),$$

as desired. □

With the help of Lemmas 2.4.1 and 2.4.2, we prove Proposition 2.4.1.

Proof of Proposition 2.4.1. In a first step, we derive an inequality from which we will conclude in the second step, using the classical framework of maximal coupling.

Step 1. We show that, for $A = (\partial D)^2 \times [0, d(D))$, there exists $c > 0$ such that

$$\inf_{(x,\tilde{x},\tilde{t})\in A} \int_{\partial D} \int_{\tilde{t}}^\infty [\mu_x(\tau, z) \wedge \mu_{\tilde{x}}(\tau - \tilde{t}, z)] d\tau dz \geq c. \tag{2.4.8}$$

We have, using Lemma 2.4.2, for any $(x, \tilde{x}, \tilde{t}) \in A$,

$$\begin{aligned} J &:= \int_{\partial D} \int_{\tilde{t}}^\infty [\mu_x(\tau, z) \wedge \mu_{\tilde{x}}(\tau - \tilde{t}, z)] d\tau dz \\ &\geq c_0 \int_{\{z\in\partial D, \|z-x\|\wedge\|z-\tilde{x}\|\geq r_1\}} \int_{b_0}^{b_1} \Big(\Big[M\Big(\frac{z-x}{\tau}\Big)\frac{1}{\tau^{n+2}}|(z-x)\cdot n_x||(z-x)\cdot n_z|\Big] \\ &\qquad \wedge \Big[M\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big)\frac{1}{(\tau-\tilde{t})^{n+2}}|(z-\tilde{x})\cdot n_{\tilde{x}}||(z-\tilde{x})\cdot n_z|\Big]\Big) d\tau dz, \end{aligned}$$

where $b_0 = d(D)(\frac{\delta_1+1}{\delta_1})$ and $b_1 = d(D)(\frac{2\delta_1+1}{\delta_1})$, recalling the definition of δ_1 from Hypothesis 2.1.1. Indeed, $b_0 \geq d(D) \geq \tilde{t}$. For $\tau \in (b_0, b_1)$, $z, y \in \partial D$ with $\|z-y\| \geq r_1$, we have

$$0 < \frac{r_1}{b_1} \leq \frac{\|z-y\|}{\tau} \leq \frac{\|z-y\|}{\tau - \tilde{t}} \leq \delta_1 \frac{\|z-y\|}{d(D)} \leq \delta_1,$$

whence, recalling the definition of $\bar{M}$ from Hypothesis 2.1.1,

$$M\Big(\frac{z-x}{\tau}\Big) \wedge M\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big) \geq \bar{M}\Big(\frac{z-x}{\tau}\Big) \wedge \bar{M}\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big) \geq \kappa_1,$$

where $\kappa_1 = \min_{\frac{r_1}{b_1} \leq \|v\| \leq \delta_1} \bar{M}(v) > 0$ not depending on $(x, \tilde{x}, \tilde{t})$. We obtain, using Tonelli's theorem, that

$$\begin{aligned} J \geq{}& c_0\kappa_1 \int_{b_0}^{b_1} \frac{1}{\tau^{n+2}} d\tau \\ &\times \int_{\{z\in\partial D, \|z-x\|\wedge\|z-\tilde{x}\|\geq r_1\}} \Big[|(z-x)\cdot n_x||(z-x)\cdot n_z|\Big] \wedge \Big[|(z-\tilde{x})\cdot n_{\tilde{x}}||(z-\tilde{x})\cdot n_z|\Big] dz. \end{aligned}$$

We conclude by applying Lemma 2.4.1.

Step 2. Recall that $x_0 \in \partial D$, $\tilde{x}_0 \in D$, $\tilde{v}_0 \in \mathbb{R}^n$ such that $\|\tilde{v}_0\| \geq 1$ are fixed. Set $x = x_0$, $\tilde{x} = q(\tilde{x}_0, \tilde{v}_0)$ and $\tilde{t} = \zeta(\tilde{x}_0, \tilde{v}_0) \leq \frac{d(D)}{\|\tilde{v}_0\|} \leq d(D)$, . Classicaly, using (2.4.8), one can couple $(S, Y) \sim \mu_x$ and $(\tilde{S}, \tilde{Y}) \sim \mu_{\tilde{x}}$ so that $\mathbb{P}(Y = \tilde{Y}, S = \tilde{S} + \tilde{t}) \geq c$. Recalling that, if $(R, \Theta) \sim \Upsilon$ and $(\tilde{R}, \tilde{\Theta}) \sim \Upsilon$, $(\xi(x, R, \Theta), q(x, \Theta)) \sim \mu_x$ and $(\tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{R}, \tilde{\Theta}) - \tilde{t}, \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\Theta})) \sim \mu_{\tilde{x}}$, the conclusion follows. □

2.4.2 Some more preliminary results.

Recall that the function $r : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, continuous, and that there exists $C > 0$ satisfying, for all $(x, y) \in (\mathbb{R}_+)^2$, $r(x+y) \leq C(r(x) + r(y))$.

Remark 2.4.1. There exist $C > 0$, $\beta > 0$ such that for all $n \geq 1$, for all $x_1, \dots, x_n \geq 0$,

$$r\Big(\sum_{i=1}^{n} x_i\Big) \leq C n^\beta \sum_{i=1}^{n} r(x_i). \tag{2.4.9}$$

Proof. If $n = 2^p$, $p \in \mathbb{N}$, we have

$$r\Big(\sum_{i=1}^{2^p} x_i\Big) \leq C^p \sum_{i=1}^{2^p} r(x_i).$$

In the general case, setting $x_j = 0$ for all $j \in \{1, \dots, 2^{[\log_2(n)]+1}\} \setminus \{1, \dots n\}$, we obtain

$$r\Big(\sum_{i=1}^{n} x_i\Big) = r\Big(\sum_{i=1}^{2^{[\log_2(n)]+1}} x_i\Big) \leq C^{[\log_2(n)]+1}\Big(\sum_{i=1}^{n} r(x_i) + (2^{[\log_2(n)]+1} - n) r(0)\Big)$$
$$\leq 2Cn^{\log_2(C)} \sum_{i=1}^{n} r(x_i),$$

where we used that $r(0) \leq r(x_i)$, that $2^{[\log_2(n)]+1} - n \leq n$, and that $C^{[\log_2(n)]+1} \leq Cn^{\log_2(C)}$. $\square$

Lemma 2.4.3. *Let $(\mathcal{G}_k)_{k\geq 0}$ be a non-decreasing family of σ-algebras, $(\tau_k)_{k\geq 1}$ a family of random times such that τ_k is $\mathcal{G}_k$-measurable for all $k \geq 1$. Let $(E_k)_{k\geq 1}$ a family of events such that for all $k \geq 1$, $E_k \in \mathcal{G}_k$ and assume there exists $c > 0$ such that a.s.*

$$\forall k \geq 1, \quad \mathbb{P}(E_k | \mathcal{G}_{k-1}) \geq c. \tag{2.4.10}$$

Set $G = \inf\{k \geq 1, E_k \text{ is realized}\}$, which is almost surely finite. Assume there exists a positive $\mathcal{G}_0$-measurable random variable L such that for all $k \geq 1$, (note that $\{G \geq k\} \in \mathcal{G}_{k-1}$),

$$\mathbf{1}_{\{G\geq k\}} \mathbb{E}[r(\tau_{k+1} - \tau_k) | \mathcal{G}_{k-1}] \leq L \quad \text{ and } \quad \mathbb{E}[r(\tau_1) | \mathcal{G}_0] \leq L. \tag{2.4.11}$$

Then

$$\mathbb{E}[r(\tau_G) | \mathcal{G}_0] \leq \kappa L,$$

for some constant $\kappa > 0$ depending only on c and the function r.

Proof. For all $j \geq 1$, on $\{G = j\}$, setting $\tau_0 = 0$, we have $\tau_G = \sum_{i=0}^{j-1} (\tau_{i+1} - \tau_i)$. Hence, using (2.4.9),

$$\mathbb{E}\Big[r(\tau_G)\Big|\mathcal{G}_0\Big] = \sum_{j=1}^{\infty} \mathbb{E}\Big[r\Big(\sum_{i=0}^{j-1}(\tau_{i+1} - \tau_i)\Big)\mathbf{1}_{\{G=j\}}\Big|\mathcal{G}_0\Big]$$
$$\leq C\sum_{j=1}^{\infty} j^{\beta} \sum_{i=0}^{j-1} \mathbb{E}\Big[r(\tau_{i+1} - \tau_i)\Big(\prod_{k=1}^{j-1} \mathbf{1}_{E_k^c}\Big)\mathbf{1}_{E_j}\Big|\mathcal{G}_0\Big] = C\sum_{j=1}^{\infty} j^{\beta} \sum_{i=0}^{j-1} u_{i,j}, \tag{2.4.12}$$

the last equality standing for the definition of $u_{i,j}$. By convention, we give the value 1 to any product indexed by the empty set. Note that for any $l \geq m \geq 1$, using (2.4.10),

$$\mathbb{E}\Big[\Big(\prod_{k=m}^{l} \mathbf{1}_{E_k^c}\Big)\Big|\mathcal{G}_{m-1}\Big] = \mathbb{E}\Big[\Big(\prod_{k=m}^{l-1} \mathbf{1}_{E_k^c}\Big)\mathbb{E}[\mathbf{1}_{E_l^c}|\mathcal{G}_{l-1}]\Big|\mathcal{G}_{m-1}\Big] \leq (1-c)\mathbb{E}\Big[\Big(\prod_{k=m}^{l-1} \mathbf{1}_{E_k^c}\Big)\Big|\mathcal{G}_{m-1}\Big].$$

Iterating the argument,

$$\mathbb{E}\Big[\Big(\prod_{k=m}^{l}\mathbf{1}_{E_k^c}\Big)\Big|\mathcal{G}_{m-1}\Big]\le(1-c)^{l-m+1}. \tag{2.4.13}$$

We first bound $u_{i,j}$ in the case where $i\ge 1$ and $j\ge i+2$. We have, using that $\mathbf{1}_{E_j}\le 1$ and that $\{G\ge i\}$ on $E_1^c\cap\cdots\cap E_{j-1}^c$,

$$\begin{aligned}
u_{i,j}&\le\mathbb{E}\Big[r(\tau_{i+1}-\tau_i)\Big(\prod_{k=i+2}^{j-1}\mathbf{1}_{E_k^c}\Big)\Big(\prod_{k=1}^{i+1}\mathbf{1}_{E_k^c}\Big)\mathbf{1}_{\{G\ge i\}}\Big|\mathcal{G}_0\Big]\\
&\le\mathbb{E}\Big[r(\tau_{i+1}-\tau_i)\Big(\prod_{k=1}^{i+1}\mathbf{1}_{E_k^c}\Big)\mathbf{1}_{\{G\ge i\}}\mathbb{E}\Big[\prod_{k=i+2}^{j-1}\mathbf{1}_{E_k^c}\Big|\mathcal{G}_{i+1}\Big]\Big|\mathcal{G}_0\Big]\\
&\le(1-c)^{j-i-2}\mathbb{E}\Big[r(\tau_{i+1}-\tau_i)\Big(\prod_{k=1}^{i+1}\mathbf{1}_{E_k^c}\Big)\mathbf{1}_{\{G\ge i\}}\Big|\mathcal{G}_0\Big]
\end{aligned}$$

by (2.4.13). Using (2.4.11), that $\mathbf{1}_{E_{i+1}^c}\mathbf{1}_{E_i^c}\le 1$ and the fact that $\{G\ge i\}\in\mathcal{G}_{i-1}$, we deduce that

$$\begin{aligned}
u_{i,j}&\le(1-c)^{j-i-2}\mathbb{E}\Big[\mathbf{1}_{\{G\ge i\}}\mathbb{E}\Big[r(\tau_{i+1}-\tau_i)\Big|\mathcal{G}_{i-1}\Big]\prod_{k=1}^{i-1}\mathbf{1}_{E_k^c}\Big|\mathcal{G}_0\Big]\\
&\le L(1-c)^{j-i-2}\mathbb{E}\Big[\prod_{k=1}^{i-1}\mathbf{1}_{E_k^c}\Big|\mathcal{G}_0\Big]\le L(1-c)^{j-3},
\end{aligned}$$

where we used (2.4.13). Using similar (easier) computations, one can show that

$$u_{0,1}\le L,\quad\text{and for } j\ge 2,\quad u_{0,j}\le L(1-c)^{j-2}\quad\text{and}\quad u_{j-1,j}\le L(1-c)^{j-2}.$$

We plug-in those results into (2.4.12) to conclude, splitting the sum over the cases $j=1$, $j=2$ and $j\ge 3$, that there exists a constant $\kappa>0$ depending only on r and c such that

$$\mathbb{E}\Big[r(\tau_G)\Big|\mathcal{G}_0\Big]\le C\Big(L+2^{\beta+1}L+\sum_{j=3}^{\infty}j^{\beta+1}L(1-c)^{j-3}\Big)\le\kappa L,$$

as desired. □

Recall, for $(x,\theta)\in\partial D\times\mathcal{A}$, the notation $\vartheta(x,\theta)$ introduced in Lemma 2.3.1. For any filtration $(\mathcal{F}_t)_{t\ge0}$, any stopping time ν we introduce the σ-algebra $\mathcal{F}_{\nu-}:=\sigma(A\cap\{t<\nu\},t\in\mathbb{R}_+,A\in\mathcal{F}_t)$, see [78, Definition 1.11]. We set $\mathcal{F}_{0-}$ to be the completion of the trivial σ-algebra.

Lemma 2.4.4. *Let $x\in\partial D$ and $V=R\vartheta(x,\Theta)$, with $(R,\Theta)\sim\Upsilon$. Let $(X_t,V_t)_{t\ge0}$ be a free-transport process (see Remark 2.3.2) with $(X_0,V_0)=(x,V)\in\partial_-G$. Set $T_0=0$ and let*

$T_{i+1} = \inf\{t > T_i, X_t \in \partial D\}$ for all $i \geq 0$. Then, for all $i \geq 1$, T_i admits a density with respect to the Lebesgue measure on $\mathbb{R}_+$.

Proof. We set for all $t \geq 0$, $\mathcal{F}_t = \sigma((X_s, V_s)_{0 \leq s \leq t})$. Let $A \in \mathcal{B}(\mathbb{R}_+)$ with $\lambda(A) = 0$, where λ is the Lebesgue measure on $\mathbb{R}_+$. We have $T_1 = \frac{\|x - q(x, \vartheta(x, \Theta))\|}{R}$, so that

$$\mathbb{P}(T_1 \in A) = \int_{\mathcal{A}} \mathbb{P}\Big(\frac{\|x - q(x, \vartheta(x, \theta))\|}{R} \in A\Big) h_\Theta(\theta) d\theta.$$

For $\theta \in \mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi]^{n-2}$, we set $A_{x,\theta} = \{s \in \mathbb{R}_+, \frac{\|x - q(x, \vartheta(x, \theta))\|}{s} \in A\}$, so that

$$\mathbb{P}(T_1 \in A) = \int_{\mathcal{A}} \mathbb{P}(R \in A_{x,\theta}) h_\Theta(\theta) d\theta.$$

Note that $\lambda(A_{x,\theta}) = 0$ for all $\theta \in \mathcal{A}$. Since R has a density h_R with respect to the Lebesgue measure on $\mathbb{R}_+$, we conclude that $\mathbb{P}(T_1 \in A) = 0$, so that T_1 admits a density with respect to the Lebesgue measure on $\mathbb{R}_+$.

Concerning T_2, we introduce the event $B = \{\text{Specular reflection at } X_{T_1}\}$. Note that B is independent of R, see Definition 2.3.1. We fix $A \in \mathcal{B}(\mathbb{R}_+)$ with $\lambda(A) = 0$.

i) On the event B, since $T_2 = T_2 - T_1 + T_1$, setting $Y = q(x, \vartheta(x, \Theta))$ and recalling (2.2.3),

$$T_2 = \frac{\Big\|Y - q(Y, \eta_Y(\vartheta(Y, \Theta))\Big\|}{R} + \frac{\|x - Y\|}{R}.$$

Proceeding as for T_1, we find, with the notation $y = q(x, \vartheta(x, \theta))$,

$$\mathbb{P}(\{T_2 \in A\} \cap B) = \int_{\mathcal{A}} (1 - \alpha(y)) \mathbb{P}\Big(\frac{\|x - y\| + \|y - q(y, \eta_y(\vartheta(y, \theta)))\|}{R} \in A\Big) h_\Theta d\theta = 0.$$

ii) On the event B^c, we introduce the process $(\tilde{X}_t, \tilde{V}_t)_{t \geq 0}$ with, $\tilde{X}_t = X_{T_1 + t}$, $\tilde{V}_t = V_{T_1 + t}$. By the strong Markov property for the process $(X_s, V_s)_{s \geq 0}$, we have that, setting

$$\tilde{T}_1 = \inf\{t > 0, \tilde{X}_t \in \partial D\} = T_2 - T_1,$$

$\tilde{T}_1$ admits a density with respect to λ, conditionally on $\mathcal{F}_{T_1-}$ on B^c. Indeed, $X_{T_1} \in \partial D$ and is $\mathcal{F}_{T_1-}$-measurable, $V_{T_1} = R_1 \vartheta(X_{T_1}, \Theta_1)$ on B^c, with $(R_1, \Theta_1) \sim \Upsilon$ independent of $\mathcal{F}_{T_1-}$, so that we can apply the previous study for T_1. We obtain, since T_1 is $\mathcal{F}_{T_1-}$ measurable.

$$\mathbb{P}(T_2 \in A \cap B^c) = \mathbb{P}(\{\tilde{T}_1 + T_1 \in A\} \cap B^c) = 0.$$

Hence, $\mathbb{P}(\{T_2 \in A\}) = 0$. The conclusion follows by induction. □

2.4.3 Construction of the coupling.

In this section, we define the coupling of the two processes that we will use to prove Theorem 2.1.1, and show two of its properties.

We recall that $\mathcal{U}$ is the uniform distribution over $[0,1]$ and $\mathcal{Q}$ is the law on $[0,1]\times\mathbb{R}_+\times\mathcal{A}$ such that $\mathcal{Q}=\mathcal{U}\otimes\Upsilon$, where Υ is defined in Lemma 2.3.1. For $x\in\partial D$, $\tilde{x}\in D$, $\tilde{v}\in\mathbb{R}^n$ with $\|\tilde{v}\|\geq 1$, recall the law $\Lambda_{x,\tilde{x},\tilde{v}}$ on $(\mathbb{R}_+\times\mathcal{A})^2$ defined in Proposition 2.4.1 .

Let $(x,v,\tilde{x},\tilde{v})$ in $(\bar{D}\times\mathbb{R}^n)^2$ with $x\in\partial D$ or $\tilde{x}\in\partial D$. We define the law $\Gamma_{x,v,\tilde{x},\tilde{v}}$ on the space $([0,1]\times\mathbb{R}_+\times\mathcal{A})^2$ by:

$$\begin{aligned}\Gamma_{x,v,\tilde{x},\tilde{v}}(du,dr,d\theta,d\tilde{u},d\tilde{r},d\tilde{\theta}) &= \mathbf{1}_{\{x=\tilde{x}\}}\mathcal{Q}(du,dr,d\theta)\delta_u(d\tilde{u})\delta_r(d\tilde{r})\delta_\theta(d\tilde{\theta}) \\ &+\mathbf{1}_{\{x\in\partial D\}}\mathbf{1}_{\{\tilde{x}\in D\}\cap\{\|v\|\geq 1,\|\tilde{v}\|\geq 1\}}\mathcal{U}(du)\Lambda_{x,\tilde{x},\tilde{v}}(dr,d\theta,d\tilde{r},d\tilde{\theta})\delta_u(d\tilde{u}) \\ &+\mathbf{1}_{\{x\neq\tilde{x}\}}\mathbf{1}_{\{\tilde{x}\in\partial D\}\cup\{\|\tilde{v}\|<1\}\cup\{\|v\|<1\}}(\mathcal{Q}\otimes\mathcal{Q})(du,dr,d\theta,d\tilde{u},d\tilde{r},d\tilde{\theta}).\end{aligned} \tag{2.4.14}$$

We can now describe the global coupling procedure with the help of this law. In order to obtain a Markov process, we introduce an additional random process $(Z_s)_{s\geq 0}$ with values in the set $\{\emptyset\}\cup([0,1]\times\mathbb{R}_+\times\mathcal{A})$.

Definition 2.4.1. We define a coupling process $(X_s,V_s,\tilde{X}_s,\tilde{V}_s,Z_s)_{s\geq 0}$ by the following steps:

Step 0: Simulate $(X_0,V_0)\sim f_0$, $(\tilde{X}_0,\tilde{V}_0)\sim\mu_\infty$, set $Z_0=\emptyset$ and $S_0=0$.

...

Step k+1: Set $S_{k+1}=S_k+\zeta(X_{S_k},V_{S_k})\wedge\zeta(\tilde{X}_{S_k},\tilde{V}_{S_k})$.

Set, for all $t\in(S_k,S_{k+1})$, $X_t=X_{S_k}+(t-S_k)V_{S_k}$, $V_t=V_{S_k}$,

$$\tilde{X}_t=\tilde{X}_{S_k}+(t-S_k)\tilde{V}_{S_k},\ \tilde{V}_t=\tilde{V}_{S_k},$$

$$Z_t=Z_{S_k}.$$

Set $X_{S_{k+1}}=X_{S_{k+1}-}$, $\tilde{X}_{S_{k+1}}=\tilde{X}_{S_{k+1}-}$.

Simulate $(Q_{k+1},\tilde{Q}_{k+1})\sim\Gamma_{X_{S_{k+1}},V_{S_{k+1}-},\tilde{X}_{S_{k+1}},\tilde{V}_{S_{k+1}-}}$.

Set $V_{S_{k+1}}=V_{S_{k+1}-}\mathbf{1}_{\{X_{S_{k+1}}\notin\partial D\}}+w(X_{S_{k+1}},V_{S_{k+1}-},Q_{k+1})\mathbf{1}_{\{X_{S_{k+1}}\in\partial D\}}$.

Set $\tilde{Q}'_{k+1}=\tilde{Q}_{k+1}\mathbf{1}_{\{Z_{S_{k+1}-}=\emptyset\}}+Z_{S_{k+1}-}\mathbf{1}_{\{Z_{S_{k+1}-}\neq\emptyset\}}$.

Set $\tilde{V}_{S_{k+1}}=\tilde{V}_{S_{k+1}-}\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\notin\partial D\}}+w(\tilde{X}_{S_{k+1}},\tilde{V}_{S_{k+1}-},\tilde{Q}'_{k+1})\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\in\partial D\}}$.

Set $Z_{S_{k+1}}=\emptyset\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\in\partial D\}}+\tilde{Q}'_{k+1}\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\notin\partial D\}}$.

Observe that the last line of Definition 2.4.1 rewrites as

$$Z_{S_{k+1}}=\emptyset\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\in\partial D\}}+Z_{S_{k+1}-}\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\notin\partial D,Z_{S_{k+1}-}\neq\emptyset\}}+\tilde{Q}_{k+1}\mathbf{1}_{\{\tilde{X}_{S_{k+1}}\notin\partial D,Z_{S_{k+1}-}=\emptyset\}}.$$

Remark 2.4.2. One can readily see from Definition 2.4.1 that $(X_s,V_s,\tilde{X}_s,\tilde{V}_s,Z_s)_{s\geq 0}$ is a strong Markov process.

Let us explain informally this definition. The sequence $(S_k)_{k\geq 1}$ is the sequence of collisions with the boundary of $(X_s, V_s)_{s\geq 0}$ and $(\tilde{X}_s, \tilde{V}_s)_{s\geq 0}$. The behavior of the coupling process is clear between S_k and S_{k+1} for all $k \geq 0$. For all $k \geq 1$, at time S_k, we set $(X, V_-) = (X_{S_k}, V_{S_k-})$, $(\tilde{X}, \tilde{V}_-) = (\tilde{X}_{S_k}, \tilde{V}_{S_k-})$, $Z_- = Z_{S_k-}$ and we have $X \in \partial D$ or $\tilde{X} \in \partial D$. We explain in Table 2.1 how we choose the new velocities $(V, \tilde{V})$ and update the value of Z. Observe that all those cases are treated in a rather concise way in Definition 2.4.1. This leads to simpler notations and hopefully allows for a clearer proof.

Lemma 2.4.5. *Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s\geq 0}$ be a coupling process. Then $(X_s, V_s)_{s\geq 0}$ is a free-transport process with initial distribution f_0 (see Definition 2.3.1). Moreover, $(\tilde{X}_s, \tilde{V}_s)_{s\geq 0}$ is a free-transport process with initial distribution μ_∞.*

Proof. We write, for all $s \geq 0$, $\mathcal{G}_s = \sigma((X_t, V_t, \tilde{X}_t, \tilde{V}_t, Z_t)_{0\leq t\leq s})$, $\mathcal{F}_s = \sigma((X_t, V_t)_{0\leq t\leq s})$ and $\tilde{\mathcal{F}}_s = \sigma((\tilde{X}_t, \tilde{V}_t)_{0\leq t\leq s})$. Note first that for all $i \geq 1$, $X_{S_i} \in \partial D$ or $\tilde{X}_{S_i} \in \partial D$. We have, a.s., recalling (2.4.14) and Proposition 2.4.1,

$$\int_{(\tilde{u},\tilde{r},\tilde{\theta})\in[0,1]\times\mathbb{R}_+\times\mathcal{A}} \Gamma_{X_{S_i}, V_{S_i-}, \tilde{X}_{S_i}, \tilde{V}_{S_i-}}(du, dr, d\theta, d\tilde{u}, d\tilde{r}, d\tilde{\theta}) = \mathcal{Q}(du, dr, d\theta). \tag{2.4.15}$$

Hence, with a similar argument for $\tilde{Q}_i$,

$$\mathcal{L}(Q_i|\mathcal{G}_{S_i-}) = \mathcal{Q}, \qquad \mathcal{L}(\tilde{Q}_i|\mathcal{G}_{S_i-}) = \mathcal{Q}. \tag{2.4.16}$$

We focus first on the process $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$. We introduce the subsequence $(\nu_k)_{k\geq 0}$ defined by $\nu_0 = 0$ and $\nu_{k+1} = \inf\{j > \nu_k, \tilde{X}_{S_j} \in \partial D\}$. Comparing Definitions 2.3.1 and 2.4.1, one realizes that the only difficulty is to verify that for all $k \geq 1$, $\tilde{Q}'_{\nu_k}$ is $\mathcal{Q}$-distributed and independent of $\tilde{\mathcal{F}}_{S_{\nu_k}-} = \tilde{\mathcal{F}}_{S_{\nu_{k-1}}}$.

Note first that, for all $k \geq 1$, $\{Z_{S_{\nu_k}-} = \emptyset\} \in \mathcal{G}_{S_{\nu_{k-1}}}$. Indeed, we have $Z_{S_{\nu_{k-1}}} = \emptyset$ a.s. and thus

$$\{Z_{S_{\nu_k}-} = \emptyset\} = \Big\{\zeta(X_{S_{\nu_{k-1}}}, V_{S_{\nu_{k-1}}}) \geq \zeta(\tilde{X}_{S_{\nu_{k-1}}}, \tilde{V}_{S_{\nu_{k-1}}})\Big\} \in \mathcal{G}_{S_{\nu_{k-1}}}. \tag{2.4.17}$$

We claim that for all $k \geq 1$,

$$\tilde{Q}'_{\nu_k} = \mathbf{1}_{\{Z_{S_{\nu_k}-}=\emptyset\}}\tilde{Q}_{\nu_k} + \mathbf{1}_{\{Z_{S_{\nu_k}-}\neq\emptyset\}}\tilde{Q}_{\nu_{k-1}+1}.$$

Table 2.1 Update when $X \in \partial D$ or $\tilde{X} \in \partial D$.

X	$\tilde{X}$	Z_-	$\|V_-\| \wedge \|\tilde{V}_-\|$	Update
$\in \partial D$	$\notin \partial D$	$\emptyset$	≥ 1	Simulate $(R, \Theta, \tilde{R}, \tilde{\Theta}) \sim \Lambda_{X,\tilde{X},\tilde{V}_-}$, $U \sim \mathcal{U}$. Set $(Q, \tilde{Q}) = ((U, R, \Theta), (U, \tilde{R}, \tilde{\Theta}))$. Update V using Q, set $\tilde{V} = \tilde{V}_-$ and store $\tilde{Q}$ in Z: $Z = \tilde{Q}$.
$\in \partial D$	$\notin \partial D$	$\emptyset$	< 1	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q, set $\tilde{V} = \tilde{V}_-$, store $\tilde{Q}$ in Z: $Z = \tilde{Q}$ (this is quite artificial since $\tilde{Q}$ is independent of Q).
$\in \partial D$	$\in \partial D$ $\tilde{X} = X$	$\emptyset$	all values	Simulate $Q \sim \mathcal{Q}$. Update V and $\tilde{V}$ using Q (if $V_- = \tilde{V}_-$ then $V = \tilde{V}$). Set $Z = \emptyset$.
$\in \partial D$	$\in \partial D$, $\tilde{X} \neq X$	$\emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q and $\tilde{V}$ using $\tilde{Q}$. Set $Z = \emptyset$.
$\in \partial D$	$\notin \partial D$	$\neq \emptyset$	≥ 1	Simulate $(R, \Theta, \tilde{R}, \tilde{\Theta}) \sim \Lambda_{X,\tilde{X},\tilde{V}_-}$, $U \sim \mathcal{U}$. Set $(Q, \tilde{Q}) = ((U, R, \Theta), (U, \tilde{R}, \tilde{\Theta})$, update V using Q. Set $\tilde{V} = \tilde{V}_-$. Leave Z unchanged: $Z = Z_-$ ($\tilde{Q}$ is useless).
$\in \partial D$	$\notin \partial D$	$\neq \emptyset$	< 1	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q, set $\tilde{V} = \tilde{V}_-$. Leave Z unchanged: $Z = Z_-$ ($\tilde{Q}$ is useless).
$\in \partial D$	$\in \partial D$ $\tilde{X} = X$	$\neq \emptyset$	all values	Simulate $Q \sim \mathcal{Q}$. Update V using Q, update $\tilde{V}$ using Z_-. Clear Z by setting $Z = \emptyset$.
$\in \partial D$	$\in \partial D$ $\tilde{X} \neq X$	$\neq \emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update V using Q, update $\tilde{V}$ using Z_-. Clear Z by setting $Z = \emptyset$ ($\tilde{Q}$ is useless).
$\notin \partial D$	$\in \partial D$	$\emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update $\tilde{V}$ using $\tilde{Q}$, set $V = V_-$. Set $Z = \emptyset$ (Q is useless).
$\notin \partial D$	$\in \partial D$	$\neq \emptyset$	all values	Simulate $(Q, \tilde{Q}) \sim \mathcal{Q} \otimes \mathcal{Q}$. Update $\tilde{V}$ using Z_-, set $V = V_-$. Clear Z by setting $Z = \emptyset$ ($Q, \tilde{Q}$ are useless).

Indeed, we clearly have $\tilde{Q}'_{\nu_k} = \tilde{Q}_{\nu_k}$ on $\{Z_{S_{\nu_k}-} = \emptyset\}$, and, by (2.4.17) and since $Z_{S_{\nu_{k-1}}} = \emptyset$ a.s.,

$$\begin{aligned}\{Z_{S_{\nu_k}-} \neq \emptyset\} &= \{\zeta(X_{S_{\nu_{k-1}}}, V_{S_{\nu_{k-1}}}) < \zeta(\tilde{X}_{S_{\nu_{k-1}}}, \tilde{V}_{S_{\nu_{k-1}}})\} \\ &\subset \{X_{S_{\nu_{k-1}+1}} \in \partial D, \tilde{X}_{S_{\nu_{k-1}+1}} \notin \partial D, Z_{S_{\nu_{k-1}+1}-} = \emptyset\} \\ &\subset \{Z_{S_{\nu_{k-1}+1}} = \tilde{Q}_{\nu_{k-1}+1}, \quad \nu_k > \nu_{k-1}+1\} \\ &\subset \{Z_{S_{\nu_k}-} = \tilde{Q}_{\nu_{k-1}+1}\}.\end{aligned}$$

This concludes the proof of the claim.

Using (2.4.16), for all $k \geq 1$, $\mathcal{L}(\tilde{Q}_{\nu_k}|\mathcal{G}_{S_{\nu_k}-}) = \mathcal{Q}$ and $\mathcal{L}(\tilde{Q}_{\nu_{k-1}+1}|\mathcal{G}_{S_{\nu_{k-1}+1}-}) = \mathcal{Q}$. Consider a function $\phi \in C_c^\infty([0,1] \times \mathbb{R}_+ \times \mathcal{A})$. For $k \geq 1$, we compute

$$\begin{aligned}\mathbb{E}[\phi(\tilde{Q}'_{\nu_k})|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}] &= \mathbb{E}[\phi(\tilde{Q}_{\nu_k})\mathbf{1}_{\{Z_{S_{\nu_k}-}=\emptyset\}}|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}] + \mathbb{E}[\phi(\tilde{Q}_{\nu_{k-1}+1})\mathbf{1}_{\{Z_{S_{\nu_k}-}\neq\emptyset\}}|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}] \\ &= \mathbb{E}\Big[\mathbf{1}_{\{Z_{S_{\nu_k}-}=\emptyset\}}\mathbb{E}[\phi(\tilde{Q}_{\nu_k})|\mathcal{G}_{S_{\nu_k}-}]\Big|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\Big] \\ &\quad + \mathbb{E}\Big[\mathbf{1}_{\{Z_{S_{\nu_k}-}\neq\emptyset\}}\mathbb{E}[\phi(\tilde{Q}_{\nu_{k-1}+1})|\mathcal{G}_{S_{\nu_{k-1}+1}-}]\Big|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}\Big],\end{aligned}$$

using (2.4.17) and the fact that $\tilde{\mathcal{F}}_{S_{\nu_{k-1}}} \subset \mathcal{G}_{S_{\nu_{k-1}}} \subset \mathcal{G}_{S_{\nu_{k-1}+1}-} \subset \mathcal{G}_{S_{\nu_k}-}$. From the previous remarks on the conditional law of $\tilde{Q}_{\nu_k}$, $\tilde{Q}_{\nu_{k-1}+1}$, we obtain

$$\mathbb{E}[\phi(\tilde{Q}'_{\nu_k})|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}] = \int_{[0,1]\times\mathbb{R}_+\times\mathcal{A}} \phi(x)\mathcal{Q}(dx)\Big(\mathbb{E}[\mathbf{1}_{\{Z_{S_{\nu_k}-}=\emptyset\}}|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}] + \mathbb{E}[\mathbf{1}_{\{Z_{S_{\nu_k}-}\neq\emptyset\}}|\tilde{\mathcal{F}}_{S_{\nu_{k-1}}}]\Big),$$

from which we conclude that $\mathcal{L}(\tilde{Q}'_{\nu_k}|\tilde{\mathcal{F}}_{S_{\nu_k}-}) = \mathcal{Q}$, as desired.

The argument for $(X_s, V_s)_{s\geq 0}$ is similar and much easier since for all $j \geq 1$ such that X_{S_j} is in ∂D, $V_{S_j} = w(X_{S_j}, V_{S_j-}, Q_j)$ with $\mathcal{L}(Q_j|\mathcal{F}_{S_j-}) = \mathcal{Q}$ using (2.4.16) and that $\mathcal{F}_{S_j-} \subset \mathcal{G}_{S_j-}$. □

Lemma 2.4.6. *Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s\geq 0}$ be a coupling process. Then for all $t \geq 0$,*

$$\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\} \subset \{(X_{t+s}, V_{t+s})_{s\geq 0} = (\tilde{X}_{t+s}, \tilde{V}_{t+s})_{s\geq 0}\}.$$

Proof. According to Definition 2.4.1, on the event $\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\}$, there exists $k \geq 1$ such that $S_k = t + \zeta(X_t, V_t) = t + \zeta(\tilde{X}_t, \tilde{V}_t)$ and we have

$$\Big\{(X_t, V_t) = (\tilde{X}_t, \tilde{V}_t), Z_t = \emptyset\Big\} \subset \Big\{(X_{S_k-}, V_{S_k-}) = (\tilde{X}_{S_k-}, \tilde{V}_{S_k-}), Z_{S_k-} = \emptyset\Big\},$$

and $(X_s, V_s)_{t\leq s<S_k} = (\tilde{X}_s, \tilde{V}_s)_{t\leq s<S_k}$. We then have, according to the definition, the equality $X_{S_k} = X_{S_k-} = \tilde{X}_{S_k-} = \tilde{X}_{S_k}$ and $Z_{S_k-} = \emptyset$. Also, by definition of $\Gamma_{X_{S_k-}, V_{S_k-}, \tilde{X}_{S_k-}, \tilde{V}_{S_k-}}$, since $X_{S_k-} = \tilde{X}_{S_k-}$, we have $Q_k = \tilde{Q}_k$ with the notations of the definition. From there we obtain

$$V_{S_k} = w(X_{S_k}, V_{S_k-}, Q_k) = w(\tilde{X}_{S_k}, \tilde{V}_{S_k-}, \tilde{Q}_k) = \tilde{V}_{S_k}, \quad \text{and} \quad Z_{S_k} = \emptyset.$$

Hence $(X_s, V_s) = (\tilde{X}_s, \tilde{V}_s)$ and $Z_s = \emptyset$ for all $s \in (S_k, S_{k+1}]$. We conclude by iterating this procedure. $\square$

2.4.4 Proof of Theorem 2.1.1 in the convex case.

We recall that the set D is a bounded C^2 domain, uniformly convex in this section. The function r defined on $\mathbb{R}_+$ is such that there exists $C > 0$ satisfying, for all $(x, y) \in (\mathbb{R}_+)^2$, $r(x + y) \leq C(r(x) + r(y))$. The function $M : \mathbb{R}^n \to (0, \infty)$ is radially symmetric and of mass 1 with $\int_{\mathbb{R}^n} \|v\| M(v)dv < \infty$. The function α defined on ∂D is uniformly bounded from below by $\alpha_0 > 0$. Finally, $\mu_\infty(dx, dv) = \frac{M(v)}{|D|} dxdv$ is the equilibrium distribution. Recall that h_R is defined by $h_R(s) = c_R s^n M(s)$ for all $s \in \mathbb{R}_+$ with c_R a normalization constant, see Lemma 2.3.1. We define the constant $C_0 > 0$ by

$$C_0 = \max\Big(\int_{D\times\mathbb{R}^n} r\Big(\frac{d(D)}{\|v\|}\Big) f_0(dx, dv), \int_{D\times\mathbb{R}^n} r\Big(\frac{d(D)}{\|v\|}\Big)\mu_\infty(dx, dv), \int_{\mathbb{R}_+} r\Big(\frac{d(D)}{s}\Big) h_R(s)ds\Big), \tag{2.4.18}$$

which is finite using (2.1.5) and since

$$\begin{aligned}\int_{\mathbb{R}_+} r\Big(\frac{d(D)}{s}\Big) h_R(s)ds &= \kappa \int_{\mathbb{R}^n} r\Big(\frac{d(D)}{\|v\|}\Big) \|v\| M(v)dv \\ &\leq \kappa \int_{\{\|v\|\leq 1\}} r\Big(\frac{d(D)}{\|v\|}\Big) M(v)dv + \kappa r(d(D)) \int_{\{\|v\|>1\}} \|v\| M(v)dv.\end{aligned}$$

In this whole subsection κ and L denote some positive constants depending on r, D and α_0, whose value is allowed to vary from line to line. Recall Remark 2.3.2 for the definition of a free-transport process with initial distribution $\delta_x \otimes \delta_v$ with $(x, v) \in \partial_+ G$.

Lemma 2.4.7. *There exists $\kappa > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in (D \times \mathbb{R}^n) \cup \partial_+ G$ and $(X_t, V_t)_{t\geq 0}$, $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ are two possibly correlated free-transport processes with initial distributions $\delta_x \otimes \delta_v$ and $\delta_{\tilde{x}} \otimes \delta_{\tilde{v}}$ respectively, setting*

$$\mathcal{T} = \inf\{t > 0, \|V_t\| \neq \|v\|, \|\tilde{V}_t\| \neq \|\tilde{v}\|\},$$

we have

$$\mathbb{E}[r(\mathcal{T})] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Proof. We introduce the sequence $(T_k)_{k\geq 0}$ by setting first $T_0 = \zeta(x, v)$ so that $T_0 = 0$ in the case where $(x, v) \in \partial_+ G$, and for $k \geq 0$, $T_{k+1} = \inf\{t > T_k, X_t \in \partial D\}$. We introduce the filtration $\mathcal{F}_t = \sigma((X_s, V_s)_{0\leq s\leq t})$. We also set $S_1 = \inf\{t > 0, \|V_t\| \neq \|v\|\}$ and introduce $\tilde{S}_1$ given by $\tilde{S}_1 = \inf\{t > 0, \|\tilde{V}_t\| \neq \|\tilde{v}\|\}$. Note that $\mathcal{T} = S_1 \vee \tilde{S}_1$.

Step 1. We prove that

$$\mathbb{E}[r(S_1)] \le \kappa\Big(r\Big(\frac{d(D)}{\|v\|}\Big) + 1\Big).$$

We write $(U_i, R_i, \Theta_i)_{i\ge 0}$ for the sequence of $\mathcal{Q}$-distributed vectors such that for all $i \ge 0$,

$$V_{T_i} = w(X_{T_i}, V_{T_i-}, U_i, R_i, \Theta_i),$$

with $V_{0-} = v$. Set $A_n = \{\|V_{T_n}\| \ne \|V_{T_n-}\|\}$ for all $n \ge 0$ and $N = \inf\{n \ge 1, A_n \text{ is realized}\}$ so that $S_1 \le T_N$ (S_1 may differ from T_N if $x \in \partial D$). We first use Lemma 2.4.3 to prove that

$$\mathbb{E}[r(T_N - T_1)|\mathcal{F}_{T_1-}] \le \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big)\Big). \tag{2.4.19}$$

1. We set for all $k \ge 0$, $\mathcal{G}_k = \mathcal{F}_{T_{k+1}-}$, and for $k \ge 1$, $\tau_k = T_{k+1} - T_1$ which is $\mathcal{G}_k$-measurable, $E_k = A_k \in \mathcal{G}_k$, so that $G = N$, with $G = \inf\{k \ge 1, E_k \text{ is realized}\}$ corresponding to the notation of Lemma 2.4.3.

2. For all $k \ge 1$, we have $\mathbb{P}(E_k|\mathcal{G}_{k-1}) = \mathbb{P}(A_k|\mathcal{F}_{T_k-}) = \mathbb{P}(U_k \le \alpha(X_{T_k})) \ge \alpha_0$, whence (2.4.10).

3. We have, by definition of C_0,

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(T_2 - T_1)|\mathcal{F}_{T_1-}] \le \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{T_1}\|}\Big)\Big|\mathcal{F}_{T_1-}\Big] \le C_0 + r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big),$$

   since $\|V_{T_1}\| = \|V_{T_1}\|\mathbf{1}_{A_1} + \|V_{T_0}\|\mathbf{1}_{A_1^c}$ with $\mathcal{L}(\|V_{T_1}\||A_1) = h_R$.

   For all $k \ge 1$, since $\|V_{T_k-}\| = \|V_{T_0}\|$ on $\{N \ge k\}$, we obtain,

$$\begin{aligned}
\mathbf{1}_{\{G\ge k\}}\mathbb{E}[r(\tau_{k+1} - \tau_k)|\mathcal{G}_{k-1}] &= \mathbf{1}_{\{N\ge k\}}\mathbb{E}\Big[r(T_{k+2} - T_{k+1})\Big|\mathcal{F}_{T_k-}\Big] \\
&\le \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{T_{k+1}}\|}\Big)\Big(\mathbf{1}_{A_{k+1}^c\cap A_k^c} + \mathbf{1}_{A_{k+1}\cap A_k^c} \\
&\quad + \mathbf{1}_{A_{k+1}^c\cap A_k} + \mathbf{1}_{A_{k+1}\cap A_k}\Big)\mathbf{1}_{\{\|V_{T_k-}\|=\|V_{T_0}\|\}}\Big|\mathcal{F}_{T_k-}\Big] \\
&\le r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big) + 3C_0,
\end{aligned}$$

   because $\|V_{T_{k+1}}\| = \|V_{T_0}\|$ on $A_k^c \cap A_{k+1}^c$ and the last three terms are bounded by C_0 since we clearly have $\mathcal{L}(\|V_{T_n}\||A_k) = h_R$ for all $n \ge k \ge 0$. We have proved (2.4.11).

Applying Lemma 2.4.3 we conclude that there exists $\kappa > 0$ such that (2.4.19) holds. To conclude this step, note that

$$\begin{aligned}\mathbb{E}[r(S_1)] &\leq C(\mathbb{E}[\mathbb{E}[r(T_N - T_1)|\mathcal{F}_{T_1-}]] + \mathbb{E}[r(T_1 - T_0)] + \mathbb{E}[r(T_0)])\\ &\leq C\Big(\mathbb{E}\Big[\kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big)\Big)\Big] + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{T_0}\|}\Big)\Big] + r\Big(\frac{d(D)}{\|v\|}\Big)\Big)\\ &\leq \kappa\Big(1 + 2C_0 + 3r\Big(\frac{d(D)}{\|v\|}\Big)\Big),\end{aligned}$$

since $\|V_{T_0}\| = \|V_{T_0}\|\mathbf{1}_{A_0} + \|v\|\mathbf{1}_{A_0^c}$ with $\mathcal{L}(\|V_{T_0}\||A_0) = h_R$.

Step 2. We apply the previous step with the process $(\tilde{X}_s, \tilde{V}_s)_{s\geq 0}$ and conclude that

$$\mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Step 3. Since $\mathcal{T} = S_1 \vee \tilde{S}_1$, we conclude that

$$\mathbb{E}[r(\mathcal{T})] \leq \mathbb{E}[r(S_1)] + \mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

□

Lemma 2.4.8. *There exists $\kappa > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in (D \times \mathbb{R}^n) \cup \partial_+ G$ and $(X_t, V_t)_{t\geq 0}$, $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ are two independent free-transport processes with initial distributions $\delta_x \otimes \delta_v$ and $\delta_{\tilde{x}} \otimes \delta_{\tilde{v}}$ respectively, setting*

$$S = \inf\{t > 0, \tilde{X}_t \in \partial D, X_t \in D, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1, \|V_{t-}\| \neq \|v\|, \|\tilde{V}_{t-}\| \neq \|\tilde{v}\|\},$$

we have

$$\mathbb{E}[r(S)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Proof. We introduce the filtration $\mathcal{F}_t = \sigma((X_s, V_s, \tilde{X}_s, \tilde{V}_s)_{0\leq s\leq t})$. We also introduce the stopping times $\mathcal{T} = \inf\{t > 0, \|V_t\| \neq \|v\|, \|\tilde{V}_t\| \neq \|\tilde{v}\|\}$ and

$$\tilde{S}_1 = \inf\{t > 0, \tilde{X}_t \in \partial D, \|\tilde{V}_{t-}\| \neq \|\tilde{v}\|, \|V_{t-}\| \neq \|v\|\}.$$

Step 1. We prove that

$$\mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Note first that $\tilde{S}_1 \leq \mathcal{T} + \zeta(\tilde{X}_{\mathcal{T}}, \tilde{V}_{\mathcal{T}})$ since for all $t \geq \mathcal{T}$, almost surely, $\|V_t\| \neq \|v\|$, $\|\tilde{V}_t\| \neq \|\tilde{v}\|$ and because $\tilde{X}_{\mathcal{T}+\sigma(\tilde{X}_{\mathcal{T}},\tilde{V}_{\mathcal{T}})} \in \partial D$.

Applying Lemma 2.4.7, we find that

$$\mathbb{E}[r(\mathcal{T})] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Hence, noting that $\mathcal{L}(\|\tilde{V}_{\mathcal{T}}\|) = h_R$, we obtain

$$\begin{aligned}\mathbb{E}[r(\tilde{S}_1)] &\leq C\Big(\mathbb{E}[r(\mathcal{T})] + \mathbb{E}[r(\zeta(\tilde{X}_{\mathcal{T}}, \tilde{V}_{\mathcal{T}}))]\Big)\\ &\leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big) + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}}\|}\Big)\Big]\Big) \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big),\end{aligned}$$

where we used that $\mathbb{E}[r(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}}\|})] \leq C_0$, see (2.4.18).

Step 2. We set $\tilde{S}_0 = 0$, define $\tilde{S}_1$ as in Step 1, and set, for $n \geq 1$,

$$\tilde{S}_{n+1} = \inf\{t > \tilde{S}_n, \tilde{X}_t \in \partial D, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_{\tilde{S}_n-}\|, \|V_{t-}\| \neq \|V_{\tilde{S}_n-}\|\}.$$

We set, for all $n \geq 1$, $B_n = \{\|V_{\tilde{S}_n-}\| \wedge \|\tilde{V}_{\tilde{S}_n-}\| \geq 1\}$ and $G := \inf\{n \geq 1 : B_n \text{ is realized}\}$. The aim of this step is to check that

$$\mathbb{E}[r(\tilde{S}_G)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

We plan to apply Lemma 2.4.3.

1. We set, for all $k \geq 0$, $\mathcal{G}_k = \mathcal{F}_{\tilde{S}_k-}$, and for all $k \geq 1$, $\tau_k = \tilde{S}_k$ which is $\mathcal{G}_k$-measurable, $E_k = B_k \in \mathcal{G}_k$ so that G corresponds to the notation in Lemma 2.4.3.

2. For all $k \geq 1$, using that $\mathcal{L}(\|V_{\tilde{S}_k-}\| | \mathcal{F}_{\tilde{S}_{k-1}-}) = \mathcal{L}(\|\tilde{V}_{\tilde{S}_k-}\| | \mathcal{F}_{\tilde{S}_{k-1}-}) = h_R$ since both processes have a diffuse reflection between $\tilde{S}_{k-1}-$ and $\tilde{S}_k-$,

   $$\mathbb{P}(E_k|\mathcal{G}_{k-1}) = \mathbb{P}(B_k|\mathcal{F}_{\tilde{S}_{k-1}-}) = \Big(\int_1^\infty h_R(r)dr\Big)^2 =: c,$$

   and $c > 0$ by hypothesis, see (2.4.2), whence (2.4.10).

3. Using the strong Markov property and Step 1, we have, for all $k \geq 0$,

   $$\mathbb{E}[r(\tilde{S}_{k+1} - \tilde{S}_k)|\mathcal{F}_{\tilde{S}_k-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{\tilde{S}_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\tilde{S}_k-}\|}\Big)\Big) =: K_k. \tag{2.4.20}$$

Moreover, $K_0 = \kappa(1 + r(\frac{d(D)}{\|v\|}) + r(\frac{d(D)}{\|\tilde{v}\|}))$ and for $k \geq 1$,

$$\mathbb{E}[r(\tau_{k+1} - \tau_k)|\mathcal{G}_{k-1}] = \mathbb{E}\Big[K_k\Big|\mathcal{F}_{\tilde{S}_{k-1}-}\Big]$$
$$\leq \kappa\mathbb{E}\Big[1 + r\Big(\frac{d(D)}{\|V_{\tilde{S}_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\tilde{S}_k-}\|}\Big)\Big|\mathcal{F}_{\tilde{S}_{k-1}-}\Big] \leq \kappa(1 + 2C_0).$$

We used again that $\mathcal{L}(\|V_{\tilde{S}_k-}\||\mathcal{F}_{\tilde{S}_{k-1}-}) = \mathcal{L}(\|\tilde{V}_{\tilde{S}_k-}\||\mathcal{F}_{\tilde{S}_{k-1}-}) = h_R$. Finally, we have

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(\tilde{S}_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

We conclude by applying Lemma 2.4.3.

Step 3. We prove that, for all $i \geq 1$, $X_{\tilde{S}_i} \notin \partial D$ almost surely. Since $\tilde{X}_{\tilde{S}_G} \in \partial D$ and $\|V_{\tilde{S}_G-}\| \wedge \|\tilde{V}_{\tilde{S}_G-}\| \geq 1$ by definition, by Step 2, this will conclude the proof. Define S_1 by $S_1 = \inf\{t > 0, X_t \in \partial D, \|V_t\| \neq \|v\|\}$ and note that $S_1 \leq \tilde{S}_1$ by definition.
We set $(X'_t, V'_t) = (X_{S_1+t}, V_{S_1+t})$, $(\tilde{X}'_t, \tilde{V}'_t) = (\tilde{X}_{S_1+t}, \tilde{V}_{S_1+t})$ for all $t \geq 0$. Set $T'_0 = 0$, and for all $i \geq 1$, $T'_{i+1} = \inf\{t > T'_i, X'_t \in \partial D\}$. Set also $\tilde{T}'_0 = 0$ and for all $i \geq 0$, define $\tilde{T}'_{i+1}$ by $\tilde{T}'_{i+1} = \inf\{t > \tilde{T}'_i, \tilde{X}'_t \in \partial D\}$. Since $(X'_t, V'_t)_{t\geq 0}$ and $(\tilde{X}'_t, \tilde{V}'_t)_{t\geq 0}$ are, conditionally on $\mathcal{F}_{S_1-}$, two independent processes, T'_i is independent of $\tilde{T}'_j$ for all $(i,j) \in (\mathbb{N}^*)^2$ conditionally on this σ-algebra. Moreover, by Lemma 2.4.4, T'_i has a density conditionally on $\mathcal{F}_{S_1-}$, since $X_{S_1} \in \partial D$ and $V_{S_1} = R\vartheta(X_{S_1}, \Theta)$ with $(R, \Theta) \sim \Upsilon$ independent of $\mathcal{F}_{S_1-}$. We thus have, for $(i,j) \in (\mathbb{N}^*)^2$,

$$\mathbb{P}(T'_i = \tilde{T}'_j|\mathcal{F}_{S_1-}) = 0.$$

Since we have $\{X_{\tilde{S}_G} \in \partial D\} \subset \cup_{i,j\geq 1}\{T'_i = \tilde{T}'_j\}$, we obtain $X_{\tilde{S}_G} \notin \partial D$ a.s. as desired. □

Let us introduce some notations for the remaining part of this section.

Notation 2.4.2. Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s\geq 0}$ a coupling process, see Definition 2.4.1. We use the same sequences $(S_i, Q_i, \tilde{Q}_i)_{i\geq 1}$ as in the definition, as well as $(\tilde{Q}'_i)_{i\geq 1}$, and we recall that, for all $i \geq 1$,

$$V_{S_i} = w(X_{S_i}, V_{S_i-}, Q_i)\mathbf{1}_{\{X_{S_i}\in\partial D\}} + V_{S_i-}\mathbf{1}_{\{X_{S_i}\notin\partial D\}},$$
$$\tilde{V}_{S_i} = w(\tilde{X}_{S_i}, \tilde{V}_{S_i-}, \tilde{Q}'_i)\mathbf{1}_{\{\tilde{X}_{S_i}\in\partial D\}} + \tilde{V}_{S_i-}\mathbf{1}_{\{\tilde{X}_{S_i}\notin\partial D\}}.$$

a) We set $T_0 = 0$, $\tilde{T}_0 = 0$ and for $k \geq 0$,

$$T_{k+1} = \inf\{t > \tilde{T}_k, X_t \in \partial D\}, \qquad \tilde{T}_{k+1} = \inf\{t > T_{k+1}, \tilde{X}_t \in \partial D\}.$$

For all $k \geq 1$, we have $Z_{T_k-} = \emptyset$ and $X_{T_k} \in \partial D$ so $Z_{T_k} \neq \emptyset$ if $\tilde{X}_{T_k} \notin \partial D$. We always have $Z_{\tilde{T}_k} = \emptyset$. For all $k \geq 1$, we write $(\underline{Q}_k, \underline{\tilde{Q}}_k) = (\underline{U}_k, \underline{R}_k, \underline{\Theta}_k, \underline{\tilde{U}}_k, \underline{\tilde{R}}_k, \underline{\tilde{\Theta}}_k)$ for the random vector such that

$$V_{T_k} = w(X_{T_k}, V_{T_k-}, \underline{Q}_k), \qquad \text{and} \qquad \tilde{V}_{\tilde{T}_k} = w(\tilde{X}_{\tilde{T}_k}, \tilde{V}_{\tilde{T}_k-}, \underline{\tilde{Q}}_k).$$

Note that $(\underline{Q}_k, \underline{\tilde{Q}}_k)_{k\geq 1}$ is a subsequence of $(Q_i, \tilde{Q}'_i)_{i\geq 1}$.
b) For all $t \geq 0$, we set

$$\mathcal{F}_t = \sigma\Big((X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{0\leq s\leq t}, (Q_i \mathbf{1}_{\{S_i\leq t\}})_{i\geq 1}, (\tilde{Q}_i \mathbf{1}_{\{S_i\leq t\}})_{i\geq 1}\Big).$$

c) We set $\sigma_1 = \inf\{t > 0, X_t = \tilde{X}_t \in \partial D, Z_{t-} = \emptyset, \|V_{t-}\| \neq \|V_0\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_0\|\}$.
d) We set $\nu_0 = 0$ and for all $k \geq 0$,

$$\nu_{k+1} = \inf\{n \geq \nu_k + 1, \tilde{X}_{T_n} \notin \partial D, \|V_{T_n-}\| \wedge \|\tilde{V}_{T_n-}\| \geq 1\}.$$

Note that, according to Definition 2.4.1, we have for all $n \geq 1$, conditionally on $\mathcal{F}_{T_{\nu_n}-}$,

$$(\underline{R}_{\nu_n}, \underline{\Theta}_{\nu_n}, \underline{\tilde{R}}_{\nu_n}, \underline{\tilde{\Theta}}_{\nu_n}) \sim \Lambda_{X_{T_{\nu_n}-}, \tilde{X}_{T_{\nu_n}-}, \tilde{V}_{T_{\nu_n}-}},$$

where we recall that Λ is defined in Proposition 2.4.1. We also have $Z_{T_{\nu_n}} \neq \emptyset$, see (a).

Lemma 2.4.9. *There exist three constants $\kappa, L, c > 0$ such that the following holds.*

i) For all $m \geq 1$,

$$\mathbf{1}_{\{T_{\nu_m} < \sigma_1\}} \mathbb{E}[r(T_{\nu_{m+1}} \wedge \sigma_1 - T_{\nu_m}) | \mathcal{F}_{T_{\nu_m}-}] \leq L.$$

ii) $\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq \kappa\Big(1 + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_0\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_0\|}\Big)\Big]\Big)$.

iii) For all $m \geq 1$, setting

$$A_m = \{\underline{U}_{\nu_m} \leq \alpha_0, X_{T_{\nu_m+1}} = \tilde{X}_{T_{\nu_m+1}}, T_{\nu_m+1} = T_{\nu_m} + \zeta(X_{T_{\nu_m}}, V_{T_{\nu_m}})\},$$

we have

$$\mathbb{P}\Big(A_m \Big| \mathcal{F}_{T_{\nu_m}-}\Big) \geq c,$$

and $A_m \subset \{\sigma_1 \leq T_{\nu_m+1}\}$ outside a $\mathbb{P}$-null set.

Proof. We prove i). Recall Remark 2.3.2 which defines a free-transport process with initial distribution $\delta_x \otimes \delta_v$, with $(x, v) \in \partial_+ G$. For all $k \geq 1$, we have $\|V_{T_{\nu_k}-}\| \wedge \|\tilde{V}_{T_{\nu_k}-}\| \geq 1$, $Z_{T_{\nu_k}-} = \emptyset$ and $X_{T_{\nu_k}} \in \partial D$, $\tilde{X}_{T_{\nu_k}} \notin \partial D$. Thus, using the strong Markov property, we only need to prove that there exists some $L > 0$ such that for all $(x, v) \in \partial_+ G$, $\tilde{x} \notin \partial D$, $\tilde{v} \in \mathbb{R}^n$ with $\|v\| \wedge \|\tilde{v}\| \geq 1$, if $(X_0, \tilde{X}_0, V_{0-}, \tilde{V}_{0-}, Z_0-) = (x, \tilde{x}, v, \tilde{v}, \emptyset)$,

$$\mathbb{E}\Big[r(T_{\nu_1} \wedge \sigma_1)\Big] \leq L. \tag{2.4.21}$$

We set $\mathcal{T} = \inf\{t > 0, \|V_t\| \neq \|v\|, \|\tilde{V}_t\| \neq \|\tilde{v}\|\}$. By Lemma 2.4.7 and since $\|v\| \wedge \|\tilde{v}\| \geq 1$, $\mathbb{E}[r(\mathcal{T})] \leq L$.

It thus suffices to prove that

$$\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1 - \mathcal{T})\mathbf{1}_{\{T_{\nu_1}\wedge\sigma_1 > \mathcal{T}\}}] \leq L.$$

To this end, we will use Lemma 2.4.8.

Set, for all $t \geq 0$, $(X'_t, V'_t) = (X_{\mathcal{T}+t}, V_{\mathcal{T}+t})$ and $(\tilde{X}'_t, \tilde{V}'_t) = (\tilde{X}_{\mathcal{T}+t}, \tilde{V}_{\mathcal{T}+t})$. Conditionally on $\mathcal{F}_{\mathcal{T}-}$, on the event $\{T_{\nu_1} \wedge \sigma_1 > \mathcal{T}\}$, the processes $(X'_t, V'_t)_{0 \leq t < T_{\nu_1}\wedge\sigma_1 - \mathcal{T}}$ and $(\tilde{X}'_t, \tilde{V}'_t)_{0 \leq t < T_{\nu_1}\wedge\sigma_1 - \mathcal{T}}$ are two independent (killed) free-transport processes with initial distributions $\delta_{X_{\mathcal{T}}} \otimes \delta_{V_{\mathcal{T}-}}$ and $\delta_{\tilde{X}_{\mathcal{T}}} \otimes \delta_{\tilde{V}_{\mathcal{T}-}}$. Indeed, by definition of σ_1 and ν_1, the first and third lines of Table 2.1 are never used during $[\mathcal{T}, T_{\nu_1} \wedge \sigma_1)$, so that the innovations $(\mathbf{Q}, \tilde{\mathbf{Q}})$ are always independent or one of them is useless.

Using Lemma 2.4.8, since $T_{\nu_1} \wedge \sigma_1 - \mathcal{T} \leq T_{\nu_1} - \mathcal{T} \leq S$ with the notation of the Lemma, we conclude that

$$\mathbf{1}_{\{T_{\nu_1}\wedge\sigma_1 > \mathcal{T}\}}\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1 - \mathcal{T})|\mathcal{F}_{\mathcal{T}-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{\mathcal{T}-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}-}\|}\Big)\Big).$$

We obtain

$$\begin{aligned}
\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1 - \mathcal{T})\mathbf{1}_{\{T_{\nu_1}\wedge\sigma_1 > \mathcal{T}\}}] &\leq \kappa\Big(\mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{\mathcal{T}-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}-}\|}\Big)\Big] + 1\Big) \qquad (2.4.22)\\
&\leq \kappa\Big(\mathbb{E}\Big[r\Big(\frac{d(D)}{\|v\|}\Big)\mathbf{1}_{\{\|V_{\mathcal{T}-}\| = \|v\|\}} + r\Big(\frac{d(D)}{\|V_{\mathcal{T}-}\|}\Big)\mathbf{1}_{\{\|V_{\mathcal{T}-}\| \neq \|v\|\}}\\
&\qquad + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\mathbf{1}_{\{\|\tilde{V}_{\mathcal{T}-}\| = \|\tilde{v}\|\}} + r\Big(\frac{d(D)}{\|\tilde{V}_{\mathcal{T}-}\|}\Big)\mathbf{1}_{\{\|\tilde{V}_{\mathcal{T}-}\| \neq \|\tilde{v}\|\}}\Big)\Big] + 1\Big)\\
&\leq L,
\end{aligned}$$

using (2.4.18), that $\mathcal{L}(\|V_{\mathcal{T}-}\| | \|V_{\mathcal{T}-}\| \neq \|v\|) = \mathcal{L}(\|\tilde{V}_{\mathcal{T}-}\| | \|\tilde{V}_{\mathcal{T}-}\| \neq \|\tilde{v}\|) = h_R$ and the fact that $\|v\| \wedge \|\tilde{v}\| \geq 1$. This concludes the proof of (2.4.21) and thus of i).

For ii), we apply the same proof as for i), replacing everywhere $(v, \tilde{v})$ by $(V_0, \tilde{V}_0)$. We conclude that

$$\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq \kappa\Big(1 + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_0\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_0\|}\Big)\Big]\Big).$$

We prove iii). Set, for all $k \geq 1$, $W_k = (\underline{U}_k, \underline{R}_k, \underline{\Theta}_k, \underline{\tilde{U}}_k, \underline{\tilde{R}}_k, \underline{\tilde{\Theta}}_k)$. Recall that $T_k < \tilde{T}_k$. We deduce that W_k is independent of $\mathcal{F}_{T_k-}$ and is $\mathcal{F}_{\tilde{T}_k}$-measurable. Also, we have $\tilde{X}_{T_{\nu_k}} \notin \partial D$ and

$\|V_{T_{\nu_k}-}\| \wedge \|\tilde{V}_{T_{\nu_k}-}\| \geq 1$ by definition of ν_k. Hence $W_{\nu_k} \sim \Gamma_{X_{T_{\nu_k}-}, V_{T_{\nu_k}-}, \tilde{X}_{T_{\nu_k}-}, \tilde{V}_{T_{\nu_k}-}}$ and its law is given by the second line of (2.4.14). Thus, conditionally on $\mathcal{F}_{T_{\nu_k}-}$,

$$(\underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k}) \sim \Lambda_{X_{T_{\nu_k}}, \tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}},$$

the random variable $\underline{U}_{\nu_k}$ satisfies $\underline{U}_{\nu_k} \sim \mathcal{U}$, is independent of $(\underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k})$ and we have $\underline{U}_{\nu_k} = \underline{\tilde{U}}_{\nu_k}$. Recall, for $(x, \tilde{x}, \tilde{v}) \in \partial D \times D \times \mathbb{R}^n$, the notation $E_{x,\tilde{x},\tilde{v}}$ from Proposition 2.4.1. We set

$$C_{x,\tilde{x},\tilde{v}} = \Big\{(u, \tilde{u}, r, \tilde{r}, \theta, \tilde{\theta}) \in [0,1]^2 \times \mathbb{R}_+^2 \times \mathcal{A}^2 : u \leq \alpha_0, \tilde{u} \leq \alpha_0, (r, \theta, \tilde{r}, \tilde{\theta}) \in E_{x,\tilde{x},\tilde{v}}\Big\}.$$

We have

$$\Big\{W_{\nu_k} \in C_{X_{T_{\nu_k}}, \tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}}\Big\} \subset A_k.$$

Indeed, if $W_{\nu_k} \in C_{X_{T_{\nu_k}}, \tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}}$, we have first $\underline{U}_{\nu_k} \leq \alpha_0$, so that $V_{T_{\nu_k}} = \underline{R}_{\nu_k} \vartheta(X_{T_{\nu_k}}, \underline{\Theta}_{\nu_k})$. In this configuration, after T_{ν_k}, (X, V) has its first collision at time $T_{\nu_k} + \zeta(X_{T_{\nu_k}}, V_{T_{\nu_k}})$ while $(\tilde{X}, \tilde{V})$ collides for the first time after T_{ν_k} at time $\tilde{T}_{\nu_k} = T_{\nu_k} + \zeta(\tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}})$. Moreover, recalling Definition 2.4.1,

$$\tilde{V}_{\tilde{T}_{\nu_k}} = \underline{\tilde{R}}_{\nu_k} \vartheta(\tilde{X}_{\tilde{T}_{\nu_k}}, \underline{\tilde{\Theta}}_{\nu_k}).$$

We obtain, recalling Notation 2.4.1 and Proposition 2.4.1, that

$$\begin{aligned} T_{\nu_{k+1}} = T_{\nu_k} + \zeta(X_{T_{\nu_k}}, V_{T_{\nu_k}}) = T_{\nu_k} + \xi(X_{T_{\nu_k}}, \underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}) &= T_{\nu_k} + \tilde{\xi}(\tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}-}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k}) \\ &= \tilde{T}_{\nu_k} + \zeta(\tilde{X}_{\tilde{T}_{\nu_k}}, \tilde{V}_{\tilde{T}_{\nu_k}}). \end{aligned}$$

and

$$X_{T_{\nu_k+1}} = q(X_{T_{\nu_k}}, V_{T_{\nu_k}}) = y(X_{T_{\nu_k}}, \underline{\Theta}_{\nu_k}) = \tilde{y}(\tilde{X}_{T_{\nu_k}}, \tilde{V}_{T_{\nu_k}}, \underline{\tilde{\Theta}}_{\nu_k}) = q(\tilde{X}_{\tilde{T}_{\nu_k}}, \tilde{V}_{\tilde{T}_{\nu_k}}) = \tilde{X}_{T_{\nu_{k+1}}}.$$

We have, for all $k \geq 1$,

$$\begin{aligned} \mathbb{P}(A_k | \mathcal{F}_{T_{\nu_k}-}) &\geq \mathbb{P}\Big(W_{\nu_k} \in C_{X_{T_{\nu_k}-}, \tilde{X}_{T_{\nu_k}-}, \tilde{V}_{T_{\nu_k}-}}\Big) \\ &= \mathbb{P}(\underline{U}_{\nu_k} \leq \alpha_0 | \mathcal{F}_{T_{\nu_k}-}) \mathbb{P}\Big((\underline{R}_{\nu_k}, \underline{\Theta}_{\nu_k}, \underline{\tilde{R}}_{\nu_k}, \underline{\tilde{\Theta}}_{\nu_k}) \in E_{X_{T_{\nu_k}-}, \tilde{X}_{T_{\nu_k}-}, \tilde{V}_{T_{\nu_k}-}} \Big| \mathcal{F}_{T_{\nu_k}-}\Big) \\ &\geq \alpha_0 c, \end{aligned}$$

with $c > 0$ given by Proposition 2.4.1.

On A_k, we have $X_{T_{\nu_k+1}} = \tilde{X}_{T_{\nu_k+1}}$, $Z_{T_{\nu_k+1}-} = \emptyset$ because, for all $i \geq 1$, $Z_{T_i-} = \emptyset$, and, since $\underline{U}_{\nu_k} = \underline{\tilde{U}}_{\nu_k} \leq \alpha_0$, and $\tilde{X}_{T_{\nu_k}} \notin \partial D$,

$$\|V_{T_{\nu_k+1}-}\| = \|V_{T_{\nu_k}}\| = \underline{R}_{\nu_k} \neq \|V_0\|,$$

$$\|\tilde V_{T_{\nu_k+1}-}\| = \|\tilde V_{\tilde T_{\nu_k}}\|\mathbf{1}_{\{\|\tilde V_{T_{\nu_k+1}-}\|=\|\tilde V_{\tilde T_{\nu_k}}\|\}} + \|\tilde V_{T_{\nu_k+1}-}\|\mathbf{1}_{\{\|\tilde V_{T_{\nu_k+1}-}\|\neq\|\tilde V_{\tilde T_{\nu_k}}\|\}},$$

with $\mathcal{L}(\|\tilde V_{T_{\nu_k+1}-}\| | \|\tilde V_{T_{\nu_k+1}-}\| \neq \|\tilde V_{\tilde T_{\nu_k}}\|) = \mathcal{L}(\|\tilde V_{\tilde T_{\nu_k}}\| | \underline{\tilde U}_{\nu_k} \leq \alpha_0) = h_R$ from which we obtain $\mathbb{P}(\|\tilde V_{T_{\nu_k+1}-}\| = \|\tilde V_0\|) = 0$. We conclude that $A_k \subset \{\sigma_1 \leq T_{\nu_{k+1}}\}$ outside a $\mathbb{P}$-null set. □

Proof of Theorem 2.1.1 in the convex case. We fix $f_0 \in \mathcal{P}(D\times\mathbb{R}^n)$. We consider the coupling $(X_s, V_s, \tilde X_s, \tilde V_s, Z_s)_{s\geq 0}$ given by Definition 2.4.1. By Lemma 2.4.5, for any $t>0$, $(X_t, V_t) \sim f_t$ and $(\tilde X_t, \tilde V_t) \sim \mu_\infty$.

We prove, with the help of Lemma 2.4.9, that, setting

$$\tau = \inf\{t>0, (X_{t+s}, V_{t+s})_{s\geq 0} = (\tilde X_{t+s}, \tilde V_{t+s})_{s\geq 0}\},$$

we have $\mathbb{E}[r(\tau)] < \infty$. We then conclude the proof of Theorem 2.1.1 in Step 4.

Step 1. Recall Notation 2.4.2 for σ_1 and for the sequence $(\nu_k)_{k\geq 0}$. We plan to apply Lemma 2.4.3 to show that $\mathbb{E}[r(\sigma_1)] \leq \kappa$.

1. Set, for $k\geq 0$, $\mathcal{G}_k = \mathcal{F}_{T_{\nu_k}\wedge\sigma_1-}$, and for $k\geq 1$, $\tau_k = T_{\nu_k}\wedge\sigma_1$, which is $\mathcal{G}_k$-measurable. Also, set
$$E_k = \{\sigma_1 \leq T_{\nu_k}\} \in \mathcal{G}_k.$$
Set $G = \inf\{k\geq 1, E_k \text{ is realized}\}$.

2. Recall, for all $k\geq 1$, the notation A_k from Lemma 2.4.9, iii). Observe that, according to the Lemma and since $\nu_{k+1} \geq \nu_k + 1$, there holds $A_{k-1} \subset \{\sigma_1 \leq T_{\nu_{k-1}+1}\} \subset \{\sigma_1 \leq T_{\nu_k}\} \subset E_k$. We have, for all $k\geq 1$, by Lemma 2.4.9 iii),
$$\mathbb{P}(E_k|\mathcal{G}_{k-1}) = \mathbb{P}(E_k|\mathcal{F}_{T_{\nu_{k-1}}\wedge\sigma_1-}) \geq \mathbb{E}\Big[\mathbb{P}(A_{k-1}|\mathcal{F}_{T_{\nu_{k-1}}-})\Big|\mathcal{F}_{T_{\nu_{k-1}}\wedge\sigma_1-}\Big] \geq c,$$
whence (2.4.10).

3. From Lemma 2.4.9 ii) and (2.4.18), we have
$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(T_{\nu_1}\wedge\sigma_1)] \leq L.$$
Moreover, by Lemma 2.4.9 i), for all $k\geq 1$, we have, using the straightforward inclusions $\mathcal{F}_{T_{\nu_{k-1}}\wedge\sigma_1-} \subset \mathcal{F}_{T_{\nu_{k-1}}-} \subset \mathcal{F}_{T_{\nu_k}-}$,
$$\begin{aligned}\mathbf{1}_{\{G\geq k\}}\mathbb{E}[r(\tau_{k+1}-\tau_k)|\mathcal{G}_{k-1}] &\leq \mathbb{E}[r(T_{\nu_{k+1}}\wedge\sigma_1 - T_{\nu_k}\wedge\sigma_1)|\mathcal{F}_{T_{\nu_{k-1}}\wedge\sigma_1-}]\\ &\leq r(0) + \mathbb{E}\Big[\mathbf{1}_{\{\sigma_1 > T_{\nu_k}\}}\mathbb{E}[r(T_{\nu_{k+1}}\wedge\sigma_1 - T_{\nu_k})|\mathcal{F}_{T_{\nu_k}-}]\Big|\mathcal{F}_{T_{\nu_{k-1}}\wedge\sigma_1-}\Big]\\ &\leq r(0) + L,\end{aligned}$$

whence (2.4.11).

We apply Lemma 2.4.3 and conclude that

$$\mathbb{E}[r(\tau_G)] \leq \kappa,$$

from which we deduce, by definition of G, that

$$\mathbb{E}[r(\sigma_1)] = \mathbb{E}[r(\sigma_1 \wedge T_{\nu_G})] = \mathbb{E}[r(\tau_G)] \leq \kappa.$$

Step 2. We introduce the sequence $(\sigma_i)_{i\geq 0}$ defined by $\sigma_0 = 0$, σ_1 defined by Notation 2.4.2, iii), and for all $k \geq 1$,

$$\sigma_{k+1} = \inf\{t > \sigma_k, X_t = \tilde{X}_t \in \partial D, Z_{t-} = \emptyset, \|V_{t-}\| \neq \|V_{\sigma_k}\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_{\sigma_k}\|\}.$$

We plan to apply Lemma 2.4.3.

1. We set $\mathcal{G}_0$ to be the completion of the trivial σ-algebra and, for $k \geq 1$, $\mathcal{G}_k = \mathcal{F}_{\sigma_{k+1}-}$. We also set, for all $k \geq 1$, $\tau_k = \sigma_{k+1}$ which is $\mathcal{G}_k$-measurable, and $E_k = \{V_{\sigma_k} = \tilde{V}_{\sigma_k}\} \in \mathcal{G}_k$. We set $N = \inf\{k \geq 1, E_k \text{ is realized }\}$.

2. Let, for all $k \geq 1$, $(\mathbf{Q}_k, \tilde{\mathbf{Q}}_k) = ((\mathbf{U}_k, \mathbf{R}_k, \mathbf{\Theta}_k), (\tilde{\mathbf{U}}_k, \tilde{\mathbf{R}}_k, \tilde{\mathbf{\Theta}}_k))$ be the couple random variables used to define V_{σ_k} and $\tilde{V}_{\sigma_k}$. Since $X_{\sigma_k} = \tilde{X}_{\sigma_k}$ and $Z_{\sigma_k-} = \emptyset$, we are in the situation of line 3 of Table 2.1, hence $\mathbf{Q}_k = \tilde{\mathbf{Q}}_k$, so that if $\mathbf{U}_k \leq \alpha_0$,

$$V_{\sigma_k} = w(X_{\sigma_k}, V_{\sigma_k-}, \mathbf{Q}_k) = w(X_{\sigma_k}, \tilde{V}_{\sigma_k-}, \mathbf{Q}_k) = \tilde{V}_{\sigma_k}.$$

   Since $\mathbf{Q}_k$ is independent of $\mathcal{F}_{\sigma_k-}$,

$$\mathbb{P}(E_k|\mathcal{G}_{k-1}) \geq \mathbb{P}(\mathbf{U}_k \leq \alpha_0|\mathcal{F}_{\sigma_k-}) = \alpha_0,$$

   whence (2.4.10).

3. Note that for $k \geq 1$,

$$\mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_{\sigma_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\sigma_k-}\|}\Big)\Big|\mathcal{F}_{\sigma_{k-1}-}\Big] \leq 2C_0, \tag{2.4.23}$$

   using that $\mathcal{L}(\|V_{\sigma_k-}\||\mathcal{F}_{\sigma_{k-1}-}) = \mathcal{L}(\|\tilde{V}_{\sigma_k-}\||\mathcal{F}_{\sigma_{k-1}-}) = h_R$, since $\|V_{\sigma_k-}\| \neq \|V_{\sigma_{k-1}}\|$ and $\|\tilde{V}_{\sigma_k-}\| \neq \|\tilde{V}_{\sigma_{k-1}}\|$ by definition of σ_k. By Step 1, Lemma 2.4.9, ii), the strong Markov property and the definition of $(\sigma_i)_{i\geq 0}$, we have, for all $k \geq 1$,

$$\mathbb{E}[r(\sigma_{k+1} - \sigma_k)|\mathcal{F}_{\sigma_k-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{\sigma_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{\sigma_k-}\|}\Big)\Big),$$

so that, using (2.4.23) and that $\mathcal{F}_{\sigma_{k-1}-} \subset \mathcal{F}_{\sigma_k-}$,

$$\mathbb{E}[r(\sigma_{k+1}-\sigma_k)|\mathcal{F}_{\sigma_{k-1}-}] \leq \kappa.$$

With this at hand, we show that (2.4.13) holds. First, by Step 1,

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(\sigma_2)] \leq C\Big(\mathbb{E}[r(\sigma_2-\sigma_1)]+\kappa\Big) \leq L.$$

Moreover, for $k \geq 1$,

$$\mathbf{1}_{\{N\geq k\}}\mathbb{E}[r(\tau_{k+1}-\tau_k)|\mathcal{G}_{k-1}] \leq \mathbb{E}[r(\sigma_{k+2}-\sigma_{k+1})|\mathcal{F}_{\sigma_k-}] \leq \kappa,$$

whence (2.4.13).

We conclude by Lemma 2.4.3 that $\mathbb{E}[r(\sigma_N)] \leq \kappa$.

Step 3 Using Lemma 2.4.6, since $(X_{\sigma_N}, V_{\sigma_N}) = (\tilde{X}_{\sigma_N}, \tilde{V}_{\sigma_N})$ and $Z_{\sigma_N-} = \emptyset$, we conclude that $\tau \leq \sigma_N$, hence $\mathbb{E}[r(\tau)] \leq \kappa$ by Step 2.

Step 4. Recall that for two probability measures μ, ν,

$$\|\mu-\nu\|_{TV} = \inf_{X\sim\mu, Y\sim\nu} \mathbb{P}(X \neq Y).$$

Hence for all $t \geq 0$,

$$\|f_t-\mu_\infty\|_{TV} \leq \mathbb{P}\Big((X_t,V_t) \neq (\tilde{X}_t,\tilde{V}_t)\Big) = \mathbb{P}(\tau > t),$$

according to our definition of τ. Finally, we use Step 3 and Markov's inequality to conclude that

$$\|f_t-\mu_\infty\|_{TV} \leq \frac{\mathbb{E}[r(\tau)]}{r(t)} \leq \frac{\kappa}{r(t)},$$

for all $t \geq 0$. □

2.5 Extension to a general regular domain

In this section, we extend the previous result on a convex bounded domain (open, connected) to the general case of a C^2 bounded domain.

2.5.1 Notations and preliminary results.

In this subsection, we introduce the notion of communication between boundary points, derive an important corollary from this definition and prove a preliminary lemma that will be key to obtain a result similar to Proposition 2.4.1 in the general setting.

We introduce first a notion of communicating boundary points taken from Evans [55].

Definition 2.5.1. We say that two points $x \in \partial D$, $y \in \partial D$ communicate, and write $x \leftrightarrow y$ if $tx + (1-t)y \in D$ for all $t \in (0,1)$, $n_x \cdot (y-x) > 0$ and $n_y \cdot (x-y) > 0$. Given a set $E \subset \partial D$ we say that $x \in \partial D$ communicates with E and write $x \leftrightarrow E$ if $x \leftrightarrow y$ for all $y \in E$. Given two sets $E_1, E_2 \subset \partial D$, we say that E_1 and E_2 communicate, and write $E_1 \leftrightarrow E_2$ if $x \leftrightarrow y$ for all $(x,y) \in E_1 \times E_2$.

Since D is regular, the condition $tx+(1-t)y \in D$ for all $t \in (0,1)$ implies that $n_x \cdot (y-x) \geq 0$. The previous notion forbids the case where $(y-x)$ is tangent to ∂D at x.

Recall that we denote by $\mathcal{H}$ the $n-1$ dimensional Hausdorff measure. The goal of this subsection is to prove the following lemma.

Lemma 2.5.1. *There exists κ_0, $d_0 > 0$, $F \subset \partial D$, $\mathcal{R} \subset \partial D$ with F, $\mathcal{R}$ compact and $F \leftrightarrow \mathcal{R}$ such that $\inf\limits_{(x,y)\in F\times\mathcal{R}} \|x-y\| \geq d_0$ and $\mathcal{H}(F) \wedge \mathcal{H}(\mathcal{R}) \geq \kappa_0$.*

Recall that $d(D)$ denotes the diameter (in the usual sense) of D and that for $x \in \mathbb{R}^n$ and $r > 0$, we write $B(x,r) = \{y \in \mathbb{R}^n, \|x-y\| < r\}$ for the Euclidian ball centered at x, with radius r, in $\mathbb{R}^n$. We denote $\bar{B}(x,r)$ the corresponding closed ball.

Notation 2.5.1. For $x \in \partial D$, $r > 0$, we set $B_{\partial D}(x,r) := B(x,r) \cap \partial D$.

Lemma 2.5.2. *Let $x, y \in \partial D$ with $x \leftrightarrow y$. There exists $\epsilon_0 > 0$ such that $B_{\partial D}(x,\epsilon_0)$ and $B_{\partial D}(y,\epsilon_0)$ communicate.*

Proof. **Step 1.** Recall first that since D is C^2, D satisfies the uniform ball condition: there exists $r_D > 0$ such that for all $z \in \partial D$, there exists B_z a ball of radius r_D with center $z + r_D n_z$ such that $B_z \subset D$ and $\bar{B}_z \cap \partial D = \{z\}$. As a consequence, for $\beta > 0$ to choose later, setting $t_0 = \frac{r_D \beta}{2d(D)} \wedge \frac{1}{4}$, there holds that for all $x, z \in \partial D$ with $n_z \cdot \frac{x-z}{\|x-z\|} \geq \frac{\beta}{2}$, $(1-t)z + tx \in B_z \subset D$. Indeed

$$\begin{aligned}\|(1-t)z + tx - z - r_D n_z\|^2 &= t^2\|x-z\|^2 + r_D^2 - 2tr_D\big(n_z \cdot (x-z)\big)\\ &\leq r_D^2 + t\|x-z\|\Big(td(D) - r_D\beta\Big)\end{aligned}$$

and since $t < t_0 < \frac{r_D\beta}{d(D)}$, the result follows.

Step 2. Let $x, y \in \partial D$ with $x \leftrightarrow y$. We have $n_x \cdot (y-x) > 0$, $n_y \cdot (x-y) > 0$ and $x \neq y$, hence $\beta := (n_y \cdot \frac{(x-y)}{\|x-y\|}) \wedge (n_x \cdot \frac{(y-x)}{\|y-x\|}) > 0$. Since $z \to n_z$ is continuous by regularity of D, there exists $\delta > 0$ such that for all $x' \in B_{\partial D}(x, \delta)$, $y' \in B_{\partial D}(y, \delta)$, $(n_{y'} \cdot \frac{(x'-y')}{\|x'-y'\|}) \wedge (n_{x'} \cdot \frac{(y'-x')}{\|y'-x'\|}) \geq \frac{\beta}{2}$. By Step 1, for all $t \in (0, t_0)$,

$$(1-t)y' + tx' \in B_{y'} \subset D,$$

and, for all $t \in (1-t_0, 1)$,

$$(1-t)y' + tx' \in B_{x'} \subset D.$$

We conclude that for all $t \in (0, t_0) \cup (1-t_0, 1)$, $tx' + (1-t)y' \in D$.

Step 3. Since $x \leftrightarrow y$ by assumption, for all $t \in [t_0, 1-t_0]$, $tx + (1-t)y \in D$. By compactness and continuity of $a \to d(a, \partial D) := \inf_{z \in \partial D} \|a - z\|$, there exists $\eta > 0$ such that for all $t \in [t_0, 1-t_0]$, $B((1-t)y + tx, \eta) \subset D$. Hence, for δ given by Step 2, for all $x' \in B_{\partial D}(x, \delta \wedge \eta)$, $y' \in B_{\partial D}(y, \delta \wedge \eta)$, for all $t \in [t_0, 1-t_0]$,

$$\|(1-t)y' + tx' - (1-t)y - tx\| \leq \max(\|y' - y\|, \|x' - x\|) < \eta,$$

so that $(1-t)y' + tx' \in B((1-t)y + tx, \eta) \subset D$. Setting $\epsilon_0 = \delta \wedge \eta$, we conclude that, for all $x' \in B_{\partial D}(x, \epsilon_0)$, $y' \in B_{\partial D}(y, \epsilon_0)$,

$$n_{y'} \cdot (x' - y') > 0 \qquad \text{and} \qquad n_{x'} \cdot (y' - x') > 0$$

by Step 2 and for all $t \in (0,1)$, $tx' + (1-t)y' \in D$ by Steps 1 and 2. □

Proof of Lemma 2.5.1. Let $x, y \in \partial D$ such that $x \leftrightarrow y$. Set $\bar{d} = \|x - y\|$. Using Lemma 2.5.2, there exists $\epsilon_0 > 0$ such that, setting $V_x := B_{\partial D}(x, \epsilon_0)$, $V_y := B_{\partial D}(y, \epsilon_0)$, $V_x \leftrightarrow V_y$. Upon reducing the value of ϵ_0, we can assume that for any $x' \in V_x, y' \in V_y$, $\|x' - y'\| \geq \frac{\bar{d}}{2}$. We conclude by setting $F = \bar{B}(y, \frac{\epsilon_0}{2}) \cap \partial D$, $\mathcal{R} = \bar{B}(x, \frac{\epsilon_0}{2}) \cap \partial D$ and $d_0 = \frac{\bar{d}}{2}$. □

2.5.2 Uniform lower bound on the density of the n_0-th collision.

We introduce the following notation.

Notation 2.5.2. Let $(x_0, v_0) \in \partial_+ G \cup (D \times \mathbb{R}^n)$. For a free-transport process $(X_t, V_t)_{t \geq 0}$ with initial condition $X_0 = x_0$, $V_{0-} = v_0$, we set $T_0 = \zeta(X_0, V_0)$ and for $i \geq 0$, $T_{i+1} = \inf\{t > T_i, X_t \in \partial D\}$. For all $k \geq 1$, we denote $P^k_{v_0}(x_0, dz)$ the law of X_{T_k}.

The goal of this section is to prove the following property.

Proposition 2.5.1. *There exist $n_0 \geq 1$, $\nu_0 > 0$ and $\delta_0 > 0$ such that, for all (x_0, v_0) in $\partial_+ G \cup (D \times \mathbb{R}^n)$,*

$$P^{n_0}_{v_0}(x_0, dz) \geq \nu_0 dz,$$

where we recall that dz stands for the $n-1$ dimensional Hausdorff measure. Moreover, for all $A \subset \partial D$, setting $O_0 = \{\|V_{T_0}\| \neq \|V_{T_0-}\|, \dots, \|V_{T_{n_0-1}}\| \neq \|V_{T_{n_0-1}-}\|\}$,

$$\mathbb{P}\Big(X_{T_{n_0}} \in A, \min_{i\in\{1,\dots,n_0\}} \|X_{T_i} - X_{T_{i-1}}\| \geq \delta_0 \Big| (X_0, V_{0-}) = (x_0, v_0), O_0\Big) \geq \nu_0 \mathcal{H}(A).$$

We recall first a result from Evans from which we will derive a key feature of our model:

Proposition 2.5.2 ([55], Proposition 2.7)**.** *For any C^1 bounded domain D, there exist an integer N and a finite set $\Delta \subset \partial D$ for which the following holds: for all $z', z'' \in \partial D$, there exist $z_0, \dots, z_N$ with $z' = z_0$, $z'' = z_N$, $\{z_1, \dots, z_{N-1}\} \subset \Delta$ and $z_k \leftrightarrow z_{k+1}$ for $0 \leq k \leq N-1$.*

Corollary 2.5.1. *There exist $\delta > 0$ and $\eta > 0$ such that for all $(x_0, y_0) \in (\partial D)^2$, there exists $z_1, \dots, z_{N+1} \in \Delta$, with N and Δ given by Proposition 2.5.2, such that, setting $z_0 = x_0$, $z_{N+2} = y_0$, $z_i \leftrightarrow z_{i+1}$ for all $i \in \{0, \dots, N+1\}$ and*

$$|(z_i - z_{i+1}) \cdot n_{z_i}||(z_i - z_{i+1}) \cdot n_{z_{i+1}}| \geq 2\delta, \tag{2.5.1}$$

moreover, for all $z'_1 \in B_{\partial D}(z_1, \eta), \dots, z'_{N+1} \in B_{\partial D}(z_{N+1}, \eta)$, setting $z'_0 = z_0$, $z'_{N+2} = z_{N+2}$, $z'_i \leftrightarrow z'_{i+1}$ for all $i \in \{0, \dots, N+1\}$ and

$$|(z'_i - z'_{i+1}) \cdot n_{z'_i}||(z'_i - z'_{i+1}) \cdot n_{z'_{i+1}}| \geq \delta. \tag{2.5.2}$$

Proof. **Step 1.** By [55, Lemma 2.3], for $z \in \partial D$, the set $U_z = \{z' \in \partial D, z' \leftrightarrow z\}$ is open in ∂D and non-empty. Using this result and the fact that D is C^1, we find that for all $z \in \Delta$,

$$x \to |(z - x) \cdot n_z||(z - x) \cdot n_x| \mathbf{1}_{U_z}(x),$$

is lower semi-continuous, and positive on U_z. Using Proposition 2.5.2, that Δ is finite, and that the maximum of two lower semi-continuous functions is lower semi-continuous, we deduce that the function $I : \partial D \to \mathbb{R}_+$ defined by

$$I(x) = \max_{z\in\Delta} \Big(|(z - x) \cdot n_z||(z - x) \cdot n_x| \mathbf{1}_{U_z}(x)\Big),$$

is lower semi-continuous. Moreover, since for all $x \in \partial D$, there exists $z \in \Delta$ such that $x \leftrightarrow z$ by Proposition 2.5.2, $I > 0$ on ∂D. We conclude by compactness that there exists $\delta_0 > 0$ such that $I(x) > 2\delta_0$ for all $x \in \partial D$.

Step 2. Set

$$\delta' := \frac{1}{2} \min_{z,z'\in\Delta, z\leftrightarrow z'} |(z - z') \cdot n_z||(z - z') \cdot n_{z'}| > 0,$$

since Δ is finite. Let $x_0, y_0 \in \partial D$. Choose z_1 such that $I(x_0) = |(z_1 - x_0) \cdot n_{z_1}||(z_1 - x_0) \cdot n_{x_0}|$, z_{N+1} such that $I(y_0) = |(z_{N+1} - y_0) \cdot n_{y_0}||(z_{N+1} - y_0) \cdot n_{z_{N+1}}|$. By Proposition 2.5.2, there exists $z_2, \dots, z_N$ such that $z_i \leftrightarrow z_{i+1}$ for all $i \in \{1, \dots, N\}$. Since $z_0 = x_0$, $z_{N+2} = y_0$, $z_i \leftrightarrow z_{i+1}$ for all $i \in \{0, \dots, N+1\}$ and for all $i \in \{1, \dots, N\}$, we have,

$$|(z_i - z_{i+1}) \cdot n_{z_i}||(z_i - z_{i+1}) \cdot n_{z_{i+1}}| \geq 2\delta',$$

while, using Step 1,

$$\Big(|(z_1 - z_0) \cdot n_{z_0}||(z_1 - z_0) \cdot n_{z_1}|\Big) \wedge \Big(|(z_{N+1} - z_{N+2}) \cdot n_{z_{N+1}}||(z_{N+1} - z_{N+2}) \wedge n_{z_{N+2}}|\Big) \geq 2\delta_0.$$

We set $\delta = \delta_0 \wedge \delta'$ to conclude the proof of (2.5.1).

Step 3. Consider the function H defined on $(\partial D)^2$ by

$$H(x, z) = \big((z - x) \cdot n_x\big)\big((x - z) \cdot n_z\big).$$

Since D is C^1, H is continuous on $(\partial D)^2$ and also uniformly continuous by compactness and Heine's theorem. Hence there exists η_0 such that,

$$\Big[(x, z) \in (\partial D)^2, (x', z') \in (\partial D)^2, \|(x, z) - (x', z')\| \leq \eta_0\Big] \implies \Big[|H(x, z) - H(x', z')| \leq \frac{\delta}{2}\Big].$$

On the other hand, for all $(x, y) \in (\partial D)^2$ with $x \leftrightarrow y$, there exists $\epsilon_{x,y} > 0$ such that we have $B_{\partial D}(x, \epsilon_{x,y}) \leftrightarrow B_{\partial D}(y, \epsilon_{x,y})$, see Lemma 2.5.2. Setting $\eta_1 = \min_{z,z' \in \Delta, z \leftrightarrow z'} \epsilon_{z,z'} > 0$, we deduce that for all $z, z' \in \Delta$ with $z \leftrightarrow z'$, $B_{\partial D}(z, \eta_1) \leftrightarrow B_{\partial D}(z', \eta_1)$. We claim that setting $\eta = \eta_1 \wedge \eta_0$ concludes the proof of (2.5.2). Indeed, for $z_1' \in B_{\partial D}(z_1, \eta)$, recalling that $z_0' = x_0$ and (2.5.1),

$$H(z_1', z_0') = H(z_1, z_0') - (H(z_1, z_0') - H(z_1', z_0')) \geq 2\delta - \frac{\delta}{2} \geq \frac{3\delta}{2},$$

and the same argument applies replacing z_1' by $z_{N+1}' \in B_{\partial D}(z_{N+1}, \eta)$ and z_0' by $z_{N+2}' = y_0$. Finally, for $i \in \{1, \dots, N\}$, $z_i' \in B_{\partial D}(z_i, \eta)$, $z_{i+1}' \in B_{\partial D}(z_{i+1}, \eta)$, we have $z_i' \leftrightarrow z_{i+1}'$ and

$$\begin{aligned} H(z_i', z_{i+1}') &= H(z_i, z_{i+1}) - (H(z_i, z_{i+1}) - H(z_i, z_{i+1}')) - (H(z_i, z_{i+1}') - H(z_i', z_{i+1}')) \\ &\geq 2\delta - \frac{\delta}{2} - \frac{\delta}{2} \geq \delta. \end{aligned}$$

□

Recall the notations ζ and q from (2.3.3) and (2.3.4).

Lemma 2.5.3. *Let $x \in \partial D$. For $V \sim c_0 M(v)|v \cdot n_x| \mathbf{1}_{\{v \cdot n_x > 0\}}$, the joint law of the couple $(\zeta(x, V), q(x, V))$ admits a density μ_x on $\mathbb{R}_+ \times \partial D$ given by*

$$\mu_x(\tau, z) = c_0 M\Big(\frac{z - x}{\tau}\Big) \frac{1}{\tau^{n+2}} |(z - x) \cdot n_x||(z - x) \cdot n_z| \mathbf{1}_{\{z \leftrightarrow x\}}.$$

Proof. The computation is the same as the one of Lemma 2.4.2. □

Proof of Corollary 2.5.1. We will show that there exist $n_0 \geq 1$, $\nu_0 > 0$ and $\delta_0 > 0$ such that for all $(x_0, v_0) \in \partial_+ G \cup (D \times \mathbb{R}^n)$, for all $A \subset \partial D$,

$$P_A := \mathbb{P}\Big(X_{T_{n_0}} \in A, \min_{i \in \{1,\dots,n_0\}} \|X_{T_i} - X_{T_{i-1}}\| \geq \delta_0, O_0 \Big| (X_0, V_{0-}) = (x_0, v_0)\Big) \geq \nu_0 \mathcal{H}(A).$$

This will imply both statements. We set, for all $x \in \partial D$, the marginal law

$$\nu_x(z) = \int_0^\infty \mu_x(\tau, z) d\tau = \mathbf{1}_{\{z \leftrightarrow x\}} c_0 |(z-x)\cdot n_x| |(x-z)\cdot n_z| \int_0^\infty M\Big(\frac{z-x}{\tau}\Big) \frac{1}{\tau^{n+2}} d\tau.$$

Let $x = q(x_0, v_0) \in \partial D$, so that $x = x_0$ if $(x_0, v_0) \in \partial_+ G$. Let Δ, N given by Proposition 2.5.2. For all $y \in \partial D$, by Corollary 2.5.1, there exist $z_1(y), \dots, z_{N+1}(y) \in \Delta$ such that, setting $z'_0 = x$, $z'_{N+2} = y$ and taking, for all $i \in \{1, \dots, N+1\}$, $z'_i \in B_{\partial D}(z_i(y), \eta)$, we have, for all $j \in \{0, \dots, N+1\}$, $z'_j \leftrightarrow z'_{j+1}$ and

$$|(z'_{j+1} - z'_j)\cdot n_{z'_j}| |(z'_j - z'_{j+1})\cdot n_{z'_{j+1}}| \geq \delta, \tag{2.5.3}$$

where $\delta > 0$ and $\eta > 0$ are given by Corollary 2.5.1. This inequality implies $\|z'_{j+1} - z'_j\| \geq \sqrt{\delta}$, and in particular we have $d(D) \geq \sqrt{\delta}$. Let $A \subset \partial D$. We introduce the event

$$O_1 = \Big\{\forall i \in \{0, \dots, N+1\}, \|V_{T_i}\| \neq \|V_{T_i-}\| \text{ and } \forall i \in \{1, \dots, N+2\}, \|X_{T_i} - X_{T_{i-1}}\| \geq \sqrt{\delta}\Big\},$$

and we have, with the choice $n_0 = N+2$, $\delta_0 = \sqrt{\delta}$,

$$P_A = \mathbb{P}\Big(\{X_{T_{N+2}} \in A\} \cap O_1 | X_0 = x_0, V_{0-} = v_0\Big).$$

Since on the event O_1, all reflections are diffuse, and recalling the definition of α_0, see Hypothesis 2.1.1, and that $X_{T_0} = x$,

$$\begin{aligned}P_A \geq \alpha_0^{N+2} \int_{y \in A} \int_{z'_1 \in B_{\partial D}(z_1(y), \eta)} \nu_x(z'_1) \int_{z'_2 \in B_{\partial D}(z_2(y), \eta)} \nu_{z'_1}(z'_2) \\ \times \dots \times \int_{z'_{N+1} \in B_{\partial D}(z_{N+1}(y), \eta)} \nu_{z'_N}(z'_{N+1}) \nu_{z'_{N+1}}(y) dz'_{N+1} \dots dz'_1 dy.\end{aligned}$$

For $\tau \in (\frac{d(D)}{\delta_1}, \frac{d(D)}{\delta_1} + 1)$ with δ_1 given by Hypothesis 2.1.1, for all $y \in A$ and for all z'_{N+1} in $B_{\partial D}(z_{N+1}(y), \eta)$,

$$\mu_{z'_{N+1}}(\tau, y) = c_0 M\Big(\frac{z'_{N+1} - y}{\tau}\Big) \frac{1}{\tau^{n+2}} |(z'_{N+1} - y)\cdot n_y| |(y - z'_{N+1})\cdot n_{z'_{N+1}}| \geq \kappa_1,$$

with, recalling (2.5.3) and that $\sqrt{\delta} \leq \|z'_{N+1} - y\| \leq d(D)$,

$$\kappa_1 = c_0 \Big(\inf_{\|v\| \in (\frac{\delta_1 \sqrt{\delta}}{d(D)+\delta_1}, \delta_1)} M(v) \Big) \Big(\frac{\delta_1}{d(D)} \Big)^{n+2} \delta > 0,$$

so that the infimum above is positive using Hypothesis 2.1.1. We thus have

$$\nu_{z'_{N+1}}(y) \geq \int_{\frac{d(D)}{\delta_1}}^{\frac{d(D)}{\delta_1}+1} \mu_{z'_{N+1}}(\tau, y) d\tau \geq \kappa_1.$$

Working similarly for the other terms, we conclude that

$$\begin{aligned} P_A &\geq \alpha_0^{N+2} \kappa_1^{N+2} \int_{y \in A} \int_{z'_1 \in B_{\partial D}(z_1(y), \eta)} \int_{z'_2 \in B_{\partial D}(z_2(y), \eta)} \\ &\quad \times \cdots \times \int_{z'_{N+1} \in B_{\partial D}(z_{N+1}(y), \eta)} dz'_{N+1} \dots dz'_2 dz'_1 dy \\ &\geq \alpha_0^{N+2} \kappa_1^{N+2} \epsilon^{N+1} \mathcal{H}(A), \end{aligned}$$

where $\epsilon = \inf_{x \in \partial D} \mathcal{H}(B_{\partial D}(x, \eta)) > 0$. This completes the proof.

□

2.5.3 Coupling of $(R, \Theta, \tilde{R}, \tilde{\Theta})$

In this subsection, we exhibit a coupling in a certain appropriate regime, to derive a result similar to Proposition 2.4.1 in the general setting. We let $d_0, \kappa_0 > 0$ and $F, \mathcal{R} \subset \partial D$ be the positive constants and compact regions of the boundary given by Lemma 2.5.1. Recall Notation 2.4.1 for the maps $\xi, \tilde{\xi}, y, \tilde{y}$. We also recall that $\mathcal{A} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \pi)^{n-2}$ and the notation Υ introduced in Lemma 2.3.1.

Proposition 2.5.3. *There exists a constant $c > 0$ such that for all $x_0 \in F$, $\tilde{x}_0 \in D$, $\tilde{v}_0 \in \mathbb{R}^n$ with $\|\tilde{v}_0\| \geq 1$ and $q(\tilde{x}_0, \tilde{v}_0) \in F$, there exists $\Lambda_{x_0, \tilde{x}_0, \tilde{v}_0} \in \mathcal{P}(((0,\infty) \times \mathcal{A})^2)$ such that if $(R, \Theta, \tilde{R}, \tilde{\Theta})$ has law $\Lambda_{x_0, \tilde{x}_0, \tilde{v}_0}$, $(R, \Theta) \sim \Upsilon$, $(\tilde{R}, \tilde{\Theta}) \sim \Upsilon$ and for $E_{x_0, \tilde{x}_0, \tilde{v}_0}$ defined by*

$$E_{x_0, \tilde{x}_0, \tilde{v}_0} := \Big\{ (r, \theta, \tilde{r}, \tilde{\theta}) \in (\mathbb{R}_+ \times \mathcal{A})^2 : y(x_0, \theta) = \tilde{y}(\tilde{x}_0, \tilde{v}_0, \tilde{\theta}), \xi(x_0, r, \theta) = \tilde{\xi}(\tilde{x}_0, \tilde{v}_0, \tilde{r}, \tilde{\theta}) \Big\},$$

we have

$$\mathbb{P}((R, \Theta, \tilde{R}, \tilde{\Theta}) \in E_{x_0, \tilde{x}_0, \tilde{v}_0}) \geq c. \tag{2.5.4}$$

Proof. **Step 1.** We prove that there exists $c > 0$ such that

$$\inf_{(x,\tilde{x},\tilde{t})\in F\times F\times[0,d(D))} \int_{\{z\in\partial D, z\leftrightarrow x, z\leftrightarrow y\}} \int_{\tilde{t}}^{\infty} \Big[\mu_x(\tau,z)\wedge\mu_{\tilde{x}}(\tau-\tilde{t},z)\Big] d\tau dz \geq c. \tag{2.5.5}$$

Note that by compactness of $\mathcal{R}\times F$, using continuity properties and that $\mathcal{R}\leftrightarrow F$, we have

$$c' := \inf_{z\in\mathcal{R}, y\in F} |(z-y)\cdot n_y| \wedge |(z-y)\cdot n_z| > 0. \tag{2.5.6}$$

For $(x,\tilde{x},\tilde{t})\in F\times F\times[0,d(D))$, we set

$$J := \int_{z\in\mathcal{R}} \int_{\tilde{t}}^{\infty} \Big[\mu_x(\tau,z)\wedge\mu_{\tilde{x}}(\tau-\tilde{t},z)\Big] d\tau dz,$$

and it suffices to verify that J is lower bounded away from 0. Recall the definition of $\bar{M}$ and δ_1 from Hypothesis 2.1.1. Using Lemma 2.5.3 and (2.5.6), we easily find

$$J \geq c' c_0 \int_{z\in\mathcal{R}} \int_{\tilde{t}}^{\infty} \Big(\frac{1}{\tau}\Big)^{n+2} \min\Big(M\Big(\frac{z-x}{\tau}\Big), M\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big)\Big) d\tau dz.$$

Note that, for $\tau \geq d(D)(1+\frac{1}{\delta_1})$, for all $z\in\mathcal{R}$, $\frac{\|z-x\|}{\tau} \vee \frac{\|z-\tilde{x}\|}{\tau-\tilde{t}} \leq \delta_1$ using that $\tilde{t}\leq d(D)$, whence

$$M\Big(\frac{z-x}{\tau}\Big) \wedge M\Big(\frac{z-\tilde{x}}{\tau-\tilde{t}}\Big) \geq \inf_{\|v\|\leq\delta_1} \bar{M}(v) =: \kappa_1 > 0.$$

We deduce that

$$J \geq \kappa_1 c' c_0 \int_{z\in\mathcal{R}} \int_{d(D)(1+\frac{1}{\delta_1})}^{+\infty} \Big(\frac{1}{\tau}\Big)^{n+2} d\tau dz \geq \kappa\mathcal{H}(\mathcal{R}) > 0,$$

with κ a positive constant not depending on $x,\tilde{x},\tilde{t}$. This concludes the proof of (2.5.5).

Step 2. We conclude as in the proof of Proposition 2.4.1, see Step 2. $\square$

2.5.4 Construction of the coupling

In comparison with the convex case, we change the definition of the law Γ on $([0,1]\times\mathbb{R}_+\times\mathcal{A})^2$ by setting, for $(x,v,\tilde{x},\tilde{v})\in(\bar{D}\times\mathbb{R}^n)^2$ with $x\in\partial D$ or $\tilde{x}\in\partial D$,

$$\begin{aligned}\Gamma_{x,v,\tilde{x},\tilde{v}}(du,dr,d\theta,d\tilde{u},d\tilde{r},d\tilde{\theta}) &= \mathbf{1}_{\{x=\tilde{x}\}}\Big(\mathcal{Q}(du,dr,d\theta)\delta_u(d\tilde{u})\delta_r(d\tilde{r})\delta_\theta(d\tilde{\theta})\Big) \\ &+ \mathbf{1}_{\{x\in F, q(\tilde{x},\tilde{v})\in F, \tilde{x}\in D, \|\tilde{v}\|\geq 1, \|v\|\geq 1\}}(\mathcal{U}\otimes\Lambda_{x,\tilde{x},\tilde{v}})(du,dr,d\theta,d\tilde{r},d\tilde{\theta})\delta_u(d\tilde{u}), \\ &+ \mathbf{1}_{\{x\neq\tilde{x}\}}\mathbf{1}_{\{x\in F, q(\tilde{x},\tilde{v})\in F, \tilde{x}\in D, \|\tilde{v}\|\geq 1, \|v\|\geq 1\}^c}(\mathcal{Q}\otimes\mathcal{Q})(du,dr,d\theta,d\tilde{u},d\tilde{r},d\tilde{\theta}),\end{aligned} \tag{2.5.7}$$

with $\Lambda_{x,\tilde{x},\tilde{v}}$ given by Proposition 2.5.3. We construct a coupling process $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s\geq 0}$ with the same definition as the one in the convex case, see Definition 2.4.1, except that we consider Γ defined by (2.5.7) rather than by (2.4.14). The statements of Lemmas 2.4.5 and 2.4.6 still hold. Indeed, the difference only relies on the law Γ.

Lemma 2.4.7 and 2.4.8 also hold with this new context, since those results do not rely on the convexity of the domain.

2.5.5 Proof of Theorem 2.1.1 in the general setting

We prove first a result on independent processes similar to Lemma 2.4.8, and conclude the proof of Theorem 2.1.1 in the general framework of C^2 bounded domains. Let $d_0, \kappa_0 > 0$ and $F, \mathcal{R} \subset \partial D$ given by Lemma 2.5.1. In this whole subsection, we denote by κ, L two positive constants depending only on $(D, r, C_0, n_0, \nu_0, \kappa_0, d_0)$ with C_0 given by (2.4.18) and (n_0, ν_0) given by Corollary 2.5.1. The values of κ and L are allowed to vary from line to line.

Lemma 2.5.4. *There exists $\kappa > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in (D \times \mathbb{R}^n) \cup \partial_+ G$ and $(X_t, V_t)_{t\geq 0}$, $(\tilde{X}_t, \tilde{V}_t)_{t\geq 0}$ are two independent free-transport processes with initial conditions $X_0 = x$, $V_{0-} = v$, $\tilde{X}_0 = \tilde{x}$, $\tilde{V}_{0-} = \tilde{v}$, setting*

$$S = \inf\{t > 0, X_t \in F, \tilde{X}_t \in D, q(\tilde{X}_t, \tilde{V}_{t-}) \in F, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1\},$$

we have

$$\mathbb{E}[r(S)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Proof. We introduce the sequence $(T_k)_{k\geq 0}$ defined by $T_0 = \zeta(x, v)$ and, where for all $k \geq 0$, we set $T_{k+1} = \inf\{t > T_k,\ X_t \in \partial D\}$, and the sequence $(\tilde{T}_k)_{k\geq 0}$ defined by $\tilde{T}_0 = \zeta(\tilde{x}, \tilde{v})$, and for $k \geq 0$, $\tilde{T}_{k+1} = \inf\{t > \tilde{T}_k,\ \tilde{X}_t \in \partial D\}$. We also introduce the filtration $\mathcal{F}_t = \sigma((X_s, V_s, \tilde{X}_s, \tilde{V}_s)_{0\leq s\leq t})$. We set $S_1 = \inf\{t \geq T_{n_0}, X_t \in \partial D, \tilde{X}_t \in D, \|V_{t-}\| \neq \|v\|, \|\tilde{V}_{t-}\| \neq \|\tilde{v}\|, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1\}$.

Step 1. We prove that

$$\mathbb{E}[r(S_1)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

By the strong Markov property, using Lemma 2.4.8, which is, as already mentioned, still valid in the non-convex case,

$$\mathbb{E}[r(S_1 - T_{n_0})|\mathcal{F}_{T_{n_0}}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{T_{n_0}}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{T_{n_0}}\|}\Big)\Big).$$

We then have, using Remark 2.4.1,

$$\begin{aligned}
\mathbb{E}[r(S_1)] &\le C\Big(\mathbb{E}[r(T_{n_0})] + \mathbb{E}\Big[\mathbb{E}[r(S_1 - T_{n_0})|\mathcal{F}_{T_{n_0}}]\Big]\Big)\\
&\le \kappa\Big(\sum_{i=0}^{n_0-1} r(T_{i+1}-T_i) + r(T_0) + 1 + r\Big(\frac{d(D)}{\|V_{T_{n_0}}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde V_{T_{n_0}}\|}\Big)\Big)\\
&\le \kappa\Big(1 + \sum_{i=0}^{n_0}\Big[r\Big(\frac{d(D)}{\|V_{T_i}\|}\Big)\Big] + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde V_{T_{n_0}}\|}\Big)\Big)\\
&\le \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde v\|}\Big)\Big),
\end{aligned}$$

since, as usual, for all $i \in \{0,\dots,n_0\}$, we have $\|V_{T_i}\| = \|V_{T_i}\|\mathbf{1}_{\{\|V_{T_i}\|\neq\|v\|\}} + \|v\|\mathbf{1}_{\{\|V_{T_i}\|=\|v\|\}}$ with $\mathcal{L}(\|V_{T_i}\|\,|\,\|V_{T_i}\| \neq \|v\|) = h_R$.

Step 2. In this step, we prove that there exists $c > 0$ such that, for all initial conditions $(x,v) \in \partial_+ G$, $(\tilde x,\tilde v) \in D \times \mathbb{R}^n$ with $\|v\| \wedge \|\tilde v\| \ge 1$,

$$\mathbb{P}(X_{S_1} \in F, q(\tilde X_{S_1}, \tilde V_{S_1}) \in F) \ge c. \tag{2.5.8}$$

Set

$$\begin{aligned}
O_0 &:= \Big\{\|V_{T_0}\| \neq \|V_{T_0-}\|, \dots, \|V_{T_{n_0-1}}\| \neq \|V_{T_{n_0-1}-}\|\Big\},\\
\tilde O_0 &:= \Big\{\|\tilde V_{T_0}\| \neq \|\tilde V_{T_0-}\|, \dots, \|\tilde V_{T_{n_0-1}}\| \neq \|\tilde V_{T_{n_0-1}-}\|\Big\},
\end{aligned}$$

and note that one has $\mathbb{P}(O_0 \cap \tilde O_0) \ge \alpha_0^{2n_0}$. We also set

$$\begin{aligned}
O_1 &:= \Big\{X_{T_{n_0}} \in F, \|V_{T_{n_0}-}\| \ge 1, \|X_{T_i} - X_{T_{i-1}}\| \ge \delta_0 \text{ for all } i \in \{1,\dots,n_0\}\Big\},\\
\tilde O_1 &:= \Big\{\tilde X_{\tilde T_{n_0}} \in F, \|\tilde V_{\tilde T_{n_0}-}\| \ge 1, \|\tilde X_{\tilde T_i} - \tilde X_{\tilde T_{i-1}}\| \ge \delta_0 \text{ for all } i \in \{1,\dots,n_0\}\Big\}.
\end{aligned}$$

We have, using that $\|V_{T_{n_0}-}\|$ is independent of the sequence $(X_{T_k})_{0\le k\le n_0}$ and has law h_R conditionally on O_0,

$$\begin{aligned}
\mathbb{P}(O_1|O_0) &= \mathbb{P}\Big(X_{T_{n_0}} \in F, \min_{i\in\{1,\dots,n_0\}} \|X_{T_i} - X_{T_{i-1}}\| \ge \delta_0 \Big| O_0\Big)\mathbb{P}(\|V_{T_{n_0}-}\| \ge 1|O_0)\\
&\ge \nu_0\kappa_0 \int_1^\infty h_R(r)dr,
\end{aligned}$$

using Proposition 2.5.1 and $\mathcal{H}(F) \ge \kappa_0$. Setting $c_0 = \nu_0\kappa_0 \int_1^\infty h_R(r)dr > 0$, we obtain similarly that $\mathbb{P}(\tilde O_1|\tilde O_0) \ge c_0$.

Moreover, we have

$$O_0 \cap O_1 \cap \tilde O_0 \cap \tilde O_1 \cap \Big\{T_{n_0} \in (\tilde T_{n_0-1}, \tilde T_{n_0})\Big\} \subset \Big\{S_1 = T_{n_0}, X_{S_1} \in F, q(\tilde X_{S_1}, \tilde V_{S_1}) = \tilde X_{\tilde T_{n_0}} \in F\Big\}.$$

To prove (2.5.8), it thus suffices to show that there exists some $\kappa > 0$ such that

$$\mathbb{P}\Big(T_{n_0} \in (\tilde{T}_{n_0-1}, \tilde{T}_{n_0})\Big| O_0 \cap \tilde{O}_0 \cap O_1 \cap \tilde{O}_1\Big) \geq \kappa. \tag{2.5.9}$$

Since all the random variables $R_i = \|V_{T_i}\|$, $i \in \{0, \dots, n_0-1\}$, and $\tilde{R}_i = \|\tilde{V}_{\tilde{T}_i}\|$, $i \in \{0, \dots, n_0-1\}$ are i.i.d. and h_R distributed on $O_0 \cap O_1 \cap \tilde{O}_0 \cap \tilde{O}_1$, and since

$$\tilde{T}_{n_0-1} = \frac{\|\tilde{X}_{\tilde{T}_0} - \tilde{x}\|}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-2} \frac{\|\tilde{X}_{\tilde{T}_{i+1}} - \tilde{X}_{\tilde{T}_i}\|}{\tilde{R}_i}, \qquad T_{n_0} = \sum_{i=0}^{n_0-1} \frac{\|X_{T_{i+1}} - X_{T_i}\|}{R_i},$$
$$\tilde{T}_{n_0} = \frac{\|\tilde{X}_{\tilde{T}_0} - \tilde{x}\|}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-1} \frac{\|\tilde{X}_{\tilde{T}_{i+1}} - \tilde{X}_{\tilde{T}_i}\|}{\tilde{R}_i},$$

we only need to prove that, for some $c_1' > 0$,

$$\inf_{\substack{\tilde{a} \in (0, d(D)) \\ a_0, \tilde{a}_0, \dots, a_{n_0-1}, \tilde{a}_{n_0-1} \in (\delta_0, d(D))}} \mathbb{P}\Big(\frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-2} \frac{\tilde{a}_i}{\tilde{R}_i} \leq \sum_{i=0}^{n_0-1} \frac{a_i}{R_i} \leq \frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-1} \frac{\tilde{a}_i}{\tilde{R}_i}\Big) \geq c_1', \tag{2.5.10}$$

with $(R_i)_{i=0,\dots,n_0-1}$, $(\tilde{R}_i)_{i=0,\dots,n_0-1}$ independent and i.i.d. of law h_R. By Hypothesis 2.1.1, for all $0 \leq \epsilon_0 < \epsilon_1 \leq \delta_1$, $\int_{\epsilon_0}^{\epsilon_1} h_R(r)dr > 0$.

We claim that there exists $0 < \tilde{\theta}_1 < \tilde{\theta}_2 < \delta_1$, $0 < \theta_1 < \theta_2 < \delta_1$, $0 < \tilde{\theta}_3 < \delta_1$, such that

$$d(D)(1 + \frac{n_0 - 1}{\tilde{\theta}_1}) \leq \frac{n_0\delta_0}{\theta_2} \leq \frac{n_0 d(D)}{\theta_1} \leq \frac{\delta_0(n_0 - 1)}{\tilde{\theta}_2} + \frac{\delta_0}{\tilde{\theta}_3}.$$

Indeed, taking $\tilde{\theta}_1 = \frac{\delta_1}{2} \wedge \frac{1}{2}$, $\theta_2 = \tilde{\theta}_1 \frac{\delta_0}{d(D)}$, we have $\tilde{\theta}_1, \theta_2 \in (0, \delta_1)$ (because $\delta_0 < d(D)$) and

$$d(D)(1 + \frac{n_0 - 1}{\tilde{\theta}_1}) \leq d(D)\frac{n_0}{\tilde{\theta}_1} = \frac{n_0\delta_0}{\theta_2}.$$

We set $\theta_1 = \frac{\theta_2}{2} \in (0, \theta_2)$, $\tilde{\theta}_2 = \frac{\tilde{\theta}_1 + \delta_1}{2} \in (\tilde{\theta}_1, \delta_1)$, and, choosing $\tilde{\theta}_3$ sufficiently small, we have $\tilde{\theta}_3 < \delta_1$ and

$$\frac{\delta_0(n_0 - 1)}{\tilde{\theta}_2} + \frac{\delta_0}{\tilde{\theta}_3} \geq \frac{n_0 d(D)}{\theta_1}.$$

We have, for all $\tilde{a} \in (0, d(D))$, for all $a_i, \tilde{a}_i \in (\delta_0, d(D))$ with $i \in \{0, \dots, n_0 - 1\}$, recalling that $\|\tilde{v}\| \geq 1$,

$$\begin{aligned}
\mathbb{P}\Big(\frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-2} \frac{\tilde{a}_i}{\tilde{R}_i} \leq \sum_{i=0}^{n_0-1} \frac{a_i}{R_i} \leq \frac{\tilde{a}}{\|\tilde{v}\|} + \sum_{i=0}^{n_0-1} \frac{\tilde{a}_i}{\tilde{R}_i}\Big) \\
&\geq \mathbb{P}(\forall i : 0 \leq i \leq n_0 - 2, R_i \in (\theta_1, \theta_2), \tilde{R}_i \in (\tilde{\theta}_1, \tilde{\theta}_2), R_{n_0-1} \in (\theta_1, \theta_2), \tilde{R}_{n_0-1} \in (0, \tilde{\theta}_3)) \\
&\geq \Big(\int_{\theta_1}^{\theta_2} h_R(r)dr \Big)^{n_0} \Big(\int_{\tilde{\theta}_1}^{\tilde{\theta}_2} h_R(r)dr \Big)^{n_0-1} \Big(\int_0^{\tilde{\theta}_3} h_R(r)dr \Big) > 0.
\end{aligned}$$

This completes the proof of (2.5.10) and thus the proof of (2.5.8).

Step 3. We set, for any stopping time τ, $T_0^\tau = \inf\{t \geq \tau, X_t \in \partial D\}$ and for all $k \geq 0$, $T_{k+1}^\tau = \inf\{t > T_k^\tau, X_t \in \partial D\}$. Note that $T_k = T_k^0$ for all $k \geq 0$. We introduce the sequence $(S_i)_{i \geq 0}$ defined by $S_0 = 0$, S_1 defined as in Step 1 and for all $k \geq 1$,

$$\begin{aligned}
S_{k+1} = \inf\{t \geq T_{n_0}^{S_k}, &X_t \in \partial D, \tilde{X}_t \in D, \\
&\|V_{t-}\| \neq \|V_{S_k-}\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_{S_k-}\|, \|V_{t-}\| \wedge \|\tilde{V}_{t-}\| \geq 1\}.
\end{aligned}$$

We set, for all $k \geq 1$,

$$B_k = \{X_{S_k} \in F, q(\tilde{X}_{S_k}, \tilde{V}_{S_k-}) \in F\}.$$

We plan to apply Lemma 2.4.3.

i) We set, for all $k \geq 0$, $\mathcal{G}_k = \mathcal{F}_{S_{k+1}-}$, and for all $k \geq 1$, $\tau_k = S_{k+1} - S_1$ which is $\mathcal{G}_k$-measurable and $E_k = B_{k+1} \in \mathcal{G}_k$. We set $G = \inf\{k \geq 1, E_k \text{ is realized}\}$.

ii) We have, for all $k \geq 1$,
$$\mathbb{P}(E_k | \mathcal{G}_{k-1}) = \mathbb{P}(B_{k+1} | \mathcal{F}_{S_k-}) \geq c$$
by Step 2, using the strong Markov property and that $\|V_{S_k-}\| \wedge \|\tilde{V}_{S_k-}\| \geq 1$, $X_{S_k} \in \partial D$, $\tilde{X}_{S_k} \in D$. Hence (2.4.10) holds.

iii) Using the strong Markov property and Step 1, we have, for all $k \geq 0$,
$$\mathbb{E}[r(S_{k+1} - S_k) | \mathcal{F}_{S_k-}] \leq \kappa \Big(1 + r\Big(\frac{d(D)}{\|V_{S_k-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{S_k-}\|}\Big)\Big) =: K_k.$$

For $k \geq 1$,

$$\begin{aligned}
\mathbb{E}[r(\tau_{k+1} - \tau_k) | \mathcal{G}_{k-1}] &= \mathbb{E}\Big[K_{k+1} \Big| \mathcal{F}_{S_k-}\Big] \\
&\leq \kappa \mathbb{E}\Big[1 + r\Big(\frac{d(D)}{\|V_{S_{k+1}-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{S_{k+1}-}\|}\Big) \Big| \mathcal{F}_{S_k-}\Big] \leq \kappa(1 + 2C_0).
\end{aligned}$$

We used that for all $k \geq 1$, $\mathcal{L}(\|V_{S_{k+1}-}\| | \mathcal{F}_{S_k-}) = \mathcal{L}(\|\tilde{V}_{S_{k+1}-}\| | \mathcal{F}_{S_k-}) = h_R$ by definition of $(S_k)_{k\geq 0}$. Note that $\tau_1 = S_2 - S_1$. We have, since $\|V_{S_1-}\| \wedge \|\tilde{V}_{S_1-}\| \geq 1$,

$$\mathbb{E}[r(\tau_1)|\mathcal{G}_0] = \mathbb{E}[r(S_2 - S_1)|\mathcal{F}_{S_1-}] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|V_{S_1-}\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_{S_1-}\|}\Big)\Big) \leq \kappa,$$

whence (2.4.11).

Setting $J = G+1$, we conclude by Lemma 2.4.3 that $\mathbb{E}[r(S_J - S_1)|\mathcal{F}_{S_1-}] = \mathbb{E}[r(\tau_G)|\mathcal{G}_0] \leq \kappa$, whence, by Step 1,

$$\mathbb{E}[r(S_J)] \leq \kappa\Big(1 + r\Big(\frac{d(D)}{\|v\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{v}\|}\Big)\Big).$$

Observe that, by definition of J, almost surely, $X_{S_J} \in F$, $\tilde{X}_{S_J} \in D$, $q(\tilde{X}_{S_J}, \tilde{V}_{S_J-}) \in F$ and $\|V_{S_J-}\| \wedge \|\tilde{V}_{S_J-}\| \geq 1$, whence $S \leq S_J$. □

We introduce some notations corresponding to Notation 2.4.2 in the general case.

Notation 2.5.3. Let $(X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{s\geq 0}$ a coupling process, see Definition 2.4.1 with Γ given by (2.5.7). We use the same sequences $(S_i, Q_i, \tilde{Q}_i)_{i\geq 1}$ as in the definition, as well as $(\tilde{Q}'_i)_{i\geq 1}$, and we recall that, for all $i \geq 1$,

$$V_{S_i} = w(X_{S_i}, V_{S_i-}, Q_i)\mathbf{1}_{\{X_{S_i}\in\partial D\}} + V_{S_i-}\mathbf{1}_{\{X_{S_i}\notin\partial D\}},$$

$$\tilde{V}_{S_i} = w(\tilde{X}_{S_i}, \tilde{V}_{S_i-}, \tilde{Q}'_i)\mathbf{1}_{\{\tilde{X}_{S_i}\in\partial D\}} + \tilde{V}_{S_i-}\mathbf{1}_{\{\tilde{X}_{S_i}\notin\partial D\}}.$$

a) We set $T_0 = 0$, $\tilde{T}_0 = 0$ and for $k \geq 0$,

$$T_{k+1} = \inf\{t > \tilde{T}_k, X_t \in \partial D\}, \qquad \tilde{T}_{k+1} = \inf\{t > T_{k+1}, \tilde{X}_t \in \partial D\}.$$

For all $k \geq 1$, we have $Z_{T_k-} = \emptyset$ and $X_{T_k} \in \partial D$ so $Z_{T_k} \neq \emptyset$ if $\tilde{X}_{T_k} \notin \partial D$. We always have $Z_{\tilde{T}_k} = \emptyset$. For all $k \geq 1$, we write $(\underline{Q}_k, \underline{\tilde{Q}}_k) = (\underline{U}_k, \underline{R}_k, \underline{\Theta}_k, \underline{\tilde{U}}_k, \underline{\tilde{R}}_k, \underline{\tilde{\Theta}}_k)$ for the random vector such that

$$V_{T_k} = w(X_{T_k}, V_{T_k-}, \underline{Q}_k), \quad \text{and} \quad \tilde{V}_{\tilde{T}_k} = w(\tilde{X}_{\tilde{T}_k}, \tilde{V}_{\tilde{T}_k-}, \underline{\tilde{Q}}_k).$$

Note that $(\underline{Q}_k, \underline{\tilde{Q}}_k)_{k\geq 1}$ is a subsequence of $(Q_i, \tilde{Q}'_i)_{i\geq 1}$.

b) For all $t \geq 0$, we set

$$\mathcal{F}_t = \sigma\Big((X_s, V_s, \tilde{X}_s, \tilde{V}_s, Z_s)_{0\leq s\leq t}, (Q_i\mathbf{1}_{\{S_i\leq t\}})_{i\geq 1}, (\tilde{Q}_i\mathbf{1}_{\{S_i\leq t\}})_{i\geq 1}\Big).$$

c) We set $\sigma_1 = \inf\{t > 0, X_t = \tilde{X}_t \in \partial D, Z_{t-} = \emptyset, \|V_{t-}\| \neq \|V_0\|, \|\tilde{V}_{t-}\| \neq \|\tilde{V}_0\|\}$.

d) We set $\nu_0 = 0$ and for all $k \geq 0$,

$$\nu_{k+1} = \inf\{n \geq \nu_k + 1, X_{T_n} \in F, \tilde{X}_{T_n} \in D, q(\tilde{X}_{T_n}, \tilde{V}_{T_n-}) \in F, \|V_{T_n-}\| \geq 1, \|\tilde{V}_{T_n-}\| \geq 1\}.$$

The only difference with Notation 2.4.2 is that Definition 2.4.1 uses (2.5.7) rather than (2.4.14), and that the sequence $(\nu_k)_{k\geq 1}$ has been slightly changed. We next update Lemma 2.4.9.

Lemma 2.5.5. *There exist three constants $\kappa, L, c > 0$ such that the following holds.*

i) For all $m \geq 1$,

$$\mathbf{1}_{\{T_{\nu_m} < \sigma_1\}} \mathbb{E}[r(T_{\nu_{m+1}} \wedge \sigma_1 - T_{\nu_m}) | \mathcal{F}_{T_{\nu_m}-}] \leq L.$$

ii) $\mathbb{E}[r(T_{\nu_1} \wedge \sigma_1)] \leq \kappa\Big(1 + \mathbb{E}\Big[r\Big(\frac{d(D)}{\|V_0\|}\Big) + r\Big(\frac{d(D)}{\|\tilde{V}_0\|}\Big)\Big]\Big).$

iii) For all $m \geq 1$, setting

$$A_m = \{\underline{U}_{\nu_m} \leq \alpha_0, X_{T_{\nu_m+1}} = \tilde{X}_{T_{\nu_m+1}}, T_{\nu_m+1} = T_{\nu_m} + \zeta(X_{T_{\nu_m}}, V_{T_{\nu_m}})\},$$

we have

$$\mathbb{P}\Big(A_m \Big| \mathcal{F}_{T_{\nu_m}-}\Big) \geq c,$$

and $A_m \subset \{\sigma_1 \leq T_{\nu_m+1}\}$ outside a $\mathbb{P}$-null set.

Proof. The proof is the same as the one of Lemma 2.4.9, using Lemma 2.5.4, Proposition 2.5.3, Notation 2.5.3, Equation (2.5.7) instead of Lemma 2.4.8, Propositon 2.4.1, Notation 2.4.2, Equation (2.4.14), and that Lemma 2.4.7 still holds when using Definition 2.4.1 with (2.5.7) instead of (2.4.14). □

Proof of Theorem 2 in the general setting. The proof is the same as the one in the convex case, using Lemma 2.5.5 instead of Lemma 2.4.9, Notation 2.5.3 instead of Notation 2.4.2 and that Lemmas 2.4.5 and 2.4.6 hold when using Definition 2.4.1 with (2.5.7) instead of (2.4.14). □

Chapter 3

Simulations of the asymptotic behavior of the free-transport process

This chapter, unpublished, gathers several numerical results on the free-transport equation with boundary conditions.

Abstract: We use the stochastic process from Definition 2.3.1 to study numerically the convergence towards equilibrium of the free-transport equation. We give numerical evidences pointing towards a polynomial decay of the total variation distance. We also present several simulations which confirm qualitatively some theoretical results from Chapter 2, in particular the modification (and its direction) of the rate of convergence when the reflection law is changed, the influence of the initial data and the fact that the rate is not widely affected by the absence of radial symmetry of the space domain.

Keywords: free-transport process, collisionless gas, subexponential convergence towards equilibrium, estimation of the total variation distance, simulation from a particle system.

3.1 Parameters

In this chapter, we present numerical results regarding the free-transport problem treated in Chapter 2. We run simulations depending on several parameters.

i) **The space domain** is either the unit disk in $\mathbb{R}^2$, or a star-shaped domain in $\mathbb{R}^2$. In the latter, there is no radial symmetry.

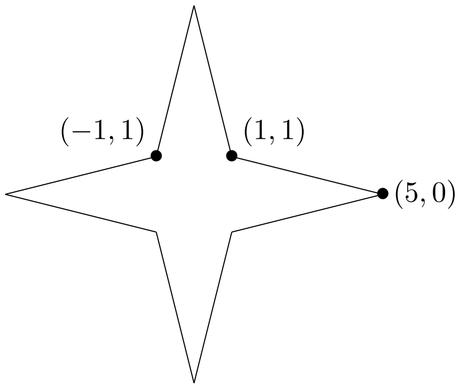

Fig. 3.1 Star-shaped domain in $\mathbb{R}^2$.

ii) **The accomodation coefficient** α is a constant whose value is between 0, corresponding to the problem with specular reflection, and 1, in which case the reflection is diffuse. For $\alpha \in (0,1)$, a fraction α of the particles reflect diffusely, the other ones reflect specularly. This corresponds to the Maxwell boundary conditions.

iii) **The coefficient** $a \in (0,3)$ is a positive constant modifying the law M of the diffuse reflection, in order to obtain more or less concentration around 0.

Recall that the standard choice of M in $\mathbb{R}^2$ is given by

$$M(v) = \frac{e^{-\frac{\|v\|^2}{2}}}{2\pi}, \qquad v \in \mathbb{R}^2.$$

The generalized function is given by

$$M_a(v) = \frac{e^{-\frac{\|v\|^{\frac{2}{a}}}{2}}}{a\pi} \frac{\|v\|^{\frac{3}{a}-3}}{2^{\frac{3}{2}-\frac{a}{2}}\Gamma(\frac{3}{2}-\frac{a}{2})}, \qquad v \in \mathbb{R}^2.$$

We have $M = M_1$ in $\mathbb{R}^2$.

Recall the notations h_R, h_Θ and ϑ defined in Section 2.3. At some point $x \in \partial D$, to simulate $V \sim M(v)|v \cdot n_x|\mathbf{1}_{\{v \cdot n_x > 0\}}$, we simulate $R \sim h_R$, $\Theta \sim h_\Theta$ and let $V = R\vartheta(x,\Theta)$. To generalize this, we take $V = R^a\vartheta(x,\Theta)$, and since R^a has a density proportional to

$M_a(r)r^2$, where, abusing notation, we write $M_a(r)$ for $M(v)$ when $\|v\| = r$, the form of M_a follows. For $a \in (1,3)$, more particles are reflected with a velocity norm close to 0. For $a \in (0,1)$, the number of particles with velocity around 0 is smaller than when $a = 1$.

iv) **The initial data** is either a Dirac mass, or corresponds to equilibrium distributions with different temperature. We introduce a further generalization of M_a by setting, for $\theta > 0$,

$$M_{a,\theta}(v) = \frac{e^{-\frac{\|v\|^{\frac{2}{a}}}{2\theta}}}{a\pi(2\theta)^{\frac{3}{2}-\frac{a}{2}}} \frac{\|v\|^{\frac{3}{a}-3}}{\Gamma(\frac{3}{2}-\frac{a}{2})}, \qquad v \in \mathbb{R}^n,$$

so that, for all $a \in (0,3)$, the equilibrium distribution corresponds to $M_{a,1}$.

Remark 3.1.1 (On the convergence rate corresponding to M_a). We consider the previous model with the choice M_a, $a \in (0,3)$ and with initial condition either equal to a Dirac mass with velocity different than 0 or to $M_{a,\theta}$ with $\theta \neq 1$. When one focuses on the convergence towards equilibrium, the convergence rate depends on a. The expected rate of convergence is $\frac{1}{t^\gamma}$ with $\gamma = \frac{3}{a} - 1$. Indeed, for any $\epsilon > 0$, the key quantity, see Theorem 2.1.1 satisfies

$$\int_{\mathbb{R}^2} M_a(v) \frac{1}{\|v\|^{\gamma-\epsilon}} dv = \int_0^{2\pi} \int_0^\infty \frac{e^{-\frac{r^{\frac{2}{a}}}{2}}}{a\pi} \frac{r^{\frac{3}{a}-2-\gamma+\epsilon}}{2^{\frac{3}{2}-\frac{a}{2}}\Gamma(\frac{3}{2}-\frac{a}{2})} dr d\theta,$$

and this last quantity is finite if

$$\frac{3}{a} - 2 - \gamma + \epsilon > -1.$$

For a difference between two solutions starting with Dirac masses with velocities different than 0, we expect the convergence towards equilibrium to occur at the rate $\frac{1}{t^{\gamma+1}}$. Indeed, we will have

$$\int_{\mathbb{R}^2} M_a(v)\|v\| \frac{1}{\|v\|^{\gamma+1-\epsilon}} dv < \infty,$$

with this choice.

3.2 Code

We present below the code (in R language) of our simulations. In a first part, we show how we obtain a sequence $(T_i, X_i, V_i)_{i\geq 1}$ of collision times, collision points and outcoming velocities, with Dirac initial conditions.

```
simulDisk <- function(alpha = 1, a = 1,n,nbrCollisions = 100,
                        dirac = 1, sd = 1){
  lst <- list()
  XV = matrix(data = rbind(rep(0,n),rep(0,n), rep(0,n), rep(.5,n),
                        rep(.5,n)), nrow = n, byrow = TRUE)
```

```
  lst[[1]] <- XV
  for (i in seq(2,nbrCollisions, by = 1)) {
    print (paste(i/nbrCollisions*100, "%"))
    lst[[i]] = collision(lst[[i-1]], n, alpha, a)   }
  print(min(lst[[nbrCollisions]][,1]))
  return(lst)   }

collision <- function(XV,n, alpha, a) {
  S = unlist(mclapply(seq(1,n), function(i) computeFirstSigma(XV[i,]),
                         mc.cores = cores))
  XV[,2:3] = XV[,2:3] + S*XV[,4:5]
  XV[,4:5] = newSpeed(XV,n, alpha, a)
  XV[,1] = S + XV[,1]
  return(XV)    }

computeFirstSigma <-function(XV){
  dotProd = dotproduct(XV[2:5])
  normX = norm_vec(XV[2:3])
  normV = norm_vec(XV[4:5])
  numRet1 = - 2*dotProd + 2*sqrt(dotProd^2 + normV^2*(1 - normX^2))
  return(numRet1/(2*(normV^2))) }
```

The functions used to generate the new velocities are the following.

```
newSpeed <- function(XV,n, alpha,a){
  return( matrix(unlist(mclapply(seq(1,n), function(i)
    simulV(XV[i,], a = a, alpha = alpha), mc.cores = cores)),
        nrow = n, ncol = 2, byrow = TRUE) )   }

simulV <- function(x,a,alpha){
  n_x = (c(-x[2],-x[3]))
  v = c(x[4],x[5])
  b = rbinom(1,1,1-alpha)
  if (b) { return(v - 2*(dotproduct(c(v,n_x)))*n_x) }
  u = 1
  v = c(0,0)
  r = norm_vec(rnorm(3))
  theta = pi/2
  while(5*u > cos(theta)) {
    theta = runif(1, -pi/2, pi/2)
    u = runif(1)        }
  t_x = matrix(c(0,-1,1,0), nrow = 2, byrow = TRUE)%*%x[2:3]
  P = t(matrix(c(n_x,t_x), nrow = 2))
  return( (r^a)*(P%*%(c(cos(theta), sin(theta))) ) }
```

To obtain the films, available at the following url: https://www.lpsm.paris/pageperso/bernou/post/simulations2/, we use the following script to obtain the images which are then combined.

```
XV = simulDisk(alpha = alpha,a = a,n=n,
                        nbrCollisions = nbrCollisions, dirac)
matrixData2 = array(as.numeric(unlist(XV)), dim=c(n,5,nbrCollisions))
compteur = 0
for(i in seq(0,50,by  = 0.01)){
    XV = getValueAtTime(i, matrixData2, n)
    compteur = compteur + 1
    filename = paste("Image/", compteur, ".png", sep = "")
    png(file = filename, width = 600, height = 525)
    draw_own_circle(1, time = i)
    points(XV[,2], XV[,3], col = "red")
    dev.off() }
```

We used the following functions:

```
getValueAtTime <-function(t = 10, dataVal,n){
XV = matrix(data = rbind(rep(0,n),rep(0,n), rep(0,n), rep(0,n),
            rep(0,n)), nrow = n, byrow = TRUE)
  for (i in seq(1,n,by=1)){  XV[i,] = getTimedData(dataVal[i,,], t) }
  return(XV[rowSums(is.na(XV)) == 0,])  }

getTimedData <-function(matrixData,t){
  index = 0
  for (j in seq(1,99,by = 1)){
    if (matrixData[1,j+1] > t){
      index = j
      break }   }
  if (index == 0) { return(NA) }
  output= c(t, matrixData[2,index]
    + (t - matrixData[1,index])*matrixData[4,index],
    matrixData[3,index]
    + (t - matrixData[1,index])*matrixData[5,index],
    matrixData[4,index], matrixData[5,index])
  return(output)  }

draw_own_circle = function(radius, center = 0, time = 0){
  x = seq(-radius,radius, .0001)
  y = sqrt(radius^2-x^2)
  z = -sqrt(radius^2-x^2)
  X_1 = matrix(cbind(x,y), ncol = 2)
  plot(X_1, type = "l", col= "red",
```

```
        xlim = c(center-radius,center+radius),
    ylim = c(center-radius,center+radius),
        main = paste("time =", time))
  X_2 = matrix(cbind(x,z), ncol = 2)
  lines(X_2, col="red")  }
```

In the case where $\alpha = 0$, we take the initial position uniform in a small disk included in the spatial domain, and the initial velocity uniform in the unit sphere, in order to emphasize the long-term asymptotics (if one takes the same Dirac mass for all particles, then one only sees one trajectory during the whole simulation).

For the simulations in the star-shaped domain we use the same code, except that we change the functions computeFirstSigma and newSpeed. Below is the code of the functions that will play the same role in this new domain, plus a function giving the unit inward normal vector of a point at the boundary and a function indicating whether a point is at the boundary of the domain. The parameter b is the length of one branch of the star.

```

isAtBoundary <-function(x,b){
  if (x[1] > 1 && (((1/(b-1))*(x[1]-1) + abs(x[2])) > 1 - 10^{-7})
    &&  (((1/(b-1))*(x[1]-1) + abs(x[2])) < 1 + 10^{-7}))
    return(TRUE)
  else if ((x[2]>1) && ((1/(b-1))*(x[2]-1) + abs(x[1])  > 1 - 10^{-7})
    && ((1/(b-1))*(x[2]-1) + abs(x[1])  < 1 + 10^{-7}) )
    return(TRUE)
  else if ((x[1] < -1)
    && (((1/(b-1))*abs(x[1]+1) + abs(x[2])) > 1 - 10^{-7})
    && (((1/(b-1))*abs(x[1]+1) + abs(x[2])) < 1  + 10^{-7}))
    return(TRUE)
  else if ((x[2]<-1) && ((1/(b-1))*abs(x[2]+1) + abs(x[1])>1-10^{-7})
    && ((1/(b-1))*abs(x[2]+1) + abs(x[1])  < 1 + 10^{-7}))
    return(TRUE)
  else return(FALSE)                    }

computeSigmaStar <- function(XV,b){
  s = vector(length = 8)
  s[1] = (1/(b-1)*(1-XV[2]) + 1 - XV[3]) / (XV[5] + (1/(b-1))* XV[4])
  s[2] = (1/(b-1)*(XV[2]-1) - 1 - XV[3]) / (XV[5] - (1/(b-1))* XV[4])
  s[3] = ( (b-1)*XV[2] - XV[3] - b ) / (XV[5] + (1-b)*XV[4])
  s[4] = ( (1-b)*XV[2] - XV[3] - b ) / (XV[5] + (b-1)*XV[4])
  s[5] = (-(1/(b-1))*(XV[2]+1) - 1 - XV[3]) / (XV[5] + (1/(b-1))*XV[4])
  s[6] = ( (1/(b-1)) *(XV[2] + 1) + 1 - XV[3] ) / (XV[5] - XV[4]/(b-1))
  s[7] = ( (b-1)*XV[2] - XV[3] + b ) / (XV[5] + (1-b)*XV[4])
  s[8] = ( (1-b)*XV[2] + b - XV[3] ) / (XV[5] + (b-1)*XV[4] )
  s[s <= 10^{-8}] = Inf
```

```
  sigma = Inf
  for (i in seq(1:8)){
    if (!is.na(s[i]) && s[i] != Inf && isAtBoundary(XV[2:3]
                   + s[i]*XV[4:5],b))
       sigma = s[i]
  }
  return(sigma)          }

 getNormalVector <-function(x, a = 3){
 ### a = b with the previous notations
  if (x[1] > 1){
    if (x[2] > 0) {return(c(-1/(a-1),-1)/norm_vec(c(-1/(a-1),-1)))}
    else {return(c(-1/(a-1),1)/norm_vec(c(-1/(a-1),1)))}
  }
  else if (x[1] < -1){
    if(x[2] > 0) {return(c(1/(a-1),-1)/norm_vec(c(1/(a-1),-1)))}
    else {return(c(1/(a-1),1)/norm_vec(c(1/(a-1),-1)))}
  }
  else if (0 < x[1] && x[1] < 1){
    if (x[2] > 0) {return(c(1-a,-1)/norm_vec(c(1-a,-1)))}
    else {return(c(1-a,1)/norm_vec(c(1-a,-1)))}
  }
  else if (-1 < x[1] && x[1] < 0){
    if(x[2] > 0) {return(c(-(1-a),-1)/norm_vec(c(1-a,-1)))}
    else {return(c(-(1-a),1)/norm_vec(c(1-a,-1)))}
  }
  else return(c(0,0))            }

newSpeedStar <- function(XV,n, alpha,a,b,cores){
  V = matrix(unlist(mclapply(seq(1,n), function(i)
    importantSamplingVStar(XV[i,], a = a, alpha = alpha, b=b)
    , mc.cores = cores)), byrow = TRUE, ncol = 2)
  return(V)
}

importantSamplingVStar <- function(x,a,alpha,b){
  n_x = getNormalVector(x[2:3],b)
  v = c(x[4],x[5])
  spec = rbinom(1,1,1-alpha)
  if (spec) return(v - 2*(dotproduct(c(v,n_x)))*n_x)
  u = 1
  v = c(0,0)

  while (sqrt(2/pi)*10*u > abs(dotproduct(c(n_x,v)))){
```

```
    v = rnorm(2)
    if (dotproduct(c(n_x,v)) < 0){v = -v}
    u = runif(1)
  }
  return(v*norm_vec(v)^(a-1))
}
```

3.3 Qualitative results in the unit disk

We present several numerical results regarding the convergence towards equilibrium in the unit disk. First, we simulate 10^6 particles with $\alpha = a = 1$ and look at the distribution at time $t = 20$ of the first space coordinate and the first velocity coordinate. We obtain the histograms of Figures 3.2 and 3.3. The theoretical density of the corresponding coordinate at equilibrium is plotted in red for comparison. The initial distribution is

$$f_0(x, v) = \delta_{(0,0)}(x)\delta_{(0.5,0.5)}(v), \qquad (x, v) \in D \times \mathbb{R}^2. \tag{3.3.1}$$

where $\delta_y(.)$ denotes the Dirac mass at the point $y \in \mathbb{R}^2$.

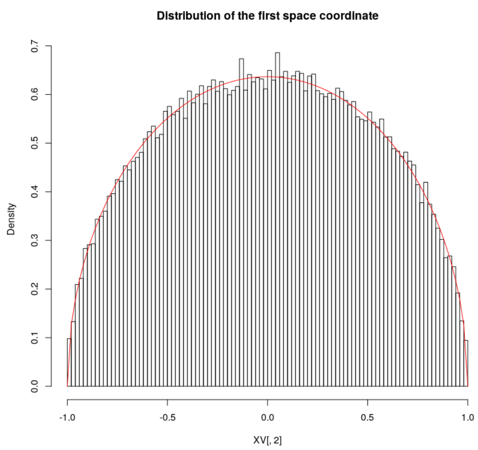

Fig. 3.2 Distribution of the first space coordinate at time t=20.

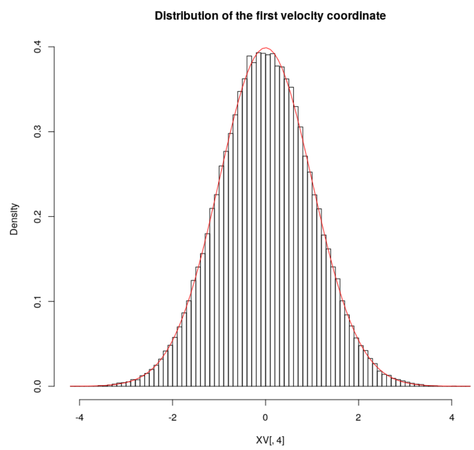

Fig. 3.3 Distribution of the first velocity coordinate at time t=20.

At this point, it seems quite clear that the equilibrium is already reached at time $t = 20$. If we run the same simulation with $\alpha = .5$, we obtain new histograms for the first coordinates in space and velocity, at time $t = 20$ and $t = 50$, see Figures 3.4-3.7 below. As one can see on those figures, at time $t = 20$, the distribution is not yet at equilibrium: there are still particles that have only been reflected with the specular component of the boundary operator, which explains why one cell of the histogram is anormally high for both variables considered. At time $t = 60$, the system seems to fit the equilibrium distribution, and we recover similar histograms

Distribution of the first variables in space and velocity with $\alpha = .5$

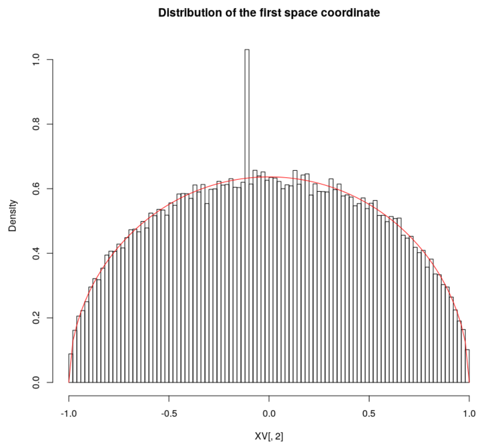

Fig. 3.4 Distribution of the first space coordinate at time t=20.

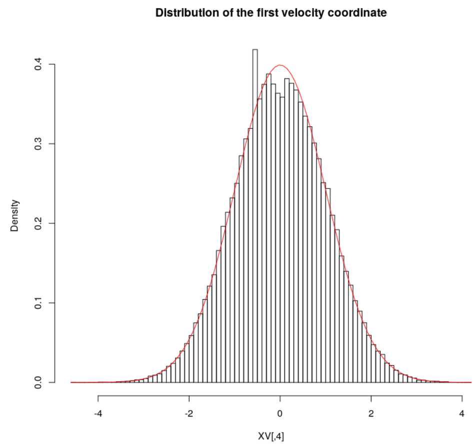

Fig. 3.5 Distribution of the first velocity coordinate at time t=20.

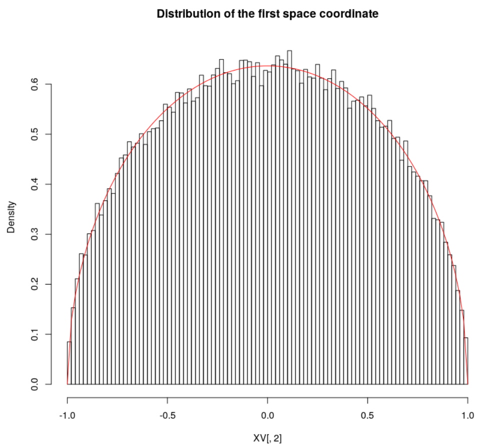

Fig. 3.6 Distribution of the first space coordinate at time t=60.

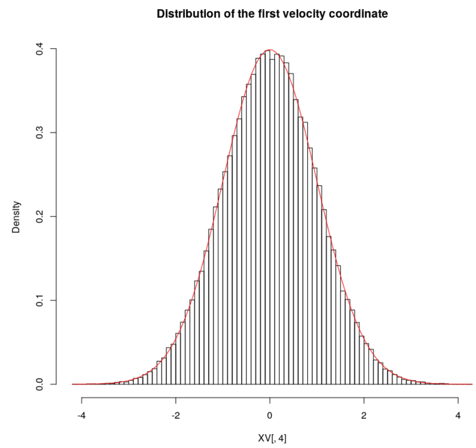

Fig. 3.7 Distribution of the first velocity coordinate at time t=60.

to those of the case $\alpha = 1$ at time $t = 20$. We also present on Figures 3.8-3.10 the histograms at time $t = 10$ and $t = 50$ (space variable) and at time $t = 10$ up to $t = 180$ (velocity variable), with $a = \alpha = 1$, in the case of an initial data given by

$$f_0(x, v) = \mathcal{U}_{B(0,1)}(dx)\mathcal{N}_2(0_2, 0.01I_2)(dv), \tag{3.3.2}$$

where $\mathcal{U}_{B(0,1)}$ is the uniform distribution in the unit disk of dimension 2, of center 0 and radius 1, $\mathcal{N}_2(\mu, \Sigma)$ is the normal distribution in $\mathbb{R}^2$ with mean $\mu \in \mathbb{R}^2$ and covariance matrix Σ. As one can observe, the difference between the distribution of the velocity and the equilibrium distribution of this component is the concentration around the velocity 0. This shall not be a surprise, as particles starting with a very small velocity require a lot of time to hit the boundary.

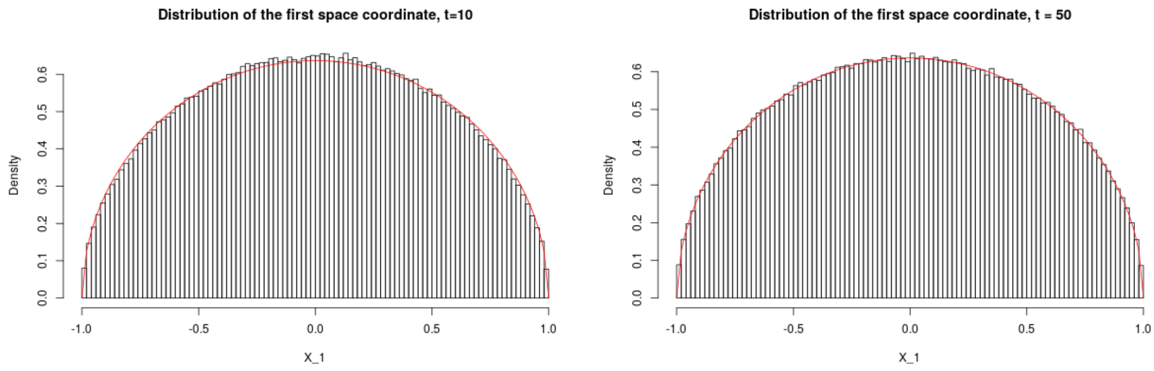

Fig. 3.8 Distribution of the first space coordinate at time $t = 10$ and $t = 50$ in the case where the initial data is given by (3.3.2).

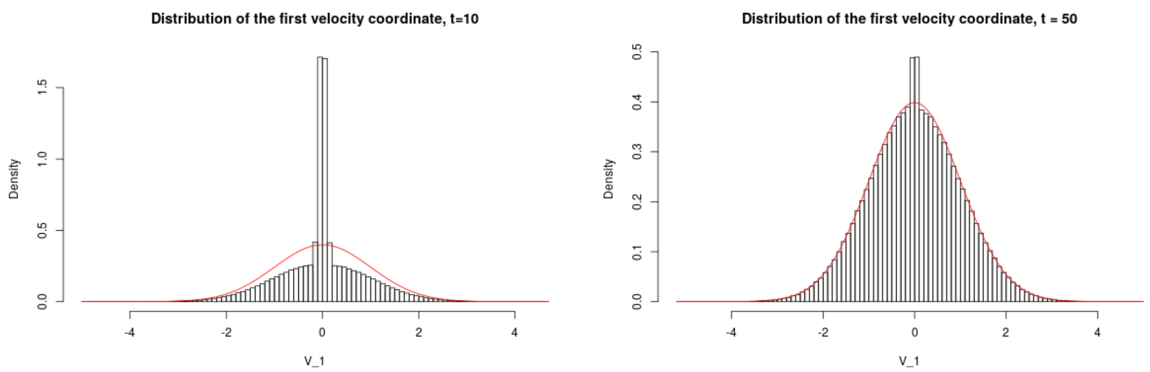

Fig. 3.9 Distribution of the first space velocity coordinate at time $t = 10$ and $t = 50$ in the case where the initial data is given by (3.3.2).

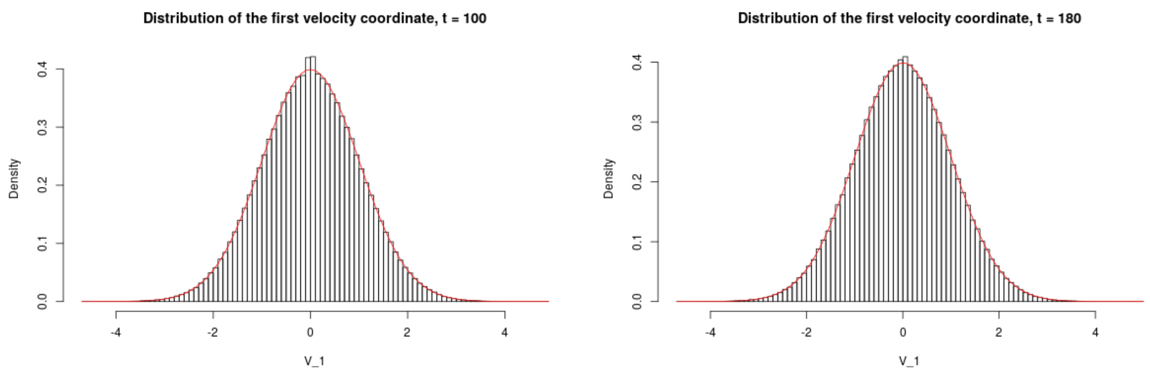

Fig. 3.10 Distribution of the first space velocity coordinate at time $t = 100$ and $t = 180$ in the case where the initial data is given by (3.3.2).

3.4 Quantitative convergence towards equilibrium in the unit disk

Limit of total variation estimates. We now turn to the quantitative estimation of the convergence towards equilibrium. We want to estimate the total variation distance between two measures: f_t, the solution at time t, and the equilibrium distribution μ_∞. To estimate the total variation distance from the samples, we build cells of radius $\epsilon > 0$ in $\mathbb{R}^4$ and count the number of points of each sample located in each cell. This also requires to compute a matrix distributed according to μ_∞. Our estimator of the total variation distance is given by the following: if $(X_i, V_i)_{1\leq i\leq n}$ has distribution $\mu \in \mathcal{M}(\mathbb{R}^4)$ and $(\tilde{X}_i, \tilde{V}_i)_{1\leq i\leq n}$ has distribution $\nu \in \mathcal{M}(\mathbb{R}^4)$, the estimate is given by (using $\sharp A$ to denote the cardinal of a finite set A):

$$\widehat{TV_4}(\mu,\nu) = \frac{1}{n}\sum_{j=1}^n \Big| \frac{\sharp\Big\{i : |X_i - \tilde{X}_j| \leq \epsilon, |V_i - \tilde{V}_j| \leq \epsilon\Big\}}{N\epsilon^4\pi^2\nu(\tilde{X}_j, \tilde{V}_j)} - 1\Big|.$$

Of course this estimate can be adapted easily to consider distributions over $\mathbb{R}^2$ rather than $\mathbb{R}^4$: if $(X_i)_{1\leq i\leq n}$ is i.i.d. of law $\mu \in \mathcal{M}(\mathbb{R}^2)$ and $(\tilde{X}_i)_{1\leq i\leq n}$ is i.i.d. of law $\nu \in \mathcal{M}(\mathbb{R}^2)$, the estimate is given by:

$$\widehat{TV_2}(\mu,\nu) = \frac{1}{n}\sum_{j=1}^n \Big| \frac{\sharp\Big\{i : |X_i - \tilde{X}_j| \leq \epsilon\Big\}}{N\epsilon^2\pi\nu(\tilde{X}_j)} - 1\Big|.$$

In Figure 3.11, we present the result of an estimation of the total variation distance at several times, with $n = 10^5$, $a = \alpha = 1$ and f_0 given by (3.3.1). As one can see, the convergence is fast at first, but the estimate remains quite large even at very large time. The fact that we use two samples to compare the two distributions makes the noise too important, as can be checked by taking two samples of μ_∞: we define the error by

$$\text{error} = \widehat{TV_4}(\mu_\infty, \mu_\infty),$$

and the previous estimate is really close to this error. The fact that this error is so large also rules out the idea of trying a different value of f_0, such as an equilibrium with a temperature different than 1.

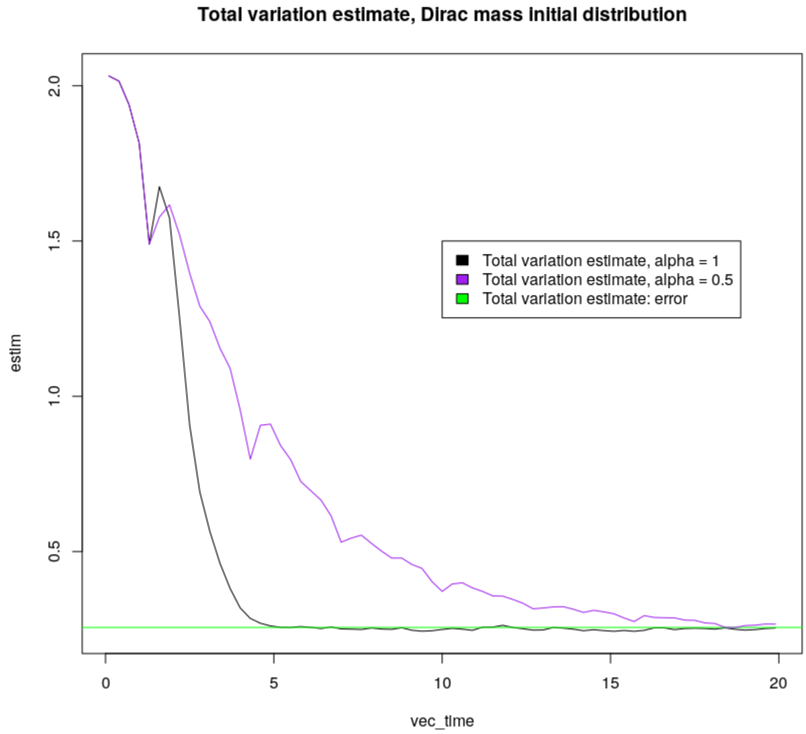

Fig. 3.11 Estimate of the total variation distance between f_t and μ_∞, with f_0 given by (3.3.1).

Although there is a clear lack of precision with this method of estimation, one can already observe some interesting features. For instance, with $\alpha = 0.5$, levels seem to appear in the total variation distance estimate: ending times of those levels correspond to the ones where the particles that have only been reflected specularly hit the boundary. On average, at those times, half of them are reflected in a diffuse manner, which accelerates the convergence towards equilibrium. To confirm this idea, we can look at the total variation estimate in the velocity distribution only, and in space distribution only, Figures 3.12 and 3.13. A first observation is that looking at those estimators does not change the previous problem: the error is still high. In the velocity variable the levels are really highlighted by the estimator, at least until the error level is reached. In the space variables the levels appear but the estimator has small fluctuations depending on the position of the fraction of particles which have only been specularly reflected: the estimator tends to be bigger when this fraction is far from the center of the disk. To confirm this idea, one can look at the behavior of the total variation distance estimator in the space coordinate when $\alpha = 0$. Here, we expect no convergence to equilibrium, but plan to observe fluctuations of the estimator because of the previously mentioned effect. As one can observe on Figure 3.14, this behavior indeed occurs.

We can also try to use the Wasserstein distance rather than the total variation distance, using the R package "transport" recently developed. The corresponding estimators, however, exhibit the same problems as the ones of the total variation distance. We do not present them here.

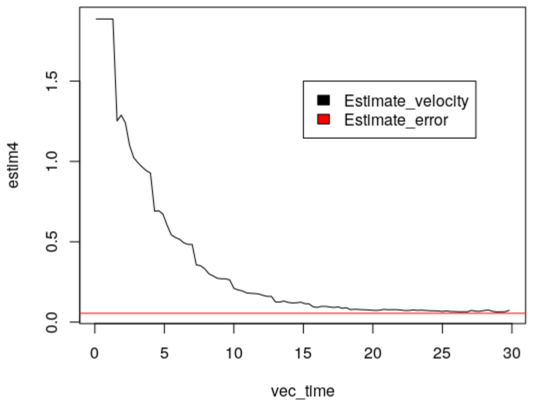

Fig. 3.12 Estimate of the total variation distance in the velocity variable, with $\alpha = 0.5$ and f_0 given by (3.3.1).

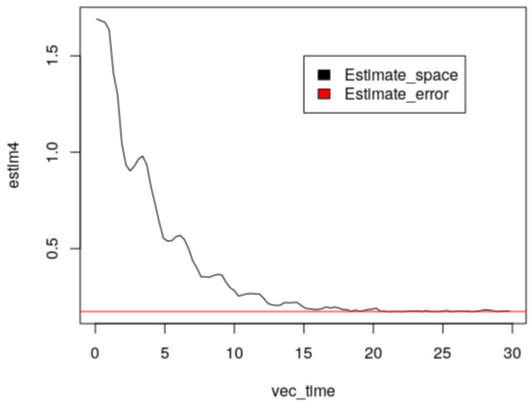

Fig. 3.13 Estimate of the total variation distance in the space variable, with $\alpha = 0.5$ and f_0 given by (3.3.1).

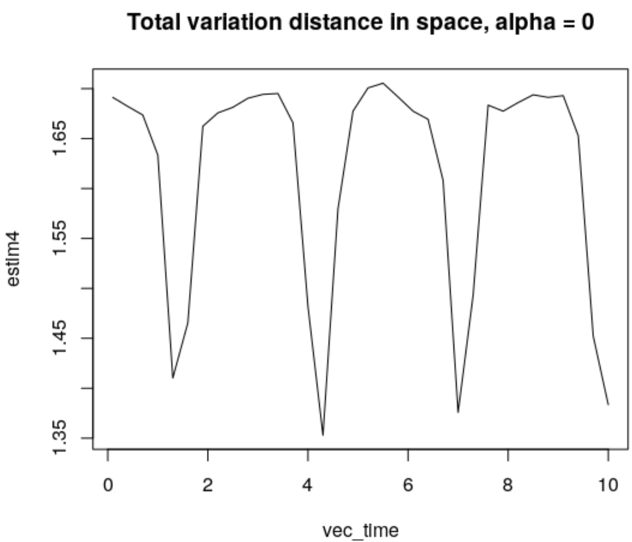

Fig. 3.14 Variation of the space estimator when $\alpha = 0$ and f_0 is given by (3.3.1). This figure confirms that the space estimator varies with the position of the fraction of particles for which only specular reflection occured. As expected, no convergence occurs for this value of α.

Estimation with a test function

Recall first that if μ, ν are two probability measures on a space E, we can write the total variation distance as:

$$TV(\mu,\nu) = \frac{1}{2} \sup_{\phi: E \to [-1,1]} \Big| \int \phi d\mu - \int \phi d\nu \Big|.$$

Following this idea, we choose a function ϕ and try to estimate, for $t \geq 0$, $|\langle f_t, \phi \rangle - \langle \mu_\infty, \phi \rangle|$, where $\langle \cdot, \cdot \rangle$ is the scalar product in the duality. This will give an estimate of the total variation distance. We pick $\phi(x,v) = \|x\|^4 + \|v\|^2$, simulate $(X_i, V_i)_{1 \leq i \leq n}$ i.i.d. of law f_t, $(\tilde{X}_i, \tilde{V}_i)_{1 \leq i \leq n}$ of law μ_∞ and use the following estimator

$$\hat{\epsilon}_n(f_t, \mu_\infty) = \frac{1}{2} \Big| \frac{1}{n} \sum_{i=1}^{n} \left(\phi(X_i, V_i) - \phi(\tilde{X}_i, \tilde{V}_i) \right) \Big|.$$

In what follows, we call this quantity the ϕ-estimate.

We also consider a different test function,

$$\phi_2(x,v) = \sqrt{\|x\|} + \sqrt{\|v\|}, \qquad x \in D, v \in \mathbb{R}^2, \tag{3.4.1}$$

We call the corresponding estimate the ϕ_2-estimate. We first compute the estimate with the following parameters: $n = 10^6$, $a = \alpha = 1$, in the unit disk. Since the estimate converges quickly, we take an initial data in the form (3.3.2), quite far from the equilibrium distribution in velocity.

In Figure 3.15, we present the result of convergence for those estimators with this choice of f_0. We draw a log-log curve to emphasize the polynomial rate of convergence, and compute the estimates of the slope of decrease for both estimators with a linear regression. Note that the values obtained for those slope, while not irrelevant, are far from the expected value of -2. After time $t = \exp(5)$ we see noise appear in both estimators.

We can also consider the behavior of this estimator for the same set of parameters as above, with different values of α. The results are displayed on Figure 3.16. One of the theoretical results is that the value of α should not change the exponent of the rate of convergence. We see on Figure 3.16 that the variation induced by the modification on α plays a role. This is not surprising: with $\alpha = 1$ the particle is thermalized immediatly, while it has to collide several times with the boundary, in many cases, when $\alpha = .5$. The absence of effet of the variation of α is probably more visible on the long run, a feature that our estimates do not capture very well since the error becomes large early.

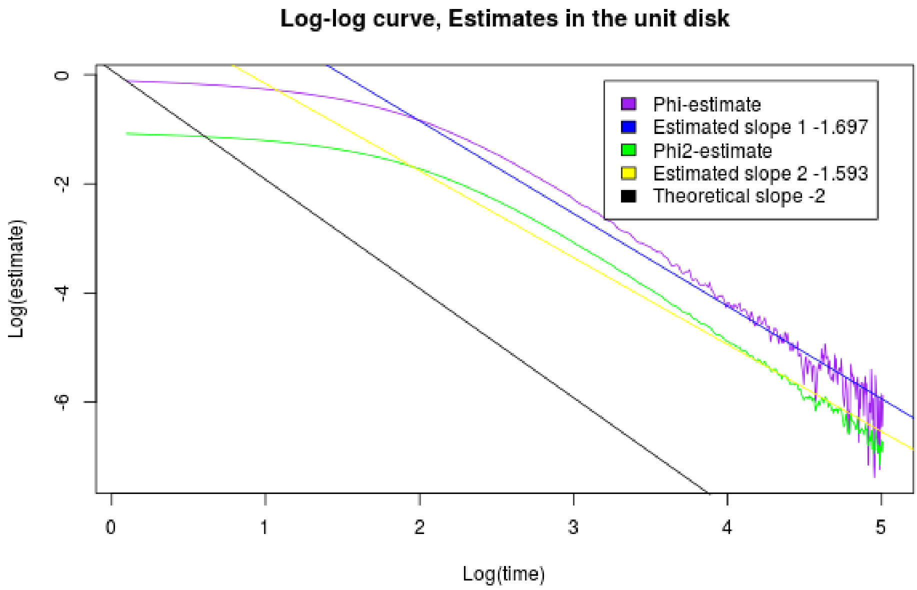

Fig. 3.15 Log-log convergence of the ϕ-estimate and the ϕ_2-estimate with f_0 given by (3.3.2).

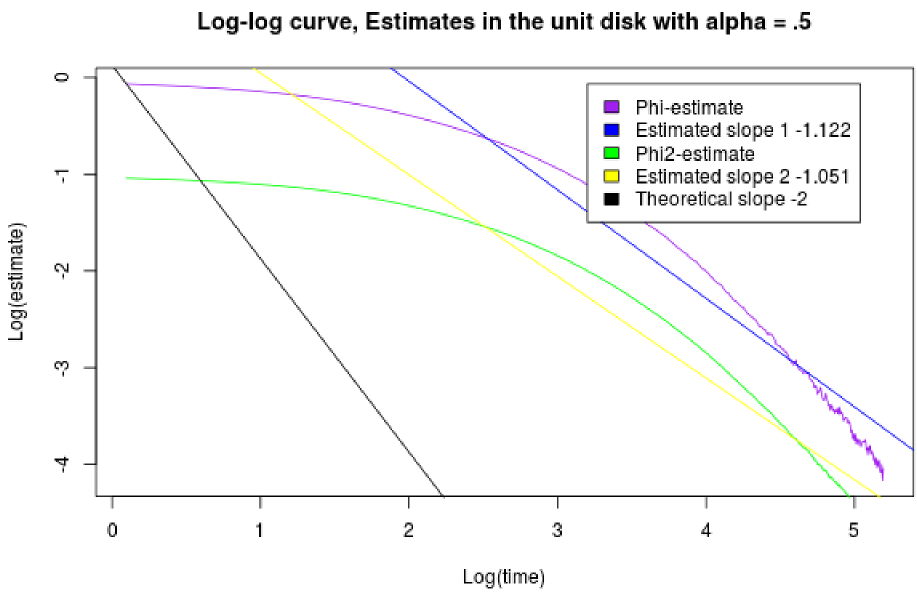

Fig. 3.16 Estimates for $\alpha = .5$, with f_0 given by (3.3.2).

Variation of the parameter a. In this part we only consider the case $\alpha = 1$. We now test different values of a. Other parameters remain unchanged. Recall that the previous case corresponds to $a = 1$.

The case where $a < 1$. In this case, the boundary operator gives a smaller weights to velocity vectors with small norms, hence we expect the convergence to be faster, provided that we take an initial data with enough inverse moments. For f_0 given by (3.3.2), this should not change the rate of convergence, and, as one can see on Figure 3.17, this is indeed the case.

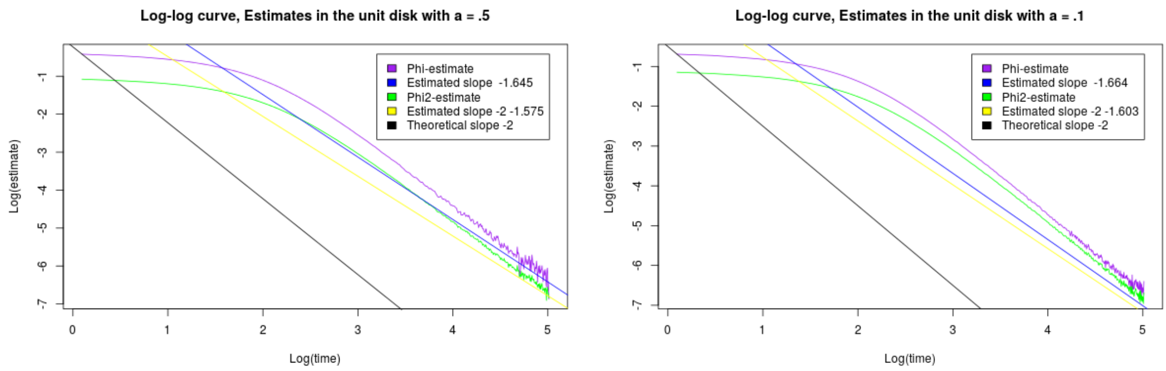

Fig. 3.17 Convergence of ϕ and ϕ_2 estimates, for f_0 given by (3.3.2) and $a = .1$ and $a = .5$.

In Figure 3.18, we test whether the rates are modified when we take a data with higher regularity. For this, we start with an initial data corresponding to the new equilibrium, and multiply its velocity by 0.1, to obtain the same variance as before. As one can see on the result, although this should not be interpreted in a quantitative way, it is clear that this time the slope is affected by this new value of a, and that the convergence occurs faster. Here, as opposed to before, the slope is computed only when the decay really starts, i.e. at time $t = 12$ when $a = .1$ and at time $t = 7$ when $a = .5$. The estimated slopes are really different from the one with $a = 1$ with an initial data given by (3.3.2).

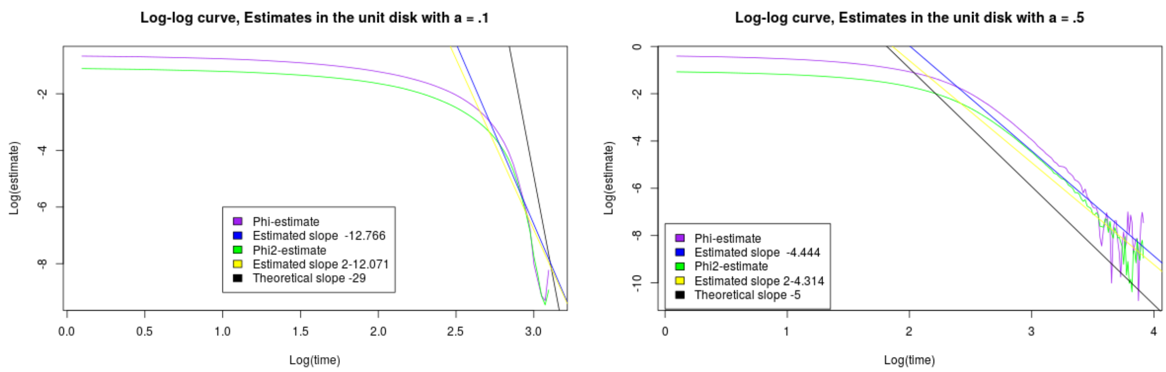

Fig. 3.18 Convergence of ϕ and ϕ_2 estimates, for f_0 given by $\frac{M_{a,0.01}}{|D|}$, with $a = .1$ and $a = .5$.

Estimates for $a > 1$. In this case, the boundary operator gives a larger weight to small velocities. We expect that this will lower the rate of convergence. The first test, see Figure 3.19, is done with $a = 1.5$, and with initial data (3.3.2) corresponding to $a = 1$, far from the equilibrium. In this case, the estimated slopes are closer to -2 than to the theoretical value -1, and in fact are very similar to the ones in the case $a = 1$, see Figure 3.15. On the other hand, if we use as initial data distribution the equilibrium corresponding to $a = 1.5$ with a different temperature, the estimated slopes become closer to the theoretical value -1. An explanation for this difference is the following: when $f_0(x, v) = \frac{M_{1,0.01}(v)}{|D|}$, particles that have not yet been thermalized are distributed according to $\frac{M_{1,0.01}(v)}{|D|}$, hence their velocity corresponds to the rate of convergence -2 that we expect in the case $a = 1$ (and this is the case even when $\alpha < 1$). On the other hand, when $f_0(x, v) = M_{1.5,0.01}(v)/|D|$, particles that have not encountered a diffuse reflection have the distribution corresponding to the rate of convergence $\frac{1}{t}$. The results for $a = 1.5$ are displayed on Figure 3.19.

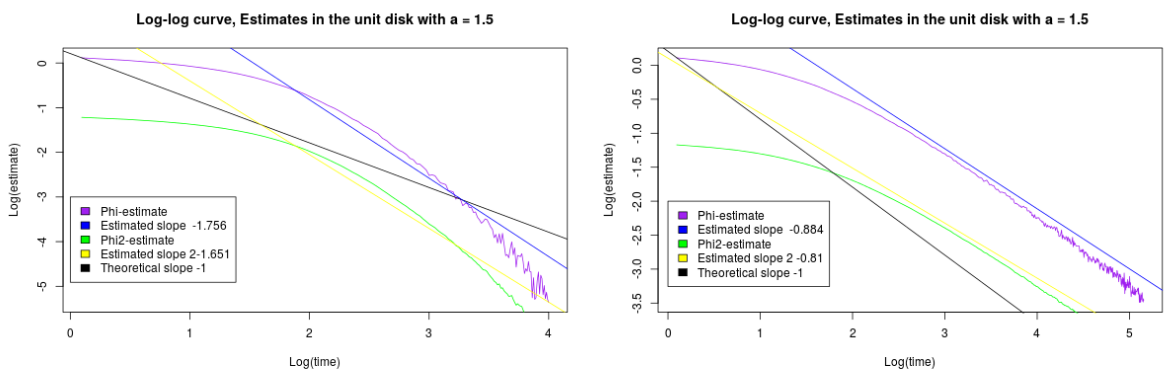

Fig. 3.19 Convergence of ϕ and ϕ_2 estimates, for a reflection law using $a = 1.5$ and f_0 given by $\frac{M_{a,0.01}}{|D|}$, with $a = 1$ (left) and $a = 1.5$ (right).

Let us expand on the surprising behavior of the case $a = 1.5$, f_0 given by (3.3.2), where the estimated slope is larger, in absolute value, than the theoretical one. One possible explanation is that the difference in variance between the initial data, centered at 0 with variance 0.01, and the reflection law is so large that it somehow counteracts the fact that M_a gives more weight to small values when $a = 1.5$ than it does when $a = 1$. To test this hypothesis, we can compute the simulation with f_0 given by (3.3.2) and a reflection law corresponding to $a = 1.5$, and multiply the outcoming velocities from this reflection law by 0.1, in order to have two variances of the same order. The result is displayed in Figure 3.20, and in this case we recover that the estimated slope is smaller, in absolute value, than the theoretical one, as expected. It is interesting to note how this use of small variances delays the convergence.

One of the reasons why our estimates on Figure 3.19 seem to fail is that, since we have a delay between the time $t = 0$ and the time at which the decay of the estimate really occurs, our estimated slopes are slightly biaised at the beginning. Moreover, we quickly reach the point where the noise is too important for our estimate to become relevant. One way to fix this second issue is to take a larger, i.e. closer to 3, to slow down the convergence and recover

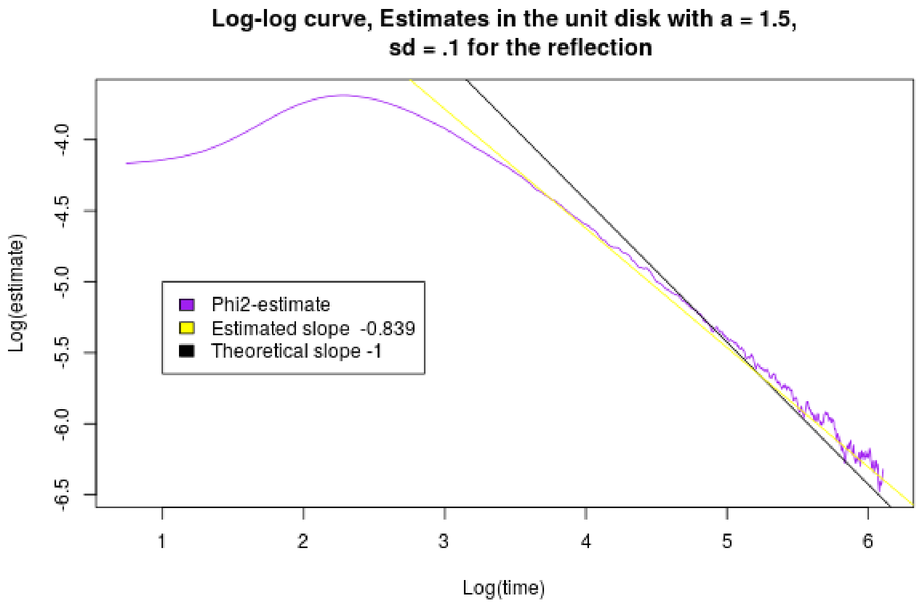

Fig. 3.20 Convergence of the ϕ_2 estimate, in the case where $a = 1.5$ with $\sigma = 0.1$ and f_0 is given by $\frac{M_{1,0.01}}{|D|}$. The slope is estimated from $t = 30$ onwards.

better estimates of the slope. We do this with $a = 2.5$, and the initial data

$$f_0(x, v) = M_{2.5,0.01}(v)/|D|.$$

The results are displayed on Figure 3.21. While the slope itself is still far from the theoretical one (although clearly strongly correlated), we see that the decay of the ϕ_2 estimate is really well approximated by the line with the corresponding slope: the decay seems to be of polynomial order. Note that the estimate starts from a very small value: this is not surprising, since the distribution $M_{a,\cdot}$ is strongly concentrated around 0, the variance plays a more restricted role compared to the previous cases. As before, the estimated slope is slightly superior, in absolute value, to the theoretical one, because of the effect of the small variance of the initial data. We only display the ϕ_2-estimate, as the ϕ-estimate provides poorer results.

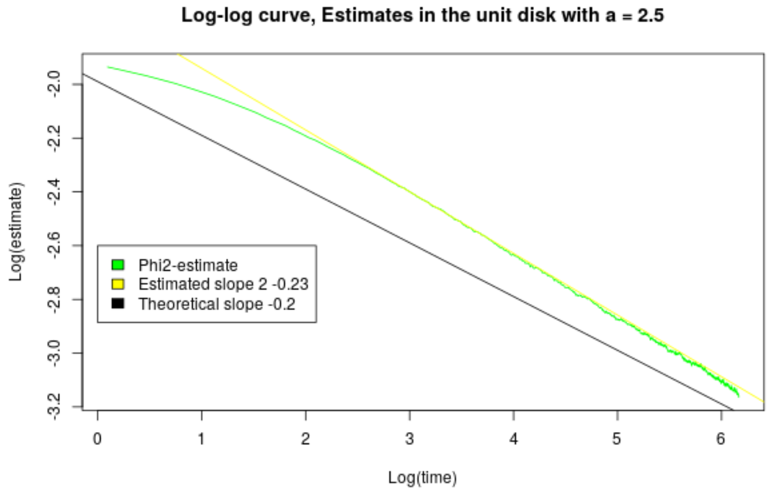

Fig. 3.21 Convergence of the ϕ_2 estimate, in the case where $a = 2.5$ and the reflection law has variance 1, with f_0 is given by $\frac{M_{2.5,0.01}}{|D|}$.

3.5 Results in the star-shaped domain

In this section, we present the result in the star-shaped domain without further comments, to show that indeed, in this non-radially symmetric domain, the problem behaves quite similarly as in the radially symmetric one.

3.5.1 Qualitative results

We present first the empirical distribution of the first space and velocity coordinates in the case where

$$f_0(x,v) = \delta_{(0,0)}(x)\delta_{(1,2)}(v), \qquad (x,v) \in D \times \mathbb{R}^2. \tag{3.5.1}$$

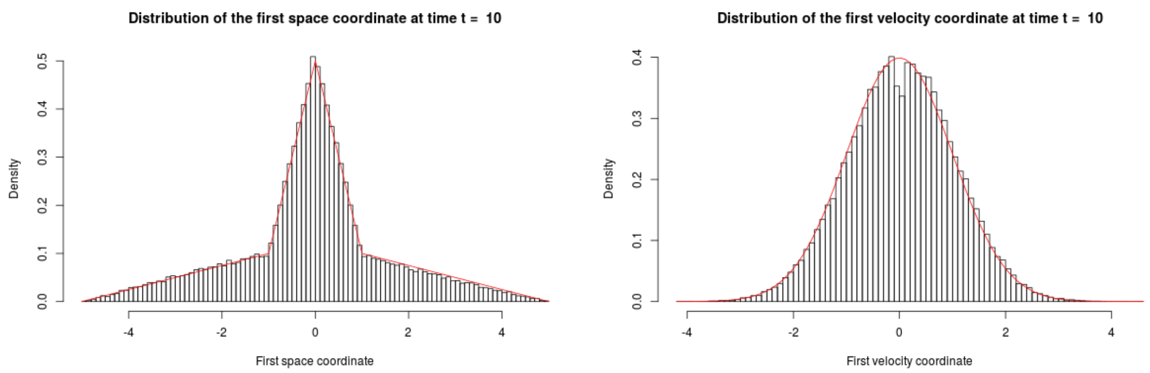

Fig. 3.22 Distribution of the first space and velocity coordinate at time $t = 10$ in the case where the initial data is given by (3.5.1).

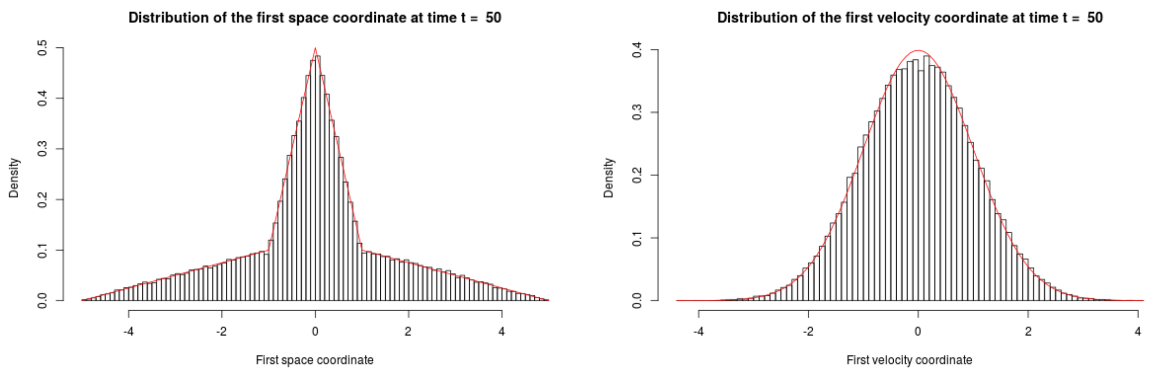

Fig. 3.23 Distribution of the first space and velocity coordinates at time $t = 50$ in the case where the initial data is given by (3.5.1).

We then present the same distribution in the case where

$$f_0(x,v) = \mathcal{U}_D(dx)\mathcal{N}_2(0_2, 0.01I_2)(dv), (x,v) \in D \times \mathbb{R}^2, \tag{3.5.2}$$

where $\mathcal{U}_D$ is the uniform distribution in the star-shaped domain $D \subset \mathbb{R}^2$.

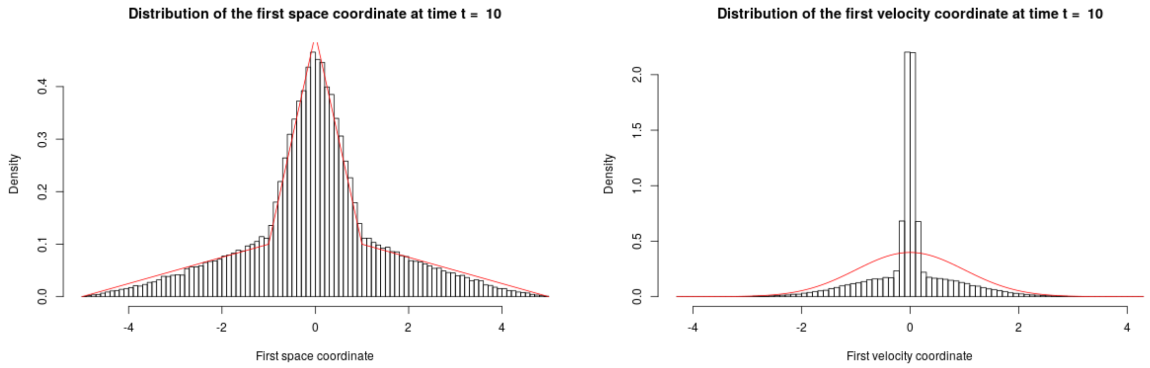

Fig. 3.24 Distribution of the first space and velocity coordinates at time $t = 10$ in the case where the initial data is given by (3.5.2).

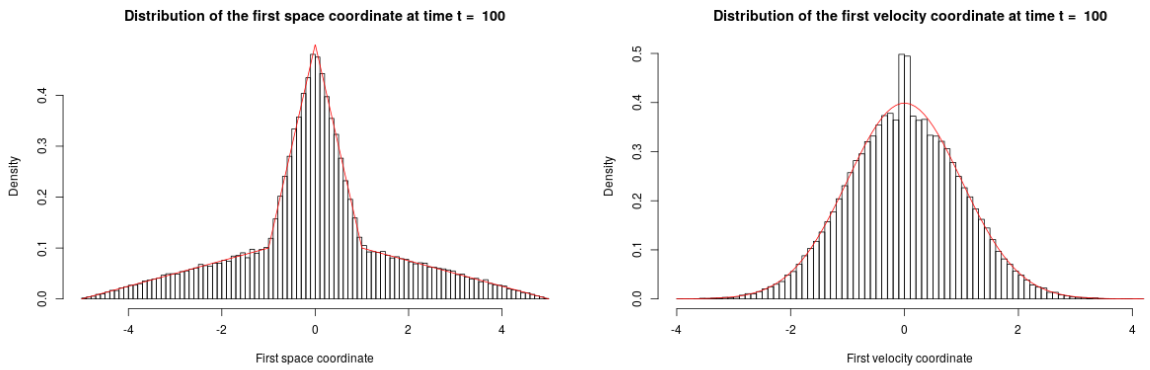

Fig. 3.25 Distribution of the first space and velocity coordinates at time $t = 100$ in the case where the initial data is given by (3.5.2).

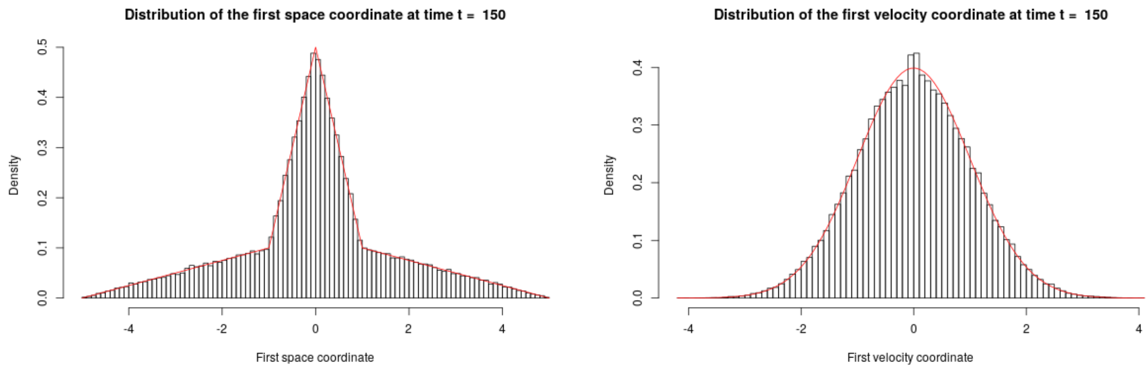

Fig. 3.26 Distribution of the first space and velocity coordinates at time $t = 150$ in the case where the initial data is given by (3.5.2).

3.5.2 Quantitative results of convergence towards equilibrium

For the study in the star domain, we will only consider the ϕ_2-estimate computed according to the function ϕ_2 (3.4.1), since the ϕ-estimate is always less efficient. We emphasize that ϕ_2 takes into account both the space and the velocity: we still take into consideration the more complicated structure of the domain. The results are quite similar to those in the unit disk.

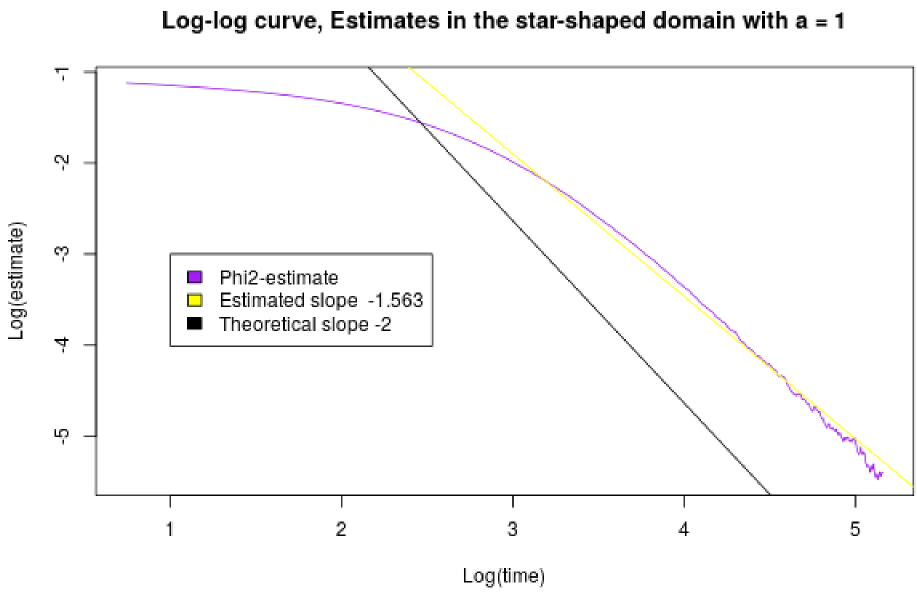

Fig. 3.27 Convergence of the ϕ_2-estimate, for f_0 given by (3.5.2), with $\alpha = a = 1$ in the star-shaped domain. The regression is computed from the values of the estimate at time $t = 30$ until time $t = 175$.

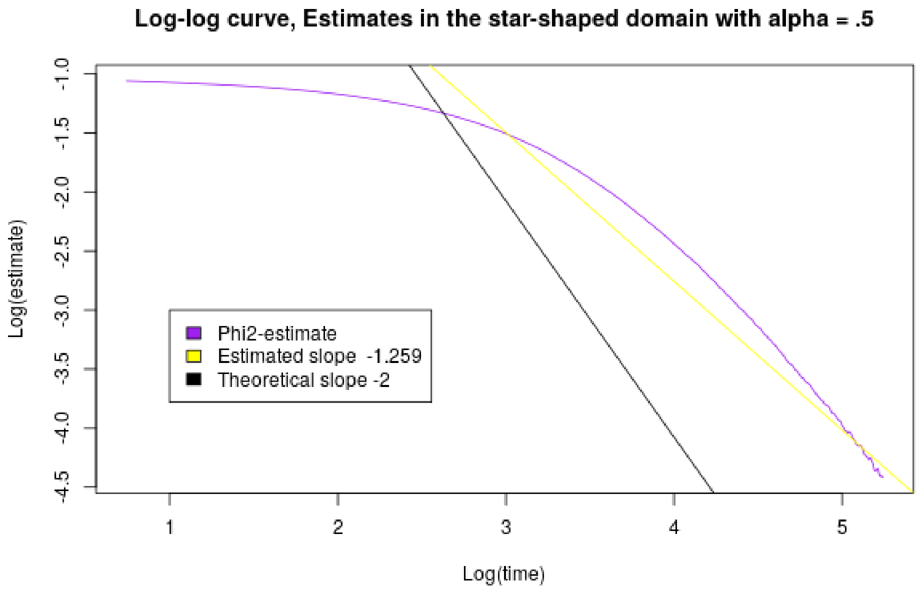

Fig. 3.28 Convergence of the ϕ_2-estimate, for f_0 given by (3.5.2), with $\alpha = 0.5$, $a = 1$ in the star-shaped domain. The regression is computed from the values of the estimate at time $t = 30$ until time $t = 190$.

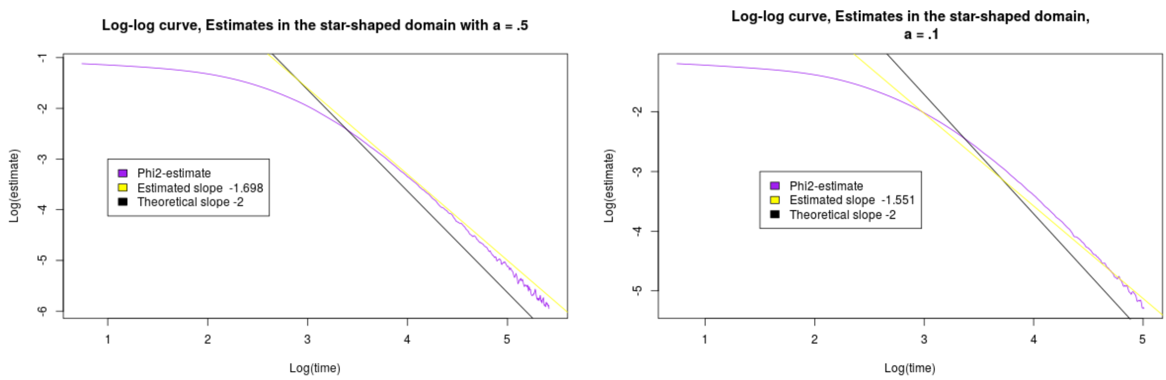

Fig. 3.29 Convergence of the ϕ_2-estimate, for f_0 given by (3.5.2), with $a = .5$ (left) and $a = .1$ (right) in the star-shaped domain. On the left (resp. right) the regression is computed from the values of the estimate at time $t = 15$ (resp. $t = 30$) until time $t = 225$ (resp $t = 190$) .

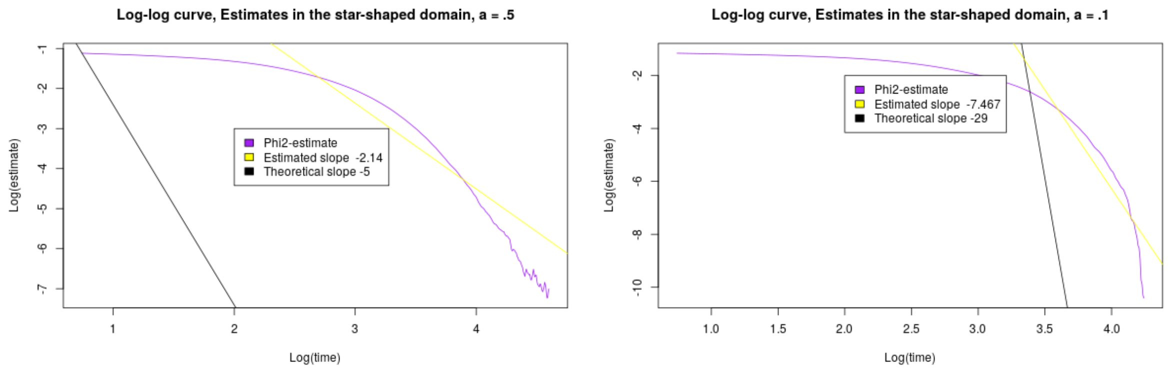

Fig. 3.30 Estimates in the case where the reflection law and the initial data have parameter $a = .5$ (left) and $a = .1$ (right). The estimated slopes are computed from time $t = 30$ onwards.

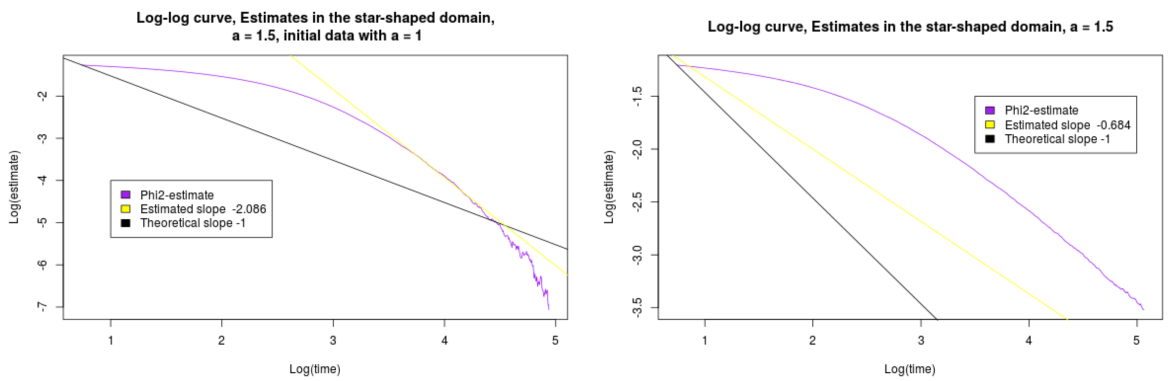

Fig. 3.31 Estimates in the case where the reflection law has parameter $a = 1.5$. The initial data is $\frac{M_{a,0.01}}{|D|}$ with $a = 1$ (left) and $a = 1.5$ (right). The estimated slopes are computed from time $t = 10$ onwards.

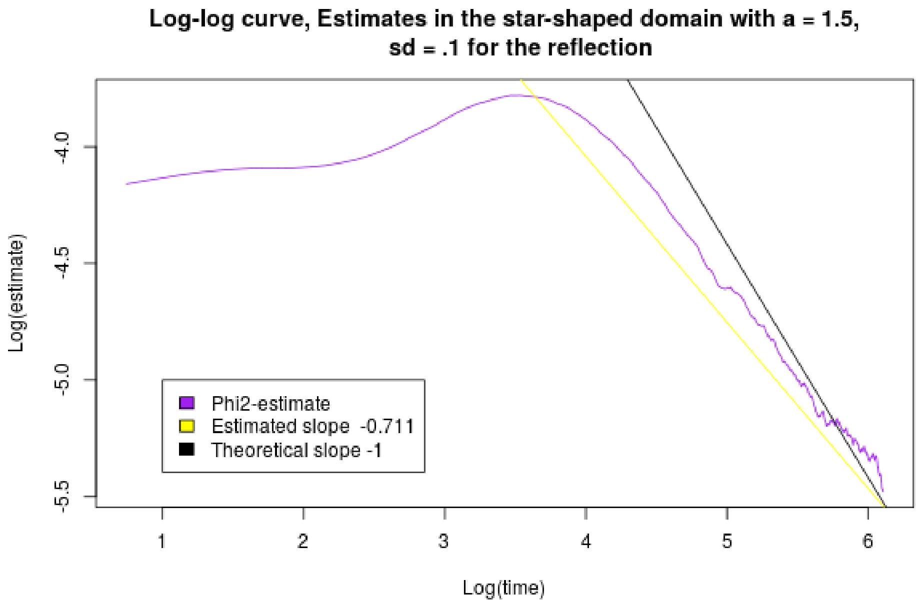

Fig. 3.32 Convergence of the ϕ_2 estimate, for f_0 given by given by $\frac{M_{1,0.01}}{|D|}$ and a reflection law with $a = 1.5$ and variance 0.01. Note that this set of parameter highly delays the convergence of the estimator. The estimated slope is computed from time $t = 30$ onwards.

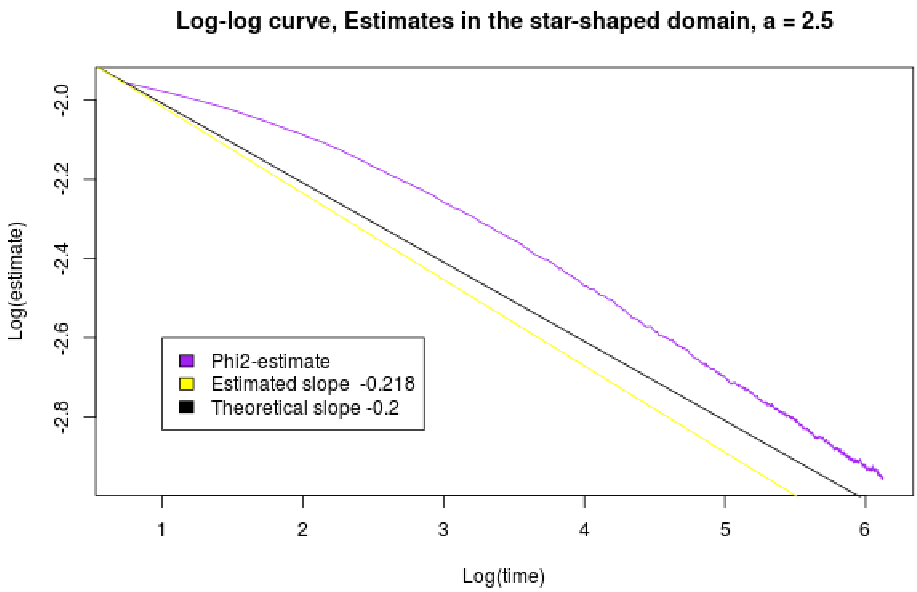

Fig. 3.33 Convergence of the ϕ_2-estimate, for f_0 given by given by $\frac{M_{2.5,0.01}}{|D|}$ and a reflection law with $a = 2.5$.

Chapter 4

A semigroup approach to the convergence rate of a collisionless gas

This chapter corresponds to the paper [8] published in Kinetic and Related Models.

Abstract: We study the rate of convergence to equilibrium for a collisionless (Knudsen) gas enclosed in a vessel in dimension $n \in \{2, 3\}$. By semigroup arguments, we prove that in the L^1 norm, the polynomial rate of convergence $\frac{1}{(t+1)^{n-}}$ given in [119], [87] and [88] can be extended to any C^2 domain, with standard assumptions on the initial data. This is to our knowledge, the first quantitative result in collisionless kinetic theory in dimension equal to or larger than 2 relying on deterministic arguments that does not require any symmetry of the domain, nor a monokinetic regime. The dependency of the rate with respect to the initial distribution is detailed. Our study includes the case where the temperature at the boundary varies. The demonstrations are adapted from a deterministic version of a subgeometric Harris' theorem recently established by Cañizo and Mischler [21]. We also compare our model with a free-transport equation with absorbing boundary.

Keywords: transport equations, Maxwellian diffusion boundary conditions, subgeometric Harris' theorem, explicit, collisionless gas.

4.1 Introduction

Model. In this paper, we study the kinetic free-transport equation with Maxwell boundary conditions inside a domain D in $\mathbb{R}^n$ having closure $\bar{D}$, with $n \in \{2,3\}$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0, & (t,x,v) \in \mathbb{R}_+^* \times G, \\ \gamma_- f(t,x,v) = K\gamma_+ f(t,x,v), & (t,x,v) \in \mathbb{R}_+ \times \partial_- G, \\ f|_{t=0}(x,v) = f_0(x,v), & (x,v) \in G, \end{cases} \tag{4.1.1}$$

where we use the notations $G := D \times \mathbb{R}^n$, and, denoting n_x the unit **inward** normal vector at $x \in \partial D$,

$$\partial_+ G := \{(x,v) \in \partial D \times \mathbb{R}^n, v \cdot n_x < 0\},$$
$$\partial_- G := \{(x,v) \in \partial D \times \mathbb{R}^n, -(v \cdot n_x) < 0\}.$$

Given a function ϕ on $(0,\infty) \times \bar{D} \times \mathbb{R}^n$, $\gamma_\pm \phi$ denotes its trace on $(0,\infty) \times \partial_\pm G$, provided this object is well-defined. The boundary operator K is defined, for all $(t,x,v) \in \mathbb{R}_+ \times \partial_- G$ and for ϕ supported on $(0,\infty) \times \partial_+ G$ such that $\phi(t,x,\cdot)$ belongs to $L^1(\{v' : v' \cdot n_x < 0\})$, by

$$\begin{aligned} K\phi(t,x,v) = \alpha(x) M(x,v) &\int_{\{v' \in \mathbb{R}^n : v' \cdot n_x < 0\}} \phi(t,x,v') |v' \cdot n_x| dv' \\ &+ (1-\alpha(x))\phi(t,x,v - 2(v \cdot n_x) n_x). \end{aligned} \tag{4.1.2}$$

In this paper, we consider the standard (and physically relevant) case of the Maxwellian distribution at the boundary ∂D,

$$M(x,v) = \frac{\tilde{c}(x)}{(2\pi\theta(x))^{\frac{n}{2}}} e^{-\frac{\|v\|^2}{2\theta(x)}}, \qquad x \in \partial D, v \in \mathbb{R}^n, \tag{4.1.3}$$

where, for all $x \in \partial D$, for $z \in \partial D$,

$$\tilde{c}(x) = \Big(\int_{\{v \cdot n_z < 0\}} \frac{1}{(2\pi\theta(x))^{\frac{n}{2}}} e^{-\frac{\|v\|^2}{2\theta(x)}} |v \cdot n_z| dv \Big)^{-1}, \tag{4.1.4}$$

which is independent of the choice of z since the integrand is radial, so that

$$\int_{\{v \cdot n_x < 0\}} M(x,v) |v \cdot n_x| dv = 1. \tag{4.1.5}$$

The parameter $\theta(x)$ corresponds physically to the temperature at the point $x \in \partial D$ of the boundary wall considered at rest.

Physical motivations. This problem models the evolution of a Knudsen (collisionless) gas enclosed in the vessel D. For such diluted gases, the Lebesgue measure of the set of collisions between particles is 0, hence the collision operator of the Boltzmann equation describing statistically the dynamic, as introduced by Maxwell [95], vanishes. Particles in D move according to the free-transport dynamic until they meet with the boundary. They are reflected at the boundary ∂D in a diffuse or specular manner, corresponding to the two terms in the definition of K: at a point $x \in \partial D$, a fraction $1-\alpha(x)$ of the gas particles is specularly reflected, i.e., if $v \in \mathbb{R}^n$ is the initial velocity, the outgoing velocity is given by

$$\eta_x(v) := v - 2(v \cdot n_x)n_x. \tag{4.1.6}$$

The remaining fraction $\alpha(x)$ is diffusively reflected (and thus thermalized). The latter corresponds, physically, to the case where the particle is adsorbed by the wall before being re-emited inside the domain according to a new velocity distribution defined through M. More details on the derivation of this boundary condition can for instance be find in the monograph of Cercignani, Illner and Pulvirenti [27, Chapter 8]. For this model, the distribution function of the gas, $f(t,x,v)$, representing the density of particles in position $x \in \bar{D}$ with velocity $v \in \mathbb{R}^n$ at time $t \geq 0$, satisfies (4.1.1).

Link with the Boltzmann equation and convergence rate for (4.1.1). We study the rate of convergence towards equilibrium of (4.1.1). Taking $\theta \equiv \Theta$ for some $\Theta > 0$ so that M only depends on v, the existence of a steady state and the convergence towards it (at least in a restricted context) is known since the work of Arkeryd and Cercignani [2]. This equilibrium is given by, assuming the initial data to be of mass 1,

$$f_\infty(x,v) = \frac{e^{-\frac{\|v\|^2}{2\Theta}}}{|D|(2\pi\Theta)^{\frac{n}{2}}}, \qquad (x,v) \in G, \tag{4.1.7}$$

where $|D|$ denotes the volume of D. In the collisional case, for instance when one studies the space-homogeneous Boltzmann equation with the same boundary condition as in (4.1.1), the famous H-theorem of Boltzmann gives a starting point from which Boltzmann [16] gave plausible arguments for the convergence towards an equilibrium as time goes to infinity. Once this convergence is established, a key question is the rate at which it occurs. Physically, one would also like to obtain an explicit form for the constant playing a role in the convergence rate, to avoid unsignificant values as one can find when working with the Poincaré recurrence theorem. Regarding Boltzmann equation with Maxwell boundary condition (or diffuse boundary condition, i.e. with $\alpha \equiv 1$) and constant temperature, there are strong reasons to believe that the convergence occurs at an exponential rate, i.e., that there exist λ, $C > 0$ such that if f_t denotes the solution at time $t > 0$, for all $t \geq 0$,

$$\|f_t - f_\infty\|_{L^1} \leq Ce^{-\lambda t},$$

where f_∞ is the Maxwellian corresponding to the equilibrium of the system. On this matter, see for instance Villani [123, 18.5], where it is established that the convergence rate is equal to, or better, than $t^{-\infty}$ in some Sobolev norm assuming some strong regularity estimates. However those estimates may not hold true in a non-convex setting, see [67] for a discussion on those issues in a general context. Guo [68, Theorem 4] proved the exponential convergence towards equilibrium, when the initial data is close to the equilibrium.

This (expected) dissipative property of the previously mentioned Boltzmann equation is a consequence of two factors: the interactions with the boundary wall and the collision operator. On the other hand the model corresponding to (4.1.1) only deals with the interactions with the boundary wall. This leads to several natural questions.

i) Can we still prove a convergence towards an equilibrium ? In particular in the case where θ is not constant ?

ii) Is the rate at which this convergence occurs exponential, as expected for the Boltzmann equation with similar boundary conditions ? If not, can we characterize this rate in a precise manner ?

iii) Can we compute the corresponding constants explicitely ?

Well-posedness and qualitative convergence. Arkeryd and Cercignani [2] established the well-posedness of (4.1.1) in the L^1 setting. This allows one to associate a semigroup $(S_t)_{t\geq 0}$ to the evolution equation, so that given an initial datum f_0, $S_t f_0$ is the solution of (4.1.1) at time $t \geq 0$. The decay property of the distance with respect to the equilibrium, and thus the answer to the three questions above, can then be read at the level of the associated semigroup. The fact that the answer to i) is positive is physically intuitive and has been established qualitatively in the convex setting and in dimension 3 by Arkeryd and Nouri [3]. In the general setting, the obtention of a rate to answer ii) will provide an *a posteriori* answer for question i) as well.

Known results for question ii). Question ii) was first addressed numerically by Tsuji, Aoki and Golse [119]. They gave strong arguments to support the intuition that the rate of convergence in the L^1 norm will no more be exponential in this case, but rather polynomial of order $\frac{1}{t^n}$, where n is the dimension of the problem. The absence of a spectral gap for the sole free-transport operator is a natural reason to think that the exponential rate cannot be reached for this model. Later, Aoki and Golse [1] proved that the rate of convergence is better than $\frac{1}{t}$ in the L^1 distance, with an additional assumption of symmetry of the domain and of the initial data, by means of Feller's renewal theory. Still with this symmetry assumption on the domain and in dimensions 1 to 3, the problem was studied *via* probabilistic methods by Kuo, Liu and Tsai [87], see also [88] for the case where θ varies, with the sole diffuse condition, which corresponds to $\alpha \equiv 1$ with our notations. The key idea is that the symmetry of the domain allows one to consider the intervals in time between two rebounds of a particle as independent and identically distributed random variables, and to deduce a law of large numbers from which one can control the flux of the solution at the boundary in the L^∞ norm. Kuo [86] later extended this result with similar tools to the case of the Maxwell boundary condition, in dimension 2. Finally let us mention that Mokhtar-Kharroubi and Seifert [102] recently obtained an explicit polynomial rate in slab geometry (dimension 1). Their proof relies on a quantified version of Ingham's tauberian theorem.

Hypothesis and main result. While the methods used in [1], [87], [88] and [86] are difficult to adapt to a nonsymmetric setting, it seems intuitive to expect that the rate of convergence will be of the same order without this assumption. In this work, we give, using a slightly modified version of the subgeometric Doeblin-Harris theory of Cañizo and Mischler [21], an answer to questions i) and ii) and a partial answer to question iii) in the larger context of C^2 domains.

Let us introduce some assumptions and key notations and present our main result. The dimension n belongs to $\{2,3\}$. We endow $\mathbb{R}^n$ and $\mathbb{R}$ with the Lebesgue measure. The symbols $dx, dv, \dots$ denote this measure. We assume that the domain (open, connected) $D \subset \mathbb{R}^n$ is bounded and C^2 with closure $\bar{D}$, and that the map $x \to n_x$ can be extended to the whole set $\bar{D}$ as a $W^{1,\infty}$ map, where $W^{1,\infty}$ denotes the corresponding Sobolev space. For any $k \in \mathbb{N}^*$, we use the Euclidian norm in $\mathbb{R}^k$. We write $d(D)$ for the diameter of D

$$d(D) = \sup_{(x,y)\in D^2} \|x-y\|.$$

On $\bar{D} \times \mathbb{R}^n$, setting

$$\partial_0 G := \{(x,v) \in \partial D \times \mathbb{R}^n, v \cdot n_x = 0\},$$

we define the map σ by:

$$\sigma(x,v) = \begin{cases} \inf\{t > 0, x + tv \in \partial D\}, & (x,v) \in \partial_- G \cup G, \\ 0, & (x,v) \in \partial_+ G \cup \partial_0 G, \end{cases} \tag{4.1.8}$$

which corresponds to the time of the first collision with the boundary for a particle in position x with velocity v at time $t = 0$. The L^1 space on G, denoted $L^1(G)$ is the space of measurable $\mathbb{R}$-valued functions f such that

$$\|f\|_{L^1} := \int_G |f(x,v)| dv dx < \infty.$$

For any non-negative measurable function w defined on G, we introduce the weighted L^1 space $L^1_w(G) = \{f \in L^1(G), \|fw\|_{L^1} < \infty\}$ endowed with the norm defined by $\|f\|_w := \|fw\|_{L^1}$. For any function $f \in L^1(G)$, we define the mean of f by

$$\langle f \rangle := \int_G f(x,v)dvdx. \tag{4.1.9}$$

For the function $\alpha : \partial D \to [0,1]$ playing a role in the boundary condition, we assume that there exists a constant $c_0 \in (0,1)$ such that

$$\alpha(x) \geq c_0, \quad \forall x \in \partial D. \tag{4.1.10}$$

This condition implies that $1-\alpha(x) \leq 1-c_0$ for all $x \in \partial D$, a fact that will allow us to control the contribution of the specular component of the reflection at the boundary.

Remark 4.1.1. In the case of the diffuse boundary condition, that is, when $\alpha \equiv 1$, our condition implies that one has to choose some $c_0 \in (0,1)$ rather than the value 1 itself. Any value $c_0 \in (0,1)$ will satisfy (4.1.10) in this case.

We define the constant $c_4 \in (0,1)$ by the equation

$$(1-c_4)^4 = (1-c_0), \tag{4.1.11}$$

so that, for $i \in [1,4]$,

$$(1-c_4)^i \geq (1-c_0).$$

Finally, we assume that the temperature function $\theta : \partial D \to \mathbb{R}_+$ is continuous, positive on ∂D compact, so that $(x,v) \to M(x,v)$ is continuous and positive. We introduce the weights ω_i for $i \in \{1,\dots,4\}$ defined by setting, for all $(x,v) \in \bar{D} \times \mathbb{R}^n$,

$$\omega_i(x,v) = \left(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\right)^i \ln\left(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\right)^{-1.6}. \tag{4.1.12}$$

Note that $2 > 1.6 > \frac{n+1}{n}$ for $n \in \{2,3\}$. The idea behind this choice is that we will be able to interpolate a first result for ω_{n+1} by considering the weight $\omega_{n+1}^{\frac{n}{n+1}}$, and that the exponent of the logarithmic factor will still be smaller than -1. Our main results are the following.

Theorem 4.1.1. *There exists a constant $C > 0$ such that for all $t \geq 0$, for all $f,g \in L^1_{\omega_{n+1}}(G)$ with $\langle f \rangle = \langle g \rangle$, there holds*

$$\|S_t(f-g)\|_{L^1} \leq \frac{C\ln(1+t)^{n+2}}{(1+t)^{n+1}}\|f-g\|_{\omega_{n+1}}.$$

For $i \in [\![n-1, n+1]\!]$, we define the weight m_i by setting, for all $(x,v) \in \bar{D} \times \mathbb{R}^n$,

$$m_i(x,v) = \left(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\right)^i \ln\left(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\right)^{-1.6\frac{n}{n+1}}. \tag{4.1.13}$$

A second theorem, which relies on similar arguments, answers question i) above even in the case where θ is not constant.

Theorem 4.1.2. *There exists a unique $f_\infty \in L^1_{m_n}(G)$ such that $0 \le f_\infty$, $\langle f_\infty \rangle = 1$ satisfying*

$$\begin{aligned} v \cdot \nabla_x f_\infty(x,v) &= 0, && (x,v) \in G, \\ \gamma_- f_\infty(x,v) &= K\gamma_+ f_\infty(x,v), && (x,v) \in \partial_- G. \end{aligned}$$

Regarding the convergence towards equilibrium, we obtain by interpolation the following corollary from Theorem 4.1.1.

Corollary 4.1.1. *There exists a constant $C' > 0$ such that for all $t \ge 0$, for all $f \in L^1_{m_n}(G)$ with $\langle f \rangle = 1$, for f_∞ given by Theorem 4.1.2,*

$$\|S_t(f - f_\infty)\|_{L^1} \le \frac{C' \ln(1+t)^{n+1}}{(1+t)^n} \|f - f_\infty\|_{m_n}.$$

We make several remarks on those results.

Remark 4.1.2 (About question *iii*)*.* The constants C, C' in Theorem 4.1.1 and Corollary 4.1.1 are explicit (constructive) in the easy case of the unit disk. We believe that for any given domain D, one may find explicit constants using the geometry of D. The measure given by Doeblin-Harris condition is the only part of the proof where one may lose the constructive property of the constants, see Remark 4.4.1 for more details.

Remark 4.1.3. In general, we do not have $f_\infty \in L^1_{\omega_{n+1}}(G)$. In particular, in the case where $\theta \equiv \Theta$ with $\Theta > 0$ constant, f_∞ is explicit and given by (4.1.7), and $f_\infty \in L^1_{m_n}(G) \setminus L^1_{\omega_{n+1}}(G)$. Therefore, one cannot apply Theorem 4.1.1 to study the convergence towards equilibrium. This limiting role of the equilibrium distribution when computing a rate of convergence is well-known in the probabilistic version of the Doeblin-Harris theory used in this paper, see for instance Douc, Fort and Guillin [44] and Hairer [71].

Remark 4.1.4. To complete the previous remark, note that $f_\infty \notin L^1_m(G)$ for a weight m such that $m(x,v) \underset{v \to 0}{\sim} \frac{C}{\|v\|^n}$ for some constant $C > 0$. This is a slight drawback of the method, which prevents us from obtaining the optimal rate $\frac{1}{(t+1)^n}$ from Kuo, Liu and Tsai [87, 88, 86]. However, the rate obtained here is almost optimal in the sense that it is better than $\frac{1}{(1+t)^{n-\epsilon}}$ for all $\epsilon > 0$. Moreover, even in the unit ball of dimension n, our result is slightly different from the one of Kuo et al.: the space of initial data is not the same. Their space of initial data $L^{\infty,\mu}(G)$ is included in $L^1_{m_n}(G)$, while the converse is not true, as examplified by the function of $L^1_{m_n}(G) \setminus L^{\infty,\mu}(G)$ given by:

$$f(x,v) = \frac{1}{|D|} \frac{|\ln(\|v\|)|^{\frac{1}{2}(1.6\frac{n}{n+1}-1)}}{1 + \|v\|^{n+1}}, \qquad (x,v) \in G.$$

Remark 4.1.5. The existence result in Theorem 4.1.2 (in particular in the case where θ varies) can also be seen as a consequence of the explicit formula for the equilibrium f_∞ obtained by Sone, see the monograph [115, Chapter 2, Section 2.5, Equation (2.48)] in the form of an infinite series. In this paper, we deduce the result from Theorem 4.1.1 and do not make use of this explicit form of f_∞.

Remark 4.1.6. The hypothesis $f \in L^1_{m_n}(G)$ is quite general even if f charges 0, since $m_n(x,v) \underset{v\to 0}{\sim} \frac{C}{\|v\|^n \ln(\|v\|)^{\frac{1.6n}{n+1}}}$ for some $C > 0$. For instance if one considers, as in [1], $\theta \equiv 1$ so that M is independent of x and $f \in L^1(G)$ with $0 \leq f(x,v) \lesssim M(v)$ on $D \times \mathbb{R}^n$, the assumption is satisfied.

Remark 4.1.7. The boundary condition prevents one from considering higher-order moments, with weight exponents of order larger than $n+1$. Hence Theorem 4.1.1 cannot be improved by considering a weight ω_{n+2} with $\omega_{n+2}(x,v) \underset{v\to 0}{\sim} \frac{C}{\|v\|^{n+2} \ln(\|v\|)^{1.6}}$ for some $C > 0$. Indeed, the boundary condition becomes a limiting factor for the Lyapunov condition that we will use (see Section 4.3 below for more details): to be compatible with our proof, a weight w must satisfy, for all $x \in \partial D$,

$$\int_{\mathbb{R}^n} M(x,v)|v \cdot n_x| w(x,v)dv < \infty.$$

In [10], Bernou and Fournier study a similar model with probabilistic tools, more precisely they use a coupling of two Markov processes to derive a rate similar (up to logarithmic factors) to the one of Corollary 4.1.1. This method allows one to treat, in the space of measures, various choices of M independent of x. Indeed, they only assume that M is radial, with a first order moment and that M is lower bounded by a continuous positive function in a ball around 0. The domain is again assumed to be C^2.

Comparison with absorbing boundary condition. We conclude the paper by studying the following close problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0, & (t,x,v) \in \mathbb{R}_+^* \times G, \\ \gamma_- f(t,x,v) = 0, & (t,x,v) \in \mathbb{R}_+ \times \partial_- G, \\ f|_{t=0}(x,v) = f_0(x,v), & (x,v) \in G, \end{cases} \tag{4.1.14}$$

i.e., we take $K \equiv 0$ in (4.1.1). We set, for $\nu > 0$,

$$r_\nu(x,v) = (1 + \sigma(x,v))^\nu, \qquad (x,v) \in \bar{G}. \tag{4.1.15}$$

We refer to Theorem 4.7.1 for precise results on (4.1.14). The rough conclusions of the comparison between the two models are the following.

1) For very regular initial data, typically if $f \in L^1(G)$ with $f\mathbf{1}_{\{\|v\|\leq\epsilon\}} = 0$ for some $\epsilon > 0$, the convergence rate is exponential in (4.1.14) while it is only of order $n+1$ (up to log factors) in (4.1.1), because of the influence of the boundary condition.

2) With the assumption f, g in $L^1_{r_{n+1}}(G)$, the convergence rate of $f_t - g_t$, with obvious notations, is polynomial with, roughly, exponent $n+1$ for both problems.

3) More generally, for $f \in L^1_{r_\nu}(G)$, $g \in L^1_{r_{\nu-\delta}}(G)$ with $\nu - \delta > 1$, $\delta > 0$, the exponent of the polynomial rate of convergence is $\nu - \delta$ in (4.1.14). In particular, if $f \in L^1_{r_{n+1-}}(G)$, the exponent of the polynomial convergence rate towards equilibrium is roughly $n+1$ in (4.1.14) since the equilibrium 0 belongs to $L^1_{r_{n+1-}}(G)$ while it is only n (up to log factors) in (4.1.1) since the equilibrium f_∞ belongs to $L^1_{r_{n-}}(G) \setminus L^1_{r_{n+1-}}(G)$.

Proof strategy. Our proof of Theorem 4.1.1 is purely deterministic. While this proof is also self contained, it is adapted from the method introduced in [21]. Let us elaborate on the strategy. The first step towards the obtention of a Harris' theorem is to prove that, setting $\mathcal{L}$ the operator such that the evolution problem (4.1.1) rewrites as a Cauchy problem,

$$\begin{aligned} \partial_t f &= \mathcal{L} f && \text{in } \bar{D} \times \mathbb{R}^n, \\ f(0,.) &= f_0(.) && \text{in } G, \end{aligned} \tag{4.1.16}$$

we have the inequality

$$\mathcal{L}^* \omega_{n+1} \leq -\omega_n + \kappa, \tag{4.1.17}$$

with $\kappa > 0$ constant and $\mathcal{L}^*$ the adjoint operator of $\mathcal{L}$, and that such inequality also holds by considering various couples of weights instead of (ω_{n+1}, ω_n). It turns out that, since in our model the whole dissipative component is localized at the boundary, (4.1.17) is very difficult and perhaps impossible to obtain. On the other hand, using that

$$v \cdot \nabla_x \sigma(x,v) = -1 \qquad \text{in } G,$$

one can prove an integrated version of (4.1.17), namely that there exist $C_1, b_1 > 0$ such that for all $T > 0$, $f \in L^1_{\omega_{n+1}}(G)$,

$$\|S_T f\|_{\omega_{n+1}} + C_1 \int_0^T \|S_s f\|_{\omega_n} ds \leq \|f\|_{\omega_{n+1}} + b_1(1+T)\|f\|_{L^1}, \tag{4.1.18}$$

and that this inequality also holds for various couples of weights.

As a second step, we prove a positivity result (Doeblin-Harris condition) for the semigroup $(S_t)_{t\geq 0}$, Theorem 4.4.1. By following the characteristics of (4.1.1) backward, we prove that there exists $R_0 > 0$ such that for all $R > R_0$, there exist $T(R) > 0$ and a non-negative measure

ν on $\bar{D} \times \mathbb{R}^n$ with $\nu \not\equiv 0$ such that for all $(x, v) \in G$,

$$S_{T(R)} f(x, v) \geq \nu(x, v) \int_{\{(y,w) \in D \times \mathbb{R}^n : \sigma(y,w) \leq R\}} f(y, w) dw dy. \tag{4.1.19}$$

The measure ν depends on D and whether or not it is constructive is the key point for question iii), see Remark 4.4.1 below. As already mentioned, if ν is explicit, the constants C, C' of Theorem 4.1.1 and Corollary 4.1.1 are constructive.

To obtain the proof of Theorem 4.1.1, we assume without loss of generality that $g = 0$ and that $f \in L^1_{\omega_{n+1}}(G)$ with $\langle f \rangle = 0$. We fix $T > 0$ large enough and introduce some modified norm

$$|||.|||_{\omega_{n+1}} = \|.\|_{L^1} + \beta \|.\|_{\omega_{n+1}} + \alpha \|.\|_{\omega_n},$$

for two well-chosen constants $\alpha, \beta > 0$ depending on T. We prove, with the help of (4.1.18) and of the Doeblin-Harris condition, that

$$|||S_T f|||_{\omega_{n+1}} \leq |||f|||_{\omega_{n+1}}. \tag{4.1.20}$$

We then introduce some further weights w_0, w_1 such that $1 \leq w_0 \leq w_1 \leq \omega_{n+1}$. With a similar argument, we find that, for some modified norm $|||.|||_{w_1}$, for T as above and $\tilde{\alpha} > 0$ well-chosen,

$$|||S_T f|||_{w_1} + 2\tilde{\alpha} \|f\|_{w_0} \leq |||f|||_{w_1}. \tag{4.1.21}$$

We use repeatedly (4.1.20) and (4.1.21), along with the inequalities satisfied by the weights, to conclude.

Theorem 4.1.2 is obtained from Theorem 4.1.1 and a refined version of (4.1.20) with the couple (m_{n+1}, m_n) instead of (ω_{n+1}, ω_n). Once Theorem 4.1.2 is established, Corollary 4.1.1 follows from Theorem 4.1.1 by an interpolation argument.

Proof strategy for the study of (4.1.14). To compute the convergence rate towards equilibrium of (4.1.14), we use a method introduced by Hairer [71] which is much more direct than the previous strategy. This proof can not be easily applied to study (4.1.1) because of the boundary condition and its impact on the derivation of the Lyapunov inequality. On the other hand the strategy to prove Theorem 4.1.1 can not be adapted easily here because the Doeblin-Harris condition, Theorem 4.4.1, does not hold. The proof in the case of an exponential weight is a straightforward application of Gronwall's lemma. In the polynomial case, i.e. when the initial data $f \in L^1_{r_\nu}(G)$ for some $\nu > 1$, the idea is to prove that,

$$\mathcal{B}^* r_\nu \leq -\phi(r_\nu),$$

where $\mathcal{B}$ is the generator of the semigroup which can be associated to (4.1.14) and where $\phi(x) = \nu x^{\frac{\nu-1}{\nu}}$ for all $x \geq 1$ is a concave function. We then define, for $t \geq 0$, $u \geq 1$,

$$\psi(t,u) = (H(u) + t + 1)^\nu,$$

with $H(u) = \int_1^u \frac{1}{\phi(s)} ds = u^{\frac{1}{\nu}} - 1$ for all $u \geq 1$ and prove that $t \to \|S_t f\|_{\psi(t,r_\nu)}$ is non-increasing in $\mathbb{R}_+$ using the differential properties of ψ, where $(S_t)_{t\geq 0}$ is now the semigroup associated to (4.1.14). To conclude, we have for all $t \geq 0$,

$$(t+1)^\nu \|S_t f\|_{L^1} \leq \|S_t f\|_{\psi(t,r_\nu)} \leq \|f\|_{\psi(0,r_\nu)} = \|f\|_{r_\nu},$$

and the polynomial rate $(t+1)^{-\nu}$ follows. In both cases, the constants are constructive.

Plan of the paper. In Section 4.2 we introduce a few notations and recall some basic properties of (4.1.1). In Section 4.3 we prove the Lyapunov inequality (4.1.18) for several couples of weights. In Section 4.4 we prove the Doeblin-Harris condition satisfied by the semigroup $(S_t)_{t\geq 0}$, (4.1.19). In Section 4.5 we recall some interpolation results for L^1-weighted space and give very slight extensions in the case of spaces defined through a projection. The proof of Theorem 4.1.1, Theorem 4.1.2 and Corollary 4.1.1 is done in Section 4.6 using the previous results. Section 4.7 is devoted to the study of the case of an absorbing boundary condition with the strategy detailed above.

4.2 Setting and first properties

4.2.1 Notations and associated semigroup

We first set some notations. For any set B, we write $\bar{B}$ for the closure of B. For any space E, we write $\mathcal{D}(E) = C_c^1(E)$ the space of test functions (C^1 with compact support) on E. We write $d\zeta(x)$ for the surface measure at $x \in \partial D$. We denote by $\mathcal{H}$ the $n-1$ dimensional Hausdorff measure.

For a function $f \in L^\infty([0,\infty); L^1(\bar{D} \times \mathbb{R}^n))$, admitting a trace γf at the boundary we write $\gamma_\pm f$ for its restriction to $(0,\infty) \times \partial_\pm G$. This corresponds to the trace obtained in Green's formula, see Mischler [101]. If f is a solution to (4.1.1) with initial data $f_0 \in L^1(G)$ the traces are well-defined, see Arkeryd and Cercignani [2, Section 3].

Lemma 4.2.1. *The boundary operator K defined by (4.1.2) is non-negative and satisfies, for all $t \geq 0$, $x \in \partial D$, for all f solution to (4.1.1) with $f_0 \in L^1(G)$, f regular enough so that both integrals are well-defined,*

$$\int_{\{v\cdot n_x > 0\}} K\gamma f(t,x,v)(v\cdot n_x) dv = \int_{\{v \cdot n_x < 0\}} \gamma f(t,x,v) |v \cdot n_x| dv. \tag{4.2.1}$$

Proof. The non-negativity of K is straightforward from (4.1.2). Since, for all $x \in \partial D$,

$$\int_{\{v \cdot n_x > 0\}} M(x,v)|v \cdot n_x| dv = 1$$

by (4.1.5), we have, for all $t \geq 0$, recalling the notation η_x from (4.1.6),

$$\begin{aligned}\int_{\{v \cdot n_x > 0\}} K\gamma f(t,x,v)(v \cdot n_x) dv = &\int_{\{v \cdot n_x > 0\}} \alpha(x) M(x,v)|v \cdot n_x| dv \\ &\times \int_{\{v' \cdot n_x < 0\}} \gamma f(t,x,v')|v' \cdot n_x| dv' \\ &+ \int_{\{v \cdot n_x > 0\}} (1-\alpha(x))\gamma f(t,x,\eta_x(v))|v \cdot n_x| dv,\end{aligned}$$

and, using the involutive change of variable $w = \eta_x(v)$ and that $w \cdot n_x = -v \cdot n_x$

$$\begin{aligned}\int_{\{v \cdot n_x > 0\}} K\gamma f(t,x,v)(v \cdot n_x) dv = \alpha(x) &\Big(\int_{\{v' \cdot n_x < 0\}} \gamma f(t,x,v')|v' \cdot n_x| dv' \Big) \\ &+ (1-\alpha(x)) \int_{\{v \cdot n_x < 0\}} \gamma f(t,x,v)|v \cdot n_x| dv.\end{aligned}$$

The result follows. □

As a consequence, $\|K\| = 1$ and (4.1.1) is well posed in the L^1 setting, see Arkeryd and Cercignani [2, Theorem 3.6]. Therefore we can associate to the equation a strongly continuous semigroup $(S_t)_{t \geq 0}$ of linear operators, such that, for $f_0 \in L^1(G)$, for all $t \geq 0$, $S_t f_0 = f(t,.)$ is the unique solution in $L^\infty([0,\infty); L^1(\bar{D} \times \mathbb{R}^n))$ to (4.1.1) taken at time t. Decay properties of the equation will be studied at the level of this semigroup.

4.2.2 Positivity and mass conservation

We gather in the next theorem several key properties of (4.1.1). For $f \in L^1(G)$, recall the notation $\langle f \rangle$ from (4.1.9).

Theorem 4.2.1. *Let $f \in L^1(G)$. For all $t \geq 0$, $\langle S_t f \rangle = \langle f \rangle$. Moreover, we have*

$$\|S_t f\|_{L^1} \leq \|f\|_{L^1},$$

and if f is non-negative, so is $S_t f$.

Proof. We sketch the proof with the additional assumption that f and the trace γf are sufficiently regular so that all the integrals are well-defined, and refer to [2] for a rigorous derivation of the result.

Step 1. We write $f(t,x,v)$ for $S_t f(x,v)$ for all $(t,x,v) \in [0,\infty) \times G$, γf for the corresponding trace on $(0,\infty) \times \partial D \times \mathbb{R}^n$. Using Green's formula, we have, for all $t \geq 0$,

$$\frac{d}{dt}\int_G f(t,x,v)dvdx = -\int_G v \cdot \nabla_x f(t,x,v)dvdx = \int_{\partial D \times \mathbb{R}^n} \gamma f(t,x,v)(v \cdot n_x)dvd\zeta(x),$$

recalling that n_x is pointing towards D for all $x \in \partial D$. Since $\gamma_- f = K\gamma_+ f$, we conclude by (4.2.1) that

$$\frac{d}{dt}\langle S_t f\rangle = 0.$$

Step 2. To establish the contraction result, note first that, by triangle inequality, for almost all $t \geq 0$, $x \in \partial D$,

$$\begin{aligned}\int_{\{v\cdot n_x>0\}} |v \cdot n_x||K\gamma_+ f|(t,x,v)dv &\leq (1-\alpha(x))\int_{\{v\cdot n_x>0\}} |v \cdot n_x||\gamma_+ f|(t,x,\eta_x(v))dv \\ &+ \alpha(x)\int_{\{v\cdot n_x>0\}} |v \cdot n_x| M(x,v)\int_{\{v'\cdot n_x<0\}} |\gamma_+ f|(t,x,v')|v' \cdot n_x|dv'.\end{aligned}$$

We deduce that

$$\int_{\{v\cdot n_x>0\}} |v \cdot n_x||K\gamma_+ f|(t,x,v)dv \leq \int_{\{v\cdot n_x>0\}} |v \cdot n_x|K|\gamma_+ f|(t,x,v)dv,$$

and applying (4.2.1) with $|\gamma_+ f|$, we conclude that

$$\int_{\{v\cdot n_x>0\}} |v \cdot n_x||K\gamma_+ f|(t,x,v)dv \leq \int_{\{v\cdot n_x<0\}} |v \cdot n_x||\gamma_+ f|(t,x,v)dv. \tag{4.2.2}$$

Step 3. With similar computations to those of Step 1, one obtains,

$$\frac{d}{dt}\int_G |S_t f|dvdx = \int_{\partial_+ G} |\gamma_+ f(t)|(v \cdot n_x)dvd\zeta(x) + \int_{\partial_- G} |\gamma_- f(t)|(v \cdot n_x)dvd\zeta(x).$$

According to the boundary condition in (4.1.1), we have $\gamma_- f(t,x,v) = K\gamma_+ f(t,x,v)$. We conclude that

$$\frac{d}{dt}\|S_t f\|_{L^1} \leq 0,$$

using (4.2.2).

Step 4. To prove the positivity property, we use the previous results and the fact that $(S_t f)_- = \frac{|S_t f| - S_t f}{2}$. Assuming $f \geq 0$, we have $f_- \equiv 0$ and, for all $t \geq 0$, since $\langle S_t f\rangle = \langle f\rangle$ and since S_t is a contraction in L^1,

$$\begin{aligned}\|(S_t f)_-\|_{L^1} &= \int_G \frac{|S_t f| - S_t f}{2} dvdx \\ &= \frac{1}{2}\Big(\|S_t f\|_{L^1} - \langle S_t f\rangle\Big) \\ &\leq \frac{1}{2}\Big(\|f\|_{L^1} - \langle f\rangle\Big) = \int_G \frac{|f|-f}{2} dvdx = \|f_-\|_{L^1} = 0\end{aligned}$$

and since $(S_t f)_- \geq 0$ almost everywhere (a.e.) on G we deduce that $(S_t f)_- = 0$ a.e. on G. $\square$

4.3 Subgeometric Lyapunov condition

In this section, we derive several subgeometric Lyapunov inequalities that will play a key role in our proof of Theorem 4.1.1. Recall the definition (4.1.8) of σ. We first introduce a notation.

Notation 4.3.1. We define the map q from $\bar{D} \times \mathbb{R}^n$ to ∂D by

$$q(x, v) := x + \sigma(x, v)v, \tag{4.3.1}$$

for all $(x, v) \in \bar{D} \times \mathbb{R}^n$.

In terms of characteristics of the free transport equation, for $(x, v) \in \bar{D} \times \mathbb{R}^n$, $q(x, v)$ corresponds to the right limit in $\bar{D}$ of the characteristic with origin x directed by v. The real number $\sigma(x, v)$ corresponds to the time at which this characteristic reaches the boundary, if it started from x at time 0 with velocity v with $x \in D$ or $x \in \partial D, v \cdot n_x > 0$. If $x \in \partial D$ and v is not pointing towards the gas region (that is, (x, v) is already the right limit of the corresponding characteristic), $q(x, v)$ simply denotes x.

We recall a result on the derivative of σ inside G from Esposito, Guo, Kim and Marra [54, Lemma 2.3]. We parametrize locally D by a C^1 map $\xi : \mathbb{R}^n \to \mathbb{R}$, and D is locally $\{x \in \mathbb{R}^n, \xi(x) < 0\}$. By definition of $\sigma(x, v)$, for all $(x, v) \in G$, $\xi(x + \sigma(x, v)v) = 0$, and using the implicit function theorem, we find, for all index $j \in \{1, \dots, n\}$,

$$\partial_j \xi + \sum_{i=1}^n \partial_i \xi \frac{\partial \sigma(x, v)}{\partial x_j} v_i = 0.$$

Rearranging the terms and by definition of $n_{q(x,v)}$, we have:

$$\frac{\partial \sigma(x, v)}{\partial x_j} = -\frac{(n_{q(x,v)})_j}{v \cdot n_{q(x,v)}},$$

so that

$$\nabla_x \sigma(x, v) = -\frac{n_{q(x,v)}}{v \cdot n_{q(x,v)}}, \quad v \cdot \nabla_x \sigma(x, v) = -1. \tag{4.3.2}$$

This minus sign can be understood in the following way: since $\sigma(x,v)$ is the time needed for a particle in position $x \in \bar{D}$ with velocity $v \in \mathbb{R}^n$ at time $t=0$ to hit the boundary, moving the particle from x along the direction v reduces this time. Recall from (4.1.11) that $c_4 \in (0,1)$ is such that $(1-c_4)^4 = (1-c_0)$. For all $(x,v) \in \bar{D} \times \mathbb{R}^n$, we set

$$\langle x, v\rangle = \Big(e^2 + \frac{d(D)}{\|v\| c_4} - \sigma(x,-v)\Big),$$

so that $e^2 \leq \langle x,v\rangle$ and $\langle x,v\rangle \underset{v\to 0}{\sim} \frac{\kappa}{\|v\|}$ for some $\kappa > 0$. Moreover, for all $(x,v) \in \partial_+ G$, since $\sigma(x,-v) \leq \frac{d(D)}{\|v\|}$ by definition of $d(D)$, c_4 is chosen is such a way that we have for all $i \in [1,4]$,

$$(1-c_0)^{\frac{1}{i}}\Big(e^2 + \frac{d(D)}{\|v\| c_4}\Big) \leq (1-c_4)\Big(e^2 + \frac{d(D)}{\|v\| c_4}\Big) \leq e^2 + \frac{d(D)}{\|v\| c_4} - \sigma(x,-v). \tag{4.3.3}$$

We prove the following:

Lemma 4.3.1. *For a couple of weights (m_1, m_0), for $\epsilon \in (0,3)$, with any of the choices*

(1) $(m_1, m_0) = (\langle x,v\rangle^i \ln(\langle x,v\rangle)^{-1-\epsilon}, \langle x,v\rangle^{i-1}\ln(\langle x,v\rangle)^{-1-\epsilon}), \quad i \in [\![2, n+1]\!]$,

(2) $(m_1, m_0) = (\langle x,v\rangle^i, \langle x,v\rangle^{i-1}), \qquad i \in \{\frac{3}{2}, 2, \frac{5}{2}, \dots, \frac{2n+1}{2}\}$,

(3) $(m_1, m_0) = (\langle x,v\rangle \ln(\langle x,v\rangle)^{0.1}, \ln(\langle x,v\rangle)^{0.1})$,

there exist $C>0$, $b>0$ explicit, depending on (m_1,m_0), such that for all $T>0$, all $f \in L^1_{m_1}(G)$,

$$\|S_T f\|_{m_1} + C\int_0^T \|S_s f\|_{m_0} ds \leq \|f\|_{m_1} + b(1+T)\|f\|_{L^1}. \tag{4.3.4}$$

Proof. **Step 1.** Note that, for all $(x,v) \in G$, according to (4.3.2) and to the definition of $\langle x,v\rangle$, $(v\cdot\nabla_x\langle x,v\rangle) = -1$. We treat case (1) first. For $i \in [\![2,n+1]\!]$, $\epsilon \in (0,3)$,

$$\begin{aligned}(v\cdot\nabla_x)m_1 &= (v\cdot\nabla_x)\big(\langle x,v\rangle^i \ln(\langle x,v\rangle)^{-(1+\epsilon)}\big)\\ &= i(v\cdot\nabla_x\langle x,v\rangle)(\langle x,v\rangle)^{i-1}\ln(\langle x,v\rangle)^{-(1+\epsilon)}\\ &\quad + (v\cdot\nabla_x\langle x,v\rangle)(-(1+\epsilon))(\langle x,v\rangle)^{i-1}\ln(\langle x,v\rangle)^{-(2+\epsilon)}\\ &= (\langle x,v\rangle)^{i-1}\ln(\langle x,v\rangle)^{-(1+\epsilon)}\Big(-i + \frac{(1+\epsilon)}{\ln(\langle x,v\rangle)}\Big).\end{aligned}$$

Finally, $\ln(\langle x,v\rangle) \geq \ln(e^2) = 2$, hence

$$(v\cdot\nabla_x)m_1 \leq \Big(-i + \frac{1+\epsilon}{2}\Big)m_0,$$

and we set $C_i = i - \frac{1+\epsilon}{2} > 0$.

Step 2. Let $f \in L^1_{m_1}(G)$. We differentiate the $L^1_{m_1}(G)$ norm of f, and use Step 1. We first have, since n_x is the unit normal vector pointing towards the gas region, for $T > 0$, by Green's formula,

$$\frac{d}{dT} \int_G |S_T f| m_1 dv dx = \int_G |S_T f| (v \cdot \nabla_x m_1) dv dx + \int_{\partial D \times \mathbb{R}^n} (n_x \cdot v) m_1 (\gamma |S_T f|) dv d\zeta(x),$$

where we recall that $d\zeta$ denotes the induced volume form on ∂D. From [101, Corollary 1],

$$|\gamma S_t f(x,v)| = \gamma |S_t f|(x,v) \quad \text{a.e. in } \big((0,\infty) \times \partial_+ G\big) \cup \big((0,\infty) \times \partial_- G\big), \tag{4.3.5}$$

hence we will not distinguish between both values in what follows.

Applying Step 1 we find, using the boundary condition and (4.3.5),

$$\begin{aligned} &\frac{d}{dT} \int_G |S_T f| m_1 dv dx \\ &\qquad \le -C_i \int_G |S_T f| m_0 dv dx + \int_{\partial D \times \mathbb{R}^n} \gamma |S_T f| m_1 (v \cdot n_x) dv d\zeta(x) \end{aligned}$$

$$\begin{aligned} &\le -C_i \int_G |S_T f| m_0 dv dx + \int_{\partial D} \alpha(x) \int_{\{v \cdot n_x > 0\}} M(x,v) m_1(x,v) (v \cdot n_x) \\ &\quad \times \Big(\int_{\{v' \cdot n_x < 0\}} \gamma |S_T f|(x,v') |v' \cdot n_x| dv' \Big) dv d\zeta(x) \\ &\quad + \int_{\partial D} (1 - \alpha(x)) \int_{\{v \cdot n_x > 0\}} m_1(x,v) (v \cdot n_x) (\gamma |S_T f|(x, \eta_x(v))) dv d\zeta(x) \\ &\quad - \int_{\partial D} \int_{\{v \cdot n_x < 0\}} m_1(x,v) |v \cdot n_x| (\gamma |S_T f|(x,v)) dv d\zeta(x), \end{aligned} \tag{4.3.6}$$

with $\eta_x(v) = v - 2(v \cdot n_x) n_x$ for all $(x,v) \in \partial D \times \mathbb{R}^n$. We focus on the third and fourth terms of the last inequality of (4.3.6). We use in the third term, for $x \in \partial D$ fixed, the involutive change of variable $w = \eta_x(v)$ in the integral in v, so that $v = \eta_x(w)$, $|w \cdot n_x| = |v \cdot n_x|$, $\|w\| = \|v\|$ and $w \cdot n_x < 0$ (since $v \cdot n_x > 0$). Hence, for all $x \in \partial D$,

$$\int_{\{v \cdot n_x > 0\}} m_1(x,v) (v \cdot n_x) (\gamma |S_T f|(x, \eta_x(v))) dv = \int_{\{v \cdot n_x < 0\}} m_1(x, \eta_x(v)) |v \cdot n_x| (\gamma |S_T f|(x,v)) dv.$$

For $(x,v) \in \partial_+ G$, $\sigma(x, -\eta_x(v)) = 0$, therefore the sum of the third and fourth terms of (4.3.6) reads

$$\begin{aligned} A := \int_{\partial D} \int_{\{v : v \cdot n_x < 0\}} &|v \cdot n_x| (\gamma |S_T f|(x,v)) \Big\{ (1 - \alpha(x)) \Big(e^2 + \frac{d(D)}{\|v\| c_4} \Big)^i \ln \Big(e^2 + \frac{d(D)}{\|v\| c_4} \Big)^{-(1+\epsilon)} \\ &- \Big(e^2 + \frac{d(D)}{\|v\| c_4} - \sigma(x,-v) \Big)^i \ln \Big(e^2 + \frac{d(D)}{\|v\| c_4} - \sigma(x,-v) \Big)^{-(1+\epsilon)} \Big\} dv d\zeta(x). \end{aligned}$$

Note that, using $1 \leq i \leq 4$, we can control for all $(x,v) \in \partial_+ G$ the quantity $I(x,v)$ defined by

$$
\begin{aligned}
I(x,v) := (1-\alpha(x))\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)^i \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)^{-(1+\epsilon)} \\
- \Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^i \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^{-(1+\epsilon)}.
\end{aligned}
$$

Indeed, by definition of c_4 and since $\alpha(x) \geq c_0$ for all $x \in \partial D$,

$$
\begin{aligned}
I(x,v) &\leq (1-c_4)^4\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)^i \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)^{-(1+\epsilon)} \\
&\quad - \Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^i \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^{-(1+\epsilon)} \\
&\leq \Big((1-c_4)\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)\Big)^i \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)^{-(1+\epsilon)} \\
&\quad - \Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^i \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^{-(1+\epsilon)}.
\end{aligned}
$$

With the obvious bound $e^2 + \frac{d(D)}{\|v\|c_4} \geq e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)$ for all $(x,v) \in \partial_+ G$, we deduce easily,

$$
\ln\Big(e^2 + \frac{d(D)}{\|v\|c_4}\Big)^{-(1+\epsilon)} \leq \ln\Big(e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x,-v)\Big)^{-(1+\epsilon)}. \tag{4.3.7}
$$

From (4.3.3) and (4.3.7) we conclude that

$$
I(x,v) \leq 0,
$$

for all $(x,v) \in \partial_+ G$, and finally that

$$
A = \int_{\partial_+ G} |v \cdot n_x| (\gamma |S_T f|(x,v)) I(x,v) dv d\zeta(x) \leq 0.
$$

Applying this result to (4.3.6) we obtain

$$
\begin{aligned}
\frac{d}{dT}\int_G |S_T f| m_1 dv dx \leq -C_i \int_G |S_T f| m_0 dv dx + \int_{\partial_- G} \alpha(x) M(x,v) m_1(x,v)(v \cdot n_x) \\
\times \Big(\int_{\{v' \cdot n_x < 0\}} (\gamma |S_T f|(x,v'))|v' \cdot n_x| dv'\Big) dv d\zeta(x).
\end{aligned} \tag{4.3.8}
$$

Step 3. We focus on the second term on the right-hand side of (4.3.8). We have, for all $T > 0$,

$$
\partial_t |f| + v \cdot \nabla_x |f| = 0, \tag{4.3.9}
$$

a.e. on $(0,T) \times D \times \mathbb{R}^n$. Recall that $n_\cdot : x \to n_x$ is a $W^{1,\infty}$ map on $\bar{D}$. We multiply (4.3.9) by $(v \cdot n_x)$ and integrate it over $(0,T) \times D \times \{v \in \mathbb{R}^n, \|v\| \leq 1\}$ to obtain, using also Green's formula,

$$\begin{aligned}
0 &= \int_0^T \int_D \int_{\{\|v\|\le 1\}} \Big((\partial_t + v\cdot\nabla_x)|S_t f(x,v)|\Big)(v\cdot n_x)dvdxdt \\
&= \Big[\int_{D\times\{\|v\|\le 1\}} |S_t f(x,v)|(v\cdot n_x)dvdx\Big]_0^T \\
&\quad - \int_0^T \int_D \int_{\{\|v\|\le 1\}} |S_t f(x,v)|\Big(v\cdot\nabla_x(v\cdot n_x)\Big)dvdxdt \\
&\quad - \int_0^T \int_{\{\|v\|\le 1\}} \int_{\partial D} (\gamma|S_t f|(x,v))(v\cdot n_x)^2 d\zeta(x)dvdt,
\end{aligned}$$

where the minus sign in the last term comes from our definition of n_x as a vector pointing towards the gas region. Using the L^1 contraction from Theorem 4.2.1, we deduce from the previous computation

$$\begin{aligned}
\int_0^T \int_{\partial D} \int_{\{v\cdot n_x>0,\|v\|\le 1\}} (\gamma|S_T f|(x,v))(v\cdot n_x)^2 dvd\zeta(x)dt &\le 2\int_G |f(x,v)|dvdx \\
&\quad + T\|n.\|_{W^{1,\infty}} \int_G |f(x,v)|dvdx.
\end{aligned} \tag{4.3.10}$$

As a consequence of the boundary condition, and since $\alpha \ge c_0$ on ∂D, we obtain,

$$\begin{aligned}
&c_0 \int_0^T \int_{\partial D} \Big(\int_{\{v\cdot n_x>0,\|v\|\le 1\}} M(x,v)(v\cdot n_x)^2 dv \\
&\times \int_{\{v'\cdot n_x<0\}} (\gamma|S_T f|(x,v'))|v'\cdot n_x|dv'\Big)d\zeta(x)dt \le (2+T\|n.\|_{W^{1,\infty}})\|f\|_{L^1}.
\end{aligned} \tag{4.3.11}$$

Note that for a fixed $x\in\partial D$, $x\to\int_{\{v\cdot n_x>0,\|v\|\le 1\}} M(x,v)(v\cdot n_x)^2dv$ is continuous and positive since $x\to M(x,v)$ and $x\to n_x$ are continuous for all $v\in\mathbb{R}^n$. Since ∂D is compact, writing $\Delta = c_0 \min_{x\in\partial D}\int_{\{v\cdot n_x>0,\|v\|\le 1\}} M(x,v)(v\cdot n_x)^2dv>0$, we deduce from (4.3.11) that

$$\Delta \int_0^T \int_{\partial_+ G} (\gamma|S_t f|(x,v))|v\cdot n_x|dvd\zeta(x)dt \le \max(2,\|n.\|_{W^{1,\infty}})(1+T)\|f\|_{L^1}. \tag{4.3.12}$$

Step 4. We use the previous steps to conclude the proof of case (1). We integrate (4.3.8) over $(0,T)$. Using (4.3.12) and $\alpha\le 1$ on ∂D, we obtain:

$$\begin{aligned}
&\|S_T f\|_{m_1} + C_i\int_0^T \|S_s f\|_{m_0} ds \\
&\qquad \le \|f\|_{m_1} + \int_0^T \int_{\partial D} \Big(\int_{\{v\cdot n_x>0\}} M(x,v)m_1(x,v)|v\cdot n_x|\Big) \\
&\qquad\qquad \times \Big(\int_{\{v'\cdot n_x<0\}} (\gamma|S_s f|(x,v'))|v'\cdot n_x|dv'\Big)dvd\zeta(x)ds.
\end{aligned}$$

Note that, for $(x,v) \in \partial D \times \mathbb{R}^n$, $\sigma(x,-v) \le \frac{d(D)}{\|v\|}$, so that

$$\begin{aligned}\int_{\{v\cdot n_x>0\}} M(x,v)|v\cdot n_x| m_1(x,v)dv \le \int_{\{v\cdot n_x>0\}} \Big(\max_{x\in\partial D} M(x,v)\Big)\|v\|\Big(e^2+\frac{d(D)}{\|v\|c_4}\Big)^i \\ \times \ln\Big(e^2+\frac{d(D)}{\|v\|c_4}-\frac{d(D)}{\|v\|}\Big)^{-(1+\epsilon)} dv := a_1,\end{aligned} \tag{4.3.13}$$

where a_1 is independent of x and f and finite by choice of i,ϵ. Hence

$$\begin{aligned}&\|S_T f\|_{m_1} + C_i \int_0^T \|S_s f\|_{m_0} ds \\ &\qquad \le \|f\|_{m_1} + a_1 \int_0^T \int_{\partial_+ G} (\gamma|S_s f|(x,v'))|v'\cdot n_x| dv' d\zeta(x) ds.\end{aligned} \tag{4.3.14}$$

To conclude, we plug (4.3.12) into (4.3.14) to find

$$\begin{aligned}&\|S_T f\|_{m_1} + C_i \int_0^T \|S_s f\|_{m_0} ds \\ &\qquad \le \|f\|_{m_1} + \frac{a_1}{\Delta}\max(2,\|n.\|_{W^{1,\infty}})(1+T)\|f\|_{L^1}.\end{aligned} \tag{4.3.15}$$

Setting $b = \frac{a_1}{\Delta}\max(2,\|n.\|_{W^{1,\infty}})$ terminates the proof of case (1).

Step 5. In case (2), for all $(x,v)\in G$, $i \in \{\frac{3}{2}, 2, \dots, \frac{2n+1}{2}\}$, we have

$$(v\cdot\nabla_x)m_1 = v\cdot\nabla_x(\langle x,v\rangle^i) = -i\langle x,v\rangle^{i-1} = -im_0,$$

so that we can replicate the previous Steps 1 to 4 with the choice $C_i = i$ and a new value a_1 for (4.3.13).

Step 6. For case (3), for $\alpha = 0.1$, for all $(x,v)\in G$,

$$\begin{aligned} v\cdot\nabla_x(m_1(x,v)) &= -\ln(\langle x,v\rangle)^\alpha - \alpha\ln(\langle x,v\rangle)^{\alpha-1} \\ &= -\ln(\langle x,v\rangle)^\alpha(1+\alpha\ln(\langle x,v\rangle)^{-1}) \\ &\le -\ln(\langle x,v\rangle)^\alpha = -m_0(x,v),\end{aligned}$$

so that again the previous proof can be replicated with the value $C=1$ and a new value a_1 for (4.3.13). □

4.4 Doeblin-Harris condition

Recall that D is a C^2 bounded domain. In this section, we prove the Doeblin-Harris condition, Theorem 4.4.1. For any two points x and y at the boundary ∂D of D, we write

$$]x,y[= \{tx+(1-t)y, t\in]0,1[\}.$$

Definition 4.4.1. For $(x, y) \in (\partial D)^2$, we write $x \leftrightarrow y$ and say that x and y see each other if $]x, y[\subset D$, $n_x \cdot (y - x) > 0$, $n_y \cdot (x - y) > 0$.

Since M is radial in the second variable, we write $M(x, r) = \frac{\tilde{c}(x)}{(2\pi\theta(x))^{\frac{n}{2}}} e^{-\frac{r^2}{2\theta(x)}}$ for all $r \in \mathbb{R}$, $x \in \partial D$ see (4.1.3) for the definition of $\tilde{c}$, so that $M(x, v) = M(x, \|v\|)$ for all vector $v \in \mathbb{R}^n$. Possible ambiguity can always be solved by checking the living space of the variable considered.

We will crucially use this result on C^1 bounded domains from Evans:

Proposition 4.4.1 (Proposition 1.7 in [55]). *For all C^1 bounded domain C, there exist an integer P and a finite set $\Delta' \subset \partial C$ for which the following holds: for all $z', z'' \in \partial C$, there exist $z_0, \dots, z_P$ with $z' = z_0$, $z'' = z_P$, $\{z_1, \dots z_{P-1}\} \subset \Delta'$, and $z_k \leftrightarrow z_{k+1}$ for $0 \leq k \leq P - 1$.*

We now state the main result of this section. Recall that $(S_t)_{t\geq 0}$ is the semigroup associated to (4.1.1) as introduced in Section 4.2.

Theorem 4.4.1 (Doeblin-Harris condition). *For any $R > 0$, there exist $T(R) > 0$ and a non-negative measure ν on G with $\nu \not\equiv 0$ such that for all (x, v) in G, for all $f_0 \in L^1(G)$, $f_0 \geq 0$,*

$$S_{T(R)} f_0(x, v) \geq \nu(x, v) \int_{B_R} f_0(y, w) dw dy, \tag{4.4.1}$$

with $B_R = \{(y, w) \in G : \sigma(y, w) \leq R\}$. Moreover there exists $\kappa > 0$ such that for all $R > 0$, $T(R) = \kappa R$.

Proof. We only treat the case $n = 3$, as the case of $n = 2$ follows from similar (and easier) computations. For all $t \geq 0$, $(x, v) \in \bar{D} \times G$, we write $f(t, x, v) = S_t f_0(x, v)$. For simplicity, we write $f(t, x, v) = \gamma f(t, x, v)$ for all $(t, x, v) \in (0, \infty) \times \partial D \times \mathbb{R}^n$. Recall that this trace is well-defined, see Section 4.2.

Step 1. We let $(t, x, v) \in (0, \infty) \times G$ and compute a first inequality for $f(t, x, v)$. Recall the definition of σ, (4.1.8) and q, (4.3.1). From the characteristic method we have

$$f(t, x, v) = f_0(x - tv, v)\mathbf{1}_{\{t<\sigma(x,-v)\}} + f(t - \sigma(x, -v), q(x, -v), v)\mathbf{1}_{\{t\geq\sigma(x,-v)\}}.$$

Set $y_0 = q(x,-v)$, $\tau_0 = \sigma(x,-v)$. We have, using the boundary condition and the characteristics of the free-transport equation, along with the positivity of f_0, with c_0 given by (4.1.10),

$$\begin{aligned}
f(t,x,v) &\geq \mathbf{1}_{\{\tau_0 \leq t\}} f(t-\tau_0, y_0, v) \\
&\geq \mathbf{1}_{\{\tau_0 \leq t\}} c_0 M(y_0,v) \int_{\{v_0 \in \mathbb{R}^n, v_0 \cdot n_{y_0} < 0\}} f(t-\tau_0, y_0, v_0) |v_0 \cdot n_{y_0}| dv_0 \\
&\geq \mathbf{1}_{\{\tau_0 \leq t\}} c_0 M(y_0,v) \int_{\{v_0 \cdot n_{y_0} < 0\}} f(t-\tau_0 - \sigma(y_0,-v_0), q(y_0,-v_0), v_0) \\
&\quad \times \mathbf{1}_{\{\tau_0 + \sigma(y_0,-v_0) \leq t\}} |v_0 \cdot n_{y_0}| dv_0 \\
&\geq \mathbf{1}_{\{\tau_0 \leq t\}} c_0^2 M(y_0,v) \int_{\{v_0 \cdot n_{y_0} < 0\}} M(q(y_0,-v_0), v_0) \mathbf{1}_{\{\tau_0 + \sigma(y_0,-v_0) \leq t\}} |v_0 \cdot n_{y_0}| \\
&\quad \times \int_{\{v_1 \cdot n_{q(y_0,-v_0)} < 0\}} f(t-\tau_0-\sigma(y_0,-v_0), q(y_0,-v_0), v_1) |v_1 \cdot n_{q(y_0,-v_0)}| dv_1 dv_0.
\end{aligned}$$

We write v_0 in spherical coordinates $(r,\phi,\theta) \in \mathbb{R}_+ \times [-\pi,\pi] \times [0,\pi]$ in the space directed by the vector n_{y_0}. We write $u = u(\phi,\theta)$ for the unit vector corresponding to the direction of v_0. The condition $v_0 \cdot n_{y_0} < 0$ is equivalent to $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and we obtain, using also that $q(y_0,-v_0) = q(y_0,-u)$ as it is independent of $\|v_0\|$,

$$\begin{aligned}
f(t,x,v) \geq{}& \mathbf{1}_{\{\tau_0 \leq t\}} c_0^2 M(y_0,v) \int_0^\infty \int_{-\pi/2}^{\pi/2} \int_0^\pi M(q(y_0,-u), r) \mathbf{1}_{\{\tau_0 + \frac{\sigma(y_0,-u)}{r} \leq t\}} |u \cdot n_{y_0}| \\
&\times \sin(\theta) r^3 \int_{\{v_1 \cdot n_{q(y_0,-u)} < 0\}} f(t-\tau_0 - \frac{\sigma(y_0,-u)}{r}, q(y_0,-u), v_1) |v_1 \cdot n_{q(y_0,-u)}| dv_1 d\theta d\phi dr.
\end{aligned}$$

We now use the change of variable $(y_1, \tau_1) = (q(y_0,-u), \sigma(y_0,-ru))$. The inverse of the determinant of the Jacobian matrix was derived by Esposito et al. [54, Lemma 2.3] and is given by (in the case where $y_1 \leftrightarrow y_0$)

$$\frac{\tau_1^3 r \sin(\theta) |\partial_3 \xi(y_1)|}{|n_{y_1} \cdot u| |\nabla_x \xi(y_1)|},$$

where ξ is the C^1 function that locally parametrizes D, i.e. $D = \{y : \xi(y) < 0\}$, with the further assumption (which can be done without loss of generality) that $\partial_3 \xi(y_1) \neq 0$. Finally u is the unit vector giving the direction going from y_1 to y_0, hence

$$u = \frac{y_0 - y_1}{\|y_0 - y_1\|} \qquad \text{and} \qquad r = \frac{\|y_1 - y_0\|}{\tau_1}.$$

Setting, for $a \in \partial D$,

$$U_a := \{y \in \partial D, y \leftrightarrow a\},$$

we obtain from this change of variables

$$
\begin{aligned}
f(t,x,v) \geq\ & \mathbf{1}_{\{\tau_0 \leq t\}} c_0^2 M(y_0,v) \int_0^{t-\tau_0} \int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big) |u \cdot n_{y_0}| \frac{\|y_1-y_0\|^2}{\tau_1^5} \\
& \times |u \cdot n_{y_1}| \frac{|\nabla_x \xi(y_1)|}{|\partial_3 \xi(y_1)|} \int_{\{v_1 \cdot n_{y_1} < 0\}} f(t-\tau_0-\tau_1, y_1, v_1) |v_1 \cdot n_{y_1}| dv_1 dy_1 d\tau_1 \\
\geq\ & \mathbf{1}_{\{\tau_0 \leq t\}} c_0^2 M(y_0,v) \int_0^{t-\tau_0} \int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big) |(y_1-y_0) \cdot n_{y_0}| \frac{|\nabla_x \xi(y_1)|}{|\partial_3 \xi(y_1)|} \\
& \times \frac{|(y_0-y_1)\cdot n_{y_1}|}{\tau_1^5} \int_{\{v_1 \cdot n_{y_1} < 0\}} f(t-\tau_0-\tau_1-\sigma(y_1,-v_1), q(y_1,-v_1), v_1) \\
& \times |v_1 \cdot n_{y_1}| \mathbf{1}_{\{\sigma(y_1,-v_1)+\tau_1+\tau_0 \leq t\}} dv_1 dy_1 d\tau_1.
\end{aligned}
$$

Using again the boundary condition, we have:

$$
\begin{aligned}
f(t,x,v) \geq\ & \mathbf{1}_{\{\tau_0 \leq t\}} c_0^3 M(y_0,v) \int_0^{t-\tau_0} \int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big) |(y_1-y_0) \cdot n_{y_0}| |(y_0-y_1)\cdot n_{y_1}| \frac{1}{\tau_1^5} \\
& \times \int_{\{v_1 \cdot n_{y_1} < 0\}} |v_1 \cdot n_{y_1}| \mathbf{1}_{\{\sigma(y_1,-v_1)+\tau_1+\tau_0 \leq t\}} M(q(y_1,-v_1), v_1) \\
& \times \Big(\int_{\{v_2 \cdot n_{q(y_1,-v_1)} < 0\}} f(t-\tau_0-\tau_1-\sigma(y_1,-v_1), q(y_1,-v_1), v_2) \\
& \times |v_2 \cdot n_{q(y_1,-v_1)}| dv_2 \Big) dv_1 d\zeta(y_1) d\tau_1,
\end{aligned}
$$

with $d\zeta$ the surface measure of ∂D, which is given by $d\zeta(y) = \frac{|\nabla_x \xi(y)|}{|\partial_3 \xi(y)|} dy$ for any $y \in \partial D$.

Step 2. We use the same method as in Step 1 $P-2$ times and make a change of variable to obtain a first integral over a subset of $D \times \mathbb{R}^n$.

Repeating the previous computation $P-2$ times, where $P \in \mathbb{Z}^+$ is given by Proposition 4.4.1, we obtain,

$$
\begin{aligned}
f(t,x,v) \geq\ & \mathbf{1}_{\{\tau_0 \leq t\}} c_0^{P+1} M(y_0,v) \int_0^{t-\tau_0} \int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big) |(y_1-y_0)\cdot n_{y_0}| \frac{1}{\tau_1^5} \\
& \times |(y_0-y_1)\cdot n_{y_1}| \int_0^{t-\tau_0-\tau_1} \int_{U_{y_1}} M\Big(y_2, \frac{y_2-y_1}{\tau_2}\Big) |(y_2-y_1)\cdot n_{y_1}| \frac{1}{\tau_2^5} \\
& \times |(y_1-y_2)\cdot n_{y_2}| \times \dots \\
& \times \int_0^{t-\tau_0-\dots-\tau_{P-1}} \int_{U_{y_{P-1}}} M\Big(y_P, \frac{y_P-y_{P-1}}{\tau_P}\Big) |(y_P-y_{P-1})\cdot n_{y_{P-1}}| \\
& \times |(y_{P-1}-y_P)\cdot n_{y_P}| \frac{1}{\tau_P^5} \\
& \times \int_{\{v_P \cdot n_{y_P} < 0\}} f(t-\tau_0-\dots-\tau_P, y_P, v_P) |v_P \cdot n_{y_P}| dv_P d\zeta(y_P) d\tau_P \dots d\zeta(y_1) d\tau_1.
\end{aligned}
$$

We then use that, on $\{t \geq \tau_0 + \cdots + \tau_P\}$,

$$\begin{aligned} &f(t-\tau_0-\cdots-\tau_P, y_P, v_P) \\ &\quad \geq f_0(y_P-(t-\tau_0-\cdots-\tau_P)v_P, v_P)\mathbf{1}_{\{t-\tau_0-\cdots-\tau_P-\sigma(y_P,-v_P)\leq 0\}}, \end{aligned}$$

and obtain from the previous inequality,

$$\begin{aligned} f(t,x,v) \geq \mathbf{1}_{\{\tau_0\leq t\}} c_0^{P+1} M(y_0,v) &\int_0^{t-\tau_0}\int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big)|(y_1-y_0)\cdot n_{y_0}|\frac{1}{\tau_1^5} \\ &\times |(y_0-y_1)\cdot n_{y_1}|\int_0^{t-\tau_0-\tau_1}\int_{U_{y_1}} M\Big(y_2, \frac{y_2-y_1}{\tau_2}\Big)|(y_2-y_1)\cdot n_{y_1}|\frac{1}{\tau_2^5} \\ &\times |(y_1-y_2)\cdot n_{y_2}| \times \dots \\ &\times \int_0^{t-\tau_0-\cdots-\tau_{P-1}}\int_{U_{y_{P-1}}} M\Big(y_P, \frac{y_P-y_{P-1}}{\tau_P}\Big)|(y_P-y_{P-1})\cdot n_{y_{P-1}}| \\ &\times |(y_{P-1}-y_P)\cdot n_{y_P}|\frac{1}{\tau_P^5}\Big(\int_{\{v_P\cdot n_{y_P}<0\}} f_0(y_P-(t-\tau_0-\cdots-\tau_P)v_P, v_P) \\ &\times |v_P\cdot n_{y_P}|\mathbf{1}_{\{\tau_0+\cdots+\tau_P+\sigma(y_P,-v_P)\geq t\}}dv_P\Big)d\zeta(y_P)d\tau_P\dots d\zeta(y_1)d\tau_1. \end{aligned}$$

We set $z=\psi(y_P,\tau_P)=y_P-(t-\tau_0-\cdots-\tau_P)v_P$ (i.e. we compute the result of the change of variable from (y_P,τ_P) to z). The map ψ is a C^1 diffeomorphism with

$$\begin{aligned} \psi : &\Big\{(y_P,\tau_P)\in\partial D\times\mathbb{R}_+ : \sigma(y_P,-v_P) > t-\tau_0-\cdots-\tau_P, y_P\leftrightarrow y_{P-1}\Big\} \\ &\to \Big\{z\in D : q(z,v_P)\leftrightarrow y_{P-1}, \sigma(z,v_P)+\tau_0+\cdots+\tau_{P-1}\leq t\Big\}. \end{aligned}$$

With this change of variable, $y_P=q(z,v_P)$. Moreover, $t-\tau_0-\cdots-\tau_P=\sigma(z,v_P)$ by definition of z, so that

$$\tau_P = t-\tau_0-\cdots-\tau_{P-1}-\sigma(z,v_P).$$

The inverse of the Jacobian is $|v_P \cdot n_{y_P}|$, see Esposito et al. [54, Lemma 2.3]. Therefore,

$$
\begin{aligned}
f(t,x,v) \geq\ & \mathbf{1}_{\{\tau_0 \leq t\}} c_0^{P+1} M(y_0,v) \int_0^{t-\tau_0} \int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big) |(y_1-y_0)\cdot n_{y_0}| \frac{1}{\tau_1^5} \\
& \times |(y_0-y_1)\cdot n_{y_1}| \int_0^{t-\tau_0-\tau_1} \int_{U_{y_1}} M\Big(y_2, \frac{y_2-y_1}{\tau_2}\Big) |(y_2-y_1)\cdot n_{y_1}| \frac{1}{\tau_2^5} \\
& \times |(y_1-y_2)\cdot n_{y_2}| \times \dots \\
& \times \int_0^{t-\tau_0-\dots-\tau_{P-2}} \int_{U_{y_{P-2}}} M\Big(y_{P-1}, \frac{y_{P-1}-y_{P-2}}{\tau_{P-1}}\Big) |(y_{P-1}-y_{P-2})\cdot n_{y_{P-2}}| \\
& \times |(y_{P-2}-y_{P-1})\cdot n_{y_{P-1}}| \frac{1}{\tau_{P-1}^5} \Big\{ \int_G \frac{|(y_{P-1}-q(z,v_P))\cdot n_{q(z,v_P)}|}{(t-\tau_0-\dots-\tau_{P-1}-\sigma(z,v_P))^5} \\
& \times M\Big(q(z,v_P), \frac{y_{P-1}-q(z,v_P)}{t-\tau_0-\dots-\tau_{P-1}-\sigma(z,v_P)}\Big) \\
& \times |(q(z,v_P)-y_{P-1})\cdot n_{y_{P-1}}| \mathbf{1}_{\{q(z,v_P)\leftrightarrow y_{P-1}\}} \mathbf{1}_{\{\sigma(z,v_P)+\tau_{P-1}+\dots+\tau_0 \leq t\}} \\
& \times f_0(z,v_P)\,dv_P\,dz \Big\} d\zeta(y_{P-1})\,d\tau_{P-1}\dots d\zeta(y_1)\,d\tau_1.
\end{aligned}
$$

Using Tonelli's theorem, we then have

$$
\begin{aligned}
f(t,x,v) \geq\ & \mathbf{1}_{\{\tau_0 \leq t\}} c_0^{P+1} M(y_0,v) \int_{D\times\mathbb{R}^n} f_0(z,v_P) \\
& \times \int_0^{t-\tau_0} \int_{U_{y_0}} M\Big(y_1, \frac{y_1-y_0}{\tau_1}\Big) |(y_1-y_0)\cdot n_{y_0}| \frac{1}{\tau_1^5} \\
& \times |(y_0-y_1)\cdot n_{y_1}| \int_0^{t-\tau_0-\tau_1} \int_{U_{y_1}} M\Big(y_2, \frac{y_2-y_1}{\tau_2}\Big) |(y_2-y_1)\cdot n_{y_1}| \frac{1}{\tau_2^5} \\
& \times |(y_1-y_2)\cdot n_{y_2}| \times \dots \\
& \times \int_0^{t-\tau_0-\dots-\tau_{P-2}} \int_{U_{y_{P-2}}} M\Big(y_{P-1}, \frac{y_{P-1}-y_{P-2}}{\tau_{P-1}}\Big) |(y_{P-1}-y_{P-2})\cdot n_{y_{P-2}}| \\
& \times |(y_{P-2}-y_{P-1})\cdot n_{y_{P-1}}| \frac{1}{\tau_{P-1}^5} \frac{|(y_{P-1}-q(z,v_P))\cdot n_{q(z,v_P)}|}{(t-\tau_0-\dots-\tau_{P-1}-\sigma(z,v_P))^5} \\
& \times M\Big(q(z,v_P), \frac{y_{P-1}-q(z,v_P)}{t-\tau_0-\dots-\tau_{P-1}-\sigma(z,v_P)}\Big) \\
& \times |(q(z,v_P)-y_{P-1})\cdot n_{y_{P-1}}| \mathbf{1}_{\{q(z,v_P)\leftrightarrow y_{P-1}\}} \mathbf{1}_{\{\sigma(z,v_P)+\tau_{P-1}+\dots+\tau_0 \leq t\}} \\
& \times d\zeta(y_{P-1})\,d\tau_{P-1}\dots d\zeta(y_1)\,d\tau_1\,dv_P\,dz,
\end{aligned}
\tag{4.4.2}
$$

Step 3. We choose the value of t and control all the time integrals in (4.4.2).

Let $R > 0$ and set $t = (2P+2)R$, $\tau_0 \in (R, 2R)$, i.e., for all $(x,v) \in G$ with $\sigma(x,-v) \notin (R,2R)$, we simply set $\nu(x,v) = 0$. Note that for any $R > 0$, one can find a couple $(x,v) \in G$ such that $\sigma(x,-v) \in (R,2R)$. For all $i \in \{1,\dots,P-1\}$, we lower bound the integral with respect to τ_i by the integral over $(R,2R)$. We also lower bound the integral with respect to (z,v_P) by an integral over B_R, where $B_R := \{(z, v_P) \in G : \sigma(z, v_P) \leq R\}$. For $\tau_0, \dots, \tau_{P-1} \in (R, 2R)$, $\sigma(z, v_P) \leq R$ and $t = (2P+2)R$, we have first

$$\begin{aligned}(2P+2)R - 2PR - R = R &\leq t - \tau_0 - \tau_1 - \dots - \tau_{P-1} - \sigma(z, v_P) \\ &\leq (2P+2)R - PR = (P+2)R,\end{aligned}$$

and thus, with those choices,

$$\mathbf{1}_{\{\tau_0 + \dots \tau_{P-1} + \sigma(z, v_P) \leq t\}} = 1.$$

Moreover, recalling that for all $i \in \{1, \dots, P-1\}$, the integration interval for τ_i in the equation (4.4.2) is $[0, t - \tau_0 - \tau_1 - \dots - \tau_{i-1}]$, and since

$$t - \tau_0 - \tau_1 - \dots - \tau_{i-1} \geq (2P+2)R - 2iR = 2R + 2(P-i)R \geq 2R,$$

the lower bound detailed above using an integral over $[R, 2R]$ for τ_i is legitimate. We set for all $a > 0$,

$$\underline{M}(a) = \min_{x \in \partial D, \tau \in [R, 2R]} M\Big(x, \frac{a}{\tau}\Big) > 0 \quad \text{and} \quad \underline{\underline{M}}(a) = \min_{\substack{x \in \partial D, \\ \tau \in [R, (P+2)R]}} M\Big(x, \frac{a}{\tau}\Big) > 0,$$

where the positivity is obtained by continuity of M and compactness. Applying those lower bounds, we obtain from (4.4.2)

$$\begin{aligned}f(t, x, v) \geq\ & \mathbf{1}_{\{\tau_0 \in [R, 2R]\}} c_0^{P+1} M(y_0, v) \int_{B_R} f_0(z, v_P) \\ & \times \int_R^{2R} \int_{U_{y_0}} \underline{M}(\|y_1 - y_0\|) |(y_1 - y_0) \cdot n_{y_0}| \frac{1}{\tau_1^5} |(y_0 - y_1) \cdot n_{y_1}| \\ & \times \int_R^{2R} \int_{U_{y_1}} \underline{M}(\|y_2 - y_1\|) |(y_2 - y_1) \cdot n_{y_1}| \frac{1}{\tau_2^5} \times |(y_1 - y_2) \cdot n_{y_2}| \times \dots \\ & \times \int_R^{2R} \int_{U_{y_{P-2}}} \underline{M}(\|y_{P-1} - y_{P-2}\|) |(y_{P-1} - y_{P-2}) \cdot n_{y_{P-2}}| \\ & \times |(y_{P-2} - y_{P-1}) \cdot n_{y_{P-1}}| \frac{1}{\tau_{P-1}^5} \frac{|(y_{P-1} - q(z, v_P)) \cdot n_{q(z, v_P)}|}{((P+2)R)^5} \\ & \times \underline{\underline{M}}(\|y_{P-1} - q(z, v_P)\|) |(q(z, v_P) - y_{P-1}) \cdot n_{y_{P-1}}| \\ & \times \mathbf{1}_{\{q(z, v_P) \leftrightarrow y_{P-1}\}} d\zeta(y_{P-1}) d\tau_{P-1} \dots d\zeta(y_1) d\tau_1 dv_P dz.\end{aligned}$$

Since, $\int_R^{2R} \frac{1}{t^5} dt < \infty$, one finds from (4.4.2), with $\delta > 0$ explicit, depending on R,

$$
\begin{aligned}
f(t,x,v) \geq \mathbf{1}_{\{\tau_0 \in [R,2R]\}} \delta M(y_0,v) \int_{B_R} & f_0(z,v_P) \\
&\times \int_{U_{y_0}} \underline{M}(\|y_1-y_0\|)|(y_1-y_0)\cdot n_{y_0}||(y_0-y_1)\cdot n_{y_1}| \\
&\times \int_{U_{y_1}} \underline{M}(\|y_2-y_1\|)|(y_2-y_1)\cdot n_{y_1}||(y_1-y_2)\cdot n_{y_2}| \times \dots \\
&\times \int_{U_{y_{P-2}}} \underline{M}(\|y_{P-1}-y_{P-2}\|)|(y_{P-1}-y_{P-2})\cdot n_{y_{P-2}}||(y_{P-2}-y_{P-1})\cdot n_{y_{P-1}}| \\
&\times |(y_{P-1}-q(z,v_P))\cdot n_{q(z,v_P)}||(q(z,v_P)-y_{P-1})\cdot n_{y_{P-1}}| \\
&\times \underline{\underline{M}}(\|y_{P-1}-q(z,v_P)\|)\mathbf{1}_{\{q(z,v_P)\leftrightarrow y_{P-1}\}} d\zeta(y_{P-1})\dots d\zeta(y_1) dv_P dz.
\end{aligned}
\tag{4.4.3}
$$

Step 4. For a couple of points $(a,b) \in (\partial D)^2$, we set

$$
\begin{aligned}
h_P(a,b) = \int_{U_a} & \underline{M}(\|y_1-a\|)|(y_1-a)\cdot n_a||(a-y_1)\cdot n_{y_1}| \\
&\times \int_{U_{y_1}} \underline{M}(\|y_2-y_1\|)|(y_2-y_1)\cdot n_{y_1}||(y_1-y_2)\cdot n_{y_2}| \times \dots \\
&\times \int_{U_{y_{P-2}}} \underline{M}(\|y_{P-1}-y_{P-2}\|)|(y_{P-1}-y_{P-2})\cdot n_{y_{P-2}}||(y_{P-2}-y_{P-1})\cdot n_{y_{P-1}}| \\
&\times |(y_{P-1}-b)\cdot n_b||(b-y_{P-1})\cdot n_{y_{P-1}}|\underline{\underline{M}}(\|y_{P-1}-b\|) \\
&\times \mathbf{1}_{\{b\leftrightarrow y_{P-1}\}} d\zeta(y_{P-1})\dots d\zeta(y_1).
\end{aligned}
$$

In this step, we want to show that, for all $y_0 \in \partial D$, $b \to h_P(y_0,b)$ is lower semicontinuous and positive. We can rewrite h_P as

$$
h_P(a,b) = \int_{\{(y_1,\dots,y_{P-1})\in \bar{D}(a,b)\}} N(a,y_1,\dots,y_{P-1},b) d\zeta(y_1)\dots d\zeta(y_{P-1}),
$$

with

$$
\bar{D}(a,b) = \{(y_1,\dots,y_{P-1}) \in (\partial D)^{P-1} : y_1 \leftrightarrow a, y_2 \leftrightarrow y_1, \dots, y_{P-1} \leftrightarrow y_{P-2}, b \leftrightarrow y_{P-1}\},
$$

and

$$
\begin{aligned}
N(a,y_1,\dots y_{P-1},b) = {} & \underline{M}(\|y_1-a\|)|(y_1-a)\cdot n_a||(a-y_1)\cdot n_{y_1}| \\
&\times \underline{M}(\|y_2-y_1\|)|(y_2-y_1)\cdot n_{y_1}||(y_1-y_2)\cdot n_{y_2}| \times \dots \\
&\times \underline{M}(\|y_{P-1}-y_{P-2}\|)|(y_{P-1}-y_{P-2})\cdot n_{y_{P-2}}||(y_{P-2}-y_{P-1})\cdot n_{y_{P-1}}| \\
&\times |(y_{P-1}-b)\cdot n_b||(b-y_{P-1})\cdot n_{y_{P-1}}|\underline{\underline{M}}(\|y_{P-1}-b\|).
\end{aligned}
$$

By regularity assumption, if $(z_1, z_2) \in (\partial D)^2$ with $z_1 \leftrightarrow z_2$, there exists $\epsilon > 0$ such that $B(z_1, \epsilon) \cap \partial D \leftrightarrow B(z_2, \epsilon) \cap \partial D$, i.e. for all $p \in B(z_1, \epsilon) \cap \partial D$, all $q \in B(z_2, \epsilon) \cap \partial D$, we have $p \leftrightarrow q$, see [10, Lemma 38]. Combining this with the statement of Proposition 4.4.1, we find that

$$\mathcal{H}(\bar{D}(a, b)) > 0, \tag{4.4.4}$$

where we recall that $\mathcal{H}$ denotes the $n-1$ dimensional Hausdorff measure.

We set, for all $a \in \partial D$,

$$\bar{D}(a) = \{(y_1, \ldots, y_{P-1}) \in (\partial D)^{P-1} : y_1 \leftrightarrow a, y_2 \leftrightarrow y_1, \ldots, y_{P-1} \leftrightarrow y_{P-2}\}.$$

For $a \in \partial D$ and $(y_1, \ldots y_{P-1}) \in \bar{D}(a)$, for all $b \in \partial D$ such that $b \leftrightarrow y_{P-1}$, we have $N(a, y_1, \ldots, y_{P-1}, b) > 0$ according to Definition 4.4.1. Using (4.4.4), one concludes that for all $(a, b) \in (\partial D)^2$, $h_P(a, b) > 0$. Moreover, the map $b \to N(a, y_1, \ldots, y_{P-1}, b)$ is continuous according to the definition of $\underline{\underline{M}}$ through M and since $z \to n_z$ is continuous.

Note that, according to [55, Lemma 2.3], for any $z \in \partial D$, the set U_z is open and non-empty. Hence for all $y_{P-1} \in \partial D$, $b \to \mathbf{1}_{U_{y_{P-1}}}(b)$ is lower semicontinuous. We conclude that for all $a \in \partial D$, $(y_1, \ldots, y_{P-1}) \in \bar{D}(a)$,

$$b \to N(a, y_1, \ldots, y_{P-1}, b)\mathbf{1}_{\{y_{P-1} \leftrightarrow b\}}$$

is lower semicontinuous. For $a \in \partial D$, $(b_n)_{n \geq 0}$ a sequence of ∂D converging towards $b \in \partial D$, we obtain

$$\begin{aligned} 0 < h_P(a, b) &\leq \int_{\bar{D}(a)} \liminf_{n \to \infty} N(a, y_1, \ldots, y_{P-1}, b_n)\mathbf{1}_{\{y_{P-1} \leftrightarrow b_n\}} d\zeta(y_1) \ldots d\zeta(y_{P-1}) \\ &\leq \liminf_{n \to \infty} h_P(a, b_n), \end{aligned}$$

using Fatou's lemma. Hence $\partial D \ni b \to h_P(a, b)$ is also lower semicontinuous and positive for all $a \in \partial D$.

Step 5. We conclude the proof using Step 4. Since ∂D is compact, we deduce from the previous step that for all $a \in \partial D$,

$$\mu(a) := \inf_{b \in \partial D} h_P(a, b) > 0.$$

With this at hand, we have

$$\begin{aligned} f(t, x, v) &\geq \mathbf{1}_{\{\tau_0 \in [R, 2R]\}} \delta M(y_0, v) \int_{(z, v_P) \in B_R} f_0(z, v_P) h_P(y_0, q(z, v_P)) dv_P dz \\ &\geq \mathbf{1}_{\{\tau_0 \in [R, 2R]\}} \delta M(y_0, v) \mu(y_0) \int_{B_R} f_0(z, w) dw dz, \end{aligned}$$

and, recalling that $\tau_0 = \sigma(x, -v)$, $y_0 = q(x, -v)$, we set

$$\nu(x, v) = \delta M(q(x, -v), v)\mu(q(x, -v))\mathbf{1}_{\{\sigma(x,-v)\in[R,2R]\}},$$

and $T(R) = t = (2P + 2)R$ to complete the proof. □

Remark 4.4.1. Although we use a compactness argument to derive μ, for a given domain D, we believe that one may find an explicit lower bound for h_P defined in Step 4 of the previous proof using the geometry of D. Note however that this computation might be very difficult. With such constructive lower bound, the constants in Theorem 4.1.1 and Corollary 4.1.1 become explicit.

As an example of an easy case where an explicit lower bound on h_P can be find, assume that $n = 2$ and D is the unit disk, so that $U_z = \partial D \setminus \{z\}$ for all $z \in \partial D$. We can clearly take $P = 2$ in Proposition 4.4.1 and we have, for all $(a, b) \in (\partial D)^2$,

$$\begin{aligned} h_P(a, b) &= \int_{\partial D} \underline{M}(\|y - a\|)|(y - a) \cdot n_a||(a - y) \cdot n_y|\underline{\underline{M}}(\|y - b\|)|(y - b) \cdot n_b||(b - y) \cdot n_y|dy \\ &\geq \int_{H_{a,b}} \underline{M}(\|y - a\|)|(y - a) \cdot n_a||(a - y) \cdot n_y|\underline{\underline{M}}(\|y - b\|)|(y - b) \cdot n_b||(b - y) \cdot n_y|dy \\ &\geq \kappa, \end{aligned}$$

where

$$H_{a,b} = \Big\{y \in \partial D, y \cdot a \wedge y \cdot b \geq \frac{\sqrt{2}}{2}\Big\},$$

is a set whose Hausdorff measure in ∂D is uniformly bounded from below by $\frac{1}{2}$, and such that for all $y \in H(a, b)$, $d(D) \geq \|y - a\|, \|y - b\| \geq \sqrt{2 + \sqrt{2}}$ so that κ is a positive constant independent of a and b.

Recall that $\langle x, v\rangle = (e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x, -v))$ for all $(x, v) \in \bar{D} \times \mathbb{R}^n$, with $c_4 < 1$. We conclude this section by stating a similar result for the level sets of $\langle ., .\rangle$.

Corollary 4.4.1. *There exists $R_0 > 0$ such that for any $R \geq R_0$, for $T(R) > 0$ and ν non-negative measure on G given by Theorem 4.4.1, for all (x, v) in G, for all $f_0 \in L^1(G)$, $f_0 \geq 0$, we have*

$$S_{T(R)} f_0(x, v) \geq \nu(x, v) \int_{\Gamma_R} f_0(y, w)dwdy, \tag{4.4.5}$$

with $\Gamma_R = \{(y, w) \in G, \langle y, w\rangle \leq R\}$. Moreover there exists $\xi > 0$ such that for all $R \geq R_0$, $T(R) = \xi R$.

Proof. Set $R_0 = e^2 + 1$, so that $\lambda(\{(y, w) \in G, \langle y, w\rangle \leq R_0\}) > 0$ where λ denotes the Lebesgue measure on G. We have, for all $(x, v) \in \bar{D} \times \mathbb{R}^n$, by definition of $\sigma(x, v)$,

$$\sigma(x, v) + \sigma(x, -v) \leq \frac{d(D)}{\|v\|},$$

and therefore, using $c_4 < 1$,

$$\langle x, v\rangle \geq \frac{d(D)}{\|v\|} - \sigma(x, -v) \geq \sigma(x, v).$$

We conclude that for all $R \geq R_0$, $\Gamma_R \subset B_R$ with $\Gamma_R \neq \emptyset$, and the result follows from Theorem 4.4.1. □

4.5 Preliminary interpolation results

In this section, we briefly present several results of interpolation theory used in the proof of Theorem 4.1.1. Those are generalizations of the Riesz-Thorin Theorem for weighted L^1 spaces and some of their subspaces. Recall that G denotes $D \times \mathbb{R}^n$. Recall also that for $(x, v) \in \bar{D} \times \mathbb{R}^n$, we write $\langle x, v\rangle = (e^2 + \frac{d(D)}{\|v\|c_4} - \sigma(x, -v))$, with c_4 given by (4.1.11). For any weight w on $\bar{D} \times \mathbb{R}^n$, we set $L^1_{w,0}(G) := \{f \in L^1_w(G), \langle f\rangle = 0\}$ that we endow with the norm $\|.\|_w$ and $L^1_0(G) := \{f \in L^1(G), \langle f\rangle = 0\}$ which inherits the norm $\|.\|_{L^1}$ from $L^1(G)$. For A, B two Banach spaces with respective norms $\|.\|_A$, $\|.\|_B$ and $T : A \to B$ a linear operator, $\|\!|T|\!\|_{A\to B}$ denotes the operator norm of T, i.e.

$$\|\!|T|\!\|_{A\to B} = \sup_{v\in A, v\neq 0} \frac{\|Tv\|_B}{\|v\|_A}.$$

We introduce the Maxwellian of temperature 1 given by

$$M_1(v) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\|v\|^2}{2}}, \qquad v \in \mathbb{R}^n.$$

Lemma 4.5.1 (Interpolation of L^1-weighted spaces)**.** *Let $\phi_1, \phi_2, \tilde{\phi}_1, \tilde{\phi}_2$ four measurable functions on G such that $\phi_1, \phi_2, \tilde{\phi}_1, \tilde{\phi}_2 > 0$ almost everywhere. Let also $A_1 = L^1_{\phi_1}(G)$, $A_2 = L^1_{\phi_2}(G)$, $\tilde{A}_1 = L^1_{\tilde{\phi}_1}(G)$, $\tilde{A}_2 = L^1_{\tilde{\phi}_2}(G)$. Then, if T is a linear operator from A_1 to $\tilde{A}_1$ and from A_2 to $\tilde{A}_2$ such that*

$$\|\!|T|\!\|_{A_1\to\tilde{A}_1} \leq N_1, \quad \|\!|T|\!\|_{A_2\to\tilde{A}_2} \leq N_2, \tag{4.5.1}$$

for some $N_1, N_2 > 0$, for any $\theta \in (0, 1)$, for ϕ_θ, $\tilde{\phi}_\theta$ defined on G by $\phi_\theta = \phi_1^\theta \phi_2^{1-\theta}$ and for $\tilde{\phi}_\theta = \tilde{\phi}_1^\theta \tilde{\phi}_2^{1-\theta}$, T is a linear operator from $A_\theta := L^1_{\phi_\theta}(G)$ to $\tilde{A}_\theta = L^1_{\tilde{\phi}_\theta}(G)$ satisfying

$$\|\!|T|\!\|_{A_\theta\to\tilde{A}_\theta} \leq N_\theta, \tag{4.5.2}$$

for $N_\theta = N_1^\theta N_2^{1-\theta} > 0$.

Proof. This is obtained by Peetre's K-method of interpolation [106] and is a particular case of the Stein-Weiss Theorem with $p = 1$, see [7, Theorem 5.4.1]. □

Corollary 4.5.1. *Let $\theta \in (0,1)$ and $A_1, A_2, \tilde{A}_1, \tilde{A}_2, A_\theta, \tilde{A}_\theta$ defined as in Lemma 4.5.1. Assume that there exists a bounded projection $P : (A_i, \tilde{A}_i) \to (A'_i, \tilde{A}'_i)$ for $i \in \{1,2\}$ with $A'_i \subset A_i$, $\tilde{A}'_i \subset \tilde{A}_i$. Let $A'_\theta = (A'_1 + A'_2) \cap A_\theta$, $\tilde{A}'_\theta = (\tilde{A}'_1 + \tilde{A}'_2) \cap \tilde{A}_\theta$. Assume that T is a linear operator from A'_1 to $\tilde{A}'_1$ and from A'_2 to $\tilde{A}'_2$ with*

$$|||T|||_{A'_1 \to \tilde{A}'_1} \leq N_1, \qquad |||T|||_{A'_2 \to \tilde{A}'_2} \leq N_2,$$

for $N_1, N_2 > 0$. Then T is a linear operator from A'_θ to $\tilde{A}'_\theta$ and there exists $C > 0$ depending only on P such that

$$|||T|||_{A'_\theta \to \tilde{A}'_\theta} \leq C N_1^\theta N_2^{1-\theta}.$$

Proof. The couple (A'_1, A'_2) is a complemented subcouple of (A_1, A_2), and as such a so-called K-subcouple for the K-method of interpolation. The same thing holds with $(\tilde{A}'_1, \tilde{A}'_2)$ which is a complemented subcouple of $(\tilde{A}_1, \tilde{A}_2)$. From [79, Section 7, Theorem 2.1 and Example 7.1], this immediatly gives the result. We refer to Janson [79] for details about those notions. $\square$

We now turn to a second type of interpolation results in L^1 weighted spaces, no more focused on polynomial interpolation.

Lemma 4.5.2. *For $(y,v) \in G$, let ϕ_1 defined by $\phi_1(y,v) = \langle y, v \rangle$. Let T be a linear operator from $L^1_{\phi_1}(G)$ to $L^1_{\phi_1}(G)$ and from $L^1(G)$ to $L^1(G)$ such that*

$$|||T|||_{L^1_{\phi_1}(G) \to L^1_{\phi_1}(G)} \leq N_1, \quad |||T|||_{L^1(G) \to L^1(G)} \leq N_2,$$

for some $N_1, N_2 > 0$. Then, for $R(y,v) = \ln(\phi_1(y,v))$, T is a linear operator from $L^1_R(G)$ to itself and there exists an explicit $C > 0$ such that

$$|||T|||_{L^1_R(G) \to L^1_R(G)} \leq C.$$

Proof. From [63, Chapter 2, Theorem 1], given a weight ϕ_1, the space $L^1_{\phi_2}(G)$ is an interpolation space (and therefore $|||T|||_{L^1_{\phi_2}(G) \to L^1_{\phi_2}(G)} \leq C$ for some constant $C > 0$) for the couple $(L^1(G), L^1_{\phi_1}(G))$ if ϕ_2 satisfies for all $(y,v) \in G$,

$$\phi_2(y,v) = \int_0^\infty \min(\phi_1(y,v), t)\gamma(dt),$$

for some positive measure γ on $(0,\infty)$, $\gamma \not\equiv 0$ and satisfying

$$\int_0^\infty \min(1,t)\gamma(dt) < \infty.$$

The constant C then depends only on N_1, N_2 and γ. In particular, we consider a measure of the form $\gamma = f\lambda$, with λ the Lebesgue measure on $(0, \infty)$, and

$$f(t) = \begin{cases} 0 & \text{if } t \in (0, e), \\ \frac{1}{t^2} & \text{if } t \in (e, \infty). \end{cases}$$

We then have, for all $(y, v) \in G$,

$$\phi_2(y, v) = \int_e^{\phi_1(y,v)} \frac{dt}{t} + \phi_1(y, v) \int_{\phi_1(y,v)}^{\infty} \frac{dt}{t^2} = \ln(\phi_1(y, v)) = R(y, v),$$

and since $\phi_1(y, v) = \langle y, v\rangle$ for all $(y, v) \in G$, the result follows. $\square$

Corollary 4.5.2. *Lemma 4.5.2 holds when replacing the space $L^1_w(G)$ by $L^1_{w,0}(G)$ for any weight w on G considered, including replacing $L^1(G)$ by $L^1_0(G)$.*

Proof. We set, for $f \in L^1(G)$, for all $(x, v) \in G$,

$$Pf(x, v) := f(x, v) - \frac{M_1(v)}{|D|} \int_G f(y, w)dwdy.$$

Then, for $\phi_1 \geq 1$ defined as in Lemma 4.5.2, we have

$$\|Pf\|_{L^1} \leq 2\|f\|_{L^1} \quad \text{and} \quad \|Pf\|_{\phi_1} \leq (1 + c)\|f\|_{\phi_1},$$

with $c = \int_G M_1(v) \frac{\phi_1(x,v)}{|D|} dvdx < \infty$. The map P is linear, and $P^2 f = Pf$. We conclude as in the proof of Corollary 4.5.1 that, setting $\phi_2(x, v) = \ln(\langle x, v\rangle)$ for all $(x, v) \in G$,

$$L^1_{\phi_2}(G) \cap (L^1_0(G) + L^1_{\phi_1,0}(G)) = L^1_{\phi_2,0}(G),$$

is the interpolation space required, i.e. is such that for any T linear from $L^1_0(G)$ to itself and from $L^1_{\phi_1,0}(G)$ to itself with

$$|||T|||_{L^1_0(G)\to L^1_0(G)} \leq N_1, \qquad |||T|||_{L^1_{\phi_1,0}(G)\to L^1_{\phi_1,0}(G)} \leq N_2,$$

for two constants $N_1, N_2 > 0$, T is a linear operator from $L^1_{\phi_2,0}(G)$ to itself and there exists $N > 0$ explicit such that

$$|||T|||_{L^1_{\phi_2,0}(G)\to L^1_{\phi_2,0}(G)} \leq N.$$

$\square$

4.6 Proof of Theorem 4.1.1, Theorem 4.1.2 and Corollary 4.1.1

This section is devoted to the proof of Theorem 4.1.1, Theorem 4.1.2 and Corollary 4.1.1. We recall the notation $\langle x, v\rangle = (e^2 + \frac{d(D)}{\|v\|_{C_4}} - \sigma(x, -v))$ for all $(x, v) \in \bar{D} \times \mathbb{R}^n$. In this section, the constants are explicit up to the fact that they depend on ν given by Corollary 4.4.1. As already stated, ν itself may not be explicit, see Remark 4.4.1.

In the first subsection, we establish some contraction property for a well-chosen norm. In the second part, we use this property and the previous results to conclude the proof of Theorem 4.1.1. Subsection 4.6.3 is devoted to the proof of Theorem 4.1.2 and Corollary 4.1.1.

4.6.1 Contraction property in well-chosen norm

This subsection is devoted to the proof of the following lemma.

Lemma 4.6.1. *For all $\epsilon \in (0,3)$, setting $\bar{\omega}_k(x,v) = \langle x,v\rangle^k \ln(\langle x,v\rangle)^{-(1+\epsilon)}$ on G with the value $k \in [\![n-1,n+1]\!]$ there exists $T_0 > 0$ such that for all $T \geq T_0$, there exist $\beta(T) > 0$, $\alpha = C_3\beta(T)T$ with $C_3 > 0$ constant such that, for all $f \in L^1_{\bar{\omega}_{n+1}}(G)$ with $\langle f\rangle = 0$, we have*

$$\|S_T f\|_{L^1} + \beta\|S_T f\|_{\bar{\omega}_{n+1}} + \alpha\|S_T f\|_{\bar{\omega}_n} \leq \|f\|_{L^1} + \beta\|f\|_{\bar{\omega}_{n+1}} + \frac{\alpha}{3}\|f\|_{\bar{\omega}_n}, \tag{4.6.1}$$

so that, setting

$$|\!|\!|.|\!|\!|_{\bar{\omega}_{n+1}} := \|.\|_{L^1} + \beta\|.\|_{\bar{\omega}_{n+1}} + \alpha\|.\|_{\bar{\omega}_n},$$

there holds $|\!|\!|S_T f|\!|\!|_{\bar{\omega}_{n+1}} \leq |\!|\!|f|\!|\!|_{\bar{\omega}_{n+1}}$. Moreover, there exists $M_{n+1} > 1$ such that for all $f \in L^1_{\bar{\omega}_{n+1}}(G)$ with $\langle f\rangle = 0$,

$$\|S_T f\|_{\bar{\omega}_{n+1}} \leq M_{n+1}\|f\|_{\bar{\omega}_{n+1}}.$$

Finally, setting $\tilde{w}_i(x,v) = \langle x,v\rangle^{i-\frac{1}{2}}$ on G with $i \in \{1,\dots,4\}$, there exists $\tilde{T}_0 > 0$ such that for all $T \geq \tilde{T}_0$, there exists $\tilde{M}_{n+1} > 0$ such that for all $f \in L^1_{\tilde{w}_{n+1}}(G)$ with $\langle f\rangle = 0$,

$$\|S_T f\|_{\tilde{\omega}_{n+1}} \leq \tilde{M}_{n+1}\|f\|_{\tilde{\omega}_{n+1}}.$$

Proof. We prove the result on $\bar{\omega}_{n+1}$ first, and explain how to adapt the argument for the second statement at the end of the proof.

Step 1. We use the Lyapunov condition, Lemma 4.3.1, case (1), with both $\bar{\omega}_{n+1}$ and $\bar{\omega}_n$ to deduce a new integral inequality. For any $T > 0$, using Lemma 4.3.1, with $C_3, C_2, \tilde{b}_3, b_2 > 0$ constant, for all $f \in L^1_{\bar{\omega}_{n+1}}(G)$,

$$\|S_T f\|_{\bar{\omega}_{n+1}} + C_3\int_0^T \|S_t f\|_{\bar{\omega}_n} dt \leq \|f\|_{\bar{\omega}_{n+1}} + \tilde{b}_3(1+T)\|f\|_{L^1}, \tag{4.6.2a}$$

$$\text{and } \|S_T f\|_{\bar{\omega}_n} + C_2\int_0^T \|S_t f\|_{\bar{\omega}_{n-1}} dt \leq \|f\|_{\bar{\omega}_n} + b_2(1+T)\|f\|_{L^1}. \tag{4.6.2b}$$

Let $t \in (0,T)$. From (4.6.2b) we deduce

$$\|S_{T-t}S_t f\|_{\bar{\omega}_n} \leq \|S_t f\|_{\bar{\omega}_n} + b_2(1+T-t)\|S_t f\|_{L^1},$$

which we rewrite as

$$\|S_T f\|_{\bar{\omega}_n} - b_2(1+T-t)\|S_t f\|_{L^1} \leq \|S_t f\|_{\bar{\omega}_n}.$$

We plug this inside (4.6.2a) to obtain

$$\begin{aligned}\|S_T f\|_{\bar{\omega}_{n+1}} + C_3 \int_0^T & \Big(\|S_T f\|_{\bar{\omega}_n} - b_2(1+T-t)\|S_t f\|_{L^1}\Big) dt \\ &\leq \|f\|_{\bar{\omega}_{n+1}} + \tilde{b}_3(1+T)\|f\|_{L^1}.\end{aligned}$$

Using the L^1 contraction result from Theorem 4.2.1, we conclude

$$\|S_T f\|_{\bar{\omega}_{n+1}} + C_3 T \|S_T f\|_{\bar{\omega}_n} \leq \|f\|_{\bar{\omega}_{n+1}} + b_3(1+T+T^2)\|f\|_{L^1}, \tag{4.6.3}$$

with $b_3 > 0$ constant.

Step 2. From the Doeblin-Harris condition, Theorem 4.4.1, and more precisely Corollary 4.4.1, for all $\rho > R_0$, there exist $T(\rho) = \xi\rho$ for some constant $\xi > 0$ and a measure ν on G with $\nu \not\equiv 0$ such that

$$S_{T(\rho)} h \geq \nu \int_{\{(x,v)\in G : \langle x,v\rangle \leq \rho\}} h \, dv dx,$$

for all $h \in L^1(G)$ with $h \geq 0$.

Recall that by assumption f is such that $f \in L^1_{\bar{\omega}_{n+1}}(G)$, and $\langle f \rangle = 0$.

Set for any $\rho \geq R_0$, $\bar{\bar{\omega}}_n(\rho) := \rho^n \ln(\rho)^{-(1+\epsilon)}$ and $\kappa(\rho) = \frac{b_3(1+T+T^2)}{T}(\rho)$. Since $T(\rho) = \xi\rho$ for some constant $\xi > 0$, $\kappa(\rho) \underset{\rho\to+\infty}{\sim} C\rho$ for some $C > 0$. Since $n \in \{2,3\}$ one can find ρ_0 such that for all $\rho \geq \rho_0$, $\bar{\bar{\omega}}_n(\rho) \geq \frac{12\kappa(\rho)}{C_3}$. We fix $\rho > \rho_0$, $T = T(\rho) > T(\rho_0)$ for the remaining part of the proof. Note that since $T(\rho) = \xi\rho$ for some given constant ξ, any choice of $T > T(\rho_0)$ is possible. We set $A := \frac{\bar{\bar{\omega}}_n(\rho)}{4}$, and define, for all $\beta > 0$, the β-norm by:

$$\|f\|_\beta := \|f\|_{L^1} + \beta\|f\|_{\bar{\omega}_{n+1}}.$$

We distinguish two cases. Indeed, we have the alternative

$$\|f\|_{\bar{\omega}_n} \leq A\|f\|_{L^1}, \tag{4.6.4a}$$

$$\text{or } \|f\|_{\bar{\omega}_n} > A\|f\|_{L^1}. \tag{4.6.4b}$$

Step 3. We prove a convergence result in the β-norm in the case of the first alternative, (4.6.4a). Recall that for all $R > 0$, $\Gamma_R = \{(x,v) \in G, \langle x,v\rangle \leq R\}$. Using $\langle f \rangle = 0$, we have for all $(x,v)\in G$,

$$\begin{aligned}
S_T f_\pm(x,v) &\geq \nu(x,v)\int_G f_\pm(x',v')dv'dx' - \nu\int_{\Gamma_\rho^c} f_\pm(x',v')dv'dx' \\
&\geq \frac{\nu(x,v)}{2}\int_G |f(x',v')|dv'dx' - \nu(x,v)\int_{\Gamma_\rho^c}|f(x',v')|dv'dx' \\
&\geq \frac{\nu(x,v)}{2}\int_G |f(x',v')|dv'dx' - \frac{\nu(x,v)}{\bar{\bar{\omega}}_n(\rho)}\int_G |f(x',v')|\bar{\omega}_n(x',v')dv'dx' \\
&\geq \frac{\nu(x,v)}{2}\int_G |f(x',v')|dv'dx' - \frac{\nu(x,v)}{4}\int_G |f(x',v')|dv'dx' \\
&= \frac{\nu(x,v)}{4}\int_G |f(x',v')|dv'dx' =: \eta(x,v),
\end{aligned}$$

where the third inequality is given by definition of Γ_ρ and $\bar{\omega}_n \geq 1$, using that $\bar{\omega}_n(x,v)\leq \bar{\bar{\omega}}_n(\rho)$ for all $(x,v)\in\Gamma_\rho$, recalling also that $\langle x,v\rangle \geq e^2$. The last inequality is obtained by condition (4.6.4a). The final equality stands for a definition of $\eta(x,v)$ for all $(x,v)\in G$. Note that $\eta\geq 0$ on G. We deduce,

$$\begin{aligned}
|S_T f| &= |S_T f_+ - \eta - (S_T f_- - \eta)| \\
&\leq |S_T f_+ - \eta| + |S_T f_- - \eta| \\
&= S_T f_+ + S_T f_- - 2\eta = S_T|f| - 2\eta,
\end{aligned}$$

and, integrating over G, we obtain, using the mass conservation, that $\eta = \frac{\nu}{4}\|f\|_{L^1}$, and that ν is non-negative,

$$\|S_T f\|_{L^1} \leq \|f\|_{L^1} - 2\|\eta\|_{L^1} = \Big(1 - \frac{\langle\nu\rangle}{2}\Big)\|f\|_{L^1} = \tilde{\eta}\|f\|_{L^1}, \tag{4.6.5}$$

with $\tilde{\eta}\in(0,1)$. Hence, S_T is a strict contraction in L^1 in the case where f satisfies (4.6.4a). We use this result along with (4.6.3) and the definition of $\kappa(\rho)$ to derive an inequality on the β-norm of $S_T f$

$$\begin{aligned}
\|S_T f\|_\beta &= \|S_T f\|_{L^1} + \beta\|S_T f\|_{\bar{\omega}_{n+1}} \\
&\leq \tilde{\eta}\|f\|_{L^1} + \beta\big(-C_3 T\|S_T f\|_{\bar{\omega}_n} + \|f\|_{\bar{\omega}_{n+1}} + \kappa(\rho)T\|f\|_{L^1}\big) \\
&\leq \beta\|f\|_{\bar{\omega}_{n+1}} + (\tilde{\eta} + \kappa(\rho)T\beta)\|f\|_{L^1} - \beta C_3 T\|S_T f\|_{\bar{\omega}_n}.
\end{aligned}$$

Finally, we choose $0<\beta\leq \frac{1-\tilde{\eta}}{\kappa(\rho)T}$ and deduce

$$\|S_T f\|_\beta + C_3\beta T\|S_T f\|_{\bar{\omega}_n} \leq \|f\|_\beta. \tag{4.6.6}$$

Step 4. We prove that a slightly different version of (4.6.6) also holds in the case (4.6.4b). From (4.6.3), using (4.6.4b), we have, for T, $\kappa(\rho)$ fixed as above

$$\|S_T f\|_{\bar{\omega}_{n+1}} + C_3 T \|S_T f\|_{\bar{\omega}_n} \leq \|f\|_{\bar{\omega}_{n+1}} + \frac{\kappa(\rho) T}{A} \|f\|_{\bar{\omega}_n}.$$

Since $A \geq \frac{3\kappa(\rho)}{C_3}$, it follows that

$$\|S_T f\|_{\bar{\omega}_{n+1}} + C_3 T \|S_T f\|_{\bar{\omega}_n} \leq \|f\|_{\bar{\omega}_{n+1}} + \frac{C_3 T}{3} \|f\|_{\bar{\omega}_n}.$$

Using this inequality and the L^1 contraction we deduce

$$\begin{aligned} \|S_T f\|_\beta + C_3 \beta T \|S_T f\|_{\bar{\omega}_n} &= \|S_T f\|_{L^1} + \beta \|S_T f\|_{\bar{\omega}_{n+1}} + C_3 \beta T \|S_T f\|_{\bar{\omega}_n} \\ &\leq \|f\|_{L^1} + \beta \|f\|_{\bar{\omega}_{n+1}} + \beta \frac{C_3 T}{3} \|f\|_{\bar{\omega}_n} \\ &= \|f\|_\beta + \beta C_3 \frac{T}{3} \|f\|_{\bar{\omega}_n}. \end{aligned} \tag{4.6.7}$$

Step 5. For β as above and $\alpha = C_3 \beta T$, we have $|||.|||_{\bar{\omega}_{n+1}} = \|.\|_\beta + \alpha \|.\|_{\bar{\omega}_n}$. Gathering (4.6.6) and (4.6.7), we conclude that (4.6.1) holds and we deduce

$$|||S_T f|||_{\bar{\omega}_{n+1}} \leq |||f|||_{\bar{\omega}_{n+1}}.$$

Since $\bar{\omega}_{n+1} \geq \bar{\omega}_n \geq 1$ on G, we conclude that for all $f \in L^1_{\bar{\omega}_{n+1}}(G)$ with $\langle f \rangle = 0$,

$$\|S_T f\|_{\bar{\omega}_{n+1}} \leq M_{n+1} \|f\|_{\bar{\omega}_{n+1}}, \tag{4.6.8}$$

for some constant $M_{n+1} \geq 1$.

The proof of the second statement follows from similar arguments, note in particular that Step 1 can be adapted by using Lemma 4.3.1 case (2) instead of case (1), and that the argument giving the existence of ρ_0 from the properties of $\bar{\omega}_n$ still applies and gives a new $\tilde{\rho}_0$ (hence a $\tilde{T}_0$ playing the role of T_0) when considering $\tilde{\omega}_n$ instead of $\bar{\omega}_n$. The remaining steps follow by straightforward adaptations. $\square$

4.6.2 Proof of Theorem 4.1.1

In this subsection, we conclude the proof of Theorem 4.1.1 using Lemma 4.6.1. We consider the weights $w_1(x,v) = \langle x, v \rangle \ln(\langle x, v \rangle)^{0.1}$, and $w_0(x,v) = \ln(\langle x, v \rangle)^{0.1}$ for all $(x,v) \in \bar{D} \times \mathbb{R}^n$. Recall the definition of the weights ω_i from (4.1.12) for all $i \in \{1, \dots, 4\}$. We want to prove a decay rate for $S_t(f-g)$ with $f, g \in L^1_{\omega_{n+1}}$, $\langle f \rangle = \langle g \rangle$. We assume without loss of generality that $g \equiv 0$ so that $f \in L^1_{\omega_{n+1}}(G)$ with $\langle f \rangle = 0$.

Step 1. Recall that we write $L^1_{w,0}(G) = \{g \in L^1_w(G), \langle g\rangle = 0\}$, and the notation M_1 from Section 4.5. We introduce the bounded projection $P : L^1(G) \to L^1_0(G)$ such that for all $h \in L^1(G)$ and $(x,v) \in G$,

$$Ph(x,v) = h(x,v) - \frac{M_1(v)\|v\|^2}{c_1|D|}\int_G h(y,w)dydw, \tag{4.6.9}$$

with $c_1 = \int_{\mathbb{R}^n} M_1(v)\|v\|^2 dv < \infty$, where we recall that $|D|$ denotes the volume of D. One can see by a simple use of hyperspherical coordinates that $Ph \in L^1_{\omega_{n+1},0}(G)$ if $h \in L^1_{\omega_{n+1}}(G)$. Note that there exists $C > 0$ so that $\|Ph\|_{\omega_{n+1}} \le C\|h\|_{\omega_{n+1}}$ for all $h \in L^1_{\omega_{n+1}}(G)$ and $\|Ph\|_{L^1} \le C\|h\|_{L^1}$, and, since $\langle h\rangle = 0$ implies $Ph = h$, P is a bounded projection as claimed. Let $T > (T_0 \vee \tilde{T}_0)$ with T_0, $\tilde{T}_0$ given by Lemma 4.6.1. From Theorem 4.2.1, we have

$$|||S_T|||_{L^1_0(G)\to L^1_0(G)} \le 1,$$

and from Lemma 4.6.1,

$$|||S_T|||_{L^1_{\tilde{\omega}_{n+1},0}(G)\to L^1_{\tilde{\omega}_{n+1},0}(G)} \le \tilde{M}_{n+1}.$$

We apply Corollary 4.5.1 with the projection P and the values:

1. $A_1 = \tilde{A}_1 = L^1(G)$, and, using the definition of P, $A_1' = \tilde{A}_1' = L^1_0(G)$,
2. $A_2 = \tilde{A}_2 = L^1_{\tilde{\omega}_{n+1}}(G)$, and, using the definition of P, $A_2' = \tilde{A}_2' = L^1_{\tilde{\omega}_{n+1},0}(G)$,
3. $\theta = \frac{2}{2n+1} \in (0,1)$ so that $A_\theta = \tilde{A}_\theta = L^1_\mu(G)$, where μ is defined on $\bar{D} \times \mathbb{R}^n$ by

$$\mu(x,v) = \langle x,v\rangle = \tilde{\omega}_{n+1}(x,v)^{\frac{2}{2n+1}},$$

   and, using the definition of P, $\tilde{A}_\theta' = A_\theta' = (A_1' + A_2') \cap A_\theta = L^1_{\mu,0}(G)$.

We conclude that there exists $C_\mu > 0$ such that

$$\|S_T f\|_\mu \le C_\mu \|f\|_\mu.$$

Using Corollary 4.5.2, we obtain, for $\mu'(x,v) = \ln(\langle x,v\rangle)$ on $\bar{D}\times\mathbb{R}^n$, using that $f \in L^1_{\mu',0}(G)$, $\|S_T f\|_{\mu'} \le C_{\mu'}\|f\|_{\mu'}$ for some constant $C_{\mu'} > 0$. Finally, since $w_0(x,v) = \mu'(x,v)^{0.1}$ for all $(x,v) \in \bar{D}\times\mathbb{R}^n$, we apply one more time Corollary 4.5.1 with the projection P to conclude that, for all $T > (T_0 \vee \tilde{T}_0)$, there exists $\tilde{W}_0 \ge 1$ such that, using $f \in L^1_{w_0,0}(G)$,

$$\|S_T f\|_{w_0} \le \tilde{W}_0\|f\|_{w_0}.$$

Since $(S_t)_{t\ge 0}$ is a strongly continuous semigroup of operators on $L^1_{w_0}(G)$, this implies, using the growth bound of the semigroup, that there exists $W_0 \ge 1$ such that for all $t \in (0,T)$, for all

$f \in L^1_{w_0,0}(G)$,

$$\|S_T f\|_{w_0} = \|S_{T-t} S_t f\|_{w_0} \le W_0 \|S_t f\|_{w_0}. \tag{4.6.10}$$

Step 2. Using Lemma 4.3.1, case (3), and (4.6.10), we obtain, for some constants $C, W_0 > 0$,

$$\|S_T f\|_{w_1} + \frac{T}{W_0}\|S_T f\|_{w_0} \le \|f\|_{w_1} + C(1+T)\|f\|_{L^1},$$

which rewrites

$$\|S_T f\|_{w_1} + \frac{T}{W_0}\|S_T f\|_{w_0} \le \|f\|_{w_1} + \kappa(\rho) T \|f\|_{L^1},$$

with, for all $\rho > 0$, $\kappa(\rho) = \frac{C(1+T(\rho))}{T(\rho)}$ so that $\kappa \le C'$ for some constant C' independent of ρ. Set $w_0(r) = \ln(r)^{0.1}$ for $r \ge 1$. Since $\frac{w_0(\rho)}{\kappa(\rho)} \to \infty$ when $\rho \to \infty$, one can replicate the arguments of Steps 2 to 4 of the proof of Lemma 4.6.1. We obtain

$$\|S_T f\|_\beta + 3\alpha \|S_T f\|_{w_0} \le \|f\|_\beta + \alpha \|f\|_{w_0} \tag{4.6.11}$$

just as (4.6.7), for $T = T(\rho)$ large enough with $T > T_0$, $T > \tilde{T}_0$ where $T_0, \tilde{T}_0$ are given by Lemma 4.6.1, with $\beta > 0$ constant, $\alpha = \frac{\beta T}{3W_0}$ and

$$\|f\|_\beta := \|f\|_{L^1} + \beta \|f\|_{w_1}. \tag{4.6.12}$$

Step 3. We have, from our definition of w_i, $i \in \{0,1\}$ and of ω_{n+1}, for $(x,v) \in G$,

$$\begin{aligned} w_1(x,v) &= \langle x,v\rangle \ln(\langle x,v\rangle)^{0.1} \\ &= \langle x,v\rangle \ln(\langle x,v\rangle)^{0.1} \mathbf{1}_{\{\langle x,v\rangle < \lambda\}} + \langle x,v\rangle \ln(\langle x,v\rangle)^{0.1} \mathbf{1}_{\{\langle x,v\rangle \ge \lambda\}} \\ &\le w_0(x,v)\lambda + \frac{\langle x,v\rangle^{n+1} \ln(\langle x,v\rangle)^{-1.6} \ln(\langle x,v\rangle)^{1.7}}{\langle x,v\rangle^n} \mathbf{1}_{\{\langle x,v\rangle \ge \lambda\}} \\ &\le w_0(x,v)\lambda + \frac{\omega_{n+1}(x,v)\ln(\lambda)^{1.7}}{\lambda^n} \\ &\le w_0(x,v)\lambda + \omega_{n+1}(x,v)\epsilon_\lambda, \end{aligned}$$

for λ large enough, with $\epsilon_\lambda = \frac{\ln(\lambda)^{1.7}}{\lambda^n} \to 0$ as $\lambda \to \infty$, where we used that $x \to \frac{\ln(x)^{1.7}}{x^n}$ is non-increasing on (e^2, ∞) and that $\langle x,v\rangle \ge e^2$ for all $(x,v) \in G$. We deduce, since $w_1(x,v) \ge 1$ for $(x,v) \in G$,

$$\frac{1}{\lambda(1+\beta)}\|f\|_\beta = \frac{1}{\lambda(1+\beta)}(\|f\|_{L^1} + \beta\|f\|_{w_1}) \le \frac{1}{\lambda}\|f\|_{w_1} \le \|f\|_{w_0} + \frac{\epsilon_\lambda}{\lambda}\|f\|_{\omega_{n+1}}. \tag{4.6.13}$$

Moreover, for $\tilde\beta$, $\tilde\alpha$ the positive values used to define $|||.|||_{\omega_{n+1}}$ when applying Lemma 4.6.1 with $\epsilon = 0.6$, one has, setting $B = \alpha/\tilde\beta$,

$$\frac{\alpha\epsilon_\lambda}{\lambda}\|S_T f\|_{\omega_{n+1}} = \frac{\alpha}{\tilde\beta}\frac{\epsilon_\lambda}{\lambda}\tilde\beta\|S_T f\|_{\omega_{n+1}} \le B\frac{\epsilon_\lambda}{\lambda}|||S_T f|||_{\omega_{n+1}}, \tag{4.6.14}$$

with the definition given in Lemma 4.6.1 for $|||.|||_{\omega_{n+1}}$. Let $\delta := \frac{\alpha}{1+\beta}$, $Z := 1+\frac{\delta}{\lambda}$ with $\lambda \ge \lambda_0 \ge 1$, λ_0 large enough so that $Z \le 2$. Then

$$\begin{aligned}
Z(\|S_T f\|_\beta + \alpha\|S_T f\|_{w_0}) &\le \|S_T f\|_\beta + \frac{\alpha}{\lambda(1+\beta)}\|S_T f\|_\beta + Z\alpha\|S_T f\|_{w_0}\\
&\le \|S_T f\|_\beta + \alpha\|S_T f\|_{w_0} + \frac{\alpha\epsilon_\lambda}{\lambda}\|S_T f\|_{\omega_{n+1}} + Z\alpha\|S_T f\|_{w_0}\\
&\le \|S_T f\|_\beta + 3\alpha\|S_T f\|_{w_0} + \frac{B\epsilon_\lambda}{\lambda}|||S_T f|||_{\omega_{n+1}}\\
&\le \|f\|_\beta + \alpha\|f\|_{w_0} + \frac{B\epsilon_\lambda}{\lambda}|||S_T f|||_{\omega_{n+1}},
\end{aligned}$$

where we used (4.6.11), (4.6.13) and (4.6.14). We introduce the norm $|||.|||_{w_1}$ defined, for all function $h \in L^1_{w_1}(G)$, by

$$|||h|||_{w_1} := \|h\|_\beta + \alpha\|h\|_{w_0},$$

so that the previous inequality rewrites

$$Z|||S_T f|||_{w_1} \le |||f|||_{w_1} + \frac{B\epsilon_\lambda}{\lambda}|||S_T f|||_{\omega_{n+1}}.$$

Step 4. We set $u_0 = |||f|||_{w_1}$, and, for $k \ge 1$, $u_k = |||S_{kT} f|||_{w_1}$. We also set $v_0 = |||f|||_{\omega_{n+1}}$ and, for $k \ge 1$, $v_k = |||S_{kT} f|||_{\omega_{n+1}}$. Note that $v_k \le v_0$ for all $k \ge 1$ by Lemma 4.6.1. Setting $Y = \frac{B\epsilon_\lambda}{\lambda}$, the previous inequality writes,

$$Zu_1 \le u_0 + Yv_1.$$

Iterating this inequality, we obtain

$$Z^k u_k \le u_0 + Y\sum_{i=1}^k Z^{i-1} v_i,$$

from which we conclude that

$$u_k \le Z^{-k}u_0 + \frac{YZ}{Z-1}\sup_{i\ge1} v_i \le Z^{-k}u_0 + \frac{YZ}{Z-1}v_0.$$

From this we deduce, recalling the definition of the β-norm (4.6.12) and that $Z \le 2$, $w_1 \le \omega_{n+1}$

$$\begin{aligned}
|||S_{kT} f|||_{w_1} &\le \frac{1}{(1+\delta/\lambda)^k}(1+\beta+\alpha)\|f\|_{w_1} + \epsilon_\lambda\frac{2B}{\delta}|||f|||_{\omega_{n+1}}\\
&\le C\Big(e^{-\frac{kT}{\lambda}\frac{\delta}{2T}} + \epsilon_\lambda\Big)\|f\|_{\omega_{n+1}},
\end{aligned}$$

with $C > 0$ explicit, where we used that $|||\cdot|||_{\omega_{n+1}} \lesssim \|\cdot\|_{\omega_{n+1}}$. We set $T_1 = kT$ and choose $\lambda = \left(\frac{T_1^{\frac{\delta}{2T}}}{\ln\left(T_1^n\right)}\right)$ with $k \geq k_0$, $k_0 \geq 1$ large enough so that $\lambda > \lambda_0$ and $T_1 > \exp(1)$ to obtain

$$|||S_{T_1} f|||_{w_1} \leq C'(n)\Big(\frac{\ln(T_1)^{n+2}}{(T_1)^n}\Big)\|f\|_{\omega_{n+1}},$$

for some $C'(n) > 0$ depending explicitely on C, independent of k, where we used the trivial inequality $\frac{T_1}{\ln(T_1)} \leq T_1$. Upon modifying the definition of $C'(n)$ so that the previous inequality also holds for $k \in \{1, \dots k_0 - 1\}$, we can rewrite it as

$$|||S_{kT} f|||_{w_1} \leq C'(n)\Theta(k)\|f\|_{\omega_{n+1}}, \tag{4.6.15}$$

with $\Theta(k) = \frac{\ln(kT)^{n+2}}{(kT)^n}$ for all $k \geq 1$.

Step 5. With the norm $|||.|||_{w_1}$, (4.6.11) rewrites

$$|||S_T f|||_{w_1} + 2\alpha\|S_T f\|_{w_0} \leq |||f|||_{w_1}.$$

By iterating this inequality and summing, we obtain, for $l \geq 1$, writing $[x]$ for the floor of $x \in \mathbb{R}$,

$$0 \leq |||S_{lT} f|||_{w_1} + 2\alpha \sum_{k=[\frac{l}{2}]+1}^{l} \|S_{kT} f\|_{w_0} \leq \Big|\Big|\Big| S_{[\frac{l}{2}]T} f \Big|\Big|\Big|_{w_1}. \tag{4.6.16}$$

Note that for any $1 \leq k \leq l$,

$$\|S_{lT} f\|_{L^1} \leq \|S_{kT} f\|_{L^1} \leq \|S_{kT} f\|_{w_0}.$$

Hence, from (4.6.15) and (4.6.16),

$$\min(1, 2\alpha)\Big(l - \Big[\frac{l}{2}\Big] + 1\Big)\|S_{lT} f\|_{L^1} \leq C'(n)\Theta\Big(\Big[\frac{l}{2}\Big]\Big)\|f\|_{\omega_{n+1}},$$

so that, for some $C > 0$

$$\|S_{lT} f\|_{L^1} \leq C\frac{\ln(lT)^{n+2}}{(lT)^{n+1}}\|f\|_{\omega_{n+1}}.$$

We conclude to the desired rate by standard semigroup properties.

4.6.3 Proof of Theorem 4.1.2 and Corollary 4.1.1

In this subsection, we prove Theorem 4.1.2 and Corollary 4.1.1 using the result of Theorem 4.1.1. We first show the following lemma, from which we will deduce both the uniqueness property in Theorem 4.1.2 and Corollary 4.1.1. Recall the definition of m_n from (4.1.13) and that $m_n \equiv \omega_{n+1}^{\frac{n}{n+1}}$ on G.

Lemma 4.6.2. *There exists an explicit constant $C' > 0$ such that for all $t \geq 0$, for all $f, g \in L^1_{\omega_n}(G)$ with $\langle f \rangle = \langle g \rangle$, there holds*

$$\|S_t(f-g)\|_{L^1} \leq \frac{C' \ln(1+t)^{n+1}}{(1+t)^n} \|f-g\|_{m_n}.$$

Proof. We set $\tilde{f} := f - g$ so that $\langle \tilde{f} \rangle = 0$ and $\tilde{f} \in L^1_{m_n,0}(G)$. From Theorem 4.2.1, we have for all $t \geq 0$,

$$|\!|\!| S_t |\!|\!|_{L^1_0(G) \to L^1_0(G)} \leq 1,$$

and from Theorem 4.1.1,

$$|\!|\!| S_t |\!|\!|_{L^1_{\omega_{n+1},0}(G) \to L^1_0(G)} \leq C \frac{\ln(1+t)^{n+2}}{t^{n+1}} = C\tilde{\Theta}(t),$$

the last equality standing for a definition of $\tilde{\Theta}(t)$, with $C > 0$ independent of t. We introduce the projection $P : L^1(G) \to L^1_0(G)$ given, for $h \in L^1(G)$ by

$$Ph(x,v) = h(x,v) - \frac{M_1(v)\|v\|^2}{|D| c_1} \int_G h(y,w) dw dy, \quad (x,v) \in G,$$

with $c_1 = \int_{v \in \mathbb{R}^n} M_1(v) \|v\|^2 dv$ a normalizing constant, see Section 4.5 for the definition of M_1. Note that if $h \in L^1_{\omega_{n+1}}(G)$, $Ph \in L^1_{\omega_{n+1},0}(G)$ as one can check using hyperspherical coordinates, and that $\langle Ph \rangle = 0$. Moreover, P sends $L^1_r(G)$ to $L^1_{r,0}(G)$ for any weight $1 \leq r \leq \omega_{n+1}$ and is bounded.

We apply Corollary 4.5.1 with the projection P and

i. $A_1 = \tilde{A}_1 = \tilde{A}_2 = L^1(G)$,

ii. $A_2 = L^1_{\omega_{n+1}}(G)$,

iii. $A'_1 = \tilde{A}'_1 = \tilde{A}'_2 = L^1_0(G)$, $A'_2 = L^1_{\omega_{n+1},0}(G)$,

iv. $\theta = \frac{n}{n+1}$ so that $A_\theta = L^1_{m_n}(G)$, $\tilde{A}_\theta = L^1(G)$,

v. $A'_\theta = (A'_1 + A'_2) \cap A_\theta = L^1_{m_n,0}(G)$ and $\tilde{A}'_\theta = (\tilde{A}'_1 + \tilde{A}'_2) \cap \tilde{A}_\theta = L^1_0(G)$.

We deduce that for some explicit constant $C' > 0$, for all $t > 0$,

$$|\!|\!| S_t |\!|\!|_{L^1_{m_n,0}(G) \to L^1_0(G)} = C' \tilde{\Theta}(t)^{\frac{n}{n+1}} = C' \frac{\ln(1+t)^{\frac{n(n+2)}{n+1}}}{(1+t)^n} \leq C' \frac{\ln(1+t)^{n+1}}{(1+t)^n}.$$

$\square$

Proof of Theorem 4.1.2. **Step 1: Uniqueness.** Assume that there exists two functions f_∞, g_∞, both belonging to $L^1_{m_n}(G)$, with the desired properties. Applying Lemma 4.6.2, we have, for some $C > 0$, for all $t \geq 0$,

$$\|S_t(f_\infty - g_\infty)\|_{L^1} \leq C \frac{\ln(1+t)^{n+1}}{(t+1)^n} \|f_\infty - g_\infty\|_{m_n}.$$

For all $t \geq 0$, we have $S_t f_\infty = f_\infty$ and $S_t g_\infty = g_\infty$. Set $\delta(t) := C\frac{\ln(1+t)^{n+1}}{(t+1)^n}$. We deduce that, for all $t \geq 0$,

$$\|f_\infty - g_\infty\|_{L^1} \leq \delta(t)\|f_\infty - g_\infty\|_{m_n}.$$

We conclude that $f_\infty = g_\infty$ a.e. on G since $\delta(t) \to 0$ as $t \to \infty$.

Step 2: Existence. Recall that for all $k \in [\![n-1, n+1]\!]$, the weight m_k is given, for all (x, v) in $D \times \mathbb{R}^n$, by $m_k(x, v) = \langle x, v\rangle^k \ln(\langle x, v\rangle)^{-1.6\frac{n}{n+1}}$. Let $g \in L^1_{m_{n+1}}(G)$ with $g \geq 0$ and $\langle g\rangle = 1$. We apply Lemma 4.6.1 with $k = n+1$ and $\epsilon = 1.6\frac{n}{n+1} - 1 \in (0, 1)$, so that $\bar{\omega}_{n+1} = m_{n+1}$ and $\bar{\omega}_n = m_n$ and fix $T \geq T_0$. We set, for all $k \geq 1$,

$$g_k := S_{Tk} g \qquad \text{and } f_k := g_{k+1} - g_k.$$

By mass conservation, for all $k \geq 1$, $\langle g_k\rangle = 1$ so that $\langle f_k\rangle = 0$ and $f_k \in L^1_{m_{n+1},0}(G)$. Applying (4.6.1), for two constants $\beta, \alpha > 0$, setting $\|.\|_\beta = \|.\|_{L^1} + \beta\|.\|_{m_{n+1}}$, for all $k \geq 1$, we have

$$\|S_T f_k\|_\beta + \alpha\|S_T f_k\|_{m_n} \leq \|f_k\|_\beta + \frac{\alpha}{3}\|f_k\|_{m_n}.$$

We introduce the modify norm $|\!|\!|.|\!|\!|_{\tilde{\alpha}}$ defined by $|\!|\!|.|\!|\!|_{\tilde{\alpha}} = \|.\|_\beta + \frac{\alpha}{3}\|.\|_{m_n}$, so that the previous inequality reads

$$|\!|\!|S_T f_k|\!|\!|_{\tilde{\alpha}} + \frac{2\alpha}{3}\|S_T f_k\|_{m_n} \leq |\!|\!|f_k|\!|\!|_{\tilde{\alpha}}. \tag{4.6.17}$$

This implies that

$$|\!|\!|f_{k+1}|\!|\!|_{\tilde{\alpha}} \leq |\!|\!|f_k|\!|\!|_{\tilde{\alpha}},$$

for all $k \geq 1$, so that the sequence $(|\!|\!|f_k|\!|\!|_{\tilde{\alpha}})_{k\geq 1}$ is non-negative, non-decreasing, and is thus a converging subsequence. We fix $\epsilon > 0$. The previous observation implies that for $N \geq 0$ large enough, $p > l \geq N$,

$$0 \leq |\!|\!|f_l|\!|\!|_{\tilde{\alpha}} - |\!|\!|f_p|\!|\!|_{\tilde{\alpha}} \leq \frac{2\alpha}{3}\epsilon.$$

Let N as before, $p > l \geq N$. We have, using (4.6.17)

$$\begin{aligned}
\frac{2\alpha}{3}\left\|g_{p+1} - g_{l+1}\right\|_{m_n} &= \frac{2\alpha}{3}\Big\|\sum_{k=l+1}^{p} f_k\Big\|_{m_n} \\
&\leq \sum_{k=l}^{p-1} \frac{2\alpha}{3}\left\|S_T f_k\right\|_{m_n} \\
&\leq \sum_{k=l}^{p-1}\left(\frac{2\alpha}{3}\|S_T f_k\|_{m_n} + |||S_T f_k|||_{\tilde{\alpha}}\right) - \sum_{k=l}^{p-1} |||S_T f_k|||_{\tilde{\alpha}} \\
&\leq \sum_{k=l}^{p-1} |||f_k|||_{\tilde{\alpha}} - \sum_{k=l}^{p-1} |||S_T f_k|||_{\tilde{\alpha}} = |||f_l|||_{\tilde{\alpha}} - |||f_p|||_{\tilde{\alpha}} \leq \frac{2\alpha}{3}\epsilon,
\end{aligned}$$

by choice of p and l. We deduce that the sequence $(g_k)_{k\geq 1}$ is a Cauchy sequence in the Banach space $L^1_{m_n}(G)$, hence converges towards a limit $f_\infty \in L^1_{m_n}(G)$ with $\langle f_\infty \rangle = \langle g \rangle = 1$ by mass conservation. A similar argument to the one in Step 1 can be used to prove that this limit is independent of the choice of $g \in L^1_{m_{n+1}}(G)$ with $\langle g \rangle = 1$. □

Proof of Corollary 4.1.1. We simply apply Lemma 4.6.2 with $g = f_\infty$, f_∞ given by Theorem 4.1.2, to obtain Corollary 4.1.1. □

4.7 Free-transport with absorbing boundary condition

We consider in this section the free transport equation with absorbing condition at the boundary, which corresponds to (4.1.14). We make the same assumptions as before on D, n and $x \to n_x$. This problem is well-posed in the L^1 setting, since the boundary operator has norm 0, see Arkeryd and Cercignani [2, Theorem 3.5]. Assuming $f_0 \in L^1(G)$, the characteristic method gives an explicit solution for all times $t \geq 0$, almost all $(x, v) \in G$,

$$f(t, x, v) = f_0(x - tv, v)\mathbf{1}_{\{t<\sigma(x,-v)\}}, \tag{4.7.1}$$

where $\sigma(x, v)$ is defined by (4.1.8) for all (x, v) in $\bar{D} \times \mathbb{R}^n$. This explicit formula makes the positivity of (4.1.14) clear. Obviously mass is not preserved by this problem. In what follows, we write $(S_t)_{t\geq 0}$ for the semigroup of linear operators corresponding to this evolution problem. For $f_0 \in L^1(G)$, and f the associated solution to (4.1.14) on $[0, \infty) \times G$, the trace $\gamma f(., ., .)$ is well-defined on $[0, T) \times \partial D \times \mathbb{R}^n$ for any $T > 0$, see Mischler [101, Theorem 1]. Moreover, from [101, Corollary 1],

$$|\gamma f(t, x, v)| = \gamma|f|(t, x, v) \quad \text{a.e. in } ((0, \infty) \times \partial_+ G) \cup ((0, \infty) \times \partial_- G). \tag{4.7.2}$$

From the explicit solution (4.7.1), one easily deduces the convergence towards the equilibrium distribution given by $f_\infty(x, v) = 0$ for all $(x, v) \in \bar{D} \times \mathbb{R}^n$. We study the convergence rate of

(4.1.14) towards equilibrium. We recall that the weights r_ν for $\nu > 0$, are given by

$$r_\nu(x,v) = (1+\sigma(x,v))^\nu, \qquad (x,v) \in \bar{D} \times \mathbb{R}^n. \tag{4.7.3}$$

Theorem 4.7.1. *For any $f \in L^1_m(G)$, $t \geq 0$,*

$$\|S_t f\|_{L^1} \leq \Theta(t)\|f\|_m,$$

with the following choices

i. $m(x,v) = e^{\sigma(x,v)}$ *in* $\bar{D} \times G$, $\Theta(t) = e^{-t}$.

ii. $m(x,v) = r_\nu(x,v)$ *in* $\bar{D} \times G$, $\Theta(t) = \frac{1}{(t+1)^\nu}$, $\nu > 1$.

Proof. Recall that $(v \cdot \nabla_x \sigma(x,v)) = -1$ for all $(x,v) \in G$, see (4.3.2). Note that, as a trivial consequence of the boundary condition, $\gamma S_t f = 0$ on $\partial_- G$ for all $f \in L^1(G)$.

For $i.$, we have, by definition of $(S_t)_{t\geq 0}$, using also (4.7.2),

$$\begin{aligned}
\frac{d}{dt}\int_G |S_t f| e^{\sigma(x,v)} dv dx &= \int_G (-v\cdot\nabla_x |S_t f|) e^{\sigma(x,v)} dv dx \\
&= -\int_G |S_t f| e^{\sigma(x,v)} dv dx + \int_{\partial D\times\mathbb{R}^n} |\gamma(S_t) f|(v\cdot n_x) e^{\sigma(x,v)} dv d\zeta(x) \\
&= -\|S_t f\|_m + 0 - \int_{\partial_+ G} |\gamma(S_t f)||v\cdot n_x| e^{\sigma(x,v)} dv d\zeta(x) \\
&\leq -\|S_t f\|_m
\end{aligned}$$

where we used Green's formula (recall that n_x is the unit inward normal vector at $x \in \partial D$) and the boundary condition. We conclude with a straightforward application of Grönwall's lemma.

To prove $ii.$, we differentiate the $L^1_{r_\nu}(G)$ norm of $S_t f$ and use the same arguments as in case $i.$ to obtain

$$\begin{aligned}
\frac{d}{dt}\int_G |S_t f|(1+\sigma(x,v))^\nu dv dx &= \int_G (-v\cdot\nabla_x |S_t f|)(1+\sigma(x,v))^\nu dv dx \\
&= -\nu\int_G |S_t f|(1+\sigma(x,v))^{\nu-1} dv dx \\
&\quad - \int_{\partial_+ G} |\gamma(S_t f)||v\cdot n_x| r_\nu(x,v) dv d\zeta(x) \\
&\leq -\nu\int_G |S_t f|(1+\sigma(x,v))^{\nu-1} dv dx.
\end{aligned}$$

Writing $\mathcal{B}$ for the generator of $(S_t)_{t\geq 0}$, we proved

$$\mathcal{B}^* r_\nu \leq -\phi(r_\nu), \tag{4.7.4}$$

with for all $x \geq 1$, $\phi(x) = \nu x^{\frac{\nu-1}{\nu}}$, so that ϕ is strictly concave. We set for all $u \geq 1$,

$$H(u) = \int_1^u \frac{ds}{\phi(s)} = (u^{1/\nu} - 1),$$

$$\text{so that } H^{-1}(y) = (y+1)^\nu, \quad \forall y \geq 0.$$

$$\text{We also set } \forall t \geq 0, u \geq 1, \qquad \psi(t,u) := H^{-1}(H(u)+t) = (H(u)+t+1)^\nu,$$

which satisfies, for all $u \geq 1$, for all $t \geq 0$,

$$\partial_t \psi(t,u) = \nu(H(u)+t+1)^{\nu-1} = \phi(\psi(t,u)), \tag{4.7.5a}$$

$$\text{and } \partial_u \psi(t,u) = H'(u)\nu(H(u)+t+1)^{\nu-1} = \frac{\phi(\psi(t,u))}{\phi(u)}. \tag{4.7.5b}$$

We have, for all $t \geq 0$,

$$\begin{aligned}
\frac{d}{dt}\int_G |S_t f|\psi(t,r_\nu)dvdx &= \int_G \mathcal{B}|S_t f|\psi(t,r_\nu) + |S_t f|\partial_t\psi(t,r_\nu)dvdx \\
&= \int_G |S_t f|\Big(\mathcal{B}^*\psi(t,r_\nu) + \partial_t\psi(t,r_\nu)\Big)dvdx \\
&= \int_G |S_t f|\Big((\mathcal{B}^* r_\nu)\partial_u\psi(t,r_\nu) + \partial_t\psi(t,r_\nu)\Big)dvdx \leq 0,
\end{aligned}$$

using (4.7.4) along with (4.7.5a) and (4.7.5b). Finally we use this inequality to conclude:

$$\begin{aligned}
(t+1)^\nu \|S_t f\|_{L^1} \leq \int_G |S_t f|(H(r_\nu)+t+1)^\nu dvdx &= \|S_t f\|_{\psi(t,r_\nu)} \\
&\leq \|f\|_{\psi(0,r_\nu)} = \|f\|_{r_\nu}.
\end{aligned}$$

□

Part II

Subgeometric convergence towards equilibrium of Markov processes

Chapter 5

On subexponential convergence to equilibrium of Markov processes

This chapter is an extended version of the paper [9], submitted. In particular, Section 5.3 is a detailed rewriting of the result of Hairer [71] which is not included in [9].

Abstract: Studying the subexponential convergence towards equilibrium of a strong Markov process, we exhibit an intermediate Lyapunov condition equivalent to the control of some moment of a hitting time. This provides a link, similar (although more intricate) to the one existing in the exponential case, between the coupling method and the approach based on the existence of a Lyapunov function for the generator, in the context of the subexponential rates found by [44] and [71]. We also present an extended version of the proof of the result on the subgeometric rate of convergence of a functional with petite sublevel sets satisfying a Foster-Lyapunov condition given by Hairer [71].

Keywords: subgeometric ergodicity, strong Markov processes, Foster-Lyapunov criteria.

5.1 Introduction

The study of the convergence towards an invariant measure of continuous-time Markov processes has generated a large literature devoted to the geometric case (also referred to as the exponential case). Meyn and Tweedie and coauthors [97, 100, 45] developed stability concepts for continuous-time Markov processes along with simple criteria for non-explosion, Harris-recurrence, positive Harris-recurrence, ergodicity and geometric convergence to equilibrium. When applying those stability concepts, the key question of the existence of verifiable conditions emerges. In the discrete-time context, development of Foster-Lyapunov-type conditions on the transition kernel has provided such criteria. In the continuous-time context, Foster-Lyapunov inequalities applied to the (extended) generator of the process play the same role. One of the key results of this theory is the equivalence of two conditions, both implying an exponential convergence towards equilibrium: the control of the moment of the hitting time of a set with appropriate properties and the existence of some test function satisfying a Foster-Lyapunov inequality with respect to the generator. Loosely speaking, considering a topological space E and a E-valued strong Markov process $(X_t)_{t\geq 0}$, with semigroup $(\mathcal{P}_t)_{t\geq 0}$, invariant probability distribution π and with appropriate properties (irreducibility, non-explosion and aperiodicity, see Section 5.2 for precise definitions), we have the following result. Roughly, a set $C \in \mathcal{B}(E)$ is said to be *petite* if there is a probability measure a on $\mathcal{B}(\mathbb{R}_+)$ and a non-trivial measure ν on $\mathcal{B}(E)$ such that $\forall x \in C, \int_0^\infty \mathcal{P}_t(x,\cdot)a(dt) \geq \nu(\cdot)$.

Theorem 5.1.1 (Informal statement in the exponential case, [97, Theorem 8])**.** *The two following conditions are equivalent.*

1. *There exist a closed petite set $C \in \mathcal{B}(E)$, some constants $\delta > 0$ and $\kappa > 1$ such that,*

$$\text{setting } \tau_C(\delta) = \inf\{t > \delta : X_t \in C\}, \text{ we have } \sup_{x\in C} \mathbb{E}_x[\kappa^{\tau_C(\delta)}] < \infty; \tag{5.1.1}$$

2. *There exist a closed petite set $C \in \mathcal{B}(E)$, some constants $b > 0$, $\beta > 0$ and a function $V : E \to [1,\infty]$, finite at some $x_0 \in E$, such that for all $x \in E$,*

$$\mathcal{L}V(x) \leq -\beta V(x) + b\mathbf{1}_C(x). \tag{5.1.2}$$

Those conditions imply that there exist $\rho < 1$ and $d > 0$ such that for all $x \in E$,

$$\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} \leq dV(x)\rho^t.$$

A more precise statement is given in Section 5.2, see Theorem 5.2.2. In the study of subgeometric rates, the situation is quite different. While a moment condition of some hitting time set similar to (5.1.1) can be found, as well as a Lyapunov condition similar to (5.1.2), there is no equivalence between them. In this note, we identify an intermediate Lyapunov condition, equivalent to the moment condition for subgeometric convergence rates, and prove the following result, with the same notations as above.

Theorem 5.1.2 (Informal statement in the subexponential case)**.** *Let $\phi : [1,\infty) \to \mathbb{R}_+$ be an increasing, strictly concave C^1 function, satisfying some additional innocuous properties. Define the function $H_\phi : [1,\infty) \to \mathbb{R}_+$ by*

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)},$$

as well as its inverse $H_\phi^{-1} : \mathbb{R}_+ \to [1,\infty)$. Consider the three following conditions.

1. *There exist a compact petite set $C \in \mathcal{B}(E)$ and $r > 0$ such that, for $\tilde{\tau}_C^r$ defined by*

$$\tilde{\tau}_C^r = \inf\left\{t > 0, \int_0^t \mathbf{1}_C(X_s)ds \geq \frac{T}{r}\right\},$$

   where T is an exponential random variable of parameter 1 independent of our process, we have

$$\sup_{x\in C} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty, \qquad \textit{and, for all } x \in E, \quad \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty. \tag{5.1.3}$$

2. *There exist a function $\psi : \mathbb{R}_+ \times E \to \mathbb{R}_+$, two constants $\kappa, \eta > 0$ and a compact petite set $C \in \mathcal{B}(E)$ such that for all $(t,x) \in \mathbb{R}_+ \times E$, $H_\phi^{-1}(t) \leq \psi(t,x)$, $\sup_{x\in C} \psi(0,x) \leq \kappa$, and such that for all $(t,x) \in \mathbb{R}_+ \times E$,*

$$(\partial_t + \mathcal{L})\psi(t,x) \leq \kappa H_\phi^{-1}(t)\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)), \qquad \mathcal{L}\psi(0,x) \leq \kappa \mathbf{1}_C(x) - \eta. \tag{5.1.4}$$

3. *There exist a compact petite set C, a continuous function $V : E \to [1,\infty)$ with precompact sublevel sets and a constant $K > 0$ such that, for all $x \in E$*

$$\mathcal{L}V(x) \leq K\mathbf{1}_C(x) - \phi(V(x)). \tag{5.1.5}$$

Conditions 1. and 2. are equivalent, and both are implied by condition 3. Moreover, any of those conditions implies that there exists an invariant measure π and that, for all $x \in E$,

$$\lim_{t\to\infty} \phi(H_\phi^{-1}(t))\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

Remark 5.1.1. It is worth noticing that the proof of the result from condition (5.1.5) is constructive, whereas the one starting from (5.1.4) is not.

The precise statement is given in Theorem 5.4.1. The fact that (5.1.5) implies the convergence was proved by Douc, Fort and Guillin [44], see also Fort and Roberts [59] for the polynomial case, and was simplified for the case of the total variation distance by Hairer [71] in a slightly stronger setting. The papers [59] and [44] also identify a moment condition similar to (5.1.3), however they do not provide an equivalence result between the two conditions.

The remaining part of this note is organized as follows. In Section 5.2, we recall the main definitions of the theory of convergence for continuous-time strong Markov processes, and define our notion of extended generator, following Davis [34]. In Section 5.3, we give an extended version of the proof of Hairer [71] showing that a slightly stronger version of condition (5.1.5) implies the convergence towards equilibrium as claimed above, with a constructive constant. In Section 5.4, we give the precise statement and prove the remaining results of Theorem 5.1.2 above.

5.2 Setting and definitions

Let $X = (X_t)_{t\geq 0}$ be a continuous-time strong Markov process with values in a Polish space E. For $x \in E$, we write $\mathbb{P}_x$ for the probability measure such that $\mathbb{P}_x(X_0 = x) = 1$, $\mathbb{E}_x$ the corresponding expectation. We denote by $(\mathcal{P}_t)_{t\geq 0}$ the corresponding semigroup: for all functions f in $\mathcal{B}_b(E)$ with $\mathcal{B}_b(E) = \{f : E \to \mathbb{R}, f \text{ measurable and bounded}\}$, for all $x \in E$, we have $\mathcal{P}_t f(x) = \mathbb{E}_x[f(X_t)]$. We set, for $f \in \mathcal{B}_b(E)$, $x \in E$, $\hat{\mathcal{L}} f(x) = \frac{d}{dt}\mathbb{E}_x[f(X_t)]|_{t=0}$ provided this object exists. We call $\hat{\mathcal{L}}$ the (strong) generator and $\mathcal{D}(\hat{\mathcal{L}})$ its domain given by

$$\mathcal{D}(\hat{\mathcal{L}}) = \Big\{f : E \to \mathbb{R}, \forall x \in E, \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} \text{ exists}\Big\}.$$

Let us recall some more definitions. We say that a continuous-time Markov process $(X_t)_{t\geq 0}$ with values in E is non-explosive if there exists a family of pre-compact open sets $(O_n)_{n\geq 0}$ such that $O_n \to E$ as $n \to \infty$, and such that, setting for all $m \geq 0$, $T_m = \inf\{t > 0, X_t \notin O_m\}$, for all $x \in E$,

$$\mathbb{P}_x\Big(\lim_{m\to\infty} T_m = \infty\Big) = 1.$$

We say that $(X_t)_{t\geq 0}$ is φ-irreducible for some σ-finite measure φ if $\varphi(B) > 0$ implies that for all $B \in \mathcal{B}(E)$, for all $x \in E$, $\mathbb{E}_x[\int_0^\infty \mathbf{1}_B(X_s)ds] > 0$. A φ-irreducible process admits a maximal irreducibility measure ψ such that μ is absolutely continuous with respect to ψ for any other irreducibility measure μ. A set $A \in \mathcal{B}(E)$ such that $\psi(A) > 0$ for some maximal irreducibility measure ψ is then said to be accessible, and full is $\psi(A^c) = 0$. A set $A \in \mathcal{B}(E)$ is said to be absorbing if $\mathbb{P}_x(X_t \in A) = 1$ for all $x \in A$, $t \geq 0$. We simply say that $(X_t)_{t\geq 0}$ is irreducible if it is φ-irreducible for some σ-finite measure φ.

A non-empty measurable set C is said to be petite if there exists a probability measure a on $\mathcal{B}(\mathbb{R}_+)$ and a non-trivial σ-finite measure ν on $\mathcal{B}(E)$ such that

$$\forall x \in C, \int_0^\infty \mathcal{P}_t(x,\cdot)a(dt) \geq \nu(\cdot).$$

We say that a process $(X_t)_{t\geq 0}$ with associated semigroup $(\mathcal{P}_t)_{t\geq 0}$ is aperiodic if there exists an $m > 0$ such that, denoting by δ_m the Dirac mass at m, there exists an accessible δ_m-petite set C (i.e. petite with measure $a = \delta_m$ on $\mathbb{R}_+$) and some $t_0 \geq 0$ such that for all $x \in C$, $t \geq t_0$, $\mathcal{P}_t(x, C) > 0$.

We assume furthermore that our process is Feller, in the sense that for all $t > 0$, all continuous bounded function $f : E \to \mathbb{R}$, the function $\mathcal{P}_t f : E \to \mathbb{R}$ is also continuous.

The (weak) Feller property implies that $(X_s)_{s\geq 0}$ has a càdlàg modification, which we will always consider from now on, see for instance [109, Theorem 2.7]. In particular, the hitting times of closed sets are stopping times.

We recall without proof the following result, due to Krylov-Bogolioubov, [85]. In this note we use the statement of Fornaro [58, Chapter 5, Theorem 5.1.6].

Theorem 5.2.1. *Assume that there exist $T_0 > 0$ and $x \in E$ such that the sequence $(\mu_T(x))_{T>T_0}$ is tight, where for all $T \geq 0$, all $A \in \mathcal{B}(E)$,*

$$\mu_T(A) = \frac{1}{T}\int_0^T \mathcal{P}_s(x, \cdot)ds.$$

Then there exists at least one invariant probability measure for $(\mathcal{P}_t)_{t\geq 0}$.

We have the following result on $\mathcal{D}(\hat{\mathcal{L}})$.

Proposition 5.2.1. *[34, Propositions 14.10 and 14.13] For $f \in \mathcal{D}(\hat{\mathcal{L}})$, for all $x \in E$, all $t \geq 0$, we have $\int_0^t |\hat{\mathcal{L}} f(X_s)|ds < \infty$ $\mathbb{P}_x$-a.s. Moreover, defining the real-valued process $(C_t^f)_{t\geq 0}$ by*

$$C_t^f = f(X_t) - f(X_0) - \int_0^t \hat{\mathcal{L}} f(X_s)ds,$$

the process $(C_t^f)_{t\geq 0}$ is a $\mathbb{P}_x$-local martingale for any $x \in E$.

Following Davis [34], we define an extension of the generator $\hat{\mathcal{L}}$ in the following way.

Definition 5.2.1. Let $\mathcal{D}(\mathcal{L})$ denote the set of measurable functions $f : E \to \mathbb{R}$ with the following property: there exists a measurable function $h : E \to \mathbb{R}$ such that for all $x \in E$, there holds $\mathbb{P}_x(\forall t \geq 0, \int_0^t |h(X_s)|ds < \infty) = 1$, and the process

$$C_t^f = f(X_t) - f(X_0) - \int_0^t h(X_s)ds,$$

is a $\mathbb{P}_x$-local martingale. In this case, we set $\mathcal{L}f := h$. We call $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ the extended generator of $(X_t)_{t\geq 0}$.

The extended generator is indeed an extension: we have $\mathcal{D}(\hat{\mathcal{L}}) \subset \mathcal{D}(\mathcal{L})$ and $\mathcal{L}$ and $\hat{\mathcal{L}}$ coincide on $\mathcal{D}(\hat{\mathcal{L}})$. Following [34] again, we introduce the following notation.

Notation 5.2.1. For $f : E \to \mathbb{R}$, for $g : E \to \mathbb{R}$ measurable such that $\int_0^t |g(X_s)|ds < \infty$ for all $t \geq 0$, $\mathbb{P}_x$-almost surely for all $x \in E$, we write

$$\mathcal{L}f \leq g$$

if the process

$$f(X_t) - f(x) - \int_0^t g(X_s)ds$$

is a $\mathbb{P}_x$-local supermartingale for all $x \in E$.

Remark 5.2.1 (Hairer [71]). It is possible to have $\mathcal{L}f \leq g$ even in situations where f does not belong to the extended domain of $\mathcal{L}$. For instance, take $f(x) = -|x|$ when $(X_t)_{t\geq 0}$ is a Brownian motion. In this case, one has $\mathcal{L}f \leq 0$, but $f \notin \mathcal{D}(\mathcal{L})$, and *a fortiori* $f \notin \mathcal{D}(\hat{\mathcal{L}})$.

Similarly, we introduce

Notation 5.2.2. If $j : \mathbb{R}_+ \times E \to \mathbb{R}$ is C^1 in its first argument, for $k : \mathbb{R}_+ \times E \to \mathbb{R}$ measurable such that for all $t \geq 0$, we have $\int_0^t |k(s, X_s)|ds < \infty$ $\mathbb{P}_x$-a.s. for all $x \in E$, we write

$$(\partial_t + \mathcal{L})j \leq k$$

if

$$M_t := j(t, X_t) - j(0, x) - \int_0^t k(s, X_s)ds$$

is a $\mathbb{P}_x$-local supermartingale for all $x \in E$.

In this note, we use the following definition of the total variation distance: for two probability measures μ, ν on E, we set

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{A \in \mathcal{B}(E)} |\mu(A) - \nu(A)|.$$

As a consequence, we have

$$\|\mu - \nu\|_{TV} = \inf_{Z \sim \mu, Z' \sim \nu} \mathbb{P}(Z \neq Z'),$$

where the infimum is taken over all couples of random variables such that Z has law μ and Z' has law ν.

To conclude this section, we give a precise statement of the informal Theorem 5.1.1 above, with the help of the previous definitions.

Theorem 5.2.2 (Exponential case, Meyn-Tweedie [97])**.** *Assume that $(X_t)_{t\geq 0}$ is non-explosive, irreducible, and aperiodic. Then the following conditions are equivalent.*

1. *There exist a closed petite set* $C \in \mathcal{B}(E)$ *and some constants* $\delta > 0$ *and* $\kappa > 1$ *such that, setting*
$$\tau_C(\delta) = \inf\{t > \delta, X_t \in C\}, \qquad \textit{we have} \qquad \sup_{x \in C} \mathbb{E}_x[\kappa^{\tau_C(\delta)}] < \infty.$$

2. *There exist a closed petite set* $C \in \mathcal{B}(E)$*, some constants* $b > 0$*,* $\beta > 0$ *and* $V : E \to [1, \infty]$ *finite at some* $x_0 \in E$ *such that, in the sense of Notation 5.2.1,*
$$\mathcal{L}V \leq -\beta V + b\mathbf{1}_C. \tag{5.2.1}$$

Any of those conditions implies that the set $S_V = \{x : V(x) < \infty\}$ *is absorbing and full for any* V *solution of (5.2.1), and that there exists* $\rho < 1$ *and* $d > 0$ *such that for all* $x \in E$*,*
$$\|\mathcal{P}_t(x, \cdot) - \pi(\cdot)\|_{TV} \leq dV(x)\rho^t.$$

5.3 From Lyapunov inequality to convergence result

In this section, we prove that when a Lyapunov inequality for a function $V : E \to \mathbb{R}_+$, associated with the appropriate properties for the sublevel sets of V, holds, the convergence towards equilibrium occurs with a subgeometric rate given by the inequality. This result corresponds to the fact that condition (5.1.5) implies the convergence towards equilibrium as stated in Theorem 5.1.2. The proof below relies on a coupling argument and is a detailed version of the one of Hairer [71].

Theorem 5.3.1 (Hairer [71], Douc-Fort-Guillin [44])**.** *Let* $V : E \to [1, +\infty)$ *continuous. Assume that* $\{V \leq M\}$ *is compact for any* $M \geq 1$ *and that there exist a constant* $K > 0$ *and a strictly increasing, strictly concave and* C^1 *function* $\phi : [1, \infty) \to \mathbb{R}_+$ *such that* $\phi(1) > 0$*, such that* $\phi(x) \leq x$ *for all* $x \geq 1$*,* $\frac{\phi(x)}{x} \downarrow 0$ *and* $\phi(x) - x\phi'(x) \uparrow \infty$ *when* $x \to \infty$ *and such that*
$$\mathcal{L}V \leq K - \phi \circ V. \tag{5.3.1}$$

Finally, we assume that for all $C > 0$*, there exist* $\alpha > 0$ *and* $T > 0$ *such that*
$$\|\mathcal{P}_T(x, \cdot) - \mathcal{P}_T(y, \cdot)\|_{TV} \leq 1 - \alpha, \tag{5.3.2}$$

for all x, y *such that* $V(x) + V(y) \leq C$*. Then we have the following conclusions.*

1. *There exists an invariant probability measure* μ *for* $(X_t)_{t \geq 0}$ *with moreover*
$$\int_E \phi(V(x))\mu(dx) \leq K;$$

2. *Define* $H_\phi : [1,\infty) \to \mathbb{R}_+$ *by*

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)}$$

and consider its inverse function $H_\phi^{-1} : \mathbb{R}_+ \to [1,\infty)$. *There exists an explicit constant* $\kappa > 0$ *such that for all* $x, y \in E$, *for all* $t > 0$,

$$\|\mathcal{P}_t(x,\cdot) - \mathcal{P}_t(y,\cdot)\|_{TV} \leq \frac{\kappa(V(x)+V(y))}{H_\phi^{-1}(t)};$$

3. *There exists an explicit constant* $\kappa > 0$ *such that for all* $x \in E$, *all* $t > 0$,

$$\|\mathcal{P}_t(x,\cdot) - \mu\|_{TV} \leq \frac{\kappa V(x)}{H_\phi^{-1}(t)} + \frac{\kappa}{\phi(H_\phi^{-1}(t))}.$$

Remark 5.3.1. The hypothesis on the limits of $\frac{\phi(x)}{x}$ and $\phi(x) - x\phi'(x)$ are not explicitely stated in [71]. While innocuous, they are necessary for the argument to hold.

Example 5.3.1. Consider ϕ given, for $u \geq 0$, by $\phi(u) = u^{1-\epsilon}$ for some $\epsilon \in (0,1)$. Then ϕ satisfies the requirements of Theorem 5.3.1. In this case, for $u \geq 1$ we have

$$H_\phi(u) = \frac{1}{\epsilon}(u^\epsilon - 1).$$

This gives, for $t \geq 0$, $H_\phi^{-1}(t) = (\epsilon t + 1)^{\frac{1}{\epsilon}}$ and finally $\phi(H_\phi^{-1}(t)) = (\epsilon t + 1)^{\frac{1-\epsilon}{\epsilon}}$.

5.3.1 Existence of an invariant probability measure

In this subsection, we prove Point 1 of Theorem 5.3.1 by applying the Krylov-Bogolioubov criterion.

Proof of Point 1. We apply Theorem 5.2.1. Fix any $x \in E$ and define, for all $T > 0$, all $A \in \mathcal{B}(E)$,

$$\mu_T(A) = \frac{1}{T}\int_0^T \mathcal{P}_s(x,A)ds.$$

Then, since the process given for all $t \geq 0$ by

$$M_t = V(X_t) - V(x) - \int_0^t [K - \phi(V(X_s))]ds,$$

is a local supermartingale starting at 0, for $(\tau_n)_{n\geq 0}$ an increasing to ∞ sequence of stopping times such that $(M_{t\wedge\tau_n})_{t\geq 0}$ is a supermartingale, we have for all $n \geq 0$,

$$\mathbb{E}_x[V(X_{t\wedge\tau_n}) - V(x)] \leq \mathbb{E}_x\Big[\int_0^{t\wedge\tau_n} [K - \phi(V(X_s))]ds\Big],$$

so that, using Fatou's lemma, the fact that V is non-negative and dividing by t,

$$\frac{1}{t}\Big(\mathbb{E}_x[V(X_t)] - V(x)\Big) \leq K - \int_E \phi(V(y))\mu_t(dy).$$

We obtain, for all $T \geq 0$,

$$\int_E \phi(V(y))\mu_T(dy) \leq K + \frac{V(x)}{T}.$$

Since $\lim_{y\to\infty} \phi(V(y)) = +\infty$, we conclude that the sequence $(\mu_T)_{T\geq 1}$ is tight and that any limit point μ satisfies $\int_E \phi(V(y))\mu(dy) < \infty$ as desired. $\square$

5.3.2 Extended generator and local martingales

As we are working in an abstract framework, we heavily use the extended generator (see Notations 5.2.1 and 5.2.2) and the inequalities of the form

$$\mathcal{L}f \leq g \qquad \text{and} \qquad (\partial_t + \mathcal{L})\psi \leq \psi_2.$$

For this reason, we will use several preliminary results that we detail below.

Proposition 5.3.1. *Let $(y_t)_{t\geq 0}$ be a real-valued càdlàg semimartingale and let $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function that is C^1 in its first argument, and C^2 and concave in its second argument. Then, the process*

$$\varphi(t, y_t) - \int_0^t \partial_x\varphi(s, y_{s-})dy_s - \int_0^t \partial_t\varphi(s, y_{s-})ds$$

is non-increasing.

Proof. As $(y_t)_{t\geq 0}$ is a semimartingale, we can write it as $y_t = A_t + M_t$, where $(A_t)_{t\geq 0}$ is a process of finite variation and $(M_t)_{t\geq 0}$ is a local martingale. From Itô's formula for càdlàg processes, see for instance [78, Theorem 4.57], we then have

$$\begin{aligned}\varphi(t, y_t) = \varphi(0, y_0) &+ \int_0^t \partial_x\varphi(s, y_{s-})dy_s + \int_0^t \partial_t\varphi(s, y_{s-})ds \\ &+ \int_0^t \partial_x^2\varphi(s, y_{s-})d\langle M\rangle_s^c + \sum_{s\in[0,t]} \Big(\varphi(s, y_s) - \varphi(s, y_{s-}) - \partial_x\varphi(s, y_{s-})\Delta y_s\Big),\end{aligned}$$

where $\langle M\rangle_t^c$ denotes the quadratic variation of the continuous part of M at time t, with Δy_s defined by $\Delta y_s = y_s - y_{s-}$. Since $\langle M\rangle_t^c$ is an increasing process, and $\partial_x^2\varphi(\cdot,\cdot) \leq 0$ by hypothesis, the claim follows. $\square$

Recall that we write $\mathcal{L}$ for the extended generator of our E-valued Markov process $(X_t)_{t\geq 0}$.

Corollary 5.3.1. *Let $F, G : E \to \mathbb{R}$ such that*

$$\mathcal{L}F \leq G$$

in the sense of Notation 5.2.1. Then, if $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function that is C^1 in its first argument, and C^2 and concave in its second argument with additionally $\partial_x \varphi \geq 0$, then for all $t \geq 0$, all $x \in E$,

$$(\partial_t + \mathcal{L})\varphi(t, F(x)) \leq \partial_t \varphi(t, F(x)) + \partial_x \varphi(t, F(x)) G(x),$$

in the sense of Notation 5.2.2.

Proof. Set $y_t = F(X_t)$ for all $t \geq 0$. We have

$$dy_t = G(X_t)dt + dN_t + dM_t,$$

with M a càdlàg local martingale such that $M_0 = 0$ and N a non-increasing process. By Proposition 5.3.1, there is a non-increasing process $(R_t)_{t\geq 0}$ such that

$$d\varphi(t, y_t) = \partial_x \varphi(t, y_{t-}) dy_t + \partial_t \varphi(t, y_{t-}) dt + dR_t,$$

so that

$$d\varphi(t, y_t) = \partial_x \varphi(t, y_{t-})(G(X_t)dt + dN_t + dM_t) + \partial_t \varphi(t, y_{t-})dt + dR_t.$$

Since $\partial_x \varphi$ is non-negative, the process

$$\varphi(t, y_t) - \varphi(0, y_0) - \int_0^t \Big(\partial_t \varphi(s, y_{s-}) + \partial_x \varphi(s, y_{s-}) G(X_s) \Big) ds$$

is indeed a local supermartingale (as sum of a local martingale and of a non-increasing process). □

5.3.3 A result on some geometric sums

The proof of Point 2 of Theorem 5.3.1 will be obtained via a coupling of two processes $(X_t)_{t\geq 0}$, $(\tilde{X}_t)_{t\geq 0}$, both having for semigroup $(\mathcal{P}_t)_{t\geq 0}$. We will use the following lemma to estimate the time required for the success of the coupling, after a sort of geometric number of trials.

Lemma 5.3.1. *Let H_ϕ^{-1} defined as in Theorem 5.3.1, with the same hypothesis on ϕ. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and let $(Y_n)_{n\geq 0}$ be an $(\mathcal{F}_n)_{n\geq 0}$-adapted sequence of non-negative random variables. Let N be an $\mathbb{N}$-valued random variable such that, for some $\alpha \in (0,1)$, for all $n \geq 0$, $\{N = n\} \in \mathcal{F}_{n+1}$ and*

$$\mathbf{1}_{\{N\geq n\}} \mathbb{P}(N = n | \mathcal{F}_n) \geq \alpha \mathbf{1}_{\{N \geq n\}}. \tag{5.3.3}$$

Assume further that, for some $C > 0$, for all $n \geq 1$,

$$\mathbf{1}_{\{N\geq n-1\}} \mathbb{E}[H_\phi^{-1}(Y_n) | \mathcal{F}_{n-1}] \leq C \mathbf{1}_{\{N\geq n-1\}} \quad a\text{-}s. \tag{5.3.4}$$

Then, there is a constant $L > 0$ depending explicitely on α, $C > 0$ and ϕ such that

$$\mathbb{E}\Big[H_\phi^{-1}\Big(\sum_{n=1}^N Y_n\Big)\Big|\mathcal{F}_0\Big] \leq L.$$

Proof. We divide the proof into several steps.

Step 1. Recall that $C > 0$ is defined in the statement. We prove here that for all $\epsilon > 0$, there exists $K \geq 0$ such that for all $u \geq K$, all $v \geq K$,

$$H_\phi^{-1}(u+v) \leq \frac{\epsilon}{C} H_\phi^{-1}(u) H_\phi^{-1}(v). \tag{5.3.5}$$

Setting $g = \log \circ H_\phi^{-1}$ and, for $u, v \geq 0$,

$$\varphi(u,v) := g(u) + g(v) - g(u+v) = g(v) - \int_0^v g'(u+s)ds,$$

it suffices to prove that $\lim_{(u,v)\to(\infty,\infty)} \varphi(u,v) = +\infty$.

Since $\partial_u H_\phi^{-1}(u) = \phi(H_\phi^{-1}(u))$, we have

$$\partial_u \varphi(u,v) = \frac{\phi(H_\phi^{-1}(u))}{H_\phi^{-1}(u)} - \frac{\phi(H_\phi^{-1}(u+v))}{H_\phi^{-1}(u+v)} > 0$$

by hypothesis on ϕ. By symmetry, we conclude that φ is increasing in both its variables.

Fix now $M > 0$. Since g increases to infinity, there exists $v_0 > 0$ such that $g(v_0) \geq M + 1$. Then, since $g'(u+s) = \frac{\phi(H_\phi^{-1}(u+s))}{H_\phi^{-1}(u+s)}$ decreases to 0 as u goes to infinity (by assumption on ϕ), there exists $u_0 > 0$ such that

$$\int_0^{v_0} g'(u_0+s)ds \leq g'(u_0)v_0 \leq 1.$$

This shows that $\varphi(u_0, v_0) \geq M$. Since M is arbitrarily large and since φ is increasing in both its variables, we conclude that $\lim_{(u,v)\to(\infty,\infty)} \varphi(u,v) = +\infty$ as desired. Note that, given ϵ, $C > 0$ and ϕ, one can find an explicit K such that (5.3.5) holds for all $u \geq K$, $v \geq K$.

Step 2. We set, for $k \geq 1$, $S_k = \sum_{i=1}^k Y_i$. We fix $\epsilon > 0$ and consider $K > 0$ as in Step 1.

Setting $A^{c,k} = \{1,\dots,k\} \setminus A$ for $A \subset \{1,\dots,k\}$, we may write

$$\begin{aligned}
H_\phi^{-1}(S_k) &= \sum_{A\subset\{1,\dots,k\}} H_\phi^{-1}(S_k) \prod_{n\in A} \mathbf{1}_{\{Y_n\le K\}} \prod_{m\in A^{c,k}} \mathbf{1}_{\{Y_m>K\}} \\
&\le \sum_{A\subset\{1,\dots,k\}} H_\phi^{-1}\Big(|A|K + \sum_{m\in A^{c,k}} Y_m\Big) \prod_{m\in A^{c,k}} \mathbf{1}_{\{Y_m>K\}} \\
&\le \sum_{A\subset\{1,\dots,k\}} H_\phi^{-1}(|A|K)\Big(\frac{\epsilon}{C}\Big)^{|A^{c,k}|} \prod_{m\in A^{c,k}} H_\phi^{-1}(Y_m) \\
&= H_\phi^{-1}(kK) \sum_{A\subset\{1,\dots,k\}} \epsilon^{k-|A|} \prod_{m\in A^{c,k}} \frac{H_\phi^{-1}(Y_m)}{C},
\end{aligned}$$

using (5.3.5). Multiplying by $\mathbf{1}_{\{N\ge k\}}$ and taking the expectation given $\mathcal{F}_0$, we find

$$\mathbb{E}[H_\phi^{-1}(S_k)\mathbf{1}_{\{N\ge k\}}|\mathcal{F}_0] \le H_\phi^{-1}(kK) \sum_{A\subset\{1,\dots,k\}} \epsilon^{k-|A|}\mathbb{E}\Big[\mathbf{1}_{\{N\ge k\}} \prod_{m\in A^{c,k}} \frac{H_\phi^{-1}(Y_m)}{C}\Big|\mathcal{F}_0\Big]. \tag{5.3.6}$$

Step 3. We claim that for any $k \ge 1$, any $A \subset \{1,\dots,k\}$,

$$\mathbb{E}\Big[\mathbf{1}_{\{N\ge k\}} \prod_{m\in A^{c,k}} \frac{H_\phi^{-1}(Y_m)}{C}\Big|\mathcal{F}_0\Big] \le (1-\alpha)^{|A|}. \tag{5.3.7}$$

It suffices to proceed by successive conditionning, observing that for any $i = 1,\dots,k$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{N\ge i\}} \prod_{m\in A^{c,k},m\le i} \frac{H_\phi^{-1}(Y_m)}{C}\Big|\mathcal{F}_{i-1}\Big] \\
\le&\mathbf{1}_{\{N\ge i-1\}}\Big(\prod_{m\in A^{c,k},m\le i-1} \frac{H_\phi^{-1}(Y_m)}{C}\Big)\Big[\mathbf{1}_{\{i\in A^{c,k}\}} + (1-\alpha)\mathbf{1}_{\{i\in A\}}\Big].
\end{aligned}$$

Indeed, if $i \in A^{c,k}$, we simply use that $\mathbf{1}_{\{N\ge i\}} \le \mathbf{1}_{\{N\ge i-1\}}$ and (5.3.4). If now $i \in A$, we use that $\mathbb{P}(N \ge i|\mathcal{F}_{i-1}) = \mathbf{1}_{\{N\ge i-1\}}\mathbb{P}(N \ge i|\mathcal{F}_{i-1}) = \mathbf{1}_{\{N\ge i-1\}}[1-\mathbb{P}(N = i-1|\mathcal{F}_{i-1})] \le (1-\alpha)\mathbf{1}_{\{N\ge i-1\}}$ by (5.3.3). We conclude that

$$\mathbb{E}\Big[\mathbf{1}_{\{N\ge k\}} \prod_{m\in A^{c,k}} \frac{H_\phi^{-1}(Y_m)}{C}\Big|\mathcal{F}_0\Big] \le \prod_{i=1}^{k} \Big[\mathbf{1}_{\{i\in A^{c,k}\}} + (1-\alpha)\mathbf{1}_{\{i\in A\}}\Big] = (1-\alpha)^{|A|}.$$

Step 4. Gathering (5.3.6) and (5.3.7), we conclude that for any $k \ge 1$,

$$\mathbb{E}[H_\phi^{-1}(S_k)\mathbf{1}_{\{N\ge k\}}|\mathcal{F}_0] \le H_\phi^{-1}(kK)\sum_{l=1}^{k} \frac{k!}{l!(k-l)!}\epsilon^{k-l}(1-\alpha)^k = H_\phi^{-1}(kK)(1+\epsilon-\alpha)^k.$$

With the choice $\epsilon = \alpha/2$ (which induces some value of $K > 0$), setting $r = 1 - \frac{\alpha}{2} \in (0,1)$, we deduce that

$$\mathbb{E}\Big[H_\phi^{-1}\Big(\sum_{i=1}^{N} Y_i\Big)\Big|\mathcal{F}_0\Big] \le \sum_{k\ge 1} \mathbb{E}[H_\phi^{-1}(S_k)\mathbf{1}_{\{N\ge k\}}|\mathcal{F}_0] \le \sum_{k\ge 1} H_\phi^{-1}(kK)r^k.$$

Note that H_ϕ^{-1} has subexponential growth, in the sense that $\lim_{u\to\infty} u^{-1}\log H_\phi^{-1}(u) = 0$, which follows from the fact that $\lim_{u\to\infty} g'(u) = 0$, see Step 1. Hence, for $\eta = \log(1/r)/(2K) > 0$, there is a constant $L' > 0$ such that, for all $u > 0$, $H_\phi^{-1}(u) \le L'\exp(\eta u)$, whence the inequality $H_\phi^{-1}(Kk) \le L'\exp(\eta K k) = L' r^{-k/2}$ holds. Finally, we have

$$\mathbb{E}\Big[H_\phi^{-1}\Big(\sum_{i=1}^{N} Y_i\Big)\Big|\mathcal{F}_0\Big] \le L'\sum_{k\ge 1} r^{k/2} = L'\frac{\sqrt{r}}{1-\sqrt{r}}.$$

Note that the constant L' can be determined explicitely given K and ϕ. Setting $L = L'\frac{\sqrt{r}}{1-\sqrt{r}}$ terminates the proof. $\square$

5.3.4 Coupling

For $(X_t)_{t\ge 0}$ and $(Y_t)_{t\ge 0}$ independant and with transition semigroup $(\mathcal{P}_t)_{t\ge 0}$, we define the semigroup $(Q_t)_{t\ge 0}$ from $\mathcal{B}(E^2) \to \mathbb{R}$

$$Q_t f(x,y) = \mathbb{E}_{(x,y)}[f(X_t, Y_t)],$$

i.e.

$$Q_t((x,y),\cdot) = \mathcal{P}_t(x,\cdot) \otimes \mathcal{P}_t(y,\cdot),$$

and we denote $\mathcal{L}_2$ the corresponding generator. Note that, if $f(x,y) = f_1(x) + f_2(y)$, we have $\mathcal{L}_2 f(x,y) = \mathcal{L}f_1(x) + \mathcal{L}f_2(y)$.

We recall that $\phi : [1,\infty) \to \mathbb{R}_+$ is strictly increasing, strictly concave and C^1, with $\phi(1) > 0$, $\phi(x) \le x$ for all $x \ge 1$, $\frac{\phi(x)}{x} \downarrow 0$ and $\phi(x) - x\phi'(x) \uparrow \infty$ when $x \to \infty$, and that we defined H_ϕ by $H_\phi(u) = \int_1^u \frac{ds}{\phi(s)}$ for all $u \in [1,\infty)$ and consider its inverse function $H_\phi^{-1} : \mathbb{R}_+ \to [1,\infty)$.

Proposition 5.3.2. *Let $\psi_2 : [0,\infty) \times E^2 \to \mathbb{R}_+$ and $W_1, W_2 : E \to [1,\infty)$ be three functions such that $\psi_2(0,x,y) = W_1(x) + W_2(y)$ and $\psi_2(t,x,y) \ge H_\phi^{-1}(t)$ for all $(x,y) \in E^2$ and all $t \ge 0$. Assume also that there exist a compact non-empty set $\mathcal{K}$ of E^2 and four constants $K, T, \kappa > 0$ and $\alpha \in (0,1)$ such that*

a) for all $t \ge 0$, for all $(x,y) \notin \mathcal{K}$, $(\partial_t + \mathcal{L}_2)\psi_2(t,x,y) \le 0$; for all $(x,y) \in \mathcal{K}$, $\psi(0,x,y) \le \kappa$,

b) for all $x \in E$, $\mathcal{L}W_1(x) \le K$, $\mathcal{L}W_2(x) \le K$, *c) for all* $(x,y) \in \mathcal{K}$,

$$\|\mathcal{P}_T(x,\cdot) - \mathcal{P}_T(y,\cdot)\|_{TV} \le 1-\alpha. \tag{5.3.8}$$

Then there is an explicit constant $L>0$ *so that for all* $t \ge 0$, *all* $x,y \in E$,

$$\|\mathcal{P}_t(x,\cdot) - \mathcal{P}_t(y,\cdot)\|_{TV} \le L\frac{W_1(x)+W_2(y)}{H_\phi^{-1}(t)}.$$

Proof. We divide the proof in several steps. We consider some initial points x and y in E and of course assume that $x \neq y$.

Step 1: Construction. We build a coupling $(X_t, \tilde{X}_t)_{t\ge0}$ using the following steps.

Step 0: We set $X_0 = x$, $\tilde{X}_0 = y$, $T_0 = 0$.

Step n: If $T_n = \infty$ we do nothing. Else,

(i) If $X_{T_n} = \tilde{X}_{T_n}$, we pick a trajectory $(Y_t)_{t\ge0}$ with transition semigroup $(\mathcal{P}_t)_{t\ge0}$ starting from X_{T_n} and set $X_{T_n+t} = \tilde{X}_{T_n+t} = Y_t$ for all $t \ge 0$, as well as $T_{n+1} = \infty$.

(ii) If $X_{T_n} \neq \tilde{X}_{T_n}$ and $(X_{T_n}, \tilde{X}_{T_n}) \notin \mathcal{K}$, we pick two independent trajectories $(Y_t)_{t\ge0}$, $(\tilde{Y}_t)_{t\ge0}$ starting respectively at X_{T_n} and $\tilde{X}_{T_n}$, with transition semigroup $(\mathcal{P}_t)_{t\ge0}$. We then set

$$T_{n+1} = T_n + \inf\{t \ge 0, (Y_t, \tilde{Y}_t) \in \mathcal{K}\}$$

and $X_{T_n+t} = Y_t$, $\tilde{X}_{T_n+t} = \tilde{Y}_t$ for all $t \in [0, T_{n+1} - T_n]$.

(iii) If $X_{T_n} \neq \tilde{X}_{T_n}$ and $(X_{T_n}, \tilde{X}_{T_n}) \in \mathcal{K}$, we use a coupling (which exists by point (c) of the statement) of $X_{T_n+T} \sim \mathcal{P}_T(X_{T_n},\cdot)$ and $\tilde{X}_{T_n+T} \sim \mathcal{P}_T(\tilde{X}_{T_n},\cdot)$ such that

$$\mathbb{P}(X_{T_n+T} = \tilde{X}_{T_n+T} | (X_t, \tilde{X}_t)_{t\in[0,T_n]}) \ge \alpha.$$

We then set $T_{n+1} = T_n + T$ and build $(X_t)_{t\in[T_n,T_{n+1}]}$ and $(\tilde{X}_t)_{t\in[T_n,T_{n+1}]}$ using two bridges (e.g. independent conditionally on $(X_{T_n}, \tilde{X}_{T_n}, X_{T_n+T}, \tilde{X}_{T_n+T})$).

It is clear that both $(X_t)_{t\ge0}$ and $(\tilde{X}_t)_{t\ge0}$ are Markov processes with semigroup $(\mathcal{P}_t)_{t\ge0}$. We introduce the filtration $\mathcal{F}_t = \sigma(X_s, \tilde{X}_s, s \in [0,t])$.

Step 2. We introduce

$$\tau_1 = \inf\{t > 0, (X_t, \tilde{X}_t) \in \mathcal{K}\},$$

which equals 0 if $(x,y) \in \mathcal{K}$ and T_1 else. We then introduce, for $n \ge 1$,

$$\tau_{n+1} = \inf\{t \ge \tau_n + T, (X_t, \tilde{X}_t) \in \mathcal{K}\},$$

as well as $G = \inf\{n \geq 1 : X_{\tau_n+T} = \tilde{X}_{\tau_n+T}\}$.

Observe that by construction, we a.s. have $\tau_n \in (T_k)_{k\geq 0}$ for all $n = 1, \dots, G+1$.

We also have $X_t = \tilde{X}_t$ for all $t \geq \tau_G + T$, and it suffices to prove that

$$\mathbb{E}[H_\phi^{-1}(\tau_G + T)] \leq L(W_1(x) + W_2(y)) \tag{5.3.9}$$

since this will imply that

$$\|\mathcal{P}_t(x,\cdot) - \mathcal{P}_t(y,\cdot)\|_{TV} \leq \mathbb{P}(X_t \neq \tilde{X}_t) \leq \mathbb{P}(\tau_G + T > t) \leq \frac{\mathbb{E}[H_\phi^{-1}(\tau_G + T)]}{H_\phi^{-1}(t)} \leq L\frac{W_1(x) + W_2(y)}{H_\phi^{-1}(t)}.$$

Step 3. In this step, we show that

$$\mathbb{E}\Big[H_\phi^{-1}(\tau_1)\Big] \leq W_1(x) + W_2(y). \tag{5.3.10}$$

If $(x, y) \in \mathcal{K}$, the result is trivial since $H_\phi^{-1}(0) \leq 1 \leq W_1(x)+W_2(y)$. Otherwise, by definition of τ_1, the law of $(X_t, \tilde{X}_t)_{t\in[0,\tau_1]}$ is then the one corresponding to case (ii) in Step 1. By hypothesis a) and by Notation 5.2.2, the process defined for all $t \geq 0$ by

$$M_t = \psi_2(t, X_t, \tilde{X}_t) - \psi_2(0, x, y),$$

is a local supermartingale, on $[0, \tau_1]$, starting from 0. Thus, there exists an increasing to infinity sequence of stopping times $(\sigma_i)_{i\geq 1}$ in $\mathbb{R}_+$ such that, for all $i \geq 1$, $(M_{t\wedge\sigma_i})_{t\geq 0}$ is a bounded supermartingale on $[0, \tau_1]$. We then have, by positivity of ψ_2 and using Fatou's lemma,

$$\mathbb{E}\Big[\psi_2(\tau_1, X_{\tau_1}, \tilde{X}_{\tau_1})\Big] \leq \lim_{i\to\infty}\Big(\mathbb{E}[\psi_2(0, x, y) + M_{\tau_1\wedge\sigma_i}]\Big) \leq W_1(x) + W_2(y). \tag{5.3.11}$$

The conclusion follows, since $\psi_2(t, x, y) \geq H_\phi^{-1}(t)$.

Step 4. In this step, we show that there is a constant $C > 0$ such that for all $n \geq 1$,

$$\mathbf{1}_{\{G\geq n\}}\mathbb{E}[H_\phi^{-1}(\tau_{n+1} - \tau_n)|\mathcal{F}_{\tau_n}] \leq C\mathbf{1}_{\{G\geq n\}}. \tag{5.3.12}$$

Observe that $\{G \geq n\} = \{G \leq n-1\}^c \in \mathcal{F}_{\tau_{n-1}+T} \subset \mathcal{F}_{\tau_n}$, since $\tau_n \geq \tau_{n-1} + T$. On the event $\{G = n\}$, $\tau_{n+1} = \tau_n + T$ and therefore,

$$\mathbb{E}[\mathbf{1}_{\{G=n\}}H_\phi^{-1}(\tau_{n+1} - \tau_n)|\mathcal{F}_{\tau_n}] \leq H_\phi^{-1}(T).$$

On the event $\{G \geq n+1\} \in \mathcal{F}_{\tau_n+T}$, $X_{\tau_n+T} \neq \tilde{X}_{\tau_n+T}$, so that, according to Step 1, the process $(X_t, \tilde{X}_t)_{t\in(\tau_n+T,\tau_{n+1})}$ is build using (ii).With the same argument as in Step 3, and obtain

$$\mathbf{1}_{\{G\geq n+1\}}\mathbb{E}[H_\phi^{-1}(\tau_{n+1} - (\tau_n + T))|\mathcal{F}_{\tau_n+T}] \leq \mathbf{1}_{\{G\geq n+1\}}(W_1(X_{\tau_n+T}) + W_2(\tilde{X}_{\tau_n+T})). \quad (5.3.13)$$

Moreover, by assumption (b) on $\mathcal{L}W_1$, on $\{G \geq n\} \in \mathcal{F}_{\tau_n}$, the process $(N_t)_{t\geq\tau_n}$ defined by

$$N_t = W_1(X_t) - W_1(X_{\tau_n}) - K(t - \tau_n),$$

is a local supermartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ on $[\tau_n, \tau_n + T]$, with $N_{\tau_n} = 0$, in both cases (ii) and (iii) of Step 1. Hence there exists an increasing to infinity sequence of stopping times $(\sigma'_i)_{i\geq 1}$ such that for all $i \geq 1$, $(N_{t\wedge\sigma'_i})_{t\geq 0}$ is a bounded supermartingale on $[\tau_n, \tau_n + T]$. We then have, using the positivity of W_1 and Fatou's lemma,

$$\begin{aligned}
\mathbf{1}_{\{G\geq n\}}\mathbb{E}[W_1(X_{\tau_n+T})|\mathcal{F}_{\tau_n}] &\leq \lim_{i\to\infty} \mathbf{1}_{\{G\geq n\}}\mathbb{E}[W_1(X_{(\tau_n+T)\wedge\sigma'_i})|\mathcal{F}_{\tau_n}] \\
&\leq \lim_{i\to\infty} \mathbf{1}_{\{G\geq n\}}\mathbb{E}[W_1(X_{\tau_n}) + N_{(\tau_n+T)\wedge\sigma'_i} + KT|\mathcal{F}_{\tau_n}] \\
&\leq \mathbf{1}_{\{G\geq n\}}\Big(KT + W_1(X_{\tau_n})\Big).
\end{aligned}$$

Similarly, we have

$$\mathbf{1}_{\{G\geq n\}}\mathbb{E}[W_2(\tilde{X}_{\tau_n+T})|\mathcal{F}_{\tau_n}] \leq \mathbf{1}_{\{G\geq n\}}\Big(KT + W_2(\tilde{X}_{\tau_n})\Big).$$

All in all, we have verified that

$$\begin{aligned}
&\mathbf{1}_{\{G\geq n\}}\mathbb{E}[H_\phi^{-1}(\tau_{n+1} - \tau_n)|\mathcal{F}_{\tau_n}] \\
&\quad = \mathbb{E}[\mathbf{1}_{\{G=n\}}H_\phi^{-1}(\tau_{n+1}-\tau_n)|\mathcal{F}_{\tau_n}] + \mathbb{E}[\mathbf{1}_{\{G\geq n+1\}}H_\phi^{-1}(\tau_{n+1}-\tau_n)|\mathcal{F}_{\tau_n}] \\
&\quad \leq H_\phi^{-1}(T) + H_\phi^{-1}(T)\mathbb{E}\Big[\mathbf{1}_{\{G\geq n+1\}}\mathbb{E}[H_\phi^{-1}(\tau_{n+1} - (\tau_n + T))|\mathcal{F}_{\tau_n+T}]\Big|\mathcal{F}_{\tau_n}\Big] \\
&\quad \leq H_\phi^{-1}(T) + H_\phi^{-1}(T)\mathbf{1}_{\{G\geq n\}}\mathbb{E}[W_1(X_{\tau_n+T}) + W_2(\tilde{X}_{\tau_n+T})|\mathcal{F}_{\tau_n}] \\
&\quad \leq H_\phi^{-1}(T) + H_\phi^{-1}(T)\mathbf{1}_{\{G\geq n\}}(\psi_2(0, X_{\tau_n}, \tilde{X}_{\tau_n}) + 2KT) \\
&\quad \leq C,
\end{aligned}$$

where $C = H_\phi^{-1}(T)(1 + \kappa + 2KT)$. At the second line we used that $H_\phi^{-1}(s+t) \leq H_\phi^{-1}(s)H_\phi^{-1}(t)$ (because $(\log H_\phi^{-1}(s))' = \phi(H_\phi^{-1}(s))/H_\phi^{-1}(s) \leq 1$). For the last two lines we used the fact that $\psi_2(0, x, y) = W_1(x) + W_2(y)$ is bounded by κ on $\mathcal{K}$.

Step 5. We now conclude the proof of (5.3.9). We make use of Lemma 5.3.1 with, abusing notation, for all $n \geq 0$, $\mathcal{F}_n = \mathcal{F}_{\tau_{n+1}}$, $Y_n = \tau_{n+1} - \tau_n$, which is $\mathcal{F}_n$-measurable and with $N = G - 1$. Observe that the event $\{N = n\} = \{G = n + 1\}$ is $\mathcal{F}_{\tau_{n+2}} = \mathcal{F}_{n+1}$-measurable and, by construction, see Step 1,

$$\mathbf{1}_{\{N\geq n\}}\mathbb{P}(N=n|\mathcal{F}_n)=\mathbf{1}_{\{G\geq n+1\}}\mathbb{P}(G=n+1|\mathcal{F}_{\tau_{n+1}})\geq \alpha\mathbf{1}_{\{G\geq n+1\}}=\alpha\mathbf{1}_{\{N\geq n\}}.$$

Finally, we deduce from (5.3.12) that

$$\mathbf{1}_{\{N\geq n-1\}}\mathbb{E}[H_\phi^{-1}(Y_n)|\mathcal{F}_{n-1}]=\mathbf{1}_{\{G\geq n\}}\mathbb{E}[H_\phi^{-1}(\tau_{n+1}-\tau_n)|\mathcal{F}_{\tau_n}]\leq \mathbf{1}_{\{G\geq n\}}C=C\mathbf{1}_{\{N\geq n-1\}},$$

for $C>0$ constant independent of n. Applying Lemma 5.3.1, we deduce that

$$\mathbb{E}[H_\phi^{-1}(\tau_G-\tau_1)|\mathcal{F}_{\tau_1}]=\mathbb{E}\Big[H_\phi^{-1}\Big(\sum_{n=1}^N Y_n\Big)\Big|\mathcal{F}_0\Big]\leq L,$$

for some constant L depending only on C,α and ϕ. To conclude, we combine this and (5.3.10) and use that $H_\phi^{-1}(s+t)\leq H_\phi^{-1}(s)H_\phi^{-1}(t)$ to find

$$\begin{aligned}\mathbb{E}[H_\phi^{-1}(\tau_G+T)]&\leq H_\phi^{-1}(T)\mathbb{E}[H_\phi^{-1}(\tau_1)H_\phi^{-1}(\tau_G-\tau_1)]\\&\leq H_\phi^{-1}(T)\mathbb{E}\Big[H_\phi^{-1}(\tau_1)\mathbb{E}[H_\phi^{-1}(\tau_G-\tau_1)|\mathcal{F}_{\tau_1}]\Big]\\&\leq LH_\phi^{-1}(T)\mathbb{E}[H_\phi^{-1}(\tau_1)]\\&\leq LH_\phi^{-1}(T)(W_1(x)+W_2(y)).\end{aligned}$$

□

5.3.5 Construction of the function ψ_2 from the function V

Lemma 5.3.2. *Under the hypothesis of Theorem 5.3.1, setting $W_1=W_2=2V:E\to[1,\infty)$ and, for all $t\geq 0$, $x,y\in E$,*

$$\psi_2(t,x,y)=H_\phi^{-1}\Big(H_\phi(2V(x)+2V(y))+t\Big),$$

ψ_2 and W_1, W_2 satisfy the hypothesis of Proposition 5.3.2 with the compact set given, for some well-chosen $M>0$, by

$$\mathcal{K}=\{(x,y):V(x)+V(y)\leq M\}.$$

Proof. Points (b) and (c) are obvious by (5.3.1) and (5.3.2). We thus only have to check (a), i.e. that there is $M>0$ such that, setting $\mathcal{K}=\{(x,y):V(x)+V(y)\leq M\}$, we have $\sup_{(x,y)\in\mathcal{K}}\psi_2(0,x,y)<\infty$ (this quantity being equal to $2M$, this is clear) and that $(\partial_t+\mathcal{L}_2)\psi_2(t,x,y)\leq 0$ for all $t\geq 0$ and all $x,y\in\mathcal{K}^c$.

We set, for all $(x,y)\in E^2$, $W(x,y)=2V(x)+2V(y)$. By assumption (see Theorem 5.3.1) and definition of $\mathcal{L}_2$, we have, for all $x,y\in E$,

$$\mathcal{L}_2W(x,y)\leq 4K-2(\phi(V(x))+\phi(V(y))). \tag{5.3.14}$$

However, $2\phi(x) - \phi(2x)$ increases to infinity as $x \to \infty$, because

$$2\phi(x) - \phi(2x) = \phi(x) - \int_x^{2x} \phi'(s)ds \geq \phi(x) - \phi'(x)x \to \infty$$

as $x \to \infty$, by hypothesis on ϕ. Thus, there exists $M > 0$ such that $V(x) + V(y) \geq M$ implies

$$2\phi(V(x) + V(y)) - 4K \geq \phi(2V(x) + 2V(y)) = \phi(W(x,y)).$$

Combining this with (5.3.14), for x, y in $\mathcal{K}^c$, we have

$$\mathcal{L}_2 W(x,y) \leq -\phi(W(x,y)). \tag{5.3.15}$$

Using now that $H'_\phi(u) = \frac{1}{\phi(u)}$, we find, for $u \geq 1$ and $t \geq 0$, for $\psi(t,x) = H_\phi^{-1}(H_\phi(x) + t)$,

$$\partial_u \psi(t,u) = (H_\phi^{-1})'\Big(H_\phi(u) + t\Big) H'_\phi(u) = \frac{H'_\phi(u)}{H'_\phi(H_\phi^{-1}(H_\phi(u) + t))} = \frac{\phi(\psi(t,u))}{\phi(u)}.$$

We also have

$$\partial_t \psi(t,u) = \frac{1}{H'_\phi(H_\phi^{-1}(H_\phi(u) + t))} = \phi(\psi(t,u)).$$

Using now (5.3.15) and Corollary 5.3.1, we conclude that, for $(x,y) \in \mathcal{K}^c$,

$$\begin{aligned}
(\partial_t + \mathcal{L}_2)\psi_2(t,x,y) &= (\partial_t + \mathcal{L}_2)\Big(\psi(t, W(x,y))\Big) \\
&\leq \partial_u \psi(t, W(x,y)) \times [-\phi(W(x,y))] + \partial_t \psi(t, W(x,y)) \\
&= -\frac{\phi(\psi(t, W(x,y)))}{\phi(W(x,y))} \phi(W(x,y)) + \phi(\psi(t, W(x,y))) = 0,
\end{aligned}$$

as desired. The boundedness of ψ_2 on $\mathcal{K}$ and the fact that $H_\phi^{-1}(t) \leq \psi_2(t,x,y)$ for all triplets $(t,x,y) \in \mathbb{R}_+ \times E^2$ are straightforward. $\square$

5.3.6 Conclusion of the proof of Theorem 5.3.1

Proof of Theorem 5.3.1. We have already verified Point 1, and Point 2 follows from Proposition 5.3.2 and Lemma 5.3.2. It only remains to prove Point 3. We fix $x \in E$ and start from the fact that, since $\mu \mathcal{P}_t = \mu$,

$$\|\mathcal{P}_t(x,\cdot) - \mu\|_{TV} \leq \int_E \|\mathcal{P}_t(x,\cdot) - \mathcal{P}_t(y,\cdot)\|_{TV} \mu(dy).$$

For any $R > 0$, we write

$$\begin{aligned}\|\mathcal{P}_t(x,\cdot)-\mu\|_{TV} &\le \int_{\{y:V(y)\le R\}} \|\mathcal{P}_t(x,\cdot)-\mathcal{P}_t(y,\cdot)\|_{TV}\mu(dy)+2\mu(\{y:V(y)>R\})\\ &\le \frac{C}{H_\phi^{-1}(t)}\Big(V(x)+\int_{\{y:V(y)\le R\}} V(y)\mu(dy)\Big)+2\mu(\{y:V(y)>R\}),\end{aligned}$$

where we used Point 2 of the theorem. Since $x \to \frac{\phi(x)}{x}$ decreases to 0, $V(y) \le \phi(V(y))\frac{R}{\phi(R)}$ for all $y \in E$ with $V(y) \le R$. Recalling that, by Point 1,

$$\Gamma := \int_E \phi(V(y))\mu(dy) < \infty,$$

we conclude that $\int_{\{y\in E, V(y)\le R\}} V(y)\mu(dy)$ is bounded by $\frac{\Gamma R}{\phi(R)}$. We also have, by Markov's inequality

$$\mu(\{y:V(y)>R\}) = \mu(\{y:\phi(V(y))>\phi(R)\}) \le \frac{\Gamma}{\phi(R)}.$$

Hence for all $R > 0$,

$$\|\mathcal{P}_t(x,\cdot)-\mu\|_{TV} \le \frac{C}{H_\phi^{-1}(t)}\Big(V(x)+\frac{\Gamma R}{\phi(R)}\Big)+\frac{\Gamma}{\phi(R)}.$$

We take $R = H_\phi^{-1}(t)$ to conclude that

$$\|\mathcal{P}_t(x,\cdot)-\mu\|_{TV} \le \frac{CV(x)}{H_\phi^{-1}(t)}+\frac{\Gamma(C+1)}{\phi(H_\phi^{-1}(t))}.$$

The proof is complete. $\square$

5.4 New formulation

5.4.1 Hypothesis and statement

In this section, we give the precise statement corresponding to the informal theorem 5.1.2 and detail the proofs. In the whole section, we consider as before a strictly increasing, strictly concave and C^1 function $\phi : [1,\infty) \to \mathbb{R}_+$ such that $\phi(1) > 0$, $\phi(x) \le x$ for all $x \ge 1$, $\frac{\phi(x)}{x} \downarrow 0$ and $\phi(x) - x\phi'(x) \uparrow \infty$ when $x \to \infty$. As before, we introduce the function H_ϕ defined for all $u \ge 1$ by

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)},$$

as well as the corresponding inverse function $H_\phi^{-1} : [0,\infty) \to [1,\infty)$. We have already checked that

$$H_\phi^{-1}(s+t) \le H_\phi^{-1}(s)H_\phi^{-1}(t) \quad \text{for all } s,t \ge 0, \tag{5.4.1}$$

and an immediate study shows that

$$\phi(\kappa x) \le \kappa \phi(x) \quad \text{for all } x \ge 0, \text{ all } \kappa \ge 1. \tag{5.4.2}$$

Theorem 5.4.1. *Assume that $(X_t)_{t\ge 0}$ is non-explosive, irreducible and aperiodic. Consider the three following conditions.*

1. *There exists a compact petite subset C of E and some $r > 0$ such that, for $\tilde{\tau}_C^r$ defined by*

$$\tilde{\tau}_C^r = \inf\Big\{t > 0, \int_0^t \mathbf{1}_C(X_s)ds \ge \frac{T}{r}\Big\},$$

where T is an exponential random variable with parameter 1 independant of everything else, we have

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty \quad \textit{for all } x \in E \quad \textit{ and } \quad \sup_{x\in C} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^r)] < \infty. \tag{5.4.3}$$

2. *There exists a compact petite subset C of E, two constants $\kappa, \eta > 0$ and a function ψ from $\mathbb{R}_+ \times E$ with values in $[1,\infty)$, continuous and non-decreasing in its first argument, continuous in its second argument, such that for all $t \ge 0$, $x \in E$,*

$$H_\phi^{-1}(t) \le \psi(t,x) \qquad \textit{and} \quad (\partial_t + \mathcal{L})\psi(t,x) \le \kappa H_\phi^{-1}(t)\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)),$$

with moreover $\psi(0,x) \le \kappa$ for all $x \in C$ and for all $x \in E$, $\mathcal{L}\psi(0,x) \le \kappa \mathbf{1}_C(x) - \eta$.

3. *There exists a compact petite subset C of E, a constant $K > 0$ and $V : E \to [1,\infty)$ continuous with precompact sublevel sets such that for all $x \in E$,*

$$\mathcal{L}V(x) \le -\phi(V(x)) + K\mathbf{1}_C(x).$$

Conditions 1. and 2. are equivalent, and both are implied by Condition 3. Moreover, in those three cases, there exists an invariant probability measure π for $(\mathcal{P}_t)_{t\ge0}$ on E and for all $x \in E$,

$$\lim_{t\to\infty} \phi(H_\phi^{-1}(t))\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} = 0.$$

We detail the proof below and split it in several subsections, each corresponding to an implication in Theorem 5.4.1. We will use several times the following expression

Remark 5.4.1. For all x and all non-decreasing C^1 function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$,

$$\mathbb{E}_x[f(\tilde{\tau}_C^r)] = \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du} f'(s)ds\Big].$$

Indeed, it suffices to use that $\mathbb{E}_x[f(\tilde{\tau}_C^r)] = \int_0^\infty \mathbb{P}_x(\tilde{\tau}_C^r \geq s) f'(s)ds$ and that

$$\mathbb{P}_x(\tilde{\tau}_C^r \geq s) = \mathbb{P}_x\Big(T \geq r \int_0^s \mathbf{1}_C(X_u)du\Big) = \mathbb{E}_x\Big[e^{-r\int_0^s \mathbf{1}_C(X_u)du}\Big]$$

5.4.2 Proof that Condition 3 implies Condition 2

We introduce $\psi_0 : \mathbb{R}_+ \times [1, \infty) \to [1, \infty)$ defined by $\psi_0(t, x) = H_\phi^{-1}(H_\phi(x) + t)$. It is C^1 in its first argument t and C^2 in its second argument. Moreover, for all $t \geq 0$, all $x \geq 1$,

$$\partial_x \psi_0(t, x) = H'_\phi(x)(H_\phi^{-1})'(H_\phi(x) + t) = \frac{\phi\Big(H_\phi^{-1}(H_\phi(x) + t)\Big)}{\phi(x)} \geq 0.$$

Next,

$$\begin{aligned}\partial_x^2 \psi_0(t, x) &= \frac{\phi'\Big(H_\phi^{-1}(H_\phi(x) + t)\Big)\phi\Big(H_\phi^{-1}(H_\phi(x) + t)\Big) - \phi'(x)\phi\Big(H_\phi^{-1}(H_\phi(x) + t)\Big)}{\phi^2(x)} \\ &= \frac{\phi\Big(H_\phi^{-1}(H_\phi(x) + t)\Big)}{\phi^2(x)}\Big(\phi'\Big(H_\phi^{-1}(H_\phi(x) + t)\Big) - \phi'(x)\Big) \leq 0,\end{aligned}$$

since the first factor is positive, while the second one is negative because ϕ' is decreasing and $x \leq H_\phi^{-1}(H_\phi(x) + t)$. We conclude that ψ_0 satisfies the assumption of Corollary 5.3.1. We set $\psi(t, x) = 2\psi_0(t, V(x)) - H_\phi^{-1}(t)$. On the one hand

$$H_\phi^{-1}(t) = 2H_\phi^{-1}(t) - H_\phi^{-1}(t) \leq 2\psi_0(t, V(x)) - H_\phi^{-1}(t) = \psi(t, x)$$

for all $t \geq 0$, all $x \in E$, and, using Corollary 5.3.1 and that $(H_\phi^{-1})' = \phi \circ H_\phi^{-1}$, one has

$$\begin{aligned}&(\partial_t + \mathcal{L})\psi(t, x) \leq 2\partial_t\psi_0(t, V(x)) + 2\partial_x\psi_0(t, V(x))\mathcal{L}V(x) - \phi(H_\phi^{-1}(t)) \\ &= 2\phi\Big(H_\phi^{-1}(H_\phi(V(x)) + t)\Big) + 2\frac{\phi\Big(H_\phi^{-1}(H_\phi(V(x)) + t)\Big)}{\phi(V(x))}\mathcal{L}V(x) - \phi(H_\phi^{-1}(t)) \\ &\leq 2\phi\Big(H_\phi^{-1}(H_\phi(V(x)) + t)\Big) + 2\frac{\phi\Big(H_\phi^{-1}(H_\phi(V(x)) + t)\Big)}{\phi(V(x))}(-\phi(V(x)) + K\mathbf{1}_C(x)) - \phi(H_\phi^{-1}(t)) \\ &\leq 2K\frac{\phi\Big(H_\phi^{-1}(H_\phi(V(x)) + t)\Big)}{\phi(V(x))}\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)),\end{aligned}$$

where we used the bound on $\mathcal{L}V$ of Condition 3. Using now (5.4.1) and then (5.4.2) (recall that $H_\phi^{-1}(t) \geq 1$), we conclude that

$$(\partial_t + \mathcal{L})\psi(t, x) \leq 2K\frac{\phi\Big(H_\phi^{-1}(t)V(x)\Big)}{\phi(V(x))}\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)) \leq 2KH_\phi^{-1}(t)\mathbf{1}_C(x) - \phi(H_\phi^{-1}(t)).$$

We also have $\psi(0,x) = 2V(x) - 1$, so that indeed $\sup_{x\in C}\psi(0,x) < \infty$ (because C is compact and V has precompact sublevel sets), and, using that $\mathcal{L}1 = 0$, recalling Condition 3, that $V \geq 1$ and that ϕ is non-decreasing,

$$\mathcal{L}\psi(0,x) = 2K\mathbf{1}_C(x) - 2\phi(V(x)) \leq 2K\mathbf{1}_C(x) - 2\phi(1),$$

which completes the proof.

5.4.3 Proof that Condition 2 implies Condition 1

Let $x \in E$ and set, for all $t \geq 0$,

$$M_t = \psi(t, X_t) - \psi(0,x) - \kappa \int_0^t \mathbf{1}_C(X_s) H_\phi^{-1}(s)ds + \int_0^t \phi(H_\phi^{-1}(s))ds,$$

then by Condition 2, $(M_t)_{t\geq 0}$ is a $\mathbb{P}_x$-local supermartingale starting at 0. Hence there exists an increasing to infinity sequence $(\sigma_i)_{i\geq 1}$ of stopping times such that for all $i \geq 1$, $(M_{t\wedge\sigma_i})_{t\geq 0}$ is a bounded supermartingale.

Step 1. We introduce the stopping time

$$\tilde{\tau}^1 = \inf\Big\{t \geq 0, \int_0^t \mathbf{1}_C(X_u)du \geq \frac{1}{2\kappa}\Big\},$$

and note that $X_{\tilde{\tau}^1} \in C$ almost surely. In this step, we show that for all $x \in E$,

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] \leq 2\psi(0,x).$$

For all $i \geq 1$, using that H_ϕ^{-1} is non-decreasing and then that $\tilde{\tau}^1 \wedge \sigma_i \leq \tilde{\tau}^1$,

$$\begin{aligned}
\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1\wedge\sigma_i)] \leq& \mathbb{E}_x[\psi(\tilde{\tau}^1\wedge\sigma_i, X_{\tilde{\tau}^1\wedge\sigma_i})]\\
=&\mathbb{E}_x\Big[\psi(0,x) + \kappa\int_0^{\tilde{\tau}^1\wedge\sigma_i}\mathbf{1}_C(X_u)H_\phi^{-1}(u)du - \int_0^{\tilde{\tau}^1\wedge\sigma_i}\phi(H_\phi^{-1}(s))ds + M_{\tilde{\tau}^1\wedge\sigma_i}\Big]\\
\leq&\psi(0,x) + \kappa\mathbb{E}_x\Big[\int_0^{\tilde{\tau}^1\wedge\sigma_i}\mathbf{1}_C(X_u)H_\phi^{-1}(u)du\Big]\\
\leq&\psi(0,x) + \kappa\mathbb{E}_x\Big[H_\phi^{-1}(\tilde{\tau}^1\wedge\sigma_i)\int_0^{\tilde{\tau}^1}\mathbf{1}_C(X_u)du\Big]\\
=&\psi(0,x) + \frac{1}{2}\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1\wedge\sigma_i)].
\end{aligned}$$

We obtain that for all $i \geq 1$,

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1\wedge\sigma_i)] \leq 2\psi(0,x),$$

and an application of the monotone convergence theorem allows us to conclude.

Step 2. We consider the quantity defined for all $x \in E$, for $\rho \geq 0$ and $r > 0$ by

$$A_{x,\rho,r} := \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)e^{-\rho s^2}ds\Big]$$

which is finite because $(H_\phi^{-1})'(s) = \phi(H_\phi^{-1}(s)) \leq H_\phi^{-1}(s)$, whence $H_\phi^{-1}(s) \leq H_\phi^{-1}(0)e^s = e^s$. We have

$$\begin{aligned}
A_{x,\rho,r} &= \mathbb{E}_x\Big[\int_0^{\tilde{\tau}^1} e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)e^{-\rho s^2}ds\Big] + \mathbb{E}_x\Big[\int_{\tilde{\tau}^1}^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)e^{-\rho s^2}ds\Big] \\
&\leq \mathbb{E}_x\Big[\int_0^{\tilde{\tau}^1}(H_\phi^{-1})'(s)ds\Big] + \mathbb{E}_x\Big[\int_{\tilde{\tau}^1}^\infty e^{-r\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}e^{-r\int_{\tilde{\tau}^1}^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)e^{-\rho s^2}ds\Big] \\
&\leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x\Big[e^{-r\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}\int_{\tilde{\tau}^1}^\infty e^{-r\int_{\tilde{\tau}^1}^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)e^{-\rho s^2}ds\Big].
\end{aligned}$$

Using the strong Markov property

$$\begin{aligned}
A_{x,\rho,r} &\leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] \\
&\qquad + \mathbb{E}_x\Big[e^{-r\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}\mathbb{E}_{X_{\tilde{\tau}^1}}\Big[\int_0^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(\tilde{\tau}^1+s)e^{-\rho(s+\tilde{\tau}^1)^2}ds\Big]\Big] \\
&\leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] \\
&\qquad + \mathbb{E}_x\Big[e^{-r\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}H_\phi^{-1}(\tilde{\tau}^1)\mathbb{E}_{X_{\tilde{\tau}^1}}\Big[\int_0^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)e^{-\rho s^2}ds\Big]\Big]
\end{aligned}$$

because $(H_\phi^{-1})'(\tilde{\tau}^1+s) = \phi(H_\phi^{-1}(\tilde{\tau}^1+s)) \leq \phi(H_\phi^{-1}(\tilde{\tau}^1)H_\phi^{-1}(s)) \leq H_\phi^{-1}(\tilde{\tau}^1)\phi(H_\phi^{-1}(s))$ by (5.4.1) and (5.4.2). Using the definition of $A_{x,\rho,r}$ and the fact that $X_{\tilde{\tau}^1} \in C$, we conclude that

$$A_{x,\rho,r} \leq \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x\Big[e^{-r\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}H_\phi^{-1}(\tilde{\tau}^1)\Big]\sup_{y\in C} A_{y,\rho,r}. \tag{5.4.4}$$

Step 3. We now prove that there is $r_0 > 0$ (large) such that

$$\sup_{x\in C}\mathbb{E}_x\Big[e^{-r_0\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}H_\phi^{-1}(\tilde{\tau}^1)\Big] \leq \frac{1}{2},$$

By definition of $\tilde{\tau}^1$, $\int_0^{\tilde{\tau}^1}\mathbf{1}_C(X_u)du = \frac{1}{2\kappa}$. Hence, for all $x \in E$,

$$\mathbb{E}_x\Big[e^{-r\int_0^{\tilde{\tau}^1} \mathbf{1}_C(X_u)du}H_\phi^{-1}(\tilde{\tau}^1)\Big] = \mathbb{E}_x\Big[e^{-\frac{r}{2\kappa}}H_\phi^{-1}(\tilde{\tau}^1)\Big] \leq 2e^{-\frac{r}{2\kappa}}\psi(0,x)$$

by Step 1. Since $\kappa = \sup_{x\in C}\psi(0,x) < \infty$ by assumption, the conclusion follows.

Step 4. Coming back to (5.4.4), choosing $r = r_0$ and taking the supremum over $x \in C$ on both sides and using Step 3, we find

$$\sup_{x\in C} A_{x,\rho,r_0} \le \sup_{x\in C} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \frac{1}{2}\sup_{x\in C} A_{x,\rho,r_0},$$

so that, using Step 1 and that $\psi(0,\cdot) \le \kappa$ on C,

$$\sup_{x\in C} A_{x,\rho,r_0} \le 4\kappa.$$

We now apply Fatou's lemma and Remark 5.4.1,

$$\sup_{x\in C} \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^{r_0})] = \sup_{x\in C} \mathbb{E}_x\Big[\int_0^\infty e^{-r_0\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)ds\Big] \le \sup_{x\in C}\liminf_{\rho\to 0} A_{x,\rho,r_0} \le 4\kappa.$$

Conclusion We come back to (5.4.4) using the results of Step 1 and Step 4. For all $x \in E$,

$$\begin{aligned} A_{x,\rho,r_0} &\le \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)] + \mathbb{E}_x\Big[e^{-r_0\int_0^{\tilde{\tau}^1}\mathbf{1}_C(X_u)du} H_\phi^{-1}(\tilde{\tau}^1)\Big]\sup_{x\in C} A_{x,\rho,r_0} \\ &\le \mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}^1)](1+4\kappa) \\ &\le 2\psi(0,x)(1+4\kappa). \end{aligned}$$

Hence, as in Step 4,

$$\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C^{r_0})] = \mathbb{E}_x\Big[\int_0^\infty e^{-r_0\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s)ds\Big] \le \liminf_{\rho\to 0} A_{x,\rho,r_0} \le 2\psi(0,x)(1+4\kappa).$$

5.4.4 Proof that Condition 1 implies Condition 2

We fix $r > 0$ so that Condition 1 holds and recall that the randomized hitting time is given by

$$\tilde{\tau}_C^r = \inf\Big\{t > 0, \int_0^t \mathbf{1}_C(X_s)ds > \frac{T}{r}\Big\},$$

where T is a random variable with exponential law of parameter 1 independent of everything else. For the sake of simplicity we will omit the superscript r in what follows and write $\tilde{\tau}_C = \tilde{\tau}_C^r$. Our goal is to show that

$$\psi(t,x) = \mathbb{E}_x\Big[H_\phi^{-1}(\tilde{\tau}_C + t)\Big] = \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(s+t)ds\Big]$$

satisfies Condition 2. The second equality follows from Remark 5.4.1.

We of course have $\psi(t,x) \ge H_\phi^{-1}(t)$ for all $t \ge 0$, all $x \in E$, and $\kappa = \sup_{x\in C}\psi(0,x)$ is finite by assumption.

Consider a sequence $(\varphi_\epsilon)_{\epsilon>0}$ of continuous functions such that $\varphi_\epsilon(x) \downarrow \mathbf{1}_C(x)$ and satisfying $\epsilon \le \varphi_\epsilon(x) \le 1$ for all $x \in E$. This is possible because C is compact. We set, for all $\epsilon > 0$,

$$\psi_\epsilon(t,x) = \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)ds\Big].$$

Step 1: Computation of $(\partial_t + \mathcal{L})\psi_\epsilon(t,x)$. We first have, for $(t,x) \in \mathbb{R}_+ \times E$,

$$\partial_t\psi_\epsilon(t,x) = \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})''(s+t)ds\Big].$$

This is easily justified, using that $\varphi_\epsilon \ge \mathbf{1}_C(x)$ and that

$$\mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})''(s+t)ds\Big] = \mathbb{E}_x[(H_\phi^{-1})'(\tilde\tau_C + t)] \le \phi(1)H_\phi^{-1}(t)\mathbb{E}_x[H_\phi^{-1}(\tilde\tau_C)] < \infty$$

by assumption. We used that

$$(H_\phi^{-1})'(s+t) = \phi(H_\phi^{-1}(s+t)) \le \phi(1)H_\phi^{-1}(s+t) \le \phi(1)H_\phi^{-1}(s)H_\phi^{-1}(t)$$

by (5.4.1).

We use the strong generator. We fix $t \ge 0$ and recall that

$$\mathcal{L}\psi_\epsilon(t,x) = \lim_{v\to 0}\frac{1}{v}(\mathbb{E}_x[\psi_\epsilon(t,X_v)] - \psi_\epsilon(t,x)).$$

For $v > 0$, we have

$$\begin{aligned}\mathbb{E}_x[\psi_\epsilon(t,X_v)] &= \mathbb{E}_x\Big(\mathbb{E}_{X_v}\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)ds\Big]\Big)\\ &= \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_{u+v})du}(H_\phi^{-1})'(s+t)ds\Big]\\ &= \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_v^{s+v} \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)ds\Big].\end{aligned}$$

Noting that

$$\int_v^{s+v}\varphi_\epsilon(X_u)du = \int_0^s \varphi_\epsilon(X_u)du - \int_0^v \varphi_\epsilon(X_u)du + \int_s^{s+v}\varphi_\epsilon(X_u)du,$$

we find

$$\begin{aligned}&\mathbb{E}_x[\psi_\epsilon(t,X_v)] - \psi_\epsilon(t,x)\\ =&\mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)\Big(e^{r\int_0^v \varphi_\epsilon(X_u)du}e^{-r\int_s^{s+v}\varphi_\epsilon(X_u)du} - 1\Big)ds\Big].\end{aligned}$$

Note that since $(X_t)_{t\geq 0}$ is càdlàg and φ_ϵ is smooth, it holds that $\lim_{v\to 0}\frac{1}{v}\int_0^v \varphi_\epsilon(X_u)du = \varphi_\epsilon(X_0)$ and $\lim_{v\to 0}\frac{1}{v}\int_s^{s+v}\varphi_\epsilon(X_u)du = \varphi_\epsilon(X_s)$ a.s. We easily conclude by dominated convergence, using that $\mathbf{1}_C \leq \varphi_\epsilon \leq 1$ and that

$$\mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)ds\Big] = \mathbb{E}_x[(H_\phi^{-1})'(\tilde{\tau}_C+t)] \leq \phi(1)H_\phi^{-1}(t)\mathbb{E}_x[H_\phi^{-1}(\tilde{\tau}_C)] < \infty,$$

that

$$\begin{aligned}\mathcal{L}\psi_\epsilon(t,x) &= \lim_{v\to 0}\frac{1}{v}\Big(\mathbb{E}_x[\psi_\epsilon(t,X_v)] - \psi_\epsilon(t,x)\Big)\\ &= r\mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)(\varphi_\epsilon(x)-\varphi_\epsilon(X_s))ds\Big]\\ &= r\varphi_\epsilon(x)\psi_\epsilon(t,x) - r\mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)\varphi_\epsilon(X_s)ds\Big].\end{aligned}$$

Note that $\partial_s(e^{-r\int_0^s \varphi_\epsilon(X_u)du}) = -r\varphi_\epsilon(X_s)e^{-r\int_0^s \varphi_\epsilon(X_u)du}$ a.s., so that, by integration by parts,

$$\begin{aligned} r\mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)\varphi_\epsilon(X_s)\Big] &= \mathbb{E}_x\Big[\big[-e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)\big]_0^\infty\Big]\\ &\quad + \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})''(s+t)ds\Big].\end{aligned}$$

Using that $\varphi_\epsilon \geq \epsilon$ and the properties of ϕ ($(H_\phi^{-1})'$ is subexponential), one can check that

$$\lim_{s\to\infty}\mathbb{E}_x\Big[e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)\Big] = 0,$$

from which we conclude that

$$\mathbb{E}_x\Big[\big[-e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(s+t)\big]_0^\infty\Big] = (H_\phi^{-1})'(t) = \phi(H_\phi^{-1}(t)).$$

We have proved that, in the sense of the strong generator (which a fortiori implies the result for the weak generator),

$$\begin{aligned}(\partial_t+\mathcal{L})\psi_\epsilon(t,x) &= r\varphi_\epsilon(x)\psi_\epsilon(t,x) - \phi(H_\phi^{-1}(t))\\ &\leq r\varphi_\epsilon(x)H_\phi^{-1}(t)\psi_\epsilon(0,x) - \phi(H_\phi^{-1}(t)).\end{aligned} \tag{5.4.5}$$

We finally used that $\psi_\epsilon(t,x) \leq H_\phi^{-1}(t)\psi_\epsilon(0,x)$, because $H_\phi^{-1}(t+s) \leq H_\phi^{-1}(t)H_\phi^{-1}(s)$.

Step 2: limit as $\epsilon \to 0$**.** By (5.4.5), we know that

$$M_t^\epsilon = \psi_\epsilon(t,X_t) - \psi_\epsilon(0,x) - r\int_0^t \varphi_\epsilon(X_s)H_\phi^{-1}(s)\psi_\epsilon(0,X_s)ds + \int_0^t \phi(H_\phi^{-1}(s))ds$$

is a local supermartingale for each $\epsilon > 0$, and we want to check that

$$M_t = \psi(t, X_t) - \psi(0, x) - r\int_0^t \mathbf{1}_C(X_s)H_\phi^{-1}(s)\psi(0, X_s)ds + \int_0^t \phi(H_\phi^{-1}(s))ds$$

is also a local supermartingale.

It classically suffices to check that for all $T > 0$, $\sup_{[0,T]} |M_t^\epsilon - M_t| \to 0$ a.s. as $\epsilon \to 0$. To this aim, the only issue is to verify that for all $T > 0$, all compact subset $K \subset E$,

$$\sup_{[0,T]\times K} |\psi_\epsilon(t, x) - \psi(t, x)| \to 0. \tag{5.4.6}$$

Recalling that $\varphi_\epsilon \geq \mathbf{1}_C$ and that $(H_\phi^{-1})'$ is non-decreasing, we observe that by definition of ψ_ϵ and ψ, it holds that

$$\sup_{[0,T]} |\psi_\epsilon(t, x) - \psi(t, x)| = \psi(T, x) - \psi_\epsilon(T, x).$$

Since now $\varphi_\epsilon \downarrow \mathbf{1}_C$ pointwise, we deduce from the monotone convergence theorem that for each $x \in E$,

$$\begin{aligned}\psi_\epsilon(T, x) = \mathbb{E}_x\Big[\int_0^\infty e^{-\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})'(T + s)ds\Big] &\overset{\epsilon\to 0}{\uparrow} \mathbb{E}_x\Big[\int_0^\infty e^{-\int_0^s \mathbf{1}_C(X_u)du}(H_\phi^{-1})'(T + s)ds\Big] \\ &= \mathbb{E}_x[H_\phi^{-1}(\tilde\tau_C + T)] = \psi(t, x).\end{aligned}$$

By [80, Theorem 17.25], it follows from the Feller property that when $y \to x$, the process $(X_t^y)_{t\geq 0}$ with semigroup $(\mathcal{P}_t)_{t\geq 0}$ and $X_0^y = y$ converges in distribution, in the Skorokhod space $\mathbb{D}([0,\infty), E)$, towards the process $(X_t^x)_{t\geq 0}$ with semigroup $(\mathcal{P}_t)_{t\geq 0}$ and $X_0^x = x$. We easily deduce the continuity in x of $\psi_\epsilon(T, x)$ and $\psi(T, x)$. We then may use Dini's theorem to conclude that, as desired,

$$\sup_{x\in K}[\psi(T, x) - \psi_\epsilon(T, x)] \to 0$$

as $\epsilon \to 0$, for any compact K of E.

Step 3 : Conclusion. It remains to verify that

$$\mathcal{L}\psi(0, x) \leq \kappa\mathbf{1}_C(x) - \eta$$

Using Step 1 with $t = 0$, we have

$$\begin{aligned}\mathcal{L}\psi_\epsilon(0, x) =& r\varphi_\epsilon(x)\psi_\epsilon(0, x) - \mathbb{E}_x\Big[\int_0^\infty e^{-r\int_0^s \varphi_\epsilon(X_u)du}(H_\phi^{-1})''(s)ds\Big] - \phi(H_\phi^{-1}(0)) \\ \leq& r\varphi_\epsilon(x)\psi_\epsilon(0, x) - \phi(1).\end{aligned}$$

We throwed away the non-negative expectation and used that $H_\phi^{-1}(0) = 1$. Using the same limit procedure as in Step 2 (through local supermartingales), we conclude that

$$\mathcal{L}\psi(0,x) \leq r\mathbf{1}_C(x)\psi(0,x) - \phi(1)$$

and conclude using that $\psi(0,x)$ is bounded on C.

5.4.5 Proof of the result from Condition 2

Existence of an invariant measure. According to [97, Theorems 5 and 6], an invariant probability measure π exists as soon as there exist a petite set C, a constant $b > 0$ and a continuous function $W : E \to [0,\infty)$ such that

$$\mathcal{L}W(x) \leq -1 + b\mathbf{1}_C(x).$$

It directly follows from Condition 2 that $W(x) := \frac{\psi(0,x)}{\eta}$ is convenient. Moreover, according to [97, Theorem 7], for all $x \in E$,

$$\|\mathcal{P}_t(x,\cdot) - \pi(\cdot)\|_{TV} \to 0, \qquad \text{as } t \to \infty. \tag{5.4.7}$$

Convergence result By [59, Theorem 1], with $f_* = 1$ and $r_*(s) = \phi(H_\phi^{-1}(s))$ for all $s \geq 0$, $\Psi_1(u) = u$ and $\Psi_2(v) = 1$, it suffices to verify the following three conditions.

(a) r_* is a rate function in the sense of [59], i.e. $\lim_{s\to\infty} \frac{1}{s}\log(r_*(s)) = 0$. Indeed, setting $g(s) = \ln(H_\phi^{-1}(s))$, $g'(s) = \frac{\phi(H_\phi^{-1}(s))}{H_\phi^{-1}(s)} \to 0$ as $s \to \infty$, by hypothesis on ϕ. Therefore $\frac{g(s)}{s} \to 0$ as $s \to \infty$. Since $\phi(H_\phi^{-1}(s)) \leq \phi(1)H_\phi^{-1}(s)$, the conclusion follows.

(b) There is $t_0 > 0$ such that the Markov chain with matrix $\mathcal{P}_{t_0}$ is irreducible. This follows from (5.4.7) and [100, Theorem 6.1].

(c) There is $\delta > 0$ such that, with the petite set C of Condition 2 and recalling that

$$\tau_C(\delta) = \inf\{t \geq \delta, X_t \in C\},$$

$$\sup_{x\in C} \mathbb{E}_x\Big[\int_0^{\tau_C(\delta)} 1ds\Big] + \sup_{x\in C} \mathbb{E}_x\Big[\int_0^{\tau_C(\delta)} \phi(H_\phi^{-1}(s))ds\Big] < \infty.$$

Since ϕ is bounded from below, it suffices to study the second term. By the usual super-martingale argument, recalling the condition on ψ, we have

$$\mathbb{E}_x[\psi(\tau_C(\delta), X_{\tau_C(\delta)})] \leq \psi(0,x) + \kappa\mathbb{E}_x\Big[\int_0^{\tau_C(\delta)} \mathbf{1}_C(X_s)H_\phi^{-1}(s)ds\Big] - \mathbb{E}_x\Big[\int_0^{\tau_C(\delta)} \phi(H_\phi^{-1}(s))ds\Big].$$

Since now $\int_0^{\tau_C(\delta)} H_\phi^{-1}(s)\mathbf{1}_C(X_s)ds = \int_0^\delta H_\phi^{-1}(s)\mathbf{1}_C(X_s)ds \leq H_\phi^{-1}(\delta)\delta$, we conclude that

$$\mathbb{E}_x\Big[\int_0^{\tau_C(\delta)} \phi(H_\phi^{-1}(s))ds\Big] \leq \psi(0,x) + \kappa\delta H_\phi^{-1}(\delta).$$

Since $\psi(0,\cdot)$ is bounded on C by assumption, we conclude with e.g. $\delta = 1$.

Part III

Hypocoercivity for linear kinetic equations with boundary conditions

Chapter 6

Hypocoercivity for kinetic linear equations in bounded domains with general Maxwell boundary conditions

This chapter corresponds to an article in preparation, written in collaboration with Kleber Carrapatoso[1], Stéphane Mischler[2] and Isabelle Tristani[3].

Abstract: We establish the exponential convergence to equilibrium for linear collisional kinetic equations with the physical local conservation laws in bounded domains with general Maxwell boundary conditions. Our proof consists in establishing the hypocoercivity of the associated operator, in other words, we exhibit a convenient Hilbert norm for which the associated operator is coercive in the orthogonal of the global conservation laws. Our result includes the case of vanishing accommodation coefficient and in particular the case of the specular reflection boundary condition.

Keywords: hypocoercivity, Maxwell boundary condition, specular reflection boundary condition, linearized Boltzmann equation, linearized Landau equation.

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6.1 Introduction

6.1.1 The equation

In this paper, we consider a linear collisional kinetic equation in a bounded domain with general Maxwell boundary conditions.

More precisely, we consider a gas confined in a smooth enough bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, and we denote by $\mathcal{O} := \Omega \times \mathbb{R}^d$ the interior set (of phase space) and $\Sigma := \partial\Omega \times \mathbb{R}^d$ the boundary set (of phase space). The state of the gas is described by the variations $f = f(t, x, v)$ of the density of particles which at time $t \geq 0$ and at position $x \in \Omega$, move with the velocity $v \in \mathbb{R}^d$, around a global equilibrium which we take as the standard (normalized and centered) Maxwellian distribution $\mu = \mu(v) := (2\pi)^{-d/2} e^{-|v|^2/2}$.

The evolution of f is governed by the system of linear equations

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \mathcal{C} f \quad \text{in} \quad (0, \infty) \times \mathcal{O}, \tag{6.1.1}$$

$$\gamma_- f = \mathcal{R} \gamma_+ f \quad \text{on} \quad (0, \infty) \times \Sigma, \tag{6.1.2}$$

where $\gamma_\pm f$ denote the trace of f at the boundary set and where $\mathcal{C}$ and $\mathcal{R}$ stand for two linear collisional operators that we describe below.

Let us first describe the boundary condition (6.1.2). For that purpose, we need to introduce regularity hypotheses on $\partial\Omega$ and some notations. We assume that the boundary $\partial\Omega$ is smooth enough so that the outward unit normal vector $n(x)$ at $x \in \partial\Omega$ is well-defined as well as $d\sigma_x$ the Lebesgue surface measure on $\partial\Omega$. We then define $\Sigma_\pm^x := \{v \in \mathbb{R}^d; \pm v \cdot n(x) > 0\}$ the sets of outgoing (Σ_+^x) and incoming (Σ_-^x) velocities at point $x \in \partial\Omega$ as well as

$$\Sigma_\pm = \Big\{(x, v) \in \Sigma; \pm n(x) \cdot v > 0\Big\} = \Big\{(x, v);\ x \in \partial\Omega,\ v \in \Sigma_\pm^x\Big\}.$$

We denote by γf the trace of f on Σ, and by $\gamma_\pm f = \mathbf{1}_{\Sigma_\pm} \gamma f$ the traces on $\Sigma_\pm$. The boundary condition (6.1.2) thus takes into account how the particles are reflected by the wall and takes the form of a balance between the values of the trace γf on the outgoing and incoming velocities subsets of the boundary. We assume that the reflection operator acts locally in time and position, namely

$$(\mathcal{R}\gamma_+ f)(t, x, v) = \mathcal{R}_x(\gamma_+ f(t, x, \cdot))(v)$$

and more specifically it is a possibly position dependent Maxwell boundary condition operator

$$\mathcal{R}_x(g(x, \cdot))(v) = (1 - \alpha(x)) g(x, R_x v) + \alpha(x) D g(x, v), \tag{6.1.3}$$

for any $(x, v) \in \Sigma_-$ and for any function $g : \Sigma_+ \to \mathbb{R}$. Here $\alpha : \partial\Omega \to [0, 1]$ is a Lipschitz function, called the accommodation coefficient, R_x is the specular reflection operator

$$R_x v = v - 2n(x)(n(x) \cdot v),$$

and D is the diffusive operator

$$Dg(x, v) = c_\mu \mu(v) \widetilde{g}(x), \quad \widetilde{g}(x) = \int_{\Sigma_+^x} g(x, w)\, n(x) \cdot w \, \mathrm{d}w, \tag{6.1.4}$$

where the constant $c_\mu \in \mathbb{R}_+$ is such that

$$\widetilde{c_\mu \mu} = c_\mu \int_{\Sigma_+^x} \mu(w)\, n(x) \cdot w \, \mathrm{d}w = 1.$$

The boundary condition (6.1.3) corresponds to the *specular reflection* boundary condition when $\alpha \equiv 0$ and it corresponds to the *pure diffusive* boundary condition when $\alpha \equiv 1$. It is worth emphasizing that when γf satisfies the boundary condition (6.1.2)–(6.1.3), for any test function $\varphi = \varphi(v)$, we have, for all $x \in \partial\Omega$, setting $\Sigma^x := \Sigma_+^x \cup \Sigma_-^x$,

$$\int_{\Sigma^x} \gamma f \varphi n(x) \cdot v \, \mathrm{d}v = \int_{\Sigma_+^x} \gamma_+ f \, n(x) \cdot v \, [\varphi - (1 - \alpha(x)) \varphi \circ R_x - \alpha(x) \widetilde{\varphi \circ R_x c_\mu \mu)}] \, \mathrm{d}v. \tag{6.1.5}$$

As a consequence, whatever is the accommodation coefficient α, making the choice $\varphi = 1$ so that $\varphi \circ R_x = \widetilde{c_\mu \varphi \mu} = 1$, we get

$$\int_{\Sigma^x} \gamma f \, n(x) \cdot v \, \mathrm{d}v = 0, \tag{6.1.6}$$

which means that there is always no flux of mass at the boundary (no particle goes out nor enters in). Assuming $\alpha \equiv 0$, making the choice $\varphi(v) = |v|^2$ and observing that $|R_x v|^2 = |v^2|$, we get

$$\int_{\Sigma^x} \gamma f \, |v|^2 \, n(x) \cdot v \, \mathrm{d}v = 0, \tag{6.1.7}$$

which means that there is no flux of energy at the boundary in the case of the pure specular reflection boundary condition.

Let us now describe the hypotheses made on the collisional linear operator $\mathcal{C}$ involved in the linear evolution equation (6.1.1). We assume that the operator acts locally in time and position, namely

$$(\mathcal{C}f)(t, x, v) = \mathcal{C}(f(t, x, \cdot))(v),$$

have the mass, velocity and energy conservation laws, namely

$$\int_{\mathbb{R}^d} (\mathcal{C}g)(v)\, \varphi(v) \, \mathrm{d}v = 0, \tag{6.1.8}$$

for $\varphi := 1, v_i, |v|^2$, $i \in \{1, \dots, d\}$ and any nice enough function g, and a spectral gap in the classical Hilbert space associated to the standard Maxwellian. In order to be more precise, we introduce the Hilbert space

$$L^2_v(\mu^{-1}) := \left\{ f : \mathbb{R}^d \to \mathbb{R} \,\Big|\, \int_{\mathbb{R}^d} f^2 \mu^{-1} \,\mathrm{d}v < +\infty \right\}$$

endowed with the scalar product

$$(f, g)_{L^2_v(\mu^{-1})} := \int_{\mathbb{R}^d} f g \mu^{-1} \,\mathrm{d}v$$

and the norm

$$\|f\|^2_{L^2_v(\mu^{-1})} := \int_{\mathbb{R}^d} f^2 \mu^{-1} \,\mathrm{d}v.$$

We assume that the operator $\mathcal{C}$ satisfies on $L^2_v(\mu^{-1})$:

(i) Its kernel is given by

$$\ker(\mathcal{C}) = \mathrm{span}\{\mu, v_1\mu, \dots, v_d\mu, |v|^2\mu\},$$

and we denote by πf the projection onto $\ker(\mathcal{C})$ given by

$$\pi f = \left(\int_{\mathbb{R}^d} f \,\mathrm{d}w \right) \mu + \left(\int_{\mathbb{R}^d} w f \,\mathrm{d}w \right) \cdot v\mu + \left(\int_{\mathbb{R}^d} \frac{|w|^2 - d}{\sqrt{2d}} f \,\mathrm{d}w \right) \frac{|v|^2 - d}{\sqrt{2d}} \mu. \tag{6.1.9}$$

(ii) The operator is self-adjoint on $L^2_v(\mu^{-1})$ and negative $(\mathcal{C}f, f)_{L^2_v(\mu^{-1})} \leq 0$, so that its spectrum is included in $\mathbb{R}_-$, and (6.1.8) holds true for any $g \in L^2_v(\mu^{-1})$. We assume further that $\mathcal{C}$ satisfies a coercivity estimate: there is a positive constant $\lambda > 0$ such that for any $f \in \mathrm{Dom}(\mathcal{C})$ one has

$$(-\mathcal{C}f, f)_{L^2_v(\mu^{-1})} \geq \lambda \|f^\perp\|^2_{L^2_v(\mu^{-1})}, \tag{6.1.10}$$

where $f^\perp := f - \pi f$.

(iii) For any polynomial function $\phi = \phi(v) : \mathbb{R}^d \to \mathbb{R}$ of degree ≤ 4, one has

$$\left| \int_{\mathbb{R}^d} \phi(v) f^\perp \,\mathrm{d}v \right| \leq C \|f^\perp\|_{L^2_v(\mu^{-1})}$$

and

$$\left| (\mathcal{C}f^\perp, \phi\mu)_{L^2_v(\mu^{-1})} \right| = \left| (f^\perp, \mathcal{C}(\phi\mu))_{L^2_v(\mu^{-1})} \right| \leq C \|f^\perp\|_{L^2_v(\mu^{-1})}.$$

It is worth mentioning that the typical examples of collision operators we have in mind are the ones corresponding to the linearized Boltzmann and Landau equations. More precisely, our assumptions cover the cases of the linearized operators associated to Boltzmann with angular cutoff operators for hard potentials and Maxwell molecules, Boltzmann without angular cutoff and Landau operators for hard potentials, Maxwell molecules and moderately soft potentials since in all those cases, the homogeneous linearized operators enjoy some strong coercivity property of type (6.1.10).

6.1.2 Conservation laws

Without loss of generality, we shall consider hereafter that the domain Ω verifies

$$\int_\Omega \mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega x \, \mathrm{d}x = 0. \tag{6.1.11}$$

One easily obtains from (6.1.8), the Stokes theorem and (6.1.6) that any solution f to equation (6.1.1)–(6.1.2) satisfies the conservation of mass

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} f \, \mathrm{d}v \, \mathrm{d}x = \int_{\mathcal{O}} (\mathcal{C}(f) - v \cdot \nabla_x f) \, \mathrm{d}v \, \mathrm{d}x = 0. \tag{6.1.12}$$

In the case of the specular reflection boundary condition, that is (6.1.2) with $\alpha \equiv 0$, some additional conservation laws appear. On the one hand, one also has the conservation of the energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}} |v|^2 f \, \mathrm{d}v \, \mathrm{d}x = \int_{\mathcal{O}} |v|^2 (\mathcal{C}(f) - v \cdot \nabla_x f) \, \mathrm{d}v \, \mathrm{d}x = 0, \tag{6.1.13}$$

because of (6.1.8) the Stokes theorem again and (6.1.7). On the other hand, if the domain Ω possesses rotational symmetry, then we also have the conservation of the corresponding angular momentum. More precisely, we define the set of all infinitesimal rigid displacement fields

$$\mathscr{R} := \{x \in \Omega \mapsto Ax + b \in \mathbb{R}^d : A \in \mathcal{M}_d^a(\mathbb{R}), \; b \in \mathbb{R}^d\}, \tag{6.1.14}$$

where $\mathcal{M}_d^a(\mathbb{R})$ denotes the set of skew-symmetric $d \times d$-matrices with real coefficients, as well as the linear manifold of *centered* infinitesimal rigid displacement fields preserving Ω

$$\mathscr{R}_\Omega = \{R \in \mathscr{R} \mid b = 0, \; R(x) \cdot n(x) = 0, \; \forall x \in \partial\Omega\}. \tag{6.1.15}$$

We observe here that, thanks to the assumption (6.1.11), we can work only with *centered* infinitesimal rigid displacement fields preserving Ω. Indeed, if R is an infinitesimal rigid displacement field preserving Ω, that is, $R(x) = A(x) + b \in \mathscr{R}$ is such that $R(x) \cdot n(x) = 0$ on $\partial\Omega$, then we obtain

$$\begin{aligned} |b|^2 &= \int_\Omega \nabla(b \cdot x) \cdot (Ax + b) \, \mathrm{d}x \\ &= -\int_\Omega (b \cdot x) \operatorname{div}(Ax + b) \, \mathrm{d}x + \int_{\partial\Omega} (b \cdot x)(Ax + b) \cdot n(x) \, \mathrm{d}\sigma_x = 0 \end{aligned}$$

and thus $b = 0$. When the set $\mathscr{R}_\Omega$ is not reduced to $\{0\}$, that is when Ω has rotational symmetry, then one has the conservation of angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} R(x)\cdot v f \,\mathrm{d}v\,\mathrm{d}x = 0, \tag{6.1.16}$$

for any $R \in \mathscr{R}_\Omega$. Indeed if $Ax \in \mathscr{R}_\Omega$ we then compute, using integration by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} Ax\cdot v f \mathrm{d}v\,\mathrm{d}x &= \int_{\mathcal{O}} Ax\cdot v(-v\cdot\nabla_x f + \mathcal{C}f)\,\mathrm{d}v\,\mathrm{d}x\\ &= \int_{\mathcal{O}} \partial_{x_k}(Ax\cdot v)v_k f\,\mathrm{d}v\,\mathrm{d}x - \int_\Sigma Ax\cdot v\,\gamma f\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x\\ &= -\int_\Sigma Ax\cdot v\,\gamma f\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x,\end{aligned}$$

thanks to property (i) of the collision operator $\mathcal{C}$ and the fact that A is skew-symmetric. For the boundary term, using (6.1.5) with $\varphi(x,v) := Ax\cdot v$ and $\alpha \equiv 0$, we get

$$\begin{aligned}\int_\Sigma Ax\cdot v\,\gamma f\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x &= \int_{\Sigma_+} Ax\cdot(v - R_x v)\gamma_+ f\,|n(x)\cdot v|\,\mathrm{d}v\,\mathrm{d}\sigma_x\\ &= 2\int_{\Sigma_+} Ax\cdot n(x)\gamma_+ f\,|n(x)\cdot v|^2\,\mathrm{d}v\,\mathrm{d}\sigma_x = 0.\end{aligned}$$

by definition of $Ax \in \mathscr{R}_\Omega$.

6.1.3 Main result

Define the Hilbert space

$$L^2_{x,v}(\mu^{-1}) := \left\{ f : \mathcal{O}\to\mathbb{R} \;\middle|\; \int_{\mathcal{O}} f^2\mu^{-1}\,\mathrm{d}v\,\mathrm{d}x < +\infty \right\}$$

endowed with the scalar product

$$\langle f, g\rangle_{L^2_{x,v}(\mu^{-1})} := \int_{\mathcal{O}} f g \mu^{-1}\,\mathrm{d}v\,\mathrm{d}x$$

and the norm

$$\|f\|^2_{L^2_{x,v}(\mu^{-1})} := \int_{\mathcal{O}} f^2\mu^{-1}\,\mathrm{d}v\,\mathrm{d}x.$$

We are now able to state the main result of this paper.

Theorem 6.1.1. *Let $f_{\mathrm{in}} \in L^2_{x,v}(\mu^{-1})$ satisfy*

i) in the specular reflection case ($\alpha \equiv 0$ in (6.1.2))

$$\int_{\mathcal{O}} f_{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int_{\mathcal{O}} |v|^2 f_{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = 0, \quad \int_{\mathcal{O}} R\cdot v f_{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v = 0,$$

for any $R \in \mathscr{R}_\Omega$*;*

ii) otherwise

$$\int_{\mathcal{O}} f_{\mathrm{in}} \, \mathrm{d}x \, \mathrm{d}v = 0.$$

There exist positive constants $\kappa, C > 0$ *such that for any solution* f *to* (6.1.1)–(6.1.2) *associated to the initial data* f_{in} *and for any* $t \geq 0$ *there holds*

$$\|f(t)\|_{L^2_{x,v}(\mu^{-1})} \leq Ce^{-\kappa t} \|f_{\mathrm{in}}\|_{L^2_{x,v}(\mu^{-1})}.$$

Our paper improves the existing results regarding the stability of the *linearized* Boltzmann and Landau equations that enjoy a spectral gap estimate (namely for not too soft potentials) in two regards:

– we study a general, smooth enough, convex or non-convex domain;

– our L^2 estimates are constructive, and our method encompasses the three boundary conditions (pure diffusive, specular reflection and Maxwell) in a single treatment. In particular, we can solve the Maxwell boundary condition in the case where the accommodation coefficient α vanishes on some subset of the boundary.

State of the art. Let us describe some well-known results concerning the trend to equilibrium for kinetic equations in bounded domains.

The Cauchy theory as well as the trend-to-equilibrium issue for the cutoff Boltzmann equation with hard potentials or hard-spheres in a perturbative regime, that is for initial data sufficiently close to the Maxwellian equilibrium, has been developed by Guo [68], who proved exponential convergence towards equilibrium in a weighted $L^\infty_{x,v}$ space considering two different cases: the specular reflection boundary condition with strictly convex and analytic domains Ω, and the pure diffusive boundary condition assuming the domain Ω is smooth and convex. We also refer to Briant [18], who obtained similar results considering more general weights. The $L^2 - L^\infty$ theory of [68] works as follows: the coercive property of the linearized collision operator is captured in the space $L^2_{x,v}$, and $L^\infty_{x,v}$ estimates are derived by an analysis of the iterated Duhamel formula. In [68, 18], the $L^2_{x,v}$ coercivity estimates rely on a non-constructive $L^2_{x,v}$ theory.

More recently, still for the cutoff Boltzmann equation with hard potentials or hard-spheres, Briant and Guo [20] derived constructive results in $L^2_{x,v}$ for positive constant accommodation coefficient $\alpha > 0$ with no convexity assumptions on Ω, and conclude to the exponential convergence in a weighted (with stretch exponential or polynomial weight) space $L^\infty_{x,v}$. Furthermore, for the specular reflection boundary condition, well-posedness and stability results relying on non-constructive L^2 estimates were derived in the convex setting, without analyticity assumptions on the domain, by Kim and Lee [81]. The authors then extended their results to periodic cylindrical domain with non-convex analytic cross-section [82].

The only results we are aware of in the case of long-range interaction, that is, for non-cutoff Boltzmann and Landau collision operators in a bounded domain, are the very recent works of Guo-Hwang-Jang-Ouyang [70] (see also [69]) for the Landau equation with specular reflection boundary condition, and Duan-Liu-Sakamoto-Strain [49] for non-cutoff Boltzmann and Landau equations in a finite channel with inflow or specular reflection boundary conditions.

Strategy of the proof. Our proof is based on a L^2-hypocoercivity approach. The challenge of hypocoercivity is to understand the interplay between the collision operator that provides dissipativity and the transport one which is conservative, in order to obtain global dissipativity for the whole problem. There are two main hypocoercivity methods, the H^1 and the L^2 ones. The H^1-hypocoercivity approach has been first introduced for hypoelliptic operators by Hérau, Nier [77] and Eckmann, Hairer [53], further developed by Helffer, Nier [74] and Villani [123] and extended to more general kinetic operators in Villani [123] and Mouhot, Neumann [103]. In summary, the idea consists in endowing the H^1 space with a new scalar product which makes coercive the considered operator and whose associated norm is equivalent to the usual H^1 norm. In order to be adapted to more general operators and geometries, the L^2-hypocoercivity technique has been next introduced by Hérau [76] and developed by Dolbeault-Mouhot-Schmeiser [43, 42]. Again the idea consists in endowing the L^2 space with a new scalar product which makes coercive the considered operator and whose associated norm is equivalent to the usual L^2 norm. This approach that we adopt heavily relies on the micro-macro decomposition of the solution of the equation.

More precisely, our strategy follows the classical line of reasoning presented hereafter. Notice first that splitting the solution as $f = f^\perp + \pi f$, where $f^\perp$ denotes the microscopic part and πf the macroscopic part defined in (6.1.9), the coercive estimate (6.1.10) on the collision operator $\mathcal{C}$ already gives us a control on $f^\perp$. However the conservative skew-symmetric transport operator $-v \cdot \nabla_x$ does not provide any dissipative property. Therefore, as explained above, in order to control the missing macroscopic part πf, the idea is to construct a new inner product on $L^2_{x,v}(\mu^{-1})$, the associated norm of which being equivalent to the usual one, for which the full operator $\mathcal{L}$ is coercive. The construction of this new inner product starts with the usual inner product of $L^2_{x,v}(\mu^{-1})$ to which we add, step by step, new terms in order to control the missing terms appearing on the macroscopic part πf. More specifically, we introduce the modified scalar product

$$\langle\!\langle f, g \rangle\!\rangle \;:=\; \langle f, g\rangle_{L^2_{x,v}(\mu^{-1})} - \varepsilon \left\langle \widetilde{\pi} f, \nabla \Delta^{-1} \pi g \right\rangle_{L^2_x(\Omega)} - \varepsilon \left\langle \nabla \Delta^{-1} \pi f, \widetilde{\pi} g \right\rangle_{L^2_x(\Omega)},$$

choosing $\varepsilon > 0$ small enough, and where the moments operator $\widetilde{\pi} : L^2_{x,v}(\mu^{-1}) \to (L^2_x(\Omega))^{d+2}$ and the inverse Laplacian type operator Δ^{-1} have to be suitably defined (see Sections 6.2 & 6.3).

Our proof is a variant of the previous proofs but differs from them by several aspects:

– The order between the ∇ operator and the Δ^{-1} operator is the one from Guo's approach rather than the one from Dolbeault-Mouhot-Schmeiser's approach. That is important in order to handle the boundary condition which leads to a rather singular operator.

– The choice of the mean operator $\widetilde{\pi} f$ differs from the one used in [68, 20, 18, 82] but looks very like the one in [47, 48]. It allows to deal with general Maxwell boundary condition (and the possibility that α vanishes somewhere or everywhere) but leads to a first natural control of the symmetric gradient of the momentum component of the macroscopic part $\nabla^s m$ instead of the full derivative ∇m as in Guo's approach.

– The definition of the Δ^{-1} operator has to be chosen wisely in order to handle the general Maxwell boundary condition and the mean operator $\widetilde{\pi} f$.

– We also need to establish natural $H^{-1} \to H^1$ and $L^2 \to H^2$ regularity estimates for some classical elliptic problems but associated with somehow unusual boundary conditions.

Let us give more details about these last two points. First, we shall introduce an auxiliary Poisson equation with Robin or Neumann boundary conditions, which are devised in order to control the *mass* and *energy* terms of πf. This result is stated in Theorem 6.2.1 and is based on Poincaré type inequalities. Then, we shall introduce a tailored Lamé-type system with mixed Robin-type boundary conditions in order to deal with the *momentum* component of the macroscopic part πf. The corresponding result is presented in Theorem 6.2.2 and is based on Korn-type inequalities, which are discussed in Section 6.2.2. For more insights on Korn inequalities we refer to Duvaut-Lions [52, Theorem 3.2 Chap. 3], and to the variant introduced by Desvillettes and Villani [38]. A recent treatment of Korn's inequality is given in Ciarlet and Ciarlet [30]. For more details concerning regularity and non-regularity issues for similar elliptic equations and systems we refer to [66, 32, 110] and the references therein.

6.2 Elliptic equations

We present some functional estimates associated to some elliptic equations related to the macroscopic quantities. In this section, we denote the classical norm on $L^2(\Omega)$ by $\|\cdot\|$ and by $(\cdot,\cdot)$ the associated scalar product. We also write $\langle f\rangle := |\Omega|^{-1}\int_\Omega f\,\mathrm{d}x$. The operators that we consider only act on the variable x so in order to lighten the notations, we will not mention it in our proofs. For the same reason, we often write ∂_i for ∂_{x_i}, $i\in\{1,\dots,d\}$.

6.2.1 Poincaré inequalities and Poisson equation

We consider the following Poisson equation

$$\begin{cases} -\Delta u = \xi & \text{in} \quad \Omega, \\ (2-\alpha(x))\nabla u\cdot n(x) + \alpha(x) u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.2.1}$$

for a scalar source term $\xi : \Omega \to \mathbb{R}$. Remark that when $\alpha \equiv 0$ then (6.2.1) corresponds to the Poisson equation with homogeneous Neumann boundary condition. Otherwise, (6.2.1) corresponds to the Poisson equation with homogeneous Robin (or mixed) boundary condition.

We define the Hilbert spaces

$$V_1 := H^1(\Omega) \quad \text{and} \quad V_0 := \left\{ u \in H^1(\Omega);\ \int_\Omega u \,\mathrm{d}x = 0 \right\}$$

endowed with the $H^1(\Omega)$-norm, we denote

$$V_\alpha := \begin{cases} V_1 & \text{if} \quad \alpha \not\equiv 0 \\ V_0 & \text{if} \quad \alpha \equiv 0 \end{cases}$$

and we remark that $H_0^1(\Omega) \subseteq V_\alpha \subseteq H^1(\Omega)$ and thus $(H^1(\Omega))' \subseteq V_\alpha' \subseteq H^{-1}(\Omega)$.

We also define on V_α the bilinear form

$$a_\alpha(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} \frac{\alpha}{2-\alpha}\, uv \,\mathrm{d}\sigma_x.$$

We recall the Poincaré-Wirtinger inequality

$$\forall\, u \in V_0, \quad \|u\|_{L^2(\Omega)} \lesssim \|\nabla_x u\|_{L^2(\Omega)}, \tag{6.2.2}$$

and when $\alpha \not\equiv 0$, the Poincaré-type inequality

$$\forall\, u \in V_1, \quad \|u\|^2_{L^2(\Omega)} \lesssim a_\alpha(u,u). \tag{6.2.3}$$

We have no precise reference for a constructive proof of this classical inequality. However, for the sake of completeness and because we will need to repeat that kind of argument in the next section, we give a sketch of a non constructive proof by contradiction based on a compactness argument. Assuming that (6.2.3) is not true, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega)$ such that

$$1 = \|u_n\|^2_{L^2(\Omega)} \geq n \left(\|\nabla u_n\|^2_{L^2(\Omega)} + \left\| \sqrt{\frac{\alpha}{2-\alpha}}\, u_n \right\|^2_{L^2(\partial\Omega)} \right).$$

As a consequence, up to the extraction of a subsequence, there exists $u \in H^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and $u_n \to u$ strongly in $L^2(\Omega)$. From the above estimate we deduce that $\|\nabla u\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^2(\Omega)} = 0$, so that $u = C$ is a constant. On the one hand, we have $\|\sqrt{\alpha/(2-\alpha)}u\|_{L^2(\partial\Omega)} = \lim_{n\to\infty} \|\sqrt{\alpha/(2-\alpha)}u_n\|_{L^2(\partial\Omega)} = 0$ so that $C = 0$. On the other hand, we get $\|u\|_{L^2(\Omega)} = \lim_{n\to\infty} \|u_n\|_{L^2(\Omega)} = 1$, which implies that $C \neq 0$ and thus a contradiction.

We now state a result on the existence, uniqueness and regularity of solutions to (6.2.1).

Theorem 6.2.1. *For any given $\xi \in V_\alpha'$, there exists a unique $u \in V_\alpha$ solution to the variational problem*

$$a_\alpha(u,w) = \langle \xi, w \rangle_{V_\alpha', V_\alpha}, \quad \forall w \in V_\alpha, \tag{6.2.4}$$

and this one satisfies

$$\|u\|_{H^1(\Omega)} \lesssim \|\xi\|_{(H^1(\Omega))'}. \tag{6.2.5}$$

If furthermore $\xi \in L^2(\Omega)$ with $\langle \xi \rangle = 0$ when $\alpha \equiv 0$, there holds $u \in H^2(\Omega)$, u verifies the elliptic problem (6.2.1) a.e. and

$$\|u\|_{H^2(\Omega)} \lesssim \|\xi\|_{L^2(\Omega)}. \tag{6.2.6}$$

We give a sketch of the proof of Theorem 6.2.1 which is very classical, except maybe the way we handle the H^2 regularity estimate. The proof will be taken up again in the next section where we deal with elliptic system of equations associated to the symmetric gradient.

Proof of Theorem 6.2.1. We split the proof into 4 steps. The first one is dedicated to the application of Lax-Milgram theorem. The last three ones are devoted to the proof of the H^2 regularity estimate: in Step 2, we develop a formal argument which leads to a directional regularity estimate supposing that the variational solution u is *a priori* smooth; we then make it rigorous in Step 3 by not supposing any smoothness assumption on u and in Step 4, we end the proof of (6.2.6).

Step 1. We first observe that there exists $\lambda > 0$ such that

$$a_\alpha(u,u) \geq \lambda \|u\|^2_{H^1(\Omega)}, \quad \forall\, u \in V_\alpha,$$

and thus a_α is coercive. The above estimate is a direct consequence of the Poincaré-Wirtinger inequality (6.2.2) in the case when $\alpha \equiv 0$ and the variant of the classical Poincaré inequality given in (6.2.3) when $\alpha \not\equiv 0$. As a consequence of the Lax-Milgram theorem, we get the existence and uniqueness of $u \in V_\alpha$ satisfying (6.2.4) and (6.2.5).

For the remainder of the proof, we suppose that $\xi \in L^2(\Omega) \subseteq (H^1(\Omega))'$ with $\langle \xi \rangle = 0$ when $\alpha \equiv 0$. We claim that (6.2.4) can be improved into the following new formulation: there exists a unique $u \in V_\alpha$ satisfying

$$a_\alpha(u,w) = \langle \xi, w \rangle_{(H^1(\Omega))', H^1(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{6.2.7}$$

When $\alpha \not\equiv 0$ formulation (6.2.7) is nothing but (6.2.4) since in that case $V_\alpha = H^1(\Omega)$. In the case $\alpha \equiv 0$, we remark that for any $w \in H^1(\Omega)$ we have $w - \langle w \rangle \in V_0$ and therefore

$$\begin{aligned} a_\alpha(u,w) &= a_\alpha(u, w - \langle w \rangle) \\ &= \int_\Omega \xi w \, \mathrm{d}x - \int_\Omega \xi \, \langle w \rangle \, \mathrm{d}x \\ &= \int_\Omega \xi w \, \mathrm{d}x \end{aligned}$$

where we have used that $\nabla(\langle w\rangle) = 0$ in the first line, formulation (6.2.4) in the second line, and the condition $\langle \xi \rangle = 0$ so that $\int_\Omega \xi \, \langle w\rangle \, \mathrm{d}x = 0$ in the third line.

Step 2. A priori directional estimate. For any small enough open set $\omega \subset \Omega$, we fix a vector field $a \in C^2(\bar\Omega)$ such that $|a| = 1$ on ω and $a \cdot n = 0$ on $\partial\Omega$, and we set $X := a \cdot \nabla$ the associated differential operator. For a smooth function u, we compute

$$\begin{aligned} \|\nabla X u\|^2 &= (\nabla u, X^* \nabla X u) + ([\nabla, X]u, \nabla X u) \\ &= (\nabla u, \nabla X^* X u) + (\nabla u, [X^*, \nabla] X u) + ([\nabla, X]u, \nabla X u) \end{aligned}$$

and where we have used that

$$(Xf, g) = (f, X^* g), \quad X^* g := -\mathrm{div}(ag),$$

because $a \cdot n = 0$ on $\partial\Omega$. On the other hand, we compute formally

$$\int_{\partial\Omega} (Xu)^2 \, \frac{\alpha}{2-\alpha} \mathrm{d}\sigma_x = \int_{\partial\Omega} \frac{\alpha}{2-\alpha} \, u(X^* X u) \, \mathrm{d}\sigma_x - \int_{\partial\Omega} \left(X \frac{\alpha}{2-\alpha} \right) u(Xu) \, \mathrm{d}\sigma_x. \tag{6.2.8}$$

Notice here that we implicitly assumed that there is no boundary term in our integration by parts. In the next step of the proof, we will work with a discrete version of the operator X which will allow us to make rigorous computations. Assuming furthermore now that $u \in V_\alpha$ satisfies (6.2.7) and that $X^* X u \in H^1(\Omega)$, we may use (6.2.7) and we deduce

$$\begin{aligned} &\|\nabla X u\|^2 + \int_{\partial\Omega} \frac{\alpha}{2-\alpha} (Xu)^2 \, \mathrm{d}\sigma_x \\ &\quad = (\xi, X^* X u) + (\nabla u, [X^*, \nabla] X u) + ([\nabla, X]u, \nabla X u) - \int_{\partial\Omega} \left(X \frac{\alpha}{2-\alpha} \right) u(Xu) \, \mathrm{d}\sigma_x. \end{aligned}$$

We easily compute for $i = 1, \dots, d$

$$[\partial_i, X] = (\partial_i a) \cdot \nabla, \quad [X^*, \partial_i] = \partial_i (\mathrm{div} a) + (\partial_i a) \cdot \nabla,$$

so that for some constant $C = C(\|a\|_{W^{2,\infty}(\Omega)})$ and any function w on Ω, we have

$$\|[\partial_i, X] w\| \le C \|\nabla w\|, \quad \|[X^*, \nabla] w\| \le C \|w\|_{H^1(\Omega)}.$$

We then deduce that for some constant $C = C(\|a\|_{W^{2,\infty}(\Omega)}, \|\alpha\|_{W^{1,\infty}(\Omega)})$, we have

$$\|\nabla X u\|^2 \le \|\xi\| \|X^* X u\| + C\|\nabla u\| \|Xu\|_{H^1} + C\|\nabla u\| \|\nabla X u\| + C\|u\|_{L^2(\partial\Omega)} \|Xu\|_{L^2(\partial\Omega)}.$$

Recalling (6.2.5) and observing that $\|X^* w\| + \|Xw\| + \|w\|_{L^2(\partial\Omega)} \lesssim \|w\|_{H^1(\Omega)}$, we obtain

$$\|\nabla X u\|^2 \lesssim \|\xi\| \|\nabla X u\| + \|\xi\|^2$$

and we conclude that

$$\|\nabla X u\| \lesssim \|\xi\|. \tag{6.2.9}$$

Step 3. Rigorous directional estimate. When we do not deal with an *a priori* smooth solution, but just with a variational solution $u \in V_\alpha$ satisfying (6.2.7), we have to modify the argument in the following way. We define $\Phi_t : \bar\Omega \to \bar\Omega$ the flow associated to the differential equation

$$\dot y = a(y), \quad y(0) = x,$$

so that $\Phi_t(x) := y(t)$, $(t,x) \mapsto \Phi_t(x)$ is C^1 and Φ_t is a diffeomorphism on both Ω and $\partial\Omega$ for any $t \in \mathbb{R}$.

We next define

$$X^h u(x) := \frac{1}{h}(u(\Phi_h(x)) - u(x)),$$

so that $X^h u \in H^1(\Omega)$ if $u \in V_\alpha$. Repeating the argument of Step 1, we get the identity

$$\begin{aligned} \|\nabla X^h u\|^2 + \int_{\partial\Omega} \frac{\alpha}{2-\alpha}(X^h u)^2 \, d\sigma_x &= (\xi, X^{h*} X^h u) + (\nabla u, [X^{h*}, \nabla] X^h u) \\ + \ ([\nabla, X^h]u, \nabla X^h u) &- \int_{\partial\Omega} u(\Phi_h(x)) \Big((X^h u)\, X^h \Big(\frac{\alpha}{2-\alpha} \Big) \Big)(x) \, d\sigma_x, \end{aligned} \tag{6.2.10}$$

where we denote

$$X^{h*} w(x) := \frac{1}{h}\Big[w(\Phi_{-h}(x))\, |\mathrm{det} D\Phi_{-h}(x)| - w(x) \Big].$$

Notice here that we used a discrete version of the integration by parts leading to (6.2.8) and it only relies on a change of variable on $\partial\Omega$, which makes our computation fully rigorous. As in the second step of the proof, we are now going to bound each term of the right-hand-side of (6.2.10). First, notice that for $|h| \le 1$, we have for some $|h_0| \le 1$:

$$X^h u(x) = \sum_j \partial_j u(\Phi_{h_0}(x)) a_j(\Phi_{h_0}(x))$$

so that there exists $C = C(\|a\|_{W^{1,\infty}(\Omega)})$ such that for any $|h| \le 1$, we have $\|X^h u\| \le C\|\nabla u\|$. We can estimate $\|X^{h*} w\|$ in a similar way using that

$$X^{h*} w(x) = \frac{1}{h}[w(\Phi_{-h}(x)) - w(x)]\, |\mathrm{det} D\Phi_{-h}(x)| + \frac{1}{h} w(x) [\, |\mathrm{det} D\Phi_{-h}(x)| - |\mathrm{det} D\Phi_0(x)| \,].$$

Consequently, we deduce that there exists $C = C(\|a\|_{W^{2,\infty}(\Omega)})$ such that for $|h| \le 1$,

$$\|X^{h*} w\| + \|X^h w\| + \|w\|_{L^2(\partial\Omega)} \le C \|w\|_{H^1(\Omega)}.$$

For $i = 1, \dots, d$, we compute

$$
\begin{aligned}
[\partial_i, X^h] w(x) &= \frac{1}{h} \sum_{j \neq i} \partial_j w(\Phi_h(x)) \partial_i \Phi_{h,j}(x) + \frac{1}{h} \partial_i w(\Phi_h(x)) \left(\partial_i \Phi_{h,i}(x) - 1 \right) \\
&= \frac{1}{h} \sum_j \partial_j w(\Phi_h(x)) \left(\partial_i \Phi_{h,j}(x) - \partial_i \Phi_{0,j}(x) \right)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
[X^{h*}, \partial_i] w(x) = \frac{1}{h} \sum_j & \partial_j w(\Phi_{-h}(x)) \left(\partial_i \Phi_{0,j}(x) - \partial_i \Phi_{-h,j}(x) \right) \left| \mathrm{det} D\Phi_{-h}(x) \right| \\
& - \frac{1}{h} w(\Phi_{-h}(x)) \partial_i \left| \mathrm{det} D\Phi_{-h}(x) \right|.
\end{aligned}
$$

As previously, we can easily bound $[\partial_i, X^h]w$ and the first term in $[X^{h*}, \partial_i]w$ by $C\|\nabla w\|$ with $C = C(\|a\|_{W^{1,\infty}(\Omega)})$ for any $|h| \leq 1$. The second term of $[X^{h*}, \partial_i]w$ can be bounded by $C\|w\|$ with $C = C(\|a\|_{W^{2,\infty}(\Omega)})$ for any $|h| \leq 1$ since for any j, we have $\partial_{ij}\Phi_0(x) = 0$. This implies that there exists $C = C(\|a\|_{W^{2,\infty}(\Omega)})$ such that for $|h| \leq 1$ and any function w in $H^1(\Omega)$, we have

$$
\|[\partial_i, X^h]w\| \leq C\|\nabla w\|, \quad \|[X^{h*}, \partial_i]w\| \leq C\|w\|_{H^1(\Omega)}.
$$

We deduce that for some $C = C(\|a\|_{W^{2,\infty}}, \|\alpha\|_{W^{1,\infty}})$, we have for any $|h| \leq 1$:

$$
\begin{aligned}
\|\nabla X^h u\|^2 \leq \|\xi\| \|X^{h*} X^h u\| &+ C\|\nabla u\| \|X^h u\|_{H^1(\Omega)} \\
&+ C\|\nabla u\| \|\nabla X^h u\| + C\|u\|_{L^2(\partial\Omega)} \|X^h u\|_{L^2(\partial\Omega)}
\end{aligned}
$$

and then

$$
\|\nabla X^h u\| \lesssim \|\xi\|. \tag{6.2.11}
$$

Passing to the limit $h \to 0$, we recover (6.2.9).

Step 4. Proof of (6.2.6). Consider any small enough open set $\omega \subset \Omega$. First, we fix $a^1, \dots, a^d$ a family of smooth vectors fields such that it is an orthonormal basis of $\mathbb{R}^d$ at any point $x \in \omega$ and $a^1(x) = n(x)$ for any $x \in \partial\Omega \cap \partial\omega$. For this, we observe that $\Omega = \{x \in \mathbb{R}^d, \delta(x) < 0\}$ with $\delta(x) := -d(x, \partial\Omega)$ if $x \in \Omega$, $\delta(x) := d(x, \partial\Omega)$ if $x \in \Omega^c$, and we assume that $\delta \in W^{3,\infty}(\Omega)$. We set

$$
b^1(x) := -\nabla\delta(x) \neq 0,
$$

since $\partial\Omega$ is an hypersurface in $\mathbb{R}^d$. Hence there exists $j \in \{1, \dots, d\}$ such that $\partial_{x_j}\delta(x) \neq 0$. We then set, for $i \in \{1, \dots, d\} \setminus \{j\}$,

$$
\bar{b}^i(x) = \partial_{x_i}\delta(x) e_j - \partial_{x_j}\delta(x) e_i,
$$

where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^d$, so that

$$\bar{b}^i(x) \cdot b^1(x) = 0,$$

for all i. We then set, if $j \neq 1$, $b^j(x) = \bar{b}^1(x)$, and for all $i \geq 2$,

$$b^i(x) = \bar{b}^i(x).$$

Finally, we apply the Gram-Schmidt process to $(b^1(x), \dots, b^d(x))$ to obtain $(a^1(x), \dots, a^d(x))$. We have $a^1(x) = n(x)$ and for all $i \in \{1, \dots, d\}$, a^i is smooth since b^i is smooth by hypothesis on δ. We set now $X_i := a^i \cdot \nabla$. From the third step, we have

$$\|\nabla X_i u\| \lesssim \|\xi\|, \quad \forall\, i = 2, \dots, d. \tag{6.2.12}$$

As a consequence of our previous construction, the matrix $A := (a^1, \dots, a^d)$ is orthonormal. We thus have that $\delta_{k\ell} = a^k \cdot a^\ell = a_k \cdot a_\ell$ where we denoted by a_m the m-th line vector of the matrix A. We thus have

$$\sum_i X_i^* X_i u = -\sum_{i,k,\ell} \partial_k (a_k^i a_\ell^i \partial_\ell u) = -\sum_{k,\ell} \partial_k (a_k \cdot a_\ell \partial_\ell u) = -\Delta u \tag{6.2.13}$$

by Fubini's theorem. Then

$$X_1^* X_1 u = \xi - \sum_{i \neq 1} X_i^* X_i u.$$

Because of (6.2.12), the above identity and $[X_1, X_1^*]u = -(a^1 \cdot \nabla \mathrm{div}(a^1))u$, we get

$$\begin{aligned} \|X_1^2 u\|^2 &= (X_1^* X_1 u, X_1^* X_1 u) + (X_1 u, [X_1^*, X_1] X_1 u) \\ &\lesssim \|\xi\|^2 + \sum_{i \neq 1} \left(\|\nabla X_i u\| \|\xi\| + \|\nabla X_i u\|^2 \right) + \|u\|_{H^1(\Omega)}^2 \lesssim \|\xi\|^2, \end{aligned}$$

and then together with (6.2.12) again, we have established

$$\|X_i X_j u\| \lesssim \|\xi\|, \quad \forall\, i, j = 1, \dots, d. \tag{6.2.14}$$

Recalling that $A = (a^1, \dots, a^d)$ and denoting $B := {}^T\!A = (a_1, \dots, a_d)$, we have $X_i = (B\nabla)_i$ and $\partial_i = (AX)_i$. As a consequence, we may write

$$\begin{aligned} \partial_i \partial_j u &= \sum_{m,\ell} A_{im} X_m A_{j\ell} X_\ell u \\ &= \sum_{m,\ell} A_{im} A_{j\ell} X_m X_\ell u + A_{im} [X_m, A_{j\ell}] X_\ell u, \end{aligned}$$

where the last operator is of order 1. Together with the starting point estimate (6.2.5) and (6.2.14), we conclude that

$$\|\partial_i\partial_j u\| \lesssim \|\xi\|, \quad \forall\, i,j = 1,\dots,d, \tag{6.2.15}$$

which ends the proof of (6.2.6). We can now conclude the proof of Theorem 6.2.1. Indeed, we now have that $u \in H^2(\Omega)$, we thus compute from (6.2.4) and the Stokes formula:

$$\begin{aligned}\int_{\partial\Omega} \left\{\frac{\partial u}{\partial n} + \frac{\alpha u}{2-\alpha}\right\} w \,\mathrm{d}\sigma_x &= \int_\Omega \Delta u\, w \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla w \,\mathrm{d}x + \int_{\partial\Omega} \frac{\alpha}{2-\alpha} uw \,\mathrm{d}\sigma_x \\ &= \int_\Omega (\Delta u + \xi) w \,\mathrm{d}x,\end{aligned}$$

for any $w \in V_\alpha$. Considering first $w \in C^1_c(\Omega)$ and next $w \in C^1(\bar\Omega)$, we get that u satisfies both equations in (6.2.1). □

6.2.2 Korn inequalities and the associated elliptic equation

For a vector field $M = (m_i)_{1\le i\le d} : \Omega \to \mathbb{R}^d$, we define its symmetric gradient through

$$\nabla_x^{\mathrm{sym}} M := \frac{1}{2}\left(\partial_j m_i + \partial_i m_j\right)_{1\le i,j\le d},$$

as well as its skew-symmetric gradient by

$$\nabla_x^{\mathrm{skew}} M := \frac{1}{2}\left(\partial_j m_i - \partial_i m_j\right)_{1\le i,j\le d}.$$

Through this section we shall also use the shorthand notation ∇^s for ∇_x^{sym}, and ∇^a for ∇_x^{skew}.

We consider the system of equations

$$\begin{cases} -\operatorname{div}(\nabla^s U) = \Xi & \text{in} \quad \Omega, \\ U \cdot n(x) = 0 & \text{on} \quad \partial\Omega, \\ (2-\alpha(x))\left[\nabla^s U n(x) - (\nabla^s U : n(x)\otimes n(x)) n(x)\right] + \alpha(x) U = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.2.16}$$

for a vector-field source term $\Xi : \Omega \to \mathbb{R}^d$. Noting that

$$\operatorname{div}(\nabla^s U) = \Delta U + \nabla \operatorname{div} U,$$

we see that (6.2.16) is nothing but a Lamé-type system. For any $\alpha : \partial\Omega \to [0,1]$ the boundary condition is a kind of homogeneous Robin (or mixed) boundary condition.

We define the Hilbert spaces

$$\mathcal{V}_1 := \left\{ W : \Omega \to \mathbb{R}^d \mid W \in H^1(\Omega),\ W \cdot n(x) = 0 \text{ on } \partial\Omega \right\}$$

and

$$\mathcal{V}_0 := \left\{ W : \Omega \to \mathbb{R}^d \mid W \in H^1(\Omega),\ W \cdot n(x) = 0 \text{ on } \partial\Omega,\ P_\Omega \left\langle \nabla^a W \right\rangle = 0 \right\}$$

where P_Ω denotes the projection onto the set $\mathcal{A}_\Omega = \{A \in \mathcal{M}_d^a(\mathbb{R}) \mid Ax \in \mathscr{R}_\Omega\}$ of all skew-symmetric matrices giving rise to a centered infinitesimal rigid displacement field preserving Ω (see (6.1.15) for the definition of $\mathscr{R}_\Omega$). Both spaces are endowed with the $H^1(\Omega)$ norm. We then denote

$$\mathcal{V}_\alpha := \begin{cases} \mathcal{V}_1 & \text{if} \quad \alpha \not\equiv 0 \\ \mathcal{V}_0 & \text{if} \quad \alpha \equiv 0 \end{cases}$$

and we remark that $H^1_0(\Omega) \subseteq \mathcal{V}_\alpha \subseteq H^1(\Omega)$ and thus $(H^1(\Omega))' \subseteq \mathcal{V}'_\alpha \subseteq H^{-1}(\Omega)$.

We also define on $\mathcal{V}_\alpha$ the bilinear form

$$A_\alpha(U, W) := \int_\Omega \nabla^s U : \nabla^s W \,\mathrm{d}x + \int_{\partial\Omega} \frac{\alpha(x)}{2 - \alpha(x)} U \cdot W \,\mathrm{d}\sigma_x.$$

The coercivity of the bilinear form A_α is related to Korn-type inequalities that we present below. A first classical version of Korn's inequality claims that there is a constant $K_1 > 0$, such that for any vector-field $U \in H^1(\Omega)$ there holds

$$\inf_{R \in \mathscr{R}} \|\nabla(U - R)\|^2_{L^2(\Omega)} \le K_1 \|\nabla^s U\|^2_{L^2(\Omega)}, \tag{6.2.17}$$

where we recall that $\mathscr{R}$ is the space of all infinitesimal rigid displacement fields defined in (6.1.14), or equivalently

$$\|\nabla U\|^2_{L^2(\Omega)} \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + |\langle \nabla^a U \rangle|^2. \tag{6.2.18}$$

For the statement of (6.2.17) and its proof, we refer to Desvillettes-Villani [38, Eq. (1)] where Friedrichs [60, Eq. (13), Second case] and Duvaut-Lions [52, Eq. (3.49)] are quoted, as well as [30, Theorem 2.2] and the references therein.

On the other hand, we claim that when $\alpha \not\equiv 0$ we have

$$|\langle \nabla^a U \rangle|^2 \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + \left\| \sqrt{\frac{\alpha}{2 - \alpha}}\, U \right\|^2_{L^2(\partial\Omega)}, \tag{6.2.19}$$

for any vector-field $U \in H^1(\Omega)$. In order to establish (6.2.19), we argue by contradiction. We assume thus that (6.2.19) is not true, so that there exists a sequence $(U_n)_{n\in\mathbb{N}}$ satisfying

$$1 = |\langle \nabla^a U_n \rangle|^2 \ge n \left(\|\nabla^s U_n\|^2_{L^2(\Omega)} + \left\| \sqrt{\frac{\alpha}{2 - \alpha}}\, U_n \right\|^2_{L^2(\partial\Omega)} \right).$$

Together with (6.2.17) and (6.2.3) applied to each component of U_n, we obtain that $(U_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Thus, up to the extraction of a subsequence, there exists $U \in H^1(\Omega)$ such that $U_n \rightharpoonup U$ weakly in $H^1(\Omega)$. Passing to the limit in the above estimates satisfied by $(U_n)_{n\in\mathbb{N}}$, we get $|\langle\nabla^a U\rangle|^2 = 1$, $\|\sqrt{\alpha/(2-\alpha)}U\|^2_{L^2(\partial\Omega)} = 0$ and $\|\nabla^s U\|_{L^2(\Omega)} = 0$. From $\nabla^s U = 0$, we first deduce that there exist an antisymmetric matrix A and a constant vector $b \in \mathbb{R}^d$ such that $U(x) = Ax + b$ on Ω, and, thanks to the estimate $\|\sqrt{\alpha/(2-\alpha)}U\|^2_{L^2(\partial\Omega)} = 0$, we deduce that

$$Ax + b = 0 \quad \text{on} \quad \Gamma := \{x \in \partial\Omega,\, \alpha(x) > 0\},$$

which has positive measure $|\Gamma| > 0$. We fix $\bar{x}$ an interior point of Γ. As in the fourth step of the proof of Theorem 6.2.1, we consider a family of smooth vectors fields $a^1, \dots, a^d$ such that it is an orthonormal basis of $\mathbb{R}^d$ and such that for any $x \in \partial\Omega$, $a^1(x) = n(x)$. We then introduce the flow $(\Phi^i_t)_{t\geq 0}$ associated to a^i for $i = 2, \dots, d$. For t small enough, $\Phi^i_t(\bar{x})$ is still in the interior of Γ so that

$$Aa^i(\bar{x}) = \frac{\mathrm{d}}{\mathrm{d}t}(A\Phi^i_t(\bar{x}) + b) = 0.$$

Therefore, for any $i \geq 2$, one has, using that $A\bar{x} + b = 0$ so that $b = -A\bar{x}$

$$a^i(\bar{x}) \cdot U(x) = a^i(\bar{x}) \cdot (Ax + b) = -Aa^i(\bar{x}) \cdot x + Aa^i(\bar{x}) \cdot \bar{x} = 0,$$

for any $x \in \Omega$, or, in other words, $U(x) \in \mathbb{R}\bar{n}$ for any $x \in \Omega$, with $\bar{n} := n(\bar{x})$. We may thus write $U(x) = \phi(x)\bar{n}$, with $\phi : \Omega \to \mathbb{R}$ an affine function, so that $\phi(x) = k \cdot x + k_0$, $k \in \mathbb{R}^d$, $k_0 \in \mathbb{R}$. There exists next at least i_0 such that $\bar{n}_{i_0} \neq 0$ because $|\bar{n}| = 1$. Using again the fact that $\nabla^s U = 0$ on Ω and observing that $(\nabla U)_{ij} = k_j \bar{n}_i$, we deduce first $k_{i_0} = 0$ because $k_{i_0}\bar{n}_{i_0} = (\nabla^s U)_{i_0 i_0} = 0$ and next $k_j = 0$ for any $j \neq i_0$ because $k_j \bar{n}_{i_0} = 2(\nabla^s U)_{i_0 j} = 0$. We have thus established that $U = n_0 := k_0\,\bar{n}$ on Ω, for some constant $n_0 \in \mathbb{R}^d$. We may alternatively prove that $\nabla U = 0$ and U is constant again by using just the claim [38, Eq. (3)]. Anyway, both arguments lead to the fact that $U = 0$ because of the boundary condition on Γ which is in contradiction with $|\langle\nabla^a U\rangle|^2 = 1$. That ends the proof of (6.2.19). Gathering (6.2.18) and (6.2.19), we then have established the (probably classical) following Korn-type inequality

$$\|\nabla U\|^2_{L^2(\Omega)} \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + \left\|\sqrt{\frac{\alpha}{2-\alpha}}U\right\|^2_{L^2(\partial\Omega)}, \tag{6.2.20}$$

for any vector-field $U \in H^1(\Omega)$ and assuming $\alpha \not\equiv 0$.

For later reference, we also mention that a similar argument (and even a bit simpler, see also [38, Eq. (2)] and [30, Theorem 2.1]) leads to the following variant of Korn's inequality: for any vector-field $U \in H^1(\Omega)$ there holds

$$\|\nabla U\|^2_{L^2(\Omega)} \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + \|U\|^2_{L^2(\Omega)}. \tag{6.2.21}$$

It is worth emphasizing that we also have the following Poincaré inequality: there exists a positive constant $C > 0$ such that for any $U \in H^1(\Omega)$ such that $U(x) \cdot n(x) = 0$ on $\partial\Omega$ there holds

$$\|U\|^2_{L^2(\Omega)} \le C\|\nabla U\|^2_{L^2(\Omega)}. \tag{6.2.22}$$

As before, we may argue by contradiction, assuming that (6.2.22) is not true, so that there exists a sequence $(U_n)_{n\in\mathbb{N}}$ in $H^1(\Omega)$ satisfying $U_n \cdot n(x) = 0$ on $\partial\Omega$ and such that

$$1 = \|U_n\|^2_{L^2(\Omega)} \ge n\|\nabla U_n\|^2_{L^2(\Omega)}.$$

We immediately deduce that there exists $U \in H^1(\Omega)$ such that $\nabla U = 0$, $\|U\|^2_{L^2(\Omega)} = 1$ and $U \cdot n(x) = 0$ which gives our contradiction. Gathering (6.2.20) and (6.2.22), we may state a last version of our first Korn inequality, namely: there exists a positive constant $C > 0$ such that for any $U \in H^1(\Omega)$ such that $U(x)\cdot n(x) = 0$ on $\partial\Omega$ there holds

$$C^{-1}\|U\|^2_{H^1(\Omega)} \le \|\nabla^s U\|^2_{L^2(\Omega)} + \left\|\sqrt{\frac{\alpha}{2-\alpha}}U\right\|^2_{L^2(\partial\Omega)}, \tag{6.2.23}$$

recalling that we have supposed $\alpha \not\equiv 0$.

On the other hand, a less classical Korn's inequality has been established by Desvillettes and Villani [38], that says there exists a positive constant $K_2 > 0$ such that for any vector-field $U \in H^1(\Omega)$ verifying $U \cdot n(x) = 0$ on $\partial\Omega$, one has

$$\inf_{R\in\mathscr{R}_\Omega} \|\nabla(U-R)\|^2_{L^2(\Omega)} \le K_2\|\nabla^s U\|^2_{L^2(\Omega)}, \tag{6.2.24}$$

where we remind that $\mathscr{R}_\Omega$ stands for the space of centered infinitesimal rigid displacement fields defined in (6.1.15), or equivalently

$$\|\nabla U\|^2_{L^2(\Omega)} \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + |P_\Omega\langle\nabla^a U\rangle|^2, \tag{6.2.25}$$

where we recall that P_Ω stands for the projection onto the space $\mathcal{A}_\Omega = \{A \in \mathcal{M}^a_d(\mathbb{R}) \mid Ax \in \mathscr{R}_\Omega\}$ of all skew-symmetric matrices giving rise to a centered infinitesimal rigid displacement field preserving Ω. In the case when $\mathscr{R}_\Omega = \{0\}$, that is when Ω has no axi-symmetry, (6.2.24) is nothing but the inequality stated in [38, Theorem 3] and for which a detailed constructive proof is provided therein. The proof of (6.2.24) in the three dimensional case is also alluded in [38, Section 5]. We do not explain how the analysis developed in [38] makes possible to get a constructive proof of (6.2.24) in the general case (whatever is the dimension d), but rather briefly explain how (6.2.24) may be established thanks to a compactness argument.

We first claim that for any vector-field $U \in H^1(\Omega)$ such that $U \cdot n(x) = 0$ on $\partial\Omega$, one has

$$\|U\|^2_{L^2(\Omega)} \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + |P_\Omega\langle\nabla^a U\rangle|^2. \tag{6.2.26}$$

Assume indeed by contradiction that (6.2.26) is not true, so that there exists a sequence $(U_n)_{n\in\mathbb{N}}$ satisfying $U_n \cdot n(x) = 0$ on $\partial\Omega$ such that

$$1 = \|U_n\|^2_{L^2(\Omega)} \geq n \left(\|\nabla^s U_n\|^2_{L^2(\Omega)} + |P_\Omega \langle \nabla^a U_n \rangle|^2 \right).$$

Together with the Korn inequality (6.2.21), we deduce that there exists $U \in H^1(\Omega)$ satisfying $U \cdot n(x) = 0$ on $\partial\Omega$ such that (up to the extraction of a subsequence) $U_n \rightharpoonup U$ weakly in $H^1(\Omega)$ and $U_n \to U$ strongly in $L^2(\Omega)$. Passing to the limit in the estimates satisfied by $(U_n)_{n\in\mathbb{N}}$, we first get $\nabla^s U = 0$ which implies that $U = Ax + b \in \mathscr{R}$. Moreover we obtain $U \cdot n(x) = (Ax+b)\cdot n(x) = 0$ on $\partial\Omega$ and thus, thanks to the remark after (6.1.15) using the assumption (6.1.11), we obtain that $b = 0$ and hence $A \in \mathcal{A}_\Omega$ or equivalently $Ax \in \mathscr{R}_\Omega$. Finally, we also have $P_\Omega \langle \nabla^a U \rangle = P_\Omega A = 0$ which implies $A \in \mathcal{A}_\Omega^\perp$ (or equivalently $Ax \in \mathscr{R}_\Omega^\perp$) and thus $A = 0$. We therefore obtain $U = 0$ which is in contradiction with the fact that $\|U\|^2_{L^2(\Omega)} = 1$. That ends the proof of (6.2.26). The proof of (6.2.25) follows by gathering (6.2.21) and (6.2.26), and gathering (6.2.25) together with (6.2.26) we finally obtain the following Korn-type inequality: for any vector-field $U \in H^1(\Omega)$ such that $U \cdot n(x) = 0$ on $\partial\Omega$ there holds

$$\|U\|^2_{H^1(\Omega)} \lesssim \|\nabla^s U\|^2_{L^2(\Omega)} + |P_\Omega \langle \nabla^a U \rangle|^2. \tag{6.2.27}$$

We can now state our result concerning the existence, uniqueness and regularity of solutions to the elliptic system (6.2.16).

Theorem 6.2.2. *For any given $\Xi \in \mathcal{V}_\alpha'$, there exists a unique solution $U \in \mathcal{V}_\alpha$ to the variational problem associated to (6.2.16), namely*

$$A_\alpha(U, W) = \langle \Xi, W \rangle_{\mathcal{V}_\alpha', \mathcal{V}_\alpha} \quad \forall W \in \mathcal{V}_\alpha, \tag{6.2.28}$$

and moreover there holds

$$\|U\|_{H^1(\Omega)} \lesssim \|\Xi\|_{\mathcal{V}_\alpha'}. \tag{6.2.29}$$

If furthermore $\Xi \in L^2(\Omega)$ with the condition $\langle \Xi, Ax \rangle = 0$ for any $Ax \in \mathscr{R}_\Omega$ when $\alpha \equiv 0$, then the variational solution U to (6.2.16) satisfies $U \in H^2(\Omega)$ with

$$\|U\|_{H^2(\Omega)} \lesssim \|\Xi\|_{L^2(\Omega)}, \tag{6.2.30}$$

and moreover U verifies (6.2.16) a.e.

The proof of Theorem 6.2.2 follows the same steps as the proof of Theorem 6.2.1. We briefly present it below.

Proof of Theorem 6.2.2. We split the proof into four steps, the three last ones being devoted to the proof of the H^2 regularity estimate.

Step 1. Thanks to the above Korn-type inequalities, more precisely (6.2.23) for the case $\alpha \not\equiv 0$ and (6.2.27) for the case $\alpha \equiv 0$, we deduce that the bilinear form A_α is coercive in $\mathcal{V}_\alpha$, that is

$$\forall U \in \mathcal{V}_\alpha, \qquad \|U\|^2_{L^2(\Omega)} + \|\nabla U\|^2_{L^2(\Omega)} \lesssim A_\alpha(U,U). \tag{6.2.31}$$

One can therefore apply Lax-Milgram theorem which gives us the existence and uniqueness of $U \in \mathcal{V}_\alpha$ satisfying (6.2.28) and (6.2.29).

For the remainder of the proof, we suppose that $\Xi \in L^2(\Omega) \subseteq \mathcal{V}_1'$ with the additional assumption $\langle \Xi, Ax\rangle = 0$ for any $Ax \in \mathscr{R}_\Omega$ when $\alpha \equiv 0$. We then claim that (6.2.28) can be improved into the following new variational-type formulation: there exists a unique $U \in \mathcal{V}_\alpha$ verifying

$$A_\alpha(U,W) = \langle \Xi, W\rangle_{\mathcal{V}_1',\mathcal{V}_1} \quad \forall W \in \mathcal{V}_1. \tag{6.2.32}$$

In the case $\alpha \not\equiv 0$ or $\alpha \equiv 0$ with a non axi-symmetric domain Ω, that is $\mathscr{R}_\Omega = \{0\}$, equation (6.2.32) is nothing but (6.2.28) since in these cases $\mathcal{V}_\alpha = \mathcal{V}_1$. When $\alpha \equiv 0$ and Ω has rotational symmetry, that is $\mathscr{R}_\Omega \neq \{0\}$, for any $W \in \mathcal{V}_1$ we have $W - P_\Omega \langle \nabla^a W\rangle x \in \mathcal{V}_0$ and therefore

$$\begin{aligned} A_\alpha(U,W) &= A_\alpha(U, W - P_\Omega \langle \nabla^a W\rangle x) \\ &= \int_\Omega \Xi \cdot W \,\mathrm{d}x - \int_\Omega \Xi \cdot (P_\Omega \langle \nabla^a W\rangle x)\,\mathrm{d}x \\ &= \int_\Omega \Xi \cdot W\,\mathrm{d}x \end{aligned}$$

where we have used that $\nabla^s(P_\Omega \langle \nabla^a W\rangle x) = 0$ in the first line, formulation (6.2.28) in the second line, and the condition $\langle \Xi, Ax\rangle = 0$ for any $Ax \in \mathscr{R}_\Omega$ in the third line, since $P_\Omega \langle \nabla^a W\rangle x \in \mathscr{R}_\Omega$ by definition.

Step 2. For any small enough open set $\omega \subset \Omega$, we fix a vector field $a \in C^2(\bar\Omega)$ such that $|a| = 1$ on ω and $a \cdot n = 0$ on $\partial\Omega$, and we set $X := a \cdot \nabla$ the associated differential operator. For a smooth solution U to (6.2.32), we compute

$$\begin{aligned} \|\nabla^s XU\|^2 &= (\nabla^s U, X^*\nabla^s XU) + ([\nabla^s, X]U, \nabla^s XU) \\ &= (\nabla^s U, \nabla^s X^* XU) + (\nabla^s U, [X^*, \nabla^s]XU) + ([\nabla^s, X]U, \nabla^s XU) \end{aligned}$$

where we have used that

$$(Xf, g) = (f, X^* g), \quad X^* g := -\mathrm{div}(ag),$$

because $a \cdot n = 0$ on $\partial\Omega$. On the other hand, we have the following formal equality

$$\int_{\partial\Omega} (XU)\cdot(XU)\,\frac{\alpha}{2-\alpha}\mathrm{d}\sigma_x = \int_{\partial\Omega} \frac{\alpha}{2-\alpha}\, U \cdot (X^* XU)\,\mathrm{d}\sigma_x - \int_{\partial\Omega}\left(X\frac{\alpha}{2-\alpha}\right) U\cdot (XU)\,\mathrm{d}\sigma_x.$$

We define

$$
\begin{aligned}
(\mathcal{A}W)_{ij} &:= \frac{1}{2}([\partial_i, X]W_j + [\partial_j, X]W_i) \\
(\mathcal{B}W)_{ij} &:= \frac{1}{2}([X^*, \partial_i]W_j + [X^*, \partial_j]W_i).
\end{aligned}
$$

Then, supposing that U is smooth enough so that $X^*XU \in \mathcal{V}_1$, using also $(\nabla^s)^*\nabla^s = \mathrm{div}(\nabla^s \cdot)$, we obtain:

$$
\begin{aligned}
&\|\nabla^s XU\|^2 + \int_{\partial\Omega} (XU)\cdot(XU)\,\frac{\alpha}{2-\alpha}\mathrm{d}\sigma_x \\
&\quad = (\Xi, X^*XU) + (\nabla^s U, \mathcal{B}XU) + (\mathcal{A}U, \nabla^s XU) - \int_{\partial\Omega}\left(X\frac{\alpha}{2-\alpha}\right) U\cdot(XU)\,\mathrm{d}\sigma_x.
\end{aligned}
$$

From the Korn inequality, we first deduce that

$$
\|\nabla XU\|^2 \lesssim \|\Xi\|\|X^*XU\| + \|\nabla U\|\|\mathcal{B}XU\| + \|\mathcal{A}U\|\|\nabla XU\| + \|U\|_{L^2(\partial\Omega)}\|XU\|_{L^2(\partial\Omega)}.
$$

Then, since

$$
[\partial_i, X] = (\partial_i a)\cdot\nabla, \quad [X^*, \partial_i] = \partial_i(\mathrm{div} a) + (\partial_i a)\cdot\nabla,
$$

we deduce that

$$
\|\mathcal{A}W\| + \|\mathcal{B}W\| \lesssim \|W\|_{H^1(\Omega)}, \quad \forall W \in \mathcal{V}_1.
$$

We also have the elementary estimates

$$
\|X^*W\| + \|XW\| \lesssim \|W\|_{H^1(\Omega)}, \quad \forall W \in \mathcal{V}_1.
$$

so that thanks to the already established estimate $\|U\|_{H^1(\Omega)} \lesssim \|\Xi\|$, we are able to deduce that

$$
\|\nabla XU\|^2 \lesssim \|\Xi\|\|\nabla XU\| + \|\Xi\|^2,
$$

and thus

$$
\|\nabla XU\| \lesssim \|\Xi\|. \tag{6.2.33}
$$

Note that as in the proof of Theorem 6.2.1, the multiplicative constants involved in our estimates depend on $\|a\|_{W^{2,\infty}(\Omega)}$ and $\|\alpha\|_{W^{1,\infty}(\Omega)}$.

Step 3. When we do not deal with an a priori smooth solution, but just with a solution $U \in \mathcal{V}_\alpha$ to (6.2.32), we modify the argument in the following way. We define $\Phi_t : \bar\Omega \to \bar\Omega$ the flow associated to the differential equation

$$
\dot y = a(y), \quad y(0) = x,
$$

so that $\Phi_t(x) := y(t)$ and $(t,x) \mapsto \Phi_t(x)$ is a C^1-diffeomorphism. We observe that for a smooth function U such that $U(x) \cdot n(x) = 0$ for any $x \in \partial\Omega$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}h}(U(\Phi_h(x)) \cdot n(\Phi_h(x)))_{|h=0} = (XU)(x) \cdot n(x) + U(x) \cdot (Xn)(x),$$

so that $XU \notin \mathcal{V}_1$ in general. We fix $a^1, \dots, a^d$ a family of smooth vectors fields such that it is an orthonormal basis of $\mathbb{R}^d$ at any point $x \in \omega$, $a^1(x) = n(x)$ for any $x \in \partial\Omega \cap \partial\omega$. We set $A := (a^1, \dots, a^d)$ the associated matrix and then $J^h(x) := A(\Phi_h(x))A(x)^{-1}$, so that in particular $J^h(x)n(x) = n(\Phi_h(x))$ for any h. We next define

$$X^h U(x) := \frac{1}{h}\left({}^T J^h(x) U(\Phi_h(x)) - U(x)\right),$$

so that $X^h U \in \mathcal{V}_1$ if $U \in \mathcal{V}_\alpha$. Repeating the argument of Step 2, we get

$$\|\nabla^s X^h U\|^2 = (\nabla^s U, \nabla^s X^{h*} X^h U) + (\nabla^s U, \mathcal{B}^h X^h U) + (\mathcal{A}^h U, \nabla^s X^h U),$$

where we denote

$$\begin{aligned} X^{h*} M(x) &:= \frac{1}{h}[|\det D\Phi_{-h}(x)| J^h(\Phi_{-h}(x)) M(\Phi_{-h}(x)) - M(x)] \\ (\mathcal{A}^h W)_{ij} &:= \frac{1}{2}([\partial_i, X^h] W_j + [\partial_j, X^h] W_i) \\ (\mathcal{B}^h W)_{ij} &:= \frac{1}{2}([X^{h*}, \partial_i] W_j + [X^{h*}, \partial_j] W_i). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_{\partial\Omega} \frac{\alpha}{2-\alpha}(x) U(x) \cdot X^{h*} X^h U(x) \mathrm{d}\sigma_x \\ &\quad = \int_{\partial\Omega} \frac{\alpha}{2-\alpha}(\Phi_h(x)) (X^h U)(x) \cdot (X^h U)(x) \mathrm{d}\sigma_x + \int_{\partial\Omega} U(x) \cdot X^h U(x) Y^h\Big(\frac{\alpha}{2-\alpha}\Big)(x) \mathrm{d}\sigma_x, \end{aligned}$$

where

$$Y^h M(x) := \frac{1}{h}\Big(M(\Phi_h(x)) - M(x)\Big).$$

We also have that if $U \in \mathcal{V}_\alpha$ then $X^{h*} X^h U \in \mathcal{V}_1$ too. Indeed, we compute

$$\begin{aligned} X^{h*} X^h U(x) = &\frac{1}{h^2} |\det D\Phi_{-h}(x)| J^h(\Phi_{-h}(x)) \left(\left({}^T J^h(\Phi_{-h}(x))\right) U(x) - U(\Phi_{-h}(x))\right) \\ &- \frac{1}{h^2}\left({}^T J^h(x) U(\Phi_h(x)) - U(x)\right) =: T_1(x) + T_2(x), \end{aligned}$$

the last equality standing for a definition of T_1 and T_2. As already noticed, if $U \in \mathcal{V}_\alpha$, then $X^h U(x) \cdot n(x) = 0$ so that $T_2(x) \cdot n(x) = 0$. Concerning T_1, we first have that

$$J^h(\Phi_{-h}(x)) \left({}^T J^h(\Phi_{-h}(x))\right) U(x) \cdot n(x) = U(x) \cdot n(x) = 0.$$

Then, we remark that $J^h(\Phi_{-h}(x)) = {}^T J^{-h}(x)$ so that

$$J^h(\Phi_{-h}(x))U(\Phi_{-h}(x)) \cdot n(x) = U(\Phi_{-h}(x)) \cdot J^{-h}(x)n(x) = U(\Phi_{-h}(x)) \cdot n(\Phi_{-h}(x)) = 0.$$

Using this, since U is a solution of (6.2.32), we deduce that

$$\|\nabla^s X^h U\|^2 + \int_{\partial\Omega} \frac{\alpha}{2-\alpha}(\Phi_h(x))(X^h U)(x) \cdot (X^h U)(x) \mathrm{d}\sigma_x$$
$$= (\Xi, X^{h*} X^h U) + (\nabla^s U, \mathcal{B}^h X^h U) + (\mathcal{A}^h U, \nabla^s X^h U) - \int_{\partial\Omega} U \cdot (X^h U) \left(Y^h \frac{\alpha}{2-\alpha}\right) \mathrm{d}\sigma_x.$$

As in the proof of Theorem 6.2.1, one can prove the following elementary estimates

$$\|X^h W\| + \|X^{h*} W\| + \|\mathcal{A}^h W\| + \|\mathcal{B}^h W\| \lesssim \|W\|_{H^1(\Omega)}, \quad \forall W \in \mathcal{V}_1.$$

Using these bounds combined with the already established estimate $\|U\|_{H^1(\Omega)} \lesssim \|\Xi\|$ and the Korn inequality, we deduce, as in the Poisson case, that

$$\|\nabla X^h U\| \lesssim \|\Xi\|, \quad \forall\, |h| \le 1.$$

Passing to the limit $h \to 0$, we then get

$$\|\nabla X^0 U\| \lesssim \|\Xi\|,$$

with $X^0 U_k = a \cdot \nabla U_k + A\,(a \cdot \nabla A^{-1}) U_k$ for $k = 1, \dots, d$. Note that as in the Poisson case, the multiplicative constants are uniform in $|h| \le 1$ and depend on $\|a\|_{W^{2,\infty}}$ and $\|\alpha\|_{W^{1,\infty}}$. We then recover (6.2.33) by observing that we have $\|A\,(a \cdot \nabla A^{-1}) U\|_{H^1} \lesssim \|\Xi\|$.

Step 4. We set now $X_i := a^i \cdot \nabla$. From the second step, we have

$$\|\nabla X_i U\| \lesssim \|\Xi\|, \quad \forall\, i = 2, \dots, d. \tag{6.2.34}$$

We first notice that

$$\partial_j = \sum_i a^i_j X_i = -\sum_i X^*_i (a^i_j \cdot).$$

Combining this with (6.2.13), we deduce that

$$\Xi_j = -\Delta U_j - \partial_j(\operatorname{div} U) = \sum_i X_i^* X_i U_j + \sum_{i,\ell,m} X_i^*(a_j^i a_\ell^m X_m U_\ell)$$
$$= X_1^* X_1 U_j + \sum_\ell X_1^*(a_j^1 a_\ell^1 X_1 U_\ell) + \sum_{i\neq 1} X_i^* X_i U_j + \sum_{(i,m)\neq(1,1)} \sum_\ell X_i^*(a_j^i a_\ell^m X_m U_\ell).$$

We notice that $X_i^*(fg) = (X_i^* f)g - f(X_i g)$. Using then (6.2.34) combined with the fact that for $i = 1, \ldots, d$, we have $a^i \in W^{2,\infty}(\Omega)$, we deduce that

$$X_1^* X_1 U_j + \sum_\ell a_j^1 a_\ell^1 X_1^* X_1 U_\ell = R_j(U, \Xi) \quad \text{with} \quad \|R_j(U,\Xi)\| \lesssim \|\Xi\|. \tag{6.2.35}$$

Multiplying the equality in (6.2.35) by a_j^1 and then summing it over j, we get

$$2 \sum_\ell a_\ell^1 X_1^* X_1 U_\ell = \sum_j R_j(U, \Xi)$$

and thus

$$\|a^1 \cdot X_1^* X_1 U\| \lesssim \|\Xi\|. \tag{6.2.36}$$

Coming back to (6.2.35) and using once more that $\delta_{j\ell} = a_j \cdot a_\ell$, so that

$$X_1^* X_1 U_j = \sum_{l,m} a_j^m a_l^m X_1^* X_1 U_l,$$

we obtain that

$$\sum_{m\neq 1, l\in\{1,\ldots,d\}} a_j^m a_\ell^m X_1^* X_1 U_\ell = R_j(U,\Xi) - 2\sum_\ell a_j^1 a_\ell^1 X_1^* X_1 U_\ell$$

which, thanks to (6.2.36) and the fact that $\|R_j(U,\Xi)\| \lesssim \|\Xi\|$, yields

$$\Big\| \sum_{\ell, m\neq 1} a_j^m a_\ell^m X_1^* X_1 U_\ell \Big\| \lesssim \|\Xi\|. \tag{6.2.37}$$

Finally, we can write that

$$X_1^* X_1 U_j = \sum_{\ell,m} a_j^m a_\ell^m X_1^* X_1 U_\ell$$

so that thanks to (6.2.36) and (6.2.37), we obtain

$$\|X_1^* X_1 U_j\| \lesssim \|\Xi\|.$$

Recalling that $[X_1, X_1^*]u = (a^1 \cdot \nabla \mathrm{div}(a^1))u$, because of (6.2.34), the above inequality implies

$$\|X_1^2 U\| \lesssim \|\Xi\|,$$

and then together with (6.2.34), we have established

$$\|X_i X_j U\| \lesssim \|\Xi\|, \quad \forall\, i,j = 1, \dots, d. \tag{6.2.38}$$

We can then conclude the proof of Theorem 6.2.2 as in the one of Theorem 6.2.1. □

6.3 Proof of the main result

Consider the operator

$$\mathcal{L} = \mathcal{C} - v \cdot \nabla_x = \mathcal{C} + \mathcal{T}.$$

For any $f \in L^2_{x,v}(\mu^{-1})$ we decompose $f = \pi f + f^\perp$ with the macroscopic part πf given by

$$\pi f(x,v) = \varrho(x)\mu(v) + m(x) \cdot v\mu(v) + \theta(x)\, \frac{(|v|^2 - d)}{\sqrt{2d}}\, \mu(v),$$

where the mass, momentum and energy are defined respectively by

$$\varrho(x) = \int_{\mathbb{R}^d} f(x,v)\,\mathrm{d}v, \quad m(x) = \int_{\mathbb{R}^d} v f(x,v)\,\mathrm{d}v \quad \text{and} \quad \theta(x) = \int_{\mathbb{R}^d} \frac{(|v|^2 - d)}{\sqrt{2d}}\, f(x,v)\,\mathrm{d}v.$$

Remark that

$$\|f\|^2_{L^2_{x,v}(\mu^{-1})} = \|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + \|\pi f\|^2_{L^2_x(\Omega)}$$

and

$$\|\pi f\|^2_{L^2_x(\Omega)} = \|\varrho\|^2_{L^2_x(\Omega)} + \|m\|^2_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)}.$$

We recall that $\langle \cdot \rangle$ is the mean on Ω, that is $\langle a \rangle := |\Omega|^{-1} \int_\Omega a\,\mathrm{d}x$.

Our main result Theorem 6.1.1 is obtained as a direct consequence of the following result:

Theorem 6.3.1. *There exists a scalar product $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ on the space $L^2_{x,v}(\mu^{-1})$ so that the associated norm $|\!|\!| \cdot |\!|\!|$ is equivalent to the usual norm $\|\cdot\|_{L^2_{x,v}(\mu^{-1})}$, and for which the linear operator $\mathcal{L}$ satisfies the following coercivity estimate: there is a positive constant $\kappa > 0$ such that one has*

$$\langle\!\langle -\mathcal{L}f, f \rangle\!\rangle \geq \kappa |\!|\!| f |\!|\!|^2 \tag{6.3.1}$$

for any $f \in \mathrm{Dom}(\mathcal{L})$ satisfying the boundary condition (6.1.2) and moreover

i) in the specular reflection case ($\alpha \equiv 0$ in (6.1.2))

$$\langle \varrho \rangle = 0, \quad \langle \theta \rangle = 0 \quad \textit{and} \quad \langle Ax \cdot m \rangle = 0 \quad \forall Ax \in \mathscr{R}_\Omega. \tag{6.3.2}$$

ii) otherwise

$$\langle \varrho \rangle = 0. \tag{6.3.3}$$

As explained in Section 6.1.3, the construction of this new inner product $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ on the space $L^2_{x,v}(\mu^{-1})$ begins with the usual inner product, which gives us a control of the microscopic part $f^\perp$, and after that, step by step, new terms are added to it in order to control all components of the macroscopic part πf. The construction of each of those terms is performed from Section 6.3.1 through Section 6.3.5, and then in Section 6.3.6 we shall complete the proof of Theorem 6.3.1.

We consider hereafter f satisfying the conditions of Theorem 6.3.1. For simplicity we introduce the notation $f_\pm := \gamma_\pm f$ and $D^\perp := \mathrm{Id} - D$, where D is given by (6.1.4).

6.3.1 Microscopic part

Lemma 6.3.1. *There exists $\lambda > 0$ such that*

$$\langle -\mathcal{L} f, f \rangle_{L^2_{x,v}(\mu^{-1})} \geq \lambda \| f^\perp \|^2_{L^2_{x,v}(\mu^{-1})} + \frac{1}{2} \| \sqrt{\alpha(2-\alpha)} D^\perp f_+ \|^2_{L^2(\Sigma_+;\mu^{-1}(v) n(x)\cdot v)}.$$

Proof. We write

$$\langle -\mathcal{L} f, f \rangle_{L^2_{x,v}(\mu^{-1})} = \langle -\mathcal{C} f, f \rangle_{L^2_{x,v}(\mu^{-1})} + \langle -\mathcal{T} f, f \rangle_{L^2_{x,v}(\mu^{-1})}.$$

Thanks to (6.1.10) one has

$$\langle -\mathcal{C} f, f \rangle_{L^2_{x,v}(\mu^{-1})} \geq \lambda \| f^\perp \|^2_{L^2_{x,v}(\mu^{-1})}.$$

For the second term we first get

$$\langle -\mathcal{T} f, f \rangle_{L^2_{x,v}(\mu^{-1})} = \int_{\mathcal{O}} (v \cdot \nabla_x f) f \mu^{-1} \,\mathrm{d}x \,\mathrm{d}v = \frac{1}{2} \int_\Sigma \gamma f^2 \mu^{-1} n(x) \cdot v \,\mathrm{d}\sigma_x \,\mathrm{d}v.$$

Writing $\gamma f^2 = f_+^2 \mathbf{1}_{\Sigma_+} + f_-^2 \mathbf{1}_{\Sigma_-}$ and using the boundary condition (6.1.2), we thus obtain

$$\begin{aligned}
\langle -\mathcal{T} f, f \rangle_{L^2_{x,v}(\mu^{-1})} &= \frac{1}{2} \int_{\Sigma_+} f_+^2 \mu^{-1} |n(x)\cdot v| \,\mathrm{d}\sigma_x \,\mathrm{d}v - \frac{1}{2} \int_{\Sigma_-} f_-^2 \mu^{-1} |n(x)\cdot v| \,\mathrm{d}\sigma_x \,\mathrm{d}v \\
&= \frac{1}{2} \int_{\Sigma_+} f_+^2 \mu^{-1} |n(x)\cdot v| \,\mathrm{d}\sigma_x \,\mathrm{d}v \\
&\quad - \frac{1}{2} \int_{\Sigma_-} \left\{ (1-\alpha(x)) f_+(x, R_x v) + \alpha(x) D f_+(x,v) \right\}^2 \mu^{-1} |n(x)\cdot v| \,\mathrm{d}\sigma_x \,\mathrm{d}v.
\end{aligned}$$

We apply the change of variables $v \mapsto R_x v$, so that Σ_- transforms into Σ_+, which yields

$$\begin{aligned}\langle -\mathcal{T} f, f\rangle_{L^2_{x,v}(\mu^{-1})} = &\frac{1}{2}\int_{\Sigma_+} f_+^2 \mu^{-1} |n(x)\cdot v| \,\mathrm{d}\sigma_x\,\mathrm{d}v \\ &- \frac{1}{2}\int_{\Sigma_+} \{(1-\alpha(x))f_+ + \alpha(x) Df_+\}^2 \mu^{-1}|n(x)\cdot v|\,\mathrm{d}\sigma_x\,\mathrm{d}v,\end{aligned}$$

since $Df_+(x, R_x v) = Df_+(x,v)$ and $|n(x)\cdot R_x v| = |n(x)\cdot v|$. Writing $f_+ = D^\perp f_+ + Df_+$, one has

$$\int_{\Sigma_+} f_+^2\mu^{-1} n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v = \int_{\Sigma_+} (Df_+)^2\mu^{-1}n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v + \int_{\Sigma_+}(D^\perp f_+)^2\mu^{-1}n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v$$

since

$$\int_{\Sigma_+} Df_+ D^\perp f_+ \mu^{-1} n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v = 0.$$

Therefore

$$\begin{aligned}&\langle -\mathcal{T} f, f\rangle_{L^2_{x,v}(\mu^{-1})} \\ &\quad = \frac{1}{2}\int_{\Sigma_+}\Big\{(Df_+)^2 + (D^\perp f_+)^2 - [(1-\alpha(x))D^\perp f_+ + Df_+]^2\Big\}\,\mu^{-1} n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v \\ &\quad = \frac{1}{2}\int_{\Sigma_+}\Big\{[1-(1-\alpha(x))^2](D^\perp f_+)^2 - 2(1-\alpha(x))Df_+D^\perp f_+\Big\}\,\mu^{-1}n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v \\ &\quad = \frac{1}{2}\int_{\Sigma_+}\alpha(x)(2-\alpha(x))(D^\perp f_+)^2\mu^{-1}n(x)\cdot v\,\mathrm{d}\sigma_x\,\mathrm{d}v.\end{aligned}$$

We finish the proof by gathering previous estimates. $\square$

6.3.2 Boundary terms

We start by stating a technical lemma which will be useful to treat the boundary terms in what follows.

Lemma 6.3.2. *Let $\psi : \partial\Omega \to \mathbb{R}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$. Then*

$$\begin{aligned}&\int_\Sigma \psi(x)\phi(v)\gamma f(x,v)\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &\quad = \int_{\Sigma_+}\psi(x)\phi(v)\alpha(x)D^\perp f_+\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &\qquad + \int_{\Sigma_+}\psi(x)\{\phi(v)-\phi(R_xv)\}\,(1-\alpha(x))D^\perp f_+\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &\qquad + \int_{\Sigma_+}\psi(x)\{\phi(v)-\phi(R_xv)\}\,Df_+\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x.\end{aligned}$$

Proof. We first write

$$\begin{aligned}
&\int_{\Sigma} \psi(x)\phi(v)\gamma f(x,v)\, n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\qquad = \int_{\Sigma_+} \psi(x)\phi(v) f_+\, n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x - \int_{\Sigma_-} \psi(x)\phi(v) f_- \,|n(x)\cdot v| \,\mathrm{d}v\,\mathrm{d}\sigma_x.
\end{aligned}$$

Applying the boundary condition (6.1.2) and then the change of variables $v \mapsto R_x v$, we hence obtain

$$\begin{aligned}
&\int_{\Sigma_-} \psi(x)\phi(v) f_- \,|n(x)\cdot v| \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\quad = \int_{\Sigma_-} \psi(x)\phi(v) \left\{(1-\alpha(x)) f_+(x, R_x v) + \alpha(x) D f_+(x,v)\right\} |n(x)\cdot v| \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\quad = \int_{\Sigma_+} \psi(x)\phi(R_x v) \left\{(1-\alpha(x)) f_+(x, v) + \alpha(x) D f_+(x,v)\right\} |n(x)\cdot v| \,\mathrm{d}v\,\mathrm{d}\sigma_x,
\end{aligned}$$

since $Df_+(x, R_x v) = Df_+(x,v)$ and $|n(x)\cdot R_x v| = |n(x)\cdot v|$. We write $f_+ = D^\perp f_+ + Df_+$ and thus

$$\begin{aligned}
&\int_{\Sigma} \psi(x)\phi(v)\gamma f(x,v)\, n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\quad = \int_{\Sigma_+} \psi(x) \left\{\phi(v) f_+ - \phi(R_x v)(1-\alpha) f_+ - \phi(R_x v)\alpha D f_+\right\} n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\quad = \int_{\Sigma_+} \psi(x)\phi(v)\alpha(x) D^\perp f_+\, n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\qquad + \int_{\Sigma_+} \psi(x) \left\{\phi(v) - \phi(R_x v)\right\} (1-\alpha(x)) D^\perp f_+\, n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\
&\qquad + \int_{\Sigma_+} \psi(x) \left\{\phi(v) - \phi(R_x v)\right\} D f_+\, n(x)\cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x,
\end{aligned}$$

which concludes the proof. $\square$

6.3.3 Energy

Denote

$$\theta[g] := \int_{\mathbb{R}^d} \frac{(|v|^2 - d)}{\sqrt{2d}}\, g \,\mathrm{d}v$$

so that one has $\theta = \theta[f]$. We also introduce the vector $p = (p_i)_{1\le i\le d}$ defined by

$$p_i(v) := v_i \frac{(|v|^2 - d - 2)}{\sqrt{2d}}. \tag{6.3.4}$$

A straightforward computation gives

$$\theta[\mathcal{L} f] = -\sqrt{\frac{2}{d}}\, \nabla_x \cdot m + \nabla_x \cdot \left(\int_{\mathbb{R}^d} p(v) f^\perp \,\mathrm{d}v \right). \tag{6.3.5}$$

In the previous computation, we have added some vanishing terms because this formulation will be more convenient in the sequel. Indeed, we have

$$\theta[\mathcal{L}f] = \theta[-v\cdot\nabla_x(\pi f)] + \theta[-v\cdot\nabla_x f^\perp] + \theta[\mathcal{C}f^\perp].$$

Moreover,

$$\theta[-v\cdot\nabla_x(\varrho\mu)] = -\partial_{x_k}\varrho\int_{\mathbb{R}^d} v_k\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\,\mathrm{d}v = 0,$$

and

$$\begin{aligned}\theta[-v\cdot\nabla_x(m\cdot v\mu)] &= -\partial_{x_k}m_i\int_{\mathbb{R}^d} v_kv_i\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\,\mathrm{d}v\\ &= -\partial_{x_k}m_k\int_{\mathbb{R}^d} v_k^2\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\,\mathrm{d}v\\ &= -\sqrt{\frac{2}{d}}\,\nabla_x\cdot m.\end{aligned}$$

We also obtain

$$\theta\left[-v\cdot\nabla_x\left(\theta\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\right)\right] = -\partial_{x_k}\theta\int_{\mathbb{R}^d} v_k\left(\frac{|v|^2-d}{\sqrt{2d}}\right)^2\mu\,\mathrm{d}v = 0,$$

and finally, for the last term we combine the fact that $\int_{\mathbb{R}^d} vf^\perp\,\mathrm{d}v = 0$ with

$$\theta[-v\cdot\nabla_x f^\perp] = -\nabla_x\cdot\int_{\mathbb{R}^d} v\frac{|v|^2-d}{\sqrt{2d}}f^\perp\,\mathrm{d}v.$$

Let $u[\theta]$ be the solution to the elliptic equation (6.2.1) associated to $\theta\in L^2_x(\Omega)$ constructed in Theorem 6.2.1, namely $u[\theta]$ satisfies the following system a.e.

$$\begin{cases} -\Delta_x u[\theta] = \theta \quad \text{in} \quad \Omega,\\ (2-\alpha(x))\nabla_x u[\theta]\cdot n(x) + \alpha(x)u[\theta] = 0 \quad \text{on} \quad \partial\Omega,\end{cases} \tag{6.3.6}$$

and

$$\|u[\theta]\|_{H^2_x(\Omega)} \lesssim \|\theta\|_{L^2_x(\Omega)}. \tag{6.3.7}$$

Remark that in the specular reflection case, that is when $\alpha\equiv 0$ in (6.1.2), we supposed $\langle\theta\rangle = 0$ so that the solution $u[\theta]$ to the Poisson equation with Neumann boundary condition is well-defined.

Thanks to (6.3.5) one has $\theta[\mathcal{L}f]\in(H^1_x(\Omega))'$. By Theorem 6.2.1, we can hence also consider the unique variational solution $u[\theta[\mathcal{L}f]]$ to (6.2.1) associated to the data $\theta[\mathcal{L}f]$, namely $u[\theta[\mathcal{L}f]]$

satisfies (6.2.4) with source term $\theta[\mathcal{L}f]$ and verifies

$$\|u[\theta[\mathcal{L}f]]\|_{H^1_x(\Omega)} \lesssim \|\theta[\mathcal{L}f]\|_{(H^1_x(\Omega))'} \lesssim \|m\|_{L^2_x(\Omega)} + \|f^\perp\|_{L^2_{x,v}(\mu^{-1})}. \tag{6.3.8}$$

Recall that $p = (p_i)_{1\le i\le d}$ is defined in (6.3.4) and define $M_p[g] = (M_{p_i}[g])_{1\le i\le d}$ with

$$M_{p_i}[g] = \int_{\mathbb{R}^d} v_i \frac{(|v|^2 - d - 2)}{\sqrt{2d}}\, g\,\mathrm{d}v. \tag{6.3.9}$$

A straightforward computation gives

$$M_p[f] = M_p[f^\perp] \tag{6.3.10}$$

as well as $M_p[\mathcal{L}f] \in (H^1_x(\Omega))'$. Indeed one has

$$M_{p_i}[-v\cdot\nabla_x(\varrho\mu)] = -\partial_{x_i}\varrho \int_{\mathbb{R}^d} v_i^2 \frac{(|v|^2-d-2)}{\sqrt{2d}}\,\mu\,\mathrm{d}v = 0,$$

as well as

$$M_{p_i}[-v\cdot\nabla_x(m\cdot v\mu)] = -\partial_{x_k} m_\ell \int_{\mathbb{R}^d} v_k v_\ell v_i \frac{(|v|^2-d-2)}{\sqrt{2d}}\,\mu\,\mathrm{d}v = 0,$$

and, finally, for the term in θ one gets

$$\begin{aligned} M_{p_i}\left[-v\cdot\nabla_x\left(\theta\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\right)\right] &= -\partial_{x_i}\theta \int_{\mathbb{R}^d} v_i^2 \frac{(|v|^2-d)}{\sqrt{2d}}\frac{(|v|^2-d-2)}{\sqrt{2d}}\,\mu\,\mathrm{d}v \\ &= -\left(1+\frac{2}{d}\right)\partial_{x_i}\theta. \end{aligned}$$

Overall, we find

$$M_{p_i}[\mathcal{L}f] = -\left(1+\frac{2}{d}\right)\partial_{x_i}\theta - \partial_{x_k}\left(\int_{\mathbb{R}^d} v_k p_i(v) f^\perp\,\mathrm{d}v\right) + \Big(f^\perp, \mathcal{C}(p_i\mu)\Big)_{L^2_v(\mu^{-1})}. \tag{6.3.11}$$

We then obtain the following result.

Lemma 6.3.3. *There are constants $\kappa_1, C > 0$ such that*

$$\begin{aligned} &\langle -\nabla_x u[\theta], M_p[\mathcal{L}f]\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \langle -\nabla_x u[\theta[\mathcal{L}f]], M_p[f]\rangle_{L^2_x(\Omega)} \\ &\qquad \ge \kappa_1\|\theta\|^2_{L^2_x(\Omega)} - C\|m\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} \\ &\qquad\quad - C\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}. \end{aligned}$$

Proof. For the second term, one has, using (6.3.8) and (6.3.10),

$$\begin{aligned}\left|\left\langle -\nabla_x u[\theta[\mathcal{L}f]], M_p[f^\perp]\right\rangle_{L^2_x(\Omega)}\right| &\lesssim \|\nabla_x u[\theta[\mathcal{L}f]]\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})}\\ &\lesssim \|m\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} + \|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})}.\end{aligned}$$

For the first term, writing $M_p[\mathcal{L}f] = M_p[-v\cdot\nabla_x f] + M_p[\mathcal{C}f^\perp]$ one obtains

$$\langle -\nabla_x u[\theta], M_p[\mathcal{L}f]\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} = T_1 + T_2$$

with

$$T_1 := \left\langle \partial_{x_i} u[\theta], \partial_{x_j}\int_{\mathbb{R}^d} p_i(v) v_j f \,\mathrm{d}v\right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'}$$

and

$$T_2 := \left\langle -\nabla_x u[\theta], \int_{\mathbb{R}^d} p(v)\mathcal{C}f^\perp \,\mathrm{d}v\right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'}.$$

For the term T_2 we remark that

$$\int_{\mathbb{R}^d} p(v)\mathcal{C}f^\perp\,\mathrm{d}v = \left(f^\perp, \mathcal{C}(p\mu)\right)_{L^2_v(\mu^{-1})}$$

so that from the property (iii) on $\mathcal{C}$ and (6.3.7),

$$|T_2| \lesssim \|\nabla_x u[\theta]\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \lesssim \|\theta\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})}.$$

For the term T_1 we write

$$\begin{aligned}T_1 &= -\left\langle \partial_{x_j}\partial_{x_i} u[\theta], \int_{\mathbb{R}^d} p_i(v) v_j f\,\mathrm{d}v\right\rangle_{L^2_x(\Omega)} + \int_{\partial\Omega} \partial_{x_i} u[\theta] n_j(x)\left(\int_{\mathbb{R}^d} p_i(v) v_j f\,\mathrm{d}v\right)\mathrm{d}\sigma_x\\ &=: A + B.\end{aligned}$$

By writing $f = \varrho\mu + m\cdot v\mu + \theta\frac{|v|^2-d}{\sqrt{2d}}\mu + f^\perp$ we get

$$\int_{\mathbb{R}^d} p_i(v) v_j f\,\mathrm{d}v = \delta_{ij}\left(1+\frac{2}{d}\right)\theta + \int_{\mathbb{R}^d} p_i(v) v_j f^\perp\,\mathrm{d}v$$

and hence

$$A = \left(1+\frac{2}{d}\right)\langle -\Delta_x u[\theta], \theta\rangle_{L^2_x(\Omega)} - \left\langle \partial_{x_j}\partial_{x_i} u[\theta], \int_{\mathbb{R}^d} p_i(v) v_j f^\perp\,\mathrm{d}v\right\rangle_{L^2_x(\Omega)}.$$

Since $-\Delta_x u[\theta] = \theta$ and because of (6.3.7),

$$\begin{aligned}\left|\left\langle \partial_{x_j}\partial_{x_i} u[\theta], \int_{\mathbb{R}^d} p_i(v) v_j f^\perp \,\mathrm{d}v \right\rangle_{L^2_x(\Omega)}\right| &\lesssim \|\nabla_x^2 u[\theta]\|_{L^2_x(\Omega)} \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \\ &\lesssim \|\theta\|_{L^2_x(\Omega)} \|f^\perp\|_{L^2_{x,v}(\mu^{-1})},\end{aligned}$$

we get thanks to Young's inequality,

$$A \geq \frac{1}{2}\left(1 + \frac{2}{d}\right) \|\theta\|^2_{L^2_x(\Omega)} - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})}.$$

We now investigate the boundary term B. Thanks to Lemma 6.3.2 we have

$$\begin{aligned} B &= \int_\Sigma \nabla_x u[\theta] \cdot p(v)(\gamma f)\, n(x) \cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &= \int_{\Sigma_+} \nabla_x u[\theta] \cdot p(v) \alpha(x) D^\perp f_+\, n(x) \cdot v \,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &\quad + \int_{\Sigma_+} \nabla_x u[\theta] \cdot [p(v) - p(R_x v)](1-\alpha(x)) D^\perp f_+\, n(x)\cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &\quad + \int_{\Sigma_+} \nabla_x u[\theta] \cdot [p(v) - p(R_x v)] D f_+\, n(x) \cdot v\,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &:= B_1 + B_2 + B_3,\end{aligned}$$

and we remark that

$$p(v) - p(R_x v) = 2n(x)(n(x)\cdot v)\frac{(|v|^2 - d - 2)}{\sqrt{2d}},$$

so that

$$\nabla_x u[\theta] \cdot [p(v) - p(R_x v)] = +2\nabla_x u[\theta] \cdot n(x)\,(n(x)\cdot v)\,\frac{(|v|^2-d-2)}{\sqrt{2d}}.$$

Therefore in the case $\alpha \equiv 0$ we already obtain that $B = 0$.

Otherwise, if $\alpha \not\equiv 0$, recalling that $Df(x,v) = c_\mu \mu(v)\widetilde{f}(x)$, we first obtain for the term B_3, that

$$\begin{aligned} B_3 &= \frac{2c_\mu}{\sqrt{2d}} \int_{\Sigma_+} \nabla_x u[\theta] \cdot n(x) \mu(v)(|v|^2 - d - 2)\widetilde{f}(x)\,(n(x)\cdot v)^2 \,\mathrm{d}v\,\mathrm{d}\sigma_x \\ &= \frac{2c_\mu}{\sqrt{2d}} \int_{\partial\Omega} \nabla_x u[\theta]\cdot n(x) \widetilde{f}(x) \left(\int_{\Sigma_+^x} (|v|^2 - d - 2)\mu(v)\,(n(x)\cdot v)^2 \mathrm{d}v\right)\mathrm{d}\sigma_x\end{aligned}$$

and the integral in v vanishes, thus $B_3 = 0$. For the term B_1, Cauchy-Schwarz inequality and (6.3.7) gives

$$\begin{aligned}|B_1| &\lesssim \|\nabla_x u[\theta]\|_{L^2_x(\partial\Omega)} \|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v) n(x)\cdot v)} \\ &\lesssim \|\nabla_x u[\theta]\|_{H^1(\Omega)} \|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v) n(x)\cdot v)} \\ &\lesssim \|\theta\|_{L^2_x(\Omega)} \|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v) n(x)\cdot v)}.\end{aligned}$$

For the term B_2 the boundary condition satisfied by $u[\theta]$ in (6.3.6) implies

$$\nabla_x u[\theta] \cdot [p(v) - p(R_x v)](1-\alpha(x)) = -\frac{1-\alpha(x)}{2-\alpha(x)}\alpha(x) u[\theta] 2(n(x)\cdot v)\frac{(|v|^2-d-2)}{\sqrt{2d}},$$

hence we obtain

$$\begin{aligned}|B_2| &= 2\left|\int_{\Sigma_+} u[\theta]\frac{(|v|^2-d-2)}{\sqrt{2d}}\alpha(x)\frac{1-\alpha(x)}{2-\alpha(x)} D^\perp f_+\,(n(x)\cdot v)^2\,\mathrm{d}v\,\mathrm{d}\sigma_x\right| \\ &\lesssim \|u[\theta]\|_{L^2_x(\partial\Omega)}\|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)} \\ &\lesssim \|\theta\|_{L^2_x(\Omega)}\|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}.\end{aligned}$$

The proof is then complete by gathering previous estimates and using Young's inequality. □

6.3.4 Momentum

Denote

$$m[g] := \int_{\mathbb{R}^d} vg\,\mathrm{d}v$$

so that $m = m[f]$. A straightforward computation gives us

$$m[\mathcal{L}f] = -\nabla_x\varrho - \sqrt{\frac{2}{d}}\nabla_x\theta - \nabla_x\cdot\left(\int_{\mathbb{R}^d}(v\otimes v - I_d)f^\perp\,\mathrm{d}v\right). \tag{6.3.12}$$

Indeed we write

$$m[\mathcal{L}f] = m\left[-v\cdot\nabla_x\left(\varrho\mu + m\cdot v\mu + \theta\left(\frac{|v|^2-d}{\sqrt{2d}}\right)\mu + f^\perp\right) + \mathcal{C}f^\perp\right],$$

and we first have

$$m_i[-v\cdot\nabla_x\varrho\mu] = -\partial_{x_k}\varrho\int_{\mathbb{R}^d} v_iv_k\mu\,\mathrm{d}v = -\partial_{x_i}\varrho,$$

as well as

$$m_i[-v\cdot\nabla_x(m\cdot v\mu)] = -\partial_{x_k}m_\ell\int_{\mathbb{R}^d} v_iv_kv_\ell\mu\,\mathrm{d}v = 0,$$

and finally

$$\begin{aligned}m_i\left[-v\cdot\nabla_x\left(\theta\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\right)\right] &= -\partial_{x_k}\theta\int_{\mathbb{R}^d} v_iv_k\frac{(|v|^2-d)}{\sqrt{2d}}\,\mu\,\mathrm{d}v \\ &= -\sqrt{\frac{2}{d}}\,\partial_{x_i}\theta.\end{aligned}$$

For the last term on the right-hand side of (6.3.12), note that $\int_{\mathbb{R}^d} f^\perp \, \mathrm{d}v = 0$ by definition of $f^\perp$ and

$$m_i[-v \cdot \nabla_x f^\perp] = -\partial_{x_k} \int_{\mathbb{R}^d} v_i v_k f^\perp \, \mathrm{d}v.$$

Let $U[m]$ be the solution to the elliptic equation (6.2.16) associated to $m \in L^2_x(\Omega)$ constructed in Theorem 6.2.2, namely $U[m]$ satisfies the following system a.e.

$$\begin{cases} -\operatorname{div}_x(\nabla_x^{\mathrm{sym}} U) = m \quad \text{in} \quad \Omega, \\ U \cdot n(x) = 0 \quad \text{on} \quad \partial\Omega, \\ (2-\alpha(x))\left[\nabla_x^{\mathrm{sym}} U n(x) - (\nabla_x^{\mathrm{sym}} U : n(x) \otimes n(x)) n(x)\right] + \alpha(x) U = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{6.3.13}$$

and

$$\|U[m]\|_{H^2_x(\Omega)} \lesssim \|m\|_{L^2_x(\Omega)}. \tag{6.3.14}$$

It is worth noting that in the specular reflection case, that is when $\alpha \equiv 0$ in (6.1.2), we have further supposed the condition $\langle m \cdot Ax \rangle = 0$ for any $Ax \in \mathscr{R}_\Omega$, and therefore the solution $U[m]$ constructed in Theorem 6.2.2 is well-defined.

Thanks to (6.3.12) one has $m[\mathcal{L}f] \in (H^1_x(\Omega))'$ with

$$\|m[\mathcal{L}f]\|_{(H^1_x(\Omega))'} \lesssim \|\varrho\|_{L^2_x(\Omega)} + \|\theta\|_{L^2_x(\Omega)} + \|f^\perp\|_{L^2_{x,v}(\mu^{-1})}.$$

By Theorem 6.2.2, we can therefore also consider the unique variational solution $U[m[\mathcal{L}f]]$ to (6.2.16) associated to the data $m[\mathcal{L}f]$, namely $U[m[\mathcal{L}f]]$ satisfies (6.2.28) and verifies

$$\|U[m[\mathcal{L}f]]\|_{H^1_x(\Omega)} \lesssim \|\varrho\|_{L^2_x(\Omega)} + \|\theta\|_{L^2_x(\Omega)} + \|f^\perp\|_{L^2_{x,v}(\mu^{-1})}. \tag{6.3.15}$$

Let $q_{ij} = (q_{ij})_{1 \le i,j \le d}$ be given by

$$q_{ij}(v) = v_i v_j - \delta_{ij}$$

and define $M_q[f] = (M_{q_{ij}}[f])_{1 \le i,j \le d}$ with

$$M_{q_{ij}}[f] = \int_{\mathbb{R}^d} (v_i v_j - \delta_{ij}) f \, \mathrm{d}v. \tag{6.3.16}$$

By a straightforward computation, one gets

$$M_q[f] = \sqrt{\frac{2}{d}}\theta I_d + M_q[f^\perp]. \tag{6.3.17}$$

as well as

$$M_{q_{ij}}[\mathcal{L}f] = -2(\nabla_x^{\rm sym} m)_{ij} - \partial_{x_k} \left(\int_{\mathbb{R}^d} v_k q_{ij}(v) f^\perp \, \mathrm{d}v \right) + \left(f^\perp, \mathcal{C}(q_{ij}\mu) \right)_{L^2_v(\mu^{-1})} \tag{6.3.18}$$

so that $M_q[\mathcal{L}f] \in (H^1_x(\Omega))'$. Indeed one has

$$M_{q_{ij}}[-v \cdot \nabla_x \varrho \mu] = -\partial_{x_k} \varrho \int_{\mathbb{R}^d} v_k (v_i v_j - \delta_{ij}) \mu \, \mathrm{d}v = 0,$$

and moreover

$$\begin{aligned} M_{q_{ij}}[-v \cdot \nabla_x (v \cdot m\mu)] &= -\partial_{x_k} m_\ell \int_{\mathbb{R}^d} v_k v_\ell (v_i v_j - \delta_{ij}) \mu \, \mathrm{d}v \\ &= -\partial_{x_j} m_i - \partial_{x_i} m_j. \end{aligned}$$

We also have

$$M_{q_{ij}} \left[-v \cdot \nabla_x \left(\theta \frac{(|v|^2 - d)}{\sqrt{2d}} \, \mu \right) \right] = -\partial_{x_k} \theta \int_{\mathbb{R}^d} v_k \frac{(|v|^2 - d)}{\sqrt{2d}} (v_i v_j - \delta_{ij}) \mu \, \mathrm{d}v = 0,$$

from which we easily deduce (6.3.18).

We are now able to establish the following result.

Lemma 6.3.4. *There are constants $\kappa_2, C > 0$ such that*

$$\begin{aligned} &\langle -\nabla_x^{\rm sym} U[m], M_q[\mathcal{L}f] \rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \langle -\nabla_x^{\rm sym} U[m[\mathcal{L}f]], M_q[f] \rangle_{L^2_x(\Omega)} \\ &\quad \geq \kappa_2 \|m\|^2_{L^2_x(\Omega)} - C \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \|\varrho\|_{L^2_x(\Omega)} - C \|\theta\|_{L^2_x(\Omega)} \|\varrho\|_{L^2_x(\Omega)} \\ &\qquad - C \|\theta\|^2_{L^2_x(\Omega)} - C \|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} - C \|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+; \mu^{-1}(v) n(x) \cdot v)}. \end{aligned}$$

Proof. For the second term, using (6.3.15) and (6.3.17) we obtain

$$\begin{aligned} &\left| \left\langle -\nabla_x^{\rm sym} U[m[\mathcal{L}f]], \sqrt{\frac{2}{d}} \theta I_d + M_q[f^\perp] \right\rangle_{L^2_x(\Omega)} \right| \\ &\lesssim \|\nabla_x^{\rm sym} U[m[\mathcal{L}f]]\|_{L^2_x(\Omega)} \left(\|\theta\|_{L^2_x(\Omega)} + \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \right) \\ &\lesssim \left(\|\varrho\|_{L^2_x(\Omega)} + \|\theta\|_{L^2_x(\Omega)} + \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \right) \left(\|\theta\|_{L^2_x(\Omega)} + \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \right). \end{aligned}$$

For the first term, we write $M_q[\mathcal{L}f] = M_q[-v \cdot \nabla_x f] + M_q[\mathcal{C} f^\perp]$ to obtain

$$\langle -\nabla_x^{\rm sym} U[m], M_q[\mathcal{L}f] \rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} = T_1 + T_2$$

with

$$T_1 := \left\langle (\nabla_x^{\rm sym} U[m])_{ij}, \partial_{x_k} \int_{\mathbb{R}^d} q_{ij}(v) v_k f \, \mathrm{d}v \right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'}$$

and

$$T_2 := \left\langle -\nabla_x^{\mathrm{sym}} U[m], \int_{\mathbb{R}^d} q(v)\mathcal{C}f^\perp \,\mathrm{d}v \right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'}.$$

Observing that

$$\int_{\mathbb{R}^d} q(v)\mathcal{C}f^\perp \,\mathrm{d}v = \left(f^\perp, \mathcal{C}(q\mu)\right)_{L^2_v(\mu^{-1})}$$

we get from (6.3.14)

$$|T_2| \lesssim \|\nabla_x^{\mathrm{sym}} U[m]\|_{L^2_x(\Omega)} \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \lesssim \|m\|_{L^2_x(\Omega)} \|f^\perp\|_{L^2_{x,v}(\mu^{-1})}.$$

For the term T_1 we write

$$\begin{aligned} T_1 &= -\left\langle \partial_{x_k}(\nabla_x^{\mathrm{sym}} U[m])_{ij}, \int_{\mathbb{R}^d} q_{ij}(v) v_k f \,\mathrm{d}v \right\rangle_{L^2_x(\Omega)} \\ &\quad + \int_{\partial\Omega} (\nabla_x^{\mathrm{sym}} U[m])_{ij} n_k(v) \left(\int_{\mathbb{R}^d} q_{ij}(v) v_k f \,\mathrm{d}v \right) \mathrm{d}\sigma_x \\ &=: A + B. \end{aligned}$$

By writing $f = \varrho\mu + m\cdot v\mu + \theta\frac{|v|^2-d}{\sqrt{2d}}\mu + f^\perp$ we get

$$\int_{\mathbb{R}^d} q_{ij}(v) v_k f \,\mathrm{d}v = \delta_{jk} m_i + \delta_{ik} m_j + \int_{\mathbb{R}^d} q_{ij}(v) v_k f^\perp \,\mathrm{d}v$$

and hence

$$A = 2\left\langle -\operatorname{div}_x(\nabla_x^{\mathrm{sym}} U[m]), m\right\rangle_{L^2_x(\Omega)} - \left\langle \partial_{x_k}(\nabla_x^{\mathrm{sym}} U[m])_{ij}, \int_{\mathbb{R}^d} q_{ij}(v) v_k f^\perp \,\mathrm{d}v \right\rangle_{L^2_x(\Omega)}.$$

Since $-\operatorname{div}_x(\nabla_x^{\mathrm{sym}} U[m]) = m$ and using (6.3.14),

$$\begin{aligned} \left| \left\langle \partial_{x_k}(\nabla_x^{\mathrm{sym}} U[m])_{ij}, \int_{\mathbb{R}^d} q_{ij}(v) v_k f^\perp \,\mathrm{d}v \right\rangle_{L^2_x(\Omega)} \right| &\lesssim \|\nabla_x^2 U[m]\|_{L^2_x(\Omega)} \|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \\ &\lesssim \|m\|_{L^2_x(\Omega)} \|f^\perp\|_{L^2_{x,v}(\mu^{-1})}, \end{aligned}$$

we get thanks to Young's inequality,

$$A \geq \|m\|^2_{L^2_x(\Omega)} - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})}.$$

We now investigate the boundary term B. Thanks to Lemma 6.3.2 we have

$$
\begin{aligned}
B &= \int_{\Sigma} \nabla_x^{\mathrm{sym}} U[m] : q(v) f \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&= \int_{\Sigma_+} \nabla_x^{\mathrm{sym}} U[m] : q(v) \alpha(x) D^{\perp} f_+ \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&\quad + \int_{\Sigma_+} \nabla_x^{\mathrm{sym}} U[m] : [q(v) - q(R_x v)](1-\alpha(x)) D^{\perp} f_+ \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&\quad + \int_{\Sigma_+} \nabla_x^{\mathrm{sym}} U[m] : [q(v) - q(R_x v)] D f_+ \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&:= B_1 + B_2 + B_3,
\end{aligned}
$$

and we remark that

$$
q(v) - q(R_x v) = 4 \left[(n(x) \otimes v)^{\mathrm{sym}} - n(x) \otimes n(x) (n(x) \cdot v) \right] (n(x) \cdot v),
$$

so that

$$
\begin{aligned}
&\nabla_x^{\mathrm{sym}} U[m] : [q(v) - q(R_x v)] \\
&\quad = +4 \Big\{ \nabla_x^{\mathrm{sym}} U[m] : (n(x) \otimes v)^{\mathrm{sym}} - \nabla_x^{\mathrm{sym}} U[m] : n(x) \otimes n(x) (n(x) \cdot v) \Big\} (n(x) \cdot v).
\end{aligned}
$$

Taking the scalar product with v in the boundary condition of (6.3.13), we see that, in the case $\alpha \equiv 0$, we already have $B = 0$.

Otherwise, if $\alpha \not\equiv 0$, we first obtain for the term B_3, making a change of variables $v \mapsto R_x v$, using also that $(R_x v \cdot n) = -(v \cdot n)$, and recalling that $Df(x,v) = c_\mu \mu(v) \widetilde{f}(x)$, that

$$
\begin{aligned}
B_3 &= 2 c_\mu \int_{\Sigma} \nabla_x^{\mathrm{sym}} U[m] : q(v) \mu(v) \widetilde{f}(x) \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&= 2 c_\mu \int_{\partial\Omega} (\nabla_x^{\mathrm{sym}} U[m])_{ij} n_k(x) \widetilde{f}(x) \left(\int_{\mathbb{R}^d} q_{ij}(v) v_k \mu(v) \, \mathrm{d}v \right) \mathrm{d}\sigma_x
\end{aligned}
$$

and the integral in v vanishes, thus $B_3 = 0$. For the term B_1, Cauchy-Schwarz inequality and (6.3.14) give

$$
\begin{aligned}
|B_1| &\lesssim \| \nabla_x^{\mathrm{sym}} U[m] \|_{L^2_x(\partial\Omega)} \| \alpha D^{\perp} f_+ \|_{L^2(\Sigma_+; \mu^{-1}(v) n(x) \cdot v)} \\
&\lesssim \| m \|_{L^2_x(\Omega)} \| \alpha D^{\perp} f_+ \|_{L^2(\Sigma_+; \mu^{-1}(v) n(x) \cdot v)}.
\end{aligned}
$$

For the term B_2 the boundary condition satisfied by $U[m]$ in (6.3.13) implies

$$
\nabla_x^{\mathrm{sym}} U[m] : [q(v) - q(R_x v)](1 - \alpha(x)) = -\frac{1-\alpha(x)}{2-\alpha(x)} 4 \alpha(x) (U[m] \cdot v)(n(x) \cdot v)
$$

hence we obtain

$$\begin{aligned}|B_2| &= 4\left|\int_{\Sigma_+}(U[m]\cdot v)\,\frac{1-\alpha(x)}{2-\alpha(x)}\alpha(x)D^\perp f\,(n(x)\cdot v)^2\,\mathrm{d}v\,\mathrm{d}\sigma_x\right| \\ &\lesssim \|U[m]\|_{L^2_x(\partial\Omega)}\|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)} \\ &\lesssim \|m\|_{L^2_x(\Omega)}\|\alpha D^\perp f_+\|_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}.\end{aligned}$$

The proof is then complete by gathering previous estimates and using Young's inequality. □

6.3.5 Mass

Denote

$$\varrho[g] := \int_{\mathbb{R}^d} g\,\mathrm{d}v$$

so that $\varrho = \varrho[f]$ and remark that, by a straightforward computation, one has

$$\varrho[\mathcal{L}f] = \int_{\mathbb{R}^d}(-v_k\partial_{x_k}f + \mathcal{C}f)\,\mathrm{d}v = -\partial_{x_k}\int_{\mathbb{R}^d} v_k f\,\mathrm{d}v = -\nabla_x\cdot m, \tag{6.3.19}$$

so that $\varrho[\mathcal{L}f] \in (H^1_x(\Omega))'$.

Let $u_{\mathrm{N}}[\varrho]$ be the solution to the Poisson equation (6.2.1) with Neumann boundary condition associated to $\varrho \in L^2_x(\Omega)$ constructed in Theorem 6.2.1, namely $u_{\mathrm{N}}[\varrho]$ satisfies a.e.

$$\begin{cases} -\Delta_x u_{\mathrm{N}}[\varrho] = \varrho \text{ in } \Omega, \\ \nabla_x u_{\mathrm{N}}[\varrho]\cdot n(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.3.20}$$

which is indeed well-defined since $\langle\varrho\rangle = 0$, and we have

$$\|u_{\mathrm{N}}[\varrho]\|_{H^2_x(\Omega)} \lesssim \|\varrho\|_{L^2_x(\Omega)}. \tag{6.3.21}$$

Thanks to (6.3.19) one has $\varrho[\mathcal{L}f] \in (H^1(\Omega))'$. By Theorem 6.2.1 we can hence also consider the unique variational solution $u_{\mathrm{N}}[\varrho[\mathcal{L}f]]$ to (6.2.1) with Neumann boundary condition associated to the data $\varrho[\mathcal{L}f]$, namely $u_{\mathrm{N}}[\varrho[\mathcal{L}f]]$ satisfies (6.2.4) and

$$\|u_{\mathrm{N}}[\varrho[\mathcal{L}f]]\|_{H^1_x(\Omega)} \lesssim \|m\|_{L^2_x(\Omega)}. \tag{6.3.22}$$

We now obtain the following result concerning the mass ϱ.

Lemma 6.3.5. *There are constants $\kappa_3, C > 0$ such that*

$$\begin{aligned}&\langle -\nabla_x u_{\mathrm{N}}[\varrho], m[\mathcal{L}f]\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \langle -\nabla_x u_{\mathrm{N}}[\varrho[\mathcal{L}f]], m[f]\rangle_{L^2_x(\Omega)} \\ &\qquad \geq \kappa_3\|\varrho\|^2_{L^2_x(\Omega)} - C\left(\|m\|^2_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)} + \|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})}\right) \\ &\qquad\quad - C\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}.\end{aligned}$$

Proof. For the second term one has from (6.3.22) that

$$\left|\langle -\nabla_x u_{\mathrm{N}}[\varrho[\mathcal{L}f]], m[f]\rangle_{L^2_x(\Omega)}\right| \lesssim \|\nabla_x u_{\mathrm{N}}[\varrho[\mathcal{L}f]]\|_{L^2_x(\Omega)}\|m\|_{L^2_x(\Omega)} \lesssim \|m\|^2_{L^2_x(\Omega)}.$$

For the first term, writing $m[\mathcal{L}f] = m[-v\cdot\nabla_x f] + m[\mathcal{C}f^\perp]$ and observing that $m[\mathcal{C}f^\perp] = 0$, one obtains

$$\langle -\nabla_x u_{\mathrm{N}}[\varrho], m[\mathcal{L}f]\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} = \left\langle \partial_{x_i} u_{\mathrm{N}}[\varrho], \partial_{x_j}\int_{\mathbb{R}^d} v_i v_j f\,\mathrm{d}v\right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'}.$$

We then write

$$\begin{aligned}&\left\langle \partial_{x_i} u_{\mathrm{N}}[\varrho], \partial_{x_j}\int_{\mathbb{R}^d} v_i v_j f\,\mathrm{d}v\right\rangle_{H^1_x(\Omega),(H^1_x(\Omega))'}\\ &\quad= -\left\langle \partial_{x_j}\partial_{x_i} u_{\mathrm{N}}[\varrho], \int_{\mathbb{R}^d} v_i v_j f\,\mathrm{d}v\right\rangle_{L^2_x(\Omega)} + \int_{\partial\Omega} \partial_{x_i} u_{\mathrm{N}}[\varrho] n_j(x)\left(\int_{\mathbb{R}^d} v_i v_j f\,\mathrm{d}v\right)\mathrm{d}\sigma_x\\ &\quad=: A + B.\end{aligned}$$

By writing $f = \varrho\mu + m\cdot v\mu + \theta\frac{|v|^2-d}{\sqrt{2d}}\mu + f^\perp$ we get

$$\int_{\mathbb{R}^d} v_i v_j f\,\mathrm{d}v = \delta_{ij}\varrho + \delta_{ij}\sqrt{\frac{2}{d}}\theta + \int_{\mathbb{R}^d} v_i v_j f^\perp\,\mathrm{d}v$$

and hence

$$A = \langle -\Delta_x u_N[\varrho], \varrho\rangle_{L^2_x(\Omega)} + \sqrt{\frac{2}{d}}\langle -\Delta_x u_N[\varrho], \theta\rangle_{L^2_x(\Omega)} - \left\langle \partial_{x_j}\partial_{x_i} u_N[\varrho], \int_{\mathbb{R}^d} v_i v_j f^\perp\,\mathrm{d}v\right\rangle_{L^2_x(\Omega)}.$$

Since $-\Delta_x u_N[\varrho] = \varrho$ and using (6.3.21)

$$\begin{aligned}\left|\left\langle \partial_{x_j}\partial_{x_i} u_N[\varrho], \int_{\mathbb{R}^d} v_i v_j f^\perp\,\mathrm{d}v\right\rangle_{L^2_x(\Omega)}\right| &\lesssim \|\nabla^2_x u_N[\varrho]\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})}\\ &\lesssim \|\varrho\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})},\end{aligned}$$

we get thanks to Young's inequality,

$$A \geq \frac{1}{2}\|\varrho\|^2_{L^2_x(\Omega)} - C\|\theta\|^2_{L^2_x(\Omega)} - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})}.$$

We now investigate the boundary term B. Thanks to Lemma 6.3.2 we have

$$
\begin{aligned}
B &= \int_{\Sigma} \nabla_x u_{\mathrm{N}}[\varrho] \cdot v f \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&= \int_{\Sigma_+} \nabla_x u_{\mathrm{N}}[\varrho] \cdot v \alpha(x) D^{\perp} f_+ \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&\quad + \int_{\Sigma_+} \nabla_x u_{\mathrm{N}}[\varrho] \cdot [v - R_x v](1 - \alpha(x)) D^{\perp} f_+ \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&\quad + \int_{\Sigma_+} \nabla_x u_{\mathrm{N}}[\varrho] \cdot [v - R_x v] D f_+ \, n(x) \cdot v \, \mathrm{d}v \, \mathrm{d}\sigma_x \\
&:= B_1 + B_2 + B_3,
\end{aligned}
$$

and we remark that

$$
v - R_x v = 2n(x)(n(x) \cdot v),
$$

so that

$$
\nabla_x u_{\mathrm{N}}[\varrho] \cdot [v - R_x v] = 2 \nabla_x u_{\mathrm{N}}[\varrho] \cdot n(x) \, (n(x) \cdot v).
$$

Therefore, thanks to the boundary condition satisfied by $u_{\mathrm{N}}[\varrho]$ in (6.3.20), we already obtain $B_2 = B_3 = 0$.

In the case $\alpha \equiv 0$ we also have $B_1 = 0$. Otherwise, if $\alpha \not\equiv 0$, Cauchy-Schwarz inequality and (6.3.21) yield for the term B_1

$$
\begin{aligned}
|B_1| &\lesssim \|\nabla_x u_{\mathrm{N}}[\varrho]\|_{L^2_x(\partial\Omega)} \|\alpha D^{\perp} f_+\|_{L^2(\Sigma_+; \mu^{-1}(v) n(x) \cdot v)} \\
&\lesssim \|\varrho\|_{L^2_x(\Omega)} \|\alpha D^{\perp} f_+\|_{L^2(\Sigma_+; \mu^{-1}(v) n(x) \cdot v)}.
\end{aligned}
$$

The proof is then complete by gathering previous estimates and using Young's inequality. □

6.3.6 Proof of Theorem 6.3.1

Let f satisfy the assumptions of Theorem 6.3.1. We define the scalar product on $L^2_{x,v}(\mu^{-1})$ by

$$
\begin{aligned}
\langle\!\langle f, g \rangle\!\rangle &:= \langle f, g \rangle_{L^2_{x,v}(\mu^{-1})} \\
&\quad + \varepsilon_1 \langle -\nabla_x u[\theta[f]], M_p[g] \rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \varepsilon_1 \langle -\nabla_x u[\theta[g]], M_p[f] \rangle_{L^2_x(\Omega)} \\
&\quad + \varepsilon_2 \langle -\nabla^{\mathrm{sym}}_x U[m[f]], M_q[g] \rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \varepsilon_2 \langle -\nabla^{\mathrm{sym}}_x U[m[g]], M_q[f] \rangle_{L^2_x(\Omega)} \\
&\quad + \varepsilon_3 \langle -\nabla_x u_{\mathrm{N}}[\varrho[f]], m[g] \rangle_{H^1_x(\Omega),(H^1_x(\Omega))'} + \varepsilon_3 \langle -\nabla_x u_{\mathrm{N}}[\varrho[g]], m[f] \rangle_{L^2_x(\Omega)}
\end{aligned}
\tag{6.3.23}
$$

with $0 \ll \varepsilon_3 \ll \varepsilon_2 \ll \varepsilon_1 \ll 1$, and where we recall that the moments M_p and M_q are defined respectively in (6.3.9) and (6.3.16); $u[\theta[f]]$ is the solution of the Poisson equation (6.3.6) with data $\theta[f]$; $U[m[f]]$ is the solution to the elliptic system (6.3.13) with data $m[f]$; $u_{\mathrm{N}}[\varrho[f]]$ is the solution to the Poisson equation with homogeneous Neumann boundary condition (6.3.20) with data $\varrho[f]$, and similarly for the terms depending on g. We denote by $|\!|\!| \cdot |\!|\!|$ the norm associated to the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, and we observe that

$$\|f\|_{L^2_{x,v}(\mu^{-1})} \lesssim |\!|\!| f |\!|\!| \lesssim \|f\|_{L^2_{x,v}(\mu^{-1})}.$$

Recalling that we denote $\varrho = \varrho[f]$, $m = m[f]$ and $\theta = \theta[f]$, noting that $\sqrt{\alpha(2-\alpha)} \geq \alpha$ since α takes values in $[0,1]$, and gathering Lemmas 6.3.1, 6.3.3, 6.3.4 and 6.3.5, one has

$$\begin{aligned}
\langle\!\langle -\mathcal{L}f, f \rangle\!\rangle \geq{}& \lambda \|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + \frac{1}{2}\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)} \\
&+ \varepsilon_1\Big(\kappa_1 \|\theta\|^2_{L^2_x(\Omega)} - C\|m\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} \\
&\qquad - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} - C\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}\Big) \\
&+ \varepsilon_2\Big(\kappa_2 \|m\|^2_{L^2_x(\Omega)} - C\|\varrho\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} - C\|\varrho\|_{L^2_x(\Omega)}\|\theta\|_{L^2_x(\Omega)} \\
&\qquad - C\|\theta\|^2_{L^2_x(\Omega)} - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} - C\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}\Big) \\
&+ \varepsilon_3\Big(\kappa_3 \|\varrho\|^2_{L^2_x(\Omega)} - C\|m\|^2_{L^2_x(\Omega)} - C\|\theta\|^2_{L^2_x(\Omega)} \\
&\qquad - C\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} - C\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}\Big).
\end{aligned}$$

Thanks to Young's inequality, observe that we have

$$\begin{aligned}
\varepsilon_1 C\|m\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} &\leq \frac{\lambda}{4}\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + C\varepsilon_1^2\|m\|^2_{L^2_x(\Omega)}, \\
\varepsilon_2 C\|\varrho\|_{L^2_x(\Omega)}\|f^\perp\|_{L^2_{x,v}(\mu^{-1})} &\leq \frac{\lambda}{4}\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + C\varepsilon_2^2\|\varrho\|^2_{L^2_x(\Omega)}, \\
\varepsilon_2 C\|\varrho\|_{L^2_x(\Omega)}\|\theta\|_{L^2_x(\Omega)} &\leq \frac{\varepsilon_1\kappa_1}{2}\|\theta\|^2_{L^2_x(\Omega)} + C\frac{\varepsilon_2^2}{\varepsilon_1}\|\varrho\|^2_{L^2_x(\Omega)}.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\langle\!\langle -\mathcal{L}f, f \rangle\!\rangle \geq{}& \left(\frac{\lambda}{2} - \varepsilon_1 C - \varepsilon_2 C - \varepsilon_3 C\right)\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} \\
&+ \left(\frac{1}{2} - \varepsilon_1 C - \varepsilon_2 C - \varepsilon_3 C\right)\|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)} \\
&+ \left(\frac{\varepsilon_1\kappa_1}{2} - \varepsilon_2 C - \varepsilon_3 C\right)\|\theta\|^2_{L^2_x(\Omega)} \\
&+ \left(\varepsilon_2\kappa_2 - \varepsilon_1^2 C - \varepsilon_3 C\right)\|m\|^2_{L^2_x(\Omega)} \\
&+ \left(\varepsilon_3\kappa_3 - \varepsilon_2^2 C - \frac{\varepsilon_2^2}{\varepsilon_1}C\right)\|\varrho\|^2_{L^2_x(\Omega)}.
\end{aligned}$$

We now choose $\varepsilon_1 := \varepsilon$, $\varepsilon_2 := \varepsilon^{\frac{3}{2}}$ and $\varepsilon_3 := \varepsilon^{\frac{7}{4}}$, therefore

$$\begin{aligned}\langle\!\langle -\mathcal{L}f, f\rangle\!\rangle \geq & \left(\frac{\lambda}{2} - \varepsilon C\right) \|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + \left(\frac{1}{2} - \varepsilon C\right) \|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)} \\ & + \varepsilon \left(\frac{\kappa_1}{2} - \varepsilon^{\frac{1}{2}} C\right) \|\theta\|^2_{L^2_x(\Omega)} + \varepsilon^{\frac{3}{2}} \left(\kappa_2 - \varepsilon^{\frac{1}{4}} C\right) \|m\|^2_{L^2_x(\Omega)} \\ & + \varepsilon^{\frac{7}{4}} \left(\kappa_3 - \varepsilon^{\frac{1}{4}} C\right) \|\varrho\|^2_{L^2_x(\Omega)}\end{aligned}$$

and choosing $0 < \varepsilon < 1$ small enough we get

$$\begin{aligned}\langle\!\langle -\mathcal{L}f, f\rangle\!\rangle \geq \kappa & \left(\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + \|\varrho\|^2_{L^2_x(\Omega)} + \|m\|^2_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)}\right) \\ & + \kappa' \|\alpha D^\perp f_+\|^2_{L^2(\Sigma_+;\mu^{-1}(v)n(x)\cdot v)}\end{aligned}$$

for some constant $\kappa, \kappa' > 0$. We concludes the proof of Theorem 6.3.1 since

$$\|f^\perp\|^2_{L^2_{x,v}(\mu^{-1})} + \|\varrho\|^2_{L^2_x(\Omega)} + \|m\|^2_{L^2_x(\Omega)} + \|\theta\|^2_{L^2_x(\Omega)} = \|f\|^2_{L^2_{x,v}(\mu^{-1})}$$

and $\|\cdot\|_{L^2_{x,v}(\mu^{-1})}$ is equivalent to $|\!|\!|\cdot|\!|\!|$. $\square$

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