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Sujet de la thèse :

## Study of singular multi-critical fixed points in the $\mathrm{O}(\mathrm{N})$ model using the functional renormalization group.

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## Chapter 1

## Introduction

> "It is not certain that everything is uncertain.", $\binom{n}{r}=\binom{n-1}{r-1}+\binom{n-1}{r}$ -Blaise Pascal $\quad l$

The universe is governed by four forces which we understand to a very high degree and yet research in physics is still quite vivid and rich. It thus seems that the whole can be larger than the sum of its parts but how does such complexity arise from so little information? In essence there are two major reasons. The first is due to the great complexity of many body systems and the second is due to the extremely large range of scales that appear in physical phenomena. Indeed our current range of exploration lies between the scales of particle physics at around $10^{-20} \mathrm{~m}$ up to the scales of our observable universe at $10^{28} \mathrm{~m}$. Between these scales several successive emergent phenomena arise. The phase diagram of ordinary water is a great example of this as the same underlying constituents obeying the same underlying rules of physics can lead, depending on the external conditions, to a gas, a liquid or one of the 18 different states of ice 11 . If we consider much larger temperatures, thereby probing the microscopic scales, the structure of the atom as an association of electrons and a nucleus becomes apparent and its dissociation takes place creating a plasma. Finally at the energies of the early stages of universe we may further explore into the underlying structure of the nucleons of the atoms thereby creating a quark-gluon plasma. Hence, the collective is indeed much more complex than the sum of its parts and full knowledge of what happens at the tiniest subatomic scales is often insufficient when one seeks to understand larger scales.

Another, perhaps simpler, example of how complexity arises from elementary rules lies in the field of cellular automata. In this case, from an initial condition of bits of zeroes and ones and a set of simple rules given by a few lines of code, emerges a huge variety of complex phenomena. For example in Conway's game of life, a few set of rules can lead to an amazing array of structures that may interact with one another leading to further structures at larger or smaller scales. This is not the only example however, in fact the field of cellular automata are known for the phenomenon of emergence where we may also cite the so-called "rule 110 " which like Conway's game of life is Turing-complete [2-5] in the sense that one can use these simple sets of rules to simulate any computer code. Thus, the same rules can lead to different "physics" depending on the setup. Reciprocally, different physics may sometimes be grouped together such that they obey the same universal rules. This offers a possibility to classify systems thereby decoupling the details of physical phenomena from their underlying common "universality class". As such, it is possible to make predictions from very little information such as symmetries. As an example of the power of symmetry we may remark that we do not expect an endpoint to liquid-solid transitions as such an endpoint would lead to a continuous crossover between the liquid phase that has a continuous rotation symmetry and the solid which usually has a discrete rotational symmetry. But this is not the case of liquid and gas as both have the same symmetries and are in fact just two states of the same matter. At high pressures the liquid state is favored and at high temperature the gas state is favored. But what happens in systems where both the temperature and pressure are large ? Along the transition line between gas and liquid we expect a special point, called the critical point, where the gas and liquid states become equivalent and the system can easily jump between these two states. This is thus an added symmetry to the problem where both states become equivalent for
all physical purposes. A similar situation happens in uni-axial ${ }^{1}$ ferromagnets where the magnetization is described by the sum of the elementary magnetizations of the atoms in the system. When the ferromagnet is uni-axial each elementary constituent can have a magnetization that can either be up along the natural axis of the magnet or down. In absence of any external magnetic field, these two situations are locally equivalent and thus globally as well, that is, it is equivalent whether the average macroscopic magnetization points up or down. These two macroscopic states are then equivalent in the same way that gas and liquid become equivalent along the transition line.

Hence the symmetries are indeed the same, that is, regardless of the details, such as the fact that the atoms are not the same, or the type of interactions, at the level of a global abstract description, both systems are composed of a macroscopic mixture of equivalent states. These systems that share common space dimensionality and symmetry breaking patterns can be gathered in "universality classes" such that all systems in a given class show common behavior when they are close to their respective critical point. More precisely, the quantities that allow us to describe the critical behavior of a system are either universal or non-universal, the universal ones being common to all systems belonging to a given universality class. Among the most famous universal quantities are the critical exponents describing the manner in which certain quantities such as susceptibilities diverge close to the critical point. The renormalization group has provided a theoretical framework explaining why systems that differ microscopically can be grouped into universality classes.

Perhaps even more surprising is that in fact this is not limited to condensed matter physics. Given a set of symmetries and a spatial dimension of a problem at hand, models in condensed matter, high energy physics or socio-economics will display the same behavior near their respective critical points. As such the study of symmetries independently of any particular model at hand is often quite fruitful. This thesis is then dedicated to a very general class of symmetries called the $O(N)$ models that will be introduced in Sec.(3.1.1). In particular we will be interested in the study of critical points where scale invariance emerges. Due to this scale invariance fluctuations at all scales emerge. In the case of the gas-liquid transition, fluctuations in density lead to fluctuations in the index of refraction resulting in a chaotic light scattering which is called critical opalescence. The renormalization group (RG) as a mapping of scales then stands out as a crucial tool as such theories correspond to fixed-points of the RG.

More broadly, the renormalization group goes hand in hand with the presence of fluctuations in a system. These fluctuations are in turn quite ubiquitous in nature. For example even the "empty" vacuum is filled with fluctuations that can lead to measurable effects such as the Casimir effect and the Lamb shift and while much less measurable, Hawking radiation. It is even possible that the primordial quantum fluctuations seen on the cosmic microwave background may have lead to the inhomogeneities needed for gravity to form galaxies and ultimately billions of years later, us. More generally, the domain of interest of fluctuations in a system include biology, glassy systems, mathematical models of economy, society, and turbulence to name but a few. Moreover, the combination of both quantum and statistical fluctuations is used in fields such as cold atom gases, superfluids, superconductors and even cosmological models of the early universe.

This great flexibility of the RG also allows us to tackle questions that are beyond the reach of other methods such as the conformal bootstrap which are unable to study first order transitions ${ }^{2}$ or more generally phase diagrams.

The outline of the present thesis is then separated into two chapters:

- In the first chapter we give a general introduction to the RG from the perspective of functional selfsimilarity. This chapter is then further divided into the following sections:
- In Sec.(2.1) we study mean field theory and in particular the study of symmetries in Landau theory and the predictions of continuous phase transitions. This section on Landau theory will allows us to introduce the concept of multicritical fixed-points of the RG in Sec.(3.1.2).

[^0]- In Sec. 2.2 we will introduce the concept of functional self-similarity (FSS) which will be crucial for our introduction to the RG. In particular, we will study how FSS is used in the framework of differential equations and how it may be applied to perturbative RG. The last subsection will also introduce a variational method that will be analogous to optimizations within the non perturbative renormalization group.
- In Sec. 2.3 we introduce the non-perturbative renormalization group from the perspective of FSS and its relationship to Wilsonian RG. This study will allow us to discuss the dispersive nature of the RG.
- In Sec.(2.4 we discuss in general the relationship between FSS and approximations. In the last sections we will explore possible approximation schemes for the functional RG, thereby introducing new approximation schemes.
- In the second chapter we apply the RG to the $O(N)$ models:
- In Sec. 3.1 we will discuss the Bardeen-Moshe-Bander phenomenon for multi-critical fixed-points and the existence of new singular fixed-points. In this study we will be mainly interested in the tricritical case. One of the results obtained will be quantitatively poor at the level of the leading approximation (the LPA) which is a great opportunity to study the convergence of various approximation methods within the NPRG framework. At moderately large $N$ we will find non trivial homotopies in the space $(N, d)$ where the action of a loop in the space $(N, d)$ leads to permutations among the fixed-points.


## Chapter 2

## Introduction to the functional renormalization framework

### 2.1 The role of correlations

### 2.1.1 Mean field theory applied to gas-liquid and uni-axial ferromagnetic systems

In this section we shall take a quick glimpse of mean field theory in order to understand the corrections that the renormalization group adds to it. Moreover, this section will also introduce very briefly Landau theory as it will be a useful tool for understanding multicritical points in Sec.(3.1.2).

Mean-field theory while being quite simple in nature has made many achievements. Just to cite a few we may mention superconductors, ferromagnets outside of the critical temperature, the Van der Waals equation and the Hartree-Fock approximation.

In the case of the Hartree-Fock approximation, the objective is to find an approximation scheme to manage the many body interactions of Z electrons within an atom. This is achieved by replacing the many body interactions between an electron and its Z-1 neighbors within the atom by an overall effective interaction of an electron with a fixed field generated by the Z-1 other electrons. This then neglects the back reaction the first electron has on the others and as quantum mechanics is a probabilistic theory, this can be framed in terms of neglected correlations within the atom. However, while being a rather crude approximation in nature, it has proven quite useful in chemistry and typically agrees reasonably with experiments.

For the Van der Waals model, the equations are obtained similarly by considering that each atom in the gas feels an effective interaction from the other atoms while neglecting correlations between atoms. This then reduces the coupled N -body problem to N decoupled 1-body problems. For a sufficiently dilute gas the energy cost of the interaction of the atom with its environment will be $u=-\frac{N}{V} a$ as it should increase in absolute value with the density $\frac{N}{V}$ of the environment and $a$ here is positive such that the interaction between atoms will be attractive, and hence stabilizing. This attractive force is due to the Van der Waals forces whose quantum origin will be mentioned very briefly in the following section and the reduction of the $N-1$ other particles into the term "environment" is the mean field approximation as the actual dynamics involved is much more complex. However, for sufficiently low densities and for pressures and temperatures outside of the critical point, where the second order transition takes place, this approximation is valid and correlations can be reasonably neglected on the large macroscopic scales. We shall thus carry on with this approximation in order to retrieve the Van der Waals equation.

The total cost in energy from the mean field approximation is $\Delta U=N u=-\frac{N^{2} a}{V}$. This cost in energy can then be seen as a cost in free energy via $F=U-T S$.

However, this is, of course, only true for distances sufficiently large with respect to the size of the atom as atoms do not penetrate. This volume exclusion leads us to modify the volume $V$ as $V^{\prime}=V-N b$ where $b$ gives the exclusion for a single atom. We shall now gather all of this into the free energy. First, let us retrieve the free energy in the case of an ideal gas where $P V=N k T$. As $\mathrm{d} F=-S \mathrm{~d} T-P \mathrm{~d} V$ we have $P=-\left.\frac{\partial F}{\partial V}\right|_{T}$ and thus using $P=\frac{N k T}{V}$ we obtain $F_{I}(V, T)=-N k T \log (V)+F_{0}(T)$ where we use $F_{I}$ to stress that this is only for the ideal gas and $F_{0}$ is a $T$ dependent integration constant as we have only integrated with respect to $V$.

If we now take into account the attractive energy term $\Delta U$ and the volume exclusion we obtain:

$$
\begin{equation*}
F(V, T)=F_{I}(V-N b, T)+\Delta U=-N k T \log (V-N b)-\frac{N^{2} a}{V}+F_{0}(T) \tag{2.1.1}
\end{equation*}
$$

Hence using again $P=-\left.\frac{\partial F}{\partial V}\right|_{T}$ and defining $x=\frac{N}{V}$ we arrive at the Van der Waals equation : $P=$ $\frac{x k T}{(1-b x)}-a x^{2}$. This equation is valid mainly for $x$ sufficiently small and in particular it is not very useful for dense liquids. As such only the first few terms really matter and in general one tends to use instead the virial expansion $P=x k T\left(1+A x+B x^{2}+\ldots\right)$ valid for a larger range of densities. However, for the sake of concreteness and clarity, we will stick with the Van der Waals equation which we rewrite using dimensionless variables :

$$
\begin{equation*}
\widehat{x}^{3}-\widehat{x}^{2}+\widehat{x}(\widehat{T}+\widehat{P})-\widehat{P}=0, \widehat{x}=x b, \widehat{T}=k T b / a, \widehat{P}=P b^{2} / a \tag{2.1.2}
\end{equation*}
$$

For a given $(\widehat{P}, \widehat{T})$, the roots of this polynomial that verify the physical criteria $\left.\frac{\partial x}{\partial P}\right|_{T}>0$ or equivalently $\left.\frac{\partial P}{\partial x}\right|_{T}>0$, that is, those where the density increases with the pressure, correspond to different phases of the fluid. The number of these roots depend on the values of $(\widehat{P}, \widehat{T})$ and as we shall see this model predicts both first order and second order phase transitions. This is rather remarkable for such a simple approximation valid a priori only for reasonably small densities in the gas phase. As we shall see later in Sec. (2.1.2), this is due to the fact that it contains the minimal ingredients to predict a transition between the disordered high temperature state where the entropic free energy $T S$ dominates and an ordered state dominated by the cohesion energy $-\frac{N a}{V}$. We study this phase transition and in general the phase diagram of the liquid gas transition using only Descartes's rule of signs for polynomial roots in Appendix A -where the Descartes's rule is stated and a short intuitive "semi proof" is given - as the rule of signs will be of use for the discussion of multicritical diagrams in section Sec.(3.1.2). Instead to finish this subsection let us just note that the analogy between gas-liquid systems and uni-axial ferromagnetic systems can be further expanded as the equation of state of a uni-axial ferromagnetic leads to similar phenomenology. Indeed, if we project the spin of a single entity in a uni-axial ferromagnet along the direction of its easy axis, this single spin bathed in the average magnetic field of its neighbors, can be up $s=+1$ with probability $p_{+}$or down $s=-1$ with probability $p_{-}$. The average magnetization of a single spin is thus given by $m=\left((+\mu) p_{+}+(-\mu) p_{-}\right)$where $p_{ \pm}=\mathcal{N} e^{-E_{ \pm} / k T}$ with $\mathcal{N}$ chosen such that $p_{+}+p_{-}=1$ and $\mu$ is Bohr magnetic constant that links spin to magnetization.

The energy of a spin is $E=-\mu s B_{\text {eff }}$ where $B_{e f f}=a z m$ is the effective magnetic field generated by the nearest $z$ neighbors which are all approximated by their average magnetization $m$ and where $a$ represents the coupling influence of the neighbors. The resulting equation of state is that given by a mean field approximation of the so called Ising model whose energy function is given by $\sum_{i, h}-(J / k T) s_{i} s_{h(i)}$ where $h(i)$ represents a nearest neighbor of the site $i$. The mean field approximation is thus obtained by taking $\sum_{h} s_{h(i)} \simeq \sum_{h}(m / \mu)=(m / \mu) \sum_{h} 1=z(m / \mu)$ and setting $a=J / \mu$ which leads to the average magnetization:

$$
\begin{equation*}
m=\frac{(+\mu) e^{a z m / k T}}{e^{a z m / k T}+e^{-a z m / k T}}+\frac{(-\mu) e^{-a z m / k T}}{e^{a z m / k T}+e^{-a z m / k T}}=\mu \tanh (a z m / k T) \tag{2.1.3}
\end{equation*}
$$

We see that $m=0$ is always a solution for this equation and for infinite $T$ we only have $m=0$. However, for $T=0$ we have two solutions $m= \pm 1$. As with the liquid gas system (see Appendix Afor reference), the high temperature solution $m=0$ is nonphysical in the low temperature phase where this time it is due to the fact that it verifies $\frac{\partial m}{\partial B}<0$. Moreover, as with the liquid gas transition there is a critical point $\left(B_{c}=0, T_{c}\right)$ where the system goes from two possible coexisting states to one state $m=0$ which is given by the Curie Temperature $T_{c}$. This temperature separates the ferromagnetic and para-magnetic phases. The liquid gas and ferromagnetic systems are thus analogous with the substitution $P \leftrightarrow B, x \leftrightarrow m$.

### 2.1.2 Landau theory

In the previous section we saw that along the coexistence line of two phases, both the liquid gas and ferromagnetic systems exhibit a transition at $T_{c}$ above which there was only one unique phase. This transition
from one to two states is completely analogous to bifurcations in dynamical systems. For example if we consider a spring whose end points lie at heights $y=0$ and $y=d$ while being free to move along one axis that we call the $x$-axis, then for $d$ large the spring will be stretched and in order to minimize its potential energy it will remain perfectly vertical at a position we call $x=0$. However, if we compress the spring by lowering $d$, the spring will be able to retrieve its rest length by sliding either to the left or to the right thus breaking the left-right symmetry of the problem. This is analogous to how up down symmetry for our magnetic problem was broken below $T_{c}$. What is interesting however is that in the case of the spring one does not need to know exactly the potential energy of the system in order to predict the phenomenology of the system. Indeed, it is sufficient to expand the potential energy about the minimum near the transition, namely $x=0$, to the lowest order necessary such that it may capture the large $d$ to small $d$ transition. More precisely, in the case of the spring, the potential energy is $k\left(l-l_{0}\right)^{2} / 2=k\left(\sqrt{d^{2}+x^{2}}-l_{0}\right)^{2} / 2$. If we then expand about $x=0$ which is the only state in the high $d$ regime we obtain:

$$
k\left(l-l_{0}\right)^{2} / 2=k\left(\sqrt{d^{2}+x^{2}}-l_{0}\right)^{2} / 2=k\left(d-l_{0}\right)^{2} / 2+k\left(1-l_{0} / d\right) x^{2} / 2+\frac{k l_{0} x^{4}}{8 d^{3}}+O\left(x^{6}\right) .
$$

If we then discard the constant term we have:

$$
\begin{equation*}
E=\frac{k}{2} x^{2}\left(l_{0} x^{2} / 4 d^{3}-\left(l_{0} / d-1\right)\right) \tag{2.1.4}
\end{equation*}
$$

Once again $x=0$ is always a solution but for $l_{0}>d$ the quadratic term shifts to a negative sign and we have two other solutions $x_{ \pm}= \pm 2 d \sqrt{\left(1-d / l_{0}\right)}$. When $d$ is arbitrarily close to $l_{0}$ the two solutions are arbitrarily close to the large $d$ minimum that is $x=0$. Hence this low $d$ to high $d$ transition is continuous with respect to the parameter that describes the phases, that is the positions $x^{*}$ that describe minima of the energy potential. This in turn is similar to second order phase transitions where the order parameter, that describe the different phases, varies continuously at the transition. This is to be contrasted with first order transitions such as the liquid gas transition of water at ambient pressure as the difference in density between the gas and liquid states is non zero even at the transition. Dynamical systems can also display first order like transitions [8] but we will not discuss this as it lies beyond the scope of this thesis. Instead let us note that similarly to bifurcation analysis, one does not need to know the entire free energy of a thermodynamic system in order to describe the phenomenology. Instead a similar Taylor expansion about the high temperature minimum is usually sufficient. This is called Landau theory and for the uni-axial ferromagnet we can expand similarly the free energy up to fourth order as $F=a m^{2}+b m^{4}$. As in the case of the spring this predicts a phase transition when $a$ goes from positive to negative. In order to relate this to phase transitions obtained by varying the temperature this transition between positive and negative values of $a$ can be modeled as $a \propto\left(1-T_{c} / T\right)$ similarly to how the term $\left(1-l_{0} / d\right) x^{2} / 2$ in the case of the spring changes sign as a function of $c$ The great advantage of the Landau approach is that we do not need to consider a specific model when discussing a phase transition and as such the same free energy describes all phase transitions that display a similar behavior. For example this free energy may represent the gas liquid or uni-axial ferromagnet in the vicinity of the critical temperature. We will now verify this statement by deriving the exact free energies of these models and Taylor expanding about the high temperature phases. This will then lead to a more convincing argument of the general applicability of the Landau approach.

For the Ising model the equation of state of Eq. 2.1 .3 ) can be derived as the minimum with respect to $m$ of the following (Gibbs) free energy per particle :

$$
\begin{equation*}
G=m^{2}-\frac{\mu k T}{a z} \log (\cosh (a z m / k T)) \tag{2.1.5}
\end{equation*}
$$

This expression can also be obtained by usual statistical methods by means of the partition function. We may thus expand this expression in powers of the minimum at the transition namely $m=0$ which leads to:

$$
\begin{equation*}
G=m^{2}\left(r m^{2}-\left(T_{c} / T-1\right)\right) \tag{2.1.6}
\end{equation*}
$$

[^1]with $r=\frac{\mu}{12}\left(\frac{a z}{k T}\right)^{3}$ and $T_{c}=\frac{\mu}{2} \frac{a z}{k}$. The system is thus indeed perfectly analogous to that of the spring where the bifurcation at $d=l_{0}$ is replaced by a phase transition at $T=T_{c}$.

Finally in the case of the liquid gas transition the free energy verifies $\frac{\partial F}{\partial V}=-P$ from which we derive the equation of state. Instead we would like to obtain the equation of state by minimizing a function as with the Ising case. This can be achieved if we distinguish the internal pressure $P$ of the system from the externally applied $P_{0}$ of the environment. If we do this and consider $\widetilde{F}=F+P_{0} V$ then we obtain $\frac{\partial}{\partial V} \widetilde{F}=\left(P_{0}-P\right)$ and the equation of state is obtained when the internal pressure is equal to the external pressure, that is, when $P_{0}-P=0 \Leftrightarrow \frac{\partial}{\partial V} \widetilde{F}=0$. However this free energy contains a bothersome $\log$ as in Eq. 2.1.1. As such it is simpler to consider instead the free energy obtained by integrating directly Eq. 2.1 .2 which will be polynomia ${ }^{2}$. In any case, the phenomenology of the number of physical states is entirely contained within the equation of state and thus simply integrating it will be sufficient for our purpose. We thus obtain our new free energy $\widetilde{G}{ }^{3}$ from Eq. 2.1.2 as:

$$
\begin{equation*}
\widetilde{G}(\widehat{x})=\widehat{x}^{4} / 4-\widehat{x}^{3} / 3+\widehat{x}^{2}(\widehat{T}+\widehat{P}) / 2-\widehat{P} \widehat{x}=0 \tag{2.1.7}
\end{equation*}
$$

We now expand about the critical point as $\widehat{x}=\widehat{x}_{c}+\delta \widehat{x}=1 / 3+\delta \widehat{x}$ (see Appendix Afor explanation of $\widehat{x}_{c}$ ) and perform the same expansions on the pressure and temperature. We then obtain:

$$
\begin{equation*}
\frac{1}{2} \delta \widehat{x}^{2}(\delta \widehat{P}+\delta \widehat{T})+\frac{1}{3} \delta \widehat{x}(\delta \widehat{T}-2 \delta \widehat{P})+\frac{1}{324}(-90 \delta \widehat{P}+18 \delta \widehat{T}-1)+\frac{\delta \widehat{x}^{4}}{4} \tag{2.1.8}
\end{equation*}
$$

Finally if we set $\delta \widehat{T}-2 \delta \widehat{P}=0$ we can remove the linear term which, after subtraction of the $\delta \widehat{x}$ independent term then leads to:

$$
\begin{equation*}
\widetilde{G}(\delta \widehat{x})=\frac{\delta \widehat{x}^{2}}{4}\left(\delta \widehat{x}^{2}+3 \delta \widehat{T}\right) \tag{2.1.9}
\end{equation*}
$$

For $\delta \widehat{T}>0$ and thus above the critical temperature there is only one phase whereas below for $\delta \widehat{T}<0$ there are two new phases given by $\delta \widehat{x}= \pm \sqrt{-3 \delta \widehat{T}}$. We have thus retrieved the same phenomenology as with the Ising model. Notice in particular the $\delta \widehat{x} \rightarrow-\delta \widehat{x}$ symmetry as with the ferromagnetic case. Thus, near the critical point, increases in density are equivalent to decreases in density. Because of this symmetry the liquid gas and Ising models are equivalent near the critical temperature up to changes in the value of the critical temperature. This equivalence can be further explained by means of the lattice gas model where each point in the lattice contains $n_{i}=0$ or $n_{i}=1$ particles and the presence of two neighboring particles lowers the energy in the system. This cohesion energy is thus an extremely simplified version of the Van der Waals force as it is non zero only for nearest neighbor sites. Moreover, the volume exclusion term is also simplified by the fact that there can only be at most one particle in a cell. The particles are still free to move however as moving to a nearest neighbor for example is equivalent to updating $n_{i}=1$ to $n_{h(i)}=1$ for $h(i)$ a nearest neighbor of site $i$. The model is thus a space discretized short range version of the typical gas model. The associated energy function is $\sum_{i, h}-(J / k T) n_{i} n_{h(i)}$. A mapping to the Ising model can be made if we set $n_{i}=\frac{1}{2}+\frac{s_{i}}{2}$ with $s_{i}= \pm 1$.

The Landau model with its generic free energy for a given $m \rightarrow-m$ symmetry thus allows us to capture a large range of physical systems with the same underlying symmetries. Moreover, this approach is not restricted only to discrete symmetries and can be used for continuous symmetries as well such as rotations among the fields. For example isotropic ferromagnets have a spin that may take any direction in space and is thus described by a vector. In the following section we will make explicit the range of application of mean field theory.

[^2]
### 2.1.3 Range of application of Landau theory

Another remarkable advantage of the Landau model is that it does not only give a correct qualitative description but it also gives a correct quantitative description, in the limit of large dimensions. The precise dimension above which mean field is sufficient for universal quantities is called the upper critical dimension ${ }^{4}$, This is the dimension above which correlations, in the thermodynamic limit, can be neglected. For models like the liquid gas transition, the isotropic ferromagnetic transition and more broadly the $O(N)$ models, that will be described in Sec.(3.1.1), this dimension is four. As three dimensions is "close" to four dimensions, the critical exponents which take into account correlations are reasonably close to the mean field ones. However, as the dimension decreases mean-field theory becomes increasingly insufficient. In the extreme case of one dimension for the Ising model, there is no second order phase transition as can be checked easily using the transfer matrix method $\sqrt[5]{5}$. In this case we say that $d=1$ is the lower critical dimension of the Ising model. This can be explained using the Peierls droplet argument but we will not discuss this here. Instead let us focus on the regime of validity of mean field theory. Consider then a magnetization field $\phi(x)$ below the critical temperature. We then have $\phi(x)=\langle\phi\rangle+\delta \phi(x)$ where $\langle\phi\rangle$ is the average magnetization in the ferromagnet. Let us then define the connected correlation function:

$$
\begin{equation*}
\mathcal{C}(r)=<\phi(r) \phi(0)>_{c}=<\phi(r) \phi(0)>-<\phi(r)><\phi(0)> \tag{2.1.10}
\end{equation*}
$$

In statistics this two point function is also called the covariance and it is zero when $<\phi(r) \phi(0)>=<$ $\phi(r)><\phi(0)>$, that is when the two variables $\phi(r)$ and $\phi(0)$ are not correlated such that the averages can be taken separately. When the system is short ranged and away from criticality, $\mathcal{C}(r)$ should decay exponentially as $e^{-r / \xi}$ where $\xi$ is the correlation length representing the typical size over which the magnetizations between two points are correlated. In Fourier space this then implies that $\mathcal{C}(r)$ has a pole at $q^{2}=-m^{2} \propto \xi^{-2}$ with :

$$
\begin{equation*}
\widehat{\mathcal{C}}(q)=\left(q^{2}+m^{2}\right)^{-1} \tag{2.1.11}
\end{equation*}
$$

which is typical of a massive Gaussian theory. Hence, in real space taking the Fourier transform of $\left(q^{2}+m^{2}\right)^{-1}$ we have more explicitly ${ }^{6}$

$$
\begin{equation*}
<\phi(r) \phi(0)>_{c} \underset{r \rightarrow \infty}{\sim} \frac{e^{-r / \xi}}{r^{d-2}} \tag{2.1.12}
\end{equation*}
$$

where the power $r^{2-d}$ can be found by dimensional analysis.
Moreover, fluctuations are small when the correlations are small within a ball of radius $\xi$. More explicitly we require that :

$$
A=\frac{\int_{0}^{\xi}<\phi(x) \phi(0)>_{c} \mathrm{~d}^{d} x}{\int_{0}^{\xi}<\phi>^{2} \mathrm{~d}^{d} x} \ll 1 .
$$

$A$ can be simplified using mean field theory as:

$$
\begin{equation*}
A=\frac{\int_{0}^{\xi}<\phi(x) \phi(0)>_{c} \mathrm{~d}^{d} x}{\int_{0}^{\xi}<\phi>^{2} \mathrm{~d}^{d} x}=\frac{d}{\left\langle\phi>^{2} \xi^{d}\right.} \int_{0}^{\xi} r^{d-1}<\phi(r) \phi(0)>_{c} \mathrm{~d} r \propto \frac{\xi^{2-d}}{\left\langle\phi>^{2}\right.} . \tag{2.1.13}
\end{equation*}
$$

In Landau theory, one takes $m^{2} \propto\left(T-T_{c}\right)$ चand thus $\xi \propto\left|T-T_{c}\right|^{-1 / 2}$. Moreover, within mean field theory, one also finds $\langle\phi\rangle^{2} \propto\left|T-T_{c}\right|^{8}$ which then allows us to obtain :

$$
\begin{equation*}
A \propto\left|T-T_{c}\right|^{(d-4) / 2} \tag{2.1.14}
\end{equation*}
$$

[^3]Hence, in terms of dimensionless quantities, fluctuations are small when $A=\left(t / \tau_{G}\right)^{\frac{d-4}{2}} \ll 1$ with $t=$ $\left|T-T_{c}\right| / T_{c}$ and where we have introduced the Ginzburg parameter $\tau_{G}$ which depends on the characteristics of the model at hand. We then notice that for $d>4$ the exponent is positive and the inequality is thus always verified. For $d<4$ the regime in temperature for which one needs to take into account correlations depends on the Ginzburg parameter. For certain superconductors such as superconducting aluminum we have $\tau_{G} \simeq 10^{-16}$ which leads to an extremely small interval near the critical point and thus in essence mean field theory is applicable for practically all temperatures. However, for certain classical fluids or uni-axial ferromagnets we have $\tau_{G} \simeq 1$ and thus correlations need to be taken into account in a fairly wide domain in the vicinity of the critical temperature. The question thus arises, how do we go beyond mean field theory?

### 2.1.4 Going beyond mean field theory

As stated above mean field theory becomes insufficient for sufficiently low dimensions. This is clear as one can easily prove that the one dimensional Ising model does not exhibit any phase transition although mean field theory does not set any constraint on the dimension.

First, in order to include correlations between points we need to include the possibility of spatially dependent fluctuations and thus we need to include a gradient term in the free energy such as:

$$
\begin{equation*}
G[m]=\int \mathrm{d}^{d} x\left((\nabla m(x))^{2} / 2+a m(x)^{2}+b m(x)^{4}\right) \tag{2.1.15}
\end{equation*}
$$

where $m$ is now a magnetization density. The functional $G$ in Eq. 2.1 .15 is called the Ginzburg-Landau model and can be understood as an energy functional whose minimization leads to an equation of state. For uniform field configurations the energy functional is the same as in the Landau theory. The novelty here is the presence of the term $(\nabla m(x))^{2} / 2$ which is large only for magnetic configurations with large gradients. Minimizing this term then adds extra weight to smooth configurations where the gradient is small and thus the spins locally align with each other. Taking into account both the gradient and potential term then requires minimizing the functional $G$ by equating its functional derivative to zero:

$$
\begin{equation*}
\triangle m=2 a m(x)+4 b m(x)^{3} \tag{2.1.16}
\end{equation*}
$$

This equation corresponds to a spatially dependent "equation of state" where the usual thermodynamic state is obtained by taking a uniform field $m$. Taking for example $a=-1 / 2$ and $b=1 / 4$, a uniform configuration is given by $m= \pm 1$. However, the space dependent equation of state also has the solution $m=\tanh (x / \sqrt{2})$ that connects the two uniform solutions $\pm 1$ at $x= \pm \infty$. This solution is called a domain wall and below the critical Curie temperature, the system is indeed made of blocks of $+m_{0}$ or $-m_{0}$ solutions separated by such domain walls ${ }^{9} 1^{10}$. However, while this equation of state now contains non uniform configurations it still does not include correlations. In order to achieve this we must consider genuine probability distributions.

We will consider a $d$ dimensional space and generically denote the position of a point by $x$. For the sake of simplicity, we also consider the probability distribution of a scalar field $\phi$ which may represent the distribution of spins of a uni-axial magnet along its natural axis. We then consider that to each field configuration, given by $x \rightarrow \phi(x)$, is associated a Boltzmann-type exponential distribution. More explicitly, momentarily discretizing space into a lattice of positions $x_{i}$, the probability density of the stochastic configuration $\phi\left(x_{i}\right) \underset{\text { def }}{\overline{=}}$

[^4]$\phi_{x_{i}}=\phi_{\text {def }}$ that associates to each position $x_{i}$, the local magnetization $\Phi_{i}$ is given by:
\[

$$
\begin{align*}
\mathcal{P}\left[\left\{\phi\left(x_{i}\right)=\Phi_{i}\right\}_{i}\right] & =\frac{\prod_{i}\left(d \phi_{i} e^{-\frac{\mathfrak{b}_{i}\left[\left\{\Phi_{j} j_{j}\right]\right.}{h}}\right)}{\int \prod_{i}\left(d \phi_{i} e^{-\frac{h_{i}\left[\left\{\Phi_{j} j_{j}\right]\right.}{h}}\right)} \\
& =\frac{\left(\prod_{i}\left(d \phi_{i}\right)\right) e^{-\sum_{i} \frac{\mathfrak{b}_{i}\left[\left\{\Phi_{j} j_{j}\right]\right.}{h}}}{\int \prod_{i}\left(d \phi_{i} e^{-\frac{b_{i}\left[\left\{\Phi_{j}\right\}_{j}\right]}{h}}\right)} \\
& =\frac{D \phi e^{-\frac{\mathcal{H}\left[\left\{\Phi_{j}\right\}_{j}\right]}{h}}}{\operatorname{def}} \int D \phi e^{-\frac{\mathcal{H}\left[\left\{\Phi_{j}\right\}_{j}\right]}{h}} \tag{2.1.17}
\end{align*}
$$
\]

where $D \phi$ thus gives the measure associated with the set $\left\{\phi_{i}\right\}_{i}, \mathcal{H}\left[\left\{\Phi_{i}\right\}_{i}\right]$ is the hamiltonian that measures how much a given field configuration $\left\{\phi\left(x_{i}\right)=\Phi_{i}\right\}_{i}$ should be suppressed as the probability $\mathcal{P}$ is small when $\mathcal{H}$ is large and $h$ is a parameter gauging the amplitude of fluctuations. Throughout this thesis $h$ will be viewed as small and will be analogous to Planck's constant.

Reverting back to a continuum space, $\mathfrak{h}$ is a hamiltonian density whose integral over $d$-dimensional space gives the hamiltonian $\mathcal{H}[\phi]=\mathcal{H}[x \rightarrow \phi(x)]$. More explicitly, the Landau Ginzburg free energy form of Eq. (2.1.15) can be used as an energy functional with hamiltonian density:

$$
\begin{equation*}
\mathfrak{h}=(\nabla m(x))^{2} / 2+a m(x)^{2}+b m(x)^{4} \tag{2.1.18}
\end{equation*}
$$

One might argue that the Landau Ginzburg free energy is already an approximation to the underlying lattice energy function of the Ising model. Hence at the macroscopic level, when all correlations are taken into account, do we truly expect that this simplified model will be equivalent to the more complicated lattice Ising model or even the mean field energy in absence of the Landau expansion obtained for uniform configurations by Eq. 2.1 .5 ? The answer is no, the models will not be exactly the same and in particular the absolute critical temperature will be different. However, because they all have the same underlying $\mathbb{Z}_{2}$ symmetry $m \rightarrow-m$, the approach to the critical temperature will be similar. This is an example of universality and in the second chapter of this thesis we will exploit this property to find results which depend solely on the symmetry and dimensionality of a given model. In particular, the results will not depend on the underlying details of the model such as the type of interaction terms involved.

### 2.2 Perturbative Renormalization without field theory: a first conceptual step towards functional renormalization

> "The art of doing mathematics is finding that special case that contains all the germs of generality.", $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m} b_{n}}{m+n}<$ $\frac{\pi \csc \left(\frac{\pi}{p}\right)\left(\sum_{m=1}^{\infty} a_{m}^{p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_{n}^{q}\right)^{\frac{1}{q}}}{- \text { David Hilbert }}$

### 2.2.1 A one loop calculation

In the previous section we studied the very fruitful approach of neglecting correlations which implies taking the limit $h \rightarrow 0$. Hence, what new features appear for non zero $h$ ? There are many but for the sake of the discussion with regard to the renormalization group we will focus on one in particular which may be illustrated with an example in electrodynamics. As we have stated in the introduction, in the presence of quantum fluctuations, the vacuum also fluctuates, because of this the charge density measured for a single
electron in an empty box of volume $V$ corresponds to the overall contribution of both the charge of the electron and of the quantum cloud of virtual electrons and positrons that appear and disappear for brief times according to the Heisenberg principle $\Delta E \Delta t \geq \frac{h}{4 \pi}$. This is somewhat similar to how the dipole moment of an atom can be classically vanishing whereas the quantum fluctuations may lead to a non zero dipole moment for brief times. This then leads to a net interaction force called the London dispersion force which is the Van der Waals force for two apolar molecules. In the present case, the positively charged virtual positrons are attracted toward the electron whereas the negatively charged virtual electrons are repulsed. This then leads to a spatial distribution of the electron charge due to the interactions at play. As such, the charge of the electron also depends on the scale of the experiment and as the electron charge corresponds to the interaction coupling entering the Coulomb force this is an example of how couplings may depend on scales. We will give a statistical version of this in the next section where the fluctuations are due to the existence of a non zero temperature $T$. The analogue of this for Eq. 2.1.18) is that the couplings ( $a, b$ ) depend on the scale at which they are observed. However, in concrete mathematical terms, what does this mean ? As the coupling is a given number such as, for example 0.4 , writing $0.4(s)$ where $s$ is the scale at which an experiment was performed is clearly a nonsensical statement. However, recalling classes in analysis, one might notice that this is not the first time that a constant has been upgraded to a function. Indeed, such an upgrade is also systematically carried out with the method of the variation of the constant. We shall hopefully make this analogy clear in the following subsections. For now we give without proof the following result derived from quantum field theory :

$$
\begin{equation*}
e_{R}^{2}\left(p^{2}\right)=e_{b}^{2}\left(1-e_{b}^{2} f\left(p^{2}\right)\right)=e_{b}^{2}-e_{b}^{4} A\left(\frac{2}{\epsilon}+\ln \left(\frac{\mu^{2}}{p^{2}}\right)+\frac{5}{3}+O(\epsilon)\right) \tag{2.2.1}
\end{equation*}
$$

This equation contains many quantities to define but much of these details will be irrelevant for the following discussion. The only parts of this equation that will interest us here is the $\ln \left(p^{2}\right)$ and $2 / \epsilon$ terms. However for the sake of clarity we define these quantities below:
$e_{b}$ : electron charge in units where the speed of light is $c=1$, to be inserted inside the action of a quantum path integra ${ }^{11}$
$A$ : a numerical constant
$p^{2}: p^{2}=p_{s}^{2}$ where $p_{s}^{2}$ is the norm of the spatial momentum with the time component set to zero for simplicity $\epsilon: \epsilon=4-d$ where $d$ is the space-time dimension
$\mu$ : an arbitrary energy scale which is used as a conversion factor due to the fact that the dimension of the charge depends on the dimension of space time as $e^{2}=\mu^{4-d} \widetilde{e}^{2}$ where $\widetilde{e}$ is dimensionless.

Moreover, Eq. 2.2.1 is only valid in the limit $m^{2} \ll p^{2}$ where $m$ is a mass inserted in the quantum action.
Eq. 2.2.1) is obtained by computing the quantum correction to the Coulomb potential in Fourier space as :

$$
\begin{equation*}
V\left(p^{2}\right)=\frac{e_{b}^{2}\left(1-e_{b}^{2} f\left(p^{2}\right)\right)}{p^{2}} \tag{2.2.2}
\end{equation*}
$$

For $f=0$ we formally have the usual Coulomb potential in Fourier space where the real space version, for three spatial dimensions, is given by $V(r)=\frac{1}{4 \pi} \frac{e_{b}^{2}}{r}$. Hence, $e_{R}^{2}$ is defined such that

$$
\begin{equation*}
V\left(p^{2}\right)=\frac{e_{R}^{2}\left(p^{2}\right)}{p^{2}} \tag{2.2.3}
\end{equation*}
$$

for all $p$. This is called a renormalization prescription where we have defined $e_{R}$ such that it resembles the usual Coulomb potential. This might seem arbitrary and, in a sense, it is, as we could have also defined

[^5]$e_{R}$ such that the quantum potential verifies $V(r)=\frac{1}{4 \pi} \frac{e_{R}^{2}(r)}{r}$ in real space. This then leads to a different renormalization prescription as the Fourier transform of a product is not a product but a convolution product. However, it is easier to work in Fourier space and once a definition has been given for what the charge is then there is no longer any ambiguity. There is nevertheless another reason that explains why we require the charge to be defined the same way at all scales but we will not be able to appreciate this thoroughly until Sec. 2.2.6). In essence however this scale dependent charge translates into concrete terms what we had already envisioned when picturing the vacuum fluctuations of positive and negative charges "screening" or "dressing" the actual "bare" charge. In fact the calculation for the potential came precisely from the so-called "vacuum polarization diagram" which can be seen as a "virtual photon" interacting with a "virtual electron-positron pair".

Yet, one of the intriguing parts of Eq. 2.2 .1 that has perhaps not gone missed is the $1 / \epsilon$ term. In essence this term should actually be $1 / 0$ as our ambient space-time is $d=3+1$ and thus the actual direct computation leads to an infinite result. Instead one usually considers the problem at a lower dimension and takes the limit $d \rightarrow 4$ only after finding a form that does not lead to any divergence. This is called a choice of regularization and as the name suggests, it is a choice, and should not influence actual physical quantities. We will explore regularization choices and how one may find finite results from naively divergent quantities in Sec. 2.2.3.

For now, let us focus however on the fact that the charge $e_{R}$, which can be seen as an electromagnetic coupling as in the Coulomb potential, depends on the scale $p^{2}$. We will address this aspect of a scale dependent coupling by looking at a few simple examples that will turn out to be crucial for understanding the Wilsonian renormalization procedure and ultimately the more modern Functional Renormalization Group (FRG) ${ }^{12}$. We will then follow up with the problem of a divergent product which will explain how one may obtain finite predictions from naively divergent quantities. This will in turn lead us to return to concepts found in 2.2 .2 in order to find an equation that will allow us to make finite predictions.

### 2.2.2 Effective scale-dependent parameters

In this section we introduce the concept of scale dependent quantities and functional self-similarity which will be crucial in the following sections.

We consider the problem of a ball in a vacuum that is initially at rest and is suddenly pushed by a force $F_{0}$. This force is maintained such that Newton's second law reads :

$$
\begin{equation*}
m \frac{\mathrm{~d} v}{\mathrm{dt}}=F_{0} \Theta(t) \tag{2.2.4}
\end{equation*}
$$

where $\Theta$ is the Heaviside function. The solution to this problem is $v(t)=\frac{F_{0}}{m} t \Theta(t)$. An experimentalist that wishes to measure the force $F_{0}$ at time $t$ may measure the velocity for two nearby times and calculate $m \frac{\delta v}{\delta t}$. The measured force will indeed be $F_{0}$. Now consider an additional fluid friction force $-\alpha v$ such that

$$
\begin{equation*}
m \frac{\mathrm{~d} v}{\mathrm{dt}}=F_{0} \Theta(t)-\alpha v \tag{2.2.5}
\end{equation*}
$$

where the solution to this new problem is $\frac{F_{0}}{\alpha}\left(1-\exp \left(-\frac{\alpha}{m} t\right)\right) \Theta(t)$. We will now use whenever convenient the renormalization group language marked by quotation marks as to make clear the analogies. For short times, in the "ultra violet"(UV) we retrieve the previous "bare" behavior $v(t)=\frac{F_{0}}{m} t \Theta(t)$ where $F_{0}$ is analogous to the "bare" coupling $e_{b}$ of the previous section. The "bare" velocity is the one obtained from Eq. 2.2 .4 ) in absence of interactions with the surrounding medium given by $\alpha$. However, for long times, that is, in the "infrared" (IR) ${ }^{13}$ we have $v(t)=\frac{F_{0}}{\alpha}$. This solution is a stationary solution of Eq. 2.2 .5 as it verifies $\frac{\mathrm{d} v}{\mathrm{dt}}=0$ and we call it a fixed-point by analogy with the renormalization group. An experimentalist who wishes to measure the "effective" force will use the same "prescription" as in the "bare" theory. That is, he/she will define the force by $m \frac{\mathrm{~d} v}{\mathrm{dt}}$ which now gives $F_{0} \exp \left(-\frac{\alpha}{m} t\right)$ where we have used the solution for $v(t)$. This is analogous to our now scale dependent $e_{r}(s)$ in Eq.2.2.1.

[^6]Furthermore, one may define a "renormalized" force $F_{r}(t)=F_{0} \exp \left(-\frac{\alpha}{m} t\right)$. This is reminiscent of the Debye screening effect given by the Poisson equation on the electric potential $\phi$ as : $\varepsilon \triangle \phi=\lambda_{D}^{-2} \phi-Q \delta(r)$, where $Q$ is the charge of a point particle, $\varepsilon$ the dielectric constant and $\lambda_{D}$ the Debye Length. $\lambda_{D}$ is due to the statistical tendency of accumulation (resp. depletion) of positively (resp. negatively) charged particles near the negatively charged point particle of charge $Q$. This is thus a somewhat statistical equivalent of our explanation for the scale dependence of the charge as given in the previous section. The end result is that $\phi=\frac{Q e^{-r / \lambda_{D}}}{4 \pi \varepsilon r}=\frac{Q(r)}{4 \pi \varepsilon r}=\frac{Q}{4 \pi \varepsilon(r) r}$ where one can absorb the exponential factor into the charge or the dielectric constant depending on the "prescription" that the experimentalist chooses. These choices allow us to obtain a regular, vacuum like $1 / r$ behavior with effective $r$-dependent "coupling constants".

We note also that in the case of the moving ball, the force may also be written as $F_{r}(t)=F_{0}\left(t_{0}\right) \exp \left(-\frac{\alpha}{m}\left(t-t_{0}\right)\right)$ where $F_{0}\left(t_{0}\right)=F_{0} \exp \left(-\frac{\alpha}{m} t_{0}\right)$ is the "bare" force at "scale" $t_{0}$. This seemingly trivial remark is actually the most important aspect of renormalization as it defines the renormalization group. In mathematical terms, it says that the transformation of the force from $t=t_{a}$ to $t=t_{b}$, that is,

$$
\begin{equation*}
F_{r}\left(t_{b}\right)=F_{0}\left(t_{a}\right) \exp \left(-\frac{\alpha}{m}\left(t_{b}-t_{a}\right)\right) \tag{2.2.6}
\end{equation*}
$$

which we will denote more generically as $T\left(F_{0}\left(t_{a}\right), t_{b}-t_{a}, t_{a}\right)$, may be obtained by first performing the calculation from $t=t_{a}$ to $t=t_{b^{\prime}}$ meaning $F_{r}\left(t_{b^{\prime}}\right)=F_{0}\left(t_{a}\right) \exp \left(-\frac{\alpha}{m}\left(t_{b^{\prime}}-t_{a}\right)\right)=T\left(F_{0}\left(t_{a}\right), t_{b^{\prime}}-t_{a}, t_{a}\right)$, then, finally, from $t=t_{b^{\prime}}$ to $t=t_{b}$. This translates to:

$$
\begin{equation*}
T\left(T\left(F_{0}\left(t_{a}\right), t_{b^{\prime}}-t_{a}, t_{a}\right), t_{b}-t_{b^{\prime}}, t_{b^{\prime}}\right)=T\left(F_{0}\left(t_{a}\right), t_{b}-t_{a}, t_{a}\right) \tag{2.2.7}
\end{equation*}
$$

or more concretely in terms of the force as

$$
\begin{equation*}
F_{r}\left(t_{b}\right)=F_{0}\left(t_{b^{\prime}}\right) \exp \left(-\frac{\alpha}{m}\left(t_{b}-t_{b^{\prime}}\right)\right)=F_{0}\left(t_{a}\right) \exp \left(-\frac{\alpha}{m}\left(t_{b}-t_{a}\right)\right) \tag{2.2.8}
\end{equation*}
$$

with

$$
\begin{equation*}
F_{0}\left(t_{b^{\prime}}\right)=F_{0}\left(t_{a}\right) \exp \left(-\frac{\alpha}{m}\left(t_{b^{\prime}}-t_{a}\right)\right) . \tag{2.2.9}
\end{equation*}
$$

This is called functional self similarity (FSS) [13] and is related to an underlying group law which is already given in Eq. 2.2.7). This notion, while admittedly trivial at this point, is crucial for understanding the renormalization group and it will be the centerpiece of the functional renormalization group (FRG). Hence, in light of its importance, we will make this group action clear by writing the group law explicitly.

The group is a one parameter Lie group $G$ and an element of the group $g_{\lambda}$ acts on the space $I \times J$, where $t \in I, F \in J$ as $g_{\lambda} \cdot\left(t_{0}, F_{0}\right)=\left(t_{0}+\lambda, F_{0} e^{-\frac{\alpha}{m} \lambda}\right)=T\left(F_{0}, \lambda, t_{0}\right)$. Hence, for $\lambda_{1}=t_{b^{\prime}}-t_{a}$ and $\lambda_{2}=t_{b}-t_{b^{\prime}}$ Eq. 2.2 .7 may be written as:

$$
g_{\lambda_{2}}\left(g_{\lambda_{1}} \cdot\left(t_{a}, F_{0}\left(t_{a}\right)\right)\right) \underset{\text { def }}{\overline{=}}\left(g_{\lambda_{2}} * g_{\lambda_{1}}\right) \cdot\left(t_{a}, F_{0}\left(t_{a}\right)\right)=g_{\lambda_{2}+\lambda_{1}} \cdot\left(t_{a}, F_{0}\left(t_{a}\right)\right), \lambda_{1}=t_{b^{\prime}}-t_{a}, \lambda_{2}=t_{b}-t_{b^{\prime}}
$$

This group may act either on $F_{r}(t)$ in which case the group action may be understood as generating the orbit of $F_{r}$, that is, the set $\left\{F_{r}(t)\right\}_{t \in \mathbb{R}}$ and the group law states that given an initial point $t_{0}$ and final point $t_{f}$ one may obtain $F_{r}\left(t_{f}\right)$ by means of the group element $g_{t_{f}-t_{0}}$ or by the composition of $g_{t_{m}-t_{0}}$ and $g_{t_{f}-t_{m}}$. If instead we see the group as acting on the initial conditions in the expression for $F_{r}(t)$ for a fixed $t$ then the group action states that $F_{r}(t)$ is an invariant of the group action as one can compute $F_{r}(t)$ from any initial condition. In general the time translation part may be left implicit and we may define the action on the second component as $y^{\prime}=K(\lambda, y)$ such that we have :

$$
\begin{equation*}
K\left(\lambda_{2}, K\left(\lambda_{1}, y\right)\right)=K\left(\lambda_{1}+\lambda_{2}, y\right) \tag{2.2.10}
\end{equation*}
$$

A natural question thus arises. What kind of functions or group actions verify such a property ? In order to answer this question let us mention the fact that the infinitesimal form of a Lie group is sufficient to reconstruct the component of the group which is connected to the identity. In essence this is clear from Eq. 2.2.10) as given a number $y$, one can always calculate $y^{\prime}=K(\lambda, y)$ by writing $\lambda=N \delta \lambda$ and using repeatedly the group law:

$$
\begin{equation*}
K(N \delta \lambda, y) \underset{\text { def }}{=} \widehat{K}_{N \delta \lambda \cdot} \cdot y=K((N-1) \delta \lambda, K(\delta \lambda, y)) \underset{\text { def }}{=}\left(\widehat{K}_{(N-1) \delta \lambda} * \widehat{K}_{\delta \lambda}\right) \cdot y=\left(\widehat{K}_{\delta \lambda}\right)^{N} \cdot y \tag{2.2.11}
\end{equation*}
$$

This ability to decompose a global action into a repeated sequence of infinitesimal actions is a convenience at this stage but it is another important step in the renormalization group with regard to approximations. In general, the global transformation $K(\lambda, y)$ with $\lambda$ large cannot be computed reliably once approximations are performed. However, when $K(\delta \lambda, y)$ is perturbatively under control, the repeated sequence $K(\delta \lambda, y)^{N}$ is better suited and can yield accurate results. This is because the global approximation, obtained by iterating infinitesimal local approximations infinitely many times will automatically verify FSS, whereas the initial global approximation will not in general. This situation arises in perturbative field theory and one of the advantages of NPRG is that functional self-similarity is built in from the very beginning.

Let us now return to our initial question with regard to the types of solutions that verify the functional self-similarity property. As was shown, local infinitesimal transformations $y^{\prime}=K(\delta \lambda, y)=y+\delta \lambda \frac{\mathrm{d} K}{\mathrm{~d} \lambda}(0, y)+$ $O\left(\delta \lambda^{2}\right)=y+\delta \lambda Z(y)+O\left(\delta \lambda^{2}\right)$, with $Z(y)=\frac{\mathrm{d} K}{\mathrm{~d} \lambda}(0, y)$ are sufficient. Indeed from this equation alone we have :

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} \lambda}(\lambda)=Z(y) \tag{2.2.12}
\end{equation*}
$$

Thus, finding the action of the group is equivalent to finding the solution of an autonomous differential equation. This in turn shows us that the term functional self-similarity refers to the lack of any explicit time in Eq. 2.2 .12 meaning that the governing laws are similar at all times. In turn, this implies that local knowledge about some scale $\lambda_{0}$, with $y\left(\lambda+\lambda_{0}\right)=y_{0}+\lambda \beta\left(y_{0}\right)+O\left(\lambda^{2}\right)$, is sufficient to deduce the behavior of $y$ at all scales if $\beta$ is known exactly. In the field theory context, this implies that it is sufficient to know how a quantity varies locally, such as the charge of an electron, to deduce the behavior for all energies both arbitrarily high and arbitrarily low ${ }^{14}$

We will investigate how renormalization makes sense of field theory calculations in the following sections.

### 2.2.3 A divergent product

We now return to the problem of the divergent charge in Eq. 2.2 .1 . In order to show on a toy model how these divergences can be handled, we consider now the factorization of the sin function initially conjectured by Euler. This factorization was found before the Weierstrass factorization theorem and Euler used the factorization of sin to take on the famous Basel problem of calculating the sum of inverse squares which was first posed in 1644 and remained open for 90 years. It is now commonly shown as an exercise in Fourier Series. The idea goes as follows. A polynomial $p$ may be written either as a sum over some basis of polynomials such as the canonical basis $x^{j}$ or as the following product:

$$
\begin{equation*}
p(x)=a_{0} \prod_{i}\left(x-r_{i}\right) \tag{2.2.13}
\end{equation*}
$$

Euler's idea was to extend this to infinite sums such as those obtained by series expansions of functions with infinite radius of convergence ${ }^{15}$. Thus, as the sin function has roots at $n \pi$ we may postulate that :

$$
\begin{equation*}
\sin (x)=a_{0} \prod_{n \in \mathbb{Z}}(x-n \pi)=a_{0} x \prod_{n \in \mathbb{N}^{*}}\left(x^{2}-n^{2} \pi^{2}\right) \tag{2.2.14}
\end{equation*}
$$

where $a_{0}$ is a normalization factor. To find it, it is sufficient to evaluate both sides of Eq. 2.2.14 at a particular value of x . We choose to compute $\sin (x) / x$ in the limit $x \rightarrow 0$ which yields :

$$
\begin{equation*}
1=a_{0} \prod_{n \in \mathbb{N}^{*}}\left(-n^{2} \pi^{2}\right) \tag{2.2.15}
\end{equation*}
$$

[^7]At this point we may notice that we broke math as the right hand side of the above Eq. 2.2 .15 is divergent. However taking the quotient of the two precedent divergent formulas we may obtain a new well defined formula:

$$
\begin{equation*}
\frac{\sin (x)}{x}=\prod_{n \in \mathbb{N}^{*}}\left(1-\frac{x^{2}}{n^{2} \pi^{2}}\right) \tag{2.2.16}
\end{equation*}
$$

which indeed turns out to be meaningful and even correct. This formula allowed Euler to evaluate the sum of inverse squares by extracting the terms involving $x^{2}$ on both sides, supplemented by two other proofs, this in turn made him a famous mathematician at age 28 . The technique may also be recursively applied to yield other sums of even powers of $\frac{1}{n}{ }^{16}$

However, obtaining Eq. 2.2 .16 we performed the usual algebra for finite quantities even though we were manipulating divergent objects. Hence, in order to make this calculation meaningful we must first regularize as was done in Eq. 2.2 .1 by considering a different dimension. This may be done in many ways such as inserting a convergent factor within the product or truncating at some finite large integer. In general, by regularizing a theory we explicitly deform the problem $P$ and the quantity to be calculated $Q$ into an adjacent nearby problem $\tilde{P}_{\epsilon}$ and quantity $\tilde{Q}_{\epsilon}$ where all quantities needed to calculate $\tilde{Q}_{\epsilon}$ are finite. If we know that $Q$ is well defined then after obtaining an expression for $\tilde{Q}_{\epsilon}$ that is well defined in the limit $\epsilon \rightarrow 0$, we will be able to obtain $\tilde{P}_{\epsilon} \rightarrow P$ and $\tilde{Q}_{\epsilon} \rightarrow Q$ with a well defined expression for $Q$. For our particular problem the simplest regularization is perhaps to make the replacement $\prod_{n \in \mathbb{N}^{*}} \rightarrow \prod_{n \leq N}$ and consider instead a "truncated sin" by truncating the product at some finite and large order: $\sin _{N}(x)=a_{0}(N) x \prod_{n \leq N}\left(x^{2}-n^{2} \pi^{2}\right)$. We can then perform the same algebra as before and take the limit $N \rightarrow \infty$ at the level of Eq. 2.2 .16 . This then gives us a first conceptual insight as how to tackle the divergent term in the equation for the charge in Eq. 2.2.1). Indeed, as we know that $\sin (x) / x$ is a well defined quantity for all $x$, the right hand side of Eq. 2.2 .16 ), which no longer depends on an unknown coefficient, should also be well-defined. This implies that the divergence involved in the definition of $a_{0}$, Eq. 2.2.15), should cancel another divergence involved in Eq. 2.2 .14 making the final result in Eq. 2.2 .16 indeed well-defined. Eq. 2.2 .16 should also give finite values and any apparent divergence should be cancelled by another hidden divergence. As the momentum dependent charge $e_{R}\left(p^{2}\right)$, defined in Eq. 2.2 .3 ), is also perfectly measurable for any $s$ it must also be well defined. In the present case of the sin function the hidden divergence may be made explicit taking the log of Eq. 2.2.14 :

$$
\begin{equation*}
\log \left(\frac{\sin (x)}{x}\right)=\log \left(a_{0}\right)+\sum \log \left(x^{2}-n^{2} \pi^{2}\right) \tag{2.2.17}
\end{equation*}
$$

The divergence from the sum $\sum \log \left(x^{2}-n^{2} \pi^{2}\right)$ is cancelled by the divergence from $\log \left(a_{0}\right)$ using Eq. 2.2.15). What is remarkable here is that, naively, the divergence from $\sum \log \left(x^{2}-n^{2} \pi^{2}\right)$ seems $x$ dependent and one would thus imagine that it needs to be compensated for each value of $x$ independently by some function $\log \left(a_{0}(x)\right)$. But this is not true here, the divergence for any value of $x$ say $x=0$, is cancelled by one and the same, $x$-independent, counter divergence coming from $a_{0}$. In the same way, if we were able to cancel the divergence in $e_{R}$ by an $p$-independent term we would have a well defined expression for $e_{R}\left(p^{2}\right)$ for all values of $p$ simply by requiring that it has a definite value for some arbitrary $p_{r e f}$ where it has been measured. This means that after performing one simple measurement at some scale $p_{\text {ref }}$ we will be able to make an infinite number of predictions for all the other scales which a priori would seem quite remarkable and exciting. Fortunately this is exactly the case. During the early days of renormalization this led to great mystery as to why all of the forces of nature except perhaps gravity was given by these mysterious and amazing so called "renormalizable theories". Why were we so fortunate ? However, these questions turned out to be overly mysticized and the explanation turned out to be both fortunately and sadly linked to a trivial remark about functional self-similarity (FSS) and dimensional analysis. In the next section, we will extensively study FSS. In the present case, let us note the following: for any $x_{0}$, the regularized $a_{0}$ may be obtained by inverting the regularized version of Eq. 2.2.14 as:

$$
\begin{equation*}
a_{0}(N)=\frac{\sin \left(x_{0}\right)}{x_{0} \prod_{n \leq N}\left(x_{0}^{2}-n^{2} \pi^{2}\right)} \tag{2.2.18}
\end{equation*}
$$

[^8]In the case where $x_{0}$ is a zero of $\sin$ this equality must be seen as the limit $x \rightarrow x_{0}$. As such in the limit $N \rightarrow \infty$ we have:

$$
\begin{equation*}
\sin (x)=\sin \left(x_{0}\right) \frac{x}{x_{0}} \prod_{n \in \mathbb{N}^{*}} \frac{x^{2}-n^{2} \pi^{2}}{x_{0}^{2}-n^{2} \pi^{2}} \tag{2.2.19}
\end{equation*}
$$

This transformation law looks quite similar to the one encountered on the force in the previous section, meaning Eq. 2.2 .6 , and we may indeed write it similarly as $\sin (x)=T\left(\sin \left(x_{0}\right), x, x_{0}\right)$. This can then also be extended by including an intermediate point $x_{m}$ in order to retrieve a functional relation that is formally similar to Eq. $2 \cdot 2.2 .7$ ). However, Eq. 2.2 .19 ) is not a FSS relation. This allows us to distinguish FSS from simple transitivity. In the case of FSS, the group action $g_{\lambda}$ acts globally on the set $I \mathrm{x} J$ with $x_{0} \in I$, $\sin \left(x_{0}\right) \in J$ where the action depends only on $\lambda$. This means that an actual FSS property would be

$$
\begin{align*}
(x, \sin (x)) & =g_{\lambda} \cdot\left(x_{0}, \sin \left(x_{0}\right)\right)=\left(x_{0}+\lambda, K\left(\lambda, \sin \left(x_{0}\right)\right)\right) \\
& =\left(x, K\left(x-x_{0}, \sin \left(x_{0}\right)\right)\right) \tag{2.2.20}
\end{align*}
$$

but this is not the form of Eq. $(2.2 .19)$ as it depends explicitly on both $x_{0}$ and $x$ whereas we require it to depend only on $x-x_{0}$ to be FSS. In turn this means that we would have to consider local $x$-dependent transformations to see Eq. 2.2 .19 ) as a group transformation and it is thus less restrictive than FSS. However the transitivity property of Eq.(2.2.19) has still allowed us to discard the initial divergence and in many cases transitivity can be sufficient both to understand the renormalization group (RG) and to obtain actual physical result: ${ }^{[17}$ In the case of the divergent charge this will be sufficient to remove the divergence. Indeed, instead of dividing we will subtract by considering a reference point $p_{\text {ref }}$ which leads to:

$$
\begin{equation*}
e_{R}^{2}\left(p^{2}\right)-e_{R}^{2}\left(p_{r e f}^{2}\right)=e_{b}^{4} A \ln \left(\frac{p^{2}}{p_{r e f}^{2}}\right) \tag{2.2.21}
\end{equation*}
$$

This new equation no longer contains any divergent terms, however, now $e_{R}\left(p^{2}\right)$ depends both on $e_{b}$ and on $e_{R}\left(p_{r e f}^{2}\right)$. We thus choose $p_{\text {ref }}$ such that $e_{b}=e_{R}\left(p_{\text {ref }}^{2}\right)^{18}$, this does not pose any problem as at the end of the calculations neither $e_{b}$ nor $p_{\text {ref }}$ will appear in our final result and thus both can be considered arbitrary quantities essential for the mathematical formalism but irrelevant for actual measurable observables. Hence we now have:

$$
\begin{equation*}
e_{R}^{2}\left(p^{2}\right)-e_{R}^{2}\left(p_{r e f}^{2}\right)=e_{R}^{4}\left(p_{r e f}^{2}\right) A \ln \left(\frac{p^{2}}{p_{r e f}^{2}}\right) . \tag{2.2.22}
\end{equation*}
$$

Yet, Eq. (2.2.22) still poses a problem when the logarithm term is large as in terms of $e_{b}$ the perturbative expansion obtained for fixed dimension $d<4$ is only consistent when $e_{b}^{4} A \ln \left(\frac{p^{2}}{p_{n e f}^{2}}\right) \ll e_{b}^{2}$. This in turn means that although $p_{\text {ref }}$ is arbitrary at this point we are limited to the region $p \simeq p_{\text {ref }}$ for the perturbative expansion to be valid. This is in principle problematic, however, we have only used transitivity to remove our divergence and we have not yet fully exploited functional self-similarity. Indeed, FSS states that it is sufficient to know how the charge varies in any interval to obtain full knowledge for all $p^{2} \in \mathbb{R}^{+}$as one can always compose infinitesimal transformations.

[^9] the perturbative expansion we have $e_{b}=e_{R}\left(p_{r e f}^{2}\right)$ exactly where both $e_{b}$ and $p_{r e f}$ are in essence arbitrary and non observable.

In the following sections we will show how this works for simpler examples such as differential equations. We will then solve the problem at hand in Sec. 2.2.6 ${ }^{19}$

### 2.2.4 An exact solution from a first order correction using the renormalization group :

In the following sections of this chapter we will study how the renormalization group can be used as an efficient tool for generating approximate solutions of differential equations. This will hopefully give the reader a first insight into the usefulness of the renormalization group when seeking approximate solutions and will also allow us to extract useful information from the divergent charge of Eq. 2.2 .1 ).

We thus consider the following equation:

$$
\begin{equation*}
F^{\prime}(x)=\epsilon+\exp (-F(x)), F\left(x_{i}\right)=F_{i} \tag{2.2.23}
\end{equation*}
$$

whose solution is

$$
\begin{equation*}
F(x)=\ln \left(\frac{e^{\epsilon\left(x-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)-1}{\epsilon}\right) \tag{2.2.24}
\end{equation*}
$$

This solution verifies FSS which we will now check as it makes explicit how to verify FSS for approximate solutions. We thus fix a point $x_{f}$ for which the solution for $F$ gives

$$
\begin{equation*}
F\left(x_{f}\right)=\ln \left(\frac{e^{\epsilon\left(x_{f}-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)-1}{\epsilon}\right) \tag{2.2.25}
\end{equation*}
$$

This solution may be interpreted as generated by $g_{x_{f}-x_{i}}$ as $\left(x_{f}, F\left(x_{f}\right)\right)=g_{x_{f}-x_{i}} .\left(x_{i}, F\left(x_{i}\right)\right)$. We now verify that $F\left(x_{f}\right)$ may be obtained from the composition of $g_{x_{m}-x_{i}}$ and $g_{x_{f}-x_{m}}$. The action of $g_{x_{m}-x_{i}}$ leads to

$$
\begin{equation*}
F\left(x_{m}\right)=\ln \left(\frac{e^{\epsilon\left(x_{m}-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)-1}{\epsilon}\right) \tag{2.2.26}
\end{equation*}
$$

and that of $g_{x_{f}-x_{m}}$ corresponds to

$$
\begin{equation*}
F\left(x_{f}\right)=\ln \left(\frac{e^{\epsilon\left(x_{f}-x_{m}\right)}\left(1+\epsilon e^{F\left(x_{m}\right)}\right)-1}{\epsilon}\right) \tag{2.2.27}
\end{equation*}
$$

Hence using Eq. 2.2.26):

$$
\begin{align*}
& F\left(x_{f}\right)=\ln \left(\frac{e^{\epsilon\left(x_{f}-x_{m}\right)}\left(1+\left(e^{\epsilon\left(x_{m}-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)-1\right)\right)-1}{\epsilon}\right) \\
& F\left(x_{f}\right)=\ln \left(\frac{e^{\epsilon\left(x_{f}-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)-1}{\epsilon}\right) \tag{2.2.28}
\end{align*}
$$

Thus the function is FSS. We may also interpret this as the fact that $F\left(x_{f}\right)$ is invariant with respect to a change in initial condition as long as initial conditions change according to the action of a group element $g_{\lambda}$. Considering infinitesimal transformations $g_{\delta \lambda} \cdot\left(x_{i}, F_{i}\right)=\left(x_{i}, F_{i}\right)+\delta \lambda\left(1, K\left(F_{i}\right)\right) \underset{\text { def }}{=}\left(x_{i}, F_{i}\right)+\left(\delta x_{i}, \delta F_{i}\right)$, this translates to:

$$
\begin{equation*}
\delta F\left(x_{f}\right)=0=\frac{-\epsilon \delta x_{i} e^{\epsilon\left(x_{f}-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)+\delta F_{i} \epsilon e^{F_{i}} e^{\epsilon\left(x_{f}-x_{i}\right)}}{\left(e^{\epsilon\left(x_{f}-x_{i}\right)}\left(1+\epsilon e^{F_{i}}\right)-1\right)} \tag{2.2.29}
\end{equation*}
$$

[^10]from which we obtain:
\[

$$
\begin{equation*}
\frac{\delta F_{i}}{\delta x_{i}}=\epsilon+\exp \left(-F_{i}\right) \tag{2.2.30}
\end{equation*}
$$

\]

Hence demanding FSS leads to a differential equation on the initial conditions that is equivalent to the one obtained on the solution function. Let us thus check what happens when we use approximations. We take $\epsilon$ as small and expand $F$ as $F(x)=F_{0}(x)+\epsilon F_{1}(x)+O\left(\epsilon^{2}\right)$. The 0th order term gives:

$$
\begin{equation*}
F_{0}^{\prime}(x)=\exp \left(-F_{0}(x)\right), F_{0}\left(x_{i}\right)=F_{i} \tag{2.2.31}
\end{equation*}
$$

For which the solution at this order is

$$
\begin{equation*}
F_{0}(x)=\ln \left(x-x_{i}+e^{F_{i}}\right) . \tag{2.2.32}
\end{equation*}
$$

At order $\epsilon$, we obtain:

$$
\begin{equation*}
F_{1}^{\prime}(x)+\frac{F_{1}(x)}{x-x_{i}+e^{F_{i}}}=1 . \tag{2.2.33}
\end{equation*}
$$

We now take the initial condition $F_{1}\left(x_{i}\right)=0$ at $x_{i}$, such that for $x=x_{i}$, the solution is given by the 0 -th order term, we obtain:

$$
\begin{equation*}
F(x) \simeq F_{0}(x)+\epsilon F_{1}(x)=\ln \left(x-x_{i}+e^{F_{i}}\right)+\epsilon \frac{\left(x-x_{i}\right)\left(e^{F_{i}}+\left(x-x_{i}\right) / 2\right)}{e^{F_{i}}+\left(x-x_{i}\right)} . \tag{2.2.34}
\end{equation*}
$$

Let us first note that for $\left(x-x_{i}\right) \gg \frac{1}{\epsilon}$ the correction term $F_{1}$ dominates the 0 -th order term. This means that perturbation theory breaks down in that regime. In perturbative field theory, this happens when one has an effective theory for which beyond a certain regime properties such as unitarity break down. In fact, the same situation arises for Eq. $(2.2 .22)$ when the reference point and the point for which we wish to calculate the charge are far away. Thus we do not expect to achieve a reasonable approximation beyond that limit and thus our approximations are only valid locally. As FSS is a global property for which $g_{\lambda}$ can send a point $x_{0}$ arbitrarily far away we do not expect such a property to be fulfilled a priori which we now verify.

For $x=x_{m}$ or $x=x_{f}$ which we denote as $x=x_{m, f}$, the action

$$
\begin{equation*}
g_{x_{f, m}-x_{i}} \cdot\left(x_{i}, F_{i}\right)=\left(x_{f, m}, K\left(x_{f, m}-x_{i}, F_{i}\right)\right) \tag{2.2.35}
\end{equation*}
$$

gives:

$$
\begin{equation*}
K\left(x_{f, m}-x_{i}, F_{i}\right)=F\left(x_{f, m}\right)=\ln \left(x_{f, m}-x_{i}+e^{F_{i}}\right)+\epsilon \frac{\left(x_{f, m}-x_{i}\right)\left(e^{F_{i}}+\left(x_{f, m}-x_{i}\right) / 2\right)}{e^{F_{i}}+\left(x_{f, m}-x_{i}\right)} \tag{2.2.36}
\end{equation*}
$$

Thus if now compute the action of $g_{x_{f}-x_{m}} \cdot\left(g_{x_{m}} \cdot\left(x_{i}, F_{i}\right)\right)=g_{x_{f}-x_{m}} \cdot\left(x_{m}, F\left(x_{m}\right)\right)$ we obtain:

$$
\begin{align*}
K\left(x_{f}-x_{m}, F\left(x_{m}\right)\right)= & \ln \left(x_{f}-x_{m}+e^{F\left(x_{m}\right)}\right)+\epsilon \frac{\left(x_{f}-x_{m}\right)\left(e^{F\left(x_{m}\right)}+\left(x_{f}-x_{m}\right) / 2\right)}{e^{F\left(x_{m}\right)}+\left(x_{f}-x_{m}\right)} \\
= & \ln \left(x_{f}-x_{m}+\left(x_{m}-x_{i}+e^{F_{i}}\right) \exp \left(\epsilon \frac{\left(x_{f, m}-x_{i}\right)\left(e^{F_{i}}+\left(x_{f, m}-x_{i}\right) / 2\right)}{e^{F_{i}}+\left(x_{f, m}-x_{i}\right)}\right)\right)+ \\
& \epsilon \frac{\left(x_{f}-x_{m}\right)\left(( x _ { m } - x _ { i } + e ^ { F _ { i } } ) \operatorname { e x p } \left(\epsilon \frac{\left(x_{f, m}-x_{i}\right)\left(e^{F_{i}}+\left(x_{f, m}-x_{i}\right) / 2\right)}{\left.\left.e^{F_{i}+\left(x_{f, m}-x_{i}\right)}\right)+\left(x_{f}-x_{m}\right) / 2\right)}\right.\right.}{\left(x_{m}-x_{i}+e^{F_{i}}\right) \exp \left(\epsilon \frac{\left(x_{f, m}-x_{i}\right)\left(e^{F_{i}}+\left(x_{f, m}-x_{i}\right) / 2\right)}{\left.e^{F_{i}+\left(x_{f, m}-x_{i}\right)}\right)+\left(x_{f}-x_{m}\right)}\right.} \tag{2.2.37}
\end{align*}
$$

Hence, the usual simplifications are no longer possible and we do not retrieve $F\left(x_{f}\right)$. In particular $x_{m}$ no longer drops out and we no longer have invariance with respect to initial conditions. To fix this, we impose that the approximate solution is invariant with respect to a change of initial conditions ( $\delta F_{i}=F_{i}^{\prime}-F_{i}, \delta x_{i}=x_{i}^{\prime}-x_{i}$ ) as :

$$
\begin{equation*}
\delta F\left(x_{f}\right)=0=\frac{-\delta x_{i}+\delta F_{i} e^{F_{i}}}{x_{f}-x_{i}+e^{F_{i}}}+\epsilon \frac{-\left(2 e^{2 F_{i}}+2 e^{F_{i}}\left(x_{f}-x_{i}\right)+\left(x_{f}-x_{i}\right)^{2}\right) \delta x_{i}+\left(x_{f}-x_{i}\right)^{2} e^{F_{i}} \delta F_{i}}{2\left(e^{F_{i}}+x_{f}-x_{i}\right)^{2}} . \tag{2.2.38}
\end{equation*}
$$

However, the equation $\delta F_{i} / \delta x_{i}$ obtained is not autonomous and thus cannot correspond to the action of a FSS group. This is in fact due to the lack of higher order terms in $\epsilon$ such as $\epsilon^{2} F_{2}(x)$. However, we may check that while $F_{1}$ is proportional to $\left(x-x_{i}\right), F_{2}$ is proportional to a higher power of $\left(x-x_{i}\right)$ hence restricting to $x \simeq x_{i}$ we will be able to neglect the contribution of $F_{2}$. This in turn is sufficient as once we have a FSS solution we can use the FSS group to generate this locally valid solution to any point we wish as we can always compose infinitesimal transformations to obtain a global transformation. Thus taking the limit $x \rightarrow x_{i}$ we obtain:

$$
\begin{equation*}
\delta F\left(x_{f}\right)=0=\left(-\delta x_{i}+\delta F_{i} e^{F_{i}}\right) e^{-F_{i}}-\epsilon \delta x_{i} \tag{2.2.39}
\end{equation*}
$$

Which then leads to :

$$
\begin{equation*}
\frac{\delta F_{i}}{\delta x_{i}}=\epsilon+e^{-F_{i}} \tag{2.2.40}
\end{equation*}
$$

Which is exactly our initial equation but stated in terms of the "flowing" initial conditions. As the action of the group on the initial conditions is the same as the action of the group on the solution function this then means that demanding that our approximation at order $\epsilon$ be FSS was enough to recover the entire function. This in turn is due to the fact that our approximate solution and our exact solution have the same first order Taylor expansion about $\left(x_{f}-x_{i}\right)$ which means that locally they behave in the same manner. This may be understood by the fact that the underlying differential equation of Eq. 2.2 .23 is linear in $\epsilon$ and thus an expansion to linear order in $\left(x-x_{i}\right)$ of the exact solution, given directly by the differential equation, is necessarily the same result as having performed first a linear expansion in $\epsilon$, as with $F_{1}$, and then an expansion about $\left(x-x_{i}\right)$. Finally, as the FSS is able to construct global solutions from local solutions the results are necessarily the same.

In general however, when the underlying exact equations are not affine or linear with respect to the small parameter, such as in field theories in general, we will not recover the exact solution by imposing FSS. Instead, demanding FSS, we will obtain an approximate renormalized, or equivalently FSS improved, solution which will be much more accurate than the one obtained from naive perturbation theory which only provides a local approximation. In particular the renormalized solution will not have the secular term $\epsilon\left(x-x_{i}\right)$ that limited the domain of validity to $\left(x-x_{i}\right) \ll 1 / \epsilon$.

The two following sections will thus be more closely related to the case of field theories.

### 2.2.5 An improved approximation using the renormalization group :

In order to show how the renormalization procedure works when the exact solution is never obtained at any order of the $\epsilon$ expansion $F=F_{0}+\epsilon F_{1}+\epsilon^{2} F_{2}+\ldots$ we will study a differential equation that is non linear in $\epsilon$ :

$$
\begin{equation*}
F^{\prime}(x)=\exp (-\epsilon F(x)), F\left(x_{i}\right)=F_{i} \tag{2.2.41}
\end{equation*}
$$

The exact solution is obtained in a simple manner by the method of separation of variables as:

$$
\begin{equation*}
F(x)=\frac{\ln \left(\epsilon\left(x-x_{i}+e^{\epsilon F_{i}} / \epsilon\right)\right)}{\epsilon} \tag{2.2.42}
\end{equation*}
$$

However, we will seek instead an approximate solution. Expanding $F$ as $F=F_{0}+\epsilon F_{1}+\epsilon^{2} F_{2}+\ldots$, the order 0 in $\epsilon$ leads to:

$$
\begin{equation*}
F_{0}^{\prime}(x)=1 \Longleftrightarrow F_{0}(x)=x-x_{i}+F_{i} . \tag{2.2.43}
\end{equation*}
$$

The first $\epsilon$ correction gives:

$$
\begin{equation*}
F_{1}^{\prime}(x)=-F_{0}(x)=-\left(x-x_{i}+F_{i}\right) . \tag{2.2.44}
\end{equation*}
$$

Integrating from $x_{i}$ to $x$ with $F_{1}\left(x_{i}\right)=0$ such that the initial condition may be contained in the 0th order term we obtain:

$$
\begin{equation*}
F(x)=F_{0}(x)+\epsilon F_{1}(x)+O\left(\epsilon^{2}\right)=x-x_{i}+F_{i}-\frac{1}{2} \epsilon\left(x-x_{i}\right)\left(x-x_{i}+2 F_{i}\right)+O\left(\epsilon^{2}\right) . \tag{2.2.45}
\end{equation*}
$$

We notice once more that the correction term is dominant for $\left(x-x_{i}\right) \gg 1 / \epsilon$ and as such the approximation is only valid for $\left(x-x_{i}\right) \ll 1 / \epsilon$. Hence we need to impose FSS in order to remove this constraint. This can be achieved by demanding $\delta F=0$ when changing initial conditions which leads to:

$$
-\delta x_{i}+\delta F_{i}+\epsilon\left(F_{i}+x-x_{i}\right) \delta x_{i}-\epsilon\left(x-x_{i}\right) \delta F_{i}=0
$$

If we now acknowledge that our approximation is only local and there are higher order terms $O\left(\left(x-x_{i}\right)^{m}\right)$ when including higher orders in $\epsilon$ we see that we must take the limit $x \rightarrow x_{i}$ in order to cancel the higher order terms. This then leads to:

$$
\begin{equation*}
\frac{\delta F_{i}}{\delta x_{i}}=1-\epsilon F_{i} \tag{2.2.46}
\end{equation*}
$$

Hence $F_{i}$ must be made a function of $x_{i}$ via Eq. (2.2.46) which is a generalization of the method of the variation of the constant and is the proper way to understand why couplings in field theories depend on the choice of scale. Using this information that $F_{i}$ is a function of $x_{i}$ we could have obtained Eq. 2.2 .46 much faster by directly promoting $F_{i}$ to a function of $x_{i}$ in Eq. 2.2 .45 and then demanding that F be invariant with respect to changes in initial conditions is equivalent to $\frac{d}{\mathrm{dx}_{i}} F(x)=0$. If we then set $x=x_{i}$ after taking the derivative we obtain Eq. 2.2 .46 as $\left.\frac{\mathrm{d}}{\mathrm{dx}_{i}} F(x)\right|_{x=x_{i}}=0$. This method is quicker as for terms such as $\left(x-x_{i}\right) p\left(x_{i}\right)$ we only have to take the derivative of $\left(x-x_{i}\right)$ as the term $\left(x-x_{i}\right) p\left(x_{i}\right)^{\prime}$ will equate to zero for $x=x_{i}$. In any case, we notice that Eq. $\sqrt[2.2 .46]{ }$ is indeed different from the original equation and is actually much simpler. Moreover, as the action of the group on the initial conditions is the same as the action of the group on the solution function, Eq. (2.2.46) can be used directly on $F$ by replacing $F_{i}$ with $F$ and $x_{i}$ with $x$. Hence solving Eq. 2.2.46) with initial condition $F^{\text {app }}\left(x_{i}\right)=F_{i}^{\text {app }}$ we obtain:

$$
\begin{equation*}
F^{a p p}(x)=\frac{1+e^{-\epsilon\left(x-x_{i}\right)}\left(-1+\epsilon F_{i}^{a p p}\right)}{\epsilon} . \tag{2.2.47}
\end{equation*}
$$

This solution is clearly different from the exact solution and thus the natural question is why and what is missing? In the previous section we saw that if the approximate solution has the same local behavior near an initial condition as the exact solution then the FSS improved solution is necessarily the exact solution. Thus any difference with the exact result is entirely due to missing terms in the local description between the exact and approximate solution. This in turn is readily seen by the fact that Eq. 2.2 .45 ) is the first order Taylor expansion of Eq.(2.2.41) in the small parameter $\epsilon$. Hence, in this case, a naive expansion of the solution followed by imposing FSS is equivalent to performing directly an expansion on the exact differential equation itself. This is an important remark as performing approximations on the differential equation rather than directly on the solution allows us to avoid secular terms which would otherwise prevent us from having reasonable global approximations.

In conclusion to this section we will compare our FSS improved and secular approximations with the exact solutions.

Taking the initial condition $F(0)=0$ we obtain from Eq. 2.2 .46 ) the renormalized approximation:

$$
\begin{equation*}
F_{\text {app }, r}(x)=\frac{1-e^{-x \epsilon}}{\epsilon} \tag{2.2.48}
\end{equation*}
$$

which, as stated, is clearly different from the exact solution:

$$
\begin{equation*}
F(x)=\frac{\ln (\epsilon x+1))}{\epsilon} \tag{2.2.49}
\end{equation*}
$$

but it is significantly better than the unrenormalized approximation:

$$
\begin{equation*}
F_{a p p}(x)=x-\frac{x^{2} \epsilon}{2} . \tag{2.2.50}
\end{equation*}
$$

Indeed, $F_{\text {app }}(x)$ diverges to $-\infty$ quadratically for $x \rightarrow \infty$ whereas both the exact solution and the renormalized approximate solution remain positive. In Fig. (2.2.1], it is apparent that $F_{\text {app,r }}(x)$ is indeed a much better approximation than the direct approach $F_{\text {app }}$.

In summary this section showed how one can find interesting approximations by imposing FSS on approximations that do not verify FSS initially. We will now apply this to Eq.(2.2.21) where the underlying exact differential equation is not known at this stage.

### 2.2.6 Charge beta function

In the previous sections we have shown how to remove secular terms $\epsilon\left(x-x_{i}\right)$ by imposing FSS. This term $e_{b}^{4} \ln \left(p^{2} / p_{r e f}^{2}\right)$ in Eq. 2.2.21) is also secular as it limits the range of applicability of $e_{R}^{2}(p)$. In order to make


Figure 2.2.1: The degree of approximation between $F_{a p p}$ and the renormalized approximation $F_{a p p, r}$ when compared to $F$ for $\epsilon=0.5$ on the left and $\epsilon=0.01$ on the right
the analogy with the previous section clear, we take an arbitrary scale $\mu$ and define $t=\ln \left(p^{2} / \mu^{2}\right), t_{0}=$ $\ln \left(p_{\text {ref }}^{2} / \mu^{2}\right)$ then Eq. 2.2 .22 may be written as:

$$
\begin{equation*}
e_{R}^{2}(t)=e_{R}^{2}\left(t_{0}\right)+e_{R}^{4}\left(t_{0}\right) A\left(t-t_{0}\right) \tag{2.2.51}
\end{equation*}
$$

This in turn is formally equivalent to a one loop approximation that gives the momentum dependence of $b$ in the $m^{4}$ Ginzburg-Landau model of Eq. 2.1.15), which is once again induced by fluctuations, and is given by:

$$
\begin{equation*}
b_{R}=b(\Lambda)+b(\Lambda)^{2} C \log \left(\Lambda^{2} / s\right)=b\left(x_{0}\right)+b\left(x_{0}\right)^{2} C\left(x-x_{0}\right) \tag{2.2.52}
\end{equation*}
$$

where $\Lambda$ represents a "UV cut-off" such that that fluctuation effects are suppressed beyond that scale as it was needed to regulate a divergence of an integral ${ }^{20}$, $s^{1 / 2}$ represents an energy scale ${ }^{21}$ and $x=\log \left(\mu^{2} / s\right), x_{0}=$ $\log \left(\mu^{2} / \Lambda^{2}\right)$. Hence, when $x-x_{0}$ is large, and therefore perturbation theory breaks down, there is a limit in the range of $x$ but also $x_{0}$. In particular this implies that the limit $x_{0} \rightarrow-\infty$ can not be taken and thus the limit $\Lambda \rightarrow \infty$ either. This is problematic as prior to regularization we had $\Lambda \rightarrow \infty$.

The subtraction technique can be used to remove this divergence using a reference point $s_{r e f}$ but this only replaces $\Lambda$ with $s_{\text {ref }}$ in $b_{R}$. However, imposing FSS allows us to take $x$ or $t$ arbitrarily far away from $x_{0}$ or $t_{0}$ without breaking perturbation theory. This is important as the breakdown of perturbation theory is usually accompanied with the breakdown of unitarity meaning that we would have probabilities for events that are larger than one. Hence, imposing FSS we then obtain:

$$
\begin{equation*}
\left.\frac{\mathrm{de}_{\mathrm{R}}^{2}(\mathrm{t})}{d t_{0}}\right|_{t_{0}=t}=0 \Longleftrightarrow \frac{\mathrm{de}_{\mathrm{R}}^{2}\left(\mathrm{t}_{0}\right)}{d t_{0}}=A e_{R}^{4}\left(t_{0}\right) \Longleftrightarrow \frac{\mathrm{de}_{\mathrm{R}}\left(\mathrm{t}_{0}\right)}{d t_{0}}=A e_{R}^{3}\left(t_{0}\right) / 2 \tag{2.2.53}
\end{equation*}
$$

This equation can be obtained equivalently by considering Eq. 2.2 .51 as an infinitesimal step of the FSS group as:

$$
\begin{align*}
\left(t, e_{R}^{2}(t)\right) & =g_{t-t_{0}} \cdot\left(t_{0}, e_{R}^{2}\left(t_{0}\right)\right)=\left(t_{0}, e_{R}^{2}\left(t_{0}\right)\right)+\left(t-t_{0}, e_{R}^{4}\left(t_{0}\right) A\left(t-t_{0}\right)\right) \\
& =\left(t_{0}, e_{R}^{2}\left(t_{0}\right)\right)+\left(t-t_{0}\right)\left(1, K\left(e_{R}^{2}\left(t_{0}\right)\right)\right) \tag{2.2.54}
\end{align*}
$$

where $K$ thus corresponds to the infinitesimal generator of the group. Any subsequent point $t$ may be obtained by composing this infinitesimal transformation, or equivalently, as in Sec. 2.2 .2 ) solving the associated autonomous differential equation given by the generator of the group. This in turn leads to Eq. 2.2.53).

In field theory, Eq. 2.2 .53 is called the Gell-Mann-Low beta function. To see the link with the usual method of absorbing divergencies into undetermined coupling constants we may insert an arbitrary scale $\xi$ into Eq. 2.2.52 as $\left(x-x_{0}\right)=\left(x-\xi+\xi-x_{0}\right)$ and absorb the divergent $\left(\xi-x_{0}\right)=\ln \left(\frac{\Lambda^{2}}{\mu^{2}} e^{\xi}\right)$ into the undetermined coupling $b(\Lambda)$, this then leads $b$ to be $\xi$ dependant and the renormalization condition is that

[^11]the physical $b_{R}$ does not depend on $\xi$, that is, $\frac{\mathrm{d} b_{R}}{\mathrm{~d} \xi}=0$. However this still constrains us to use $x=\xi$ at the end to neglect higher order corrections $O\left((x-\xi)^{n}\right) \sqrt{22}$. In any case the solution of Eq. 2.2.53) is:
\[

$$
\begin{equation*}
e_{R}(t)=\frac{e_{R}\left(t_{i}\right)}{\left(-e_{R}^{2}\left(t_{0}\right) A\left(t-t_{i}\right)+1\right)^{1 / 2}} \tag{2.2.55}
\end{equation*}
$$

\]

or in terms of $p$ as

$$
\begin{equation*}
e_{R}\left(p^{2}\right)=\frac{e_{R}\left(p_{i}^{2}\right)}{\left(-e_{R}^{2}\left(p_{i}^{2}\right) A \log \left(\frac{p^{2}}{p_{i}^{2}}\right)+1\right)^{1 / 2}} \tag{2.2.56}
\end{equation*}
$$

Once again we have achieved our goal of removing the secular term by imposing FSS. Moreover, this last equation may then be expanded in terms of $e_{R}^{2}\left(p_{i}^{2}\right) \log \left(\frac{p^{2}}{p_{i}^{2}}\right)$ showing us that, similarly to Eq. 2.2.48, , Eq. 2.2 .56 ) is a rather subtle re-summation of an infinite number of corrections. As it was the case for Eq. 2.2.48, Eq. 2.2.56, only retrieves a subset of the exact expansion of $e_{R}$ but these are indeed the dominant contributions corresponding to powers $e_{R}^{2 k+2}\left(p_{i}^{2}\right) \log \left(\frac{\mu}{\Lambda}\right)^{k}$. This is the famous "leading log resummation". The terms that have been discarded are of the form $e_{R}^{m} \log \left(\frac{\mu}{\Lambda}\right)^{k}$ for $m>k+1 \mid 14$ and are thus analogous to the terms $\epsilon^{m}\left(x-x_{i}\right)$ for $m>1$ that were discarded in Sec. 2.2.5. As such, we expect similarly to Sec. 2.2.5, that the approximation will remain valid as long as the renormalized charge is small. This might seem problematic as Eq. 2.2 .56 actually diverges for a specific value of the momentum called the Landau Pole. However, using the infrared value of the charge of $1 / 137$, in the appropriate units, this pole is located at energies of $10^{286} \mathrm{ev}$ far beyond the reach of the large hadron collider at $10^{13} \mathrm{ev}$ or even that of quantum gravity at $10^{28} \mathrm{ev}$. Thus for all practical purposes, using the starting infrared value of $1 / 137$, QED is well determined by renormalized perturbation theory.

Hence, we have succeeded in not only removing the divergence but also in obtaining a very good approximation at all scales using a seemingly trivial remark about functional self-similarity. Moreover, we remark that the initial renormalization prescription of defining the charge such that the Coulomb potential has the same form at all scales was already a first step in the FSS procedure.

This, in essence, is sufficient for us to understand renormalization but we wish to consider one last example that shows a different way of obtaining approximate solutions that will have the advantage of introducing a parameter into the exact equation such that the exact solution is trivially independent of the parameter but the approximate section is not. This will shed some light on how to understand the concept of the regulator in functional renormalization.

### 2.2.7 A comparison of the renormalization group and the variational approach on an approximation of a non trivial second order differential equation

We consider the following example of a non linear oscillator:

$$
\begin{equation*}
y^{\prime \prime}(t)+y(t)+\epsilon y(t)^{3}=0 \tag{2.2.57}
\end{equation*}
$$

This equation can be set into a set of two first order equations that are linear in $\epsilon$. Hence a direct renormalization approach would lead us nowhere. Instead we reparametrize the problem in the following way: the 0 th term in $\epsilon$ gives a simple linear oscillator equation whose solution is $y_{0}(t)=A \cos (t-\phi)$, hence instead of performing an expansion on $y(t)$ and $y^{\prime}(t)$ we may trade variables to $A, \phi$. Notice here that the exact underlying equations for $A$ and $\phi$ in terms of Eq. 2.2 .57 are not known here and are not even clear how to define at this stage.

We thus proceed in the usual manner by expanding $y=y_{0}+\epsilon y_{1}+\ldots$ where the first order in $\epsilon$ gives:

$$
\begin{equation*}
y_{1}^{\prime \prime}(t)+y_{1}(t)+y_{0}(t)^{3}=0 \tag{2.2.58}
\end{equation*}
$$

[^12]We remove the homogeneous solution which may be absorbed into the 0th order via a trivial re-parametrization of constants. This then leads us to:

$$
\begin{align*}
y_{1}(t)= & \frac{1}{32}\left(-12 A^{3} t \sin (t-\phi)-\right.  \tag{2.2.59}\\
& \left.6 A\left(t_{0}\right)^{3} \cos (t-\phi)+A^{3} \cos (3(t-\phi))\right)
\end{align*}
$$

Our aim now is to renormalize the secular term $\epsilon t$ because it diverges for $t \rightarrow \infty$ contrary to the exact solution. In this case we will not impose that $y_{1}\left(t_{0}\right)=0$ as it clutters the equations with unimportant phase arguments and renormalizing would force us to artificially include the non $\operatorname{secular} \cos \left(3\left(t-\phi\left(t_{0}\right)\right)\right)$ term into the 0 -th order term even though it has a higher frequency. We instead write $t \sin \left(t-\phi\left(t_{0}\right)\right)=\left(t-t_{0}+t_{0}\right) \sin \left(t-\phi\left(t_{0}\right)\right)$ where the term $t_{0} \sin \left(t-\phi\left(t_{0}\right)\right)$ can be absorbed into the 0 -th order term via redefinitions of $A$ and $\phi$ along with the homogeneous solution of $y_{1}$. Furthermore, we absorb the term $\frac{1}{32} 6 A^{3} \cos (t-\phi)$ into the 0 -th term as well. Hence, we have:

$$
\begin{equation*}
y_{1}(t)=\frac{1}{32}\left(-12 A^{3}\left(t-t_{0}\right) \sin (t-\phi)+A^{3} \cos (3(t-\phi))\right) \tag{2.2.60}
\end{equation*}
$$

where only the secular term vanishes at $t=t_{0}$. This is a more flexible method where we choose the initial conditions such that only the secular terms equate to zero at $t=t_{0}$.

We then require that $y=y_{0}+\epsilon y_{1}$ be independent of $t_{0}$. Collecting the terms proportional to $\cos (t-\phi)$ and $\sin (t-\phi)$ and recalling that $A$ and $\phi$ are also functions of the initial time $t_{0}$ we obtain in the limit $t \rightarrow t_{0}$ :

$$
\begin{equation*}
A^{\prime}\left(t_{0}\right)=0, \phi^{\prime}\left(t_{0}\right)=-\frac{3 \epsilon A\left(t_{0}\right)^{2}}{8} \tag{2.2.61}
\end{equation*}
$$

The first equation shows that $A$ is a constant at this order of the approximation. The second equation shows that $\phi(t)=\frac{-3}{8} A^{2} t \epsilon+\phi_{0}$ at order $\epsilon$. Thus, using the expressions for $A(t)$ and $\phi(t)$, the general solution at order $\epsilon$ is :

$$
\begin{equation*}
y(t)=A \cos \left(t\left(1+\frac{3}{8} A^{2} \epsilon\right)-\phi_{0}\right)+\frac{\epsilon}{32} A^{3} \cos \left(3\left(t-\phi_{0}\right)\right) \tag{2.2.62}
\end{equation*}
$$

We see here that the renormalization of the phase $\phi$ may be reinterpreted as a correction to the frequency at this order in $\epsilon$. Moreover, we note here that this solution is not exact and in fact infinitely many corrections in $\epsilon$ need to be added to get an exact result. This is reminiscent of the complications of field theories where one needs to calculate infinitely many loop orders to get exact results.

Let us also notice that this change of frequency shows that we could have performed our perturbation differently. Indeed, we could have written the differential equation as:

$$
\begin{equation*}
y^{\prime \prime}(t)+r^{2} y(t)+\left(\left(1-r^{2}\right) y+\epsilon y(t)^{3}\right)=0 \tag{2.2.63}
\end{equation*}
$$

and considered the 0 th order term as $y_{0}^{\prime \prime}(t)+r^{2} y_{0}(t)=0$. This is what is done in optimized perturbation theory 1523 and is quite reminiscent of both variational approximations and of the counter terms method used in perturbation theory as in Appendix C.

However, one might feel slightly at unease as to why the term $\left(1-r^{2}\right) y+\epsilon y(t)^{3}$ in Eq. 2.2 .63 is small and how should one write the expansion of $y$ if $\epsilon$ is no longer the small parameter in which we expand. The answer to this question lies within the framework of the homotopy analysis method $24-26$. Within this very general approximation scheme we may continuously deform a difficult problem into a simpler problem without the presence of a small parameter. We will see that the $\Gamma$ flow, which will be the formulation of the functional renormalization group that we will use in this thesis, is also a homotopy transformation. In the present case the homotopy or deformation may be obtained by introducing a new variable $s$ such that the deformed $y_{s}$ verifies the equation:

$$
\begin{equation*}
\left(y_{s}^{\prime \prime}(t)+r^{2} y_{s}(t)\right)(1-s)+s\left(y_{s}^{\prime \prime}(t)+y_{s}(t)+\epsilon y_{s}(t)^{3}\right)=y_{s}^{\prime \prime}(t)+r^{2} y_{s}(t)+s\left(\left(1-r^{2}\right) y_{s}(t)+\epsilon y_{s}(t)^{3}\right)=0 \tag{2.2.64}
\end{equation*}
$$

with $0 \leq s \leq 1$. We see that at $s=0$ we obtain the 0 th order term previously discussed and at $s=1$ we have the problem which we would like to solve. Moreover an expansion in $s$ is in fact an expansion in $s\left(\left(1-r^{2}\right) y_{s}(t)+\epsilon y_{s}(t)^{3}\right)$ which is the term we now wish to consider as small. As such, the expansion on
$y_{s}$ will be written as $y_{s}=y^{(0)}+s y^{(1)}+s^{2} y^{(2)} \ldots$ and $y_{s=1}$, which is the solution we seek, will be given as a truncated sum $y_{s=1}=\sum_{n<N} y^{(n)}$.

The first correction in $s$ of Eq. 2.2 .64 gives:

$$
\begin{equation*}
y_{1}(t)=\frac{A^{3} \epsilon \cos (3 r t-3 \phi)-2 A\left(3 A^{2} \epsilon-4 r^{2}+4\right)(2 r t \sin (r t-\phi)+\cos (r t-\phi))}{32 r^{2}} . \tag{2.2.65}
\end{equation*}
$$

We thus see that both $y_{0}$ and $y_{1}$ have spurious dependence on the parameter $r$ which the exact solution does not have. This may naively seem as a problem as results would depend on the value of $r$. However, we see that by choosing

$$
\begin{equation*}
3 A^{2} \epsilon-4 r^{2}+4=0 \tag{2.2.66}
\end{equation*}
$$

we eliminate the secular term without having to use the renormalization technique. Moreover, even if this was not noticed we could have observed, plotting the solution for different values of $r$, that the long time divergence would disappear for some optimal value of $r$. This could have been noticed also by computing the energy which should be conserved but would diverge because of the secular term. For small $\epsilon$ we retrieve the perturbative result at order $\epsilon$. The following powers of $\epsilon$ differ however from the exact results as we have neglected the contribution from the higher order $y_{2}$ but the solution is still quite accurate for a large range of $\epsilon$. First of all it gives the right sign and order of magnitude for the first corrections in $\epsilon$ of the frequency of this an-harmonic oscillator even though the Taylor expansion in $\epsilon$ is not the same. Moreover the value obtained for the frequency using Eq. 2.2 .66 is, for $\epsilon=1$, precise up to a relative correction of order $10^{-3}$ when compared to the exact solution. This is actually slightly better than the results obtained from the perturbative expansion to order 3 in $\epsilon$ using the usual renormalization techniques explained in this text [27]. Finally, for $\epsilon=10$ we have a relative error on the order of $10^{-2}$ while the perturbative result at order 3 is about 1.7 times larger than the exact result. This is the power of the variational approach where one may add some parameter $r$ that does not change the exact result but is capable of optimizing approximations. We will see that the functional renormalization group (FRG) also contains such a parameter that does not change the exact solution but may be used to optimize approximate solutions. Moreover, we will also see that the non perturbative approximations, while dependent on this spurious parameter, may give similar results to perturbation theory in certain limits while giving reasonable results outside of the perturbative regime.

### 2.3 Non perturbative Renormalization

$$
\begin{aligned}
& \text { "Truth is ever to be found in the } \\
& \text { simplicity, and not in the } \\
& \text { multiplicity and confusion of } \\
& \text { things.", } \\
& \pi=\frac{3 \sqrt{3}}{4}- \\
& \begin{array}{r}
24 \sum_{n=0}^{\infty} \frac{(2 n)!}{2^{4 n+2}(n!)^{2}(2 n-1)(2 n+3)} \\
- \text { Isaac Newton }
\end{array}
\end{aligned}
$$

### 2.3.1 Exact RG equations

As we have explained in the previous chapter, renormalisable interacting field theories naturally lead to large logarithms $\log \left(p^{2} / p_{r e f}^{2}\right)$ which then limit the range of validity of approximations. This is fixed by using the so-called "renormalization procedure" which is simply the act of imposing FSS on approximate solutions. The infinitesimal form of this self-similarity is given by the beta-functions $\frac{\mathrm{dg}_{i}}{\mathrm{dt}}=\beta_{i}\left(\left\{g_{j}\right\}\right)$ of the different couplings that appear in the initial action as in Sec 2.2.6. Failure to impose this leads to non nonsensical divergences for quantities that can be measured and are thus finite. However, at this stage we do not have the exact
equation for the renormalization flow of a given theory and thus it is difficult to construct approximation schemes that go beyond perturbation theory. Hence, in this chapter we will show how the exact equation may be found by looking for a transitivity property of the path integral itself. Doing this we adapt the Wilsonian renormalization picture where instead of locating the divergencies and removing them by demanding that the couplings vary with scale, we seek an FSS relation directly on the partition function, or equivalently, the free energy, as it contains all of the information of the theory. Once an FSS relation is found we may seek an infinitesimal form of the group action which will be as usual an autonomous differential equation, but it will now act on functionals such as the free energy. By imposing FSS and also choosing from the beginning a regime of validity given by UV and IR cutoffs we may evade all previous discussions on divergences or large logarithms.

Hence we now begin our derivation of the RG flows. We first need to consider an object that contains all of the information of the theory both perturbative and non perturbative. This can be the effective action but it is much easier to first consider the partition function:

$$
\begin{equation*}
Z[J]=\int D \phi \mathcal{P}^{u}[J, \phi]=\int D \phi e^{-(S(\phi)-J \cdot \phi)} \tag{2.3.1}
\end{equation*}
$$

where $\mathcal{P}^{u}[J, \phi]=Z \mathcal{P}[J, \phi]$ is the "un-normalized probability" given directly by the Boltzmann factor and we used the generalized scalar product:

$$
\begin{equation*}
J \cdot \phi=\sum_{\alpha} J_{\alpha} \phi_{\alpha}=\int_{x} \sum_{a} J_{a}(x) \varphi_{a}(x)=\sum_{a} \int J_{a}(x) \varphi_{a}(x) \mathrm{d}^{d} x \tag{2.3.2}
\end{equation*}
$$

where we consider N fields indexed by latin letters $a$ while greek letters represent both indices for fields and the position index $x$ of the fields. From the partition function we may compute correlation functions at vanishing external source as:

$$
\begin{equation*}
<\phi_{\alpha_{1}} \phi_{\alpha_{2}} \ldots \phi_{\alpha_{s}}>\underset{\text { def }}{=} \int D \phi \phi_{\alpha_{1}} \phi_{\alpha_{2}} \ldots \phi_{\alpha_{s}}\left(\frac{e^{-S(\phi)}}{Z[0]}\right)=\left.\partial_{J_{\alpha_{1}} J_{\alpha_{2}} \ldots J_{\alpha_{s}}}\left(\frac{Z[J]}{Z[0]}\right)\right|_{J=0} . \tag{2.3.3}
\end{equation*}
$$

Our next step is to define a notion of scale, or domain of applicability, within our theory. There are a few reasons for doing this, first any reasonable theory should have a domain of applicability as there should always be room for potential improvements at smaller or larger scales. These limits might not be explicit within the theory and so we consider generic lower and upper bounds $(k, \Lambda)$. These bounds would need to be determined by experiments. The second reason for this is that in general, as in the previous chapter, the theory leads to divergencies when bounds are not made explicit. The third and final reason is that the renormalization group corresponds to a mapping of scales and so the scales at which the theory applies must be made explicit. A natural question is thus how do we concretely define these scales within the partition function? The most natural way is simply to define sharp limits as :

$$
\begin{equation*}
Z[J]=\int_{k, \Lambda} D \phi e^{-(S(\phi)-J \cdot \phi)} \tag{2.3.4}
\end{equation*}
$$

where the subscript $k, \Lambda$ implies that fluctuations $\phi(q)$, where $q^{2}>\Lambda^{2}$ or $q^{2}<k^{2}$, do not contribute to the path integral. This can be implemented also by modifying the measure $D \phi$ to $D \phi \mu_{k, \Lambda}$ where $\mu_{k, \Lambda}$ is zero for $q^{2} \in \mathbb{R} \backslash\left[k^{2}, \Lambda^{2}\right]$ and one for $q^{2} \in\left[k^{2}, \Lambda^{2}\right]$. Finally, one may also view this modification of the measure as a modification of the action as $\mu_{k, \Lambda}=e^{-\widetilde{S}_{k, \Lambda}}$ where $\widetilde{S}_{k, \Lambda}=\infty$ for $q^{2} \in \mathbb{R} \backslash\left[k^{2}, \Lambda^{2}\right]$ and $\widetilde{S}_{k, \Lambda}=0$ for $q \in\left[k^{2}, \Lambda^{2}\right]$. This is indeed a valid way to obtain an exact non perturbative flow but it is not the only way. Instead of such a singular behavior for $\widetilde{S}_{k, \Lambda}$ one may consider an action that is simply negligible for $q^{2} \in\left[k^{2}, \Lambda^{2}\right]$ and large elsewhere while smoothly interpolating between the regimes. Such an action also defines a domain of interest $\left[k^{2}, \Lambda^{2}\right]$ but due to the smooth behavior it is less clear cut where $k$ and $\Lambda$ are positioned.

However, is there any interest in considering one or the other ? A priori, no, as long as we work with the exact flow equations. Nevertheless, if we wish to obtain concrete numbers for some given model, an approximation scheme is usually necessary. In Sec.(3.1.5) we shall see that at a given level of approximation, numerical results depend on our choice of $\widetilde{S}_{k, \Lambda}$ and in fact the sharp cut-off gives rather poor results. The



Figure 2.3.1: $K_{s}$ the high momentum cut-off function on the left and $K_{s^{\prime}}-K_{s^{\prime}}$ with $s^{\prime}<s$ that acts both as a high momentum cut-off and a low momentum one which thus enters the definition $\Delta_{s^{\prime}, s}$ as $\Delta_{s^{\prime}, s}=$ $\frac{K_{s}\left(q^{2}\right)-K_{s^{\prime}}\left(q^{2}\right)}{q^{2}}$
fact that results depend on $\widetilde{S}_{k, \Lambda}$ is in no way problematic and is in fact desirable. Indeed in Sec. 2.2.7) we explicitly added the parameter $r$ precisely so that approximate results may depend on the choice of $r$ thereby allowing us to optimize results. For now it suffices to say that one should keep $\widetilde{S}_{k, \Lambda}$ as general as possible. In principle this is a rather difficult task as $\widetilde{S}_{k, \Lambda}$ can be arbitrarily complicated with many parameters to vary to find an optimal solution. This would then lead to difficult calculations with lengthy optimizations in other to obtain coherent results. Instead we restrict our study to the simplest choice: a Gaussian action. Indeed Gaussian theories are easy to manipulate and will lead to simple exact equations. Hence, we consider the choice:

$$
\begin{equation*}
\widetilde{S}_{k, \Lambda}[\phi]=\frac{\phi \cdot H_{k, \Lambda} \cdot \phi}{2} \tag{2.3.5}
\end{equation*}
$$

where we used the notation:

$$
\begin{align*}
\phi \cdot H \cdot \phi & =\sum_{\alpha, \beta} \phi_{\alpha} H_{\alpha, \beta} \phi_{\beta}=\int_{x, y} \sum_{a, b} \phi_{a}(x) H_{a, b}(x-y) \phi_{b}(y) \\
& =\sum_{a, b} \int \phi_{a}(x) H_{a, b}(x-y) \phi_{b}(y) \mathrm{d}^{d} x \mathrm{~d}^{d} y . \tag{2.3.6}
\end{align*}
$$

At this point one may derive an exact flow equation as in Appendix D. However, as that derivation is purely mathematical we would like to instead derive the exact RG equations in a way that is longer but has the advantage of making explicit the Wilsonian idea of using FSS to generate an RG flow. We regard this as important for historic and conceptual reasons. The conceptual insights learned along the way will be summarized at the end of this subsection. Moreover, having a clear idea of how the Wilsonian RG is related to the FSS framework will allow us to study simpler examples in Sec. 2.4 which will explain why FSS, and thus the RG, is a good framework for obtaining approximate results. The derivation here follows that of [28] and $[29]$.

We recall that the action $S$ usually contains a Gaussian kinetic term, allowing us to separate $S$ as $S=S_{G}+V$. We may then consider separately the Gaussian term $S_{G}+\widetilde{S}_{k, \Lambda}$ where in momentum space the bosonic gaussian kernel can be written as $q^{2}+\widehat{h}_{k, \Lambda}\left(q^{2}\right)$ where $\widehat{h}_{k, \Lambda}$ is the kernel of $\widetilde{S}_{k, \Lambda}$. We may then reparametrize $q^{2}+\widehat{h}_{k, \Lambda}\left(q^{2}\right)$ to $q^{2} / K_{k, \Lambda}\left(q^{2}\right)$ as it will be a more convenient expression in this part of the derivation. As such, we have $K_{k, \Lambda} \simeq 1$ for $q^{2} \in\left[k^{2}, \Lambda^{2}\right]$ while for $q^{2} \in \mathbb{R} \backslash\left[k^{2}, \Lambda^{2}\right], \widetilde{S}_{k, \Lambda}$ is large and thus $K_{k, \Lambda}$ must be small. These restrictions can be implemented by taking $K_{k, \Lambda}=K_{\Lambda}-K_{k}=K\left(q^{2} / \Lambda^{2}\right)-K\left(q^{2} / k^{2}\right)$ where $K_{s} \simeq 1$ for $q^{2}<s^{2}$ and $K_{s}$ small for $q^{2}>s^{2}$ as in Fig. 2.3.1.

Hence, our choice for the functional $\widetilde{S}_{k, \Lambda}$ has now been reduced to that of the function $K$ which greatly simplifies the space on which we seek to optimize results. We may then obtain a flow equation by inserting $\mu_{k, \Lambda}=e^{-S_{G}-\widetilde{S}_{k, \Lambda}}$ in $Z$ as:

$$
\begin{equation*}
Z[J]=\int D \phi \mu_{k, \Lambda}(\phi) e^{-(V(\phi)-J \cdot \phi)} \tag{2.3.7}
\end{equation*}
$$

Defining $e^{-W_{k, \Lambda}[J]}=Z$ we may derive the flow equation of $W_{k, \Lambda}[J]$ of Appendix $D$ from here. However we wish to obtain a flow equation where the argument of the functional we consider plays the role of a field variable rather than a source term $J$. To see how this may come about let us consider what the inclusion of the current $J$ does to a $\mathbb{Z}_{2}$ symmetric theory with potential $V(\phi)=m^{2} \phi^{2} / 2+\lambda \phi^{4}$ Introducing a current $J$ creates a preference in the system to align along the direction $J$, in other terms we break the $\mathbb{Z}_{2}$ symmetry $\phi \rightarrow-\phi$ of the model. This is not the only way to create an asymmetry however, we could also replace $\phi$ with $\Phi+\phi$. If we then define a new type of partition function as:

$$
\begin{equation*}
\widetilde{Z}[\Phi]=\int D \phi \mu_{k, \Lambda}(\phi) e^{-V(\phi+\Phi)} . \tag{2.3.8}
\end{equation*}
$$

This partition function may be interpretated as corresponding to the sum of magnetic fluctuations $\phi$ about the background magnetization field $\Phi$. From the point of view of an experimentalist, instead of imposing the external magnetic field $J$ we impose the internal magnetization $\Phi$ and let the system fluctuate around this background field. In turn, it turns out that these two methods are equivalent as the change of variables $\phi+\Phi=\varphi$ leads to:

$$
\begin{align*}
\widetilde{Z}_{k, \Lambda}[\Phi] & =\int D \phi \mu_{k, \Lambda}(\phi) e^{-V(\phi+\Phi)}=\int D \phi e^{-V(\phi+\Phi)} e^{-\frac{\phi \cdot H_{k, \Lambda} \cdot \phi}{2}} \\
& =\int D \phi e^{-V(\varphi)} e^{-\frac{(\varphi-\Phi) \cdot H_{k, \Lambda} \cdot(\varphi-\Phi)}{2}}=e^{-\frac{\Phi \cdot H_{k, \Lambda} \cdot \Phi}{2}} \int D \phi \mu_{k, \Lambda}(\varphi) e^{-V(\varphi)} e^{\Phi \cdot H_{k, \Lambda} \cdot \varphi}  \tag{2.3.9}\\
& =e^{-\frac{\Phi \cdot H_{k, \Lambda} \cdot \Phi}{2}} Z_{k, \Lambda}\left[\Phi \cdot H_{k, \Lambda}\right] .
\end{align*}
$$

Finally, for notational convenience, in the following we rewrite Eq. (2.3.8) as:

$$
\begin{equation*}
\widetilde{Z}_{k, \Lambda}[\Phi]=\int D \phi \mu_{k, \Lambda}(\phi) e^{-V_{\Lambda, \Lambda}(\phi+\Phi)}=\int D \phi \mu_{k, \Lambda}(\phi) \Psi_{\Lambda, \Lambda}(\phi+\Phi) . \tag{2.3.10}
\end{equation*}
$$

Our objective will now be to find an FSS group. This will allow us to obtain an RG flow equation from the infinitesimal FSS generator of the group. We thus recall the FSS group law:

$$
\begin{equation*}
G\left(\lambda_{2}, G\left(\lambda_{1}, y\right)\right)=G\left(\lambda_{1}+\lambda_{2}, y\right) . \tag{2.3.11}
\end{equation*}
$$

Notice that this equation is written in terms of translations $t \rightarrow t+\lambda$. However, $k$ and $\Lambda$ are in units of $q$ and correspond to scales as is apparent from $K_{k}=K\left(q^{2} / k^{2}\right)$. The appropriate transformations in the RG context is then scale transformations $q \rightarrow \lambda q$, where Eq.(2.3.11) then translates to:

$$
\begin{equation*}
G\left(\lambda_{2}, G\left(\lambda_{1}, y\right)\right)=G\left(\lambda_{1} \lambda_{2}, y\right) . \tag{2.3.12}
\end{equation*}
$$

This means that the group is written in terms of products rather than sums. This can be easily fixed by considering $\lambda=e^{t}$ which implies that we replace $k / \Lambda$ by $e^{t}$. Our first objective is then to find the transitivity property:

$$
\begin{equation*}
\mathrm{RG}\left[t_{3}, t_{1}\right]=\mathrm{RG}\left[t_{3}, t_{2}\right] \circ \mathrm{RG}\left[t_{2}, t_{1}\right] \tag{2.3.13}
\end{equation*}
$$

which then requires us to introduce an intermediate scale into the problem. This can easily be done as:

$$
\begin{equation*}
K_{k, \Lambda}=K_{\Lambda}-K_{k}=K_{\Lambda}-K_{\Lambda_{-}}+K_{\Lambda_{-}}-K_{k}=K_{\Lambda_{-}, \Lambda}+K_{k, \Lambda_{-}} . \tag{2.3.14}
\end{equation*}
$$

Defining the "propagator" $\Delta_{k, \Lambda}=K_{k, \Lambda}\left(q^{2}\right) / q^{2}$, which is the inverse of the Gaussian kernel, the same decomposition holds for $\Delta_{k, \Lambda}$. Now we may use the most important part of this derivation of the RG flow: the convolution product of two Gaussians $\mathcal{P}_{\Delta_{1}}$ and $\mathcal{P}_{\Delta_{2}}$ with propagators $\Delta_{1}$ and $\Delta_{2}$ is a Gaussian $\mathcal{P}_{\Delta_{1}+\Delta_{2}}$ ${ }^{23}$. In fact the central limit theorem which shall be discussed shortly is a consequence of the fact that the functional form of the Gaussian is a fixed-point for the convolution operator. Hence, using the decomposition

[^13]$\Delta_{k, \Lambda}=\Delta_{k, \Lambda_{-}}+\Delta_{\Lambda_{-}, \Lambda}$ this then translates to $\mu_{k, \Lambda_{-}} * \mu_{\Lambda_{-}, \Lambda}=\mu_{k, \Lambda}$ where $*$ is a convolution product. We thus have :
\[

$$
\begin{equation*}
\widetilde{Z}_{k, \Lambda}[\Phi]=\int D \phi \mu_{k, \Lambda}(\phi) \Psi_{\Lambda, \Lambda}(\phi+\Phi)=\int D \phi\left(\int D \phi_{<} \mu_{k, \Lambda_{-}}\left(\phi_{<}\right) \mu_{\Lambda_{-}, \Lambda}\left(\phi-\phi_{<}\right)\right) \Psi_{\Lambda, \Lambda}(\phi+\Phi) \tag{2.3.15}
\end{equation*}
$$

\]

If we now consider the change of variables $\phi=\phi_{>}+\phi_{<}$we obtain :

$$
\begin{align*}
\widetilde{Z}_{k, \Lambda}[\Phi] & =\int D \phi_{<} D \phi_{>} \mu_{k, \Lambda_{-}}\left(\phi_{<}\right) \mu_{\Lambda_{-}, \Lambda}\left(\phi_{>}\right) \Psi_{\Lambda, \Lambda}\left(\phi_{>}+\phi_{<}+\Phi\right) \\
& =\int D \phi_{<} \mu_{k, \Lambda_{-}}\left(\phi_{<}\right)\left(\int D \phi_{>} \mu_{\Lambda_{-}, \Lambda}\left(\phi_{>}\right) \Psi_{\Lambda, \Lambda}\left(\phi_{>}+\phi_{<}+\Phi\right)\right) \tag{2.3.16}
\end{align*}
$$

At this point we will define a renormalization prescription. Similar to how we defined the charge in Sec. 2.2 .1 ) such that the potential maintained a Coulomb like form at all scales, we will define the term within the parenthesis in the above equation to be $\Psi$ but in the domain $\Lambda_{-}^{2}<p^{2}<\Lambda^{2}$ which we will denote as $\Psi_{\Lambda_{-}, \Lambda}$. This means that we have both :

$$
\begin{equation*}
\widetilde{Z}_{k, \Lambda}[\Phi]=\int D \phi_{<} \mu_{k, \Lambda_{-}}\left(\phi_{<}\right) \Psi_{\Lambda_{-}, \Lambda}\left(\phi_{<}+\Phi\right) \tag{2.3.17}
\end{equation*}
$$

with:

$$
\begin{equation*}
\Psi_{\Lambda_{-}, \Lambda}(\Phi)=\int D \phi_{>} \mu_{\Lambda_{-}, \Lambda}\left(\phi_{>}\right) \Psi_{\Lambda, \Lambda}\left(\phi_{>}+\Phi\right) \tag{2.3.18}
\end{equation*}
$$

and the original

$$
\begin{equation*}
\widetilde{Z}_{k, \Lambda}[\Phi]=\int D \phi \mu_{k, \Lambda}(\phi) \Psi_{\Lambda, \Lambda}(\phi+\Phi) \tag{2.3.19}
\end{equation*}
$$

Eq. 2.3.18 then motivates us to write

$$
\begin{equation*}
\Psi_{k, \Lambda}(\Phi)=\int D \phi \mu_{k, \Lambda}(\phi) \Psi_{\Lambda, \Lambda}(\phi+\Phi) \tag{2.3.20}
\end{equation*}
$$

which means that we took $\Psi_{k, \Lambda}(\Phi)=\widetilde{Z}_{k, \Lambda}[\Phi]$. If we also keep the definition of $\Psi$ in terms of potentials as $\Psi_{k_{i}, k_{j}}(\Phi)=e^{-V_{k_{i}, k_{j}}}$ then we have:

$$
\begin{equation*}
e^{-V_{k, \Lambda}(\Phi)}=\int D \phi \mu_{k, \Lambda}(\phi) e^{-V_{\Lambda, \Lambda}(\phi+\Phi)} \tag{2.3.21}
\end{equation*}
$$

This is the usual definition of free energy in terms of the partition function in statistical mechanics. This in turn gives us a convenient thermodynamic interpretation we shall exploit in the following. For now, for notational convenience, let us return to the $\Psi$ notation. From the previous equations we may deduce the general scale transformation rule:

$$
\begin{equation*}
\Psi_{k_{i}, k_{f}}(\Phi)=\int D \phi \mu_{k_{i}, k_{m}}\left(\phi_{>}\right) \Psi_{k_{m}, k_{f}}\left(\phi_{>}+\Phi\right) \underset{\operatorname{def}}{=} T_{k_{i}, k_{m}}\left(\Psi_{k_{m}, k_{f}}\right) \tag{2.3.22}
\end{equation*}
$$

This is true for any $k_{i}<k_{m}<k_{f}$ and thus as promised we have obtained a formula showing how one may compute physics within the interval of scales $\left[k_{i}, k_{f}\right]$ by first computing the physics at $\left[k_{m}, k_{f}\right]$ and using this as an initial condition to further integrate the fluctuations down to the scale $k_{i}$. One can then further extend this as

$$
\begin{aligned}
\Psi_{k_{i}, k_{f}}(\Phi) & =T_{k_{i}, k_{m}}\left(\Psi_{k_{m}, k_{f}}\right) \\
& =T_{k_{i}, k_{m^{\prime}}}\left(\Psi_{k_{m^{\prime}}, k_{f}}\right) \\
& =T_{k_{i}, k_{m^{\prime}}}\left(T_{k_{m^{\prime}}, k_{m}}\left(\Psi_{k_{m}, k_{f}}\right)\right) \\
& =\left(T_{k_{i}, k_{m^{\prime}}} \circ T_{k_{m^{\prime}}, k_{m}}\right)\left(\Psi_{k_{m}, k_{f}}\right)
\end{aligned}
$$

which gives us the functional equation ${ }^{24}$

$$
\begin{equation*}
T_{k_{i}, k_{m}}=T_{k_{i}, k_{m^{\prime}}} \circ T_{k_{m^{\prime}}, k_{m}} \tag{2.3.23}
\end{equation*}
$$

corresponding to Eq. $(\sqrt{2.3 .13)}$ in terms of the variables $k$ rather than $t$. This is not the Wilsonian RG however which should verify FSS and not just transitivity. As such a Wilsonian RG which we write as WRG should verify :

$$
\begin{equation*}
\mathrm{WRG}\left[t_{a}+t_{b}\right]=\mathrm{WRG}\left[t_{a}\right] \circ \mathrm{WRG}\left[t_{b}\right] \tag{2.3.24}
\end{equation*}
$$

which is more restrictive than Eq. 2.3 .13 . But why does this matter ? Wouldn't we be able to obtain a differential equation from $\mathrm{RG}[t+\delta t, t]$ ? The answer to these questions lie in the fact that differentiating a FSS group law gives an autonomous differential equation while differentiating non FSS transitivity gives us a differential equation which is in general time dependent. There is much we can do already with a non autonomous equation but autonomous equations come prepackaged with notions of fixed-points, by setting the operator $\partial_{t}$ to zero, and stability analysis, by replacing the time operator with an eigenvalue. This is not crucial, for example we may obtain critical exponents by other means ( see for example appendix in 30 ), but working with WRG is both numerically and theoretically convenient as will be discussed in Sec.(2.3.5). For now the natural question arises: How do we go from RG to WRG ? Before answering this directly let us consider what we expect from WRG. We expect WRG to converge to a fixed-point theory when the theory is scale invariant. Thus there is an inherent notion of convergence such that we expect the theories along the RG flow to look more and more similar as we approach the fixed-point. This cannot happen with our current RG flow as it is simply a mapping of scales and thus the theories along the RG flow cannot look similar as each theory has a different scale. This in turn shows us that our first task in obtaining a WRG flow should be to rescale the theory after each step so that a fixed-point solution is possible. This is indeed the approach taken in Appendix Ewhere we derive an FSS RG flow. For now, as it is simpler and sufficient for our current purposes, let us remain with the transitive RG flow and consider an infinitesimal transformation $T_{k, k+\delta k}$ in order to obtain the generator of the transformation and thus a differential equation on $k$ for $\Psi$. Doing this, we obtain:

$$
\begin{equation*}
\Psi_{k-\delta k, \Lambda}(\Phi)=\int D \phi \mu_{k-\delta k, k}(\phi) \Psi_{k, \Lambda}(\phi+\Phi) . \tag{2.3.25}
\end{equation*}
$$

The propagator associated with $\mu$ is thus $\Delta_{k+\delta k}-\Delta_{k}=\partial_{k} \Delta_{k} \delta k$ so we have:

$$
\begin{equation*}
D \mu_{k-\delta k, k}(\phi)=\mathcal{N}_{k-\delta k, k} D \phi e^{-\frac{\phi \cdot\left(\partial_{k} \Delta_{k}\right)^{-1} \cdot \phi}{\delta k}} \tag{2.3.26}
\end{equation*}
$$

where $\mathcal{N}_{k-\delta k, k}$ is a normalization factor. Thus only small field values of the order of $\delta k^{1 / 2}$ contribute while larger fields are exponentially suppressed. We may thus Taylor expand $\Psi$ about $\Phi$ as it is equivalent to expanding in powers of $\delta k$. Moreover, we will derive the equation for $\Psi$ in dimension zero as arbitrary integer dimensions simply require considering more variables and integrals that clutter the main ideas. However, at the end, when we consider the $\Gamma$ flow we will return to arbitrary dimensions. As such we have:

$$
\begin{equation*}
\Psi_{k-\delta k, \Lambda}(\Phi)=\Psi_{k, \Lambda}(\Phi)<1>+\partial_{\Phi} \Psi_{k, \Lambda}(\Phi)<\phi>+\partial_{\Phi} \partial_{\Phi} \Psi_{k, \Lambda}(\Phi)<\phi \phi>+\ldots \tag{2.3.27}
\end{equation*}
$$

where the <> represents the integration over all fields. Using the Gaussian measure of Eq. $(2.3 .26$ ) to calculate the averages and using a Taylor expansion on the left hand side about $k$ we obtain :

$$
\begin{equation*}
\partial_{k} \Psi_{k, \Lambda}(\Phi) \delta k=-\partial_{k} \Delta_{k} \partial_{\Phi} \partial_{\Phi} \Psi_{k, \Lambda}(\Phi) \delta k+O\left(\delta k^{2}\right) \tag{2.3.28}
\end{equation*}
$$

Simplifying by $\delta k$ and taking the limit $\delta k \rightarrow 0$ leads to a heat equation where $-\partial_{k} \Delta_{k}>0$ 31] 33. We thus see that information of the UV physics is diffused as we arrive at the infrared. We will discuss this further in Sec.(2.3.4).

We may also re-formulate equation 2.3 .28 in terms of $V_{k, \Lambda}$, where $\Psi_{k_{i}, k_{j}}(\Phi)=e^{-V_{k_{i}, k_{j}}}$ :

$$
\begin{equation*}
\partial_{k} e^{-V_{k, \Lambda}(\Phi)}=-\partial_{k} \Delta_{k} \partial_{\Phi} \partial_{\Phi} e^{-V_{k, \Lambda}(\Phi)}=\partial_{\Phi}\left(\partial_{k} \Delta_{k} \partial_{\Phi} V_{k, \Lambda}(\Phi) e^{-V_{k, \Lambda}(\Phi)}\right)=\partial_{\Phi}\left(\Theta_{k, \Lambda} e^{-V_{k, \Lambda}(\Phi)}\right) . \tag{2.3.29}
\end{equation*}
$$

[^14]If we now integrate over $\Phi$ we obtain:

$$
\begin{equation*}
\partial_{k} Z=\partial_{k} \int D \phi e^{-V_{k, \Lambda}(\phi)}=\int D \phi \partial_{\phi}\left(\Theta_{k, \Lambda} e^{-V_{k, \Lambda}(\phi)}\right) . \tag{2.3.30}
\end{equation*}
$$

The last term is zero as it is the integral of a total derivative. The partition function is thus indeed conserved along the renormalization flow which is crucial as the partition function contains all of the information of the underlying physics. Thus, the different actions along the renormalization flow do not represent different physics but simply different descriptions, or parametrizations, of the same underlying phenomena. For example, the different values of the charge in the previous chapter did not represent different particles of different charges but the same particle via effective charges at different scales. The reason why the charge changes when looked at different scales is an artifact of how we defined the charge in the first place, meaning our renormalization prescription in terms of the Coulomb potential $e^{2}=V\left(p^{2}\right) p^{2}$ motivated by the fact that it "looks" like the usual Coulomb potential. In turn as it was not the usual Coulomb potential, or rather it is dressed by interactions, we naturally expect that the effect of these interactions depend on the scale at which we look. This is similar to how the screened Debye charge looks different depending on whether it is looked from a microscopic view or a macroscopic view when considering all of the screening effects. Similarly, the potential here is scale dependent due to our renormalization prescription where we chose to define the potential as $V_{k, \Lambda}=-\log \left(\int D \phi \mu_{k, \Lambda}(\phi) e^{-V_{\Lambda, \Lambda}(\phi+\Phi)}\right)$, which was motivated by the fact that we want the integrated term to "look" like the usual Boltzmann probability density. Moreover, as with the charge, imposing the same functional form at all scales is our first step in our FSS program and the condition $\partial_{k} Z=0$ can be understood in the same way as $\partial_{t_{0}} F(t)=0$ for the solution of the differential equations in Sec. 2.2 . Somewhat surprisingly however, the fact that these different actions simply represent different ways to describe the same underlying physics is actually due to fact that these different actions may be related to one another by $k$-dependent change of variables $[34]$. Let us thus show that the renormalization transformation of Eq. 2.3 .29 can be viewed as an infinitesimal change of variables $\phi^{\prime}=\phi+\delta k \Theta_{k, \Lambda}=\phi+\delta \phi$. We have 25

$$
\begin{align*}
\int D \phi^{\prime} e^{-V_{k, \Lambda}\left(\phi^{\prime}\right)} & =\int D \phi e^{-V_{k, \Lambda}(\phi)}+\int \delta(D \phi) e^{-V_{k, \Lambda}(\phi)}+\int D \phi \delta\left(e^{-V_{k, \Lambda}(\phi)}\right) \\
& =\int D \phi e^{-V_{k, \Lambda}(\phi)}+\int D \phi \partial_{\phi}\left(\delta k \Theta_{k, \Lambda}\right) e^{-V_{k, \Lambda}(\phi)}+\int D \phi \delta k \Theta_{k, \Lambda}\left(\partial_{\phi}\left(e^{-V_{k, \Lambda}(\phi)}\right)\right)  \tag{2.3.31}\\
& =\int D \phi e^{-V_{k, \Lambda}(\phi)}+\delta k \int D \phi \partial_{\phi}\left(\Theta_{k, \Lambda} e^{-V_{k, \Lambda}(\phi)}\right) \\
& =Z+\delta k \partial_{k} Z .
\end{align*}
$$

In the perturbative case the beta functions can be seen as infinitesimal $k$-dependent change of variables of the couplings. Moreover, all Wilsonian renormalization transformations may be viewed as a change of variables including the original Kadanoff blocking procedure of recursively averaging over spins as $\phi(x)=$ $b_{k}\left[\phi_{0}\right](x)=\int_{y} f_{k}(x-y) \phi_{0}(y)$ where $f_{k}$ is a smearing function that operates the average. The Kadanoff procedure repeats this averaging over and over in order to go from the UV to the infrared. We give an explicit example of this in Sec. 2.3 .4 with the central limit theorem. As such, at each step or scale $k$ we obtain a new field $\phi$ and as a result a new action depending on this field $\phi$. Indeed, we may write 35, 36:

$$
\begin{equation*}
e^{-V_{k, \Lambda}[\phi]}=\int D \phi_{0} \delta\left[\phi-b_{k}\left[\phi_{0}\right]\right] e^{-V_{\Lambda, \Lambda}\left[\phi_{0}\right]} \tag{2.3.32}
\end{equation*}
$$

Differentiating with respect to $k$ we may identify $\Theta_{k, \Lambda}$ as

$$
\begin{equation*}
\Theta_{k, \Lambda} e^{-V_{k, \Lambda}[\phi]}=\int D \phi_{0} \delta\left[\phi-b_{k}\left[\phi_{0}\right]\right] \partial_{k} b_{k}\left[\phi_{0}\right] e^{-V_{\Lambda, \Lambda}\left[\phi_{0}\right]} \tag{2.3.33}
\end{equation*}
$$

which indeed is simply a change of variables.
However, Eq. 2.3 .28 is not the equation we use to perform practical approximations within the NPRG framework as $\Psi_{k, \Lambda}=e^{-V_{k, \Lambda}(\Phi)}$ is non local in the fields even in the UV bare theory as it is the exponential

[^15]of an integral over space. Instead, in order to perform approximations such as the derivative expansion of Sec.(2.4.3) a better starting point is an equation on $V_{k, \Lambda}$ that is directly the integral of a local density function in the UV and is quasi-local along the RG flow, at least above the critical temperature [37, 38].

Doing this we obtain the Polchinski equation from Eq. $\sqrt{2.3 .29}$ by expanding out the last derivative with respect to $\Phi$ and simplifying the exponential factors which gives us, in 0 dimensions:

$$
\begin{equation*}
\partial_{k} V_{k, \Lambda}(\Phi)=-\partial_{k} \Delta_{k}\left(\partial_{\Phi} \partial_{\Phi} V_{k, \Lambda}(\Phi)+\partial_{\Phi} V_{k, \Lambda}(\Phi) \partial_{\Phi} V_{k, \Lambda}(\Phi)\right) . \tag{2.3.34}
\end{equation*}
$$

This equation is mainly useful for theoretical considerations such as proving renormalizability of theories [39]. However, the lowest order approximation, called the local potential approximation which consists of discarding all momentum dependence in $V_{k, \Lambda}$, is indeed useful for approximations. Indeed, this approximation will be used in section (3.1.1).

In general, however, one generally uses the flow obtained from the Legendre transform of Polchinski equation which we will call $\Gamma$ flow in this thesis and which has proven to be better suited to approximations. The $\Gamma$ flow may be obtained from the Polchinski equation using a Legendre transformation [40]:

$$
\begin{equation*}
U_{k, \Lambda}[\varphi]=V_{k, \Lambda}[\phi]-\frac{1}{2}(\varphi-\phi) \Delta_{k}^{-1}(\varphi-\phi) . \tag{2.3.35}
\end{equation*}
$$

The effective action $\Gamma_{k, \Lambda}$ is then obtained from $U_{k, \Lambda}$ by including a Gaussian action. Taking into account all of these changes of variables and definitions on arrives at the following $\Gamma$ flow [40, 41]:

$$
\begin{equation*}
\partial_{k} \Gamma_{k}=\frac{1}{2} \operatorname{Tr}\left(\frac{\partial_{k} R_{k}}{\Gamma_{k}^{(2)}+R_{k}}\right) \tag{2.3.36}
\end{equation*}
$$

where we return to $d$ dimensions with $\Gamma_{k}^{(2)}=\frac{\delta^{2}}{\delta \phi \delta \phi} \Gamma$ and $\operatorname{Tr}=\sum_{\alpha}\langle\alpha| .|\alpha\rangle$ is a trace over both internal latin indices, such as field labels $\phi_{a}$, and spatial indices. $R_{k}$ is related to $\Delta_{k}$ but the precise relationship is unimportant and we may view $R_{k}$ as a new function. We have also replaced $\Gamma_{k, \Lambda}$ by $\Gamma_{k}$ as the term $\partial_{k} R_{k}$ will be chosen such that high-momentum contributions where $q^{2} \gg k^{2}$ are suppressed and so the cut-off $\Lambda$ may be sent to infinity at this point. Moreover, the $R_{k}$ in the denominator will serve as an infrared cut-off function acting as a mass that forbids the system from entering the massless regime for non zero $k$. We have thus arrived at the main equation of this thesis which shows precisely the manner in which physics, in general, both statistical and quantum, evolve with scales.

Let us remind the reader however that our objective in this section was not to derive Eq.(2.3.36) in a prolonged or complicated manner but rather to uncover the structure of the RG flow. We thus summarize here what was uncovered from this derivation of the flow equations as opposed to the shorter one in Appendix D

1. A Wilsonian mapping between scales showing explicitly the relationship between the RG, transitivity and FSS (see Appendix Efor FSS derivation).
2. The RG prescription $V_{k, \Lambda}=-\log \left(\int D \phi \mu_{k, \Lambda}(\phi) e^{-V_{\Lambda, \Lambda}(\phi+\Phi)}\right)$ was made explicit.
3. The underlying heat equation making explicit that the RG flow is diffusive in nature.
4. Relationship with the original Kadanoff blocking procedure.
5. An explanation as to why the simpler heat equation is not used in practical applications.
6. The Polchinski equation that will be used in Sec.(3.1.4).

The first point is important in understanding how the RG equations are related to the much more general framework of FSS and how the RG is applied elsewhere for example in real space Monte Carlo simulations or in artificial intelligence through the many scales of neuron layers $42-53$. This will also allow us in Sec. 2.4 .2 ) to study simple examples where the errors in approximation schemes are made clearer. The second point allows us to see more explicitly the link with RG prescriptions in perturbative field theory. The third point will be discussed in Sec. 2.3 .4 where we will discuss universality and the entropic nature of the RG flow. The fourth is conceptually interesting as it shows that the theory is not changed and rather it is only the effective descriptions used that changes.

In the following sections we will focus more on the $\Gamma$ flow which may be interpreted in many ways.

### 2.3.2 The $\Gamma$ flow as an interpolation function

The $\Gamma$ flow has the peculiarity of being both a renormalization group flow and a continuous deformation between theories ${ }^{26}$. Indeed, it is often viewed as an interpolation between the UV and the IR as the solution of the $\Gamma$ flow may be written as (see Appendix $D$ for explanation) ${ }^{27}$

$$
\begin{equation*}
\exp \left(-\Gamma_{k}[\varphi]\right)=\int D \phi e^{-S[\phi+\varphi]-\frac{\phi \cdot R_{k} \cdot \phi}{2}+\frac{\delta \Gamma_{k}[\varphi]}{\delta \varphi} \cdot \phi} \tag{2.3.37}
\end{equation*}
$$

Hence, if we define $R_{k}=k^{2} r\left(q^{2} / k^{2}\right)$ then for $k \rightarrow \infty$, since $r(0)>0$, we have $R_{k} \rightarrow \infty$ such that $\frac{e^{-\frac{\phi \cdot R_{k} \cdot \phi}{2}}}{\int D \phi e^{-\frac{\phi \cdot R_{k} \cdot \phi}{2}}} \rightarrow \delta(\phi)$ where $\delta$ is the delta Dirac functional 28 and thus we obtain $\Gamma_{\infty}=S$. In the opposite limit for $k \rightarrow 0$ we have $R_{k} \rightarrow 0$ thereby retrieving the effective action of the original problem under study. The $\Gamma$ flow is thus a rather interesting (homotopy) transformation between UV and IR as a simpler transformation could be $\exp \left(-\Gamma_{t}[\varphi]\right)=(1-t) \exp (-S[\varphi])+t \exp (-\Gamma[\varphi])$ for $t \in[0,1]$. However such an interpolation between the UV action $S$ and the infrared action $\Gamma$ would not involve a trace over momenta as in the $\Gamma$ flow and as such would not be useful for approximations involving locality such as the derivative expansion of Sec. 2.4.3. To obtain such a trace it is thus necessary to insert an operator into the path integral as is the case here via $e^{-\frac{\phi \cdot R_{k} \cdot \phi}{2}}$ or the probability measure $\mu_{k, \Lambda}$ in the Polchinski case.

We also note that the $\Gamma$ flow need not correspond to an interpolation between UV and IR. For example, one can choose the regulator to depend on dimensionless parameter $s$ such that at $s=0 R_{s}$ cancels the kinetic term thereby leading to a one body problem. In this case the starting point is exactly solvable. We may then consider that at $s=1$ we have $R_{s}=0$ thereby retrieving the original infrared theory. This approach was used in [54 to obtain non universal quantities such as the critical temperature of a lattice model.

### 2.3.3 The $\Gamma$ flow as an RG improved one loop calculation

One may obtain a one loop expression for the effective action by means of a saddle point approximation in the limit of small $h$. Making the $h$ dependence in Eq. 2.3.37) explicit we have:

$$
\begin{equation*}
\exp \left(-\Gamma_{k}[\varphi] / h\right)=\int D \phi e^{-\frac{1}{h}\left(S[\phi+\varphi]+\frac{\phi \cdot R_{k} \cdot \phi}{2}-\frac{\delta \Gamma_{k}[\varphi]}{\delta \varphi} \cdot \phi\right)}=\int D \phi e^{-\frac{1}{h} \widetilde{S}[\phi, \varphi]} . \tag{2.3.38}
\end{equation*}
$$

In the limit $h \rightarrow 0$, which corresponds to the limit of small fluctuations, using the saddle point approximation, we may expand about the fluctuation $\phi$ up to quadratic order as $\phi=0$ corresponds to a minimum of $\widetilde{S}$. After integrating the resulting Gaussian function and using the identity $\operatorname{det}\left(e^{A}\right)=e^{\operatorname{Tr}(A)}$ we obtain:

$$
\begin{equation*}
\Gamma_{k}[\varphi]=S[\varphi]+\frac{1}{2} h \operatorname{Tr}\left(\log \left(S^{(2)}[\varphi]+R_{k}\right)\right) \tag{2.3.39}
\end{equation*}
$$

This expression is in general UV divergent and requires regularization. However, we may bypass this step by subtracting $\Gamma_{k^{\prime}}$ as we did for the charge in Sec. 2.2 .3 . This then leads to:

$$
\begin{equation*}
\Gamma_{k}[\varphi]-\Gamma_{k^{\prime}}[\varphi]=\frac{1}{2} h \operatorname{Tr}\left(\log \left(\frac{S^{(2)}[\varphi]+R_{k}}{S^{(2)}[\varphi]+R_{k^{\prime}}}\right)\right) \tag{2.3.40}
\end{equation*}
$$

In fact we may take $k^{\prime}$ arbitrarily close to $k$ which then leads to the differential equation:

$$
\begin{equation*}
\partial_{k} \Gamma_{k}[\varphi]=\frac{1}{2} h \operatorname{Tr}\left(\frac{\partial_{k} R_{k}}{S^{(2)}[\varphi]+R_{k}[\varphi]}\right) \tag{2.3.41}
\end{equation*}
$$

[^16]However, even after considering the rescaling operator this equation cannot be made autonomous as it depends explicitly on the UV action $S=\Gamma_{k=\Lambda}$. Hence, this equation must be considered as a local approximation around the UV scale $\Lambda$ where the above equation corresponds to the non linear part of the FSS generator ${ }^{29}$,

If we consider the re-scaled variable $\widetilde{\Gamma}_{t}$ with the usual definition $\frac{k}{\Lambda}=e^{t}$ we have:

$$
\widetilde{\Gamma}_{t+\delta t}=\widetilde{\Gamma}_{t}+\mathcal{L}\left(\widetilde{\Gamma}_{t}\right)+\frac{1}{2} \delta \operatorname{th} \operatorname{Tr}\left(\frac{\partial_{t} r_{t}}{\widetilde{S}^{(2)}[\varphi]+r_{t}}\right)
$$

where $\mathcal{L}$ is the rescaling operator. If our initial condition is $\widetilde{\Gamma}_{t}=S$ then this equation can be interpreted as the action of the FSS group at $k=\Lambda$ as:

$$
\left(t+\delta t, \widetilde{\Gamma}_{t+\delta t}\right)=\left(t, \widetilde{\Gamma}_{t}\right)+\delta t\left(1, K\left(\widetilde{\Gamma}_{t}\right)=\mathcal{L}\left(\widetilde{\Gamma}_{t}\right)+\frac{1}{2} \delta \operatorname{th} \operatorname{Tr}\left(\frac{\partial_{t} r_{t}}{\widetilde{\Gamma}_{t}^{(2)}[\varphi]+r_{t}}\right)\right)
$$

where $K$ is the generator of the FSS group. In turn, as the FSS group allows us to extend any local formula to a global one by simply iterating the FSS group, the functional form of $K$ does not depend on $t$ and the FSS group improved one loop consists in replacing $\widetilde{S}=\widetilde{\Gamma}_{t}$ from which we obtain the $\Gamma$ flow in its dimensionless form 30 .

Moreover, as the $\Gamma$ flow has a one loop structure one may easily obtain one-loop results by performing perturbation theory on the $\Gamma$ flow and obtain the further loops by inserting the approximation from the previous order into the exact equation in an iterative fashion $[55-60$.

Finally, once a regulator has been chosen to render the integral finite one is allowed to calculate physics in arbitrary dimension and in particular it is possible to calculate critical exponents directly at $d=3$ rather than performing the $\epsilon$-expansion at $4-\epsilon$ dimensions of perturbative field theory or having to resort to re-summations 31 .

### 2.3.4 Diffusive nature of the RG flow

In Sec. 2.3.1 we saw that the RG equation can be set into a heat equation thereby showing its diffusive nature. A concrete example of this may be given by the Gaussian fixed-point through the central limit theorem which we now show using Kadanoff blocking.

Consider $N$ identical and independent stochastic variables $x_{k}$, with probability distribution $p$, where each variable $x_{k}$ is associated to a stochastic object at position $k$ on a one dimensional lattice. The Kadanoff RG blocking procedure applied to this problem consists in defining a hierarchy of probability distributions $p_{l}$ as we move from the microscopic variables $x_{k}$ to macroscopic variables $x_{k, l}$. This can be done by defining $p_{0}=p, x_{k, 0}=x_{k}$ then successively taking averages of nearest neighbors. More precisely at level $l=1$ we take $\widehat{x}_{k, 1}=\frac{x_{k}+x_{k+1}}{2}$ where the index $k$ in $x_{k, 1}$ only takes macroscopic steps $k=1,3,5, \ldots$ as opposed to the microscopic steps of $x_{k}$. We then "rescale" $k$ such that $x_{k, 1}=\widehat{x}_{2 k-1,1}$. The probability distribution of $x_{k, 1}$ is then $p_{1}=L . p \underset{\text { def }}{=} p * p$ where $*$ represents a convolution product. At level $l=2$ we iterate this procedure on the "blocks" $x_{k, 1}$ defining $\widehat{x}_{k, 2}=\frac{x_{k, 1}+x_{k+2,1}}{2}=\frac{x_{k}+x_{k+1}+x_{k+2}+x_{k+3}}{4}$ where we may once more rescale $k$ such that we may define $x_{k, 2}$ with probability distribution $p_{2}=L . p_{1}=p_{1} \otimes p_{1}=p \otimes p \otimes p \otimes p$.

Zooming out by taking larger and larger levels $l$, the probability distribution of $p_{l}$ for $l \rightarrow \infty$ tends towards a fixed-point of the $L$ operator which is the Gaussian function. Hence, regardless of the microscopic probability distribution $p$, as one zooms out taking averages of the stochastic variables, the probability distribution of the average becomes Gaussian. This is regardless of the microscopic probability distribution $p$ as long as it verifies the rather general set of hypotheses of the central limit theorem.

More concretely for condensed matter systems, if we consider a system of nearest neighbor interactions, then at sufficiently high temperatures the system is weakly correlated. Hence, we expect the system to be

[^17]composed of small blocks of the size of the correlation length $\xi$ where the components of the system remain correlated. Each block may then be regarded as a random variable and if the system is homogeneous these random variables fit the criteria of the central limit theorem which then allows us to predict a Gaussian probability distribution in the infrared limit without any information on the type of interactions at hand. This is a somewhat trivial renormalization flow which leads to the high temperature fixed-point where the system is completely decorrelated. A more interesting renormalization flow is the flow between different scale invariant theories. In this case the system loses information as well but the probability distribution is not necessarily Gaussian as scale invariant theories have infinite variance and thus the central limit theorem does not apply. Hence, the renormalization group can be seen as a generalization of the central limit theorem. This is most easily seen in two dimensions via the c-theorem. The c-theorem states that an object called the central charge decreases along the renormalization flow. This object in turn measures, in some sense, the number of degrees of freedom in the system. Hence, as we expect the information content of a theory to diminish with the number of degrees of freedom this is another example of the diffusive nature of the RG flow ${ }^{32}$

For example, a system of $N$ free Majorona fermions has central charge $\mathrm{N} / 2$ and hence via bosonization, where two free Majorona fermions can be seen as a system of one boson we may deduce that a system of N free bosons has central charge N . This then leads to the fact that a theory of one massless free boson can, through the c-theorem, lead to a theory of one free massless majorona fermion via the renormalization group as one would have $c_{U V}-c_{I R}=1-1 / 2=1 / 2>0$. In fact, this situation does arise with uni-axial ferromagnets, that is, the Ising model. Both the massless boson and massless fermion theory are scale invariant and correspond to adjusting the temperature of the system to the critical temperature $T=T_{c}$ but the boson fixed-point is unstable whereas the free fermion one is stable.

As the RG flows from smaller to larger scales, this dissipative effect is a fortunate feature when working on statistical mechanics as one does not need to know great detail of the UV physics. However, this poses a real problem when the objective is to instead decipher the UV from the IR. Indeed, moving in the opposite direction one never knows when the flow of the couplings will lead to "new physics".

However, everything is not lost along this river flow as, for example, the symmetries of the initial UV problem are also conserved as long as they act linearly on the fields ${ }^{34}$. Moreover, not all quantities are universal as, for example the critical temperature at which a phase transition happens depends on the physics at the UV scale. Examples of universal quantities instead are critical exponents or ratios of correlation amplitudes or masses. Fortunately however, non universal quantities can and have 54 been obtained in the framework of the non perturbative renormalization group (NPRG).

### 2.3.5 Phase transition and stability analysis

Now that we have obtained an exact equation the question is : how can we obtain quantitative results ? Perhaps the simplest quantitative results one can obtain are critical exponents. There are quite a few critical exponents but fortunately they are not all independent and in the simplest type of second order transition there are just two independent exponents commonly obtained from experiments ${ }^{35}$. The two which are commonly discussed are the exponents $\nu$ and $\eta$. Both exponents are related to the connected correlation function:

$$
\begin{equation*}
<\phi(r) \phi(0)>_{c}=<\phi(r) \phi(0)>-<\phi(r)><\phi(0)> \tag{2.3.42}
\end{equation*}
$$

[^18]As mentioned in Sec. 2.1.3 $<\phi(r) \phi(0)>_{c}$ decays exponentially like $e^{-r / \xi}$ at large distances where $\xi$ is the correlation length representing the typical size over which spins are correlated. For a scale invariant theory $\xi$ is then infinite. As $T \rightarrow T_{c}$ the correlation length grows and it diverges right at $T_{c}$ making the system scale invariant. The correlation length then behaves as $\xi \propto\left|T / T_{c}-1\right|^{-\nu}$. What is remarkable is that $\nu$ is universal in the sense that it depends only on the symmetries and dimensionality of the system when interactions are short ranged.

The second critical exponent is obtained at $T=T_{c}$. As we stated above, $\xi$ is infinite for $T=T_{c}$ and in this case the two point function no longer decays exponentially but rather it decays as a power law as:

$$
<\phi(r) \phi(0)>_{c} \propto r^{-d+2-\eta}
$$

where $-d+2$ would be the exponent in absence of interactions for a scaleless Gaussian theory and $\eta$ then measures the distance from "gaussianity". This last exponent may be related to the exponent that describes the divergence of the susceptibility in the neighborhood of the transition as $\chi=\left|T / T_{c}-1\right|^{-\gamma}$ where one finds the scaling relation $\gamma=(2-\eta) \nu$.

We will not need to calculate $\eta$ in this thesis so instead we focus on the exponent $\nu$. There are in fact many ways that one can extract this exponent from calculations (see appendix of 30$]$ ) but we will focus on the one linked to the so called stability matrix as it is perhaps the closest to the RG philosophy. This method then relates critical exponents to eigenvalues of a stability analysis. In general however, a stability analysis can be of interest independently of critical exponents. To see this let us consider, for the sake of simplicity, the following RG flow ${ }^{36}$ :

$$
\begin{equation*}
\beta\left(\lambda_{t}\right)=\frac{\mathrm{d} \lambda_{t}}{\mathrm{dt}}=\frac{1}{N} \lambda_{t}\left(-2 \alpha+12 \lambda_{t}-\pi^{2} \lambda_{t}^{2} / 2\right) \tag{2.3.43}
\end{equation*}
$$

where $\lambda_{t}$ is the coupling constant of the model and $N$ and $\alpha$ are two parameters to be defined in Sec.(3.1.4). This RG flow will turn out to be important in the following of this thesis but we consider it for now as a toy model to illustrate the notion of stability of a fixed-point which will be important in calculating $\nu$ later on.

First, let us notice that for $N=\infty$ we have $\beta\left(\lambda_{t}\right)=0$ and thus the coupling $\lambda$ does not renormalize. In this case we say that the coupling is exactly marginal and it then plays the role of a parameter, as $\alpha$ or $N$. An exactly marginal direction implies a line of fixed-points. Let us then look at the less trivial case when $N$ is finite.

A fixed-point solution to Eq. 2.3 .43 is obtained by setting the beta function to 0 . This leads to an equation with three solutions $\lambda^{*}=0, \lambda_{-}^{*}(\alpha), \lambda_{+}^{*}(\alpha)$ with $\lambda_{-}^{*}<\lambda_{+}^{*}$. When obtaining stationary solutions to an autonomous equation one usually also studies the stability of these solutions with respect to perturbations. We then take $\lambda_{t}$ infinitesimally close to $\lambda^{*}$ meaning we take $\lambda_{t}=\lambda^{*}+\delta \lambda_{t}$ and neglect terms on the order of $\delta \lambda^{2}$ in the beta function. In other words, we linearize the beta function as $\beta\left(\lambda_{t}\right)=\beta\left(\lambda^{*}+\delta \lambda_{t}\right)=$ $\beta\left(\lambda^{*}\right)+\delta \lambda_{t} \frac{\mathrm{~d} \beta\left(\lambda^{*}\right)}{d t}=\delta \lambda_{t} \frac{\mathrm{~d} \beta\left(\lambda^{*}\right)}{d \lambda}$. This means that locally around the fixed-points we have:

$$
\begin{equation*}
\frac{\mathrm{d} \delta \lambda_{t}}{\mathrm{~d} t}=\delta \lambda_{t} \frac{\mathrm{~d} \beta\left(\lambda^{*}\right)}{d \lambda} \underset{d e f}{=} \delta \lambda_{t} y\left(\lambda^{*}\right) \tag{2.3.44}
\end{equation*}
$$

The solution to this linear ordinary differential equation is $\delta \lambda_{t}=\delta \lambda(0) e^{y t}=\left(\frac{k}{\Lambda}\right)^{y}$. Hence as $k / \Lambda<1$ or equivalently $t<0, \delta \lambda_{t}$ decreases if $y>0$ and increases when $y<0$. When the fixed-point (FP) is unstable, where does the flow go? In general such a question is not always simple and requires computing a map of the $\beta$ functions as a function of their couplings. However, in this simple one dimensional case it is rather straightforward. First, let us calculate $y$ for the three fixed-points. We have:

$$
\begin{equation*}
y\left(\lambda^{*}\right)=-2 \alpha+24 \lambda^{*}-3 \pi^{2}\left(\lambda^{*}\right)^{2} / 2 \tag{2.3.45}
\end{equation*}
$$

For $\lambda^{*}=0$, we have $y(0)=-2 \alpha$. Hence for $\alpha>0$, which will be the only case of interest when we discuss the physics of this model, we have an unstable fixed-point as $y<0$. Taking a positive initial perturbation, the coupling then increases to stronger values and thus towards $\lambda_{-}^{*}$ with $0<\lambda_{-}^{*}<\lambda_{+}^{*}$. Schematically, we

[^19]may represent this flow as $\left(0 \longrightarrow \lambda_{-}^{*}\right)$. At this point it is perhaps clear that we must have $y\left(\lambda_{-} *\right) \geq 0$ such that the flow may converge towards this FP. If $y\left(\lambda_{-}^{*}\right)$ is non zero, then it must also be stable in the opposite direction, and thus we have: $\left(0 \longrightarrow \lambda_{-}^{*} \longleftarrow \lambda_{+}^{*}\right){ }^{37}$. We conclude that $\lambda_{-}^{*}$ is a stable fixed-point while $\lambda^{*}=0$ and $\lambda^{*}=\lambda_{+}^{*}$ are unstable fixed-points that, after perturbation, converge either to $\lambda_{-}^{*}$ or run off to infinity. Another interesting aspect of this equation is that $\lambda_{-}^{*} \rightarrow 0$ when $\alpha \rightarrow 0$ as an extra factor of $\lambda$ can be factorized from $\beta$. This shows that this FP collapses with the gaussian FP in this limit. This is only possible between a stable fixed-point and unstable fixed-point in order to be compatible with the RG flow [68]. This is an important fact which will be used later on in Sec.(3.1.4).

A somewhat more complicated flow is given by the following approximate flow for the $\phi^{4}$, or equivalently Ising, model in $d$ dimensions:

$$
\begin{align*}
& \frac{\mathrm{d} \kappa}{\mathrm{~d} t}=\beta_{\kappa}(\kappa, \lambda)=-(d-2) \kappa+\frac{1}{(1+2 \kappa \lambda)^{2}} \\
& \frac{\mathrm{~d} \lambda}{\mathrm{~d} t}=\beta_{\lambda}(\kappa, \lambda)=-(4-d) \lambda+\frac{6 \lambda^{2}}{(1+2 \kappa \lambda)^{3}} \tag{2.3.46}
\end{align*}
$$

This system was obtained by taking the following approximation for the effective potential :

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int_{x}\left(\frac{\bar{\lambda}_{k}}{2}\left(\frac{1}{2} \phi(x)^{2}-\bar{\kappa}_{k}\right)^{2}+\frac{1}{2} \partial \phi(x) \partial \phi(x)\right) \tag{2.3.47}
\end{equation*}
$$

that is, we consider the usual action of $\phi^{4}$ theory and we take all couplings to depend on the scale $k$. This is similar to what was done with the $\beta$ function of the charge in Sec.(2.2.6) but in general along the RG flow all couplings not protected by symmetry are generated and thus this is a rather crude approximation.

To obtain a fixed-point we then rescale the dimensionful $(\bar{\kappa}, \bar{\lambda})$ to the dimensionless $(\kappa, \lambda)$ as $\lambda_{k}=$ $k^{-(4-d)} \bar{\lambda}_{k}$ and $\kappa_{k}=k^{-(d-2)} \bar{\kappa}_{k}{ }^{38}$. When the theory is scale invariant a fixed-point solution is then obtained as the only length scale $k$ has been factored out.

The system in Eq. (2.3.46) has 2 fixed-points given by $\vec{\beta}=\left(\beta_{\kappa}, \beta_{\lambda}\right)=0$. The first is the Gaussian fixed-point with :

$$
\begin{equation*}
(\kappa, \lambda)=\left(\frac{1}{d-2}, 0\right) \tag{2.3.48}
\end{equation*}
$$

and the second is a non trivial fixed-point corresponding to the Wilson-Fisher (WF) fixed-point from which one obtains the scaling features of critical phenomena that have a $\phi \rightarrow-\phi$ symmetric potential. For example, this is the case of the continuum limit of the Ising model. This WF fixed-point is given by :

$$
\begin{align*}
& \kappa=\frac{4(5-2 d)^{2}}{9(d-2)^{3}}  \tag{2.3.49}\\
& \lambda=\frac{9(4-d)(d-2)^{3}}{16(2 d-5)^{3}} .
\end{align*}
$$

Let us then notice that $\lambda$ is small for $d=4-\sqrt{39}$ In this case, it is perturbatively under control and in fact one may retrieve the exact one loop $\beta$ function for $\lambda$ in this limit. The system in Eq. $(2.3 .46$ ) is however non perturbative when $d$ is no longer close to 4 . In particular, within the NPRG framework we do not expand the denominator in $(1+2 \kappa \lambda)^{-1}$. In fact this expression $(1+2 \kappa \lambda)^{-1}$ is important as in terms of dimensionful

[^20]variables it corresponds to $\left(1+2 \bar{\kappa} \bar{\lambda} k^{-2}\right)^{-1}$. One may then check that $\bar{\kappa} \bar{\lambda}$ gives the curvature of the potential at the minimum $\bar{\kappa}$. Hence, $\left(1+2 \bar{\kappa} \bar{\lambda} k^{-2}\right)^{-1}$ becomes small when the curvature, or equivalently, the mass squared, is larger than $k^{2}$. This means that fluctuation modes with mass larger than the scale $k$ at which we observe decouple from the theory as they should.

We may then obtain a flow diagram for this system of autonomous equations by computing the vector $\vec{\beta}=$ $\left(\beta_{k}, \beta_{\lambda}\right)$ for every point $(\kappa, \lambda)$ as in Fig. 2.3 .2 . In this figure we notice that there is a flow from the Gaussian fixed-point towards the Wilson-Fisher fixed-point along a critical line. Along this critical line the Gaussian fixed-point is repulsive whereas the Wilson-Fisher fixed-point is attractive. However, in an orthogonal direction both fixed-points are repulsive. This implies that the system has to be fine-tuned along this orthogonal direction to find the critical line. Once this critical line is found the RG flow will naturally converge towards the Wilson-Fisher fixed-point. In the case of the Gaussian fixed-point however both directions have to be fine-tuned as it is repulsive in both directions. This means that the Gaussian fixed-point has more than one relevant eigendirection in which case we say that it is multicritical. In principle, one needs to consider more couplings than our simple truncation at hand to see whether there are more relevant directions. For example, one could include a further term in the potential as $U_{k}=\frac{\bar{\lambda}_{k}}{2}\left(\frac{1}{2} \phi^{2}-\bar{\kappa}_{k}\right)^{2}+\frac{\bar{\tau}_{k}}{6}\left(\frac{1}{2} \phi^{2}-\bar{\kappa}_{k}\right)^{3}$. However, it is known that the Wilson-Fisher fixed-point has only one relevant eigendirection. This implies that adding the term $\frac{\bar{\tau}_{\Lambda}}{6}\left(\frac{1}{2} \phi^{2}-\bar{\kappa}_{\Lambda}\right)^{3}$ in the initial condition of the flow will not change qualitatively the flow around the Wilson-Fisher fixed-point and in particular the number of relevant directions will not change. This does not imply however that considering $U_{k}=\frac{\bar{\lambda}_{k}}{2}\left(\frac{1}{2} \phi^{2}-\bar{\kappa}_{k}\right)^{2}+\frac{\bar{\tau}_{k}}{6}\left(\frac{1}{2} \phi^{2}-\bar{\kappa}_{k}\right)^{3}$ at all $k$ does not change anything as $U_{k}$ is non-polynomial from the first RG step and thus any truncation to a polynomial subspace constitutes an approximation. In the exact theory, in the space of all couplings, the fact that the Wilson-Fisher fixed-point has only one relevant eigendirection implies that the critical line becomes a critical surface of co-dimension 1. In other terms, it is sufficient to vary a single coupling orthogonal to the critical surface as an initial condition in the UV to obtain in the IR the Wilson-Fisher fixed-point.

In general within an experiment, these different couplings depend on the experimental setup. To reproduce an experiment exactly one has to have the same system but also the same environment. We may denote the set of all environmental factors as the set $\left\{e_{n}\right\}_{n}$. The couplings $\bar{\lambda}_{k}$ within the theory depend on the environmental factors as $\bar{\lambda}_{k}\left(\left\{e_{n}\right\}_{n}\right)$. Thus, varying $\left\{e_{n}\right\}_{n}$ is equivalent to varying the couplings $\bar{\lambda}_{k}$. That there is only one coupling to vary in the initial condition of the flow then implies that there is only one external control parameter to vary. In the Ising model at zero magnetization this is the temperature and thus one may interpret the existence of a critical surface of codimension 1 as the experimental fact that it is sufficient to vary only the temperature of the system ignoring all other environmental factors.

One can also verify using dimensional analysis as in Sec. 3.1 .2 .2 that the Gaussian fixed-point has exactly two relevant eigendirections for $3<d<4$ in which case it is said to be tricritical. This then implies that there is also a critical surface for this fixed-point but it has codimension 2. Hence, given an experiment, one has to fine-tune two external parameters to find such a fixed-point. Finding the zero of a function in 2 dimensions is substantially more difficult than in one dimension where it is possible to use dichotomy. Because of this it is more difficult to find multicritical fixed-points and they are also significantly rarer while not unheard of. In Sec. 3.1 .2 we will give examples of systems that have a tricritical point in their phase diagram.

To compute these eigendirections and eigenvalues it is sufficient to linearize the system around a fixedpoint solution. In the case of the simplified model in Eq. 2.3.46, this leads to:

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \kappa=\frac{\mathrm{d}}{\mathrm{~d} t}\left(\kappa^{*}+\delta \kappa\right)=\frac{\mathrm{d}}{\mathrm{~d} t} \delta \kappa=\delta \kappa \partial_{\kappa} \beta_{\kappa}\left(\kappa^{*}, \lambda^{*}\right)+\delta \lambda \partial_{\lambda} \beta_{\kappa}\left(\kappa^{*}, \lambda^{*}\right)  \tag{2.3.50}\\
& \frac{\mathrm{d}}{\mathrm{~d} t} \kappa=\frac{\mathrm{d}}{\mathrm{~d} t}\left(\lambda^{*}+\delta \lambda\right)=\frac{\mathrm{d}}{\mathrm{~d} t} \delta \lambda=\delta \kappa \partial_{\kappa} \beta_{\lambda}\left(\kappa^{*}, \lambda^{*}\right)+\delta \lambda \partial_{\lambda} \beta_{\lambda}\left(\kappa^{*}, \lambda^{*}\right)
\end{align*}
$$

Taking $\widetilde{g}_{1}=\kappa, \widetilde{g}_{2}=\lambda$, the matrix $\partial_{\widetilde{g}_{i}} \beta_{\widetilde{g}_{j}}$ is called the stability matrix and its eigenvectors and eigenvalues correspond to the generalization of the previous 1 dimension case given by Eq. 2.3.43). In particular, once more, the relevant directions are those for which the associated eigenvalues are negative. Computing the eigendirections for the Gaussian and Wilson-Fisher fixed-point we indeed retrieve the RG flow of Fig. 2.3 .2 in the vicinity of these two fixed-points.

Let us then generalize this procedure. We recall that the $\Gamma$ flow is a functional equation with an infinite number of degrees of freedom and as such it is far more complex than our previous examples. We will see


Figure 2.3.2: Flow diagram of $\phi^{4}$ model in $d=3$. The arrows indicate the direction of the flow as given by $\vec{\beta}=\left(\beta_{\kappa}, \beta_{\lambda}\right)$. Both the Gaussian and Wilson-Fisher fixed-points are given by $\vec{\beta}=\overrightarrow{0}$ and are indicated by the blue dots. The Gaussian is located at the bottom with coordinates $(\kappa, \lambda)=(1,0)$ while the Wilson-Fisher fixed-point is located at $(\kappa, \lambda) \simeq(0.44,0.56)$. The color code indicates the norm of the vector $\vec{\beta}$ and we thereby see that there exists a critical line given in red where the $\beta$ function is very small. Along this critical line the Gaussian fixed-point is repulsive whereas the Wilson Fisher fixed-point is attractive. This plot was obtained by the function StreamDensityPlot in Mathematica.
in Sec.(2.4) that it is often possible to set up a hierarchy between these degrees of freedom and to consider only the leading ones in the framework of approximations.

The flow of the potential is obtained by taking a uniform field $\phi(x)=\varphi \in \mathbb{R}$ as then all derivative terms of the action are removed and we have $\Gamma[\varphi]=\mathcal{V} U(\varphi)$ where $U$ is by definition the potential and $\mathcal{V}$ is the volume of the system. We may then Taylor expand this potential as $U=\sum_{i} \lambda_{i} \phi^{i}$, or consider $U=\sum_{i} h_{i} L_{i}(\phi)$ where $L_{i}$ is a Legendre polynomial or more generally $U=\sum_{i} r_{i} f_{i}(\phi)$ where $f_{i}$ is a basis function to be specified and $r_{i}$ represents a coordinate set on the space spanned by these $f_{i}$. However, as we shall see in Sec.(2.4.3), one generally considers more than the coordinate set of $U$. For example, one of the simplest approximations consists in considering:

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int_{x}\left(U_{k}(\phi)+\frac{Z_{k}(\phi)}{2} \partial \phi \partial \phi+O\left(\partial^{4}\right)\right) \tag{2.3.51}
\end{equation*}
$$

This is an expansion in powers of the derivatives $\partial^{2}$ where $O\left(\partial^{4}\right)$ means that we omit terms involving four derivative terms or more such as $\partial^{2} \phi \partial^{2} \phi$. This expansion will be justified in Sec.(2.4.3). For now it suffices to remark that $Z$ can then be decomposed into a basis set as well. Thus, in order to keep the following discussion general we decompose $\Gamma_{k}$ as:

$$
\begin{equation*}
\Gamma_{k}=\sum_{i} g_{i} \gamma_{i} \tag{2.3.52}
\end{equation*}
$$

where in Eq. (2.3.51) the coordinates $g_{i}$ correspond to a coordinate basis of $U$ and $Z$ but we also allow general operators $\gamma_{i}$ such as basis functions of $\int_{x} \partial \phi Z\left(\partial^{2}\right) \partial \phi$. If we then consider a hierarchy scheme where neglecting some couplings $g_{i}$ leads to negligible error, it is in general possible to truncate the infinite sum $\Gamma_{k}=\sum_{i} g_{i} \gamma_{i}$ into a finite one. This is the case for example in Eq.(2.3.51) when a finite basis $f_{i}$ is chosen to compute $(U, Z)$. Our infinite dimensional theory space is then reduced to a finite dimensional approximation space. Such a scenario is typical within perturbation theory and this indeed extends to the non perturbative
case. This finite dimensional coordinate system $\left\{g_{i}\right\}_{i}=\left\{g_{1}, g_{2}, \ldots\right\}$ then allows us to project the functional form of the $\Gamma$ flow onto a finite set of real valued coupled differential equations as:

$$
\begin{equation*}
\frac{\mathrm{d} g_{i}}{\mathrm{dt}}=F_{i}\left(\left\{g_{j}\right\}_{j}\right) \tag{2.3.53}
\end{equation*}
$$

Once an approximation scheme is chosen we also need to work with dimensionless variables to find a fixedpoint as was discussed in Sec. 2.3 . For example, we may rescale the potential as $U_{k}(\phi)=k^{d} \widetilde{U}_{k}(\widetilde{\phi})=$ $k^{d} \widetilde{U}\left(\phi / k^{\Delta_{\phi}(k)}\right) 4^{40}$ which is equivalent to rescaling the couplings in terms of their dimensions as $\lambda_{i}(k)=$ $k^{\Delta_{\lambda_{i}}(k)} \tilde{\lambda}_{i}(k)$ such that we have $\lambda_{m} \phi^{m}=k^{d} \widetilde{\lambda}_{m} \widetilde{\phi}^{m}$.

Once this rescaling is performed we will have a new set of equations on the dimensionless quantities $\widetilde{g}_{i}$ defined by $g_{i}=k^{\Delta_{i}} \widetilde{g}_{i}$. We then have :

$$
\begin{equation*}
\partial_{t} g_{i}=\partial_{t}\left(k^{\Delta_{i}} \widetilde{g}_{i}\right)=k^{\Delta_{i}}\left(\Delta_{i} \widetilde{g}_{i}+\partial_{t} \widetilde{g}_{i}\right) \tag{2.3.54}
\end{equation*}
$$

Simplifying by $k^{\Delta_{i}}$ on both sides of $\partial_{t} g_{i}=F_{i}\left(\left\{g_{j}\right\}_{j}\right)$ then leads to:

$$
\begin{equation*}
\partial_{t} \widetilde{g}_{i}=-\Delta_{i} \widetilde{g}_{i}+F_{i}\left(\left\{\widetilde{g}_{j}\right\}_{j}\right)=\beta_{i}\left(\left\{\widetilde{g}_{j}\right\}_{j}\right) \tag{2.3.55}
\end{equation*}
$$

Written in this way, when the left hand side is zero we have $F_{i}\left(\left\{\widetilde{g}_{j}^{*}\right\}_{j}\right)=\Delta_{i} \widetilde{g}_{i}^{*}$ meaning the action of the renormalization group $F_{i}$ becomes equivalent to a rescaling of $\tilde{g}_{4}^{41}$. At this point the theory becomes scale invariant. For example, in terms of the potential and for $k$ sufficiently small we have 42

$$
\begin{align*}
U_{k}(\phi) & =k^{d} \widetilde{U}_{k}(\widetilde{\phi})=k^{d} \widetilde{U}(k, \widetilde{\phi})=k^{d} \widetilde{U}(0, \widetilde{\phi})+O\left(k^{d+1}\right) \\
& =k^{d} \widetilde{U}^{*}(\widetilde{\phi})+O\left(k^{d+1}\right)=k^{d} \widetilde{U}^{*}\left(\phi / k^{\Delta_{\phi}}\right)+O\left(k^{d+1}\right) \tag{2.3.56}
\end{align*}
$$

Hence, in the neighborhood of the fixed-point solution, the dimensionful $U_{k}$ simply scales with $k$ at leading order. This is a further reason why it is usually convenient to rescale all couplings as the scaling behavior $k^{d} \widetilde{U}^{*}\left(\phi / k^{\Delta_{\phi}}\right)$ implies that for $\phi=O\left(k^{\Delta_{\phi}}\right)$ the potential is diminished by powers of $k^{d}$ where $k$ is small near the fixed-point. In turn, $\phi=O\left(k^{\Delta_{\phi}}\right)$ implies $\widetilde{\phi}=O(1)$ which is the range of field values where the interesting features of $\widetilde{U}$ appear. It is then clear that working with the dimensionful variables in the vicinity of a second order transition is numerically unstable as it is always possible to take $k$ sufficiently small such that the physical features of $U_{k}$ are suppressed by powers of $k$. Working directly with $\widetilde{U}$ instead this problem is avoided.

This scaling behavior for $k$ small is far from generic however and requires fine tuning the initial conditions of the RG flow ${ }^{43}$. This is synonymous to how the temperature must be fine tuned to obtain the critical Curie temperature of a ferromagnet. However, what happens if the initial conditions are not fine tuned? To answer this, consider taking the second derivative of Eq. 2.3 .56 with respect to $\phi$ as $U_{k}^{\prime \prime}(0) \underset{\text { def }}{=} m_{k}^{2}$ where $m_{0}=m$ is the physical mass in the infrared when the regulator $R_{0}$ is equal to zero.

Eq. 2.3.56) then translates to $m_{k}=k^{\Delta_{\widetilde{m}}} m^{*}$ where $m^{*}$ is a fixed number. This means that in the limit where $k$ goes to zero we have $m_{0}=0$ which we would expect from a scale invariant theory. When the theory has only one unstable direction, such as the case of a uni-axial ferromagnet ${ }^{44}$, this zero mass in the infrared

[^21]

Figure 2.3.3: Approach to convexity. Left: RG flow of Potential $U_{t}(\phi)$ in the regime $T<T_{c}$ shown at increasing RG times $t=|\log (k / \Lambda)|$. The flow leads to the ferromagnetic phase where the potential is flat between two magnetization values $\phi_{ \pm}$. Right: RG flow of the potential $U_{t}(\phi)$ in the regime $T>T_{c}$. The flow leads to the paramagnetic phase with a single minimum at $\phi=0$.
is obtained for an initial condition $m_{\Lambda}$ in the UV, that must be fine tuned to a critical value $m_{\Lambda, c}$. This is equivalent to how one needs to adjust $\kappa$ in Fig. 2.3 .2 to encounter the critical line. Within Landau theory, neglecting fluctuations, we would have $m_{\Lambda, c}=0$ and for $m_{\Lambda}^{2}<0$ we would then expect the model to be at $T<T_{c}$ where the up and down spins coexist along a first order transition line. However, fluctuations tend to disorder the system such that we may have $m_{\Lambda}^{2}<0$ and still obtain at $k=0$ a disordered system that is then above the critical temperature in the paramagnetic phase. This is indeed the case in Fig. 2.3.2). To see this, first let us note that for a fixed initial condition $\lambda_{\Lambda}>0$, there exists $\kappa_{\Lambda}(\lambda)$ such that the point $\left(\kappa_{\Lambda}\left(\lambda_{\Lambda}\right), \lambda_{\Lambda}\right)$ lies on the critical surface. For $\kappa<\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ the system flows to negative $\kappa$ which implies that the potential $U_{k}(\phi)$ has no minima for non zero $\phi$. Moreover, as we have:

$$
\begin{equation*}
\frac{\lambda}{2}\left(\frac{1}{2} \phi^{2}-\kappa\right)^{2}=\frac{\lambda \kappa}{2}-\frac{\lambda \kappa}{2} \phi^{2}+\frac{\lambda}{8} \phi^{4} \tag{2.3.57}
\end{equation*}
$$

and thus $m^{2}=-\lambda \kappa$, for $\kappa<0$ the mass at the origin is indeed positive which is what we expect from a potential with a single minimum at the origin. Thus, for $m_{\Lambda}^{2}=-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right) \lambda_{\Lambda}<0$ it is indeed possible to retrieve the paramagnetic phase with a single minimum at $\phi=0$. It is then necessary to decrease $m_{\Lambda}^{2}$ so that it may be sufficiently negative that it counterbalances the fluctuations in the infrared thereby leading to the regime $T<T_{c}$. This also implies that at $T=T_{c}$, the fixed-point solution in the infrared, has a negative dimensionless mass. Hence, because of this disordering effect of fluctuations, the fixed-point potential typically has the shape we would naively expect to correspond to the ordered phase within Landau theory. For example, the fixed-point potential $\widetilde{U}^{*}$ in the Ising model has the characteristic double well shape "w" that one would expect from Landau theory for $T<T_{c}$.

Equivalently, for fixed $\lambda>0$ it is necessary to take $\kappa$ sufficiently large to have $T<T_{c}$. This is indeed what occurs for $\kappa>\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ in which case $\kappa_{k}$ diverges to $+\infty$ for $k \rightarrow 0$. Such a situation may occur when the dimensionful $\bar{\kappa}_{k}$ converges to a fixed positive value $\bar{\kappa}_{\infty}$. In this case we indeed have $\kappa_{k}=\bar{\kappa}_{k} k^{-(d-2)} \underset{k \rightarrow 0}{\sim}$ $\bar{\kappa}_{\infty} k^{-(d-2)} \rightarrow+\infty$ 45. Due to convexity of the effective potential at $k=0$ 69 71 , when the dimensionful field $\frac{1}{2} \bar{\phi}^{2}$ is between 0 and $\bar{\kappa}_{\infty}$ the potential is strictly flat rather than concave as we would have with the above polynomial approximation. This approach towards convexity can be achieved when considering approximations that retain the full field dependence of $U_{k}$ thereby working with partial differential equations rather than the ordinary differential equations we have considered here ${ }^{46}$

The two values $\phi_{ \pm}$that bound the flat regime of the potential correspond to two phases in the system that coexist when $T<T_{c}$. The flat region then corresponds to phase mixing where the different phases are separated by domain walls in a physical system. Such a potential is exactly what one would expect

[^22]from the common tangent or Maxwell equal area construction which we recall is due to the equal chemical potential between the two phases. However, for non zero $k$ this is not the case and the potential indeed has two minima as in Fig. 2.3.3 where each minima represents a phase in the system. Thus, it is clear that the sign of $\kappa-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ determines whether the RG flow converges to the ordered ferromagnetic or disordered paramagnetic phase. This then implies that among the many environmental factors $\left\{e_{n}\right\}_{n}, \kappa$ depends on the temperature as :
\[

$$
\begin{equation*}
\kappa-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)=\left(T-T_{c}\right) f\left(\left\{e_{n}\right\}_{n}\right) \tag{2.3.58}
\end{equation*}
$$

\]

where $T_{c}$ is the critical temperature. Let us then consider $\kappa-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ small such that $T-T_{c}$ is also small. Hence, for $k$ sufficiently small in the infrared we have $\widetilde{U}_{k} \simeq \widetilde{U}^{*}$ such that the dimensionful potential scales with $k$ in this regime. In this case, $\widetilde{U}_{k}=\sum_{i} r_{i}(k) f_{i}$ may be regarded as a perturbation of the fixed-point $\widetilde{U}^{*}=\sum_{i} r_{i}^{*} f_{i}$ with $r_{i}(k)=r_{i}^{*}+\delta r_{i}(k)$. We may then again linearize the RG flow around the fixed-point solution to study the behavior of the flow in this regime. The linearization of Eq. 2.3.55 leads to:

$$
\begin{equation*}
\partial_{t} \widetilde{g}_{i}=\partial_{t}\left(\widetilde{g}_{i}^{*}+\delta \widetilde{g}_{i}\right)=\partial_{t}\left(\delta \widetilde{g}_{i}\right)=\sum_{p} \delta \widetilde{g}_{p} . \partial_{\widetilde{g}_{p}} \beta_{i}\left(\left\{\widetilde{g}_{j}\right\}_{j}\right)=\sum_{p} \delta \widetilde{g}_{p} M_{p, i} \tag{2.3.59}
\end{equation*}
$$

Where we retrieve once more the stability matrix $M_{p, i}$. Diagonalizing Eq. 2.3 .59 leads to a system of the form $\frac{\mathrm{d}}{\mathrm{d} t} E_{i}=y_{i} E_{i}$ and thus $E_{i} \propto e^{y_{i} t} \propto k^{y_{i}}$. Hence, as before, the sign of $y_{i}$ tells us whether the direction $E_{i}$ is stable or unstable with respect to perturbations. Certain perturbations lead to other scale invariant theories, as between the Gaussian fixed-point and the Wilson-Fisher fixed-point, while others may remove the system from the critical surface. In the simple case of a fixed-point with just one unstable direction the sign of the perturbation determines whether the system flows to the high temperature or low temperature phase. Hence, taking both $\kappa-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ and $k$ to be sufficiently small, one has $\widetilde{U}_{k}=\widetilde{U}^{*}+\delta \widetilde{U}$ and $r_{i}(k)=r_{i}^{*}+\delta r_{i}(k)$, where we recall $\widetilde{U}_{k}=\sum_{i} r_{i}(k) f_{i}$. The perturbation vector $\left\{\delta r_{i}\right\}_{i}$ can be decomposed onto the eigenstates $E_{i}$ leading to $\delta r_{i}(t)=\sum_{j} c_{i, j} e^{y_{j} t}$. Thus we obtain:

$$
\begin{align*}
\widetilde{U}_{k} & =\sum_{i} r_{i}(t) f_{i}=\sum_{i} r_{i}^{*} f_{i}+\sum_{i} \delta r_{i}(t) f_{i} \\
& =\widetilde{U}^{*}+\sum_{i} u_{i} e^{y_{i} t} \tag{2.3.60}
\end{align*}
$$

with,

$$
\begin{equation*}
u_{i}=\sum_{j} c_{j, i} f_{j} \tag{2.3.61}
\end{equation*}
$$

Among these eigenvalues one is negative which we call $y_{1}$ while the others are positive. We then have:

$$
\begin{equation*}
\widetilde{U}_{k}=\widetilde{U}^{*}+u_{1} e^{y_{1} t}+u_{2} e^{\omega t}+\ldots \tag{2.3.62}
\end{equation*}
$$

where we also considered explicitly $\omega$ which is the smallest positive eigenvalue. In the flowing it will be sufficient to focus on the mass at the origin $\widetilde{m}_{k}^{2}=\widetilde{U}_{k}^{\prime \prime}(0)$ and thus we have:

$$
\begin{equation*}
\widetilde{m}_{k}^{2}=\widetilde{m}^{* 2}+\widehat{m}_{1}^{2} e^{y_{1} t}+\widehat{m}_{2}^{2} e^{\omega t}+\ldots \tag{2.3.63}
\end{equation*}
$$

with

$$
\begin{equation*}
\widehat{m}_{i}^{2}=u_{i}^{\prime \prime}(0) \tag{2.3.64}
\end{equation*}
$$

The same argument can be carried out in the neighborhood of the Gaussian fixed-point:

$$
\begin{equation*}
\widetilde{m}_{k, G}^{2}=\widehat{m}_{1, G}^{2} e^{y_{1}^{G} t}+\widehat{m}_{2, G}^{2} e^{y_{2}^{G} t}+\widehat{m}_{3, G}^{2} e^{\omega^{G} t} \ldots \tag{2.3.65}
\end{equation*}
$$

where both $y_{1}^{G}$ and $y_{2}^{G}$ are negative while $\omega^{G}>0$. Our objective will now be to compute the critical exponent $\nu$ and more generally we wish to obtain the correlation length $\xi$ in the neighborhood of any scale invariant theory which may or may not be multi-critical. Let us then consider a perturbation of the Gaussian fixed-point that leads to the Wilson-Fisher fixed-point. We may then consider the initial condition $\widetilde{U}_{\Lambda}=\frac{\widetilde{\lambda}_{\Lambda}}{2}\left(\frac{1}{2} \widetilde{\phi}^{2}-\widetilde{\kappa}_{\Lambda}\right)^{2}$ for $\widetilde{\lambda}_{\Lambda}$ small as a perturbation of the Gaussian fixed-point. Once again there exists
$\widetilde{\kappa}_{\Lambda}\left(\widetilde{\lambda}_{\Lambda}\right)$ such that the RG flow leads to the Wilson-Fisher fixed-point. Thus, as $\widetilde{\lambda}_{\Lambda}$ is small, for $k \simeq \Lambda$, the RG flows in the vicinity of the Gaussian fixed-point according to $\operatorname{Eq} 2.3 .65$. Moreover, for $\tilde{\lambda}_{\Lambda}=0$ we retrieve the Gaussian fixed-point so we have both $\widehat{m}_{1, G}^{2} \propto \widetilde{\lambda}_{\Lambda}$ and $\widehat{m}_{2, G}^{2} \propto \lambda_{\Lambda}$. The system then leaves the vicinity of the Gaussian fixed-point when the perturbation becomes $O(1)$. When this takes place we thus have:

$$
\begin{equation*}
\tilde{\lambda}_{\Lambda}\left(\frac{k}{\Lambda}\right)^{-\left|y_{1}^{G}\right|}=O(1), \tilde{\lambda}_{\Lambda}\left(\frac{k}{\Lambda}\right)^{-\left|y_{2}^{G}\right|}=O(1) \tag{2.3.66}
\end{equation*}
$$

The eigenvalues of the Gaussian fixed-point are given by dimensional analysis as we show in Sec. 3.1.2.2. . For now let us simply state that we have $y_{1}^{G}=-2$ and $y_{2}^{G}=-(4-d)$. For $2<d<4\left|y_{1}^{G}\right|$ is larger than $\left|y_{2}^{G}\right|$. Hence, if $\widetilde{\lambda}_{\Lambda}\left(\frac{k}{\Lambda}\right)^{-\left|y_{2}^{G}\right|}=O(1)$ then we also have $\widetilde{\lambda}_{\Lambda}\left(\frac{k}{\Lambda}\right)^{-\left|y_{1}^{G}\right|}=O(1)$. As such, the scale at which we leave the vicinity of the Gaussian fixed-point is given by:

$$
\begin{equation*}
k_{G}=\Lambda \widetilde{\lambda}_{\Lambda}^{1 /(d-4)} \tag{2.3.67}
\end{equation*}
$$

One may check that by adding a one loop contribution to the potential, perturbation theory breaks down when the renormalized mass is smaller than this scale $\sqrt[72]{ }$. Taking the usual Landau theory dependence of the mass $m_{R}^{2} \propto\left(T-T_{c}\right)$, perturbation then breaks down when :

$$
\begin{equation*}
\left|T / T_{c}-1\right|^{\frac{(d-4)}{2}} \ll A \widetilde{\lambda}_{\Lambda} \tag{2.3.68}
\end{equation*}
$$

where A is a numerical factor. We thus retrieve the Ginzburg criterion of Sec. 2.1 .3 which implies that fluctuations become large when the RG flows away from the vicinity of Gaussian fixed-point and perturbation theory breaks down. If $\kappa_{\Lambda}-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ is small, then the system flows in the neighborhood of the critical line as in Fig. 2.3.2 where up to small corrections the potential scales with $k$ as $U_{k}(\phi)=k^{d} \widetilde{U}^{*}\left(\phi / k^{\Delta_{\phi}}\right)$. For $\kappa_{\Lambda}-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ sufficiently small the system then enters the vicinity of the WF fixed-point where the RG flow is given by Eq. 2.3.63). If $\kappa_{\Lambda}=\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$, then we have $\widehat{m}_{1}^{2}\left(\kappa_{\Lambda}, \lambda_{\Lambda}\right)=0$ in Eq. 2.3.62) and the system converges to the Wilson Fisher fixed-point at a rate given by $\omega$. If instead $\kappa_{\Lambda}-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ is small and non zero the system flows away when $\widehat{m}_{1}^{2} e^{y_{1} t}=O(1)$ which gives us a new scale :

$$
\begin{equation*}
k_{c}=\Lambda\left|\widehat{m}_{1}^{2}\right|^{1 /\left|y_{1}\right|} \tag{2.3.69}
\end{equation*}
$$

For $k \ll k_{c}$, the system flows away from the critical surface as in Fig. 2.3 .2 . This is due to the fact that the dimensionful mass $m_{k}$ converges to a fixed value $m_{\infty}$ given by the inverse of the correlation length. Hence for sufficiently small $k$ we have ${ }^{47}$ :

$$
\begin{equation*}
\tilde{m}_{k}=\frac{m_{\infty}}{k} \propto \frac{1}{k \xi} \tag{2.3.70}
\end{equation*}
$$

Thus starting from the divergent regime where $m_{k} \propto \frac{1}{k \xi}, m_{k}$ become of order one when :

$$
\begin{equation*}
k_{\xi}=O\left(\xi^{-1}\right) . \tag{2.3.71}
\end{equation*}
$$

This is then roughly the scale where the $m_{k}$ enters the scaling regime in the neighborhood of the critical surface. Thus we expect $k_{\xi}=O\left(k_{c}\right)$ such that the RG flow converges to a scaling regime for $k_{G} \gg k \gg \xi^{-1}$ and dimensionless couplings diverge for $k \ll \xi^{-1}$ as shown in Fig. 2.3.4. We also note that this divergence for $k \ll \xi^{-1}$ is related to a convergence of the associated dimensionful quantities, see Fig. 2.3.4. This implies that when working only with dimensionful quantities the flow slows down when the RG scale $k^{-1}$ is larger than the correlation length as the correlation length then replaces $k^{-1}$ as the infrared regulator. Hence, using $k_{\xi}=\xi^{-1}=O\left(k_{c}\right)$ and Eq. 2.3.69 we find:

$$
\begin{equation*}
\xi \propto\left|\widehat{m}_{1}^{2}\right|^{-1 /\left|y_{1}\right|} \tag{2.3.72}
\end{equation*}
$$

[^23]We now recall that $\widehat{m}_{1}^{2}=0$ for $\kappa_{\Lambda}=\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ such that we also have

$$
\begin{equation*}
\widehat{m}_{1}^{2} \propto \kappa_{\Lambda}-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right) \tag{2.3.73}
\end{equation*}
$$

Finally let us recall that the sign of $\kappa_{\Lambda}-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right)$ determines whether the system enters the low temperature or high temperature phase such that :

$$
\begin{equation*}
\kappa_{\Lambda}-\kappa_{\Lambda}\left(\lambda_{\Lambda}\right) \propto\left|T-T_{c}\right| \tag{2.3.74}
\end{equation*}
$$

Thus, using Eq. 2.3 .72 and Eq. 2.3 .74 we obtain :

$$
\begin{equation*}
\xi \propto\left|T-T_{c}\right|^{-1 /\left|y_{1}\right|} \tag{2.3.75}
\end{equation*}
$$

which allows us to retrieve the critical exponent $\nu$ as ${ }^{48}$.

$$
\begin{equation*}
\nu=1 /\left|y_{1}\right| \tag{2.3.76}
\end{equation*}
$$

However, as this was derived in a somewhat imprecise manner we wish to give a more precise proof using general RG arguments. For the following proof we consider instead the Polchinski flow. As the $\Gamma$ flow is simply a Legendre transform of the Polchinski flow the eigenvalues of the stability matrix and the relationship between critical exponents and eigenvalues remain the same.

Let us then recall that the partition function remained invariant along the coarse graining flow such that the various effective coarse grained descriptions were solely $k$-dependent change of variables. This implies that the underlying physics are maintained along the flow and what varies are the tools we use to describe the phenomena at hand. Hence, as for a given temperature $T$, the correlation length is measurable, it is also physical and must therefore remain invariant through the coarse graining procedure. However, the Wilsonian RG flow involves both coarse graining and scaling and as the correlation is measured in units of length it must also scale. More concretely, considering a rescaling $x \rightarrow l x$ of the spatial variables we must also have $\xi \rightarrow l \xi \boxed{49}^{49}$. If we then consider the definition of the exponent $\nu:$

$$
\begin{equation*}
\xi \propto \check{T}^{-\nu} \tag{2.3.77}
\end{equation*}
$$

with $\check{T}=\left|T / T_{c}-1\right|$ the reduced temperature, it is apparent that for $\check{T}$ sufficiently small it must scale along the RG flow as $\check{T} \rightarrow l^{-1 / \nu} \check{T}{ }^{50}$. Moreover, in the neighborhood of the phase transition the RG flows linearly such that the couplings associated to the different eigendirections simply scale with $l$. In particular the dimensionless coupling associated to the negative eigenvalue $y_{1}$, which we will call $g_{-}$also scales. If $g_{-}=g_{-}^{*}$ where $g_{-}^{*}$ is the value of $g_{-}$at the fixed-point, the RG flows to the fixed-point regardless of the values of the irrelevant couplings which are stable to perturbations. However, for $g_{-}=g_{-}^{*}+\delta g_{-}$the sign of $\delta g_{-}$ determines whether the system enters the disordered state for $T>T_{c}$ or the broken state for $T_{c}<T$. Thus, once more, we have $\delta g_{-} \propto \check{T}$ such that the scaling of $\check{T}$ is given by the scaling of $\delta g_{-}$. We then retrieve the same result that $\nu=\left|y_{1}\right|^{-1}$. This argument can be made more general however. Consider a scale invariant theory given by coordinates $\left\{g_{i}^{*}\right\}$ in theory space where the $g_{i}$ are couplings associated to eigendirections of a fixed-point of the renormalization operator $\mathcal{R}$. Consider now a relevant perturbation $\delta g_{i_{0}}$ of a possibly multicritical scale invariant theory ${ }^{51}$. As $\delta g_{i_{0}}$ is non zero and relevant the physical correlation length of the system must be nonzero. We then have:

$$
\xi^{-1}=f\left(\delta g_{i_{0}}\right)
$$

[^24]


Figure 2.3.4: RG flow of the dimensionless (left) $\widetilde{U}_{t}^{\prime}(0)$ and dimensionful (right) $U_{t}^{\prime}(0)$ mass $\left.U^{\prime}(\rho)\right|_{\rho=\phi^{2} / 2=0}$ between $k_{G}\left(\widetilde{\lambda}_{\Lambda}\right)$ and $k_{\xi}\left(\widetilde{\lambda}_{\Lambda}\right)<k_{G}\left(\widetilde{\lambda}_{\Lambda}\right)\left(\left|t_{G}\right|<\left|t_{\xi}\right|\right)$ indicated by vertical dashed lines for $\widetilde{\lambda}_{\Lambda}=0.1$ and $\widetilde{\lambda}_{\Lambda}=0.2$. The scaling regime is obtained for constant $\widetilde{U}_{t}^{\prime}(0)$ where the value of the constant is independent of the initial condition (universality). This takes place when $k_{G} \gg k>k_{\xi}$. The flow then diverges for $k \ll k_{\xi}$ when the dimensionful $U_{t}^{\prime}(0)$ reaches a fixed value.
where $f(0)=0$. Thus for $\delta g_{i_{0}}$ small we have $f(x) \simeq x^{\alpha}$. Moreover, when $\delta g_{i_{0}}$ is small it scales as $\delta g_{i_{0}} \rightarrow$ $l^{-\left|y_{g_{i_{0}}}\right|} \delta g_{i_{0}}$ while we have $\xi^{-1} \rightarrow l^{-1} \xi^{-1}$. Hence, we have $-\alpha\left|y_{g_{i_{0}}}\right|=-1$, which leads to

$$
\begin{equation*}
\Lambda \xi=A\left(\delta g_{i_{0}}\right)^{-\frac{1}{\left|\lg _{i_{0}}\right|}} \tag{2.3.78}
\end{equation*}
$$

If we then consider an external control parameter $b-b_{c}$ that governs the sign of $\delta g_{i_{0}}$, then one finds the associated critical exponent $\nu_{b}$. However, Eq. 2.3 .78 is interesting outside of second order phase transitions as it implies that the correlation length of a system is large when it is near a fixed-point of the RG. For example, as mentioned in [6], Eq. $\sqrt[2.3 .78]{ }$ ) can be useful for explaining hierarchy in quantum field theories where $\xi^{-1}$ can be a particle mass. In that case, fine tuning $\delta g_{i_{0}}$ in Eq. 2.3 .78 can explain a large ratio between the UV scale $\Lambda$ and the mass of particles in the system. The degree of fine tuning necessary then depends on the value $y_{g_{i_{0}}}$ and can be rather small when $y_{g_{i_{0}}}$ is also small. Another interesting aspect is that the system can be close to a scale invariant theory which might in fact be complex. Such a situation often happens near the collapse of two fixed-points in parameter space such that for the physical parameters of interest the system is "near" a complex fixed-point. In this case, one may obtain weak first order phase transitions where the correlation is large but finite [6]. Much of this thesis will be concerned with the collapse of fixed-points at non integer dimensions where the dimensionality of the problem will also be regarded as a continuous parameter. If a fixed-point collapse takes place near an integer dimension in parameter space, then one expects a complex scale invariant theory with small imaginary parts at the nearest integer dimension. This may then imply the existence of weak first order transitions at that integer dimension.

Let us then summarize this section, the RG flow written in terms of dimensionless variables is a dynamical system of autonomous equations. This system usually has a countable set of fixed-point solutions except when there are exactly marginal. In this case one may obtain fixed-point manifolds. Fixed-point solutions are scale invariant theories which often correspond to conformal field theories. As in any dynamical system, it is usually interesting to study the stability of these fixed-points. In theory space, these fixed-points have an infinity of stable directions and a finite number of unstable directions. These unstable directions are given by negative eigenvalues of the stability matrix. All of the eigenvalues of a stability matrix correspond to scaling dimensions of the corresponding scale-invariant theory. When these scale-invariant theories are perturbed, the RG may flow to other scale-invariant theories or instead the RG may converge to a theory with a finite correlation length. In this case, the perturbation may be controlled by an external control variable and one may obtain critical exponents from the negative eigenvalues of the stability matrix. The choice of eigenvalue for a multi-critical fixed-point depends on the projections of the perturbation in theory space with respect to the eigendirections of the stability matrix.

### 2.4 Approximation schemes

"It would appear that we have reached the limits of what it is possible to achieve with computer technology, although one should be careful with such statements, as they tend to sound pretty silly in 5 years.", $S=-\operatorname{Tr}(\rho \log \rho)$

- John von Neumann

The $\Gamma$ flow is a non-linear functional integro-differential equation, hence, an exact solution is usually unknown and one therefore has to rely on approximations. The very first approximation scheme that may come to mind is perturbation theory which indeed may be retrieved from the exact renormalization scheme [55-60]. However, we would like to obtain insights into physics that are not limited to the small coupling regime. The purpose of this section is therefore to gain insights on methods for obtaining approximate results of non perturbative and non universal physics. In order to highlight the efficiency of FSS or renormalization techniques in the framework of approximations we will first consider two examples outside of the field theory domain.

### 2.4.1 FSS as a convergence accelerator

In [73] it was found that imposing FSS on sequences leads to faster convergence. This in turn can be understood by minimizing the Cauchy difference $\left|f_{n+p}-f_{n}\right|$ for the sequence at hand. We will give a perturbative example here but the method can be extended to achieve non perturbative results, in particular using variational techniques and minimal sensitivity as is done in NPRG. Hence following [73] we consider the function:

$$
\frac{1}{1-g}
$$

The first terms in the Taylor expansion are $f_{0}(g)=1+g$ and $f_{1}(g)=1+g+g^{2}$. We can express $f_{1}$ as a function of $f_{0}$ as $g=\left(f_{0}-1\right)$ thus we have:

$$
\begin{equation*}
f_{1}=f_{0}+\left(1-f_{0}\right)^{2} \tag{2.4.1}
\end{equation*}
$$

Our objective is then to turn this relationship between $f_{0}$ and $f_{1}$ to a recursion formula which will be the discrete version of FSS. From this recursion we will then take the continuum limit to retrieve the continuum version of FSS which we will use to obtain our approximation.

Hence, consider a time dependent function $s(t){ }^{52}$ such that $s(\tau)=f_{1}$ and $s(0)=f_{0}$. Thus, Eq. 2.4.1) gives the rule to compute $s(\tau)$ knowing $s(0)$ on a coarse grid of two points $\{0, \tau\}$ where the rule is $s(\tau)=$ $s(0)+(1-s(0))^{2}=Z(s(0))$. Let us now consider a finer grid with $\tau^{\prime}=\tau / 4$ for which we now have 5 grid points $\left\{k \tau^{\prime}\right\}_{0 \leq k \leq 4}$ between $t=0$ and $t=\tau$ at $t_{0}=0, t_{1}=\tau / 4, t_{2}=\tau / 2, t_{3}=3 \tau / 4, t_{4}=\tau$ and we choose the recurrence rule to compute $s\left(t_{k+1}\right)$ as a function of $s\left(t_{k}\right)$ to be the same as the rule we used to compute $s(\tau)$ as a function of $s(0)$, that is, $Z(x)$. Hence, the rule to compute $s\left(t_{k}\right)$ at this new scale with a finer grid is the same as the rule at the initial scale with a coarse grid. One might start noticing the FSS philosophy behind this and indeed this leads to:

$$
s(\tau)=s\left(t_{4}\right)=Z\left(s\left(t_{3}\right)\right)=Z\left(Z\left(s\left(t_{2}\right)\right)\right) \underset{d e f}{=} Z^{2}\left(s\left(t_{2}\right)\right)=Z^{3}\left(s\left(t_{1}\right)\right)=Z^{4}\left(s\left(t_{0}\right)\right)
$$

What is interesting about this is that it works for all points in our 5 point grid and we thus have:

$$
s\left(t_{k+p}\right)=Z^{k+p}\left(s\left(t_{0}\right)\right) \underset{\text { def }}{=} L\left(k+p, s\left(t_{0}\right)\right)=Z^{k}\left(Z^{p}\left(s\left(t_{0}\right)\right)\right)=L\left(k, L\left(p, s\left(t_{0}\right)\right)\right)
$$

[^25]Thus, our choice to extend the rule between $s(\tau)$ and $s(0)$ to all points on our finer grid is indeed equivalent to imposing FSS on our function $s$. Of course this can then be extended to $N$ points on a much finer grid with $\tau^{\prime}=\tau / N$. On this new set of grid points we then have:

$$
s\left(t_{k+1}\right)=Z\left(s\left(t_{k}\right)\right)=s\left(t_{k}\right)+\left(1-s\left(t_{k}\right)\right)^{2}, t_{k}=k \tau / N
$$

which is the recursion formula we sought. In the limit $N \rightarrow \infty, \tau^{\prime}=\tau / N$ is arbitrarily small and we have $s\left(t_{k+1}\right)=s\left(t_{k}+\tau^{\prime}\right)=s\left(t_{k}\right)+\tau^{\prime} \frac{\mathrm{d} s}{\mathrm{dt}}$ which then leads to:

$$
\begin{equation*}
\tau^{\prime} \frac{\mathrm{d} s}{\mathrm{~d} t}(t)=(1-s(t))^{2} \tag{2.4.2}
\end{equation*}
$$

This is then the usual continuum version of FSS given here by an autonomous differential equation. Integrating this equation with initial condition $s(0)=f_{0}=1+g$ then leads to:

$$
\begin{equation*}
s(t)=1+\frac{g}{1-g t / \tau^{\prime}} \tag{2.4.3}
\end{equation*}
$$

The Taylor expansion of $s$ in powers of $g$ is $s(t)=1+g+g^{2} t / \tau^{\prime}+O\left(g^{3}\right)$ and if we want to obtain the same expansion as $f_{1}=1+g+g^{2}$ we must take $t=\tau^{\prime}$ from which we find:

$$
\begin{equation*}
s\left(\tau^{\prime}\right)=f_{R}=\frac{1}{1-g} \tag{2.4.4}
\end{equation*}
$$

We have thus retrieved our initial function using only the two first terms of a Taylor expansion which we then used to generate an entire "zoomed" sequence by scaling and imposing FSS at all scales. This is admittedly quite remarkable. Let us now consider the case of a function where our approximation scheme will not converge to an exact result. One such example is given by:

$$
\begin{equation*}
\sin (g) \tag{2.4.5}
\end{equation*}
$$

Our two first Taylor expansions are $f_{0}=g$ and $f_{1}=g-g^{3} / 6$ and thus considering again a function $s$ such that $s(0)=f_{0}$ and $s(\tau)=f_{1}$ we have the rule

$$
\begin{equation*}
s(\tau)=s(0)-s(0)^{3} / 6 \tag{2.4.6}
\end{equation*}
$$

Rescaling and imposing FSS we obtain the recursion formula:

$$
\begin{equation*}
s\left(t_{k+1}\right)=s\left(t_{k}\right)-s\left(t_{k}\right)^{3} / 6, t_{k}=k \tau / N=k \tau^{\prime} \tag{2.4.7}
\end{equation*}
$$

If we then take the limit $N \rightarrow \infty$ we obtain:

$$
\begin{equation*}
\tau^{\prime} \frac{\mathrm{d} s}{\mathrm{~d} t}=-s^{3} / 6 \tag{2.4.8}
\end{equation*}
$$

Using the initial condition $s(0)=f_{0}=g$ we have:

$$
\begin{equation*}
s(t)=\frac{g}{\left(1+g^{2}\left(\frac{t}{3 \tau^{\prime}}\right)\right)^{1 / 2}} \tag{2.4.9}
\end{equation*}
$$

Taylor expanding $s$ about $g$ we obtain $s=g-g^{3} t /(6 \tau)+O\left(g^{5}\right)$ hence we take $t=\tau^{\prime}$ so that the expansion is consistent with $f_{1}=g-g^{3} / 6$. the result is then

$$
\begin{equation*}
s\left(\tau^{\prime}\right)=f_{R}=\frac{g}{\left(1+\frac{g^{2}}{3}\right)^{1 / 2}} \tag{2.4.10}
\end{equation*}
$$

This function is rather remarkable as it remains bounded for large $g$ which means that the error with respect to the sin function is never arbitrarily large as is the case for Taylor expansions. Indeed, Fig. 2.4.1) shows that $f_{R}$ gives a reasonable approximation for small $g$ while never diverging too far away from the curve. This


Figure 2.4.1: Comparison of approximations. Left: $\sin (x)$ and its approximations as functions of their arguments. Right: Relative error $\log _{10}(|\operatorname{app}(x) / \sin (x)-1|)$ for different approximations app. The peaks are due to zeroes of the sin function
approximation is thus quite similar to the renormalized solution in our perturbative analysis of differential equations Eq. 2.2 .48 as it minimizes the error when moving away from the domain of best accuracy. The same FSS improved approximation replacing sin with cos leads to the (2,2) Padé approximant of the cos function. $(n, n)$ Padé approximants are known to be well adapted to approximations and thus the error in the case of a cos function is indeed quite small for small $g$ while remaining bounded for large $g$. These methods can be further extended to include non perturbative approximations by introducing so called "governing functions" to ensure that the first few approximations, to be FSS improved, are reasonably good [74]. This is similar to the NPRG framework where one has to optimize the regulator to find accurate approximations.

In summary, this subsection showed how FSS can be used to enhance approximations even when a notion of scale is not apparent. We hope this convinces the reader that FSS, and thus renormalization, is a very convenient tool that is highly adapted to the framework of approximations. In the following subsection we hope to elucidate this within a non perturbative scheme.

### 2.4.2 Applying FSS in absence of a small coupling

In the present section we shall study how FSS can be used in absence of an explicit small parameter ${ }^{53}$ This in turn is more closely related to the NPRG framework as the latter also lacks such a parameter. More precisely we will study the logistic map which is known to be closely related to the renormalization group. We will not dwell into the many fascinating aspects of this map, instead we will focus on an FSS approximation scheme to derive universal quantities. This section follows closely 75 where we have added more details.

The logistic map is defined by the following equation:

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right)=r x_{n}\left(1-x_{n}\right) \tag{2.4.11}
\end{equation*}
$$

For an initial condition $0<x_{0}<1$ and $r<3=r_{1}$ the sequence converges to the fixed-point $x^{*}=1-1 / r$. However, for slightly larger values of $r$ this fixed-point becomes unstable and the system oscillates between two values $x_{1}^{* *}$ and $x_{2}^{* *}$ as the sequence $x_{1}^{* *} \rightarrow f\left(x_{1}^{* *}\right)=x_{2}^{* *} \rightarrow f\left(x_{2}^{* *}\right)=x_{1}^{* *} \rightarrow f\left(x_{1}^{* *}\right)=x_{2}^{* *} \rightarrow$ etc. Thus, $x_{1}^{* *}=f\left(x_{2}^{* *}\right)=f\left(f\left(x_{1}^{* *}\right)\right) \underset{\text { def }}{=} f^{2}\left(x_{1}^{* *}\right)$ such that $\left(x_{1}^{* *}, x_{2}^{* *}\right)$ are fixed-points of $f^{2}(x)=f(f(x))$. This is called period-doubling and for slightly larger values of $r$, above a value $r_{2}$, this period doubling happens once more creating a sequence of period 4 :

$$
\begin{equation*}
x_{1}^{* * * *} \rightarrow f\left(x_{1}^{* * * *}\right)=x_{2}^{* * * *} \rightarrow f\left(x_{2}^{* * * *}\right)=x_{3}^{* * * *} \rightarrow f\left(x_{3}^{* * * *}\right)=x_{4}^{* * * *} \rightarrow f\left(x_{4}^{* * * *}\right)=x_{1}^{* * * *} \rightarrow \text { etc. } \tag{2.4.12}
\end{equation*}
$$

In the same manner one may deduce that these are fixed-points of $f^{4}(x)$. This scenario repeats itself such that for each new value $r_{n}$ there is a new period doubling leading to a sequence of points $x_{k}^{* * * \ldots * * *}=x_{k}^{*(n)}$

[^26]whose period is of length $p_{n+1}=2 p_{n}=2^{n}$. However, the sequence $r_{n}$ does not grow indefinitely and instead converges to a finite value $r_{\infty}$. Beyond this point the system becomes chaotic. What is interesting however is that even though the value of $r_{\infty}$ depends on the details of recursion relationship, the approach to chaos displays universality such that many chaotic systems approach the chaotic regime in a similar way. This is analogous to how many physical systems approach a second order transition in a similar fashion. Quantitatively, for second order transitions this is given by the critical exponents and systems that have similar critical exponents are said to belong to the same universality class. As mentioned in Sec.(2.3) this is due to the diffusive nature of the renormalization group that washes away many details and the distinction between universality classes depend on minimal aspects such as symmetry, dimension and whether the underlying interaction is long range or short range ${ }^{54}$. In the case of the logistic map the function $f$ may be replaced by a relatively large range of functions that need to verify only somewhat minimal conditions to belong to the same universality class. Moreover, the analogue of the critical exponents here will be the Feigenbaum's constants. We will be interested mainly by the first one as it is easier to define but the calculations will naturally lead to the second one as well and as such it will be defined at that point. The first constant is defined by:
\[

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{r_{n-1}-r_{n-2}}{r_{n}-r_{n-1}}=\delta \simeq 4.6692 \ldots \tag{2.4.13}
\end{equation*}
$$

\]

The link with critical exponents can be made apparent as the above limit entails that the sequence $\Delta_{n}=$ $r_{n+1}-r_{n}$ is near geometric for large $n$ as $\Delta_{n+1} \simeq \delta^{-1} \Delta_{n} \propto \delta^{-n}=e^{-n \ln (\delta)}$. This is thus indeed reminiscent of the behavior of the RG flow around a fixed-point as $g(t) \propto e^{\sigma t}$ for an eigendirection of the RG flow given by a coupling $g$. We will now seek an approximation for $\delta$ where we may notice that there is no small parameter here in which we may expand. Our strategy here will be to obtain a renormalization flow and thus the first step will be to find a blocking transformation that allows us to move from one scale to another. As we now know this blocking must verify FSS. We have a natural FSS mapping procedure here which is the "squaring function" $\mathcal{R}(g)=g \circ g=g^{2}$ as we know that at each bifurcation $r_{l+1}$ the $x_{k}^{*(l+1)}$ are fixed-points of $f_{l+1}=\mathcal{R}\left(f_{l}\right)=f_{l}^{2}$. This then leads to the idea of mapping between scales on longer and longer periods where we omit the "microscopic" details between $x_{n+1}=f\left(x_{n}\right)$ and $x_{n}$ and instead focus on a broader "macroscopic" range between $x_{n+p_{l}}=f_{l}\left(x_{n}\right)$ and $x_{n}$ where $p_{l}$ is the period of the $x_{k}^{*(l)}$. Moreover defining $g_{s} .\left(l, f_{l}\right)=\left(l+s, f_{l+s}=\mathcal{R}\left(\mathcal{R}\left(\ldots \mathcal{R}\left(f_{l}\right)\right)\right)\right) \underset{\text { def }}{\overline{=}}\left(l+s, \mathcal{R}^{s}\left(f_{l}\right)\right) \underset{\text { def }}{\overline{=}}\left(l+s, K\left(s, f_{l}\right)\right)$ we indeed have the usual FSS condition on $K$ due to the transitivity of compositions. Our objective will thus be to follow the system along this flow mapping to larger and larger scales as we approach the chaotic regime. Indeed, in the same spirit as the RG, instead of trying to perform direct calculations with a specific model, it will prove sufficient to know how models map along this flow to obtain the Feigenbaum's constants. Hence, the space on which we now work is the space of models where the function $f$ verifies the universality condition of the logistic map which is that $f$ admits a quadratic maximum ${ }^{55}$. We thus consider the more general polynomial equation:

$$
\begin{equation*}
x_{n+1}=-(1+r) x_{n}+a x_{n}^{2}=f_{0}\left(x_{n}\right) \tag{2.4.14}
\end{equation*}
$$

where we have used the notation $f_{0}$ to make explicit that this is only the initial condition in a flow of functions $f_{l}=\mathcal{R}^{l}\left(f_{0}\right)$. Let us now notice that by rescaling $x \rightarrow x / a$ the factor of $a$ may be removed which yields:

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right)=-(1+r) x_{n}+x_{n}^{2} \tag{2.4.15}
\end{equation*}
$$

We now move to the scale $l=1$ given by the function $f_{1}=\mathcal{R}\left(f_{0}\right)$ :

$$
\begin{equation*}
x_{n+2}=f_{1}\left(x_{n}\right)=f\left(f\left(x_{n}\right)\right) \tag{2.4.16}
\end{equation*}
$$

We then define $\hat{x}_{n}^{(1)}=x_{n}$ and $\hat{x}_{n+1}^{(1)}=x_{n+2}$ which should be seen as rescaling the "time" axis given by the integers $\{n\}$. Indeed, we omit the intermediate $x_{n+1}$ and are now interested only on longer time scales of length 2 which we rescale to length 1 via $x_{n+2}=\hat{x}_{n+1}$. This then defines the sequences at scale $l=1$ :

$$
\begin{equation*}
\hat{x}_{n+1}^{(1)}=f_{1}\left(\hat{x}_{n}^{(1)}\right) \tag{2.4.17}
\end{equation*}
$$

[^27]This last equation describes a new model with a new function $f_{1}$ which we can then consider as a new initial condition. If we were to iterate this procedure the functions $f_{l}$ would grow in complexity rather quickly as each iteration increases rather dramatically the degree of the polynomial function by composition. We thus have recourse to an approximation scheme. At each new scale, instead of considering the exact composition $\mathcal{R}(f)=f \circ f$ we will project the function onto a model subspace that will hopefully be sufficient to capture enough information about the exact calculations. This is our first glimpse into the FRG approximation philosophy as the same idea is commonly used in that framework and we will see examples of this in the following sections. In the present case we will just try the simplest model space given by the initial condition of second degree polynomials and hope for the best. Within the NPRG schemes however we will give arguments and guides to estimate the error and optimize approximations. Hence, for now let us consider a Taylor expansion of $f_{1}$, which is a fourth degree polynomial, about some point $\bar{x}^{(1)}$. This leads us to define $\hat{x}_{n}^{(1)}=\bar{x}^{(1)}+\xi_{n}^{(1)}$ and thus:

$$
\begin{equation*}
\bar{x}^{(1)}+\xi_{n+1}^{(1)}=f_{1}\left(\bar{x}^{(1)}+\xi_{n}^{(1)}\right) \simeq f_{1}\left(\bar{x}^{(1)}\right)+f_{1}^{\prime}\left(\bar{x}^{(1)}\right) \xi_{n}^{(1)}+f_{1}^{\prime \prime}\left(\bar{x}^{(1)}\right) \xi_{n}^{(1)^{2}} / 2 . \tag{2.4.18}
\end{equation*}
$$

We then notice that the new sequence is still not exactly like the original sequence, in particular neither the original sequence nor the logistic map have a constant term. Instead $x_{n}=0$ was always a viable fixed-point. Hence in the spirit of FSS we will impose that at this new scale $l=1$, the new map between $\xi_{n+1}^{(1)}$ and $\xi_{n}^{(1)}$ has the same functional form as at $l=0$. This leads us to impose that $f_{1}\left(\bar{x}^{(1)}\right)=f\left(f\left(\bar{x}^{(1)}\right)\right)=\bar{x}^{(1)}$ and to define new coupling constants at scale $l=1$ as $-(1+\tilde{r})=f_{1}^{\prime}\left(\bar{x}^{(1)}\right)$ and $a_{1}=f_{1}^{\prime \prime}\left(\bar{x}^{(1)}\right) / 2$. This last redefinition is interesting as we may notice that although we initially rescaled $x_{n}$ such that the coefficient of $x_{n}^{2}$ was 1 , the RG flow has generated a non trivial $a_{1}$ at scale $l=1$. A similar situation occurs in field theories where even if we set the kinetic term to the canonical $\partial \phi \partial \phi / 2$ the RG changes the prefactor. We may also notice that setting the kinetic term to $\partial \phi \partial \phi / 2$ to is a choice of normalization for the field $\phi$ as re-parametrization invariance allows us to change $\phi \rightarrow Z^{1 / 2} \phi$ without changing the physics. This means that the choice of normalization is a redundancy in the model. However, as the renormalization flow creates a continuous variation between models it is necessary to pick a reference normalization such that we can compare the field $\phi$ along an RG flow between different scales. For one infinitesimal step of the RG flow, we will obtain a term $Z_{\delta k} \partial \phi \partial \phi / 2$ which is usually fixed by rescaling the field $\phi$ again or "renormalizing" such that the kinetic term is once more canonical and of the form $\partial \tilde{\phi} \partial \tilde{\phi} / 2$ with now $\tilde{\phi}=\phi Z_{\delta k}^{1 / 2}$. As $Z_{\delta k}$ changes at each step of the renormalization flow this leads to a running renormalization and near the critical point it verifies $Z_{k} \propto k^{-\eta}$ where $\eta$ is called the anomalous dimension as near the second order transition we have $\tilde{\phi}_{k}=\phi_{k} Z_{k}^{1 / 2} \propto k^{d_{\phi}-\eta / 2}$ where $d_{\phi}$ is the dimension obtained from dimensional analysis.

We now recall our objective. We wish to find a mapping between scales using the renormalization operator $\mathcal{R}(g)=g \circ g$. We have found that by projecting the model onto a subspace at each renormalization step it is possible to obtain an approximate RG flow. This was done using the following procedure:

1. Shifting with $\hat{x}_{n}^{(1)}=\bar{x}^{(1)}+\xi_{n}^{(1)}$ where $\mathcal{R}\left(f_{0}=f\right)\left(\bar{x}^{(1)}\right)=f\left(f\left(\bar{x}^{(1)}\right)\right)=\bar{x}^{(1)}$.
2. Rescaling $\xi_{n}^{(1)}$ to keep the coefficient of $\xi_{n}^{(1)^{2}}$ set to 1 .
3. Re-defining new coupling constants, $-(1+\tilde{r})=f_{1}^{\prime}\left(\bar{x}^{(1)}\right)$ and $a_{1}=f_{1}^{\prime \prime}\left(\bar{x}^{(1)}\right) / 2$ at scale $l=1$

This is the essence of our RG procedure as it is sufficient to ensure FSS by imposing that the equations look similar up to re-definitions at the new scale. The first step is thus to compute $\bar{x}^{(1)}$. Hence, returning to the initial problem at scale $l=1$, the sequence given by Eq. 2.4 .15 ) undergoes a period doubling bifurcation at $r=0$ with $f\left(x_{1}^{* *}\right)=x_{2}^{* *}$ and $f\left(x_{2}^{* *}\right)=x_{1}^{* *}$ and thus $f\left(f\left(x_{i}^{* *}\right)\right)=x_{i}^{* *}$. This leads to the system :

$$
\begin{align*}
& x_{1}^{* *}=-x_{2}^{* *}(1+r)+x_{2}^{* *^{2}} \\
& x_{2}^{* *}=-x_{1}^{* *}(1+r)+x_{1}^{* *^{2}} \tag{2.4.19}
\end{align*}
$$

The difference of these two equations leads to:

$$
x_{1}^{* *}-x_{2}^{* *}=(1+r)\left(x_{1}^{* *}-x_{2}^{* *}\right)-\left(x_{1}^{* *^{2}}-x_{2}^{* *^{2}}\right)=\left(x_{1}^{* *}-x_{2}^{* *}\right)\left((1+r)-\left(x_{1}^{* *}+x_{2}^{* *}\right)\right)
$$

thus:

$$
\begin{equation*}
x_{1}^{* *}+x_{2}^{* *}=r . \tag{2.4.20}
\end{equation*}
$$

Moreover we have $x_{1}^{* *} x_{2}^{* *}=x_{1}^{* *} x_{2}^{* *}\left(x_{1}^{* *}-(1+r)\right)\left(x_{2}^{* *}-(1+r)\right)$ thus inserting the value for $r$ given by Eq. 2.4.20) we obtain:

$$
\begin{equation*}
1=\left(-x_{2}^{* *}-1\right)\left(-x_{1}^{* *}-1\right)=x_{1}^{* *} x_{2}^{* *}+x_{1}^{* *}+x_{1}^{* *}+1 \tag{2.4.21}
\end{equation*}
$$

Inserting again Eq. 2.4 .20 we obtain:

$$
\begin{equation*}
x_{1}^{* *} x_{2}^{* *}=-r \tag{2.4.22}
\end{equation*}
$$

Knowing the product and sum, the $x_{i}^{* *}$ are then roots of the polynomial $X^{2}-r X-r$, which leads to:

$$
\begin{equation*}
x_{i}^{* *}=\frac{r}{2} \pm \frac{\sqrt{r^{2}+4 r}}{2} \tag{2.4.23}
\end{equation*}
$$

We can now Taylor expand $f(f(x))$ about $x_{1}^{* *}=\frac{r}{2}+\frac{\sqrt{r^{2}+4 r}}{2}$ (for example) as $\hat{x}_{n}^{(1)}=x_{1}^{* *}+\xi_{n}^{(1)}$ which allows us to obtain :

$$
\begin{equation*}
\xi_{n+1}^{(1)}=(1-r(4+r)) \xi_{n}^{(1)}+(r(4+r)-3 \sqrt{r(r+4)}) \xi_{n}^{(1)^{2}} . \tag{2.4.24}
\end{equation*}
$$

If we now rescale by

$$
\begin{equation*}
a_{1}=(r(4+r)-3 \sqrt{r(r+4)}) \tag{2.4.25}
\end{equation*}
$$

and define

$$
\begin{equation*}
-(1+\tilde{r})=(1-r(4+r)) \tag{2.4.26}
\end{equation*}
$$

we obtain:

$$
\begin{equation*}
\xi_{n+1}^{(1)}=-(1+\tilde{r}) \xi_{n}^{(1)}+\xi_{n}^{(1)^{2}} \tag{2.4.27}
\end{equation*}
$$

which is indeed similar to the initial Eq.(2.4.15). More precisely, if we now wish to know when the bifurcation to period four will happen it will be given exactly for $\tilde{r}=0$ for the same reason that the first bifurcation took place at $r=0$ in Eq. 2.4.15. From Eq. 2.4.26 we find:

$$
\begin{equation*}
r=-2+\sqrt{6+\tilde{r}} \tag{2.4.28}
\end{equation*}
$$

which allows us to deduce for $\tilde{r}=0$ the exact value for the bifurcation of period 45 56

$$
\begin{equation*}
r_{1}=-2+\sqrt{6} \tag{2.4.29}
\end{equation*}
$$

We may then use Eq. 2.4 .28 ) as a general rule at all scales similarly to what we did in the previous section with the function $s$ and the Taylor expansion of $\sin (g)$. Hence the general approximate FSS group mapping for all scales is

$$
\begin{equation*}
r_{l}=-2+\sqrt{6+r_{l-1}} \tag{2.4.30}
\end{equation*}
$$

where Eq. 2.4 .28 is obtained for $r_{0}=0$. Our objective is thus to calculate $\delta$ by taking the limit of the quotient of differences. This can be done expanding about the fixed-point solution as:

$$
\begin{equation*}
r_{l+1}=b\left(r_{l}\right)=b\left(r_{\infty}+\rho_{l}\right) \simeq_{l \rightarrow \infty} r_{\infty}+b^{\prime}\left(r_{\infty}\right) \rho_{l} \tag{2.4.31}
\end{equation*}
$$

Then taking the difference:

$$
\begin{equation*}
r_{l+1}-r_{l} \simeq b^{\prime}\left(r_{\infty}\right)\left(\rho_{l}-\rho_{l-1}\right)=b^{\prime}\left(r_{\infty}\right)\left(r_{l}-r_{l-1}\right) \tag{2.4.32}
\end{equation*}
$$

leads to :

$$
\begin{equation*}
\delta=b^{\prime}\left(r_{\infty}\right)^{-1} \tag{2.4.33}
\end{equation*}
$$

We then find $\delta_{a p p} \simeq 5.12$ instead of the exact value $\delta=4.669 \ldots$ which leads to a relative error of 9.7 percent. This error is reasonable for such a crude second degree polynomial approximation of $f_{l}$ and it is the typical

[^28]order of magnitude of the leading approximation in the NPRG framework which is called the local potential approximation .

The second Feigenbaum constant, which is commonly denoted as $\alpha$, is related to the scaling factor that we used to set the coefficient of $\xi^{(1)^{2}}$ to one. More precisely the scaling factor in the infinite $l$ limit would be equal to $\alpha$ if our calculations were exact. Instead we expect our truncations to lead to an approximate value for $\alpha$. To obtain this scaling factor we extend the relation between $a_{1}$ and $r=r_{0}$ given by Eq. 2.4 .25 to all scales as $a_{l}=r_{l-1}\left(4+r_{l-1}\right)-3 \sqrt{r_{l-1}\left(r_{l-1}+4\right)}$ giving in the infinite $l$ limit $\alpha_{a p p}=-2.24$ instead of the exact result $\alpha=-2.50 \ldots$ which leads to a relative error of $10.4 \%$. Another renormalization scheme 76 is to consider the map:

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right)=1-r x_{n}^{2} \tag{2.4.34}
\end{equation*}
$$

where $f$ maps the interval $[-1,1]$ onto itself for $1<r<2$ and has a maximum at $x=0$ where $f(0)=1$. If we now use the renormalization operator $\mathcal{R}(f)=f \circ f=f_{1}$, the function will not verify the same properties but one may show [76] that there exists $a_{1}$ such that $\mathcal{R}_{2}(f)=a_{1} f\left(f\left(x / a_{1}\right)\right)$ where $\mathcal{R}_{2}(f)$ verifies the same properties. This definition allows the mapped function $\mathcal{R}_{2}(f)$ at a larger scale $l$ to "look similar' 57 , Instead of computing the flow and following the initial condition to a fixed-point at $l=\infty$, we may compute directly approximations at $l=\infty$. This is possible because iterating $\mathcal{R}_{2}$ on Eq. 2.4 .34 leads to a fixed-point function

$$
\begin{equation*}
\mathcal{R}_{2}\left(f^{*}\right)=\alpha f^{*}(x / \alpha)=f^{*}(x) \tag{2.4.35}
\end{equation*}
$$

where $\alpha$ is the converged sequence of the scaling factors $a_{l}$ and corresponds to the exact second Feigenbaum constant. By Taylor expanding the function $f^{*}$ one may obtain its coefficients and the value of $\alpha$ by keeping the normalization $f(0)=1$ and the parity of the functions which are conserved along the RG flow. Truncating $f^{*}$ to a second order polynomial then leads once more to an error around $10 \%$ for $\alpha$ but the approximate value of $\alpha$ is not the same as the approximation we found before. This is an example of how different RG schemes can lead to different results when approximations are involved. In the case of the NPRG formalism, different choices of the regulator $R_{k}$ in the $\Gamma$ flow Eq. 2.3 .36 correspond to different RG schemes as well which also leads to different results when approximations are involved. Let us also note that the functional equation that $f^{*}$ verifies is independent of the details of our logistic map. Hence it has the potential to reappear in seemingly unrelated problems and is a concrete example of universality. In many body systems, such as in condensed matter, the situation is perfectly analogous where there are fixed-point functionals and universal numbers which play the role of the Feigenbaum constants. These are the critical exponents and more complicated numbers that constitute the so called "conformal field theory data".

Finally, let us note that in the case of Eq. 2.4.35, higher order polynomial truncations lead to rather rapidly converging results. For instance, using polynomials of order 6 instead of 2 leads to an error on $\alpha$ of order $10^{-6}$ [76]. This in turn shows that the approximations scheme converges quickly without an explicit small parameter in the model. The small parameter is obtained instead only after having performed approximations and is given by the inverse of the radius of convergence of the universal function in Eq. 2.4.35. . This is analogous to the situation in FRG where the small parameter can only be estimated a posteriori and it is also related to a radius of convergence 77 .

### 2.4.3 NPRG approximation schemes

In the present section we will derive the derivative expansion ansatz from an explicit quasi-locality hypothesis which will allow us to discuss the validity of this approximation. We shall also explore the space of all possible approximation schemes. From these approximations we may retrieve the well known derivative expansion, perturbation theory, BMW and LPA" approximations. We hope that such a minimal scheme will be of use for further investigations of convergent approximation schemes. In particular it will be interesting to check whether the real space RG formulations that are known to have nice convergence properties are also contained within this formulation.

The $\Gamma$ flow is a functional equation and thus our first task is to choose a field configuration $\phi^{a}(x) \underset{\text { def }}{=} \phi_{\alpha}$ where $\alpha$ includes both spatial and internal indices. As long as the flow is kept exact the choice of $\phi_{\alpha}$ does not matter and any choice would allow us to obtain exact results for quantities such as critical exponents,

[^29]mass ratios, etc. In particular, we may choose the simplest field configuration which is a uniform field configuration $\phi_{0}^{a}$. When approximations are involved it is possible that some field configurations might be better than others but in general a uniform field often provides sufficiently accurate results.

Moreover, as the $\Gamma$ flow contains a functional derivative $\frac{\delta^{2}}{\delta \phi \delta \phi} \Gamma_{k}$, it is necessary to consider the neighborhood of our base field as $\phi^{a}(x)=\varphi^{a}(x)+\phi_{0}^{a} \underset{\text { def }}{=} \varphi_{\alpha}+\phi_{0}^{a}$, such that we may calculate derivatives. For notational reasons, we now consider the $k$ dependence of $\Gamma_{k}$ implicit and we Taylor expand $\Gamma$ about our base field as:

$$
\begin{equation*}
\Gamma[\phi(x)]=\Gamma\left[\phi_{0}^{a}\right]+D \Gamma_{\phi_{0}}(\varphi)+\frac{1}{2!} D^{2} \Gamma_{\phi_{0}}(\varphi \otimes \varphi)+\frac{1}{3!} D^{3} \Gamma_{\phi_{0}}(\varphi \otimes \varphi \otimes \varphi)+\ldots \frac{1}{n!} D^{n} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n}}\right) \tag{2.4.36}
\end{equation*}
$$

where:

$$
\begin{align*}
D^{n} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n}}\right) & =\left.\sum_{\alpha_{1}, \ldots, \alpha_{n}} \varphi_{\alpha_{1}} \varphi_{\alpha_{2}} \ldots \varphi_{\alpha_{n}} \frac{\delta^{n}}{\delta \phi_{\alpha_{1}} \delta \phi_{\alpha_{2}} \ldots \delta \phi_{\alpha_{n}}} \Gamma\right|_{\phi_{0}^{a}} \\
& =\left.\int_{x_{1}, x_{2}, \ldots, x_{n}} \sum_{a_{1}, \ldots, a_{n}} \varphi^{a_{1}}\left(x_{1}\right) \varphi^{a_{2}}\left(x_{2}\right) \ldots \varphi^{a_{n}}\left(x_{n}\right) \frac{\delta^{n}}{\delta \phi^{a_{1}}\left(x_{1}\right) \ldots \delta \phi^{a_{1}}\left(x_{n}\right)} \Gamma\right|_{\phi_{0}^{a}} \tag{2.4.37}
\end{align*}
$$

Let us now consider,

$$
\begin{equation*}
D^{2} \Gamma_{\phi_{0}}(\varphi \otimes \varphi)=\int_{x, y} \sum_{a, b} \varphi^{a}(x) \varphi^{b}(y) \Gamma_{a, b}^{(2)}\left(x, y ; \phi_{0}\right) \tag{2.4.38}
\end{equation*}
$$

with

$$
\Gamma_{a_{1}, a_{2}, \ldots, a_{n}}^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)=\left.\frac{\delta^{n}}{\delta \phi^{a_{1}}\left(x_{1}\right) \ldots \delta \phi^{a_{1}}\left(x_{n}\right)} \Gamma\right|_{\phi_{0}^{a}}
$$

To simplify the following let us also consider a 1-dimensional model with a single field variable $\phi^{a}=\phi$. In the following we shall consider that the system is sufficiently far from a phase transition that the $\Gamma^{(n)}$ are quasi-local.

Let us then make this notion of quasi-locality more explicit. Using translation invariance we have $\Gamma^{(2)}\left(x, y ; \phi_{0}\right)=\Gamma^{(2)}\left(x-y, 0 ; \phi_{0}\right)=\gamma_{2}\left(x-y ; \phi_{0}\right)$. If $\gamma_{2}$ is local then it must be $P\left(\partial^{2}\right) \delta(x-y)$ with $P$ a polynomial. Instead we say that $\gamma_{2}$ is quasi-local if it admits a polynomial approximation as $P\left(\partial^{2}\right) \delta(x-y)$ when the field $\varphi$ is chosen to be sufficiently smooth. An equivalent definition is that $\gamma_{2}$ is peaked about the origin. Indeed, if $\gamma_{2}$ is peaked then we may Taylor expand the fields $\varphi$ as :

$$
\begin{align*}
D^{2} \Gamma_{\phi_{0}}(\varphi \otimes \varphi) & =\int_{x, y} \varphi(x) \varphi(y) \gamma_{2}\left(x-y ; \phi_{0}\right)=\int_{z, y} \varphi(z+y) \varphi(y) \gamma_{2}\left(z ; \phi_{0}\right) \\
& =\int_{z, y} \varphi(y) \gamma_{2}\left(z ; \phi_{0}\right)\left(\varphi(y)+z \varphi^{\prime}(y)+\frac{z^{2}}{2} \varphi^{\prime \prime}(y)+\ldots\right)  \tag{2.4.39}\\
& =\int_{y} \varphi(y)\left(u_{2,0} \varphi(y)+u_{2,1} \varphi^{\prime}(y)+u_{2,2} \varphi^{\prime \prime}(y)+\ldots\right) \\
& =\int_{z, y} \varphi(y) \varphi(z)\left(u_{2,0} \delta(z-y)-u_{2,1} \delta^{\prime}(z-y)+u_{2,2} \delta^{\prime \prime}(z-y)+\ldots\right)
\end{align*}
$$

where

$$
\begin{equation*}
u_{2, i}=\int_{z} \gamma_{2}\left(z ; \phi_{0}\right) z^{i} \tag{2.4.40}
\end{equation*}
$$

The term $\varphi(y) \varphi^{\prime}(y)$ is a total derivative and can be removed.

In the case of $\Gamma^{(3)}\left(x, y, z ; \phi_{0}\right)=\gamma_{3}\left(x-z, y-z ; \phi_{0}\right)$ one must impose quasi-locality on both variables. This then allows us to write:

$$
\begin{align*}
D^{3} \Gamma_{\phi_{0}}(\varphi \otimes \varphi \otimes \varphi)= & \int_{x, y, z} \varphi(x) \varphi(y) \varphi(z) \gamma_{3}\left(x-z, y-z ; \phi_{0}\right)=\int_{h, l, z} \varphi(z+h) \varphi(z+l) \varphi(z) \gamma_{3}\left(h, l ; \phi_{0}\right) \\
= & \int_{h, l, z} \varphi(z) \gamma_{3}\left(h, l ; \phi_{0}\right)\left(\varphi(z) \varphi(z)+\varphi(z) \varphi^{\prime}(z)(h+l)+\right. \\
& \left.\frac{\left(h^{2}+l^{2}\right)}{2} \varphi(z) \varphi^{\prime \prime}(z)+h l \varphi^{\prime}(z)^{2}+\ldots\right) \\
= & \int_{z} u_{3,0} \varphi(z)^{3}+u_{3,1} \varphi(z)^{2} \varphi^{\prime}(z)+u_{3,2} \varphi(z) \varphi^{\prime}(z)^{2}+\ldots \tag{2.4.41}
\end{align*}
$$

where we have used an integration by parts on the last line to go from $\varphi(z)^{2} \varphi^{\prime \prime}(z)$ to $\varphi(z) \varphi^{\prime}(z)^{2}$ and we have grouped the integrals over $h, l$ into the coefficients $u_{3, i}$. This can be carried about for all $\Gamma^{(n)}$. Inserting these expressions into Eq. 2.4.36) allows us to group the non derivative terms as $u_{0,0}\left(\phi_{0}\right)+u_{1,0}\left(\phi_{0}\right) \varphi(z)+$ $u_{2,0}\left(\phi_{0}\right) \varphi(z)^{2}+\ldots$. This expression can then be resummed to a potential $U\left(\phi_{0}+\varphi(z)\right)=U(\phi(z))$. The same resummation can be carried about grouping all of the $\varphi^{\prime}(z)^{2}$ terms. If we then stop this expansion in derivatives to terms involving at most 2 derivatives, our approximate $\Gamma_{k}$ function is:

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int_{x} U_{k}(\phi)+\frac{Z_{k}(\phi)}{2} \phi^{\prime 2} \tag{2.4.42}
\end{equation*}
$$

or more generally in $d$ dimensions :

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int_{x} U_{k}(\phi)+\frac{Z_{k}(\phi)}{2}(\nabla \phi)^{2} \tag{2.4.43}
\end{equation*}
$$

where we notice that $\Gamma_{k}$ is then the most general expression containing at most two gradients 5 ,
This approximation, which is called the order 2 of the derivative expansion, can be both simplified and improved. First, concerning the simplification, one may notice a posteriori that $Z$ usually has a weak field dependence as in Fig. 2.4.2. One may then replace $Z_{k}(\phi)$ by a $k$-dependent constant $Z_{k}$. This approximation allows us to estimate the anamolous dimension $\eta$ rather simply. However, for theories where $\eta$ can be neglected ${ }^{59}$ it is possible to consider a further approximation setting $Z_{k}$ to 1 . This approximation is called the local potential approximation (LPA) and has the merit of being often qualitatively and sometimes quantitatively accurate. The approximation where we consider the $k$-dependent constant $Z_{k}$ is called the LPA'.

Next, concerning the improvements of the order 2 of the derivative expansion, the natural extension is to go further in the expansion of $\partial^{n} \delta$ in the $\Gamma^{(r)}$. For a rotationally symmetric theory the next order involves including terms $\partial^{4} \delta$. This can be carried out as before using the quasi-locality expansion of the $\Gamma^{(r)}$ or equivalently including all possible terms up to order $\partial^{4}$ directly into the expression of $\Gamma_{k}$. One then obtains 78]:

$$
\begin{align*}
\Gamma_{k}= & \int U(\phi)+\frac{1}{2} Z(\phi) \partial_{\nu} \phi \partial_{\nu} \phi+\frac{1}{2} W_{1}(\phi)\left(\partial_{\nu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \partial_{\mu} \phi\right)+\frac{1}{2} W_{2}(\phi)\left(\partial_{\mu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)+  \tag{2.4.44}\\
& \frac{1}{4} W_{3}(\phi)\left(\partial_{\mu} \phi\right)\left(\partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)
\end{align*}
$$

In the following we will focus mainly on $d=1$ to avoid taking into account the different ways the derivative operators may contract but our final results will be valid in all dimensions. We thus state the form of the

[^30]effective action at order $\partial^{6}$ in $d=1$ 77]:
\[

$$
\begin{align*}
\Gamma_{k}= & \int U(\phi)+\frac{1}{2} Z(\phi) \phi^{\prime 2}+\frac{1}{2} W_{1}(\phi) \phi^{\prime 2}+\frac{1}{4!} W_{3}(\phi) \phi^{\prime 4}+\frac{1}{2} X_{1}(\phi) \phi^{\prime \prime \prime 2}+  \tag{2.4.45}\\
& \frac{1}{6} X_{2}(\phi) \phi^{3}+\frac{1}{4} X_{3}(\phi) \phi^{\prime 2} \phi^{\prime 2}+\frac{1}{6!} X_{4}(\phi) \phi^{\prime 6}
\end{align*}
$$
\]

We have only written independent terms here that are not related using integration by parts.
One might wonder however what are the sources of error from such an approximation scheme. Consider then the flow of the potential obtained by evaluating the $\Gamma$ flow on a uniform field configuration $\phi(x)=\phi_{0}$ and evaluating the trace in Fourier space. $\Gamma^{(2)}$ in the $\Gamma$ flow is derived from the Ansatz of Eq. 2.4.45 by taking functional derivatives and evaluating on a uniform background $\phi=\phi_{0}$. In Fourier space we obtain:

$$
\begin{equation*}
\Gamma^{(2)}\left(q,-q ; \phi_{0}\right)=U^{\prime \prime}\left(\phi_{0}\right)+Z\left(\phi_{0}\right) q^{2}+W_{1}\left(\phi_{0}\right) q^{4}+X_{1}\left(\phi_{0}\right) q^{6} \tag{2.4.46}
\end{equation*}
$$

The flow for the potential is then:

$$
\begin{equation*}
\partial_{t} U_{t}=\frac{1}{2} \int_{q} \frac{k \partial_{k} R_{k}(q)}{R_{k}(q)+U^{\prime \prime}\left(\phi_{0}\right)+Z\left(\phi_{0}\right) q^{2}+W_{1}\left(\phi_{0}\right) q^{4}+X_{1}\left(\phi_{0}\right) q^{6}} \tag{2.4.47}
\end{equation*}
$$

This is to be compared with the exact flow of the potential:

$$
\begin{equation*}
\partial_{t} U_{t}=\frac{1}{2} \int_{q} \frac{k \partial_{k} R_{k}(q)}{R_{k}(q)+U^{\prime \prime}\left(\phi_{0}\right)+\widehat{Z}\left(\phi_{0}, q^{2}\right) q^{2}} \tag{2.4.48}
\end{equation*}
$$

where $U^{\prime \prime}\left(\phi_{0}\right)+\widehat{Z}\left(\phi_{0}, q^{2}\right) q^{2}$ is simply a rewriting of $\Gamma^{(2)}\left(q,-q ; \phi_{0}\right)$. It is then clear that the derivative expansion leads to a Taylor expansion of $\Gamma^{(2)}\left(q,-q ; \phi_{0}\right)$ about the momentum $q^{2}$. However such a polynomial expansion is valid only within the radius of convergence of $\widehat{Z}\left(\phi_{0}, q^{2}\right)$ with respect to $q^{2}$ which we denote as $k_{2, I R}^{2}$. Let us also denote $k_{U V}$ the range in which the integral over the internal momentum $q$ in the $\Gamma$ flow effectively contributes. More precisely, $k_{U V}$ is defined such that the rest term in the integral of the $\Gamma$ flow $\int_{q>k_{U V}}$ can be omitted as the error produced by discarding the high momenta is smaller than the error of the approximation scheme. $k_{U V}$ is then essentially controlled by the UV cut-off term $\partial_{t} R$ in the $\Gamma$ flow which is then negligible for $q^{2}>k_{U V}^{2}$. The Taylor expansion of $\widehat{Z}\left(\phi_{0}, q^{2}\right)$ in $q^{2}-$ thus the derivative expansion - is then meaningful only if the ratio $k_{U V} / k_{2, I R}$ is small such that the large error from the Taylor expansion of $\widehat{Z}\left(\phi_{0}, q^{2}\right)$ outside of $k_{I R}$ is effectively suppressed. However, this is not the only source of error as $Z\left(\phi_{0}\right), W_{1}\left(\phi_{0}\right)$ and $X_{1}\left(\phi_{0}\right)$ are not determined at this point. As such we must compute the flow of these quantities. The flow of $Z$ may be obtained by noting that:

$$
\begin{equation*}
Z\left(\phi_{0}\right)=\left.\partial_{p^{2}} \Gamma^{(2)}\left(p,-p ; \phi_{0}\right)\right|_{p=0} \tag{2.4.49}
\end{equation*}
$$

The flow of $Z$ may then be obtained from that of $\Gamma^{(2)}\left(p,-p ; \phi_{0}\right)$ which reads:

$$
\begin{align*}
\partial_{k} \Gamma_{k}^{(2)}(p,-p)= & \int \frac{\mathrm{d}^{\mathrm{d}} \mathrm{q}}{(2 \pi)^{\mathrm{d}}} \partial_{\mathrm{k}} \mathrm{R}_{\mathrm{k}}(\mathrm{q})\left(\mathrm{G}_{\mathrm{k}}(\mathrm{q}) \Gamma_{\mathrm{k}}^{(3)}(\mathrm{p}, \mathrm{q},-\mathrm{p}-\mathrm{q}) \mathrm{G}_{\mathrm{k}}(\mathrm{q}+\mathrm{p}) \Gamma_{\mathrm{k}}^{(3)}(-\mathrm{p}, \mathrm{p}+\mathrm{q},-\mathrm{q}) \mathrm{G}_{\mathrm{k}}(\mathrm{q})-\right. \\
& \left.\frac{1}{2} G_{k}(q) \Gamma_{k}^{(4)}(p,-p, q,-q) G_{k}(q)\right) \tag{2.4.50}
\end{align*}
$$

where $G_{k}=\left(\Gamma_{k}^{(2)}+R_{k}\right)^{-1}$. It is then sufficient to expand the flow of $\Gamma_{k}^{(2)}(p,-p)$ to order $p^{2}$ to obtain the exact flow of $Z$. However, the derivative expansion replaces $\Gamma_{k}^{(3)}$ and $\Gamma_{k}^{(4)}$ with polynomial expressions which then also leads to an expansion in the momentum $q$ of the integral. Thus once more, $k_{U V}$ should be sufficiently small to suppress the errors outside of the domain of validity of these polynomial expansions. One might also wonder to what extent is this useful for scale invariant theories which will be the main object of interest in the following. Indeed scale invariant theories are not quasi local as the correlation functions behave as power laws rather than exponentials. However, in presence of the regulator, the system is never critical and the correlation functions do not scale like power laws. Instead the branch cut singularities at
the origin of momenta, due to the non integer exponents in the power laws, are now pushed to some value of order $k$ where $k$ is defined by $R_{k}=k^{2} r\left(q^{2} / k^{2}\right)$. For example, in momentum space, 79 considers the following Ansatz for the two-point function $\Gamma_{k}^{(2)}\left(p,-p ; \phi_{0}\right)=A p^{2}\left(p^{2}+b k^{2} c\left(\phi_{0}\right)\right)^{-\eta_{k} / 2}+k^{2} f\left(\phi_{0}\right)$ such that the branch-cut singularity at $p=0$ for $k=0$ is now at $p^{2}=-b k^{2} c\left(\phi_{0}\right)$ for finite $k$. The absence of any singularity at the origin then allows us to Taylor expand the $\Gamma^{(n)} 60$ in powers of momenta about the zero momentum origin. However, the momentum expansion of $\Gamma^{(n)}$ is a valid approximation only within a ball of finite radius in $\mathbb{R}^{n d}$ centered about the origin where $d$ is the space dimension. Due to the infrared regularization from $R_{k}$ in the propagator, the radius of this ball should be of order $k$ when the system is at criticality and of the order of the smallest intrinsic length scale outside of criticality. We denote by $k_{n, I R}$ the radius of this ball and we consider $k_{I R}=\min _{n} k_{I R}^{(n)}$. The momentum expansion is then valid only for $q_{i}<k_{I R}$ where the $q_{i}$ are the momenta in which the expansion in momenta of the $\Gamma^{(n)}$ is performed. Hence, within a momentum expansion, the ratio $k_{U V} / k_{I R}$ must be small to obtain reasonable approximations. $k_{U V}$ is given mostly by the profile of the regulator but $k_{I R}$ needs to be determined a posteriori. If this ratio $k_{U V} / k_{I R}$ is sufficiently small one may obtain rapidly converging results such as in [77.

Nonetheless, it is well known that higher order Taylor expansions tend to be dramatically inaccurate outside of their domain of convergence. This then adds extra pressure on the regulator to suppress these extreme errors by decaying sufficiently fast for large momentum. One might then be tempted to choose a regulator of the form $s\left(q^{2} / k_{U V}^{2}\right) \Theta\left(k_{U V}^{2}-q^{2}\right)$, where $\Theta$ is the Heaviside step function, and choose $k_{U V}$ sufficiently small such that the domain in which the error is large is set to zero. However, rescaling $k_{U V}$ simply changes the definition of $k$ and thus $k_{I R}$ is rescaled by the same factor thereby conserving the ratio $k_{I R} / k_{U V}$. Moreover, the derivative expansion requires Taylor expanding both the internal momentum within the integral and external momenta that appear via functional derivatives of the $\Gamma$ flow. Expanding in the external momenta requires Taylor expanding the propagator about some point $q$ in the integral. As we have $G_{k}=\left(\Gamma_{k}^{(2)}+R_{k}\right)^{-1}$ this implies that the regulator must also be Taylor expanded. The derivative expansion then adds a stringent condition of analycity or at least $C^{n} \sqrt{61}$ on the regulator. Thus, for a regulator of the form $s\left(q^{2} / k_{U V}^{2}\right) \Theta\left(k_{U V}^{2}-q^{2}\right), s$ would have to behave as $\left(1-q^{2} / k_{U V}^{2}\right)^{n}$ near $q=k_{U V}$ to smooth the discontinuity at $q=k_{U V}$ of the $\Theta$ function ${ }^{62}$. The problem is then that one may rescale $k_{U V}$ such that in the neighborhood of $q=k_{U V}$ the regulator behaves as $k^{2}\left(1-\frac{q^{2}}{n k^{2}}\right)^{n} \Theta\left(1-\frac{q^{2}}{n k^{2}}\right)$ and as one goes to higher and higher orders of the derivative expansion, $n$ increases and becomes close to the limit function $\lim _{n \rightarrow \infty} k^{2}\left(1-\frac{q^{2}}{n k^{2}}\right)^{n} \Theta\left(1-\frac{q^{2}}{n k^{2}}\right)=k^{2} \exp \left(-q^{2} / k^{2}\right){ }^{64}$. Thus, even for sharp cutoff functions, the large momenta regime of the $\Gamma^{(n)}$ can not be removed and the large errors of high order Taylor expansions may ultimately deteriorate results. Other hints that the derivative expansion might behave badly at sufficiently high orders reside on how results become more and more sensitive to optimization 77 .

Moreover, the derivative expansion is ill-equipped to answer momentum dependent questions such as the existence of bound states or the dependence on momenta of correlation functions. Our objective in this section is then to consider the space of all possible approximation schemes that preserve full momentum dependence.

The expansion in $\partial^{n} \delta$ of the $\Gamma^{(r)}$ is equivalent to an expansion in momenta in Fourier space. We shall show this by computing the $\Gamma^{(r)}$ directly in Fourier space rather than relying on an effective action ansatz. While this will be interesting in its own respect it will in fact be crucial in understanding how we may generalize this approximation scheme. First let us compute the functional derivatives of Eq. 2.4 .44 to have a clear view of the kind of momentum expansion we expect.

Let us consider Eq. 2.4 .44 ) in the neighborhood of a uniform field configuration. The functional derivatives in Fourier space about this point lead to :

[^31]\[

$$
\begin{align*}
\Gamma\left[\phi_{0}\right]= & \mathcal{V} U\left(\phi_{0}\right) \\
\Gamma^{(1)}\left(p ; \phi_{0}\right)= & U^{\prime}\left(\phi_{0}\right) \\
\Gamma^{(2)}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)-Z\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2} \\
\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{\prime \prime \prime}\left(\phi_{0}\right)-Z^{\prime}\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}^{\prime}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2}+W_{2}\left(\phi_{0}\right) p_{1}^{2} p_{2} \cdot p_{3}+\text { perm }  \tag{2.4.51}\\
\Gamma^{(4)}\left(p_{1}, p_{2}, p_{3}, p_{4} ; \phi_{0}\right)= & U^{\prime \prime \prime \prime}\left(\phi_{0}\right)-Z^{\prime \prime}\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}^{\prime \prime}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2}+W_{2}^{\prime}\left(\phi_{0}\right) p_{1}^{2} p_{2} \cdot p_{3}+ \\
& W_{3}\left(\phi_{0}\right) p_{1} \cdot p_{2} p_{3} \cdot p_{4}+\operatorname{perm} \\
\Gamma^{(5)}\left(p_{1}, p_{2}, p_{3}, p_{4}, p_{5} ; \phi_{0}\right)= & U^{(5)}\left(\phi_{0}\right)-Z^{\prime \prime \prime}\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}^{\prime \prime \prime}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2}+W_{2}^{\prime \prime}\left(\phi_{0}\right) p_{1}^{2} p_{2} \cdot p_{3}+ \\
& W_{3}^{\prime}\left(\phi_{0}\right) p_{1} \cdot p_{2} p_{3} \cdot p_{4}+\operatorname{perm}
\end{align*}
$$
\]

where the term perm implies permutations of the indices in momenta and "." refers to the usual dot product. To impose momentum conservation we explicitly take $p_{n}=-\sum_{1}^{n-1} p_{i}$. This allows us to define :

$$
\begin{equation*}
\gamma_{n}\left(p_{1}, p_{2}, \ldots, p_{n-1} ; \phi_{0}\right)=\Gamma^{(n)}\left(p_{1}, p_{2}, \ldots, p_{n-1},-\sum_{i=1}^{n-1} p_{i} ; \phi_{0}\right) \tag{2.4.52}
\end{equation*}
$$

with $\gamma_{0}=\mathcal{V} U\left(\phi_{0}\right)$ and $\gamma_{1}=U^{\prime}\left(\phi_{0}\right)$. Thus, imposing momentum conservation we obtain :

$$
\begin{align*}
\gamma_{0}= & \mathcal{V} U\left(\phi_{0}\right) \\
\gamma_{1}= & U^{\prime}\left(\phi_{0}\right) \\
\gamma_{2}\left(p ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)+Z(\phi) p^{2}+W_{1}(\phi) p^{4} \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{\prime \prime \prime}\left(\phi_{0}\right)+Z^{\prime}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}+p_{1} \cdot p_{2}\right)+ \\
& W_{1}^{\prime}\left(\phi_{0}\right)\left(p_{1}^{4}+2 p_{1}^{2} p_{1} \cdot p_{2}+3\left(p_{1} \cdot p_{2}\right)^{2}+2 p_{2}^{2} p_{1} \cdot p_{2}+p_{2}^{4}\right)+ \\
& W_{2}\left(\phi_{0}\right)\left(\left(p_{1} \cdot p_{2}\right)^{2}-p_{1}^{2} p_{2}^{2}\right) \\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{\prime \prime \prime \prime}\left(\phi_{0}\right)+Z^{\prime \prime}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}+p_{3}^{2}+p_{1} \cdot p_{2}+p_{1} \cdot p_{3}+p_{3} \cdot p_{2}\right)+ \\
& W_{1}^{\prime \prime}\left(\phi_{0}\right)\left(p_{1}^{4}+2 p_{1}^{2} p_{1} \cdot p_{2}+2 p_{1} \cdot p_{2} p_{1} \cdot p_{3}+3\left(p_{1} \cdot p_{2}\right)^{2}+\text { perm }\right)+  \tag{2.4.53}\\
& W_{2}^{\prime}\left(\phi_{0}\right)\left(\left(p_{1} \cdot p_{2}\right)^{2}-p_{1}^{2} p_{2}^{2}+2\left(p_{1} \cdot p_{2}\right)\left(p_{2} \cdot p_{3}\right)+\text { perm }\right)+ \\
& -W_{3}\left(\phi_{0}\right)\left(p_{1}^{2} p_{2} \cdot p_{3}+p_{2}^{2} p_{1} \cdot p_{3}+p_{3}^{2} p_{1} \cdot p_{2}+\right. \\
& \left.2\left(p_{1} \cdot p_{2}\right)\left(p_{1} \cdot p_{3}\right)+2\left(p_{1} \cdot p_{2}\right)\left(p_{2} \cdot p_{3}\right)+2\left(p_{1} \cdot p_{3}\right)\left(p_{2} \cdot p_{3}\right)\right) \\
\gamma_{5}\left(p_{1}, p_{2}, p_{3}, p_{4} ; \phi_{0}\right)= & U^{(5)}\left(\phi_{0}\right)+Z^{\prime \prime \prime}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{1} \cdot p_{2}+\text { perm }\right)+ \\
& W_{1}^{\prime \prime \prime}\left(\phi_{0}\right)\left(p_{1}^{4}+2 p_{1}^{2} p_{1} \cdot p_{2}+2 p_{1} \cdot p_{2} p_{1} \cdot p_{3}+3\left(p_{1} \cdot p_{2}\right)^{2}+\text { perm }\right)+ \\
& W_{2}^{\prime \prime}\left(\phi_{0}\right)\left(\left(p_{1} \cdot p_{2}\right)^{2}-p_{1}^{2} p_{2}^{2}+2\left(p_{1} \cdot p_{2}\right)\left(p_{2} \cdot p_{3}\right)+\text { perm }\right)+ \\
& -W_{3}^{\prime}\left(\phi_{0}\right)\left(p_{1}^{2} p_{2} \cdot p_{3}+2\left(p_{1} \cdot p_{2}\right)\left(p_{1} \cdot p_{3}\right)+\right.\text { perm }
\end{align*}
$$

Now let us show that it is in fact possible to retrieve these expressions in a much simpler manner. For simplicity we shall mainly consider $d=1$ but we will discuss the novelties at $d \geq 2$ in Appendix F $\mathrm{F}^{65}$. To start let us consider only monomials in momenta up to total degree 2. The total degree of a monomial $X_{1}^{m_{1}} X_{2}^{m_{2}} \ldots X_{n}^{m_{n}}$ is equal to $\sum_{i} m_{i}$ and the total degree of a polynomial is the maximum of the total degrees of its monomials. The first important point is that $\Gamma^{(n)}$ is symmetric in its $n$ arguments due to the Schwarz property of partial derivatives. This implies that any polynomial expression of $\gamma_{n}$ must be of the form $P\left(p_{1}, p_{2}, \ldots, p_{n-1},-\sum_{i=1}^{n-1} p_{i}\right)$ where $P$ is a polynomial symmetric in its $n$ arguments.

[^32]Using the fundamental theorem of symmetric polynomials there exists a symmetric polynomial $Q$ such that $P\left(X_{1}, X_{2}, \ldots, X_{n}\right)=Q\left(\sigma_{1, n}, \sigma_{2, n}, \ldots, \sigma_{n, n}\right)$ where the $\sigma_{s, n}$ are the elementary symmetric polynomials.

$$
\begin{equation*}
\sigma_{s, n}\left(X_{1}, X_{2}, \ldots, X_{n}\right)=\sum_{1 \leq i_{1}<i_{2} \ldots<i_{s} \leq n} X_{i_{1}} X_{i_{2}} \ldots X_{i_{s}} \tag{2.4.54}
\end{equation*}
$$

As examples:

$$
\begin{align*}
& \sigma_{1,3}(X, Y, Z)=X+Y+Z \\
& \sigma_{2,3}(X, Y, Z)=X Y+Y Z+X Z  \tag{2.4.55}\\
& \sigma_{3,3}(X, Y, Z)=X Y Z
\end{align*}
$$

Thus, for $P_{2}\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ a polynomial of total degree 2 we have :

$$
\begin{equation*}
P_{2}=A_{n} \sigma_{1, n}+B_{n} \sigma_{1, n}^{2}+C_{n} \sigma_{2, n} \tag{2.4.56}
\end{equation*}
$$

where we constructed all possible terms of total degree 2. However, $\sigma_{1, n}=\sum X_{i}$ does not contribute to $\gamma_{n}$ due to conservation of momentum. Thus we have:

$$
\begin{equation*}
P_{2}\left(p_{1}, p_{2}, \ldots, p_{n-1},-\sum_{i=1}^{n-1} p_{i}\right)=C_{n} \sigma_{2, n}\left(p_{1}, p_{2}, \ldots, p_{n-1},-\sum_{i=1}^{n-1} p_{i}\right) \tag{2.4.57}
\end{equation*}
$$

where

$$
\begin{align*}
\sigma_{2,2}(p,-p) & =-p^{2} \\
\sigma_{2,3}\left(p_{1}, p_{2},-p_{1}-p_{2}\right) & =-\left(p_{1}^{2}+p_{2}^{2}+p_{1} p_{2}\right) \\
\sigma_{2,4}\left(p_{1}, p_{2}, p_{3},-p_{1}-p_{2}-p_{3}\right) & =-\left(p_{1}^{2}+p_{2}^{2}+p_{3}^{2}+p_{1} p_{2}+p_{1} p_{3}+p_{2} p_{3}\right)  \tag{2.4.58}\\
\sigma_{2,5}\left(p_{1}, p_{2}, p_{3}, p_{4},-p_{1}-p_{2}-p_{3}\right) & =-\left(p_{1}^{2}+p_{1} p_{2}+\text { perm }\right)
\end{align*}
$$

This then severely restricts the possible monomials in $\left(p_{1}, p_{2}, \ldots, p_{n-1}\right)$ one should consider. For example we have

$$
\begin{align*}
\gamma_{2}\left(p ; \phi_{0}\right) & =c_{2,0}\left(\phi_{0}\right)+c_{2,1}\left(\phi_{0}\right) p^{2} \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) & =c_{3,0}\left(\phi_{0}\right)+c_{3,1}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}+p_{1} p_{2}\right) \\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right) & =c_{4,0}\left(\phi_{0}\right)+c_{4,1}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}+p_{3}^{2}+p_{1} p_{2}+p_{1} p_{3}+p_{3} p_{2}\right)  \tag{2.4.59}\\
\gamma_{5}\left(p_{1}, p_{2}, p_{3}, p_{4} ; \phi_{0}\right) & =c_{5,0}\left(\phi_{0}\right)+c_{5,1}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{1} p_{2}+\text { perm }\right)
\end{align*}
$$

It is then important to stress that if we had only imposed symmetry with respect to $\left(p_{1}, p_{2}, \ldots, p_{n-1}\right)$ in $\gamma_{n}$ we would have more independent terms. For example, imposing symmetry only in ( $p_{1}, p_{2}$ ) we would have :

$$
\begin{equation*}
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)=c_{3,0}\left(\phi_{0}\right)+c_{3,1}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}\right)+c_{3,2}\left(\phi_{0}\right) p_{1} p_{2} \tag{2.4.60}
\end{equation*}
$$

and there would be no reason why we should have $c_{3,2}\left(\phi_{0}\right)=c_{3,1}\left(\phi_{0}\right)$. This is then an important symmetry to consider when constructing the $\gamma_{n}$ without an underlying effective action and it will be crucial in the following. We may then notice that the expressions in Eq. 2.4 .59 are structurally similar to Eq. 2.4 .53 ) where the generalization to $d$ dimensions here is trivial and consists of simply replacing product with dot products. We remark however that in Eq. 2.4.53 the coefficients are in fact related to one another by derivatives with respect to $\phi_{0}$. Computing the inverse Fourier transforms of the $\gamma_{n}$ to real space we may insert the corresponding values of $\Gamma^{(n)}$ into the functional Taylor expansion in Eq. (2.4.36) to see once more that the $c_{n, 0}$ correspond to coefficients of the Taylor expansion of a function $U$ and the $c_{n, 1}\left(\phi_{0}\right)$ to that of a function $Z$. The signature of this in momentum space is that we have :

$$
\begin{equation*}
\gamma_{n}\left(p_{1}=0, p_{2}=0, \ldots, p_{n-1}=0\right)=U^{(n)}\left(\phi_{0}\right)=\frac{\mathrm{d}^{n}}{\mathrm{~d} \phi_{0}^{n}} \Gamma\left(\phi_{0}\right) \tag{2.4.61}
\end{equation*}
$$

and

$$
\begin{align*}
\gamma_{n}\left(p_{1}=0, p_{2}=0, \ldots p_{i}, \ldots, p_{n-1}=0\right) & =\Gamma^{(n)}\left(p_{1}=0, p_{2}=0, \ldots p_{i}, \ldots, p_{n-1}=0\right) \\
& =\frac{\mathrm{d}^{n-2}}{\mathrm{~d} \phi_{0}^{n-2}} \Gamma^{(2)}\left(p_{i},-p_{i}\right)  \tag{2.4.62}\\
& =\frac{\mathrm{d}^{n-2}}{\mathrm{~d} \phi_{0}^{n-2}} \gamma_{2}\left(p_{i} ; \phi_{0}\right)
\end{align*}
$$

One may verify these relations in Eq. 2.4 .53 but the above relations hold outside of a momentum expansion. This may be generalized as follows. Consider as in 81 an operator $\mathcal{M}_{q_{i}}$ such that:

$$
\begin{equation*}
\mathcal{M}_{q_{i}} \gamma_{n}\left(q_{1}, q_{2}, \ldots q_{i}, \ldots, q_{n-1}\right)=\gamma_{n}\left(q_{1}, q_{2}, \ldots, 0, \ldots, q_{n-1}\right) \tag{2.4.63}
\end{equation*}
$$

The operator form of $\mathcal{M}_{q_{i}}$ will be of use later. We then have:

$$
\begin{equation*}
\mathcal{M}_{q_{i}} \gamma_{n}\left(q_{1}, q_{2}, \ldots, q_{i-1}, q_{i}, q_{i+1}, \ldots, q_{n-1}\right)=\frac{\mathrm{d}}{\mathrm{~d} \phi_{0}} \gamma_{n-1}\left(q_{1}, q_{2}, \ldots,, q_{i-1}, q_{i+1}, \ldots, q_{n-1}\right) \tag{2.4.64}
\end{equation*}
$$

This is a well-known property for which we give a proof in Appendix $G$ This may iterated to obtain :

$$
\begin{equation*}
\gamma_{n}\left(q_{1}, q_{2}, \ldots, q_{i}, 0,0, \ldots, 0\right)=\frac{\mathrm{d}^{n-i-1}}{\mathrm{~d} \phi_{0}^{n-i-1}} \gamma_{i+1}\left(q_{1}, q_{2}, \ldots,, q_{i}\right) \tag{2.4.65}
\end{equation*}
$$

In the Polchinski version these relations were called fluctuation dissipation (FD) relations in 81 and we shall use the same terminology in the present section. We may then apply this to Eq. 2.4 .59 to express the relationship between the coefficients. For example from $\gamma_{3}\left(0, p_{2}\right)=\frac{\mathrm{d}}{\mathrm{d} \phi_{0}} \gamma_{2}\left(p_{2}\right)$ we obtain $c_{3,1}=c_{2,1}^{\prime}$ and $c_{3,0}=c_{2,0}^{\prime}$. Recursively applying this to all $\gamma_{n}$ we retrieve the $\gamma_{n}$ of the order 2 of the derivative expansion by identifying $c_{2,1}=Z$ and $c_{2,0}=U^{\prime \prime}$.

The important point here is that an ansatz for the effective action is not a necessary starting point and it is possible to consider directly polynomial truncations of the $\Gamma^{(n)}$ if FD and permutation symmetry of momenta is imposed. In fact, although the explanation of the previous derivation was rather lengthy, in practice this method is much faster than considering an Ansatz and performing functional derivatives especially for higher order approximations of the derivative expansion. This will be further explained in the following and in Appendix F

To see how this formalism may be used let us notice the following:

$$
\begin{equation*}
p_{1}^{2}+p_{2}^{2}+p_{1} p_{2}=\frac{1}{2}\left(p_{1}^{2}+p_{2}^{2}+\left(p_{1}+p_{2}\right)^{2}\right) \tag{2.4.66}
\end{equation*}
$$

Let us then consider $\widetilde{\sigma}_{r, n}=\sum_{i} p_{i}^{r}$ where from Eq. 2.4.58 we have :

$$
\widetilde{\sigma}_{2,3}\left(p_{1}, p_{2},-\left(p_{1}+p_{2}\right)\right)=-2 \sigma_{2,3}\left(p_{1}, p_{2},-\left(p_{1}+p_{2}\right)\right) .
$$

This is in fact a general result where

$$
\begin{equation*}
\widetilde{\sigma}_{2, n}\left(p_{1}, p_{2}, \ldots, p_{n}=-\sum_{i}^{n-1} p_{i}\right)=-2 \sigma_{2, n}\left(p_{1}, p_{2}, \ldots, p_{n}=-\sum_{i}^{n-1} p_{i}\right) \tag{2.4.67}
\end{equation*}
$$

To see this let us notice that $\widetilde{\sigma}_{2, n}$ is a polynomial of total degree 2 and thus according to Eq. 2.4 .57 ):

$$
\begin{equation*}
\tilde{\sigma}_{2, n}\left(p_{1}, p_{2}, \ldots, p_{n}=-\sum_{i}^{n-1} p_{i}\right)=C_{n} \sigma_{2, n}\left(p_{1}, p_{2}, \ldots, p_{n}=-\sum_{i}^{n-1} p_{i}\right) \tag{2.4.68}
\end{equation*}
$$

We may determine $C_{n}$ by taking $\left(p_{2}, \ldots, p_{n-1}=0\right)$ which then leads to $2=-C_{n}$.
Eq. 2.4.66) allows us to rewrite the first two equations of Eq. 2.4.59) as :

$$
\begin{align*}
\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right) & =Z\left(\phi_{0}\right) p^{2} \\
\gamma_{2}\left(p ; \phi_{0}\right) & =U^{\prime \prime}\left(\phi_{0}\right)+\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right)  \tag{2.4.69}\\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) & =U^{(3)}\left(\phi_{0}\right)+\frac{1}{2}\left[\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(p_{2}^{2} ; \phi_{0}\right)+\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)\right]
\end{align*}
$$

and the more general Eq. 2.4 .68 allows us to deduce:

$$
\begin{align*}
\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right)= & Z\left(\phi_{0}\right) p^{2} \\
\gamma_{2}\left(p ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)+\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right) \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{(3)}\left(\phi_{0}\right)+\frac{1}{2}\left[\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(p_{2}^{2} ; \phi_{0}\right)+\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)\right] \\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{(4)}\left(\phi_{0}\right)+\frac{1}{2}\left[\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{2}^{2} ; \phi_{0}\right)+\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{3}^{2} ; \phi_{0}\right)+\right.  \tag{2.4.70}\\
& \left.\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}+p_{3}\right)^{2} ; \phi_{0}\right)\right]
\end{align*}
$$

The important aspect in this rewriting is that the polynomial truncation $\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right)=Z\left(\phi_{0}\right) p^{2}$ is not necessary in the above equations as $\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right)$ can be defined as $\gamma_{2}\left(p ; \phi_{0}\right)-U^{\prime \prime}\left(\phi_{0}\right)$. Moreover, while the expressions for $\gamma_{3}$ and $\gamma_{4}$ in terms of $\widehat{\gamma}_{2}$ were found at the level of the order 2 of the derivative expansion we may wonder whether they provide reasonable approximations when the full momentum dependence of $\widehat{\gamma}_{2}$ is considered. It is important to note however that $\gamma_{3}$ and $\gamma_{4}$ are independent from $\gamma_{2}$ when approximations are not involved but that it is necessary to have such a dependence to obtain a closed system of equations. Indeed, as mentioned previously, the flow of the potential $U\left(\phi_{0}\right)$ is obtained by simply replacing $\phi(x)$ by the uniform field $\phi_{0}$ in the $\Gamma$ flow. However, this equation contains $\Gamma_{k}^{(2)}\left(q,-q, \phi_{0}\right)$ which then requires us to calculate the flow of $\Gamma_{k}^{(2)}$ as :

$$
\begin{align*}
\partial_{k} \Gamma_{k}^{(2)}(p,-p)= & \int \frac{\mathrm{d}^{\mathrm{d}} \mathrm{q}}{(2 \pi)^{\mathrm{d}}} \partial_{\mathrm{k}} \mathrm{R}_{\mathrm{k}}(\mathrm{q})\left(\mathrm{G}_{\mathrm{k}}(\mathrm{q}) \Gamma_{\mathrm{k}}^{(3)}(\mathrm{p}, \mathrm{q},-\mathrm{p}-\mathrm{q}) \mathrm{G}_{\mathrm{k}}(\mathrm{q}+\mathrm{p}) \Gamma_{\mathrm{k}}^{(3)}(-\mathrm{p}, \mathrm{p}+\mathrm{q},-\mathrm{q}) \mathrm{G}_{\mathrm{k}}(\mathrm{q})-\right. \\
& \left.\frac{1}{2} G_{k}(q) \Gamma_{k}^{(4)}(p,-p, q,-q) G_{k}(q)\right) . \tag{2.4.71}
\end{align*}
$$

In turn, this then requires us to calculate the flow of $\Gamma_{k}^{(3)}$ and $\Gamma_{k}^{(4)}$. Without any approximation this procedure never ends as at each step the flow of $\Gamma_{k}^{(n)}$ involves $\Gamma_{k}^{(n+1)}$ and $\Gamma_{k}^{(n+2)}$. This is called the ( $n+1, n+2$ ) problem which makes the system of flow equations of the $\Gamma^{(n)}$ not closed. However, if an approximation of the $\Gamma_{k}^{(n+1)}$ and $\Gamma_{k}^{(n+2)}$ sets these quantities to be functions of the $\Gamma_{k}^{(m)}$ for $m \leq n$ then the system may be closed. For example, in the approximation scheme of Eq. 2.4.70 we have :

$$
\begin{align*}
\Gamma_{k}^{(3)}(p, q,-p-q)=\gamma_{k}^{(3)}(p, q)= & U^{(3)}\left(\phi_{0}\right)+\frac{1}{2}\left(\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right)+\partial_{\phi_{0}} \widehat{\gamma}_{2}\left(q^{2} ; \phi_{0}\right)+\right. \\
& \left.\partial_{\phi_{0}} \widehat{\gamma}_{2}\left((p+q)^{2} ; \phi_{0}\right)\right) \\
= & -\frac{1}{2} U^{(3)}\left(\phi_{0}\right)+\frac{1}{2}\left(\partial_{\phi_{0}} \Gamma_{k}^{(2)}\left(p^{2} ; \phi_{0}\right)+\partial_{\phi_{0}} \Gamma_{k}^{(2)}\left(q^{2} ; \phi_{0}\right)+\right.  \tag{2.4.72}\\
& \left.\partial_{\phi_{0}} \Gamma_{k}^{(2)}\left((p+q)^{2} ; \phi_{0}\right)\right)
\end{align*}
$$

and

$$
\begin{align*}
\Gamma_{k}^{(4)}(p,-p, q,-q)=\gamma_{k}^{(4)}(p,-p, q) & =U^{(4)}\left(\phi_{0}\right)+\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right)+\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(q^{2} ; \phi_{0}\right) \\
& =\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}\left(p^{2} ; \phi_{0}\right)+\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}\left(q^{2} ; \phi_{0}\right)-U^{(4)} \tag{2.4.73}
\end{align*}
$$

Let us then note that Taylor expanding these expressions in the internal momentum $q$, which is the variable that enters the integral in Eq. 2.4.71, and using $\widehat{\gamma}_{2}\left(0 ; \phi_{0}\right)=\Gamma_{k}^{(2)}\left(0 ; \phi_{0}\right)-U^{(2)}=0$ from the FD relations, one retrieves the Blaizot-Mendez-Wschebor (BMW) approximation 82:

$$
\begin{equation*}
\Gamma_{k}^{(3)}(p, q,-p-q) \simeq \Gamma_{k}^{(3)}(p, 0,-p)=\partial_{\phi_{0}} \Gamma_{k}^{(2)}(p,-p) \tag{2.4.74}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma_{k}^{(4)}(p,-p, q,-q) \simeq \Gamma_{k}^{(4)}(p,-p, 0,0)=\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}(p,-p) \tag{2.4.75}
\end{equation*}
$$

This approximation is justified by the fact that $\partial_{k} R_{k}$ selects momentum $q<k$ and that this scheme becomes better at large momenta $p$. As such Eq. 2.4.70 contains both the derivative expansion and the BMW approximation the two most widely used approximation schemes. Let us note however that in the appendix of 79 the authors found exactly the same approximation for $\Gamma_{k}^{(3)}(p, q,-p-q)$ but the expression for $\Gamma_{k}^{(4)}$ was:

$$
\begin{equation*}
\Gamma_{k}^{(4)}(p,-p, q,-q)=\frac{1}{2}\left(\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}(p+q,-p-q)+\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}(p-q,-p+q)\right) \tag{2.4.76}
\end{equation*}
$$

In 79 these expressions were called "improved BMW" and were motivated by a one loop calculation of $\Gamma_{k}^{(4)}$ where the general expression was:

$$
\begin{equation*}
\Gamma_{k}^{(4)}\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=\frac{1}{2}\left(\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}\left(p_{1}+p_{2}\right)+\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}\left(p_{1}+p_{3}\right)+\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}\left(p_{1}+p_{4}\right)-\partial_{\phi_{0}}^{2} \Gamma_{k}^{(2)}(0)\right) \tag{2.4.77}
\end{equation*}
$$

One may then recognize the typical $s, t, u$ channels of the crossing symmetry of a four point function in perturbation theory. Thus, while this expression is not explicitly invariant with respect to permutations of $\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ it is within the subspace $p_{1}+p_{2}+p_{3}+p_{4}=0$. For example, we have $\Gamma_{k}^{(2)}\left(p_{1}+p_{2}\right)=$ $\frac{1}{2}\left(\Gamma_{k}^{(2)}\left(p_{1}+p_{2}\right)+\Gamma_{k}^{(2)}\left(p_{3}+p_{4}\right)\right)$ and equivalent expressions for the other terms. As such, we will call any symmetry with respect to external momenta crossing symmetry (CS). This CS may always be implemented via a symmetry operator $S_{n}$ where the index $n$ refers to the number of momenta. For example,

$$
\begin{align*}
S_{3} \cdot f\left(p_{1}\right) & =f\left(p_{1}\right)+f\left(p_{2}\right)+f\left(p_{3}\right) \\
S_{4} \cdot f\left(p_{1}+p_{2}\right) & =f\left(p_{1}+p_{2}\right)+f\left(p_{2}+p_{3}\right)+f\left(p_{3}+p_{4}\right)+f\left(p_{1}+p_{3}\right)+f\left(p_{1}+p_{4}\right)+f\left(p_{2}+p_{4}\right)  \tag{2.4.78}\\
S_{5} \cdot f\left(p_{1}, p_{2}\right) & =f\left(p_{1}, p_{2}\right)+f\left(p_{1}, p_{3}\right)+f\left(p_{4}, p_{5}\right)+f\left(p_{2}, p_{3}\right)+\operatorname{perm}
\end{align*}
$$

In general we have :

$$
\begin{equation*}
S_{n} f\left(p_{1}, p_{2}, \ldots, p_{h}\right)=\sum_{r \in I[\{1, \ldots, h\},\{1, \ldots, n\}]} f\left(p_{r(1)}, p_{r(2)}, \ldots, p_{r(h)}\right) \tag{2.4.79}
\end{equation*}
$$

where $I[\{1, \ldots, h\},\{1, \ldots, n\}]$ is the set of all injective functions from $\{1, \ldots, h\}$ to $\{1, \ldots, n\}$. Thus, we have two expressions for $\Gamma_{k}^{(4)}$, Eq. 2.4.73 and Eq. 2.4.76, which we rewrite in terms of $S_{4}$ :

$$
\begin{align*}
& \Gamma_{k}^{(4)}\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=U^{(4)}+\frac{1}{2} S_{4} \cdot \partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{1}\right)  \tag{2.4.80}\\
& \Gamma_{k}^{(4)}\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=U^{(4)}+\frac{1}{4} S_{4} \cdot \partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)
\end{align*}
$$

It is then interesting to notice that there is no other combination of momenta in $\widehat{\gamma}_{2}$ from which $S_{4} \cdot \partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}$ would lead to a new expression of $\Gamma_{k}^{(4)}$ compatible with FD ${ }^{66}$. For example, in the space $p_{1}+p_{2}+p_{3}+p_{4}=0$ we have $\widehat{\gamma}_{2}\left(p_{1}+p_{2}+p_{3}\right)=\widehat{\gamma}_{2}\left(-p_{4}\right)=\widehat{\gamma}_{2}\left(p_{4}\right)$ and $\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{1}-p_{2}\right)=\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(2 p_{1}+p_{3}+p_{4}\right)$ where the factor 2 is not compatible with FD as setting $p_{3}$ and $p_{4}$ to zero we do not obtain $\partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{1}\right)$. This is also true for $\Gamma_{k}^{(3)}$ where there is only one combination of momenta compatible with FD. We may also check that both expressions for $\Gamma_{k}^{(4)}$ retrieve the order 2 of the derivative expansion but this is in fact trivial as Eq. 2.4.57) shows that all symmetric polynomials of total degree 2 are proportional to $\sigma_{2, n}$ once momentum conservation is imposed. Thus, any approximation of $\Gamma_{k}^{(4)}$ and $\Gamma_{k}^{(3)}$ that depends linearly on $\Gamma_{k}^{(2)}$ must be a linear combination of Eq. 2.4.80) as :

$$
\begin{align*}
\gamma_{4}\left(p_{1}, p_{2}, p_{3}\right)= & U^{(4)}+\frac{1}{2} \partial_{\phi}^{2}\left[c_{1}\left(\widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)+\widehat{\gamma}_{2}\left(p_{1}+p_{3}\right)+\widehat{\gamma}_{2}\left(p_{2}+p_{3}\right)\right)\right.  \tag{2.4.81}\\
& \left.\left.+c_{2}\left(\widehat{\gamma}_{2}\left(p_{1}\right)+\widehat{\gamma}_{2}\left(p_{2}\right)+\widehat{\gamma}_{2}\left(p_{3}\right)+\widehat{\gamma}_{2}\left(p_{1}+p_{2}+p_{3}\right)\right)\right)\right] .
\end{align*}
$$

[^33]Acting on $\gamma_{4}$ with $\mathcal{M}_{p_{1}}$. We then obtain :

$$
\begin{align*}
\mathcal{M}_{p_{1}} \gamma_{4}\left(p_{1}, p_{2}, p_{3}\right)=\gamma_{4}\left(0, p_{2}, p_{3}\right)= & U^{(4)}+\frac{1}{2} \partial_{\phi}^{2}\left[c_{1}\left(\widehat{\gamma}_{2}\left(p_{2}\right)+\widehat{\gamma}_{2}\left(p_{3}\right)+\widehat{\gamma}_{2}\left(p_{2}+p_{3}\right)\right)\right.  \tag{2.4.82}\\
& \left.+c_{2}\left(\widehat{\gamma}_{2}\left(p_{2}\right)+\widehat{\gamma}_{2}\left(p_{3}\right)+\widehat{\gamma}_{2}\left(p_{2}+p_{3}\right)\right)\right]
\end{align*}
$$

from which FD imposes $c_{1}+c_{2}=1$. Hence, in general we have :

$$
\begin{align*}
\gamma_{4}\left(p_{1}, p_{2}, p_{3}\right)= & U^{(4)}+\frac{1}{2} \partial_{\phi}^{2}\left[c\left(\widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)+\widehat{\gamma}_{2}\left(p_{1}+p_{3}\right)+\widehat{\gamma}_{2}\left(p_{2}+p_{3}\right)\right)\right.  \tag{2.4.83}\\
& \left.+(1-c)\left(\widehat{\gamma}_{2}\left(p_{1}\right)+\widehat{\gamma}_{2}\left(p_{2}\right)+\widehat{\gamma}_{2}\left(p_{3}\right)+\widehat{\gamma}_{2}\left(p_{1}+p_{2}+p_{3}\right)\right)\right]
\end{align*}
$$

The parameter $c$ must then be varied along with the parameters of the regulator to obtain a PMS in the augmented parameter spact ${ }^{67}$. These expressions then constitute the most general leading order approximation scheme where the dependence of $\Gamma^{(3)}$ and $\Gamma^{(4)}$ on $\Gamma^{(2)}$ is linear. Let us then consider how one may generalize this. First let us consider the rest terms $R_{3}$ and $R_{4}$ that were neglected. We then have:

$$
\begin{align*}
\gamma_{2}\left(p ; \phi_{0}\right) & =U^{\prime \prime}\left(\phi_{0}\right)+\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right) \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) & =U^{(3)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}} S_{3} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+R_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) \\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right) & =U^{(4)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}}^{2} S_{4} \cdot\left[\frac{1}{2} c \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)+(1-c) \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)\right]+R_{4}\left(p_{1}, p_{2}, p_{3}\right) \tag{2.4.84}
\end{align*}
$$

where in $S_{3}$ the index 3 refers to $p_{3}=-p_{1}-p_{2}$ and in $S_{4}$ we have $p_{4}=-p_{1}-p_{2}-p_{3}$. Applying $\mathcal{M}_{p_{1}}$ to $\gamma_{3}$ we see that $\mathcal{M}_{p_{1}} R_{3}=0$ and thus there is no constraint from FD. Moreover, due to the symmetry with $p_{3}$ we must also have $\mathcal{M}_{p_{3}=-p_{1}-p_{2}} R_{3}=0$ and thus any approximation of $R_{3}$ involving only $U$ and $\widehat{\gamma}_{2}$ must satisfy this constraint. Consider then an approximation $F$ of $R_{3}$ that closes the system. This then implies that we have :

$$
\begin{equation*}
R_{3}=F\left(U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, \widehat{\gamma}_{2}, \partial_{\phi_{0}} \widehat{\gamma}_{2}, R_{k}, p_{1}, p_{2}, p_{3}\right) \tag{2.4.85}
\end{equation*}
$$

In the following we omit the possibility of explicit momentum dependence as the $\Gamma$-flow has explicit momentum dependence only from $R_{k}$ and this would thus spoil that property. The simplest possibility is then to consider $R_{3}=0$ as there is no a priori constraint on $R_{3}$. However, it is possible that other choices of $R_{3}$ lead to better approximations. One such possibility is:

$$
\begin{equation*}
R_{3}\left(p_{1}, p_{2}, p_{3}\right)=\partial_{\phi_{0}}\left(\widehat{\gamma}_{2}\left(p_{1}\right)\right) \widehat{\gamma}_{2}\left(p_{2}\right) \widehat{\gamma}_{2}\left(p_{3}\right) \int_{q} \partial_{k} R_{k}\left(\gamma_{k}^{(2)}(q)+R_{k}(q)\right)^{-3} \tag{2.4.86}
\end{equation*}
$$

where we have included three factors of $\widehat{\gamma}_{2}\left(p_{3}\right)$ such that $\mathcal{M}_{p_{i}} R_{3}=0$ and the three propagators are based on dimensional analysis.

Let us now consider the case of $\gamma_{4}$. We may check that setting $p_{1}$ to zero in $\gamma_{4}$ leads us to deduce that

$$
\begin{equation*}
R_{4}\left(0, p_{2}, p_{3}\right)=\partial_{\phi_{0}} R_{3}\left(p_{2}, p_{3}\right) \tag{2.4.87}
\end{equation*}
$$

Thus $R_{4}$ is constrained contrarily to $R_{3}$. However, $R_{4}$ can be decomposed as $R_{4,1}+R_{4,2}$ where :

$$
\begin{equation*}
\mathcal{M}_{p_{1}} R_{4,1}=\partial_{\phi_{0}} R_{3}\left(p_{2}, p_{3}\right) \tag{2.4.88}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{M}_{p_{i}} R_{4,2}=0 \tag{2.4.89}
\end{equation*}
$$

[^34]This is an important point as the terms that do not have any constraint from FD or CS are the novel terms which do not allow the system to be closed. For example $W_{2}$ is a novel independent term at the level of $\gamma_{2}$ and $W_{3}$ is a novel independent term in $\gamma_{4}$. It is then natural to consider decomposing $\gamma_{4}$ as:

$$
\begin{align*}
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{(4)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}}^{2} S_{4}\left[\frac{1}{2} c \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)+(1-c) \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)\right]+  \tag{2.4.90}\\
& b S_{4} \cdot \partial_{\phi_{0}} R_{3}\left(p_{1}, p_{2}\right)+R_{4,2}\left(p_{1}, p_{2}, p_{3}\right)
\end{align*}
$$

where the form of $\partial_{\phi_{0}} R_{3}\left(p_{1}, p_{2}\right)$ is motivated by FD, the operator $S_{4}$ by CS and the coefficient $b$ must be determined by FD. Let us then write the action of $S_{4}$ explicitly:

$$
\begin{align*}
S_{4} \cdot R_{3}\left(p_{1}, p_{2}\right)= & R_{3}\left(p_{1}, p_{2}\right)+R_{3}\left(p_{1}, p_{3}\right)+R_{3}\left(p_{1}, p_{4}\right)+R_{3}\left(p_{2}, p_{3}\right)+R_{3}\left(p_{2}, p_{4}\right)+R_{3}\left(p_{3}, p_{4}\right) \\
= & R_{3}\left(p_{1}, p_{2}\right)+R_{3}\left(p_{1}, p_{3}\right)+R_{3}\left(p_{1},-p_{1}-p_{2}-p_{3}\right)+  \tag{2.4.91}\\
& R_{3}\left(p_{2}, p_{3}\right)+R_{3}\left(p_{2},-p_{1}-p_{2}-p_{3}\right)+R_{3}\left(p_{3},-p_{1}-p_{2}-p_{3}\right)
\end{align*}
$$

Setting $p_{1}$ to zero we find :

$$
\begin{equation*}
\mathcal{M}_{p_{1}} S_{4} \cdot R_{3}\left(p_{1}, p_{2}\right)=R_{3}\left(p_{2}, p_{3}\right)+R_{3}\left(p_{2},-p_{2}-p_{3}\right)+R_{3}\left(p_{3},-p_{2}-p_{3}\right) \tag{2.4.92}
\end{equation*}
$$

Let us then recall that $R_{3}\left(p_{1}, p_{2}\right)$ is symmetric with respect to $p_{1}, p_{2}, p_{3}=-p_{1}-p_{2}$ as it is the case of $\gamma_{3}$ and the terms on the right hand side of Eq. 2.4.84). Thus there exists $f$ such that $R_{3}\left(p_{1}, p_{2}\right)=f\left(p_{1}, p_{2},-p_{1}-p_{2}\right)$ where $f$ is symmetric in its three variables. As such, we have $R_{3}\left(p_{1}, p_{2}\right)=R_{3}\left(p_{2},-p_{1}-p_{2}\right)$. Thus, Eq. 2.4.92 may be simplified as:

$$
\begin{equation*}
\mathcal{M}_{p_{1}} S_{4} \cdot R_{3}\left(p_{1}, p_{2}\right)=3 R_{3}\left(p_{2}, p_{3}\right) \tag{2.4.93}
\end{equation*}
$$

Hence, imposing FD we find $b=\frac{1}{3}$. Finally, changing notation to $\widehat{\gamma}_{3}=R_{3}$ and $\widehat{\gamma}_{4}=R_{4,2}$ we have:

$$
\begin{align*}
\gamma_{2}\left(p ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)+\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right) \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{(3)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}} S_{3} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+\widehat{\gamma}_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) \\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{(4)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}}^{2} S_{4}\left[\frac{1}{2} c \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)+(1-c) \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)\right]+  \tag{2.4.94}\\
& \frac{1}{3} \partial_{\phi_{0}} S_{4} \widehat{\gamma}_{3}\left(p_{1}, p_{2}\right)+\widehat{\gamma}_{4}\left(p_{1}, p_{2}, p_{3}\right)
\end{align*}
$$

We may notice that FD then implies that:

$$
\begin{align*}
& 0=\mathcal{M}_{p_{1}} \widehat{\gamma}_{4} \\
& 0=\mathcal{M}_{p_{1}} \mathcal{M}_{p_{2}} S_{4} \partial_{\phi_{0}} \widehat{\gamma}_{3}\left(p_{1}, p_{2}\right) \\
& 0=\mathcal{M}_{p_{1}} \mathcal{M}_{p_{2}} \mathcal{M}_{p_{3}} \frac{1}{2} \partial_{\phi_{0}}^{2} S_{4}\left[\frac{1}{2} c \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)+(1-c) \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)\right] \tag{2.4.95}
\end{align*}
$$

This hierarchy is then closely related to that in 81 where according to the prescription of the author of that paper we would have:

$$
\begin{align*}
\gamma_{2}\left(p ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)+\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right) \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{(3)}\left(\phi_{0}\right)+\partial_{\phi_{0}} S_{2} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+\widehat{\gamma}_{G, 3}\left(p_{1}, p_{2} ; \phi_{0}\right)  \tag{2.4.96}\\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{(4)}\left(\phi_{0}\right)+\partial_{\phi_{0}}^{2} S_{3} \widehat{\gamma}_{G, 2}\left(p_{1}^{2} ; \phi_{0}\right)+ \\
& S_{3} \cdot \partial_{\phi_{0}} \widehat{\gamma}_{G, 3}\left(p_{1}, p_{2}\right)+\widehat{\gamma}_{G, 4}\left(p_{1}, p_{2}, p_{3}\right)
\end{align*}
$$

with again:

$$
\begin{aligned}
& 0=\mathcal{M}_{p_{1}} \widehat{\gamma}_{G, 4} \\
& 0=\mathcal{M}_{p_{1}} \mathcal{M}_{p_{2}} \partial_{\phi_{0}}^{2}\left(S_{3} \widehat{\gamma}_{G, 2}\left(p_{1}^{2} ; \phi_{0}\right)\right) \\
& 0=\mathcal{M}_{p_{1}} \mathcal{M}_{p_{2}} \mathcal{M}_{p_{3}} \partial_{\phi_{0}}^{2}\left(S_{3} \widehat{\gamma}_{G, 2}\left(p_{1}^{2} ; \phi_{0}\right)\right)
\end{aligned}
$$

To find this decomposition consider

$$
\begin{equation*}
\mathcal{N}_{q_{i}}=1-\mathcal{M}_{q_{i}} \tag{2.4.97}
\end{equation*}
$$

and

$$
\begin{equation*}
\widehat{\mathcal{N}}_{n}=\prod_{i=1}^{n} \mathcal{N}_{q_{i}} \tag{2.4.98}
\end{equation*}
$$

from which we have:

$$
\begin{equation*}
\mathcal{M}_{q_{h}} \widehat{\mathcal{N}}_{n}=\prod_{i \neq h}\left(\mathcal{N}_{q_{i}}\right) \mathcal{M}_{q_{h}} \mathcal{N}_{q_{h}}=0 \tag{2.4.99}
\end{equation*}
$$

as $\mathcal{M}_{q_{h}} \mathcal{N}_{q_{h}}=\mathcal{M}_{q_{h}}-\mathcal{M}_{q_{h}}^{2}=0$. We then define :

$$
\begin{equation*}
\widehat{\gamma}_{G, n}=\widehat{\mathcal{N}}_{n-1} \gamma_{n} \tag{2.4.100}
\end{equation*}
$$

such that

$$
\begin{equation*}
\mathcal{M}_{q_{i}} \widehat{\gamma}_{G, n}=0 \tag{2.4.101}
\end{equation*}
$$

Thus following 81 we may decompose the $\gamma_{n}$ into the $\widehat{\gamma}_{G, n}$ by expanding $1=\prod_{i=1}^{n-1}\left(\mathcal{M}_{p_{i}}+\mathcal{N}_{p_{i}}\right)$ and using FD. As an example consider:

$$
\begin{align*}
\gamma_{3}\left(p_{1}, p_{2}\right) & =\left(\mathcal{M}_{p_{1}}+\mathcal{N}_{p_{1}}\right)\left(\mathcal{M}_{p_{2}}+\mathcal{N}_{p_{2}}\right) \gamma_{3}\left(p_{1}, p_{2}\right) \\
& =\left(\mathcal{M}_{p_{1}} \mathcal{M}_{p_{2}}+\mathcal{M}_{p_{1}} \mathcal{N}_{p_{2}}+\mathcal{M}_{p_{2}} \mathcal{N}_{p_{1}}+\mathcal{N}_{p_{1}} \mathcal{N}_{p_{2}}\right) \gamma_{3}\left(p_{1}, p_{2}\right) \\
& =\gamma_{3}(0,0)+\mathcal{N}_{p_{2}} \gamma_{3}\left(0, p_{2}\right)+\mathcal{N}_{p_{1}} \gamma_{3}\left(p_{1}, 0\right)+\widehat{\gamma}_{G, 3}\left(p_{1}, p_{2}\right)  \tag{2.4.102}\\
& =U^{(3)}+\mathcal{N}_{p_{2}} \partial_{\phi_{0}} \gamma_{2}\left(p_{2}\right)+\mathcal{N}_{p_{1}} \partial_{\phi_{0}} \gamma_{2}\left(p_{1}\right)+\widehat{\gamma}_{3}\left(p_{1}, p_{2}\right) \\
& =U^{(3)}+\partial_{\phi_{0}} \widehat{\gamma}_{G, 2}\left(p_{2}\right)+\partial_{\phi_{0}} \widehat{\gamma}_{G, 2}\left(p_{1}\right)+\widehat{\gamma}_{G, 3}\left(p_{1}, p_{2}\right)
\end{align*}
$$

The novelty in Eq. 2.4.94 is that we extract from $\widehat{\gamma}_{G, 3}$ terms involving $\widehat{\gamma}_{G, 2}$ which allow us to decompose $\gamma_{3}$ as a sum of terms symmetric in $p_{1}, p_{2}$ and $p_{3}=-p_{2}-p_{1}$, rather than just $\left(p_{1}, p_{2}\right)$, while keeping the FD property. The same is true for $\widehat{\gamma}_{G, 4}$. In the case of $\gamma_{3}$, one may retrieve Eq. 2.4.94) from Eq. 2.4 .96 by considering ${ }^{68}$ :

$$
\begin{equation*}
w=\frac{1}{2} \widehat{\mathcal{N}}_{2} \widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)=\frac{1}{2}\left(\widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)-\widehat{\gamma}_{2}\left(p_{1}\right)-\widehat{\gamma}_{2}\left(p_{2}\right)\right) \tag{2.4.103}
\end{equation*}
$$

which is a combination of $\widehat{\gamma}_{2}$ which belongs to $\widehat{\gamma}_{G, 3}$. Defining $\widehat{\gamma}_{3}=\widehat{\gamma}_{G, 3}-w$ then allows us to retrieve

$$
\begin{equation*}
\gamma_{3}=U^{(3)}+\frac{1}{2} \partial_{\phi_{0}}\left[\widehat{\gamma}_{2}\left(p_{1}\right)+\widehat{\gamma}_{2}\left(p_{2}\right)+\widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)\right]+\widehat{\gamma}_{3}\left(p_{1}, p_{2}\right) \tag{2.4.104}
\end{equation*}
$$

which is a decomposition that has explicit CS. Another advantage of our decomposition is that we may set the remainder $\widehat{\gamma}_{n}$ to zero to obtain the derivative expansion. In the case of $\gamma_{3}$ this is due to the fact that it verifies $\mathcal{M}_{p_{i}} \widehat{\gamma}_{3}=0$ for $i \in\{1,2,3\}$ which implies that $\widehat{\gamma}_{3}\left(p_{1}, p_{2}\right) \propto p_{1} p_{2} p_{3}$. As such, it may then be discarded at order 2 of the derivative expansion ${ }^{69}$.

The generalization to $\gamma_{n}$ then consists of extracting from the rest terms symmetric combinations of the $\gamma_{n-1}$ in an iterative manner where the coefficients of these symmetric combinations are found by imposing FD analogously to what was done with $\gamma_{4}$. We remark that this hierarchy is rather interesting as it implies that any approximation scheme of $\gamma_{n}$ that is non linear in the $\widehat{\gamma}_{h}$ for $h<n$ would be an approximation of $\widehat{\gamma}_{n}$ and would thus be higher order in the momenta. As such we may expect that the inclusion of non linearities in the $\widehat{\gamma}_{h}$ would lead to a subdominant effect on approximation schemes. Nevertheless, we wish to show an interesting feature of non linear approximations in the case of $\gamma_{4}$. Let us recall that for $\gamma_{3}$ there was no

[^35]linear combination of the $\widehat{\gamma}_{2}$ that belonged to $\widehat{\gamma}_{3}$. This is not the case of $\gamma_{4}$ where the subtraction of the two possible approximation schemes:
\[

$$
\begin{align*}
\frac{1}{2} S_{4}\left(\widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)-\widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)\right)= & \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)+\widehat{\gamma}_{2}\left(\left(p_{1}+p_{3}\right)^{2} ; \phi_{0}\right)+\widehat{\gamma}_{2}\left(\left(p_{2}+p_{3}\right)^{2} ; \phi_{0}\right) \\
& -\widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)-\widehat{\gamma}_{2}\left(p_{2}^{2} ; \phi_{0}\right)-\widehat{\gamma}_{2}\left(p_{3}^{2} ; \phi_{0}\right) \\
& -\widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}+p_{3}\right)^{2} ; \phi_{0}\right) \tag{2.4.105}
\end{align*}
$$
\]

satisfies CS and taking $p_{1}$ to zero sets this term to zero. This is due to the fact that such a term can be obtained from

$$
\begin{equation*}
\widehat{\mathcal{N}}_{3} \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}+p_{3}\right)^{2} ; \phi_{0}\right) \tag{2.4.106}
\end{equation*}
$$

and that acting later with $S_{4}$ simply changes an overall factor in the expression. This was not the case of $\widehat{\mathcal{N}}_{2} \widehat{\gamma}_{2}\left(p_{1}+p_{2}\right)$ for $\gamma_{3}$. This then allows us to consider a non linear approximation of $\widehat{\gamma}_{4}$ as, for example :

$$
\begin{equation*}
\widehat{\gamma}_{4}=\partial_{\phi_{0}}^{2}\left(\widehat{\mathcal{N}}_{3} \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}+p_{3}\right)^{2}\right)\right) F\left(\frac{S_{4} \widehat{\gamma}_{2}}{U^{(2)}}\right) \tag{2.4.107}
\end{equation*}
$$

with $F$ bounded over $\mathbb{R}$. The choice of $F$ seems undetermined here but it is likely constrained by high momentum asymptotics, unitarity and conformal symmetry at a fixed-point within the exact theory ${ }^{70}$, Perhaps applying the principle of minimal sensitivity to $F$, by introducing a parametrization of this function, would be a good guiding principle to achieve some of these properties or at least minimize their effects.

We expect however that simply setting the highest order $\widehat{\gamma}_{n}$ to zero will lead to results with a reasonable accuracy as such an approximation scheme contains the most widely used approximation schemes in field theory which are perturbation theory, BMW and the derivative expansion. In appendix F we show how to retrieve the order 6 and order 4 of the derivative expansion working directly with the $\gamma_{n}$ rather than using an effective action. We also show how the terms involved relate to the $\widehat{\gamma}_{n}$. Let us emphasize that we believe that it is much easier to work directly with the $\gamma_{n}$ by imposing CS and FD rather than searching for an effective action as that method has the following disadvantages:

- The effective action approach requires taking functional derivatives which can be difficult to implement using computer software.
- The effective action approach requires the user to determine how many terms are independent via integration by parts. This will likely be cumbersome for high order derivative expansions. Instead, at least at $d=1$, expanding in the elementary symmetric polynomials is straightforward. For $d \geq 2$ we discuss approaches in Appendix $F$.
- Once the functional derivatives are obtained from an ansatz, one usually expands in momenta which leads to rather complicated formulas. Instead working with the elementary symmetric polynomials we have compact expressions which would be more computer friendly in terms of efficient numerical compilation.

In summary it seems that the most natural approach is to work directly with approximations of the $\gamma_{n}$ using FD and CS rather than systematically calling upon functional analysis via an effective action.

Let us also remark however that the BMW approximation is exact for the $O(N)$ models of Sec. (3.1.1) in the limit $N \rightarrow \infty$. As such it is natural to wonder whether the extra terms in the above approximations are really necessary. We may also wonder whether the BMW approximation contains the derivative expansion as well. Technically the answer is no but one might argue that this is because the true derivative expansion is not the one commonly used as was defended in [79]. Indeed, in the usual derivative expansion one calculates $Z$ from the flow equation of $\Gamma^{(2)}$ by taking a derivative with respect to $p^{2}$ and setting $p$ to zero. However, this leads to inconsistencies as $(p . q)^{2}$ and $p^{2} q^{2}$ terms, which come from the product of $\Gamma^{(3)}$ or from the propagator with a $\Gamma^{(3)}$, then contribute to the flow of $Z$. Such terms are of the same order as the $\partial^{4}$ terms

[^36]

Figure 2.4.2: $Z\left(\rho=\phi^{2}\right)$ for the Wilson-Fisher fixed-point at $d=3$ showing how the field dependence of $Z$ may be discarded in a first approximation as it deviates very slightly from 1 in between $\rho=0$ and $\rho=10$
that were neglected in the effective action and thus should also be discarded. These terms are exactly those discarded in the BMW approach by neglecting the $q$ dependence from the integral in $\Gamma^{(3)}$ and $\Gamma^{(4)}$. Hence, setting this $q$ dependence to zero and then calculating the flow of $Z$ via $\partial_{p^{2}} \Gamma^{(2)}$ can be considered the "true" derivative expansion which the BMW approximation then contains. Nonetheless, one disadvantage of the BMW approximation is that space-time symmetries are not always conserved by this approximation scheme and as one may check from Table. 2.1) it is not as good as the derivative expansion in the $O(N)$ models for $N$ small. However, It is not clear to us whether this is simply a numerical coincidence. Concerning the point of spacetime symmetries we remark that prior to setting $\widehat{\gamma}_{n}$ to zero in the above methods the decomposition of $\gamma_{n}$ in $\widehat{\gamma}_{s}$ is exact and in principle one may always extract from $\widehat{\gamma}_{n}$ any term needed to preserve spacetime symmetries then set the rest term to zero.

Moreover, let us note that another approximation scheme exists in the literature called the LPA". This approximation scheme takes into account that within the derivative expansion the leading contribution of $\widehat{\gamma}_{2}\left(p ; \phi_{0}\right)$ is that of $Z\left(\phi_{0}\right)$ which depends weakly on the field $\phi_{0}$. As such we expect the same of the full $\widehat{\gamma}_{2}\left(p ; \phi_{0}\right)$ which then allows us to neglect the field dependence of this function leading to a function of just $p^{2}$. This is equivalent to considering the following effective action $84,85,71$,

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int \frac{1}{2} \partial \phi Z_{k}\left(\partial^{2}\right) \partial \phi+U(\phi) \tag{2.4.108}
\end{equation*}
$$

According to 84 this approximation scheme is not much more difficult to implement than the derivative expansion at order 2 while maintaining full momentum dependence of the $\Gamma^{(n)}$. This approximation can be seen as an improvement of the LPA' where instead of simply considering $Z$ to be a $k$ dependent constant we consider it to be a function of only momentum. We may also compare this approximation scheme to the derivative expansion as Eq. 2.4 .108 ) is the leading order of a Taylor expansion in fields, about $\phi_{0}=0{ }^{72}$, of $\widehat{\gamma}_{2}$. This may be contrasted with the expansion in momenta of the derivative expansion. The choice between these two expansions, on momenta or on fields, depends on the model at hand but a Taylor expansion in fields can not always be justified. For example, as we mentioned, Eq. 2.4 .108 depends on whether the dependence on fields of $\widehat{\Gamma}^{(2)}\left(p_{1}, p_{2} ; \phi_{0}\right)$ and thus also of $Z\left(\phi_{0}\right)$ is weak. This is not necessarily the case if the potential $U$ is non perturbative and has multiple extrema as this may lead to non trivial field dependence of $Z\left(\phi_{0}\right)$ via its flow equation which contains derivatives of $U$. Another example where the field dependence of $Z\left(\phi_{0}\right)$ may be non trivial is when the fixed-point at hand has a cusp or strong derivatives for a particular field value. This last scenario takes place with singular fixed-points to be studied in Sec.(3.1.4. As such, an expansion in derivatives is more robust than in expansion in fields as it relies mainly on the choice of regulator rather than on assumptions of weak field dependence which is model dependent.

[^37]We also remark that in 79 the approximation scheme

$$
\begin{equation*}
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)=U^{(4)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}}^{2} S_{4} \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right) \tag{2.4.109}
\end{equation*}
$$

with $c=1$ in Eq. $(2.4 .83)$ was considered and led to numerical instabilities. The authors then concluded that more advanced numerical schemes are needed. We are not aware of the numerical scheme or what kind of instabilities occurred but as this approximation contains the derivative expansion we expect that for a regulator that decays exponentially for large momenta that there should exist a numerical scheme that displays nice results. If such a numerical scheme does not exist due to some intrinsic properties of the system then it would be fruitful to find the source of this instability to motivate further approximations to this scheme. Indeed, the derivative expansion and the BMW approximations are two approximations choices of this scheme that yield nice results and as such it would be interesting to explore other possible approximations derived from this general scheme.

To conclude, all approximation schemes in the NPRG formalism work with a uniform background field. When the effective action is evaluated at a uniform field, one obtains the effective potential. The RG flow of this effective potential depends on $\Gamma^{(2)}$ which then requires us to compute the flow of $\Gamma^{(2)}$. In turn, the flow of $\Gamma^{(2)}$ depends on $\Gamma^{(3)}$ and $\Gamma^{(4)}$ and so on. This leads to an infinite hierarchy of equations which is unsolvable in general. The usage of approximations that close the system of equations is then necessary. In this section we have explored the landscape of approximations by searching for the minimal set of hypotheses which allow the system to be closed. This novel minimal approximation scheme then leads to the more well known approximation schemes such as the derivative expansion, the BMW apprximation and the LPA".

There are many interests in having a framework that encompases all these approximation schemes. First, it is conceptually interesting to have a general framework. Second, this general framework makes more explicit the manner in which these different approximation schemes differ and what are their source of errors. Third, the BMW approximation scheme is known to not always preserve spacetime symmetries. While the above framework considered the equilibrium case for the sake of simplicity, it would be interesting to generalize this to the out of equilibrium case in order to check whether it is possible to consider momentum dependent approximation schemes that always preserve spacetime symmetries. Finally, the above general framework can be used directly for practical purposes and it will be interesting to implement this method in concrete models during future research.

Table 2.1: Taken from 30 with permission: Critical exponents $\nu, \eta$ and $\omega$ for the three-dimensional $\mathrm{O}(N)$ universality class obtained in the FRG approach from DE to second [86, 87], fourth [88] and sixth $|77|$ orders, LPA" 84. 85 and BMW approximation [89, 90, compared to Monte Carlo (MC) simulations $91-96, d=3$ perturbative $\mathrm{RG}(\mathrm{PT})[97], \epsilon$-expansion at order $\epsilon^{6}(\epsilon-\exp )[98$ and conformal bootstrap (CB) 99103 (when several estimates are available in the literature, we show the one with the smallest error bar).

| Correlation-length exponent $\nu$ |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | LPA | $\mathrm{DE}_{2}$ | $\mathrm{DE}_{4}$ | $\mathrm{DE}_{6}$ | LPA ${ }^{\prime \prime}$ | BMW | MC | PT | $\epsilon$-exp | CB |
| 0 | 0.5925 | 0.5879(13) | 0.5876(2) | - | - | 0.589 | 0.58759700(40) | 0.5882(11) | 0.5874(3) | 0.5876(12) |
| 1 | 0.650 | 0.6308(27) | 0.62989(25) | 0.63012(16) | 0.631 | 0.632 | 0.63002(10) | 0.6304(13) | 0.6292(5) | 0.629971(4) |
| 2 | 0.7090 | 0.6725(52) | 0.6716(6) | - | 0.679 | 0.674 | 0.67169(7) | $0.6703(15)$ | 0.6690(10) | 0.6718(1) |
| 3 | 0.7620 | 0.7125(71) | 0.7114(9) | - | 0.725 | 0.715 | 0.7112(5) | 0.7073(35) | 0.7059(20) | $0.7120(23)$ |
| 4 | 0.805 | 0.749(8) | 0.7478(9) | - | 0.765 | 0.754 | $0.7477(8)$ | 0.741(6) | $0.7397(35)$ | $0.7472(87)$ |

Anomalous dimension $\eta$

| $N$ | $\mathrm{DE}_{2}$ | $\mathrm{DE}_{4}$ | $\mathrm{DE}_{6}$ | $\mathrm{LPA}^{\prime \prime}$ | BMW | MC | PT | $\epsilon-\exp$ | CB |
| :--- | :---: | :---: | :---: | :---: | :---: | :--- | :---: | :---: | :---: |
| 0 | $0.0326(47)$ | $0.0312(9)$ | - | - | 0.034 | $0.0310434(30)$ | $0.0284(25)$ | $0.0310(7)$ | $0.0282(4)$ |
| 1 | $0.0387(55)$ | $0.0362(12)$ | $0.0361(11)$ | 0.0506 | 0.039 | $0.03627(10)$ | $0.0335(25)$ | $0.0362(6)$ | $0.0362978(20)$ |
| 2 | $0.0410(59)$ | $0.0380(13)$ | - | 0.0491 | 0.041 | $0.03810(8)$ | $0.0354(25)$ | $0.0380(6)$ | $0.03818(4)$ |
| 3 | $0.0408(58)$ | $0.0376(13)$ | - | 0.0459 | 0.040 | $0.0375(5)$ | $0.0355(25)$ | $0.0378(5)$ | $0.0385(13)$ |
| 4 | $0.0389(56)$ | $0.0360(12)$ | - | 0.0420 | 0.038 | $0.0360(4)$ | $0.0350(45)$ | $0.0366(4)$ | $0.0378(32)$ |


|  | Correction-to-scaling exponent $\omega$ |  |  |  |  |  |  |  |  |
| :--- | :--- | :---: | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | LPA | $\mathrm{DE}_{2}$ | $\mathrm{DE}_{4}$ | BMW | MC | PT | $\epsilon$-exp | CB |  |
| 0 | 0.66 | $1.00(19)$ | $0.901(24)$ | 0.83 | $0.899(14)$ | $0.812(16)$ | $0.841(13)$ | - |  |
| 1 | 0.654 | $0.870(55)$ | $0.832(14)$ | 0.78 | $0.832(6)$ | $0.799(11)$ | $0.820(7)$ | $0.82968(23)$ |  |
| 2 | 0.672 | $0.798(34)$ | $0.791(8)$ | 0.75 | $0.789(4)$ | $0.789(11)$ | $0.804(3)$ | $0.794(8)$ |  |
| 3 | 0.702 | $0.754(34)$ | $0.769(11)$ | 0.73 | 0.773 | $0.782(13)$ | $0.795(7)$ | $0.791(22)$ |  |
| 4 | 0.737 | $0.731(34)$ | $0.761(12)$ | 0.72 | 0.765 | $0.774(20)$ | $0.794(9)$ | $0.817(30)$ |  |

## Chapter 3

## Application of the functional renormalisation group to models

## 3.1 $\mathrm{O}(\mathrm{N})$ models and the Bardeen-Moshe-Bander phenomenon

### 3.1.1 $\mathrm{O}(\mathrm{N})$ models

In Sec. 2.1.2 we studied the Ising and liquid gas systems and showed that, at a phase transition, they are both described by the same Landau potential of the form:

$$
\begin{equation*}
G=a \phi^{2}+b \phi^{4} \tag{3.1.1}
\end{equation*}
$$

where the mean field free energy has a $\mathbb{Z}_{2}$ symmetry $\phi \rightarrow-\phi$ reflecting the equivalence of the two phases at the phase transition. This symmetry is preserved by the renormalization group and at a second order phase transition the critical exponents of these two models are the same. This in turn leads to a significant simplification in our description of physical models since the symmetries of the model are often sufficient to characterize universal quantities. From this perspective a natural question is: what other symmetries can we consider and how does the physics differ between these universality classes? Perhaps a natural first attempt at answering this question is to consider two fields $\phi_{1}$ and $\phi_{2}$ each having a $\mathbb{Z}_{2}$ symmetry. A possible Landau potential of this model would be:

$$
\begin{equation*}
G\left(\phi_{1}, \phi_{2}\right)=a_{1} \phi_{1}^{2}+a_{2} \phi_{2}^{2}+b_{1} \phi_{1}^{4}+b_{2} \phi_{2}^{4}+b_{3} \phi_{1}^{2} \phi_{2}^{2} \tag{3.1.2}
\end{equation*}
$$

This Landau potential has a $\mathbb{Z}_{2}$ symmetry with respect to both the field $\phi_{1}$ and the field $\phi_{2}$ and thus it has, by definition, a $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ symmetry where each factor of the Cartesian product $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ acts on a different field. If we also include a swapping symmetry $\phi_{1} \leftrightarrow \phi_{2}$ we obtain:

$$
\begin{equation*}
G\left(\phi_{1}, \phi_{2}\right)=a\left(\phi_{1}^{2}+\phi_{2}^{2}\right)+b_{1}\left(\phi_{1}^{4}+\phi_{2}^{4}\right)+b_{2} \phi_{1}^{2} \phi_{2}^{2} \tag{3.1.3}
\end{equation*}
$$

At the special point in parameter space where $b_{2}=2 b_{1}=2 b$ we have an extended $O(2)$ symmetry:

$$
\begin{equation*}
G\left(\phi_{1}, \phi_{2}\right)=a\left(\phi_{1}^{2}+\phi_{2}^{2}\right)+b\left(\phi_{1}^{2}+\phi_{2}^{2}\right)^{2} \tag{3.1.4}
\end{equation*}
$$

What is interesting about this model is that contrary to the previous cases it has a continuous symmetry where a rotation of any angle leads to the same model. In fact, as reflections about an axis such as $\phi_{1} \rightarrow \phi_{1}, \phi_{2} \rightarrow-\phi_{2}$ also preserve the potential, the model in fact has symmetry with respect to the orthogonal group $O(2)$. We may then extend this analysis to include $N$ fields with $\mathbb{Z}_{2}$ symmetry and permutation symmetry for which, at particular configurations of the coefficients we have an enhanced $O(N)$ vector symmetry ${ }^{11}$. All of these models have continuous symmetry as opposed to the discrete $\mathbb{Z}_{2}$ symmetries introduced at the beginning of this section. Models with continuous symmetries are particularly interesting when the symmetry is broken at a phase transition as they lead to massless modes, called Goldstone modes. These

[^38]massless modes often explain, however via extra mechanisms, the presence of very small masses in a system. In high energy physics a typical example of this is the pions, which are the mediators of the nuclear force.
$O(3)$ models can naturally occur in our 3d isotropic world. An example of this is the Heisenberg model for isotropic ferromagnets and anti-ferromagnets $[10]^{2}$. Examples of $O(2)$ symmetries include superfluid $\mathrm{He}^{4}$, liquid crystals, and anisotropic magnets with an easy plane 10. The case of the $O(2)$ model in 2 dimensions is particularly interesting because it leads to the Berezinskii-Kosterlitz-Thouless transition which is a topological phase transition [105, 106]. It is also possible to consider real world applications of $O(N)$ symmetry for any $N>3$ as long as the system has at least $N$ degrees of freedom which happen to be in a very symmetric configuration. For example, $O$ (4) can be used as an effective model in quantum chromodynamics in the limit of two quark flavors [10]. There are also anti-ferromagnets that are described by as much as eight degrees of freedom [107] but usually such models do not have a full $O(N)$ symmetry. We may expect that systems with large symmetry groups are rare but the field of cold atom gases seems promising at least for the $S U(N)$ symmetry for $N \lesssim 10[108,109]$. In the case of the Wilson-Fisher fixedpoint, it is also possible to consider $O(N)$ models for non integer $N$ via loop $O(N)$ models [110, 111] where the symmetry needs to be understood within a more abstract category theory [111]. These loop $O(N)$ models also show that the $O(N)$ model in the limit $N \rightarrow 0$ can be understood as a model of self avoiding polymers [110, $1122^{3}$. In fact the $O(N)$ model in the limit $N \rightarrow \infty$ can also be seen as a model of branching polymers 116 . 118 where it becomes solvable [28]. Finally, the $O(N)$ model for $N=-2$ can be used to describe the depinning transition of charged density waves in disordered systems where it is also related to loop erased random walks 119 121. It is thus clear that a full understanding of the physics of the $O(N)$ models for all $N$ is important in $d=2,3,4{ }^{4}$ dimensions. Is there any interest in extending these models to non integer $d$ ? There are many scenarios where an interpolation to non integer $d$ is interesting. In particular when the physics at two consecutive integer dimensions differs substantially. An example of this is that at $d=4$ the only fixed-point of the $O(N)$ models is the gaussian fixed-point while at $d=3$ there is also the Wilson-Fisher fixed-point. This is the basis of the $\epsilon$ expansion which, in essence, is an interpolation of the Wilson-Fisher fixed-point between 3 and 4 dimensions. Another example is how the Mermin-Wagner theorem, that forbids long range order from short range interactions at non zero temperature, sets in at 2 dimensions whereas it does not exist in 3 dimensions [122. Moreover, models with non integer dimension $d$ can be related to statistical models in inhomogeneous systems $[123-128]^{5}$ and are conjectured to be related to long range models at integer dimension [129-131. However, in this last case, two loop results dismay the conjectured relationship 132 without forbidding the possibility of a more complicated correspondence. Nonetheless, even in absence of these direct applications of non integer dimensions, our study will require exploring non integer dimensions and will ultimately lead to the necessity of new fixed-points in three dimensions.

As the analysis will also involve many multi-critical fixed-points, we will explain the particularities of such models in the following section.

### 3.1.2 Multi-critical points of the $\mathrm{O}(N)$ model

### 3.1.2.1 Multi-critical points within the framework of Landau theory

A critical point/manifold in a phase diagram is obtained by fine tuning external control parameters such as temperature, pressure, external magnetic field, etc. In the case of the Ising model with zero external source, a critical point is obtained by fine tuning the temperature of the system. In the RG language, this implies that the RG fixed-point, coresponding to the associated scale invariant theory, has exactly one relevant eigendirection. Thus, in the space of all possible couplings, the critical hypersurface is of codimension one. There are models however that have critical points/manifolds which require fine tuning more than

[^39]one external control parameter. The associated theory is again scale invariant which implies that there is a corresponding fixed-point of the RG. The novelty is that the requirement to fix more than one external parameter implies the existence of more than one relevant eigendirection. More precisely, there are as many relevant eigendirections as external parameters to be fixed. These are called multi-critical fixed-points. In the following we will give examples of such models.

The best known example of a multicritcal fixed-point is that of the tricritical case. The following sections of this thesis will be devoted to these types of models. A classic example of a model that has a tricritical point is that of a $\mathrm{He}^{3}-\mathrm{He}^{4}$ mixture when considering the transition from the normal to super-fluid phase [133. Other examples include anti-ferromagnets $134-137$ and quantum chromodynamics at zero fermion mass $138-141$. In the case of $\mathrm{He}^{3}-\mathrm{He}^{4}$, the new macroscopic parameter to consider is the relative fraction $x$ of the $\mathrm{He}^{3}$ impurity within the $\mathrm{He}^{4}$ medium. This leads to a new relevant parameter that can destabilize the critical scale invariant behavior.

A simplified model of $\mathrm{He}^{3}-\mathrm{He}^{4}$ mixtures is the Blume-Emery-Griffiths model with Hamiltonian $133{ }^{6}$,

$$
\begin{equation*}
\mathcal{H}=-J \sum_{<i j>} S_{i} S_{j}+\Delta \sum_{i} S_{i}^{2} \tag{3.1.5}
\end{equation*}
$$

where $S_{i}= \pm 1$ corresponds to the superfluid $\mathrm{He}^{4}$ and $S_{i}=0$ corresponds to the $\mathrm{He}^{3}$ impurity. We will not dwell into the significance of the parameters $(J, \Delta)$ or the physics of the model. It suffices to say that for $m=<S_{i}>$, the mean Gibbs free energy $G$ may be expanded about the high temperature minimum at $m=0$ as in Landau theory. The result is :

$$
\begin{equation*}
G(m)=a m^{2}+b m^{4}+c m^{6}+\ldots \tag{3.1.6}
\end{equation*}
$$

where:

$$
\begin{equation*}
a=\frac{1}{2}(\delta-z \beta J), b=\frac{\delta^{2}}{8}\left(1-\frac{\delta}{3}\right), c=\frac{\delta^{3}}{6}\left(1-\frac{3}{8} \delta+\frac{3}{40} \delta^{2}\right) \tag{3.1.7}
\end{equation*}
$$

with $\delta=1+e^{\beta \Delta} / 2, \beta=1 /\left(k_{B} T\right)$ and $z$ the number of nearest neighbors of a given site. A more physical interpretation of $\delta$ may be obtained by noticing that in the high temperature disordered phase, that is, in absence of the superfluid phase, we have $x=1-\frac{1}{\delta}$ where $x$ is the fraction of impurity $\mathrm{He}^{3}$.

Within mean field theory, the usual second order phase transition is obtained for $a=0$ and $b>0, c>0$ while a tricritical point occurs for $a=0, b=0$ and $c>0$. It is thus clear that the tricritical phenomenology is distinct from ordinary phase transitions.

Along the ideas of Landau theory, we thus omit any details of the model at hand and consider instead a general degree 6 polynomial:

$$
\begin{equation*}
G=a(T, x) m^{2}+b(T, x) m^{4}+c(T, x) m^{6} \tag{3.1.8}
\end{equation*}
$$

where $c$ must be positive so that the free energy is bounded below but now $a$ and $b$ can take any sign. We will also omit the dependence on the control parameters $(T, x)$ and consider instead directly the parameters $(a, b)$ as independent variables in $\mathbb{R}^{2}$. An analysis of the phase diagram is given in Appendix $H$. The discussion of that appendix is summarized in Fig. 3.1.1 and in Fig. 3.1.2 we have drawn the corresponding phase diagram. From Fig. (3.1.2) we conclude that the space of coupling constants and more precisely the critical surface is divided into two parts: one that corresponds to continuous transitions and the complementary part to discontinuous transitions. The boundary between these two regions corresponds to what is called tricriticality: the behavior of the system is still scale invariant, characterized by power laws, but with critical exponents that are different from those of the second order transition of this system. Notice that the tricritical hypersurface is of codimension two in the space of coupling constants because it is a boundary of a region of codimension one. In RG terms this means that the corresponding fixed-point is not the usual Wilson-Fisher fixed-point and that this tricritical fixed-point has two relevant eigendirections. This implies that to reach it, two parameters have to be fine-tuned and not only one.

[^40]

Figure 3.1.1: Landau potentials for various values of the coefficients ( $a, b$ ) in Eq. (3.1.8). The plots contain a list of different configurations that the Landau free energy, up to order $m^{6}$, might have depending on the values of the coefficients. On the left, the degree $n$ of the free energy polynomial curves, indicated on the right, is indicated by the leading term $m^{n}$. For a given $m^{n}$ the plots indicate the possible scenarios depending on the signs of the coefficients as indicated at the bottom of each plot where for example $(a>0, b<0)=(+,-)$. Double or triple signs schematically refer to the magnitude of the ratio $a / b$. As examples, $(++,-)$ imply $a>0, b<0$ and $|a| /|b|$ is large while $(+,---)$ imply that we have $(a>0, b<0)$ and that $|a| /|b|$ is "very" small. The sign of $c$ is never indicated as it should always be positive.


Figure 3.1.2: Tricritical phase diagram. The phase diagram contains the form of the Landau free energy for various values of its coefficients $(a, b)$. The number of minima of the potential is indicated by the letters S,D and T denoting Single minimum, Double minima and Triple minima. All local minima have been included thus meta-stable states are also present and are delimited by the spinodals. The first order transition occurs when the shape of the potential looks like the one shown in Fig. 3.1.1) and called (,+-- ). It occurs by deformation of the potential in such a way that the metastable states become the stable states. On this first order line there is coexistence of three phases and as such it is a triple line similar to the liquid, gas, solid point of ordinary matter. The tricritical point lies at the intersection between the first and second order region and corresponds here to the point $(a, b)=(0,0)$. When allowing the inclusion of terms that break the $m \rightarrow-m$ symmetry such as $m^{3}$, the space of parameters becomes larger but the tricritical point remains at the intersection of a first order and second order transition and can be seen as the intersection of three critical manifolds (lines or surfaces) hence the name tricritical.

We may extend this to the tetracritical case where we must consider a polynomial of higher degree:

$$
G=a(T, x, y) m^{2}+b(T, x, y) m^{2}+c(T, x, y) m^{6}+d(T, x, y) m^{8}
$$

and $d$ must be positive while all other coefficients can take any sign. In the following sections we will discuss new fixed-points in the $O(N)$ model some being tetratrical and even pentacritical. While these new fixedpoints can not be understood in terms of polynomial expansions, the meaning of a tetracritical point in a phase diagram, that is the necessity of fixing three external control paremeters, still applies. Hence, it is of value to have an idea of the simplest kind of tetracrical phase diagram where the free energy is given by the above degree 8 polynomial. The following paragraphs then describe the new features at play.


Figure 3.1.3: Tetracritical free energies

For $c>0$ there will be no new physics when compared to the tricritical case. This is analogue to how the $m^{6}$ polynomial is qualitatively the same as the $m^{4}$ case when the coefficient of the $m^{4}$ term remains positive. However, when $c \leq 0$ there are many configurations in which the local extrema may be positioned with respect to each other. The most distinct configurations are given in Fig.(3.1.3) where we have not included special points where two extrema are at the same height. However, if we are only interested in the positions of the global minima, then the two diagrams on the first row suffice and there is one first order phase transition between the two as the global minima transfers from outer minima to inner minima. We will thus now complete the phase diagram including all possible sign changes and relative amplitudes of the coefficients. This is summarized in Table.(3.1). This table gives the domain of existence of the different types of potentials. We then use this table to find the phase diagram of Fig.(3.1.4) which illustrates the different types of phase transitions that may take place with tetracritical physics. In particular we see that at the boundary of the second order transition we have two tricritical lines between a second order transition surface and a first order one. When these two tricritical lines meet we obtain the tetracritical point at $(a, b, c)=(0,0,0)$. Hence, as this point requires adjusting three parameters, the associated linearized RG flow should have three unstable directions given by three negative eigenvalues.

|  | a | b | c | d | Positive extrema | Total number of global minima | Type of free energy |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| A | + | + | + | + | 0 | 1 | S |
| D | + | + | - | + | 2,0 | 2,1 | S,T |
| B | + | - | + | + | 2,0 | 2,1 | S,T |
| C | + | - | - | + | 2,0 | 2,1 | S,T |
| E | - | + | + | + | 1 | 2 | D |
| H | - | + | - | + | 3,1 | 2,1 | Q,D |
| F | - | - | + | + | 1 | 2 | D |
| G | - | - | - | + | 1 | 2 | D |



Table 3.1: All possible signs of the coefficients of a tetracritical Landau free energy. The table at the top gives all possible signs of the coefficients. For each given combination of signs, the first column gives the corresponding regions within the cube at the bottom. The positive extrema column corresponds to the number of possible extrema for positive $m$ as given by Descartes rule of signs (see Appendix A). The total number of global minima column corresponds to the number of minima for both $m>0$ and $m<0$ while discarding the extrema that are maxima. The special cases where meta-stable states are at the same height as stable states are not considered. The last column gives the nature of the shape of the free energy using the usual nomenclature as given by previous figures where $Q$ refers to quadruple local minima. When a given block in the cube at the bottom corresponds to a region where the number of minima depends on the magnitude of the coefficients of the free energy and is not directly given by the signs, we must consider that there exists a surface within that region that separates the two possible scenarios. This surface will be a spinodal announcing a first order transition as in the tricritical case.

Finally we see that if we define the second order surface as $\mathcal{M}$, then we have a tricritical line at the boundary of $\mathcal{M}$, that is, $\partial \mathcal{M}$ and a tetracritical point at the boundary of $\partial \mathcal{M}$ meaning $\partial \partial \mathcal{M}$. In general a $n$-th order multicritical point will be at the $(n-2)^{\text {th }}$ order boundary of $\mathcal{M}$.


Figure 3.1.4: Tetracritical phase diagram. The diagram is given in (a,b,c) space as explained in the main text. The letters S,D,T,Q refer to the number of minima of the potential as Single, Double,Triple,Quadruple respectively. In each S,T block there is a Single-Triple first order transition as in the tricritical case but the lines are now surfaces that are not shown and are left implicit. In the light blue shaded area there is a second order phase transition from a single minima to a double minima which can be regarded as a 2D extension of the 2 nd order line in the tricritical case. At the boundary of this 2D critical surfaces are tricritical line at the intersection between a 2 nd order phase transition surface and first order transition surfaces that are not shown. At the intersection of these two tricritical lines is located the tetracritical point.

### 3.1.2.2 Multicritical fixed-points in the $O(N)$ model

As before, we consider a Landau potential which we now generalize to the case of $N$ fields having $O(N)$ symmetry. The $\mathrm{O}(N)$ symmetry implies that the potential $V$ of the model is a function of only $\vec{\phi}^{2}=\sum_{i=1}^{N} \phi_{i}^{2}$ where $\vec{\phi}$ is the field variable of the model. For example for $N=1,2,3, \vec{\phi}$ represents the magnetization of a ferromagnet. As explained in Sec.(2.1.4), adding a kinetic term $K$ to this potential $V$ one can consider fluctuations in the system if one considers Boltzmann weights with energy functionals $K+V$. The motivation for this is the same as before: although a specific model with $O(N)$ symmetry might have a different energy functional then the choice we take here, universal quantities will be the same regardless of the microscopic details of the model being considered ${ }^{7}$. While non universal quantities such as the critical temperature of a system are also interesting, our focus here will mainly reside on the existence of new fixed-points and thus also the existence of new phase diagrams and physics.

Hence, as in the case of Landau theory, we expand the potential $V$ in powers of the fields where we will now consider an expansion to all orders as:

$$
\begin{equation*}
V\left(\phi^{2}\right)=a_{2} \vec{\phi}^{2}+a_{4}\left(\vec{\phi}^{2}\right)^{2}+a_{6}\left(\vec{\phi}^{2}\right)^{3}+\ldots a_{2 p}\left(\vec{\phi}^{2}\right)^{p}+\ldots \tag{3.1.9}
\end{equation*}
$$

Where should we truncate this sum ? In general for a non perturbative fixed-point this will depend on the radius of convergence of $V$ as a function of $\phi^{2}$ and on the accuracy sought ${ }^{8}$. However, there are cases where the RG analysis is genuinely perturbative and the order at which we may truncate is easily obtained. This happens in the neighborhood of critical dimensions. To see this, let us consider the Gaussian fixed-point where $V=0$ and the hamiltonian $\mathcal{H}$ only has a kinetic term which we have not written here. In this case, the complicated RG flow of Eq. $(2.3 .55$ which we rewrite here as:

$$
\begin{equation*}
\partial_{t} \widetilde{a}_{i}=-\Delta_{i} \widetilde{a}_{i}+F_{i}\left(\left\{\widetilde{a}_{j}\right\}_{j}\right), \tag{3.1.10}
\end{equation*}
$$

[^41]where the dimensionless $\widetilde{a}_{i}$ represent the couplings in the potential, can be simplified to:
\[

$$
\begin{equation*}
\partial_{t} \widetilde{a}_{i}=-\Delta_{i} \widetilde{a}_{i} \tag{3.1.11}
\end{equation*}
$$

\]

These $\Delta_{i}$ are in turn given by dimensional analysis through the hamiltonian $S$ :

$$
\begin{equation*}
S=\int \mathrm{d}^{d} x\left(\frac{1}{2} \partial \phi \partial \phi+V\left(\phi^{2}\right)\right) \tag{3.1.12}
\end{equation*}
$$

which must be dimensionless. Thus, scaling $x$ as $x \rightarrow \sigma x$ leads to $V \rightarrow \sigma^{-d} V$ in order to compensate the effect of the measure. This is true for the kinetic term as well which leads to $\partial \rightarrow \sigma^{-1} \partial$ and $\phi \rightarrow \sigma^{-(d-2) / 2} \phi$. This then allows us to deduce how the $a_{i}$ scale in the neighborhood of the Gaussian fixed-point. The result is $\Delta_{2 m}=2 m-d(m-1)$ where for $m=2$, we have $-\Delta_{4}=d-4$ which changes sign at $d=4$. Thus, for $d<4$, the eigendirection associated to the term $\phi^{4}$ becomes relevant. In this case the Gaussian fixed-point is unstable to a perturbation $a_{4} \phi^{4}$ as well as a perturbation $a_{2} \phi^{2}$. The Gaussian fixed-point is then tricritical below $d=4$. As was shown in Sec. (2.3.5) in the case of $(d=3, N=1)$, a perturbation of the Gaussian FP may lead the RG to flow to a new fixed-point : the Wilson-Fisher fixed-point. In turn, as was explained in Sec. 2.3.5), the Wilson-Fisher fixed-point has only one unstable eigendirection.

More precisely, the $\beta$ function in the neighborhood of $d=4$ can be found in, for example, [28, 67] :

$$
\begin{equation*}
\beta\left(a_{4}\right)=-(4-d) a_{4}+(N+8) a_{4}^{2}+o\left(a_{4}^{2}\right) \tag{3.1.13}
\end{equation*}
$$

It is then apparent that the above flow has two fixed-points: the Gaussian and the Wilson-Fisher (WF). The Gaussian fixed-point has $a_{4}^{G, *}=0$ while the WF fixed-point has a non trivial $a_{4}^{\mathrm{WF}, *}$ where $a_{4}^{\mathrm{WF}, *}=O(4-d)$ in the neighborhood of $d=4$. This latter fixed-point (FP) then bifurcates from the Gaussian FP in $d=4$. The value of $a_{4}^{\mathrm{WF}, *}$ continues to grow between $d=4$ and $d=3$. To summarize we have the following scenario:
$d>4$ : There is one fixed-point: the Gaussian fixed-point. The Gaussian fixed-point is obtained by fine tuning one external control parameter.
$d<4$ : There are two fixed-points: the Gaussian and the Wilson-Fisher fixed-point. As was the case for the Gaussian fixed-point for $d>4$, the Wilson-Fisher fixed-point is obtained by fine tuning one external control parameter. The Gaussian fixed-point is obtained by fine tuning two external control parameters: it is tricritical.

One might then wonder about the other multicritical Landau potentials of the previous section, how do they manifest themselves within the RG? To answer this question, consider the next eigenvalue $-\Delta_{6}=2(d-3)$ which becomes relevant for $d<3$. This eigenvalue corresponds to a $a_{6} \phi^{6}$ perturbation which is now relevant and leads to an entirely new FP that bifurcates from the Gaussian. As with the WF FP, this new FP acquires the unstable directions of the Gaussian FP below the critical dimension $d=3$, that is : $\left(a_{2}, a_{4}\right)$. In turn, for $d<3$, the Gaussian FP is now unstable with respect to $\left(a_{2}, a_{4}, a_{6}\right)$ and is thus tetracritical. Hence, summarizing, we have the following:
$d>3$ : There are two fixed-points: the Gaussian and the Wilson-Fisher fixed-point. As was the case for the Gaussian fixed-point for $d>4$, the Wilson-Fisher fixed-point is obtained by fine tuning one external control parameter. The Gaussian fixed-point is obtained by fine tuning two external control parameters: it is tricritical.
$d<3$ : There are three fixed-points: the Gaussian, the Wilson-Fisher and the tricritical fixed-point. As was the case for the Gaussian fixed-point for $d>3$, the tricritical fixed-point is obtained by fine tuning two external control parameters. The Gaussian fixed-point is obtained by fine tuning three external control parameters: it is tetracritical ${ }^{9}$.

[^42]As was the case for the WF FP, it becomes increasingly difficult to quantitatively characterize this tricritical FP as the dimension is lowered. For $d=2$ and $N=1$, it becomes a member of the two dimensional Virasoro minimal models. In fact, in two dimensions, it is the simplest model that is also superconformal as it possesses both super-symmetry and conformal symmetry [144, 145.

This discussion with regard to bifurcations of the Gaussian FP can be generalized for all multi-critical fixed-points, for all $N>0$, where each multicritical fixed-point bifurcates from the Gaussian fixed-point at $d_{c}(p)=2+2 / p$ where $a_{2(p+1)}$ becomes relevant.

At $d=2$ and for $N=1$, all multicritical points exist and are part of the Virasoro minimal models in two dimensions. Furthermore, all FPs naturally have an analytical extension to non-integer ( $N, d$ ) via the $\beta$ functions. However for generic, not necessarily integer, $N$ and $d$, it is not always clear whether a fixed-point exists as a real valued analytical fixed-point. Indeed our analysis of the eigenvalues of the Gaussian fixedpoint only allows us to deduce that a fixed-point should exist in a neighborhood of the critical dimension but it does not guarantee its existence arbitrarily far from the critical dimension. For example the Wilson-Fisher fixed-point cannot exist below 2 dimensions for $N>2$ because of the Mermin-Wagner theorem. It is often believed however that these multicritical FPs should exist for generic $(N, d)$ at least above $d=2$. Indeed this seems to be the case for $N=1,2,3,4[122,146]$ so it is quite tempting to generalize this for all $N$. However, one of the objectives of this thesis is to show that this is not true for $N$ sufficiently large. That this is the case at least for large values of $N$ seems already plausible considering that, as we shall see in Sec.(3.1.3.2), only the Gaussian and WF FPs exist in generic dimension $4>d>2$ at $N=\infty$. This is clearly in stark contrast to the multitude of multicritical FP for small $N$ such as $N=1$. Another intriguing fact is that at the special critical dimension $d_{c}(p)$ a line of FPs is found for $p>1$. The case $d_{c}(2)=3$ will be of particular interest in the following.

### 3.1.3 Bardeen-Moshe-Bander phenomenon using standard field theory techniques

The rest of this thesis will be devoted to the study of new multicritical fixed-points in the $\mathrm{O}(N)$ model. The presence of these new fixed-points will prove to be intimately related to the existence of a bounded line of fixed-points at $(d=3, N=\infty)$. This line is known in the litterature and has been given the name "Bardeen-Moshe-Bander phenomenon" (BMB) [147. As this line only exists at $N=\infty$, it was believed that this was simply a mathematical curiosity at an unhysical value of $N$. However, one of the key messages of this thesis is that this is not true and that the existence of this line implies, by intricate consistency arguments, the existence of new fixed-points at finite $N$ and at $d=3$. Hence, as the BMB line will play a pivotal role in the following, the first stages of the analysis will review the methods commonly used to highlight its existence.

### 3.1.3.1 Large $N$ analysis: leading order

In this section we will explain how to retrieve the peculiarities of the case $N=\infty$ using the standard large $N$ procedure. This analysis follows [28, 148, 149].

Let us thus consider the partition function:

$$
\begin{equation*}
Z=\int D \vec{\phi} e^{-S_{K}[\vec{\phi}]-\int V\left(\vec{\phi}^{2}\right)} \tag{3.1.14}
\end{equation*}
$$

where $\vec{\phi}$ is a vector and we have separated the usual Gaussian kinetic term $S_{K}$ and the potential term $V$. If we then use 10

$$
\begin{align*}
e^{-V\left(\vec{\phi}^{2}\right)} & =\int D \chi \delta\left(\chi-\vec{\phi}^{2}\right) e^{-V(\chi)}  \tag{3.1.15}\\
& =\int D \chi \int_{i \mathbb{R}} D \sigma e^{-V(\chi)+\sigma \cdot\left(\chi-\vec{\phi}^{2}\right) / 2}
\end{align*}
$$

we obtain:

$$
\begin{equation*}
Z=\int D \vec{\phi} D \chi D \sigma e^{-S_{K}[\vec{\phi}]-\int V(\chi)+\sigma\left(\chi-\vec{\phi}^{2}\right) / 2} . \tag{3.1.16}
\end{equation*}
$$

[^43]We now separate one field $\psi$ from the $N-1$ remaining fields and write $\vec{\phi}=(\psi, \vec{\varphi})$ as this will be useful later on. Hence, we may collect the quadratic terms as:

$$
\begin{equation*}
S_{K}[\vec{\phi}]+\sigma \vec{\phi}^{2} / 2=\frac{1}{2} \psi\left(-\partial^{2}+\sigma\right) \psi+\frac{1}{2} \vec{\varphi} \cdot\left(-\partial^{2}+\sigma\right) \cdot \vec{\varphi} \underset{\text { def }}{=} \frac{1}{2} \psi \cdot\left(-\partial^{2}+\sigma\right) \cdot \psi+\frac{1}{2} \vec{\varphi} \cdot \mathcal{G} \cdot \vec{\varphi} \tag{3.1.17}
\end{equation*}
$$

We then integrate over the fields $\vec{\varphi}$ which leads to

$$
\begin{equation*}
Z=\int D \chi D \sigma D \psi e^{-\frac{1}{2} \psi \cdot\left(-\partial^{2}+\sigma\right) \cdot \psi-\int V(\chi)+\sigma \chi / 2-\frac{N-1}{2} \operatorname{Tr}(\log (\mathcal{G}))}=\int D \chi D \sigma D \psi e^{-\widehat{S}[\psi, \chi, \sigma]} \tag{3.1.18}
\end{equation*}
$$

Rescaling the fields as $\chi=(N-1) \bar{\chi}, V[\chi]=(N-1) \bar{V}[\bar{\chi}], \psi=(N-1)^{1 / 2} \bar{\psi}$, we obtain an overall factor of $(N-1)$ in $\widehat{S}$. This allows us to obtain a non trivial limit in the limit $N \rightarrow \infty$ by means of the saddle point approximation which after writing the $\operatorname{Tr}(\log (\mathcal{G}))$ in Fourier space leads to:

$$
\begin{equation*}
\bar{V}_{e f f}[\bar{\chi}, \sigma, \bar{\psi}]=\bar{V}[\bar{\chi}]-\sigma\left(\bar{\chi}-\bar{\psi}^{2}\right) / 2+\frac{1}{2} \int_{p^{2}<\Lambda^{2}} \frac{\mathrm{~d}^{d} p}{(2 \pi)^{d}} \log \left(p^{2}+\sigma\right) \tag{3.1.19}
\end{equation*}
$$

where $\Lambda$ is a UV regularization and $(\bar{\chi}, \sigma)$ are solutions of the saddle point equation $\vec{\nabla} \bar{V}_{e f f}=\overrightarrow{0}$. The integral term may be computed analytically and leads to:

$$
\begin{equation*}
\frac{1}{2} \int_{p^{2}<\Lambda^{2}} \frac{\mathrm{~d}^{d} p}{(2 \pi)^{d}} \log \left(p^{2}+\sigma\right)=h_{1}(\Lambda, d)+\sigma h_{2}(\Lambda, d)+\sigma^{d / 2} h_{3}(d)+\frac{\sigma^{2}}{\Lambda^{4}} h_{4}\left(d, \sigma / \Lambda^{2}\right) \tag{3.1.20}
\end{equation*}
$$

with,

$$
\begin{aligned}
h_{1}(\Lambda, d) & =\frac{1}{2} \int_{p^{2}<\Lambda^{2}} \frac{\mathrm{~d}^{d} p}{(2 \pi)^{d}} \log \left(p^{2}\right) \\
h_{2}(\Lambda, d) & =\frac{\Lambda^{d-2}}{(2 \pi)^{d}} \frac{d}{d-2} \\
h_{3}(d) & =\frac{\pi}{d(2 \pi)^{d} \sin \left(\frac{d \pi}{2}\right)} \\
h_{4}\left(d, \sigma / \Lambda^{2}\right) & =\frac{\Lambda^{4}}{(2 \pi)^{d} \sigma^{2}}\left(\ln \left(1+\sigma / \Lambda^{2}\right)-\sigma / \Lambda^{2}+\frac{2 \sigma}{(d-2) \Lambda^{2}}\left({ }_{2} F_{1}\left(1,1-\frac{d}{2} ; 2-\frac{d}{2} ;-\frac{\sigma}{\Lambda^{2}}\right)-1\right)\right)
\end{aligned}
$$

The term $h_{1}$ can easily be discarded as an overall constant and the term $\sigma h_{2}(\Lambda, d)$ which is divergent in the limit $\Lambda \rightarrow \infty$ can be removed by shifting $\chi$ as $\bar{\chi}=\widehat{\chi}+2 h_{2}(\Lambda, d)$ and redefining the potential as $\widehat{V}(\widehat{\chi})=\bar{V}\left(\widehat{\chi}+2 h_{2}(\Lambda, d)\right)-\bar{V}\left(2 h_{2}(\Lambda, d)\right)$. This then leads to :

$$
\begin{equation*}
\widehat{V}_{e f f}(\widehat{\chi}, \sigma, \bar{\psi})=\widehat{V}(\widehat{\chi})-\sigma\left(\widehat{\chi}-\bar{\psi}^{2}\right) / 2+\sigma^{d / 2} h_{3}(d)+\frac{\sigma^{2}}{\Lambda^{4}} h_{4}\left(d, \sigma / \Lambda^{2}\right) \tag{3.1.21}
\end{equation*}
$$

In the limit of large $\Lambda$ we may further simplify this equation to:

$$
\begin{equation*}
\widehat{V}_{e f f}(\widehat{\chi}, \sigma, \psi)=\widehat{V}(\widehat{\chi})-\sigma\left(\widehat{\chi}-\bar{\psi}^{2}\right) / 2+\sigma^{d / 2} h_{3}(d) \tag{3.1.22}
\end{equation*}
$$

If we then consider the saddle point equation for $\sigma$ we arrive at:

$$
\begin{equation*}
\partial_{\sigma} \widehat{V}_{e f f}=-\left(\widehat{\chi}-\bar{\psi}^{2}\right) / 2+\left(\frac{d}{2}\right) \sigma^{(d-2) / 2} h_{3}(d)=0 \Longleftrightarrow \sigma=\left(\frac{\widehat{\chi}-\bar{\psi}^{2}}{d h_{3}(d)}\right)^{\frac{2}{d-2}} \tag{3.1.23}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
\widehat{V}_{e f f}(\widehat{\chi}, \sigma(\widehat{\chi}))=\widehat{V}[\widehat{\chi}]-c_{d}\left(\widehat{\chi}-\bar{\psi}^{2}\right)^{\frac{d}{d-2}} \tag{3.1.24}
\end{equation*}
$$

where,

$$
\begin{equation*}
c_{d}=\frac{1}{2 d^{\frac{d}{d-2}} h_{3}(d)^{\frac{2}{d-2}}}(d-2) . \tag{3.1.25}
\end{equation*}
$$

Considering now the critical dimension $d_{c}(p)=2+2 / p$ we have:

$$
\begin{equation*}
c_{d_{c}(p)}=\frac{(-1)^{p}}{p(2+2 / p)^{p+1}\left|h_{3}(2+2 / p)\right|^{p}} \tag{3.1.26}
\end{equation*}
$$

where we used the fact that $h_{3}(d) \propto_{+} \frac{1}{\sin (d \pi / 2)}{ }^{11}$ with $\sin (d \pi / 2)<0$ for $2<d<3$. For $d=d_{c}(p)$ the potential may then be written as :

$$
\begin{equation*}
\widehat{V}_{e f f}(\widehat{\chi}, \sigma(\widehat{\chi}), \psi)=\widehat{V}(\widehat{\chi})-c_{d_{c}(p)}\left(\widehat{\chi}-\bar{\psi}^{2}\right)^{p+1} \tag{3.1.27}
\end{equation*}
$$

Hence we notice that at the critical dimension $d=d_{c}(p)$, the potential remains analytic as a function of $\widehat{\chi}$. We will find a similar result when performing the RG analysis. Moreover, we notice that the coefficient $c_{d_{c}}$ is positive when $p$ is even, this in turn will lead to the upper limit on the line of fixed-points mentioned in the previous subsection. To see this recall that we used the following shift $\chi=\widehat{\chi}+2 h_{2}(\Lambda, d) \Longleftrightarrow \widehat{\chi}=$ $\chi-2 h_{2}(\Lambda, d)$. Hence knowing that $h_{2}$ is in fact positive and diverges in the limit $\Lambda \rightarrow \infty, \widehat{\chi}$ can be arbitrarily negative. This then leads to a constraint in the couplings of the potential so that the effective potential may remain bounded. For example, in the case of a tricritical potential $\widehat{V}(\widehat{\chi})=\widehat{m}^{2} \widehat{\chi} / 2+\widehat{\lambda} \widehat{\chi}^{2} / 4+\widehat{\tau} \widehat{\chi}^{3} / 3$ and for $p=2$, that is, $d=3$, we have:

$$
\begin{equation*}
\widehat{V}_{e f f}(\widehat{\chi}, \sigma(\widehat{\chi}), \psi)=\widehat{m}^{2} \widehat{\chi} / 2+\widehat{\lambda} \widehat{\chi}^{2} / 4+\widehat{\tau} \widehat{\chi}^{3} / 3-c_{3}\left(\widehat{\chi}-\bar{\psi}^{2}\right)^{3} \tag{3.1.28}
\end{equation*}
$$

Hence, as $\chi$ can be arbitrarily negative, we have in the limit $\chi \rightarrow-\infty$ :

$$
\begin{equation*}
\widehat{V}_{e f f}(\chi, \sigma(\chi), \psi) \simeq \frac{1}{3}|\widehat{\chi}|\left(c_{3}-\widehat{\tau}\right) . \tag{3.1.29}
\end{equation*}
$$

Thus, $\widehat{V}_{\text {eff }}$ is bounded from below if $\widehat{\tau}<c_{3} \sqrt{12}$. This is the BMB phenomenon which can be generalized to all even multicritical fixed-points where one replaces the constraint on $\widehat{\tau}$ by a constraint on $a_{2(p+1)}$. In the tricritical case, it has been shown that the RG fixed-point associated to the endpoint of this line exhibits a massless $O(N)$ scalar bound-state $[147,148,153,154$. In 147$]$, the authors identify this bound-state excitation with a dilaton and claims that the model exibits spontaneous breaking of scale invariance and dimensional transmutation. These features of the BMB endpoint attracts much attention as a toy model example of spontaneous breaking of scale invariance. The aim in the following sections will be to understand the finite $N$ origin of this BMB endpoint. In particular, the analysis will show that the functional nature of the FRG framework is crucial to fully understand the implications of the BMB line at finite $N$. Nonetheless, there is still more information that can be extracted if we consider the following $1 / N$ correction of the above analysis within perturbation theory.

### 3.1.3.2 Large $N$ analysis: order $1 / N$

The $\beta$ function for the $\vec{\phi}^{6}$ coupling $\tau$ in the neighborhood of $d=3, N=\infty$ has been known since [155] and is given by :

$$
\begin{equation*}
N \beta(\tau)=\tau\left(-2 N(3-d)+12 \tau-\pi^{2} \tau^{2} / 2\right)+O(1 / N) \tag{3.1.30}
\end{equation*}
$$

[^44]Taking $\alpha=(3-d) N$, this was the toy model given in Eq. 2.3.43). We recall that the zeroes of this $\beta$ function correspond to three scale invariant solutions which we now call $\left(G, A_{2}(\alpha), \widetilde{A}_{3}(\alpha)\right)$ with $\tau_{G}^{*}=$ $0, \tau_{A_{2}}^{*}=\tau_{-}(\alpha, N), \tau_{\widetilde{A}_{3}}^{*}=\tau_{+}(\alpha, N)$ and $\tau_{-}<\tau_{+}$. The subscripts here refer to the number of relevant infrared eigendirections of the FP. The $A_{2}$ FP is perturbatively connected to the Gaussian, that is, we have $\tau_{-}(\alpha=0, N)=0$ and it is thus the ordinary tricritical FP discussed in Sec.(3.1.2.2). As such, we indeed expect this FP to have two relevant infrared eigendirections as a tricritical FP. Moreover, we saw in Sec. 2.3 .5 ) that the FP associated with $\tau_{+}$has one extra relevant eigenvalue which is why we denote this FP as $\vec{A}_{3}{ }^{13}$. The $\beta$-function in Eq. 3.1.30 was derived from an expansion in $3-d$ where leading and subleading terms have been computed. Importantly, according to the authors in 67 , Eq. 3.1 .30 is exact in the limit $67 N \rightarrow \infty$, that is, higher powers of $\tau$ are of order $\mathrm{O}(1 / N)$. It is therefore very tempting to consider that its second root corresponding to the $\widetilde{A}_{3} \mathrm{FP}$ is physical although it is not Gaussian in $d=3$ and therefore its existence deduced from perturbation theory would be, a priori, doubtful. We will see in the following that its existence is retrieved within the NPRG at least in a finite interval of dimensions. We also explain in Appendix how the same beta function can be retrieved within the usual $1 / N$ expansion rather than with an expansion in $3-d$.

Thus, fully considering the existence of all three fixed-points, we obtain the leading behavior of the three solutions as $\tau_{G}^{*}=0, \tau_{A}^{*}=\tau_{-}(\alpha)+O(1 / N), \tau_{\widetilde{A}}^{*}=\tau_{+}(\alpha)+O(1 / N)$. Hence, for fixed $\alpha$ and sufficiently large $N$, along the path $d(N)=3-\frac{\alpha}{N}$ in the $(d, N)$ plane, the leading order of the coupling $\tau$ is constant and given by $\tau_{x}^{*}(\alpha), x \in\{G, A, \widetilde{A}\}$. Taking then the limit $N \rightarrow \infty$ along the path $d(N)=3-\frac{\alpha}{N}$, we arrive at the point $(d=3, N=\infty)$ where the corresponding value of $\tau$ at $d=3$ and $N=\infty$ is given exactly by $\tau_{x}^{*}(\alpha)$, see Fig. 3.1.5. In other words, to each hyperbolic branch $d(N)=3-\frac{\alpha}{N}$, parameterized by the value of $\alpha$, there is an associated fixed-point at $(d=3, N=\infty)$ parameterized $\tau_{x}^{*}(\alpha)$. Thus, at $d=3, N=\infty$ there is in fact a line of fixed-points, which will be called the BMB line in the following, that are in a one to one correspondence with the different values $\alpha$. We stress here that this is not the generic scenario and a line of fixed-points appears only for multicritical fixed-points in critical dimensions $d_{c}(p)$. We may compare Eq. 3.1.30) with the $\beta$ functions in the neighborhood of $(d=4, N=\infty)$, $\left(d=d_{c}(p=3), N=\infty\right)($ Tetracritical $)$ and $\left(d=d_{c}(p=4), N=\infty\right)$ (pentacritical) found in 67, 156 ${ }^{14}$.

$$
\begin{align*}
\beta\left(a_{4}\right) & =-(4-d) a_{4}+a_{4}^{2}+O(1 / N) \\
N \beta\left(a_{6}\right) & =-2 N(3-d) a_{6}+12 a_{6}^{2}-\pi^{2} a_{6}^{3} / 2+O(1 / N) \\
N \beta\left(a_{8}\right) & =-3 N\left(d_{c}(3)-d\right) a_{8}+\frac{9}{4} a_{8}^{2}+O(1 / N)  \tag{3.1.31}\\
N \beta\left(a_{10}\right) & =-4 N\left(d_{c}(4)-d\right) a_{10}+160 a_{10}^{2}+\sqrt{2 \pi} \frac{80}{3} \Gamma\left(\frac{1}{4}\right)^{2}\left(1-\frac{32}{9 \pi}\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)^{2}\right) a_{10}^{3}+O(1 / N)
\end{align*}
$$

where all couplings have been re-scaled both in terms of $N$ and of the renormalization scale $k$ according to $U\left(\phi^{2}\right)=N k^{d} \widehat{U}\left(\frac{\phi^{2}}{N k^{2 \Delta_{\phi}}}\right)$ 15. We notice that there is no line of fixed-points for the case $d=4$ as $\beta\left(a_{4}\right)$ is not written in terms of $\alpha=N\left(d_{c}(p)-d\right)$. As such, in this case we find only the Gaussian fixed-point at $d=4$. When $4-d$ is small but non zero, the beta function $\beta\left(a_{4}\right)$ has another root given by the Wilson-Fisher fixed-point. However, for multicritical fixed-points near their respective critical dimensions the situation is quite different. For example, in the tricritical case for fixed, small but non zero $3-d$, the beta function behaves as $\beta\left(a_{6}\right)=-2(3-d) a_{6}+O(1 / N)$ and thus only the Gaussian fixed-point makes this $\beta$ function vanishing. Therefore, at least in the neighborhood of $d=3, N=\infty$ there is no tricritical fixed-point for $d \neq 3$ at $N=\infty$. In fact, for generic $d$ and $N=\infty$ there is only the Gaussian fixed-point and the Wilson-Fisher fixed-point for $4>d>2$.

[^45]

Figure 3.1.5: Correspondence between the slope $\alpha$ of $d(N)=3-\alpha / N$ where the value $\tau^{*}(\alpha, N)=\tau^{*}(\alpha)+$ $O(1 / N)$ is conserved along this path, to leading order, and a value of $\tau_{d=3, N=\infty}^{*}$ on the BMB line at $(d=3, N=\infty)$ comprised of the couplings $\tau_{x}^{*}(\alpha), x \in\{G, A, \widetilde{A}\}$. The purple BMB endpoint is the maximal value the tricritical coupling can take at $(d=3, N=\infty)$ according to the BMB phenomenon.

However, as remarked before, if we instead consider $(3-d) N=O(1)$ then the term $-2 N(3-d) a_{6}$ is no longer leading and one may obtain new non trivial fixed-points at large $N$. These fixed-points may be parameterized by $\alpha=(3-d) N$ where each fixed-point $a_{6}(\alpha)$ then corresponds to a FP along the BMB line at $(d=3, N=\infty)$. This generalizes easily to the tetracritical and pentacritical case and we suspect this might be the case for all multicritical fixed-points.

Let us also notice that for large $N$ and at least up to the pentacritical FP, the beta function is a polynomial of degree three in the case of $p$ even whereas in the odd case the beta function is of degree two. In this latter case, as for the tetracritical FP, the $\beta$ function predicts only two FPs, the Gaussian and the perturbative multicritical FP. However, in the case where the $\beta$ function is of degree three, one needs to verify whether the new root is in the physical domain $a_{2(p+1)}>0$. In the tricritical case, the new FP $\widetilde{A}_{3}$ at large $N$ is within the physical domain for $\alpha \in]-\infty, \alpha_{c}$ ] where above $\alpha_{c}, A_{2}$ and $\widetilde{A}_{3}$ collapse and become complex, see Fig. 3.1.6). A similar scenario occurs in the pentacritical case.

However, this leads to many paradoxes. First, if $\widetilde{A}_{3}(\alpha)$ exists for the entire domain $\left.\alpha \in\right]-\infty, 0$ ] and we have $\alpha=(3-d) N$, then does this imply that $\widetilde{A}_{3}(\alpha)$ can be followed to arbitrarily large values of $d$ even beyond $d=4$ ? It is well known that at $d=4$ there is only the Gaussian fixed-point so this would be in contradiction with a well known fact. Second, what would happen if instead we decreased the value of $N$ to $N=1,2,3$ ? These lower values of $N$ have been thoroughly investigated and no fixed-point with the characteristics of $\widetilde{A}_{3}$ have been found. Finally, in the case of the fixed-point $\widetilde{A}_{3}(\alpha)$, a $a_{6}(\alpha)$ grows without bound as $\alpha$ decreases. As there is a map between values of $\alpha$ and points on the BMB line, this is in contradiction with the fact that there is a maximal value of $a_{6}$ on the BMB line given by the BMB endpoint. A partial answer to this last paradox can be found in [147], as the authors claim that above the maximal value of $a_{6}$, that we denote as $a_{6, \mathrm{BMB}}$, perturbation theory breaks down. In fact at $a_{6}=a_{6, \mathrm{BMB}}$ and $(d=3, N=\infty)$ the effective FP potential is singular at small field which is not apparent when focusing only on the coupling $a_{6}$ which remains finite. This suggests that a functional approach, that incorporates all field values at once, is necessary and that perhaps the $\beta$ function when $\tau(\alpha)>\tau\left(\alpha_{B M B}\right)=\tau_{B M B}$, is no longer sufficient to have a full understanding of the fixed-point solutions. As $\beta\left(a_{6}\right)$ should be exact to order $O(1 / N)$ and is in no way peculiar at $a_{6}=a_{6, \mathrm{BMB}}$, we would then expect that there are subtle limits and perhaps non analyticities at play. In the following section we will see that these are indeed present.

### 3.1.4 BMB phenomenon at the level of the LPA

In this section we will retrieve the previous discussion at the level of the Local Potential Approximation (LPA) and extend the analysis using the functional capabilities of the FRG in order to answer the question: what happens to the FP $\widetilde{A}_{3}(\alpha)$ for $\alpha>\alpha_{B M B}$ where we have $\tau(\alpha)>\tau\left(\alpha_{B M B}\right)=\tau_{B M B}$. This section reproduces my work performed with my collaborators B. Delamotte and S. Yabunaka in 157 .


Figure 3.1.6: Zeroes of the tricritical beta function of Eq. 3.1.30) as a function of $\alpha$. The Gaussian fixedpoint is given by $\tau=0$ for all $\alpha$. The upper limit $\tau_{B M B}$ is the maximal value the $\vec{\phi}^{6}$ coupling can take at $(d=3, N=\infty)$ according to the BMB phenomenon. As we also have $\tau_{d=3, N=\infty}^{*}=\tau_{x}^{*}(\alpha), x \in\{G, A, \widetilde{A}\}$ to leading order this also fixes an upper limit of $\tau^{*}(\alpha)$ where $\tau^{*}\left(\alpha_{\mathrm{BMB}}\right)=\tau_{\mathrm{BMB}}$.


Figure 3.1.7: Summary of the current section: FPs existing at $N=\infty$ (left panel) and large $N$ (right panel) in $d<3$ (the Wilson-Fisher FP is not shown although it exists everywhere). Right panel: Straight lines represent the leading order of the critical lines $d(N)=3-\alpha / N+O\left(1 / N^{2}\right)$ where two FPs collapse. The horizontal line where $G=A_{2}$ corresponds to $\alpha=0$ where we have added,by convention, a subscript indicating the number of infrared eigenvalues when $N$ is finite ( 2 as a tricritical fixed-point requires setting two external control parameters to zero which can be identified with $\left.\left(a_{2}, a_{4}\right)\right)$. Left panel: the BMB line made of regular FPs between the Gaussian and the BMB FPs. This line is made of two parts corresponding to the limits when $N \rightarrow \infty$ of two different FPs: the $A, \widetilde{A}$ which are respectively the limits of the $A_{2}$ and $\widetilde{A}_{3}$ (as was verified in Sec. 2.3.5 $\widetilde{A}$ is unstable with respect to the $\tau$ direction and thus has an extra infrared eigenvalue). The dashed line between the right and left panels shows the limit of the remarkable FPs the FP where $A=\widetilde{A}$ is the limit of $A_{2}=\widetilde{A}_{3}$ along the line indexed by $\alpha_{c r}$.

As mentioned in Sec. 2.4.3), the local potential approximation (LPA) is usually the leading order in a wide variety of approximation schemes. The LPA approximation to the flow of the potential also has the advantage of being exact in certain limits one of them being the limit $N \rightarrow \infty$ [158. Moreover, as discussed in Sec.(2.4.3), the LPA can be formulated in terms of the following Ansatz ${ }^{16}$

$$
\begin{equation*}
\Gamma_{k}^{\mathrm{LPA}}[\phi]=\int_{x}\left(\frac{1}{2}\left(\nabla \phi_{i}\right)^{2}+U_{k}(\rho)\right) \tag{3.1.32}
\end{equation*}
$$

where $\rho=\phi_{i} \phi_{i} / 2$. It was shown in 159,160 that the Litim regulator $R_{k}\left(q^{2}\right)=\left(k^{2}-q^{2}\right) \Theta\left(k^{2}-q^{2}\right){ }^{17}$ is optimal for the LPA ${ }^{18}$. Therefore, we will only consider this regulator in the present section.

Moreover, when at criticality, the system is self-similar and the RG flow converges to a FP once all dimensionful quantities have been rescaled in terms of $k$. We thus proceed as usual by rescaling fields and coordinates according to $\tilde{x}=k x$, and $\tilde{\phi}(\tilde{x})=v_{d}^{-\frac{1}{2}} k^{(2-d) / 2} \phi(x)$ with $v_{d}^{-1}=2^{d-1} d \pi^{d / 2} \Gamma\left(\frac{d}{2}\right)$ corresponding to the volume factor from the $d$-dimensional integration of the flow. The potential is then also re-scaled according to its canonical dimension $\tilde{U}_{k}(\tilde{\rho})=v_{d}^{-1} k^{-d} U_{k}(\rho)$, where the volume factor $v_{d}^{-1}$ has been included once more in order to remove this quantity from the flow equations. Notice that there is no field renormalization at the LPA level, that is, $\bar{Z}_{k}=1$, which implies that the rescaling of $\phi$ is performed according to its canonical dimension only and the anomalous dimension at the fixed-point is vanishing. In fact for the Wilson-Fisher fixed-point it is order $O(1 / N)$ while in the tricritical case it is order $O\left(1 / N^{2}\right)$ along the paths $d(N)=3-\alpha / N 67$.

In practice, computing the flow of the potential $U_{k}$ requires several steps. First, the potential is defined by: $U_{k}(\phi)=\mathcal{V}^{-1} \Gamma_{k}[\phi]$ where $\phi$ is a constant field and $\mathcal{V}$ is the volume of the system. Then, the flow of $U_{k}(\phi)$ is obtained by acting with $\partial_{t}$ on both sides of the above definition of $U_{k}$ using the $\Gamma$ flow. Finally, $\Gamma_{k}^{(2)}$ in the right hand side of the $\Gamma$ flow is computed from the LPA ansatz, Eq. 3.1.32. A more detailed calculation is given in Appendix J

It will be very convenient for the following to work mainly with the Polchinski equation at the LPA level. Fortunately, at the level of the LPA and for the Litim regulator, one can easily go from the Wetterich to the Polchinski equation using a Legendre transform that is the 0 -dimensional equivalent to the functional transform in Sec. 22.3) 161. In essence this is simply due to the fact that the LPA is trivially exact in 0 dimensions and the regulator $R_{k}=\left(k^{2}-q^{2}\right) \Theta\left(k^{2}-q^{2}\right)$ removes the kinetic term for $q<k$, thereby leading to a purely potential 0 -dimension like action, while $\partial_{t} R$ decouples the modes where $q>k$. At the LPA we thus define: $\tilde{V}(\tilde{\varrho})=\tilde{U}(\tilde{\rho})+\left(\tilde{\phi}_{i}-\tilde{\Phi}_{i}\right)^{2} / 2$ with $\tilde{\varrho}=\tilde{\Phi}_{i} \tilde{\Phi}_{i} / 2=\tilde{\Phi}^{2} / 2$ and $\tilde{\phi}_{i}-\tilde{\Phi}_{i}=-\tilde{\Phi}_{i} \tilde{V}^{\prime}(\tilde{\varrho})=-\tilde{\phi}_{i} \tilde{U}^{\prime}(\tilde{\rho})$. It is convenient to rescale $\tilde{\varrho}$ and $\tilde{V}(\tilde{\varrho})$ as usual: $\varrho=\tilde{\varrho} / N, \bar{V}=\tilde{V} / N$. In Appendix $\int$ Je compute the flow equation for $\bar{V}(\bar{\varrho})$ which reads 162164 :

$$
\begin{equation*}
\partial_{t} \bar{V}=1-d \bar{V}+(d-2) \varrho \bar{\varrho} \bar{V}^{\prime}+2 \bar{\varrho} \bar{V}^{\prime 2}-\bar{V}^{\prime}-\frac{2}{N} \bar{\varrho} \overline{V^{\prime \prime}} \tag{3.1.33}
\end{equation*}
$$

where the primes represent derivatives with respect to $\bar{\varrho}$. The flow equation of the Wetterich effective potential instead reads (also see Appendix (J):

$$
\begin{equation*}
\partial_{t} \bar{U}=-d \bar{U}+\frac{1}{2}(d-2) \bar{\phi} \bar{U}^{\prime}+\left(\frac{N-1}{N}\right) \frac{\bar{\phi}}{\bar{\phi}+\bar{U}^{\prime}}+\frac{1}{N} \frac{1}{1+\bar{U}^{\prime \prime}} \tag{3.1.34}
\end{equation*}
$$

where the primes represent derivatives with respect to $\bar{\phi}$. In terms of the variable $\bar{\rho}$ this last equation reads:

$$
\begin{equation*}
\partial_{t} \bar{U}=-d \bar{U}+(d-2) \bar{\rho} \bar{U}^{\prime}+\left(\frac{N-1}{N}\right) \frac{1}{1+\bar{U}^{\prime}}+\frac{1}{N} \frac{1}{1+\bar{U}^{\prime}+2 \bar{\rho} \bar{U}^{\prime}} \tag{3.1.35}
\end{equation*}
$$

where now the primes represent derivatives with respect to $\bar{\rho}$. The usual $N \rightarrow \infty$ limit consists in assuming that $\bar{V}$ and $\bar{U}$ are regular for all field values which allows us to discard the last term in Eq. (3.1.33) and

[^46]

Figure 3.1.8: Phase diagram at $(d=3, N=\infty)$ on the critical surface where $\kappa=1$. The light blue point represents the attractive infrared Wilson-Fisher fixed-point. The light blue line represents the UV fixedpoints given by the BMB line $\left(a_{4}=0,0<a_{6}<a_{6}^{B M B}\right)$. At the level of the local expansion about the minimum at $\bar{\rho}=\kappa$ this line is infinite and given by the dashed line. The color code of the phase diagram represents the amplitude of the vector $\left(\beta_{a_{4}}, \beta_{a_{6}}\right)$.

Eq. 3.1 .34 because of its $1 / N$ prefactor. In this limit, one may gain valuable information on the RG flow by expanding the potential $\bar{U}(\bar{\rho})$ as $\bar{U}=\sum_{n} a_{2 n}(\bar{\rho}-\kappa)^{n} / n$ ! in Eq. 3.1.35 with $\kappa$ the (running ${ }^{19}$ minimum of the (running) effective potential. Discarding the last term in Eq. (3.1.36), we obtain the following system of flow equations, valid at $N=\infty$ and $d=3$ 165]:

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{dt}} \kappa & =1-\kappa \\
\frac{\mathrm{d}}{\mathrm{dt}} a_{4} & =-a_{4}\left(1-2 a_{4}\right)  \tag{3.1.36}\\
\frac{\mathrm{d}}{\mathrm{dt}} a_{6} & =-6 a_{4}\left(a_{4}^{2}-a_{6}\right)
\end{align*}
$$

The FPs are found by imposing that $\partial_{t} U=0$, that is, $\frac{\mathrm{d}}{\mathrm{dt}} \kappa=0$ and $d a_{n} / d t=0$ for all $n$. The first solution $\left(\kappa^{*}, a_{4}^{*}, a_{6}^{*}, \ldots\right)=(1,1 / 2,1 / 4, \ldots)$ is the Wilson-Fisher fixed-point for which $\kappa$ is the only relevant eigendirection. This solution indeed has only one relevant infrared direction with respect to $\kappa$ as $\partial_{\kappa} \beta_{\kappa}=-1$ and the $2 \times 2$ matrix below $\partial_{a_{2 j}} \beta_{a_{2 i}}$ has only positive eigenvalues ${ }^{20}$. That the eigenvalue for $\kappa$ is negative regardless of the fixed-point (FP) solution allows us to infer that it depends directly on the external temperature. The next fixed-point solution is given by $\left(\kappa^{*}, a_{4}^{*}, a_{6}^{*}, \ldots\right)=\left(1,0, a_{6}, \ldots\right)$ where $\beta_{a_{6}}=0$ regardless of the value of $a_{6}$. In this case the coupling $a_{6}$ is exactly marginal in the sense that it does not renormalize and has a 0 infrared eigenvalue. This forms the line of fixed-point solutions mentioned in the previous section. Moreover, in this case, the coupling $a_{4}$ also becomes relevant as can be checked directly from $\partial_{a_{4}} \beta_{a_{4}}$ and thus these FPs are tricritical. These features are visible in Fig. 3.1.8. .

However, what about the BMB phenomenon? There does not seem to be any constraint on $a_{6}$ contrary to what we said in the previous section. In fact to obtain this constraint we will see that one needs to obtain

[^47]information about the neighborhood of $\bar{\rho}=0$ whereas up to this point we have only expanded the potential about its minimum at $\bar{\rho}=\kappa$. It is then most convenient to obtain the entire field dependence of $U$ or $V$. This is possible via an implicit solution taking the derivative of Eq. 3.1 .33 with $\partial_{t} \bar{V}^{\prime}=0$ and writing it as a differential equation for the reciprocal function $\bar{\varrho}\left(V^{\prime}\right) 162,165$. In the case of the Polchinski equation the solution $\bar{\varrho}\left(V^{\prime}\right)$ may be written as:
\[

$$
\begin{equation*}
\bar{\varrho}_{ \pm}=1+\frac{\bar{V}^{\prime}\left(\frac{5}{2}-\bar{V}^{\prime}\right)}{\left(1-\bar{V}^{\prime}\right)^{2}}+\frac{\frac{3}{2} \arcsin \left(\sqrt{\bar{V}^{\prime}}\right) \pm \sqrt{2 / \tau}}{\left(\bar{V}^{\prime}\right)^{-1 / 2}\left(1-\bar{V}^{\prime}\right)^{5 / 2}} \tag{3.1.37}
\end{equation*}
$$

\]

where $\bar{\varrho}_{+}\left(\bar{V}^{\prime}\right)$ and $\bar{\varrho}_{-}\left(\bar{V}^{\prime}\right)$ correspond to the two branches $\bar{\varrho}>1$ and $\bar{\varrho}<1$ respectively, and $\tau$ is an integration constant here but via the duality with the effective action it plays the role of the $\vec{\phi}^{6}$ coupling $\tau \vec{\phi}^{6} 162$. Hence, along constant lines $\bar{V}^{\prime}=v$ for $v \in \mathbb{R}$, we may plot $\bar{V}^{\prime}(\bar{\varrho})$ as the parametric plot $(\bar{\varrho}(v), v)$ but we may also obtain $\bar{V}(\bar{\varrho})$ via Eq. 3.1.33) as :

$$
\begin{equation*}
\bar{V}(v)=\frac{1+\bar{\varrho}(v) v+2 \bar{\varrho}(v) v^{2}-v}{3} . \tag{3.1.38}
\end{equation*}
$$

This then allows us to obtain $\bar{V}(\bar{\varrho})$ as the parametric plot $(\bar{\varrho}(v), \bar{V}(v))$. A detailed analysis of Eq. 3.1.37) shows that (i) the Gaussian FP $G$ for which $\bar{V}^{\prime}(\bar{\varrho})=0$ is obtained for $\tau=0$, (ii) a well-defined solution $\bar{V}(\bar{\varrho})$ exists for all $\tau \in\left[0, \tau_{\text {BMB }}=32 /(3 \pi)^{2}\right]$ which therefore corresponds to the BMB line of FPs, denoted here by $\mathcal{A}(\tau)$, with the BMB FP being the endpoint obtained for $\tau=\tau_{\text {BMB }}$ as in [147, 153, 154, 162, 165, (iii) for $\tau>\tau_{\text {BMB }}$ the solutions of Eq. (3.1.37) are not defined on the whole interval $\bar{\varrho} \in[0, \infty[\mid 162]$, (iv) an isolated solution exists for $\sqrt{2 / \tau}={ }^{21}$ which corresponds to the Wilson-Fisher FP associated with the usual second order phase transition of the $\mathrm{O}(N=\infty)$ model (an analytic continuation is needed when $V^{\prime}<0$ ).

We plot these FP potentials in Fig. 3.1.9). One observes that, for all $\tau<\tau_{\text {BMB }}$, the FP potentials $\bar{V}_{\tau}(\bar{\varrho})$ along the BMB line are regular for all values of the field. Approaching $\tau_{\text {BMB }}$, the FP potentials approach a limiting shape which shows a singularity at a value $\bar{\varrho}_{0}$ in its second derivative ${ }^{22}$. In the Wetterich version of the flow, the potentials are qualitatively similar to the Polchinski case in the neighborhood of the minimum of the potential but for the BMB FP $\bar{U}^{\prime}(\bar{\rho})$ shows a singularity at vanishing field as $\bar{U}^{\prime}(\bar{\rho}) \propto \frac{1}{\sqrt{\bar{\rho}}} 165$.

Let us now look for the finite $N$ origin of the BMB line within our functional framework. Just as in perturbation theory, we take the limit $N \rightarrow \infty$ and $d \rightarrow 3$ at fixed $\alpha=(3-d) N$. Our aim is to show that to each FP $\mathcal{A}(\tau)$ with $\tau \in\left[0, \tau_{\mathrm{BMB}}\right]$ on the BMB line, there is one FP at finite $N$, either $A_{2}(\alpha)$ or $\tilde{A}_{3}(\alpha)$, that converges to $\mathcal{A}(\tau)$ when $N \rightarrow \infty$. The problem is therefore to relate admissible values of $\tau$, that is, values for which a FP on the BMB line exists, to admissible values of $\alpha$ where $A_{2}(\alpha)$ or $\tilde{A}_{3}(\alpha)$ exist.

Within the LPA, the proof goes as follows. We assume that at large $N$, the FP potentials can be expanded as:

$$
\begin{equation*}
\bar{V}_{\alpha, N}(\bar{\varrho})=\bar{V}_{\alpha, N=\infty}(\bar{\varrho})+\bar{V}_{1, \alpha}(\bar{\varrho}) / N+O\left(1 / N^{2}\right) \tag{3.1.39}
\end{equation*}
$$

and that $\bar{V}_{\alpha, N}(\bar{\varrho}), \bar{V}_{1, \alpha}(\bar{\varrho})$ and $\bar{V}_{\alpha, N=\infty}(\bar{\varrho})$ are regular functions of $\bar{\varrho}$. The potential $\bar{V}_{\alpha, N=\infty}(\bar{\varrho})$ must therefore correspond to one of the FPs on the BMB line. Thus, this potential corresponds to a solution of Eq. (3.1.37) with a definite value of $\tau \in\left[0, \tau_{\text {BMB }}\right]$ :

$$
\begin{equation*}
\bar{V}_{\alpha, N=\infty}(\bar{\varrho})=\bar{V}_{\tau}(\bar{\varrho}) . \tag{3.1.40}
\end{equation*}
$$

We therefore conclude that the regularity of $\bar{V}_{1, \alpha}(\bar{\varrho})$ together with Eq. 3.1.33) and Eq. 3.1.39) determines the relation between $\tau$ and $\alpha$.

As in Eq.(3.1.36), the eigenvalues and eigendirections can be obtained directly from an expansion about the minimum of the potentia ${ }^{23}$. It is then natural to impose analycity at the minimum at finite but large $N$

[^48]

Figure 3.1.9: Left: Potentials $\bar{V}(\bar{\varrho})$ of the tricritical $\mathrm{FPs} \mathcal{A}(\tau)$ of the BMB line (blue) together with the Wilson-Fisher FP (red). The BMB FP is the endpoint of the BMB line (purple). All these potentials are given by Eq. 3.1.37) (in the Wilson-Polchinski version of the LPA flow). The Gaussian FP $G$ corresponds to the horizontal line. The BMB FP potential shows a discontinuity in its second derivative at $\bar{\varrho}=\bar{\varrho}_{0}$. Right: Potentials $\bar{V}^{\prime}(\bar{\varrho})$ of the tricritical FPs $\mathcal{A}(\tau)=\{A(\tau), \tilde{A}(\tau)\}$ of the BMB line (blue). The BMB FP is the endpoint of the BMB line (purple). The second derivative $\bar{V}^{\prime \prime}(\bar{\varrho})$ of the potential of the BMB FP shows a discontinuity in its second derivative at $\bar{\varrho}=\bar{\varrho}_{0}$.
by Taylor expanding about this point. However it can be easily shown from Eq. (3.1.33) that the minimum behaves as $\bar{\varrho}_{\text {min }}=1+O\left(\frac{1}{\sqrt{N}}\right)$ and that the presence of the $1 / \sqrt{N}$ makes the analysis a little complicated. It is thus more convenient to expand around the point $\bar{\varrho}=1$ which is the minimum at $N=\infty$. The quickest method is then to first perform the $1 / N$ expansion of Eq.(3.1.39) and then Taylor expand the full solution about $\bar{\varrho}_{\text {min }}(\infty)$ which leads to a moderately small system of equations involving the coefficients of $\bar{V}_{\alpha, N=\infty}(\bar{\varrho})$ and $\bar{V}_{1, \alpha}(\bar{\varrho})$, see Appendix K and K for more details. However, it is also interesting to take a more functional approach and instead insert Eq.(3.1.39) into Eq.(3.1.33) thereby solving the differential equations in an iterative manner as was done in the perturbative expansions of differential equations in Sec. 2.2 .5 . Looking at the neighborhood of $\bar{\varrho}_{\text {min }}=1$, we find that generically, a non analytic logarithmic behavior shows up at this point. Requiring that its prefactor vanishes imposes (see Appendix K):

$$
\begin{equation*}
\alpha-36 \tau+96 \tau^{2}=0 . \tag{3.1.41}
\end{equation*}
$$

This equation has two solutions $\tau_{1}(\alpha)$ and $\tau_{2}(\alpha)$ that we choose such that $\tau_{1}(\alpha) \leq \tau_{2}(\alpha)$ for all $\alpha$. Moreover, as explained above, $\tau$ is bounded by $\tau_{B M B}$. Hence, at finite $N$, for each value of $\alpha \in\left[0, \alpha_{c}\right]$ corresponds either one or two FPs: for $\alpha \in\left[0, \alpha_{B M B}\right]$ only $A_{2}(\alpha, N)$ exists while for $\alpha \in\left[\alpha_{B M B}, \alpha_{c}\right]$ both $A_{2}(\alpha, N)$ and $\tilde{A}_{3}(\alpha, N)$ exist, see Fig. 3.1 .10 . These FPs that exist at finite $N$ tend to two different FPs on the BMB line at $(d=3, N=\infty)$ that are $A_{2}(\alpha, \infty)=\mathcal{A}\left(\tau_{1}(\alpha)\right)$ and $\widetilde{A}_{3}(\alpha, \infty)=\mathcal{A}\left(\tau_{2}(\alpha)\right)$. While the line $\mathcal{A}(\tau)$ constitutes a continuous line we find it convenient in the following to divide this line at ( $d=3, N=\infty$ ) into two parts: $A(\tau)$ for $0<\tau<\tau_{c}$ and $\widetilde{A}(\tau)$ for $\tau_{c}<\tau<\tau_{B M B}$. Nothing special happens to FP $\mathcal{A}(\tau)$ at $\tau=\tau_{c}$ but this allows us to refer to each point along this line by their finite $N$ antecedent as $A_{2}(\alpha, \infty)=A\left(\tau_{1}(\alpha)\right)$ and $\widetilde{A}_{3}(\alpha, \infty)=\widetilde{A}\left(\tau_{1}(\alpha)\right)$ where by convention the FPS at $N=\infty$ are without subscripts in order to distinguish them from their finite $N$ counterpart. This then constitutes a graph $\tau(\alpha)$ where $\tau$ parameterizes the BMB line but also represents the leading order in $1 / N$ of the tricritical coupling $\tau(\bar{\varrho}-1)^{3}$ in the Taylor expansion of $\bar{V}_{\alpha, N}(\bar{\varrho})$. This graph $\tau(\alpha)$ given by Eq. 3.1.41) is qualitatively similar to the graph in the perturbative case in Fig.(3.1.6) but the values $\alpha_{c}$ and $\alpha_{B M B}$ differ from those obtained from the $3-d$ expansion of the $\beta$ function in Sec.(3.1.3.2) as the LPA is not exact at order $1 / N$.

At the level of the LPA approximation, the value of $\alpha_{c}$, that is, the value of $\alpha$ for which $\tau_{1}\left(\alpha_{c}\right)=$ $\tau_{2}\left(\alpha_{c}\right) \equiv \tau_{\mathrm{c}}$ is $\alpha_{\mathrm{c}}^{\mathrm{LPA}}=27 / 8=3.375$ instead of the exact result $36 / \pi^{2} \simeq 3.65$ which may be obtained from the previous section. This result has been obtained numerically in 166]. For values of $\tau$ larger than $\tau_{\mathrm{c}}$, that is, $\tau=\tau_{2}(\alpha) \in\left[\tau_{\mathrm{c}}, \tau_{\mathrm{BMB}}\right]$, the $\widetilde{A}(\tau)$ FPs on the BMB line are the limits of $\widetilde{A}_{3}(\alpha)$. Using Eq.(3.1.41 we find for $\tau$ that its upper bound $\tau_{\text {BMB }}$ translates into a lower bound on $\alpha: \alpha_{\text {BMB }}=\alpha\left(\tau_{\text {BMB }}\right)$. At the LPA, we find from Eq. 3.1.41): $\alpha_{\text {BMB }}^{\text {LPA }} \simeq 0.51$.


Figure 3.1.10: Zeroes of the tricritical beta function of Eq. $3 \cdot 1.30$ as a function of $\alpha$. The Gaussian fixedpoint is given by $\tau=0$ for all $\alpha$. The upper limit $\tau_{B M B}$ is the maximal value the $\vec{\phi}^{6}$ coupling can take at $(d=3, N=\infty)$ according to the BMB phenomenon. As we also have $\tau_{d=3, N=\infty}^{*}=\tau_{x}^{*}(\alpha), x \in\{G, A, \widetilde{A}\}$ to leading order this also fixes an upper limit of $\tau^{*}(\alpha)$ where $\tau^{*}\left(\alpha_{\mathrm{BMB}}\right)=\tau_{\mathrm{BMB}}$.

Our analysis of Eq. (3.1.33) raises a paradox however. The first one is related to the question: How is it possible that $\widetilde{A}_{3}$ disappears at finite $N$ for $\alpha<\alpha_{\mathrm{BMB}}$ ? Usually, a FP disappears by colliding with another one but if this is the case what would be the limit $N \rightarrow \infty, d \rightarrow 3$ of such a fixed-point ? It cannot be any of the solutions described by Eq. 3.1 .37 as these correspond to the $\left(A_{2}, \widetilde{A}_{3}\right)$ FPs at finite $N$. Thus, if there is indeed a FP with which $\tilde{A}_{3}$ collapses for $\alpha=\alpha_{B M B}$ then its infinite $N$ limit is missing in our list of FPs found as solutions of Eq. (3.1.33) or Eq. 3.1.35) at infinite $N$. We show in the following that there are indeed solutions to Eq. 3.1.33) that are missed in the usual large $N$ approach. This is one of the central results of this thesis.

To see how this may take place let us notice that the limit $\tau \rightarrow \tau_{B M B}$ of $\bar{V}^{\prime}$ is cusped (see Fig. 3 .1.9 ) and may be viewed as the concatenation of two functions: a linear function $\bar{V}(\bar{\varrho})=\bar{\varrho}$ followed abruptly at $\bar{\varrho}=\bar{\varrho}_{0}$ by a curvilinear function $\bar{V}_{\tau_{\text {BMB }}}\left(\bar{\varrho}>\bar{\varrho}_{0}\right)$. The function $\bar{V}(\bar{\varrho})=\bar{\varrho}$ is a solution to the Polchinski equation and thus this piecewise defined function verifies the fixed-point equation both on $\left[0, \bar{\varrho}_{0}\right]$ and on [ $\bar{\varrho}_{0},+\infty[$. It is possible to generalize this construction by concatenating the function $\bar{V}(\bar{\varrho})=\bar{\varrho}$ with any of the solutions $\bar{V}_{\tau}(\bar{\varrho})$ of Eq. 3 3.1.13). More precisely, for a given $\tau$, we can construct a potential that is equal to $\bar{V}(\bar{\varrho})=\bar{\varrho}$ for $\bar{\varrho} \in\left[0, \bar{\varrho}_{0}(\tau)\right]$ and to $\bar{V}(\bar{\varrho})=\bar{V}_{\tau}(\bar{\varrho})$ for $\bar{\varrho} \in\left[\bar{\varrho}_{0}(\tau),+\infty\left[\right.\right.$ as in Fig. 3.1.12]. The value $\bar{\varrho}_{0}(\tau)$ is then defined by the point of intersection of the functions $\bar{V}(\bar{\varrho})=\bar{\varrho}$ and $\bar{V}_{\tau}(\bar{\varrho})$. The question is then whether such piecewise defined functions have a finite $N$ counterpart and the answer is yes. As we shall show, the finite $N$ extension of these piecewise defined potentials are indeed the missing potentials which resolve the paradox of $\widetilde{A}_{3}$ at $\alpha_{\text {BMB }}$.

To see why this is the case, we must first explain how the above piecewise functions at $N=\infty$ can lead to analytical functions at finite $N$. Consider then a simpler differential equation given by:

$$
\begin{equation*}
1-u^{\prime 2}+\epsilon u^{\prime \prime}=0 \tag{3.1.42}
\end{equation*}
$$

Let us first notice that the similarity with Eq. $(3.1 .33)$ is that the small parameter multiplies the term with the highest derivative. Because of this the limit $\epsilon \rightarrow 0$ is singular in the sense that the very nature of the differential equation changes from second to first order which then changes the number of initial conditions required to specify a solution. From such a singular limit on the level of the differential equation one should expect that this limit is also singular for the space of all solutions. This is indeed the case as the initial conditions:

$$
u(1)=u(-1)=0
$$



Figure 3.1.11: Boundary layer formation of Eq. 3.1.42
leads to the solution :

$$
\begin{equation*}
u(x)=\epsilon\left(\log \left(\cosh \left(\frac{1}{\epsilon}\right)\right)-\log \left(\cosh \left(\frac{x}{\epsilon}\right)\right)\right) . \tag{3.1.43}
\end{equation*}
$$

Notice then that we have:

$$
\begin{align*}
\epsilon \log \left(\cosh \left(\frac{x}{\epsilon}\right)\right) & =\epsilon \log \left(\frac{e^{\frac{x}{\epsilon}}+e^{-\frac{x}{\epsilon}}}{2}\right)=\epsilon \log \left(e^{\frac{|x|}{\epsilon}} \frac{e^{\frac{x-|x|}{\epsilon}}+e^{-\frac{x+|x|}{\epsilon}}}{2}\right)  \tag{3.1.44}\\
& =\epsilon \log \left(e^{\frac{x-|x|}{\epsilon}}+e^{-\frac{x+|x|}{\epsilon}}\right)+|x|-\epsilon \log (2)
\end{align*}
$$

Thus, for $x>0$ we have $e^{\frac{x-|x|}{\epsilon}}=1$ and

$$
\begin{equation*}
\epsilon \log \left(e^{\frac{x-|x|}{\epsilon}}+e^{-\frac{x+|x|}{\epsilon}}\right)=\epsilon \log \left(1+e^{-2 \frac{x}{\epsilon}}\right) \sim_{\epsilon \rightarrow 0} \epsilon e^{-2 \frac{|x|}{\epsilon}} \tag{3.1.45}
\end{equation*}
$$

The same result holds for $x<0$ for similar reasons. In turn this means that for small $\epsilon$ the solution to Eq. 3.1.42 behaves as $1-|x|+O\left(\epsilon e^{-\frac{2|x|}{\epsilon}}\right)$ as can be seen in Fig. 3.1.11.

We may then notice that the solution at $\epsilon=0$ is the concatenation of the two solutions $u^{\prime}=1$ and $u^{\prime}=-1$ which are indeed solutions of Eq. 3.1 .42 for $\epsilon=0$ where the term $\epsilon u^{\prime \prime}$ is dropped. Moreover for finite $\epsilon$ and for $|x|<\epsilon$ the correction is of order $\epsilon e^{0}=\epsilon$ while for $|x| \gg \epsilon$ the correction is exponentially vanishing. Hence the correction at finite $\epsilon$ is significant only within a boundary layer $|x|<\epsilon$ where we have $\epsilon u^{\prime \prime}=O\left(e^{-\frac{2|x|}{\epsilon}}\right)=O(1)$ instead of $O(\epsilon)$ as one would naively expect.

The same scenario plays out with the Polchinski Eq. 3.1 .33 where the two solutions to concatenate are $\bar{V}(\bar{\varrho})=\bar{\varrho}$ and $\bar{V}_{\tau}(\bar{\varrho})$. Joining these two solutions leads to a cusp which then forms a boundary layer at finite $N$ as shown in Fig. 3.1.13. Moreover, we have verified that within the boundary layer, these solutions verify $\frac{2}{N} \bar{\varrho} \bar{V}^{\prime \prime}=O(1)$ which is in perfect analogy with was found from the analysis of Eq. 3.1.43).

Finding this boundary layer is easier done with $\bar{V}^{\prime}(\bar{\varrho})$ rather than with $\bar{V}(\bar{\varrho})$ (see Appendix M for details). We define the scaled variable: $\tilde{\varrho}=N\left(\bar{\varrho}-\bar{\varrho}_{0}\right)$ inside the layer. Then, we find that in terms of this variable, $F(\tilde{\varrho})=\bar{V}^{\prime}(\bar{\varrho})$ satisfies at leading order in $1 / N:$

$$
\begin{equation*}
0=1-3 \bar{V}\left(\bar{\varrho}_{0}\right)+\bar{\varrho}_{0} F+2 \bar{\varrho}_{0}\left(F^{2}-F^{\prime}\right)-F \tag{3.1.46}
\end{equation*}
$$

The solution to this equation reads: $F(\tilde{\varrho})=V_{1}-V_{2} \tanh \left(V_{2} \tilde{\varrho}\right)$ with $2 V_{i}=V^{\prime}\left(\overline{\varrho_{0}}{ }^{-}\right) \pm V^{\prime}\left(\bar{\varrho}_{0}{ }^{+}\right)$where the plus sign goes with $i=1$ and the minus sign with $i=2$. It is then straightforward to show that this solution connects smoothly the two values $\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{-}\right)$and $\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{+}\right)$across the boundary layer, as expected.

Notice that a boundary layer cannot be found with the usual $1 / N$ expansion or perturbation theory. Indeed, the usual $1 / N$ expansion fixes a scaling in $N$ for all field values but the scaling in $N$ is actually different within the boundary layer and outside of it. Moreover, as for Eq. 3.1.43), outside of the boundary layer, the difference between these solutions and their regular, non cuspy, counterparts is exponentially small


Figure 3.1.12: Construction of singular solutions by concatenating the linear solution in black with any of the tricritical solutions in blue at $(d=3, N=\infty)$.


Figure 3.1.13: $N=\infty$ and $d=3$ : Singular potential of $S \mathcal{A}(\tau=0.33)$ from the potential of $\mathcal{A}(\tau=0.33)$ given by the red and dashed red curves, Eq. (3.1.37). The green and dashed green curves show $\bar{V}(\bar{\varrho})=\bar{\varrho}$. The potential of $S \mathcal{A}(\tau=0.33)$ is made of the plain green and red curves that meet at $\bar{\varrho}_{0}(\tau=0.33)$. Inset: zoom of the region around the cusp and its rounding at finite $N$ within the boundary layer.
and thus subleading to all powers in $1 / N /{ }^{24}$. Perturbation theory is not capable of finding such non trivial field dependence either as they rely on an implicit criteria of smooth analyticity whereas the boundary layer is singular in nature as it leads to a non analytic cusp at $N=\infty$.

This construction then allows us to build a "singular copy" of the BMB line ${ }^{25}$. We call these fixed-points $S \mathcal{A}(\tau)$ where the $S$ means singular. Thus, the usual BMB line is actually only half of the true line of FPs at $N=\infty$. In the construction above, the BMB FP plays a pivotal role since all singular FPs are obtained by continuously deforming its potential. Notice as well that the endpoint of the singular part of the BMB line $S \mathcal{A}(0)$ is the singular counterpart of the Gaussian FP: It is made of the linear part $\bar{V}(\bar{\varrho})=\bar{\varrho}$ at small field followed by a horizontal part which is identical to the potential of the Gaussian FP. We call $S G$ this FP for this reason.

Now that we have shown that the potentials of the $S \mathcal{A}(\tau)$ FPs have an extension at finite $N$, we have to study on which interval of dimensions $d=3-\alpha / N$ these FPs exist and how many are there for a given $(N, d)$. Concerning the regular solutions we know that there is the Gaussian G, the Wilson-Fisher WF, the perturbative tricritical $A_{2}(\alpha)$ and the non perturbative fixed-point $\widetilde{A}_{3}(\alpha)$. We recall that Eq. 3.1.41 can

[^49]be obtained by a Taylor expansion about $\bar{\varrho}=1{ }^{26}$ which is above the position of the cusps in Fig. 3.1.12. Moreover, we saw that in the case of the solution of Eq.(3.1.42), the correction to the solution is exponentially suppressed beyond the boundary layer. In Appendix Lwe show that this is also the case of the fixed-point $S G=S \mathcal{A}(0)$ corresponding to the singular gaussian. By continuity with respect to $\tau$, we expect this to be true for the entire line $S \mathcal{A}(\tau)$ except perhaps the endpoint corresponding to $\tau_{B M B}$. Hence, as $\bar{\varrho}=1$ is outside of the boundary layer we expect the difference between the regular and singular fixed-points to be of order $O\left(e^{-N}\right)$ at this point which is then smaller than any power of $1 / N$. Thus, the $1 / N$ expansion in Eq. 3.1 .39 ) and the result Eq. 3.1.41 that links $\tau \propto \bar{V}^{\prime \prime \prime}(1)$ to $\alpha$ is also valid for singular FPS at leading order. Hence, there are once more two FPS $S A(\alpha)$ and $S \widetilde{A}(\alpha)$ which correspond to the "singularized" versions of $A_{2}(\alpha)$ and $\widetilde{A}_{3}(\alpha)$. How many unstable eigendirections do these new FPs have? In Sec. 3.1.9 we will show that the eigenvalue set of the singular version of a FP is equal to the union of eigenvalue set of the linear part $\bar{V}=\bar{\varrho}$ and of the regular part. This then leads to the fact that a singularized FP has one more unstable eigendirection than the regular part. Thus, we can already claim that the singular version of $A_{2}$ is three times unstable and that of $\widetilde{A}_{3}$ is four times unstable. We call them $S A_{3}(\alpha, N)$ and $S \widetilde{A}_{4}(\alpha, N)$ where the relationship between $\alpha$ and $\tau$ is the same as that of the regular potentials.

In turn, as $S \widetilde{A}_{4}$ has one extra unstable eigendirection than $\widetilde{A}_{3}(\alpha)$, it is possible that it may collapse with this fixed-point at $\alpha=\alpha_{\mathrm{BMB}}$ as was expained in Sec. 2.3.5 and as is the case of $A_{2}$ and $\widetilde{A}_{3}$. This is not the main reason why these two FPs might collapse, rather it is the fact that at $\tau=\tau_{\text {BMB }}$ they are indeed equal at least at $N=\infty$ where we indeed have $S \mathcal{A}\left(\tau\left(\alpha_{\mathrm{BMB}}\right)=\tau_{\mathrm{BMB}}\right)=\mathcal{A}\left(\tau\left(\alpha_{\mathrm{BMB}}\right)=\tau_{\mathrm{BMB}}\right)$. Thus, we might also expect $S \widetilde{A}_{4}\left(\alpha_{\mathrm{BMB}}, N\right)=\widetilde{A}_{3}\left(\alpha_{\mathrm{BMB}}, N\right)$ as we expect from a FP collapse. However, we were not able to prove analytically that these fixed-points do indeed collapse along the line $\alpha_{B M B}$ for finite $N$ but numerically we find that this is the case and that it takes place along the line $d=3-\alpha_{B M B} / N$ as it should. Thus, at large $N$, both $\widetilde{A}_{3}(\alpha)$ and $S \widetilde{A}_{4}(\alpha)$ exist only within the interval: $\alpha \in\left[\alpha_{\mathrm{BMB}}, \alpha_{\mathrm{c}}\right]$ and they collapse with each other at $\alpha_{\mathrm{BMB}}{ }^{27}$. These results are summarized in Fig. 3.1.14.

Hence, to conclude this section, we have found at $N=\infty$ and $d=3$ that the usual, regular, BMB line represents only half of the full BMB line which is made of both regular and singular FPs. In the WilsonPolchinski RG framework, the singular branch of this line consists of FPs whose potentials are identical to those of the regular part of the BMB line except at small field where it is replaced by a linear part ${ }^{28}$. At the points $\bar{\varrho}_{0}(\tau)$ where these two parts connect, these singular FP potentials show a cusp. The BMB FP is the pivotal point between the regular branch of the BMB line and the singular branch. All FPs of the BMB line, either regular or singular, are the limits of FPs existing at finite $N$ with the subtlety that the $N \rightarrow \infty$ limit should be taken together with $d \rightarrow 3$, letting $\alpha=(3-d) N$ fixed. More precisely, the regular branch of the BMB line is obtained as the limit of two sets of FPs, $A_{2}(\alpha)$ and $\widetilde{A}_{3}(\alpha)$. The singular branch is the limit of two other sets of FPs, namely $S A_{3}(\alpha)$ and $S \widetilde{A}_{4}(\alpha)$, whose potentials show boundary layers at finite $N$ that become cusps at $N=\infty$. At large $N$, all these FPs exist on finite intervals of $d$ except, seemingly, $S A_{3}(\alpha)$ which continues to exist for $\alpha<0$ and thus $d>3$. Thus, it seems that we have traded one paradox for another as once more we do not expect any non trivial fixed-point to exist at $d=4$. Moreover, what happens if we follow these new fixed-points to smaller values of $N$ ? We do not expect these FPs to exist at $N=1,2,3$ as they have never been found previously in the litterature and the $\mathrm{O}(N)$ model has been studied extensively for these values of $N$. The answer to these questions will be given in the final section of this thesis. For now, let us notice that the exact value of $\alpha_{\mathrm{BMB}}$ can be computed from the $N=\infty$ analysis.

[^50]

Figure 3.1.14: FPs existing at $N=\infty$ (left panel) and large $N$ (right panel) in $d \leq 3$ (the Wilson-Fisher FP is not shown although it exists everywhere). Right panel: Straight lines represent the leading order of the critical lines $d(N)=3-\alpha / N+O\left(1 / N^{2}\right)$ where two FPs collapse. The horizontal line where $G=A_{2}$ corresponds to $\alpha=0$ and the line where $\tilde{A}_{3}=S \tilde{A}_{4}$ to $\alpha=\alpha_{\text {BMB }}$. The lines where $S A_{3}=S \widetilde{A}_{4}$ and $A_{2}=\tilde{A}_{3}$ are superimposed since for both of them $\alpha=\alpha_{\text {c }}$ These two lines are therefore represented by a single line with alternating colors: dark pink for the line where $S A_{3}=S \tilde{A}_{4}$ and blue for the line where $A_{2}=\tilde{A}_{3}$. In reality however they only converge towards one another as we have omitted the corrections of order $e^{-N}$ in $S A_{3}$ and $S \widetilde{A}_{4}$ which then slightly modifies the path $d_{c}(N)$, as compared to the case $A_{2}=\tilde{A}_{3}$, along which they collapse. Left panel: the full BMB line made of regular FPs between the Gaussian and the BMB FPs and of singular FPs between the BMB and the singular Gaussian $S G$ FPs. This line is made of four parts corresponding to the limits when $N \rightarrow \infty$ of four different kinds of FPs: the $A, \tilde{A}, S \tilde{A}$ and $S A$ FPs are respectively the limits of the $A_{2}, \tilde{A}_{3}, S \tilde{A}_{4}$ and $S A_{3}$ FPs. The dashed lines between the right and left panels show the limits of the remarkable FPs: The Gaussian FP $G$ on the BMB line is the limit of the Gaussian FP at finite $N$, the FP where $A=\tilde{A}$ is the limit of $A_{2}=\tilde{A}_{3}$ along the line indexed by $\alpha_{c}$, the BMB FP is the limit of $\tilde{A}_{3}=S \tilde{A}_{4}$ along the line indexed by $\alpha_{\mathrm{BMB}}$, the FP where $S A=S \tilde{A}$ is the limit of $S A_{3}=S \tilde{A}_{4}$ along the line indexed by $\alpha_{\mathrm{c}}$ and finally $S G$ is the limit of $S A_{3}$ along the line indexed by $\alpha=0$.

The effective potentials of the FPs along the BMB line are all regular at small $\tau[153,154,165]$ and it is only from the BMB FP that the potentials start showing a singularity at small fields. Within the usual large N framework this has been shown to occur for $\tau_{B M B}=2[147,153,154]$. Provided that the beta function of the previous section is exact at order $1 / N$, the corresponding exact value of $\alpha$ is $\alpha_{\mathrm{BMB}}=12-\pi^{2} \simeq 2.13$. Let us notice that whereas the LPA value of $\alpha_{\mathrm{c}}$ is not too far from the exact value -- 3.375 instead of 3.65 -- the LPA value of $\alpha_{\mathrm{BMB}}$ is quantitatively off by a factor 4 : It is 0.51 instead of 2.13 , this is quantitatively rather poor compared to other quantities determined at the same order of approximation.

There seems to be a few reasons for this. First let us note that the collapse of fixed-points at $\alpha_{B M B}$ is highly non trivial. Indeed, the very existence of the FP $\widetilde{A}_{3}$ requires computation up to four loops within perturbation theory $\sqrt[67]]{ }$. In contrast the LPA is able to retrieve perturbation theory only at one loop for example in $4-\epsilon$. From this perspective it is remarkable that the LPA is even able to predict the existence of $\widetilde{A}_{3}$ let alone find a quantitatively reasonable result for $\alpha_{c}$. Second, $\alpha_{B M B}$ is obtained in a rather subtle way by consistency with exact results at $N=\infty$ but the collapse of fixed-ppoints is not predicted by perturbation theory which is to be expected from the non perturbative functional nature of the BMB FP. In turn this also leads to the fact that contrarily to critical exponents and $\alpha_{c}$, local knowledge of the FP in the neighborhood of the minimum of the potential is insufficient ${ }^{29}$. Instead for $\alpha_{B M B}$ the functional nature of the FP for all fields becomes important which perhaps sets this quantity apart from the more standard critical exponent

[^51]calculations that often lead to good quantitative results. At order two of the derivative expansion, the value of $\alpha_{B M B}$ is significantly improved however as will be shown in a forth coming publication. A similar improvement at the second order of the derivative expansion was obtained for the Ising model at $d=1$ where the potential also develops a boundary layer [168]. More surprisingly however is the result of the following section that it is in fact possible to improve greatly on the value of $\alpha_{B M B}$ even at the level of the LPA!

### 3.1.5 Improving the LPA result

The above discussion can be generalized in a rather straightforward way. First let us note that taking the derivative of the $\Gamma$-flow and defining $w=\bar{U}^{\prime}$ one obtains:

$$
\begin{equation*}
\partial_{t} w=-2 w+(d-2) \bar{\rho} w-\left(\frac{N-1}{N}\right) \frac{w^{\prime}}{(1+w)^{2}}-\frac{1}{N} \frac{3 w^{\prime}+2 \bar{\rho} w^{\prime \prime}}{\left(1+w+2 \bar{\rho} w^{\prime}\right)^{2}} \tag{3.1.47}
\end{equation*}
$$

The functional form of the terms $(1+w)^{-2}$ and $\left(1+w+2 \bar{\rho} w^{\prime}\right)^{-2}$ depend on the choice of the regulator. In general the LPA flow may be written as:

$$
\begin{equation*}
\partial_{t} w=-2 w+(d-2) \bar{\rho} w+\left(\frac{N-1}{N}\right) w^{\prime} f(w)+\left(3 w^{\prime}+2 \bar{\rho} w^{\prime \prime}\right) \frac{1}{N} f\left(w+2 \bar{\rho} w^{\prime}\right) \tag{3.1.48}
\end{equation*}
$$

wher $\underbrace{30}$

$$
\begin{equation*}
f(x)=-k^{-d} \int \mathrm{~d}^{d} q \frac{k^{2} \partial_{t} R_{k}\left(q^{2}\right)}{\left(q^{2}+R_{k}+x k^{2}\right)^{2}} \overline{\overline{d e f}}-\int \mathrm{d}^{d} q \frac{s\left(q^{2}\right)}{\left(q^{2}+r\left(q^{2}\right)+x\right)^{2}} \tag{3.1.49}
\end{equation*}
$$

with,

$$
\begin{equation*}
R_{k}\left(q^{2}\right)=k^{2} r\left(q^{2} / k^{2}\right), s\left(q^{2}\right)=k^{-2} \partial_{t} R_{k}\left(q^{2}\right) \tag{3.1.50}
\end{equation*}
$$

In the literature $f$ is known as a threshold function as it decays for large masses either $x=w$ or $x=w+2 \bar{\rho} w^{\prime}$. At large $N$, we expand $w$ as:

$$
\begin{equation*}
w_{\alpha, N}(\bar{\varrho})=w_{\alpha, N=\infty}(\bar{\varrho})+w_{1, \alpha}(\bar{\varrho}) / N+O\left(1 / N^{2}\right) . \tag{3.1.51}
\end{equation*}
$$

We then normalize $f$ to $f(0)=-1$, which can be implemented by rescaling $\varrho \bar{\varrho}$ and $w$. Next, Taylor expanding $w_{1, \alpha}(\bar{\varrho})$ as in the previous section we arrive at a system of equations which corresponds to the generalization of Eq. 3.1.41) for arbitrary $f$ :

$$
\begin{equation*}
\alpha=18 \tau f^{\prime}(0)+2 \tau^{2}\left(2 f^{\prime \prime}(0)-9 f^{\prime}(0)^{2}\right) \tag{3.1.52}
\end{equation*}
$$

In the case of the Litim regulator, we have $f(x)=-\frac{1}{(1+x)^{2}}$ which leads to $f^{\prime}(0)=2, f^{\prime \prime}(0)=-6$ thereby retrieving the polynomial equation of the previous section.

Does the Litim regulator lead to the best possible values for $\alpha_{c}$ and $\alpha_{B M B}$ ? First let us recall that $\alpha_{c}$ corresponds to the collapse of two roots of Eq. 3.1.52 and may then be obtained from the discriminant of Eq. (3.1.52) as :

$$
\begin{equation*}
\alpha_{c}=\frac{9}{2} \frac{1}{1-\frac{2}{9} \frac{f^{\prime \prime}(0)}{f^{\prime}(0)^{2}}} \tag{3.1.53}
\end{equation*}
$$

Finally reinserting $f(0)$ via $f(x) \rightarrow f(x) /(-f(0))$ and thus $f^{\prime}(0) \rightarrow f^{\prime}(0) /(-f(0))$ and $f^{\prime \prime}(0) \rightarrow f^{\prime \prime}(0) /(-f(0))$, we obtain:

$$
\begin{equation*}
\alpha_{c}=\frac{9}{2} \frac{1}{1+\frac{2}{9} \frac{f^{\prime \prime}(0) f(0)}{f^{\prime}(0)^{2}}} \tag{3.1.54}
\end{equation*}
$$

Applying the principle of minimal sensitivity (PMS) then leads to maximizing or minimizing the block $\frac{f^{\prime \prime}(0) f(0)}{f^{\prime}(0)^{2}}$ which depends on the choice of regulator. For the Litim regulator, $R_{k}\left(q^{2}\right)=k^{2}\left(1-\frac{q^{2}}{k^{2}}\right) \Theta\left(1-\frac{q^{2}}{k^{2}}\right)$ we find $\frac{f^{\prime \prime}(0) f(0)}{f^{\prime}(0)^{2}}=3 / 2$. Let us thus prove that this is in fact the minimal value for the class of threshold

[^52]${ }^{30}$ In the second equation we used the change of variables $q \rightarrow k q$
functions given by Eq. 3.1.49. We then need to show $z=f^{\prime \prime}(0) f(0)-\frac{3}{2} f^{\prime}(0)^{2}>0$. Using Eq. 3.1.49. we find:
\[

$$
\begin{align*}
z= & 6 \int \mathrm{~d}^{d} q \mathrm{~d}^{d} q^{\prime} s(q) s\left(q^{\prime}\right)\left(\frac{1}{\left(q^{2}+r\left(q^{2}\right)\right)^{2}\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{4}}-\frac{1}{\left(q^{2}+r\left(q^{2}\right)\right)^{3}\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{3}}\right) \\
& =6 \int \mathrm{~d}^{d} q \mathrm{~d}^{d} q^{\prime} \frac{s(q) s\left(q^{\prime}\right)}{\left(q^{2}+r\left(q^{2}\right)\right)^{2}\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{2}}\left(\frac{1}{\left(q^{\prime 2}+r\left(q^{2}\right)\right)^{2}}-\frac{1}{\left(q^{2}+r\left(q^{2}\right)\right)\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)}\right) \\
= & 6 \int \mathrm{~d}^{d} q \mathrm{~d}^{d} q^{\prime} \frac{s(q) s\left(q^{\prime}\right)}{\left(q^{2}+r\left(q^{2}\right)\right)^{2}\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{2}}\left(\frac{1}{2}\left(\frac{1}{\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{2}}+\frac{1}{\left(q^{2}+r\left(q^{2}\right)\right)^{2}}\right)-\right. \\
& \left.\frac{1}{\left(q^{2}+r\left(q^{2}\right)\right)\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)}\right)  \tag{3.1.55}\\
= & 3 \int \mathrm{~d}^{d} q \mathrm{~d}^{d} q^{\prime} \frac{s(q) s\left(q^{\prime}\right)}{\left(q^{2}+r\left(q^{2}\right)\right)^{4}\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{4}}\left(\left(q^{2}+r\left(q^{2}\right)\right)^{2}+\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{2}-\right. \\
& \left.\left.2\left(q^{2}+r r^{2}\right)\right)\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)\right) \\
= & 3 \int \mathrm{~d}^{d} q \mathrm{~d}^{d} q^{\prime} \frac{s(q) s\left(q^{\prime}\right)}{\left(q^{2}+r\left(q^{2}\right)\right)^{4}\left(q^{\prime 2}+r\left(q^{\prime 2}\right)\right)^{4}}\left(q^{2}-q^{\prime 2}+r\left(q^{2}\right)-r\left(q^{\prime 2}\right)\right)^{2}>0
\end{align*}
$$
\]

For the Litim regulator for $q^{2}, q^{2}<1$ we have $q^{2}+r\left(q^{2}\right)=q^{2}+1-q^{2}=1$ such that $q^{2}-q^{2}+r\left(q^{2}\right)-r\left(q^{2}\right)$ is exactly zero while for $q>1 s(q)=0$. Thus, indeed $z=0$ for the Litim regulator. This inequality is a rare opportunity to explicitly see what constitutes an optimal regulator at the level of the LPA.

To see that the Litim regulator is optimal more explicitly we may consider the class of threshold functions given by

$$
\begin{equation*}
f(x)=-\frac{1}{(1+x)^{\gamma}} \tag{3.1.56}
\end{equation*}
$$

where the value of $\gamma$ depends on the choice of regulator and a few known values are given in Table. (3.2) [165. In this case we obtain:

$$
\begin{equation*}
\alpha_{c}=\frac{81}{22}\left(1-\frac{1}{1+\frac{11}{2} \gamma}\right) \tag{3.1.57}
\end{equation*}
$$

As $\alpha_{c}(\gamma)$ is then an increasing function of $\gamma$ it is clear that among the regulators in Table $3.2, \gamma=2$ leads to the maximal $\alpha_{c}$ while $\gamma=0$ leads to the non nonsensical case where $\alpha=0$. Hence, the optimal value of $\alpha_{c}$ should be obtained for the maximal value of $\gamma$ rather than the minimal value. According to Eq. (3.1.49) we must have $\gamma<2$ as such a threshold function cannot decay faster than $1 / x^{2}$ for large $x$. However, from Eq. 3.1.57) it is apparent that $\gamma=2$ is not special and that $\alpha_{c}$ continues to increase reaching its maximal value when $\gamma \rightarrow \infty$. Applying the principle of minimal sensitivity without any prejudice should then lead us to take the limit $\gamma \rightarrow \infty$.

There are many reasons to consider this limit regardless of the fact that it can not be obtained from the threshold function given in Eq. (3.1.49). The first is that the LPA is not exact itself and thus the constraint $\gamma<2$ is not strictly necessary at this level of approximation. It is then possible to consider a "generalized PMS" which we now explain.

Consider an approximation scheme for which, at every order $n$ of the approximation, the RG flow depends on a parameter, say $\gamma$, such that for $\gamma \in\left[\gamma_{\min }(n), \gamma_{\max }(n)\right]$, there exists a regulator that corresponds to such a flow using the $\Gamma$-flow Eq. 2.3 .36 . We also consider the possibility that outside of this interval $\left[\gamma_{\min }(n), \gamma_{\max }(n)\right]$ there are no regulators that correspond to such an RG flow. Suppose that we then search for a PMS of $\gamma$ which may or may not lie within $\left[\gamma_{\min }(n), \gamma_{\max }(n)\right]$. Then the only real constraint that exists is that the PMS value of $\gamma$ which we call $\gamma_{n}$ at order $n$ converges to a value $\gamma^{*}$ that belongs to the interval $\left[\gamma_{\min }(\infty), \gamma_{\max }(\infty)\right]$. However, at any finite order $n$ there are no real requirements that $\gamma \in$ $\left[\gamma_{\min }(n), \gamma_{\max }(n)\right]$ for all $n$. If the PMS is a good guide for obtaining optimal results it should be automatic to have $\gamma^{*} \in\left[\gamma_{\min }(\infty), \gamma_{\max }(\infty)\right]$ without artificially forcing $\gamma$ to lie within $\left[\gamma_{\min }(n), \gamma_{\max }(n)\right]$ for every $n$. This is conceptually quite different from what is usually done. One may view this as approximate $R G$ with approximate flows that need not be a truncation of an exact flow at the level of approximation considered but rather one wishes to obtain the optimal renormalization scheme given a predefined approximation.

| $\gamma$ | $R_{k}$ |
| :---: | :---: |
| 2 | $\left(k^{2}-q^{2}\right) \theta\left(k^{2}-q^{2}\right)$ |
| $3 / 2$ | $q^{4} / k^{2}$ |
| 1 | $\lim _{a \rightarrow \infty} a \theta\left(k^{2}-q^{2}\right)$ |
| 0 | $k^{2}$ |

Table 3.2: Some regulators corresponding to different values of the parameter $\gamma$ defined in Eq. 3.1.56).

We call this method "generalized PMS" as we optimize the functional form of the RG equations as well as the regulator. This is only meaningful if at every order $n$ of approximation we allow the possibility that there exists $\left[\gamma_{\min }(n), \gamma_{\max }(n)\right]$ where the RG flow becomes a truncation of an exact flow. In fact we may simply modify by hand the exact $\Gamma$ flow to, for example :

$$
\begin{equation*}
\partial_{k} \Gamma_{k}=\frac{1}{2} \operatorname{Tr}\left(\frac{k \partial_{k}\left(R_{k}\right)}{R_{k}\left(1+\Gamma_{k}^{(2)} / R_{k}\right)^{\gamma}}\right) \tag{3.1.58}
\end{equation*}
$$

In this case, we take the hypothesis that the PMS of $\gamma$ will converge to 1 with appropriate approximation schemes. Of course this depends on whether the PMS can be trusted to such a high degree or whether other criteria should be considered. Also this might depend on the type of approximation scheme considered and whether it breaks symmetries. Hence, clearly the weak point of such a method is that one would have to trust that higher order approximations would lead $\gamma$ to converge to 1 . Because of this we prefer to postpone this question of functionally optimized approximate RG flows to a later time.

Instead let us remark that in fact the LPA flow with generic $\gamma$ given by threshold functions $f(x)=$ $-\frac{1}{(1+x)^{\gamma}}$ are the LPA approximation of a different type of RG flow called "proper time flows" $169-173$. These RG flows were shown to be exact within a background field approach for the effective action 174 177. However justifying the simplest approximation schemes within that framework also requires more work than with the usual $\Gamma$ flow. Instead, let us remark that it has been recently found in [178] that proper time flows are indeed exact but for a Wilsonian action, as the action in the Polchinski equation, rather than an effective action. The difference between the two is that that a Wilsonian action is to be used within a path integral in order to compute expectation values whereas the effective action can be used directly in the tree expansion of correlation functions. The flow equation of a proper time flow is:

$$
\begin{equation*}
\partial_{k} S_{k}=\frac{1}{2} \operatorname{Tr}\left(\frac{k^{2 \gamma}}{\left(k^{2}+S_{k}^{(2)}\right)^{\gamma}}\right) \tag{3.1.59}
\end{equation*}
$$

Which at the level of the LPA retrieves the previous LPA equations but where the action has to be understood as Wilsonian.

Hence, now that we know that there is an underlying exact equation let us verify whether the PMS at $\gamma \rightarrow \infty$ indeed leads to better results. In the limit $\gamma \rightarrow \infty$ we obtain from Eq. 3.1.57) :

$$
\begin{equation*}
\alpha_{c}=\frac{81}{22} \simeq 3.68 \tag{3.1.60}
\end{equation*}
$$

Comparing this to the exact value $\frac{36}{\pi^{2}} \simeq 3.65$ we find a relative error of $0.94 \%$. This result is significantly better than what was obtained with the Litim regulator where we found a relative error of $7.5 \%$. Does the determination of $\alpha_{B M B}$ also improve with the proper time flow? Yes, in fact, surprisingly, the improvement is even better! In order to calculate $\alpha_{B M B}$ for generic $\gamma$ we need to be able to know $\tau_{B M B}$ for all $\gamma$ which, fortunately, has already been calculated in [165] and is given by:

$$
\begin{equation*}
\tau_{B M B}(\gamma)=\frac{2 \Gamma(\gamma)^{2}}{\Gamma(1 / 2)^{2} \Gamma(1 / 2+\gamma)^{2}} \tag{3.1.61}
\end{equation*}
$$



Figure 3.1.15: BMB endpoint for multicritical fixed-points at ( $\left.d_{c}(p)=2+2 / p, N=\infty\right)$ with $p$ even. All BMB endpoints show cusps.

Inserting this expression into Eq. 3.1.52) using $f^{\prime}(0)=\gamma, f^{\prime \prime}(0)=-\gamma(\gamma+1)$ we arrive at:

$$
\begin{equation*}
\alpha_{B M B}(\gamma)=2 \tau_{B M B}(\gamma) \gamma\left(9-\tau_{B M B}(\gamma)(11 \gamma+2)\right) \tag{3.1.62}
\end{equation*}
$$

While admittedly not obvious from the above expression, $\alpha_{B M B}$ is also a steadily increasing function of $\gamma$. For $\gamma \rightarrow \infty, \tau_{B M B}$ converges to zero as $\tau_{B M B} \sim \frac{2}{\pi \gamma}$. This does not mean anything by itself as one can always rescale couplings. Instead what should be physical is the critical dimension $d=3-\alpha_{B M B} / N$ where in the limit $\gamma \rightarrow \infty$ we have

$$
\begin{equation*}
\alpha_{B M B} \sim 2 \tau_{B M B}(\gamma) \gamma\left(9-\tau_{B M B}(\gamma) 11 \gamma\right)=\frac{4}{\pi}\left(9-\frac{22}{\pi}\right) \simeq 2.54 \tag{3.1.63}
\end{equation*}
$$

It is then apparent that this result is much closer to the exact value $\alpha_{\mathrm{BMB}}=12-\pi^{2} \simeq 2.13$ than what was obtained from the Litim regulator with $\alpha_{B M B}=0.51$. Why are these results better ? One reason is likely due to Litim's gap maximization criteria $159,160,179$ where these proper-time flows seem to naturally lead us to consider the limit $\gamma \rightarrow \infty$. A second perhaps related feature for the BMB FP is that the singularity at small fields is now $165 u^{\prime}(\rho) \propto 1 / \rho^{1 / \gamma}$ such that larger values of $\gamma$ lead to a weaker singularity. Having a weaker singularity perhaps leads to better convergence properties with respect to approximations.

It would be interesting to continue this analysis at higher orders of approximation within the framework of the generalized PMS using Eq. 3.1 .58 ) or perhaps the safer exact Eq. $(3.1 .59$ ). We note however that Eq. (3.1.59) does not have the UV cut-off function $\partial_{t} R_{k}$ as does Eq. (3.1.58). Hence Eq. 3.1 .58 might be better suited to approximations such as the derivative expansion which is able to efficiently remove the physics at large momenta while the proper-time flow might be better suited to approximations such as the BMW or momentum cluster approximation of Sec.(2.4.3) that do not strongly depend on the UV cut-off function $\partial_{t} R_{k}$.

### 3.1.6 Generalization to all upper multicritical dimensions

The BMB analysis of this chapter can be generalized straightforwardly to all upper multicritical dimensions. Let us first note that at $\left(d_{c}(p)=2+2 / p, N=\infty\right)$ with $p$ even, the potentials are qualitatively similar to the tricritical case, with the difference that they are flatter at the minimum due to the fact that $V_{d_{c}(p), N=\infty}^{(j)}\left(\rho_{\min }(p)\right)=0, j<p+1$, see Fig. 3.1.15.

However, in the case of odd $p$, for example the tetracritical case in Fig.(3.1.16), there is no analog to the BMB FP potential because the potentials are decreasing at small field, see Fig.(3.1.16), instead of increasing, see Fig.(3.1.9). Hence, these tetracritical potentials never approach the linear behavior $\bar{V}(\bar{\varrho})=\bar{\varrho}$ observed in the tricritical case. This does not imply that singular FPs, formed by the concatenation of the linear solution with a regular tetracritical FP, do not exist as the singular FP construction is independent of the BMB phenomenon. Indeed, it was shown in 164,166 that a singular Wilson-Fisher FP exists independently of the BMB phenomenon. Nonetheless, as there is no BMB FP potential that bridges these singular FPs with the regular ones, these two FPs sets are disconnected. Moreover, as there is a relationship between


Figure 3.1.16: tetracritical potentials for $\left(d_{c}(3)=2+2 / 3, N=\infty\right)$ interpolating between the flat Gaussian in black and the Wilson-Fisher fixed-point in red.
$\alpha=\left(d_{c}(p)-d\right) N$ and FPs along the $\left(d_{c}(p), N=\infty\right)$ line, these FPs remain also disconnected as functions of ( $d, N$ ), in the sense they never collapse, at least for $N$ sufficiently large. Moreover, from the tetracritical beta function we see that there is no FP collapse between regular FPs and we thus expect the same from their singular counterparts. Let us then consider generalizing the $(\alpha \leftrightarrow \tau)$ relationship for all multicritical FPs in order to verify whether we retrieve a qualitatively similar relationship to that predicted by the perturbative $\beta$ functions.

Expanding the potential in powers of $1 / N$ and then Taylor expanding about the minimum of the potential as usual, we obtain in the tetracritical case an expected linear relationship between ( $\alpha \leftrightarrow a_{8}$ ). More precisely, taking the first non zero derivative of the potential at $N=\infty$ to be equal to $a_{8}$ we find $\alpha=81 a_{8} f^{\prime}(0)^{31}$ which does not predict any FP collapse and is thus consistent with the Tetracritical $\beta$ function. However, in the pentacritical case we find $\alpha=0$ at order $1 / N$ while at order $1 / N^{2}$ with $\alpha_{2}=\left(d_{c}(4)-d\right) N^{2}$ we find $\alpha_{2}=57600 a_{1}{ }^{32}$ where we have used the Litim regulator. This is clearly unexpected as setting the perturbative beta function to zero and discarding the Gaussian solution leads to :

$$
\begin{equation*}
-4 N\left(d_{c}(4)-d\right)+160 a_{10}+\sqrt{2 \pi} \frac{80}{3} \Gamma\left(\frac{1}{4}\right)^{2}\left(1-\frac{32}{9 \pi}\left(\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)^{2}\right) a_{10}^{2}=0 \tag{3.1.64}
\end{equation*}
$$

which is a second degree polynomial which is then in stark contrast with the linear relationship $\alpha_{2}=57600 a_{10}$ at the level of the LPA. Moreover, one expects a relationship between $N\left(d_{c}(4)-d\right)$ and $a_{10}$ while the LPA leads to a relationship between $N^{2}\left(d_{c}(4)-d\right)$ and $a_{10}$. This also seems to be in contradiction with 156 that conjectured a similar $d=d_{c}+\alpha / N$ behavior in the pentacritical case for the LPA. Perhaps this is due to an effective $1 / N$ fit in the numerical results of that paper.

The explanation to this failure of the LPA is that there are many coefficients in the Taylor expansion of these higher order multicritical potentials that are null. We then expect that the leading order approximation to the coefficients of the polynomial relationship between $\alpha$ and $a_{2 p}$ are given by terms obtained at higher order approximations and that are set to zero within the LPA.

Interestingly enough however, in the pentacritical case when $a_{10}=a_{10, B M B}$, we still expect a FP collapse at $\alpha_{2, B M B}=57600 a_{10, B M B}$ such that the BMB endpoint at ( $d_{c}(4), N=\infty$ ) is still the endpoint of a curve in the ( $d, N$ ) plane where a regular FP collapses with its singular counterpart. It is then likely that the consistency of the LPA will always reproduce the finite $N$ origin of the BMB phenomenon in this manner regardless of whether the results are quantitatively accurate.

In conclusion, we argue that the LPA is not very trustworthy at least starting from the pentacritical if not also the tetracritical. Thus higher order approximations are necessary.

[^53]
### 3.1.7 Exact order $1 / \mathrm{N}$ equations

As noticed in the previous sections, the LPA does not always give quantitatively satisfying results. The objective of this section is then to discuss the structure of a $1 / N$ expansion within the exact formalism to have a better view of the landscape of approximations that can be realized .

Let us then note that in [158] it was found that in the limit $N=\infty$ the effective action is of the form $\Gamma_{k}[\vec{\phi}]=\int_{x} \frac{1}{2} \partial \phi^{a}(x) \partial \phi^{a}(x)+\widehat{\Gamma}_{k}[\rho(x)]$ with $\rho=\phi^{a} \phi^{a} / 2$. The notation $\widehat{\Gamma}_{k}[\rho(x)]$ then infers that other than the potential of the effective action, all derivative terms in $\widehat{\Gamma}_{k}$ are functionals of $\rho$ rather than simply $\vec{\phi}$. An example of ansatz for $\widehat{\Gamma}_{k}$ will be given in the next section. We then extend this to finite $N$ as :

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int_{x} \frac{1}{2} \partial \phi \partial \phi+\widehat{\Gamma}_{k}[\rho]+\frac{1}{N} \widetilde{\Gamma}_{k}[\phi] \tag{3.1.65}
\end{equation*}
$$

where $\widehat{\Gamma}_{k}[\rho]$ is given at $N=\infty$ and $\widetilde{\Gamma}_{k}[\phi]$ is a generic functional of $\vec{\phi}$.
At this point, only singular solutions are not considered in Eq. 3.1.65. Indeed, $\widehat{\Gamma}_{k}[\rho]$ was obtained in 158 by taking a regularity hypothesis which, as we shall see, is not valid for singular FPs. Moreover, taking a uniform field configuration $\vec{\phi}(x)=\vec{\phi}_{0}$, the expansion $\widehat{\Gamma}_{k}[\rho]+\frac{1}{N} \widetilde{\Gamma}_{k}[\phi]$ is the basis of the $1 / N$ expansion of the regular potentials in the large $N$ LPA study of Sec.(3.1.4). Instead, for a study of singular FPs it is necessary to construct the finite $N$ solution starting within the boundary layer via an appropriate scaling with $N$. Hence, the leading order term $\widehat{\Gamma}_{k}[\rho]$ does not capture the particularities of these singular FPs. Finally, the expansion in $1 / N$ of Eq. 3.1.65 omits the typical $e^{-N}$ corrections of singular FPs outside of the boundary layer. We will discuss equations that contain these singular FPs at the end of the present section. For now, let us continue to focus on the regular FPs. The expansion in Eq. 3.1.65 leads to :

$$
\begin{equation*}
\frac{\delta^{2} \Gamma_{k}[\phi]}{\delta \phi_{a, q} \delta \phi_{b,-q}} \underset{\operatorname{def}}{=} \Gamma_{k}^{(2)}(a, b ; q,-q ; \phi)=\left(q^{2}+V_{k, N=\infty}^{\prime}(\rho)\right) \delta_{a, b}+\phi_{a} \phi_{b} \frac{\delta^{2} \widehat{\Gamma}_{k}[\rho]}{\delta \rho_{q} \delta \rho_{-q}}+\frac{1}{N} \widetilde{\Gamma}_{k}^{(2)}(a, b ; q,-q ; \phi) \tag{3.1.66}
\end{equation*}
$$

where $V_{k, N}$ is the effective potential of the effective action $\Gamma_{k}[\phi]$. Including the regulator $R_{k}$ then leads to :

$$
\begin{align*}
\Gamma_{k}^{(2)}(a, b ; q,-q ; \phi)+\delta_{a, b} R_{k}\left(q^{2}\right) & =\left(q^{2}+R_{k}\left(q^{2}\right)+V_{k, N=\infty}^{\prime}(\rho)\right) \delta_{a, b}+\phi_{a} \phi_{b} \frac{\delta^{2} \widehat{\Gamma}_{k}[\rho]}{\delta \rho_{q} \delta \rho_{-q}}+\frac{1}{N} \widetilde{\Gamma}_{k}^{(2)}(a, b ; q,-q ; \phi) \\
& =\overline{\operatorname{def}}\left(\gamma_{1, k}\right)_{a, b}+\frac{1}{N}\left(\gamma_{2, k}\right)_{a, b} \tag{3.1.67}
\end{align*}
$$

Inserting the above expression into the $\Gamma$ flow we find:

$$
\begin{equation*}
\partial_{t} \Gamma_{k}[\phi]=\frac{1}{2} \operatorname{Tr}\left(\frac{\partial_{t} R_{k}}{\gamma_{1, k}+\frac{1}{N} \gamma_{2, k}}\right)=\frac{1}{2} \operatorname{Tr}\left(\partial_{t} R_{k} \gamma_{1, k}^{-1}-\frac{1}{N} \partial_{t} R_{k} \gamma_{1, k}^{-1} \gamma_{2, k} \gamma_{1, k}^{-1}+O\left(N^{-2}\right)\right) \tag{3.1.68}
\end{equation*}
$$

We may further decompose $\gamma_{1, k}^{-1}$ as:

$$
\begin{equation*}
\gamma_{1, k}^{-1}\left(a, b ; q^{2} ; \phi\right)=G_{T, k}\left(q^{2} ; \rho\right)\left(\delta_{a, b}-\frac{\phi_{a} \phi_{b}}{2 \rho}\right)+G_{L}\left(q^{2} ; \rho\right) \frac{\phi_{a} \phi_{b}}{2 \rho} \tag{3.1.69}
\end{equation*}
$$

with,

$$
\begin{equation*}
G_{T, k}\left(q^{2} ; \rho\right)^{-1}=q^{2}+V_{k, N=\infty}^{\prime}(\rho)+R_{k}\left(q^{2}\right) \tag{3.1.70}
\end{equation*}
$$

and,

$$
\begin{equation*}
G_{L, k}\left(q^{2} ; \rho\right)^{-1}=q^{2}+V_{k, N=\infty}^{\prime}(\rho)+R_{k}\left(q^{2}\right)+2 \rho \frac{\delta^{2} \widehat{\Gamma}_{k}[\rho]}{\delta \rho_{q} \delta \rho_{-q}} \tag{3.1.71}
\end{equation*}
$$

We will now consider, for simplicity, the flow of the potential $V_{k, N}$. This is easily obtained by considering a uniform field configuration $\vec{\phi}(x)=\vec{\phi}_{0}$. We may then replace $\partial_{t} \Gamma_{k}[\phi]$ by $\partial_{t} V_{k, N}\left(\phi_{0}\right)$ where the trace of the

[^54]first term in Eq. 3.1.68 leads to:
$\partial_{t} V_{k, N}\left(\phi_{0}\right)=\frac{N-1}{2} \int_{q} \partial_{t} R_{k} G_{T, k}\left(q^{2} ; \rho_{0}\right)+\frac{1}{2} \int_{q} \partial_{t} R_{k} G_{L, k}\left(q^{2} ; \rho_{0}\right)-\frac{1}{2 N} \operatorname{Tr}\left(\partial_{t} R_{k} \gamma_{1, k}^{-1} \gamma_{2, k} \gamma_{1, k}^{-1}\right)+O\left(N^{-2}\right)$.
We then further decompose $\gamma_{2, k}$ as:
\[

$$
\begin{equation*}
\gamma_{2, k}\left(a, b ; q^{2} ; \phi\right)=\delta_{a, b} \sigma_{k}\left(q^{2} ; \rho\right)+\phi_{a} \phi_{b} \lambda_{k}\left(a, b ; q^{2} ; \phi\right) \tag{3.1.73}
\end{equation*}
$$

\]

Collecting all $\delta_{a, b}$ terms in $\frac{1}{2 N} \operatorname{Tr}\left(\partial_{t} R_{k} \gamma_{1, k}^{-1} \gamma_{2, k} \gamma_{1, k}^{-1}\right)$ the trace leads to a factor $N$ that compensates the denominator. The neglected terms are then $O\left(N^{-1}\right)$ which leads to :

$$
\partial_{t} V_{k, N}(\phi)=\frac{N-1}{2} \int_{q} \partial_{t} R_{k} G_{T, k}\left(q^{2} ; \rho\right)+\frac{1}{2} \int_{q} \partial_{t} R_{k} G_{L, k}\left(q^{2} ; \rho\right)-\frac{1}{2} \int_{q} \partial_{t} R_{k} \sigma_{k}\left(q^{2} ; \rho\right) G_{T, k}\left(q^{2} ; \rho\right)^{2}+O\left(N^{-1}\right) .
$$

Finally, the usual rescaling $\Gamma_{k}[\phi] \rightarrow(N-1) \Gamma_{k}[\phi], \phi \rightarrow(N-1)^{1 / 2} \phi$ leads to:

$$
\begin{align*}
\partial_{t} V_{k, N}(\phi)=\partial_{t} V_{k, N=\infty}(\phi)+\frac{1}{N} \partial_{t} \delta V_{k}= & \frac{1}{2} \int_{q} \partial_{t} R_{k} G_{T, k}\left(q^{2} ; \rho\right)+\frac{1}{2 N} \int_{q} \partial_{t} R_{k}\left(G_{L, k}\left(q^{2} ; \rho\right)-\right.  \tag{3.1.74}\\
& \left.-\sigma_{k}\left(q^{2} ; \rho\right) G_{T, k}\left(q^{2} ; \rho\right)^{2}\right)+O\left(N^{-2}\right) .
\end{align*}
$$

This should be compared with the LPA version which reads :

$$
\begin{align*}
\partial_{t} V_{k, N}(\phi)=\partial_{t} V_{k, N=\infty}(\phi)+\frac{1}{N} \partial_{t} \delta V_{k}= & \frac{1}{2} \int_{q} \partial_{t} R_{k} G_{T, k}\left(q^{2} ; \rho\right)+\frac{1}{2 N} \int_{q} \partial_{t} R_{k}\left(G_{L, k}^{L P A}\left(q^{2} ; \rho\right)-\right.  \tag{3.1.75}\\
& \left.-\delta V_{k}(\rho) G_{T, k}\left(q^{2} ; \rho\right)^{2}\right)+O\left(N^{-2}\right)
\end{align*}
$$

with,

$$
\begin{equation*}
G_{L, k}^{L P A}\left(q^{2} ; \rho\right)^{-1}=q^{2}+V_{k, N=\infty}^{\prime}(\rho)+R_{k}\left(q^{2}\right)+2 \rho V_{k, N=\infty}^{\prime \prime}(\rho) \tag{3.1.76}
\end{equation*}
$$

The novelties lie in the momentum dependence of the term $\sigma_{k}\left(q^{2} ; \rho\right)$ but also in the momentum dependence of $\frac{\delta^{2} \widehat{\Gamma}_{k}[\rho]}{\delta \rho_{q} \delta \rho_{-q}}=\widehat{\Gamma}_{k}^{(2)}(q,-q ; \rho)$ in $G_{L, k}$ as we have $\widehat{\Gamma}_{k}^{(2)}(0,0 ; \rho)=V_{k, N=\infty}^{\prime \prime}(\rho)$ as in Sec. 2.4.3. As $\widehat{\Gamma}_{k}^{(2)}(q,-q ; \rho)$ is defined at $N=\infty$, it may be calculated from the flow of $\frac{\delta^{2} \Gamma_{k}[\phi]}{\delta \phi_{a, q} \delta \phi_{b,-q}}$ in the limit $N \rightarrow \infty$. Recalling that the flow of $\frac{\delta^{2} \Gamma_{k}[\phi]}{\delta \phi_{a, q} \delta \phi_{b,-q}}$ is:

$$
\begin{align*}
\partial_{k} \Gamma_{k}^{(2)}(p,-p ; \phi)=\operatorname{Tr} & \int{ }_{\mathrm{q}} \partial_{\mathrm{k}} \mathrm{R}_{\mathrm{k}}(\mathrm{q})\left(\mathrm{G}_{\mathrm{k}, \mathrm{~N}}(\mathrm{q}) \Gamma_{\mathrm{k}}^{(3)}(\mathrm{p}, \mathrm{q},-\mathrm{p}-\mathrm{q}) \mathrm{G}_{\mathrm{k}, \mathrm{~N}}(\mathrm{q}+\mathrm{p}) \Gamma_{\mathrm{k}}^{(3)}(-\mathrm{p}, \mathrm{p}+\mathrm{q},-\mathrm{q}) \mathrm{G}_{\mathrm{k}, \mathrm{~N}}(\mathrm{q})\right. \\
& \left.-\frac{1}{2} G_{k, N}(q) \widetilde{\Gamma}_{k}^{(4)}(p,-p, q,-q) G_{k, N}(q)\right) \tag{3.1.77}
\end{align*}
$$

where in the large $N$ limit $G_{k, N}$ can be replaced by $\gamma_{1, k}^{-1}$. It is again sufficient to retain only the transverse part $G_{T, k}\left(q^{2} ; \rho\right)$ of $\gamma_{1, k}^{-1}$ such that there is an overall $\delta_{a, b}$. The end result is 82:

$$
\partial_{k} \widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho)=\int_{q} \partial_{k} R_{k}\left(q^{2}\right) G_{T, k}\left(q^{2} ; \rho\right)^{2}\left(\widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho)^{2} G_{T, k}\left((p+q)^{2} ; \rho\right)-\frac{1}{2} \partial_{\rho} \widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho)\right)
$$

This last equation with that of the potential can then be solved taking $\rho_{k}=f\left(v=V_{k, N=\infty}^{\prime}\right)$ similarly to
what was done with the LPA in Eq. (3.1.37) [82]. One then obtains :

$$
\begin{align*}
G_{T, k}\left(q^{2} ; v\right) & =\left(q^{2}+v+R_{k}\left(q^{2}\right)\right)^{-1} \\
G_{T, \Lambda}\left(q^{2} ; v\right) & =\left(q^{2}+v+R_{\Lambda}\left(q^{2}\right)\right)^{-1} \simeq 0 \\
\rho_{k}-\rho_{\Lambda} & =-\int_{q} G_{T, k}\left(q^{2} ; v\right)  \tag{3.1.78}\\
\widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho) & =\frac{\widehat{\Gamma}_{\Lambda}^{(2)}(p,-p ; \rho)}{1+\frac{\hat{\Gamma}_{\Lambda}^{(2)}(p,-p ; \rho)}{2} \int_{q} G_{T, k}\left(q^{2} ; v\right) G_{T, k}\left((p+q)^{2} ; v\right)} .
\end{align*}
$$

However, $\sigma_{k}\left(q^{2} ; \rho\right)$ is obtained at order $1 / N$ and the flow equations for this quantity do not yield closed equations due to the ( $n+1, n+2$ ) problem discussed in Sec. 2.4 .3 ) ${ }^{34}$. This does not imply however that no progress has been made as the $\Gamma$ flow is now linear and it is perhaps possible to functionally solve the equations. Moreover, as the exact relation between $\alpha=\left(d_{c}(p)-d\right) N$ and $a_{2(p+1)}$ is obtained at a finite order of perturbation theory in $d_{c}(p)-d$, this relationship should also be found at a finite order of the BMW or momentum cluster approximations as they contain perturbation theory [81, 82].

Of course, this cannot be the case of the derivative expansion as it is unable to retrieve $\widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho)$ at any finite order of the expansion in $\partial^{2}$ and is unable to retrieve perturbation theory beyond 1 -loop. Nonetheless, if the regulator is chosen such that the error in

$$
\begin{equation*}
\int_{q} \partial_{t} R_{k} G_{L, k}\left(q^{2} ; \rho\right) \tag{3.1.79}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{q} \partial_{t} R_{k} \sigma_{k}\left(q^{2} ; \rho\right) G_{T, k}\left(q^{2} ; \rho\right)^{2} \tag{3.1.80}
\end{equation*}
$$

are made small we may hope to achieve a reasonable accuracy in our determination of $\alpha_{B M B}$ and $\alpha_{c}$. One could also perform a hybrid derivative expansion retaining the full momentum dependence of the exact $\widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho)$ and performing a derivative expansion only on $\sigma_{k}$. However, it is interesting to instead perform a "naive" derivative expansion performing also a derivative expansion on $\widehat{\Gamma}_{k}^{(2)}(p,-p ; \rho)$ to have an idea of the magnitude of the error this approximation leads to. This will be done in the following section.

As for the singular solutions, it is not presently clear how to obtain an exact solution even at $N=\infty$. Indeed, let us note that Eq. 3.1 .75 can be obtained from an expansion $V_{k, N}(\phi)=V_{k, N=\infty}(\phi)+\frac{1}{N} \delta V_{k}$ of the finite $N$ LPA of Sec.(3.1.4):

$$
\begin{equation*}
\partial_{t} V_{k, N}(\phi)=\frac{1}{2} \int_{q} \partial_{t} R_{k} G_{T, k}\left(q^{2} ; \rho\right)+\frac{1}{2 N} \int_{q} \partial_{t} R_{k} G_{L, k}^{L P A}\left(q^{2} ; \rho\right) \tag{3.1.81}
\end{equation*}
$$

where Eq.(3.1.81) does contain singular fixed-point solutions in its dimensionless form. Indeed, with the choice of the Litim regulator and working with dimensionless variables, the fixed-point solution of Eq.(3.1.81) becomes:

$$
\begin{equation*}
0=-d \bar{V}+(d-2) \bar{\rho} \bar{V}^{\prime}+\frac{1}{1+\bar{V}^{\prime}}+\frac{1}{N} \frac{1}{1+\bar{V}^{\prime}+2 \bar{\rho} \bar{V}^{\prime}} \tag{3.1.82}
\end{equation*}
$$

which as we saw in Sec.(3.1.4) is dual to the Polchinski flow which indeed contains singular fixed-point solutions. In Appendix 0 we show how to map the singular solutions of the Polchinski equation to that of the LPA $\Gamma$ flow. Here we note that the the signature of singular fixed-points in the Polchinski flow was the existence of a point $\rho_{0}$ where in the neighborhood of this point, the naively negligible term $\frac{1}{N} \rho V^{\prime \prime}$ was actually of order 1. In Appendix $\square$ we show that the signature of singular fixed-points in the LPA $\Gamma$ flow is that the term $\frac{1}{N} \frac{1}{1+V^{\prime}+2 \bar{\rho} V^{\prime}}$ from the longitudinal propagator is order 1. As shown in Appendix O singular solutions of Eq. $\sqrt{3.1 .82}$ then have an entire interval where $\frac{1}{N} \frac{1}{1+V^{\prime}+2 \bar{\rho} V^{\prime}}$ is non negligeable. Thus, although we have not studied singular fixed-points outside of the choice of the Litim regulator, we expect that in general

[^55]the hallmark of singular FPs in the $\Gamma$-flow is that the longitudinal modes can not be neglected such that $\int_{q} \partial_{t} R_{k} G_{L, k}^{L P A}\left(q^{2} ; \rho\right)$ is of order $N$. In analogy with the LPA we then expect that within the exact flow it is not possible to neglect $\int_{q} \partial_{t} R_{k} G_{L, k}\left(q^{2} ; \rho\right)$ where $G_{L, k}$ is the exact propagator. This is rather unfortunate as this was the basis to obtain exact solutions at $N=\infty$. As such it is presently unclear whether it is possible to obtain exact solutions for these singular solutions. The singularities of these solutions also obscures any convergence analysis of approximation schemes but we do expect that these fixed-points are real features of the exact equations rather than approximation artifacts as they lead to a fully consistent picture regarding the finite $N$ origin of the BMB phenomenon.

### 3.1.8 BMB phenomenon at order 2 of the derivative expansion

In the previous section we found that the novelties in the flow at order $1 / N$ when compared to the LPA lie in the momentum dependence of $\widehat{\Gamma}_{k}^{(2)}$ and $\sigma_{k}$. The following section is then an application of the previous section to the order 2 of derivative expansion. The numerical results from this analyis will be published in a forthcoming paper.

The order two of the derivative expansion consists in considering field dependence from the kinetic term as $\int_{x} \frac{1}{2} \zeta_{a, b}(\phi) \partial \phi_{a} \partial \phi_{b}$. Using the $O(N)$ invariance of our model we can decompose the tensor $\zeta_{a, b}(\phi)$ on a covariant basis as $\zeta_{a, b}(\phi)=Z(\rho) \delta_{a, b}+Y(\rho) \phi_{a} \phi_{b}$ with $\rho=\phi_{a} \phi_{a} / 2$ as usual. Hence, using $\phi_{a} \partial \phi_{a}=\partial \frac{\phi_{a} \phi_{a}}{2}$, the order $O\left(\partial^{2}\right)$ approximation of the effective action reads:

$$
\begin{equation*}
\Gamma_{k}[\phi]=\int_{x}\left(\frac{1}{2} Z(\rho(x)) \partial \phi_{a} \partial \phi_{a}+\frac{1}{2} Y(\rho(x)) \partial \rho \partial \rho+U(\phi(x))\right) . \tag{3.1.83}
\end{equation*}
$$

If we expand $Z, Y, U$ as $Z=Z_{0}+\frac{1}{N} Z_{1}+O\left(1 / N^{2}\right), Y=Y_{0}+\frac{1}{N} Y_{1}$ and $U=U_{0}+\frac{1}{N} U_{1}$ and we recall the expansion $\int_{x} \frac{1}{2} \partial \phi \partial \phi+\widehat{\Gamma}_{k}[\rho]+\frac{1}{N} \widetilde{\Gamma}_{k}[\phi]$ of the previous section, we obtain the following identifications:

$$
\begin{align*}
& Z_{0}=1 \\
& \int_{x} \frac{1}{2} Y_{0}(\rho) \partial \rho \partial \rho+\int_{x} U_{0}(\rho)=\left.\widehat{\Gamma}_{k}[\rho]\right|_{\text {order } \partial^{2}} \\
& Y_{0, k}(\rho) q^{2}+U_{0, k}(\rho)=\left.\widehat{\Gamma}_{k}^{(2)}(q,-q ; \rho)\right|_{\text {order } \partial^{2}}  \tag{3.1.84}\\
& \int_{x} \frac{1}{2} Z_{1} \partial \phi_{a} \partial \phi_{a}+\int_{x} \frac{1}{2} Y_{1} \partial \rho \partial \rho+\int_{x} U_{1}(\rho)=\left.\widetilde{\Gamma}_{k}[\phi]\right|_{\text {order }} \partial^{2} \\
& Z_{1, k} q^{2}+U_{1, k}^{\prime}=\sigma_{k}\left(q^{2} ; \rho\right) .
\end{align*}
$$

The approximate flow of $U_{1}$ is then obtained by inserting these expressions into the exact flow of the previous section or considering the derivative expansion at finite $N$ and then taking the limit $N \rightarrow \infty$. The flow of $Z_{1}$ and $Y_{0}$ can also be obtained from the derivative expansion at finite $N$ and then taking the large $N$ limit by rescaling $(U, Y, Z, \rho)$ in terms of $N$ according to $\Gamma_{k}[\phi] \rightarrow(N-1) \Gamma_{k}[\phi], \phi \rightarrow(N-1)^{1 / 2} \phi$. It is also convenient to consider instead $W=U^{\prime}(\rho)$ and expand $W$ as $W=W_{0}+\frac{1}{N} W_{1}$. This then leads to a system of equations given in Appendix $\mathbb{R}$ that have the following triangular form :

$$
\begin{align*}
\partial_{k} W_{0} & =F_{1}\left[W_{0}\right] \\
\partial_{k} Y_{0} & =F_{2}\left[W_{0}, Y_{0}\right] \\
\partial_{k} Z_{1} & =F_{3}\left[W_{0}, Y_{0}, Z_{1}\right]  \tag{3.1.85}\\
\partial_{k} W_{1} & =F_{4}\left[W_{0}, Y_{0}, Z_{1}, W_{1}\right] .
\end{align*}
$$

If we then rescale the field $\rho$ such that as usual $W(\rho=1)=0$ we may expand the variables $W_{0}, Y_{0}, Z_{1}$, $W_{1}$ in powers of $(\rho-1)$. This leads to a system of equations which can be solved straightforwardly due to the triangular nature of the flow equations. The last step in the resolution is evidently that of $W_{1}$ where as in the LPA for the tricritical case one arrives at an equation that relates $\alpha=(3-d) N$ to $\tau=U_{0}^{\prime \prime \prime}$. This equation is again a polynomial of degree two with coefficients that depend on the choice of regulator. We have then used the Wetterich regulator $R_{k}=\frac{A q^{2}}{\exp \left(q^{2} / k^{2}\right)-1}$ and varied $A$ in order to obtain a PMS in $\alpha_{c}$ and $\alpha_{B M B}$. The results from this analysis will be given in a forthcoming paper.

### 3.1.9 Physical interpretation of cusped fixed-points

In this section we shall discuss the physics behind the existence of singular fixed-points.
Our first insight into the physics of singular FPs will come from an understanding as to why singular FPs have one more infrared eigenvalue than their regular counterparts. In fact, the eigenvalues of a singular FP is given by the union of the eigenvalues of the regular part and of the linear part $\bar{V}(\bar{\varrho})=\bar{\varrho}$. We have not found a fully rigorous proof but we have studied this precisely in the case of the FP $S \mathcal{A}(\tau=0)$, which we call Singular Gaussian, and by continuity we expect this picture to hold for all $\tau<\tau_{B M B}{ }^{35}$. In the following we will give an intuition as to why this is the case.

The point of view we will defend is that the part below the cusp is independent from the part above and thus one may consider perturbing each part independently. Of course, one might argue that while the potentials do have a discontinuity in their derivative, the potential is still continuous and perturbing each side independently would create a discontinuity in the resulting perturbed potential. Yet, the eigenvalues from perturbations to the derivative of the potential are the same as those from the potential but without the eigenvalue associated to constant zero point energy perturbations. As such, from the point of view of the derivatives of the potential, it seems somewhat more meaningful to consider perturbing either side independently. The important point however is that for $N$ finite discontinuities of the potential are translated into boundary layers and as such any discussion on discontinuities are in fact reserved to the case of $N=\infty$.

Hence, from the perspective that both sides are independent, the eigenspace is then given by the Cartesian product of the eigenspace of each part of the potential. The canonical basis for a Cartesian product of two vector spaces $E$ and $F$ with respective basis $\left(e_{i}\right)_{i \in\{1,2, \ldots, \operatorname{dim} E\}}$ and $\left(f_{j}\right)_{j \in\{1,2, \ldots, \operatorname{dim} F\}}$ is $\left(\left(e_{1}, 0\right),\left(e_{2}, 0\right), \ldots,\left(e_{\operatorname{dim} E}\right),\left(0, f_{1}\right),\left(0, f_{2}\right), \ldots\left(0, f_{\operatorname{dim} F}\right)\right)$. Consider then the eigenfunctions from the linear part on the left, which we denote as $L_{i}(\bar{\varrho})$ for $\bar{\rho}<\bar{\varrho}_{0}$, and the regular part on the right, which we denote $R_{j}(\bar{\varrho})$ for $\bar{\varrho}>\bar{\varrho}_{0}$. In analogy with the above construction of the canonical basis we consider the following constructions of global piecewise eigenfunctions $S L_{m}(\bar{\varrho}), S R_{m}(\bar{\varrho})$ :

$$
\begin{align*}
& S L_{m}(\bar{\varrho})= \begin{cases}L_{m}(\bar{\varrho}) & \text { for } \bar{\varrho}<\bar{\varrho}_{0} \\
0 & \text { for } \bar{\varrho}>\bar{\varrho}_{0}\end{cases} \\
& S R_{m}(\bar{\varrho})= \begin{cases}0 & \text { for } \bar{\varrho}<\bar{\varrho}_{0} \\
R_{m}(\bar{\varrho}) & \text { for } \bar{\varrho}>\bar{\varrho}_{0}\end{cases} \tag{3.1.86}
\end{align*}
$$

We now explain why this is a reasonable construction.
First let us recall the Polchinski equation:

$$
\begin{equation*}
\partial_{t} \bar{V}=1-d \bar{V}+(d-2) \varrho \bar{\varrho} \bar{V}^{\prime}+2 \bar{\varrho} \bar{V}^{\prime 2}-\bar{V}^{\prime} \tag{3.1.87}
\end{equation*}
$$

A fixed-point solution verifies:

$$
\begin{equation*}
0=1-d \bar{V}^{*}+(d-2) \bar{\varrho} \bar{V}^{\prime *}+2 \bar{\varrho} \bar{V}^{\prime * 2}-\bar{V}^{\prime *} \tag{3.1.88}
\end{equation*}
$$

If we then insert $\bar{V}(\bar{\varrho}, t)=\bar{V}^{*}+\delta V(\bar{\varrho}) e^{\lambda t}$ into Eq. 3.1.87, where $\delta V$ is a small perturbation, then we obtain after linearization:

$$
\begin{equation*}
0=(\lambda+d) \delta V+(d-2) \varrho \bar{\varrho} \delta V^{\prime}+4 \bar{\varrho} \bar{V}^{\prime *} \delta V^{\prime}-\delta V^{\prime} . \tag{3.1.89}
\end{equation*}
$$

Consider then an eigenfunction $R_{m}(\bar{\varrho})$ with an associated eigenvalue $\lambda_{m}$ for the regular potential on the right of a singular potential. Then, by defintion of $R_{m}$ and $\lambda_{m}$ we have

$$
\begin{equation*}
0=\left(\lambda_{m}+d\right) R_{m}+(d-2) \bar{\varrho} R_{m}^{\prime}+4 \varrho \bar{\varrho} \bar{V}^{\prime} R_{m}^{\prime}-R_{m}^{\prime} \tag{3.1.90}
\end{equation*}
$$

and $R_{m}$ then satisfies the eigenequation for $\bar{\varrho}>\bar{\varrho}_{0}$. Notice in particular that $\bar{\varrho}_{0}(\tau)<1$ and that the eigenvalues of $R_{m}$ can be determined from the neighborhood of $\bar{\varrho}=1$ as was shown in Sec.(3.1.4) and further explained in [162, 165]. Thus, the behavior of $R_{m}$ in the interval $\bar{\varrho}<\bar{\varrho}_{0}$ is irrelevant for the computation of the eigenvalues and it is thus possible to compute these values from the interval $\bar{\varrho}>\bar{\varrho}_{0}$

[^56]

Figure 3.1.17: Plots of the normalized eigenfunctions of the FP $S G$ (singular Gaussian) as a function of $\bar{\varrho}$ for $d=3.2$ and $N=82$. The abscissa of the eigenfunctions have not been rescaled and thus the minimum of the potential is not at $\bar{\varrho}=1$. We observe that the eigenfunctions on the top panels are flat for $\bar{\varrho}>\bar{\varrho}_{0}$, to the right of the boundary layer, and thus they belong to the finite $N$ extensions of $S L_{m}(\bar{\varrho})$. The eigenfunctions aon the bottom panels are flat instead for small $\bar{\varrho}$ and thus belong to the finite $N$ extensions of $S R_{m}(\bar{\varrho})$. Similar plots are given in Appendix $P$.
independently of the fact that the eigenperturbation $S R_{m}$ is null for $\bar{\varrho}<\varrho_{0}$. The same is true for $S L_{n}$ as the behavior of the fixed-point $\bar{V}=\bar{\varrho}$ is trivial and thus any interval is sufficient to compute the spectrum of this potential.

However, if $\lambda_{m}$ does not belong to the spectrum of $\bar{V}=\bar{\varrho}$ for $\bar{\varrho}<\bar{\varrho}_{0}$ than the only choice is to take $\delta V=0$ for $\bar{\varrho}<\bar{\varrho}_{0}$ as it is the only "perturbation" of $\bar{V}=\bar{\varrho}$ which verifies

$$
\begin{equation*}
0=\left(\lambda_{m}+d\right) \delta V+(d-2) \varrho \bar{\varrho} \delta V^{\prime}+4 \bar{\varrho} \bar{V}^{\prime *} \delta V^{\prime}-\delta V^{\prime} \tag{3.1.91}
\end{equation*}
$$

with $\bar{V}^{*}=\bar{\varrho}$ and $\lambda_{m}$ belonging to the spectrum $R_{m}$ but not $L_{n}$. This is why $S R_{m}$ is set to zero for $\bar{\varrho}<\bar{\varrho}_{0}$. The eigenvalues are then readily obtained from the right part of the eigenfunction for $S R_{m}$ and from the left part of $S L_{m}$. Hence, the eigenvalues of the singular FP is formed by the union of the eigenvalues of the regular part and of the linear part. However, while this might seem like a reasonable construction for $N=\infty$ the natural question is whether the discontinuity in these eigenfunctions will lead to a boundary layer as was the case of the FP potential. We have not been able to prove that a boundary layer is necessary from this construction but this is indeed what we have found numerically as in Fig. 3.1.17).

The eigenvalues of the linear part are easily obtained from Eq. 3.1 .89 by taking $V^{*}=\bar{\varrho}$ :

$$
\begin{equation*}
0=-(d+\lambda) \delta V+(d+2) \bar{\varrho} \delta V^{\prime}-\delta V^{\prime} \tag{3.1.92}
\end{equation*}
$$

This may be compared to the linearization of the Gaussian solution:

$$
\begin{equation*}
0=-(d+\lambda) \delta V+(d-2) \varrho \bar{\varrho} \delta V^{\prime}-\delta V^{\prime} \tag{3.1.93}
\end{equation*}
$$

whose eigenvalues are known to be [162]:

$$
\begin{equation*}
\lambda_{n}=(d-2) n-d \tag{3.1.94}
\end{equation*}
$$

Eq. 3.1.93 can be mapped to Eq. 3.1.92 by using the substitution $d \rightarrow d+4, \lambda \rightarrow \lambda-4$ which allows us to deduce the eigenvalues from Eq. 3.1.92) as:

$$
\begin{equation*}
\lambda_{n}=(d+2) n-d \tag{3.1.95}
\end{equation*}
$$

Notice then that the only negative eigenvalue is obtained for $n=0$ which is $\lambda_{0}=-d$. Normally this eigenvalue would be omitted as it corresponds to perturbing the potential by an unphysical constant however there are now two eigenvalues $-d$ which correspond to adding a constant to the left or to the right. As such, the eigenspace associated to the eigenvalue $-d$ is of dimension 2 with basis $S L_{0}$ and $S R_{0}$. $S R_{0}$ can be normalized as $\Theta\left(\bar{\varrho}-\bar{\varrho}_{0}\right)$ for $\bar{\varrho}>0$ where $\Theta$ is the Heaviside function. In the same way we have $S L_{0}=\Theta\left(\bar{\varrho}_{0}-\bar{\varrho}\right)$. However, it is always possible to consider a linear combination of eigenfunctions as a change of basis in which case we may also consider the eigenfunction $S L_{0}+S R_{0}=1$. This eigenfunction is then simply the usual constant eigenfunction and it can be omitted. As such there is only one non trivial eigenfunction $S L_{0}$ or $S R_{0} . S L_{0}$ and $S R_{0}$ are equivalent at $N=\infty$ however for $N$ finite the degeneracy is lifted. It is not clear to us why but it seems that only the finite $N$ extension of $S L_{0}$ exists where an example is given in the top left plot of Fig. (3.1.17). The important conclusion of this analysis is that singular FPs have an eigenvalue $-d$ at $N=\infty$ which then receives corrections at finite $N$. A more detailed study is given in Appendix P where we study the case of $S \mathcal{A}(0)$.

It is then interesting to notice the similarities between the above analysis and an analysis of so called "discontinuity fixed-points" in an article by Fisher and Berker in 1982 [181], see also 182. In 181 it is mentioned that there is always an eigenvalue $-d$ from the zero point energy which is conjugate to the density of the phase in the system however when there are two coexisting phases along a first order transition there are two densities and thus two eigenvalues - $d$. These arguments are then nearly the same as what was shown above that singular FPs have two eigenvalues $-d$ with one being associated to a trivial 0 point energy. The authors also state that the combinations between basis vectors in the two dimensional eigenspace corresponds to different ratios of phase mixing in a first order transition. Moreover, they demonstrate that an eigenvalue of $-d$ in a first order transition is expected as a limit case of the scaling relations. For example, in the case of the Ising model, they show that the eigenvalue $\lambda$ associated with a perturbation of the external magnetization verifies

$$
\begin{equation*}
-\frac{d}{\lambda}=1-\frac{1}{\delta} \tag{3.1.96}
\end{equation*}
$$

with the critical exponent $\delta$ defined by

$$
\begin{equation*}
M \propto H^{1 / \delta} \tag{3.1.97}
\end{equation*}
$$

Thus, if $\lambda=-d$, we have $\delta \rightarrow \infty$ which leads to a discontinuity of the magnetization as expected of a first order transition. The authors also show that from the scaling relation:

$$
\begin{equation*}
2-\eta=\frac{d(\delta-1)}{\delta+1} \tag{3.1.98}
\end{equation*}
$$

that the anomalous dimension $\eta$ verifies:

$$
\begin{equation*}
\eta=2-d \tag{3.1.99}
\end{equation*}
$$

in the limit $\delta \rightarrow \infty$. Hence from

$$
\begin{equation*}
<M(0) M(r)>\propto \frac{1}{r^{2-d-\eta}} \tag{3.1.100}
\end{equation*}
$$

we deduce long range order.
Fixed-points with two eigenvalues $-d$ that entail the discontinuity of internal thermodynamic variables such as the magnetization of the field are called discontinuity fixed-points. These discontinuity fixed-points are associated to zero temperature fixed-points 150,183 , that is, fixed-points towards which the RG may converge and that yield the characteristic that the effective scale dependent temperature of the system flows towards zero. As such, one might expect that such discontinuity fixed-points would appear when following a FP to its lower critical dimension where the critical temperature goes to zero. As an example, it is known that the Ising 0 temperature fixed-point in $\mathrm{d}=1$ has $\eta=1$ which is indeed equal to $2-d=2-1$. Another interesting case where only 0 temperature fixed-points are allowed are for $d=2$ when $N>2$ according to the Mermin-Wagner theorem. In [122], it was indeed found that $2-d-\eta \rightarrow 0$ as FPs are followed to $d=2$
for $N>2$. Moreover, in 146 it was shown that the tricritical FP has an eigenvalue that behaves as $-d$ in the neighborhood of $d=2$. We believe that the eigenvalue $-d$ is readily visible in the tricritical case, that is without having to introduce any source term as in the Ising model, as one of the eigendirections is conjugate to an external parameter capable of performing first order transitions. This was shown within Landau theory in Sec. (3.1.2) for the case $N=1$. For $N=2$ the plots should be understood as a slice of a "Mexican hat" with a dip in the middle, which we may call a "cowboy hat". The natural question is then whether in the NPRG the existence of two eigenvalues $-d$ in the tricritical case when $d \rightarrow 2$ is associated to the existence of a singular FP as well. While we do not have a definitive answer to this, unpublished preliminary work in [184 seems to indicate that this is indeed the case. "Cuspy" FPs have also been found in the random Ising model $185-187$ where at zero temperature there is a cusp which is related to the existence of metastable states and leads to avalanche phenomena which share characteristics with first order transitions.

However, in the case of the $\mathrm{O}(N)$ model for $N \rightarrow \infty$ the singular FPs do not seem to be related to a zero temperature fixed-point. In particular, the anomalous dimension $\eta$ goes to zero rather than $2-d$. Nevertheless, in this section we shall discuss the plausibility that these singular FPs are indeed related to the existence of a first order surface. To see why, consider the phase diagram at $N=\infty$ in Fig. 3.1.18 which was obtain in $[153,154$ from the saddle point equations of Sec.(3.1.3.1).

Below the BMB point $P$, Fig. 3.1.18 has the typical phase diagram of a tricritical diagram as in Sec.(3.1.2). However, when the tricritical coupling $a_{6, R}$ is beyond the BMB point, the line $t$ extend to a novel line $l$ which is an end-line of a first order surface $X^{\prime}$ given in Fig.(3.1.89). In Sec.(3.1.3.1), we had separated the vector field as $\vec{\phi}=(\psi, \vec{\varphi})$ and integrated only the $N-1$ fields in $\vec{\varphi}$. This is because the saddle point from $\psi$ leads to :

$$
\begin{equation*}
\psi \sigma=0 \tag{3.1.101}
\end{equation*}
$$

In the broken phase we have $\psi \neq 0$ and $\sigma=0$ while in the symmetric phase we have $\psi=0$ and $\sigma \neq 0$. The first order surface $X$ in Fig. 3.1 .89 corresponds to a discontinuity in $\psi$ while in the first order surface $X^{\prime}$ the system remains in the symmetric phase and instead there is a discontinuity in $\sigma$ and in $<\vec{\phi}^{2}>153$ 154. As $<\vec{\phi}^{2}>$ is an $O(N)$ singlet, the authors in 153,154 associated this transition to a liquid-gas type transition similar to the discontinuity of the scalar density in a fluid along the first order line. The end-line $l$ is then related to a second order transition which is related to this scalar order parameter. Along this line the vector particles are massive, as can be checked in Fig. (3.1.18) while there is a massless $O(N)$ bound-state $\left.<\vec{\phi}^{2}>\mid 153,154\right]$. As $l$ is related to a second order transition it is natural to wonder whether the NPRG is able to find the associated FP and in particular its extension at finite $N$. Let us then note that if such a FP were to exist, as the vector particles are massive the rescaled mass would necessarily be infinite at the FP. As this is the case of the singular FPs in the $\Gamma$ flow formulation it is thus natural to wonder whether this line is related to these singular FPs. Moreover, [153, 154] explains that much of the particularities of the BMB FP is inherited from the line $l$ which seems similar to the case of the singular FPs where the BMB endpoint also displays a singularity. It is also interesting to wonder whether the eigenvalue $-d$ of the singular FPs is related to the first order surface $X^{\prime}$ as the potential is written in terms of the massive vector particles rather than the singlet state $<\vec{\phi}^{2}>$. Finally, in 153,154$]$ it was shown that the line $l$ is no longer connected to the line $t$ for $d>3$ in the same way the FP $S G$ is no longer connected to the FP $G$ for $d>3{ }^{36}$. All these elements point towards the possibility that the line $l$ is related to the singular FPs. A phase diagram within the NPRG and a search for a massless scalar bound-state would be necessary to conclude.

### 3.1.10 Extension of the BMB phenomenon to moderate $N$ and non trivial homotopies in $(N, d)$ space

This section is based on my work performed with my collaborators B. Delamotte and S. Yabunaka in 188 .
In the previous sections we discussed, from a large $N$ analysis, the finite $N$ extension of the lines of multicritical fixed-points at $d=d_{c}(p)=2+2 / p$ for $p \geq 2$. In the tricritical case, we found the following fixed-points $A_{2}, \widetilde{A}_{3}, S A_{3}, S \widetilde{A}_{4}$, where for sufficiently large $N$, all four exist between $d_{c}(N)=3-\alpha_{c} / N$ and $d_{\mathrm{BMB}}(N)=3-\alpha_{\mathrm{BMB}} / N$. This was summarized in the last figure of Sec.(3.1.3.1) and we have reproduced this figure in Fig. 3.1.19 for convenience.

[^57]

Figure 3.1.18: Sketch of phase diagram in 153, 154. The couplings $a_{2, R}, a_{4, R}, a_{6, R}$ are the renormalized couplings from a typical Landau tricritical potential $\frac{1}{6} a_{6}\left(\vec{\phi}^{2}\right)^{3}+\frac{1}{4} a_{4}\left(\vec{\phi}^{2}\right)^{2}+\frac{1}{2} a_{2} \vec{\phi}^{2}$. The surface $H$ is the Heisenberg surface obtained by fine tuning $a_{2, R}=0$. Hence, the Wilson-Fisher FP for second order phase transitions belongs to the surface $H$. The line $t$ is the tricritical line at $a_{2}=0$ and $a_{4}=0$ as one would expect from Sec.(3.1.2). The Gaussian point $G$ is located at $a_{2}=0, a_{4}=0, a_{6}=0$. The tricritical line is located between the second order surface $H$ and the first order surface $X$. The BMB end point is located at the point $P$. Beyond the point $P$ the tricritical line extends to a line $l$ which is an end-line of a new first order surface $X^{\prime}$.

For $d=d_{\text {BMB }}(N), \widetilde{A}_{3}$ collapses with $S \widetilde{A}_{4}$ and both cease to exist as real valued fixed-points for $d>$ $d_{\mathrm{BMB}}(N)$. If instead we consider $d=d_{\mathrm{c}}(N), A_{2}$ collapses with $\widetilde{A}_{3}$ and $S A_{3}$ collapses with $S \widetilde{A}_{4}$. All four fixed-points cease to exist as real valued FPs for $d<d_{\mathrm{c}}(N)$. This is in stark contrast with what is known for $N=1,2,3,4$ as it was found that $A_{2}$ exists for all $2<d<3$ [122, 146. Thus what would happen if we were to consider $A_{2}$ at, say, $(d=2.2, N=1)$ and we were to follow this FP by continuity for fixed $d$ and increasing $N$ ? One possibility is that the lines $d_{\mathrm{c}}(N)$ and $d_{\text {BMB }}(N)$ extend to $d=2$ for some $N^{*}>4$. In this case, the $A_{2}$ FP would hit this line when $N$ is increased at fixed $2<d<3$ which would then explain why it is not found at $N=\infty$ for $d<3$. In such a scenario it would also be possible to follow the FP $S A_{3}$ in the opposite direction from large to small $N$ and it would then exist for $2<d<3$ and $N=1,2,3,4$ as does $A_{2}$.

However, the FPs $\widetilde{A}_{3}, S A_{3}, S \widetilde{A}_{4}$ have never been found for small $N$ which would imply that the above scenario where $d_{\mathrm{c}}(N)$ and $d_{\text {BMB }}(N)$ extend to $d=2$ is insufficient. One might argue that this is because in the $\Gamma$ - flow formulation the singular fixed-point potentials have very large curvatures at the origin according to Appendix $O$ in which case they are not likely to be found in numerical explorations. However, as shown in Appendix $O$ the curvature at the origin of the potential is of order $e^{N}$ and for $N$ sufficiently small the curvature should be of the same order as the other fixed-points. This is indeed what was found by numerically integrating the LPA fixed-point equation at finite $N$. Furthermore, if novel fixed-points were to exist at $N=1$, for example, and if $d_{\mathrm{c}}(N)$ and $d_{\mathrm{BMB}}(N)$ do not extend to $N=1$ as well, then one would be able to follow these fixed-points to $d=2$ were exact results are known and where there is no room for new


Figure 3.1.19: FPs existing at $N=\infty$ (left panel) and large $N$ (right panel) in $d \leq 3$ (the Wilson-Fisher FP is not shown although it exists everywhere). Right panel: Straight lines represent the leading order of the critical lines $d(N)=3-\alpha / N+O\left(1 / N^{2}\right)$ where two FPs collapse. The horizontal line where $G=A_{2}$ corresponds to $\alpha=0$ and the line where $\tilde{A}_{3}=S \tilde{A}_{4}$ to $\alpha=\alpha_{\text {BMB }}$. The lines where $S A_{3}=S \widetilde{A}_{4}$ and $A_{2}=\tilde{A}_{3}$ are almost superimposed since for both of them $\alpha=\alpha_{\mathrm{c}}$. These two lines are therefore represented by a single line with alternating colors: dark pink for the line where $S A_{3}=S \tilde{A}_{4}$ and blue for the line where $A_{2}=\tilde{A}_{3}$. Left panel: the full BMB line made of regular FPs between the Gaussian and the BMB FPs and of singular FPs between the BMB and the singular Gaussian $S G$ FPs. This line is made of four parts corresponding to the limits when $N \rightarrow \infty$ of four different kinds of FPs: the $A, \tilde{A}, S \tilde{A}$ and $S A$ FPs are respectively the limits of the $A_{2}, \tilde{A}_{3}, S \tilde{A}_{4}$ and $S A_{3}$ FPs. The dashed lines between the right and left panels show the limits of the remarkable FPs: The Gaussian FP $G$ on the BMB line is the limit of the Gaussian FP at finite $N$, the FP where $A=\tilde{A}$ is the limit of $A_{2}=\tilde{A}_{3}$ along the line indexed by $\alpha_{c}$, the BMB FP is the limit of $\tilde{A}_{3}=S \tilde{A}_{4}$ along the line indexed by $\alpha_{\mathrm{BMB}}$, the FP where $S A=S \tilde{A}$ is the limit of $S A_{3}=S \tilde{A}_{4}$ along the line indexed by $\alpha_{\mathrm{c}}$ and finally $S G$ is the limit of $S A_{3}$ along the line indexed by $\alpha=0$.
fixed-points. The simplest hypothesis is then that these three fixed-points $\widetilde{A}_{3}, S A_{3}, S \widetilde{A}_{4}$ collapse in some way when $N$ is decreased. This is indeed what was found by numerically integrating the LPA equation. However, we would like to emphasize that the following discussion will hold within the exact theory if one accepts the two following hypotheses:

- $\left(\mathcal{H}_{1}\right)$ : The mechanism for the finite $N$ origin of the BMB line given in the previous sections holds within the exact theory. In particular that there exists singular FPs at $N=\infty$ that have an extension at finite $N$.
- $\left(\mathcal{H}_{2}\right)$ : The only fixed-points at $N=1,2,3,4$ are those that are commonly cited within the literature, that is the Gaussian, Wilson-Fisher, and the perturbative multicritical FPs that bifurcate from the Gaussian at $d=d_{c}(p)=2+2 / p$ for $p \geq 2$.

To show that these hypotheses are indeed sufficient we will refer to numerical results from the LPA for the sole purpose of concreteness.

To start, let us recall that the curves along which we have $A_{2}=\widetilde{A}_{3}$ and $S A_{3}=S \widetilde{A}_{4}$ are one and the same at order $1 / N$. The difference between these two curves is imperceptible within a $1 / N$ expansion due to the fact that the singular and regular fixed-points differ by $e^{-N}$ outside of the boundary layer and in particular at the minimum of the potential where the relationship $(\alpha=N(3-d)) \leftrightarrow\left(\tau \propto V^{\prime \prime \prime}(1)\right)$ is deduced. However, as $N$ is decreased, $e^{-N}$ becomes of order one in which case the two curves noticeably split. In the following, the curve defined by $S A_{3}=S \widetilde{A}_{4}$ will be called $N_{c, S^{\prime}}(d)$ where the notation $S^{\prime}$ will be explained shortly. Moreover, we shall call $N_{\mathrm{c}, \mathrm{S}}(d)$ the curve $A_{2}=\widetilde{A}_{3}$ and the curve $\widetilde{A}_{3}=S \widetilde{A}_{4}$ will be called $N_{c, S^{\prime}}^{\prime}(d)$. Thus in total there are three curves at this point :


Figure 3.1.20: Intersection of two critical lines $A=B$ and $B=C$. The notation $\widehat{A}, \widehat{B}$ on the left of the curve $A=B$ refers to the fact that the FPs $A$ and $B$ become complex while $C$ remains real. Below $B=C$, $B$ and $C$ become complex, as $\widehat{B}, \widehat{C}$ and $A$ remains real. The rectangle $T$, in blue, represents a path in $(d, N)$ space where the FPs are followed as explained in the main text. The points $p_{1}, p_{2}, p_{3}$ and $p_{4}$ represent "checkpoints" were the form of the potential of the FP is evaluated.

| Notation | Fixed-points that collapse |
| :---: | :---: |
| $N_{\mathrm{c}, \mathrm{S}}(d)$ | $A_{2}=\widetilde{A}_{3}$ |
| $N_{c, S^{\prime}}(d)$ | $S A_{3}=S \widetilde{A}_{4}$ |
| $N_{c, S^{\prime}}^{\prime}(d)$ | $\widetilde{A}_{3}=S \widetilde{A}_{4}$ |

Table 3.3: Critical curves

The question is then that of their fate as $N$ is decreased.
Let us note that neither of these lines have been found at $N=1,2,3,4$ which would imply that there exists some critical values of $N$ where these lines suddenly cease to exist. However, there is no a priori reason why a critical line, which corresponds to the equality of two fixed-points, would suddenly cease to exist unless there was some mechanism capable of removing one, or two, of the fixed-points in the equality. The usual mechanism for a FP to disappear is to collide with another FP in which case there would be an intersection of two critical curves such that one of the FPs in the equality collides with another FP from the other critical curve as in Fig. $3.1 .20{ }^{37}$. In the following we shall call such a scenario a collapse of critical curves.

A priori, such a scenario would be possible between $N_{\mathrm{c}}(d)$ and $N_{c, S^{\prime}}^{\prime}(d)$ in which case the FP $\widetilde{A}_{3}$ would cease to exist outside of the domain delimited by these two curves as is the case of the FP $B$ in Fig.(3.1.20). However, the intersection of critical curves in Fig. 3.1.20 also leads to an interesting homotopy, or more precisely monodromy, between FPs which forbids this scenario. Indeed, consider following by continuity the FP $A$ in Fig. 3.1.20 along the rectangle $T$ starting at point $p_{1}$. Crossing the line $B=C$ from $p_{1}$ to $p_{2}, A$ remains real. As there are no critical lines between $p_{2}$ and $p_{4}$, this FP remains real along this path and is therefore the only real valued FP at $p_{4}$ which is $C^{38}$. Hence, along the path between $p_{2}$ and $p_{4}, A$ continuously deforms to $C$ where at $p_{3}$ it is in an intermediate state. Crossing the line $A=B$ from $p_{4}$ to

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Figure 3.1.21: Diagram as in Fig. 3.1.20 where space is now confined to a circle. The states $A, B, C$ are the roots of $P_{\theta}(x)=x^{3}-\cos (\theta) x+\sin (\theta)$ as given in Fig. 3.1.22. These roots are functions of the angle along the circle. The roots $B$ and $C$ (resp. $A$ and $B$ ) become complex at $L_{2}(B=C)\left(\right.$ resp. $\left.L_{1}(A=B)\right)$ when the angle is increased from $p_{1}(\theta=0)$ to $p_{2}\left(\theta=\theta_{p_{2}}\right)$ (resp. decreased from $p_{1}(\theta=0)$ to $\left.p_{4}\left(\theta=\theta_{p_{4}}\right)\right)$ analogously to Fig. 3.1 .20


Figure 3.1.22: Left: Polynomial function $P_{\theta}(x)$ as a function of $x$ for different values of $\theta$ given by the angle of $p_{i}$ in Fig. 3.1.21. The roots represent the states $A, B, C$ where by convention $A$ is on the left, $B$ in the middle and $C$ is on the right. Right: Positions of the roots of $P_{\theta}(x)$ in the complex plane where the position of $\widehat{B}$ is arbitrarily chosen to be at the bottom.
$p_{1}$, the FP that was initially $A$ at $p_{1}$ is now $C \boxed{39}$. It is important to note that these arguments are purely topological and this non trivial homotopy would take place in any space were it is possible to wrap around the two critical curves. As a more simple example consider the following polynomial:

$$
\begin{equation*}
P_{\theta}(x)=x^{3}-\cos (\theta) x+\sin (\theta) \tag{3.1.102}
\end{equation*}
$$

where the angle $\theta$ corresponds to the coordinate on the circle of Fig.(3.1.21) and the critical curves $A=B$ and $B=C$ are reduced to the critical points of $L_{1}$ and $L_{2}$ of Fig.(3.1.21).

In Fig. (3.1.22) we show the positions of the roots of $P_{\theta}$ as the angle $\theta$ is varied between $[0,2 \pi]$. We take as convention that the root on the left is $A$, that $B$ is in the middle and that $C$ is on the right. Moreover, we consider that $p_{1}$ in Fig. 3.1.21) is at position $\theta=0$. If we then increase $\theta$ to $\theta\left(p_{2}\right), B$ and $C$ collapse and become complex valued. $B$ and $C$ remain complex valued until they cross the real axis for negative real values as in Fig. 3.1 .22 . During that time $A$ moves from left to right where it then has the position of $C$. Once a full turn has been completed the root $A$ followed by continuity has swapped positions with $C$.

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Figure 3.1.23: Point $S^{\prime}$ and the lines $N_{c, S}(d)$ (violet diamonds), $N_{c, S^{\prime}}^{\prime}(d)$ (green crosses) and $N_{c, S^{\prime}}(d)$ (orange squares). Starting from $P, S A_{3}$ is followed on a clockwise closed path surrounding $S^{\prime} . S A_{3}$ remains real all along the path but back to the point $P$, it is $\widetilde{A}_{3}$.

Hence, if this were to take place with $N_{\mathrm{c}}(d)$ and $N_{c, S^{\prime}}^{\prime}(d)$ there would be a continuous deformation of the potential of $S \widetilde{A}_{4}$ which would lead $S \widetilde{A}_{4}$ to loose two infrared eigenvalues to become $A_{2}$. However, generally, the number of eigendirections of a FP is conserved in the ( $d, N$ ) plane unless it collapses with another FP. Thus, the scenario where $N_{\mathrm{c}}(d)$ and $N_{c, S^{\prime}}^{\prime}(d)$ intersect is unlikely. A more reasonable intersection to consider is instead between $N_{c, S^{\prime}}(d)$ and $N_{c, S^{\prime}}^{\prime}(d)$ where in this case $S \widetilde{A}_{4}$ ceases to exist outside of the domain delimited by these curves. Indeed, in this case $S A_{3}$ and $\widetilde{A}_{3}$ have the same number of infrared eigendirections and it is conceivable that they may swap after a full turn along the square $T$ of Fig. 3.1.20). This is the scenario we found numerically by integrating the LPA FP equations where the collapse of critical lines at the point $S^{\prime}$ of coordinates $\left(d_{S^{\prime}} \simeq 3.1, N_{S^{\prime}} \simeq 55\right)$ is shown in Fig. 3.1.23).

Thus, for $N<N_{S^{\prime}}$, the FP $S \widetilde{A}_{4}$ ceases to exist as a real valued FP and the FPs $\widetilde{A}_{3}$ and $S A_{3}$ are identified onto an intermediate state. We will call this intermediate state $A_{3}$. It is also interesting to observe that although the lines $N_{c, S^{\prime}}(d)$ and $N_{c, S^{\prime}}^{\prime}(d)$ do not cross the $d=3$ axis at large $N$, we find at the level of the LPA that they do cross it respectively around $N=55$ and around $N=70$. This means that for $55<N<70$ and at the level of the LPA, the very exotic $\widetilde{A}_{3}$ and $S \widetilde{A}_{4}$ multicritical FPs of the $\mathrm{O}(N)$ model exist in $d=3$. This is a very intriguing possibility that could be checked by testing the robustness of our results by going to higher orders of the derivative expansion (and varying the regulator function $R_{k}$ ). Of course, the bounds $N=55$ and $N=70$ could drastically change when going to higher orders of the derivative expansion. It could even turn out that the exact location of the point $S^{\prime}$ is on the left of the $d=3$ axis so that the lines $N_{c, S^{\prime}}(d)$ and $N_{c, S^{\prime}}^{\prime}(d)$ would not cross the $d=3$ axis and neither $S A_{3}$ nor $S \widetilde{A}_{4}$ would exist in $d=34$.

Interestingly however, we shall see that our analysis predicts one more multicritical fixed-point which will in fact be necessary and will exist at $d=3$. Indeed, at this stage there is a remaining critical curve $N_{\mathrm{c}, S}$ which should cease to exist for sufficiently small $N$. The natural mechanism for this is again that of Fig. 3.1.20 however in that case there is a missing critical curve with which it may collapse.

To locate this curve consider following $A_{3}$ by continuity towards the right of $d=3$. Surely we would not expect this FP to exist in $d=4$ were it is well known that the only viable FP is the Gaussian and indeed the FP $A_{3}$ does not exist at $d=4$ as has been checked within the LPA 164, 166. As usual, the mechanism for this is the collapse of $A_{3}$ with another FP. In order to introduce this FP consider following $A_{3}$ for $N \rightarrow \infty$ and $d>3$. According to what is commonly known there is only the Gaussian and Wilson-Fisher fixed-point

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Figure 3.1.24: The curve $N_{c^{\prime}}(d)$ defined by $\mathrm{SG}_{3}=\mathrm{SWF}_{2}$ for $d>3$ and $N$ large. The extension of the curve below $d=3$ has been momentarily removed for clarity and will be given below.


Figure 3.1.25: The $\mathrm{SWF}_{2} \mathrm{FP}$ in $d=3.2$ at $N=\infty$. It is shown as a solid line and is made of two parts that match at $\bar{\varrho}_{0}=0.347$. For $\bar{\varrho}<\bar{\varrho}_{0}$, it is the linear solution $\bar{V}(\bar{\varrho})=\bar{\varrho}$ and for $\bar{\varrho}>\bar{\varrho}_{0}$, it is identical to the WF FP.
at $N=\infty$ for $3<d<4$ but as we have shown before this relies on the assumption that no singularities build up as $N$ increases which need not be verified. It is in fact possible to build once more cusped solutions by joining the linear $\bar{V}(\bar{\rho})=\bar{\rho}$ solution of the Polchinski equation with one of the exact solutions at $N=\infty$, either the Gaussian or the Wilson-Fisher fixed-point. This construction leads to two new singular FPs at $N=\infty$ and $d \in\left[3,4\left[\right.\right.$ that we call $\mathrm{SG}_{3}$ for singular Gaussian and $\mathrm{SWF}_{2}$ for singular Wilson-Fisher, their index giving their degree of instability. The construction of the FP $\mathrm{SWF}_{2}$ is given in Fig. (3.1.25). The number of infrared eigendirections of the singular FPs is again one more than their regular counterparts due to the explanation of Sec.(3.1.9).

We then notice that $A_{3}$ may be continuously deformed to this new FP $S G_{3}$ and depending on the path one chooses, $A_{3}$ may in fact lead to $S A_{3}, \widetilde{A}_{3}$ or $S G_{3}$. As a convention, we call this FP $A_{3}$ when $d<3$ and $N<N_{S^{\prime}}, S G_{3}$ when $d>3$ and $\widetilde{A}_{3}$ ( resp. $S A_{3}$ ) when $A_{3}$ is close to $N_{c, S^{\prime}}^{\prime}(d)$, (resp. $N_{c, S^{\prime}}(d)$ ). As shown in [164 166], within the LPA, $S G_{3}$ collapses with $\mathrm{SWF}_{2}$ along a line $N_{c, S}^{\prime}(d)$, given in Fig. 3.1.24), such that


Figure 3.1.26: The two curves $N_{c}(d)$ and $N_{c, S}^{\prime}(d)$ respectively defined by $A_{2}=A_{3}$ and $\mathrm{SWF}_{2}=S G_{3} \sim A_{3}$. $N_{c}(d)$ is calculated with the LPA (red circles) and at order 2 of the derivative expansion (blue squares).


Figure 3.1.27: Special topological point $S$ and the two lines $N_{c}(d)$ (red squares) and $N_{c}^{\prime}(d)$ (blue stars). The line $N_{c}^{\prime}(d)$ crosses the $d=3$ axis for $N \simeq 28$. Starting from $P$, the FP $\mathrm{A}_{2}$ is followed along a clockwise closed path surrounding $S$. On this path, $\mathrm{A}_{2}$ becomes $\mathrm{SWF}_{2}$ after a full rotation.
these FPs do not exist as real valued FPs for $N<N_{c}^{\prime}(d)$. This is then consistent with hypothesis $\mathcal{H}_{2}$ that no new fixed-points are to be found at small $N$. Moreover this line $N_{c, S}^{\prime}(d)$ connects to $d=4$ and $N=\infty$ in such a way that if we follow $A_{3}$ from $\left(N_{A}, d_{A}<3\right)$ by increasing $d$ at fixed $N$, it becomes $S G_{3}$ then collapses with $\mathrm{SWF}_{2}$ before attaining $d=4$.

Hence, we now have one more critical curve, $N_{c, S}^{\prime}(d)\left[\mathrm{SWF}_{2}=S G_{3}\right]$, with which $N_{c, S}(d)$ may collapse. This is indeed what was found within the LPA as was shown in [164, 166], see Fig. [3.1.26]. These two curves intersect at a point $S$ of coordinates ( $d_{S} \simeq 2.8, N_{S} \simeq 20$ ), such that $A_{3}$ ceases to exist as a real valued FP for $N<N_{S}$. Moreover, as usual, along the path of Fig.(3.1.27), $A_{2}$ continuously deforms to $\mathrm{SWF}_{2}$ which is indeed compatible with the number of their infrared eigendirections. Thus, for $N<N_{S}$, the only surviving FP is $A_{2}$ which is able to continuously deform to $\mathrm{SWF}_{2}$ for $d<d_{S}$ and $N \simeq N_{S}$ as shown in Fig. 3.1.28).


Figure 3.1.28: Evolution of the FP potential $\bar{V}(\bar{\varrho})$ in the Polchinski approach when $\mathrm{A}_{2}$ is followed from the point $a=(d=2.94, N=30)$ as a continuous function of $(N, d)$ along the clockwise closed path shown in the right panel (identical to the path shown in Fig. (3.1.27). In the left panel we show how the $\mathrm{A}_{2}$ potential changes along the path $(a, b, c, d, a)$ shown in the right panel. In $a$, the $\mathrm{A}_{2}$ potential is very flat because $a$ is close to the $d=3$ axis where it is the Gaussian FP. It remains so at point $b$ and deforms slightly in $c$. Then, it changes drastically between $c$ and $d$ which is the region where the double-valued structure plays an important role. Finally, it evolves slightly between $d$ and $a$ where it is clearly very different from the initial potential: It has become the $\mathrm{SWF}_{2} \mathrm{FP}$.

It is interesting to note that we must have $N_{S^{\prime}}>N_{S}$. Indeed, the critical curve collapse between $N_{\mathrm{c}, S}(d)$ and $N_{c, S}^{\prime}(d)$ was possible only because $\widetilde{A}_{3}$ is able to deform to $S A_{3}$ then to $S G_{3}$ for $N<N_{S^{\prime}}$. This is to be contrasted with what takes place at very large $N$ where $\widetilde{A}_{3}$ is a regular FP while $S A_{3}$ is a singular FP which then never collides with $\widetilde{A}_{3}$. As such for large $N, \widetilde{A}_{3}$ and $S A_{3}$ become similar only at the point $S^{\prime}$ where we have $\widetilde{A}_{3}=S \widetilde{A}_{4}=S A_{3}$ and it is only below this point that the collapse of critical lines between $N_{\mathrm{c}, S}(d)$ and $N_{c, S}^{\prime}(d)$ is conceivable.

This also implies that $N_{S}$ and $N_{S^{\prime}}$ can not be arbitrarily large as $N_{S}<N_{S^{\prime}}$ and at the point $S^{\prime}$ we have $\widetilde{A}_{3}=S A_{3}$ which is only possible for $N$ sufficiently small as at the origin of the potential, the curvature of $\widetilde{A}_{3}$ and $S \widetilde{A}_{4}$ is smaller than that of $S A_{3}$ at large $N$ according to our numerical studies. Hence, it is necessary that $N$ be sufficiently small that the curvature of $S A_{3}$ which is of order $e^{N}$ may be of order 1 so that $\widetilde{A}_{3}=S A_{3}$ is conceivable. Thus, in other words, it is necessary that the singular nature of $S A_{3}$ dissipates for $N$ sufficiently small. A summary of these critical lines is given in Fig. (3.1.29).

This then answers the questions mentioned both at the beginning of this chapter and of this section concerning the existence of $A_{2}$ for all $d$ when $N$ is small and the fate of this FP when $N$ is increased. Indeed, consider the starting point $A_{2}\left(d_{0}, N_{0}\right)$ for $N_{0}=1,2,3,4$. Then if $d_{0}>d_{S}$, by increasing the value of $N$ the FP $A_{2}$ collapses with $A_{3}$ according to Fig. 3.1 .29 . If instead $d_{0}<d_{S}$ then increasing $N, A_{2}\left(d_{0}, N\right)$ continuously deforms to $\mathrm{SWF}_{2}\left(d_{0}, N\right)$. Taking the limit $N \rightarrow \infty, \mathrm{SWF}_{2}$ then develops a cusp ${ }^{41}$. These two mechanisms then explain why there is no tricritical FP at $N=\infty$ for $d<3$.

We also remark that it is quite interesting that the FP $A_{2}$ that is perturbatively connected to the Gaussian at $d=3$ continuously deforms to a highly non perturbative FP for $d<d_{S}$ and $N>N_{S}$. Hence, we expect that the point $S$ is beyond the reach of perturbation theory as its existence is intimately tied to that of the non perturbative $\mathrm{SWF}_{2}$. This is even clearer in the case of $S^{\prime}$ as none of the $\mathrm{FPs} \widetilde{A}_{3}, S A_{3}, S \widetilde{A}_{4}$ are connected to the Gaussian for finite $N$.

Finally, we would like to emphasize that it is quite remarkable that the sole existence of an endpoint to the BMB line at $(d=3, N=\infty)$ and of the relationship $(\alpha=N(3-d)) \leftrightarrow\left(\tau \propto V^{\prime \prime \prime}(1)\right)$, necessarily predicts, if $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ are valid, a plethora of new FPs and a mosaic of critical lines where these FPs collapse leading

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Figure 3.1.29: The four curves $N_{c}(d)\left(\mathrm{A}_{2}=\mathrm{A}_{3}\right.$ violet stars $), N_{c}^{\prime}(d)\left(\mathrm{SWF}_{2}=\mathrm{SG}_{3}\right.$ light blue stars $), N_{c, S^{\prime}}^{\prime}(d)$ $\left(\mathrm{SG}_{3}=\mathrm{S} \tilde{\mathrm{A}}_{4}\right.$ green crosses) and $N_{c, S^{\prime}}(d)\left(\mathrm{SA}_{3}=\mathrm{S} \tilde{\mathrm{A}}_{4}\right.$ orange crosses $)$.
to non trivial homotopy between subsets of the FPs. In particular this analysis leads to the necessity of two new FPs, $\mathrm{SWF}_{2}$ and $\mathrm{SG}_{3}$, at $d=3$.

## Chapter 4

## Conclusion

The present thesis has posed and answered a puzzling aspect concerning one of the most well known and understood models in condensed matter and high energy physics: the $O(N)$ models. We recall here the essence of this question: If multicritical fixed-points (FP) bifurcate from the Gaussian at dimensions $d_{c}(p)=$ $2+2 / p$ for all $N$, then why are there no multicritical FPs at $N=\infty$ except for $d=d_{c}(p)=2+2 / p$ where one obtains a line of FPs? While this question might seem somewhat simple, the answer reveals an intricate story. Indeed, in the tricritical case, the cast at play is a set of five fixed-points whose critical lines at large $N$ meet in such a way that, other than the perturbative tricritical, their existence as real valued FPs are not found within explorations of the $O(N)$ models at more sober integers such as $N=1,2,3$. These five FPs are called : $A_{2}, \widetilde{A}_{3}, S \widetilde{A}_{4}, S G_{3}$ and $\mathrm{SWF}_{2}$. The FPs $A_{2}$ and $\widetilde{A}_{3}$ can be found within the large $N$ expansion or perturbation theory in the limit $N \rightarrow \infty$ but the potentials of the FPs $S \widetilde{A}_{4}, S G_{3}$ and $\mathrm{SWF}_{2}$ develop a cusp in this limit and can not be found within perturbation theory or the usual $1 / N$ expansion. The functional and non perturbative aspects of the NPRG framework are then crucial to study these FPs. In Sec.(2.4.3), we explored the landscape of possible approximation schemes where the leading order was always the local potential approximation (LPA) scheme.

The renormalization group flow of the potential obtained within this approximation scheme becomes exact in the limit $N \rightarrow \infty$ when the FPs are analytical in this limit. This is the case of the FPs $A_{2}$ and $\widetilde{A}_{3}$ where, in the $\Gamma$ flow formulation, the longitudinal propagator in the flow may be neglected. However, in the case of the singular FPs such as $S \widetilde{A}_{4}, S G_{3}$ or $\mathrm{SWF}_{2}$, the longitudinal propagator is non negligible in an entire range of fields. As such, the LPA is a priori an approximation in the case of the singular FPs even in the limit $N \rightarrow \infty$. This would imply that the LPA does not offer a trustworthy starting point for these FPs even in the limit $N \rightarrow \infty$ but this is only within a finite range of fields where singularities affect the shape of these FPs. For the purpose of this thesis the qualitative features of the singular FPs was sufficient. In particular, it was sufficient to know that the potential has a boundary layer where at large fields the shape of the potential is given by its regular potential counterpart up to exponentially small corrections in $N$. We also expect that at $N=\infty$, the vector excitations of these FPS are massive which implies a singularity of the dimensionless $U^{\prime}(\phi)$ at zero field. This singularity then suggests that $U^{\prime \prime}(0)$ is large for finite and large $N$. These singular FPs "completes" the usual line of FPs at ( $d=3, N=\infty$ ) where the BMB endpoint plays the role of a bridge between these FPs.

The sole information of the existence of an endpoint to the line of FPs in the tricritical case and the relationship between the $\left(\vec{\phi}^{2}\right)^{3}$ coupling $\tau$ and the composite parameter $\alpha=(3-d) N$ given in Sec. 3.1.4 allows us to deduce the qualitative aspects of the final diagram in Sec.(3.1.10). The quantitative aspects of this last diagram was verified with the LPA. This diagram then leads to a fully consistent picture of the large $N$ regime of the $O(N)$ model.

It is interesting to note that while much of the analysis of this thesis was done in non integer dimensions, consistency between the large $N$ limit and the absence of most of these FPs when $N=1,2,3$, requires at least two new FPs $S G_{3}$ and $\mathrm{SWF}_{2}$ to exist at $d=3$ for sufficiently large $N$. If the point $S^{\prime}$ is above $d=3$ within the exact theory as well, then the FPs $\widetilde{A}_{3}$ and $S \widetilde{A}_{4}$ will also exist within a finite range of $N$ at $d=3$. Importantly, the final diagram of Sec.(3.1.10) explains the puzzle as to why no tricritical FP was found for generic $d$ when $N=\infty$. Indeed, for $3>d>d_{S}$, increasing $N$ leads $A_{2}$ to collapse with $\widetilde{A}_{3}$ while considering
$d<d_{S}$ and increasing $N$, leads $A_{2}$ to develop a cusp as it becomes SWF $_{2}$.
There are numerous future directions that can be pursued from the above $O(N)$ model study. In particular, one can include supersymmetry or investigate other symmetry groups. A study in the case of the random $O(N)$ models with disorder [185] would be interesting as, within perturbation theory, disordered systems at $d+2$ dimensions and at 0 temperature are directly related to their standard, or clean, version in $d$ dimensions at finite temperature $[189,190]$. However, this property is known to be spoiled by the presence of cusps in the disordered model 185 . Thus, from a consistency point of view, it would be interesting to investigate what happens to disordered models for the values of $N$ where cusped FPs appear in the clean $O(N)$ models.

## Appendix A

## Van der Waals Phase diagram

We recall the Van der Waals equation:

$$
\widehat{x}^{3}-\widehat{x}^{2}+\widehat{x}(\widehat{T}+\widehat{P})-\widehat{P}=0, \widehat{x}=x b, \widehat{T}=k T b / a, \widehat{P}=P b^{2} / a
$$

In order to find the number of positive roots to this equation we will use Descartes rule of signs which we now state: the maximal number of positive roots of a polynomial is equal to the number of sign changes between successive non zero coefficients. The number of roots are counted with multiplicity and when the number of roots is not maximal it is less then the maximal value by an even number. This rule will be particularly useful when studying the phase diagram of a tetracritical point. We give a heuristic proof below:

Proof. This can be understood intuitively for perturbative perturbations such as $Q_{n+1}(X)=Q_{n}(X)+$ $\epsilon X^{n+1)}$ with $Q_{n}$ a polynomial of degree $n$ containing $m$ positive roots and $\epsilon \ll 1$. Indeed the term $\epsilon X^{(n+1)}$ is negligible for $X \ll 1 / \epsilon^{1 /(n+1)}$, thus as we have $\epsilon \ll 1$, this upper bound can be very large. Hence for the maximal position of the roots $X_{M}=\max \left(\left\{x_{r}, Q_{n}\left(x_{r}\right)=0\right\}\right)$, we can choose this upper bound to be much larger than $X_{M}$ such that there exists $L>0$ such that we have $Q_{n+1}(X) \approx Q_{n}(X), 0 \leq X \leq X_{M}+L$. Hence, the roots of $Q_{n+1}$ in $\left[0, X_{M}+L\right]$ are practically the same as those of $Q_{n}(X)$ and the number of roots in this interval is still $m$. If the dominant coefficient of $Q_{n}$ is positive and $\epsilon$ is also positive then we will have $Q_{n}\left(X_{M}+L\right) \simeq Q_{n+1}\left(X_{M}+L\right)>0$ and the added positive $\epsilon X^{n+1}$ term will ensure that the function remains positive for all $X>X_{M}+L$. However, if $\epsilon<0$ then we are guaranteed that the polynomial will eventually change sign as it will be dominated by $-|\epsilon| X^{n+1}$ for $X$ sufficiently large. Thus as we still have $Q_{n+1}\left(X_{M}+L\right)>0$, by use of the intermediate value theorem, we are guaranteed, in this perturbative regime, a new root. This shows us how a sign change can imply a destabilization of the system leading to a new root. Let us thus add a new perturbation $Q_{n+2}(X)=Q_{n+1}(X)+s X^{(n+2)}$. If we have $s \ll \epsilon$ then the discussion is equivalent to before and in particular if the dominant coefficient of $Q_{n}$ is positive and if $\epsilon<0$ and $s>0$, thus two new sign changes, we have two new roots such that the total number of positive roots is $m+2$. However, if we have $1 \gg s \gg \epsilon$ then the $s X^{(n+2)}$ term may become dominant while the $\epsilon X^{n+1}$ term is still negligible and thus the previous destabilization due to the negative $\epsilon$ will be completely overshadowed by the dominant positive $s X^{(n+2)}$ term and thus the polynomial will retain a positive sign for $X>X_{M}+L$. In this case there are no new roots and thus the total number of roots is $m$. Thus for the same sign change there are two scenarios depending on the relative amplitude of the coefficients. If we then recursively apply this we are led to the result of the theorem ${ }^{11}$

For positive $\widehat{P}$ and $\widehat{T}$, the number of positive roots of Eq. 2.1 .2 is 3 or 1 as the number of sign changes is 3. The precise number of positive roots will depend on the relative magnitude between coefficients and thus we generically expect a transition between 3 and 1 roots somewhere in $(\widehat{T}, \widehat{P})$ parameter space. Moreover considering for fixed $\widehat{T}$ the function $\widehat{P}(x)$, when they exist, the three roots for a given $\widehat{P}_{0}$ correspond to

[^62]three zeroes of the function $\widehat{P}(x)-\widehat{P}_{0}$, meaning that the roots are given for the same pressure $\widehat{P}_{0}$. Thus if for a given $\left(\widehat{P}_{0}, \widehat{T}\right)$ the slope $\frac{\mathrm{d} \widehat{P}(x)}{\mathrm{d} x}$ is always non zero at the zeroes of $\widehat{P}(x)-\widehat{P}_{0}$, then between two consecutive zeroes of $\widehat{P}(x)-\widehat{P}_{0}$ there will be an extrema, meaning a zero of the slope $\frac{\mathrm{d} \widehat{P}(x)}{\mathrm{d} x}$, and thus we will observe a sign alternation of $\frac{\mathrm{d} \widehat{P}(x)}{\mathrm{d} x}$ at the zeroes of $\widehat{P}(x)-\widehat{P}_{0}$. This in turns means that that the zeroes of $\widehat{P}(x)-\widehat{P}_{0}$ will alternate between being physical states and nonphysical states and as such there cannot be three physical phases at the same time for this system. The case where one zero is a double root of $\widehat{P}(x)-\widehat{P}_{0}$, that is, it is a zero of $\widehat{P}(x)-\widehat{P}_{0}$ and of its derivative will correspond to a double root in the polynomial equation and thus if the three roots are distinct it can easily be discarded. Moreover, there is always at least one physical state in the low density gas regime. Hence, in the case where there are 3 roots one is necessarily nonphysical. Thus, the transition from three roots to one root is actually a transition from 2 physical phases to one physical phase.

Moreover, in the case where there are two phases $x_{A}$ and $x_{B}$, the phase with the lightest density will be called a gas whereas the phase with the larger density will be called a liquid. In order to grasp an idea of the physics at hand we take two extreme cases.

First we consider the case $\widehat{P} \gg \widehat{T}$, then Eq. 2.1 .2 ) becomes $\widehat{x}^{2}(\widehat{x}-1)=\widehat{P}(1-\widehat{x})$ and thus there is only one positive root $\widehat{x}=1$ as the other two $\widehat{x}^{2}=-\widehat{P}$ are imaginary for positive $\widehat{P}$. Moreover, $\widehat{x}=1$ corresponds to $V=N b$ and thus the system is tightly packed which thus corresponds to a dense liquid. In another scenario where $\widehat{P} \ll 1$, we have for $\widehat{P}=0$

$$
\begin{equation*}
\widehat{x}\left(\widehat{x}^{2}-\widehat{x}+\widehat{T}\right)=0, \tag{A.0.1}
\end{equation*}
$$

which leads to the solutions $\widehat{x}_{1}(\widehat{P}=0, \widehat{T})=0, \widehat{x}_{2,3}(\widehat{P}=0, \widehat{T})=(1 \pm \sqrt{1-4 \widehat{T}}) / 2$. Hence if $\widehat{T}>$ $\widehat{T}_{c}(\widehat{P}=0)=1 / 4$ there is only one root but there are three below this critical temperature ${ }^{2}$. For small non zero $\widehat{P}$ we may expand $\widehat{x}_{1}$ to linear order in $\widehat{P}$, inserting this expression in Eq.2.1.2 and neglecting powers of $\widehat{P}^{2}$ or higher we obtain $\widehat{x}_{1}(\widehat{P}, \widehat{T})=\widehat{P} / \widehat{T}+O\left(\widehat{P}^{2}\right)$. This solution increases linearly with the pressure and thus in the more familiar $(V, P, T)$ variables it corresponds to the equation of state of the ideal gas. We then expect the next root to have a negative slope $\frac{\mathrm{d} \widehat{P}(x)}{\mathrm{d} x}$ as long as it does not form a double root for $\widehat{T}=1 / 4$. The third largest root is then the liquid phase. Thus for $\widehat{P} \ll 1$ and above $\widehat{T}=1 / 4$ there is only one gaseous phase and below there is both a gas and a liquid phase.

Notice here however that for a given $\widehat{P}$ there is an entire interval of $\widehat{T}$ for which there is coexistence of two phases. This contradicts actual phase diagrams where $T$ must be determined by $P$. This is due to the fact that there is a missing equation which is the equality of chemical potentials between coexisting phases. We will imagine that this equation has been solved and we will now keep in mind that when both phases exists, that actually $T$ is predetermined by the value of $P$. This is of course not true outside of the coexistence regime as there would no longer be a constraint of equal chemical potentials and thus both $P$ and $T$ will be independent variables.

Outside of the small $\widehat{P}$ regime the branches $\widehat{x}_{1}(\widehat{P}, \widehat{T})$ (resp. $\widehat{x}_{3}(\widehat{P}, \widehat{T})$ ) will continue to exist for $\widehat{T}$ sufficiently small as as in the small $\widehat{P}$ case. Moreover, as $(\widehat{P}, \widehat{T})$ are not independent variables in that case we can take $\widehat{T}$ as a function of $\widehat{P}$ and hence write $\widehat{x}_{1}(\widehat{P})\left(\right.$ resp. $\widehat{x}_{3}(\widehat{P})$ ) where for all $\widehat{P}$ we have by definition $\widehat{x}_{1} \leq \widehat{x}_{3}$. However, as we saw for sufficiently large $\widehat{P}$, there is only one phase hence there must be a critical point $\left(P_{c}, T_{c}\right)$ where both branches $\widehat{x}_{1}(\widehat{P})$ and $\widehat{x}_{2}(\widehat{P})$ collide to a single $\widehat{x}_{c}$ in order to form a single phase in the large pressure domain. This means that as we increase $\widehat{P},\left|\widehat{x}_{3}-\widehat{x}_{1}\right|$ diminishes and thus the gas is becoming more dense and the liquid less dense. As the intermediate nonphysical root $\widehat{x}_{2}$ sits in between $\widehat{x}_{1}$ and $\widehat{x}_{3}$, it must also converge to this critical $\widehat{x}_{c}$ such that $\widehat{x}_{1}=\widehat{x}_{2}=\widehat{x}_{3}=\widehat{x}_{c}$ and thus Eq.(2.1.2) has a triple root. This then means that $\widehat{x}_{c}$ is also a root of the first and second derivative of Eq.(2.1.2). From the second derivative we obtain $\widehat{x}_{c}=1 / 3$ and we may then use this value in Eq.(2.1.2) and its derivative to determine

[^63]$\left(\widehat{P}_{c}=\frac{1}{27}, \widehat{T}_{c}=\frac{8}{27}\right)$. What is interesting is that if we differentiate the polynomial $f(\widehat{x}, \widehat{P})$ on the left hand side of Eq.2.1.2 with respect to $\widehat{x}$ and $\widehat{P}$ we obtain $0=\mathrm{d} f(\widehat{x}, \widehat{P})=\frac{\partial}{\partial \widehat{x}} f \mathrm{~d} \widehat{x}+\frac{\partial}{\partial \widehat{P}} f \mathrm{~d} \widehat{P}$ so that $\frac{\mathrm{d} \widehat{x}}{\mathrm{~d} \widehat{P}}=-\frac{\frac{\partial}{\partial P}}{\frac{\partial}{\partial \widehat{x}}} f$.

However, since $\widehat{x}_{c}$ is also a root of the derivative of the polynomial at $(\widehat{x}, \widehat{P})=\left(\widehat{x}_{c}, \widehat{P}_{c}\right)$ we have $\frac{\partial}{\partial \widehat{x}} f\left(\widehat{x}_{c}, \widehat{P}_{c}\right)=0$ such that $\frac{\mathrm{d} \widehat{x}}{\mathrm{~d} \widehat{P}}=\infty$. This also implies that the compressibility of the fluid $-\frac{1}{V} \frac{\partial V}{\partial P}$ diverges which means that the density of the fluid is highly dependent on small changes of pressure. This is completely analogous to the divergent susceptibility of uni-axial ferromagnetic systems at second order phase transitions as discussed in the introduction. More precisely, $\frac{\partial V}{\partial P}=\frac{\partial^{2} G}{\partial P^{2}}$ where $G$ is the Gibbs energy and thus the point where $\frac{\partial V}{\partial P}=\infty$ indeed corresponds to a second order phase transition. Moreover, as in the case of a divergent susceptibility, a small fluctuation in density at a point $A$ leads to an externally felt modification in pressure at point $B$ which then leads to a dramatic change in density at that point. Hence, a large compressibility is indeed a signal of large correlations in the medium.

## Appendix B

## Discussion on Euler product

In this section we address the question: why can the sin be written simply as an infinite product of it's zeroes in the first place? This discussion will give us an opportunity to discuss effective field theories and non perturbative renormalizability.

This question addresses the concern as to why the initial hypothesis that sin is simply a product of simple functions containing it's zeros can be made. Indeed $\exp (x) \sin (x)$ contains the same amount of zeroes and obviously cannot be written in the same way. As far as the author is aware of, this issue was not mentioned by Euler but is often mentioned in modern critiques of the derivation. However, he did seek an alternative proof to show that the product formula is correct later on in his life.

Fortunately however, there is a modern exact theorem for these sorts of problems which is called the Weierstrass product formula. It essentially says that what Euler did was correct up to an overall factor, which we will call $a_{0}(x)$, and that is entire with no zeroes. Furthermore, regularization factors for the products $\left(1-\frac{x}{r_{i}}\right)$, where $r_{i}$ is a non zero root, is also needed in order to deal with convergence issues.

This result is thus similar to the remark that $\exp (x) \sin (x)$ contains the same amount of zeroes and in general $\exp (f(x)) \sin (x)=a_{0}(x) \sin (x)$ with $f$ entire does as well. As we now have a function $a_{0}(x)$ one might fear that the problem is no longer predictive for arbitrary $x$ as knowing $a_{0}\left(x_{r e f}\right)$ for any particular $x_{r e f}$ is insufficient to obtain the entire function $a_{0}(x)$. This is true for any finite number of reference points or knowing any finite number of derivatives of $a_{0}$ at reference points. However one may Taylor expand $a_{0}$ in order to obtain an approximation for $x<x_{\max }$ where $x_{\max }$ might be given by a zero of the Taylor expansion or the radius of convergence when dealing with functions that are not entire. This is similar to the manner in which we use non renormalizable effective field theories which may naively seem useless as they require an infinite number of parameters to be fixed in order to cancel all the divergencies of perturbation theory but they are in fact quite predictive within their regime of validity. Moreover, they can be much more useful than renormalizable theories. Indeed, in renormalizable theories the divergencies may be cancelled but at any finite order of the perturbative loop expansion, the solutions obtained are usually not exact, but rather, they are approximate solutions. As such one sometimes needs to perform rather tedious calculations at many loop orders to obtain an acceptable solution whereas an effective theory may give acceptable results already at the level of the most basic approximation, that is, at tree level. A drastic example of this is the theory of pions. Although quantum chromodynamics, the theory of the strong force, is renormalizable, interactions of boundstates such as pions, which are composed of the elementary quarks used in the quantum chromodynamics Lagrangian, require non perturbative calculations incorporating all loop orders. As such, it is impossible to obtain accurate predictions of pion-pion scattering using perturbative quantum chromodynamics. This is to be contrasted with the Lagrangian of pions which is an effective Lagrangian based solely on the symmetries of the bound-states and on a few couplings that may be obtained by experiment. This allows us to obtain quick predictions for the scattering of pions within the regime of validity of the approximation.

A few examples of very useful non renormalizable theories include the 4-Fermi theory, the theory of pions, the Schrodinger equation for the electron and quantum gravity up to the Planck scale [191]. These four theories are four effective theories allowing us to describe the nature of the weak force, the strong force, electromagnetism and gravity in that order. However, in the case of gravity it is not clear whether the theory is actually non-renormalizable. Indeed, even though the theory is not perturbatively renormalizable it might be non perturbatively renormalizable and thus the functional renormalization group offers us a great
opportunity to test this hypothesis $192-200$.
Returning to the Weierstrass factorization theorem, the last mentioned case of a non-perturbatively renormalizable theory is actually the case here for functions that are entire and that behave as $\exp \left(A x^{m}\right)$ for large $x$ and integer $m$. In this case, according to the Hadamard factorization theorem, $a_{0}(x)$ may be written as $a_{0}(x)=\exp (P(x))$ where $P$ is a polynomial of degree $m$. As such determining the coefficients of $P$ is sufficient to obtain the function $a_{0}(x)$ for all $x$ even though it was impossible with the Taylor expansion. We may use this in the case of the $\sin$ function. As $\sin (x) \sim \exp (i x) /(2 i)$ for large $x=-i|x|$, we have $P(x)=p_{0}+p_{1} x$ but as $\sin (x) / x$ is even, $p_{1}=0$ which thus justifies the product obtained by Euler.

## Appendix C

## Counter terms

In the counter term method we seek to write the Lagrangian in terms of measurable quantities with the counter terms chosen to cancel divergencies. More explicitly, one distinguishes between the bare fields $\phi$ and the renormalized fields $\phi_{r}=\phi / Z^{1 / 2}$, the bare mass $m$ and the renormalized mass $m=m_{r}-\delta m$ and finally the bare and renormalized couplings $\lambda=Z_{\lambda} \lambda_{r} / Z^{2}$. The reason for the factors $Z$ and $\delta m$ is that in general the two point function $\langle\phi \phi\rangle$, has a pole in the complex plane at the physical measurable mass $m_{r}$ and near this pole in the complex plane, $\phi$ behaves as a free propagator with mass $m_{r}$ multiplied by some factor $Z$ [201]. This factor $Z$ can be removed by "renormalizing" the field $\phi$ such that $\phi_{r} Z^{1 / 2}=\phi$ . The renormalization factor $Z_{\lambda}$ is chosen in order to cancel divergencies in loop diagrams and $\lambda_{r}$ is the physical measurable coupling of the interacting theory. We may thus reparametrize Eq. 2.1 .18 in terms of the physical quantities. For the sake of generality we also add a mass term and we write the field variable in terms of $\phi$ :

$$
\begin{align*}
\frac{1}{2} \partial_{\mu} \phi(x) \partial^{\mu} \phi(x)+\frac{m^{2}}{2} \phi(x)^{4}+\frac{\lambda}{4!} \phi(x)^{4}= & \frac{Z}{2} \partial_{\mu} \phi_{r}(x) \partial^{\mu} \phi_{r}(x)+\left(m_{r}-\delta m\right) Z \phi_{r}^{2}+Z_{\lambda} \lambda_{r} \phi_{r}(x)^{4} \\
= & \frac{1}{2} \partial_{\mu} \phi_{r}(x) \partial^{\mu} \phi_{r}(x)+\frac{1}{2} m_{r} \phi_{r}^{2}+ \\
& \frac{\lambda_{r}}{4!} \phi_{r}(x)^{4}+(Z-1)\left(\frac{1}{2} \partial_{\mu} \phi_{r}(x) \partial^{\mu} \phi_{r}(x)+\frac{1}{2} m_{r} \phi_{r}^{2}\right)  \tag{C.0.1}\\
& -\frac{1}{2} Z \delta m \phi_{r}^{2}+\frac{Z_{\lambda}-1}{4!} \lambda_{r} \phi_{r}(x)^{4} \\
= & \mathcal{L}_{0}+\mathcal{L}_{1}+\mathcal{L}_{2}
\end{align*}
$$

with,

$$
\begin{aligned}
\mathcal{L}_{0} & =\frac{1}{2} \partial_{\mu} \phi_{r}(x) \partial^{\mu} \phi_{r}(x)+\frac{1}{2} m_{r} \phi_{r}^{2} \\
\mathcal{L}_{1} & =\frac{\lambda_{r}}{4!} \phi_{r}(x)^{4} \\
\mathcal{L}_{2} & =(Z-1)\left(\frac{1}{2} \partial_{\mu} \phi_{r}(x) \partial^{\mu} \phi_{r}(x)+\frac{1}{2} m_{r} \phi_{r}^{2}\right)-\frac{1}{2} Z \delta m \phi_{r}^{2}+\frac{Z_{\lambda}-1}{4!} \lambda_{r} \phi_{r}(x)^{4}
\end{aligned}
$$

$\mathcal{L}_{0}$ is then taken as the Gaussian part of the Lagrangian around which the perturbation expansion is performed. Notice that the mass term is now the correct physically measurable mass $m_{r}$ containing all the interactions with the background medium which we define as the vacuum. $\mathcal{L}_{1}$ is the interacting part written in terms of the physical measurable interaction coupling given by $\lambda_{r}$ and $\mathcal{L}_{2}$ contains so called "counter terms" involving $(Z-1)$ and $\delta m$ that are determined by choosing that the full two point function has the correct form and $Z_{\lambda}-1$ is used to remove divergencies within loop diagrams when the term $e^{-\int \mathcal{L}_{1}+\mathcal{L}_{2}}$ is expanded in the path integral. Hence, in essence, we have done nothing except to reparametrize the problem in terms of variables that can be measured. Doing this and adjusting the counter terms to keep all quantities finite we obtain a perfectly reasonable theory with only finite quantities for which, at least at the given order of approximation, we may "forget" about the infinities. In essence this is analogous to Sec. 2.2.3 where
the divergencies were removed by simply reparametrizing the problem. Meaning considering our starting point for polynomials as $p(x)=\Pi\left(1-x / x_{i}\right) \rrbracket^{\top}$ and then taking the limit of this expression to obtain the sin function directly without ever encountering any divergence.

This is also analogous to the manner in which Eq. $\sqrt{2.2 .63)}$ is written. We are writing the equation in terms of the actual measurable frequency $r$ which will be determined by requiring that the divergencies disappear as we shall see. We have thus modified the 0th order term such that its contributions from the 0th order term cancel secular terms coming from the first order term. This is similar to what we did in previous examples reabsorbing divergencies into integration constants obtained at 0th order. In the framework of field theory the 0th order corresponds to the "tree level" whereas the first order correspond to the "one-loop" order. The difference here however is that one cannot cancel all divergences in this manner by simply tuning the constant $r$. Indeed the reason why it is possible to remove the divergence is due to the fact that at the first order in $\epsilon$ we we do not need to renormalize the amplitude and the renormalization of the phase gives a linear contribution in $t$ that the frequency $r$ will reproduce. However, higher order correction will lead to higher powers of $t$ that cannot be modeled by a simple frequency parameter. What we can do however is cancel the divergencies using the usual renormalization procedure and then vary $r$ to obtain an optimized expression by requiring that physical quantities depend the least as possible on the external parameter as is done in optimized perturbation theory $15-23]$.

[^64]
## Appendix D

## Formal derivation of the $\Gamma$ flow

In this section we will derive the $\Gamma$ - flow in a more direct formal manner.
For a euclidean statistical theory, all of the physics is given by the partition function in the presence of an arbitrary external source or equivalently the statistical free energy 1 . When the system is coupled to a source $J$ such as the exterior magnetic field, the statistical free energy $F$ is a function of $J$ which leads to the thermodynamic identity $\mathrm{d} F=-\Phi \mathrm{d} J-S \mathrm{~d} T$ where $\Phi=\langle\phi\rangle$ and $\phi$ is a fluctuating field. If we wish to have a functional that depends instead directly on $\Phi$ we may, as in thermodynamics, add to $F$ the term $\Phi J$ which leads to the Legendre transformed free energy $\Gamma$ that verifies $\mathrm{d} \Gamma=\mathrm{Jd} \Phi$-SdT. In the following, this free energy will be called the effective action. The reason for this name is that it verifies all of the properties of the action within mean field approximation while incorporating collective behavior effects. More explicitly, the effective action contains all the statistical information of the theory and verifies $\Gamma=S+O(h)$ where the term $O(h)$ can be obtained from loop diagrams as those used to compute the first correction in $e_{R}$ in Eq. (2.2.1). In fact Eq.(2.2.1) and all of its higher order corrections may be obtained from $\Gamma=S+O(h)$ by taking functional derivatives.

Hence, let us consider a system given by the following partition function:

$$
\begin{equation*}
Z=\int D \varphi e^{-S[\varphi]} \tag{D.0.1}
\end{equation*}
$$

where $\varphi$ should be seen as a list $\left\{\varphi_{a}\right\}_{a}$ containing various fields in the theory. In the case of the uni-axial ferromagnet, that is the Ising model, studied in the Landau section Sec (2.1.2) there is just one field. As an example of a system that contains many fields one may cite quantum electrodynamics where there is both a photon field and an electron field. Next we define the dot notation:

$$
\begin{equation*}
J \cdot \varphi=\sum_{\alpha} J_{\alpha} \varphi_{\alpha}=\int_{x} \sum_{a} J_{a}(x) \varphi_{a}(x)=\sum_{a} \int J_{a}(x) \varphi_{a}(x) \mathrm{d}^{d} x \tag{D.0.2}
\end{equation*}
$$

and

$$
\begin{align*}
\varphi \cdot R_{k} \cdot \varphi & =\sum_{\alpha, \beta} \varphi_{\alpha} R_{\alpha, \beta}(k) \varphi_{\beta}=\int_{x, y} \sum_{a, b} \varphi_{a} R_{a, b}(k) \varphi_{b}  \tag{D.0.3}\\
& =\sum_{a, b} \int \varphi_{a}(x) R_{a, b}(k)(x-y) \varphi_{b}(y) \mathrm{d}^{d} x \mathrm{~d}^{d} y
\end{align*}
$$

where latin letters represent field indices while greek letters represent both field indices and the position $x$ of the fields. We may then add a source term $J \cdot \phi$ to the action which allows us to define correlation functions by means of functional derivatives. Moreover, we add a regulator term $\chi \cdot R_{k} \cdot \chi$ that regulates any divergence for small momenta by remaining strictly positive. The regulator term is then particularly important near at a second order phase transition as the absence of intrinsic length scales leads to infrared divergencies. Finally, we define a UV cut-off which we will take to be a sharp cut-off such that momenta $q>\Lambda$ do not contribute to the partition function. This last UV regularization $\Lambda$ will be sent to infinity at the end and

[^65]thus the precise value of the cut-off is largely irrelevant. This thus allows us to define the partition function as:
\[

$$
\begin{equation*}
Z[J, \Lambda, k]=\int D \varphi e^{-S[\varphi]+J \cdot \varphi-\frac{1}{2} \varphi \cdot R_{k} \cdot \varphi} \overline{\overline{d e f}} \int D \varphi e^{-\widetilde{S}[\varphi, J, R]} \tag{D.0.4}
\end{equation*}
$$

\]

As is done with the Polchinski flow, we may now define the following macroscopic quantity:

$$
\begin{equation*}
e^{-W[J, R]}=\int D \varphi e^{-\widetilde{S}[\varphi, J, R]} \tag{D.0.5}
\end{equation*}
$$

$W[J, R]$ is the free energy depending on the external source $J$ and on the regulator $R$. Taking a functional derivative of Eq. D.0.5 with respect to $J$ we find :

$$
\begin{equation*}
-\frac{\delta W[J, R]}{\delta J}=\int D \varphi \varphi\left(\frac{e^{-\widetilde{S}[\varphi, J, R]}}{\int D \varphi e^{-\widetilde{S}[\varphi, J, R]}}\right) \underset{\overline{d e f}}{ } \int D \varphi \varphi \mathcal{P}(\varphi, J, R) \underset{d e f}{\bar{e}}<\varphi>_{J, R} \tag{D.0.6}
\end{equation*}
$$

$\frac{\delta W[J, R]}{\delta J}=-<\varphi>_{J, R}$ is equivalent to $\frac{\partial F}{\partial B}=-M$ in thermodynamics where $M$ is the averaged internal variable whereas $B$ is an external control parameter. However, as with Landau theory, it is easier to understand the underlying physics in terms of internal variables. The usual technique to trade variables in thermodynamics is by means of Legendre-transforms. The same is true here and we thus define:

$$
\begin{equation*}
\widehat{\Gamma}\left[\phi, R_{k}\right]=W[J, R]+J \cdot \phi \tag{D.0.7}
\end{equation*}
$$

The relationship between $\widehat{\Gamma}\left[\phi, R_{k}\right]$ and $W[J, R]$ is the same as the usual thermodynamic relationship between the Gibbs energy G and the free energy $F$ as $G(B, T)=F(M, T)+M B$. Using $\frac{\delta \widehat{\Gamma}}{\delta \phi}=J$ we then have:

$$
\begin{align*}
\exp \left(-\frac{1}{h} \widehat{\Gamma}_{k}[\phi]\right) & =\int D \varphi e^{-\frac{1}{h}\left(S[\varphi]+\frac{\varphi \cdot R_{k} \cdot \varphi}{2}-\frac{\delta \widehat{\Gamma}}{\delta \phi}[\phi] \cdot(\varphi-\phi)\right)}  \tag{D.0.8}\\
& =\int D \tilde{\varphi} e^{-\frac{1}{h}\left(S_{k}[\tilde{\varphi}+\phi]-\frac{\delta \widehat{\Gamma}}{\delta \phi}[\phi] \cdot \tilde{\varphi}\right)}
\end{align*}
$$

where

$$
\begin{equation*}
S_{k}[\varphi]=S[\varphi]+\frac{\varphi \cdot R_{k} \cdot \varphi}{2} \tag{D.0.9}
\end{equation*}
$$

and where we have made explicit the "Planck constant $h$ " which gauges the strenght of fluctuations. If we then chose $\phi$ to minimise the action $S_{k}$ then in the limit $h \rightarrow 0$ we have the saddle point approximation

$$
\begin{aligned}
\widehat{\Gamma}_{k}[\phi] & =S_{k}[\phi]+O(h) \\
& =S[\phi]+\frac{\phi \cdot R_{k} \cdot \phi}{2}+O(h)
\end{aligned}
$$

The regulator then appears explicitly in the definition of $\widehat{\Gamma}_{k}[\phi]$. In Sec. 2.4.3) we will consider approximations based on a derivative expansion and thus it is convenient to remove all trivial derivative terms from the choice dependent $R_{k}$. We then define :

$$
\begin{equation*}
\Gamma_{k}[\phi]=\widehat{\Gamma}_{k}[\phi]-\frac{\phi \cdot R_{k} \cdot \phi}{2} \tag{D.0.10}
\end{equation*}
$$

Other than the motivation from the derivative expansion, the choice to work with $\Gamma_{k}[\phi]$ rather than $\widehat{\Gamma}_{k}[\phi]$ will lead to two other advantages. The first is that we have:

$$
\begin{equation*}
\frac{\delta \widehat{\Gamma}}{\delta \phi}=\frac{\delta \Gamma}{\delta \phi}+\phi \cdot R_{k}=J \tag{D.0.11}
\end{equation*}
$$

which leads to:

$$
\begin{equation*}
\exp \left(-\Gamma_{k}[\varphi]\right)=\int D \phi e^{-S[\phi+\varphi]-\frac{\phi \cdot R_{k} \cdot \phi}{2}+\frac{\delta \Gamma_{k}[\varphi]}{\delta \varphi} \cdot \phi} \tag{D.0.12}
\end{equation*}
$$

Now taking the limit $k \rightarrow \Lambda$ with $R_{k} \propto k^{2}$ we have:

$$
\frac{e^{-\frac{\phi \cdot R_{k} \cdot \phi}{2}}}{\int D \phi e^{-\frac{\phi \cdot R_{k} \cdot \phi}{2}}} \rightarrow \delta(\phi)
$$

and thus $\Gamma_{\Lambda}[\varphi] \simeq S[\varphi]$ up to a trivial normalization factor. The choice $\Gamma_{k}[\phi]=\widehat{\Gamma}_{k}[\phi]-\frac{\phi \cdot R_{k} \cdot \phi}{2}$ then allows a convenient regulator independant initial condition to the flow.

The second reason for extracting the $\phi \cdot R_{k} \cdot \phi$ term will be that this definition allows us to have a very clean flow equation as we shall now see. Let us calculate the flow of $\Gamma_{k}=\Gamma\left[\phi, R_{k}\right]$ by taking the derivative with respect to $k$ of $\Gamma_{k}$. We have:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dk}} \Gamma_{k}=\frac{\mathrm{d}}{\mathrm{dk}} W\left[J, R_{k}\right]-\frac{\mathrm{d}}{\mathrm{dk}}(J) \cdot \phi-\frac{1}{2} \phi \cdot \frac{\mathrm{~d}}{\mathrm{dk}} R_{k} \cdot \phi \tag{D.0.13}
\end{equation*}
$$

In this equation the derivative does not act on $\phi$ as it should be regarded as an external parameter after taking the Legendre transform from $W$ to $\Gamma_{k}$. However, this is not the case for $J$ as it now defined implicitly as the solution $J=J^{*}(\phi)$ of the equation $\frac{\delta W\left[J^{*}, R_{k}\right]}{\delta J}=\phi$. Hence as $W$ depends on $k, J$ must also depend on $k$ as the implicit solution of a $k$ dependent equation. We thus have:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dk}} \Gamma_{k}=\frac{\mathrm{d}}{\mathrm{dk}}(J) \cdot \frac{\delta}{\delta J} W+\frac{\mathrm{d}}{\mathrm{dk}}\left(R_{k}\right) \cdot \frac{\delta}{\delta R_{k}} W-\frac{\mathrm{d}}{\mathrm{dk}}(J) \cdot \phi-\frac{1}{2} \phi \cdot \frac{\mathrm{~d}}{\mathrm{dk}} R_{k} \cdot \phi \tag{D.0.14}
\end{equation*}
$$

where $\frac{\mathrm{d}}{\mathrm{dk}}\left(R_{k}\right) \cdot \frac{\delta}{\delta R_{k}} W=\sum_{\alpha, \beta} \frac{\mathrm{d}}{\mathrm{dk}}\left(R_{k}\right)_{\alpha, \beta}\left(\frac{\delta}{\delta R_{k}} W\right)_{\alpha, \beta}$. Hence, from $\frac{\delta W}{\delta J}=\phi$ we deduce:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dk}} \Gamma_{k}=\frac{\mathrm{d}}{\mathrm{dk}}\left(R_{k}\right) \cdot \frac{\delta}{\delta R_{k}} W-\frac{1}{2} \phi \cdot \frac{\mathrm{~d}}{\mathrm{dk}} R_{k} \cdot \phi \tag{D.0.15}
\end{equation*}
$$

Moreover we have:

$$
\begin{equation*}
\frac{\delta}{\delta R_{\alpha, \beta}} W=\frac{1}{2} \int D \varphi \varphi_{\alpha} \varphi_{\beta} \mathcal{P}(\chi, J, R)=\frac{1}{2}<\varphi_{\alpha} \varphi_{\beta}> \tag{D.0.16}
\end{equation*}
$$

If we then recall that $\phi_{a}=<\varphi_{a}>$ we obtain:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dk}} \Gamma_{k}=\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{dk}}\left(R_{k}\right) \cdot \chi \tag{D.0.17}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi_{\alpha, \beta}=<\varphi_{\alpha} \varphi_{\beta}>-<\varphi_{\alpha}><\varphi_{\beta}>=<\left(\varphi_{\alpha}-<\varphi_{\alpha}>\right)\left(\varphi_{\beta}-<\varphi_{\beta}>\right)> \tag{D.0.18}
\end{equation*}
$$

is the covariance of the random variables $\varphi_{\alpha}$ and $\varphi_{b}$. If we define $\alpha=(x, a)$ and $\beta=(y, b)$, then in the case where $a=b, \chi_{\alpha, \beta}$ is called the connected correlation function and it is in a sense the proper generalization of the correlation function $<\varphi_{\alpha} \varphi_{\beta}>$ when $<\varphi_{\alpha}>$ or $<\varphi_{\beta}>$ is non zero. Indeed, it is zero if and only if $<\varphi_{\alpha} \varphi_{\beta}>=<\varphi_{\alpha}><\varphi_{\beta}>$ which means that the two variables are not correlated and one may take the averages separately.

However, at this point the NPRG equation is still not finished as we would have to explicitly calculate the covariance of the variables. Our objective is thus to express this covariance in terms of the effective action.

Correlations in the system are related to the susceptibility which in turn is related to a second derivative of the free energy. In fact, we may check from Eq. D.0.5 that we have:

$$
\begin{equation*}
\chi_{\alpha, \beta}=-\frac{\delta^{2}}{\delta J_{\alpha} \delta J_{\beta}} W=-\frac{\delta}{\delta J_{\alpha}} \phi_{\beta} \tag{D.0.19}
\end{equation*}
$$

Hence, we see that $\chi$ is also a generalized susceptibility as the usual susceptibility verifies $\chi=\frac{\partial M}{\partial B}=-\frac{\partial^{2} F}{\partial B \partial B}$ . In thermodynamics, we also have $\chi^{-1}=\frac{\partial B}{\partial M}=\frac{\partial G}{\partial M \partial M}$ where $\widehat{\Gamma}=\Gamma+\phi \cdot R \cdot \phi$ plays the role of $G$ here via the usual Legendre transform. In the slightly more complicated case at hand where we have multiple indices, $\chi$ is the Jacobian of the mapping $M_{\alpha}\left(\left\{B_{\beta}\right\}_{\beta}\right)$ and $\chi^{-1}$ is the Jacobian of the inverse transformation. Thus, $\chi^{-1}$ is the inverse matrix of $\chi$ such that we find:

$$
\begin{equation*}
\chi=\left(\frac{\delta^{2}}{\delta \phi_{\alpha} \delta \phi_{\beta}} \widehat{\Gamma}\right)^{-1} \underset{\operatorname{def}}{=}\left(\widehat{\Gamma}^{(2)}\right)^{-1}=\left(\Gamma^{(2)}+R\right)^{-1} \tag{D.0.20}
\end{equation*}
$$

which then leads to the $\Gamma$ flow:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{dk}} \Gamma_{k}=\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{dk}}\left(R_{k}\right) \cdot\left(\Gamma_{k}^{(2)}+R_{k}\right)^{-1} \tag{D.0.21}
\end{equation*}
$$

## Appendix E

## FSS RG via rescalings

In this section we show how to derive an FSS form of the flow equation.
We introduce the scaling operator:

$$
\begin{equation*}
\Xi_{t} \cdot(x, \phi(x))=\left(e^{t} x, e^{d_{\phi} t} \phi\left(e^{t} x\right)\right) \tag{E.0.1}
\end{equation*}
$$

where $d_{\phi}$ corresponds to the scaling dimension of the field $\phi$. In Sec.(2.3.1), we saw that it is necessary to implement rescalings in the theory to have at least fixed-points let alone FSS. These rescalings can be naturally implemented through the measure $D \phi \mu_{k_{1}, k_{2}}(\phi)$ of the path integral. First, we trade $\mu_{k_{1}, k_{2}}$ for $\mu_{t_{1}, t_{2}}$ using the usual change of variables $k_{i} / \Lambda=e^{t_{i}}$. This then defines $\mu$ as $\mu_{t_{1}, t_{2}}=e^{-\phi \cdot \Delta_{t_{1}, t_{2}}^{-1} \cdot \phi / 2}$ where in Fourier space we have $\Delta_{t_{1}, t_{2}}(q)=\frac{C_{t_{2}} K_{t_{2}}(q)-C_{t_{1}} K_{t_{1}}(q)}{q^{2}}=\frac{C_{t_{2}} K\left(e^{t_{2}} q\right)-C_{t_{1}} K\left(e^{t_{1}} q\right)}{q^{2}}$ where we added a factor $C_{t}$ which transforms by scale transformations in such a way that $-\phi \cdot \Delta_{t_{1}, t_{2}}^{-1} \cdot \phi / 2$ is scale invariant. This factor $C$ then absorbs any scaling factors which may appear with scale transformations. We may then remove the $e^{t_{2}}$ dependence in $K\left(e^{t_{2}} q\right)$ by using the change of variables $\widetilde{q}=e^{t_{2}} q$, or equivalently in real space, $\widetilde{x}=e^{-t_{2}} x$. We then effectively induce a rescaling $x \rightarrow e^{t_{2}} x$ which then transforms $\phi$ to $\Xi_{-t_{2}} \cdot \phi$ in $\mu$. This may then be corrected by changing variables to $\widetilde{\phi}=\Xi_{-t_{2}} \cdot \phi$ in the path integral which ultimately induces the transformation $V_{t_{2}, t_{3}}(\phi) \rightarrow V_{t_{2}, t_{3}}\left(\Xi_{t_{2}} \phi\right)=\Xi_{-t_{2}} V_{t_{2}, t_{3}}(\phi)$ in the potential. We thus have the following:

$$
\begin{align*}
\Xi_{-t_{2}} \Psi_{t_{1}, t_{3}}(\Phi) \underset{d e f}{=} \Psi_{t_{1}, t_{3}}\left(\Xi_{t_{2}} \Phi\right) & =\int D \phi \mu_{t_{1}, t_{2}}(\phi) e^{-V_{t_{2}, t_{3}}\left(\phi+\Xi_{t_{2}} \Phi\right)} \\
& =\int D \phi \mu_{t_{1}-t_{2}, 0}(\phi) e^{-V_{t_{2}, t_{3}}\left(\Xi_{t_{2}} \phi+\Xi_{t_{2}} \Phi\right)}  \tag{E.0.2}\\
& =\int D \phi \mu_{t_{1}-t_{2}, 0}(\phi) e^{-\Xi_{-t_{2}} V_{t_{2}, t_{3}}(\phi+\Phi)}=\int D \phi \mu_{t_{1}-t_{2}, 0}(\phi) \Xi_{-t_{2}} \Psi_{t_{2}, t_{3}} \\
& =T_{t_{1}-t_{2}, 0}\left(\Xi_{-t_{2}} \Psi_{t_{2}, t_{3}}\right)=\left(T_{t_{1}-t_{2}, 0} \circ \Xi_{-t_{2}}\right)\left(\Psi_{t_{2}, t_{3}}\right)
\end{align*}
$$

If we then further compose with $\Xi_{t_{2}}$ using

$$
\begin{equation*}
\left(\Xi_{t_{2}} \circ \Xi_{-t_{2}}\right)\left(\Psi_{t_{1}, t_{3}}(\Phi)\right)=\Psi_{t_{1}, t_{3}}(\Phi) \tag{E.0.3}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\Psi_{t_{1}, t_{3}}=T_{t_{1}, t_{2}}\left(\Psi_{t_{2}, t_{3}}\right)=\left(\Xi_{t_{2}} \circ T_{t_{1}-t_{2}, 0} \circ \Xi_{-t_{2}}\right)\left(\Psi_{t_{2}, t_{3}}\right) \tag{E.0.4}
\end{equation*}
$$

This expression is almost a function of $\tau=t_{1}-t_{2}$ which is what we need to define the FSS group. However, it has an asymmetry where $t_{2}$ is composed on both sides. Let us thus rewrite this equation using $t_{2}=-\tau+t_{1}$ and thus $\Xi_{t_{2}}=\Xi_{t_{1}-\tau}=\Xi_{t_{1}} \circ \Xi_{-\tau}$. We then obtain:

$$
\begin{equation*}
T_{t_{1}, t_{2}}=\Xi_{t_{1}} \circ \Xi_{-\tau} \circ T_{\tau, 0} \circ \Xi_{-t_{2}}=\Xi_{t_{1}} \circ \widehat{T}_{\tau} \circ \Xi_{-t_{2}} \tag{E.0.5}
\end{equation*}
$$

This last equation thus looks more balanced with $t_{2}$ on the left and $t_{1}$ on the right similar to how $T_{t_{1}, t_{2}}$ is arranged. A natural question at this point is thus: how does the transitivity of $T$ affect $\widehat{T}$ ? We thus
compose two transformations:

$$
\begin{align*}
T_{t_{1}, t_{2}} \circ T_{t_{2}, t_{3}} & =\left(\Xi_{t_{1}} \circ \widehat{T}_{t_{1}-t_{2}} \circ \Xi_{-t_{2}}\right) \circ\left(\Xi_{t_{2}} \circ \widehat{T}_{t_{2}-t_{3}} \circ \Xi_{-t_{3}}\right)  \tag{E.0.6}\\
& =\Xi_{t_{1}} \circ \widehat{T}_{t_{1}-t_{2}} \circ \widehat{T}_{t_{2}-t_{3}} \circ \Xi_{-t_{3}}
\end{align*}
$$

However, we also have by transitivity:

$$
\begin{equation*}
T_{t_{1}, t_{2}} \circ T_{t_{2}, t_{3}}=T_{t_{1}, t_{3}}=\Xi_{t_{1}} \circ \widehat{T}_{t_{1}-t_{3}} \circ \Xi_{-t_{3}} \tag{E.0.7}
\end{equation*}
$$

which then leads us to conclude:

$$
\begin{equation*}
\widehat{T}_{t_{1}-t_{2}} \circ \widehat{T}_{t_{2}-t_{3}}=\widehat{T}_{t_{1}-t_{3}} \tag{E.0.8}
\end{equation*}
$$

which is indeed an FSS group law. We may also notice that the definition of $\widehat{T}_{t}=\Xi_{-t} \circ T_{t, 0}$ is what we might have expected as it performs an RG map followed by a rescaling such that the theory is always defined on the same domain of application. It is then possible to compare theories along the RG flow and in particular fixed-points are then conceivable. As such, we have proved that field theories have an inherent FSS property. We may then iterate this new FSS group law an arbitrary number of times $t_{j}=t_{i}+j\left(t_{f}-t_{i}\right) / N=t_{i}+j \delta t / N$ which would then allow us to compute $\widehat{\Psi}_{t_{i}, t_{f}}$ as $\widehat{\Psi}_{t_{i}, t_{f}}(\Phi)=T_{\delta}^{N} \widehat{\Psi}_{t_{N-1}, t_{N}}$ and $t_{i}=k_{i}$ and $k_{N}=k_{f}$. In turn the infinitesimal generator of $\widehat{T}_{\delta}$ can be decomposed into the generator of $T$ followed by a linear rescaling term. In order to avoid dwelling onto more generalities we will consider the linear rescaling terms once we start looking at concrete examples.

## Appendix F

## Derivative expansion without an underlying effective action

In this section we show how the $\widehat{\gamma}_{n}$ of Sec. 2.4.3) are related to the terms in the derivative expansion. First let us recall the notations of Sec. 2.4 .3 :

$$
\begin{gather*}
\sigma_{s, n}\left(X_{1}, X_{2}, \ldots, X_{n}\right)=\sum_{1 \leq i_{1}<i_{2} \ldots<i_{s} \leq n} X_{i_{1}} X_{i_{2}} \ldots X_{i_{s}}  \tag{F.0.1}\\
\gamma_{2}\left(p ; \phi_{0}\right)=U^{\prime \prime}\left(\phi_{0}\right)+\widehat{\gamma}_{2}\left(p^{2} ; \phi_{0}\right) \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= \\
U^{(3)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}}\left(S_{3} \widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)\right)+\widehat{\gamma}_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)  \tag{F.0.2}\\
\left.\gamma_{4}, p_{3} ; \phi_{0}\right)= \\
=U^{(4)}\left(\phi_{0}\right)+\frac{1}{2} \partial_{\phi_{0}}^{2}\left(S_{4}\left(\widehat{\gamma}_{2}\left(p_{1}^{2} ; \phi_{0}\right)+\frac{1}{2}(1-c) \widehat{\gamma}_{2}\left(\left(p_{1}+p_{2}\right)^{2} ; \phi_{0}\right)\right)\right)+  \tag{F.0.3}\\
\\
\frac{1}{3} \partial_{\phi_{0}} S_{4} \widehat{\gamma}_{3}\left(p_{1}, p_{2}\right)+\widehat{\gamma}_{4}\left(p_{1}, p_{2}, p_{3}\right) \\
\mathcal{S}_{n} f\left(p_{1}, p_{2}, \ldots, p_{h}\right)=\sum_{r \in I[\{1, \ldots, h\},\{1, \ldots, n\}]} f\left(p_{\left.r(1), p_{r(2)}, \ldots, p_{r(h)}\right)}\right.
\end{gather*}
$$

where we have changed notation from $S_{n}$ to $\mathcal{S}_{n}$ as it will be convenient later. We also recall the results from the effective action ansatz method. The order 4 ansatz is:

$$
\begin{align*}
\Gamma_{k}= & \int U(\phi)+\frac{1}{2} Z(\phi) \partial_{\nu} \phi \partial_{\nu} \phi+\frac{1}{2} W_{1}(\phi)\left(\partial_{\nu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \partial_{\mu} \phi\right)+\frac{1}{2} W_{2}(\phi)\left(\partial_{\mu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)+  \tag{F.0.4}\\
& \frac{1}{4} W_{3}(\phi)\left(\partial_{\mu} \phi\right)\left(\partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)
\end{align*}
$$

Prior to imposing momentum conservation, the functional derivatives of $\Gamma_{k}$ reads:

$$
\begin{align*}
\Gamma\left[\phi_{0}\right]= & \mathcal{V} U\left(\phi_{0}\right) \\
\Gamma^{(1)}\left(p ; \phi_{0}\right)= & U^{\prime}\left(\phi_{0}\right) \\
\Gamma^{(2)}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)-Z\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2} \\
\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{\prime \prime \prime}\left(\phi_{0}\right)-Z^{\prime}\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}^{\prime}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2}+W_{2}\left(\phi_{0}\right) p_{1}^{2} p_{2} \cdot p_{3}+\mathrm{perm}  \tag{F.0.5}\\
\Gamma^{(4)}\left(p_{1}, p_{2}, p_{3}, p_{4} ; \phi_{0}\right)= & U^{\prime \prime \prime \prime}\left(\phi_{0}\right)-Z^{\prime \prime}\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}^{\prime \prime}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2}+W_{2}^{\prime}\left(\phi_{0}\right) p_{1}^{2} p_{2} \cdot p_{3}+ \\
& W_{3}\left(\phi_{0}\right) p_{1} \cdot p_{2} p_{3} \cdot p_{4}+\operatorname{perm} \\
\Gamma^{(5)}\left(p_{1}, p_{2}, p_{3}, p_{4}, p_{5} ; \phi_{0}\right)= & U^{(5)}\left(\phi_{0}\right)-Z^{\prime \prime \prime}\left(\phi_{0}\right) p_{1} \cdot p_{2}+W_{1}^{\prime \prime \prime}\left(\phi_{0}\right)\left(p_{1} \cdot p_{2}\right)^{2}+W_{2}^{\prime \prime}\left(\phi_{0}\right) p_{1}^{2} p_{2} \cdot p_{3}+ \\
& W_{3}^{\prime}\left(\phi_{0}\right) p_{1} \cdot p_{2} p_{3} \cdot p_{4}+\operatorname{perm}
\end{align*}
$$

After imposing momentum conservation we obtain:

$$
\begin{align*}
\gamma_{0}= & \mathcal{V} U\left(\phi_{0}\right) \\
\gamma_{1}= & U^{\prime}\left(\phi_{0}\right) \\
\gamma_{2}\left(p ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)+Z(\phi) p^{2}+W_{1}(\phi) p^{4} \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{\prime \prime \prime}\left(\phi_{0}\right)+Z^{\prime}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}+p_{1} \cdot p_{2}\right)+ \\
& W_{1}^{\prime}\left(\phi_{0}\right)\left(p_{1}^{4}+2 p_{1}^{2} p_{1} \cdot p_{2}+3\left(p_{1} \cdot p_{2}\right)^{2}+2 p_{2}^{2} p_{1} \cdot p_{2}+p_{2}^{4}\right)+ \\
& W_{2}\left(\phi_{0}\right)\left(\left(p_{1} \cdot p_{2}\right)^{2}-p_{1}^{2} p_{2}^{2}\right)  \tag{F.0.6}\\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{\prime \prime \prime \prime}\left(\phi_{0}\right)+Z^{\prime \prime}\left(\phi_{0}\right)\left(p_{1}^{2}+p_{2}^{2}+p_{3}^{2}+p_{1} \cdot p_{2}+p_{1} \cdot p_{3}+p_{3} \cdot p_{2}\right)+ \\
& W_{1}^{\prime \prime}\left(\phi_{0}\right)\left(p_{1}^{4}+2 p_{1}^{2} p_{1} \cdot p_{2}+2 p_{1} \cdot p_{2} p_{1} \cdot p_{3}+3\left(p_{1} \cdot p_{2}\right)^{2}+\text { perm }\right)+ \\
& W_{2}^{\prime}\left(\phi_{0}\right)\left(\left(p_{1} \cdot p_{2}\right)^{2}-p_{1}^{2} p_{2}^{2}+2\left(p_{1} \cdot p_{2}\right)\left(p_{2} \cdot p_{3}\right)+\text { perm }\right)+ \\
& -W_{3}\left(\phi_{0}\right)\left(p_{1}^{2} p_{2} \cdot p_{3}+p_{2}^{2} p_{1} \cdot p_{3}+p_{3}^{2} p_{1} \cdot p_{2}+\right. \\
& \left.2\left(p_{1} \cdot p_{2}\right)\left(p_{1} \cdot p_{3}\right)+2\left(p_{1} \cdot p_{2}\right)\left(p_{2} \cdot p_{3}\right)+2\left(p_{1} \cdot p_{3}\right)\left(p_{2} \cdot p_{3}\right)\right)
\end{align*}
$$

The objective is then to retrieve these terms without invoking an ansatz for the effective action.
Let us then consider a natural extension of the symmetric polynomials for $d>1$. First notice that:

$$
\begin{equation*}
\sigma_{h, n}=\mathcal{S}_{n} \cdot X_{1} X_{2} \ldots X_{h} \tag{F.0.7}
\end{equation*}
$$

In particular we have:

$$
\begin{align*}
& \sigma_{1,3}=\mathcal{S}_{3} \cdot p_{1} \\
& \sigma_{2,3}=\mathcal{S}_{3} \cdot p_{1} p_{2}  \tag{F.0.8}\\
& \sigma_{3,3}=p_{1} p_{2} p_{3} .
\end{align*}
$$

The natural extension of $\sigma_{1, n}$ to $d>1$ is :

$$
\begin{equation*}
\vec{\sigma}_{1, n}\left(\vec{p}_{1}, \vec{p}_{2}, \ldots, \vec{p}_{n}\right)=\sum_{i} \vec{p}_{i} \tag{F.0.9}
\end{equation*}
$$

Such that $\vec{\sigma}_{1, n}=0$ implies momentum conservation. In the case of $\sigma_{2, n}$, the natural extension is :

$$
\begin{equation*}
\dot{\sigma}_{2, n}\left(\vec{p}_{1}, \vec{p}_{2}, \ldots, \vec{p}_{n}\right)=\mathcal{S}_{n} \cdot \vec{p}_{1} \cdot \vec{p}_{2} \tag{F.0.10}
\end{equation*}
$$

where "." represents the Euclidean scalar product. In the case of $\sigma_{3,3}$ there is an ambiguity concerning which multiplication operator should be promoted to a scalar product. We then define a promotion operator $\mathcal{P}_{n}$ which gives the set of all possible contraction of indices. For $\sigma_{3,3}$ we have:

$$
\mathcal{P}_{3} \cdot \sigma_{3,3}=\left\{\vec{p}_{1} \cdot \overrightarrow{p_{2}} \overrightarrow{p_{3}},\left(\vec{p}_{1} \cdot \vec{p}_{3}\right) \vec{p}_{2},\left(\vec{p}_{2} \cdot \vec{p}_{3}\right) \vec{p}_{1}\right\}
$$

which we denote more synthetically as:

$$
\mathcal{P}_{3} \cdot \sigma_{3,3}=\mathcal{S}_{3} \cdot\left\{\vec{p}_{1} \cdot \overrightarrow{p_{2}} \overrightarrow{p_{3}}\right\}
$$

As such the natural extension of $\sigma_{3,3}$ for $d>1$ is to consider the average of all promotions:

$$
\begin{equation*}
\vec{\sigma}_{3,3}\left(\vec{p}_{1}, \vec{p}_{2}, \vec{p}_{3}\right)=\frac{1}{3}\left(\left(\vec{p}_{1} \cdot \vec{p}_{2}\right) \vec{p}_{3}+\left(\vec{p}_{1} \cdot \vec{p}_{3}\right) \vec{p}_{2}+\left(\vec{p}_{2} \cdot \vec{p}_{3}\right) \vec{p}_{1}\right) \tag{F.0.11}
\end{equation*}
$$

The factor $1 / 3$ is needed so that we retrieve the usual $\sigma_{3,3}$ in the limit $d \rightarrow 1$. In the following we will leave normalization factors to be implicit and we will write $\vec{\sigma}_{3,3}$ more synthetically as:

$$
\begin{equation*}
\vec{\sigma}_{3,3}\left(\vec{p}_{1}, \vec{p}_{2}, \vec{p}_{3}\right)=\mathcal{S}_{3} \cdot\left(\vec{p}_{1} \cdot \vec{p}_{2}\right) \vec{p}_{3} \tag{F.0.12}
\end{equation*}
$$

We then define $\mathcal{S P}{ }_{n}$ which takes the average of promotions of a $d=1$ monomial. Thus we have:

$$
\begin{equation*}
\vec{\sigma}_{3,3}=\mathcal{S} \mathcal{P}_{3} \sigma_{3,3} \tag{F.0.13}
\end{equation*}
$$

In general, for $d=1$ we have :

$$
\sigma_{3, n}\left(p_{1}, p_{2}, \ldots, p_{n}\right)=\mathcal{S}_{n} \cdot p_{1} p_{2} p_{3}
$$

Thus the generalization to $d>1$ is then:

$$
\vec{\sigma}_{3, n}\left(\vec{p}_{1}, \vec{p}_{2}, \ldots, \vec{p}_{n}\right)=\mathcal{S}_{n} \cdot\left(\vec{p}_{1} \cdot \vec{p}_{2}\right) \vec{p}_{3}=\mathcal{S} \mathcal{P}_{3} \sigma_{3, n}
$$

where in $\mathcal{S}_{n} \cdot\left(\vec{p}_{1} \cdot \vec{p}_{2}\right) \vec{p}_{3}$ the normalization factor is again left implicit.
Similarly we have:

$$
\begin{equation*}
\dot{\sigma}_{4, n}\left(\vec{p}_{1}, \vec{p}_{2}, \ldots, \vec{p}_{n}\right)=\mathcal{S}_{n} \cdot\left(\vec{p}_{1} \cdot \vec{p}_{2}\right)\left(\vec{p}_{3} \cdot \vec{p}_{4}\right)=\mathcal{S} \mathcal{P}_{4} \sigma_{4, n} \tag{F.0.14}
\end{equation*}
$$

and

$$
\vec{\sigma}_{5, n}\left(\vec{p}_{1}, \vec{p}_{2}, \ldots, \vec{p}_{n}\right)=\mathcal{S}_{n} \cdot\left(\vec{p}_{1} \cdot \vec{p}_{2}\right)\left(\vec{p}_{3} \cdot \vec{p}_{4}\right) \vec{p}_{5}=\mathcal{S} \mathcal{P}_{5} \sigma_{5, n}
$$

It is then apparent that the promotion of $\sigma_{h, n}$ is a vector when $h$ is odd and a scalar when $h$ is even ${ }^{1}$, When the promotion of a polynomial from $d=1$ to $d>1$ leads to a vector (resp. a scalar) we shall call it a v-polynomial (resp. a s-polynomial). The $\gamma_{n}$ are of course s-polynomials but they may be obtained from scalar products of v-polynomials. When $d=1$, any polynomial truncation $P_{n}$ of the $\gamma_{n}$ may be obtained as a polynomial $Q_{n}$ in the $\sigma_{h, n}$. To generalize this we introduce a "demotion" operator $\mathcal{D}$ that replaces all vectors by scalars and all scalar products by multiplication operators. The operator $\mathcal{D}$ thereby essentially takes the limit $d \rightarrow 1$ of a s-polynomial or v-polynomial.

Consider then a generic s-polynomial $A$ of $n$ variables. According to the fundamental theorem of symmetric polynomials there exists a polynomial $Q$ such that:

$$
\begin{equation*}
\mathcal{D} A=Q\left(\sigma_{1, n}, \sigma_{2, n}, \ldots, \sigma_{n, n}\right) \tag{F.0.15}
\end{equation*}
$$

We then consider the decomposition:

$$
\begin{equation*}
A=\mathcal{S} \mathcal{P}_{\sigma} Q+R \tag{F.0.16}
\end{equation*}
$$

where $\mathcal{S} \mathcal{P}_{\sigma}$ gives an averaged promotion of $Q$ where the averages are obtained directly in terms of the $\sigma_{h, n}$ rather than from the underlying monomials $p_{1}^{m_{1}} p_{2}^{m_{2}} \ldots p_{n}^{m_{n}}$. For example:

$$
\begin{equation*}
\mathcal{S} \mathcal{P}_{\sigma} \sigma_{2, n}^{2} \sigma_{3, n}^{2} \sigma_{5, n}^{2}=\dot{\sigma}_{2, n}^{2} \mathcal{S} \mathcal{P}_{\sigma}\left(\sigma_{3, n}^{2} \sigma_{5, n}^{2}\right)=\frac{\dot{\sigma}_{2, n}^{2}}{2}\left(\left(\vec{\sigma}_{3, n} \cdot \vec{\sigma}_{5, n}\right)^{2}+\left(\vec{\sigma}_{3, n} \cdot \vec{\sigma}_{3, n}\right)\left(\vec{\sigma}_{5, n} \cdot \vec{\sigma}_{5, n}\right)\right) \tag{F.0.17}
\end{equation*}
$$

If we then apply $\mathcal{D}$ to the decomposition of $A$ we obtain:

$$
\begin{equation*}
\mathcal{D} R=0 \tag{F.0.18}
\end{equation*}
$$

$R$ is then non zero when there is a mixing between scalar products and multiplication operators. In the following we will denote basis elements of $R$ as $\bar{\sigma}_{m, n}$ where $m$ refers to the degree of the polynomial and $n$ to the number of variables. A typical example is

$$
\begin{equation*}
\bar{\sigma}_{4,2}=\left(p_{1} \cdot p_{2}\right)^{2}-\left(p_{1} \cdot p_{1}\right)\left(p_{2} \cdot p_{2}\right) \tag{F.0.19}
\end{equation*}
$$

which is indeed null when $d=1$. From the perspective of the derivative expansion with an effective action ansatz, s- or v-polynomials that vanish under $\mathcal{D}$ are obtained from terms in the effective action which become total derivatives in the limit $d \rightarrow 1$. An example is $\left(\partial_{\mu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)$ which in the limit $d \rightarrow 1$ leads to $\phi^{\prime \prime} \phi^{\prime 2}=\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} x}\left(\phi^{\prime 3}\right)$. Notice then that when $d=1$, an order 4 truncation of $\widehat{\gamma}_{3}$ is of the form:

$$
\begin{equation*}
A \sigma_{1,3} \sigma_{3,3} \tag{F.0.20}
\end{equation*}
$$

Indeed, by defintion of $\widehat{\gamma}_{3}$, it should be set to zero when replacing any of the momenta by zero and thus it must be proportional to $\sigma_{3,3}$. A term $\sigma_{1,3} \sigma_{3,3}$ is null after momentum conservation and thus when $d>1$ any contribution to $\widehat{\gamma}_{3}$ must be set to zero when $\mathcal{D}$ is applied.

The s-monomials of degree four which may belong to $\widehat{\gamma}_{3}\left(p_{1}, p_{2}\right)$ are

$$
\begin{equation*}
\left\{\left(p_{1} \cdot p_{1}\right)\left(p_{2} \cdot p_{2}\right),\left(p_{1} \cdot p_{2}\right)^{2}\right\} \tag{F.0.21}
\end{equation*}
$$

[^66]as they are set to zero by replacing $p_{1}$ or $p_{2}$ by zero. However, while necessary this condition is insufficient. Prior to momentum conservation a polynomial of $\widehat{\gamma}_{3}$ is of the form:
\[

$$
\begin{equation*}
P\left(p_{1}, p_{2}, p_{3}\right) \tag{F.0.22}
\end{equation*}
$$

\]

where $P$ is symmetric with respect to $\left(p_{1}, p_{2}, p_{3}\right)$ and setting $p_{1}, p_{2}$ or $p_{3}$ to zero should set this polynomial to zero. Thus, after momentum conservation, elements of $\widehat{\gamma}_{3}$ are of the form :

$$
\begin{equation*}
P\left(p_{1}, p_{2},-p_{1}-p_{2}\right) \tag{F.0.23}
\end{equation*}
$$

Which may also be set to zero by taking $p_{1}=-p_{2}$. This is the necessary and suficient condition for s-polynomials that may arise in the decomposition of $\widehat{\gamma}_{3}$. This is not the case of $\left(p_{1} \cdot p_{1}\right)\left(p_{2} . p_{2}\right)$ or $\left(p_{1} \cdot p_{2}\right)^{2}$ but it may be the case of a combination of the two. As mentioned before, any element of $\widehat{\gamma}_{3}$ of degree four is set to zero by $\mathcal{D}$ and thus the only viable combination is :

$$
\begin{equation*}
\bar{\sigma}_{4,2}=\left(p_{1} \cdot p_{1}\right)\left(p_{2} \cdot p_{2}\right)-\left(p_{1} \cdot p_{2}\right)^{2} \tag{F.0.24}
\end{equation*}
$$

If we apply $\mathcal{S}_{3}$ to this combination then impose momentum conservation we retrieve the same combination up to a normalization. Moreover, setting $p_{1}=-p_{2}$ indeed sets this term to zero such that it indeed belongs to $\widehat{\gamma}_{3}$. Thus, we have the following expressions at order 4 of the derivative expansion:

$$
\begin{align*}
\widehat{\gamma}_{2}\left(p ; \phi_{0}\right) & =Z\left(\phi_{0}\right) p^{2}+W_{1}\left(\phi_{0}\right) p^{4} \\
\widehat{\gamma}_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) & =W_{2}\left(\phi_{0}\right)\left(p_{1}^{2} p_{2}^{2}-\left(p_{1} \cdot p_{2}\right)^{2}\right) \tag{F.0.25}
\end{align*}
$$

Using Eq. F.0.2 we then deduce:

$$
\begin{align*}
\gamma_{2}\left(p ; \phi_{0}\right) & =U^{\prime \prime}\left(\phi_{0}\right)+Z\left(\phi_{0}\right) p^{2}+W_{1}\left(\phi_{0}\right) p^{4} \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right) & =U^{(3)}\left(\phi_{0}\right)+\frac{1}{2}\left(Z^{\prime}\left(\phi_{0}\right) S_{3} p_{1}^{2}+W_{1}^{\prime}\left(\phi_{0}\right) S_{3} p_{1}^{4}\right)+W_{2}\left(\phi_{0}\right)\left(p_{1}^{2} p_{2}^{2}-\left(p_{1} \cdot p_{2}\right)^{2}\right) \tag{F.0.26}
\end{align*}
$$

The above expansion leads to that of Eq. F.0.6 by a simple redefinition of $W_{2}$ as $W_{2} \rightarrow-W_{2}-W_{1}^{\prime}$. Of course, there is no need to take such a redefinition and one may simply compute the flow equations using Eq. F.0.26 to find equivalent results. As $\widehat{\gamma}_{4} \propto p_{1} p_{2} p_{3} p_{4}=\sigma_{4,4}$ and $\widehat{\gamma}_{5} \propto p_{1} p_{2} p_{3} p_{4} p_{5}$ it is sufficient to compute the Taylor expansion of $\widehat{\gamma}_{4}$ and set the $\widehat{\gamma}_{h>4}$ to zero in order to retrieve the order 4 of the derivative expansion. For $d=1, \widehat{\gamma}_{4}$ is readily given by $\sigma_{4,4}$ which is then promoted to :

$$
\begin{equation*}
\dot{\sigma}_{4,4}=\mathcal{S}_{4} p_{1} \cdot p_{2} p_{3} \cdot p_{4}=\frac{1}{3}\left(p_{1} \cdot p_{2} p_{3} \cdot p_{4}+p_{1} \cdot p_{3} p_{2} \cdot p_{4}+p_{1} \cdot p_{4} p_{2} \cdot p_{3}\right)=\mathcal{S} \mathcal{P}_{4} \sigma_{4,4} \tag{F.0.27}
\end{equation*}
$$

There are no other symmetric combinations of momenta for $d>1$ and thus we find the following expansion:

$$
\begin{align*}
\gamma_{2}\left(p ; \phi_{0}\right)= & U^{\prime \prime}\left(\phi_{0}\right)+Z\left(\phi_{0}\right) p^{2}+W_{1}\left(\phi_{0}\right) p^{4} \\
\gamma_{3}\left(p_{1}, p_{2} ; \phi_{0}\right)= & U^{(3)}\left(\phi_{0}\right)+\frac{1}{2}\left(Z^{\prime}\left(\phi_{0}\right) S_{3} p_{1}^{2}+W_{1}^{\prime}\left(\phi_{0}\right) S_{3} p_{1}^{4}\right)+W_{2}\left(\phi_{0}\right)\left(p_{1}^{2} p_{2}^{2}-\left(p_{1} \cdot p_{2}\right)^{2}\right) \\
= & U^{(3)}\left(\phi_{0}\right)-Z^{\prime}\left(\phi_{0}\right) \sigma_{2,3}+W_{1}^{\prime}\left(\phi_{0}\right) \sigma_{2,3}^{2}+W_{2}\left(\phi_{0}\right) \bar{\sigma}_{4,3} \\
\gamma_{4}\left(p_{1}, p_{2}, p_{3} ; \phi_{0}\right)= & U^{(4)}\left(\phi_{0}\right)+\frac{1}{2}\left(Z^{\prime \prime}\left(\phi_{0}\right) S_{4} p_{1}^{2}+W_{1}^{\prime \prime}\left(\phi_{0}\right) S_{4} p_{1}^{4}\right)+\frac{1}{3} W_{2}^{\prime}\left(\phi_{0}\right) S_{4}\left(p_{1}^{2} p_{2}^{2}-\left(p_{1} \cdot p_{2}\right)^{2}\right)  \tag{F.0.28}\\
& W_{3}\left(\phi_{0}\right) \sigma_{4,4}\left(p_{1}, p_{2}, p_{3},-p_{1}-p_{2}-p_{3}\right) \\
= & U^{(4)}\left(\phi_{0}\right)-Z^{\prime \prime}\left(\phi_{0}\right) \sigma_{2,4}+W_{1}^{\prime \prime}\left(\phi_{0}\right) \sigma_{2,4}^{2}+W_{3}\left(\phi_{0}\right) \bar{\sigma}_{4,4}+W_{4}\left(\phi_{0}\right) \sigma_{4,4}
\end{align*}
$$

where we have introduced the notation :

$$
\begin{equation*}
\bar{\sigma}_{4, n}=\frac{1}{3} \mathcal{S}_{n}\left(p_{1}^{2} p_{2}^{2}-\left(p_{1} \cdot p_{2}\right)^{2}\right) \tag{F.0.29}
\end{equation*}
$$

Eq. (F.0.28 is equivalent to Eq. F.0.6) up to redefinitions $W_{2} \rightarrow-W_{2}-W_{1}^{\prime}$ and $W_{3} \rightarrow 3 W_{3}-2 W_{2}^{\prime}$.
Above the order six of the derivative expansion, the number of independent terms becomes quite large and a formal systematic procedure is needed to find the polynomial basis of the $\widehat{\gamma}_{n}$. In the list below we define the notations we shall use, some of which have already been introduced:
$\mathfrak{S}_{n}:$ the space of symmetric $\mathrm{s}, \mathrm{v}$-polynomials of $n$ variables.
$\mathcal{H}_{D} \mathfrak{S}_{n}$ : the space of symmetric s,v-polynomials of $n$ variables and total degree D that are homogeneous in the sense that all monomials in the decomposition of the polynomial have the same total degree $D$.
$\mathcal{M}_{X_{i}}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n-1}:$ the projector operator which sets the variable $X_{i}$ to zero.
$\mathcal{C} \mathfrak{S}_{n}$ : the space of symmetric s,v-polynomials of $n$ variables where all monomials are complete in the sense that they contain all $n$ variables and are thereby canceled by $\mathcal{M}_{X_{i}}$.
$\mathcal{C H} \mathcal{D}_{D} \mathfrak{S}_{n}$ : the space of symmetric homogeneous s,v-polynomials of $n$ variables and total degree $D$ where all monomials are complete in the sense that they contain all $n$ variables .
$\mathcal{M}_{n}^{c}=\mathcal{M}_{\sum_{i=1}^{n} X_{i}}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n-1}:$ the momentum conservation operator which sets the variable $X_{n}$ to $-\sum_{i=1}^{n-1} X_{i}$.
$\widehat{\mathcal{N}}=\prod_{i=1}^{n}\left(I_{n}-\mathcal{M}_{X_{i}}\right): \mathcal{H}_{d} \mathfrak{S}_{n} \rightarrow \mathcal{C} \mathcal{H}_{d} \mathfrak{S}_{n}: I_{n}$ is the identity operator and $\widehat{\mathcal{N}}$ removes all non complete terms from a polynomial.
$\mathcal{S}_{n+1}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n+1}$ : the symmetrization operator which symmetrizes with respect to an independent variable $X_{n+1}$.
$S_{n+1}=\mathcal{M}_{n+1}^{c} \mathcal{S}_{n+1}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ : the operator which symmetrizes with respect to $X_{n}=-\sum_{i=1}^{n-1} X_{i}$.
$\mathbb{S}_{n}$ : the space of regular $(d=1)$ symmetric polynomials.
$\mathcal{D}_{n}: \mathfrak{S}_{n} \rightarrow \mathbb{S}_{n}:$ the demotion operator which replaces vectors by scalars and dot products by multiplication.
$\mathfrak{P}\left(\mathfrak{S}_{n}\right)$ : the space of all subsets of the elements of $\mathfrak{S}_{n}$.
$\mathcal{P}_{n}: \mathbb{S}_{n} \rightarrow \mathfrak{P}\left(\mathfrak{S}_{n}\right)$ : inverse application of the demotion operator which retrieves the pre-image of an element of $\mathbb{S}_{n}$.
$\mathcal{S} \mathcal{P}_{n}: \mathbb{S}_{n} \rightarrow \mathfrak{S}_{n}:$ the average of the elements formed by $\mathcal{P}_{n}$.
The polynomial truncations of the $\widehat{\gamma}_{n}$ belong to the space $\mathcal{C} \Im_{n}$. Each order of the derivative expansion introduces new polynomial terms. At order $D$ of the derivative expansion and prior to momentum conservation, the novel terms belong to the vector space $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}$. After momentum conservation they belong to the vector space $\operatorname{Im}\left(\left.\mathcal{M}_{n}^{c}\right|_{\mathcal{C H}} ^{D} \mathfrak{S}_{n}\right) \subset \mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n-1}$, where $\left.\mathcal{M}_{n}^{c}\right|_{\mathcal{C H}} ^{D} \mathfrak{S}_{n}$ is the restriction of $\mathcal{M}_{n}^{c}$ to $\mathcal{C H} \mathcal{H}_{D} \mathfrak{S}_{n}$. The number of independent terms to include at a given order of the derivative expansion is then given by the rank of the linear application $\left.\mathcal{M}_{n}^{c}\right|_{\mathcal{C H}_{D} \mathfrak{S}_{n}}$. In the following as we shall mainly work with the restriction of $\mathcal{M}_{n}^{c}$ to $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}$, we will use the notation $\mathcal{M}_{n}^{c}$ rather than $\left.\mathcal{M}_{n}^{c}\right|_{\mathcal{C H}} ^{D} \mathfrak{S}_{n}$ in order to simplify notations.

As the spaces $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}$ and $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n-1}$ are finite dimensional vector spaces there are numerous ways to find the rank of $\mathcal{M}_{n}^{c}$. The most straightforward method is to construct the matrix of $\mathcal{M}_{n}^{c}$ by applying $\mathcal{M}_{n}^{c}$ to a basis set of monomials of $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}$. This can be implemented straightforwardly on a computer but it can be quite tedious to implement by hand. The following two methods generally involve less calculations. First we remark that the dimension of $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}$ increases for $n<D / 2$ then decreases for $n>D / 2$. Thus, for $n>D / 2$ the dimension of the base space of $\mathcal{M}_{n}^{c}$, that is $\mathcal{C H} \mathcal{H}_{n}$ is smaller than the dimension of the target space $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n-1}$. It is then less computationally intensive to find the rank of $\mathcal{M}_{n}^{c}$ by computing the dimension of $\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)$ using the rank-nullity theorem.

Within the framework of the ansatz method, $\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)$ is given by the set of total derivatives which after imposing momentum conservation leads to the zero vector. Let us also note that when the background field configuration is uniform, the terms in the effective action that contribute to $\widehat{\gamma}_{n}$ are those which involve $n$ field derivative terms. For example, $\partial^{2} \phi \partial^{4} \phi(\partial \phi)^{4}$ has 6 field derivative terms and contributes only to $\widehat{\gamma}_{6}$. Let us then define $J_{n}^{\mu}(\phi)$ to be a purely derivative term with $n$ derivative field terms. For example $J_{6}^{\mu}=\partial^{2} \phi \partial^{2} \partial^{\mu} \phi(\partial \phi)^{4}$, then we have:

$$
\begin{equation*}
\partial_{\mu}\left(F(\phi(x)) J_{n}^{\mu}(\phi)\right)=F^{\prime}(\phi(x)) \partial_{\mu}(\phi(x)) J_{n}^{\mu}(\phi)+F(\phi(x)) \partial_{\mu} J_{n}^{\mu}(\phi) \tag{F.0.30}
\end{equation*}
$$

The term $F^{\prime}(\phi(x)) \partial_{\mu}(\phi(x)) J_{n}^{\mu}(\phi)$ has more field derivative terms and so it contributes to $\widehat{\gamma}_{n+1}$ rather than $\widehat{\gamma}_{n}$. As such it is sufficient to focus on $F(\phi(x)) \partial_{\mu} J_{n}^{\mu}(\phi)$ to find the kernel of $\mathcal{M}_{n}^{c}$. Moreover, as the
background field is uniform, the $n$ functional derivatives must operate only on $\partial_{\mu} J_{n}^{\mu}(\phi)$ or else any remaining derivative term from $J_{n}^{\mu}(\phi)$ would lead to zero after taking $\phi$ to be uniform. Thus the $x$ dependence in $F(\phi(x))$ is irrelevant and as such we will set $F(\phi(x))$ directly to $F\left(\phi_{0}\right)$. Consider now as an example $J_{n}^{\mu}(\phi)=\partial^{2} \partial^{\mu} \phi \partial^{4} \phi \partial^{2} \phi$. Then we have:

$$
\begin{equation*}
F\left(\phi_{0}\right) \partial_{\mu} J_{n}^{\mu}(\phi)=F\left(\phi_{0}\right)\left(\left(\partial^{2} \partial_{\mu} \partial^{\mu} \phi\right) \partial^{4} \phi \partial^{2} \phi+\partial^{2} \partial^{\mu} \phi \partial^{4} \partial_{\mu} \phi \partial^{2} \phi+\partial^{2} \partial^{\mu} \phi \partial^{4} \phi \partial^{2} \partial_{\mu} \phi\right) \tag{F.0.31}
\end{equation*}
$$

Inserting this expression into the effective action and taking a 3rd order functional derivative in Fourier space leads to:

$$
\begin{align*}
\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3}\right)= & -W\left(\phi_{0}\right)\left(p_{1}^{4} p_{2}^{4} p_{3}^{2}+p_{1} \cdot p_{2} p_{1}^{2} p_{2}^{4} p_{3}^{2}+p_{1} \cdot p_{3} p_{1}^{2} p_{2}^{4} p_{3}^{2}\right) \\
& -W\left(\phi_{0}\right)\left(p_{1}^{4} p_{3}^{4} p_{2}^{2}+p_{1} \cdot p_{3} p_{1}^{2} p_{3}^{4} p_{2}^{2}+p_{1} \cdot p_{2} p_{1}^{2} p_{3}^{4} p_{2}^{2}\right), \quad(2 \longleftrightarrow 3) \\
& -W\left(\phi_{0}\right)\left(p_{1}^{4} p_{2}^{4} p_{3}^{2}+p_{1} \cdot p_{2} p_{2}^{2} p_{1}^{4} p_{3}^{2}+p_{2} \cdot p_{3} p_{2}^{2} p_{1}^{4} p_{3}^{2}\right), \quad(1 \longleftrightarrow 2)  \tag{F.0.32}\\
& -W\left(\phi_{0}\right)\left(p_{3}^{4} p_{2}^{4} p_{1}^{2}+p_{3} \cdot p_{2} p_{3}^{2} p_{2}^{4} p_{1}^{2}+p_{1} \cdot p_{3} p_{1}^{2} p_{2}^{4} p_{3}^{2}\right), \quad(1 \longleftrightarrow 3) \\
& -W\left(\phi_{0}\right)\left(p_{2}^{4} p_{3}^{4} p_{1}^{2}+p_{2} \cdot p_{3} p_{2}^{2} p_{3}^{4} p_{1}^{2}+p_{2} \cdot p_{1} p_{2}^{2} p_{3}^{4} p_{1}^{2}\right), \quad(1 \rightarrow 2,2 \rightarrow 3,3 \rightarrow 1) \\
& -W\left(\phi_{0}\right)\left(p_{3}^{4} p_{1}^{4} p_{2}^{2}+p_{3} \cdot p_{1} p_{3}^{2} p_{1}^{4} p_{2}^{2}+p_{3} \cdot p_{2} p_{3}^{2} p_{1}^{4} p_{2}^{2}\right) \quad(1 \rightarrow 3,2 \rightarrow 1,3 \rightarrow 2)
\end{align*}
$$

where the notation $(2 \longleftrightarrow 3)$ refers to a permutation of $p_{2}$ and $p_{3}$. We may then factorize $p_{1}$ as :

$$
\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3}\right)=-W\left(\phi_{0}\right) p_{1} \cdot\left(p_{1}^{2} p_{2}^{4} p_{3}^{2} p_{1}+p_{1}^{2} p_{3}^{4} p_{2}^{2} p_{1}+p_{2}^{2} p_{1}^{4} p_{3}^{2} p_{2}+p_{3}^{2} p_{2}^{4} p_{1}^{2} p_{3}+p_{2}^{2} p_{3}^{4} p_{1}^{2} p_{2}+p_{3}^{2} p_{1}^{4} p_{2}^{2} p_{3}\right)+\ldots
$$

which was obtained by factoring the colored terms:

$$
\begin{align*}
\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3}\right)= & -W\left(\phi_{0}\right)\left(p_{1}^{4} p_{2}^{4} p_{3}^{2}+p_{1} \cdot p_{2} p_{1}^{2} p_{2}^{4} p_{3}^{2}+p_{1} \cdot p_{3} p_{1}^{2} p_{2}^{4} p_{3}^{2}\right) \\
& -W\left(\phi_{0}\right)\left(p_{1}^{4} p_{3}^{4} p_{2}^{2}+p_{1} \cdot p_{3} p_{1}^{2} p_{3}^{4} p_{2}^{2}+p_{1} \cdot p_{2} p_{1}^{2} p_{3}^{4} p_{2}^{2}\right),(2 \longleftrightarrow 3) \\
& -W\left(\phi_{0}\right)\left(p_{1}^{4} p_{2}^{4} p_{3}^{2}+p_{1} \cdot p_{2} p_{2}^{2} p_{1}^{4} p_{3}^{2}+p_{2} \cdot p_{3} p_{2}^{2} p_{1}^{4} p_{3}^{2}\right),(1 \longleftrightarrow 2)  \tag{F.0.33}\\
& -W\left(\phi_{0}\right)\left(p_{3}^{4} p_{2}^{4} p_{1}^{2}+p_{3} \cdot p_{2} p_{3}^{2} p_{2}^{4} p_{1}^{2}+p_{1} \cdot p_{3} p_{1}^{2} p_{2}^{4} p_{3}^{2}\right),(1 \longleftrightarrow 3) \\
& -W\left(\phi_{0}\right)\left(p_{2}^{4} p_{3}^{4} p_{1}^{2}+p_{2} \cdot p_{3} p_{2}^{2} p_{3}^{4} p_{1}^{2}+p_{2} \cdot p_{1} p_{2}^{2} p_{3}^{4} p_{1}^{2}\right),(1 \rightarrow 2,2 \rightarrow 3,3 \rightarrow 1) \\
& -W\left(\phi_{0}\right)\left(p_{3}^{4} p_{1}^{4} p_{2}^{2}+p_{3} \cdot p_{1} p_{3}^{2} p_{1}^{4} p_{2}^{2}+p_{3} \cdot p_{2} p_{3}^{2} p_{1}^{4} p_{2}^{2}\right)(1 \rightarrow 3,2 \rightarrow 1,3 \rightarrow 2)
\end{align*}
$$

The important point is that the term

$$
\begin{equation*}
p_{1}^{2} p_{2}^{4} p_{3}^{2} p_{1}+p_{1}^{2} p_{3}^{4} p_{2}^{2} p_{1}+p_{2}^{2} p_{1}^{4} p_{3}^{2} p_{2}+p_{3}^{2} p_{2}^{4} p_{1}^{2} p_{3}+p_{2}^{2} p_{3}^{4} p_{1}^{2} p_{2}+p_{3}^{2} p_{1}^{4} p_{2}^{2} p_{3} \tag{F.0.34}
\end{equation*}
$$

is symmetric with respect to permutations of $\left(p_{1}, p_{2}, p_{3}\right)$. Moreover as $\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3}\right)$ is also symmetric with respect to $\left(p_{1}, p_{2}, p_{3}\right)$ it is then necessary that this symmetric term multiplies the symmetrization of $p_{1}$, that is $p_{1}+p_{2}+p_{3}$. Thus we have:

$$
\begin{align*}
\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3}\right)= & -F\left(\phi_{0}\right)\left(p_{1}+p_{2}+p_{3}\right) \cdot\left(p_{1}^{2} p_{2}^{4} p_{3}^{2} p_{1}+p_{1}^{2} p_{3}^{4} p_{2}^{2} p_{1}+p_{2}^{2} p_{1}^{4} p_{3}^{2} p_{2}\right. \\
& \left.p_{3}^{2} p_{2}^{4} p_{1}^{2} p_{3}+p_{2}^{2} p_{3}^{4} p_{1}^{2} p_{2}+p_{3}^{2} p_{1}^{4} p_{2}^{2} p_{3}\right) \tag{F.0.35}
\end{align*}
$$

The total number of terms of this expression is $3^{*} 6$ and thus this expression retains all terms and is an exact rewriting of $\Gamma^{(3)}\left(p_{1}, p_{2}, p_{3}\right)$.

It is then clear that this expression cancels to zero after imposing momentum conservation $p_{1}+p_{2}+p_{3}=0$. We thus wish to prove that this is a general result. Consider then a general term

$$
\begin{equation*}
J^{\mu}=g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} \partial^{\alpha_{1}} \partial^{\mu} \phi \partial^{\alpha_{2}} \phi \ldots \partial^{\alpha_{n}} \phi \tag{F.0.36}
\end{equation*}
$$

where we have used multi-index notation $\alpha_{i}=\left(\alpha_{i, 1}, \alpha_{i, 2}, \ldots, \alpha_{i, m_{i}}\right)$ and $g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}}$ is a tensor that operates contractions between the indices. Then:

$$
\begin{equation*}
\partial_{\mu} J^{\mu}=g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}}\left(\partial^{\alpha_{1}} \partial^{\mu} \partial_{\mu} \phi \partial^{\alpha_{2}} \phi \ldots \partial^{\alpha_{n}} \phi+\partial^{\alpha_{1}} \partial^{\mu} \phi \partial^{\alpha_{2}} \partial_{\mu} \phi \ldots \partial^{\alpha_{n}} \phi+\ldots \partial^{\alpha_{1}} \partial^{\mu} \phi \partial^{\alpha_{2}} \phi \ldots \partial^{\alpha_{n}} \phi\right) \tag{F.0.37}
\end{equation*}
$$

Taking $n^{\text {nth }}$ order functional derivatives, going into Fourier space and omitting powers of the imaginary number $i^{2}=-1$ we have :

$$
\begin{aligned}
\Gamma^{(n)}\left(p_{1}, p_{2}, \ldots, p_{n}\right)= & g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}}\left(p_{1}^{\alpha_{1}} p_{1}^{\mu} p_{1}^{\mu} p_{2}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+p_{1}^{\alpha_{1}} p_{1}^{\mu} p_{2}^{\alpha_{2}} p_{2}^{\mu} \ldots p_{n}^{\alpha_{n}}+\ldots p_{1}^{\alpha_{1}} p_{1}^{\mu} p_{2}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}} p_{n}^{\mu}\right) \\
& g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}}\left(p_{2}^{\alpha_{1}} p_{2}^{\mu} p_{2}^{\mu} p_{1}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+p_{2}^{\alpha_{1}} p_{2}^{\mu} p_{1}^{\alpha_{2}} p_{1}^{\mu} \ldots p_{n}^{\alpha_{n}}+\ldots p_{2}^{\alpha_{1}} p_{2}^{\mu} p_{1}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}} p_{n}^{\mu}\right),(1 \longleftrightarrow 2) \\
& +\ldots
\end{aligned}
$$

We may then factorize $p_{1}^{\mu}$ as:

$$
\Gamma^{(n)}\left(p_{1}, p_{2}, \ldots, p_{n}\right)=g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} p_{1}^{\mu}\left(p_{1}^{\alpha_{1}} p_{1}^{\mu} p_{2}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+p_{2}^{\alpha_{1}} p_{2}^{\mu} p_{1}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+\ldots\right)+\ldots
$$

where we factorized terms such that $p_{1}^{\alpha_{1}} p_{1}^{\mu} p_{2}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+p_{2}^{\alpha_{1}} p_{2}^{\mu} p_{1}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+\ldots$ is symmetric. Thus, symmetrizing with respect to the factor $p_{1}^{\mu}$ we find:

$$
\Gamma^{(n)}\left(p_{1}, p_{2}, \ldots, p_{n}\right)=g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}}\left(p_{1}^{\mu}+p_{2}^{\mu}+\ldots+p_{n}^{\mu}\right)\left(p_{1}^{\alpha_{1}} p_{1}^{\mu} p_{2}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+p_{2}^{\alpha_{1}} p_{2}^{\mu} p_{1}^{\alpha_{2}} \ldots p_{n}^{\alpha_{n}}+\ldots\right)
$$

Formally, defining the permutation group as $\mathscr{S}_{n}$, the steps above can be written as:

$$
\begin{align*}
\Gamma^{(n)}\left(p_{1}, p_{2}, \ldots, p_{n}\right) & =\sum_{\left\{\alpha_{r}\right\}} g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} \sum_{s \in \mathscr{S}_{n}} \sum_{k=1}^{n} \prod_{i=1}^{n}\left(p_{s(i)}^{\alpha_{i}}\right) p_{s(1)}^{\mu} p_{s(k)}^{\mu} \\
& =\sum_{\left\{\alpha_{r}\right\}} g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} \sum_{k=1}^{n} \sum_{s \in \mathscr{S}_{n}} \prod_{i=1}^{n}\left(p_{s(i)}^{\alpha_{i}}\right) p_{s(1)}^{\mu} p_{s(k)}^{\mu} \\
& =\sum_{\left\{\alpha_{r}\right\}} g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{s \in \mathscr{S}_{n}, s(k)=l} \prod_{i}\left(p_{s(i)}^{\alpha_{i}}\right) p_{s(1)}^{\mu} p_{l}^{\mu} \\
= & \sum_{\left\{\alpha_{r}\right\}} g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} \sum_{l=1}^{n} p_{l}^{\mu} \sum_{k=1}^{n} \sum_{s \in \mathscr{S}_{n}, k=s^{-1}(l)} \prod_{i}\left(p_{s(i)}^{\alpha_{i}}\right) p_{s(1)}^{\mu}  \tag{F.0.38}\\
= & \sum_{\left\{\alpha_{r}\right\}} g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}} \sum_{l=1}^{n} p_{l}^{\mu} \sum_{s \in \mathscr{S}_{n}} \prod_{i}\left(p_{s(i)}^{\alpha_{i}}\right) p_{s(1)}^{\mu} \\
= & \sum_{\left\{\alpha_{r}\right\}} g_{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}}\left(\sum_{l=1}^{n} p_{l}^{\mu}\right)\left(\sum_{s \in \mathscr{S}_{n}} \prod_{i}\left(p_{s(i)}^{\alpha_{i}}\right) p_{s(1)}^{\mu}\right) .
\end{align*}
$$

Thus in general, total derivative terms are of the form $\vec{\sigma}_{1} \cdot \vec{P}$ in Fourier space with $\vec{\sigma}_{1}=\sum_{n} \vec{p}_{i}$. This implies that

$$
\begin{equation*}
\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)=\left\{\overrightarrow{\sigma_{1}} \cdot \vec{P}, \vec{P} \in \mathcal{H}_{D-1} \mathfrak{S}_{n}\right\} \tag{F.0.39}
\end{equation*}
$$

If we define the application $\mathcal{G}: \mathcal{H}_{D-1} \mathfrak{S}_{n} \rightarrow \mathcal{H}_{D} \mathfrak{S}_{n}$ with $\mathcal{G}(\vec{P})=\overrightarrow{\sigma_{1}} \cdot \vec{P}$ then we have

$$
\begin{equation*}
\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)=\operatorname{Im}(\mathcal{G}) \tag{F.0.40}
\end{equation*}
$$

and it is then sufficient to calculate the rank of $\mathcal{G}$.
The number of independent terms to include at order $D$ of the derivative expansion is then given by the sum of $\operatorname{dim}\left(\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}\right)-\operatorname{dim}\left(\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)\right)$. This sum is to be carried to $n=D$. This method is essentially the same as the usual ansatz method which consists of finding terms independent with respect to integration by parts. Indeed, $\operatorname{dim}\left(\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}\right)-\operatorname{dim}\left(\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)\right)$ is the dimension of the quotient space $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n} / \operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)$ which consists of identifying terms that differ by elements of $\operatorname{Ker}\left(\mathcal{M}_{n}^{c}\right)$. This is the same as retaining independent derivative terms up to an integration by parts. The novel feature of this formalism is that it is possible to instead work directly in the target space of $\mathcal{M}_{n}^{c}$, that is $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n-1}$, rather than the base space $\mathcal{C H}{ }_{D} \mathfrak{S}_{n}$. This is especially convenient for $n<D / 2$ when $\operatorname{dim}\left(\mathcal{C H}_{D} \mathfrak{S}_{n}\right)<\operatorname{dim}\left(\mathcal{C H} \mathcal{S}_{D+1}\right)$. To characterize $\operatorname{Im}\left(\mathcal{M}_{n}^{c}\right)$ in $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n-1}$ let us consider applying $\mathcal{M}_{n}^{c}$ to a symmetric polynomial $P\left(p_{1}, p_{2}, \ldots, p_{n}\right)$. Then we have :

$$
\begin{equation*}
\mathcal{M}_{n}^{c} P\left(p_{1}, p_{2}, \ldots, p_{n}\right)=P\left(p_{1}, p_{2}, \ldots,-\sum_{i=1}^{n-1} p_{i}\right) \tag{F.0.41}
\end{equation*}
$$

The particularities of this polynomial is that

$$
\begin{aligned}
S_{n} \cdot P\left(p_{1}, p_{2}, \ldots,-\sum_{i=1}^{n-1} p_{i}\right) & =P\left(p_{1}, p_{2}, \ldots,-\sum_{i=1}^{n-1} p_{i}\right) \\
\mathcal{M}_{\sum_{i=1}^{n-1} p_{i}} \cdot P\left(p_{1}, p_{2}, \ldots,-\sum_{i=1}^{n-1} p_{i}\right) & =0
\end{aligned}
$$

Thus $\operatorname{Im}\left(\mathcal{M}_{n}^{c}\right)$ is given by the intersection of the invariants of $S_{n}$ and the kernel of $\mathcal{M}_{\sum_{i=1}^{n-1} p_{i}}$. When working with a computer it is more convenient to characterize this as:

$$
\begin{equation*}
\operatorname{Ker}\left(I_{n}-S_{n}\right) \cap \operatorname{Ker}\left(\mathcal{M}_{\sum_{i=1}^{n-1} p_{i}}\right)=\operatorname{Im}\left(\mathcal{M}_{n}^{c}\right) \tag{F.0.42}
\end{equation*}
$$

The advantage of this rewriting is that we may define the application:

$$
\begin{equation*}
H=\left(I_{n}-S_{n}\right)^{\mathrm{T}}\left(I_{n}-S_{n}\right)+\mathcal{M}_{\sum_{i=1}^{\mathrm{T}} p_{i}}^{\mathrm{T}} \mathcal{M}_{\sum_{i=1}^{n-1} p_{i}} \tag{F.0.43}
\end{equation*}
$$

where T is the transpose operator such that:

$$
\langle x| H|x\rangle=\|H x\|^{2}=\left\|\left(I_{n}-S_{n}\right) x\right\|^{2}+\left\|\mathcal{M}_{\sum_{i=1}^{n-1} p_{i}} x\right\|^{2}
$$

and thus,

$$
H x=0 \Leftrightarrow\|H x\|^{2}=0 \Leftrightarrow\left\|\left(I_{n}-S_{n}\right) x\right\|^{2}=0,\left\|\mathcal{M}_{\sum_{i=1}^{n-1} p_{i}} x\right\|^{2}=0 .
$$

This then allows a more computer friendly characterization :

$$
\operatorname{Im}\left(\mathcal{M}_{n}^{c}\right)=\operatorname{Ker}(H)
$$

Another possibility is to consider the matrix :

$$
\begin{equation*}
\binom{I_{n}-S_{n}}{\mathcal{M}_{\sum_{i=1}^{n-1} p_{i}}} \tag{F.0.44}
\end{equation*}
$$

where $\binom{A_{l, h}}{B_{l, h}}$ is the matrix of size $2 l \mathrm{x} h$ formed by stacking $A_{l, h}$ on top of $B_{l, h}$. Then

$$
\begin{align*}
& A_{l, h} \cdot x_{h}=0_{l} \\
& B_{l, h} \cdot x_{h}=0_{l} \tag{F.0.45}
\end{align*}
$$

is equivalent to

$$
\begin{equation*}
\binom{A_{l, h}}{B_{l, h}} x_{h}=0_{2 l} . \tag{F.0.46}
\end{equation*}
$$

Finally, it is often convenient as before to determine a basis of $\operatorname{Im}\left(\mathcal{M}_{n}^{c}\right)$ by first considering $d=1$, applying $\mathcal{S P}{ }_{n}$ then considering the other basis elements of $\operatorname{Im}\left(\mathcal{M}_{n}^{c}\right)$ to be contained in $\operatorname{Ker}(\mathcal{D})$.

Now that we have developed the formalism, we may compute the order 6 of the derivative expansion in order to compare with the standard ansatz method. Within the ansatz method, the effective action at order 6 is given by 77]:

$$
\begin{align*}
\Gamma_{k}= & \int U(\phi)+\frac{1}{2} Z(\phi) \partial_{\nu} \phi \partial_{\nu} \phi+\frac{1}{2} W_{1}(\phi)\left(\partial_{\nu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \partial_{\mu} \phi\right)+\frac{1}{2} W_{2}(\phi)\left(\partial_{\mu} \partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)+ \\
& \frac{1}{4} W_{3}(\phi)\left(\partial_{\mu} \phi\right)\left(\partial_{\mu} \phi\right)\left(\partial_{\nu} \phi\right)\left(\partial_{\nu} \phi\right)+\frac{1}{2} X_{1}\left(\partial_{\mu} \partial_{\nu} \partial_{\rho} \phi\right)^{2}+\frac{1}{2} X_{2}(\phi)\left(\partial_{\mu} \partial_{\nu} \phi\right)\left(\partial_{\nu} \partial_{\rho} \phi\right)\left(\partial_{\mu} \partial_{\rho} \phi\right)+  \tag{F.0.47}\\
& \frac{1}{2} X_{3}(\phi)\left(\partial^{2} \phi\right)^{3}+\frac{1}{2} X_{4}(\phi)\left(\partial^{2} \phi\right)^{2}(\partial \phi)^{2}+\frac{1}{2} X_{5}(\phi)(\partial \phi)^{2}\left(\partial_{\mu} \phi\right)\left(\partial^{2} \partial_{\mu} \phi\right)+ \\
& \frac{1}{2} X_{6}(\phi)(\partial \phi)^{2}\left(\partial_{\mu} \partial_{\nu} \phi\right)^{2}+\frac{1}{2} X_{7}(\phi)\left(\partial^{2} \phi\right)\left((\partial \phi)^{2}\right)^{2}+\frac{1}{2} X_{8}(\phi)\left((\partial \phi)^{2}\right)^{3}
\end{align*}
$$

The expansion of $\widehat{\gamma}_{2}$ is straightforward:

$$
\begin{equation*}
\widehat{\gamma}_{2}\left(p, \phi_{0}\right)=Z\left(\phi_{0}\right) p^{2}+W_{1}\left(\phi_{0}\right) p^{4}+X_{1}\left(\phi_{0}\right) p^{6} \tag{F.0.48}
\end{equation*}
$$

Consider now for $d=1$ the expansion of a homogeneous symmetric polynomial of total degree 6 :

$$
\begin{equation*}
P\left(X_{1}, X_{2}, \ldots, X_{n}\right)=A \sigma_{2, n}^{3}+B \sigma_{3, n}^{2}+C \sigma_{4, n} \sigma_{2, n}+D \sigma_{6, n}+\sigma_{1, n} R\left(X_{1}, X_{2}, \ldots, X_{n}\right) \tag{F.0.49}
\end{equation*}
$$

For $\widehat{\gamma}_{3}$ the novel term is then $\sigma_{3, n}^{2}$ such that we have:

$$
\begin{equation*}
\widehat{\gamma}_{3}=X_{2}\left(\phi_{0}\right) \sigma_{3,3}^{2} \tag{F.0.50}
\end{equation*}
$$

Within the ansatz approach this term comes from the inclusion of a $X_{2}(\phi) \phi^{\prime \prime}{ }^{3}$ in the effective action. If $d>1$ then applying $\mathcal{S} \mathcal{P}_{\sigma}$ to $\widehat{\gamma}_{3}$ leads to $X_{2}\left(\phi_{0}\right) \vec{\sigma}_{3,3}^{2}$. Let us now find a basis in $\operatorname{Ker} \mathcal{D}$. Once more there are two possibilities.

The first is to consider symmetric s-polynomials in three variables $\left(\vec{p}_{1}, \vec{p}_{2}, \vec{p}_{3}\right)$ then impose momentum conservation. Up to symmetrizations, the monomials to consider are :

$$
\begin{array}{r}
\left(p_{1} \cdot p_{2}\right)^{2} p_{1} \cdot p_{3}, p_{1} \cdot p_{2} p_{1} \cdot p_{3} p_{2} \cdot p_{3}, \\
p_{1} \cdot p_{2} p_{1} \cdot p_{3} p_{1}^{2}, p_{1} \cdot p_{2} p_{2} \cdot p_{3} p_{1}^{2},\left(p_{2} \cdot p_{3}\right)^{2} p_{1}^{2}  \tag{F.0.51}\\
p_{2} \cdot p_{3} p_{1}^{4}, p_{1} \cdot p_{3} p_{1}^{2} p_{2}^{2}, p_{1}^{2} p_{2}^{2} p_{3}^{2}
\end{array}
$$

Thus there are 8 in total. In general, the monomial basis of $\mathcal{C H}{ }_{D} \mathfrak{S}_{n}$ may be conveniently obtained from the expansion of

$$
\begin{equation*}
\widehat{\mathcal{N}}_{n}\left(\left(p_{1}+p_{2}+\ldots p_{n}\right) \cdot\left(p_{1}+p_{2}+\ldots p_{n}\right)\right)^{D / 2} \tag{F.0.52}
\end{equation*}
$$

Indeed, the expansion of

$$
\begin{equation*}
\left(\left(p_{1}+p_{2}+\ldots p_{n}\right) \cdot\left(p_{1}+p_{2}+\ldots p_{n}\right)\right)^{D / 2} \tag{F.0.53}
\end{equation*}
$$

contains all possible s-polynomials of degree $D$ and $\widehat{\mathcal{N}}_{n}$ selects those that belong to $\mathcal{C} \mathcal{H}_{D} \mathfrak{S}_{n}$. From the basis elements of $\mathcal{C} \mathcal{H}_{6} \mathfrak{S}_{3}$ one then has to consider all polynomials of degree 6 of the form $\vec{\sigma}_{1} \cdot \vec{P}$ where $\vec{P}$ is then a v-polynomial of total degree five. The counting is then about as difficult as finding terms that are independent by integration by parts and the advantages gained over the usual ansatz method are then mild if non existent. Instead we consider the second method which consists of computing the rank of $\mathcal{M}_{n}^{c}$ by working directly in the target space. These are then polynomials of 2 variables. When $d=1$ the monomials to consider are:

$$
\begin{equation*}
p_{1}^{4} p_{2}^{2}, p_{1}^{3} p_{2}^{3}, p_{1}^{5} p_{2} \tag{F.0.54}
\end{equation*}
$$

The promotions of these monomials are:

$$
\begin{align*}
& p_{1}^{4} p_{2}^{2} \rightarrow\left\{p_{1}^{4} p_{2}^{2},\left(p_{1} \cdot p_{2}\right)^{2} p_{1}^{2}\right\} \\
& p_{1}^{5} p_{2} \rightarrow\left\{p_{1} \cdot p_{2} p_{1}^{4}\right\}  \tag{F.0.55}\\
& p_{1}^{3} p_{2}^{3} \rightarrow\left\{p_{1} \cdot p_{2} p_{1}^{2} p_{2}^{2},\left(p_{1} \cdot p_{2}\right)^{3}\right\}
\end{align*}
$$

As there is only one element in the promotions of $p_{1}^{5} p_{2}$ it is not possible to construct an element in $\operatorname{Ker\mathcal {D}}$. Taking the difference of the promotions of $p_{1}^{4} p_{2}^{2}$ we find:

$$
\begin{equation*}
p_{1}^{2}\left(\left(p_{1} \cdot p_{2}\right)^{2}-p_{1}^{2} p_{2}^{2}\right)=p_{1}^{2} \bar{\sigma}_{4,3} \tag{F.0.56}
\end{equation*}
$$

As $\bar{\sigma}_{4,3}$ is already symmetric with respect to $S_{3}$ it is sufficient to symmetrize $p_{1}$ which leads to :

$$
\begin{equation*}
2 \dot{\sigma}_{2,3} \bar{\sigma}_{4,3} \tag{F.0.57}
\end{equation*}
$$

The next element to consider is the difference of the promotions of $p_{1}^{3} p_{2}^{3}$ :

$$
\begin{equation*}
\left(p_{1} \cdot p_{2}\right)\left(\left(p_{1} \cdot p_{2}\right)-p_{1}^{2} p_{2}^{2}\right) \tag{F.0.58}
\end{equation*}
$$

Symmetrizing then leads once more to $\sigma_{2,3} \bar{\sigma}_{4,3}$ and thus there are only two independent terms in $\widehat{\gamma}_{3}$ at order 6 of the derivative expansion. This leads to the expansion:

$$
\begin{equation*}
\widehat{\gamma}_{3}\left(p_{1}, p_{2}\right)=W_{2}\left(\phi_{0}\right) \bar{\sigma}_{4,3}+X_{2}\left(\phi_{0}\right) \vec{\sigma}_{3,3} \cdot \vec{\sigma}_{3,3}-X_{3}\left(\phi_{0}\right) \dot{\sigma}_{2,3} \bar{\sigma}_{4,3} \tag{F.0.59}
\end{equation*}
$$

The terms $X_{2}$ and $X_{3}$ are related to $X_{a}$ and $X_{b}$ in Eq. F.0.51) by:

$$
\begin{align*}
& X_{1}=3 X_{a}+\frac{8}{3} X_{b}  \tag{F.0.60}\\
& X_{2}=\frac{1}{3}\left(X_{a}-X_{b}\right)
\end{align*}
$$

In the following we shall no longer give the precise correspondence between our derivation and Eq. F.0.47) but we will check that the total number of independent terms are the same. In $d=1$ the order 6 term of $\widehat{\gamma}_{4}$ is:

$$
\begin{equation*}
\sigma_{4,4} \sigma_{2,4} \tag{F.0.61}
\end{equation*}
$$

We shall now compute the elements of $\operatorname{Ker} \mathcal{D}$. The monomial basis of $\mathcal{C H} \mathcal{H}_{6} \mathfrak{S}_{4}$ is given by:

$$
\mathcal{S}_{4} \cdot\left\{p_{1} \cdot p_{2} p_{1} \cdot p_{3} p_{1} \cdot p_{4}, p_{1} \cdot p_{3} p_{1} \cdot p_{4} p_{2}^{2}, p_{1} \cdot p_{4} p_{2} \cdot p_{3} p_{1}^{2}, p_{1} \cdot p_{2} p_{1} \cdot p_{4} p_{2} \cdot p_{3},\left(p_{1} \cdot p_{4}\right)^{2} p_{2} \cdot p_{3}, p_{1} \cdot p_{4} p_{2}^{2} p_{3}^{2}\right\}
$$

Thus we have $\operatorname{dim}\left(\mathcal{C H} \mathcal{S}_{6}\right)=6$ and $\operatorname{dim}\left(\mathcal{C H}_{6} \mathfrak{S}_{3}\right)=8$. A priori the simplest method is then to deduce the rank of $\mathcal{M}_{n}^{c}$ from its kernel. As we have shown this is given by the image of $\mathcal{G}: \mathcal{C H}_{D-1} \mathfrak{S}_{n} \rightarrow \mathcal{C H}_{D} \mathfrak{S}_{n}$ with $\mathcal{G}(\vec{P})=\overrightarrow{\sigma_{1}} \cdot \vec{P}$. A monomial basis of $\mathcal{C H}_{5} \mathfrak{S}_{4}$ is given by:

$$
\mathcal{S}_{4} \cdot\left\{p_{1} \cdot p_{4} p_{2} \cdot p_{3} p_{1}, p_{2} \cdot p_{3} p_{2} \cdot p_{4} p_{1}, p_{2} \cdot p_{4} p_{1} p_{3}^{2}\right\}
$$

and thus $\operatorname{dimKer}\left(\mathcal{M}_{4}^{c}\right) \leq 3$. If we then apply $\mathcal{G}$ we find:

$$
\begin{array}{r}
\mathcal{S}_{4} \cdot\left\{p_{2} \cdot p_{3}\left(p_{1} \cdot p_{4}\right)^{2}+p_{2} \cdot p_{3} p_{1}^{2} p_{1} \cdot p_{4}+p_{1} \cdot p_{2} p_{2} \cdot p_{3} p_{1} \cdot p_{4}+p_{1} \cdot p_{3} p_{2} \cdot p_{3} p_{1} \cdot p_{4}\right. \\
p_{2} \cdot p_{3} p_{2} \cdot p_{4} p_{1}^{2}+p_{1} \cdot p_{2} p_{2} \cdot p_{3} p_{2} \cdot p_{4}+p_{1} \cdot p_{3} p_{2} \cdot p_{3} p_{2} \cdot p_{4}+p_{1} \cdot p_{4} p_{2} \cdot p_{3} p_{2} \cdot p_{4} \\
\left.p_{2} \cdot p_{4} p_{1}^{2} p_{3}^{2}+p_{1} \cdot p_{2} p_{2} \cdot p_{4} p_{3}^{2}+p_{1} \cdot p_{3} p_{2} \cdot p_{4} p_{3}^{2}+p_{1} \cdot p_{4} p_{2} \cdot p_{4} p_{3}^{2}\right\}
\end{array}
$$

The first element is the only one to contain a s-monomial of the form $\left(p_{1} \cdot p_{4}\right)^{2}$ which implies that it is independent from the two others. The same is true of the third element which contains a $p_{1}^{2} p_{3}^{2}$. Thus there is no linear combination of the first and third which would lead to the second element. As such, we have $\operatorname{dimKer}\left(\mathcal{M}_{4}^{c}\right)=3$. Using the rank-nullity theorem, the rank of $\mathcal{M}_{4}^{c}$ is $6-3=3$. The three independent terms are given in Eq. F.0.47) by $\left(X_{4}, X_{5}, X_{6}\right)$. We may then take two linear independent terms from $\operatorname{Ker}(\mathcal{D})$ to complete the basis:

$$
\begin{align*}
& \bar{\sigma}_{6,4, A}=\mathcal{S}_{4} \cdot\left(\left(p_{1} \cdot p_{4}\right)^{2} p_{2} \cdot p_{3}-p_{1} \cdot p_{4} p_{1} \cdot p_{2} p_{4} \cdot p_{3}\right)  \tag{F.0.62}\\
& \bar{\sigma}_{6,4, B}=\mathcal{S}_{4} \cdot\left(p_{1} \cdot p_{2} p_{1} \cdot p_{3} p_{1} \cdot p_{4}-p_{1} \cdot p_{4} p_{2} \cdot p_{3} p_{1}^{2}\right)
\end{align*}
$$

After applying momentum conservation, their images by $\mathcal{M}_{4}^{c}$ are independent as well.
As $\operatorname{dim}\left(\mathcal{C H}_{D} \mathfrak{S}_{D / 2}\right)$ is the largest dimension, we expect that the following $\widehat{\gamma}_{n}$ will have less independent terms. First notice that elements of $\mathcal{C} \mathcal{H}_{n} \mathfrak{S}_{n}$ are all proportional to $\sigma_{n}$ as there is only one symmetric way to contract the indices. Hence the rank of $\mathcal{M}_{n}^{c}$ is necessarily one in these spaces. In particular there is only one term to consider for $\widehat{\gamma}_{6}$ given by $\dot{\sigma}_{6,6}$ which comes from $X_{8}$ in Eq. F.0.47). Next, notice that for $n$ even, elements of $\mathcal{C} \mathcal{H}_{n} \mathfrak{S}_{n-1}$ are of the form:

$$
\begin{equation*}
\mathcal{S}_{n} \cdot\left\{p_{1}^{2} p_{2} \cdot p_{3} p_{4} \cdot p_{5} \ldots p_{n-1} \cdot p_{n}, p_{1} \cdot p_{2} p_{1} \cdot p_{3} p_{4} \cdot p_{5} \ldots p_{n-1} \cdot p_{n}\right\} \tag{F.0.63}
\end{equation*}
$$

Thus it is a two dimensional space when $d>1$ and a one dimensional space when $d=1$. As elements of $\mathcal{C H} H_{n-1} \mathfrak{S}_{n-1}$ are all proportional to $\sigma_{n-1}$ we have $\operatorname{dim}(\mathcal{G})=1$. As such the rank of $\mathcal{M}_{n}^{c}$ is $2-1=1$ when $d>1$ and 0 when $d=1$. In particular there is only one term to consider for $\widehat{\gamma}_{5}$ given by the difference:

$$
\begin{equation*}
p_{1}^{2} p_{2} \cdot p_{3} p_{4} \cdot p_{5}-p_{1} \cdot p_{2} p_{1} \cdot p_{3} p_{4} \cdot p_{5} \tag{F.0.64}
\end{equation*}
$$



Figure F.0.1: Total number of field dependent functions as a function of the order of the derivative expansion where order 0 is the LPA. The plot is given in $\log$ scale showing a potentially exponential growth in the number of terms.
which belongs to $\operatorname{Ker}(\mathcal{D})$.
Thus we have retrieved all of the terms of the derivative expansion at order 6 . These methods may be used at order 8 and order 10 of the derivative expansions to find a polynomial basis of the $\widehat{\gamma}_{n}$. We find that there are 23 more terms to include at order 8 of the derivative expansion. The order 10 of the derivative expansion requires 66 more terms then the order 8 . Thus the order 8 of the derivative expansion is a system of non linear pde's with a total of 36 equations. The order 10 is a formidable system of size 102 as shown in Fig. F.0.1. It is then clear that the question of convergence of the derivative expansion will require a formal derivation or more sophisticated numerical methods.

## Appendix G

## Fluctuation dissipation relations

In this section we shall derive the fluctuation dissipation relations in a model with a single field $\phi$. The proof follows 82 and utilizes the functional Taylor expansion of the effective action in real space:
$\Gamma\left[\phi(x)=\phi_{0}+\varphi(x)\right]=\Gamma\left[\phi_{0}\right]+D \Gamma_{\phi_{0}}(\varphi)+\frac{1}{2!} D^{2} \Gamma_{\phi_{0}}(\varphi \otimes \varphi)+\frac{1}{3!} D^{3} \Gamma_{\phi_{0}}(\varphi \otimes \varphi \otimes \varphi)+\ldots \frac{1}{n!} D^{n} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n}}\right)$
where:

$$
\begin{align*}
D^{n} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n}}\right) & =\left.\sum_{\alpha_{1}, \ldots, \alpha_{n}} \varphi_{\alpha_{1}} \varphi_{\alpha_{2}} \ldots \varphi_{\alpha_{n}} \frac{\delta^{n}}{\delta \phi_{\alpha_{1}} \delta \phi_{\alpha_{2}} \ldots \delta \phi_{\alpha_{n}}} \Gamma\right|_{\phi_{0}}  \tag{G.0.2}\\
& =\int_{x_{1}, x_{2}, \ldots, x_{n}} \varphi\left(x_{1}\right) \varphi\left(x_{2}\right) \ldots \varphi\left(x_{n}\right) \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)
\end{align*}
$$

$\Gamma$ is independent of the choice of the background field thus $\frac{\partial \Gamma}{\partial \phi_{0}}=0$. Hence, after applying $\partial_{\phi_{0}}$ to Eq. G.0.1 we may set each term of order $O\left(\varphi^{n}\right)$ to zero. Consider then for example:

$$
\begin{align*}
\partial_{\phi_{0}} D \Gamma_{\phi_{0}}(\varphi)= & \partial_{\phi_{0}} \int_{x}\left(\phi(x)-\phi_{0}\right) \Gamma^{(1)}\left(x ; \phi_{0}\right)  \tag{G.0.3}\\
& -\int_{x} \Gamma^{(1)}\left(x ; \phi_{0}\right)+\int_{x} \varphi(x) \partial_{\phi_{0}} \Gamma^{(1)}\left(x ; \phi_{0}\right)
\end{align*}
$$

Then both $\partial_{\phi_{0}} \Gamma\left[\phi_{0}\right]$ and $-\int_{x} \Gamma^{(1)}\left(x ; \phi_{0}\right)$ are of order $O\left(\varphi^{0}\right)$ such that the sum is null leading to:

$$
\begin{equation*}
\partial_{\phi_{0}} \Gamma\left[\phi_{0}\right]=\int_{x} \Gamma^{(1)}\left(x ; \phi_{0}\right) \tag{G.0.4}
\end{equation*}
$$

We may consider as well :

$$
\begin{aligned}
\frac{1}{2} \partial_{\phi_{0}} D^{2} \Gamma_{\phi_{0}}(\varphi)= & \frac{1}{2} \partial_{\phi_{0}} \int_{x, y}\left(\phi(x)-\phi_{0}\right)\left(\phi(y)-\phi_{0}\right) \Gamma^{(2)}\left(x, y ; \phi_{0}\right) \\
= & \frac{1}{2} \int_{x, y}\left(-\left(\phi(y)-\phi_{0}\right) \Gamma^{(2)}\left(x, y ; \phi_{0}\right)-\left(\phi(x)-\phi_{0}\right) \Gamma^{(2)}\left(x, y ; \phi_{0}\right)+\right. \\
& \left.\left(\phi(x)-\phi_{0}\right)\left(\phi(y)-\phi_{0}\right) \partial_{\phi_{0}} \Gamma^{(2)}\left(x, y ; \phi_{0}\right)\right) \\
= & \int_{x, y}-\varphi(x) \Gamma^{(2)}\left(x, y ; \phi_{0}\right)+\frac{1}{2} \varphi(x) \varphi(y) \partial_{\phi_{0}} \Gamma^{(2)}\left(x, y ; \phi_{0}\right)
\end{aligned}
$$

where we used the $(x, y)$ symmetry of $\Gamma^{(2)}\left(x, y ; \phi_{0}\right)$. Then the sum of the $O(\varphi)$ terms from $\frac{1}{2} \partial_{\phi_{0}} D^{2} \Gamma_{\phi_{0}}(\varphi)$ and $\partial_{\phi_{0}} D \Gamma_{\phi_{0}}(\varphi)$ may be equated to zero leading to:

$$
\begin{equation*}
\int_{y} \Gamma^{(2)}\left(x, y ; \phi_{0}\right)=\partial_{\phi_{0}} \Gamma^{(1)}\left(x ; \phi_{0}\right) \tag{G.0.5}
\end{equation*}
$$

In general we have :

$$
\begin{aligned}
\frac{1}{n!} \partial_{\phi_{0}} D^{n} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n}}\right)= & \frac{1}{n!} \partial_{\phi_{0}} \int_{x_{1}, x_{2}, \ldots, x_{n}} \varphi\left(x_{1}\right) \varphi\left(x_{2}\right) \ldots \varphi\left(x_{n}\right) \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right) \\
= & \frac{1}{n!} \int_{x_{1}, x_{2}, \ldots, x_{n}}\left(-\sum_{i=1}^{n} \prod_{j \neq i}\left(\varphi\left(x_{j}\right)\right) \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)+\right. \\
& \left.\prod_{i}\left(\varphi\left(x_{i}\right)\right) \partial_{\phi_{0}} \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)\right) \\
& \int_{x_{1}, x_{2}, \ldots, x_{n}}\left(-\frac{1}{(n-1)!} \prod_{j=1}^{n-1}\left(\varphi\left(x_{j}\right)\right) \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)+\right. \\
& \left.\frac{1}{n!} \prod_{i}\left(\varphi\left(x_{i}\right)\right) \partial_{\phi_{0}} \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)\right)
\end{aligned}
$$

From the sum of $\frac{1}{(n-1)!} \partial_{\phi_{0}} D^{n-1} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n-1}}\right)$ and $\frac{1}{n!} \partial_{\phi_{0}} D^{n} \Gamma_{\phi_{0}}\left(\varphi^{\otimes_{n}}\right)$ we then obtain:

$$
\begin{equation*}
\int_{x_{n}} \Gamma^{(n)}\left(x_{1}, x_{2}, \ldots, x_{n} ; \phi_{0}\right)=\partial_{\phi_{0}} \Gamma^{(n-1)}\left(x_{1}, x_{2}, \ldots, x_{n-1} ; \phi_{0}\right) . \tag{G.0.6}
\end{equation*}
$$

Taking the Fourier transform of this relationship we retrieve the fluctuation dissipation relations.

## Appendix H

## Multicritical phase diagram

In this section we show how to derive the tricritical and tetracritical phase diagram.
Let us start by finding the number of minima of the free energy as they correspond to the number of phases in the system, either stable or meta-stabl ${ }^{1}$. This may be achieved by analyzing the number of sign changes of the derivative of the free energy as was done in appendix A when analyzing the Van der Waals equation.

Let us then first consider the tricritical case. In order to find the minima, we first calculate the derivative of the free energy as :

$$
\begin{equation*}
\frac{\mathrm{d} G}{\mathrm{dm}}=m\left(\widetilde{a}+\widetilde{b} m^{2}+\widetilde{c} m^{4}\right) \tag{H.0.1}
\end{equation*}
$$

where we have absorbed all numerical factors onto a redefinition of the constants. We now drop the previous definitions of $(a, b)$ and we set $\widetilde{a}=a, \widetilde{b}=b, \widetilde{c}=1$.
$m=0$ will always be a root of the above equation but it is a minimum only if the curvature at that point is positive. For $a$ non zero, the curvature is given by $a$ which then implies a phase transition at $a=0$ independently of the sign of $b{ }^{2}$. If we now wish to see how $b$ might affect the phase diagram we may consider the possible number of sign changes between coefficients. The possibilities are 0,1 or 2 . If there are 0 sign changes the free energy is strictly positive and thus there can be no strictly positive roots. As $F$ is also an even function of $m$ the same is true for $m<0$. Hence, $F$ has a unique extrema which is then necessarily a minimum as the system should be stable. In this scenario we thus obtain the familiar high temperature phase with one minimum. If there is one sign change then we are guaranteed that there is exactly one strictly positive root. As $\widetilde{c}>0$ the only possibility is that $a<0$ and thus $m=0$ is a local maxima with two non zero extrema $\pm m_{0}$ using parity. The extrema must be minima as the curvature must alternate in sign. As $m_{0}$ cannot be non zero when we change $a$ to positive values, by continuity, when increasing $a, m_{0}$ must continuously approach $m=0$ until it reaches a triple root at $a=0$ as we can then factor the term $\widetilde{b} m^{2}$ in Eq. H.0.1). This is thus the usual second order phase transition we saw for the Ising model but it now corresponds to a line as for any positive value for $b$, changing the sign of $a$ leads to a phase transition.

This line then corresponds to $(a=0, b>0)$. Let us now consider the novel case of two sign changes. This situation corresponds to 2 or 0 positive roots. In the case of 0 positive roots we return to the case of only one minimum. If there are two distinct positive roots, then one root must be unstable for the same reasons we explained in the case of the Van der Waals gas, meaning that the curvature of $F$ must alternate in sign between extrema. Hence, in that case there is only one non zero positive minimum. Moreover, this case of maximal sign change can only occur in the situation $(a>0, b<0, \widetilde{c}=1)=(+,-,+)$ thus the curvature at $m=0$ given by the sign of $a$ is positive. Therefore, when including the negative roots, we have three local minima. However, the case where all minima are located at exactly the same height cannot be the generic case. Then, which one of these minima is the global one ? To answer this let us return to the two cases of sign changes. There is either one minimum or three minima, and one must be able to interpolate

[^67]between these cases by changing the coefficients. if we take $b=-10^{-10}$ and $a=10^{10}$ we expect the potential to be indistinguishable from the case of one minimum when $b=0$ and the polynomial is always positive. Thus $b<0$ but relatively small as compared to $a$ which leads to a single minimum. The case $a=-10^{10}$ and $b=10^{10}$ is indistinguishable from the case $a=0$ and thus $\frac{\mathrm{d} F}{\mathrm{dm}}=m^{3}\left(-|\mathrm{b}|+m^{2}\right)$ where there is a global minimum for $m= \pm|b|^{1 / 2}$. Thus as we vary $b$ from small to very large we continuously move from the case of one global minimum at $m=0$ to two global minima and one local minimum at $m=0$. For intermediate $b$, by continuity, there must also be a point where one goes from a unique minimum to three local minima with $m=0$ still being the global one. Hence for fixed $a$ and varying $b$ there is only one very particular case where there are three global minima at the same height. This case then represents coexistence of three phases in the medium but in all other other cases there is only one or two global minima and thus one or two phases.

In the tetracritical case, we consider directly the case where $c<0$ and all signs alternate which allows us to consider the case with the most minima. As the sign changes do not differ when taking the derivative we may readily deduce the possible number of local minima. We may count 3 sign changes thus there are 3 or 1 strictly positive roots. The case of 1 root is the usual double minima and in the case of three positive roots the intermediate one must be a maximum. Moreover, in order to have three sign changes we must have $a<0$ and as this coefficient gives the curvature at $m=0$, the extrema at $m=0$ is in fact a maximum. Thus counting both positive and negative roots we have four local minima.

## Appendix I

## $1 / \mathrm{N}$ expansion for the tricritical $\beta$ function

We recall the partition function:

$$
\begin{equation*}
Z=\int D \chi D \sigma D \psi e^{-\frac{1}{2} \psi \cdot\left(-\partial^{2}+\sigma\right) \cdot \psi-\int V(\chi)+\sigma \chi / 2-\frac{N-1}{2} \operatorname{Tr}(\log (\mathcal{G}))}=\int D \chi D \sigma e^{-\widehat{S}\lceil\chi, \sigma, \psi]} \tag{I.0.1}
\end{equation*}
$$

For which, after rescaling the fields as $\chi=(N-1) \widehat{\chi}, V[\chi]=(N-1) \widehat{V}[\widehat{\chi}], \psi=(N-1)^{1 / 2} \widetilde{\psi}$, we obtain an overall factor of $(N-1)$ in $\widehat{S}$. Hence, making explicit the Planck constant $h$, the factor $N-1$ can be seen as an effective Planck constant $\widehat{h}=\frac{h}{N-1}$. As was mentioned in Sec. 2.3 the first correction in the $\widehat{h}$ expansion is obtained as a trace log of the second derivative of the action. As such, our first task in computing the first $1 / N$ correction is to calculate the Hessian of $\widehat{S}[\chi, \sigma, \psi]$ where we follow [148, 149 202] decomposing our fields as $(\chi, \sigma, \psi) \rightarrow(\chi, \sigma, \psi)+(\delta \chi, i \delta \sigma, \delta \psi)$. The quadratic terms may be regrouped into the following Hessian:

$$
\left(\begin{array}{ccc}
-\partial^{2}+\sigma & 0 & i \phi  \tag{I.0.2}\\
0 & V^{\prime \prime}(\chi) & -i / 2 \\
i \phi & -i / 2 & \frac{1}{-\partial^{2}+\sigma}
\end{array}\right)
$$

Following [202], the trace $\log$ of the Hessian is subtracted by $\frac{1}{2} \operatorname{Tr}\left(\log \left(-\partial^{2}+\sigma\right)\right){ }^{1}$ to obtain the correction to the effective potential. The trace log can then be written as a log det which after factorizing the $-\partial^{2}+\sigma$ term and writing the trace in terms of Fourier variables leads to the following correction to the potential:

$$
\begin{equation*}
V_{e f f}^{(1)}=\frac{1}{2} \int \frac{\mathrm{~d}^{d} p}{(2 \pi)^{d}} \log \left(1+2 V^{\prime \prime}(\chi) \int \frac{\mathrm{d}^{d} q}{(2 \pi)^{d}} \frac{1}{(p+q)^{2}} \frac{1}{q^{2}+\sigma}+\frac{4 \phi^{2} V^{\prime \prime}(\chi)}{p^{2}+\sigma}\right) \tag{I.0.3}
\end{equation*}
$$

Let us now notice that along the path $d=d(N)=3-\alpha / N$ the term $\alpha / N$ in the measure $\mathrm{d}^{d(N)} q$ of the integral can be neglected as this would be higher order in $1 / N$. The same is true for the renormalization scale that should be introduced to render the couplings in $V_{\text {eff }}$ dimensionless. Hence, we may replace $d$ by 3 in $V_{\text {eff }}^{(1)}$. The beta function can then be written as:

$$
\begin{equation*}
\beta(\tau)=\beta_{L}(\tau)+\beta_{N L}(\tau) \tag{I.0.4}
\end{equation*}
$$

where $\beta_{L}$ is the linear term in the beta function $\beta_{L}=-2(3-d) \tau=-2 \frac{\alpha}{N} \tau$ which is then obtained by rescaling $\tau$ in terms of the renormalization scale. $\beta_{N L}(\tau)=\beta_{d=3}(\tau)$ was obtained in [148, 149, 202. We refer the reader to [148, 202] for more detailed discussions, in the following we will only discuss a few elements of their calculations.

Reference [148] argues that $\sigma$ should be a divergent function of the cutoff which then perhaps justifies [202] in their expansion in large momenta or large $\sigma$ required to expand the logarithm in Eq.(I.0.3). From this expansion 202 obtains a loop expansion where we write here two of the four divergent terms:

$$
\begin{equation*}
2 V^{\prime \prime}(\chi) \psi^{2} \int \frac{\mathrm{~d}^{3} p}{(2 \pi)^{3}} \frac{1}{p^{2}+\sigma},-4 V^{\prime \prime}(\chi)^{2} \psi^{2} \int \frac{\mathrm{~d}^{3} p}{(2 \pi)^{3}} \frac{\mathrm{~d}^{3} q}{(2 \pi)^{3}} \frac{1}{p^{2}+\sigma} \frac{1}{q^{2}+\sigma} \frac{1}{(p+q)^{2}+\sigma} \tag{I.0.5}
\end{equation*}
$$

[^68]The first term diverges with the cut off like $\Lambda$ whereas the second one is logarithmically divergent. Hence these terms may be removed by the following counter terms:

$$
2 V^{\prime \prime}(\chi) \psi^{2} \int \frac{\mathrm{~d}^{3} p}{(2 \pi)^{3}} \frac{1}{p^{2}},-4 V^{\prime \prime}(\chi)^{2} \psi^{2} \int \frac{\mathrm{~d}^{3} p}{(2 \pi)^{3}} \frac{\mathrm{~d}^{3} q}{(2 \pi)^{3}} \frac{1}{p^{2}+\mu} \frac{1}{q^{2}+\mu} \frac{1}{(p+q)^{2}+\mu}
$$

where $\mu$ is a renormalization scale introduced to remove the logarithmic divergence. This introduction of a renormalization scale then leads the potential to depend on the scale $\mu$ for which promoting the couplings to functions of $\mu$, after including all counter terms, allows us to make the potential independent of this scale. Hence taking $\frac{\mathrm{d} V}{\mathrm{~d} \mu}=0$ leads to the desired beta function at $d=3$ which up to rescalings is equivalent to the perturbative beta function at $d=3$ in the limit $N \rightarrow \infty$.

As such along the path $d(N)=3-\frac{\alpha}{N}$ and in the limit $N \rightarrow \infty$ we retrieve, up to numerical rescalings of the couplings and the renormalization scale, the $\beta$ function:

$$
\begin{equation*}
\beta(\tau)=\beta_{L}(\tau)+\beta_{N L}(\tau)=-2 \frac{\alpha}{N} \tau+12 \frac{\tau^{2}}{N}\left(1-\pi^{2} \tau / 24\right) \tag{I.0.6}
\end{equation*}
$$

where indeed both $\beta_{L}$ and $\beta_{N L}$ are of order $1 / N$ and $\beta_{N L, d}=\beta_{N L, d=3}$ as terms involving $\alpha / N$ in $\beta_{N L, d}$ would lead to terms sub-leading in the $1 / N$ expansion of $\beta(\tau)$.

## Appendix J

## LPA Polchinski and $\Gamma$ flow equation

The exact Polchinski flow in $d$ dimensions for $N$ fields is given by:

$$
\begin{equation*}
\partial_{t} S_{k}[\boldsymbol{\phi}]=\frac{1}{2} \operatorname{Tr}\left(\partial_{t} \Delta_{k}(q)\left(S_{k}^{(1)}(q) S_{k}^{(1)}(-q)-S_{k}^{(2)}(q,-q)\right)\right) \tag{J.0.1}
\end{equation*}
$$

with $\boldsymbol{\phi}=\left(\phi_{a}\right)_{a}, \Delta_{k}(q)=\frac{K\left(q^{2} / k^{2}\right)}{q^{2}}$ where $K$ is given in Sec. 2.3.1,,$S_{k}^{(n)}\left(q_{1}, q_{2}, \ldots, q_{n}\right)=\frac{\delta^{n} S_{k}}{\delta \boldsymbol{\phi}\left(q_{1}\right) \delta \phi\left(q_{2}\right) \delta \boldsymbol{\phi}\left(q_{3}\right) \ldots \delta \boldsymbol{\sigma}\left(q_{n}\right)}$ and the trace acts both over spatial and internal indices. The LPA approximation then consists of taking $S_{k}$ to be

$$
\begin{equation*}
S_{k}=\int_{x} V_{k} \tag{J.0.2}
\end{equation*}
$$

Taking a uniform field one then obtains :

$$
\begin{equation*}
\partial_{t} V_{k}=\frac{1}{k^{2}}\left(\gamma\left(\frac{\partial}{\partial \phi_{a}} V_{k}\right)^{2}-\alpha \frac{\partial^{2} V_{k}}{\partial \phi_{a}^{2}}\right) \tag{J.0.3}
\end{equation*}
$$

where the summation over $a$ is left implicit and

$$
\begin{equation*}
\gamma=-K^{\prime}(0), \alpha=-\int_{q} K^{\prime}\left(q^{2} / k^{2}\right) \tag{J.0.4}
\end{equation*}
$$

The Litim regulator for the Polchinski flow is

$$
\begin{equation*}
K\left(q^{2} / k^{2}\right)=\left(1-q^{2} / k^{2}\right) \Theta\left(1-q^{2} / k^{2}\right) \tag{J.0.5}
\end{equation*}
$$

which then leads to :

$$
\begin{equation*}
\gamma=1, \alpha=\frac{2}{d L_{d}} k^{d} \tag{J.0.6}
\end{equation*}
$$

with $L_{d}=(4 \pi)^{d / 2} \Gamma(d / 2)$. If we rescale the potential and field as $V_{k}=\alpha N k^{d} \bar{V}_{k}$, and $\phi_{a}=\sqrt{\alpha N} k^{(d-2) / 2} \bar{\phi}_{a}$, set $\bar{\phi}_{a}=\bar{\phi} \delta_{1, a}($ a choice of axis $)$ and define $\bar{\varrho}=\bar{\phi}^{2} / 2$ we obtain :

$$
\begin{equation*}
\partial_{t} \bar{V}_{k}=1-d \bar{V}_{k}+(d-2) \bar{\varrho} \bar{V}_{k}^{\prime}+2 \bar{\varrho} \bar{V}_{k}^{\prime 2}-\bar{V}_{k}^{\prime}-\frac{2}{N} \bar{\varrho} \bar{V}_{k}^{\prime \prime} \tag{J.0.7}
\end{equation*}
$$

In the case of the $\Gamma$ flow:

$$
\begin{equation*}
\partial_{k} \Gamma_{k}=\frac{1}{2} \operatorname{Tr}\left(\frac{\partial_{t} R_{k}}{\Gamma_{k}^{(2)}+R_{k}}\right) \tag{J.0.8}
\end{equation*}
$$

The LPA of the $O(N)$ model consists in taking:

$$
\begin{equation*}
\Gamma_{k}=\int_{x} \frac{1}{2} \partial \varphi^{a} \partial \varphi^{a}+U_{k}(\varrho) \tag{J.0.9}
\end{equation*}
$$

with $\varrho=\varphi^{a} \varphi^{a} / 2$. Taking a uniform field and setting $\varphi_{a}=\varphi \delta_{1, a}$ leads to :

$$
\begin{equation*}
\partial_{t} U_{k}=(N-1) I\left[U_{k}^{\prime}(\varrho)\right]+I\left[U_{k}^{\prime}(\varrho)+2 \varrho U_{k}^{\prime \prime}(\varrho)\right] \tag{J.0.10}
\end{equation*}
$$

with

$$
\begin{equation*}
I[x]=\frac{1}{2} k^{-d} \int_{q} \frac{\partial_{t} R_{k}\left(q^{2}\right)}{q^{2}+R_{k}\left(q^{2}\right)+x k^{2}} . \tag{J.0.11}
\end{equation*}
$$

Choosing the Litim regulator $R_{k}\left(q^{2}\right)=\left(k^{2}-q^{2}\right) \Theta\left(k^{2}-q^{2}\right)$ and rescaling potential and field as $U_{k}=$ $d L_{d} N k^{d} \bar{U}_{k} / 2$ and $\varrho=d L_{d} N k^{d-2} \varrho$ $\varrho$ then leads to:

$$
\begin{equation*}
\partial_{t} \bar{U}=-d \bar{U}+(d-2) \bar{\rho} \bar{U}^{\prime}+\left(\frac{N-1}{N}\right) \frac{1}{1+\bar{U}^{\prime}}+\frac{1}{N} \frac{1}{1+\bar{U}^{\prime}+2 \bar{\rho} \bar{U}^{\prime}} . \tag{J.0.12}
\end{equation*}
$$

## Appendix K

## Derivation of $\tau(\alpha)$ at LPA

We recall the LPA differential equation in the Wilson-Polchinski formulation :

$$
\begin{equation*}
1-d \bar{V}+(d-2) \bar{\varrho} \bar{V}^{\prime}+2 \bar{\varrho} \bar{V}^{\prime 2}-\bar{V}^{\prime}-\frac{2}{N} \bar{\varrho} \bar{V}^{\prime \prime}=0 \tag{K.0.1}
\end{equation*}
$$

Differentiating this equation with respect to $\varrho$ and writing $v=\bar{V}^{\prime}$ and $1 / N=\epsilon$ we obtain:

$$
\begin{equation*}
((d-2) \bar{\varrho}-2 \epsilon-1) v^{\prime}-2 \bar{\varrho} \epsilon v^{\prime \prime}+2 v\left(2 \bar{\varrho} v^{\prime}+v-1\right)=0 . \tag{K.0.2}
\end{equation*}
$$

We now present two methods to obtain the relation between $\alpha$ and $\tau$. The first method is straightforward and requires expanding the potential in powers of $\epsilon$ and $\bar{\varrho}-1 \rrbracket$. However, this method gives information only in the vicinity of $\bar{\varrho}=1$ and does not explain the type of non analytical behavior obtained when one refuses to choose the relation between $\alpha$ and $\tau$. Thus, we will also use a fully functional method. This method has the advantage of yielding the potential at finite and large $N$ up to $1 / N^{2}$ corrections and is therefore useful to get the behavior near $\bar{\varrho}=0$ where a divergence appears at $N=\infty$ and $\tau=\tau_{B M B}$.

The first method consists in Taylor expanding Eq. K.0.2) about $\bar{\varrho}=1$ (the inflexion point) with $v=$ $\sum a_{k}(\bar{\varrho}-1)^{k}$. We moreover expand the couplings $a_{k}$ in powers of $\epsilon$ as $a_{k}=a_{k}^{0}+\epsilon a_{k}^{1}+O\left(\epsilon^{2}\right)$ where the $a_{k}^{0}$ 's are the couplings involved in the expansion of the FP potential given at $N=\infty$. The system of equations obtained by independently setting equal to 0 the coefficients of $\epsilon^{n}(\bar{\varrho}-1)^{p}$ yields the relation between $\alpha$ and $\tau$. Notice that in this method the $\tau$ dependence comes from the $a_{k}^{0}$ s.

The functional method consists in expanding $v$ as $v=v_{0}+\epsilon v_{1}+O\left(\epsilon^{2}\right)$ in Eq. (K.0.2). At order $\epsilon$, this yields a differential equation on $v_{1}$ that depends on $\bar{\varrho}, v_{0}, v_{0}^{\prime}$ and $v_{0}^{\prime \prime}$. Using Eq. K.0.2 and its derivative both evaluated at $\epsilon=0, v_{0}^{\prime \prime}$ and $v_{0}^{\prime}$ can be eliminated in terms of $v_{0}$. This leads to:

$$
\begin{array}{r}
32 \bar{\varrho}^{2} v_{0}^{4}\left(\alpha \bar{\varrho}+4 \bar{\varrho}\left(2 \bar{\varrho} v_{1}^{\prime}+v_{1}\right)-1\right)+16 \bar{\varrho}^{2} v_{0}^{3}\left(-\alpha(\bar{\varrho}+1)+8(\bar{\varrho}-1)\left(2 \bar{\varrho} v_{1}^{\prime}+v_{1}\right)+1\right)+ \\
2 v_{0}^{2}\left(\bar{\varrho}\left(-7 \alpha \bar{\varrho}^{2}+6(\alpha+2) \bar{\varrho}+\alpha-6\right)+4(\bar{\varrho}-1) \bar{\varrho}\left(12(\bar{\varrho}-1) \bar{\varrho} v_{1}^{\prime}+(\bar{\varrho}-5) v_{1}\right)+2\right) \\
2(\bar{\varrho}-1) v_{0}\left(\bar{\varrho}(\alpha(-\bar{\varrho})+\alpha-4)+8 \bar{\varrho}(\bar{\varrho}-1)^{2} v_{1}^{\prime}+\left(-6 \bar{\varrho}^{2}+4 \bar{\varrho}+2\right) v_{1}+2\right)+(\bar{\varrho}-1)^{3}\left((\bar{\varrho}-1) v_{1}^{\prime}-2 v_{1}\right)=0 . \tag{K.0.3}
\end{array}
$$

We then assume that $v_{1}$ is analytic at $h=\bar{\varrho}-1=0$. The Taylor expansion of $v_{0}$ at $h=0$ is:

$$
\begin{array}{r}
v_{0}=\frac{h^{2} \tau}{2}-2 h^{3} \tau^{2}+\frac{5}{4} h^{4} \tau^{2}(8 \tau-1)+h^{5}(13-56 \tau) \tau^{3}+ \\
\frac{7}{8} h^{6} \tau^{3}(128 \tau(3 \tau-1)+5)+h^{7}\left(-2112 \tau^{6}+912 \tau^{5}-\frac{383 \tau^{4}}{5}\right)+O\left(h^{8}\right) . \tag{K.0.4}
\end{array}
$$

Inserting Eq. $\overline{\text { K.0.4 }}$ into Eq. K.0.3), neglecting terms of order 5, and dividing by $h^{3}$ gives :

$$
\begin{equation*}
-\tau h^{2}(\alpha+6 \tau(12 \tau-5))+\left(8 h^{2} \tau+h\right) v_{1}^{\prime}+\left(6 h^{2} \tau(4 \tau-1)-8 h \tau-2\right) v_{1}-h \tau(\alpha-12 \tau+4)-2 \tau=0 . \tag{K.0.5}
\end{equation*}
$$

[^69]Finally, replacing $v_{1}$ in Eq. K.0.5 by its Taylor expansion: $v_{1}=v_{1}^{(0)}+v_{1}^{(1)} h+v_{1}^{(2)} h^{2}+O(h)^{3}$ leads to:

$$
\begin{align*}
v_{1}^{(0)} & =-\tau  \tag{K.0.6a}\\
v_{1}^{(1)} & =\tau(-\alpha+20 \tau-4)  \tag{K.0.6b}\\
\alpha & =36 \tau-96 \tau^{2} . \tag{K.0.6c}
\end{align*}
$$

Notice that it is because the $v_{1}^{(2)}$ term cancels in Eq. K.0.6c that we can obtain a relation between $\alpha$ and $\tau$ only.

Let us finally notice that Eq. $\overline{\mathrm{K} .0 .6 \mathrm{c}}$ can be retrieved in a functional way. The solution of Eq. $\overline{\mathrm{K} .0 .3}$ is:

$$
\begin{array}{r}
v_{1}(\bar{\varrho})=\exp (K(\bar{\varrho}))\left(C-\int_{1}^{\bar{\varrho}} 2 e^{-K(\chi)}\left(\frac{16 \alpha \chi^{3} v_{0}(\chi)^{4}-8 \alpha \chi^{3} v_{0}(\chi)^{3}-7 \alpha \chi^{3} v_{0}(\chi)^{2}-\alpha \chi^{3} v_{0}(\chi)-8 \alpha \chi^{2} v_{0}(\chi)^{3}}{\left(4 \chi v_{0}(\chi)+\chi-1\right)^{4}}+\right.\right. \\
\frac{6 \alpha \chi^{2} v_{0}(\chi)^{2}+2 \alpha \chi^{2} v_{0}(\chi)+\alpha \chi v_{0}(\chi)^{2}-\alpha \chi v_{0}(\chi)-16 \chi^{2} v_{0}(\chi)^{4}}{\left(4 \chi v_{0}(\chi)+\chi-1\right)^{4}}+ \\
 \tag{K.0.7}\\
\left.\frac{\left.8 \chi^{2} v_{0}(\chi)^{3}+12 \chi^{2} v_{0}(\chi)^{2}-4 \chi^{2} v_{0}(\chi)-6 \chi v_{0}(\chi)^{2}+2 v_{0}(\chi)^{2}+6 \chi v_{0}(\chi)-2 v_{0}(\chi)\right)}{\left(4 \chi v_{0}(\chi)+\chi-1\right)^{4}} d \chi\right)
\end{array}
$$

where $C$ is an integration constant and

$$
\begin{gather*}
K(\bar{\varrho})=\int_{1}^{\bar{\varrho}}-2\left(\frac{64 \chi^{3} v_{0}(\chi)^{4}+64 \chi^{3} v_{0}(\chi)^{3}+4 \chi^{3} v_{0}(\chi)^{2}-6 \chi^{3} v_{0}(\chi)-64 \chi^{2} v_{0}(\chi)^{3}-24 \chi^{2} v_{0}(\chi)^{2}}{\left(4 \chi v_{0}(\chi)+\chi-1\right)^{4}}+\right. \\
\left.\frac{10 \chi^{2} v_{0}(\chi)+20 \chi v_{0}(\chi)^{2}-2 \chi v_{0}(\chi)-2 v_{0}(\chi)-\chi^{3}+3 \chi^{2}-3 \chi+1}{\left(4 \chi v_{0}(\chi)+\chi-1\right)^{4}} d \chi\right) . \tag{K.0.8}
\end{gather*}
$$

Replacing $v_{0}$ in Eq. (K.0.7) by its Taylor expansion (K.0.4) yields:
$v_{1}(\bar{\varrho})=-\tau-h \tau(\alpha-20 \tau+4)+h^{2}\left(\tau^{2}(8 \alpha-156 \tau+37)+\tau(\alpha+12 \tau(8 \tau-3)) \log (h)+C\right)+O\left(h^{3} \log (h)\right)$.
(K.0.9)

The analyticity of $v_{1}$ implies that the log term in Eq. K.0.9) is absent. This requires that its prefactor vanishes, that is, $\alpha=36 \tau-96 \tau^{2}$. To all orders checked (up to 5th order) this also eliminates the following log terms.

Notice that the expression (K.0.7) giving $v_{1}(\bar{\varrho})$ is ill-conditioned for a numerical plot of this function because of the poles of the integrands of $K$ in Eq. K.0.8) and in $v_{1}$ in Eq. K.0.7). Although the final expression for $v_{1}$ is well-defined it is tricky to get rid of apparent divergencies showing up because of the poles within the integrands: This requires adding and subtracting divergencies and making some integration by parts. For this reason, it is simpler to numerically integrate Eq.(K.0.3).

## Appendix L

## LPA singular solutions as weak solutions

In this section we show that the singular solutions $S \mathcal{A}(\tau)$ are weak solutions at $N=\infty$. We will use the notion of weak solutions in the sense of distributions. There is also a definition in terms of viscosity solutions but this last definition is more abstract 1

To see how this works consider a strong solution $\bar{V}$ which is simply the usual definition of a solution, that is, $\bar{V}$ verifies the fixed-point differential equation :

$$
\begin{equation*}
1-d \bar{V}+(d-2) \bar{\varrho} \bar{V}^{\prime}+2 \bar{\varrho} \bar{V}^{\prime 2}-\bar{V}^{\prime}=0 \tag{L.0.1}
\end{equation*}
$$

Let us then differentiate this equation with respect to $\bar{\varrho}$ and set $v=\bar{V}^{\prime}$ which leads to :

$$
\begin{equation*}
((d-2) \bar{\varrho}-1) v^{\prime}+2 v\left(2 \bar{\varrho} v^{\prime}+v-1\right)=0 . \tag{L.0.2}
\end{equation*}
$$

Consider then multiplying this equation by a test function $\lambda(\bar{\varrho})$ and integrating $\bar{\varrho}$ over $\mathbb{R}^{+}$as :

$$
\begin{equation*}
\int_{\mathbb{R}^{+}}\left(((d-2) \bar{\varrho}-1) v^{\prime}+2 v\left(2 \bar{\varrho} v^{\prime}+v-1\right)\right) \lambda(\bar{\varrho})=0 . \tag{L.0.3}
\end{equation*}
$$

We may then integrate by parts, considering that $\lambda(\bar{\varrho})$ decays at the boundaries sufficiently fast that the boundary terms may be discarded. Then, using $2 v v^{\prime}=\left(v^{2}\right)^{\prime}$ we find:

$$
\begin{equation*}
\int_{\mathbb{R}^{+}} v\left(-d \lambda+\lambda^{\prime}(1-(d-2) \bar{\varrho}-2 v \bar{\varrho})\right)=0 . \tag{L.0.4}
\end{equation*}
$$

The advantage of this last expression is that cusped solutions which are only piecewise continuous lead to a well defined integral whereas they lead to ill-defined differential equations. Consider then a piecewise function $v$ which is discontinuous at $\bar{\varrho}=\bar{\varrho}_{0}$ and is a strong solution of Eq. L.0.2) for $\bar{\varrho} \neq \bar{\varrho}_{0}$. We may then separate the integral as $\int_{\mathbb{R}^{+}}=\int_{\left[0, \bar{\varrho}_{0}[ \right.}+\int_{] \bar{\varrho}_{0},+\infty[ }$ and perform integration by parts on each integral. As $v$ is a strong solution for $\bar{\varrho} \neq \bar{\varrho}_{0}$, the integral terms are null by definition and one is left with only the boundary terms. As usual the boundary terms at 0 and $+\infty$ are null and one is then left with the boundary terms at $\bar{\varrho}=\bar{\varrho}_{0}^{ \pm}$. The integral over $\int_{\mathbb{R}^{+}}$being zero this implies that the boundary terms must sum to zero, that is:

$$
\begin{equation*}
\left(1-(d-2) \bar{\varrho}_{0}-2 v^{+} \bar{\varrho}_{0}\right) v^{+}=\left(1-(d-2) \bar{\varrho}_{0}-2 v^{-} \bar{\varrho}_{0}\right) v^{-} \tag{L.0.5}
\end{equation*}
$$

where $v^{ \pm}=\lim _{\bar{\varrho} \rightarrow \bar{\varrho}_{0}^{ \pm}} v(\bar{\varrho})$. This adds a constraint for a function that is piecewise strong to be a weak solution. In the framework of time dependent shock solutions this is called the Rankine-Hugoniot jump condition. Let us verify that for $d=3, S \mathcal{A}(\tau)$ satisfies this condition.

From Eq. M.0.3) in Appendix 0 we show that

$$
\begin{equation*}
\left(1-\bar{\varrho}_{0}-2 v^{ \pm} \bar{\varrho}_{0}\right) v^{ \pm}=1-3 \bar{\varrho}_{0} \tag{L.0.6}
\end{equation*}
$$

and thus Eq. L.0.5 is verified such that $S \mathcal{A}(\tau)$ is indeed a weak solution.

[^70]
## Appendix M

## Singular perturbation theory for the LPA

We now show how to compute the boundary layers of the singular FPs $S A_{3}$ and $S \tilde{A}_{4}$. We use the following change of variables: $\bar{\varrho}=\frac{\tilde{\varrho}}{N}+\bar{\varrho}_{0}, \bar{V}=\frac{\tilde{V}}{N}+\bar{V}_{0}$, where the point $\left(\bar{\varrho}_{0}, \bar{V}_{0}\right)$ corresponds to the location of the cusp. As we are working here at the lowest order in $\epsilon=3-d$, one may check that $d(N)=3-\alpha / N$ can be replaced in Eq. K.0.1 by $d(N)=3$. Moreover, the potential behaves linearly as $\bar{V}=\bar{\varrho}$ towards the left of the cusp, thus at the cusp $\bar{\varrho}_{0}$ we have: $\bar{V}_{0}=\bar{\varrho}_{0}$. Inserting these elements in Eq. K.0.1), we obtain to leading order in $\epsilon$ :

$$
\begin{equation*}
0=1-3 \bar{\varrho}_{0}+\bar{\varrho}_{0} \tilde{V}^{\prime}+2 \bar{\varrho}_{0} \tilde{V}^{\prime^{2}}-\tilde{V}^{\prime}-2 \bar{\varrho}_{0} \tilde{V}^{\prime \prime} \tag{M.0.1}
\end{equation*}
$$

whose solution is

$$
\begin{equation*}
\tilde{V}^{\prime}(\tilde{\varrho})=\frac{\left(5 \bar{\varrho}_{0}-1\right) \tanh \left(\frac{\left(1-5 \bar{\varrho}_{0}\right)(\tilde{\varrho}-C)}{4 \bar{\varrho}_{0}}\right)-\bar{\varrho}_{0}+1}{4 \bar{\varrho}_{0}} \tag{M.0.2}
\end{equation*}
$$

where $C$ is some integration constant.
Moreover, using K.0.1 for $1 / N=\epsilon=0$ and $d=3$ without the previous change of variables $\bar{\varrho}=\frac{\tilde{\varrho}}{N}+\bar{\varrho}_{0}$, $\bar{V}=\frac{\tilde{V}}{N}+\bar{V}_{0}$, we obtain:

$$
\begin{equation*}
\bar{V}^{\prime}(\bar{\varrho})\left(\bar{\varrho}+2 \bar{\varrho} \bar{V}^{\prime}(\bar{\varrho})-1\right)-3 \bar{V}(\bar{\varrho})+1=0 . \tag{M.0.3}
\end{equation*}
$$

By evaluating Eq. M.0.3) at $\bar{\varrho}=\bar{\varrho}_{0}=\bar{V}_{0}$, we obtain two possible values for $\bar{V}^{\prime}\left(\varrho_{0}\right)$ :

$$
\left\{\begin{array}{l}
\bar{V}_{-}^{\prime}\left(\bar{\varrho}_{0}\right)=1  \tag{M.0.4}\\
\bar{V}_{+}^{\prime}\left(\bar{\varrho}_{0}\right)=\frac{1}{2}\left(\frac{1}{\bar{\varrho}_{0}}-3\right) .
\end{array}\right.
$$

These are the two distinct derivatives at the cusp with $\bar{V}_{-}^{\prime}\left(\bar{\varrho}_{0}\right)=\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{-}\right)$and $\bar{V}_{+}^{\prime}\left(\bar{\varrho}_{0}\right)=\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{+}\right)$. Using this we may rewrite $\tilde{V}^{\prime}(\tilde{\varrho})$ as

$$
\begin{equation*}
\tilde{V}^{\prime}(\tilde{\varrho})=\frac{-\left(\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{-}\right)-\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{+}\right)\right) \tanh \left(\frac{\left(\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{-}\right)-\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{+}\right)\right)(\tilde{\varrho}-C)}{2}\right)+\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{+}\right)+\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{-}\right)}{2} . \tag{M.0.5}
\end{equation*}
$$

Finally we choose $C=0$ such that for $\bar{\varrho}=\bar{\varrho}_{0}($ and thus $\tilde{\varrho}=0)$ we have $\tilde{V}^{\prime}(0)=\frac{\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{+}\right)+\bar{V}^{\prime}\left(\bar{\varrho}_{0}^{-}\right)}{2}$.

## Appendix N

## Boundary layer analysis of fixed-point SG

The leading order of the boundary layer analysis is capable of describing the rounding of the cusp but it is unable to reproduce the non trivial variations of the regular part of the FP to the right of the cusp. The next correction to the boundary layer analysis leads to a solution in terms of an integral from which we have not been able to extract much information. Let us then consider instead the FP $S G=S \mathcal{A}(\tau=0)$ whose nature above the cusp is trivially flat. In this case the boundary layer analysis can be extended to all $\bar{\varrho}$ in a meaningful way. Indeed, from the boundary layer analysis we find the following expression for $S G$ in $d=3$ :

$$
\begin{equation*}
\bar{V}(\bar{\varrho}, \epsilon)=\frac{1}{2}\left(-2 \epsilon \log \left(\operatorname{sech}\left(\frac{1}{3 \epsilon}\right) \cosh \left(\frac{1-3 \bar{\varrho}}{6 \epsilon}\right)\right)+\bar{\varrho}-1\right)+\frac{1}{3} \tag{N.0.1}
\end{equation*}
$$

with $\epsilon=1 / N$. The plot of this function is given in Fig. (N.0.1).
Inserting Eq. N.0.1 into the Polchinski flow equation we do not find zero as Eq. N.0.1) is only an approximate solution. Nonetheless, the function obtained from this insertion may be used as a measure of the error of the approximation:

$$
\begin{equation*}
\operatorname{error}(\bar{\varrho})=\frac{1}{2}(3 \bar{\varrho}-1) \tanh \left(\frac{1-3 \bar{\varrho}}{6 \epsilon}\right)+3 \epsilon \log \left(\operatorname{sech}\left(\frac{1}{3 \epsilon}\right) \cosh \left(\frac{1-3 \bar{\varrho}}{6 \epsilon}\right)\right)+1 \tag{N.0.2}
\end{equation*}
$$

This error function is plotted in Fig. N.0.2 and shows that the approximation is very good except at the boundary layer where the error is $\mathrm{O}(1 / N)$. Hence, we may consider that the exact $S G$ potential, at the level of the LPA, is accurately described by Eq. N.0.1 outside of the boundary layer ${ }^{1}$. In particular the large $\bar{\varrho}, N$ asymptotics of Eq. N.0.1 leads to:

[^71]

Figure N.0.1: Plot of the $S G$ FP potential $\bar{V}(\bar{\varrho})$ at $(d=3, N=100)$ as given by the approximation in Eq. N.0.1.


Figure N.0.2: Error function given in Eq. (N.0.2) which measures the accuracy of Eq. N.0.1 for $N=100$ (left) and $N=1000$ (right).

$$
\begin{equation*}
\bar{V}(\bar{\varrho}, \epsilon) \simeq \frac{1}{3}-\epsilon\left(e^{\frac{1-3 \bar{\varrho}}{3 \epsilon}}-e^{\frac{-2}{3 \epsilon}}\right) \simeq \frac{1}{3}+\epsilon e^{\frac{-2}{3 \epsilon}} \tag{N.0.3}
\end{equation*}
$$

such that $\bar{V}(\bar{\varrho}, \epsilon)-\bar{V}(\bar{\varrho}, 0)$ is exponentially suppressed for $\bar{\varrho}$ large. In the case of $\bar{\varrho}=0$ and $N$ large we obtain:

$$
\begin{equation*}
\bar{V}(0)=-\epsilon e^{-\frac{1}{3 \epsilon}} \tag{N.0.4}
\end{equation*}
$$

Thus, outside of the boundary layer, the corrections to $\bar{V}(\bar{\varrho}, 0)$ at small $\epsilon$ are indeed exponentially suppressed.

## Appendix O

## LPA equivalence of Polchinski and $\Gamma$ flow for Litim regulator

From the map:

$$
\begin{equation*}
\bar{V}\left(\bar{\varrho}=\frac{\varphi_{a} \varphi_{a}}{2}\right)=\bar{U}\left(\bar{\rho}=\frac{\phi_{a} \phi_{a}}{2}\right)+\frac{1}{2}\left(\varphi_{a}-\phi_{a}\right)^{2} \tag{O.0.1}
\end{equation*}
$$

we find:

$$
\begin{equation*}
\sqrt{\frac{\overline{\bar{\rho}}}{\bar{\varrho}}}=1-\bar{V}^{\prime}=\frac{1}{1+\bar{U}^{\prime}} \tag{O.0.2}
\end{equation*}
$$

Eq. O.0.2 implies that the small $\bar{\varrho}$ region in which the singular potentials at large $N$ verify $\bar{V}^{\prime} \simeq 1$ in the Polchinski formulation is mapped to the small $\bar{\rho}$ region where $\bar{U}^{\prime} \gg 1$ in the $\Gamma$ flow formulation. More precisely, from Appendix P in the case of the FP $S G=S \mathcal{A}(\tau=0)$ one may check that we have $\bar{V}^{\prime}=$ $1+\mathrm{O}\left(e^{-N / 6}\right)$ in the small $\bar{\varrho}$ regimes which then implies:

$$
\begin{equation*}
\bar{U}^{\prime}(0)=\mathrm{O}\left(e^{N / 6}\right) \tag{O.0.3}
\end{equation*}
$$

Thus in the neighborhood of the fixed-point and in dimensional variables we have:

$$
\begin{equation*}
m_{k}^{2}=k^{2} e^{N / 6} \tag{O.0.4}
\end{equation*}
$$

meaning that the mass becomes small only for very small values of $k$. By continuity, we expect that the finite extension of the FPs $S \mathcal{A}(\tau)$ will have masses in the neighborhood of their respective fixed-points as:

$$
\begin{equation*}
m_{k}^{2}(\tau)=k^{2} e^{N f(\tau)} \tag{O.0.5}
\end{equation*}
$$

If instead we consider $\bar{U}$ as a function of $\bar{\phi}=\sqrt{2 \bar{\rho}}$ then from :

$$
\begin{equation*}
\frac{\mathrm{d} \bar{U}}{\mathrm{~d} \bar{\phi}}=\bar{\phi} \frac{\mathrm{d} \bar{U}}{\mathrm{~d} \bar{\rho}} \tag{O.0.6}
\end{equation*}
$$

we see that $\bar{U}(\bar{\phi})$ has a large slope at the origin in the case of the singular FPs. In the case of the FP $S G$ we indeed find such a large slope at the origin as in Fig. O.0.1 ${ }^{1}$. In the large field domain the singular FP behaves as a regular FP and in particular for the $S G \overline{\mathrm{FP}}$ it is completely flat for large fields both in the Polchinski and in the $\Gamma$ flow version. For the other singular FPs the mapping at large fields is given by Eq. O.0.2 where the qualitative shape of the potentials are similar as shown for $(N=\infty, d=3)$ in [162]. In between the small $\bar{\phi}$ and large $\bar{\phi}$ regions there is an entire interval where $\bar{U}^{\prime \prime}(\bar{\phi}) \simeq-1$ such that the contribution from the longitudinal propagator in the LPA equation:

$$
\begin{equation*}
\frac{1}{N} \frac{1}{1+\bar{U}^{\prime \prime}(\bar{\phi})} \tag{O.0.7}
\end{equation*}
$$

[^72]

Figure O.0.1: $\bar{U}^{\prime}(\bar{\phi})$ for the FP $S G$ in the $\Gamma$ flow version at $d=3.2$ for different values of $N$.
is of order one. This is to be paralleled with :

$$
\begin{equation*}
\frac{1}{N} \bar{\varrho} \bar{V}^{\prime \prime}=\mathrm{O}(1) \tag{O.0.8}
\end{equation*}
$$

in the Polchinski version of the flow. The precise mapping between the two is 162 :

$$
\begin{equation*}
\frac{1}{N} \frac{1}{1+\bar{U}^{\prime \prime}(\bar{\phi})}=1-\bar{V}^{\prime}-\frac{2}{N} \bar{\varrho} \bar{V}^{\prime \prime} \tag{O.0.9}
\end{equation*}
$$

The novelty in the $\Gamma$ flow version however is that the region in which Eq. O.0.7 is of order 1 converges to an interval of finite length in the limit $N \rightarrow \infty$ as can be seen in Fig. O.0.1). This is to be contrasted with the Polchinski version where the region in which we have $\frac{1}{N} \bar{\varrho} \bar{V}^{\prime \prime}=\mathrm{O}(1)$ converges to a single point forming the cusp. These extra complications in the construction of the singular potentials in the $\Gamma$ flow formulation are the reasons why we prefered to study these potentials within the Polchinski formulation. In fact, currently we are unaware how to to find the location of the singular points in the $\Gamma$ flow formulation without first constructing these potentials in the Polchinski version and then using the map between the two.

## Appendix P

## SG eigenvalues

To compute eigenvalues one has to first find a solution to the FP equation. Unfortunately we have found a global approximation only in the case of the FP $S G$. The approximation for the potential of the FP SG at $d=3$ was given in Appendix P and we display the result here:

$$
\begin{equation*}
\bar{V}^{*}(\bar{\varrho}, \epsilon)=\frac{1}{2}\left(-2 \epsilon \log \left(\operatorname{sech}\left(\frac{1}{3 \epsilon}\right) \cosh \left(\frac{1-3 \bar{\varrho}}{6 \epsilon}\right)\right)+\bar{\varrho}-1\right)+\frac{1}{3} \tag{P.0.1}
\end{equation*}
$$

where $\epsilon=1 / N$. To find the eigenvalues associated to this FP potential we consider a perturbation of this solution as $V=\bar{V}^{*}+\delta V(\bar{\varrho}) e^{\lambda t}$ which we insert into the time dependent LPA equation and linearize omitting terms of order $\delta V^{2}$. Doing this with the FP $S G$ we find:

$$
\begin{equation*}
-2 \bar{\varrho} \epsilon \delta \mathrm{~V}^{\prime \prime}+2 \bar{\varrho} \delta \mathrm{~V}^{\prime}\left(\tanh \left(\frac{1-3 \bar{\varrho}}{6 \epsilon}\right)+1\right)+\bar{\varrho} \delta \mathrm{V}^{\prime}-\delta \mathrm{V}^{\prime}-(3+\lambda) \delta \mathrm{V}=0 \tag{P.0.2}
\end{equation*}
$$

We expect a boundary layer from the eigenfunctions and thus as in Appendix $O$ we parameterize the solutions as :

$$
\begin{align*}
\delta \mathrm{V} & =\delta \mathrm{V}_{0}+\epsilon \delta \mathrm{V}_{1}\left(\frac{\bar{\varrho}-\frac{1}{3}}{\epsilon}\right)  \tag{P.0.3}\\
\bar{\varrho} & =\frac{1}{3}+\epsilon \widetilde{\varrho}
\end{align*}
$$

where $\delta \mathrm{V}_{0}$ is a constant and $\bar{\varrho}=\frac{1}{3}$ is the position of the cusp for the FP $S G$ in $d=3$. Inserting Eq. P.0.3 into Eq.(P.0.2) we obtain:-

$$
\begin{equation*}
-(\lambda+3) \epsilon \delta \mathrm{V}_{1}-(\lambda+3) \delta \mathrm{V}_{0}-\frac{2}{3}(3 \widetilde{\varrho} \epsilon+1) \delta \mathrm{V}_{1}{ }^{\prime \prime}+\frac{1}{3} \delta \mathrm{~V}_{1}{ }^{\prime}\left(9 \widetilde{\varrho} \epsilon-2(3 \widetilde{\varrho} \epsilon+1) \tanh \left(\frac{\widetilde{\varrho}}{2}\right)\right)=0 \tag{P.0.4}
\end{equation*}
$$

Consider then the limit $\epsilon \rightarrow 0$ :

$$
\begin{equation*}
-(\lambda+3) \delta \mathrm{V}_{0,0}-\frac{2}{3} \delta \mathrm{~V}_{1,0}{ }^{\prime \prime}-\frac{2}{3} \delta \mathrm{~V}_{1,0}{ }^{\prime} \tanh \left(\frac{\widetilde{\varrho}}{2}\right)=0 \tag{P.0.5}
\end{equation*}
$$

whose solution is :

$$
\begin{equation*}
\delta \mathrm{V}_{1,0}=\frac{1}{2} \tanh \left(\frac{\widetilde{\varrho}}{2}\right)\left(-3(\lambda+3) \delta \mathrm{V}_{0,0} \widetilde{\varrho}+A\right)+B . \tag{P.0.6}
\end{equation*}
$$

Consider then the eigenfunction that is approximately constant at small $\varrho$ and null at large $\varrho$ 解ich was motivated in Sec.(3.1.9). As $\delta \mathrm{V}_{1}$ is then bounded we should have $\delta \mathrm{V}_{0}=0$ such that:

$$
\begin{equation*}
\delta \mathrm{V}_{1,0}=\frac{A}{2} \tanh \left(\frac{\widetilde{\varrho}}{2}\right)+B \tag{P.0.7}
\end{equation*}
$$

We may then consider $B=-A / 2$ which leads to:

$$
\begin{equation*}
\delta \mathrm{V}_{1,0}=\frac{A}{2}\left(\tanh \left(\frac{3 \bar{\varrho}-1}{2 \epsilon}\right)-1\right) \tag{P.0.8}
\end{equation*}
$$

The constant $A$ is then just a choice for the normalization of the eigenfunction and so we we may consider $A=2$. To find the eigenvalue associated to this eigenfunction we need to consider the next order in $\epsilon$ as:

$$
\begin{equation*}
\delta \mathrm{V}_{1}=\delta \mathrm{V}_{1,0}+\epsilon \delta \mathrm{V}_{1,1} \tag{P.0.9}
\end{equation*}
$$

Inserting this into Eq. $(\overline{\text { P.0.4 }}$ ) one obtains:

$$
\begin{equation*}
3 \operatorname{sech}^{2}\left(\frac{\widetilde{\varrho}}{2}\right)((\lambda+3) \sinh (\widetilde{\varrho})-(\lambda+3) \cosh (\widetilde{\varrho})-3 \widetilde{\varrho}-(3+\lambda))-4 \delta \mathrm{~V}_{1,1}^{\prime \prime}-4 \tanh \left(\frac{\widetilde{\varrho}}{2}\right) \delta \mathrm{V}_{1,1}^{\prime}=0 \tag{P.0.10}
\end{equation*}
$$

The solution to this differential equation is:

$$
\begin{align*}
\delta \mathrm{V}_{1,1}= & B+\frac{1}{4}\left(3 \widetilde{\varrho}\left(2(\lambda+3)+3 \widetilde{\varrho}+12 \log \left(e^{-\widetilde{\varrho}}+1\right)\right)+\right. \\
& \left.-36 \operatorname{Li}_{2}\left(-e^{-\widetilde{\varrho}}\right)+\tanh \left(\frac{\widetilde{\varrho}}{2}\right)\left(-3\left(2(\lambda+3)+3 \widetilde{\varrho}^{2}\right)-6(\lambda+3) \widetilde{\varrho}+8 A\right)\right) \tag{P.0.11}
\end{align*}
$$

The integration constants may be determined by imposing $\delta \mathrm{V}_{1,1}(\infty)=0$ which leads to:

$$
\begin{equation*}
\delta \mathrm{V}_{1,1}=B+\frac{1}{4}(-6(\lambda+3)+8 A) \tag{P.0.12}
\end{equation*}
$$

Consider now the limit $\widetilde{\varrho} \rightarrow-\infty$ :

$$
\begin{equation*}
\delta \mathrm{V}_{1,1} \sim B+\frac{1}{4}\left(12 \widetilde{\varrho}(\lambda+3)+6 \pi^{2}+6(\lambda+3)+8 A\right) \tag{P.0.13}
\end{equation*}
$$

It is then necessary to impose $\lambda+3=0$ to obtain a bounded perturbation which allows us to retrieve the eigenvalue $\lambda=-3$ where the associated eigenfunction is constant then negligeable above the boundary layer. As such we also have:

$$
\begin{equation*}
B=2 A \tag{P.0.14}
\end{equation*}
$$

If we were to impose a normalization condition such as $\left\|\delta \mathrm{V}_{1}\right\|_{2}^{2}=1$ where $\left\|\|_{2}\right.$ is the usual $L_{2}$ norm for functions then we would change the value of $A$ in $\delta \mathrm{V}_{1,0}$ and $B$ would be determined from this constraint as well.

To consider instead the other eigenfunctions we have resorted to a numerical analysis of Eq. $(\bar{P} .0 .4$ from which we obtain Fig.(P.0.1). This figure may be compared to Fig.(3.1.17).


Figure P.0.1: Eigenfunctions of the SG FP at $(d=3, N=100)$. Top: Eigenfunctions that become flat above the boundary layer. The yellow curve on the top right corresponds to an eigenfunction of the potential $\bar{V}=\bar{\varrho}$ which then shows that it is nearly identical to the eigenfunction of $S G$ below the boundary layer. Bottom: Eigenfunctions that become flat below the boundary layer. The yellow curves correspond to eigenfunctions of the Gaussian potential showing that it is nearly identical to the eigenfunction of $S G$ above the boundary layer.

## Appendix Q

## Coupling to $(d, N)$ space mapping for all multicritical dimensions

In the following we shall derive the analogue of the $\tau(\alpha)$ function from the tricritical analysis to the case of all critical dimensions $d_{c}(p)=2+2 / p$ for $p>1$. We will restrict ourselves to the LPA in the Polchinski formulation as the equations are simpler. In the following we shall write $V$ and $\rho$ rather than $\bar{V}$ and $\bar{\varrho}$. The LPA flow equation in terms of $v=V^{\prime}$ is obtained by taking a derivative with respect to $\rho$ of the LPA potential equation. This leads to :

$$
\begin{equation*}
-2 v(\rho)+(d-2) \rho v^{\prime}(\rho)-(1+2 \epsilon) v^{\prime}(\rho)+2 v(\rho)^{2}+4 \rho v(\rho) v^{\prime}(\rho)-2 \rho \epsilon v^{\prime \prime}(\rho)=0 \tag{Q.0.1}
\end{equation*}
$$

where $\epsilon=1 / N$. We then expand $v$ in powers of $\epsilon$ as :

$$
\begin{equation*}
v=v_{0}+\epsilon v_{1}+\epsilon^{2} v_{2} \tag{Q.0.2}
\end{equation*}
$$

and $d$ as

$$
d=2+2 / p-\epsilon \alpha-\epsilon^{2} \alpha_{2}
$$

From the zeroth order we find the following series expansion ${ }^{1}$

$$
\begin{align*}
& v_{0}=h^{2}\left(-896 h^{3} \tau_{2}^{4}+8 h^{2}(13 h+10) \tau_{2}^{3}-h(5 h+8) \tau_{2}^{2}+\tau_{2}\right) \quad(p=2) \\
& v_{0}=\frac{81}{5}(16 h+15) h^{7} \tau_{3}^{3}-\frac{1}{2}(14 h+27) h^{5} \tau_{3}^{2}+h^{3} \tau_{3} \quad(p=3)  \tag{Q.0.3}\\
& v_{0}=\frac{1}{63} h^{4} \tau_{4}\left(64(531 h+616) h^{6} \tau_{4}^{2}-21(27 h+64) h^{3} \tau_{4}+63\right) \quad(p=4)
\end{align*}
$$

where $h=\rho-\frac{p}{2}$. The first order correction in $\epsilon$ leads to:

$$
\begin{equation*}
\frac{\left(4 p \rho v_{0}-p+2 \rho\right) v_{1}^{\prime}}{p}-(\alpha \rho+2) v_{0}^{\prime}+v_{1}\left(4 \rho v_{0}^{\prime}+4 v_{0}-2\right)-2 \rho v_{0}^{\prime \prime}=0 \tag{Q.0.4}
\end{equation*}
$$

Expanding $v_{0}$ and $v_{1}$ about $h=\rho-\frac{p}{2}$ leads to a system of equations from which we may determine $\alpha$ as a function of $\tau_{p}$. In the case of the tetracritical $(p=3)$ we obtain the following system:

$$
\begin{align*}
v_{1,0} & =0 \\
-\frac{4 v_{1,1}}{3}-18 \tau_{3} & =0 \\
-\frac{9}{2} \tau_{3}\left(-4 v_{1,0}+\alpha+4\right)-\frac{2 v_{1,2}}{3} & =0  \tag{Q.0.5}\\
\tau_{3}\left(16 v_{1,0}+24 v_{1,1}-3 \alpha+810 \tau_{3}\right) & =0
\end{align*}
$$

[^73]where $v_{1}=\sum_{i} v_{1, i} h^{i}$. This system leads to $\alpha=162 \tau_{3}$ which indeed corresponds to the $\alpha=81 f^{\prime}(0) \tau_{3}$ of Sec.(3.1.6) for $f^{\prime}(0)=2$ where the regulator is then chosen to be the Litim regulator. For the pentacritical $p=5$ we obtain:
\[

$$
\begin{align*}
v_{1,0} & =0 \\
v_{1,1} & =0 \\
-v_{1,2}-48 \tau_{4} & =0  \tag{Q.0.6}\\
-8 \tau_{4}\left(-4 v_{1,0}+\alpha+4\right)-\frac{v_{1,3}}{2} & =0 \\
-4 \tau_{4}\left(\alpha-5\left(v_{1,0}+2 v_{1,1}\right)\right) & =0
\end{align*}
$$
\]

It is then clear that from $v_{1,0}=0$ and $v_{1,1}=0$, that we also have $\alpha=0$ from the last equation. We also find $\alpha=0$ in the case $p=5$ and we imagine that this will be the case for $p \geq 4$. Hence, it is necessary to consider the next order in $\epsilon$ which then implies that $\tau_{p}$ is a function of $\left(d_{c}(p)-d\right) N^{l}$ for $l>1$. This is not the case of perturbation theory and is thus an artifact of the LPA which implies that the LPA is no longer trustworthy in the large $N$ limit for $p \geq 4$. The next order in $\epsilon$ leads to:

$$
\begin{align*}
& -\alpha_{2} \rho v_{0}^{\prime}+v_{2}\left(4 \rho v_{0}^{\prime}+4 v_{0}-2\right)-(\alpha \rho+2) v_{1}^{\prime}-v_{2}^{\prime}+ \\
& 2\left(\frac{\left(2 p \rho v_{0}+\rho\right) v_{2}^{\prime}}{p}+2 \rho v_{1} v_{1}^{\prime}-\rho v_{1}^{\prime \prime}+v_{1}^{2}\right)=0 \tag{Q.0.7}
\end{align*}
$$

Expanding $v_{0}, v_{1}$ and $v_{2}$ in powers of $h$ we arrive at another system of equations. In the pentacritical case it is necessary to consider two more equations at order $\epsilon$ :

$$
\begin{align*}
v_{1,0} & =0 \\
v_{1,1} & =0 \\
-v_{1,2}-48 \tau_{4} & =0 \\
-8 \tau_{4}\left(-4 v_{1,0}+\alpha+4\right)-\frac{v_{1,3}}{2} & =0  \tag{Q.0.8}\\
-4 \tau_{4}\left(\alpha-5\left(v_{1,0}+2 v_{1,1}\right)\right) & =0 \\
\frac{1}{2}\left(48 \tau_{4}\left(v_{1,1}+2 v_{1,2}\right)+v_{1,5}+7168 \tau_{4}^{2}\right) & =0 \\
\frac{224}{3} \tau_{4}^{2}\left(-16 v_{1,0}+4 \alpha+55\right)+28 \tau_{4}\left(v_{1,2}+2 v_{1,3}\right)+v_{1,6} & =0
\end{align*}
$$

Solving this system and substituting the $v_{1, i}$ into the system obtained from Eq. Q.0.7) we obtain:

$$
\begin{align*}
v_{2,0}-192 \tau_{4} & =0 \\
1920 \tau_{4}-\frac{3 v_{2,1}}{2} & =0 \\
-48 v_{1,4}-v_{2,2}+1152 \tau_{4} & =0  \tag{Q.0.9}\\
32 \tau_{4} v_{2,0}-32 v_{1,4}-\frac{v_{2,3}}{2}-8 \alpha_{2} \tau_{4}+241664 \tau_{4}^{2} & =0 \\
2\left(10 \tau_{4} v_{2,0}+20 \tau_{4} v_{2,1}-2 \alpha_{2} \tau_{4}+87680 \tau_{4}^{2}\right) & =0
\end{align*}
$$

where $v_{2}=\sum_{i} v_{2, i} h^{i}$. This last system of equations leads to $\alpha_{2}=57600 \tau_{4}$.

## Appendix R

## Large $N$ flow equations at order 2 of the derivative expansion

In the following section we give the flow equations for $W=U^{\prime}, Z$ and $Y$ in the large $N$ limit of the $\Gamma$ flow formulation. We shall omit the anomalous dimension as it is of order $\frac{1}{N^{2}}$ along the path $d=3-\alpha / N \mid 67$.

The flow equation of the potential may be obtained directly from the large $N$ analysis of Sec.(3.1.7) followed by a derivative expansion. Another option is to instead consider the derivative expansion at finite $N$ and then take the large $N$ limit. The flow equation for $W$ is then obtained by taking a derivative with respect to $\rho$ of the flow equation of the potential $U$. In terms of rescaled dimensionless variables we have :

$$
\begin{align*}
W & =k^{2} \widetilde{W} \\
\rho & =k^{d-2} N \widetilde{\rho} \\
Z & =\widetilde{Z}  \tag{R.0.1}\\
Y & =\frac{1}{N k^{d-2}} \widetilde{Y}
\end{align*}
$$

We then expand in powers of $1 / N$ as:

$$
\begin{align*}
\widetilde{W} & =\widetilde{W}_{0}+\frac{1}{N} \widetilde{W}_{1} \\
\widetilde{Z} & =1+\frac{1}{N} \widetilde{Z}_{1}  \tag{R.0.2}\\
\widetilde{Y} & =\widetilde{Y}_{0}+\frac{1}{N} \widetilde{Y}_{1}
\end{align*}
$$

In the following we shall omit the tilde notation. We take the notations $\epsilon=1 / N, R_{k}=q^{2} r\left(q^{2} / k^{2}\right), y=q^{2}$. Moreover, we take $\mathrm{v}_{d}$ to be the volume factor from the angular integration. Then, in terms of these notations, the flow equations of $W=W_{0}+\epsilon W_{1}, Z=1+\epsilon Z_{1}$ and $Y=Y_{0}+\epsilon Y_{1}$ are :

$$
\begin{align*}
\partial_{t} W= & -2 W+(d-2) \rho W^{\prime}+\int\left(\frac{2 \mathrm{v}_{d} y^{\frac{d}{2}+1} r^{\prime}(y) \mathrm{W}_{0}^{\prime}}{\left(y r(y)+\mathrm{W}_{0}+y\right)^{2}}+\right. \\
& 2 \mathrm{vd} \epsilon y^{\frac{d}{2}+1} r^{\prime}(y)\left(-\frac{2 \mathrm{~W}_{0}^{\prime}\left(\mathrm{W}_{1}+y \mathrm{Z}_{1}\right)}{\left(y r(y)+\mathrm{W}_{0}+y\right)^{3}}+\right.  \tag{R.0.3}\\
& \frac{2 \rho\left(\mathrm{~W}_{0}^{\prime \prime}(\rho)+y \mathrm{Y}_{0}^{\prime}(\rho)\right)+3 \mathrm{~W}_{0}^{\prime}(\rho)+2 y \mathrm{Y}_{0}}{\left(y(1+r(y))+\mathrm{W}_{0}+2 \rho\left(y \mathrm{Y}_{0}+\mathrm{W}_{0}^{\prime}\right)+y\right)^{2}}+ \\
& \left.\left.\frac{\mathrm{W}_{1}^{\prime}+y \mathrm{Z}_{1}^{\prime}}{\left(y r(y)+\mathrm{W}_{0}+y\right)^{2}}\right)\right)
\end{align*}
$$

$$
\begin{align*}
\partial_{t} Y_{0}= & (d-2) Y_{0}+(d-2) \rho Y_{0}^{\prime}+\int \frac{\mathrm{v}_{d} y^{\frac{d-2}{2}+2} r^{\prime}(y)}{d}\left(\frac{4 d \rho \mathrm{Y}_{0}^{\prime \prime}}{\left(y\left(r(y)+2 \rho \mathrm{Y}_{0}+1\right)+2 \rho W_{0}^{\prime}+W_{0}\right)^{2}}+\right. \\
- & \frac{8 d \mathrm{Y}_{0} W_{0}^{\prime}-2\left(y r(y)+W_{0}+y\right)}{\left(y r(y)+W_{0}+y\right)^{3}}\left(2 ( W _ { 0 } ^ { \prime } ) ^ { 2 } \left(\frac{d\left(y r^{\prime}(y)+r(y)+1\right)}{\left(y r(y)+W_{0}+y\right)^{2}}+\right.\right.  \tag{R.0.4}\\
& \left.\left.\frac{2 y\left(\left(y r(y)+W_{0}+y\right)\left(y r^{\prime \prime}(y)+2 r^{\prime}(y)\right)-2\left(y r^{\prime}(y)+r(y)+1\right)^{2}\right)}{\left(y r(y)+W_{0}+y\right)^{3}}\right)+d \mathrm{Y}_{0}^{\prime}\right) \\
\partial_{t} Z_{1}= & (d-2) \rho Z_{1}^{\prime}-\int \frac{2 \mathrm{v}_{d} y^{\frac{d}{2}+1} r^{\prime}(y)}{d\left(y r(y)+W_{0}+y\right)^{3}\left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)^{3}}( \\
& 4 \rho\left(W_{0}^{\prime}+y Y_{0}\right)^{2}\left(-y\left(r(y)+2 \rho Y_{0}+1\right)-2 \rho W_{0}^{\prime}-W_{0}\right)( \\
& \left(y r^{\prime}(y)+r(y)+1\right)\left(y\left(d-4 y r^{\prime}(y)-4\right)+(d-4) y r(y)+d W_{0}\right)+ \\
& \left.2 y\left(y r(y)+W_{0}+y\right)\left(y r^{\prime \prime}(y)+2 r^{\prime}(y)\right)\right)+ \\
& \left(y r(y)+W_{0}+y\right)\left(-2\left(2 \rho\left(W_{0}^{\prime}\right)^{2}\left(y r^{\prime}(y)+r(y)+2 \rho Y_{0}+1\right)( \right.\right. \\
& \left.d\left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)-4 y\left(y r^{\prime}(y)+r(y)+2 \rho Y_{0}+1\right)\right)+Y_{0}( \\
& d\left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)(  \tag{R.0.5}\\
& \left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}-2 \rho y Y_{0}+y\right)\left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)+ \\
& \left.2 \rho y^{2} Y_{0}\left(y r^{\prime}(y)+r(y)+2 \rho Y_{0}+1\right)\right)-4 \rho\left(y^{2}\left(-r^{\prime}(y)\right)+2 \rho W_{0}^{\prime}+W_{0}\right)( \\
& -2 y Y_{0}\left(y^{2} r^{\prime}(y)-2 \rho W_{0}^{\prime}-W_{0}\right) \\
& \left.\left.\left.W_{0}^{\prime}\left(d\left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)-4 y\left(y r^{\prime}(y)+r(y)+2 \rho Y_{0}+1\right)\right)\right)\right)\right)+ \\
& \left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)\left(-d Z_{1}^{\prime}\left(y r(y)+2 \rho W_{0}^{\prime}+W_{0}+2 \rho y Y_{0}+y\right)^{2}+\right. \\
& \left.\left.\left.-8 \rho y\left(y r^{\prime \prime}(y)+2 r^{\prime}(y)\right)\left(W_{0}^{\prime}+y Y_{0}\right)^{2}\right)\right)\right)
\end{align*}
$$

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## Sujet : Study of singular multi-critical fixed points in the $\mathrm{O}(\mathrm{N})$ model using the functional renormalization group.

Résumé : Cette thèse a pour but l'étude de nouveaux points fixes du groupe de renormalisation dans le modèle $O(N)$ en théorie des champs. En particulier, nous étudions des points fixes qui deviennent singuliers dans limite N grand. Pour N modérément grand nous trouvons une homotopie non triviale qui échange des points fixes en fonction de la dimension et de la valeur de N. Enfin, nous explorons le champ de possibilités pour les approximations du groupe de renormalisation non perturbatif.

Mots clés : renormalization non perturbatif; point fixe cuspé; modèle $\mathrm{O}(\mathrm{N})$; multicritique; tricritique; homotopie

## Subject : Study of multicritical singular fixed points in the $\mathrm{O}(\mathrm{N})$ model and of a reaction diffusion problem using the functional renormalisation group


#### Abstract

This thesis studies new fixed points of the renormalization group of the $\mathrm{O}(\mathrm{N})$ model within field theory. In particular, we study fixed points which become singular in limit N large. For moderately large N, we find a non-trivial homotopy which exchanges fixed points when the dimension and the value of N is varied together. Finally, we explore the field of possibilities for the approximations of the non-perturbative renormalization group.


Keywords : Non perturbative renormalization; cusped fixed points; $\mathrm{O}(\mathrm{N})$ model; multicritical; tricritical; homotopy


[^0]:    ${ }^{1}$ A uni-axial ferromagnet is a ferromagnet that has a natural axis along which the system is easily magnetized.
    ${ }^{2}$ While the conformal bootstrap can not study precisely a first order transition it can still predict the necessity of such a transition when there is no conformal field theory possible. In particular, the conformal bootstrap can, in principle, study complex conformal field theories which can imply weak first order transitions where the correlation length in the system is large (6).

[^1]:    ${ }^{1}$ Phase transitions are only possible in the thermodynamic limit. This is due to the fact that a phase transition implies a non analycity of the partition function which is not possible when the partition function is a finite sum of Boltzman factors. In the bifucation analysis of dynamical systems, well defined transitions exist only in the limit of asymptotic time 8 , 9

[^2]:    ${ }^{2}$ These two energies differ so drastically because the equation of state written as $r(x, t, p)=0$ can always be written in many ways such as $r(x, t, p)^{m} / y(x, t, p)=0$ where $m$ and $y$ are arbitrary but non zero. Integrating these equations of states with respect to $x$ then leads to different free energies that describe the same system up to a change in entropy as the entropy is obtained from an independent variable $T$ as $\left.\frac{\partial F}{\partial T}\right|_{x}=-S$. However, as we also have $\frac{\partial S}{\partial V}=-\frac{\partial F}{\partial T \partial V}=\frac{\partial P}{\partial T}$ in the original model, we may obtain the $V$ or equivalently $x$ dependence of the entropy from the temperature dependence of the pressure obtained from the equation of state. Integrating over $x$ we then obtain the entropy up to a $T$ dependent, but not $x$ dependent term.
    ${ }^{3}$ At this point the density $x$ is viewed as an internal fluctuating variable whereas the pressure plays the role of an external control parameter (pressiostat). The thermodynamic potential is then the Gibbs free energy.

[^3]:    ${ }^{4}$ Universal quantities are those which do not depend explicitly on the underlying model and the types of interactions. Examples will be given throughout the present thesis. In the case of non universal quantities, mean field is correct only at infinite dimension.
    ${ }^{5}$ In the case of continuous symmetries at equilibrium this is the case already at $d=2$. An example of this is given by the $O(N)$ models for $N \geq 2$ due to the Mermin-Wagner theorem.
    ${ }^{6}$ Let us note that actual correlation functions may be substantially different than what is predicted here by mean field. In particular for the $O(N)$ models for $N \geq 2$. See for example $10+12$
    ${ }^{7}\left(q^{2}+m^{2}\right)^{-1}$ can be found using Gaussian integration with the Landau-Ginzburg free energy functional in the next section. One then finds that $m^{2}$ is the coefficient of quadratic term in Eq. 2.1 .6 which is indeed proportional to $T-T_{c}$.
    ${ }^{8}$ This can be found by computing the minimum of Eq. 2.1 .6 as a tunction of T .

[^4]:    ${ }^{9}$ If we focus only on the energy functional it might seem clear that one should simply return to the usual uniform configuration to minimize the potential. However one also needs to take into account entropic effects as there are many ways to form domain walls within a system. Taking into account this fact, the low temperature regime is indeed dominated by a proliferation of domain walls. However, one might check that domain wall solutions exist only below the critical temperature. As such the high temperature regime is still given by the uniform paramagnetic configuration $m=0$.
    ${ }^{10}$ Let us note that this discusion is a simplification where only non uniformity in the $x$ direction is taken into account. For a one dimensional magnet the Curie temperature is zero and there is no low temperature phase.

[^5]:    ${ }^{11}$ A reader unfamiliar with the Minkowskian geometry or the quantum path integral can consider, for the sake of simplicity, that the geometry is euclidean as in the rest of this thesis and that the probability distribution is given as in Eq. 2.1.17. The actual quantum path integral is formally similar by replacing the energy functional $\mathcal{H}$ by $i S$, where $S$ is called the action in quantum mechanics, and replacing the kinetic terms with a Minkowskian version using the so-called Wick rotation $t \rightarrow i t$ where t is the time dimension.

[^6]:    ${ }^{12}$ Within this thesis we will alternate when convenient to either the term Non perturbative renormalization or Functional renormalization group but both names refer to the same concept and equations
    ${ }^{13}$ The terminology UV and IR in field theory refer to the scale at which the physics under consideration take place. UV may be replaced by microscopic and IR with macroscopic.

[^7]:    ${ }^{14}$ Actually this is only true mathematically, physically our observations and measurements come with finite precision and thus we generally never know the underlying laws to arbitrary precision. This in turn is irrelevant when flowing from a high energy UV theory to a low energy IR theory due to universality as we shall discuss later but in the opposite direction the problem is ill conditioned in the sense that finite precision at the low energy scale leads to many possible UV theories that are capable of giving the same result. Hence, quantum field theory is not actually predictive in that direction (even though the standard model is renormalizable and one generally associates renormalizability with predictive power). This is clear by the constant expectations of new physics as we increase the energies of colliders. This is not as visible within perturbation theory as one considers only a small subspace of the infinite dimensional space of couplings.
    ${ }^{15}$ As rigorous discussions on convergence were not as popular at the time, the notion of radius of convergence is probably an anachronism here

[^8]:    ${ }^{16}$ We stress here however that the author of the present thesis is not aware in which way Euler initially formulated the factorization of the sin function although, at least in modern times, the parametrization of Eq. 2.2 .14 seems the most readily natural.

[^9]:    ${ }^{17}$ We will see in Sec. 2.3 that one may retrieve an FSS RG from a non FSS RG by rescaling and we will discuss when is it useful to have the FSS condition
    ${ }^{18}$ This should be read as a definition of $e_{R}\left(p_{r e f}^{2}\right)$ and is an exact equation that needs to be updated for each new quantum correction considered where here we took the leading correction coming from a "one-loop" calculation. This is to be contrasted with the usual perturbative arguments of inverting Eq. ( 2.2 .21 ) to obtain $e_{b}$ considering both $e_{b}$ and $e_{R}$ small which are problematic as it requires $e_{R}$ to be small at all scales which at the level of the present equation is contradictory as the $\ln \left(\frac{p_{r e f}^{2}}{p^{2}}\right)$ term can be arbitrarily large. Moreover, when renormalization is fully implemented $e_{R}$ diverges at a pole called the Landau pole. Finally, $e_{b}$ can only be considered as small for the fixed dimension in which it was used in the perturbative expansion but when we take $d \rightarrow 4, e_{b}$ diverges as it must compensate the $2 / \epsilon$ term in Eq. 2.2.1 so that the physical $e_{R}$ remains finite. All of these reasons imply that inverting Eq. 2.2 .21 considering $e_{b}$ and $e_{R}$ as small is problematic. However, defining $p_{r e f}$ such that $e_{b}=e_{R}\left(p_{r e f}^{2}\right)$ evades these issues while in essence obtaining the same result where instead of $e_{b} \simeq e_{R}$ due to

[^10]:    ${ }^{19}$ One might wonder whether it is really justified to consider that sin can be written as such a simple product as was done here and in fact it is not. The interested reader can find a discussion on this in Appendix B. That discussion will also lead to a discussion on effective field theories and non perturbative renormalizability.

[^11]:    ${ }^{20}$ More precisely, the equation was obtained with the Pauli-Villars regularization scheme.
    ${ }^{21}$ There are actually other similar contributions from so called " $t$ " and " $u$ " channels and are due to the fact that from the quantum field theory perspective $b_{R}$ corresponds to a quantum amplitude of the inelastic scattering of 4 identical particles where the other channels are due to the indiscernability of the particles.

[^12]:    ${ }^{22}$ In the framework of counter terms the idea is similar, that is, we use the 0 -th order term to absorb a divergence coming from a correction. In dimensional regularization the large logarithms are always added to the $1 / \epsilon$ poles as was the case of the charge in this manuscript, this in turn allows us to shortcut the discussion on scales and UV cut-offs by simply extracting the divergent $1 / \epsilon$ parts but this is only a convenient technical shortcut and in principle the UV dependence is still there until we take the coupling to be a solution of the beta function as is the case of the scale dependent charge.

[^13]:    ${ }^{23}$ Quick proof: Consider a convolution product $*$, a Fourier Transformation $F T$ and a Gaussian function $\mathcal{P}^{a}$ with propagator $a^{-1}$ such that $\mathcal{P}^{a}=\mathcal{P}_{a^{-1}}($ notation choice $)$ then we have $\mathcal{P}^{a} * \mathcal{P}^{b}=F T^{-1}\left(F T\left(\mathcal{P}^{a}\right) . F T\left(\mathcal{P}^{b}\right)\right)=F T^{-1}\left(\hat{\mathcal{P}}^{a^{-1}} . \hat{\mathcal{P}}^{b}\right)=$ $F T^{-1}\left(\hat{\mathcal{P}}^{a^{-1}+b^{-1}}\right)=\mathcal{P}^{\left(a^{-1}+b^{-1}\right)^{-1}}$ and thus we indeed have $\mathcal{P}_{a-1} * \mathcal{P}_{b^{-1}}=\mathcal{P}_{a}{ }^{-1}+b^{-1}$

[^14]:    ${ }^{24}$ Actually $\Psi$ is already a functional that takes as arguments functions $\phi(x)$, thus T is a functional of a functional

[^15]:    ${ }^{25}$ Note that we are currently omitting discusions of field renormalization

[^16]:    ${ }^{26}$ In fact it can be understood in the more general framework of homotopy transformations
    ${ }^{27}$ In the following we shall refer to the energy functional $\mathcal{H}$ as an action $S$ and we will call $\Gamma_{k}$ the effective averaged action and $\Gamma_{k=0}$ the effective action as they are commonly given these names in the litterature - even outside of the Minkowskian geometry context- for historical reasons. In any case, in the equilibrium setting with euclidean geometry, $\Gamma_{k=0}$ is the usual Gibbs free energy albeit promoted to a functional (for the field variable) rather than a function.
    ${ }^{28}$ the normalization factor $\int D \phi e^{-\phi \cdot R_{k} \cdot \phi}$ for the convergence to the Dirac function is unimportant for all practical purposes

[^17]:    ${ }^{29}$ The linear part corresponding to the rescaling operator
    ${ }^{30}$ Notice that the rescaling operation was also needed so that the group action leads to an autonomous differential equation.
    ${ }^{31}$ At least the lowest order approximations usually do not require re-summation even though it is possible that very high order approximations will become asymptotic and require re-summations.

[^18]:    ${ }^{32}$ Another interesting image is that of the smoothening effect of the RG flow where the microscopic details are rubbed away. This is particularly visible with the non linear sigma model $S=\int g_{a, b}(\phi)\left(\partial^{\mu} \phi^{a} \partial^{\mu} \phi^{b}\right) / 2$ whose one loop renormalization corresponds to a Ricci flow which is a kind of mean curvature flow. Another example of mean curvature flow is that given by bubble surfaces that tend toward the minimal surface configuration irrespective of their initial condition. In the case of the Ricci flow with metric $g_{a, b} d \phi^{a} d \phi^{b}=e^{p(\phi)} \delta_{a, b} d \phi^{a} d \phi^{b}$, the flow of $p$ is exactly that of a diffusion equation $\partial_{t} p=\Delta p$. The Ricci flow in presence of a potential term $V$ in the action $S$ can be seen as a flow convected by the potential derivative $V^{\prime}$ using a co-variant derivative 61. This quantum field theory version of the Ricci flow inspired Perelman in his proof of the Poincare conjecture by using this flow instead of the original one which can lead to singularities 61-66.
    ${ }^{33}$ The boson should be compactified as well, that is, it should verify $X\left(\sigma_{1}, \sigma_{2}+2 \pi R\right)=X\left(\sigma_{1}, \sigma_{2}\right)+2 \pi$.
    ${ }^{34}$ In the case of approximate RG flows the choice of regularization affects the end results. It is then necessary to choose a regularization that preserves the symmetries of the UV theory or, at least, one should verify that the symmetry breaking effect is small via Ward Identities.
    ${ }^{35}$ We omit here universal corrections to scaling.

[^19]:    ${ }^{36}$ This $\beta$ function is taken from 67] where it was obtained by perturbative methods. As such the time $t$ here is not given by $e^{t}=\frac{k}{\Lambda}$ where $k$ controls the amplitude of the regulator $R_{k}$. This is irrelevant however for the pedagogical purpose of this section.

[^20]:    ${ }^{37}$ If $y\left(\lambda_{-}^{*}\right)=0$ this would not be true. The direction of the RG flow is given by the sign of the beta function. Setting 3 points as zeroes with an initial negative slope at the first zero $\lambda^{*}=0$, the sign of $\beta$ will change above the second root unless it is a local maximum in which case the sign remains negative which then leads the coupling to increase beyond this point. However in this particular case it is not possible as $y\left(\lambda_{-}^{*}\right)=0$ means a double root which would mean a total of 4 roots counted with multiplicity for a polynomial of degree 3 which is not possible.
    ${ }^{38}$ In this example we neglected the anomalous dimension for simplicity
    ${ }^{39} \lambda$ seems to be small for $d=2$ as well but starting at $d=4$ where it becomes Gaussian and lowering the dimension it first crosses a pole at $d=5 / 2=2.5$. There is no such singularity at $d=2.5$ within the exact theory and we may perhaps regard $d=2.5$ as the lower limit of validity of this approximation. Indeed, it is known that even higher order approximations become less accurate as the dimension decreases and fluctuations become stronger. Thus we can no longer trust this expression below $d=2.5$ and in particular the approximation is also insufficient at $d=2$.

[^21]:    ${ }^{40}$ We have $\Delta_{\phi}(k)=\Delta_{\phi}^{c}-\delta \Delta_{\phi}(k)$ where $\Delta_{\phi}^{c}$ is the canonical dimension from dimensional analysis. For a bosonic theory $\Delta_{\phi}^{c}=(d-2) / 2$ while $\delta \Delta_{\phi}(k)$ is obtained from the renormalized prefactor of the kinetic term and corresponds to a running critical exponent $\eta_{k} / 2$. At a fixed-point $\delta \Delta_{\phi}(k)$ becomes independent of $k$ and it is then equal to the critical exponent $\eta / 2$. The running $\eta_{k}$ for non zero $k$ has no physical meaning however as it depends on our choice of renormalization scheme even within the exact formalism.
    ${ }^{41}$ However, the scaling dimension of $g_{i}$ is not given by $\Delta_{i}$ if the fixed-point is non Gaussian. In general the scaling dimension is not even defined as $g_{i}$ has to be an eigendirection where the scaling dimension can then be obtained from the corresponding eigenvalue. When the fixed-point is also conformal and not just scale invariant, these scaling dimensions can be found within the conformal bootstrap approach as well.
    ${ }^{42}$ Outside of quantum gravity the potentials are defined up to a constant thus we may impose for example that $U_{k}(0)=0$ by looking at the flow of $U_{k}(\phi)-U_{k}(0)$.
    ${ }^{43}$ At least for many systems at equilibrium, there are for example out of equilibrium systems that are self driven into criticality.
    ${ }^{44}$ We choose to always keep the system at zero external field but the external magnetization is also an unstable direction.

[^22]:    ${ }^{45}$ It might seem that this is no longer true for $d<2$ but we recall that we have neglected the anomalous dimension $\eta$. When $\eta$ is taken into account nothing special happens at $d=2$, at least for the Ising model. This is an example as to why it becomes increasingly important to consider the anomalous dimension as the dimension of the system is decreased.
    ${ }^{46}$ Of course, these partial differential equations can always be written into a system of ordinary differential equations by supplying a set of basis functions $f_{i}$ which then corresponds to a choice of numerical scheme.

[^23]:    ${ }^{47}$ We do not need to consider the anomalous dimension here as it converges to zero outside of criticality. This can be seen by including the anomalous dimension $\eta_{k}=\frac{4 \kappa \lambda^{2}}{(1+2 \lambda \kappa)^{2}}$ in Eq. 2.3 .46 which then indeed converges to 0 when $\kappa$ and $\lambda$ diverge to infinity.

[^24]:    ${ }^{48}$ We remark here that the long development to reach to this result was used as an opportunity to discuss various aspects of the RG. Thus, as was the case in the section on the exact RG equations, what was important in this section was the story and elements discussed along the way rather than the end result which can be proved in a more direct mathematical fashion.
    ${ }^{49}$ Note that this should not be confused with $m_{k}$ along the $\Gamma$-flow or any other singularity of the two point function where such a linear scaling takes place only when $T-T_{c}$ is small and for $k_{G} \gg k \gg k_{\xi}$. This is because the mass $R_{k}$ itself varies within the $\Gamma$-flow thereby changing the physical effective action itself. In fact, as we have mentioned before in Sec. 2.3 .2 , it is not necessary to take as initial condition a classical action and rather we may consider a decoupled one body theory where the kinetic term of the classical action is removed as in 54 . This is why we preferred to give a more intuitive argument in the case of the $\Gamma$-flow to avoid confusion rather than relying only on the more standard approach given here.
    ${ }^{50}$ The reduced temperature here is the effective temperature "felt" by the system which is different from the temperature imposed by an experimentalist which is a fixed value corresponding to an initial condition in the temperature flow.
    ${ }^{51}$ In which case the system has multiple relevant directions.

[^25]:    ${ }^{52}$ At this point we have taken the liberty to perhaps change somewhat the original explanation of this method as the present explanation is perhaps somewhat easier to understand.

[^26]:    ${ }^{53}$ Of course, as with any convergent or asymptotic approximation scheme there is always an underlying small parameter. This small parameter will be made explicit at the end of this section and will hopefully give some insight into the convergence of non-perturbative schemes in the NPRG framework

[^27]:    ${ }^{54}$ However the exact form of the interaction is irrelevant and instead there is simply a coarse definition of what is long range such as having a power law that exceeds a value that depends on the symmetries and dimension of the problem
    ${ }^{55}$ One can also impose parity centered about the maximum in which case the link with $\mathbb{Z}_{2}$ theories such as the Ising model is more apparent.

[^28]:    ${ }^{56}$ The fact $\tilde{r}=0$ gives the exact value is due to the fact that $x_{1}^{* *}$ is exact and the expansion of $f_{1}$ was taken to order two about the fixed-point of $f_{1}$ when the linear order is already sufficient to obtain the stability of a fixed-point.

[^29]:    ${ }^{57}$ More precisely we wish $\mathcal{R}_{2}(f)$ to have one quadratic maximum and to map $[-1,1]$ onto itself.

[^30]:    ${ }^{58}$ One may check that the inclusion of a term $G(\phi) \triangle \phi$ can by removed by an integration by parts and a redefinition of $Z$.
    ${ }^{59}$ It is not always easy to know when $\eta$ can be neglected. For example, for the $O(N)$ models to be discussed in Sec.3.1.1. If $N \geq 2$, then as $d \rightarrow 2, \eta \rightarrow 0$ but if we simply neglect $\eta$ we do not find the correct $\beta$ function at $d=2+\epsilon$. In general, it becomes increasingly important to consider $\eta$ as the dimension is lowered from the upper critical dimension.

[^31]:    ${ }^{60}$ The correlation functions are related to the $\Gamma^{(n)}$ by the tree expansion.
    ${ }^{61}$ Notation: $C^{n}$ means $n$ times differentiable with the $n^{\text {th }}$ derivative being continuous
    ${ }^{62}$ It is in fact possible to bypass this difficulty as in 80 but regulators that omit this $C^{n}$ constraint lead to a loss of accuracy.
    ${ }^{63}$ The prefactor $n k^{2}$ can also be rescaled using a prefactor $\alpha$ as $R_{k}=\alpha s\left(q^{2} / k_{U V}^{2}\right) \theta\left(k_{U V}^{2}-q^{2}\right)$. The principle of minimal sensitivity naturally leads to such a scaling in $\alpha$ as has been checked numerically at order 6 of the derivative expansion.
    ${ }^{64}$ This has also been confirmed numerically at order 6 of the derivative expansion 77

[^32]:    ${ }^{65}$ It is best to read this section prior to looking at Appendix $F$ as we will refer to certain quantities that will be given later in this section.

[^33]:    ${ }^{66}$ We have discarded the possibility of including explicit momentum dependent terms such as $k^{-4} p_{1} p_{2} p_{3} p_{4} S_{4} \partial_{\phi_{0}}^{2} \widehat{\gamma}_{2}\left(p_{1}\right)$, which are not constrained by FD, because the $\Gamma$-flow has only explicit momentum dependence from the regulator and this would spoil that property.

[^34]:    ${ }^{67}$ One might argue that it is necessary to take $c=1$ in order to retrieve the one loop structure of the four point function. However, our approximation scheme at order $n$ will consist of performing approximations only on $\Gamma^{(n+1)}$ and $\Gamma^{(n+2)}$ and thus the one loop structure of $\Gamma^{(4)}$ will be retrieved at higher order approximations. Therefore, while retrieving the one loop structure does, a priori, seem like a nice feature, it is not necessary at this level of approximation and we esteem that it is best to let the system choose the value of $c$ it "prefers".

[^35]:    ${ }^{68}$ We recall that $\widehat{\gamma}_{2}(0)=0$ as $\widehat{\gamma}_{2}\left(p_{1}\right)=\gamma_{2}\left(p_{1}\right)-U^{(2)}$
    ${ }^{69}$ In fact due to rotational symmetry, $\widehat{\gamma}_{3}$ is order 4 in the momenta. In out of equilibrium systems it would be order 3 in the frequency and this would also be true for equilibrium systems with sharp momentum cut-off as the expansion is then in $\left(p^{2}\right)^{1 / 2} 80$.

[^36]:    ${ }^{70}$ Let us recall that within the derivative expansion conformal symmetry is broken but the broken Ward identities can be made small by using the principle of minimal sensitivity 83 .

[^37]:    ${ }^{71}$ In 8485 the focus was mainly on the $O(N)$ models for $N>2$ where one also includes a term $\partial \rho Y\left(\partial^{2}\right) \partial \rho$ in analogy with the LPA'. However, in this section we focused mainly on a theory of a single scalar for the sake of simplicity.
    ${ }^{72}$ When the potential is analytic such an expansion is meaningful as for $\phi_{0} \rightarrow \infty$ the background dependent mass $U^{\prime \prime}\left(\phi_{0}\right)$ becomes large and the threshold functions (see 72 for definition) in the $\Gamma$ flow become small.

[^38]:    ${ }^{1}$ We use the term vector as there are other representations of $O(N)$ symmetry such as the $O(N)$ matrix models.

[^39]:    ${ }^{2}$ As mentioned in 10 real magnets tend to also have cubic anisotropy due to their lattice structure and dipolar interactions. However, although the most stable fixed-point is the cubic model when $N=3$, the critical exponents from these two models differ very little. For a recent review and study of the cubic instability see [104]. Also see 68] for RG dynamical theory arguments against the usual picture of an exchange of stability between the $O(N)$ and cubic model at $N=N_{c}$.
    ${ }^{3}$ The case $N=0$ can also be seen as $N-N$ by introducing anti-commuting fields which then realizes a supersymmetry as in $113-115$
    ${ }^{4}$ The case of four dimensions includes the dimension of time in a Minkowski relativistic space
    ${ }^{5}$ For a lattice of non integer dimensions there many different definitions one can find for the notion of dimension. These different definitions are equivalent only in special cases. The references given defend the position that the spectral dimension is the one relevant to field theories

[^40]:    ${ }^{6}$ In the actual Blume-Emery-Griffiths model there is another term $-K \sum_{<i j>} S_{i}^{2} S_{j}^{2}$ which has been set to zero here as a tricritical point is already visible for $K=0$. This model with $K=0$ is also called the Blume-Capel model 142143 .

[^41]:    ${ }^{7}$ Of course this is only true if both models also have the same degree of criticality for example they are both tricritical.
    ${ }^{8}$ In general however, in the NPRG framework one usually prefers to evaluate the potential on a grid of points rather than to Taylor expand the potential. The advantage is then that the approximation scheme is no longer limited by the radius of convergence of $V$ with respect to the field variable. This point will be particularly true when we investigate singular fixed-point solutions.

[^42]:    ${ }^{9}$ At $\mathrm{d}=3$ the eigenvalue for the Gaussian fixed-point along the $a_{6}$ direction is 0 at linear order but is in fact positive at higher order such that it actually remains stable in this direction

[^43]:    ${ }^{10}$ We have omitted factoring numbers in passing from the first to the second line.

[^44]:    ${ }^{11} \propto_{+}$means here that it is proportional to a positive constant
    ${ }^{12}$ We have reproduced what is commonly done in the literature but this point seems a bit subtle as one might argue "why not just add a positive term $\chi^{4 "}$ in which case the potential remains bounded. Usually one would not consider such a term as it should be irrelevant but perhaps it is a dangerously irrelevant term $150-152$. In any case, from this analysis it seems that something new might happen above this critical coupling. We will see using the renormalization group that indeed the potential behaves differently above the critical coupling but after an infinite number of RG steps the potential remains bounded and instead it becomes double valued. That the RG induces non analyticities above the critical coupling was already noticed in 153 .

[^45]:    ${ }^{13}$ It is a rather special feature here that we can discuss the number of eigendirections by looking only at the $\beta$ function of the coupling $\tau$. This is possible because at this order of approximation $\beta(\tau)$ is independent of the other couplings in the theory as is visible in Eq. 3.1.30.
    ${ }^{14}$ We have added a $O(1 / N)$ for the penta critical $\beta\left(a_{10}\right)$ but the reference 156 did not explicitly add this. Furthermore, we have corrected a sign in the pentacritical $\beta$ function as the large $N$ limit given in 156 was not the correct limit of the finite $N$ expression of the $\beta$ function given in the appendix of that paper.
    ${ }^{15}$ The effect of the anomalous dimension can be neglected at this order

[^46]:    ${ }^{16}$ Note however that while the flow of the potential is exact in the limit $N \rightarrow \infty$, the Ansatz below is not. More details will be given in Sec.3.1.7.
    ${ }^{17}$ We recall that the function $\Theta$ is the Heaviside function.
    ${ }^{18}$ Here the word optimal refers to the fact that in the space of all possible regulator functions, the results obtained with the Litim regulator minimizes the error of the LPA approximation. This will be shown explicitly in Sec. 3.1.5.

[^47]:    ${ }^{19}$ The use of running here implies that it is $k$ dependent and thus it runs along the RG flow.
    ${ }^{20}$ In principle one should check that there are no negative eigenvalue for all couplings including higher order couplings but higher order couplings tend to be increasingly irrelevant in the RG sense.

[^48]:    ${ }^{21}$ In this case $\tau$ no longer represents $V^{\prime \prime \prime}\left(\rho_{\min }\right)$ with $V^{\prime}\left(\rho_{\min }\right)=0$.
    ${ }^{22}$ The linear part of the BMB FP potential corresponding to $\bar{\varrho} \in\left[0, \varrho_{0}\right]$, see Fig. 3.1.9, can be replaced by a smooth analytic continuation where there is no longer any discontinuity in $\bar{V}^{\prime \prime}$. There is then a distinction between the FP solution obtained as the the limit $\tau \rightarrow \tau_{\text {BMB }}$ and the analytical prolongation of the FP potential obtained exactly at $\tau=\tau_{\text {BMB }}$. Considering either solution has no physical consequence because in both cases, the interval $\bar{\varrho} \in\left[\bar{\varrho}_{0}, \infty[\right.$ is entirely mapped onto $\bar{\rho} \geq 0$ in the $\Gamma$ flow version 162] which then implies that the interval of values of $\bar{\varrho}$ : $0<\varrho<\varrho_{0}$ does not play any physical role for this particular FP.
    ${ }^{23}$ There we used the effective potential $U$ but the same is true for the Polchinski potential $V$.

[^49]:    ${ }^{24}$ See below for a more detailed explanation
    ${ }^{25}$ One might wonder whether it is possible to know whether piecewise constructions are admissible solutions to a differential equation without having to consider their "finite $\epsilon$ " extensions. Such piecewise solutions are called weak solutions and in Appendix $L$ we study whether the above singular FP solutions of the Polchinski equation can be viewed as weak solutions.

[^50]:    ${ }^{26}$ The expansion was performed on the potential $U$ but an equivalent system can be obtain for $V$.
    ${ }^{27}$ We recall here the paradox with respect to the perturbative beta function which should be exact to order $1 / N$ but is unable to predict any FP collapse at $\alpha_{B M B}$. We see here that the mechanism involves a collapse between $\widetilde{A}_{3}$ and the FP $S \widetilde{A}_{4}$ which involves a boundary layer analysis leading to exponentially vanishing terms $e^{-V_{2} N\left(\bar{\varrho}-\varrho_{0}\right)}$ at the minimum $\bar{\varrho}=1$. It is then likely that the exact beta function contains exponentially vanishing terms $e^{-N \ldots}$ that become non negligible precisely for $\alpha=\alpha_{B M B}$. We then note that this is perhaps somewhat analogous to the information paradox in quantum gravity as in that case general relativity predicts that nothing should happen at the horizon, as does the perturbative $\beta$ function here, whereas calculations involving quantum mechanics lead to a firewall. The calculations involving quantum mechanics can then be seen as an independent consistency requirement, as the requirement $\tau<\tau_{B M B}$ here, while the $\beta$ function represents an equation of motion for the couplings as the geodesic equation in general relativity. Here $\alpha_{B M B}$ then plays the role of the horizon and in a recent paper 167 it was suggested that the information paradox may be solved by taking into account exponentially suppressed correlations in the Hawking radiation .
    ${ }^{28}$ The construction of these singular FPs within the $\Gamma$-flow formulation rather than the Polchinski formulation is given in Appendix O

[^51]:    ${ }^{29}$ In the case of critical exponents we showed that at $N=\infty$ the local behavior in the neighborhood of the minimum is sufficient to calculate the critical exponents. Moreover, as the finite N case is obtained by simply adding a higher order diffusion operator to the flow of the potential, we expect that the critical exponents still depend only on the neighborhood of the minimum as seems to be apparent when performing field expansions about the running minimum of the potential which

[^52]:    give better results than at $\phi=0$.

[^53]:    ${ }^{31}$ In Appendix $Q$ we derive the case of the Litim regulator in the Polchinski formulation
    ${ }^{32}$ This is derived in Appendix Q in the Polchinski formulation.

[^54]:    ${ }^{33}$ We omit here the unimportant volume factor.

[^55]:    ${ }^{34}$ A similar conclusion was found in 180

[^56]:    ${ }^{35}$ We have also checked that the eigenvalues of the singular Wilson-Fisher, a FP we will introduce in Sec. 3.1.10, align with this picture.

[^57]:    ${ }^{36}$ For $d>3$ tricritical physics is given by the Gaussian and without the BMB FP there is no bridge between regular and singular FPs.

[^58]:    ${ }^{37}$ In principle, it is also possible that the FPs in the equality become singular. We do not expect a singularity to arise by decreasing $N$ as the term $\frac{1}{N} \bar{\rho} V^{\prime \prime}$ acts like a diffusion operator which thus smooths the potential solution as $N$ decreases but it is possible that a singularity would arise at $d=2$. Nonetheless, considering that all critical lines end at $d=2$ would not explain why all three FPS $\widetilde{A}_{3}, S A_{3}, S \widetilde{A}_{4}$ do not exist for $N$ small while $A_{2}$ does.
    ${ }^{38}$ This argument considers however that all FPs not involved in the FP collapses of the two critical lines remain the same, that is, there is no permutations among FPs not involved in the collapses after a full turn around the rectangle $T$. This is indeed reasonable as such FPs should, a priori, be unaware of the FP collapses of other FPs involved in Fig. 3.1.20). It would then be rather peculiar that FPs not involved in the mechanisms of Fig. 3.1 .20 swap without any a priori reason.

[^59]:    ${ }^{39} C$ must remain unchanged as $C$ when crossing $A=B$ as it would be the case in reverse from $p_{1}$ to $p_{4}$ which is analogous to $A$ between $p_{1}$ and $p_{2}$.

[^60]:    ${ }^{40}$ However, if $d_{S^{\prime}}<3$ then this would imply that there exists a path where $S A_{3}$ becomes $S G_{3}$ without crossing $d=3$. While we do not see why this would not be possible it would add unecessary information the the $(d, N)$ diagram as within the LPA, $S A_{3}$ becomes $S G_{3}$ only when crossing the axis $d=3$.

[^61]:    ${ }^{41}$ While a cusp exists only at $N=\infty$ the singular nature of $\mathrm{SWF}_{2}$ becomes quite clear even for moderate $N$ as it develops cusps much faster than any other FP when $N$ is increased.

[^62]:    ${ }^{1}$ The dual polynomial $\widetilde{Q}_{n}(X)=X^{n} Q_{n}(1 / X)$ has the same number of strictly positive roots as $Q_{n}$ and inverses the order of the coefficients thus the perturbative expansion performed on the coefficient of highest degree is dual to an expansion of the coefficient of lowest degree. This thus widens the scope of our analysis.

[^63]:    ${ }^{2}$ We use the term critical here but this is not the critical temperature for the second order phase transition

[^64]:    ${ }^{1}$ If the polynomial has a root at zero then it would be $p(x)=x^{a} \prod\left(1-x / x_{i}\right)$

[^65]:    ${ }^{1}$ In this thesis the term statistical free energy refers to $e^{-F / k T}=Z$ where $Z$ is the partition function.

[^66]:    ${ }^{1}$ This also explains the notations $\vec{a}$ for vectors and $\dot{a}$ for scalars.

[^67]:    ${ }^{1} \mathrm{~A}$ system is meta-stable when it is at a local minimum of the free energy that is not a global one. If this minimum is deep enough the system can stay within that local minimum as the energy barrier to overcome the nearby intermediate local maximum costs energy. The probability of overcoming the maximum is $\propto e^{-E_{b a r r i e r} / k T}$ which can be quite small and may require the need of an external energy input in order to find the global minimum in a reasonable amount of time.
    ${ }^{2}$ At least if we include the possibility of meta-stable states that are not global minima.

[^68]:    ${ }^{1}$ In 202 the authors subtract the $N$ fields $\phi$ in the original action but then it is not clear where the $\frac{1}{-\partial^{2}+\sigma}$ term in the Hessian matrix came from as at least $N-1$ fields needed to have been integrated over prior to that. To evade these issues one can also take the approach of 148 working with the current instead of the fields.

[^69]:    ${ }^{1}$ One may also expand around the finite $N$ minimum $\bar{\varrho}_{m}$ of $v$. However, this point being a function of $\epsilon$, the expansion turns out to be in powers of $\epsilon^{1 / 2}$ which makes the calculations a little more difficult.

[^70]:    ${ }^{1}$ We note however that the $1 / N$ term which multiplies ${ }^{-}{ }^{-} \underline{b}^{\prime \prime}$ in the Polchinski equation can be regarded as a viscosity term 203, 204.

[^71]:    ${ }^{1}$ Actually the large $\bar{\varrho}$ asymptotics need not be flat but we may consider that this is a reasonable approximation for intermediate and small $\bar{\varrho}$ outside of the boundary layer.

[^72]:    ${ }^{1}$ In Fig. O.0.1 the singular gaussian is given at $d=3.2$ but in the Polchinski formulation it is still defined at $N=\infty$ by a linear $\bar{V}=\bar{\rho}$ potential followed by a flat gaussian potential. The FP $S G$ for $d>3$ shall be discussed in Sec. 3.1.10

[^73]:    ${ }^{1}$ We have only included a few terms here but in the following derivations we have include terms up to $h^{20}$ and more for the function $v_{0}$. Such high orders were done in particular for the pentacritical case in order to be sure that it is truly different from the tricritical case. We have not checked what is the minimal order in $h$ necessary to retrieve our results.

