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# Supercuspidal representations of $GL(n)$ over a non-archimedean local field: distinction by a unitary or orthogonal subgroup, base change and automorphic induction

Jiandi Zou

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Représentations supercuspidales de  
 $GL(n)$  sur un corps local non archimédien:  
distinction par un sous-groupe unitaire  
ou orthogonal, changement de base et  
induction automorphe.

*Supercuspidal representations of  $GL(n)$  over a non-archimedean local field : distinction by a unitary or orthogonal subgroup, base change and automorphic inductions*

**Thèse de doctorat de l'université Paris-Saclay**

Ecole Doctorale n°574, Mathématique Hadamard (EDMH)  
Spécialité de doctorat : Mathématiques fondamentales  
Unité de recherche : Université Paris-Saclay, UVSQ, CNRS, Laboratoire de  
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**Thèse présentée et soutenue à Paris-Saclay,  
le 13/07/2021, par**

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*Dedicate to my grandparents*

*I wish for this night-time to last for a lifetime  
The darkness around me, shores of a solar sea  
Oh how I wish to go down with the sun  
Sleeping, weeping, with you  
“Sleeping sun”, Nightwish*

*At the end of the river the sundown beams  
All the relics of a life long lived  
Here, weary traveller rest your wand  
Sleep the journey from your eyes  
“Turn loose the mermaids”, Nightwish*



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# Introduction (English version)

This thesis contains three parts. In this introductory chapter, we will explain the general background in the first section, and then in the following three sections we will focus on each part and provide specific introductions.

## 0.1 General context

Let  $F_0$  be a non-archimedean locally compact field of residue characteristic  $p$  and let  $R$  be an algebraically closed field of characteristic  $l \neq p$ , and in particular when  $l > 0$  we are in the “ $l$ -modular case”. For instance, we will mainly focus on the following three cases:  $R$  being the complex number field  $\mathbb{C}$ , the algebraic closure of the field of  $l$ -adic numbers denoted by  $\overline{\mathbb{Q}_l}$ , or the algebraic closure of the finite field of  $l$  elements denoted by  $\overline{\mathbb{F}_l}$  when  $l \neq 0$ . Let  $\mathbf{G}$  be a reductive group<sup>1</sup> over  $F_0$  and let  $G$  be the locally profinite group consisting of the  $F_0$ -rational points of  $\mathbf{G}$ . We are interested in the category of smooth irreducible representations of a locally profinite group and we denote by  $\text{Irr}_R(G)$  the set of isomorphism classes of smooth irreducible representations of  $G$  over  $R$ .

### 0.1.1 Local Langlands correspondence

First of all let us consider the case where  $R = \mathbb{C}$ . We fix a separable closure  $\overline{F_0}$  of  $F_0$ , we denote by  $\mathcal{W}_{F_0}$  the Weil group of  $F_0$  with respect to  $\overline{F_0}$  and by  $\text{WD}_{F_0} = \mathcal{W}_{F_0} \times \text{SL}_2(\mathbb{C})$  the Weil-Deligne group of  $F_0$ . We define the dual group of  $\mathbf{G}$ , denoted by  $\hat{G}$ , as the complex reductive group (identified with the complex topological group of its rational points by abuse of notation) determined by the dual of the root datum of  $\mathbf{G}$ . Since the root datum of  $\mathbf{G}$  is endowed with a  $\mathcal{W}_{F_0}$ -action, so is the group  $\hat{G}$  after fixing a pinning of the root datum, and we denote by  ${}^L G = \hat{G} \rtimes \mathcal{W}_{F_0}$  the  $L$ -group of  $\mathbf{G}$ .

**Definition 0.1.1.** An  $L$ -parameter of  $\mathbf{G}$  (over  $\mathbb{C}$ ) is a homomorphism  $\phi : \text{WD}_{F_0} \rightarrow {}^L G$  such that

- The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{W}_{F_0} \times \text{SL}_2(\mathbb{C}) = \text{WD}_{F_0} & \xrightarrow{\phi} & {}^L G = \hat{G} \rtimes \mathcal{W}_{F_0} \\ & \searrow & \swarrow \\ & \mathcal{W}_{F_0} & \end{array}$$

where the two unmarked arrows are canonical projections.

- $\phi|_{\mathcal{W}_{F_0} \times \{1\}}$  is continuous with image consisting of semisimple elements<sup>2</sup>, and  $\phi|_{\{1\} \times \text{SL}_2(\mathbb{C})}$  is algebraic<sup>3</sup> with image consisting of unipotent elements in  $\hat{G}$ .

<sup>1</sup>We will only consider *connected* reductive groups.

<sup>2</sup>An element  $(g, w)$  in  ${}^L G$  is semisimple if for any  $r$  as a finite dimensional representation of  ${}^L G$ , the image  $r((g, w))$  is semisimple.

<sup>3</sup>That is, it is an algebraic representation from the complex algebraic group  $\text{SL}_2$  into the complex algebraic group  $\hat{G}$ .

Two L-parameters are said to be isomorphic if they are in the same  $\hat{G}$ -conjugacy class, and we denote by  $\Phi(G)$  the isomorphism classes of Langlands parameters of  $\mathbf{G}$ . The famous local Langlands correspondence is stated as follows.

**Conjecture 0.1.2.** *There is a unique finite-to-one surjection*

$$\text{LLC} : \text{Irr}_R(G) \longrightarrow \Phi(G)$$

*satisfying certain desiderata.*

**Definition 0.1.3.** *For  $\phi \in \Phi(G)$ , we call  $\Pi_\phi := \text{LLC}^{-1}(\phi)$  the L-packet of  $\phi$  as a finite set of irreducible representations of  $G$ .*

Here we won't specify what exactly do these desiderata mean (compatible with parabolic induction, transfer L-factors and  $\epsilon$ -factors, etc.) but refer to [Bor79] for an expository introduction. The local Langlands correspondence for certain reductive groups is known, such as  $\mathbf{G}$  being a torus,  $\text{GL}_n$  or certain classical groups, due to the work of Langlands [Lan97], Harris-Taylor [HT01], Arthur [Art13], etc.

Moreover, although the original conjecture of Langlands is only proposed for representations over  $\mathbb{C}$ , for other  $R$  under our settings it is still possible to give a definition for L-parameters and to propose the corresponding conjecture with the corresponding desiderata being adapted to the new situations. For example, there are pioneering work of Vignéras [Vig01] for  $\mathbf{G} = \text{GL}_n$ , and also recent work of Dat-Helm-Kurinczuk-Moss [DHKM20] for general reductive groups and representations over an integral domain with  $p$  being invertible within instead of over a field  $R$ . Finally we mention the recent result of Fargues-Scholze [FS21]. Using geometric method and under general enough settings (more general than ours), they constructed  $\Phi(G)$  (indeed as a stack) and the local Langlands correspondence, and verified the corresponding desiderata under their settings (*cf. ibid.* Theorem IX.0.5).

### 0.1.2 Local Langlands functoriality

Now we discuss the local Langlands functoriality and we still assume that  $R = \mathbb{C}$ . Let  $\mathbf{G}_0$  be another reductive group over  $F_0$ , and let  $G_0$  be the group of  $F_0$ -rational points of  $\mathbf{G}_0$ . As in the previous subsection, we may similarly define its dual group  $\hat{G}_0$ , its L-group  ${}^L G_0 = \hat{G}_0 \rtimes \mathcal{W}_{F_0}$  and the isomorphism classes  $\Phi(G_0)$ .

**Definition 0.1.4.** *A group homomorphism*

$$\iota : {}^L G_0 \longrightarrow {}^L G$$

*is called an L-homomorphism, if*

- *it is continuous, and its restriction to  $\hat{G}_0$  is an algebraic representation of  $\hat{G}_0$  into  $\hat{G}$ .*
- *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{W}_{F_0} \rtimes \hat{G}_0 = {}^L G_0 & \xrightarrow{\iota} & {}^L G = \hat{G} \rtimes \mathcal{W}_{F_0} \\ & \searrow & \swarrow \\ & \mathcal{W}_{F_0} & \end{array}$$

*where the two unmarked arrows are canonical projections.*

By definition, given an L-parameter  $\phi_0$  in  $\Phi(G_0)$ , the composition  $\iota \circ \phi_0$  is an L-parameter in  $\Phi(G)$ . Thus we construct a map between isomorphism classes of L-parameters:

$$\Phi(\iota) : \Phi(G_0) \longrightarrow \Phi(G), \quad \phi_0 \longmapsto \iota \circ \phi_0.$$

If we admit the local Langlands correspondence for both  $\mathbf{G}_0$  and  $\mathbf{G}$ , we have the following diagram

$$\begin{array}{ccc} \mathrm{Irr}_R(G_0) & \xrightarrow{\mathrm{LLC}} & \Phi(G_0) \\ \Pi(\iota) \downarrow & & \downarrow \Phi(\iota) \\ \mathrm{Irr}_R(G) & \xrightarrow{\mathrm{LLC}} & \Phi(G) \end{array}$$

The local Langlands functoriality predicts the existence of a map  $\Pi(\iota)$ , called *local lifting* with respect to  $\iota$ , such that the above diagram is commutative. Moreover this map  $\Pi(\iota)$  is expected to be constructed independently of this diagram, but using other technical tools such as trace formula or L-function. On the one hand, understanding different local liftings maps forms an important part of the local Langlands program. On the other hand, it also plays a crucial role in the inductive strategy, proposed by Langlands-Shelstad [LS87] and called the method of endoscopy, of constructing the local Langlands correspondence for general reductive group, which has become a prosperous area in recent years with fruitful results, including the work of Arthur, Kottwitz, Langlands, Laumon, Ngô, Shelstad, Waldspurger, etc.

Still we need not confine ourself in complex representations, instead we consider possible local lifting over  $R$ . For example, one expectation for the expected local lifting over  $\overline{\mathbb{F}}_l$  is that, it is supposed to be compatible with the local lifting over  $\overline{\mathbb{Q}}_l$ , after we identify  $\mathbb{C}$  with  $\overline{\mathbb{Q}}_l$  via a certain algebraic isomorphism and implement the modulo  $l$  reduction. One typical result is about the Jacquet-Langlands correspondence as one natural enough lifting between  $\mathrm{GL}_n$  and its inner form. Over  $\overline{\mathbb{F}}_l$ , the construction of this map and also its compatibility with the usual Jacquet-Langlands correspondence was studied by Dat [Dat12] for special case, and then generalized by Mínguez-Sécherre [MS17] for general case.

### 0.1.3 Problem of distinction

Let  $\mathbf{H} \subset \mathbf{G}$  be a closed algebraic subgroup over  $F_0$  and we denote by  $H$  the group of  $F_0$ -rational points of  $\mathbf{H}$ . For  $\pi \in \mathrm{Irr}_R(G)$  and  $\rho \in \mathrm{Irr}_R(H)$ , we say that  $\pi$  is  $(H, \rho)$ -*distinguished* if

$$\mathrm{Hom}_H(\pi, \rho) \neq 0,$$

or in other words, the restriction of  $\pi$  to  $H$  admits  $\rho$  as a quotient. In particular, when  $\rho$  is trivial, we call  $\pi$  *distinguished by  $H$*  or  *$H$ -distinguished*. Still for simplicity we temporarily assume  $R = \mathbb{C}$ .

The problem of distinction is ubiquitous and plays an important role in the representation theory of  $p$ -adic groups. For example, if  $\mathbf{G}$  is quasisplit, we let  $\mathbf{H} = \mathbf{U}$  be the unipotent radical of a Borel subgroup of  $\mathbf{G}$  and we let  $\psi$  be a non-degenerate character of  $H = U$ , that is, its restriction to any unipotent subgroup  $U_\alpha$  of  $U$  related to a simple root  $\alpha$  is non-trivial. One well-known result [Sha74] is that the vector space

$$\mathrm{Hom}_U(\pi, \psi)$$

is of dimension smaller than or equal to one. Those  $\pi$  with the corresponding dimension equalling one are called generic representations. By the Frobenius reciprocity, such  $\pi$  can be embedded into the space of  $(U, \psi)$ -invariant  $G$ -linear forms, which is called the Whittaker model of  $\pi$  and plays an prominent role in the local and global theory of L-functions. In another example we consider  $V$  as

a finite dimensional vector space over  $F_0$  endowed with a sesquilinear form, and  $W$  as a subspace of  $V$ . We denote by  $\mathbf{G}$  the group of  $F_0$ -automorphisms of  $V$  and by  $\mathbf{H}$  the group of  $F_0$ -automorphisms of  $W$ , preserving the sesquilinear form. Then the corresponding problem of distinction is related to the “branching laws”, which dates back to the representation theory of complex algebraic groups and has been performing as an active area in decades because of the initiation and breakthrough of the Gan-Gross-Prasad conjecture [GGP12] and its variants.

Under good settings, the problem of distinction is closely related to the local Langlands correspondence and its functoriality. In the remarkable book [SV17a], Sakellaridis and Venkatesh proposed a general framework to study the problem of distinction, in which they assume  $\mathbf{G}$  to be split and  $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$  to be a spherical variety with  $X$  denoting its  $F_0$ -rational points. Their starting point is the construction of the dual group  $\hat{G}_X$  for  $\mathbf{X}$  as a complex reductive group, under an assumption on the roots of  $\mathbf{X}$ , together with a canonical algebraic representation

$$\iota_X : \hat{G}_X \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \hat{G}.$$

Under their conjectural proposal, roughly speaking, the  $H$ -distinguished representations of  $G$  correspond to the  $X$ -distinguished Arthur parameters via the local Langlands correspondence, where Arthur parameters are the analogue of  $L$ -parameters with a corresponding version of local Langlands correspondence related, and those Arthur parameters factoring through  $\iota_X$  are called  $X$ -distinguished, for which we leave *ibid.* section 16 for more details. So the idea behind is that, under good circumstances, the property of being distinguished is transferred by the local Langlands correspondence. In [Pra15], Prasad considered the case where  $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$  is a symmetric space with respect to a Galois involution. He constructed a quasisplit subgroup  $\mathbf{G}_0$  (denoted by  $\mathbf{G}^{op}$  in *loc. cit.*) over  $F_0$ , a natural  $L$ -homomorphism  $\iota : {}^L G_0 \rightarrow {}^L G$  which simply comes from the restriction, and a character  $\omega_H$  of  $H$ . Finally he conjectured that, for  $\pi$  an irreducible representation of  $G$  distinguished by  $(H, \omega_H)$ , the  $L$ -packet of  $\pi$  is derived from the local lifting related to  $\iota$ , or more precisely there exists  $\phi_0 \in \Phi(G_0)$  such that  $\pi \in \Pi(\iota \circ \phi_0)$ . Moreover a conjectural formula for the dimension of the space of distinction has been given. These two general frameworks, combining with various concrete examples being verified, should be regarded as our guideline of the results we should expect under the language of local Langlands correspondence and its functoriality.

We briefly introduce some known methods of dealing with problem of distinction. One important method, initiated by Jacquet and developed by himself, his students and other followers, is called the relative trace formula method, for which we name a few articles [JLR93], [JY96], [Guo96], [Mao98]. The idea, roughly speaking, is first to solve the corresponding problem over a global field, and then to realize our local field  $F_0$  as a component of the ring of adèles of a global field and to use a global-to-local argument. Then one compares two different trace formulae as distributions on two spaces of test functions, one of which relates exactly to our global problem. After verifying the fundamental lemma and the existence of smooth transfer, one obtains sufficient many pairs of matching test functions such that two trace formulae coincide. If the other trace formula is well understood, we get the information to solve the global problem of distinction. In addition, to solve the local Gan-Gross-Prasad conjecture for orthogonal groups, Waldspurger [Wal10], [Wal12] initiated a new method with the consideration of a local relative trace formula, such that the dimension of the space of distinction can be expressed inside, and then he used sophisticated techniques in harmonic analysis over  $p$ -adic reductive groups to reformulate the trace formula and to obtain the result. In the last decade this method has been developed and applied to different situations by some people including Beuzart-Plessis and C. Wan. For example in [BP18] using the similar method, Beuzart-Plessis solved part of the above conjecture proposed by Prasad for essentially square integrable representations.

Another possible method to study the problem of distinction is algebraic, which first studies the same problem for supercuspidal representations as the starting point, and then applies parabolic

induction to study more general irreducible representations. For  $\pi$  a supercuspidal representation of  $G$ , a general belief is that it can be written down as the compact induction of a finite dimensional smooth irreducible representation, more precisely, there exists a pair  $(\mathbf{J}, \Lambda)$  such that  $\mathbf{J}$  is a compact subgroup of  $G$  modulo the centre, and  $\Lambda$  is a smooth irreducible finite dimensional representation of  $\mathbf{J}$  such that  $\pi \cong \text{ind}_{\mathbf{J}}^G \Lambda$ . This belief is verified for many cases, including tame supercuspidal representations [Yu01], [Fin21] for tamely ramified reductive group  $\mathbf{G}$ , and also general supercuspidal representations for classical groups [BK93], [Ste08]. Then if we focus on the study of  $H$ -distinguished supercuspidal representation  $\pi$ , using the Mackey formula and the Frobenius reciprocity, it is easily seen that

$$\text{Hom}_H(\pi, 1) \cong \text{Hom}_H(\text{ind}_{\mathbf{J}}^G \Lambda, 1) \cong \prod_{g \in \mathbf{J} \backslash G/H} \text{Hom}_{\mathbf{J}^g \cap H}(\Lambda^g, 1).$$

Thus we only need to study those  $g \in \mathbf{J} \backslash G/H$  such that the  $R$ -vector space  $\text{Hom}_{\mathbf{J}^g \cap H}(\Lambda^g, 1)$  is non-zero, and then to study corresponding dimension. To that aim, we date back to the detailed construction of  $(\mathbf{J}, \Lambda)$ . One typical work is [HM08], where the authors studied, for  $\mathbf{G}/\mathbf{H}$  as a symmetric space, tame supercuspidal representations  $\pi$  of  $G$  distinguished by  $H$  using the idea mentioned above and the structural result of J.-K. Yu [Yu01] for such representations.

Still we are not necessarily confined in the case where  $R = \mathbb{C}$ , but we focus on the general  $R$  in our settings. In this case the two analytic methods mentioned above become invalid. By contrast the algebraic method remains valid, since the structural result for the  $(\mathbf{J}, \Lambda)$ , once being established, usually works for general  $R$  rather than just  $R = \mathbb{C}$ , such as [Vig96], [MS14b] and [Fin19]. To sum up, searching the possible relation between the problem of distinction and the local Langlands correspondence and its functoriality for general  $R$  should be regarded as the original motivation for this thesis.

#### 0.1.4 Our concrete settings

Although the context above is quite general, the aim of this thesis is humble, which focuses on the understanding of several special examples. Fix  $n$  as a positive integer. Let  $F/F_0$  be a finite cyclic extension of non-archimedean locally compact fields of residue characteristic  $p$  of degree  $r$ , and let  $\mathbf{G}$  be the Weil restriction of the reductive group  $\text{GL}_n$  over  $F$ , which is a reductive group over  $F_0$ . In particular we have  $G = \text{GL}_n(F)$ . For most of the time, we will concentrate on cuspidal or supercuspidal representations of  $G$  over  $R$ , which should be regarded as the building blocks for general irreducible representations. Recall that an irreducible representation of  $G$  is cuspidal (*resp.* supercuspidal) if it doesn't occur as a subrepresentation (*resp.* subquotient) of the parabolic induction of an irreducible representation of a proper Levi subgroup of  $G$ . When  $\text{char}(R) = 0$  the two concepts above are equivalent, however when  $\text{char}(R) = l > 0$ , a supercuspidal representation must be cuspidal, but the existence of counter-example manifests that the converse is false in general.

To study a cuspidal representation  $\pi$  of  $G$  over  $R$ , our main tool is the simple type theory established by Bushnell-Kutzko [BK93] when  $\text{char}(R) = 0$ , and further generalized by Vignéras [Vig96] to the  $l$ -modular case. We refer to chapter 1, section 3 or chapter 3, section 2 for a detailed introduction for the theory, but here we also give a brief introduction for ease of giving more details.

As indicated above, the idea of simple type theory is to realize  $\pi$  as the compact induction of a finite dimensional irreducible representation  $\Lambda$  of  $\mathbf{J}$ , which is an open subgroup of  $G$  compact modulo the centre. Such a pair  $(\mathbf{J}, \Lambda)$  is called an *extended maximal simple type* which we will abbreviate to *simple type* for short. The main theorem says that, any  $\pi$  can be constructed in this way, and the corresponding simple type  $(\mathbf{J}, \Lambda)$  is unique up to  $G$ -conjugacy. We also mention the following main properties of  $(\mathbf{J}, \Lambda)$ :



- (1) The group  $\mathbf{J}$  contains a unique maximal open compact subgroup  $J$  which contains a unique maximal normal pro- $p$ -subgroup  $J^1$ ;
- (2) We have  $J/J^1 \cong \mathrm{GL}_m(\mathfrak{l})$ . Here  $\mathfrak{l}$  is the residue field of  $E$ , where  $E$  is a field extension over  $F$  of degree  $d$ . Moreover we have  $n = md$ , where  $m$  and  $d$  are integers determined by  $\pi$ ;
- (3) We may write  $\Lambda = \kappa \otimes \rho$ , where  $\kappa$  and  $\rho$  are irreducible representations of  $\mathbf{J}$  such that the restriction  $\kappa|_{J^1} = \eta$  is an irreducible representation of  $J^1$ , called a *Heisenberg representation*, and  $\rho|_J$  is the inflation of a cuspidal representation of  $\mathrm{GL}_m(\mathfrak{l}) \cong J/J^1$ ;
- (4) There exists a pro- $p$ -subgroup of  $J^1$  denoted by  $H^1$ , and a character of  $H^1$  denoted by  $\theta$  and called a simple character, such that the restriction of  $\eta$  to  $H^1$  equals the direct sum of  $(J^1 : H^1)^{1/2}$  copies of  $\theta$ .

Finally we enter the introduction for our concrete work. For the first part, we study the problem of distinction related to a unitary subgroup of  $G$  and its relation with the Langlands functoriality, or embodied as the quadratic base change lift in our settings; For the second part, we study the problem of distinction related to an orthogonal subgroup of  $G$ , and we focus only on supercuspidal representations over  $R = \mathbb{C}$ , which is the first step towards the understanding of more general irreducible representations; For the final part for  $R = \mathbb{C}$  we give explicit constructions for two special local liftings, the base change lift and the automorphic induction, for supercuspidal representations.

## 0.2 Problem of distinction related to unitary subgroups of $\mathrm{GL}_n(F)$ and $l$ -modular base change lift

### 0.2.1 Background

The first eight sections of chapter 1 is based on the preprint [\[Zou19\]](#). In this subsection we assume  $F/F_0$  to be a quadratic extension of  $p$ -adic fields of residue characteristic  $p$ , and we let  $\sigma$  denote its non-trivial automorphism. For  $\mathbf{G}$  and  $G$  as above, we write  $\varepsilon$  for a *hermitian matrix* in  $G$ , that is,  $\sigma({}^t\varepsilon) = \varepsilon$  with  ${}^t$  denoting the transpose of matrices. We define

$$\tau_\varepsilon(x) = \varepsilon \sigma({}^t x^{-1}) \varepsilon^{-1}$$

for any  $x \in G$ , called a *unitary involution* on  $G$ , which also induces an  $F_0$ -automorphism on  $\mathbf{G}$ . We fix one  $\tau = \tau_\varepsilon$ , and we denote by  $\mathbf{G}^\tau$  the subgroup of  $\mathbf{G}$  over  $F_0$ , such that  $G^\tau$  is the subgroup of  $G$  consisting of the elements fixed by  $\tau$ . Such  $\mathbf{G}^\tau$  (resp.  $G^\tau$ ) is called the *unitary subgroup* of  $\mathbf{G}$  (resp.  $G$ ) with respect to  $\tau$ .

For  $\pi$  a smooth irreducible representation of  $G$  over  $\mathbb{C}$ , Jacquet proposed to study the problem of distinction related to the pair  $(G, G^\tau)$  as above, that is, to study the space of  $G^\tau$ -invariant linear forms

$$\mathrm{Hom}_{G^\tau}(\pi, 1)$$

and its dimension as a complex vector space. For  $n = 3$  and  $\pi$  supercuspidal, he proved in [\[Jac01\]](#) by using global argument, that  $\pi$  is distinguished by  $G^\tau$  if and only if  $\pi$  is  $\sigma$ -invariant, that is,  $\pi^\sigma \cong \pi$ , where  $\pi^\sigma := \pi \circ \sigma$ . Moreover he showed that this space is of dimension one as a complex vector space when the condition above is satisfied. Besides in *ibid.*, he also sketched a similar proof when  $n = 2$  and  $\pi$  is supercuspidal, to give the same criterion of distinction and the same dimension one theorem. Based on these results as one of the main reasons, he conjectured that in general,  $\pi$  is distinguished by  $G^\tau$  if and only if  $\pi$  is  $\sigma$ -invariant. Moreover, the dimension of the space of  $G^\tau$ -invariant linear forms is not necessary to be one in general. Under the assumption that  $\pi$  is  $\sigma$ -invariant and supercuspidal Jacquet further conjectured that the dimension is one.

In addition, an irreducible representation  $\pi$  of  $G$  is contained in the image of quadratic base change lift with respect to  $F/F_0$  if and only if it is  $\sigma$ -invariant ([AC89]). Thus for irreducible representations, the conjecture of Jacquet gives a connection between quadratic base change lift and  $G^\tau$ -distinction.

Besides the special case mentioned above, there are two more evidences which support the conjecture. First we consider the analogue of the conjecture in the finite field case. For  $\bar{\rho}$  an irreducible complex representation of  $\mathrm{GL}_n(\mathbb{F}_{q^2})$ , Gow [Gow84] proved that  $\bar{\rho}$  is distinguished by the unitary subgroup  $U_n(\mathbb{F}_q)$  if and only if  $\bar{\rho}$  is isomorphic to its twist under the non-trivial element of  $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ . Under this condition, he also showed that the space of  $U_n(\mathbb{F}_q)$ -invariant linear forms is of dimension one as a complex vector space. In addition, Shintani [Shi76] showed that there is a one-to-one correspondence between the set of irreducible representations of  $\mathrm{GL}_n(\mathbb{F}_q)$  and that of Galois invariant irreducible representations of  $\mathrm{GL}_n(\mathbb{F}_{q^2})$ , where the correspondence, called the *base change map*, is characterized by a trace identity. These two results give us a clear feature between base change map and  $U_n(\mathbb{F}_q)$ -distinction. Finally, when  $\bar{\rho}$  is generic and Galois-invariant, Anandavardhanan and Matringe [AM18] recently showed that the  $U_n(\mathbb{F}_q)$ -average of Bessel function of  $\bar{\rho}$  on the Whittaker model as a  $U_n(\mathbb{F}_q)$ -invariant linear form is non-zero. Since the space of  $U_n(\mathbb{F}_q)$ -invariant linear forms is of dimension one, this result gives us a concrete characterization of the space of distinction.

The other evidence for the Jacquet conjecture is its global analogue. We assume  $\mathcal{K}/\mathcal{K}_0$  to be a quadratic extension of number fields and we denote by  $\sigma$  its non-trivial automorphism. We consider  $\tau$  to be a unitary involution on  $\mathrm{GL}_n(\mathcal{K})$ , which also gives us an involution on  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ , still denoted by  $\tau$  by abuse of notation, where  $\mathbb{A}_{\mathcal{K}}$  denotes the ring of adèles of  $\mathcal{K}$ . We denote by  $\mathrm{GL}_n(\mathcal{K})^\tau$  (resp.  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$ ) the unitary subgroup of  $\mathrm{GL}_n(\mathcal{K})$  (resp.  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ ) with respect to  $\tau$ . For  $\phi$  a cusp form of  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ , we define

$$\mathcal{P}_\tau(\phi) = \int_{\mathrm{GL}_n(\mathcal{K})^\tau \backslash \mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau} \phi(h) dh$$

to be the *unitary period integral* of  $\phi$  with respect to  $\tau$ . We say that a cuspidal automorphic representation  $\Pi$  of  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$  is  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$ -distinguished if there exists a cusp form in the space of  $\Pi$  such that  $\mathcal{P}_\tau(\phi) \neq 0$ . In 1990's, Jacquet and Ye began to study the relation between  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$ -distinction and global base change lift (see for example [JY96] when  $n = 3$ ). For general  $n$ , Jacquet [Jac05] showed that  $\Pi$  is contained in the image of quadratic base change lift (or equivalently  $\Pi$  is  $\sigma$ -invariant [AC89]) with respect to  $\mathcal{K}/\mathcal{K}_0$  if and only if there exists a unitary involution  $\tau$  such that  $\Pi$  is  $G^\tau$ -distinguished. This result may be viewed as the global version of Jacquet conjecture for supercuspidal representations.

In fact, for the special case of the Jacquet conjecture in [Jac01], Jacquet used the global analogue of the same conjecture and relative trace formula as two main techniques to finish the proof. To say it simple, he first proved the global analogue of the conjecture. Then he used the relative trace formula to write a non-zero unitary period integral as the product of its local components at each place of  $\mathcal{K}_0$ , where each local component characterizes the distinction of the local component of  $\Pi$  with respect to the corresponding unitary group over local fields. When  $\pi$  is  $\sigma$ -invariant, he chose  $\Pi$  as a  $\sigma$ -invariant cuspidal automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$  and  $v_0$  as a non-archimedean place of  $\mathcal{K}_0$  such that  $(G^\tau, \pi) = (\mathrm{GL}_n(\mathcal{K}_{v_0})^\tau, \Pi_{v_0})$ . Then the product decomposition leads to the proof of the “if” part of the conjecture. The “only if” part of the conjecture, which will be discussed in chapter 1, section 4, requires the application of globalization theorem. His method was generalized by Feigon-Lapid-Offen in [FLO12] to general  $n$  and more general family of representations. They showed that the Jacquet conjecture works for generic representations of  $G$ . Moreover for the same family of representations, they were able to give a lower bound for the dimension of  $\mathrm{Hom}_{G^\tau}(\pi, 1)$  and they further conjectured that the inequality they gave is actually an equality. Finally, Beuzart-Plessis [BP20] recently verified the equality based on the work of Feigon-Lapid-Offen and the relative local trace formula. Thus for generic representations of  $G$ , the Jacquet conjecture was settled.

Instead of using global-to-local argument, there are also partial results based on the algebraic

method we explained before. In [HM98] Hakim-Mao verified the conjecture when  $\pi$  is supercuspidal of level zero, that is,  $\pi$  is supercuspidal such that  $\pi^{1+\mathfrak{p}_F M_n(\mathfrak{o}_F)} \neq 0$ , where  $\mathfrak{o}_F$  denotes the ring of integers of  $F$  and  $\mathfrak{p}_F$  denotes its maximal ideal. When  $\pi$  is supercuspidal and  $F/F_0$  is unramified, Prasad [Pra01] proved the conjecture by applying the simple type theory developed by Bushnell-Kutzko in [BK93]. When  $\pi$  is tame supercuspidal, that is,  $\pi$  is a supercuspidal representation arising from the construction of Howe [How77], Hakim-Murnaghan [HM02b] verified the conjecture. Noting that in the results of Hakim-Mao and Hakim-Murnaghan, they need the additional assumption that the residue characteristic  $p \neq 2$ .

The discussion above leaves us an open question: *Is there any local and algebraic method that leads to a proof of the Jacquet conjecture which works for all supercuspidal representations of  $G$ ?* First, this method will generalize the results of Hakim-Mao, Prasad and Hakim-Murnaghan which we mentioned in the last paragraph. Secondly, we are willing to consider  $F/F_0$  to be a quadratic extension of non-archimedean locally compact fields instead of  $p$ -adic fields. Since the result of Feigon-Lapid-Offen heavily relies on the fact that the characteristic of  $F$  equals 0, their method fails when considering non-archimedean locally compact fields of positive characteristic. Finally, instead of considering complex representations, we are also willing to study  $l$ -modular representations with  $l \neq p$ . One hopes to prove an analogue of the Jacquet conjecture for  $l$ -modular supercuspidal representations, which will generalize the result of Feigon-Lapid-Offen for supercuspidal representations. Noting that they use global method in their proof, which strongly relies on the assumption that all the representations are complex. Thus their method doesn't work anymore for  $l$ -modular representations.

The aim of chapter 1 is first to address the question above, and then to explore the problem of distinction for more general irreducible representations in the  $l$ -modular case and its relation with the “ $l$ -modular” base change lift whose construction will be given.

## 0.2.2 Main results

To begin with, from now on we assume  $F/F_0$  to be a quadratic extension of non-archimedean locally compact fields of residue characteristic  $p$  instead of  $p$ -adic fields, and we assume that  $p \neq 2$ . We fix  $R$  an algebraically closed field of characteristic  $l \neq p$ , allowing that  $l = 0$ . We assume  $\pi$  to be an irreducible representation of  $G = \mathrm{GL}_n(F)$  over  $R$ . Now we state our first main theorem.

**Theorem 0.2.1.** *For  $\pi$  a supercuspidal representation of  $G$  and  $\tau$  a unitary involution,  $\pi$  is distinguished by  $G^\tau$  if and only if  $\pi^\sigma \cong \pi$ .*

Moreover, we may also calculate the dimension of the space of  $G^\tau$ -invariant linear forms.

**Theorem 0.2.2.** *For  $\pi$  a  $\sigma$ -invariant supercuspidal representation of  $G$ , we have*

$$\dim_R \mathrm{Hom}_{G^\tau}(\pi, 1) = 1.$$

One important corollary of Theorem [0.2.1] relates to the  $\overline{\mathbb{Q}}_l$ -lift of a  $\sigma$ -invariant supercuspidal representation of  $G$  over  $\overline{\mathbb{F}}_l$  when  $l > 0$ , where we denote by  $\overline{\mathbb{Q}}_l$ ,  $\overline{\mathbb{Z}}_l$  and  $\overline{\mathbb{F}}_l$  the algebraic closure of an  $l$ -adic field, its ring of integers and the algebraic closure of the finite field of  $l$  elements respectively. For  $(\tilde{\pi}, V)$  a smooth irreducible representation of  $G$  over  $\overline{\mathbb{Q}}_l$ , we call it *integral* if it admits an integral structure, that is, a  $\overline{\mathbb{Z}}_l[G]$ -submodule  $L_V$  of  $V$  such that  $L_V \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = V$ . For such a representation, the semi-simplification of  $L_V \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  doesn't depend on the choice of  $L_V$ , which we denote by  $r_l(\tilde{\pi})$  a representation of  $G$  over  $\overline{\mathbb{F}}_l$ , called the *modulo  $l$  reduction* of  $\pi$  (see [Vig96] for more details). The following theorem which will be proved at the end of chapter 1, section 8, says that it is always possible to find a  $\sigma$ -invariant  $\overline{\mathbb{Q}}_l$ -lift for a  $\sigma$ -invariant supercuspidal representation of  $G$  over  $\overline{\mathbb{F}}_l$ .

**Theorem 0.2.3.** *For  $\pi$  a  $\sigma$ -invariant supercuspidal representation of  $G$  over  $\overline{\mathbb{F}_l}$ , there exists an integral  $\sigma$ -invariant supercuspidal representation  $\tilde{\pi}$  of  $G$  over  $\overline{\mathbb{Q}_l}$ , such that  $r_l(\tilde{\pi}) = \pi$ .*

For irreducible generic representations, we are able to prove one direction of the Jacquet conjecture, which is new only if  $\text{char}(R) = l > 0$ .

**Theorem 0.2.4** (see Theorem [1.9.1](#)). *Let  $\pi$  be an irreducible generic representation of  $G$  over  $R$ . If  $\pi$  is distinguished by  $G^\tau$ , then  $\pi$  is  $\sigma$ -invariant.*

Our next goal is to characterize  $l$ -modular distinguished representations via local Langlands functoriality, or base change lift in our settings. To that aim, first we need to construct an  $l$ -modular base change lift. The upshot is the following theorem:

**Theorem 0.2.5** (see Theorem [1.10.17](#)). *We may define the  $l$ -modular cyclic base change lift*

$$\text{BC}_{\overline{\mathbb{F}_l}} : \text{Irr}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F_0)) \longrightarrow \text{Irr}_{\overline{\mathbb{F}_l}}^{\sigma\text{-inv}}(\text{GL}_n(F))$$

which satisfies and is determined by the following commutative diagram

$$\begin{array}{ccc} \text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F_0)) & \xrightarrow{\text{BC}_{\overline{\mathbb{Q}_l}}} & \text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}, \sigma\text{-inv}}(\text{GL}_n(F)) \\ \downarrow J_l & & \downarrow J_l \\ \text{Irr}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F_0)) & \xrightarrow{\text{BC}_{\overline{\mathbb{F}_l}}} & \text{Irr}_{\overline{\mathbb{F}_l}}^{\sigma\text{-inv}}(\text{GL}_n(F)) \end{array}$$

We briefly explain the notations and leave the corresponding section for more details. Here the superscripts Int and  $\sigma$ -inv represent integral and  $\sigma$ -invariant respectively,  $\text{BC}_{\overline{\mathbb{Q}_l}}$  represents the base change lift of Arthur-Clozel being transferred to representations over  $\overline{\mathbb{Q}_l}$  via a certain algebraic isomorphism  $\mathbb{C} \cong \overline{\mathbb{Q}_l}$ , and for  $\tilde{\pi}_0$  (resp.  $\tilde{\pi}$ ) in  $\text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F_0))$  (resp.  $\text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}, \sigma\text{-inv}}(\text{GL}_n(F))$ ), the image  $J_l(\tilde{\pi}_0)$  (resp.  $J_l(\tilde{\pi})$ ) is the unique irreducible constituent in  $r_l(\tilde{\pi}_0)$  (resp.  $r_l(\tilde{\pi})$ ) having the highest derivative sequence. Finally as an application, we explore the distinguished cuspidal (but not necessarily supercuspidal) representations in the  $l$ -modular case.

### 0.2.3 Organization of the chapter 1

Let us outline the content of chapter 1. We introduce our settings in section 1 and basic knowledge about hermitian matrices and unitary subgroups in section 2. Our main tool to prove the theorems will be the simple type theory developed by Bushnell-Kutzko in [\[BK93\]](#), and further generalized by Vignéras [\[Vig96\]](#) to the  $l$ -modular case. In section 3 we will give a detailed introduction of this theory.

For a given supercuspidal representation  $\pi$  of  $G$ , our starting point is to prove the “only if” part of Theorem [0.2.1](#). When  $R = \mathbb{C}$  and  $\text{char}(F) = 0$ , it is a standard result by using global argument, especially the globalization theorem ([\[HM02a\]](#), Theorem 1). When  $\text{char}(F) = p > 0$ , we may keep the original proof except that we need a characteristic  $p$  version of the globalization theorem. Fortunately we can use a more general result due to Gan-Lomelí [\[GL18\]](#) to get the result we need. Since any supercuspidal representation of  $G$  over a characteristic 0 algebraically closed field can be realized as a representation over  $\overline{\mathbb{Q}}$  up to twisting by an unramified character, we finish the proof when  $\text{char}(R) = 0$ . When  $R = \overline{\mathbb{F}_l}$ , we consider the projective envelope  $P_{\Lambda|_J}$  of  $\Lambda|_J$  and we use the results in [\[Vig96\]](#) to study its irreducible components and the irreducible components of its  $\overline{\mathbb{Q}_l}$ -lift. Finally we show that there exists a  $\overline{\mathbb{Q}_l}$ -lift of  $\pi$  which is supercuspidal and  $G^\tau$ -distinguished. Thus by using the characteristic 0 case we finish the proof for the “only if” part for any  $R$  under our settings. The details will be presented in section 4.

In section 5, we prove the  $\tau$ -selfdual type theorem, which says that for any given unitary involution  $\tau$  and a  $\sigma$ -invariant cuspidal representation of  $G$  with a technical condition (see Theorem 1.5.3), which is automatically true at least in the supercuspidal case, we may find a simple type  $(\mathbf{J}, \Lambda)$  contained in  $\pi$  such that  $\tau(\mathbf{J}) = \mathbf{J}$  and  $\Lambda^\tau \cong \Lambda^\vee$ , where  $\vee$  denotes the contragredient. In other words, we find a “symmetric” simple type contained in  $\pi$  with respect to  $\tau$ . Our strategy follows from [AKM<sup>+</sup>19], section 4. First we consider the case where  $E/F$  is totally wildly ramified and  $n = d$ . Then for  $E/F$  in general with  $n = d$ , we make use of the techniques about endo-class and tame lifting developed in [BH96] to prove the theorem by reducing it to the former case. Finally by using the  $n = d$  case, we prove the general theorem.

In section 6, for a given  $\sigma$ -invariant cuspidal representation  $\pi$  and a certain unitary involution  $\tau$  satisfying the technical condition, we first use our results in section 5 to choose a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  contained in  $\pi$ . The main result of section 6, which we call the *distinguished type theorem*, says that  $\pi$  is distinguished by  $G^\tau$  if and only if there exists a  $\tau$ -selfdual and distinguished simple type of  $\pi$ . More specifically, by the Frobenius reciprocity and the Mackey formula, we have

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

We concentrate on those  $g$  in the double coset such that  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ . The proof of the distinguished type theorem also shows that there are at most two such double cosets which can be written down explicitly. Moreover for those  $g$  we have

$$\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \cong \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\kappa}^g, \chi^{-1}) \otimes_R \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\rho}^g, \chi),$$

where  $\boldsymbol{\kappa}^\tau \cong \boldsymbol{\kappa}^\vee$  and  $\chi$  is a quadratic character of  $\mathbf{J}^g \cap G^\tau$  which is trivial when restricting to  $J^{1g} \cap G^\tau$ . In the tensor product, the first term  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\kappa}^g, \chi^{-1})$  is of dimension one as an  $R$ -vector space. So essentially we only need to study the second term. If we denote by  $\overline{\boldsymbol{\rho}^g}$  the cuspidal representation of  $\mathrm{GL}_m(\mathbf{l}) \cong J^g / J^{1g}$  whose inflation equals  $\boldsymbol{\rho}^g|_{J^g}$ , and by  $\overline{\chi}$  the character of  $H := J^g \cap G^\tau / J^{1g} \cap G^\tau$  whose inflation equals  $\chi|_{J^g \cap G^\tau}$ , then we further have

$$\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\rho}^g, \chi) \cong \mathrm{Hom}_H(\overline{\boldsymbol{\rho}^g}, \overline{\chi}).$$

Here  $H$  could be a unitary subgroup, an orthogonal subgroup or a symplectic subgroup of  $\mathrm{GL}_m(\mathbf{l})$ . So we reduce our problem to study the  $H$ -distinction of a supercuspidal representation of  $\mathrm{GL}_m(\mathbf{l})$ .

Now we assume that  $\pi$  is supercuspidal. At the beginning of section 6, we use the result in section 5 to extend  $\sigma$  to a non-trivial involution on  $E$ . We write  $E_0 = E^\sigma$ , where  $E/E_0$  is a quadratic extension. When  $E/E_0$  is unramified,  $H$  is a unitary subgroup. We first use the result of Gow [Gow84] to deal with the characteristic 0 case. For  $\mathrm{char}(R) > 0$ , we use the same method as in section 4. When  $E/E_0$  is ramified,  $H$  is either an orthogonal subgroup or a symplectic subgroup. When  $H$  is orthogonal, we use Deligne-Lusztig theory [DL76], precisely a formula given by Hakim-Lansky [HL12] to calculate the dimension of  $\mathrm{Hom}_H(\overline{\boldsymbol{\rho}^g}, \overline{\chi})$  when  $\mathrm{char}(R) = 0$ . For  $\mathrm{char}(R) > 0$ , we use again the same method as in section 4 to finish the proof. When  $H$  is symplectic, we show that the space is always 0. These two cases will be dealt with in section 7 and section 8 separately. As a result, we finish the proof of Theorem 0.2.1, Theorem 0.2.2 and Theorem 0.2.3.

The section 9 is dedicated to the proof of Theorem 0.2.4. We first deal with the cuspidal case, whose strategy follows from the same argument in section 5-8. In particular, we also give a new proof of the main result of section 4, which is purely local and doesn't depend on the globalization theorem. And then using the parabolic induction and following the similar argument of Feigon-Lapid-Offen, we finish the proof for the generic case.

Finally in section 10, we construct the  $l$ -modular base change map as promised in Theorem 0.2.5. The strategy of construction is quite naive. We first construct the  $l$ -modular base change lift from

the Galois side, which corresponds to a restriction map. Then we use the  $l$ -modular local Langlands correspondence developed by Vignéras [Vig01] to transfer this map to the  $\mathrm{GL}_n$  side, such that it is compatible with the desired  $l$ -modular local Langlands functoriality. What remains to show is the compatibility of the constructed map with the usual base change lift of Arthur-Clozel, which relies on the local Langlands correspondence over  $\overline{\mathbb{Q}}_l$  and  $\overline{\mathbb{F}}_l$  and their compatibility, and the local Langlands functoriality for base change lift over  $\overline{\mathbb{Q}}_l$ . However it should be pointed out that our  $l$ -modular base change lift is in some sense “artificial”, since in the theorem the map  $J_l$  is not the usual modulo  $l$  reduction  $r_l$ , and in general we cannot ensure that the modulo  $l$  reduction of an irreducible representation is irreducible. But for cuspidal representations, the definition of  $r_l$  and  $J_l$  coincides, thus we could make use of our  $l$ -modular base change lift to study the distinction of  $l$ -modular cuspidal representations, which will be displayed in the final subsection.

It is worth mentioning that in [Séc19], Sécherre studied the  $\sigma$ -selfdual supercuspidal representations of  $G$  over  $R$ , with the same notation unchanged as before. He proved the following *Dichotomy Theorem* and *Disjunction Theorem*: For  $\pi$  a supercuspidal representation of  $G$ , it is  $\sigma$ -selfdual (that is,  $\pi^\sigma \cong \pi^\vee$ ) if and only if  $\pi$  is either distinguished by  $\mathrm{GL}_n(F_0)$  or  $\omega$ -distinguished, where  $\omega$  denotes the unique non-trivial character of  $F_0^\times$  which is trivial on  $N_{F/F_0}(F^\times)$ . The method we use in this chapter is the same as that was developed in *ibid*. For example, our section 5 corresponds to section 4 of [AKM<sup>+</sup>19] and our section 6 corresponds to section 6 of [Séc19], etc.

To point out the main differences in our case as the end of the introduction, first in section 5 we will find that in a certain case, it is even impossible to find a hereditary order  $\mathfrak{a}$  such that  $\tau(\mathfrak{a}) = \mathfrak{a}$ , which isn't a problem in section 4 of [AKM<sup>+</sup>19]. That's why we need to add a technical condition in the main theorem of section 5 and finally verify it for supercuspidal representations. Precisely, for a  $\sigma$ -invariant supercuspidal representation, we first consider the unitary involution  $\tau = \tau_1$  corresponding to the identity hermitian matrix  $I_n$ . In this case, we may use our discussion in section 5 to find a  $\tau$ -selfdual type contained in  $\pi$  and we may further use our discussion in section 6 and section 7 to show that  $m$  is odd when  $E/E_0$  is unramified. This affirms the technical condition we need, thus we may repeat the procedure of section 5 and section 6 for general unitary involutions. This detouring argument also indicates that a  $\sigma$ -invariant cuspidal not supercuspidal representation does not always contain a  $\tau$ -selfdual simple type. Moreover in section 9 we also provide another method to deal with this difficult. The rough idea is to regard a general unitary involution as a twist of a special unitary involution. This idea enable us to prove Theorem [0.2.4] for cuspidal representations.

Furthermore in section 8, we may find out that the character  $\chi$  mentioned above cannot always be realized as a character of  $\mathbf{J}$ , thus cannot be assumed to be trivial a priori as in [Séc19]. It means that we need to consider a supercuspidal representation of the general linear group over finite field distinguished by a non-trivial character of an orthogonal subgroup instead of the trivial one. That's why the result of Hakim-Lansky ([HL12], Theorem 3.11) shows up.

Last but not least, in section 6 a large part of our results are stated and proved for a general involution instead of a unitary one. This provides the possibility to generalize this method to study the distinction of supercuspidal representations of  $G$  by other involutions. For instance, the similar problem for orthogonal subgroups is explored in chapter 2 of the thesis.

## 0.3 Problem of distinction related to orthogonal subgroups of $\mathrm{GL}_n(F)$

### 0.3.1 Background

This chapter is based on the preprint [Zou20]. Let  $F = F_0$  be a non-archimedean locally compact field of residue characteristic  $p$ . We will only consider the case where  $R = \mathbb{C}$ , although the main results of this chapter are also expected to be true for  $R$  in general. As before we let  $\mathbf{G} = \mathrm{GL}_n$  be as an

algebraic group over  $F$  and we have  $G = \mathrm{GL}_n(F)$ . For  $\varepsilon$  a symmetric matrix in  $G$ , we denote by

$$\tau_\varepsilon(x) = \varepsilon^{-1} {}^t x^{-1} \varepsilon \quad \text{for any } x \in G$$

the orthogonal involution with respect to  $\varepsilon$ , and by  $\mathbf{G}^{\tau_\varepsilon}$  the orthogonal subgroup of  $\mathbf{G}$ , such that the group of its  $F_0$ -rational points, denoted by  $G^{\tau_\varepsilon}$  and called the orthogonal subgroup of  $G$ , is the subgroup of  $G$  consisting of the elements fixed by  $\tau_\varepsilon$ . In this settings, we are interested in the problem of distinction related to the pair  $(\mathbf{G}, \mathbf{G}^{\tau_\varepsilon})$ , and its relation with the local Langlands correspondence and its functoriality.

If we write  $\mathcal{S}$  for the set of invertible symmetric matrices as a topological subspace of  $G$ , which is endowed with a continuous right  $G$ -action

$$\varepsilon \cdot g := {}^t g \varepsilon g, \quad g \in G, \varepsilon \in \mathcal{S},$$

then we have the following decomposition as  $G$ -spaces

$$\mathcal{S} = \bigsqcup_{[\varepsilon]} G^{\tau_\varepsilon} \backslash G,$$

where  $[\varepsilon]$  ranges over  $\mathcal{S}/G$ , and  $G^{\tau_\varepsilon}$  is the orthogonal group defined by a certain representative  $\varepsilon$  in the class  $[\varepsilon]$ . A more uniformed version of the above problem is to study the space

$$\mathrm{Hom}_G(\pi, \mathcal{C}^\infty(\mathcal{S})) \cong \bigoplus_{[\varepsilon]} \mathrm{Hom}_G(\pi, \mathrm{Ind}_{G^{\tau_\varepsilon}}^G 1) \cong \bigoplus_{[\varepsilon]} \mathrm{Hom}_{G^{\tau_\varepsilon}}(\pi, 1), \quad (0.3.1)$$

for irreducible representation  $\pi$  of  $G$ , and to determine a criterion for the space being non-zero and to study the corresponding dimension, where  $\mathcal{C}^\infty(\mathcal{S})$  denotes the space of uniformly locally constant functions on  $\mathcal{S}$  with complex values.

The study of this problem was first proposed by Jacquet [Jac91]. The method, as we already introduced before, is first to consider its global analogue, and then to initiate a global-to-local argument, and the key point is to compare two relative trace formulae: one relates to the relative trace formula for the symmetric matrices or orthogonal groups, and the other relates to the Kuznetsov trace formula for the two-fold metaplectic covering of  $\mathrm{GL}_n$  (see [Mao98] for a brief introduction).

We provide a brief summary for the known results. In [Off05], Offen followed Jacquet's argument [Jac03] to consider the Kloosterman-Fourier transform for orbital integrals with respect to symmetric matrices, which might be a partial step to prove the existence for smooth transfer in the non-archimedean case, and the corresponding archimedean case remains a mystery. For the fundamental lemma for unit Hecke elements, Mao [Mao98] gave a proof, for  $n = 3$ , by direct calculation and Do first proved, for general  $n$ , for local fields of positive characteristic via geometric method [Do15], and then he transferred the result to  $p$ -adic fields for  $p$  large enough [Do18]. However for ease of later application, a stronger version of fundamental lemma working with general Hecke elements is needed but remains unknown. The spectral sides of both trace formulae are less studied. Partial results due to Chinta and Offen [CO12], [CO13], on the one hand, shed some light on the spectral expansions, but on the other hand, indicate the difficult of solving the full question. In particular, since the local Whittaker model for the two-fold metaplectic covering group of  $\mathrm{GL}_n$  is not unique, the terms of the spectral side of Kuznetsov trace formula are not factorizable, adding the difficulty to the global-to-local argument.

One subtle point of this problem is that, it is out of the reach of known general proposals, for example the one proposed by Sakellaridis-Venkatesh as we introduced before. In fact for  $\mathbf{X} = \mathbf{G}^\tau \backslash \mathbf{G}$  with  $\tau$  a given orthogonal involution, even if  $\mathbf{X}$  is a spherical variety, the assumption on the roots

of  $\mathbf{X}$  is not satisfied, which prevents us from constructing the dual group  $\hat{G}_X$ . If we believe in the existence of the connection between distinction and the local Langlands functoriality as we explained before, then an expected local lifting should be constructed

$$\mathrm{Irr}_{\mathbb{C}}(G_0) \longrightarrow \mathrm{Irr}_{\mathbb{C}}(G),$$

for  $G_0$  as at least a locally profinite group, such that those distinguished representations are exactly in its image. As already indicated in the work of Jacquet and his successors above, a general belief is that, the group  $G_0$  should be the two-fold metaplectic covering of  $\mathrm{GL}_n(F)$ , and the corresponding lifting should be the metaplectic correspondence studied by Flicker-Kazhdan [FK86]. However to the knowledge of the author, no precise statement or conjecture has been made, which is probably due to the lack of known cases of the problem of distinction itself. Instead of understanding the full problem, it should also be cheerful if enlightening partial results or even reasonable guesses could be made.

Another strategy starts from studying the distinction of supercuspidal representations, and then uses parabolic induction to get at least some partial results for more general representations. For the study of a supercuspidal representation  $\pi$ , as we introduced before, the rough idea is first to regard it as the compact induction of a finite dimensional representation  $\Lambda$  of an open subgroup  $\mathbf{J}$  of  $G$  which is compact modulo its centre, and then to use the Mackey formula and the Frobenius reciprocity to write the original distinguished space as direct product, ranging over the double cosets in  $\mathbf{J} \backslash G/H$ , of distinguished spaces with respect to  $\Lambda$ . Under the assumption that  $p \neq 2$ , the question is completely addressed by Hakim and Mao [HM99] when  $\pi$  is of level 0 and by Hakim and Lansky [HL12] and Hakim [Hak13] when  $\pi$  is tamely ramified. The goal of this chapter is to generalize their results to all supercuspidal representations of  $G$ , which we explain in the following subsection.

### 0.3.2 Statement of the main theorems

From now on we further assume that  $p \neq 2$ . For  $\pi$  a supercuspidal representation of  $G$ , we recall several invariants given by the simple type theory of Bushnell-Kutzko [BK93] and the theory of endo-class of Bushnell-Henniart [BH96], which we refer to §2.1.2 for more details. First of all, there is a unique tamely ramified extension  $T/F$  up to  $F$ -isomorphism, called the tame parameter field of  $\pi$ . We write  $d$  for the degree of the endo-class of  $\pi$  which divides  $n$  and is divided by  $[T : F]$ . We write  $m$  for the integer such that  $n = md$ . Let  $T_m$  be the unramified extension of degree  $m$  over  $T$ . Here  $T$ ,  $d$ ,  $m$ ,  $T_m$  are intrinsically determined by  $\pi$ .

To give an impression of what these invariants should be, we let  $\varphi_\pi$  be the irreducible representation of the Weil group  $\mathcal{W}_F$  corresponding to  $\pi$  via the local Langlands correspondence. Then the restriction of  $\varphi_\pi$  to the wild inertia subgroup  $\mathcal{P}_F$  of  $\mathcal{W}_F$  is semisimple and can be written as direct sum of irreducible representations with each irreducible component of multiplicity exactly  $m$ . We choose  $\alpha$  to be any irreducible component of  $\varphi_\pi|_{\mathcal{P}_F}$ , then there exists a finite tamely ramified extension  $T/F$  such that

$$N_F(\alpha) := \{g \in \mathcal{W}_F \mid \alpha^g \cong \alpha\}$$

as a subgroup of  $\mathcal{W}_F$  equals  $\mathcal{W}_T$ . And it turns out that  $T/F$  is uniquely determined up to an  $F$ -isomorphism and independent of the choice of  $\alpha$ . We let  $n = \dim(\varphi_\pi)$ ,  $d = n/m$  and  $T_m$  be as above. Then  $T$ ,  $d$ ,  $m$ ,  $T_m$  defined here from the Galois side match with those defined from the  $\mathrm{GL}_n$  side mentioned in the last paragraph (see [BH14b] for more details).

The following theorem gives a criterion for distinction.

**Theorem 0.3.1.** *Let  $\pi$  be a supercuspidal representation of  $G$  and let  $T$ ,  $d$ ,  $m$ ,  $T_m$  be as above. Then  $\pi$  is distinguished by an orthogonal group  $H$  if and only if the following two conditions hold:*

1.  $\omega_\pi(-1) = 1$ , where  $\omega_\pi$  denotes the central character of  $\pi$ ;



2. Precisely one of the following conditions holds:

- $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$  and  $H$  is split;
- $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  is a subgroup of  $F^\times/F^{\times 2}$  of order 2 and  $H$  is either split or  $H = G^{\tau_\varepsilon}$  which is quasisplit but not split, where  $\varepsilon$  is a symmetric matrix such that  $(-1)^{n(n-1)/2}\det(\varepsilon) \in N_{T_m/F}(T_m^\times) - F^{\times 2}$ ;
- $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = \{1\}$  and  $H$  is either split or not quasisplit.

In particular, it is easily seen that:

**Corollary 0.3.2.** *When  $H$  is split,  $\pi$  is distinguished by  $H$  if and only if  $\omega_\pi(-1) = 1$ .*

Moreover, the following theorem calculates the dimension of the distinguished space.

**Theorem 0.3.3.** *Let  $\pi$  be a supercuspidal representation of  $G$  such that  $\omega_\pi(-1) = 1$  and let  $H$  be an orthogonal group satisfying the condition 2 of Theorem [0.3.1](#)*

1. *If  $H$  is not split, then  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 1$ ;*

2. *If  $H$  is split, then*

- *If  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$ , then  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 1$ ;*
- *If  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  is a subgroup of  $F^\times/F^{\times 2}$  of order 2, then  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 2$ ;*
- *If  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = \{1\}$ , then  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 3$ .*

Finally using [\(0.3.1\)](#) and the same argument in [\[Hak13\]](#), the following theorem holds as a corollary of Theorem [0.3.3](#).

**Theorem 0.3.4.** *For  $\pi$  a supercuspidal representation of  $G$ , it is distinguished by a certain orthogonal subgroup if and only if  $\omega_\pi(-1) = 1$ . Moreover, if this condition holds, then*

$$\dim_{\mathbb{C}}\mathrm{Hom}_G(\pi, \mathcal{C}^\infty(\mathcal{S})) = 4.$$

Thus for  $p \neq 2$  and any supercuspidal representation  $\pi$  of  $G = \mathrm{GL}_n(F)$ , the problem of distinction for orthogonal subgroups is fully settled. The only restriction on  $\pi$ , being the triviality of its central character on  $-1$ , can also be rephrased as the triviality of the determinant character of its Langlands parameter on  $-1$  via the local Langlands correspondence for  $\mathrm{GL}_n$ .

### 0.3.3 Sketch of the proof and the structure of chapter 2

We sketch the proof and the structure of chapter 2. We briefly recall the simple type theory we need in section 1, which is indeed a proper subset of chapter 1, section 2. In section 2 we build up necessary results for symmetric matrices, orthogonal involutions and orthogonal groups for future use.

In section 3 we prove our first main theorem, the tau-selfdual type theorem, which says that for a certain well-chosen orthogonal involution  $\tau_0$  depending on  $\pi$ , there exists a simple type  $(\mathbf{J}, \Lambda)$  compactly inducing  $\pi$  such that  $\tau_0(\mathbf{J}) = \mathbf{J}$  and  $\Lambda \circ \tau_0 = \Lambda^\vee$ , where  $\Lambda^\vee$  denotes the contragredient of  $\Lambda$ . In fact, for each orthogonal group  $H$  satisfying Theorem [0.3.1](#), condition 2, we may find a  $\tau_0$  satisfying  $H = G^{\tau_0}$  and the tau-selfdual theorem. Such a simple type is called  $\tau_0$ -selfdual and will be regarded as the starting point to pursue the problem of distinction.

In section 4, we study the distinction with respect to an arbitrary orthogonal involution  $\tau$  and the corresponding orthogonal group  $G^\tau$ . We fix a  $\tau_0$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  and we may use the Mackey formula and the Frobenius reciprocity to write the distinguished space as follows:

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

The distinguished type theorem says that for those double cosets  $g \in \mathbf{J} \backslash G / G^\tau$  contributing to the distinction, the simple type  $(\mathbf{J}^g, \Lambda^g)$  is  $\tau$ -selfdual. In particular, when  $\tau = \tau_0$  we may also give out all the possible  $\mathbf{J}\text{-}G^{\tau_0}$  double cosets contributing to the distinction.

Finally in section 5, we continue to study the distinguished space  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1)$ . The techniques developed in section 4 enable us to further study the distinguished space via the more delicate structure given by the simple type theory, and finally reduce the question to study the distinguished space  $\mathrm{Hom}_{\overline{H}}(\overline{\rho}, \overline{\chi})$ , where  $\overline{H}$  is an orthogonal subgroup of a finite general linear group  $\overline{G} = \mathrm{GL}_m(\mathbb{F}_q)$ , and  $\overline{\rho}$  is a supercuspidal representation of  $\overline{G}$ , and  $\overline{\chi}$  is a character of  $\overline{H}$  of order 1 or 2. Using the Deligne-Lusztig theory, the condition for the space being non-zero is given and the dimension is at most one. The condition turns out to be the central character of  $\pi$  being trivial at  $-1$ . Thus for those special  $\tau_0$  in section 4, we fully study the distinguished space and the corresponding dimension. Since those  $\tau_0$  correspond exactly to the orthogonal groups in Theorem 0.3.1 and Theorem 0.3.3, we prove the “if” part of Theorem 0.3.1 and Theorem 0.3.3.

It remains the “only if” part of Theorem 0.3.1, of which we take advantage to explain the condition for the orthogonal groups or corresponding orthogonal involutions in the theorem. For  $E_m/F$  an extension of degree  $n$  and  $\tau$  an orthogonal involution, we call  $E_m$   $\tau$ -split if there exists an embedding  $\iota : E_m^\times \hookrightarrow \mathrm{GL}_n(F)$  such that  $\tau(\iota(x)) = \iota(x)^{-1}$  for any  $x \in E_m^\times$ . The following intermediate proposition gives important information for  $\pi$  being distinguished by  $G^\tau$ :

**Proposition 0.3.5.** *For  $\pi$  a given supercuspidal representation of  $G$  with  $\omega_\pi(-1) = 1$ , there exists a field  $E_m$  of degree  $n$  over  $F$  which is totally wildly ramified over  $T_m$ , such that if  $\pi$  is distinguished by  $G^\tau$ , then  $E_m$  is  $\tau$ -split.*

The construction of  $E_m$  is derived from the construction of  $\tau_0$ -selfdual simple type given in section 3. In particular, when  $\tau_0$  corresponds to a split orthogonal group, from the “if” part of Theorem 0.3.1,  $E_m$  is  $\tau_0$ -split. Once knowing this, it is not hard to study all the involutions  $\tau$  such that  $E_m$  is  $\tau$ -split, which turn out to be those involutions satisfying the condition of Theorem 0.3.1, proving the “only if” part of the theorem.

When  $T_m/F$  is of degree  $n$ , or equivalently when  $\pi$  is essentially tame in the sense of Bushnell-Henniart [BH05a], which is the same as being tamely ramified in the context of Hakim [Hak13] thanks to the work of Mayeux [May20], our result gives another proof for the result of Hakim by using the simple type theory instead of Howe’s construction for tamely ramified representations. It is worth mentioning that we also borrow many lemmas from [HM99], [HL12], [Hak13], which effectively help us to reduce our task.

As in chapter 1, it should also be pointed out that the method we use here is not new. It has first been developed by Sécherre to solve the similar problem where  $\tau$  is a Galois involution [AKM<sup>+</sup>19], [Séc19], and then by us for the case where  $\tau$  is a unitary involution (*cf.* chapter 1), and then by Sécherre for the case where  $\tau$  is an inner involution [Séc20] (there  $G$  can also be an inner form of  $\mathrm{GL}_n(F)$ ). The sketches of the proof in different cases are similar, but one major difference in the current case is worth to be mentioned, that is, we need to consider those involution  $\tau$  not contributing to the distinction. In this case we cannot construct a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  using the method in section 3. The novelty of our argument is first to consider a special involution  $\tau_0$ , and then to regard  $\tau$  as another involution which differs from  $\tau_0$  up to a  $G$ -conjugation. Thus we choose  $(\mathbf{J}, \Lambda)$

to be a  $\tau_0$ -selfdual simple type and, using the general results built up in chapter 1, we can still study those  $\mathbf{J}\text{-}G^\tau$  double cosets contributing to the distinction. If one wants to fit the method in the above cases to a general involution  $\tau$ , one major problem encountered is to construct a  $\tau$ -selfdual simple type, which, as we explained, may be impossible if  $G^\tau$  does not contribute to the distinction. The strategy we explained above gives a possible solution, which helps to consider the same question for an abstract involution.

## 0.4 Explicit base change lift and automorphic induction for supercuspidal representations

### 0.4.1 Background

Let  $F/F_0$  be as in §0.1.4 and we only consider the case  $R = \mathbb{C}$  in this chapter. We will focus on two special local liftings, say base change lift and automorphic induction with respect to  $F/F_0$ . More precisely, when  $F/F_0$  is tamely ramified and for supercuspidal representations, we will study these two maps via the simple type theory.

First we give a brief introduction for the local Langlands correspondence for general linear groups, whose existence and properties have been known for a while ([LRS93], [HT01], [Hen00], [Sch13]). For  $n'$  a certain positive integer and  $\mathbf{G}_0 = \mathrm{GL}_{n'}$  as a reductive group over  $F_0$ , the local Langlands correspondence is a bijection

$$\mathrm{LLC}_{F_0} : \mathrm{Irr}_{\mathbb{C}}(G_0) \longrightarrow \Phi(G_0).$$

Here we keep the notations of §0.1.1 and in this case  $\Phi(G_0)$  consists of  $\mathrm{GL}_{n'}(\mathbb{C})$ -conjugacy classes of homomorphisms

$$\phi_0 = (\varphi_0, \lambda_0) : \mathcal{W}_{F_0} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_{n'}(\mathbb{C}),$$

such that  $\varphi_0 := \phi_0|_{\mathcal{W}_{F_0} \times \{1\}}$  is a smooth representation of  $\mathcal{W}_{F_0}$ , and  $\lambda_0 := \phi_0|_{\{1\} \times \mathrm{SL}_2(\mathbb{C})}$  is an algebraic representation of  $\mathrm{SL}_2(\mathbb{C})$  of dimension  $n'$ . For  $n$  a positive integer, let  $\mathbf{G}$  be the Weil restriction of the reductive group  $\mathrm{GL}_n$  over  $F$ , which is a reductive group over  $F_0$  with  $G = \mathrm{GL}_n(F)$ . Still the local Langlands correspondence is a bijection

$$\mathrm{LLC}_F : \mathrm{Irr}_{\mathbb{C}}(G) \longrightarrow \Phi(G).$$

Here  $\Phi(G)$  is the isomorphism classes of L-parameters related to  $\mathbf{G}$ , which can be naturally identified with the isomorphism classes of L-parameters related to  $\mathrm{GL}_n$  over  $F$ . Using this identification,  $\Phi(G)$  consists of  $\mathrm{GL}_n(\mathbb{C})$ -conjugacy classes of homomorphisms

$$\phi = (\varphi, \lambda) : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

such that  $\varphi := \phi|_{\mathcal{W}_F \times \{1\}}$  is a smooth representation of  $\mathcal{W}_F$ , and  $\lambda := \phi|_{\{1\} \times \mathrm{SL}_2(\mathbb{C})}$  is an algebraic representation of  $\mathrm{SL}_2(\mathbb{C})$  of dimension  $n$ .

Now we introduce the base change lift and automorphic induction related to  $F/F_0$ . First we assume  $n' = n$  and we define the restriction map

$$\mathrm{Res}_{F/F_0} : \Phi(G_0) \longrightarrow \Phi(G), \quad \phi_0 = (\varphi_0, \lambda_0) \longmapsto \phi = (\varphi_0|_{\mathcal{W}_F}, \lambda_0),$$

where we notice that  $\mathcal{W}_F$  is a subgroup of  $\mathcal{W}_{F_0}$ . Thus the base change lift is the expected local lifting  $\mathrm{BC}_{F/F_0} : \mathrm{Irr}_{\mathbb{C}}(G_0) \rightarrow \mathrm{Irr}_{\mathbb{C}}(G)$  such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Irr}_{\mathbb{C}}(G_0) & \xrightarrow{\mathrm{LLC}_{F_0}} & \Phi(G_0) \\ \mathrm{BC}_{F/F_0} \downarrow & & \downarrow \mathrm{Res}_{F/F_0} \\ \mathrm{Irr}_{\mathbb{C}}(G) & \xrightarrow{\mathrm{LLC}_F} & \Phi(G) \end{array}$$

Secondly we assume  $n' = nr$  and we define the induction map

$$\mathrm{Ind}_{F/F_0} : \Phi(G) \longrightarrow \Phi(G_0), \quad \phi = (\varphi, \lambda) \longmapsto \phi'_0 = (\mathrm{Ind}_{\mathcal{W}_F}^{\mathcal{W}_{F_0}} \varphi, i \circ \lambda),$$

where  $i : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}_{nr}(\mathbb{C})$  is a group embedding<sup>4</sup>. Thus the automorphic induction is the expected local lifting  $\mathrm{A}_{F/F_0} : \mathrm{Irr}_{\mathbb{C}}(G) \rightarrow \mathrm{Irr}_{\mathbb{C}}(G_0)$  such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Irr}_{\mathbb{C}}(G_0) & \xrightarrow{\mathrm{LLC}_{F_0}} & \Phi(G_0) \\ \mathrm{A}_{F/F_0} \uparrow \cdots & & \uparrow \mathrm{Ind}_{F/F_0} \\ \mathrm{Irr}_{\mathbb{C}}(G) & \xrightarrow{\mathrm{LLC}_F} & \Phi(G) \end{array}$$

In [AC89], [HH95] and [HL11], the base change lift for all irreducible representations, and the automorphic induction for at least essentially unitary generic representations have been constructed via the method of trace formula without the utilisation of the local Langlands correspondence, and the functoriality above have been verified.

Although for  $\mathrm{GL}_n$  the local Langlands correspondence has already been constructed as a bijection with desiderata being verified, it seems that the information extracted from the two sides are not equal. Let us focus on supercuspidal representations, then for any  $n \in \mathbb{N}$  the correspondence can be realized as a bijection

$$\mathrm{LLC}_F : \mathcal{A}_n^0(F) \longrightarrow \mathcal{G}_n^0(F)$$

from the set of equivalence classes of supercuspidal representations of  $\mathrm{GL}_n(F)$ , to the set of equivalence classes of smooth irreducible representations of the Weil group  $\mathcal{W}_F$  of dimension  $n$ , denoted by  $\mathcal{A}_n^0(F)$  and  $\mathcal{G}_n^0(F)$  respectively. Usually we get few concrete information for irreducible representations of  $\mathcal{W}_F$  from the representation theory, but by contrast we have the classification theory for supercuspidal representations of  $\mathrm{GL}_n(F)$ , the so-called simple type theory built up by Bushnell-Kutzko [BK93], which is down-to-earth and sophisticated. So one natural question is, can we characterize the  $\mathrm{LLC}_F$  above using the structural theory for supercuspidal representations of  $\mathrm{GL}_n(F)$ ?

To answer this question, Bushnell and Henniart initiate a long-running project with the outcome contained in a series of articles [BH96], [BH99], [BH05c], [BH03], [BH05a], [BH05b], [BH10], [BH14b], [BH17], [BH19], etc. Especially, in [BH05a], [BH05b], [BH10] they fully addressed the question above for a special class of supercuspidal representations, say essentially tame supercuspidal representations. To do that, they first constructed an algebraic version of local Langlands correspondence, which they called “naïve local Langlands correspondence”, as a bijection between same sets as  $\mathrm{LLC}_F$  and denoted by  $\mathrm{NLC}_F$ . For  $\varphi \in \mathcal{G}_n^0(F)$ , let  $T_m$  be defined as the tamely ramified extension of  $F$  related to  $\varphi$  as in the last section. For  $\mu$  a tamely ramified character of  $T_m^\times$ , they constructed a certain “twist” of  $\varphi$  by  $\mu$ , denoted by  $\varphi \odot \mu$ , which is another representation in  $\mathcal{G}_n^0(F)$ . The upshot is the comparison theorem, which predicts the existence of a tamely ramified character  $\mu_\varphi$  of  $T_m^\times$ , such that  $\mathrm{LLC}_F^{-1}(\varphi)$  is isomorphic to  $\mathrm{NLC}_F^{-1}(\varphi \odot \mu_\varphi)$ . In the essentially tame case in the sense of [BH10], the character  $\mu_\varphi$  of  $T_m^\times$  is of order dividing 4 which can be calculated explicitly, thus in this case the local Langlands correspondence is fully understand in an algebraic way. In the general case as in [BH14b], the construction of the naïve local Langlands correspondence relies on the local Langlands correspondence in the wildly ramified case as a “black box”, and the full characterization of the character  $\mu_\varphi$  remains unknown. So to fully understand the  $\mathrm{LLC}_F$  in general, we first need to understand its special case for wildly ramified supercuspidal representations, which seems to be a deep enough question, and then we need to calculate the character  $\mu_\varphi$  above, which will be discussed in chapter 3 later on.

<sup>4</sup>Using basic argument in linear algebra we may show that the definition doesn't depend on the choice of  $i$ .

The principal aim of this chapter is to adapt the idea of Bushnell-Henniart above to the base change lift and automorphic induction, that is, we will construct algebraic versions of the two maps, and then will compare them with the original maps correspondingly. More details will be given in the subsection below.

### 0.4.2 Main results

To give a detailed introduction, we use the basic terminologies and properties of the simple type theory and the theory of cyclic base change and automorphic induction, for which the readers may refer to chapter 3, section 2 and section 4 respectively. For  $F/F_0$  as before, let  $\Sigma = \text{Gal}(F/F_0)$  be the Galois group and fix  $\sigma \in \Sigma$  a generator. For  $\pi_0$  a supercuspidal representation of  $\text{GL}_n(F_0)$ , we define its base change  $\pi := \text{BC}_{F/F_0}(\pi_0)$  as an irreducible representation of  $\text{GL}_n(F)$ . We further assume that either  $\pi$  is supercuspidal, or  $r$  divides  $n$  and there exists a supercuspidal representation  $\pi'$  of  $\text{GL}_{n/r}(F)$  such that  $\pi$  is isomorphic to the parabolic induction

$$\pi' \times \pi'^{\sigma} \times \dots \times \pi'^{\sigma^{r-1}}.$$

Equivalently we have  $\pi_0 = A_{F/F_0}(\pi')$  as the automorphic induction of  $\pi'$  in the latter case. Our aim is to give an explicit construction of  $\pi$  and  $\pi'$  in the two cases respectively, using the simple type theory and the information from  $\pi_0$ . To that end, we need to assume the additional condition that  $F/F_0$  is tamely ramified.

Let  $[\mathfrak{a}_0, \beta]$  be a maximal simple stratum in  $M_n(F_0)$ , and let  $\theta_0$  be a simple character of  $H^1(\mathfrak{a}, \beta)$  contained in  $\pi_0$ . We choose  $\kappa_0$  to be a full Heisenberg representation of  $\theta_0$  as a representation of  $\mathbf{J}(\mathfrak{a}_0, \beta)$ , and then there exists a unique representation  $\rho_0$  of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  trivial on  $J^1(\mathfrak{a}_0, \beta)$ , such that  $\pi_0$  is isomorphic to the compact induction  $\text{ind}_{\mathbf{J}(\mathfrak{a}_0, \beta)}^{\text{GL}_n(F_0)}(\kappa_0 \otimes \rho_0)$ . Here  $E_0 = F_0[\beta]$  is a field of degree  $d$  over  $F_0$  with  $n = md$  for  $m$  an integer, and we denote by  $T_0$  its maximal tamely ramified subextension over  $F_0$  and by  $T_{0,m}$  the unramified extension of degree  $m$  over  $T_0$ . Then the representation  $\rho_0$  can be characterized by the  $\Delta_0$ -orbit of a  $\Delta_0$ -regular tamely ramified character  $\xi_0$  of  $T_{0,m}^\times$ , where  $\Delta_0 = \text{Gal}(T_{0,m}/T_0)$ .

We first consider the case where  $\pi$  is supercuspidal. Using the tame lifting result in [BH96] and [BH03], we construct  $[\mathfrak{a}, \beta]$  as a maximal simple stratum in  $M_n(F)$ , and  $\theta_b$  as a simple character of  $H^1(\mathfrak{a}, \beta)$  contained in  $\pi$ . And such  $\theta_b$  may be regarded as the base change lift of  $\theta_0$  for simple characters. Then we determine a full Heisenberg representation  $\kappa_b$  of  $\theta_b$  in an algebraic way, which relies only on  $\theta_0$  and  $\kappa_0$ , but not  $\pi_0$ . Then there exists a unique representation  $\rho_b$  of  $\mathbf{J}(\mathfrak{a}, \beta)$  trivial on  $J^1(\mathfrak{a}, \beta)$ , such that  $\pi$  is isomorphic to the compact induction  $\text{ind}_{\mathbf{J}(\mathfrak{a}, \beta)}^{\text{GL}_n(F)}(\kappa_b \otimes \rho_b)$ . Such  $\rho_b$  is characterized by the  $\Delta$ -orbit of a  $\Delta$ -regular tamely ramified character  $\xi_b$  of  $T_m^\times$ , where  $T_m = F \otimes_{F_0} T_{0,m}$  and  $T = F \otimes_{F_0} T_0$  are fields over  $F$  and  $\Delta = \text{Gal}(T_m/T)$ . The following comparison theorem is recorded as the main theorem for base change.

**Theorem 0.4.1** (See Theorem 3.6.2). *There exists a tamely ramified character  ${}_b\phi_{\theta_0}^{F/F_0}$  of  $T_m^\times$ , such that  $\xi_b \cdot {}_b\phi_{\theta_0}^{F/F_0}$  and  $\xi_0 \circ N_{T_m/T_{0,m}}$  are in the same  $\Delta$ -orbit.*

Similarly we consider the case where  $\pi_0$  equals the automorphic induction of a supercuspidal representation  $\pi'$  of  $\text{GL}_{n/r}(F)$ . To proceed, we further assume that either  $F$  is identified with a subfield of  $E_0$  via an embedding, which corresponds to the interior automorphic induction case, or  $E = F \otimes_{F_0} E_0$  is a field of degree  $r$  over  $E_0$ , which corresponds to the exterior automorphic induction case, and we divide the latter case into two subcases depending on  $E/E_0$  unramified or totally ramified, since in different cases the corresponding situations and methods are different. Still using the tame lifting method in [BH96] and [BH03], we construct  $[\mathfrak{c}, \beta]$  as a simple stratum in  $M_{n/r}(F)$  and  $\theta_a$  as a

simple character of  $H^1(\mathfrak{c}, \beta)$  contained in  $\pi'$ . And  $\theta_0$  may be regarded as the automorphic induction of  $\theta_a$  for simple characters. Like the base change case, we determine a full Heisenberg representation  $\kappa_a$  of  $\theta_a$  in an algebraic way depending only on  $\theta_0$  and  $\kappa_0$ , and then there exists a unique representation  $\rho_a$  of  $J(\mathfrak{c}, \beta)$  trivial on  $J^1(\mathfrak{c}, \beta)$ , such that  $\pi'$  is isomorphic to the compact induction  $\text{ind}_{J(\mathfrak{c}, \beta)}^{\text{GL}_{n/r}(F)}(\kappa_a \otimes \rho_a)$ . In the interior automorphic induction case the representation  $\rho_a$  is characterized by the  $\Delta_0$ -orbit of a  $\Delta_0$ -regular tamely ramified character  $\xi_a$  of  $T_{0,m}^\times$ . In the exterior automorphic induction case we write  $T = F \otimes_{F_0} T_0$  as a field and we let  $T_{m/r}$  be the unramified extension of degree  $m/r$  over  $T$ , then for  $\Delta = \text{Gal}(T_{m/r}/T)$  the representation  $\rho_a$  is characterized by the  $\Delta$ -orbit of a  $\Delta$ -regular tamely ramified character of  $T_{m/r}^\times$ , still denoted by  $\xi_a$ . In this case if  $E/E_0$  is unramified,  $T_{m/r}$  is identified with  $T_{0,m}$  and  $\Delta$  is identified with a subgroup of  $\Delta_0$ , and if  $E/E_0$  is totally ramified,  $T_m = F \otimes_{F_0} T_{0,m}$  is a field of degree  $m$  over  $T$  with  $T_{m/r}$  being regarded as its subfield, and we write  $\Delta' = \text{Gal}(T_m/T)$ . The following comparison theorem is recorded as the main theorem for automorphic induction.

**Theorem 0.4.2** (See Theorem 3.6.4, Theorem 3.6.6 and Theorem 3.6.8). (1) In the interior automorphic induction case, there exists a tamely ramified character  ${}_a\phi_{\theta_0}^{F/F_0}$  of  $T_{0,m}^\times$  such that  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0$  are in the same  $\Delta_0$ -orbit;

(2) In the exterior automorphic induction case, there exists a tamely ramified character  ${}_a\phi_{\theta_0}^{F/F_0}$  of  $T_{m/r}^\times$  such that

- if  $E/E_0$  is unramified,  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0$  are in the same  $\Delta_0$ -orbit;
- if  $E/E_0$  is totally ramified,  $(\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}) \circ N_{T_m/T_{m/r}}$  and  $\xi_0 \circ N_{T_m/T_{0,m}}$  are in the same  $\Delta'$ -orbit.

We mention three applications of the above two theorems to end this subsection. The first application relates to the study of  $\bar{\rho}_b$  and  $\bar{\rho}_a$  via  $\bar{\rho}_0$ , where  $\bar{\rho}_0$  is the supercuspidal representation of  $\text{GL}_m(\mathbf{k}_{E_0}) \cong J(\mathfrak{a}_0, \beta)/J^1(\mathfrak{a}_0, \beta)$  whose inflation equals  $\rho_0|_{J(\mathfrak{a}_0, \beta)}$  with  $\mathbf{k}_{E_0}$  denoting the residue field of  $E_0$ , and  $\bar{\rho}_b$  is the supercuspidal representation of  $\text{GL}_m(\mathbf{k}_E) \cong J(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta)$  whose inflation equals  $\rho_b|_{J(\mathfrak{a}, \beta)}$  with  $\mathbf{k}_E$  denoting the residue field of  $E$  in the base change case, and  $\bar{\rho}_a$  is the supercuspidal representation of  $\text{GL}_m(\mathbf{k}_{E_0}) \cong J(\mathfrak{c}, \beta)/J^1(\mathfrak{c}, \beta)$  in the interior automorphic induction case, of  $\text{GL}_{m/r}(\mathbf{k}_E) \cong J(\mathfrak{c}, \beta)/J^1(\mathfrak{c}, \beta)$  in the exterior automorphic induction case, whose inflation equals  $\rho_a|_{J(\mathfrak{c}, \beta)}$ . Then the restriction of  ${}_b\phi_{\theta_0}^{F/F_0}$  and  ${}_a\phi_{\theta_0}^{F/F_0}$  to those elements in the ring of integers are quadratic characters which can be fully characterized, the representations  $\bar{\rho}_b$  and  $\bar{\rho}_a$  are understood via the theory of Green (cf. [Gre55]). In particular for  $E/E_0$  as an unramified extension of degree  $r$  in the base change case or exterior automorphic induction case, this gives a relation between the Arthur-Clozel base change lift and the Shintani base change map (cf. [Shi76]) for supercuspidal representations, see Corollary 3.6.3 and Corollary 3.6.9.

The second application relates to the  $l$ -modular representations. For  $l$  a prime number different from  $p$ , we fix an algebraic isomorphism  $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ , thus all the representations above can be realized as representations over  $\overline{\mathbb{Q}}_l$ . We further assume that  $\pi_0$  is integral, saying that it comes from a representation over a  $\overline{\mathbb{Z}}_l$ -lattice by extension of scalar. Using the two theorems, we may prove that tamely ramified cyclic base change lift and automorphic induction are compatible with modulo  $l$  reduction for supercuspidal representations. More precisely, for one such  $\pi_0$  with its base change  $\pi$  supercuspidal,  $\pi$  is also integral, and if we change  $\pi_0$  by another integral supercuspidal representation with its modulo  $l$  reduction  $r_l(\pi_0)$  unchanged as a cuspidal representation over  $\overline{\mathbb{F}}_l$ , then the corresponding  $\pi$  is still integral supercuspidal with its modulo  $l$  reduction  $r_l(\pi)$  unchanged. Similarly for one such  $\pi_0$  as the automorphic induction of  $\pi'$  as a supercuspidal representation,  $\pi'$  is integral, and if we change  $\pi'$  by another integral supercuspidal representation with its modulo  $l$  reduction  $r_l(\pi')$  unchanged as a cuspidal representation over  $\overline{\mathbb{F}}_l$ , then the corresponding  $\pi_0$  is still integral supercuspidal with its modulo  $l$

reduction  $r_l(\pi_0)$  unchanged. The proof is direct and will not be given in this chapter, but the readers may consult [BH14a] for a similar idea.

The final application relates to the calculation of the character  $\mu_\varphi$  related to the comparison theorem in [BH14b] mentioned in the last subsection. The strategy is to consider a certain unramified base change lift, and then to compare the corresponding characters “mu” related the two base fields, which has already been used for the essentially tame case in [BH05a]. To that end we need to study  ${}_b\phi_{\theta_0}^{F/F_0}$  in the case where  $F/F_0$  is unramified.

**Theorem 0.4.3** (See Theorem [3.9.1]). *When  $F/F_0$  is unramified, the character  ${}_b\phi_{\theta_0}^{F/F_0}$  is unramified, and  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = (-1)^{(t-1)(r-1)}$ , where  $\varpi_{T_0}$  denotes a uniformizer of  $T_m^\times$ , and  $K_0$  denotes the maximal unramified subextension of  $T_{0,m}$  over  $F_0$ , and  $t = [T_{0,m} : K_0]$  and  $[E_0 : T_0] = p^s$ .*

Using a special case of the theorem, that is Proposition [3.9.9], we may update the values of  $\mu_\varphi$  which will be discussed in the last section. Our result there is obviously incomplete and not satisfactory enough.

### 0.4.3 Structure of the chapter 3

We sketch the structure of chapter 3. The sections 1-4 are preliminaries, including a brief introduction and summary of the simple type theory, symplectic signs, and base change lift and automorphic induction respectively. After the first elementary discussion in section 5, in section 6 we formulate our algebraic construction of tamely ramified cyclic base change and automorphic induction following the sketch mentioned in the last subsection and state Theorem [0.4.1] and Theorem [0.4.2]. But the construction of the corresponding full Heisenberg representations remains to be done until section 7, whose strategy relies on the idea of a series of results of Bushnell-Henniart which will be recalled and reformulated there. The proof of Theorem [0.4.1] and Theorem [0.4.2] will be given in section 8 which seems to be surprisingly easy, which actually relies on two highly-nontrivial ingredients: the local Langlands functoriality for base change lift and automorphic induction, and the comparison theorem of Bushnell-Henniart. The section 9 is devoted to prove Theorem [0.4.3] and the section 10 is its application to calculate the character  $\mu_\varphi$  related to the comparison theorem.

The author would like to take advantage of this place to thank Colin J. Bushnell and Guy Henniart for their enormous influence on the author and on this chapter. Actually it is better to regard this part as a (clumsy) continuation of their work rather than an independent work, since almost all the important ideas and techniques are originated from their articles listed above. Moreover the author would like to thank them for their generous encouragements, which indeed helped a lot to a young PhD student who was not confident and was even suspicious with the necessity of his work.

# Introduction (version française)

Cette thèse contient trois parties. Dans ce chapitre introductif, nous expliquerons le contexte général dans la première section, puis dans les trois sections suivantes, nous nous concentrerons sur chaque partie et fournirons des introductions spécifiques.

## 0.1 Contexte général

Soient  $F_0$  un corps localement compact non archimédien de caractéristique résiduelle  $p$  et  $R$  un corps algébriquement clos de caractéristique  $l \neq p$ , et en particulier lorsque  $l > 0$  nous dirons que nous sommes dans le cas “ $l$ -modulaire”. Par exemple, nous nous concentrerons principalement sur les trois cas suivants:  $R$  est le corps des nombres complexes  $\mathbb{C}$ , la clôture algébrique du corps  $l$ -adique noté  $\overline{\mathbb{Q}_l}$ , ou la clôture algébrique du corps fini avec  $l$  éléments noté  $\overline{\mathbb{F}_l}$  quand  $l \neq 0$ . Soient  $\mathbf{G}$  un groupe réductif <sup>5</sup> sur  $F_0$  et  $G$  le groupe localement profini constitué des  $F_0$ -points rationnels de  $\mathbf{G}$ . Nous nous intéressons à la catégorie des représentations irréductibles lisses d’un groupe localement profini et nous désignons par  $\text{Irr}_R(G)$  l’ensemble des classes d’isomorphisme de représentations irréductibles lisses de  $G$  sur  $R$ .

### 0.1.1 Correspondance de Langlands locale

Considérons tout d’abord le cas où  $R = \mathbb{C}$ . Fixons une clôture séparable  $\overline{F_0}$  de  $F_0$ , et notons  $\mathcal{W}_{F_0}$  le groupe de Weil de  $F_0$  par rapport à  $\overline{F_0}$  et  $\text{WD}_{F_0} = \mathcal{W}_{F_0} \times \text{SL}_2(\mathbb{C})$  le groupe de Weil-Deligne de  $F_0$ . Définissons le groupe dual de  $\mathbf{G}$ , noté  $\hat{\mathbf{G}}$ , comme le groupe réductif complexe (identifié au groupe topologique complexe de ses points rationnels par abus de notation) déterminé par le dual de la donnée radicielle de  $\mathbf{G}$ . Puisque la donnée radicielle de  $\mathbf{G}$  est munie d’une action par  $\mathcal{W}_{F_0}$ , il en est de même pour le groupe  $\hat{\mathbf{G}}$  en fixant un épingleage, et nous notons  ${}^L G = \hat{\mathbf{G}} \rtimes \mathcal{W}_{F_0}$  le  $L$ -groupe de  $\mathbf{G}$ .

**Définition 0.1.1.** *Un  $L$ -paramètre de  $\mathbf{G}$  (sur  $\mathbb{C}$ ) est un homomorphisme  $\phi : \text{WD}_{F_0} \rightarrow {}^L G$  tel que*

- *Le diagramme suivant est commutatif:*

$$\begin{array}{ccc} \mathcal{W}_{F_0} \times \text{SL}_2(\mathbb{C}) = \text{WD}_{F_0} & \xrightarrow{\phi} & {}^L G = \hat{\mathbf{G}} \rtimes \mathcal{W}_{F_0} \\ & \searrow & \swarrow \\ & \mathcal{W}_{F_0} & \end{array}$$

*où les deux flèches transversales sont les projections canoniques.*

---

<sup>5</sup>Nous ne considérerons que les groupes réductifs *connexes*.



- $\phi|_{\mathcal{W}_{F_0} \times \{1\}}$  est continu et d'image constituée d'éléments semi-simples<sup>6</sup>, et  $\phi|_{\{1\} \times \mathrm{SL}_2(\mathbb{C})}$  est algébrique<sup>7</sup> et d'image constituée d'éléments unipotents dans  $\hat{G}$ .

Deux L-paramètres sont dits isomorphes s'ils sont dans la même  $\hat{G}$ -classe de conjugaison, et nous notons  $\Phi(G)$  l'ensemble des classes d'isomorphisme des paramètres de Langlands de  $\mathbf{G}$ . La célèbre correspondance de Langlands locale est énoncée comme suit.

**Conjecture 0.1.2.** *Il y a une surjection unique*

$$\mathrm{LLC} : \mathrm{Irr}_{\mathbb{C}}(G) \longrightarrow \Phi(G)$$

avec fibres finies et satisfaisant certaines desiderata.

**Définition 0.1.3.** *Pour  $\phi \in \Phi(G)$ , nous appelons  $\Pi_{\phi} := \mathrm{LLC}^{-1}(\phi)$  le L-paquet de  $\phi$ . C'est un ensemble fini de représentations irréductibles de  $G$ .*

Ici, nous ne précisons pas ce que signifient exactement ces desiderata (compatibilité avec l'induction parabolique, transfert des L-facteurs et des  $\epsilon$ -facteurs, etc.) mais nous nous référons à [Bor79] pour une introduction explicative. La correspondance de Langlands locale pour certains groupes réductifs est connue, comme pour les tores,  $\mathrm{GL}_n$  ou certains groupes classiques, grâce aux travaux de Langlands [Lan97], Harris-Taylor [HT01], Arthur [Art13], etc.

De plus, bien que la conjecture de Langlands ne soit énoncée que pour les représentations sur  $\mathbb{C}$ , il est possible de l'étendre aux représentations à coefficients dans  $R$ , en adaptant les définitions des L-paramètres et des desiderata. Par exemple, il existe des travaux novateurs de Vignéras [Vig01] pour  $\mathbf{G} = \mathrm{GL}_n$ , ainsi que des travaux récents de Dat-Helm-Kurinczuk-Moss [DHKM20] pour les groupes réductifs généraux et représentations sur un anneau intègre dans lequel  $p$  est inversible au lieu de sur un corps  $R$ . Enfin, nous mentionnons le résultat récent de Fargues-Scholze [FS21]. En utilisant des méthodes géométriques et sous des paramètres assez généraux (plus généraux que les nôtres), ils ont construit  $\Phi(G)$  (en fait comme un champ) et la correspondance de Langlands locale, et vérifié les desiderata correspondants sous leurs paramètres (*cf. ibid.* Theorem IX.0.5).

## 0.1.2 Fonctorialité de Langlands locale

Maintenant nous discutons la fonctorialité de Langlands locale et nous supposons que  $R = \mathbb{C}$ . Soient  $\mathbf{G}_0$  un autre groupe réductif sur  $F_0$  et  $G_0$  le groupe de  $F_0$ -points rationnels de  $\mathbf{G}_0$ . Comme dans la sous-section précédente, nous pouvons définir de la même manière son groupe dual  $\hat{G}_0$ , son L-groupe  ${}^L G_0 = \hat{G}_0 \rtimes \mathcal{W}_{F_0}$  et l'ensemble des classes d'isomorphisme des L-paramètres  $\Phi(G_0)$ .

**Définition 0.1.4.** *Un homomorphisme de groupe*

$$\iota : {}^L G_0 \longrightarrow {}^L G$$

est appelé un L-homomorphisme, si

- il est continu, et sa restriction à  $\hat{G}_0$  est une représentation algébrique de  $\hat{G}_0$  dans  $\hat{G}$ .

<sup>6</sup>Un élément  $(g, w)$  dans  ${}^L G$  est semi-simple si pour tout  $r$  comme une représentation de dimension finie de  ${}^L G$ , l'image  $r((g, w))$  est semi-simple.

<sup>7</sup>C'est-à-dire qu'il s'agit d'une représentation algébrique du groupe algébrique complexe  $\mathrm{SL}_2$  dans le groupe algébrique complexe  $\hat{G}$ .

- Le diagramme suivant est commutatif

$$\begin{array}{ccc} \mathcal{W}_{F_0} \times \hat{G}_0 = {}^L G_0 & \xrightarrow{\iota} & {}^L G = \hat{G} \times \mathcal{W}_{F_0} \\ & \searrow & \swarrow \\ & \mathcal{W}_{F_0} & \end{array}$$

où les deux flèches transversales sont des projections canoniques.

Étant donné un  $L$ -paramètre  $\phi_0$  dans  $\Phi(G_0)$ , la composition  $\iota \circ \phi_0$  est un  $L$ -paramètre dans  $\Phi(G)$ . Ainsi, nous construisons une application entre les classes d'isomorphisme des  $L$ -paramètres:

$$\Phi(\iota) : \Phi(G_0) \longrightarrow \Phi(G), \quad \phi_0 \longmapsto \iota \circ \phi_0.$$

Si nous admettons la correspondance de Langlands locale pour  $\mathbf{G}_0$  et  $\mathbf{G}$ , nous avons le diagramme suivant:

$$\begin{array}{ccc} \mathrm{Irr}_R(G_0) & \xrightarrow{\mathrm{LLC}} & \Phi(G_0) \\ \Pi(\iota) \downarrow \text{dotted} & & \downarrow \Phi(\iota) \\ \mathrm{Irr}_R(G) & \xrightarrow{\mathrm{LLC}} & \Phi(G) \end{array}$$

La functorialité de Langlands locale prédit l'existence d'une application  $\Pi(\iota)$ , appelée *relèvement local* par rapport à  $\iota$ , telle que le diagramme ci-dessus est commutatif. De plus, cette application  $\Pi(\iota)$  devrait être construite indépendamment de ce diagramme, mais en utilisant d'autres outils techniques comme la formule des traces ou les fonctions L. D'une part, la compréhension des relèvements locaux constitue une partie importante du programme de Langlands local. D'autre part, il joue également un rôle crucial dans la stratégie inductive, proposée par Langlands-Shelstad [LS87] et appelée la méthode d'endoscopie, pour construire la correspondance de Langlands locale pour un groupe réductif général, qui est devenu un domaine prospère depuis quelques décennies avec des résultats fructueux, notamment les travaux d'Arthur, Kottwitz, Langlands, Laumon, Ngô, Shelstad, Waldspurger, etc.

Néanmoins, nous n'avons pas besoin de nous confiner aux représentations complexes, mais nous pouvons aussi considérer un relèvement local possible sur  $R$ . Par exemple, une attente pour un relèvement local sur  $\overline{\mathbb{F}}_l$  est que, il soit compatible avec le relèvement local sur  $\overline{\mathbb{Q}}_l$ , en identifiant  $\mathbb{C}$  avec  $\overline{\mathbb{Q}}_l$  via un certain isomorphisme algébrique et appliquant la réduction modulo  $l$ . Un résultat typique est la correspondance de Jacquet-Langlands qui est un relèvement assez naturel entre  $\mathrm{GL}_n$  et ses formes intérieures. Sur  $\overline{\mathbb{F}}_l$ , la construction de cette application et aussi sa compatibilité avec la correspondance de Jacquet-Langlands habituelle a été étudiée par Dat [Dat12] dans un cas particulier, puis généralisée par Mínguez-Sécherre [MS17] dans le cas général.

### 0.1.3 Problème de la distinction

Soit  $\mathbf{H} \subset \mathbf{G}$  un sous-groupe algébrique fermé sur  $F_0$  et nous notons  $H$  le groupe de  $F_0$ -points rationnels de  $\mathbf{H}$ . Pour  $\pi \in \mathrm{Irr}_R(G)$  et  $\rho \in \mathrm{Irr}_R(H)$ , nous disons que  $\pi$  est  $(H, \rho)$ -distinguée si

$$\mathrm{Hom}_H(\pi, \rho) \neq 0,$$

ou en d'autres termes, la restriction de  $\pi$  à  $H$  admet  $\rho$  comme un quotient. En particulier, lorsque  $\rho$  est triviale, nous appelons  $\pi$  *distinguée par  $H$*  ou  $H$ -distinguée. Pour simplifier, nous supposons temporairement  $R = \mathbb{C}$ .

Le problème de la distinction est omniprésent et joue un rôle important dans la théorie des représentations des groupes  $p$ -adiques. Par exemple, si  $\mathbf{G}$  est quasi-déployé, nous notons  $\mathbf{H} = \mathbf{U}$

le radical unipotent d'un sous-groupe Borel de  $\mathbf{G}$  et nous choisissons  $\psi$  comme un caractère non dégénéré de  $H = U$ , c'est-à-dire que sa restriction à tout sous-groupe unipotent  $U_\alpha$  de  $U$  lié à une racine simple  $\alpha$  n'est pas triviale. Un résultat bien connu [Sha74] est que l'espace vectoriel

$$\mathrm{Hom}_U(\pi, \psi)$$

est de dimension inférieure à un. Ces  $\pi$  pour lesquelles cette dimension vaut un sont appelés les représentations génériques. Par la réciprocity de Frobenius, une telle  $\pi$  peut être plongée dans l'espace des formes linéaires  $(U, \psi)$ -invariantes  $G$ , qui est appelé le modèle de Whittaker de  $\pi$  et joue un rôle important dans la théorie locale et globale des L-fonctions. Dans un autre exemple, nous considérons  $V$  comme un espace vectoriel de dimension finie sur  $F_0$  muni d'une forme sesquilinéaire, et  $W$  comme un sous-espace de  $V$ . On note  $\mathbf{G}$  le groupe de  $F_0$ -automorphismes de  $V$  et  $\mathbf{H}$  le groupe de  $F_0$ -automorphismes de  $W$ , en préservant la forme sesquilinéaire. Ensuite, le problème de la distinction correspondant est lié aux "lois de branchement", qui remontent à la théorie des représentations des groupes algébriques complexes et se sont comportées comme un domaine actif depuis des décennies en raison de l'initiation et de la percée de la conjecture de Gan-Gross-Prasad [GGP12] et ses variantes.

Dans de bonnes conditions, le problème de la distinction est étroitement lié à la correspondance de Langlands locale et à sa functorialité. Dans le livre remarquable [SV17a], Sakellaridis et Venkatesh ont proposé un cadre général pour étudier le problème de la distinction, dans lequel ils supposent que  $\mathbf{G}$  soit déployé et  $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$  soit une variété sphérique avec  $X$  désignant ses  $F_0$ -points rationnels. Leur point de départ est la construction du groupe dual  $\hat{G}_X$  pour  $\mathbf{X}$  comme un groupe réductif complexe, sous une hypothèse sur les racines de  $\mathbf{X}$ , avec une représentation algébrique canonique

$$\iota_X : \hat{G}_X \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \hat{G}.$$

Selon leur suggestion conjecturale, grosso modo, les représentations de  $G$  distinguées par  $H$  correspondent aux paramètres d'Arthur  $X$ -distingués via la correspondance de Langlands locale, où les paramètres d'Arthur sont l'analogie des  $L$ -paramètres avec une version correspondante de la correspondance de Langlands locale, et ces paramètres d'Arthur factorisés via  $\iota_X$  sont appelés  $X$ -distingués, pour lesquels nous laissons *ibid.* section 16 pour plus de détails. L'idée sous-jacente est donc que, dans de bonnes circonstances, la propriété d'être distinguée est préservée par la correspondance de Langlands locale. Dans [Pra15], Prasad a considéré le cas où  $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$  est un espace symétrique par rapport à une involution galoisienne. Il a construit un sous-groupe quasi-déployé  $\mathbf{G}_0$  (noté  $\mathbf{G}^{op}$  dans *loc. cit.*) sur  $F_0$ , un  $L$ -homomorphisme naturel  $\iota : {}^L\mathbf{G}_0 \rightarrow {}^L\mathbf{G}$  qui vient simplement de la restriction, et un caractère  $\omega_H$  de  $H$ . Finalement, il a conjecturé que, pour  $\pi$  une représentation irréductible de  $G$  distinguée par  $(H, \omega_H)$ , le  $L$ -paquet de  $\pi$  vient du relèvement local lié à  $\iota$ , ou plus précisément il existe  $\phi_0 \in \Phi(G_0)$  tel que  $\pi \in \Pi(\iota \circ \phi_0)$ . De plus, une formule conjecturale pour la dimension de l'espace de la distinction a été donnée. Ces deux cadres généraux, combinés aux exemples divers concrets, doivent être considérés comme notre ligne directrice des résultats auxquels nous devrions nous attendre dans le langage de la correspondance de Langlands locale et de sa functorialité.

Nous présentons brièvement quelques méthodes connues pour traiter le problème de la distinction. Une méthode importante, initiée par Jacquet et développée par lui-même, ses étudiants et d'autres adeptes, est appelée la méthode de la formule de trace relative, pour laquelle nous nommons quelques articles [JLR93], [JY96], [Guo96], [Mao98]. L'idée, grosso modo, est d'abord de résoudre le problème correspondant sur un corps global, puis de réaliser notre corps local  $F_0$  comme un composant de l'anneau des adèles d'un corps global et d'utiliser un argument global-local. Ensuite, nous comparons deux formules de trace différentes comme des distributions sur deux espaces de fonctions de test, dont l'une se rapporte exactement à notre problème global. Après avoir vérifié le lemme fondamental et l'existence d'un transfert lisse, nous obtenons suffisamment de paires de fonctions de test correspondantes pour que deux formules de trace coïncident. Si l'autre formule de trace est bien comprise, nous

obtenons les informations pour résoudre le problème global de la distinction. De plus, pour résoudre la conjecture locale de Gan-Gross-Prasad pour les groupes orthogonaux, Waldspurger [Wal10], [Wal12] a lancé une nouvelle méthode avec la considération d'une formule de trace relative locale, telle que la dimension de l'espace de la distinction peut être exprimée, puis il a utilisé des techniques sophistiquées d'analyse harmonique sur des groupes réductifs  $p$ -adiques pour reformuler la formule de trace et obtenir le résultat. Au cours de la dernière décennie, cette méthode a été développée et appliquée à différentes situations par certaines personnes dont Beuzart-Plessis et C. Wan. Par exemple dans [BP18] en utilisant la méthode similaire, Beuzart-Plessis a résolu une partie de la conjecture ci-dessus proposée par Prasad pour des représentations essentiellement de carré intégrable.

Une autre méthode possible pour étudier le problème de la distinction est algébrique: on étudie d'abord le problème pour les représentations supercuspidales, puis on applique l'induction parabolique pour étudier des représentations irréductibles plus générales. Pour  $\pi$  une représentation supercuspidale de  $G$ , une croyance générale est que, elle peut être écrite comme l'induction compacte d'une représentation lisse irréductible de dimension finie. Plus précisément, il existe une paire  $(\mathbf{J}, \Lambda)$  telle que  $\mathbf{J}$  est un sous-groupe compact de  $G$  modulo le centre, et  $\Lambda$  est une représentation irréductible lisse de dimension finie de  $\mathbf{J}$  telle que  $\pi \cong \text{ind}_{\mathbf{J}}^G \Lambda$ . Cette croyance est vérifiée dans de nombreux cas, y compris les représentations supercuspidales modérées [Yu01], [Fin21] pour un groupe réductif modérément ramifié  $G$ , et aussi les représentations supercuspidales générales pour les groupes classiques [BK93], [Ste08]. En ce moment si nous nous concentrons sur l'étude de la représentation supercuspidale  $\pi$  distinguée par  $H$ , en utilisant la formule de Mackey et la réciprocity de Frobenius, on voit facilement que

$$\text{Hom}_H(\pi, 1) \cong \text{Hom}_H(\text{ind}_{\mathbf{J}}^G \Lambda, 1) \cong \prod_{g \in \mathbf{J} \backslash G/H} \text{Hom}_{\mathbf{J}^g \cap H}(\Lambda^g, 1).$$

Il suffit donc d'étudier les  $g \in \mathbf{J} \backslash G/H$  tels que  $\text{Hom}_{\mathbf{J}^g \cap H}(\Lambda^g, 1)$  est différent de zéro, puis d'étudier la dimension correspondante. Pour cela, nous remontons à la construction détaillée de  $(\mathbf{J}, \Lambda)$ . Un travail typique est [HM08], où les auteurs ont étudié, pour  $G/H$  un espace symétrique, les représentations supercuspidales modérées  $\pi$  de  $G$  distinguées par  $H$  en utilisant l'idée mentionnée ci-dessus et le résultat structurel de J.-K. Yu [Yu01] pour de telles représentations.

Pourtant, nous ne sommes pas forcément confinés au cas où  $R = \mathbb{C}$ , mais nous nous concentrons sur le général  $R$  dans nos paramètres. Les deux méthodes analytiques mentionnées ci-dessus deviennent invalides. En revanche, la méthode algébrique reste valide, puisque le résultat structurel pour  $(\mathbf{J}, \Lambda)$ , une fois établi, fonctionne généralement pour  $R$  général plutôt que juste  $R = \mathbb{C}$ , comme [Vig96], [MS14b] et [Fin19]. En résumé, la recherche de la relation possible entre le problème de la distinction et la correspondance de Langlands locale et sa functorialité pour le  $R$  général doit être considérée comme la motivation originelle de cette thèse.

#### 0.1.4 Notre paramètres concrets

Bien que le contexte ci-dessus soit assez général, le but de cette thèse est humble, qui se concentre sur la compréhension de quelques exemples particuliers. Fixons  $n$  un entier positif. Soit  $F/F_0$  une extension cyclique finie de corps localement compacts non archimédiens de caractéristique résiduelle  $p$  de degré  $r$ , et soit  $G$  la restriction de Weil du groupe réductif  $\text{GL}_n/F$ , qui est un groupe réductif sur  $F_0$ . En particulier, nous avons  $G = \text{GL}_n(F)$ . La plupart du temps, nous nous concentrerons sur les représentations cuspidales ou supercuspidales de  $G$  sur  $R$ , qui devraient être considérées comme les blocs de construction des représentations irréductibles générales. Rappelons qu'une représentation irréductible de  $G$  est cuspidale (*resp.* supercuspidale) si elle ne se produit pas comme une sous-représentation (*resp.* sous-quotient) de l'induction parabolique d'une représentation irréductible d'un

sous-groupe de Levi propre de  $G$ . Quand  $\text{char}(R) = 0$  les deux concepts ci-dessus sont équivalents, cependant quand  $\text{char}(R) = l > 0$ , une représentation supercuspidale est cuspidale, mais l'existence de contre-exemples montre que l'inverse est faux en général.

Pour étudier une représentation cuspidale  $\pi$  de  $G$  sur  $R$ , notre outil principal est la théorie des types simples établie par Bushnell-Kutzko [BK93] lorsque  $\text{char}(R) = 0$ , et généralisée par Vignéras [Vig96] au cas  $l$ -modulaire. Nous nous référons au chapitre 1, section 3 ou au chapitre 3, section 2 pour une introduction détaillée de la théorie, mais ici nous donnons également une brève introduction pour faciliter les détails.

Comme indiqué ci-dessus, l'idée de la théorie des types simples est de réaliser  $\pi$  comme l'induction compacte d'une représentation irréductible de dimension finie  $\Lambda$  de  $\mathbf{J}$ , qui est un sous-groupe ouvert de  $G$  compact modulo le centre. Une telle paire  $(\mathbf{J}, \Lambda)$  est appelée un *type simple maximal étendu* que nous abrègerons en *type simple*. Le théorème principal dit que toute  $\pi$  peut être construite de cette manière, et le type simple correspondant  $(\mathbf{J}, \Lambda)$  est unique à  $G$ -conjugaison près. Nous mentionnons également les principales propriétés suivantes de  $(\mathbf{J}, \Lambda)$ :

- (1) Le groupe  $\mathbf{J}$  contient un unique maximal sous-groupe compact ouvert  $J$  qui contient un unique maximal pro- $p$ -sous-groupe distingué  $J^1$ ;
- (2) Nous avons  $J/J^1 \cong \text{GL}_m(\mathbf{l})$ . Ici  $\mathbf{l}$  est le corps résiduel de  $E$ , où  $E$  est une extension de corps sur  $F$  de degré  $d$ . De plus, nous avons  $n = md$ , où  $m$  et  $d$  sont entières déterminés par  $\pi$ ;
- (3) On peut écrire  $\Lambda = \kappa \otimes \rho$ , où  $\kappa$  et  $\rho$  sont des représentations irréductibles de  $\mathbf{J}$  tel que la restriction  $\kappa|_{J^1} = \eta$  est une représentation irréductible de  $J^1$ , appelée *représentation de Heisenberg*, et  $\rho|_J$  est l'inflation d'une représentation cuspidale de  $\text{GL}_m(\mathbf{l}) \cong J/J^1$ ;
- (4) Il existe un pro- $p$ -sous-groupe de  $J^1$  noté  $H^1$ , et un caractère de  $H^1$  noté  $\theta$  et appelé un caractère simple, tels que la restriction de  $\eta$  à  $H^1$  égale la somme directe de  $(J^1 : H^1)^{1/2}$  copies de  $\theta$ .

Enfin, nous entrons dans l'introduction pour nos travaux concrets. Pour la première partie, nous étudions le problème de la distinction lié à un sous-groupe unitaire de  $G$  et sa relation avec la fonctorialité de Langlands, ou incarné comme le changement de base quadratique dans nos contextes; Pour la deuxième partie, nous étudions le problème de la distinction lié à un sous-groupe orthogonal de  $G$ , et nous nous concentrons uniquement sur les représentations supercuspidales sur  $R = \mathbb{C}$ , ce qui est la première étape vers la compréhension de représentations irréductibles plus générales; Pour la partie finale, nous donnons des constructions explicites pour deux relèvements locaux particuliers, le changement de base et l'induction automorphe, pour les représentations supercuspidales sur  $R = \mathbb{C}$ .

## 0.2 Le problème de la distinction pour le sous-groupe unitaire de $\text{GL}_n(F)$ et le changement de base $l$ -modulaire

### 0.2.1 Contexte général

Les huit premières sections du chapitre 1 sont basées sur la prépublication [Zou19]. Dans cette sous-section, nous supposons que  $F/F_0$  est une extension quadratique de corps  $p$ -adiques de caractéristique résiduelle  $p$ , et nous notons  $\sigma$  son automorphisme non trivial. Pour  $\mathbf{G}$  et  $G$  comme ci-dessus, nous notons  $\varepsilon$  pour une *matrice hermitienne* dans  $G$ , c'est-à-dire  $\sigma({}^t\varepsilon) = \varepsilon$  avec  ${}^t$  désignant la transposition des matrices. Nous définissons

$$\tau_\varepsilon(x) = \varepsilon \sigma({}^t x^{-1}) \varepsilon^{-1}$$

pour tout  $x \in G$ , appelée une *involution unitaire* sur  $G$ , qui induit également un  $F_0$ -automorphisme sur  $\mathbf{G}$ . Nous fixons une  $\tau = \tau_\varepsilon$ , et nous notons  $\mathbf{G}^\tau$  le sous-groupe de  $\mathbf{G}$  sur  $F_0$ , tel que  $G^\tau$  est le sous-groupe de  $G$  constitué des éléments fixés par  $\tau$ . Un tel  $\mathbf{G}^\tau$  (resp.  $G^\tau$ ) est appelé le *sous-groupe unitaire* de  $\mathbf{G}$  (resp.  $G$ ) par rapport à  $\tau$ .

Pour  $\pi$  une représentation lisse irréductible de  $G$  sur  $\mathbb{C}$ , Jacquet a proposé d'étudier le problème de la distinction lié au couple  $(G, G^\tau)$  comme ci-dessus, c'est-à-dire, d'étudier l'espace des formes linéaires  $G^\tau$ -invariantes

$$\mathrm{Hom}_{G^\tau}(\pi, 1)$$

et sa dimension en tant qu'espace vectoriel complexe. Pour  $n = 3$  et  $\pi$  supercuspidale, il a prouvé dans [Jac01] en utilisant un argument global, que  $\pi$  est distinguée par  $G^\tau$  si et seulement si  $\pi$  est  $\sigma$ -invariante, c'est-à-dire  $\pi^\sigma \cong \pi$  où  $\pi^\sigma := \pi \circ \sigma$ . De plus, il a montré que cet espace est de dimension un en tant qu'un espace vectoriel complexe lorsque la condition ci-dessus est satisfaite. D'ailleurs dans *ibid.*, Il a aussi esquissé une preuve similaire lorsque  $n = 2$  et  $\pi$  est supercuspidale, en donnant le même critère de la distinction et le même théorème de dimension un. Il a conjecturé qu'en général,  $\pi$  est distinguée par  $G^\tau$  si et seulement si  $\pi$  est  $\sigma$ -invariante. De plus, la dimension de l'espace des formes linéaires  $G^\tau$ -invariantes n'est pas nécessairement un en général. Sous l'hypothèse que  $\pi$  est  $\sigma$ -invariante et supercuspidale, Jacquet a conjecturé que la dimension est un.

De plus, une représentation irréductible  $\pi$  de  $G$  est contenue dans l'image du changement de base quadratique par rapport à  $F/F_0$  si et seulement si elle est  $\sigma$ -invariante ([AC89]). Ainsi pour les représentations irréductibles, la conjecture de Jacquet donne un lien entre le changement de base quadratique et  $G^\tau$ -distinction.

Outre le cas particulier mentionné ci-dessus, il existe deux autres motivations qui corroborent la conjecture. Nous considérons d'abord l'analogie de la conjecture dans le cas de corps finis. Pour  $\bar{\rho}$  une représentation complexe irréductible de  $\mathrm{GL}_n(\mathbb{F}_{q^2})$ , Gow [Gow84] a prouvé que  $\bar{\rho}$  est distinguée par le sous-groupe unitaire  $U_n(\mathbb{F}_q)$  si et seulement si  $\bar{\rho}$  est isomorphe à sa torsion par l'élément non trivial de  $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ . Sous cette condition, il a également montré que l'espace des formes linéaires  $U_n(\mathbb{F}_q)$ -invariantes est de dimension un en tant qu'espace vectoriel complexe. De plus, Shintani [Shi76] a montré qu'il existe une bijection entre l'ensemble des représentations irréductibles de  $\mathrm{GL}_n(\mathbb{F}_q)$  et celle des représentations irréductibles Galois-invariantes de  $\mathrm{GL}_n(\mathbb{F}_{q^2})$ , où la correspondance, appelée *l'application de changement de base*, est caractérisée par une identité de traces. Ces deux résultats nous donnent une caractérisation claire entre le changement de base et la distinction par  $U_n(\mathbb{F}_q)$ . Enfin, lorsque  $\bar{\rho}$  est générique et Galois-invariante, Anandavardhanan et Matringe [AM18] ont récemment montré que la  $U_n(\mathbb{F}_q)$ -moyenne de la fonction de Bessel de  $\bar{\rho}$  sur le modèle de Whittaker en tant que une forme linéaire  $U_n(\mathbb{F}_q)$ -invariante est non nulle. Puisque l'espace des formes linéaires  $U_n(\mathbb{F}_q)$ -invariantes est de dimension un, ce résultat nous donne une caractérisation concrète de l'espace de la distinction.

L'autre motivation de la conjecture de Jacquet est son analogue global. Nous supposons que  $\mathcal{K}/\mathcal{K}_0$  est une extension quadratique de corps de nombres et nous notons  $\sigma$  son automorphisme non trivial. Nous considérons  $\tau$  comme une involution unitaire sur  $\mathrm{GL}_n(\mathcal{K})$ , ce qui nous donne aussi une involution sur  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ , notée  $\tau$  par abus de notation, où  $\mathbb{A}_{\mathcal{K}}$  désigne l'anneau des adèles de  $\mathcal{K}$ . Nous notons  $\mathrm{GL}_n(\mathcal{K})^\tau$  (resp.  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$ ) le sous-groupe unitaire de  $\mathrm{GL}_n(\mathcal{K})$  (resp.  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ ) par rapport à  $\tau$ . Pour  $\phi$  une forme automorphe cuspidale de  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$ , nous définissons

$$\mathcal{P}_\tau(\phi) = \int_{\mathrm{GL}_n(\mathcal{K})^\tau \backslash \mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau} \phi(h) dh$$

comme l'intégrale de période unitaire de  $\phi$  par rapport à  $\tau$ . Nous disons qu'une représentation automorphe cuspidale  $\Pi$  de  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$  est  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$ -distinguée s'il existe une forme automorphe cuspidale dans l'espace de  $\Pi$  telle que  $\mathcal{P}_\tau(\phi) \neq 0$ . Dans les années 1990, Jacquet et Ye ont commencé à étudier la relation entre  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})^\tau$ -distinction et le changement de base globale (voir par exemple [JY96] quand  $n = 3$ ). Pour  $n$  général, Jacquet [Jac05] a montré que  $\Pi$  est contenu dans l'image du changement de base quadratique (ou de manière équivalente  $\Pi$  est  $\sigma$ -invariant [AC89]) par rapport à  $\mathcal{K}/\mathcal{K}_0$  si et seule-

ment s'il existe une involution unitaire  $\tau$  telle que  $\Pi$  est  $G^\tau$ -distinguée. Ce résultat peut être considéré comme la version globale de la conjecture de Jacquet pour les représentations supercuspidales.

En fait, pour le cas particulier de la conjecture de Jacquet dans [Jac01], Jacquet a utilisé l'analogie globale de la même conjecture et la formule des traces relative comme deux techniques principales pour terminer la démonstration. Pour le dire simple, il a d'abord prouvé l'analogie globale de la conjecture. Puis il a utilisé la formule des traces relative pour écrire une intégrale de période unitaire non nulle comme le produit de ses composantes locales à chaque place de  $\mathcal{K}_0$ , où chaque composante locale caractérise la distinction de chaque composante locale de  $\Pi$  par le sous-groupe unitaire correspondant. Lorsque  $\pi$  est  $\sigma$ -invariante, il choisit  $\Pi$  comme une représentation automorphe cuspidale  $\sigma$ -invariante de  $\mathrm{GL}_n(\mathbb{A}_{\mathcal{K}})$  et  $v_0$  comme une place non archimédienne de  $\mathcal{K}_0$  telles que  $(G^\tau, \pi) = (\mathrm{GL}_n(\mathcal{K}_{v_0})^\tau, \Pi_{v_0})$ . Ensuite, la décomposition du produit conduit à la preuve de la partie "si" de la conjecture. La partie "seulement si" de la conjecture, qui sera discutée dans chapitre 1, section 4, a besoins de l'application du théorème de la globalisation. Sa méthode a été généralisée par Feigon-Lapid-Offen dans [FLO12] aux  $n$  général et représentations plus générales. Ils ont montré que la conjecture de Jacquet fonctionne pour des représentations génériques de  $G$ . De plus pour la même famille de représentations, ils ont donné une borne inférieure pour la dimension de  $\mathrm{Hom}_{G^\tau}(\pi, 1)$  et ils ont en outre conjecturé que l'inégalité qu'ils ont donnée est en fait une égalité. Enfin, Beuzart-Plessis [BP20] a récemment vérifié l'égalité sur la base des travaux de Feigon-Lapid-Offen et de la formule de trace locale relative. Ainsi pour les représentations génériques de  $G$ , la conjecture de Jacquet a été résolue.

Au lieu d'utiliser l'argument global-local, il existe également des résultats partiels basés sur la méthode algébrique que nous avons expliquée précédemment. Dans [HM98] Hakim-Mao a vérifié la conjecture lorsque  $\pi$  est supercuspidale de niveau zéro, c'est-à-dire que  $\pi$  est supercuspidale telle que  $\pi^{1+\mathfrak{p}_F \mathcal{M}_n(\mathfrak{o}_F)} \neq 0$ , où  $\mathfrak{o}_F$  désigne l'anneau des entiers de  $F$  et  $\mathfrak{p}_F$  désigne son idéal maximal. Quand  $\pi$  est supercuspidale et  $F/F_0$  est non ramifiée, Prasad [Pra01] a prouvé la conjecture en appliquant la théorie des types simples développée par Bushnell-Kutzko dans [BK93]. Quand  $\pi$  est supercuspidale modérée, c'est-à-dire que  $\pi$  est une représentation supercuspidale donnée par la construction de Howe [How77], Hakim-Murnaghan [HM02b] a vérifié la conjecture. Notant que dans les résultats de Hakim-Mao et Hakim-Murnaghan, ils ont besoin de l'hypothèse supplémentaire que la caractéristique résiduelle  $p \neq 2$ .

La discussion ci-dessus nous laisse une question ouverte: *Y a-t-il une méthode locale et algébrique qui mène à une preuve de la conjecture de Jacquet qui fonctionne pour toutes les représentations supercuspidales de  $G$ ?* Premièrement, cette méthode généralisera les résultats de Hakim-Mao, Prasad et Hakim-Murnaghan dont nous avons parlé dans le dernier paragraphe. Deuxièmement, nous considérons  $F/F_0$  comme une extension quadratique de corps localement compacts non archimédiens au lieu de corps  $p$ -adiques. Puisque le résultat de Feigon-Lapid-Offen repose fortement sur le fait que la caractéristique de  $F$  est nulle, leur méthode échoue lorsqu'on considère des corps localement compacts non archimédiens de caractéristique positive. Enfin, au lieu de considérer des représentations complexes, nous sommes également disposés à étudier les représentations  $l$ -modulaires avec  $l \neq p$ . Nous espérons prouver un analogue de la conjecture de Jacquet pour les représentations supercuspidales  $l$ -modulaires, qui généralisera le résultat de Feigon-Lapid-Offen pour les représentations supercuspidales. Notons qu'ils utilisent les méthodes globales dans leur preuve, qui repose fortement sur l'hypothèse que toutes les représentations sont complexes. Ainsi leur méthode ne fonctionne plus pour les représentations  $l$ -modulaires.

Le but du chapitre 1 est d'abord de répondre la question ci-dessus, puis d'explorer le problème de la distinction pour des représentations irréductibles plus générales dans le cas  $l$ -modulaire et sa relation avec le changement de base " $l$ -modulaire" dont la construction sera donnée.

## 0.2.2 Principaux résultats

Pour commencer, nous supposons désormais que  $F/F_0$  est une extension quadratique de corps localement compacts non archimédiens de caractéristique résiduelle  $p$  au lieu de corps  $p$ -adiques, et nous supposons que  $p \neq 2$ . Nous fixons  $R$  un corps algébriquement clos de caractéristique  $l \neq p$ , permettant que  $l = 0$ . Nous supposons que  $\pi$  est une représentation irréductible de  $G = \mathrm{GL}_n(F)$  sur  $R$ . Maintenant, nous énonçons notre premier théorème principal.

**Théorème 0.2.1.** *Pour  $\pi$  une représentation supercuspidale de  $G$  et  $\tau$  une involution unitaire,  $\pi$  est distinguée par  $G^\tau$  si et seulement si  $\pi^\sigma \cong \pi$ .*

De plus, nous pouvons aussi calculer la dimension de l'espace de formes linéaires  $G^\tau$ -invariantes.

**Théorème 0.2.2.** *Pour  $\pi$  une représentation supercuspidale  $\sigma$ -invariante de  $G$ ,*

$$\dim_R \mathrm{Hom}_{G^\tau}(\pi, 1) = 1.$$

Un corollaire important du Théorème [0.2.1](#) concerne le  $\overline{\mathbb{Q}_l}$ -relèvement d'une représentation supercuspidale  $\sigma$ -invariante de  $G$  sur  $\overline{\mathbb{F}_l}$  quand  $l > 0$ , où nous notons  $\overline{\mathbb{Q}_l}$ ,  $\overline{\mathbb{Z}_l}$  et  $\overline{\mathbb{F}_l}$  la clôture algébrique d'un corps  $l$ -adique, son anneau des entiers et la clôture algébrique d'un corps fini à  $l$  éléments respectivement. Pour  $(\tilde{\pi}, V)$  une représentation lisse et irréductible de  $G$  sur  $\overline{\mathbb{Q}_l}$ , nous l'appelons *entière* si elle admet une structure entière, c'est-à-dire un  $\overline{\mathbb{Z}_l}[G]$ -sous-module  $L_V$  de  $V$  tel que  $L_V \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{Q}_l} = V$ . Pour une telle représentation, la semi-simplification de  $L_V \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l}$  ne dépend pas du choix de  $L_V$ , que nous notons  $r_l(\tilde{\pi})$  une représentation de  $G$  sur  $\overline{\mathbb{F}_l}$ , appelé la *réduction modulo  $l$*  de  $\pi$  (voir [Vig96](#) pour plus de détails). Le théorème suivant qui sera prouvé à la fin du chapitre 1, section 8, dit qu'il est toujours possible de trouver un  $\overline{\mathbb{Q}_l}$ -relèvement  $\sigma$ -invariant pour une représentation supercuspidale  $\sigma$ -invariante de  $G$  sur  $\overline{\mathbb{F}_l}$ .

**Théorème 0.2.3.** *Pour  $\pi$  une représentation supercuspidale  $\sigma$ -invariante de  $G$  sur  $\overline{\mathbb{F}_l}$ , il existe une représentation entière  $\sigma$ -invariante  $\tilde{\pi}$  de  $G$  sur  $\overline{\mathbb{Q}_l}$ , telle que  $r_l(\tilde{\pi}) = \pi$ .*

Pour les représentations génériques irréductibles, nous pouvons prouver une direction de la conjecture de Jacquet, qui n'est nouvelle que si  $\mathrm{char}(R) = l > 0$ .

**Théorème 0.2.4** (Voir Theorem [1.9.1](#)). *Soit  $\pi$  une représentation irréductible générique de  $G$  sur  $R$ . Si  $\pi$  est distinguée par  $G^\tau$ , alors  $\pi$  est  $\sigma$ -invariante.*

Notre prochain objectif est de caractériser les représentations distinguées  $l$ -modulaires via la functorialité de Langlands locale, ou le changement de base dans notre paramètre. Pour ce faire, nous devons d'abord construire un changement de base  $l$ -modulaire. Le résultat est le théorème suivant:

**Théorème 0.2.5** (Voir Theorem [1.10.17](#)). *Nous pouvons définir le changement de base cyclique  $l$ -modulaire*

$$\mathrm{BC}_{\overline{\mathbb{F}_l}} : \mathrm{Irr}_{\overline{\mathbb{F}_l}}(\mathrm{GL}_n(F_0)) \longrightarrow \mathrm{Irr}_{\overline{\mathbb{F}_l}}^{\sigma\text{-inv}}(\mathrm{GL}_n(F))$$

qui satisfait et est déterminé par le diagramme commutatif suivant

$$\begin{array}{ccc} \mathrm{Irr}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathrm{GL}_n(F_0)) & \xrightarrow{\mathrm{BC}_{\overline{\mathbb{Q}_l}}} & \mathrm{Irr}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}, \sigma\text{-inv}}(\mathrm{GL}_n(F)) \\ \downarrow J_l & & \downarrow J_l \\ \mathrm{Irr}_{\overline{\mathbb{F}_l}}(\mathrm{GL}_n(F_0)) & \xrightarrow{\mathrm{BC}_{\overline{\mathbb{F}_l}}} & \mathrm{Irr}_{\overline{\mathbb{F}_l}}^{\sigma\text{-inv}}(\mathrm{GL}_n(F)) \end{array}$$



Nous expliquons brièvement les notations et laissons la section correspondante pour plus de détails. Ici, les exposants  $\text{Int}$  et  $\sigma\text{-inv}$  représentent respectivement entier et  $\sigma$ -invariant,  $\text{BC}_{\overline{\mathbb{Q}}_l}$  représente le changement de base d'Arthur-Clozel aux représentations sur  $\overline{\mathbb{Q}}_l$  via un certain isomorphisme algébrique  $\mathbb{C} \cong \overline{\mathbb{Q}}_l$ , et pour  $\tilde{\pi}_0$  (resp.  $\tilde{\pi}$ ) dans  $\text{Irr}_{\overline{\mathbb{Q}}_l}^{\text{Int}}(\text{GL}_n(F_0))$  (resp.  $\text{Irr}_{\overline{\mathbb{Q}}_l}^{\text{Int}, \sigma\text{-inv}}(\text{GL}_n(F))$ ), l'image  $J_l(\tilde{\pi}_0)$  (resp.  $J_l(\tilde{\pi})$ ) est l'unique composant irréductible dans  $r_l(\tilde{\pi}_0)$  (resp.  $r_l(\tilde{\pi})$ ) ayant la séquence dérivée la plus haute. Enfin, en tant qu'une application, nous explorons les représentations cuspidales distinguées (mais pas nécessairement supercuspidales) dans le cas  $l$ -modulaire.

### 0.2.3 Organisation du chapitre 1

Décrivons le contenu du chapitre 1. Nous introduisons nos paramètres dans la section 1 et les connaissances de base sur les matrices hermitiennes et les sous-groupes unitaires dans la section 2. Notre principal outil pour prouver les théorèmes sera la théorie des types simples développée par Bushnell-Kutzko dans [BK93], et généralisé par Vignéras [Vig96] au cas  $l$ -modulaire. Dans la section 3, nous donnerons une introduction détaillée de cette théorie.

Pour une représentation supercuspidale donnée  $\pi$  de  $G$ , notre point de départ est de prouver la partie “seulement si” du Théorème 0.2.1. Quand  $R = \mathbb{C}$  et  $\text{char}(F) = 0$ , c'est un résultat standard en utilisant un argument global, en particulier le théorème de globalisation ([HM02a], Theorem 1). Quand  $\text{char}(F) = p > 0$ , nous pouvons garder la preuve originale sauf que nous avons besoin d'une version en caractéristique  $p$  du théorème de globalisation. Heureusement, nous pouvons utiliser un résultat plus général en raison de Gan-Lomeli [GL18] pour obtenir le résultat dont nous avons besoin. Comme toute représentation supercuspidale de  $G$  sur un corps algébriquement clos de caractéristique 0 peut être réalisée comme une représentation sur  $\overline{\mathbb{Q}}$  à torsion près par un caractère non ramifié, nous terminons la démonstration lorsque  $\text{char}(R) = 0$ . Quand  $R = \overline{\mathbb{F}}_l$ , nous considérons l'enveloppe projective  $P_{\Lambda|_J}$  de  $\Lambda|_J$  et nous utilisons les résultats de [Vig96] pour étudier ses composants irréductibles et les composants irréductibles de son  $\overline{\mathbb{Q}}_l$ -relèvement. Enfin nous montrons qu'il existe un  $\overline{\mathbb{Q}}_l$ -relèvement de  $\pi$  qui est supercuspidal et  $G^\tau$ -distingué. Ainsi, en utilisant le cas de caractéristique 0, nous finissons la preuve de la partie “seulement si” pour tout  $R$  sous nos conditions. Les détails seront présentés dans la section 4.

Dans la section 5, nous prouvons le *théorème du type  $\tau$ -autodual*, qui dit que pour toute involution unitaire  $\tau$  et toute représentation cuspidale  $\sigma$ -invariante de  $G$  avec une condition technique (voir Theorem 1.5.3) qui est automatiquement vraie au moins dans le cas supercuspidal, on peut trouver un type simple  $(\mathbf{J}, \Lambda)$  contenu dans  $\pi$  tel que  $\tau(\mathbf{J}) = \mathbf{J}$  et  $\Lambda^\tau \cong \Lambda^\vee$ , où  $\vee$  désigne la contragrédiente. En d'autres termes, nous trouvons un type simple “symétrique” contenu dans  $\pi$  par rapport à  $\tau$ . Notre stratégie découle de [AKM<sup>+</sup>19], section 4. Nous considérons d'abord le cas où  $E/F$  est totalement ramifiée et  $n = d$ . Ensuite, pour  $E/F$  quelconque avec  $n = d$ , nous utilisons les techniques sur l'endoclasse et le relèvement modéré développées dans [BH96] pour prouver le théorème en le réduisant au cas précédent. Enfin en utilisant le cas  $n = d$ , nous prouvons le théorème général.

Dans la section 6, pour une représentation cuspidale  $\sigma$ -invariante  $\pi$  et une certaine involution unitaire  $\tau$  satisfaisant la condition technique, nous utilisons d'abord nos résultats dans la section 5 pour choisir un type simple  $\tau$ -autodual  $(\mathbf{J}, \Lambda)$  contenu dans  $\pi$ . Le résultat principal de la section 6, que nous appelons le *théorème du type distingué*, dit que  $\pi$  est distingué par  $G^\tau$  si et seulement s'il existe un type simple de  $\pi$  qui est  $\tau$ -autodual et distingué. Plus précisément, par la réciprocity de Frobenius et la formule de Mackey, nous avons

$$\text{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

Nous nous concentrons les  $g$  tels que  $\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ . La preuve du théorème du type distin-

gué montre également qu'il existe au plus deux doubles classes de ce type qui peuvent être décrites explicitement. De plus pour ces  $g$  nous avons

$$\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \cong \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1}) \otimes_R \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi),$$

où  $\kappa^\tau \cong \kappa^\vee$  et  $\chi$  est un caractère quadratique de  $\mathbf{J}^g \cap G^\tau$  qui est trivial quand on le restreint à  $J^{1g} \cap G^\tau$ . Dans le produit tensoriel, le premier terme  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1})$  est de dimension un comme un espace vectoriel sur  $R$ . Donc essentiellement, nous n'avons besoin d'étudier que le deuxième terme. Si nous notons  $\bar{\rho}^g$  la représentation cuspidale de  $\mathrm{GL}_m(\mathbf{l}) \cong J^g/J^{1g}$  dont l'inflation est égale à  $\rho^g|_{J^g}$ , et  $\bar{\chi}$  le caractère de  $H := J^g \cap G^\tau/J^{1g} \cap G^\tau$  dont l'inflation est égale à  $\chi|_{J^g \cap G^\tau}$ , alors nous avons encore

$$\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi) \cong \mathrm{Hom}_H(\bar{\rho}^g, \bar{\chi}).$$

Ici  $H$  est un sous-groupe unitaire, ou un sous-groupe orthogonal ou un sous-groupe symplectique de  $\mathrm{GL}_m(\mathbf{l})$ . Nous réduisons donc notre problème à étudier la distinction d'une représentation supercuspidale de  $\mathrm{GL}_m(\mathbf{l})$  par  $H$ .

Nous supposons maintenant que  $\pi$  est supercuspidale. Au début de la section 6, nous utilisons le résultat de la section 5 pour étendre  $\sigma$  à une involution non triviale sur  $E$ . Nous écrivons  $E_0 = E^\sigma$ , où  $E/E_0$  est une extension quadratique. Lorsque  $E/E_0$  n'est pas ramifiée,  $H$  est un sous-groupe unitaire. Nous utilisons d'abord le résultat de Gow [Gow84] pour traiter le cas liée de caractéristique 0. Pour  $\mathrm{char}(R) > 0$ , nous utilisons la même méthode que dans la section 4. Lorsque  $E/E_0$  est ramifiée,  $H$  est soit un sous-groupe orthogonal soit un sous-groupe symplectique. Lorsque  $H$  est orthogonal, nous utilisons la théorie de Deligne-Lusztig [DL76], précisément une formule donnée par Hakim-Lansky [HL12] pour calculer la dimension de  $\mathrm{Hom}_H(\bar{\rho}^g, \bar{\chi})$  quand  $\mathrm{char}(R) = 0$ . Pour  $\mathrm{char}(R) > 0$  nous utilisons à nouveau la même méthode que dans la section 4 pour terminer la preuve. Lorsque  $H$  est symplectique, nous montrons que l'espace est toujours 0. Ces deux cas seront traités séparément aux sections 7 et 8. En conséquence, nous terminons la démonstration du Théorème [0.2.1], du Théorème [0.2.2] et du Théorème [0.2.3].

La section 9 est dédiée à la preuve du Théorème [0.2.4]. Nous traitons d'abord le cas cuspidal, dont la stratégie découle du même argument dans les sections 5-8. En particulier, nous donnons également une nouvelle preuve du résultat principal de la section 4, qui est purement local et ne dépend pas du théorème de globalisation. Puis en utilisant l'induction parabolique et en suivant l'argument similaire de Feigon-Lapid-Offen, nous terminons la preuve pour le cas générique.

Enfin dans la section 10, nous construisons le changement de base  $l$ -modulaire comme promis dans le Théorème [0.2.5]. La stratégie de construction est assez naïve. Nous construisons d'abord le changement de base  $l$ -modulaire du côté galoisien, ce qui correspond à l'application de restriction. Ensuite, nous utilisons la correspondance locale de Langlands  $l$ -modulaire développée par Vignéras [Vig01] pour transférer cette application au côté de  $\mathrm{GL}_n$ , de sorte qu'elle soit compatible avec la functorialité de Langlands local  $l$ -modulaire. Ce qui reste à montrer est la compatibilité de l'application construite avec le changement de base d'Arthur-Clozel, qui repose sur la correspondance de Langlands locale sur  $\overline{\mathbb{Q}}_l$  et  $\overline{\mathbb{F}}_l$  et leur compatibilité, et la functorialité de Langlands locale pour le changement de base sur  $\overline{\mathbb{Q}}_l$ . Cependant, il faut souligner que notre changement de base  $l$ -modulaire est en quelque sorte "artificiel", puisque dans le théorème l'application  $J_l$  n'est pas  $r_l$ , la réduction modulo  $l$  habituelle, et en général la réduction modulo  $l$  d'une représentation irréductible n'est pas irréductible. Mais pour les représentations cuspidales, la définition de  $r_l$  et  $J_l$  coïncide, donc nous pourrions utiliser notre changement de base  $l$ -modulaire pour étudier la distinction des représentations cuspidales  $l$ -modulaires, qui seront exposés dans la sous-section finale.

Il est à noter que dans [Séc19], Sécherre a étudié les représentations supercuspidales  $\sigma$ -auto-duales de  $G$  sur  $R$ , avec la même notation que précédemment. Il a prouvé le *Théorème de Dichotomie* et

*Théorème de Disjonction:* Pour  $\pi$  une représentation supercuspidale de  $G$ , elle est  $\sigma$ -autoduale (c'est-à-dire  $\pi^\sigma \cong \pi^\vee$ ) si et seulement si  $\pi$  est soit distinguée par  $\mathrm{GL}_n(F_0)$  soit  $\omega$ -distinguée, où  $\omega$  désigne le caractère non trivial unique de  $F_0^\times$  qui est trivial sur  $N_{F/F_0}(F^\times)$ . La méthode que nous utilisons dans ce chapitre est la même que celle développée dans *ibid.* Par exemple, notre section 5 correspond à la section 4 de [AKM<sup>+</sup>19] et notre section 6 correspond à la section 6 de [Séc19], etc.

Pour signaler les principales différences dans notre cas pour finir cette introduction, d'abord dans la section 5 nous verrons que dans un certain cas, il est impossible de trouver un ordre héréditaire  $\mathfrak{a}$  tel que  $\tau(\mathfrak{a}) = \mathfrak{a}$ , ce qui n'est pas un problème dans la section 4 de [AKM<sup>+</sup>19]. C'est pourquoi nous devons ajouter une condition technique dans le théorème principal de la section 5 et enfin la vérifier pour les représentations supercuspidales. Précisément, pour une représentation supercuspidale  $\sigma$ -invariante, nous considérons d'abord l'involution unitaire  $\tau = \tau_1$  correspondant à la matrice hermitienne  $I_n$ . Dans ce cas, nous pouvons utiliser notre discussion dans la section 5 pour trouver un type simple  $\tau$ -autodual contenu dans  $\pi$  et nous pouvons utiliser notre discussion dans les sections 6 et 7 pour montrer que  $m$  est impair quand  $E/E_0$  n'est pas ramifiée. Cela confirme la condition technique dont nous avons besoin, nous pouvons donc répéter la procédure des sections 5 et 6 pour les involutions unitaires générales. Cet argument de détour indique également qu'une représentation cuspidale non supercuspidale  $\sigma$ -invariante ne contient pas toujours un type simple  $\tau$ -autodual. De plus, dans la section 9, nous fournissons également une autre méthode pour résoudre ce problème. L'idée est de considérer une involution unitaire générale comme une torsion d'une involution unitaire particulière. Cette idée nous permet de prouver Théorème 0.2.4 pour les représentations cuspidales.

De plus dans la section 8, nous pouvons découvrir que le caractère  $\chi$  mentionné ci-dessus ne peut pas toujours être réalisé comme un caractère de  $\mathbf{J}$ , donc ne peut pas être supposé trivial a priori comme dans [Séc19]. Cela signifie que nous devons considérer une représentation supercuspidale du groupe linéaire général sur un corps fini distinguée par un caractère non trivial d'un sous-groupe orthogonal au lieu du caractère trivial. C'est pourquoi le résultat de Hakim-Lansky ([HL12], Theorem 3.11) apparaît.

Enfin, dans la section 6, une grande partie de nos résultats sont énoncés et prouvés pour une involution générale au lieu d'une involution unitaire. Ceci offre la possibilité de généraliser cette méthode pour étudier la distinction des représentations supercuspidales de  $G$  par d'autres involutions. Par exemple, le problème similaire pour les sous-groupes orthogonaux est exploré au chapitre 2 de la thèse.

## 0.3 Problème de la distinction pour le sous-groupe orthogonal de $\mathrm{GL}_n(F)$

### 0.3.1 Contexte général

Ce chapitre est basé sur la prépublication [Zou20]. Soit  $F = F_0$  un corps localement compact non archimédien de caractéristique résiduelle  $p$ . Nous ne considérerons que le cas où  $R = \mathbb{C}$ , bien que les principaux résultats de ce chapitre devraient également être vrais pour  $R$  en général. Comme précédemment, soient  $\mathbf{G} = \mathrm{GL}_n$  comme un groupe algébrique sur  $F$  et  $G = \mathrm{GL}_n(F)$ . Pour  $\varepsilon$  une matrice symétrique dans  $G$ , nous notons

$$\tau_\varepsilon(x) = \varepsilon^{-1} {}^t x^{-1} \varepsilon \quad \text{pour tout } x \in G$$

l'involution orthogonale par rapport à  $\varepsilon$ , et  $\mathbf{G}^{\tau_\varepsilon}$  le sous-groupe orthogonal de  $\mathbf{G}$ , tel que le groupe de ses  $F_0$ -points rationnels, noté  $G^{\tau_\varepsilon}$  et appelé le sous-groupe orthogonal de  $G$ , est le sous-groupe de  $G$  constitué des éléments fixés par  $\tau_\varepsilon$ . Dans ce cadre, nous nous intéressons au problème de la distinction lié au couple  $(\mathbf{G}, \mathbf{G}^{\tau_\varepsilon})$ , et sa relation avec la correspondance de Langlands locale et sa fonctorialité.

Si nous écrivons  $\mathcal{S}$  pour l'ensemble des matrices symétriques inversibles comme un sous-espace topologique de  $G$ , qui est doté d'une  $G$ -action continue à droite

$$\varepsilon \cdot g := {}^t g \varepsilon g, \quad g \in G, \varepsilon \in \mathcal{S},$$

alors nous avons la décomposition suivante comme  $G$ -espaces

$$\mathcal{S} = \bigsqcup_{[\varepsilon]} G^{\tau_\varepsilon} \backslash G,$$

où  $[\varepsilon]$  parcourt  $\mathcal{S}/G$ , et  $G^{\tau_\varepsilon}$  est le groupe orthogonal défini par un certain représentant  $\varepsilon$  dans la classe  $[\varepsilon]$ . Une version plus uniforme du problème ci-dessus consiste à étudier l'espace

$$\mathrm{Hom}_G(\pi, \mathcal{C}^\infty(\mathcal{S})) \cong \bigoplus_{[\varepsilon]} \mathrm{Hom}_G(\pi, \mathrm{Ind}_{G^{\tau_\varepsilon}}^G 1) \cong \bigoplus_{[\varepsilon]} \mathrm{Hom}_{G^{\tau_\varepsilon}}(\pi, 1), \quad (0.3.1)$$

pour la représentation irréductible  $\pi$  de  $G$ , et déterminer un critère pour que l'espace soit non nul et pour étudier la dimension correspondante, où  $\mathcal{C}^\infty(\mathcal{S})$  désigne l'espace des fonctions uniformément localement constantes sur  $\mathcal{S}$  à valeurs complexes.

L'étude de ce problème a d'abord été proposée par Jacquet [Jac91]. La méthode, comme nous l'avons déjà introduit précédemment, consiste d'abord à considérer son analogue global, puis à initier un argument global-local, et le point clé est de comparer deux formules de trace relatives: l'une concerne la formule de trace relative pour les matrices symétriques ou groupes orthogonaux, et l'autre se rapporte à la formule de trace de Kuznetsov pour le revêtement métaplectique double de  $\mathrm{GL}_n$  (voir [Mao98] pour une brève introduction).

Nous fournissons un bref résumé des résultats connus. Dans [Off05], Offen a suivi l'argument de Jacquet [Jac03] pour considérer la transformation de Kloosterman-Fourier pour les intégrales orbitales par rapport aux matrices symétriques, ce qui pourrait être une étape partielle pour prouver l'existence d'un transfert lisse dans le cas non-archimédien, et le cas archimédien correspondant reste un mystère. Pour le lemme fondamental pour l'unité de l'algèbre de Hecke, Mao [Mao98] a donné une preuve, pour  $n = 3$ , par calcul direct et Do a d'abord prouvé, pour  $n$  général, pour les corps locaux de caractéristique positive via la méthode géométrique [Do15], puis il a transféré le résultat aux corps  $p$ -adiques pour  $p$  assez grand [Do18]. Cependant, pour faciliter l'application ultérieure, une version plus forte du lemme fondamental fonctionnant pour des éléments généraux dans l'algèbre de Hecke est nécessaire mais reste inconnu. Les côtés spectraux des deux formules de trace sont moins étudiés. Des résultats partiels dus à Chinta et Offen [CO12], [CO13], d'une part, jettent un peu de lumière sur les expansions spectrales, mais d'autre part, indiquent la difficulté de résoudre la question complète. En particulier, comme le modèle de Whittaker local pour le revêtement métaplectique double de  $\mathrm{GL}_n$  n'est pas unique, les termes du côté spectral de la formule de trace de Kuznetsov ne sont pas factorisables, ajoutant la difficulté à un argument global-local.

Un point subtil de ce problème est qu'il est hors de portée des propositions générales connues, par exemple celle proposée par Sakellaridis-Venkatesh comme nous l'avons présenté précédemment. En fait pour  $\mathbf{X} = G^\tau \backslash G$  avec  $\tau$  une involution orthogonale donnée, même si  $\mathbf{X}$  est une variété sphérique, l'hypothèse sur les racines de  $\mathbf{X}$  n'est pas satisfaite, ce qui nous empêche de construire le groupe dual  $\hat{G}_X$ . Si nous croyons en l'existence du lien entre la distinction et la fonctorialité de Langlands locale comme nous l'avons expliqué précédemment, alors un relèvement local attendu doit être construit:

$$\mathrm{Irr}_{\mathbb{C}}(G_0) \longrightarrow \mathrm{Irr}_{\mathbb{C}}(G),$$

pour  $G_0$  comme au moins un groupe localement profini, de sorte que les représentations distinguées soient exactement dans son image. Comme déjà indiqué dans les travaux de Jacquet et de ses successeurs ci-dessus, une croyance générale est que, le groupe  $G_0$  devrait être le revêtement métaplectique

de  $\mathrm{GL}_n(F)$ , et le relèvement correspondant doit être la correspondance métaplectique étudiée par Flicker-Kazhdan [FK86]. Cependant, à la connaissance de l'auteur, aucune déclaration ou conjecture précise n'a été faite, ce qui est probablement dû à l'absence de cas connus du problème de la distinction elle-même. Au lieu de comprendre le problème complètement, il est également joyeux si des résultats partiels éclairants ou même des suppositions raisonnables pouvaient être faites.

Une autre stratégie commence par étudier la distinction des représentations supercuspidales, puis utilise l'induction parabolique pour obtenir au moins des résultats partiels pour les représentations plus générales. Pour l'étude d'une représentation supercuspidale  $\pi$ , comme nous l'avons présenté précédemment, l'idée approximative est d'abord de la considérer comme l'induction compacte d'une représentation de dimension finie  $\Lambda$  d'un sous-groupe ouvert  $\mathbf{J}$  de  $G$  qui est compact modulo son centre, puis d'utiliser la formule de Mackey et la réciprocity de Frobenius pour écrire l'espace de la distinction comme produit direct sur les doubles cosets dans  $\mathbf{J}\backslash G/H$ , des espaces de la distinction par rapport à  $\Lambda$ . Sous l'hypothèse que  $p \neq 2$ , la question est complètement traitée par Hakim et Mao [HM99] lorsque  $\pi$  est de niveau 0 et par Hakim et Lansky [HL12] et Hakim [Hak13] lorsque  $\pi$  est modérément ramifiée. Le but de ce chapitre est de généraliser leurs résultats à toutes les représentations supercuspidales de  $G$ , ce que nous expliquons dans la sous-section suivante.

### 0.3.2 Énoncé des principaux théorèmes

À partir de maintenant, nous supposons en outre que  $p \neq 2$ . Pour  $\pi$  une représentation supercuspidale de  $G$ , nous rappelons plusieurs invariants donnés par la théorie des types simples de Bushnell-Kutzko [BK93] et la théorie de l'endo-classe de Bushnell-Henniart [BH96], qui on se réfère à §2.1.2 pour plus de détails. Tout d'abord, il existe une unique extension modérément ramifiée  $T/F$  à  $F$ -isomorphisme après, appelée le corps de paramètre modéré de  $\pi$ . Nous écrivons  $d$  pour le degré de l'endo-classe de  $\pi$  qui divise  $n$  et est divisé par  $[T : F]$ . Nous écrivons  $m$  pour l'entier tel que  $n = md$ . Soit  $T_m$  l'extension non ramifiée du degré  $m$  sur  $T$ . Ici,  $T$ ,  $d$ ,  $m$ ,  $T_m$  sont intrinsèquement déterminés par  $\pi$ .

Pour donner une idée de ce que devraient être ces invariants, nous notons  $\varphi_\pi$  la représentation irréductible du groupe de Weil  $\mathcal{W}_F$  correspondante à  $\pi$  via la correspondance de Langlands locale. Alors la restriction de  $\varphi_\pi$  au sous-groupe d'inertie sauvage  $\mathcal{P}_F$  de  $\mathcal{W}_F$  est semi-simple et peut être écrite comme somme directe de représentations irréductibles avec chaque composante irréductible de multiplicité exactement  $m$ . Soit  $\alpha$  n'importe quel composant irréductible de  $\varphi_\pi|_{\mathcal{P}_F}$ , alors il existe une extension finie modérément ramifiée  $T/F$  telle que

$$N_F(\alpha) := \{g \in \mathcal{W}_F \mid \alpha^g \cong \alpha\}$$

en tant que sous-groupe de  $\mathcal{W}_F$  est égal à  $\mathcal{W}_T$ . Et il s'avère que  $T/F$  est uniquement déterminée à  $F$ -isomorphisme près et indépendante du choix de  $\alpha$ . Soient  $n = \dim(\varphi_\pi)$ ,  $d = n/m$  et  $T_m$  comme ci-dessus. Alors  $T$ ,  $d$ ,  $m$ ,  $T_m$  définis ici à partir du côté galoisien correspondent à ceux définis du côté de  $\mathrm{GL}_n$  mentionné dans le dernier paragraphe (voir [BH14b] pour plus de détails).

Le théorème suivant donne un critère de distinction.

**Théorème 0.3.1.** *Soit  $\pi$  une représentation supercuspidale de  $G$  et soient  $T$ ,  $d$ ,  $m$ ,  $T_m$  comme ci-dessus. Alors  $\pi$  est distinguée par un sous-groupe orthogonal  $H$  si et seulement si les deux conditions suivantes sont valables:*

1.  $\omega_\pi(-1) = 1$ , où  $\omega_\pi$  désigne le caractère central de  $\pi$ ;
2. Précisément une des trois conditions suivantes est valable:

- $N_{T_m/F}(T_m^\times F^{\times 2}/F^{\times 2}) = F^\times/F^{\times 2}$  et  $H$  est déployé;

- $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  est un sous-groupe de  $F^\times/F^{\times 2}$  d'ordre 2 et  $H$  est soit déployé ou  $H = G^{\tau_\varepsilon}$  qui est quasi-déployé mais pas déployé, où  $\varepsilon$  est une matrice symétrique telle que  $(-1)^{n(n-1)/2}\det(\varepsilon) \in N_{T_m/F}(T_m^\times) - F^{\times 2}$ ;
- $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = \{1\}$  et  $H$  est soit déployé ou non-quasi-déployé.

En particulier, c'est facile de voir que:

**Corollaire 0.3.1.** *Quand  $H$  est déployé,  $\pi$  est distinguée par  $H$  si et seulement si  $\omega_\pi(-1) = 1$ .*

De plus, le théoème suivant calcule la dimension de l'espace de la distinction.

**Théorème 0.3.2.** *Soit  $\pi$  une représentation supercuspidale de  $G$  telle que  $\omega_\pi(-1) = 1$  et soit  $H$  un sous-groupe orthogonal satisfaisant la condition 2 du Théorème [0.3.1](#).*

1. Si  $H$  n'est pas déployé, alors  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 1$ ;
2. Si  $H$  est déployé, alors
  - Si  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$ , alors  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 1$ ;
  - Si  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  est un sous-groupe de  $F^\times/F^{\times 2}$  de l'ordre 2, alors  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 2$ ;
  - Si  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = \{1\}$ , alors  $\dim_{\mathbb{C}}\mathrm{Hom}_H(\pi, 1) = 3$ .

Finalement en utilisant [\(0.3.1\)](#) et le même argument dans [\[Hak13\]](#), le théorème suivant est valable comme un corollaire du Théorème [0.3.3](#).

**Théorème 0.3.3.** *Pour  $\pi$  une représentation supercuspidale de  $G$ , elle est distinguée par un certain sous-groupe orthogonal si et seulement si  $\omega_\pi(-1) = 1$ . De plus, si cette condition est valable, alors*

$$\dim_{\mathbb{C}}\mathrm{Hom}_G(\pi, \mathcal{C}^\infty(\mathcal{S})) = 4.$$

Ainsi pour  $p \neq 2$  et toute représentation supercuspidale  $\pi$  de  $G = \mathrm{GL}_n(F)$ , le problème de la distinction pour les sous-groupes orthogonaux est entièrement résolu. La seule restriction sur  $\pi$ , étant la trivialeité de son caractère central sur  $-1$ , peut également être reformulée comme la trivialeité du caractère déterminant de son paramètre de Langlands sur  $-1$  via la correspondance de Langlands locale pour  $\mathrm{GL}_n$ .

### 0.3.3 Esquisse de la preuve et de la structure du chapitre 2

Nous esquissons la preuve et la structure du chapitre 2. Nous rappelons brièvement la théorie des types simples dont nous avons besoin dans la section 1, qui est en effet un sous-ensemble propre du chapitre 1, section 2. Dans la section 2, nous construisons les résultats nécessaires pour les matrices symétriques, involutions orthogonales et groupes orthogonaux pour une utilisation future.

Dans la section 3, nous prouvons notre premier théorème principal, le théorème du type tau-autodual, qui dit que pour une certaine involution orthogonale bien choisie  $\tau_0$  dépendant de  $\pi$ , il existe un type simple  $(\mathbf{J}, \Lambda)$  compactement induisant  $\pi$  tel que  $\tau_0(\mathbf{J}) = \mathbf{J}$  et  $\Lambda \circ \tau_0 = \Lambda^\vee$ , où  $\Lambda^\vee$  désigne le contragrédient de  $\Lambda$ . En fait, pour chaque groupe orthogonal  $H$  satisfaisant le théorème [0.3.1](#), condition 2, on peut trouver une  $\tau_0$  satisfaisant  $H = G^{\tau_0}$  et le théorème du type tau-autodual. Un tel type simple est appelé  $\tau_0$ -autodual et sera considéré comme le point de départ pour poursuivre le problème de la distinction.

Dans la section 4, nous étudions la distinction par rapport à une involution orthogonale arbitraire  $\tau$  et le groupe orthogonal correspondant  $G^\tau$ . Nous fixons un  $\tau_0$ -autodual type simple  $(\mathbf{J}, \Lambda)$  et nous

pouvons utiliser la formule de Mackey et la réciprocity de Frobenius pour écrire l'espace de la distinction comme suit:

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

Le théorème du type distingué dit que pour les doubles classes  $g \in \mathbf{J} \backslash G / G^\tau$  contribuant à la distinction, le type simple  $(\mathbf{J}^g, \Lambda^g)$  est  $\tau$ -autodual. En particulier, quand  $\tau = \tau_0$  nous pouvons aussi donner toutes les  $\mathbf{J}$ - $G^{\tau_0}$  doubles classes contribuant à la distinction.

Enfin dans la section 5, nous continuons à étudier l'espace de la distinction  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1)$ . Les techniques développées dans la section 4 nous permettent d'étudier plus avant l'espace de la distinction via la structure plus délicate donnée par la théorie des types simples, et enfin de réduire la question à étudier l'espace de la distinction  $\mathrm{Hom}_{\overline{H}}(\overline{\rho}, \overline{\chi})$ , où  $\overline{H}$  est un sous-groupe orthogonal d'un groupe linéaire général fini  $\overline{G} = \mathrm{GL}_m(\mathbb{F}_q)$ , et  $\overline{\rho}$  est une représentation supercuspidale de  $\overline{G}$ , et  $\overline{\chi}$  est un caractère de  $\overline{H}$  d'ordre 1 ou 2. En utilisant la théorie de Deligne-Lusztig, la condition pour que l'espace soit non nul est donnée et la dimension est au plus un. La condition s'avère être le caractère central de  $\pi$  étant trivial à  $-1$ . Ainsi pour ces  $\tau_0$  spéciales dans la section 4, nous étudions entièrement l'espace de la distinction et la dimension correspondante. Puisque ces  $\tau_0$  correspondent exactement aux groupes orthogonaux dans les Théorème 0.3.1 et Théorème 0.3.2, nous prouvons la partie "si" du Théorème 0.3.1 et Théorème 0.3.2.

Il reste la partie "seulement si" du théorème 0.3.1, dont nous profitons pour expliquer la condition des groupes orthogonaux ou des involutions orthogonales correspondantes dans le théorème. Pour  $E_m/F$  une extension de degré  $n$  et  $\tau$  une involution orthogonale, nous appelons  $E_m$   $\tau$ -déployé s'il existe un plongement  $\iota : E_m^\times \hookrightarrow \mathrm{GL}_n(F)$  tel que  $\tau(\iota(x)) = \iota(x)^{-1}$  pour tout  $x \in E_m^\times$ . La proposition intermédiaire suivante donne des informations importantes pour que  $\pi$  soit distinguée par  $G^\tau$ :

**Proposition 0.3.2.** *Pour  $\pi$  une représentation supercuspidale de  $G$  avec  $\omega_\pi(-1) = 1$ , il existe un corps  $E_m$  de degré  $n$  sur  $F$  qui est totalement sauvagement ramifié sur  $T_m$ , tel que si  $\pi$  est distinguée par  $G^\tau$ , alors  $E_m$  est  $\tau$ -déployé.*

La construction de  $E_m$  provient de la construction de type simple  $\tau_0$ -autodual donnée dans la section 3. En particulier, lorsque  $\tau_0$  correspond à un groupe orthogonal déployé, de la partie "si" du Théorème 0.3.1,  $E_m$  est  $\tau_0$ -déployé. En sachant cela, il n'est pas difficile d'étudier toutes les involutions  $\tau$  telles que  $E_m$  est  $\tau$ -déployé, qui s'avèrent être involutions satisfaisant la condition du théorème 0.3.1, prouvant la partie "seulement si" du théorème.

Lorsque  $T_m/F$  est de degré  $n$ , ou de manière équivalente lorsque  $\pi$  est essentiellement modérée au sens de Bushnell-Henniart [BH05a], ce qui revient à être modérément ramifiée dans le contexte de Hakim [Hak13] grâce au travail de Mayeux [May20], notre résultat donne une autre preuve du résultat de Hakim en utilisant la théorie des types simples au lieu de la construction de Howe pour des représentations modérément ramifiées. Notons que nous empruntons également nombreux lemmes à [HM99], [HL12], [Hak13], qui nous aident à réduire notre tâche.

Comme dans le chapitre 1, il convient également de souligner que la méthode que nous utilisons ici n'est pas nouvelle. Elle a d'abord été développée par Sécherre pour résoudre le problème similaire où  $\tau$  est une involution galoisienne [AKM<sup>+</sup>19], [Séc19], puis par l'auteur pour le cas où  $\tau$  est une involution unitaire (cf. chapitre 1), puis par Sécherre pour le cas où  $\tau$  est une involution intérieure [Séc20] (là  $G$  peut aussi être une forme intérieure de  $\mathrm{GL}_n(F)$ ). Les stratégies de preuves dans ces différents cas sont similaires, mais une différence majeure dans le cas actuel mérite d'être mentionnée, c'est-à-dire que nous devons considérer ces involutions  $\tau$  ne contribuant pas à la distinction. En ce moment, nous ne pouvons pas construire un type simple  $\tau$ -autodual  $(\mathbf{J}, \Lambda)$  en utilisant la méthode de la section 3. La nouveauté de notre argument est d'abord de considérer une involution spéciale  $\tau_0$ , puis de considérer  $\tau$  comme une autre involution qui diffère de  $\tau_0$  à  $G$ -conjugaison près. Ainsi, nous choisissons  $(\mathbf{J}, \Lambda)$

comme un type simple  $\tau_0$ -autodual et, en utilisant les résultats généraux construits au chapitre 1, nous pouvons encore étudier les  $\mathbf{J}\text{-}G^\tau$  doubles classes contribuant à la distinction. Si l'on veut adapter la méthode dans les cas ci-dessus à une involution générale  $\tau$ , un problème majeur est de construire un type simple  $\tau$ -autodual, ce qui, comme nous l'avons expliqué, peut être impossible si  $G^\tau$  ne contribue pas à la distinction. La stratégie que nous avons expliquée ci-dessus donne une solution possible, ce qui permet de considérer la même question pour une involution abstraite.

## 0.4 Changement de base et induction automorphe explicites pour les représentations supercuspidales

### 0.4.1 Contexte général

Soit  $F/F_0$  comme dans §0.1.4, et nous ne considérons que le cas  $R = \mathbb{C}$  dans ce chapitre. Nous nous concentrerons sur deux relèvements locaux spéciaux, disons le changement de base et l'induction automorphe par rapport à  $F/F_0$ . Plus précisément, lorsque  $F/F_0$  est modérément ramifiée et pour les représentations supercuspidales, nous étudierons ces deux applications via la théorie des types simples.

Nous donnons d'abord une brève introduction pour la correspondance de Langlands locale pour les groupes linéaires généraux, dont l'existence et les propriétés sont connues depuis un certain temps ([LRS93], [HT01], [Hen00], [Sch13]). Pour  $n'$  un certain entier positif et  $\mathbf{G}_0 = \mathrm{GL}_{n'}$  comme un groupe réductif sur  $F_0$ , la correspondance de Langlands locale est une bijection

$$\mathrm{LLC}_{F_0} : \mathrm{Irr}_{\mathbb{C}}(G_0) \longrightarrow \Phi(G_0).$$

Ici nous gardons les notations de §0.1.1 et  $\Phi(G_0)$  est constitué de  $\mathrm{GL}_{n'}(\mathbb{C})$ -classes de conjugaison d'homomorphismes

$$\phi_0 = (\varphi_0, \lambda_0) : \mathcal{W}_{F_0} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_{n'}(\mathbb{C}),$$

tels que  $\varphi_0 := \phi_0|_{\mathcal{W}_{F_0} \times \{1\}}$  est une représentation lisse de  $\mathcal{W}_{F_0}$ , et  $\lambda_0 := \phi_0|_{\{1\} \times \mathrm{SL}_2(\mathbb{C})}$  est une représentation algébrique de  $\mathrm{SL}_2(\mathbb{C})$  de dimension  $n'$ . Pour  $n$  un entier positif, soit  $\mathbf{G}$  la restriction de Weil du groupe réductif  $\mathrm{GL}_n$  sur  $F$ , qui est un groupe réductif sur  $F_0$  avec  $G = \mathrm{GL}_n(F)$ . La correspondance de Langlands locale est une bijection

$$\mathrm{LLC}_F : \mathrm{Irr}_{\mathbb{C}}(G) \longrightarrow \Phi(G).$$

Ici  $\Phi(G)$  est constitué des classes d'isomorphisme des L-paramètres liés à  $\mathbf{G}$ , qui peuvent être naturellement identifiés avec les classes d'isomorphisme des L-paramètres liés à  $\mathrm{GL}_n$  over  $F$ . En utilisant cette identification,  $\Phi(G)$  est constitué des  $\mathrm{GL}_n(\mathbb{C})$ -classes de conjugaison d'homomorphismes

$$\phi = (\varphi, \lambda) : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}),$$

tels que  $\varphi := \phi|_{\mathcal{W}_F \times \{1\}}$  est une représentation lisse de  $\mathcal{W}_F$ , et  $\lambda := \phi|_{\{1\} \times \mathrm{SL}_2(\mathbb{C})}$  est une représentation algébrique de  $\mathrm{SL}_2(\mathbb{C})$  de dimension  $n$ .

Nous introduisons maintenant le changement de base et l'induction automorphe liés à  $F/F_0$ . D'abord nous supposons  $n' = n$  et nous définissons l'application de restriction

$$\mathrm{Res}_{F/F_0} : \Phi(G_0) \longrightarrow \Phi(G), \quad \phi_0 = (\varphi_0, \lambda_0) \longmapsto \phi = (\varphi_0|_{\mathcal{W}_F}, \lambda_0),$$

où nous remarquons que  $\mathcal{W}_F$  est un sous-groupe de  $\mathcal{W}_{F_0}$ . Ainsi, le changement de base est le relèvement



local  $\text{BC}_{F/F_0} : \text{Irr}_{\mathbb{C}}(G_0) \rightarrow \text{Irr}_{\mathbb{C}}(G)$  tel que le diagramme suivant est commutatif:

$$\begin{array}{ccc} \text{Irr}_{\mathbb{C}}(G_0) & \xrightarrow{\text{LLC}_{F_0}} & \Phi(G_0) \\ \text{BC}_{F/F_0} \downarrow \text{dotted} & & \downarrow \text{Res}_{F/F_0} \\ \text{Irr}_{\mathbb{C}}(G) & \xrightarrow{\text{LLC}_F} & \Phi(G) \end{array}$$

Deuxièmement, nous supposons  $n' = nr$  et nous définissons l'application d'induction

$$\text{Ind}_{F/F_0} : \Phi(G) \longrightarrow \Phi(G_0), \quad \phi = (\varphi, \lambda) \longmapsto \phi'_0 = (\text{Ind}_{\mathcal{W}_F}^{\mathcal{W}_{F_0}} \varphi, i \circ \lambda),$$

où  $i : \text{GL}_n(\mathbb{C}) \rightarrow \text{GL}_{nr}(\mathbb{C})$  est un plongement des groupes [\[8\]](#). Ainsi l'induction automorphe est le relèvement local  $\text{A}_{F/F_0} : \text{Irr}_{\mathbb{C}}(G) \rightarrow \text{Irr}_{\mathbb{C}}(G_0)$  tel que le diagramme suivant est commutatif:

$$\begin{array}{ccc} \text{Irr}_{\mathbb{C}}(G_0) & \xrightarrow{\text{LLC}_{F_0}} & \Phi(G_0) \\ \text{A}_{F/F_0} \uparrow \text{dotted} & & \uparrow \text{Ind}_{F/F_0} \\ \text{Irr}_{\mathbb{C}}(G) & \xrightarrow{\text{LLC}_F} & \Phi(G) \end{array}$$

Dans [\[AC89\]](#), [\[HH95\]](#) et [\[HL11\]](#), le changement de base pour toutes les représentations irréductibles, et l'induction automorphe pour au moins les représentations génériques essentiellement unitaires ont été construits via la méthode de la formule des traces sans l'utilisation de la correspondance de Langlands locale, et la fonctorialité ci-dessus a été vérifiée.

Bien que pour  $\text{GL}_n$  la correspondance de Langlands locale ait déjà été construite comme une bijection avec les desiderata vérifiés, il semble que les informations extraites des deux côtés ne soient pas égales. Concentrons-nous sur les représentations supercuspidales, alors pour tout  $n \in \mathbb{N}$  la correspondance peut être réalisée comme une bijection

$$\text{LLC}_F : \mathcal{A}_n^0(F) \longrightarrow \mathcal{G}_n^0(F)$$

de l'ensemble des classes équivalentes de représentations supercuspidales de  $\text{GL}_n(F)$ , dans l'ensemble des classes équivalentes de représentations lisses irréductibles du groupe de Weil  $\mathcal{W}_F$  de dimension  $n$ , notés respectivement  $\mathcal{A}_n^0(F)$  et  $\mathcal{G}_n^0(F)$ . Habituellement, nous obtenons peu d'informations concrètes pour les représentations irréductibles de  $\mathcal{W}_F$  à partir de la théorie des représentations, mais en revanche nous avons la théorie de classification pour les représentations supercuspidales de  $\text{GL}_n(F)$ , la théorie des types simples construite par Bushnell-Kutzko [\[BK93\]](#), qui est terre-à-terre et sophistiquée. Une question naturelle est donc la suivante: pouvons-nous caractériser le  $\text{LLC}_F$  ci-dessus en utilisant la théorie structurale pour les représentations supercuspidales de  $\text{GL}_n(F)$ ?

Pour répondre à cette question, Bushnell et Henniart lancent un projet de longue haleine dont le résultat est contenu dans une série d'articles [\[BH96\]](#), [\[BH99\]](#), [\[BH05c\]](#), [\[BH03\]](#), [\[BH05a\]](#), [\[BH05b\]](#), [\[BH10\]](#), [\[BH14b\]](#), [\[BH17\]](#), [\[BH19\]](#), etc. Surtout, dans [\[BH05a\]](#), [\[BH05b\]](#), [\[BH10\]](#) ils ont entièrement répondu à la question ci-dessus pour une classe spéciale de représentations supercuspidales, les représentations supercuspidales essentiellement modérées. Pour ce faire, ils ont d'abord construit une version algébrique de la correspondance de Langlands locale, qu'ils ont appelée "correspondance naïve de Langlands locale", comme une bijection entre les mêmes ensembles que  $\text{LLC}_F$  et notée par  $\text{NLC}_F$ . Pour  $\varphi \in \mathcal{G}_n^0(F)$ , définissons  $T_m$  comme l'extension modérément ramifiée de  $F$  liée à  $\varphi$  comme

<sup>8</sup>En utilisant un argument de base en algèbre linéaire, nous pouvons montrer que la définition ne dépend pas du choix de  $i$ .

dans la dernière section. Pour  $\mu$  un caractère modérément ramifié de  $T_m^\times$ , ils ont construit une certaine “torsion” de  $\varphi$  par  $\mu$ , notée  $\varphi \odot \mu$ , qui est une autre représentation dans  $\mathcal{G}_n^0(F)$ . Le résultat final est le théorème de comparaison, qui prédit l’existence d’un caractère modérément ramifié  $\mu_\varphi$  de  $T_m^\times$ , tel que  $\text{LLC}_F^{-1}(\varphi)$  est isomorphe à  $\text{NLC}_F^{-1}(\varphi \odot \mu_\varphi)$ . Dans le cas essentiellement modéré au sens de [BH10], le caractère  $\mu_\varphi$  de  $T_m^\times$  est d’ordre divisant 4 et peut être calculé explicitement, donc dans ce cas la correspondance de Langlands locale est pleinement comprise de manière algébrique. Dans le cas général comme dans [BH14b], la construction de la correspondance naïve de Langlands local s’appuyait sur la correspondance de Langlands locale dans le cas sauvagement ramifié comme une “boîte noire”, et la caractérisation complète du caractère  $\mu_\varphi$  reste inconnue. Donc, pour bien comprendre le  $\text{LLC}_F$  en général, nous devons d’abord comprendre le cas particulier pour les représentations supercuspidales sauvagement ramifiées, ce qui est une question assez profonde, puis nous devons calculer le caractère  $\mu_\varphi$  ci-dessus, qui sera abordé dans le chapitre 3 plus tard.

L’objectif principal de ce chapitre est d’adapter l’idée de Bushnell-Henniart ci-dessus aux changements de base et induction automorphe, c’est-à-dire que nous construirons des versions algébriques des deux applications, puis les comparerons avec les applications originales respectivement. Plus de détails seront donnés dans la sous-section ci-dessous.

### 0.4.2 Principaux résultats

Pour donner une introduction détaillée, nous utilisons les terminologies et propriétés de base de la théorie des types simples et de la théorie du changement de base et l’induction automorphe cycliques, pour lesquelles les lecteurs peuvent se référer respectivement au chapitre 3, section 2 et section 4. Pour  $F/F_0$  comme précédemment, soient  $\Sigma = \text{Gal}(F/F_0)$  le groupe de Galois et  $\sigma \in \Sigma$  un générateur. Pour  $\pi_0$  une représentation supercuspidale de  $\text{GL}_n(F_0)$ , nous définissons son changement de base  $\pi := \text{BC}_{F/F_0}(\pi_0)$  comme une représentation irréductible de  $\text{GL}_n(F)$ . Nous supposons en outre que soit  $\pi$  est supercuspidale, ou soit  $r$  divise  $n$  et qu’il existe une représentation supercuspidale  $\pi'$  de  $\text{GL}_{n/r}(F)$  telle que  $\pi$  est isomorphe à l’induction parabolique

$$\pi' \times \pi'^{\sigma} \times \dots \times \pi'^{\sigma^{r-1}}.$$

De manière équivalente, nous avons  $\pi_0 = A_{F/F_0}(\pi')$  comme l’induction automorphe de  $\pi'$  dans le dernier cas. Notre but est de donner une construction explicite de  $\pi$  et  $\pi'$  respectivement dans les deux cas, en utilisant la théorie des types simples et les informations de  $\pi_0$ . Pour cela, nous devons supposer la condition supplémentaire que  $F/F_0$  est modérément ramifiée.

Soit  $[\mathfrak{a}_0, \beta]$  une strate simple maximale dans  $M_n(F_0)$ , et soit  $\theta_0$  un caractère simple de  $H^1(\mathfrak{a}, \beta)$  contenu dans  $\pi_0$ . Nous choisissons  $\kappa_0$  comme une représentation de Heisenberg complète (full Heisenberg representation) de  $\theta_0$  comme une représentation de  $\mathbf{J}(\mathfrak{a}_0, \beta)$ , et en ce moment il existe une représentation unique  $\rho_0$  de  $\mathbf{J}(\mathfrak{a}_0, \beta)$  triviale sur  $J^1(\mathfrak{a}_0, \beta)$ , telle que  $\pi_0$  est isomorphe à l’induction compacte  $\text{ind}_{\mathbf{J}(\mathfrak{a}_0, \beta)}^{\text{GL}_n(F_0)}(\kappa_0 \otimes \rho_0)$ . Ici  $E_0 = F_0[\beta]$  est un corps de degré  $d$  sur  $F_0$  avec  $n = md$  pour  $m$  un entier, et nous notons  $T_0$  sa sous-extension maximale modérément ramifiée sur  $F_0$  et  $T_{0,m}$  l’extension non ramifiée de degré  $m$  sur  $T_0$ . En ce moment, la représentation  $\rho_0$  peut être caractérisée par  $\Delta_0$ -orbite d’un caractère  $\xi_0$  de  $T_{0,m}^\times$  qui est  $\Delta_0$ -régulier et modérément ramifié, où  $\Delta_0 = \text{Gal}(T_{0,m}/T_0)$ .

Nous considérons d’abord le cas où  $\pi$  est supercuspidale. En utilisant le résultat de relèvement modéré dans [BH96] et [BH03], nous construisons  $[\mathfrak{a}, \beta]$  comme une strate simple maximale dans  $M_n(F)$ , et  $\theta_b$  comme un caractère simple de  $H^1(\mathfrak{a}, \beta)$  contenu dans  $\pi$ . Et un tel  $\theta_b$  peut être considéré comme le changement de base de  $\theta_0$  pour des caractères simples. Ensuite, nous déterminons une représentation de Heisenberg complète  $\kappa_b$  de  $\theta_b$  de manière algébrique, qui ne dépend que de  $\theta_0$  et  $\kappa_0$ , mais pas de  $\pi_0$ . Alors il existe une représentation unique  $\rho_b$  de  $\mathbf{J}(\mathfrak{a}, \beta)$  trivial sur  $J^1(\mathfrak{a}, \beta)$ , telle que  $\pi$  est isomorphe à l’induction compacte  $\text{ind}_{\mathbf{J}(\mathfrak{a}, \beta)}^{\text{GL}_n(F)}(\kappa_b \otimes \rho_b)$ . Une telle  $\rho_b$  est caractérisée par la  $\Delta$ -orbite

d'un caractère  $\Delta$ -régulier modérément ramifié  $\xi_b$  de  $T_m^\times$ , où  $T_m = F \otimes_{F_0} T_{0,m}$  et  $T = F \otimes_{F_0} T_0$  sont des corps sur  $F$  et  $\Delta = \text{Gal}(T_m/T)$ . Le théorème de comparaison suivant est le théorème principal pour le changement de base.

**Théorème 0.4.1** (Voir Theorem 3.6.2). *Il existe un caractère modérément ramifié  ${}_b\phi_{\theta_0}^{F/F_0}$  de  $T_m^\times$ , tel que  $\xi_b \cdot {}_b\phi_{\theta_0}^{F/F_0}$  et  $\xi_0 \circ N_{T_m/T_{0,m}}$  sont dans la même  $\Delta$ -orbite.*

De même, nous considérons le cas où  $\pi_0$  est égale à l'induction automorphe d'une représentation supercuspidale  $\pi'$  de  $\text{GL}_{n/r}(F)$ . Pour continuer, nous supposons en outre que soit  $F$  est identifié avec un sous-corps de  $E_0$  via un plongement, ce qui correspond au cas de l'induction automorphe intérieure, soit  $E = F \otimes_{F_0} E_0$  est un corps de degré  $r$  sur  $E_0$ , qui correspond au cas de l'induction automorphe extérieure, et nous divisons ce dernier cas en deux sous-cas en fonction de  $E/E_0$  non ramifiée ou totalement ramifiée, car dans les différents cas, les situations et les méthodes correspondantes sont différentes. En utilisant la méthode de relèvement modéré dans [BH96] et [BH03], nous construisons  $[\mathfrak{c}, \beta]$  comme une strate simple dans  $M_{n/r}(F)$  et  $\theta_a$  comme un caractère simple de  $H^1(\mathfrak{c}, \beta)$  contenu dans  $\pi'$ , et  $\theta_0$  peut être considéré comme l'induction automorphe de  $\theta_a$  pour les caractères simples. Comme dans le cas de changement de base, nous déterminons une représentation de Heisenberg complète  $\kappa_a$  de  $\theta_a$  de manière algébrique qui dépend seulement de  $\theta_0$  et  $\kappa_0$ , et en ce moment il existe une représentation unique  $\rho_a$  de  $J(\mathfrak{c}, \beta)$  triviale sur  $J^1(\mathfrak{c}, \beta)$ , telle que  $\pi'$  est isomorphe à l'induction compacte  $\text{ind}_{J(\mathfrak{c}, \beta)}^{\text{GL}_{n/r}(F)}(\kappa_a \otimes \rho_a)$ . Dans le cas de l'induction automorphe intérieure, la représentation  $\rho_a$  est caractérisée par la  $\Delta_0$ -orbite d'un caractère  $\Delta_0$ -régulier modérément ramifié  $\xi_a$  sur  $T_{0,m}^\times$ . Dans le cas de l'induction automorphe extérieure, nous écrivons  $T = F \otimes_{F_0} T_0$  comme un corps et nous notons  $T_{m/r}$  l'extension non ramifiée de degré  $m/r$  sur  $T$ , alors pour  $\Delta = \text{Gal}(T_{m/r}/T)$  la représentation  $\rho_a$  est caractérisée par la  $\Delta$ -orbite d'un caractère  $\Delta$ -régulier modérément ramifié de  $T_{m/r}^\times$ , encore désigné par  $\xi_a$ . En ce moment, si  $E/E_0$  n'est pas ramifiée,  $T_{m/r}$  est identifié avec  $T_{0,m}$  et  $\Delta$  est identifié avec un sous-groupe de  $\Delta_0$ , et si  $E/E_0$  est totalement ramifiée,  $T_m = F \otimes_{F_0} T_{0,m}$  est un corps de degré  $m$  sur  $T$  avec  $T_{m/r}$  étant considéré comme son sous-corps, et nous écrivons  $\Delta' = \text{Gal}(T_m/T)$ . Le théorème de comparaison suivant est le principal théorème pour l'induction automorphe.

**Théorème 0.4.2** (Voir Theorem 3.6.4, Theorem 3.6.6, Theorem 3.6.8). (1) *Dans le cas de l'induction automorphe intérieure, il existe un caractère modérément ramifié  ${}_a\phi_{\theta_0}^{F/F_0}$  de  $T_{0,m}^\times$  tel que  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  et  $\xi_0$  sont dans la même  $\Delta_0$ -orbite;*

(2) *Dans le cas de l'induction automorphe extérieure, il existe un caractère modérément ramifié  ${}_a\phi_{\theta_0}^{F/F_0}$  de  $T_{m/r}^\times$  tel que*

- *Si  $E/E_0$  est non-ramifiée,  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  et  $\xi_0$  sont dans la même  $\Delta_0$ -orbite;*
- *Si  $E/E_0$  est totalement ramifiée,  $(\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}) \circ N_{T_m/T_{m/r}}$  et  $\xi_0 \circ N_{T_m/T_{0,m}}$  sont dans la même  $\Delta'$ -orbite.*

Nous mentionnons trois applications des deux théorèmes ci-dessus pour terminer cette sous-section. La première application concerne l'étude de  $\bar{\rho}_b$  et  $\bar{\rho}_a$  par  $\bar{\rho}_0$ , où  $\bar{\rho}_0$  est la représentation supercuspidale de  $\text{GL}_m(\mathbf{k}_{E_0}) \cong J(\mathfrak{a}_0, \beta)/J^1(\mathfrak{a}_0, \beta)$  dont l'inflation est égale à  $\rho_0|_{J(\mathfrak{a}_0, \beta)}$  avec  $\mathbf{k}_{E_0}$  désignant le corps résiduel de  $E_0$ , et  $\bar{\rho}_b$  est la représentation supercuspidale de  $\text{GL}_m(\mathbf{k}_E) \cong J(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta)$  dont l'inflation est égale à  $\rho_b|_{J(\mathfrak{a}, \beta)}$  avec  $\mathbf{k}_E$  désignant le corps résiduel de  $E$  dans le cas de changement de base, et  $\bar{\rho}_a$  est la représentation supercuspidale de  $\text{GL}_m(\mathbf{k}_{E_0}) \cong J(\mathfrak{c}, \beta)/J^1(\mathfrak{c}, \beta)$  dans le cas de l'induction automorphe intérieure, de  $\text{GL}_{m/r}(\mathbf{k}_E) \cong J(\mathfrak{c}, \beta)/J^1(\mathfrak{c}, \beta)$  dans le cas de l'induction automorphe extérieure, dont l'inflation est égale à  $\rho_a|_{J(\mathfrak{c}, \beta)}$ . En ce moment, les restrictions de  ${}_b\phi_{\theta_0}^{F/F_0}$

et  ${}_a\phi_{\theta_0}^{F/F_0}$  à ces éléments dans l'anneau des entiers sont des caractères quadratiques qui peuvent être entièrement caractérisés, les représentations  $\bar{\rho}_b$  et  $\bar{\rho}_a$  sont compris via la théorie de Green (cf. [Gre55]). En particulier pour  $E/E_0$  en tant qu'une extension non ramifiée de degré  $r$  dans le cas du changement de base ou dans le cas de l'induction automorphe extérieure, cela donne une relation entre le changement de base d'Arthur-Clozel et l'application de changement de base de Shintani (cf. [Shi76]) pour les représentations supercuspidales, voir Corollary 3.6.3 et Corollary 3.6.9.

La seconde application concerne les représentations  $l$ -modulaires. Pour  $l$  un nombre premier différent de  $p$ , nous fixons un isomorphisme algébrique  $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ , ainsi toutes les représentations ci-dessus peuvent être réalisées comme des représentations sur  $\overline{\mathbb{Q}}_l$ . Nous supposons en outre que  $\pi_0$  est entière, c'est-à-dire qu'il provient d'une représentation sur un  $\overline{\mathbb{Z}}_l$ -réseau par extension des scalaires. En utilisant les deux théorèmes, nous pouvons prouver que le changement de base et l'induction automorphe cycliques modérément ramifiés sont compatibles avec la réduction modulo  $l$  pour les représentations supercuspidales. Plus précisément, pour une telle  $\pi_0$  avec son changement de base  $\pi$  supercuspidal,  $\pi$  est également entière, et si nous changeons  $\pi_0$  par une autre représentation supercuspidale entière avec sa modulo  $l$  réduction  $r_l(\pi_0)$  inchangée comme une représentation cuspidale sur  $\overline{\mathbb{F}}_l$ , alors la  $\pi$  correspondante est encore supercuspidale entière avec sa modulo  $l$  réduction  $r_l(\pi)$  inchangée. De même pour une telle  $\pi_0$  comme l'induction automorphe de  $\pi'$  comme une représentation supercuspidale entière,  $\pi_0$  est également entière, et si nous changeons  $\pi'$  par une autre représentation supercuspidale entière avec sa modulo  $l$  réduction  $r_l(\pi')$  inchangée comme une représentation cuspidale sur  $\overline{\mathbb{F}}_l$ , alors la  $\pi_0$  correspondante est supercuspidale entière avec sa modulo  $l$  réduction  $r_l(\pi_0)$  inchangée. La preuve est directe et ne sera pas donnée dans ce chapitre, mais les lecteurs peuvent consulter [BH14a] pour une idée similaire.

L'application finale concerne le calcul du caractère  $\mu_\varphi$  lié au théorème de comparaison dans [BH14b] mentionné dans la dernière sous-section. La stratégie est de considérer un certain changement de base non ramifiée, puis de comparer les caractères correspondants "mu" liés aux deux corps de base, qui a déjà été utilisé pour le cas essentiellement modéré dans [BH05a]. Pour cela, nous avons besoin d'étudier  ${}_b\phi_{\theta_0}^{F/F_0}$  dans le cas où  $F/F_0$  est non-ramifiée.

**Théorème 0.4.3** (Voir Theorem 3.9.1). *Quand  $F/F_0$  est non-ramifiée, le caractère  ${}_b\phi_{\theta_0}^{F/F_0}$  est non-ramifié, et  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = (-1)^{(t-1)(r-1)}$ , où  $\varpi_{T_0}$  désigne une uniformisante de  $T_m^\times$ , et  $K_0$  désigne la sous-extension maximale de  $T_{0,m}$  sur  $F_0$ , et  $t = [T_{0,m} : K_0]$  et  $[E_0 : T_0] = p^s$ .*

En utilisant un cas particulier du théorème, c'est-à-dire la proposition 3.9.9 nous pouvons mettre à jour les valeurs de  $\mu_\varphi$  qui seront discutées dans la dernière section. Notre résultat est évidemment incomplet et pas assez satisfaisant.

### 0.4.3 La structure du chapitre 3

Nous esquissons la structure du chapitre 3. Les sections 1-4 sont des préliminaires, y compris une brève introduction et un résumé de la théorie des types simples, des signes symplectiques, du changement de base et de l'induction automorphe respectivement. Après la première discussion élémentaire de la section 5, dans la section 6 nous formulons notre construction algébrique du changement de base cyclique et de l'induction automorphe modérément ramifiés en suivant l'esquisse mentionnée dans la dernière sous-section et énonçons le théorème 0.4.1 et le théorème 0.4.2. Mais la construction des représentations de Heisenberg complètes correspondantes reste à faire jusqu'à la section 7, dont la stratégie repose sur l'idée d'une série de résultats de Bushnell-Henniart qui y seront rappelés et reformulés. La preuve du théorème 0.4.1 et théorème 0.4.2 sera donnée dans la section 8 qui semble étonnamment simple, qui repose en fait sur deux ingrédients hautement non triviaux: la functorialité de Langlands locale pour le changement de base et l'induction automorphe, et le théorème de comparaison

de Bushnell-Henniart. La section 9 est consacrée à prouver le théorème [0.4.3](#) et la section 10 est son application pour calculer le caractère  $\mu_\varphi$  lié au théorème de comparaison.

L'auteur tient à remercier Colin J. Bushnell et Guy Henniart pour leur énorme influence sur l'auteur et sur ce chapitre. En fait, il est préférable de considérer cette partie comme une continuation (maladroite) de leur travail plutôt que comme un travail indépendant, puisque presque toutes les idées et techniques importantes proviennent de leurs articles énumérés ci-dessus. De plus, l'auteur tient à les remercier pour leurs généreux encouragements, qui ont en effet beaucoup aidé un jeune doctorant qui n'était pas confiant et même était méfiant à la nécessité de son travail.

# Chapter 1

## Problem of distinction related to unitary subgroups of $\mathrm{GL}_n(F)$ and $l$ -modular base change lift

### 1.1 Notation and basic definitions

Let  $F/F_0$  be a quadratic extension of non-archimedean locally compact fields of residue characteristic  $p \neq 2$  and let  $\sigma$  be the unique non-trivial involution in the Galois group. Write  $\mathfrak{o}_F$  and  $\mathfrak{o}_{F_0}$  for the ring of integers of  $F$  and  $F_0$  and write  $\mathbf{k}$  and  $\mathbf{k}_0$  for the residue field of  $F$  and  $F_0$  respectively. The involution  $\sigma$  induces a  $\mathbf{k}_0$ -automorphism of  $\mathbf{k}$  generating  $\mathrm{Gal}(\mathbf{k}/\mathbf{k}_0)$ , still denoted by  $\sigma$ .

Let  $R$  be an algebraically closed field of characteristic  $l \geq 0$  different from  $p$ . If  $l > 0$ , then we are in the “modular case”.

We fix a character

$$\psi_0 : F_0 \rightarrow R^\times$$

trivial on the maximal ideal of  $\mathfrak{o}_{F_0}$  but not on  $\mathfrak{o}_{F_0}$ , and we define  $\psi = \psi_0 \circ \mathrm{tr}_{F/F_0}$ .

Let  $G$  be the locally profinite group  $\mathrm{GL}_n(F)$  with  $n \geq 1$ , equipped with the involution  $\sigma$  acting componentwise. Let  $\varepsilon$  be a *hermitian matrix* in  $M_n(F)$ , which means that  $\varepsilon^* = \varepsilon$ . Here  $x^* := \sigma({}^t x)$  for any  $x \in M_n(F)$  with  ${}^t$  denoting the transpose operator. Sometimes we write  $\sigma_1(x) := x^*$  for any  $x \in M_n(F)$  to emphasize that  $\sigma_1$  is an anti-involution on  $M_n(F)$  extending  $\sigma$ . For  $\varepsilon$  hermitian and  $g \in G$ , we define  $\tau_\varepsilon(g) = \varepsilon \sigma({}^t g^{-1}) \varepsilon^{-1}$ , called the *unitary involution* corresponding to  $\varepsilon$ . For  $\tau = \tau_\varepsilon$  a fixed unitary involution, we denote by  $G^\tau$  the corresponding *unitary subgroup*, which consists of the elements of  $G$  fixed by  $\tau$ .

By *representations* of a locally profinite group, we always mean smooth representations on an  $R$ -module. Given a representation  $\pi$  of a closed subgroup  $H$  of  $G$ , we write  $\pi^\vee$  for the smooth contragredient of  $\pi$ . We write  $\pi^\sigma$  and  $\pi^\tau$  for the representations  $\pi \circ \sigma$  and  $\pi \circ \tau$  of groups  $\sigma(H)$  and  $\tau(H)$  respectively. We say that  $\pi$  is  $\tau$ -*selfdual* if  $H$  is  $\tau$ -stable and  $\pi^\tau$  is isomorphic to  $\pi^\vee$ . We say that  $\pi$  is  $\sigma$ -*invariant* if  $H$  is  $\sigma$ -stable and  $\pi^\sigma$  is isomorphic to  $\pi$ . For  $g \in G$ , we write  $H^g = \{g^{-1}hg | h \in H\}$  as a closed subgroup and we write  $\pi^g : x \mapsto \pi(gxg^{-1})$  as a representation of  $H^g$ .

For  $\mathfrak{a}$  an  $\mathfrak{o}_F$ -subalgebra of  $M_n(F)$  and  $\tau = \tau_\varepsilon$  a unitary involution, we denote by

$$\tau(\mathfrak{a}) := \sigma_\varepsilon(\mathfrak{a}) = \{\sigma_\varepsilon(x) | x \in \mathfrak{a}\}$$

an  $\mathfrak{o}_F$ -subalgebra of  $M_n(F)$ , where  $\sigma_\varepsilon(x) := \varepsilon \sigma({}^t x) \varepsilon^{-1}$  is an anti-involution for any  $x \in M_n(F)$ . We say that  $\mathfrak{a}$  is  $\tau$ -*stable* if  $\tau(\mathfrak{a}) = \mathfrak{a}$ . Moreover for  $g \in G$ , we obtain

$$\tau(\mathfrak{a})^g = g^{-1} \sigma_\varepsilon(\mathfrak{a}) g = \sigma_\varepsilon(\sigma_\varepsilon(g) \mathfrak{a} \sigma_\varepsilon(g^{-1})) = \sigma_\varepsilon(\tau(g)^{-1} \mathfrak{a} \tau(g)) = \tau(\mathfrak{a}^{\tau(g)})$$

In other words, the notation  $\tau(\mathfrak{a})$  is compatible with  $G$ -conjugacy.

For  $\tau$  a unitary involution and  $\pi$  a representation of  $H$  as above, we say that  $\pi$  is  $H \cap G^\tau$ -*distinguished*, or just *distinguished*, if the space  $\text{Hom}_{H \cap G^\tau}(\pi, 1)$  is non-zero.

An irreducible representation of  $G$  is called *cuspidal* if it doesn't occur as a subrepresentation of a proper parabolically induced representation. It is called *supercuspidal* if it doesn't occur as a subquotient of a proper parabolically induced representation.

## 1.2 Hermitian matrices and unitary groups

We make use of this subsection to introduce basic knowledge of hermitian matrices and unitary groups. The references will be [HM98] and [Jac62].

Let  $E/E_0$  be a quadratic extension of non-archimedean locally compact fields which are algebraic extension of  $F$  and  $F_0$  respectively. Write  $\mathfrak{o}_E$  for the ring of integers of  $E$  and  $\mathfrak{o}_{E_0}$  for that of  $E_0$ . Let  $\sigma' \in \text{Gal}(E/E_0)$  be the unique non-trivial involution in the Galois group. For  $\varepsilon' \in \text{GL}_m(E)$ , just as in the last subsection, we say that  $\varepsilon'$  is a hermitian matrix if  $(\varepsilon')^* = \varepsilon'$ , where we consider  $(\cdot)^*$  as before with  $n, F, F_0, \sigma$  replaced by  $m, E, E_0, \sigma'$  respectively. Write  $\varpi_E$  for a uniformizer of  $E$  such that

$$\sigma'(\varpi_E) = \begin{cases} \varpi_E & \text{if } E/E_0 \text{ is unramified,} \\ -\varpi_E & \text{if } E/E_0 \text{ is ramified.} \end{cases}$$

Let  $\mathcal{X}$  denote the set of all the hermitian matrices. The group  $G$  acts on  $\mathcal{X}$  by  $g \cdot x = gxg^*$ . We have the following proposition:

**Proposition 1.2.1** ([Jac62], Theorem 3.1). *There are exactly two  $\text{GL}_m(E)$ -orbits of  $\mathcal{X}$  with respect to the action given above. Furthermore, the elements in each orbit are exactly determined by the classes of their determinants in  $E_0^\times / \text{N}_{E/E_0}(E^\times)$ .*

We may also consider the  $\text{GL}_m(\mathfrak{o}_E)$ -orbits of  $\mathcal{X}$ . We consider sequences  $\alpha = (\alpha_1, \dots, \alpha_r)$  of certain triples  $\alpha_i = (a_i, m_i, \delta_i)$ , such that  $a_1 > \dots > a_r$  is a decreasing sequence of integers, and  $m_1 + \dots + m_r = m$  is a partition of  $m$  by positive integers, and  $\delta_1, \dots, \delta_r$  are elements of  $E$  such that:

- (1) If  $E/E_0$  is unramified, then  $\delta_i = 1$ ;
- (2) If  $E/E_0$  is ramified and  $a_i$  is odd, then  $\delta_i = 1$  and  $m_i$  is even;
- (3) If  $E/E_0$  is ramified and  $a_i$  is even, then  $\delta_i$  is either 1 or  $\epsilon$ , with  $\epsilon \in \mathfrak{o}_{E_0}^\times \setminus \text{N}_{E/E_0}(\mathfrak{o}_E^\times)$  fixed.

Let  $\mathcal{A}$  be the set of all sequences  $\alpha$  satisfying these requirements. For each  $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathcal{A}$ , we introduce a hermitian matrix  $\varpi_E^\alpha = \varpi_E^{\alpha_1} \oplus \dots \oplus \varpi_E^{\alpha_r}$ , where  $\varpi_E^{\alpha_i} \in \text{GL}_{m_i}(E)$  is a hermitian matrix, such that:

- (i) In the case (1),  $\varpi_E^{\alpha_i} = \varpi_E^{\alpha_i} I_{m_i}$ ;
- (ii) In the case (2),  $\varpi_E^{\alpha_i} = \varpi_E^{\alpha_i} J_{m_i/2}$ , where  $J_{m_i/2} = \begin{pmatrix} 0 & I_{m_i/2} \\ -I_{m_i/2} & 0 \end{pmatrix}$ ;
- (iii) In the case (3),  $\varpi_E^{\alpha_i} = \varpi_E^{\alpha_i} \text{diag}(1, \dots, 1, \delta_i)$ , where  $\text{diag}(*, \dots, *)$  denotes the diagonal matrix with corresponding diagonal elements.

We state the following proposition which classifies all the  $\text{GL}_m(\mathfrak{o}_E)$ -orbits of  $\mathcal{X}$ .

**Proposition 1.2.2** ([Jac62], Theorem 7.1, Theorem 8.2). *Each class of the  $\text{GL}_m(\mathfrak{o}_E)$ -orbits of  $\mathcal{X}$  contains a unique representative of the form  $\varpi_E^\alpha$  for a certain  $\alpha \in \mathcal{A}$ .*

Now we study unitary groups. For  $\varepsilon' \in \mathcal{X}$ , we denote by  $U_m(\varepsilon')$  the unitary group consisting of those  $g \in \text{GL}_m(E)$  such that  $g\varepsilon'g^* = \varepsilon'$ . We say that two unitary groups are *equivalent* if and only if they are conjugate by some  $g \in G$ . Since it is easy to check that  $gU_m(\varepsilon')g^{-1} = U_m(g\varepsilon'g^*)$ , by Proposition [1.2.1], there are at most two equivalence classes of unitary groups, which are represented

by  $U_m(E/E_0) := U_m(I_m)$  and  $U'_m(E/E_0) := U_m(\varepsilon)$  for  $\varepsilon = \text{diag}(1, \dots, 1, \varepsilon)$ , where  $\varepsilon \in E_0^\times \setminus N_{E/E_0}(E^\times)$  is fixed.

**Remark 1.2.3.** *We list the following result for completeness:  $U_m(E/E_0)$  is equivalent to  $U'_m(E/E_0)$  if and only if  $m$  is odd. Since we will never use it in the future, we omit the proof.*

**Remark 1.2.4.** *In the future, we only consider the following two cases. First, we consider  $E = F$ ,  $E_0 = F_0$ ,  $m = n$  and  $\sigma' = \sigma$ . For any two unitary involutions with the corresponding hermitian matrices in the same  $\text{GL}_n(F)$ -orbit, we already showed that the corresponding two unitary groups are equivalent. Since distinction is a property invariant up to equivalence of unitary groups, we may choose a hermitian matrix in its  $G$ -orbit such that the corresponding unitary involution  $\tau$  is simple enough to simplify the problem. Secondly, we consider  $E$  as a finite field extension of  $F$  determined by a cuspidal representation  $\pi$  such that  $n = m[E : F]$ . We will find out that if  $\pi^\sigma \cong \pi$ , then we may find an involution  $\sigma'$  on  $E$  such that  $E_0 = E^{\sigma'}$  and  $\sigma'|_F = \sigma$ . So we may make use of the propositions in this subsection to study hermitian matrices and unitary groups of  $\text{GL}_m(E)$ .*

### 1.3 Preliminaries on simple types

In this section, we recall the main results we will need on simple strata, characters and types [BK93], [BH96], [BH14b], [MS14b]. We mainly follow the structure of [AKM<sup>+</sup>19] and [Séc19].

#### 1.3.1 Simple strata and characters

Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $M_n(F)$  for a certain  $n \geq 1$ . Recall that  $\mathfrak{a}$  is a hereditary order of  $M_n(F)$  and  $\beta$  is in  $G = \text{GL}_n(F)$  such that:

- (1) the  $F$ -algebra  $E = F[\beta]$  is a field with degree  $d$  over  $F$ ;
- (2)  $E^\times$  normalizes  $\mathfrak{a}^\times$ .

The centralizer of  $E$  in  $M_n(F)$ , denoted by  $B$ , is an  $E$ -algebra isomorphic to  $M_m(E)$  with  $n = md$ . The intersection  $\mathfrak{b} := \mathfrak{a} \cap B$  is a hereditary order of  $B$ .

We denote by  $\mathfrak{p}_\mathfrak{a}$  the Jacobson radical of  $\mathfrak{a}$ , and  $U^1(\mathfrak{a})$  the compact open pro- $p$ -subgroup  $1 + \mathfrak{p}_\mathfrak{a}$  of  $G$ . Similarly, we denote by  $\mathfrak{p}_\mathfrak{b}$  the Jacobson radical of  $\mathfrak{b}$  and  $U^1(\mathfrak{b})$  the compact open pro- $p$ -subgroup  $1 + \mathfrak{p}_\mathfrak{b}$  of  $B^\times$ . For any  $x \in B^\times$ , we have ([BK93], Theorem 1.6.1)

$$U^1(\mathfrak{a})xU^1(\mathfrak{a}) \cap B^\times = U^1(\mathfrak{b})xU^1(\mathfrak{b}). \quad (1.3.1)$$

Associated with  $[\mathfrak{a}, \beta]$ , there are open compact subgroups

$$H^1(\mathfrak{a}, \beta) \subset J^1(\mathfrak{a}, \beta) \subset J(\mathfrak{a}, \beta)$$

of  $\mathfrak{a}^\times$  and a finite set  $\mathcal{C}(\mathfrak{a}, \beta)$  of *simple characters* of  $H^1(\mathfrak{a}, \beta)$  depending on the choice of  $\psi$ . We denote by  $\mathbf{J}(\mathfrak{a}, \beta)$  a subgroup of  $G$  generated by  $J(\mathfrak{a}, \beta)$  and the normalizer of  $\mathfrak{b}^\times$  in  $B^\times$ .

The above definition excludes the “null” case, which we explain here. In this case for a simple stratum  $[\mathfrak{a}, \beta]$ , conventionally we write  $\beta = 0$ ,  $E = F$ ,  $A = B$ ,  $\mathfrak{a} = \mathfrak{b}$  and  $H^1(\mathfrak{a}, \beta) = J^1(\mathfrak{a}, \beta) = 1 + \mathfrak{p}_\mathfrak{a}$ . Moreover, the set  $\mathcal{C}(\mathfrak{a}, \beta)$  is a singleton consisting of the trivial character of  $H^1(\mathfrak{a}, \beta)$ . Later on all the simple strata we consider should also include this case.

**Proposition 1.3.1** ([Séc19], Proposition 5.1). *We have the following properties:*

- (1) The group  $J(\mathfrak{a}, \beta)$  is the unique maximal compact subgroup of  $\mathbf{J}(\mathfrak{a}, \beta)$ ;
- (2) The group  $J^1(\mathfrak{a}, \beta)$  is the unique maximal normal pro- $p$ -subgroup of  $J(\mathfrak{a}, \beta)$ ;
- (3) The group  $J(\mathfrak{a}, \beta)$  is generated by  $J^1(\mathfrak{a}, \beta)$  and  $\mathfrak{b}^\times$ , and we have

$$J(\mathfrak{a}, \beta) \cap B^\times = \mathfrak{b}^\times, J^1(\mathfrak{a}, \beta) \cap B^\times = U^1(\mathfrak{b}); \quad (1.3.2)$$



- (4) The normalizer of any simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  in  $G$  is equal to  $\mathbf{J}(\mathfrak{a}, \beta)$ ;  
(5) The intertwining set of any  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  in  $G$ , which we denote by  $I_G(\theta)$ , is equal to

$$J^1(\mathfrak{a}, \beta)B^\times J^1(\mathfrak{a}, \beta) = J(\mathfrak{a}, \beta)B^\times J(\mathfrak{a}, \beta).$$

**Remark 1.3.2.** We write for short  $J$ ,  $J^1$ ,  $H^1$  for  $J(\mathfrak{a}, \beta)$ ,  $J^1(\mathfrak{a}, \beta)$ ,  $H^1(\mathfrak{a}, \beta)$  respectively if  $\mathfrak{a}$  and  $\beta$  are clear to us.

When  $\mathfrak{b}$  is a maximal order in  $B$ , we call the simple stratum  $[\mathfrak{a}, \beta]$  and the simple characters in  $\mathcal{C}(\mathfrak{a}, \beta)$  *maximal*. In this case we may find an isomorphism of  $E$ -algebras  $B \cong M_m(E)$  which identifies  $\mathfrak{b}$  with the standard maximal order, and moreover we have group isomorphisms

$$J(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta) \cong \mathfrak{b}^\times/U^1(\mathfrak{b}) \cong \mathrm{GL}_m(\mathfrak{l}), \quad (1.3.3)$$

where  $\mathfrak{l}$  denotes the residue field of  $E$ .

### 1.3.2 Simple types and cuspidal representations

A pair  $(\mathbf{J}, \Lambda)$ , called an *extended maximal simple type* in  $G$  (we always write *simple type* for short), is made of a subgroup  $\mathbf{J}$  of  $G$  which is open and compact modulo centre, and an irreducible representation  $\Lambda$  of  $\mathbf{J}$ . It has been constructed in [BK93] in the characteristic 0 case and in [Vig96], [MS14b] in the modular case.

Given a simple type  $(\mathbf{J}, \Lambda)$  in  $G$ , there are a maximal simple stratum  $[\mathfrak{a}, \beta]$  in  $M_n(F)$  and a maximal simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , such that  $\mathbf{J}(\mathfrak{a}, \beta) = \mathbf{J}$  and  $\theta$  is contained in the restriction of  $\Lambda$  to  $H^1(\mathfrak{a}, \beta)$ . Such a character  $\theta$  is said to be *attached* to  $\Lambda$ . By [BK93], Proposition 5.1.1 (or [MS14b], Proposition 2.1 in the modular case), the group  $J^1(\mathfrak{a}, \beta)$  has, up to isomorphism, a unique irreducible representation  $\eta$  whose restriction to  $H^1(\mathfrak{a}, \beta)$  contains  $\theta$ . Such a representation  $\eta$ , called the *Heisenberg representation* associated to  $\theta$ , has the following properties:

- (1) the restriction of  $\eta$  to  $H^1(\mathfrak{a}, \beta)$  is made of  $(J^1(\mathfrak{a}, \beta) : H^1(\mathfrak{a}, \beta))^{1/2}$  copies of  $\theta$ . Here  $(J^1(\mathfrak{a}, \beta) : H^1(\mathfrak{a}, \beta))^{1/2}$  is a power of  $p$ ;
- (2) the direct sum of  $(J^1(\mathfrak{a}, \beta) : H^1(\mathfrak{a}, \beta))^{1/2}$  copies of  $\eta$ , which we denote by  $\eta^{(J^1(\mathfrak{a}, \beta) : H^1(\mathfrak{a}, \beta))^{1/2}}$ , is isomorphic to  $\mathrm{Ind}_{H^1}^{J^1} \theta$ ;
- (3) the representation  $\eta$  extends to  $\mathbf{J}$ ;
- (4) the intertwining set of  $\eta$ , which we denote by  $I_G(\eta)$ , equals  $I_G(\theta)$ ;
- (5) for  $h \in I_G(\eta)$ , we have  $\dim_R(\mathrm{Hom}_{J^1 \cap J^{1h}}(\eta^h, \eta)) = 1$ .

For any representation  $\kappa$  of  $\mathbf{J}$  extending  $\eta$ , there exists a unique irreducible representation  $\rho$  of  $\mathbf{J}$  trivial on  $J^1(\mathfrak{a}, \beta)$  such that  $\Lambda \cong \kappa \otimes \rho$ . Through (1.3.3), the restriction of  $\rho$  to  $J = J(\mathfrak{a}, \beta)$  identifies with the inflation of a cuspidal representation of  $\mathrm{GL}_m(\mathfrak{l})$ .

**Remark 1.3.3.** Recall that in [BK93], Bushnell and Kutzko also assume  $\kappa^0 = \kappa|_{J(\mathfrak{a}, \beta)}$  to be a so called beta-extension, which means that:

- (1)  $\kappa^0$  is an extension of  $\eta$ ;
- (2) if we denote by  $I_G(\kappa^0)$  the intertwining set of  $\kappa^0$ , then  $I_G(\kappa^0) = I_G(\eta) = I_G(\theta)$ .

However in our case, since  $\mathrm{GL}_m(\mathfrak{l})$  is not isomorphic to  $\mathrm{GL}_2(\mathbb{F}_2)$  ( $p \neq 2$ ), any character of  $\mathrm{GL}_m(\mathfrak{l})$  factors through the determinant. It follows that any representation of  $J$  extending  $\eta$  is a beta-extension. So finally our consideration of  $\kappa^0$  coincides with the original assumption of Bushnell and Kutzko.

We now give the classification of irreducible cuspidal representations of  $G$  in terms of simple types (see [BK93], 6.2, 8.4 and [MS14b], Section 3 in the modular case).

**Proposition 1.3.4** ([BK93], [MS14b]). *Let  $\pi$  be a cuspidal representation of  $G$ .*

(1) *There is a simple type  $(\mathbf{J}, \Lambda)$  such that  $\Lambda$  is a subrepresentation of the restriction of  $\pi$  to  $\mathbf{J}$ . It is unique up to  $G$ -conjugacy;*

(2) *Compact induction  $c\text{-Ind}_G^G$  gives a bijection between the  $G$ -conjugacy classes of simple types and the isomorphism classes of cuspidal representations of  $G$ .*

### 1.3.3 Endo-classes, tame parameter fields and tame lifting

In this subsection, we introduce the concepts of endo-classes, tame parameter fields and tame lifting. The main references will be [BK93], [BH96] and [BH14b].

For  $[\mathfrak{a}, \beta]$  a simple stratum in  $M_n(F)$  and  $[\mathfrak{a}', \beta']$  a simple stratum in  $M_{n'}(F)$  with  $n, n' \geq 1$ , if we have an isomorphism of  $F$ -algebras  $\phi : F[\beta] \rightarrow F[\beta']$  such that  $\phi(\beta) = \beta'$ , then there exists a canonical bijection

$$t_{\mathfrak{a}, \mathfrak{a}'}^{\beta, \beta'} : \mathcal{C}(\mathfrak{a}, \beta) \rightarrow \mathcal{C}(\mathfrak{a}', \beta'),$$

called the *transfer map* (see [BK93], Theorem 3.6.14).

Now let  $[\mathfrak{a}_1, \beta_1]$  and  $[\mathfrak{a}_2, \beta_2]$  be simple strata of  $M_{n_1}(F)$  and  $M_{n_2}(F)$  respectively with  $n_1, n_2 \geq 1$ . We call two simple characters  $\theta_1 \in \mathcal{C}(\mathfrak{a}_1, \beta_1)$  and  $\theta_2 \in \mathcal{C}(\mathfrak{a}_2, \beta_2)$  *endo-equivalent*, if there are simple strata  $[\mathfrak{a}', \beta'_1]$  and  $[\mathfrak{a}', \beta'_2]$  of  $M_{n'}(F)$  for some  $n' \geq 1$  such that  $\theta_1$  and  $\theta_2$  transfer to two simple characters  $\theta'_1 \in \mathcal{C}(\mathfrak{a}', \beta'_1)$  and  $\theta'_2 \in \mathcal{C}(\mathfrak{a}', \beta'_2)$  respectively which intertwine (or by [BK93], Theorem 3.5.11 which are  $\text{GL}_{n'}(F)$ -conjugate). This defines an equivalence relation on

$$\bigcup_{[\mathfrak{a}, \beta]} \mathcal{C}(\mathfrak{a}, \beta),$$

where the union runs over all simple strata in  $M_n(F)$  for all  $n \geq 1$  (see [BH96], section 8). An equivalence class for this equivalence relation is called an *endo-class*.

For  $\pi$  a cuspidal representation of  $G = \text{GL}_n(F)$ , there exist a simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  contained in  $\pi$ . The set of simple characters  $\theta$  contained in  $\pi$  constitutes a  $G$ -conjugacy class, thus those simple characters are endo-equivalent. So we may denote by  $\Theta_\pi$  the endo-class of  $\pi$  which is the endo-class determined by any  $\theta$  contained in  $\pi$ .

Given  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , the degree of  $E/F$ , its ramification index and its residue class degree depend only on the endo-class of  $\theta$ . They are called the degree, ramification index and residue class degree of this endo-class. Although the field extension  $E/F$  is not uniquely determined, its maximal tamely ramified subextension is uniquely determined by the endo-class of  $\theta$  up to  $F$ -isomorphisms. This field is called a *tame parameter field* of the endo-class (see [BH14b], 2.2, 2.4).

We denote by  $\mathcal{E}(F)$  the set of endo-classes of simple characters over  $F$ . Given a finite tamely ramified extension  $T$  of  $F$ , we have a surjection

$$\mathcal{E}(T) \rightarrow \mathcal{E}(F)$$

with finite fibers, which is called *restriction map* (see [BH14b], 2.3). Given  $\Theta \in \mathcal{E}(F)$ , the endo-classes  $\Psi \in \mathcal{E}(T)$  restricting to  $\Theta$  are called the  $T/F$ -lifts of  $\Theta$ . If  $\Theta$  has a tame parameter field  $T$ , then  $\text{Aut}_F(T)$  acts faithfully and transitively on the set of  $T/F$ -lifts of  $\Theta$  (see [BH14b], 2.3, 2.4).

Let  $[\mathfrak{a}, \beta]$  be a simple stratum and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character. Let  $T$  be the maximal tamely ramified extension of  $F$  in  $E$ . Let  $\Theta$  be the endo-class of  $\theta$ , then  $T$  is a tame parameter field of  $\Theta$ . Let  $C \cong M_{n/t}(T)$  denote the centralizer of  $T$  in  $M_n(F)$ , where  $t = [T : F]$ . The intersection  $\mathfrak{c} = \mathfrak{a} \cap C$  is an order in  $C$  which gives rise to a simple stratum  $[\mathfrak{c}, \beta]$ . The restriction of  $\theta$  to  $H^1(\mathfrak{c}, \beta)$ , denoted by  $\theta_T$ , is a simple character associated to this simple stratum, called the *interior  $T/F$ -lift* of  $\theta$ . Its endo-class, denoted by  $\Psi$ , is a  $T/F$ -lift of  $\Theta$ . For the origin and details of the construction of  $\Psi$  by using interior  $T/F$ -lift of  $\theta$ , see [BH96].

We may change our choice of simple stratum  $[\mathfrak{a}, \beta]$  but fix  $T \hookrightarrow M_n(F)$  unchanged, then the map

$$\mathfrak{a} \mapsto \mathfrak{a} \cap C$$

is injective from the set of hereditary orders of  $M_n(F)$  normalized by  $T^\times$  to the set of hereditary orders of  $C$  (see [BH96], section 2). For  $[\mathfrak{a}, \beta_1], [\mathfrak{a}_2, \theta_2]$  two simple strata, and  $\theta_1 \in \mathcal{C}(\mathfrak{a}, \beta_1), \theta_2 \in \mathcal{C}(\mathfrak{a}, \beta_2)$  two simple characters, such that  $\theta_1$  and  $\theta_2$  have the same tame parameter field  $T$ , if

$$\mathcal{C}(\mathfrak{c}, \beta_1) = \mathcal{C}(\mathfrak{c}, \beta_2) \quad \text{and} \quad (\theta_1)_T = (\theta_2)_T,$$

then (see [BH96], Theorem 7.10, Theorem 7.15)

$$\mathcal{C}(\mathfrak{a}, \beta_1) = \mathcal{C}(\mathfrak{a}, \beta_2) \quad \text{and} \quad \theta_1 = \theta_2.$$

In particular, when  $\beta_1 = \beta_2 = \beta$ , the interior  $T/F$ -lift is injective from  $\mathcal{C}(\mathfrak{a}, \beta)$  to  $\mathcal{C}(\mathfrak{c}, \beta)$ .

### 1.3.4 Supercuspidal representations

Let  $\pi$  be a cuspidal representation of  $G$ . By Proposition [1.3.4], it contains a simple type  $(\mathbf{J}, \Lambda)$ . Fix a maximal simple stratum  $[\mathfrak{a}, \beta]$  such that  $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ , and write  $\Lambda = \kappa \otimes \rho$  as in subsection [1.3.2]. Let  $\bar{\rho}$  be the cuspidal representation of  $J/J^1 \cong \mathrm{GL}_m(\mathfrak{l})$  whose inflation equals  $\rho|_J$ . We have the following proposition:

**Proposition 1.3.5** ([MS14a], Proposition 6.10). *The representation  $\pi$  is supercuspidal if and only if  $\bar{\rho}$  is supercuspidal.*

## 1.4 One direction of Theorem 0.2.1 for a supercuspidal representation

Let  $G = \mathrm{GL}_n(F)$  and let  $G^\tau$  be the unitary group corresponding to a unitary involution  $\tau$ . We state the following theorem which is well-known when  $R = \mathbb{C}$  and  $\mathrm{char}(F) = 0$  (see for example [HM02a], section 4, corollary or more ancient paper [HLR86] which illustrates the idea).

**Theorem 1.4.1.** *Let  $\pi$  be a supercuspidal representation of  $G$ . If  $\pi$  is distinguished by  $G^\tau$ , then  $\pi$  is  $\sigma$ -invariant.*

Before proving Theorem [1.4.1], we state a useful lemma which will be used not only in the proof of the theorem, but also in the latter sections.

**Lemma 1.4.2.** *For  $\delta$  a unitary involution on  $G$  and for  $(\mathbf{J}, \Lambda)$  a simple type in  $G$ , we have  $\mathbf{J} \cap G^\delta = J \cap G^\delta$ .*

*Proof.* For  $x \in \mathbf{J} \cap G^\delta$ , we have  $\delta(x) = x$  which implies that  $\sigma(\det(x))\det(x) = 1$ , where we denote by  $\det(\cdot)$  the determinant function defined on  $G$ . Thus we have  $\det(x) \in \mathfrak{o}_F^\times$ . Since  $\mathbf{J} = E^\times J$ , we get  $x \in \mathfrak{o}_E^\times J \cap G^\delta = J \cap G^\delta$ . Since  $x$  is arbitrary, we finish the proof. □

Moreover, we need the following lemma which says that the properties of distinction and  $\sigma$ -invariance are maintained up to change of base fields.

**Lemma 1.4.3.** *Let  $R_1 \hookrightarrow R_2$  be a fixed embedding of two algebraically closed fields of characteristic  $l \geq 0$ . Let  $\pi_0$  be a supercuspidal representation of  $G$  over  $R_1$ . Let  $\pi = \pi_0 \otimes_{R_1} R_2$  be the corresponding representation of  $G$  over  $R_2$ . Then:*

- (1)  $\pi_0$  is distinguished by  $G^\tau$  if and only if  $\pi$  is distinguished by  $G^\tau$ ;
- (2)  $\pi_0^\sigma \cong \pi_0$  if and only if  $\pi^\sigma \cong \pi$ .

*Proof.* For (1), let  $(\mathbf{J}, \Lambda_0)$  be a simple type of  $\pi_0$ . Then  $(\mathbf{J}, \Lambda) := (\mathbf{J}, \Lambda_0 \otimes_{R_1} R_2)$  is a simple type of  $\pi$  and thus  $\pi$  is also supercuspidal. Using the Frobenius reciprocity and the Mackey formula<sup>[1]</sup>, we have

$$\mathrm{Hom}_{R_1[G^\tau]}(\pi_0, 1) \neq 0 \iff \text{There exists } g \in G \text{ such that } \mathrm{Hom}_{R_1[\mathbf{J}^g \cap G^\tau]}(\Lambda_0^g, 1) \neq 0$$

and

$$\mathrm{Hom}_{R_2[G^\tau]}(\pi, 1) \neq 0 \iff \text{There exists } g \in G \text{ such that } \mathrm{Hom}_{R_2[\mathbf{J}^g \cap G^\tau]}(\Lambda^g, 1) \neq 0$$

By Lemma 1.4.2,  $\mathbf{J}^g \cap G^\tau = J^g \cap G^\tau$  is a compact group, and  $\Lambda_0^g$  is a representation of finite dimension. Thus

$$\mathrm{Hom}_{R_1[\mathbf{J}^g \cap G^\tau]}(\Lambda_0^g, 1) \otimes_{R_1} R_2 \cong \mathrm{Hom}_{R_2[J^g \cap G^\tau]}(\Lambda^g, 1)$$

which finishes the proof of (1). For (2), from [Vig96], Chapitre I, 6.13 we know that  $\pi_0$  is isomorphic to  $\pi_0^\sigma$  if and only if their trace characters are equal up to a scalar in  $R_1^\times$ , which works similarly for  $\pi$  and  $\pi^\sigma$ . Since the trace characters of  $\pi_0$  and  $\pi$  are equal up to the change of scalars, which works similarly for  $\pi_0^\sigma$  and  $\pi^\sigma$ , we finish the proof of (2). □

*Proof of Theorem 1.4.1.* First we consider  $R = \mathbb{C}$ . For  $\mathrm{char}(F) = 0$ , it is a standard result proved by using global method ([HM02a], section 4, Corollary). Especially, their result is based on the globalization theorem, saying a distinguished  $\pi$  under our settings can be realized as a local component of a cuspidal automorphic representation  $\Pi$  of  $\mathrm{GL}_n(\mathbb{A}_K)$ , which is distinguished by a unitary subgroup of  $\mathrm{GL}_n(\mathbb{A}_K)$  with respect to a quadratic extension of number fields  $K/K_0$  (see *ibid.*, Theorem 1). If  $\mathrm{char}(F) > 0$ , in order to use the proof of Hakim-Murnaghan, we only need a variant of globalization theorem for characteristic positive case. Fortunately, Gan-Lomelí already built up such kind of result for general reductive groups over function fields and locally compact fields of characteristic positive (see [GL18], Theorem 1.3). Following their settings, we choose the reductive group  $H$  to be  $R_{K/K_0}(\mathrm{GL}_n(K))$ , where  $K/K_0$  is a quadratic extension of function fields, and  $R_{K/K_0}$  is the Weil restriction. We choose  $V$  to be  $M_n(K)$  as a  $K_0$ -vector space and  $\iota : H \rightarrow \mathrm{GL}(V)$  to be a representation over  $K_0$  defined by

$$\iota(h)x = hx\sigma({}^t h), \quad x \in V, h \in H,$$

where  $\sigma$  denotes the non-trivial involution in  $\mathrm{Gal}(K/K_0)$ . If we choose  $x_0 \in V$  to be a hermitian matrix in  $M_n(K)$  and  $H^{x_0}$  to be the stabilizer of  $x_0$ , then  $H^{x_0}$  becomes a unitary subgroup of  $H$  which satisfies the condition of *loc. cit.* In order to use their result, we only need to verify the condition (a) and (b) in their theorem. For condition (a),  $\iota$  is semisimple since it is the direct sum of two irreducible subrepresentations, composed of hermitian matrices and anti-hermitian matrices respectively<sup>[2]</sup>. For condition (b), since we only care about the case where  $\chi = 1$ , it is automatically satisfied. Thus, if we use [GL18], Theorem 1.3 to replace [HM02a], Theorem 1 and follow the proof in [HM02a], then we finish the proof when  $R = \mathbb{C}$  and  $F/F_0$  is a quadratic extension of locally compact fields of characteristic  $p$ .

<sup>1</sup>This argument will occur several times in this section, so we refer to the reader for more details in the proof of Theorem 1.4.1

<sup>2</sup>Here we need the assumption  $p \neq 2$ .

For  $\text{char}(R) = 0$  in general, a supercuspidal representation of  $G$  can be realized as a representation over  $\overline{\mathbb{Q}}$  up to twisted by an unramified character, where  $\overline{\mathbb{Q}}$  is the algebraic closure of  $\mathbb{Q}$ . More precisely, there exists a character  $\chi : F^\times \rightarrow R^\times$  such that  $\chi|_{\mathfrak{o}_F^\times} = 1$  and  $\pi\chi$  can be realized as a representation over  $\overline{\mathbb{Q}}$ . Since  $G^\tau \cap F^\times = G^\tau \cap \mathfrak{o}_F^\times$ , we deduce that  $\pi$  is  $G^\tau$ -distinguished if and only if  $\pi\chi$  is, as a representation over  $R$ , and also as a representation over  $\overline{\mathbb{Q}}$  or  $\mathbb{C}$  by Lemma 1.4.3(1). Using the complex case,  $\pi\chi$  is  $\sigma$ -invariant as a representation over  $\mathbb{C}$ , and also as a representation over  $\overline{\mathbb{Q}}$  or  $R$  by Lemma 1.4.3(2). By definition,  $\chi$  is  $\sigma$ -invariant, thus  $\pi$  is also  $\sigma$ -invariant.

For  $R = \overline{\mathbb{F}}_l$ , we write  $\pi \cong \text{c-Ind}_J^G \Lambda$  for a simple type  $(\mathbf{J}, \Lambda)$ . Using the Mackey formula and the Frobenius reciprocity, we have

$$0 \neq \text{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

Thus  $\pi$  is distinguished if and only if there exists  $g \in G$  such that  $\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ . Let  $\gamma = \tau(g)g^{-1}$  and let  $\delta(x) = \gamma^{-1}\tau(x)\gamma$  for  $x \in G$  which is also a unitary involution, then we have

$$0 \neq \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \cong \text{Hom}_{\mathbf{J} \cap G^\delta}(\Lambda, 1) = \text{Hom}_{\mathbf{J} \cap G^\delta}(\Lambda^0, 1) \cong \text{Hom}_J(\Lambda^0, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{F}}_l),$$

where  $\Lambda^0 = \Lambda|_J$  and we use the fact that  $\mathbf{J} \cap G^\delta = J \cap G^\delta$  by Lemma 1.4.2.

Since  $\pi$  is supercuspidal, if we consider  $P_{\Lambda^0}$  as the projective envelope of  $\Lambda^0$  as a  $\overline{\mathbb{Z}}_l[J]$ -module, where we denote by  $\overline{\mathbb{Z}}_l$  the ring of integers of  $\overline{\mathbb{Q}}_l$ , then we have ( [Vig96], chapitre III, 4.28 and [Ser77], chapter 14, Proposition 42 for finite group case. Since  $\Lambda^0$  is a smooth representation of the compact group  $J$  of finite dimension, it can be regarded as a representation of a finite group.):

(1)  $P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  is the projective envelope of  $\Lambda^0$  as a  $\overline{\mathbb{F}}_l[J]$ -module, which is indecomposable of finite length, with each irreducible component isomorphic to  $\Lambda^0$ . Thus  $\text{Hom}_{\overline{\mathbb{F}}_l[J]}(P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{F}}_l) \neq 0$ ;

(2) If we write  $\widetilde{P}_{\Lambda^0} = P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  as the  $\overline{\mathbb{Q}}_l$ -lift of  $P_{\Lambda^0}$ , then  $\widetilde{P}_{\Lambda^0} \cong \bigoplus \widetilde{\Lambda}^0$ , where  $\widetilde{\Lambda}^0$  in the direct sum are  $\overline{\mathbb{Q}}_l$ -lifts of  $\Lambda^0$  of multiplicity 1 (The multiplicity 1 statement is derived from counting the length of  $P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ , and the number of different  $\widetilde{\Lambda}^0$  in  $\widetilde{P}_{\Lambda^0}$ , and then showing that they are equal. The argument is indicated in the proof of [Vig96], chapitre III, 4.28, or more precisely, *ibid.*, chapitre III, Théorème 2.2 and Théorème 2.9);

(3) In (2), each  $(\mathbf{J}, \widetilde{\Lambda}^0)$  can be extended to a simple type  $(\mathbf{J}, \widetilde{\Lambda})$  of  $G$  as a  $\overline{\mathbb{Q}}_l$ -lift of  $(\mathbf{J}, \Lambda)$ .

Using (1), we have  $\text{Hom}_{\overline{\mathbb{F}}_l[J]}(P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{F}}_l) \neq 0$ . Since  $P_{\Lambda^0}$  is a projective  $\overline{\mathbb{Z}}_l[J]$ -module, it is a free  $\overline{\mathbb{Z}}_l$ -module. Since  $\text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Z}}_l$  is a free  $\overline{\mathbb{Z}}_l$ -module,

$$\text{Hom}_{\overline{\mathbb{Z}}_l[J]}(P_{\Lambda^0}, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Z}}_l)$$

is a free  $\overline{\mathbb{Z}}_l$ -module. As a result,

$$\text{Hom}_{\overline{\mathbb{F}}_l[J]}(P_{\Lambda^0} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{F}}_l) \cong \text{Hom}_{\overline{\mathbb{Z}}_l[J]}(P_{\Lambda^0}, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Z}}_l) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \neq 0$$

if and only if

$$\text{Hom}_{\overline{\mathbb{Z}}_l[J]}(P_{\Lambda^0}, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Z}}_l) \neq 0$$

if and only if

$$\text{Hom}_{\overline{\mathbb{Q}}_l[J]}(\widetilde{P}_{\Lambda^0}, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Q}}_l) \cong \text{Hom}_{\overline{\mathbb{Z}}_l[J]}(P_{\Lambda^0}, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Z}}_l) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \neq 0,$$

So there exists  $\widetilde{\Lambda}^0$  as in condition (2) such that  $\text{Hom}_{\overline{\mathbb{Q}}_l[J]}(\widetilde{\Lambda}^0, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Q}}_l) \neq 0$ . Using (3), we may choose  $(\mathbf{J}, \widetilde{\Lambda})$  as an extension of  $(\mathbf{J}, \widetilde{\Lambda}^0)$ . We write  $\widetilde{\pi} = \text{c-Ind}_J^G \widetilde{\Lambda}$  which is a supercuspidal representation of  $G$  over  $\overline{\mathbb{Q}}_l$ . By using

$$\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\widetilde{\Lambda}^g, 1) \cong \text{Hom}_{\mathbf{J} \cap G^\delta}(\widetilde{\Lambda}, 1) = \text{Hom}_{\mathbf{J} \cap G^\delta}(\widetilde{\Lambda}^0, 1) \cong \text{Hom}_J(\widetilde{\Lambda}^0, \text{Ind}_{\mathbf{J} \cap G^\delta}^J \overline{\mathbb{Q}}_l) \neq 0$$

and the Mackey formula and the Frobenius reciprocity as before,  $\tilde{\pi}$  is  $G^\tau$ -distinguished. Using the result of characteristic 0 case, we have  $\tilde{\pi}^\sigma \cong \tilde{\pi}$ . Using (3),  $\tilde{\Lambda}$  is a  $\overline{\mathbb{Q}_l}$ -lift of  $\Lambda$ . So  $\tilde{\pi}$  is a  $\overline{\mathbb{Q}_l}$ -lift of  $\pi$ . So we have  $\pi^\sigma \cong \pi$ .

For  $\text{char}(R) = l > 0$  in general, as in the characteristic zero case, there exists a character  $\chi : F^\times \rightarrow R^\times$  such that  $\chi|_{\mathfrak{o}_F^\times} = 1$  and  $\pi\chi$  can be realized as a representation over  $\overline{\mathbb{F}_l}$ . Since  $G^\tau \cap F^\times = G^\tau \cap \mathfrak{o}_F^\times$ , we deduce that  $\pi$  is  $G^\tau$ -distinguished if and only if  $\pi\chi$  is, as a representation over  $R$ , and also as a representation over  $\overline{\mathbb{F}_l}$  by Lemma 1.4.3(1). Using the case above,  $\pi\chi$  is  $\sigma$ -invariant, as a representation over  $\overline{\mathbb{F}_l}$ , and also as a representation over  $R$  by Lemma 1.4.3(2). By definition,  $\chi$  is  $\sigma$ -invariant, thus  $\pi$  is also  $\sigma$ -invariant. □

**Remark 1.4.4.** *In section 9, we will give a purely local proof (without using the result of complex supercuspidal case) for this theorem which also works for cuspidal case.*

## 1.5 The $\tau$ -selfdual type theorem

Let  $G = \text{GL}_n(F)$  and let  $\tau$  be the unitary involution of  $G$  corresponding to a hermitian matrix  $\varepsilon$ . Let  $\pi$  be a cuspidal representation of  $G$ . From our settings of section 3, there exist a maximal simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  contained in  $\pi$ . First of all, we have the following lemma:

**Lemma 1.5.1.** *If  $\pi$  is  $\sigma$ -invariant, then we may choose the simple stratum above such that  $\sigma({}^t\beta) = \beta$ . As a result,  $\sigma_1$  (see section 2) is an involution defined on  $E$  whose restriction to  $F$  is  $\sigma$ .*

Let  $E_0 = E^{\sigma_1}$ , where  $E = F[\beta]$  and  $\beta$  is chosen as in Lemma 1.5.1. We state the following important theorem.

**Theorem 1.5.2.** *Let  $\pi$  be a  $\sigma$ -invariant cuspidal representation of  $G$  and let  $\tau$  be a unitary involution. We also assume the following additional condition:*

If the hermitian matrix corresponding to  $\tau$  is not in the same  $G$ -class as  $I_n$  in  $\mathcal{X}$  and if there exists a maximal simple stratum  $[\mathfrak{a}, \beta]$  as in Lemma 1.5.1 with a  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  contained in  $\pi$ , such that the corresponding  $E/E_0$  is unramified, then  $m$  is odd.

*Then there exist a maximal simple stratum  $[\mathfrak{a}', \beta']$  and a simple character  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  contained in  $\pi$  such that:*

- (1)  $\tau(\beta') = \beta'^{-1}$ ;
- (2)  $\tau(\mathfrak{a}') = \mathfrak{a}'$  and  $\tau(H^1(\mathfrak{a}', \beta')) = H^1(\mathfrak{a}', \beta')$ ;
- (3)  $\theta' \circ \tau = \theta'^{-1}$ .

As a corollary of Theorem 1.5.2, we state the following theorem as the main theorem of this section:

**Theorem 1.5.3** (The  $\tau$ -selfdual type theorem). *Under the same condition of Theorem 1.5.2, there exists a simple type  $(\mathbf{J}, \Lambda)$  contained in  $\pi$  such that  $\tau(\mathbf{J}) = \mathbf{J}$  and  $\Lambda^\tau \cong \Lambda^\vee$ .*

In the following subsections, we will focus on the proof of the results stated.

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<sup>3</sup>For the definition of  $\tau(\mathfrak{a}')$ , see §1.1. Same notations work for Theorem 1.5.5 and further proofs.

### 1.5.1 Endo-class version of main results

To prove Theorem [1.5.2](#) and Theorem [1.5.3](#), we state their corresponding endo-class versions. Let  $\Theta$  be an endo-class over  $F$ . As mentioned in section 3, we write  $d = \deg(\Theta)$ . Moreover, its tame parameter field  $T$  as a tamely ramified extension over  $F$  is unique up to  $F$ -isomorphism.

From the definition of endo-class, we may choose a maximal simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  such that  $\theta \in \Theta$ . We denote by  $\Theta^\sigma$  the endo-class of  $\theta^\sigma$  which doesn't depend on the choice of  $\theta$ . We denote by  $n$  the size of  $\mathfrak{a}$ , that is,  $\mathfrak{a} \hookrightarrow M_n(F)$  as a hereditary order. We write  $n = md$  with  $m$  a positive integer. First of all, we have the following lemma as the endo-class version of Lemma [1.5.1](#) which will be proved in subsection 5.4.

**Lemma 1.5.4.** *If  $\Theta^\sigma = \Theta$ , then we may choose the simple stratum above such that  $\sigma({}^t\beta) = \beta$ . As a result,  $\sigma_1$  is an involution defined on  $E$  whose restriction to  $F$  is  $\sigma$ .*

Let  $E_0 = E^{\sigma_1}$ , where  $E = F[\beta]$  and  $\beta$  is chosen as in Lemma [1.5.4](#). The following theorem as an endo-class version of Theorem [1.5.2](#) says that we may adjust our choice of simple stratum and simple character such that they are  $\tau$ -selfdual with respect to a unitary involution  $\tau$ :

**Theorem 1.5.5.** *Let  $\Theta \in \mathcal{E}(F)$  be an endo-class over  $F$  such that  $\Theta^\sigma = \Theta$ . Let  $\tau$  be a unitary involution of  $G$ . We also assume the following additional condition:*

If the hermitian matrix corresponding to  $\tau$  is not in the same  $G$ -class as  $I_n$  in  $\mathcal{X}$  and if there exists a maximal simple stratum  $[\mathfrak{a}, \beta]$  as in Lemma [1.5.4](#) with a  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  contained in  $\Theta$ , such that the corresponding  $E/E_0$  is unramified, then  $m = n/d$  is odd.

*Then there exist a maximal simple stratum  $[\mathfrak{a}', \beta']$  in  $M_n(F)$  and a simple character  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  such that:*

- (1)  $\tau(\beta') = \beta'^{-1}$ ;
- (2)  $\tau(\mathfrak{a}') = \mathfrak{a}'$  and  $\tau(H^1(\mathfrak{a}', \beta')) = H^1(\mathfrak{a}', \beta')$ ;
- (3)  $\theta' \in \Theta$  and  $\theta' \circ \tau = \theta'^{-1}$ .

Later we will focus on the proof of Theorem [1.5.5](#). So before we begin our proof, it is necessary to illustrate how does this theorem imply Theorem [1.5.2](#) and Theorem [1.5.3](#). First, we have the following important result due to Gelfand and Kazhdan (see [\[BZ76\]](#), Theorem 7.3 for complex case and [\[SV17b\]](#), Proposition 8.4 for  $l$ -modular case):

**Proposition 1.5.6.** *For  $\pi$  an irreducible representation of  $GL_n(F)$ , the representation defined by  $g \mapsto \pi({}^t g^{-1})$  is isomorphic to  $\pi^\vee$ .*

For  $\pi$  given as in Lemma [1.5.1](#), if we denote by  $\Theta_\pi$  the endo-class corresponding to  $\pi$ , then we get  $\Theta_\pi^\sigma = \Theta_\pi$ . So we may use Lemma [1.5.4](#) to get Lemma [1.5.1](#) and use Theorem [1.5.5](#) to get Theorem [1.5.2](#).

Now we show that Theorem [1.5.2](#) implies Theorem [1.5.3](#). Using Proposition [1.5.6](#), we have  $\pi^{\tau^\vee} \cong \pi^\sigma \cong \pi$ . Let  $(\mathbf{J}, \Lambda)$  be a simple type of  $\pi$  containing  $\theta'$ , where  $\theta'$  is obtained from Theorem [1.5.2](#) such that  $\theta' \circ \tau = \theta'^{-1}$ . Thus  $\tau(\mathbf{J}) = \mathbf{J}$  since they are the  $G$ -normalizers of  $\theta' \circ \tau$  and  $\theta'^{-1}$  respectively. Since  $\pi^{\tau^\vee} \cong \pi$ , it contains both  $(\mathbf{J}, \Lambda)$  and  $(\mathbf{J}, \Lambda^{\tau^\vee})$ . By Proposition [1.3.4](#), there exists  $g \in G$  such that  $(\mathbf{J}, \Lambda^{\tau^\vee}) = (\mathbf{J}^g, \Lambda^g)$ . Since  $\Lambda^{\tau^\vee} \cong \Lambda^g$  contains both  $(\theta' \circ \tau)^{-1} = \theta'$  and  $\theta'^g$  as simple characters, the restriction of  $\Lambda^g$  to the intersection

$$H^1(\mathfrak{a}', \beta') \cap H^1(\mathfrak{a}', \beta')^g, \tag{1.5.1}$$

which is a direct sum of copies of  $\theta'^g$  restricting to [\(1.5.1\)](#), contains the restriction of  $\theta'$  to [\(1.5.1\)](#). It follows that  $g$  intertwines  $\theta'$ . By Proposition [1.3.1](#)(5), we know that  $g \in J(\mathfrak{a}', \beta')B'^\times J(\mathfrak{a}', \beta')$  with  $B'$  the centralizer of  $E'$  in  $M_n(F)$ . Thus we may assume  $g \in B'^\times$ . From the uniqueness of the maximal

compact subgroup in  $\mathbf{J}$ , we deduce that  $\mathbf{J}^g = \mathbf{J}$  implies  $J(\alpha', \beta')^g = J(\alpha', \beta')$ . Intersecting it with  $B'^{\times}$  implies that  $\mathfrak{b}'^{\times g} = \mathfrak{b}'^{\times}$ . Since  $\mathfrak{b}'^{\times}$  is a maximal compact subgroup of  $B'^{\times} \cong \mathrm{GL}_m(E')$  and  $g \in B'^{\times}$ , we deduce that  $g \in E'^{\times} \mathfrak{b}'^{\times} \subset \mathbf{J}$ . Thus  $(\mathbf{J}^g, \Lambda^g) = (\mathbf{J}, \Lambda)$ , which finishes the proof of Theorem [1.5.3](#).

Finally we state the following two lemmas which will be useful in our further proof:

**Lemma 1.5.7.** *Let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum of  $M_n(F)$  and let  $\Theta$  be a  $\sigma$ -invariant endo-class over  $F$ , such that there exists a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  in  $\Theta$ . Then  $\theta \circ \tau$  and  $\theta^{-1}$  are in the same endo-class. In particular, if the hereditary order  $\mathfrak{a}$  is  $\tau$ -invariant, then  $\theta \circ \tau$  is conjugate to  $\theta^{-1}$  by an element in  $U(\mathfrak{a})$ .*

*Proof.* We choose  $\pi$  as a cuspidal representation of  $G$  containing  $\theta$ . Thus by definition, we have  $\Theta_{\pi} = \Theta$ . Using Proposition [1.5.6](#), we have  $\pi^{\tau} \cong \pi^{\sigma v}$ . So  $\theta \circ \tau \in \Theta_{\pi^{\tau}} = \Theta_{\pi^{\sigma v}} = \Theta_{\pi}^{\sigma v}$  and  $\theta^{-1} \in \Theta_{\pi^v}$ . Since  $\Theta^{\sigma} = \Theta$ , we have  $\Theta_{\pi^{\sigma v}} = \Theta_{\pi^v}$ , which means that  $\theta \circ \tau$  and  $\theta^{-1}$  are in the same endo-class. If  $\tau(\mathfrak{a}) = \mathfrak{a}$ , then by definition and construction of endo-equivalence ([BH96](#), Theorem 8.7),  $\theta \circ \tau$  intertwines with  $\theta^{-1}$ . By [BK93](#), Theorem 3.5.11,  $\theta \circ \tau$  is conjugate to  $\theta^{-1}$  by an element in  $U(\mathfrak{a})$ .  $\square$

The following lemma will be used to change the choice of unitary involution up to  $G$ -action on its corresponding hermitian matrix.

**Lemma 1.5.8.** *Let  $\tau = \tau_{\varepsilon}$  be the unitary involution on  $\mathrm{GL}_n(F)$  corresponding to a hermitian matrix  $\varepsilon$ , let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $M_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character, such that*

$$\tau(\mathfrak{a}) = \mathfrak{a}, \quad \theta \circ \tau = \theta^{-1} \quad (\text{and } \tau(\beta) = \beta^{-1}).$$

*Then for  $\tau' = \tau_{\varepsilon'}$  the unitary involution on  $\mathrm{GL}_n(F)$  corresponding to a hermitian matrix  $\varepsilon' = g^{-1}\varepsilon\sigma({}^t g^{-1})$ , we have*

$$\tau'(\mathfrak{a}^g) = \mathfrak{a}^g, \quad \theta^g \circ \tau' = (\theta^g)^{-1} \quad (\text{and } \tau'(\beta^g) = (\beta^g)^{-1}).$$

*Proof.* The proof is just a simple calculation. We have

$$\tau'(\mathfrak{a}^g) = \tau'(g^{-1})\tau'(\mathfrak{a})\tau'(g) = \tau'(g^{-1})\varepsilon'\varepsilon^{-1}\tau(\mathfrak{a})(\varepsilon'\varepsilon^{-1})^{-1}\tau'(g) = g^{-1}\tau(\mathfrak{a})g,$$

where in the last step we use

$$(\varepsilon'\varepsilon^{-1})^{-1}\tau'(g) = \varepsilon\sigma({}^t g^{-1})\varepsilon'^{-1} = g.$$

Since  $\tau(\mathfrak{a}) = \mathfrak{a}$ , we get  $\tau'(\mathfrak{a}^g) = \mathfrak{a}^g$ . The other two equations can be proved in a similar way.  $\square$

## 1.5.2 The maximal and totally wildly ramified case

Now we focus on the proof of Theorem [1.5.5](#). We imitate the strategy in [AKM<sup>+</sup>19](#), section 4 which first considered special case, and then used tame lifting developed by Bushnell and Henniart [BH96](#) and other tools developed by Bushnell and Kutzko [BK93](#) to generalize their result. In this subsection, we prove the following proposition as a special case of (2) and (3) of Theorem [1.5.5](#):

**Proposition 1.5.9.** *Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $M_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  such that  $\theta \in \Theta$  with  $\Theta$  a  $\sigma$ -invariant endo-class. Let  $E/F$  be totally wildly ramified of degree  $n$ . Let  $\tau = \tau_1$  with  $\tau_1(x) := \sigma({}^t x^{-1})$  for any  $x \in G$ . Then there exist a simple stratum  $[\mathfrak{a}'', \beta'']$  and a simple character  $\theta'' \in \mathcal{C}(\mathfrak{a}'', \beta'')$  such that  $(\mathfrak{a}'', \theta'')$  is  $G$ -conjugate to  $(\mathfrak{a}, \theta)$  with the property  $\tau(\mathfrak{a}'') = \mathfrak{a}''$  and  $\theta'' \circ \tau = \theta''^{-1}$ .*

**Remark 1.5.10.** *We have  $[E : F] = d = n$ , which is a power of  $p$  as an odd number.*



Up to  $G$ -conjugacy, we may and will assume that  $\mathfrak{a}$  is standard (that is,  $\mathfrak{a}$  is made of matrices with upper triangular elements in  $\mathfrak{o}_F$  and other elements in  $\mathfrak{p}_F$ ). First we prove the following lemma:

**Lemma 1.5.11.** *There exist  $g_1 \in G$  and  $a_1, \dots, a_n \in \mathfrak{o}_F^\times$  such that*

$$\tau(g_1)g_1^{-1} = A := \begin{pmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & \ddots & \ddots & a_2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_{n-1} & \ddots & \ddots & 0 \\ a_n & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Moreover, if we define  $\mathfrak{a}' := \mathfrak{a}^{g_1}$ , then we have  $\tau(\mathfrak{a}') = \mathfrak{a}'$ .

*Proof.* First we claim that we may choose  $a_i \in \mathfrak{o}_F^\times$  such that  $A$  is a hermitian matrix and  $\det(A) \in \mathbb{N}_{F/F_0}(F^\times)$ . To do this, noting that  $A^* = A$  if and only if  $a_i = \sigma(a_{n+1-i})$  for  $i = 1, 2, \dots, n$ . So we choose  $a_i = \sigma(a_{n+1-i}) \in \mathfrak{o}_F^\times$  for  $i = 1, 2, \dots, (n-1)/2$  and  $a_{(n+1)/2} \in \mathfrak{o}_{F_0}^\times$  to make sure that  $\det(A) \in \mathbb{N}_{F/F_0}(F^\times)$ . So we finish the proof of the claim.

Since  $A$  is a hermitian matrix which is in the same  $G$ -orbit as  $I_n$  by considering the determinant, using Proposition 1.2.1, there exists an element  $g_1 \in G$  such that  $(g_1^{-1})^*g_1^{-1} = A$ , which means that  $\tau(g_1)g_1^{-1} = A$ . By definition  $\tau(\mathfrak{a}') = \mathfrak{a}'$  if and only if  $\tau(g_1^{-1})\tau(\mathfrak{a})\tau(g_1) = g_1^{-1}\mathfrak{a}g_1$ . Since  $\mathfrak{a}^* = {}^t\mathfrak{a}$ , we deduce that  $\tau(\mathfrak{a}') = \mathfrak{a}'$  if and only if  $A^{-1}{}^t\mathfrak{a}A = (\tau(g_1)g_1^{-1})^{-1}{}^t\mathfrak{a}(\tau(g_1)g_1^{-1}) = \mathfrak{a}$ . From our choice of  $A$  and the definition of  $\mathfrak{a}$ , this can be verified directly. So we finish the proof.  $\square$

Now fix  $g_1$  as in Lemma 1.5.11. We write  $\theta' = \theta^{g_1}$  and  $\beta' = \beta^{g_1}$ . Since  $\mathfrak{a}' = \mathfrak{a}^{g_1}$ , we also have:

- (1)  $U^i := U^i(\mathfrak{a}') = U^i(\mathfrak{a})^{g_1}$ , where  $U^i(\mathfrak{a}) := 1 + \mathfrak{p}_\mathfrak{a}^i$  for  $i \geq 1$ ;
- (2)  $J' := J(\mathfrak{a}', \beta') = J(\mathfrak{a}, \beta)^{g_1}$ ;
- (3)  $J'^1 := J^1(\mathfrak{a}', \beta') = J^1(\mathfrak{a}, \beta)^{g_1}$ ;
- (4)  $\mathbf{J}' := \mathbf{J}(\mathfrak{a}', \beta') = \mathbf{J}(\mathfrak{a}, \beta)^{g_1}$ ;
- (5)  $H'^1 := H^1(\mathfrak{a}', \beta') = H^1(\mathfrak{a}, \beta)^{g_1}$ ;
- (6)  $M' := M'^{g_1}$ , where  $M = \mathfrak{o}_F^\times \times \dots \times \mathfrak{o}_F^\times$  is the subgroup of diagonal matrices contained in  $\mathfrak{a}$ .

Since  $\mathfrak{a}'$  is  $\tau$ -stable and  $\Theta^\sigma = \Theta$ , using Lemma 1.5.7, there exists  $u' \in U(\mathfrak{a}')$  such that  $\theta' \circ \tau = (\theta'^{-1})u'$ . Since  $\theta' = \theta' \circ \tau \circ \tau = (\theta'^{-1})u' \circ \tau = \theta'^{u'\tau(u')}$ , we deduce that  $u'\tau(u')$  normalizes  $\theta'$ , which means that  $u'\tau(u') \in \mathbf{J}' \cap U(\mathfrak{a}') = \mathbf{J}'$  by using Proposition 1.3.1 (4). To prove Proposition 1.5.9, we only need to find  $x' \in G$  such that  $\mathfrak{a}'' := \mathfrak{a}'^{x'}$  and  $\theta'' := \theta'^{x'}$  have the desired property. By direct calculation, it means that  $\tau(x')x'^{-1}$  normalizes  $\mathfrak{a}'$  and  $u'\tau(x')x'^{-1}$  normalizes  $\theta'$ , so using Proposition 1.3.1 (4) and the fact that  $u'^{-1}\mathbf{J}'$  is contained in the normalizer of  $\mathfrak{a}'$ , it suffices to choose  $x'$  such that  $u'\tau(x')x'^{-1} \in \mathbf{J}'$ .

First we prove the following lemma:

**Lemma 1.5.12.** *There exists  $y' \in M'$  such that  $u'\tau(y')y'^{-1} \in J(\mathfrak{a}', \beta')U^1(\mathfrak{a}') = \mathfrak{o}_F^\times U^1(\mathfrak{a}')$ .*

*Proof.* First we write  $u' = g_1^{-1}ug_1$  for a certain  $u \in U(\mathfrak{a})$ . Then  $u'\tau(u') \in J(\mathfrak{a}', \beta')$  implies that  $uA^{-1}(u^{-1})^*A \in J(\mathfrak{a}, \beta) \subset \mathfrak{o}_F^\times U^1(\mathfrak{a})$  by direct calculation, where  $A$  is defined as in Lemma 1.5.11.

We choose  $y' = g_1^{-1}yg_1$  with  $y = \text{diag}(y_1, \dots, y_n) \in M = \mathfrak{o}_F^\times \times \dots \times \mathfrak{o}_F^\times$  to be determined. By direct calculation,  $u'\tau(y')y'^{-1} \in J(\mathfrak{a}', \beta')U^1(\mathfrak{a}')$  if and only if  $uA^{-1}(y^{-1})^*Ay^{-1} \in J(\mathfrak{a}, \beta)U^1(\mathfrak{a}) = \mathfrak{o}_F^\times U^1(\mathfrak{a})$ . We use  $\bar{u}_i, \bar{a}, \bar{y}_i$  and  $\bar{b}$  to denote the image of  $u_i, a, y_i, b$  in  $k_F \cong \mathfrak{o}_F/\mathfrak{p}_F$  respectively, where  $u_i, a, b \in \mathfrak{o}_F$  will be defined in the future.

$$\text{We write } A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & \ddots & \ddots & a_2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_{n-1} & \ddots & \ddots & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix} \text{ and } u = \begin{pmatrix} u_1 & *_{\mathfrak{o}_F} & \cdots & \cdots & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & u_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1} & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & \cdots & \cdots & *_{\mathfrak{p}_F} & u_n \end{pmatrix}, \text{ where } *_{\mathfrak{o}_F} \text{ and}$$

$*_{\mathfrak{p}_F}$  represent elements in  $\mathfrak{o}_F$  and  $\mathfrak{p}_F$  respectively. By direct calculation, we have

$$uA^{-1}(u^{-1})^*A = \begin{pmatrix} u_1\sigma(u_n^{-1}) & *_{\mathfrak{o}_F} & \cdots & \cdots & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & u_2\sigma(u_{n-1}^{-1}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1}\sigma(u_2^{-1}) & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & \cdots & \cdots & *_{\mathfrak{p}_F} & u_n\sigma(u_1^{-1}) \end{pmatrix} \in \mathfrak{o}_F^\times U^1(\mathfrak{a}),$$

which means that there exists  $a \in \mathfrak{o}_F^\times$  such that

$$u_1\sigma(u_n^{-1}), u_2\sigma(u_{n-1}^{-1}), \dots, u_n\sigma(u_1^{-1}) \in a(1 + \mathfrak{p}_F). \quad (1.5.2)$$

Also by direct calculation, we have

$$uA^{-1}(y^{-1})^*Ay^{-1} = \begin{pmatrix} u_1y_1^{-1}\sigma(y_n^{-1}) & *_{\mathfrak{o}_F} & \cdots & \cdots & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & u_2y_2^{-1}\sigma(y_{n-1}^{-1}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{n-1}y_{n-1}^{-1}\sigma(y_2^{-1}) & *_{\mathfrak{o}_F} \\ *_{\mathfrak{p}_F} & \cdots & \cdots & *_{\mathfrak{p}_F} & u_ny_n^{-1}\sigma(y_1^{-1}) \end{pmatrix},$$

which means that the lemma is true if and only if there exists  $b \in \mathfrak{o}_F^\times$  such that

$$u_1y_1^{-1}\sigma(y_n^{-1}), u_2y_2^{-1}\sigma(y_{n-1}^{-1}), \dots, u_ny_n^{-1}\sigma(y_1^{-1}) \in b(1 + \mathfrak{p}_F). \quad (1.5.3)$$

If we consider modulo  $\mathfrak{p}_F$ , then the condition (1.5.2) becomes

$$\overline{u_1}\sigma(\overline{u_n^{-1}}) = \overline{u_2}\sigma(\overline{u_{n-1}^{-1}}) = \dots = \overline{u_n}\sigma(\overline{u_1^{-1}}) = \overline{a}. \quad (1.5.4)$$

Moreover, if we consider modulo  $U^1(\mathfrak{a})$ , then  $uA^{-1}(y^{-1})^*Ay^{-1} \in \mathfrak{o}_F^\times U^1(\mathfrak{a})$  if and only if there exist  $y_i \in \mathfrak{o}_F^\times$  such that there exists  $b \in \mathfrak{o}_F^\times$  in the condition (1.5.3) such that

$$\overline{u_1y_1^{-1}}\sigma(\overline{y_n^{-1}}) = \overline{u_2y_2^{-1}}\sigma(\overline{y_{n-1}^{-1}}) = \dots = \overline{u_ny_n^{-1}}\sigma(\overline{y_1^{-1}}) = \overline{b}. \quad (1.5.5)$$

We choose  $b = u_{(n+1)/2}$ , thus we have  $\overline{b}\sigma(\overline{b^{-1}}) = \overline{a}$ . Furthermore we choose  $y_i = b^{-1}u_i$  when  $i = 1, 2, \dots, (n-1)/2$  and  $y_i = 1$  when  $i = (n+1)/2, \dots, n$ . Combining with the equation (1.5.4), the equation (1.5.5) is satisfied. So we finish the proof.  $\square$

Let us write  $z'u'\tau(y')y'^{-1} \in U^1$  for some  $y' \in M'$  and  $z' \in \mathfrak{o}_F^\times$  given by Lemma 1.5.12. By replacing the simple stratum  $[\alpha', \beta']$  with  $[\alpha'^{y'}, \beta'^{y'}]$ , the simple character  $\theta'$  with  $\theta'^{y'}$  and  $u'$  with  $y'^{-1}z'u'\tau(y')$ , which doesn't affect the fact that the order is  $\tau$ -stable, we can and will assume that  $u' \in U^1$ . We write  $J'^i = J' \cap U'^i$  for  $i \geq 1$ . We state the following two lemmas which correspond to Lemma 4.16 and Lemma 4.17 in [AKM<sup>+</sup>19]. Actually the same proofs work when one replaces the Galois involution  $\sigma$  in the original lemmas with any involution  $\tau$  on  $G$ .

**Lemma 1.5.13.** *Let  $v' \in U^i$  for some  $i \geq 1$  and assume that  $v'\tau(v') \in J^i$ . Then there exist  $j' \in J^i$  and  $x' \in U^i$  such that  $j'v'\tau(x')x'^{-1} \in U^{i+1}$ .*

Using Lemma 1.5.13 to replace Lemma 4.16 in AKM<sup>+</sup>19, we may prove the following lemma:

**Lemma 1.5.14.** *There exists a sequence of  $(x'_i, j'_i, v'_i) \in U^i \times J^i \times U^{i+1}$  for  $i \geq 0$ , satisfying the following conditions:*

- (1)  $(x'_0, j'_0, v'_0) = (1, 1, u')$ ;
- (2) for all  $i \geq 0$ , if we set  $y'_i = x'_0 x'_1 \dots x'_i \in U^i$ , then the simple character  $\theta'_i = \theta^{y'_i} \in \mathcal{C}(\alpha', \beta^{y'_i})$  satisfies  $\theta'_i \circ \tau = (\theta'^{-1})^{v'_i}$ ;
- (3) for all  $i \geq 1$ , we have  $y'_i v'_i = j'_i y'_{i-1} v'_{i-1} \tau(x'_i)$ .

Let  $x' \in U^1$  be the limit of  $y'_i = x'_0 x'_1 \dots x'_i$  and let  $h' \in J^1$  be that of  $j'_i \dots j'_1 j'_0$  when  $i$  tends to infinity. By Lemma 1.5.14(3), we have

$$y'_i v'_i \tau(y'^{-1}) = j'_i y'_{i-1} v'_{i-1} \tau(y'^{-1}) = \dots = j'_i \dots j'_1 j'_0 u'.$$

Passing to the limit, we get  $x'\tau(x')^{-1} = h'u'$ , which implies that  $u'\tau(x')x'^{-1} = h'^{-1} \in J'$ . Let  $(\alpha'', \theta'') = (\alpha^{x'}, \theta^{x'})$ , which finishes the proof of Proposition 1.5.9.

### 1.5.3 The maximal case

In this subsection, we generalize Proposition 1.5.9 to the following situation:

**Proposition 1.5.15.** *Let  $[\alpha, \beta]$  be a simple stratum in  $M_n(F)$  such that  $[E : F] = n$  and let  $\theta \in \mathcal{C}(\alpha, \beta)$  such that  $\theta \in \Theta$  with  $\Theta$  a  $\sigma$ -invariant endo-class. Let  $\tau$  be a given unitary involution. Then there exist a simple stratum  $[\alpha'', \beta'']$  and a simple character  $\theta'' \in \mathcal{C}(\alpha'', \beta'')$  such that  $(\alpha'', \theta'')$  is  $G$ -conjugate to  $(\alpha, \theta)$  with the property  $\tau(\alpha'') = \alpha''$  and  $\theta'' \circ \tau = \theta''^{-1}$ .*

Let  $T$  be a tame parameter field of  $\Theta$ . First we have the following lemma:

**Lemma 1.5.16.** *Let  $\Theta$  be a  $\sigma$ -invariant endo-class and let  $T/F$  be its tame parameter field. Then given a  $T/F$ -lift  $\Psi$  of  $\Theta$ , there is a unique involution  $\alpha$  of  $T$  extending  $\sigma$  such that  $\Psi^\alpha = \Psi$ .*

*Proof.* The proof of Lemma 4.8 in AKM<sup>+</sup>19 can be used almost unchanged to our lemma. We only need to consider  $\Theta$  instead of  $\Theta^\vee$  and  $\Psi$  instead of  $\Psi^\vee$ . □

Let  $\alpha$  be the involution of  $T$  given by Lemma 1.5.16, and let  $T_0$  be the subfield of  $T$  fixed by  $\alpha$ . Thus  $T_0 \cap F = F_0$ . We write  $t = [T : F] = [T_0 : F_0]$ . We need the following proposition due to Hakim and Murnaghan (see HM02b, Proposition 2.1):

**Proposition 1.5.17.** *There exists an embedding  $\iota : T \hookrightarrow M_t(F)$  of  $F$ -algebras such that for  $x \in T$ , we have  $\iota(\alpha(x)) = \iota(x)^* := \sigma({}^t \iota(x))$ . Consequently,  $\iota(T_0)$  is contained in the set of hermitian matrices.*

*Proof of Proposition 1.5.15.* Let  $E = F[\beta]$  and let  $T$  be the maximal tamely ramified extension of  $F$  in  $E$ . It is a tame parameter field of the endo-class  $\Theta$ . The simple character  $\theta$  gives  $\Psi$ , the endo-class of the interior  $T/F$ -lift of  $\Theta$ , as we introduced in §1.3.3. Let  $\alpha$  be defined as in Lemma 1.5.16 and let  $\iota$  be defined as in Proposition 1.5.17. By abuse of notation, we define

$$\iota : M_{n/t}(T) \hookrightarrow M_{n/t}(M_t(F)) = M_n(F)$$

with each block defined by the original  $\iota$ . First we consider  $\tau(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1}$  for any  $x \in G$  with  $\varepsilon = I_n$  or  $\text{diag}(\iota(\epsilon), \dots, \iota(\epsilon), \iota(\epsilon))$ , where  $\epsilon \in T_0^\times \setminus N_{T/T_0}(T^\times)$ . The determinant of the latter matrix is  $N_{T_0/F_0}(\epsilon)^{n/t}$ . Since

$$N_{T_0/F_0} : T_0^\times \rightarrow F_0^\times$$

is a homomorphism which maps  $N_{T/T_0}(T^\times)$  to  $N_{F/F_0}(F^\times)$ , it leads to a group homomorphism

$$N_{T_0/F_0} : T_0^\times / N_{T/T_0}(T^\times) \rightarrow F_0^\times / N_{F/F_0}(F^\times)$$

between two groups of order 2. We state and proof the following lemma in general:

**Lemma 1.5.18.** *Let  $F, F_0$  be defined as before. Let  $L_0/F_0$  be a finite extension such that  $L = L_0F$  is a field with  $[L : L_0] = 2$  and  $F_0 = L_0 \cap F$ . Then the group homomorphism*

$$N_{L_0/F_0} : L_0^\times \rightarrow F_0^\times$$

*induces an isomorphism*

$$N_{L_0/F_0} : L_0^\times / N_{L/L_0}(L^\times) \rightarrow F_0^\times / N_{F/F_0}(F^\times)$$

*of groups of order 2.*

*Proof.* We first consider the case where  $L_0/F_0$  is abelian. If on the contrary the induced homomorphism is not an isomorphism, then we get  $N_{L_0/F_0}(L_0^\times) \subset N_{F/F_0}(F^\times)$  which means that  $F$  is contained in  $L_0$  by Local Class Field Theory ([Ser79], Chapter 14, Theorem 1), which is absurd.

When  $L_0/F_0$  is Galois, we may write  $F_0 = L_0^0 \subsetneq \dots \subsetneq L_0^r = L_0$ , such that  $L_0^{i+1}/L_0^i$  is abelian for  $i = 0, \dots, r-1$  ([Ser79], Chapter 4, Proposition 7). We write  $L^i = L_0^i F$ . Thus it is easy to show that  $L^i/L_0^i$  is quadratic,  $L_0^i = L_0^{i+1} \cap L^i$  and  $L_0^{i+1}L^i = L^{i+1}$  for  $i = 0, \dots, r-1$ . Using the abelian case,

$$N_{L_0^{i+1}/L_0^i} : L_0^{i+1 \times} / N_{L^{i+1}/L_0^{i+1}}(L^{i+1 \times}) \rightarrow L_0^{i \times} / N_{L^i/L_0^i}(L^{i \times})$$

is an isomorphism for  $i = 0, 1, \dots, r-1$ . Composing them together, we finish the proof.

When  $L_0/F_0$  is separable, we write  $L'_0$  as the normal closure of  $L_0$  over  $F_0$ . Thus  $L'_0$  contains  $L_0$  and  $L'_0/F_0$  is a finite Galois extension. We write  $L' = L'_0 F$ . Using the Galois case,

$$N_{L'_0/F_0} : L_0'^{\times} / N_{L'/L'_0}(L'^{\times}) \rightarrow F_0^\times / N_{F/F_0}(F^\times)$$

is an isomorphism. Since  $N_{L'_0/F_0}(L_0'^{\times}) \subset N_{L_0/F_0}(L_0^\times)$ ,

$$N_{L_0/F_0} : L_0^\times / N_{L/L_0}(L^\times) \rightarrow F_0^\times / N_{F/F_0}(F^\times)$$

is also an isomorphism.

In the characteristic  $p$  case in general, we write  $L_0^{sep}$  for the maximal separable subextension of  $F_0$  contained in  $L_0$ , thus  $L_0/L_0^{sep}$  is purely inseparable. Thus  $N_{L_0/L_0^{sep}}(x) = x^{p^{[L_0:L_0^{sep}]}}$  for any  $x \in L_0^\times$ . Since  $p \neq 2$  and  $L_0^\times / N_{L/L_0}(L^\times)$  is of order 2, we deduce that

$$N_{L_0/L_0^{sep}} : L_0^\times / N_{L/L_0}(L^\times) \rightarrow L_0^{sep \times} / N_{L^{sep}/L_0^{sep}}(L^{sep \times})$$

is an isomorphism, where  $L^{sep} := LL_0^{sep}$ . So we come back to the separable case which finishes the proof.  $\square$

Using Lemma 1.5.18 for  $L_0 = T_0$ , the homomorphism above is actually an isomorphism. Since  $n/t$  is odd and  $\epsilon \in T_0^\times \setminus N_{T/T_0}(T^\times)$ , we have  $\det(\epsilon) = N_{T_0/F_0}(\epsilon)^{n/t} \in F_0^\times \setminus N_{F/F_0}(F^\times)$ . So indeed these two involutions represent all the two  $G$ -classes of hermitian matrices. Thus using Lemma 1.5.8, we may from now on assume  $\tau$  to be the two unitary involutions we mentioned above. Furthermore,  $\iota(T)^\times$  is normalized by  $\tau$  from the exact construction of  $\tau$  and Proposition 1.5.17, where we regard  $T$  as an  $F$ -subalgebra of  $M_{n/t}(T)$  given by the diagonal embedding.

Since  $T$  and  $\iota(T)$  are isomorphic as  $F$ -subalgebras contained in  $M_n(F)$ , by the Skolem-Noether theorem, there exists  $g \in G$  such that  $\iota(T) = T^g$ . Thus, if we write  $[\mathfrak{a}', \beta'] = [\mathfrak{a}^g, \beta^g]$ ,  $\theta' = \theta^g$  and  $E' = F[\beta']$ , then  $\theta' \in \Theta$  such that its tame parameter field equals  $\iota(T)$ . Since  $\tau$  normalizes  $\iota(T)^\times$ , we deduce that  $\theta' \circ \tau$  and  $\theta'^{-1}$  have the same parameter field  $\iota(T)$ . If we write  $\Psi'$  as the endo-class of the interior  $\iota(T)/F$ -lift corresponding to  $\theta'$ , and if we choose  $\alpha' = \iota|_T \circ \alpha \circ \iota|_{\iota(T)}^{-1}$ , then we have  $\Psi'^{\alpha'} = \Psi'$ .

Let  $C' = M_{n/t}(\iota(T))$  denote the centralizer of  $\iota(T)$  in  $M_n(F)$ . For  $c \in M_{n/t}(T)$ , we have

$$\tau(\iota(c)) = \varepsilon \sigma({}^t \iota(c)^{-1}) \varepsilon^{-1} = \varepsilon({}^{t_{C'}} \iota(\alpha(c))^{-1}) \varepsilon^{-1} = \varepsilon(\alpha'({}^{t_{C'}} \iota(c))^{-1}) \varepsilon^{-1} = \tau'(\iota(c)),$$

where we denote by  $t_{C'}$  the transpose on  $C' = M_{n/t}(\iota(T))$  and  $\tau'(c') = \varepsilon(\alpha'({}^{t_{C'}} c'^{-1})) \varepsilon^{-1}$  for any  $c' \in C'^\times$ . Thus  $\tau'$ , the restriction of  $\tau$  to  $C'^\times$ , is the unitary involution  $\tau_1$  on  $C'^\times = \text{GL}_{n/t}(\iota(T))$  with respect to the Galois involution  $\alpha' \in \text{Gal}(\iota(T)/F)$ . The intersection  $\mathfrak{c}' = \mathfrak{a}' \cap C'$  gives rise to a simple stratum  $[\mathfrak{c}', \beta']$ . The restriction of  $\theta'$  to  $H^1(\mathfrak{c}', \beta')$ , denoted by  $\theta'_{\iota(T)}$ , is a simple character associated to this simple stratum with endo-class  $\Psi'$ . Since  $E'/\iota(T)$  is totally wildly ramified, using Proposition 1.5.9 with  $G$ ,  $\theta$ ,  $\Theta$ ,  $\sigma$  and  $\tau$  replaced by  $C'^\times$ ,  $\theta'_{\iota(T)}$ ,  $\Psi'$ ,  $\alpha'$  and  $\tau'$  respectively, there exists  $c' \in C'^\times$  such that  $\tau'(c'^{c'}) = c'^{c'}$  and  $\theta'_{\iota(T)}^{c'} \circ \tau' = (\theta'_{\iota(T)}^{c'})^{-1}$ .

By the injectivity of  $\mathfrak{a} \mapsto \mathfrak{a} \cap C'$  between sets of hereditary orders as mentioned in §1.3.3, we know that  $\mathfrak{a}'' := \mathfrak{a}'^{c'}$  is  $\tau$ -stable. Moreover if we write  $\theta'' = \theta'^{c'}$ , then from our construction of  $\tau$  and the definition of  $\iota(T)/F$ -lift,

$$(\theta'' \circ \tau)_{\iota(T)} = \theta'' \circ \tau|_{H^1(\tau(c'), \tau(\beta'))} = \theta'' \circ \tau'|_{H^1(\tau(c'), \tau(\beta'))} = \theta''_{\iota(T)} \circ \tau'$$

and

$$(\theta''^{-1})_{\iota(T)} = \theta''_{\iota(T)}^{-1}$$

are equal. Since the interior  $\iota(T)/F$ -lift  $\theta'' \mapsto \theta''_{\iota(T)}$  is injective between sets of simple characters as mentioned in §1.3.3, the simple character  $\theta''$  satisfies the property  $\theta'' \circ \tau = \theta''^{-1}$ .  $\square$

#### 1.5.4 The general case

In this subsection, we finish the proof of Lemma 1.5.4 and Theorem 1.5.5. First of all, we recall the following result similar to that appearing in the paper of Stevens:

**Proposition 1.5.19** ([Ste01], Theorem 6.3). *Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $M_n(F)$  with  $\sigma_1(\mathfrak{a}) = \mathfrak{a}$ . Suppose that there exists a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  such that  $H^1(\mathfrak{a}, \beta)$  is  $\sigma_1$ -stable and  $\theta \circ \sigma_1 = \theta$ . Then there exists a simple stratum  $[\mathfrak{a}, \gamma]$  such that  $\theta \in \mathcal{C}(\mathfrak{a}, \gamma)$  and  $\sigma_1(\gamma) = \gamma$ .*

*Proof.* The original proof of [Ste01], Theorem 6.3 can be modified a little bit as follows, thus can be used in our case without difficulty. For any  $x \in M_n(F)$ , we use  $-\sigma_1(x)$  to replace  $\bar{x}$ ; we use  $\sigma_1$  to replace  $\sigma$ ; for  $[\mathfrak{a}, \beta]$  a simple stratum, we say that it is  $\sigma_1$ -invariant if  $\sigma_1(\mathfrak{a}) = \mathfrak{a}$ , and  $\sigma_1(\beta) = \beta$  and we use this concept to replace the concept *skew simple stratum* in the original proof. With these replacements, the original proof can be used directly.  $\square$

We choose  $[\mathfrak{a}_0, \beta_0]$  as a maximal simple stratum in  $M_d(F)$  and  $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \beta_0)$  such that  $\theta_0 \in \Theta$ . By Proposition 1.5.15, there are a maximal simple stratum  $[\mathfrak{a}'_0, \beta'_0]$  and a simple character  $\theta'_0 \in \mathcal{C}(\mathfrak{a}'_0, \beta'_0)$  which is  $\mathrm{GL}_d(F)$ -conjugate to  $\theta_0$  such that:

- (1) the order  $\mathfrak{a}'_0$  is  $\tau_1$ -stable;
- (2) the group  $H^1(\mathfrak{a}'_0, \beta'_0)$  is  $\tau_1$ -stable and  $\theta'_0 \circ \tau_1 = \theta'^{-1}_0$ ;

Furthermore, using Proposition 1.5.19 we may assume that:

- (3)  $\sigma_1(\beta'_0) = \beta'_0$ .

We embed  $M_d(F)$  diagonally into the  $F$ -algebra  $M_n(F)$ . This gives an  $F$ -algebra homomorphism  $\iota' : F[\beta'_0] \hookrightarrow M_n(F)$ . Write  $\beta' = \iota'(\beta'_0) = \beta'_0 \otimes \dots \otimes \beta'_0$  and  $E' = F[\beta']$ . The centralizer  $B'$  of  $E'$  in  $M_n(F)$  is naturally identified with  $M_m(E')$ . We regard  $\sigma_1$  as an involution on  $E'$  extending  $\sigma$  and we write  $E'_0 = E'^{\sigma_1}$ . Let  $\mathfrak{b}'$  be a maximal standard hereditary order in  $B'$  which may be identified with  $M_m(\mathfrak{o}_{E'})$ , and let  $\mathfrak{a}' = M_m(\mathfrak{a}'_0)$  be the unique hereditary order of  $M_n(F)$  normalized by  $E'^{\times}$  such that  $\mathfrak{a}' \cap B' = \mathfrak{b}'$ . Then we have  $[\mathfrak{a}', \beta']$  in  $M_n(F)$  as a simple stratum which satisfies the requirement of Lemma 1.5.4, finishing its proof.

Now we focus on the proof of Theorem 1.5.5. By Lemma 1.5.8, we may change  $\tau$  up to  $G$ -action on its corresponding hermitian matrix which doesn't change the content of the theorem. So if  $\varepsilon$  is in the same  $G$ -class as  $I_n$ , we may simply choose  $\tau = \tau_1$ , where  $\tau_1(x) = \sigma({}^t x^{-1})$  for any  $x \in G$ . If not, we fix  $\varepsilon \in E_0^{\times} \setminus N_{E'/E'_0}(E'^{\times})$ . Regarding  $\varepsilon$  as an element in  $M_d(F)$ , we have  $\det(\varepsilon) = N_{E'_0/F_0}(\varepsilon)$ . Since

$$N_{E'_0/F_0} : E_0^{\times} \rightarrow F_0^{\times}$$

is a homomorphism which maps  $N_{E'/E'_0}(E'^{\times})$  to  $N_{F/F_0}(F^{\times})$ , by Lemma 1.5.18 with  $L_0 = E'_0$ , it leads to an isomorphism

$$N_{E'_0/F_0} : E_0^{\times} / N_{E'/E'_0}(E'^{\times}) \rightarrow F_0^{\times} / N_{F/F_0}(F^{\times})$$

of the two groups of order 2. Thus we have  $N_{E'_0/F_0}(\varepsilon) \in F_0^{\times} \setminus N_{F/F_0}(F^{\times})$ . If  $E'/E'_0$  is unramified, we write  $\varepsilon = \mathrm{diag}(\varepsilon, \dots, \varepsilon)$ . We deduce that  $\det(\varepsilon) = N_{E'_0/F_0}(\varepsilon)^m \in F_0^{\times} \setminus N_{F/F_0}(F^{\times})$ , which is because  $F_0^{\times} / N_{F/F_0}(F^{\times})$  is a group of order 2, and  $m$  is odd from the condition of the theorem. If  $E'/E'_0$  is ramified, we may further assume that  $\varepsilon \in \mathfrak{o}_{E'_0}^{\times}$ . We choose  $\varepsilon = \mathrm{diag}(I_d, \dots, I_d, \varepsilon)$  and we have  $\det(\varepsilon) = N_{E'_0/F_0}(\varepsilon) \in F_0^{\times} \setminus N_{F/F_0}(F^{\times})$ . For both two cases,  $\tau_{\varepsilon}$  is a unitary involution whose corresponding hermitian matrix is not in the same  $G$ -class as  $I_n$ . So from now on, we only consider the three unitary involutions above. From our assumption of  $\tau$ , the restriction of  $\tau$  on  $\mathrm{GL}_m(E')$  is also a unitary involution  $\tau' = \tau_1$  or  $\tau_{\varepsilon}$  with  $\varepsilon = \mathrm{diag}(1, \dots, 1, \varepsilon)$ . In particular, since  $\varepsilon$  is an element in  $E'$ , we know that  $\varepsilon$  commutes with elements in  $E'$  and we have  $\tau(\beta') = \beta'^{-1}$ .

Since  $\mathfrak{a}'_0$  is  $\tau_1$ -stable and  $\mathfrak{b}'$  is  $\tau'$ -stable, from our assumption of  $\tau$  we deduce that  $\mathfrak{a}'$  is  $\tau$ -stable, or by definition  $\varepsilon \sigma_1(\mathfrak{a}') \varepsilon^{-1} = \mathfrak{a}'$ . Since  $\sigma_1(\beta') = \beta'$ , by direct calculation we have

$$\tau(H^1(\mathfrak{a}', \beta')) = \varepsilon H^1(\sigma_1(\mathfrak{a}'), \sigma_1(\beta'))^{-1} \varepsilon^{-1} = H^1(\sigma_1(\mathfrak{a}')^{\varepsilon^{-1}}, \beta'^{\varepsilon^{-1}}) = H^1(\mathfrak{a}', \beta'^{\varepsilon^{-1}}) = H^1(\mathfrak{a}', \beta').$$

Let  $M$  be the standard Levi subgroup of  $G$  isomorphic to  $\mathrm{GL}_d(F) \times \dots \times \mathrm{GL}_d(F)$ , let  $P$  be the standard parabolic subgroup of  $G$  generated by  $M$  and upper triangular matrices, and let  $N$  be its unipotent radical. Let  $N^-$  be the unipotent radical of the parabolic subgroup opposite to  $P$  with respect to  $M$ . By [SS08], Théorème 2.17, we have

$$H^1(\mathfrak{a}', \beta') = (H^1(\mathfrak{a}', \beta') \cap N^-) \cdot (H^1(\mathfrak{a}', \beta') \cap M) \cdot (H^1(\mathfrak{a}', \beta') \cap N), \quad (1.5.6)$$

$$H^1(\mathfrak{a}', \beta') \cap M = H^1(\mathfrak{a}'_0, \beta'_0) \times \dots \times H^1(\mathfrak{a}'_0, \beta'_0). \quad (1.5.7)$$

Let  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  be the transfer of  $\theta'_0$ . By *loc. cit.*, the character  $\theta'$  is trivial on  $H^1(\mathfrak{a}', \beta') \cap N^-$  and  $H^1(\mathfrak{a}', \beta') \cap N$ , and the restriction of  $\theta'$  on  $H^1(\mathfrak{a}', \beta') \cap M$  equals  $\theta'_0 \otimes \dots \otimes \theta'_0$ . We have

$$\theta' \circ \tau|_{H^1(\mathfrak{a}', \beta') \cap N^-} = \theta' \circ \tau|_{H^1(\mathfrak{a}', \beta') \cap N} = \theta'^{-1}|_{H^1(\mathfrak{a}', \beta') \cap N^-} = \theta'^{-1}|_{H^1(\mathfrak{a}', \beta') \cap N} = 1$$

and

$$\theta' \circ \tau|_{H^1(\mathfrak{a}', \beta') \cap M} = \theta'_0 \circ \tau_1 \otimes \dots \otimes \theta'_0 \circ \tau_1 = \theta'^{-1}_0 \otimes \dots \otimes \theta'^{-1}_0 = \theta'^{-1}|_{H^1(\mathfrak{a}', \beta') \cap M}$$

for  $\tau = \tau_1$  or  $\tau_\varepsilon$  with  $\varepsilon = \text{diag}(\epsilon, \dots, \epsilon)$  or  $\text{diag}(1, \dots, 1, \epsilon)$ , since  $\epsilon \in F[\beta'_0]^\times$  normalizes  $\theta'_0$ . Thus by equation (1.5.6), we have  $\theta' \circ \tau = \theta'^{-1}$ .

**Remark 1.5.20.** From the proof of Theorem 1.5.5, we observe that if  $\tau$  is chosen as one of the three unitary involutions mentioned in the proof, then we may choose the same simple stratum and simple character which satisfy the conclusion of the theorem.

**Remark 1.5.21.** We give a counter-example to show that the condition in Theorem 1.5.5 is necessary. Let  $n = 2$ ,  $F/F_0$  unramified,  $\Theta$  is trivial and  $\varepsilon = \text{diag}(1, \varpi_{F_0})$ . Then  $d = 1$ ,  $m = n = 2$ ,  $E = F$  and  $E_0 = F_0$ . If the theorem is true, then  $\mathfrak{a} = M_2(\mathfrak{o}_F)^g$  for some  $g \in \text{GL}_2(F)$  and  $\tau(\mathfrak{a}) = \mathfrak{a}$ . By direct calculation  $\sigma({}^t g^{-1})\varepsilon^{-1}g^{-1}$  normalizes  $M_2(\mathfrak{o}_F)$ , which means that  $\sigma({}^t g^{-1})\varepsilon^{-1}g^{-1} \in F^\times \text{GL}_2(\mathfrak{o}_F)$ . It is impossible since  $\det(\sigma({}^t g^{-1})\varepsilon^{-1}g^{-1}) \in \varpi_{F_0} N_{F/F_0}(F^\times)$ , while  $\det(F^\times \text{GL}_2(\mathfrak{o}_F)) \subset N_{F/F_0}(F^\times)$ .

## 1.6 The distinguished type theorem

Let  $\pi$  be a cuspidal representation of  $G$  such that  $\pi^\sigma \cong \pi$ . From the statements and proofs of Theorem 1.5.2, 1.5.3 and 1.5.5, we may assume the following conditions as a remark of section 5:

**Remark 1.6.1.** (1) For  $\tau = \tau_1$ , there exist a simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  contained in  $\pi$  such that  $\tau(\mathfrak{a}) = \mathfrak{a}$ ,  $\tau(H^1(\mathfrak{a}, \beta)) = H^1(\mathfrak{a}, \beta)$ ,  $\theta \circ \tau = \theta^{-1}$  and  $\tau(\beta) = \beta^{-1}$ , where  $\tau_1(x) := \sigma({}^t x^{-1})$  for any  $x \in \text{GL}_n(F)$ ;

(2) For  $\tau = \tau_1$ , there exists a simple type  $(\mathbf{J}, \Lambda)$  containing  $\theta$  and contained in  $\pi$  such that  $\tau(\mathbf{J}) = \mathbf{J}$  and  $\Lambda^\tau \cong \Lambda^\vee$ ;

(3)  $\sigma_1$  is an involution on  $E = F[\beta]$ , whose restriction to  $F$  equals  $\sigma$ . So by abuse of notation, we identify  $\sigma$  with  $\sigma_t$ . Let  $E_0 = E^\sigma$ . **We assume further in this section that if  $E/E_0$  is unramified, then  $m$  is odd<sup>4</sup>;**

(4) Write  $\tau(x) = \varepsilon \sigma({}^t x^{-1})\varepsilon^{-1}$  for any  $x \in G$  such that: when  $E/E_0$  is unramified, we assume  $\varepsilon = I_n$  or  $\text{diag}(\varpi_E, \dots, \varpi_E) \in \text{GL}_m(E) \hookrightarrow \text{GL}_{md}(F) = \text{GL}_n(F)$ ; when  $E/E_0$  is ramified, we assume  $\varepsilon = I_n$  or  $\text{diag}(1, \dots, 1, \epsilon) \in \text{GL}_m(E) \hookrightarrow \text{GL}_n(F)$  with  $\epsilon \in \mathfrak{o}_{E_0}^\times \setminus N_{E/E_0}(\mathfrak{o}_E^\times)$ . By Remark 1.5.20, we assume further that for these three unitary involutions, condition (1) and (2) are also satisfied. **From now on until the end of this section, we assume  $\varepsilon$  to be one of these three hermitian matrices and  $\tau$  to be one of these three corresponding involutions.**

(5) the element  $\beta$  has the block diagonal form:

$$\beta = \text{diag}(\beta_0, \dots, \beta_0) \in M_m(M_d(F)) = M_n(F)$$

for some  $\beta_0 \in M_d(F)$ , where  $d$  is the degree of  $\beta$  over  $F$  and  $n = md$ . The centralizer  $B$  of  $E$  in  $M_n(F)$  is identified with  $M_m(E)$ . If we regard  $\tau$  as the restriction of the original involution to  $B^\times$ , then it is a unitary involution with respect to  $B^\times = \text{GL}_m(E)$ ,  $E/E_0$  and  $\sigma \in \text{Gal}(E/E_0)$ ;

(6) the order  $\mathfrak{b} = \mathfrak{a} \cap B$  is the standard maximal order  $M_m(\mathfrak{o}_E)$  of  $M_m(E)$ . Thus if we write  $\mathfrak{a}_0$  as the hereditary order of  $M_d(F)$  normalized by  $E$ , then  $\mathfrak{a}$  is identified with  $M_m(\mathfrak{a}_0)$ ;

(7)  $\varpi_E$  is a uniformizer of  $E$  such that:

$$\sigma(\varpi_E) = \begin{cases} \varpi_E & \text{if } E \text{ is unramified over } E_0; \\ -\varpi_E & \text{if } E \text{ is ramified over } E_0. \end{cases}$$

<sup>4</sup> Although this condition seems a little bit annoying, finally in §1.7 we may find out that this condition is automatically satisfied for  $\pi$  a  $\sigma$ -invariant supercuspidal representation.

Now we state the main theorem of this section:

**Theorem 1.6.2** (distinguished type theorem). *For  $\pi$  a  $\sigma$ -invariant cuspidal representation, it is  $G^\tau$ -distinguished if and only if it contains a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  such that  $\mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1) \neq 0$ .*

**Remark 1.6.3.** *Since every hermitian matrix is equivalent to one of the hermitian matrices mentioned in Remark 1.6.1(4) up to  $G$ -action, and the property of distinction is invariant up to equivalence of unitary groups, the theorem works for every unitary involution, although we only consider those occurring in loc. cit.*

Choose  $(\mathbf{J}, \Lambda)$  as in Remark 1.6.1, using the Mackey formula and the Frobenius reciprocity, we have

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_g \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1),$$

where  $g$  ranges over a set of representatives of  $(\mathbf{J}, G^\tau)$ -double cosets in  $G$ . So  $\pi$  is  $G^\tau$ -distinguished if and only if there exists  $g$  as a representative of a  $(\mathbf{J}, G^\tau)$ -double coset such that  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ . We will study such  $g$  and will show that  $(\mathbf{J}^g, \Lambda^g)$  is actually  $\tau$ -selfdual. This will finish the proof of this theorem.

### 1.6.1 Double cosets contributing to the distinction of $\theta$

First we have the following proposition:

**Proposition 1.6.4.** *For  $g \in G$ , the character  $\theta^g$  is trivial on  $H^{1g} \cap G^\tau$  if and only if  $\tau(g)g^{-1} \in JB^\times J$ .*

*Proof.* We only need to use the same proof of [Séc19], Proposition 6.6, with  $\sigma$  replaced by  $\tau$ .  $\square$

As a result, since  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$  implies that  $\mathrm{Hom}_{H^{1g} \cap G^\tau}(\theta^g, 1) \neq 0$ , using Proposition 1.6.4 we have  $\gamma := \tau(g)g^{-1} \in JB^\times J$ .

### 1.6.2 The double coset lemma

The next step is to prove the following double coset lemma:

**Lemma 1.6.5.** *Let  $g \in G$ . Then  $\gamma = \tau(g)g^{-1} \in JB^\times J$  if and only if  $g \in JB^\times G^\tau$ .*

*Proof.* If  $g \in JB^\times G^\tau$ , one verifies immediately that  $\gamma \in JB^\times J$ . Conversely, suppose that  $\gamma \in JB^\times J$ , first we need the following lemma:

**Lemma 1.6.6.** *There exists an element  $b \in B^\times$  such that  $\gamma \in JbJ$  and  $b\tau(b) = 1$ .*

*Proof.* Since  $B^\times \cap J = \mathfrak{b}^\times$  is a maximal compact subgroup of  $B^\times$ , using the Cartan decomposition over  $B^\times \cong \mathrm{GL}_m(E)$ , we write  $\gamma = xcy$  with  $x, y \in J$  and  $c = \mathrm{diag}(\varpi_E^{a_1} I_{m_1}, \dots, \varpi_E^{a_r} I_{m_r})$ , where  $a_1 > \dots > a_r$  as integers and  $m_1 + \dots + m_r = m$ .

If  $E/E_0$  is unramified, then by definition  $c^* = c$ . So if we choose  $b = c\varepsilon^{-1}$ , then  $b\varepsilon(b^*)^{-1}\varepsilon^{-1} = c(c^*)^{-1} = 1$ , that is,  $b\tau(b) = 1$ .

If  $E/E_0$  is ramified, since  $\tau(\gamma)\gamma = 1$ , we know that  $xcy = \varepsilon y^* c^* x^* \varepsilon^{-1}$  which is equivalent to  $(y^*)^{-1} \varepsilon^{-1} x c = c^* x^* \varepsilon^{-1} y^{-1}$ . Let  $z = x^* \varepsilon^{-1} y^{-1} \in J$ , then we have  $z^* c = c^* z$ . We regard  $z$  and  $c$  as matrices in  $M_m(M_d(F))$ . Denote by  $z^{(j)} \in M_{m_j}(M_d(F))$  the block matrix in  $z$  which is at the same place as  $\varpi_E^{a_j} I_{m_j}$  in  $c$ . Since  $z^* c = c^* z$ , by direct calculation

$$(z^{(j)})^* \varpi_E^{a_j} = (-1)^{a_j} \varpi_E^{a_j} z^{(j)} \quad \text{for } j = 1, \dots, r. \quad (1.6.1)$$



By considering the following embedding

$$\begin{aligned} M_{m_j}(M_d(F)) &\hookrightarrow M_m(M_d(F)) \\ h &\mapsto \text{diag}(0_{m_1d}, \dots, 0_{m_{j-1}d}, h, 0_{m_{j+1}d}, \dots, 0_{m_r d}), \end{aligned}$$

we may regard  $M_{m_jd}(F)$  as a subalgebra of  $M_{md}(F)$  which we denote by  $A^{(j)}$ , where  $0_{m_jd}$  represents the zero matrix of size  $m_jd \times m_jd$ . We write  $\mathfrak{a}^{(j)} = \mathfrak{a} \cap A^{(j)}$ . By abuse of notation, we identify the element  $\beta_0 \otimes \dots \otimes \beta_0$ , which consists of  $m_j$  copies of  $\beta_0$  and is contained in  $M_{m_j}(M_d(F))$ , with  $\beta$ . By [SS08], Théorème 2.17, since  $z \in J(\mathfrak{a}, \beta)$ , we get  $z^{(j)} \in J(\mathfrak{a}^{(j)}, \beta)$  for  $j = 1, \dots, r$ . By *loc. cit.*, if we denote by

$$M = \text{GL}_{m_1d}(F) \times \dots \times \text{GL}_{m_r d}(F)$$

the Levi subgroup of  $G$  corresponding to the partition  $n = m_1d + \dots + m_r d$ , then

$$M \cap J = J(\mathfrak{a}^{(1)}, \beta) \times \dots \times J(\mathfrak{a}^{(r)}, \beta)$$

and

$$M \cap J^1 = J^1(\mathfrak{a}^{(1)}, \beta) \times \dots \times J^1(\mathfrak{a}^{(r)}, \beta).$$

Thus we get  $\text{diag}(z^{(1)}, \dots, z^{(r)}) \in M \cap J$ . And further we have

$$M \cap J / M \cap J^1 \cong J(\mathfrak{a}^{(1)}, \beta) / J^1(\mathfrak{a}^{(1)}, \beta) \times \dots \times J(\mathfrak{a}^{(r)}, \beta) / J^1(\mathfrak{a}^{(r)}, \beta) \cong \text{GL}_{m_1}(\mathbf{l}) \times \dots \times \text{GL}_{m_r}(\mathbf{l}).$$

Since  $(\cdot)^*$  fixes  $M \cap J$  and  $M \cap J^1$ , we know that  $(\cdot)^*$  induces a map

$$\begin{aligned} M \cap J / M \cap J^1 &\cong \text{GL}_{m_1}(\mathbf{l}) \times \dots \times \text{GL}_{m_r}(\mathbf{l}) \longrightarrow \text{GL}_{m_1}(\mathbf{l}) \times \dots \times \text{GL}_{m_r}(\mathbf{l}) \cong M \cap J / M \cap J^1, \\ (\overline{z^{(1)}}, \dots, \overline{z^{(r)}}) &\longmapsto ((z^{(1)})^*, \dots, (z^{(r)})^*), \end{aligned}$$

where  $\mathbf{l}$  is the residue field of  $E$  and  $E_0$ , and  $\overline{z^{(j)}} \in J(\mathfrak{a}^{(j)}, \beta) / J^1(\mathfrak{a}^{(j)}, \beta) \cong \text{GL}_{m_j}(\mathbf{l})$  is the image of  $z^{(j)}$ .

We show that for any  $i$  such that  $2 \nmid a_i$ , we have  $2 \mid m_i$ . Consider  $j = i$  in equation (1.6.1), we get  $(z^{(i)})^* = -\varpi_E^{a_i} z^{(i)} \varpi_E^{-a_i}$ . Since  $J/J^1 \cong U(\mathfrak{b})/U^1(\mathfrak{b})$  on which  $E^\times$  acts trivially by conjugation, we get  $\overline{z^{(i)}} = \overline{\varpi_E^{a_i} z^{(i)} \varpi_E^{-a_i}} = -\overline{(z^{(i)})^*} = -\overline{z^{(i)}}$ . Since there exists no anti-symmetric invertible matrix of odd dimension, we must have  $2 \mid m_i$ . Now for  $\alpha_j = (a_j, m_j)$ , define

$$\varpi_E^{\alpha_j} = \begin{cases} \varpi_E^{a_j} I_{m_j} & \text{if } 2 \mid a_j; \\ \varpi_E^{a_j} J_{m_j/2} & \text{if } 2 \nmid a_j. \end{cases}$$

and  $c' = \text{diag}(\varpi_E^{\alpha_1}, \dots, \varpi_E^{\alpha_r})$ , where  $J_{m_j/2} := \begin{pmatrix} 0 & I_{m_j/2} \\ -I_{m_j/2} & 0 \end{pmatrix}$ . We have  $c' = c'^*$  and  $c'$  is in the same  $J$ - $J$  double coset as  $c$ . Let  $b = c'\varepsilon^{-1}$ , we get  $b\tau(b) = 1$ . □

Now we write  $\gamma = x'bx$  with  $x, x' \in J$  and  $b \in B^\times$  as in Lemma 1.6.6. Replacing  $g$  by  $\tau(x')^{-1}g$  does not change the double coset  $JgG^\tau$  but changes  $\gamma$  into  $bx\tau(x')$ . So from now on, we will assume that

$$\gamma = bx, \quad b\tau(b) = 1, \quad x \in J, \quad b \text{ is of the form in the proof of Lemma 1.6.6.} \quad (1.6.2)$$

Write  $K$  for the group  $J \cap b^{-1}Jb$ . Since  $\tau(b) = b^{-1}$  and  $J$  is  $\tau$ -stable, we have  $x \in K$ . We have the following corollary of Lemma 1.6.6

**Corollary 1.6.7.** *The map  $\delta_b : k \mapsto b^{-1}\tau(k)b$  is an involution on  $K$ .*

Now for  $a_1 > \dots > a_r$  as in the proof of Lemma 1.6.6, and  $M = \mathrm{GL}_{m_1 d}(F) \times \dots \times \mathrm{GL}_{m_r d}(F) \subseteq G$ , we write  $P$  for the standard parabolic subgroup of  $G$  generated by  $M$  and upper triangular matrices. We write  $N$  and  $N^-$  for the unipotent radicals of  $P$  and its opposite parabolic subgroup respectively. By definition of  $b$ , it normalizes  $M$  and we have

$$K = (K \cap N^-) \cdot (K \cap M) \cdot (K \cap N).$$

For  $V = K \cap B^\times = U \cap b^{-1}Ub$  as a subgroup of  $B^\times$ , similarly we have

$$V = (V \cap N^-) \cdot (V \cap M) \cdot (V \cap N),$$

where  $U = U(\mathfrak{b})$  and  $U^1 = J^1 \cap B^\times = U^1(\mathfrak{b})$ . By definition,  $V$  is also fixed by  $\delta_b$ .

**Lemma 1.6.8.** *The subset*

$$K^1 = (K \cap N^-) \cdot (J^1 \cap M) \cdot (K \cap N)$$

*is a  $\delta_b$ -stable normal pro- $p$ -subgroup of  $K$ , and we have  $K = VK^1$ .*

*Proof.* The proof is the same as that in [Séc19], Lemma 6.10. □

**Lemma 1.6.9.** *Let  $y \in K$  such that  $y\delta_b(y) = 1$ , then there are  $k \in K$  and  $v \in V$  such that:*

- (1) *the element  $v$  is in  $\mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \dots \times \mathrm{GL}_{m_r}(\mathfrak{o}_E) \subseteq B^\times$  such that  $v\delta_b(v) = 1$ ;*
- (2) *one has  $\delta_b(k)yk^{-1} \in vK^1$ .*

*Proof.* Let  $V^1 = V \cap K^1$ . We have

$$V^1 = (V \cap N^-) \cdot (U^1 \cap M) \cdot (V \cap N).$$

Thus we have canonical  $\delta_b$ -equivariant group isomorphisms

$$K/K^1 \cong V/V^1 \cong (U \cap M)/(U^1 \cap M). \quad (1.6.3)$$

Since  $B^\times \cap M = \mathrm{GL}_{m_1}(E) \times \dots \times \mathrm{GL}_{m_r}(E)$ , the right side of (1.6.3) identifies with  $\mathcal{M} = \mathrm{GL}_{m_1}(\mathfrak{l}) \times \dots \times \mathrm{GL}_{m_r}(\mathfrak{l})$ , where  $\mathfrak{l}$  denotes the residue field of  $E$ . As in the proof of Lemma 1.6.6, we may write  $\varepsilon^{-1}b = \mathrm{diag}(\varpi_E^{a_1}c_1, \dots, \varpi_E^{a_r}c_r)$  with  $c_j \in \mathrm{GL}_{m_j}(\mathfrak{o}_E)$ . Moreover, the involution  $\delta_b$  acts on  $\mathcal{M}$  by

$$(g_1, \dots, g_r) \mapsto (\overline{c_1}^{-1}\sigma({}^t g_1^{-1})\overline{c_1}, \dots, \overline{c_r}^{-1}\sigma({}^t g_r^{-1})\overline{c_r}),$$

where we denote by  $\overline{c_j}$  the image of  $c_j$  in  $\mathrm{GL}_{m_j}(\mathfrak{l})$ . We denote by  $(g_1, \dots, g_r)$  the image of  $y$  in  $\mathcal{M} = \mathrm{GL}_{m_1}(\mathfrak{l}) \times \dots \times \mathrm{GL}_{m_r}(\mathfrak{l})$ .

When  $E/E_0$  is unramified, we denote by  $\mathfrak{l}_0$  the residue field of  $E_0$ . So  $\mathfrak{l}/\mathfrak{l}_0$  is quadratic and the restriction of  $\sigma$  to  $\mathfrak{l}$  is the non-trivial involution in  $\mathrm{Gal}(\mathfrak{l}/\mathfrak{l}_0)$ . Since  $(b^{-1}\varepsilon)^* = \varepsilon(b^*)^{-1}\varepsilon^{-1}\varepsilon = \tau(b)\varepsilon = b^{-1}\varepsilon$ , we get  $\overline{c_j}^* = \overline{c_j}$ . If  $y\delta_b(y) = 1$ , then we get  $(\overline{c_j}g_j)^* = g_j^*\overline{c_j} = \overline{c_j}g_j$ . We need the following lemma:

**Lemma 1.6.10** ([KL90], Proposition 2.3.1). *For  $x = x^*$  in  $\mathrm{GL}_s(\mathfrak{l})$ , there exists  $A \in \mathrm{GL}_s(\mathfrak{l})$  such that  $AxA^* = I_s$ .*

Using Lemma 1.6.10, we choose  $k_j \in \mathrm{GL}_{m_j}(\mathfrak{o}_E)$  such that its image  $\overline{k_j}$  in  $\mathrm{GL}_{m_j}(\mathfrak{l})$  satisfies  $(\overline{k_j}^*)^{-1}\overline{c_j}g_j\overline{k_j}^{-1} = I_{m_j}$ . Choose  $k = \mathrm{diag}(k_1, \dots, k_r)$  and  $v = \mathrm{diag}(v_1, \dots, v_r) = \mathrm{diag}(c_1^{-1}, \dots, c_r^{-1})$ , we get  $\delta_b(k)yk^{-1} \in vV^1$  and  $\delta_b(v)v = \mathrm{diag}(c_1^{-1}c_1^*c_1c_1^{-1}, \dots, c_r^{-1}c_r^*c_rc_r^{-1}) = 1$ .

When  $E/E_0$  is ramified, the restriction of  $\sigma$  on  $\mathfrak{l}$  is trivial. Since  $(b^{-1}\varepsilon)^* = b^{-1}\varepsilon$ , we get  $c_j^* = (-1)^{a_j}c_j$  and  ${}^t\overline{c_j} = (-1)^{a_j}\overline{c_j}$ . We need the following two lemmas:

**Lemma 1.6.11** ([KL90], Proposition 2.5.4). For  $x = {}^t x$  in  $\mathrm{GL}_s(\mathbf{l})$ , there exists  $A \in \mathrm{GL}_s(\mathbf{l})$  such that  $Ax {}^t A$  is either  $I_s$  or  $\bar{\varepsilon}_s = \mathrm{diag}(1, \dots, 1, \bar{\varepsilon})$ , where  $\bar{\varepsilon} \in \mathbf{l}^\times \setminus \mathbf{l}^{\times 2}$  with  $\mathbf{l}^{\times 2}$  denoting the group of square elements of  $\mathbf{l}^\times$ .

**Lemma 1.6.12** ([KL90], Proposition 2.4.1). For  $x = -{}^t x$  in  $\mathrm{GL}_s(\mathbf{l})$  and  $s$  even, there exists  $A \in \mathrm{GL}_s(\mathbf{l})$  such that  $Ax {}^t A = J_{s/2}$ .

When  $a_j$  is even, using Lemma 1.6.11 we may choose  $k_j \in \mathrm{GL}_{m_j}(\mathfrak{o}_E)$  such that its image  $\bar{k}_j$  in  $\mathrm{GL}_{m_j}(\mathbf{l})$  satisfies that  $({}^t \bar{k}_j)^{-1} \bar{c}_j g_j \bar{k}_j^{-1}$  equals either  $I_{m_j}$  or  $\bar{\varepsilon}_{m_j}$ , where we choose  $\varepsilon_{m_j} = \mathrm{diag}(1, \dots, 1, \epsilon) \in \mathrm{GL}_{m_j}(\mathfrak{o}_E)$  such that its image  $\bar{\varepsilon}_{m_j}$  in  $\mathrm{GL}_{m_j}(\mathbf{l})$  is  $\mathrm{diag}(1, \dots, 1, \bar{\varepsilon})$  as in Lemma 1.6.11. Let  $v_j$  be  $c_j^{-1}$  or  $c_j^{-1} \varepsilon_{m_j}$  in the two cases respectively.

When  $a_j$  is odd  $m_j$  is even from the proof of Lemma 1.6.6. Using Lemma 1.6.12, we may choose  $k_j \in \mathrm{GL}_{m_j}(\mathfrak{o}_E)$  such that its image  $\bar{k}_j$  in  $\mathrm{GL}_{m_j}(\mathbf{l})$  satisfies  $({}^t \bar{k}_j)^{-1} \bar{c}_j g_j \bar{k}_j^{-1} = J_{m_j/2}$ . We choose  $v_j = c_j^{-1} J_{m_j/2}$ .

Choose  $k = \mathrm{diag}(k_1, \dots, k_r)$  and  $v = \mathrm{diag}(v_1, \dots, v_r)$ , we know that

$$\delta_b(k) y k^{-1} \in v V^1$$

and

$$\delta_b(v) v = \mathrm{diag}(c_1^{-1} (v_1^*)^{-1} c_1 v_1, \dots, c_r^{-1} (v_r^*)^{-1} c_r v_r) = 1$$

by direct calculation in the two cases respectively. So no matter  $E/E_0$  is ramified or not, we finish the proof.  $\square$

Now we finish the proof of Lemma 1.6.5. Applying Lemma 1.6.9 to  $x$  gives us  $k \in K$  and  $v \in V$ , such that  $bv\tau(bv) = 1$  and  $\delta_b(k) x k^{-1} \in v K^1$ . Thus we have  $\tau(k) \gamma k^{-1} \in bv K^1$ . Therefore replacing  $g$  by  $kg$  and  $b$  by  $bv$ , we assume that  $\gamma$  can be written as:

$$\gamma = bx, \quad b\tau(b) = 1, \quad x \in K^1, \quad b \in \varpi_E^{a_1} \mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \dots \times \varpi_E^{a_r} \mathrm{GL}_{m_r}(\mathfrak{o}_E). \quad (1.6.4)$$

Furthermore, we have  $\delta_b(x)x = 1$ .

Since  $K^1$  is a  $\delta_b$ -stable pro- $p$ -group and  $p$  is odd, the first cohomology set of  $\delta_b$  on  $K^1$  is trivial. Thus  $x = \delta_b(y)y^{-1}$  for some  $y \in K^1$ , hence  $\gamma = \tau(y)by^{-1}$ . Consider the determinant of this equation, we have  $\det(b) \in N_{F/F_0}(F^\times)$ . If we denote by  $\det_B$  the determinant function defined on  $B^\times = \mathrm{GL}_m(E)$ , then we have  $\det(b) = N_{E/F}(\det_B(b))$ . Using Lemma 1.5.18 for  $L = E$ , we get  $\det_B(b) \in N_{E/E_0}(E^\times)$  and  $\det_B(\varepsilon^{-1}b) \in \det_B(\varepsilon^{-1})N_{E/E_0}(E^\times)$ . Since  $\tau(b)b = 1$ , we have  $(\varepsilon^{-1}b)^* = \varepsilon^{-1}b$ . Using Proposition 1.2.1, there exists  $h \in B^\times$  such that  $\varepsilon^{-1}b = (h^*)^{-1} \varepsilon^{-1} h^{-1}$ . So we have  $b = \tau(h)h^{-1}$ . Thus  $g \in yhG^\tau \subseteq JB^\times G^\tau$ , which finishes the proof of Lemma 1.6.5.  $\square$

### 1.6.3 Distinction of the Heisenberg representation

Now let  $\eta$  be the Heisenberg representation of  $J^1$  associated to  $\theta$ . We have the following result similar to [Séc19], Proposition 6.12. by replacing  $\sigma$  with  $\tau$ :

**Proposition 1.6.13.** Given  $g \in G$ , we have:

$$\dim_R \mathrm{Hom}_{J^1 g \cap G^\tau}(\eta^g, 1) = \begin{cases} 1 & \text{if } g \in JB^\times G^\tau, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* It is useful to recall the detail of the proof of this proposition, which will be used in the next subsection. We write  $\delta(x) := \gamma^{-1}\tau(x)\gamma$  for any  $x \in G$  which is an involution on  $G$ . And for any subgroup  $H \subset G$ , we have  $H^g \cap G^\tau = (H \cap G^\delta)^g$ .

When  $g \notin JB^\times G^\tau$ , restricting  $\eta^g$  to  $H^{1g}$  and using Proposition 1.6.4 and Lemma 1.6.5, we know that the dimension equals 0. When  $g \in JB^\times G^\tau$ , we need to prove that  $\text{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \text{Hom}_{J^1 \cap G^\delta}(\eta, 1)$  is of dimension 1. We state the following general proposition which works for a general involution on  $G$ :

**Proposition 1.6.14.** *Let  $\delta$  be an involution on  $G$  such that  $\delta(H^1) = H^{1\gamma}$  and  $\theta \circ \delta = \theta^{-1\gamma}$ , where  $\gamma \in B^\times$  such that  $\delta(\gamma)\gamma = 1$ . Then we have*

$$\dim_R \text{Hom}_{J^1 \cap G^\delta}(\eta, 1) = 1.$$

Since Proposition 1.6.14 in our special case implies Proposition 1.6.13, we only need to focus on the proof of this proposition. We only need to prove that the space

$$\text{Hom}_{J^1 \cap G^\delta}(\eta^{(J^1 : H^1)^{1/2}}, 1) \cong \text{Hom}_{J^1 \cap G^\delta}(\text{Ind}_{H^1}^{J^1}(\theta), 1)$$

is of dimension  $(J^1 : H^1)^{1/2}$ . First we prove the following lemmas which will also be used in the next subsection:

**Lemma 1.6.15.** *For  $H$  a subgroup of  $G$  such that  $\delta(H) = H^\gamma$  with  $\delta$  and  $\gamma$  as in Proposition 1.6.14, we have*

$$H \cap G^\delta = H^\gamma \cap G^\delta = H \cap H^\gamma \cap G^\delta.$$

*Proof.* We have  $H \cap G^\delta = \delta(H \cap G^\delta) = \delta(H) \cap \delta(G^\delta) = H^\gamma \cap G^\delta$  which proves the lemma.  $\square$

**Lemma 1.6.16.** *Let  $\delta$  and  $\gamma$  be as in Proposition 1.6.14, then we have the following isomorphisms of finite dimensional representations:*

- (1)  $\text{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap J^{1\gamma}} \cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta;$
- (2)  $\text{Ind}_{H^{1\gamma}}^{J^{1\gamma}} \theta^\gamma|_{J^1 \cap J^{1\gamma}} \cong \bigoplus_{H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}} \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma;$
- (3)  $\text{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap G^\delta} \cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \bigoplus_{H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} \theta;$
- (4)  $\text{Ind}_{H^{1\gamma}}^{J^{1\gamma}} \theta^\gamma|_{J^{1\gamma} \cap G^\delta} \cong \bigoplus_{H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}} \bigoplus_{J^1 \cap H^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^{1\gamma} \cap G^\delta} \text{Ind}_{H^{1\gamma} \cap G^\delta}^{J^{1\gamma} \cap G^\delta} \theta.$

*Proof.* We only prove (1) and (3), since the proofs of (2) and (4) are similar to the proofs of (1) and (3) respectively.

For (1), using the Mackey formula, we have

$$\begin{aligned} \text{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap J^{1\gamma}} &\cong \bigoplus_{x \in H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{H^{1x} \cap (J^1 \cap J^{1\gamma})}^{J^1 \cap J^{1\gamma}} \theta^x \\ &\cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta. \end{aligned}$$

The last step is because  $x \in J^1$  normalizes  $H^1$  and  $\theta$ .

For (3), using the Mackey formula again, we have

$$\begin{aligned}
\text{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap G^\delta} &\cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta|_{J^1 \cap G^\delta} \\
&\cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \bigoplus_{y \in H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Ind}_{(H^1 \cap J^{1\gamma})^y \cap (J^1 \cap G^\delta)}^{J^1 \cap G^\delta} \theta^y \\
&\cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \bigoplus_{H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} \theta.
\end{aligned}$$

The last step is because  $y \in J^1 \cap J^{1\gamma}$  normalizes  $H^1 \cap J^{1\gamma}$  and  $\theta$ , and  $H^1 \cap J^{1\gamma} \cap J^1 \cap G^\delta = H^1 \cap G^\delta$  by Lemma [1.6.15](#)(2) for  $H = J^1$ . So we finish the proof.  $\square$

**Lemma 1.6.17.** *Let  $\delta$  and  $\gamma$  be as in Proposition [1.6.14](#), then we have:*

- (1)  $|H^1 \backslash J^1 / J^1 \cap J^{1\gamma}| \cdot |H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta| = (J^1 : H^1)^{1/2}$ ;
- (2)  $|H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}| \cdot |J^1 \cap H^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^{1\gamma} \cap G^\delta| = (J^{1\gamma} : H^{1\gamma})^{1/2}$ ;
- (3)  $(J^1 : H^1)^{1/2} = (J^{1\gamma} : H^{1\gamma})^{1/2} = (J^1 \cap G^\delta : H^1 \cap G^\delta)$ .

*Proof.* For (3), we refer to [Séc19](#) subsection 6.3 for a proof, by noting that all the results and proofs from Lemma 6.14 to the end of subsection 6.3 in *ibid.* can be generalized to a general involution  $\delta$  of  $G$ , with  $\tau$  in *loc. cit.* replaced by  $\delta$  in our settings. For (1), since  $J^1$  normalizes  $H^1$  and  $J^1 \cap J^{1\gamma}$  normalizes  $H^1 \cap J^{1\gamma}$ , we have

$$\begin{aligned}
\text{left hand side of (1)} &= (J^1 : H^1(J^1 \cap J^{1\gamma})) \cdot (J^1 \cap J^{1\gamma} : (H^1 \cap J^{1\gamma})(J^1 \cap G^\delta)) \\
&= (J^1 : H^1) \cdot (J^1 \cap J^{1\gamma} : H^1 \cap J^{1\gamma})^{-1} \\
&\quad \cdot (J^1 \cap J^{1\gamma} : H^1 \cap J^{1\gamma}) \cdot (J^1 \cap G^\delta : H^1 \cap J^{1\gamma} \cap G^\delta)^{-1} \\
&= (J^1 : H^1) \cdot (J^1 \cap G^\delta : H^1 \cap G^\delta)^{-1} \\
&= (J^1 : H^1)^{1/2},
\end{aligned}$$

where we use Lemma [1.6.15](#) for  $H = J^{1\gamma}$  and (3) in the last two equations. So we finish the proof of (1), and the proof of (2) is similar.  $\square$

Combining Lemma [1.6.16](#)(3) with Lemma [1.6.17](#)(1)(3), we have

$$\begin{aligned}
\dim_R \text{Hom}_{J^1 \cap G^\delta}(\text{Ind}_{H^1}^{J^1} \theta, 1) &= \dim_R \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \bigoplus_{H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Hom}_{J^1 \cap G^\delta}(\text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} \theta, 1) \\
&= (J^1 : H^1)^{1/2} \dim_R \text{Hom}_{H^1 \cap G^\delta}(\theta|_{H^1 \cap G^\delta}, 1) \\
&= (J^1 : H^1)^{1/2}.
\end{aligned}$$

For the last step, since  $\gamma$  intertwines  $\theta^{-1}$  and since  $\theta \circ \delta = \theta^{-1\gamma}$ , we know that  $\theta$  is trivial on

$$\{y\delta(y) | y \in H^1 \cap H^{1\gamma}\}.$$

This set equals  $H^1 \cap G^\delta$  since the the first cohomology group of  $\delta^{-1}$ -action on  $H^1 \cap H^{1\gamma}$  is trivial. Thus  $\theta|_{H^1 \cap G^\delta}$  is a trivial character. So we finish the proof.  $\square$

### 1.6.4 Distinction of extensions of the Heisenberg representation

Let  $\kappa$  be an irreducible representation of  $\mathbf{J}$  extending  $\eta$ . There is a unique irreducible representation  $\rho$  of  $\mathbf{J}$ , which is trivial on  $J^1$  satisfying  $\Lambda \cong \kappa \otimes \rho$ . First we have the following lemma:

**Lemma 1.6.18.** *Let  $g \in JB^\times G^\tau$ .*

(1) *There is a unique character  $\chi$  of  $\mathbf{J}^g \cap G^\tau$  trivial on  $J^{1g} \cap G^\tau$  such that*

$$\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1}).$$

(2) *The canonical linear map*

$$\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \otimes \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi) \rightarrow \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

*is an isomorphism.*

*Proof.* The proof is the same as that in [Séc19], Lemma 6.20. □

For  $g \in JB^\times G^\tau$ , we have  $\tau(g) \in \tau(JB^\times G^\tau) = JB^\times G^\tau$ , which means that we may consider the similar thing for  $\tau(g)$  as for  $g$  in Lemma 1.6.18. Thus, there exists a unique character  $\chi'$  of  $\mathbf{J}^{\tau(g)} \cap G^\tau$  trivial on  $J^{1\tau(g)} \cap G^\tau$  such that

$$\mathrm{Hom}_{J^{1\tau(g)} \cap G^\tau}(\eta^{\tau(g)}, 1) \cong \mathrm{Hom}_{\mathbf{J}^{\tau(g)} \cap G^\tau}(\kappa^{\tau(g)}, \chi'^{-1}).$$

Moreover, we know that  $\tau(\mathbf{J}) = \mathbf{J}$ ,  $\tau(J) = J$ ,  $\tau(J^1) = J^1$  and  $\tau(H^1) = H^1$ , thus using Lemma 1.4.2 and Lemma 1.6.15, we have  $\mathbf{J}^g \cap G^\tau = \mathbf{J}^{\tau(g)} \cap G^\tau = J^g \cap G^\tau = J^{\tau(g)} \cap G^\tau$ ,  $J^{1g} \cap G^\tau = J^{1\tau(g)} \cap G^\tau$  and  $H^{1g} \cap G^\tau = H^{1\tau(g)} \cap G^\tau$ . As a result,  $\chi$  and  $\chi'$  are characters defined on the same group  $\mathbf{J}^g \cap G^\tau = \mathbf{J}^{\tau(g)} \cap G^\tau$ . A natural idea is to compare them. For the rest of this subsection, we focus on the proof of the following proposition:

**Proposition 1.6.19.** *For  $\chi$  and  $\chi'$  defined above as characters of  $\mathbf{J}^g \cap G^\tau = \mathbf{J}^{\tau(g)} \cap G^\tau$ , we have  $\chi = \chi'$ .*

We write  $\delta(x) = \gamma^{-1}\tau(x)\gamma$  for any  $x \in G$  with  $\gamma = \tau(g)g^{-1}$ . From §1.3.1, we have  $\gamma \in I_G(\eta) = I_G(\kappa^0)$ , where  $\kappa^0 = \kappa|_J$ . Moreover we have

$$\dim_R(\mathrm{Hom}_{J \cap J^\gamma}(\kappa^{0\gamma}, \kappa^0)) = \dim_R(\mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta)) = 1.$$

Using Lemma 1.6.15, we have  $J^1 \cap G^\delta = J^{1\gamma} \cap G^\delta$  as a subgroup of  $J^1 \cap J^{1\gamma}$  and  $H^1 \cap G^\delta = H^{1\gamma} \cap G^\delta$ . We claim the following proposition which works for general  $\gamma$  and  $\delta$ :

**Proposition 1.6.20.** *Let  $\delta$  and  $\gamma$  be as in Proposition 1.6.14, then for a non-zero homomorphism  $\varphi \in \mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta) = \mathrm{Hom}_{J \cap J^\gamma}(\kappa^{0\gamma}, \kappa^0)$ , it naturally induces an  $R$ -vector space isomorphism:*

$$\begin{aligned} f_\varphi : \mathrm{Hom}_{J^1 \cap G^\delta}(\eta, 1) &\rightarrow \mathrm{Hom}_{J^{1\gamma} \cap G^\delta}(\eta^\gamma, 1), \\ \lambda &\mapsto \lambda \circ \varphi \end{aligned}$$

First we show that how does Proposition 1.6.20 imply Proposition 1.6.19. Using Proposition 1.6.13 for  $g$  and  $\tau(g)$  respectively, we have  $\dim_R \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \dim_R \mathrm{Hom}_{J^{1\tau(g)} \cap G^\tau}(\eta^{\tau(g)}, 1) = 1$ . By Proposition 1.6.20, we deduce that

$$\begin{aligned} f_\varphi : \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) &\rightarrow \mathrm{Hom}_{J^{1\tau(g)} \cap G^\tau}(\eta^{\tau(g)}, 1), \\ \lambda &\mapsto \lambda \circ \varphi, \end{aligned}$$

is bijective. If we choose

$$0 \neq \lambda \in \text{Hom}_{J^g \cap G^\tau}(\eta^g, 1) \quad \text{and} \quad 0 \neq \lambda' := f_\varphi(\lambda) = \lambda \circ \varphi \in \text{Hom}_{J^{\tau(g)} \cap G^\tau}(\eta^{\tau(g)}, 1),$$

then for any  $v$  in the representation space of  $\eta$  and any  $x \in J^g \cap G^\tau = J^{\tau(g)} \cap G^\tau$ , we have

$$\begin{aligned} \chi'(x)^{-1} \lambda'(v) &= \lambda'(\kappa^{0\tau(g)}(x)v) && \text{(by Proposition \ref{1.6.18})(1)} \\ &= \lambda(\varphi(\kappa^{0\tau(g)}(x)v)) && \text{(by definition of } \lambda') \\ &= \lambda(\kappa^{0g}(x)\varphi(v)) && \text{(since } \varphi \in \text{Hom}_{J^g \cap J^{\tau(g)}}(\kappa^{0\tau(g)}, \kappa^{0g})) \\ &= \chi(x)^{-1} \lambda(\varphi(v)) && \text{(by Proposition \ref{1.6.18})(1)} \\ &= \chi(x)^{-1} \lambda'(v) && \text{(by definition of } \lambda'). \end{aligned}$$

Since  $v$  and  $x \in J^g \cap G^\tau = J^{\tau(g)} \cap G^\tau$  are arbitrary, we have  $\chi'|_{J^{\tau(g)} \cap G^\tau} = \chi|_{J^g \cap G^\tau}$  which is Proposition [1.6.19](#).

So we only need to focus on the proof of Proposition [1.6.20](#). First of all, we need the following important lemma:

**Lemma 1.6.21.** *Let  $\delta$  and  $\gamma$  be as in Proposition [1.6.14](#), then there exist an  $R[J^1 \cap J^{1\gamma}]$ -module homomorphism*

$$\Phi : \eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}} \cong \text{Ind}_{H^{1\gamma}}^{J^{1\gamma}} \theta^\gamma|_{J^1 \cap J^{1\gamma}} \rightarrow \text{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap J^{1\gamma}} \cong \eta^{(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}}$$

and a linear form  $\widetilde{L}_0 \in \text{Hom}_{J^1 \cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1)$ , such that

$$0 \neq \widetilde{L}_0 \circ \Phi \in \text{Hom}_{J^1 \cap G^\delta}(\eta^{\gamma(J^{1\gamma}:H^{1\gamma})^{1/2}}, 1).$$

*Proof.* We prove this lemma by giving a direct construction of  $\Phi$  and  $\widetilde{L}_0$ . First we choose our  $\widetilde{L}_0$ . We choose  $\lambda_0 \in \text{Hom}_{J^1 \cap G^\delta}(\text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} 1, 1) \cong R$  with the isomorphism given by the Frobenius reciprocity, such that its corresponding image in  $R$  equals 1. Then we choose  $\widetilde{L}_0 = (\lambda_0, \dots, \lambda_0)$  as a element in

$$\bigoplus_{H^1 \setminus J^1 / J^1 \cap J^{1\gamma}} \bigoplus_{H^1 \cap J^{1\gamma} \setminus J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Hom}_{J^1 \cap G^\delta}(\text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} 1, 1) \cong \text{Hom}_{J^1 \cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1),$$

where the isomorphism is determined by Lemma [1.6.16](#)(3), and by Lemma [1.6.17](#) the number of copies equals  $(J^1 : H^1)^{1/2}$ .

Now we focus on the construction of  $\Phi$ . We define

$$f_0(g) := \begin{cases} \theta^\gamma(g_1)\theta(g_2) & \text{if } g = g_1g_2 \in (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) \\ 0 & \text{if } g \in J^1 \cap J^{1\gamma} \setminus (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}). \end{cases} \quad (1.6.5)$$

as a continuous function defined on  $J^1 \cap J^{1\gamma}$  with values in  $R$ . Since  $(J^1 \cap H^{1\gamma}) \cap (H^1 \cap J^{1\gamma}) = H^1 \cap H^{1\gamma}$  and  $\theta^\gamma = \theta$  on  $H^1 \cap H^{1\gamma}$ , we know that  $f_0$  is well-defined.

We want to verify that  $f_0 \in \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$  and  $f_0 \in \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$ . Since  $J^1$  normalizes  $H^1$  and  $J^{1\gamma}$  normalizes  $H^{1\gamma}$ , by direct calculation we deduce that  $J^1 \cap J^{1\gamma}$  normalizes  $J^1 \cap H^{1\gamma}$  and  $H^1 \cap J^{1\gamma}$ . In particular, we have  $(J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) = (H^1 \cap J^{1\gamma})(J^1 \cap H^{1\gamma})$ . Moreover, since  $J^1$  and  $J^{1\gamma}$  normalize  $\theta$  and  $\theta^\gamma$  respectively, we deduce that  $(J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma}) = (H^1 \cap J^{1\gamma})(J^1 \cap H^{1\gamma})$  normalizes  $\theta$  and  $\theta^\gamma$ .

For  $g'_1 \in J^1 \cap H^{1\gamma}$ ,  $g'_2 \in H^1 \cap J^{1\gamma}$  and  $g \in J^1 \cap J^{1\gamma}$ , if  $g \notin (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma})$ , then we have  $g'_1g, g'_2g \notin (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma})$ , thus

$$f_0(g'_1g) = f_0(g'_2g) = 0;$$

if  $g = g_1 g_2 \in (J^1 \cap H^{1\gamma})(H^1 \cap J^{1\gamma})$ , we have

$$f_0(g'_1 g) = \theta^\gamma(g'_1) \theta^\gamma(g_1) \theta(g_2) = \theta^\gamma(g'_1) f_0(g)$$

and

$$f_0(g'_2 g) = f_0(g'_2 g_1 g_2'^{-1} g'_2 g_2) = \theta^\gamma(g'_2 g_1 g_2'^{-1}) \theta(g'_2) \theta(g_2) = \theta(g'_2) \theta^\gamma(g_1) \theta(g_2) = \theta(g'_2) f_0(g).$$

Considering these facts, we have  $f_0 \in \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$  and  $f_0 \in \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$ .

We consider  $J^1 \cap J^{1\gamma}$ -action on  $f_0$  given by the right translation and we let  $\langle f_0 \rangle$  be the  $R[J^1 \cap J^{1\gamma}]$ -subspace of both  $\text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$  and  $\text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$  generated by  $f_0$ . We choose  $V_{f_0}$  to be a  $R[J^1 \cap J^{1\gamma}]$ -invariant subspace of  $\text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma$  such that  $\text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma = \langle f_0 \rangle \oplus V_{f_0}$ .

We define an  $R[J^1 \cap J^{1\gamma}]$ -module homomorphism

$$\Phi_1 : \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma \rightarrow \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$$

such that  $\Phi_1(f_0) = f_0$  and  $\Phi_1|_{V_{f_0}} = 0$ . And we define

$$\Phi : \bigoplus_{H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}} \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma \rightarrow \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta$$

given by  $\Phi = \text{diag}(\Phi_1, 0, \dots, 0) \in M_{N_1}(\text{Hom}_{R[J^1 \cap J^{1\gamma}]}(\text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma, \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta))$ , where the coordinates are indexed by  $N_1 := |H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}| = |H^1 \backslash J^1 / J^1 \cap J^{1\gamma}|$ . In particular, we let the first coordinate correspond to the trivial double cosets  $H^{1\gamma}(J^1 \cap J^{1\gamma})$  and  $H^1(J^1 \cap J^{1\gamma})$  respectively. As a result,  $\Phi$  gives an  $R[J^1 \cap J^{1\gamma}]$ -module homomorphism. By Lemma [1.6.16](#) we have

$$\eta^{(J^1:H^1)^{1/2}} \cong \text{Ind}_{H^1}^{J^1} \theta|_{J^1 \cap J^{1\gamma}} \cong \bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta \quad (1.6.6)$$

and

$$\eta^{\gamma(J^1:H^1)^{1/2}} \cong \text{Ind}_{H^{1\gamma}}^{J^{1\gamma}} \theta^\gamma|_{J^1 \cap J^{1\gamma}} \cong \bigoplus_{H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}} \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma. \quad (1.6.7)$$

With these two isomorphisms, we may regard  $\Phi$  as a homomorphism from  $\eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}}$  to  $\eta^{(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}}$ .

Finally, we study  $\widetilde{L}_0 \circ \Phi$ . First we calculate

$$\Phi_1 : \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma|_{J^1 \cap G^\delta} \rightarrow \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta|_{J^1 \cap G^\delta}.$$

We have the following isomorphism

$$\text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta|_{J^1 \cap G^\delta} \cong \bigoplus_{H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} \mathbf{1}. \quad (1.6.8)$$

By definition of  $\Phi_1$  and [\(1.6.5\)](#), [\(1.6.8\)](#),  $\Phi_1(f_0|_{J^1 \cap G^\delta}) = f_0|_{J^1 \cap G^\delta}$  equals

$$(\mathbf{1}_{H^1 \cap G^\delta}, \dots, \mathbf{1}_{H^1 \cap G^\delta}, 0, \dots, 0) \in \bigoplus_{H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta} \text{Ind}_{H^1 \cap G^\delta}^{J^1 \cap G^\delta} \mathbf{1}, \quad (1.6.9)$$

where the coordinates are indexed by the double coset  $H^1 \cap J^{1\gamma} \backslash J^1 \cap J^{1\gamma} / J^1 \cap G^\delta$ , and those coordinates which equal the characteristic function  $\mathbf{1}_{H^1 \cap G^\delta}$  are exactly indexed by the subset  $H^1 \cap J^{1\gamma} \backslash (J^1 \cap H^{1\gamma})(J^1 \cap H^{1\gamma}) / J^1 \cap G^\delta$ .



We define  $v_0 = (f_0|_{J^1 \cap G^\delta}, 0, \dots, 0)$  as an element in both

$$\bigoplus_{H^1 \backslash J^1 / J^1 \cap J^{1\gamma}} \text{Ind}_{J^1 \cap H^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta^\gamma|_{J^1 \cap G^\delta}$$

and

$$\bigoplus_{H^{1\gamma} \backslash J^{1\gamma} / J^1 \cap J^{1\gamma}} \text{Ind}_{H^1 \cap J^{1\gamma}}^{J^1 \cap J^{1\gamma}} \theta|_{J^1 \cap G^\delta},$$

where the first coordinate corresponds to the trivial double cosets  $H^1(J^1 \cap J^{1\gamma})$  and  $H^{1\gamma}(J^1 \cap J^{1\gamma})$  respectively as in our definition of  $\Phi$ . Thus we have

$$\begin{aligned} (\widetilde{L}_0 \circ \Phi)(v_0) &= \widetilde{L}_0((\Phi_1(f_0|_{J^1 \cap G^\delta}), 0, \dots, 0)) = \widetilde{L}_0((f_0|_{J^1 \cap G^\delta}, 0, \dots, 0)) \\ &= |H^1 \cap J^{1\gamma} \backslash (H^1 \cap J^{1\gamma})(J^1 \cap H^{1\gamma}) / J^1 \cap G^\delta| \lambda_0(\mathbf{1}_{H^1 \cap G^\delta}) \neq 0, \end{aligned}$$

where we use the definition of  $\widetilde{L}_0$  and [\(1.6.9\)](#) for the last equation. Thus we get  $\widetilde{L}_0 \circ \Phi \neq 0$  which finishes the proof.  $\square$

We also need the following lemma:

**Lemma 1.6.22.** *We keep the same notations as in Proposition [1.6.20](#) and we fix*

$$0 \neq \lambda'_0 \in \text{Hom}_{J^1 \cap G^\delta}(\eta, 1) \quad \text{and} \quad 0 \neq \lambda''_0 \in \text{Hom}_{J^1 \cap G^\delta}(\eta^\gamma, 1).$$

Then:

(1) *For any  $\widetilde{L} \in \text{Hom}_{J^1 \cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1)$ , there exists an  $R[J^1 \cap J^{1\gamma}]$ -homomorphism*

$$\text{Pr} : \eta^{(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}} \rightarrow \eta|_{J^1 \cap J^{1\gamma}}$$

such that  $\widetilde{L} = \lambda'_0 \circ \text{Pr}$ ;

(2) *For any  $\widetilde{L} \in \text{Hom}_{J^1 \cap G^\delta}(\eta^{\gamma(J^1:H^1)^{1/2}}, 1)$ , there exists an  $R[J^1 \cap J^{1\gamma}]$ -homomorphism*

$$s : \eta^\gamma|_{J^1 \cap J^{1\gamma}} \rightarrow \eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}}$$

such that  $\lambda''_0 = \widetilde{L} \circ s$ .

*Proof.* The proof is just a simple application of linear algebra. We write  $N = (J^1 : H^1)^{1/2}$ . For (1), we define  $\text{pr}_i : \eta^{(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}} \rightarrow \eta|_{J^1 \cap J^{1\gamma}}$  as the projection with respect to the  $i$ -th coordinate. Since  $\lambda'_0 \circ \text{pr}_1, \dots, \lambda'_0 \circ \text{pr}_N$  are linearly independent, and  $\dim_R \text{Hom}_{J^1 \cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1) = N$  as an  $R$ -vector space by Proposition [1.6.13](#),  $\lambda'_0 \circ \text{pr}_1, \dots, \lambda'_0 \circ \text{pr}_N$  generate  $\text{Hom}_{J^1 \cap G^\delta}(\eta^{(J^1:H^1)^{1/2}}, 1)$ . So we may choose  $\text{Pr}$  to be a linear combination of  $\text{pr}_j$  which proves (1). The proof of (2) is similar.  $\square$

Now we finish the proof of Proposition [1.6.20](#). Using Lemma [1.6.22](#)(1) we choose  $\text{Pr}$  such that  $\widetilde{L}_0 = \lambda'_0 \circ \text{Pr}$ , where  $\widetilde{L}_0$  is defined as in the statement of Lemma [1.6.21](#). Using Lemma [1.6.21](#), there exists  $\Phi$  such that  $\widetilde{L}_0 \circ \Phi \neq 0$ . Using Lemma [1.6.22](#)(2) we choose  $s$  such that  $\widetilde{L}_0 \circ \Phi \circ s = \lambda''_0 \neq 0$ . We define  $\varphi' = \text{Pr} \circ \Phi \circ s$  and we have the following commutative diagram

$$\begin{array}{ccc} \eta^{\gamma(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}} & \xrightarrow{\Phi} & \eta^{(J^1:H^1)^{1/2}}|_{J^1 \cap J^{1\gamma}} \\ \uparrow s & & \downarrow \text{Pr} \\ \eta^\gamma|_{J^1 \cap J^{1\gamma}} & \xrightarrow{\varphi'} & \eta|_{J^1 \cap J^{1\gamma}} \end{array}$$

By definition we have  $\lambda'_0 \circ \varphi' = \lambda'_0 \circ \text{Pro}\Phi \circ s = \lambda''_0 \neq 0$ , which means that  $\varphi' \neq 0$ . Since  $\text{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta)$  is of dimension 1, we deduce that  $\varphi$  equals  $\varphi'$  multiplying with a non-zero scalar, which means that  $\lambda'_0 \circ \varphi \neq 0$ . Since  $\text{Hom}_{J^1 \cap G^\delta}(\eta, 1)$  and  $\text{Hom}_{J^1 \cap G^\delta}(\eta^\gamma, 1)$  are of dimension 1, we know that  $f_\varphi$  is an  $R$ -vector space isomorphism which proves Proposition [1.6.20](#).

### 1.6.5 Existence of a $\tau$ -selfdual extension of $\eta$

Now our aim is to choose a simple  $\kappa$  as an extension of  $\eta$ . Specifically, under the condition of Remark [1.6.1](#), we show that we may assume  $\kappa$  to be  $\tau$ -selfdual, which means that  $\kappa^\tau \cong \kappa^\vee$ . First of all, we have the following lemma whose proof is the same as that in [\[Séc19\]](#), Lemma 5.21:

**Lemma 1.6.23.** *There exists a unique character  $\mu$  of  $\mathbf{J}$  trivial on  $J^1$  such that  $\kappa^{\tau^\vee} \cong \kappa\mu$ . It satisfies the identity  $\mu \circ \tau = \mu$ .*

Now we are going to prove the following important proposition:

**Proposition 1.6.24.** *When  $\text{char}(R) = 0$ , there exists a character  $\phi$  of  $\mathbf{J}$  trivial on  $J^1$  such that  $\mu = \phi(\phi \circ \tau)$ . Moreover for any  $R$ , we may choose  $\kappa$  as an extension of  $\eta$  such that  $\kappa^{\tau^\vee} \cong \kappa$ .*

*Proof.* First we consider the case where  $\text{char}(R) = 0$ . we need the following elementary lemma:

**Lemma 1.6.25.** *Assume  $\text{char}(R) = 0$ . For  $N$  odd and  $A \in \text{GL}_N(R)$  such that  $A^{2^s} = cI_N$  for  $s \in \mathbb{N}$  and  $c \in R^\times$ , we have  $\text{Tr}(A) \neq 0$ .*

*Proof.*  $s = 0$  is trivial, so from now on we assume  $s \geq 1$ . Let  $\zeta_{2^s}$  be a primitive  $2^s$ -th root of 1 in  $R$  and let  $c^{1/2^s}$  be a  $2^s$ -th root of  $c$  in  $R$ , then we get  $\text{Tr}(A) = c^{1/2^s} \sum_{i=1}^N \zeta_{2^s}^{n_i}$  with  $n_i \in \{0, 1, 2, \dots, 2^s - 1\}$ . We know that  $P(x) = x^{2^{s-1}} + 1$  is the minimal polynomial of  $\zeta_{2^s}$  in  $\mathbb{Q}[x]$ . If  $\text{Tr}(A) = 0$ , then for  $Q(x) = \sum_{i=1}^N x^{n_i}$ , we have  $Q(\zeta_{2^s}) = 0$ . As a result,  $P(x) \mid Q(x)$  in  $\mathbb{Q}[x]$  thus in  $\mathbb{Z}[x]$  by the Gauss lemma. However, the sum of all the coefficients of  $P(x)$  is even and the sum of all the coefficients of  $Q(x)$  equals  $N$  which is odd. We get a contradiction. So  $\text{Tr}(A) \neq 0$ . □

Come back to our proof. We choose  $\kappa$  as any extension of  $\eta$ , thus as in Lemma [1.6.23](#), there exists  $\phi$  as a character of  $\mathbf{J}$  such that  $\kappa^{\tau^\vee} \cong \kappa\mu$ . If  $E/E_0$  is unramified, we let

$$\bar{\mu} : \text{GL}_m(\mathfrak{l}) \cong J/J^1 \rightarrow R^\times$$

be the character whose inflation is  $\mu|_J$ . There exists a character  $\varphi : \mathfrak{l}^\times \rightarrow R^\times$  such that  $\bar{\mu} = \varphi \circ \det$ . Since  $\bar{\mu} \circ \tau = \bar{\mu}$ , we get  $(\varphi \circ \sigma)\varphi = 1$ , or equivalently  $\varphi|_{\mathfrak{l}_0^\times} = 1$ , where  $\mathfrak{l}_0$  is the residue field of  $E_0$ , and  $\sigma$  acts on  $\mathfrak{l}$  as the Frobenius map corresponding to  $\mathfrak{l}_0$ . Let  $Q$  be the cardinality of  $\mathfrak{l}_0$ , then the cardinality of  $\mathfrak{l}$  is  $Q^2$ . If we fix  $\zeta_{\mathfrak{l}}$  a generator of  $\mathfrak{l}^\times$ , then  $\zeta_{\mathfrak{l}}^{Q+1}$  is a generator of  $\mathfrak{l}_0^\times$ . So we have  $\varphi(\zeta_{\mathfrak{l}})^{Q+1} = 1$ . Choose  $\alpha : \mathfrak{l}^\times \rightarrow R^\times$  as a character such that

$$\alpha(\zeta_{\mathfrak{l}}^m)^{Q-1} = \varphi(\zeta_{\mathfrak{l}})^{-m} \text{ for } m \in \mathbb{Z}.$$

Since

$$\alpha(\zeta_{\mathfrak{l}})^{Q^2-1} = \varphi(\zeta_{\mathfrak{l}})^{-Q-1} = 1,$$

$\alpha$  is well-defined as a character of  $\mathfrak{l}^\times$ . Moreover, we get  $\varphi = \alpha(\alpha \circ \sigma)^{-1}$ . Choose  $\phi^0 : J \rightarrow R^\times$  as the inflation of  $\alpha \circ \det$ , we get  $\mu|_J = \phi^0(\phi^0 \circ \tau)$ .

Since  $\varpi_E$  and  $J$  generate  $\mathbf{J}$ , to choose  $\phi$  as a character of  $\mathbf{J}$  extending  $\phi^0$ , it suffices to show that  $\mu(\varpi_E) = 1$ . Since  $\mu = \mu \circ \tau$ , we get

$$\mu(\varpi_E) = \mu(\tau(\varpi_E)) = \mu(\varpi_E)^{-1}, \text{ thus } \mu(\varpi_E) \in \{1, -1\}.$$

Let  $e$  be the ramification index of  $E/F$ , and let  $\varpi_E^e = a_0 \varpi_F$  for a certain  $a_0 \in \mathfrak{o}_E^\times$ . We have

$$\varpi_E^{e(Q-1)} = a_0^{Q-1} \varpi_F^{Q-1} \text{ with } a_0^{Q-1} \in 1 + \mathfrak{p}_E \subset H^1(\mathfrak{a}, \beta).$$

We write  $e(Q-1) = 2^s u$  for  $2 \nmid u$  and  $s \in \mathbb{N}$ . For  $A = \kappa(\varpi_E^u)$ , we have

$$A^{2^s} = \kappa(a_0^{Q-1} \varpi_F^{Q-1}) = \theta(a_0^{Q-1}) \omega_\kappa(\varpi_F^{Q-1}) I_N,$$

where we use the fact that the restriction of  $\kappa$  to  $H^1(\mathfrak{a}, \beta)$  equals  $N$ -copies of  $\theta$  with  $N = (J^1 : H^1)^{1/2}$ , and  $\omega_\kappa$  is the central character of  $\kappa$ . Using Lemma 1.6.25 with  $A$  and  $c = \theta(a_0^{Q-1}) \omega_\kappa(\varpi_F^{Q-1})$ , we get  $\text{Tr}(\kappa(\varpi_E^u)) \neq 0$ . Since  $\kappa^{\tau^\vee} \cong \kappa\mu$ , considering the trace of both sides at  $\varpi_E^u$ , we get

$$\text{Tr}(\kappa(\varpi_E^u)) = \text{Tr}(\kappa(\varpi_E^u)) \mu(\varpi_E^u),$$

thus  $\mu(\varpi_E^u) = 1$ . Since  $u$  is odd and  $\mu(\varpi_E)$  equals either 1 or  $-1$ , we get  $\mu(\varpi_E) = 1$  which finishes the proof of this case.

If  $E/E_0$  is ramified, first we show that  $\mu|_{\mathfrak{l}^\times} = 1$ , where we consider the embedding  $\mathfrak{l}^\times \hookrightarrow E^\times$ . Let  $Q$  be the cardinality of  $\mathfrak{l} = \mathfrak{l}_0$  and let  $\zeta_{\mathfrak{l}}$  be a generator of  $\mathfrak{l}^\times$ , then we want to show that  $\mu(\zeta_{\mathfrak{l}}) = 1$ . Write  $Q-1 = 2^s u$  with  $2 \nmid u$  and use Lemma 1.6.25 with  $A = \kappa(\zeta_{\mathfrak{l}}^u)$  and  $c = 1$ , we get  $\text{Tr}(\kappa(\zeta_{\mathfrak{l}}^u)) \neq 0$ . Since  $\kappa^{\tau^\vee} \cong \kappa\mu$ , we get

$$\text{Tr}(\kappa(\zeta_{\mathfrak{l}}^u)) = \text{Tr}(\kappa(\zeta_{\mathfrak{l}}^u)) \mu(\zeta_{\mathfrak{l}}^u)$$

after considering the trace of the isomorphism. Thus  $\mu(\zeta_{\mathfrak{l}}^u) = 1$ . Since  $\mu(\zeta_{\mathfrak{l}})$  equals either 1 or  $-1$  which can be proved as the former case and  $u$  is odd, we get  $\mu(\zeta_{\mathfrak{l}}) = 1$ . Thus  $\mu|_J = 1$ .

To finish the definition of  $\phi : \mathbf{J} \rightarrow R^\times$  such that  $\mu = \phi(\phi \circ \tau)$ , we only need to verify the equation

$$\mu(\varpi_E) = \phi(\varpi_E) \phi(\tau(\varpi_E)) = \phi(\varpi_E) \phi(-\varpi_E))^{-1} = \phi(-1)^{-1}.$$

Since we have already showed that  $\mu(-1) = 1$ , using the relation  $\mu = \mu \circ \tau$ , we get  $\mu(\varpi_E^2) = \mu(-\varpi_E^2) = \mu(\varpi_E) \mu(\tau(\varpi_E))^{-1} = 1$ , so we deduce that  $\mu(\varpi_E)$  equals either 1 or  $-1$ . Choose  $\phi(-1) = \mu(\varpi_E)$  which is well defined, we finish the definition of  $\phi$  such that  $\mu = \phi(\phi \circ \tau)$ . Let  $\kappa' = \kappa\phi$ , then  $\kappa'$  is  $\tau$ -selfdual.

Now we suppose  $R = \overline{\mathbb{F}}_l$ . Let  $\tilde{\theta}$  be the lift of  $\theta$  to  $\overline{\mathbb{Q}}_l$  given by the canonical embedding  $\overline{\mathbb{F}}_l^\times \hookrightarrow \overline{\mathbb{Q}}_l^\times$ , then  $\tilde{\theta}$  is a simple character and  $\tilde{\theta} \circ \tau = \tilde{\theta}^{-1}$ . There is a  $\tau$ -selfdual representation  $\tilde{\kappa}$  of  $\mathbf{J}$  extending the Heisenberg representation  $\tilde{\eta}$  of  $J^1$  corresponding to  $\tilde{\theta}$ . Moreover we can further choose  $\tilde{\kappa}$  such that the central character of  $\tilde{\kappa}$  is integral. To do this, first we choose  $\tilde{\kappa}^0$  as a representation of  $J$  extending  $\eta$ . We extend  $\tilde{\kappa}^0$  to a representation of  $F^\times J$ . This requires us to choose a quasi-character  $\tilde{\omega} : F^\times \rightarrow \overline{\mathbb{Q}}_l^\times$  extending  $\tilde{\omega}_{\tilde{\kappa}^0}$ . We choose  $\tilde{\omega}$  such that it is integral. If we further extend this representation to  $\tilde{\kappa}$  as a representation of  $J = E^\times J$ , then  $\tilde{\kappa}$  is also integral. From the proof of characteristic 0 case, we may further assume  $\tilde{\kappa}^{\tau^\vee} \cong \tilde{\kappa}$  without losing the property that  $\tilde{\kappa}$  is integral. By [MS14b], 2.11, the reduction of  $\tilde{\kappa}$  to  $\overline{\mathbb{F}}_l$ , denoted by  $\kappa$ , is thus a  $\tau$ -selfdual representation of  $\mathbf{J}$  extending  $\eta$ .

For  $\text{char}(R) = l > 0$  in general, we fix  $\iota : \overline{\mathbb{F}}_l \hookrightarrow R$  an embedding. For  $\theta$  a simple character over  $R$  as before which is of finite image, there exists a simple character  $\theta_0$  over  $\overline{\mathbb{F}}_l$  corresponding to the same simple stratum  $[\mathfrak{a}, \beta]$ , such that  $\theta = \iota \circ \theta_0$  and  $\theta_0 \circ \tau = \theta_0^{-1}$ . Let  $\eta_0$  be the Heisenberg representation of  $\theta_0$  and choose  $\kappa_0$  to be a  $\tau$ -selfdual extension of  $\eta_0$  by the former case. Then  $\kappa = \kappa_0 \otimes_{\overline{\mathbb{F}}_l} R$  is what we want. □

### 1.6.6 Proof of Theorem 1.6.2

Using Proposition 1.6.24, we may assume that  $\kappa$  is  $\tau$ -selfdual. From its proof, when  $R = \overline{\mathbb{F}}_l$ , we assume further that  $\kappa$  is the reduction of a  $\tau$ -selfdual representation  $\tilde{\kappa}$  of  $\mathbf{J}$  over  $\overline{\mathbb{Q}}_l$ , and when  $\text{char}(R) = l > 0$  in general, we assume  $\kappa$  to be realized as a  $\overline{\mathbb{F}}_l$ -representation via a certain field embedding  $\overline{\mathbb{F}}_l \hookrightarrow R$ .

**Proposition 1.6.26.** *The character  $\chi$  defined by Lemma 1.6.18(1) is quadratic when restricting to  $J^g \cap G^\tau$ , that is,  $\chi^2|_{J^g \cap G^\tau} = 1$ .*

*Proof.* First we assume that  $\text{char}(R) = 0$ . We have the following isomorphisms

$$\begin{aligned} \text{Hom}_{J^{1\tau(g)} \cap G^\tau}(\eta^{\tau(g)}, 1) &\cong \text{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \\ &\cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\kappa}^g, \chi^{-1}) \\ &\cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\chi, \boldsymbol{\kappa}^{g^\vee}) \quad (\text{by the duality of contragredient}) \\ &\cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\kappa}^{g^\vee}, \chi) \quad (\text{since } \text{char}(R) = 0) \\ &\cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\kappa}^{g^\vee} \circ \tau, \chi \circ \tau) \\ &\cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}((\boldsymbol{\kappa}^{\tau^\vee})^{\tau(g)}, \chi \circ \tau) \\ &\cong \text{Hom}_{\mathbf{J}^{\tau(g)} \cap G^\tau}(\boldsymbol{\kappa}^{\tau(g)}, \chi \circ \tau) \quad (\text{since } \boldsymbol{\kappa} \text{ is } \tau\text{-selfdual}). \end{aligned}$$

Using Proposition 1.6.19 and the uniqueness of  $\chi'$  in the *loc. cit.*, we have  $\chi \circ \tau = \chi^{-1}$ . Since  $\chi$  is defined on  $\mathbf{J}^g \cap G^\tau$  which is  $\tau$ -invariant, we have  $\chi \circ \tau = \chi$ . Thus  $\chi^2 = \chi(\chi \circ \tau) = 1$ .

If  $R = \overline{\mathbb{F}}_l$ , we denote by  $\tilde{\boldsymbol{\kappa}}$  a  $\tau$ -selfdual  $\overline{\mathbb{Q}}_l$ -lift of  $\boldsymbol{\kappa}$  and we denote by  $\tilde{\chi}$  the character defined by Lemma 1.6.18(1) with respect to  $\tilde{\boldsymbol{\kappa}}$  and  $\tilde{\eta}$ , where  $\tilde{\eta}$  is a  $J^1 \cap G^\tau$ -distinguished  $\overline{\mathbb{Q}}_l$ -lift of  $\eta$ . Using this proposition for  $\overline{\mathbb{Q}}_l$ -representations, we get  $\tilde{\chi}^2 = 1$ . From the uniqueness of  $\chi$ , we know that  $\tilde{\chi}$  is a  $\overline{\mathbb{Q}}_l$ -lift of  $\chi$ . As a result, we get  $\chi^2 = 1$ .

If  $\text{char}(R) = l > 0$  in general, from the assumption of  $\boldsymbol{\kappa}$  mentioned at the beginning of this subsection, via a field embedding  $\overline{\mathbb{F}}_l \hookrightarrow R$ , we may realize all the representations mentioned in this proposition as representations over  $\overline{\mathbb{F}}_l$ , so we finish the proof by using the former case.  $\square$

As in the proof of Lemma 1.6.5, we assume  $g \in B^\times$  and

$$\gamma = bx, \quad b\tau(b) = 1, \quad x \in K^1, \quad b \in \varpi_E^{a_1} \text{GL}_{m_1}(\mathfrak{o}_E) \times \dots \times \varpi_E^{a_r} \text{GL}_{m_r}(\mathfrak{o}_E). \quad (1.6.10)$$

There exists a unique standard hereditary order  $\mathfrak{b}_m \subseteq \mathfrak{b}$  such that

$$U^1(\mathfrak{b}_m) = (U \cap \delta(U^1))U^1 = (U \cap U^{1\gamma})U^1,$$

where we define  $\delta(y) = \gamma^{-1}\tau(y)\gamma$  for any  $y \in G$  as an involution on  $G$ . First we have the following lemma whose proof is the same as that in [Séc19], Lemma 6.22, inspired by [HM08], Proposition 5.20:

**Lemma 1.6.27.** *We have  $U^1(\mathfrak{b}_m) = (U^1(\mathfrak{b}_m) \cap G^\delta)U^1$ .*

Now we state and prove the following important theorem:

**Theorem 1.6.28.** *Let  $g \in G$  and suppose that  $\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1)$  is non-zero. Then  $\tau(g)g^{-1} \in \mathbf{J}$ .*

*Proof.* It is enough to show that  $r = 1$  in (1.6.10). If not,  $\mathfrak{b}_m$  by definition is a proper suborder of  $\mathfrak{b}$ . Furthermore,  $\overline{U^1(\mathfrak{b}_m)} := U^1(\mathfrak{b}_m)/U^1$  is a non-trivial unipotent subgroup of  $U/U^1 \cong \text{GL}_m(\mathfrak{l})$ . Using Lemma 1.6.18(2), we have

$$\text{Hom}_{\mathbf{J} \cap G^\delta}(\boldsymbol{\rho}, \chi^{g^{-1}}) \cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\boldsymbol{\rho}^g, \chi) \neq 0.$$

Restricting to  $U^1(\mathfrak{b}_m) \cap G^\delta$ , we have

$$\text{Hom}_{U^1(\mathfrak{b}_m) \cap G^\delta}(\boldsymbol{\rho}, \chi^{g^{-1}}) \neq 0. \quad (1.6.11)$$

Using Lemma [1.6.27](#), we have the isomorphism

$$(U^1(\mathfrak{b}_m) \cap G^\delta)U^1/U^1 \cong U^1(\mathfrak{b}_m)/U^1.$$

We denote by  $\bar{\rho}$  the cuspidal representation of  $U^0/U^1 \cong \mathrm{GL}_m(\mathfrak{l})$  whose inflation is  $\rho|_{U^0}$ , and  $\overline{\chi^{g^{-1}}}$  the character of  $\overline{U^1(\mathfrak{b}_m)}$  whose inflation is  $\chi^{g^{-1}}$ . So if we consider the equation [\(1.6.11\)](#) modulo  $U^1$ , then we get

$$\mathrm{Hom}_{\overline{U^1(\mathfrak{b}_m)}}(\bar{\rho}, \overline{\chi^{g^{-1}}}) \neq 0.$$

Since  $\chi^{g^{-1}}|_{J \cap G^\delta}$  is quadratic and  $\overline{U^1(\mathfrak{b}_m)}$  is a  $p$ -group with  $p \neq 2$ , we get  $\overline{\chi^{g^{-1}}} = 1$ , thus

$$\mathrm{Hom}_{\overline{U^1(\mathfrak{b}_m)}}(\bar{\rho}, 1) \neq 0,$$

which contradicts the fact that  $\bar{\rho}$  is cuspidal. So we finish the proof.  $\square$

*Proof of Theorem [1.6.2](#).* If there exists a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  in  $\pi$  such that  $\mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1)$  is non-zero, then  $\pi$  is  $G^\tau$ -distinguished. Conversely, there exists  $g \in G$  such that  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ . Using Theorem [1.6.28](#), we conclude that  $(\mathbf{J}^g, \Lambda^g)$  is a  $\tau$ -selfdual simple type.  $\square$

Finally we state the following corollary of Theorem [1.6.28](#) as the end of this section:

**Corollary 1.6.29.** *Under the assumption of Theorem [1.6.28](#), we have  $g \in \mathbf{J}G^\tau$  or  $g \in \mathbf{J}g_1G^\tau$ , where the latter case exists only if  $m$  is even, and  $g_1 \in B^\times$  is fixed such that*

$$\tau(g_1)g_1^{-1} = \begin{cases} \varpi_E I_m & \text{if } E/E_0 \text{ is unramified.} \\ \varpi_E J_{m/2} & \text{if } E/E_0 \text{ is ramified.} \end{cases}$$

As a result,

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1) \oplus \mathrm{Hom}_{\mathbf{J}^{g_1} \cap G^\tau}(\Lambda^{g_1}, 1).$$

*Proof.* Recall that we have already assumed that  $g \in B^\times$ . Since  $\tau(g)g^{-1} \in \mathbf{J} \cap B^\times = E^\times \mathfrak{b}^\times$ , by changing  $g$  up to multiplying an element in  $E^\times$  which doesn't change the double coset it represents, we may assume  $(g^*)^{-1}\varepsilon^{-1}g^{-1} \in \mathfrak{b}^\times$  or  $\varpi_E \mathfrak{b}^\times$ , where  $\varepsilon$  equals  $I_m$  for  $E/E_0$  unramified<sup>5</sup> and  $\varepsilon$  equals  $I_m$  or  $\mathrm{diag}(1, \dots, 1, \varepsilon)$  with  $\varepsilon \in \mathfrak{o}_{E_0}^\times \setminus N_{E/E_0}(\mathfrak{o}_E^\times)$  for  $E/E_0$  ramified. Using Proposition [1.2.2](#), we may change  $g^{-1}$  up to multiplying an element in  $\mathfrak{b}^\times$  on the right, thus we may write  $(g^*)^{-1}\varepsilon^{-1}g^{-1} = \varpi_E^\alpha$ , where  $\varpi_E^\alpha$  is defined as in [§1.2](#). Thus we get  $\det_B(\varpi_E^\alpha)/\det_B(\varepsilon^{-1}) \in N_{E/E_0}(E^\times)$ .

If  $(g^*)^{-1}\varepsilon^{-1}g^{-1} \in \mathfrak{b}^\times$ , from the definition and the uniqueness of  $\varpi_E^\alpha$  in Proposition [1.2.2](#), we get  $\varpi_E^\alpha = \varepsilon$ . We may further change  $g^{-1}$  up to multiplying an element in  $\mathfrak{b}^\times$  on the right, such that  $(g^*)^{-1}\varepsilon^{-1}g^{-1} = \varepsilon^{-1}$ . Thus we get  $\tau(g) = \varepsilon(g^*)^{-1}\varepsilon^{-1} = g$ , which means that  $g \in G^\tau$ .

If  $(g^*)^{-1}\varepsilon^{-1}g^{-1} \in \varpi_E \mathfrak{b}^\times$ , by considering the determinant we deduce that  $\det_B((g^*)^{-1}\varepsilon^{-1}g^{-1}) \in E^\times$  is of even order with respect to the discrete valuation of  $E$ . Since the determinant of elements in  $\varpi_E \mathfrak{b}^\times$  is of order  $m$ , we know that  $m$  is even. Thus from the definition and the uniqueness of  $\varpi_E^\alpha$  in Proposition [1.2.2](#), we get  $\varpi_E^\alpha = \varpi_E \varepsilon$  when  $E/E_0$  is unramified and  $\varpi_E^\alpha = \varpi_E J_{m/2}$  when  $E/E_0$  is ramified. For the former case, we have  $\varepsilon = I_m$ . Using Proposition [1.2.1](#), we may choose  $g_1 \in B^\times$  such that  $(g_1^*)^{-1}g_1^{-1} = \varpi_E I_m = (g^*)^{-1}g^{-1}$ . Thus  $g \in g_1 G^\tau$ . For the latter case, considering the determinant we must have  $\det_B(\varepsilon) \in N_{E/E_0}(E^\times)$ , thus  $\varepsilon = I_m$ . Using Proposition [1.2.1](#), we may choose  $g_1 \in B^\times$  such that  $(g_1^*)^{-1}g_1^{-1} = \varpi_E J_{m/2} = (g^*)^{-1}g^{-1}$ , thus  $g \in g_1 G^\tau$ .  $\square$

<sup>5</sup>It is also possible in the unramified case that  $\varepsilon = \mathrm{diag}(\varpi_E, \dots, \varpi_E)$ . However  $\varepsilon \in E^\times$  which commutes with  $B^\times$ , thus this case can be combined into the case where  $\varepsilon = I_m$ .

## 1.7 The supercuspidal unramified case

In this section, we study the distinction of  $\sigma$ -invariant supercuspidal representations of  $G$  in the case where  $E/E_0$  is unramified.

### 1.7.1 The finite field case

In this subsection, we assume  $\mathbf{l}/\mathbf{l}_0$  to be a quadratic extension of finite fields with characteristic  $p \neq 2$ . Let  $|\mathbf{l}_0| = Q$ , then we have  $|\mathbf{l}| = Q^2$ . Let  $\sigma$  be the non-trivial involution in  $\text{Gal}(\mathbf{l}/\mathbf{l}_0)$ . For  $m$  a positive integer, we first consider the  $\sigma$ -invariant supercuspidal representation of  $\text{GL}_m(\mathbf{l})$ .

**Lemma 1.7.1.** (1) *If there exists a  $\sigma$ -invariant supercuspidal representation of  $\text{GL}_m(\mathbf{l})$ , then  $m$  is odd.*

(2) *When  $\text{char}(R) = 0$ , the converse of (1) is true.*

*Proof.* Let  $\mathbf{t}$  be an extension of degree  $m$  over  $\mathbf{l}$ . We identify  $\mathbf{t}^\times$  with a maximal torus of  $\text{GL}_m(\mathbf{l})$ . By a  $\text{Gal}(\mathbf{t}/\mathbf{l})$ -regular (or regular for short) character  $\xi : \mathbf{t}^\times \rightarrow R^\times$ , we mean  $\xi^{|\mathbf{l}^i|} \neq \xi$  for any  $i = 1, \dots, m-1$ . By Green [Gre55] when  $\text{char}(R) = 0$  and James [Jam86] when  $\text{char}(R) = l > 0$  is relatively prime to  $p$ , there is a surjective map

$$\xi \mapsto \overline{\rho}_\xi$$

between regular characters of  $\mathbf{t}^\times$  and isomorphism classes of supercuspidal representations of  $\text{GL}_m(\mathbf{l})$ , whose fibers are  $\text{Gal}(\mathbf{t}/\mathbf{l})$ -orbits. Thus, for  $\overline{\rho}$  a supercuspidal representation of  $\text{GL}_m(\mathbf{l})$ , we choose  $\xi$  as a regular character of  $\mathbf{t}^\times$  such that  $\overline{\rho} = \overline{\rho}_\xi$ .

The representation  $\overline{\rho}$  is  $\sigma$ -invariant if and only if

$$\xi^{Q^{2i}} = \xi^Q \quad \text{for a certain } i \in \{1, \dots, m\}.$$

Applying this equality twice gives us the equality  $\xi^{Q^{4i-2}} = \xi$ . The regularity assumption on  $\xi$  implies that  $2m$  divides  $4i - 2$ . Since  $1 \leq i \leq m$ , we get  $m = 2i - 1$  as an odd number.

Conversely, for  $m = 2i - 1$  and  $\text{char}(R) = 0$ , we pick a character  $\xi$  of  $\mathbf{t}^\times$  of order  $Q^m - 1$ . Thus we have  $\xi^{Q^{2i}} = \xi^{Q^{m+1}} = \xi^Q$  and  $\xi$  is regular. Thus the corresponding  $\overline{\rho}_\xi$  is a  $\sigma$ -invariant supercuspidal representation. □

Let  $H = \text{U}_m(\mathbf{l}/\mathbf{l}_0) := \text{U}_m(I_m)$  be the unitary subgroup of  $\text{GL}_m(\mathbf{l})$  corresponding to the hermitian matrix  $I_m$ . We have the following lemma:

**Lemma 1.7.2.** *Suppose that  $m = 2i - 1$  with  $i \geq 1$ , and let  $\overline{\rho}$  be a supercuspidal representation of  $\text{GL}_m(\mathbf{l})$ . The following assertions are equivalent:*

- (1) *The representation  $\overline{\rho}$  is  $\sigma$ -invariant;*
- (2) *The representation  $\overline{\rho}$  is  $H$ -distinguished;*
- (3) *The  $R$ -vector space  $\text{Hom}_H(\overline{\rho}, 1)$  has dimension 1.*

*Proof.* When  $R$  has characteristic 0, this is [Gow84], Theorem 2.1, Theorem 2.4. Suppose now that  $R = \overline{\mathbb{F}}_l$ . First we prove that (1) is equivalent to (2).

For  $\overline{\rho}$  a supercuspidal representation of  $\text{GL}_m(\mathbf{l})$ , we denote by  $P_{\overline{\rho}}$  the projective envelope of  $\overline{\rho}$  as a  $\overline{\mathbb{Z}}_l[\text{GL}_m(\mathbf{l})]$ -module, where  $\overline{\mathbb{Z}}_l$  is the ring of integers of  $\overline{\mathbb{Q}}_l$ . Using [Vig96], chapitre III, Théorème 2.9 and [Ser77], Proposition 42, we have:

(1)  $P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  is the projective envelope of  $\overline{\rho}$  as a  $\overline{\mathbb{F}}_l[\text{GL}_m(\mathbf{l})]$ -module, which is indecomposable of finite length, with each irreducible component isomorphic to  $\overline{\rho}$ ;

(2) If we write  $\widetilde{P}_{\bar{\rho}} = P_{\bar{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  as the  $\overline{\mathbb{Q}}_l$ -lift of  $P_{\bar{\rho}}$ , then  $\widetilde{P}_{\bar{\rho}} \cong \bigoplus \widetilde{\bar{\rho}}$ , where  $\widetilde{\bar{\rho}}$  in the direct sum are supercuspidal as  $\overline{\mathbb{Q}}_l$ -lifts of  $\bar{\rho}$  of multiplicity 1.

Using the result above, we have

$$\begin{aligned}
\mathrm{Hom}_H(\bar{\rho}, 1) \neq 0; &\iff \mathrm{Hom}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(\bar{\rho}, \overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})]) \neq 0; \\
&\iff \mathrm{Hom}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(P_{\bar{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})]) \neq 0; \\
&\iff \mathrm{Hom}_{\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathfrak{l})]}(P_{\bar{\rho}}, \overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})]) \neq 0; \\
&\iff \mathrm{Hom}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathfrak{l})]}(\widetilde{P}_{\bar{\rho}}, \overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})]) \neq 0; \\
&\iff \text{There exists } \widetilde{\bar{\rho}} \text{ as above such that } \mathrm{Hom}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathfrak{l})]}(\widetilde{\bar{\rho}}, \overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})]) \neq 0; \\
&\iff \text{There exists } \widetilde{\bar{\rho}} \text{ as above such that } \widetilde{\bar{\rho}}^\sigma = \widetilde{\bar{\rho}}; \\
&\iff \bar{\rho}^\sigma = \bar{\rho}.
\end{aligned}$$

For the former equivalences, they are of the similar reason to in the proof of Theorem [1.4.1](#). For the second last equivalence we use the result for characteristic 0 case. For the last equivalence, we use the construction of supercuspidal representation given by Green and James in Lemma [1.7.1](#). Since it is always possible to lift a  $\sigma$ -invariant regular character over  $\overline{\mathbb{F}}_l$  to a  $\sigma$ -invariant regular character over  $\overline{\mathbb{Q}}_l$ , it is always possible to find a  $\sigma$ -invariant  $\overline{\mathbb{Q}}_l$ -lift  $\widetilde{\bar{\rho}}$  for a  $\sigma$ -invariant supercuspidal representation  $\bar{\rho}$ .

Since (3) implies (2) by definition, we only need to prove (2) implies (3). We sum up the proof occurring in [Séc19](#), Lemma 2.19. We have the following  $\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]$ -module decomposition

$$\overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})] = V_{\bar{\rho}} \oplus V',$$

where  $V_{\bar{\rho}}$  is composed of irreducible components isomorphic to  $\bar{\rho}$ , and  $V'$  has no irreducible component isomorphic to  $\bar{\rho}$ . First we verify that  $\mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(V_{\bar{\rho}})$  is commutative. By [Gow84](#), Theorem 2.1, the convolution algebra  $\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})/H]$  is commutative. Modulo  $l$  we deduce that

$$\overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})/H] \cong \mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(\overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathfrak{l})]) \cong \mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(V_{\bar{\rho}}) \oplus \mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(V')$$

is commutative, thus  $\mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(V_{\bar{\rho}})$  is commutative.

If we write  $P = P_{\bar{\rho}}$ , then there exists a nilpotent endomorphism  $N \in \mathrm{End}_{\overline{\mathbb{F}}_l}(P)$  such that  $\mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(P) = \overline{\mathbb{F}}_l[N]$ . And there exist  $r \geq 1$  and  $n_1, \dots, n_r$  positive integers such that

$$V_{\bar{\rho}} \cong \bigoplus_{i=1}^r P/N^{n_i}P.$$

Since  $\mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathfrak{l})]}(V_{\bar{\rho}})$  is commutative, we have  $r = 1$  and  $V_{\bar{\rho}} = P/N^{n_1}P$ . Thus

$$\mathrm{Hom}_H(\bar{\rho}, 1) \cong \mathrm{Hom}_{\mathrm{GL}_m(\mathfrak{l})}(\bar{\rho}, V_{\bar{\rho}}) = \mathrm{Hom}_{\mathrm{GL}_m(\mathfrak{l})}(\bar{\rho}, P/N^{n_1}P) \cong \overline{\mathbb{F}}_l.$$

Now for  $\mathrm{char}(R) = l > 0$  in general, there exists an equivalence of categories between representations of  $\mathrm{GL}_m(\mathfrak{l})$  over  $\overline{\mathbb{F}}_l$ , and representations of  $\mathrm{GL}_m(\mathfrak{l})$  over  $R$ , which is given by tensor product  $\bar{\rho}_0 \mapsto \bar{\rho}_0 \otimes_{\overline{\mathbb{F}}_l} R$  for  $\bar{\rho}_0$  a representation of  $\mathrm{GL}_m(\mathfrak{l})$  over  $\overline{\mathbb{F}}_l$ . Thus we may use the former result to finish the proof.  $\square$

**Remark 1.7.3.** We give an example of a  $\sigma$ -invariant cuspidal non supercuspidal representation of  $\mathrm{GL}_m(\mathfrak{l})$  over  $\overline{\mathbb{F}}_l$  which is not distinguished by  $H$ . Assume  $m = 2$  and  $l \neq 2$  such that  $l|Q^2 + 1$ . Let  $B$  be

the subgroup of  $\mathrm{GL}_2(\mathfrak{l})$  consisting of upper triangular matrices. For  $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathfrak{l})} 1$ , it is a representation of length 3, with each component of dimension 1,  $Q^2 - 1$ , 1. Denote by  $\bar{\rho}$  the representation of dimension  $Q^2 - 1$  as the subquotient of  $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathfrak{l})} 1$ . It is thus cuspidal (not supercuspidal) and  $\sigma$ -invariant. Let  $\tilde{\rho}$  be a  $\overline{\mathbb{Q}_l}$ -lift of  $\bar{\rho}$  which is an irreducible cuspidal representation. We write  $\tilde{\rho}|_H = V_1 \oplus \dots \oplus V_r$  its decomposition of irreducible components. Since  $|H| = Q(Q+1)(Q^2-1)$  is prime to  $l$ , reduction modulo  $l$  preserves irreducibility. So  $\bar{\rho}|_H$  decomposes as  $W_1 \oplus \dots \oplus W_r$ , where the irreducible representation  $W_i$  is the reduction of  $V_i$  modulo  $l$  for each  $i = 1, \dots, r$ . Suppose that  $\bar{\rho}$  is distinguished. Then  $W_i = \overline{\mathbb{F}_l}$  for some  $i$ . Thus  $V_i$  is a character which must be trivial. It implies that  $\tilde{\rho}$  is distinguished. It is impossible by Lemma 1.7.1 and Lemma 1.7.2, since  $m = 2$  is even.

Finally, we need the following finite group version of Proposition 1.5.6:

**Proposition 1.7.4.** *For  $\bar{\rho}$  an irreducible representation of  $\mathrm{GL}_m(\mathfrak{l})$ , we have  $\bar{\rho}^\vee \cong \bar{\rho}({}^t \cdot^{-1})$ , where  $\bar{\rho}({}^t \cdot^{-1}) : x \mapsto \bar{\rho}({}^t x^{-1})$  for any  $x \in \mathrm{GL}_m(\mathfrak{l})$ .*

*Proof.* By definition, the Brauer characters of  $\bar{\rho}^\vee$  and  $\bar{\rho}({}^t \cdot^{-1})$  are the same. Thus we finish the proof.  $\square$

## 1.7.2 Distinction criterion in the unramified case

Let  $\pi$  be a  $\sigma$ -invariant supercuspidal representation of  $G$ . In this subsection we want to prove Theorem 0.2.1 and Theorem 0.2.2 in the case where  $E/E_0$  is unramified. Combining with Theorem 1.4.1, we only need to show that  $\pi$  is distinguished by any unitary subgroup to finish the proof of Theorem 0.2.1. Since changing  $\tau$  up to a  $G$ -action doesn't change the content of the theorem, we only need to consider two special unitary involutions as representatives of  $G$ -orbits of hermitian matrices mentioned in Remark 1.6.1. To ensure that, first we prove the following lemma:

**Lemma 1.7.5.** *For any  $\sigma$ -invariant supercuspidal representation  $\pi$  with  $E/E_0$  unramified,  $m$  is odd.*

*Proof.* We consider  $\tau = \tau_1$ , where  $\tau_1(x) = \sigma({}^t x^{-1})$  for any  $x \in G$ . We follow the settings of Remark 1.6.1. For  $(\mathbf{J}, \Lambda)$  a simple type as in Remark 1.6.1 (2), we may write  $\Lambda \cong \kappa \otimes \rho$  as before. Using Proposition 1.6.24, we may further assume  $\kappa^{\tau^\vee} \cong \kappa$ . Since  $\Lambda$  and  $\kappa$  are  $\tau$ -selfdual, we deduce that  $\rho$  is  $\tau$ -selfdual. Let  $\bar{\rho}$  be the supercuspidal representation of  $\mathrm{GL}_m(\mathfrak{l}) \cong J/J^1$  whose inflation equals  $\rho|_J$ , then  $\bar{\rho}^{\tau^\vee} \cong \bar{\rho}$  when regarding  $\tau$  as a unitary involution of  $\mathrm{GL}_m(\mathfrak{l})$ . Using Proposition 1.7.4, we have  $\bar{\rho} \circ \sigma \cong \bar{\rho}$ . Using Lemma 1.7.1, we conclude that  $m$  is odd.  $\square$

With the aid of Lemma 1.7.5, we may assume as in Remark 1.6.1 (4) that  $\tau(x) = \varepsilon \sigma({}^t x^{-1}) \varepsilon^{-1}$  for any  $x \in G$  with  $\varepsilon$  equalling  $I_n$  or  $\mathrm{diag}(\varpi_E, \dots, \varpi_E)$ , representing the two classes of unitary involutions. For  $(\mathbf{J}, \Lambda)$  a simple type as in Remark 1.6.1 (2), we may write  $\Lambda \cong \kappa \otimes \rho$  as before. Using Proposition 1.6.24, we may further assume  $\kappa^{\tau^\vee} \cong \kappa$ . Using Lemma 1.6.18 with  $g = 1$ , there exists a quadratic character  $\chi : \mathbf{J} \cap G^\tau \rightarrow R^\times$  such that

$$\dim_R \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, \chi^{-1}) = 1$$

and

$$\mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1) \cong \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, \chi^{-1}) \otimes_R \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\rho, \chi).$$

We want to show that  $\chi = 1$ . First we need the following lemma:

**Lemma 1.7.6.** *The character  $\chi$  can be extended to a character  $\chi'$  of  $\mathbf{J}$ .*



*Proof.* Using Lemma [1.4.2](#), we have  $\mathbf{J} \cap G^\tau = J \cap G^\tau$ . Write  $\bar{\chi}$  as the character of  $U_m(\mathfrak{l}/\mathfrak{l}_0) \cong J \cap G^\tau / J^1 \cap G^\tau$  whose inflation equals  $\chi$ . Since it is well-known that the derived subgroup of  $U_m(\mathfrak{l}/\mathfrak{l}_0)$  is  $SU_m(\mathfrak{l}/\mathfrak{l}_0) := \{g \in U_m(\mathfrak{l}/\mathfrak{l}_0) \mid \det(g) = 1\}$  (see [Cam00](#), Theorem 5.4 and Theorem 5.5), there exists  $\bar{\phi}$  as a quadratic character of  $\det(U_m(\mathfrak{l}/\mathfrak{l}_0)) = \{x \in \mathfrak{l}^\times \mid x\sigma(x) = x^{Q+1} = 1\}$ , such that  $\bar{\chi} = \bar{\phi} \circ \det|_{U_m(\mathfrak{l}/\mathfrak{l}_0)}$ . We extend  $\bar{\phi}$  to a character of  $\mathfrak{l}^\times$  and we write  $\bar{\chi}' = \bar{\phi} \circ \det$  which is a character of  $GL_m(\mathfrak{l})$  extending  $\bar{\chi}$ . Write  $\chi^0$  as the inflation of  $\bar{\chi}'$  with respect to the isomorphism  $GL_m(\mathfrak{l}) \cong J/J^1$ . Finally we choose  $\chi'$  as a character of  $\mathbf{J}$  extending  $\chi^0$  by choosing  $\chi'(\varpi_E) \neq 0$  randomly. By construction,  $\chi'|_{J \cap G^\tau} = \chi$ .  $\square$

**Proposition 1.7.7.** (1) When  $\text{char}(R) = 0$ , for any  $\chi'$  extending  $\chi$  we have  $\chi'(\chi' \circ \tau) = 1$ .  
(2) Furthermore, for any  $R$  we have  $\chi = 1$ .

*Proof.* First we consider  $\text{char}(R) = 0$ . Since  $m$  is odd, Lemma [1.7.1](#) implies that  $GL_m(\mathfrak{l})$  possesses a  $\sigma$ -invariant supercuspidal representation  $\bar{\rho}$ . Using Proposition [1.7.4](#), we get  $\bar{\rho}^{\tau^\vee} \cong \bar{\rho}$ . We denote by  $\rho'$  a representation of  $\mathbf{J}$  trivial on  $J^1$  such that its restriction to  $J$  is the inflation of  $\bar{\rho}$ . Since  $\sigma(\varpi_E) = \varpi_E$ , we have  $\rho'(\tau(\varpi_E)) = \rho'(\varpi_E)^{-1}$  which means that  $\rho'$  is  $\tau$ -selfdual. By Lemma [1.7.2](#), it is also distinguished.

Let  $\Lambda'$  denote the  $\tau$ -selfdual simple type  $\kappa \otimes \rho'$ . The natural isomorphism

$$\text{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda', \chi^{-1}) \cong \text{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, \chi^{-1}) \otimes_R \text{Hom}_{\mathbf{J} \cap G^\tau}(\rho', 1)$$

shows that  $\Lambda'$  is  $\chi^{-1}$ -distinguished.

By Lemma [1.7.6](#), there exists a character  $\chi'$  extending  $\chi$ . The representation  $\Lambda'' = \Lambda' \chi'$  is thus a distinguished simple type. Let  $\pi''$  be the supercuspidal representation of  $G$  compactly induced by  $(\mathbf{J}, \Lambda'')$ . It is distinguished, thus  $\tau$ -selfdual by Theorem [1.4.1](#) and Proposition [1.5.6](#). Since  $\Lambda''$  and  $\Lambda''^{\tau^\vee} \cong \Lambda'' \chi'^{-1}(\chi'^{-1} \circ \tau)$  are both contained in  $\pi''$ , it follows that  $\chi'(\chi' \circ \tau)$  is trivial.

We write  $\bar{\chi} = \bar{\phi} \circ \det$  with the same notation as in the proof of Lemma [1.7.6](#). Since  $\chi'(\chi' \circ \tau) = 1$ , we get  $\bar{\phi}(\bar{\phi} \circ \sigma)^{-1} = \bar{\phi}^{1-Q} = 1$ . Choose  $\zeta_{\mathfrak{l}}$  as a primitive root of  $\mathfrak{l}^\times$ , we know that  $\zeta_{\mathfrak{l}}^{Q-1}$  generates the group  $\det(U_m(\mathfrak{l}/\mathfrak{l}_0)) = \{x \in \mathfrak{l}^\times \mid x\sigma(x) = x^{Q+1} = 1\}$ . Since  $\bar{\phi}(\zeta_{\mathfrak{l}}^{1-Q}) = 1$ , we deduce that  $\bar{\phi}|_{\det(U_m(\mathfrak{l}/\mathfrak{l}_0))}$  is trivial, which means that  $\bar{\chi}$  is trivial. Thus  $\chi$  as the inflation of  $\bar{\chi}$  is also trivial.

Now we consider  $R = \overline{\mathbb{F}}_l$ . As already mentioned in the proof of Proposition [1.6.26](#), if we denote by  $\tilde{\kappa}$  the  $\overline{\mathbb{Q}}_l$ -lift of  $\kappa$  and if we denote by  $\tilde{\chi}$  the character defined by Lemma [1.6.18](#) (1) with respect to  $\tilde{\kappa}$  and  $\tilde{\eta}$ , then we know that  $\tilde{\chi}$  is a  $\overline{\mathbb{Q}}_l$ -lift of  $\chi$ . By using the characteristic 0 case we already proved, we get  $\tilde{\chi} = 1$  which implies that  $\chi = 1$ .

When  $R = l > 0$  in general, we just follow the same logic as in Proposition [1.6.26](#) to finish the proof.  $\square$

**Remark 1.7.8.** In fact in Proposition [1.7.7](#), we proved that when  $m$  is odd and  $E/E_0$  is unramified, any  $\tau$ -selfdual  $\kappa$  constructed in Proposition [1.6.24](#) as an extension of a  $J^1 \cap G^\tau$ -distinguished Heisenberg representation  $\eta$  is  $\mathbf{J} \cap G^\tau$ -distinguished.

Now we come back to the proof of our main theorem. We have

$$\text{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1) \cong \text{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, 1) \otimes_R \text{Hom}_{\mathbf{J} \cap G^\tau}(\rho, 1),$$

where  $\text{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, 1)$  is of dimension 1, and  $\text{Hom}_{\mathbf{J} \cap G^\tau}(\rho, 1) \cong \text{Hom}_{U_m(\mathfrak{l}/\mathfrak{l}_0)}(\bar{\rho}, 1)$  is also of dimension 1 by Lemma [1.4.2](#), Lemma [1.7.2](#) and Proposition [1.7.4](#). So  $\text{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1)$  is of dimension 1, which implies that  $\pi$  is  $G^\tau$ -distinguished. Thus we finish the proof of Theorem [0.2.1](#) when  $E/E_0$  is unramified. Using Corollary [1.6.29](#) and the fact that  $m$  is odd, we deduce that  $\text{Hom}_{G^\tau}(\pi, 1)$  is of dimension 1, which finishes the proof of Theorem [0.2.2](#) when  $E/E_0$  is unramified.

## 1.8 The supercuspidal ramified case

In this section, we study the distinction of  $\sigma$ -invariant supercuspidal representations of  $G$  in the case where  $E/E_0$  is ramified. This finishes the proof of our main theorem.

### 1.8.1 The finite field case

Let  $\mathbf{l}$  be a finite field of characteristic  $p \neq 2$  and let  $|\mathbf{l}| = Q$ . For  $m$  a positive integer, we denote by  $\mathbf{G}$  the reductive group  $\mathrm{GL}_m$  over  $\mathbf{l}$ . Thus by definition,  $\mathbf{G}(\mathbf{l}) = \mathrm{GL}_m(\mathbf{l})$ . For  $\bar{\varepsilon}$  a matrix in  $\mathbf{G}(\mathbf{l})$  such that  ${}^t\bar{\varepsilon} = \bar{\varepsilon}$ , the automorphism defined by  $\tau(x) = \bar{\varepsilon}^t x^{-1} \bar{\varepsilon}^{-1}$  for any  $x \in \mathrm{GL}_m(\mathbf{l})$  gives an involution on  $\mathrm{GL}_m(\mathbf{l})$ , which induces an involution on  $\mathbf{G}$ . Thus  $\mathbf{G}^\tau$  is the orthogonal group corresponding to  $\tau$ , which is a reductive group over  $\mathbf{l}$ , and  $\mathbf{G}^\tau(\mathbf{l}) = \mathrm{GL}_m(\mathbf{l})^\tau$  which is a subgroup of  $\mathrm{GL}_m(\mathbf{l})$ . In this subsection, for  $\bar{\rho}$  a supercuspidal representation of  $\mathrm{GL}_m(\mathbf{l})$  and  $\bar{\chi}$  a character of  $\mathrm{GL}_m(\mathbf{l})^\tau$ , we state the result mentioned in [HL12] which gives a criterion for  $\bar{\rho}$  distinguished by  $\bar{\chi}$ .

First of all, we assume  $R = \overline{\mathbb{Q}_l}$ . We recall a little bit of Deligne-Lusztig theory (see [DL76]). Let  $\mathbf{T}$  be an elliptic maximal  $\mathbf{l}$ -torus in  $\mathbf{G}$ , where elliptic means that  $\mathbf{T}(\mathbf{l}) = \mathbf{t}^\times$  and  $\mathbf{t}/\mathbf{l}$  is the field extension of degree  $m$ . Let  $\xi$  be a regular character of  $\mathbf{T}(\mathbf{l})$ , where regularity means the same as in the construction of Green and James in subsection 7.1. Using [DL76], Theorem 8.3, there is a virtual character  $R_{\mathbf{T},\xi}$  as the character of a cuspidal representation of  $\mathrm{GL}_m(\mathbf{l})$ . Moreover if we fix  $\mathbf{T}$ , we know that  $\xi \mapsto R_{\mathbf{T},\xi}$  gives a bijection from the set of Galois orbits of regular characters of  $\mathbf{T}$  to the set of cuspidal representations of  $\mathrm{GL}_m(\mathbf{l})$ . So we may choose  $\xi$  such that  $\mathrm{Tr}(\bar{\rho}) = R_{\mathbf{T},\xi}$ . Moreover, using [DL76], Theorem 4.2, we get  $R_{\mathbf{T},\xi}(-1) = \dim(\bar{\rho})\xi(-1)$  with  $\dim(\bar{\rho}) = (Q-1)(Q^2-1)\dots(Q^{m-1}-1)$ . So if we denote by  $\omega_{\bar{\rho}}$  the central character of  $\bar{\rho}$ , we get  $\omega_{\bar{\rho}}(-1) = \xi(-1)$ .

Under the same settings, we have the following proposition due to Hakim and Lansky ([HL12], Proposition 6.7):

**Proposition 1.8.1.** *For  $\tau$ ,  $\bar{\rho}$ ,  $\mathbf{T}$  and  $\xi$  above, we have:*

$$\dim_R(\mathrm{Hom}_{\mathbf{G}^\tau(\mathbf{l})}(\bar{\rho}, \bar{\chi})) = \begin{cases} 1 & \text{if } \omega_{\bar{\rho}}(-1) = \xi(-1) = \bar{\chi}(-1), \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the  $l$ -modular case and assume  $\mathrm{char}(R) = l > 0$ .

**Proposition 1.8.2.** *For  $\tau$  above and  $\bar{\rho}$  a supercuspidal representation of  $\mathrm{GL}_m(\mathbf{l})$  over  $R$ , the space  $\mathrm{Hom}_{\mathrm{GL}_m(\mathbf{l})^\tau}(\bar{\rho}, \bar{\chi}) \neq 0$  if and only if  $\omega_{\bar{\rho}}(-1) = \bar{\chi}(-1)$ . Moreover if the condition is satisfied, then we have  $\dim_R(\mathrm{Hom}_{\mathrm{GL}_m(\mathbf{l})^\tau}(\bar{\rho}, \bar{\chi})) = 1$ .*

*Proof.* First we assume  $R = \overline{\mathbb{F}_l}$ . We use the similar proof to that in Lemma [1.7.2]. Let  $H = \mathrm{GL}_m(\mathbf{l})^\tau$  with other notations unchanged. We choose  $\tilde{\chi}$  as a character of  $H$  lifting  $\bar{\chi}$ , which is defined over  $\overline{\mathbb{Z}_l}$  or  $\overline{\mathbb{Q}_l}$  by abuse of notation. For  $S = \overline{\mathbb{Z}_l}, \overline{\mathbb{Q}_l}$ , we define

$$S[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}} := \{f|f : \mathrm{GL}_m(\mathbf{l}) \rightarrow S, f(hg) = \tilde{\chi}(h)f(g) \text{ for any } h \in H, g \in \mathrm{GL}_m(\mathbf{l})\}.$$

Especially we have

$$\overline{\mathbb{Q}_l}[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}} = \mathrm{Ind}_H^{\mathrm{GL}_m(\mathbf{l})} \tilde{\chi}$$

as a representation of  $\mathrm{GL}_m(\mathbf{l})$  over  $\overline{\mathbb{Q}_l}$ , and we have  $\overline{\mathbb{Z}_l}[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}$  as a free  $\overline{\mathbb{Z}_l}$ -module. If we further define

$$\overline{\mathbb{F}_l}[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}} = \mathrm{Ind}_H^{\mathrm{GL}_m(\mathbf{l})} \tilde{\chi},$$

then we have

$$\overline{\mathbb{Z}_l}[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}} \otimes_{\overline{\mathbb{Z}_l}} \overline{\mathbb{F}_l} = \overline{\mathbb{F}_l}[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}$$

and

$$\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = \overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}.$$

We deduce that

$$\begin{aligned} \mathrm{Hom}_H(\overline{\rho}, \overline{\chi}) \neq 0; &\iff \mathrm{Hom}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathbf{l})]}(\overline{\rho}, \overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\overline{\chi}}) \neq 0; \\ &\iff \mathrm{Hom}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathbf{l})]}(P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\overline{\chi}}) \neq 0; \\ &\iff \mathrm{Hom}_{\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]}(P_{\overline{\rho}}, \overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}) \neq 0; \\ &\iff \mathrm{Hom}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathbf{l})]}(\widetilde{P}_{\overline{\rho}}, \overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}) \neq 0; \\ &\iff \text{There exists } \tilde{\rho} \text{ lifting } \overline{\rho} \text{ such that } \mathrm{Hom}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathbf{l})]}(\tilde{\rho}, \overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}) \neq 0; \\ &\iff \text{There exists } \tilde{\rho} \text{ lifting } \overline{\rho} \text{ such that } \omega_{\tilde{\rho}}(-1) = \tilde{\chi}(-1); \\ &\iff \omega_{\overline{\rho}}(-1) = \overline{\chi}(-1). \end{aligned}$$

The former equivalences are of the same reason as in the proof of Lemma [1.7.2](#), and we use Proposition [1.8.1](#) for the second last equivalence. For the last equivalence, the “ $\Rightarrow$ ” direction is trivial. For the other direction, when  $l \neq 2$ , we choose  $\tilde{\rho}$  to be any supercuspidal  $\overline{\mathbb{Q}}_l$ -lift of  $\overline{\rho}$ . Thus we have  $\omega_{\tilde{\rho}}(-1) = \omega_{\overline{\rho}}(-1) = \overline{\chi}(-1) = \tilde{\chi}(-1)$ . When  $l = 2$ , using the construction of Green and James, for  $\xi$  a regular character over  $\overline{\mathbb{F}}_l$  corresponding to  $\overline{\rho}$ , we may always find a  $\overline{\mathbb{Q}}_l$ -lift  $\tilde{\xi}$  which is regular and satisfies  $\tilde{\xi}(-1) = \tilde{\chi}(-1)$ . Thus the supercuspidal representation  $\tilde{\rho}$  corresponding to  $\tilde{\xi}$  as a lift of  $\overline{\rho}$  satisfies  $\omega_{\tilde{\rho}}(-1) = \tilde{\chi}(-1)$ . So we finish the proof of the first part.

To calculate the dimension, as in the proof of Lemma [1.7.2](#) if we write

$$\overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\overline{\chi}} = V_{\overline{\rho}} \oplus V',$$

where  $V_{\overline{\rho}}$  is composed of irreducible components isomorphic to  $\overline{\rho}$ , and  $V'$  has no irreducible component isomorphic to  $\overline{\rho}$ , then we only need to show that  $\mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathbf{l})]}(V_{\overline{\rho}})$  is commutative. We consider the following  $\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]$ -module decomposition

$$\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}} = \widetilde{V}_{\overline{\rho}} \oplus \widetilde{V}',$$

where  $\widetilde{V}_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = \bigoplus_{\tilde{\rho}} \tilde{\rho}$  with the direct sum ranging over all the irreducible representations  $\tilde{\rho}$  over  $\overline{\mathbb{Q}}_l$  occurring in  $\widetilde{V}_{\overline{\rho}}$  counting the multiplicity, and  $\widetilde{V}'$  denotes a  $\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]$ -complement of  $\widetilde{V}_{\overline{\rho}}$ , such that  $\widetilde{V}' \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  contains no irreducible component of  $\tilde{\rho}$ . Using Proposition [1.8.1](#), we know that  $\widetilde{V}_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  is multiplicity free, which means that  $\mathrm{End}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathbf{l})]}(\widetilde{V}_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$  is commutative. Since the canonical embedding from  $\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}$  to  $\overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}$  induces the following ring monomorphism

$$\mathrm{End}_{\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]}(\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}) \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathbf{l})]}(\overline{\mathbb{Q}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}})$$

given by tensoring  $\overline{\mathbb{Q}}_l$ , which leads to the ring monomorphism

$$\mathrm{End}_{\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]}(\widetilde{V}_{\overline{\rho}}) \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_l[\mathrm{GL}_m(\mathbf{l})]}(\widetilde{V}_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l),$$

thus  $\mathrm{End}_{\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]}(\widetilde{V}_{\overline{\rho}})$  is also commutative.

The modulo  $l$  map from  $\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}$  to  $\overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\overline{\chi}}$  induces the following ring epimorphism

$$\mathrm{End}_{\overline{\mathbb{Z}}_l[\mathrm{GL}_m(\mathbf{l})]}(\overline{\mathbb{Z}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\tilde{\chi}}) \twoheadrightarrow \mathrm{End}_{\overline{\mathbb{F}}_l[\mathrm{GL}_m(\mathbf{l})]}(\overline{\mathbb{F}}_l[H \backslash \mathrm{GL}_m(\mathbf{l})]_{\overline{\chi}}),$$

which leads to the ring epimorphism

$$\mathrm{End}_{\mathbb{Z}_l[\mathrm{GL}_m(\mathfrak{l})]}(\widetilde{V}_{\bar{\rho}}) \rightarrow \mathrm{End}_{\mathbb{F}_l[\mathrm{GL}_m(\mathfrak{l})]}(V_{\bar{\rho}}).$$

Since  $\mathrm{End}_{\mathbb{Z}_l[\mathrm{GL}_m(\mathfrak{l})]}(\widetilde{V}_{\bar{\rho}})$  is commutative, we know that  $\mathrm{End}_{\mathbb{F}_l[\mathrm{GL}_m(\mathfrak{l})]}(V_{\bar{\rho}})$  is also commutative. Thus we may use the same proof as in Lemma 1.7.2 to show that

$$\dim_{\mathbb{F}_l}(\mathrm{Hom}_{\mathrm{GL}_m(\mathfrak{l})}(\bar{\rho}, \bar{\chi})) = 1.$$

Finally for  $\mathrm{char}(R) = l > 0$  in general, we follow the corresponding proof in Lemma 1.7.2 □

**Remark 1.8.3.** For  $\mathbf{G}^\tau(\mathfrak{l})$  an orthogonal group with  $m \geq 2$ , it is well-known that its derived group is always a subgroup of  $\mathbf{G}^{\tau^0}(\mathfrak{l})$  of index 2 (see [Cam00], Proposition 6.5), which means that there exists a character of  $\mathbf{G}^\tau(\mathfrak{l})$  which isn't trivial on  $\mathbf{G}^{\tau^0}(\mathfrak{l})$ . It means that we cannot expect  $\bar{\chi}$  to be trivial on  $\mathbf{G}^{\tau^0}(\mathfrak{l})$  in general. However, for those  $\bar{\chi}$  occurring in the next subsection, it is highly possible that  $\bar{\chi}$  is trivial on  $\mathbf{G}^{\tau^0}(\mathfrak{l})$ . For example, [HL12], Proposition 6.4 gives an evidence in the case where  $\pi$  is tame supercuspidal. However, I don't know how to prove it.

Now we assume that  $m$  is even. We write  $J_{m/2} = \begin{pmatrix} 0 & I_{m/2} \\ -I_{m/2} & 0 \end{pmatrix}$  and we denote by

$$\mathrm{Sp}_m(\mathfrak{l}) = \{x \in \mathrm{GL}_m(\mathfrak{l}) \mid {}^t x J_{m/2} x = J_{m/2}\}$$

the symplectic subgroup of  $\mathrm{GL}_m(\mathfrak{l})$ .

**Proposition 1.8.4.** For  $\bar{\rho}$  a cuspidal representation of  $\mathrm{GL}_m(\mathfrak{l})$ , we have  $\mathrm{Hom}_{\mathrm{Sp}_m(\mathfrak{l})}(\bar{\rho}, 1) = 0$ .

*Proof.* Using [Kly84], Corollary 1.4. whose proof also works for the  $l$ -modular case, we know that an irreducible generic representation cannot be distinguished by a symplectic subgroup. Since a cuspidal representation is generic, we finish the proof. □

## 1.8.2 Distinction criterion in the ramified case

Still let  $\pi$  be a  $\sigma$ -invariant supercuspidal representation of  $G$ . In this subsection we prove Theorem 0.2.1 and Theorem 0.2.2 in the case where  $E/E_0$  is ramified. Using Theorem 1.4.1, we only need to show that  $\pi$  is distinguished by any unitary subgroup to finish the proof of Theorem 0.2.1. We may change  $\tau$  up to a  $G$ -action which doesn't change the property of distinction. Thus using Remark 1.6.1(4), we may assume  $\tau(x) = \varepsilon \sigma({}^t x^{-1}) \varepsilon^{-1}$  for any  $x \in G$ , where  $\varepsilon$  equals  $I_n$  or  $\mathrm{diag}(I_d, \dots, I_d, \epsilon)$  with  $\epsilon \in \mathfrak{o}_{E_0}^\times \setminus \mathcal{N}_{E/E_0}(\mathfrak{o}_E^\times)$ , representing the two classes of unitary involutions. We denote by  $\bar{\varepsilon}$  the image of  $\varepsilon$  in  $\mathrm{GL}_m(\mathfrak{l})$ .

For  $(\mathbf{J}, \Lambda)$  a simple type in Remark 1.6.1(2), we write  $\Lambda \cong \kappa \otimes \rho$ . Using Proposition 1.6.24, we may further assume  $\kappa^{\tau^\vee} \cong \kappa$ . Using Lemma 1.6.18 with  $g = 1$ , there exists a quadratic character  $\chi : \mathbf{J} \cap G^\tau \rightarrow R^\times$  such that

$$\dim_R \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, \chi^{-1}) = 1 \tag{1.8.1}$$

and

$$\mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1) \cong \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\kappa, \chi^{-1}) \otimes_R \mathrm{Hom}_{\mathbf{J} \cap G^\tau}(\rho, \chi). \tag{1.8.2}$$

If we denote by  $\omega_\kappa$  the central character of  $\kappa$  defined on  $F^\times$ , using (1.8.1), we get  $\omega_\kappa = \chi^{-1}$  as characters of  $F^\times \cap (\mathbf{J} \cap G^\tau)$ . In particular,  $\omega_\kappa(-1) = \chi^{-1}(-1)$ . Since  $\kappa^{\tau^\vee} \cong \kappa$ , we get  $\omega_\kappa \circ \tau = \omega_\kappa^{-1}$ . In particular we have

$$\omega_\kappa(\varpi_F)^{-1} = \omega_\kappa(\tau(\varpi_F)) = \omega_\kappa(\varpi_F)^{-1} \omega_\kappa(-1)^{-1},$$

where we use the fact that  $\sigma(\varpi_F) = -\varpi_F$ . Thus we get  $\omega_{\kappa}(-1) = \chi(-1) = 1$ .

Since  $\Lambda$  and  $\kappa$  are  $\tau$ -selfdual,  $\rho$  is  $\tau$ -selfdual. Using the same proof as that for  $\kappa$ , we get  $\omega_{\rho}(-1) = 1$ . Let  $\bar{\rho}$  be the supercuspidal representation of  $\mathrm{GL}_m(\mathfrak{l}) \cong J/J^1$  whose inflation equals  $\rho|_J$  and let  $\bar{\chi}$  be the character of

$$\mathbf{G}^{\tau}(\mathfrak{l}) \cong J \cap G^{\tau} / J^1 \cap G^{\tau}$$

whose inflation equals  $\chi$ , where  $\tau$  naturally induces an orthogonal involution on  $\mathbf{G}$  with respect to  $\bar{\varepsilon}$  with the notation in subsection 1.8.1. By definition and Lemma 1.4.2 we get

$$\mathrm{Hom}_{J \cap G^{\tau}}(\rho, \chi) \cong \mathrm{Hom}_{\mathbf{G}^{\tau}(\mathfrak{l})}(\bar{\rho}, \bar{\chi}).$$

Since  $\omega_{\bar{\rho}}(-1) = \bar{\chi}(-1) = 1$ , using Proposition 1.8.1 and Proposition 1.8.2, the space above is non-zero. Thus by 1.8.2 we have

$$\mathrm{Hom}_{J \cap G^{\tau}}(\Lambda, 1) \neq 0$$

which means that  $\pi$  is distinguished by  $G^{\tau}$ , finishing the proof of Theorem 0.2.1. Moreover using Proposition 1.8.1, Proposition 1.8.2, 1.8.1 and 1.8.2, we get

$$\dim_R \mathrm{Hom}_{J \cap G^{\tau}}(\Lambda, 1) = 1.$$

Now if  $m$  is even and  $\varepsilon = I_m$ , we also need to study the space  $\mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\Lambda^{g_1}, 1)$ , where  $g_1$  is defined in Corollary 1.6.29 such that  $\tau(g_1)g_1^{-1} = \varpi_E J_{m/2} \in B^{\times}$ . Using Lemma 1.6.18, there exists a quadratic character  $\chi_1 : J^{g_1} \cap G^{\tau} \rightarrow R^{\times}$  such that

$$\dim_R \mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\kappa^{g_1}, \chi_1^{-1}) = 1. \quad (1.8.3)$$

and

$$\mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\Lambda^{g_1}, 1) \cong \mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\kappa^{g_1}, \chi_1^{-1}) \otimes_R \mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\rho^{g_1}, \chi_1). \quad (1.8.4)$$

So we only need to study the space  $\mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\rho^{g_1}, \chi_1) \cong \mathrm{Hom}_{J \cap G^{\delta_{g_1}}}(\rho, \chi_1^{g_1^{-1}})$ , where

$$\delta_{g_1}(x) := (\tau(g_1)g_1^{-1})^{-1}\tau(x)(\tau(g_1)g_1^{-1}) = (\varpi_E J_{m/2})^{-1}\tau(x)\varpi_E J_{m/2}$$

for any  $x \in G$  as an involution on  $G$ .

Let  $\bar{\rho}$  be the supercuspidal representation of  $\mathrm{GL}_m(\mathfrak{l}) \cong J/J^1$  whose inflation equals  $\rho|_J$  and let  $\bar{\chi}_1^{g_1^{-1}}$  be the character of

$$\mathrm{Sp}_m(\mathfrak{l}) \cong J \cap G^{\delta_{g_1}} / J^1 \cap G^{\delta_{g_1}}$$

whose inflation equals  $\chi_1^{g_1^{-1}}$ , then we get

$$\mathrm{Hom}_{J \cap G^{\delta_{g_1}}}(\rho, \chi_1^{g_1^{-1}}) \cong \mathrm{Hom}_{\mathrm{Sp}_m(\mathfrak{l})}(\bar{\rho}, \bar{\chi}_1^{g_1^{-1}}) = \mathrm{Hom}_{\mathrm{Sp}_m(\mathfrak{l})}(\bar{\rho}, 1),$$

where the last equation is because of the well-known fact that  $\mathrm{Sp}_m(\mathfrak{l})$  equals its derived group ([Cam00], Lemma 4.8), thus  $\bar{\chi}_1^{g_1^{-1}}|_{\mathrm{Sp}_m(\mathfrak{l})}$  is trivial. Using Proposition 1.8.4, we get  $\mathrm{Hom}_{\mathrm{Sp}_m(\mathfrak{l})}(\bar{\rho}, 1) = 0$ . Thus  $\mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\Lambda^{g_1}, 1) = 0$ .

Using Corollary 1.6.29, we get

$$\dim_R \mathrm{Hom}_{G^{\tau}}(\pi, 1) = \dim_R \mathrm{Hom}_{J \cap G^{\tau}}(\Lambda, 1) + \dim_R \mathrm{Hom}_{J^{g_1} \cap G^{\tau}}(\Lambda^{g_1}, 1) = 1,$$

which finishes the proof of Theorem 0.2.2 when  $E/E_0$  is ramified.

**Remark 1.8.5.** *From the proof above, we may find out that when  $E/E_0$  is ramified, for the two  $\tau$ -selfdual simple types mentioned in Corollary 1.6.29, it is always the simple type  $(\mathbf{J}, \Lambda)$  which contributes to the distinction, and  $(\mathbf{J}^{g_1}, \Lambda^{g_1})$  never contributes to the distinction.*

**Remark 1.8.6.** *In general, if we weaken the condition for the representation  $\pi$  such that it is only cuspidal, and if we still assume that it is  $\sigma$ -invariant such that  $E/E_0$  is totally ramified, then the argument above still works and we have*

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \mathrm{Hom}_{G^\tau(\mathbf{l})}(\bar{\rho}, \bar{\chi})$$

as  $R$ -vector space, where  $G^\tau(\mathbf{l})$  is an orthogonal subgroup of  $\mathrm{GL}_m(\mathbf{l})$ ,  $\bar{\rho}$  is a cuspidal representation of  $\mathrm{GL}_m(\mathbf{l})$  and  $\bar{\chi}$  is a quadratic character of  $G^\tau(\mathbf{l})$  such that  $\omega_{\bar{\rho}}(-1) = \bar{\chi}(-1)$ . Furthermore we may prove that this space is non-zero. When  $\mathrm{char}(R) = 0$  it has been proved and when  $\mathrm{char}(R) = l > 0$  we assume  $R = \overline{\mathbb{F}}_l$  without loss of generality. Using the argument of Proposition 1.8.2 we may find a cuspidal representation  $\tilde{\rho}$  of  $\mathrm{GL}_m(\mathbf{l})$  as a  $\overline{\mathbb{Q}}_l$ -lift of  $\bar{\rho}$  and  $\tilde{\chi}$  as a quadratic character of  $G^\tau(\mathbf{l})$  as a  $\overline{\mathbb{Q}}_l$ -lift of  $\bar{\chi}$ , such that  $\omega_{\tilde{\rho}}(-1) = \tilde{\chi}(-1)$ . Then by Proposition 1.8.1 we get  $\mathrm{Hom}_{G^\tau(\mathbf{l})}(\tilde{\rho}, \tilde{\chi}) \neq 0$  and thus  $\mathrm{Hom}_{G^\tau(\mathbf{l})}(\bar{\rho}, \bar{\chi}) \neq 0$ . Thus even for a  $\sigma$ -invariant cuspidal representation  $\pi$  as above, it is distinguished by a given unitary subgroup, however in this case we don't know the multiplicity one result.

### 1.8.3 Proof of Theorem 0.2.3

Let  $\pi$  be a  $\sigma$ -invariant supercuspidal representation of  $G$  over  $\overline{\mathbb{F}}_l$ . For  $\tau$  a unitary involution, by Theorem 0.2.1,  $\pi$  is distinguished by  $G^\tau$ . From the proof of Theorem 1.4.1, there exists  $\tilde{\pi}$  as a distinguished integral  $\sigma$ -invariant supercuspidal representation of  $G$  over  $\overline{\mathbb{Q}}_l$  which lifts  $\pi$ . So we finish the proof of Theorem 0.2.3.

## 1.9 Generalization of Theorem 1.4.1

In this section, we generalize Theorem 1.4.1 to the irreducible generic representations, while in the meantime we also give another proof of the original theorem which is purely local. Precisely, we prove the following theorem:

**Theorem 1.9.1.** *Let  $\pi$  be an irreducible generic representation of  $G$  over  $R$ . If  $\pi$  is distinguished by  $G^\tau$ , then  $\pi$  is  $\sigma$ -invariant.*

### 1.9.1 The finite analogue

To begin with, we first study the finite analogue of the theorem:

**Proposition 1.9.2.** *Let  $\mathbf{l}/\mathbf{l}_0$  be a quadratic extension of finite fields of characteristic  $p$  and let  $\bar{\rho}$  be an irreducible generic representation of  $\overline{G} = \mathrm{GL}_m(\mathbf{l})$  over  $R$ . If  $\bar{\rho}$  is distinguished by the unitary subgroup  $\overline{H}$  of  $\overline{G}$ , then it is  $\sigma$ -invariant.*

*Proof.* When  $\mathrm{char}(R) = 0$ , the proposition was proved by Gow [Gow84] for any irreducible representations. So we only consider the  $l$ -modular case and without loss of generality we assume  $R = \overline{\mathbb{F}}_l$ . We write  $P_{\bar{\rho}}$  for the projective envelope of  $\bar{\rho}$  as a  $\overline{\mathbb{Z}}_l[G]$ -module. Thus  $P_{\bar{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  is a projective  $\overline{\mathbb{F}}_l[G]$ -module, and moreover

$$\mathrm{Hom}_{\overline{\mathbb{F}}_l[\overline{H}]}(\bar{\rho}, 1) \cong \mathrm{Hom}_{\overline{\mathbb{F}}_l[G]}(\bar{\rho}, \overline{\mathbb{F}}_l[\overline{H} \backslash \overline{G}]) \neq 0$$

implies that

$$\mathrm{Hom}_{\overline{\mathbb{F}}_l[\overline{G}]}(P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \overline{\mathbb{F}}_l[\overline{H} \backslash \overline{G}]) \neq 0.$$

From the property of projective envelope, equivalently we have

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\overline{G}]}(P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}_l[\overline{H} \backslash \overline{G}]) \neq 0,$$

and thus there exists  $\rho$  as an irreducible constituent of  $P_{\overline{\rho}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  such that

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_l[\overline{G}]}(\rho, \overline{\mathbb{Q}}_l[\overline{H} \backslash \overline{G}]) \neq 0.$$

By [Ser77], §14.5, §15.4,  $\overline{\rho}$  is a constituent of  $r_l(\rho)$ . Since  $\rho$  is  $\overline{H}$ -distinguished, it is  $\sigma$ -invariant and so is  $r_l(\rho)$ . We choose  $\rho_1, \dots, \rho_k$  to be cuspidal representations of  $\overline{G}$  over  $\overline{\mathbb{Q}}_l$ , such that  $\rho$  is a subrepresentation of the parabolic induction  $\rho_1 \times \dots \times \rho_k$ . We write  $\overline{\rho}_i = r_l(\rho_i)$  as cuspidal representations of  $\overline{G}$  over  $\overline{\mathbb{F}}_l$ , then all the irreducible constituents of  $r_l(\rho)$  are subquotients of  $\overline{\rho}_1 \times \dots \times \overline{\rho}_k$ , and in particular so is  $\overline{\rho}$ . Since  $\overline{\rho}$  is generic (or non-degenerate), by [Vig96], III.1.10, it is the unique non-degenerate subquotient contained in  $\overline{\rho}_1 \times \dots \times \overline{\rho}_k$ , thus it is the unique non-degenerate constituent in  $r_l(\rho)$ . Thus it is  $\sigma$ -invariant. □

## 1.9.2 The cuspidal case

In this subsection, we first prove the case where  $\pi$  is cuspidal. We choose  $(\mathbf{J}, \Lambda)$  to be a simple type of  $\pi$ , then by the Frobenius reciprocity and the Mackey formula, there exists  $g \in G$  such that

$$\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0. \quad (1.9.1)$$

Let  $H^1$  be the corresponding subgroup of  $\mathbf{J}$ , let  $\theta$  be the simple character of  $H^1$  contained in  $\Lambda$  and let  $\eta$  be the Heisenberg representation of  $\theta$ . Restricting (1.9.1) to  $H^{1g} \cap G^\tau$  we get  $\theta^g|_{H^{1g} \cap G^\tau} = 1$ . Following the proof of [Séc19], Lemma 6.5, we have

$$(\theta \circ \tau)^{\tau(g)}|_{\tau(H^{1g}) \cap H^{1g}} = \theta^g \circ \tau|_{\tau(H^{1g}) \cap H^{1g}} = (\theta^g)^{-1}|_{\tau(H^{1g}) \cap H^{1g}}, \quad (1.9.2)$$

or in other words,  $\theta \circ \tau$  intertwines with  $\theta^{-1}$ . Using the Intertwining Theorem (cf. [BH13]),  $\theta \circ \tau$  and  $\theta^{-1}$  are endo-equivalent, which, from the argument of Lemma 1.5.7, is equivalent to  $\Theta^\sigma = \Theta$ , where  $\Theta$  denotes the endo-class of  $\theta$ .

We let  $\tau_1$  be the unitary involution corresponding to  $I_n$ , which in particular satisfies the condition of Theorem 1.5.5. Since  $\Theta^\sigma = \Theta$ , by *loc. cit.*, we may choose a simple stratum  $[\mathfrak{a}, \beta]$  and  $\theta' \in H^1(\mathfrak{a}, \beta)$  with  $\theta' \in \Theta$ , such that

$$\tau_1(\beta) = \beta^{-1}, \quad \tau_1(\mathfrak{a}) = \mathfrak{a} \quad \text{and} \quad \theta' \circ \tau_1 = \theta'^{-1}.$$

Up to  $G$ -conjugacy, we may and will assume that  $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$  and  $\theta' = \theta$ . We write  $E = F[\beta]$  and  $B \cong M_m(E)$  as the centralizer of  $E$  in  $M_n(F)$ . Using Proposition 1.6.24, we write  $\Lambda = \kappa \otimes \rho$  with  $\kappa$  an extension of the Heisenberg representation  $\eta$  such that  $\kappa^{\tau_1} \cong \kappa^\vee$ . Let  $\varepsilon$  be an hermitian matrix such that  $\tau(x) = \varepsilon \sigma({}^t x^{-1}) \varepsilon^{-1} = \tau_1(x) \varepsilon^{-1}$  for any  $x \in G$ . For a fixed  $g \in G$ , we define  $\gamma = \varepsilon^{-1} \tau(g) g^{-1} = \tau_1(g) \varepsilon^{-1} g^{-1}$  and by direct calculation we have  $\tau_1(\gamma) = \gamma$ .

**Proposition 1.9.3.** *Let  $g \in G$  such that  $\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ .*

(1) *Changing  $g$  up to another representative in the same  $\mathbf{J}$ - $G^\tau$  double coset, we may assume  $\gamma \in B^\times$ ;*

(2) *The dimension  $\dim_R \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\eta^g, 1) = 1$ ;*

(3) There is a unique quadratic character  $\chi$  of  $\mathbf{J}^g \cap G^\tau$  trivial on  $J^{1g} \cap G^\tau$ , such that

$$\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \cong \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1}) \cong R.$$

Moreover

$$\mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \cong \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1}) \otimes \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi);$$

(4) The element  $\gamma \in \mathbf{J}$ , thus under the assumption of (1)  $\gamma \in B^\times \cap \mathbf{J} = E^\times \mathfrak{b}^\times$ .

*Proof.* We sketch the proof which follows from that of Theorem [1.6.2](#) (actually the same theorem if  $\tau = \tau_1$ ). Using [\(1.9.2\)](#) and the fact that  $\tau(H^{1g}) = \tau_1(H^1)^{\varepsilon^{-1}\tau(g)} = H^{1\varepsilon^{-1}\tau(g)}$  and  $(\theta \circ \tau)^{\tau(g)} = (\theta \circ \tau_1)^{\varepsilon^{-1}\tau(g)} = (\theta^{-1})^{\varepsilon^{-1}\tau(g)}$ , we have

$$(\theta^{\varepsilon^{-1}\tau(g)})^{-1}|_{H^{1\varepsilon^{-1}\tau(g)} \cap H^{1g}} = (\theta \circ \tau)^{\tau(g)}|_{\tau(H^{1g}) \cap H^{1g}} = \theta^g \circ \tau|_{\tau(H^{1g}) \cap H^{1g}} = (\theta^g)^{-1}|_{H^{1\varepsilon^{-1}\tau(g)} \cap H^{1g}},$$

which means that  $\gamma$  intertwines  $\theta$ , or in other words  $\gamma \in JB^\times J$ . The following lemma follows from the same proof of Lemma [1.6.5](#), once we replace the  $\gamma$  there with our  $\gamma$  here and  $\tau$  there with  $\tau_1$ .

**Lemma 1.9.4.** *There exist  $y \in J = J(\mathfrak{a}, \beta)$  and  $b \in B^\times$ , such that  $\gamma = \tau_1(y)b$ .*

Thus we change  $g$  by  $y^{-1}g$  and then the corresponding  $\gamma = b \in B^\times$ , which proves (1). For (2), we denote

$$\delta(x) := (\tau(g)g^{-1})^{-1}\tau(x)\tau(g)g^{-1} = \gamma^{-1}\tau_1(x)\gamma \quad \text{for any } x \in G$$

as an involution on  $G$ , then by definition we have

$$\mathrm{Hom}_{G^\tau}(\eta^g, 1) \cong \mathrm{Hom}_{G^\delta}(\eta, 1),$$

and

$$\gamma\delta(\gamma) = \gamma\gamma^{-1}\tau_1(\gamma)\gamma = 1.$$

Moreover, by direct calculation we have

$$\delta(H^1) = (\tau(g)g^{-1})^{-1}H^{1\varepsilon^{-1}}\tau(g)g^{-1} = H^{1\gamma} \quad \text{and} \quad \theta \circ \delta = (\theta^{-1})^{\varepsilon^{-1}\tau(g)g^{-1}} = (\theta^{-1})\gamma.$$

So using Proposition [1.6.14](#), we finish the proof of (2).

Using (2) and the same argument of Proposition [1.6.18](#) we get the statement (3), except the part  $\chi$  being quadratic. To finish that part, since

$$\tau_1(\tau_1(g)\varepsilon^{-1})\varepsilon^{-1}(\tau_1(g)\varepsilon^{-1})^{-1} = g\varepsilon\tau_1(g)^{-1} = (\tau_1(g)\varepsilon^{-1}g^{-1})^{-1} = \gamma^{-1} \in B^\times,$$

we may replace  $g$  with  $\varepsilon^{-1}\tau(g) = \tau_1(g)\varepsilon^{-1}$  in the statement (3) to get a unique character  $\chi'$  of  $\mathbf{J}^{\varepsilon^{-1}\tau(g)} \cap G^\tau$  trivial on  $J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau$ . Moreover, using the facts  $\tau(\mathbf{J}) = \mathbf{J}^{\varepsilon^{-1}}$ ,  $\tau(J) = J^{\varepsilon^{-1}}$ ,  $\tau(J^1) = J^{1\varepsilon^{-1}}$  and  $\tau(H^1) = H^{1\varepsilon^{-1}}$  and Lemma [1.4.2](#), it is easy to show that

$$\mathbf{J}^g \cap G^\tau = \mathbf{J}^{\varepsilon^{-1}\tau(g)} \cap G^\tau = J^g \cap G^\tau = J^{\varepsilon^{-1}\tau(g)} \cap G^\tau \tag{1.9.3}$$

As a result,  $\chi$  and  $\chi'$  are characters defined on the same group  $\mathbf{J}^g \cap G^\tau = \mathbf{J}^{\varepsilon^{-1}\tau(g)} \cap G^\tau$ . We have the following lemma similar to Proposition [1.6.19](#):

**Lemma 1.9.5.** *We have  $\chi = \chi'$ .*



*Proof.* We write  $\delta$  for the involution defined as above. By §1.3.2, we have  $\gamma \in I_G(\eta) = I_G(\kappa^0)$  and

$$\dim_R(\mathrm{Hom}_{J \cap J^\gamma}(\kappa^{0\gamma}, \kappa^0)) = \dim_R(\mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta)) = 1,$$

where  $\kappa^0 = \kappa|_J$ . By direct calculation, we have  $J^1 \cap G^\delta = J^{1\gamma} \cap G^\delta$  as a subgroup of  $J^1 \cap J^{1\gamma}$  and  $H^1 \cap G^\delta = H^{1\gamma} \cap G^\delta$ . Using statement (2) for  $g$  and  $\varepsilon^{-1}\tau(g)$  respectively, we get

$$\dim_R \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \dim_R \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)}, 1) = 1.$$

By Proposition 1.6.20, for  $0 \neq \varphi \in \mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta) = \mathrm{Hom}_{J^{1g} \cap J^{1\varepsilon^{-1}\tau(g)}}(\eta^{\varepsilon^{-1}\tau(g)}, \eta^g)$ , the map

$$\begin{aligned} f_\varphi : \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) &\longrightarrow \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)}, 1), \\ \lambda &\longmapsto \lambda \circ \varphi, \end{aligned}$$

is bijective<sup>6</sup>. If we choose

$$0 \neq \lambda \in \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \quad \text{and} \quad 0 \neq \lambda' := f_\varphi(\lambda) = \lambda \circ \varphi \in \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)}, 1),$$

then for any  $v$  in the representation space of  $\eta$  and any  $x \in J^g \cap G^\tau = J^{\varepsilon^{-1}\tau(g)} \cap G^\tau$ , we have

$$\begin{aligned} \chi'(x)^{-1} \lambda'(v) &= \lambda'(\kappa^{0\varepsilon^{-1}\tau(g)}(x)v) && \text{(by the statement (3))} \\ &= \lambda(\varphi(\kappa^{0\varepsilon^{-1}\tau(g)}(x)v)) && \text{(by definition of } \lambda') \\ &= \lambda(\kappa^{0g}(x)\varphi(v)) && \text{(since } \varphi \in \mathrm{Hom}_{J^g \cap J^{\varepsilon^{-1}\tau(g)}}(\kappa^{0\varepsilon^{-1}\tau(g)}, \kappa^{0g})) \\ &= \chi(x)^{-1} \lambda(\varphi(v)) && \text{(by the statement (3))} \\ &= \chi(x)^{-1} \lambda'(v) && \text{(by definition of } \lambda'). \end{aligned}$$

Since  $v$  and  $x \in J^g \cap G^\tau = J^{\varepsilon^{-1}\tau(g)} \cap G^\tau$  are arbitrary, we have  $\chi'|_{J^{\varepsilon^{-1}\tau(g)} \cap G^\tau} = \chi|_{J^g \cap G^\tau}$ . Combining with (1.9.3) we finish the proof of the lemma.  $\square$

To prove that  $\chi$  is quadratic, we first assume that  $\mathrm{char}(R) = 0$ . We have the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)}, 1) &\cong \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\kappa^g, \chi^{-1}) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\chi, \kappa^{g\vee}) && \text{(by the duality of contragredient)} \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\kappa^{g\vee}, \chi) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\kappa^{g\vee} \circ \tau, \chi \circ \tau) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}((\kappa^\vee \circ \tau_1)^{\varepsilon^{-1}\tau(g)}, \chi \circ \tau) \\ &\cong \mathrm{Hom}_{J^{\varepsilon^{-1}\tau(g)} \cap G^\tau}(\kappa^{\varepsilon^{-1}\tau(g)}, \chi \circ \tau) && \text{(since } \kappa \text{ is } \tau_1\text{-selfdual).} \end{aligned}$$

Using the above lemma and the uniqueness of  $\chi'$ , we have  $\chi \circ \tau = \chi^{-1}$ . Since  $\chi$  is defined on  $J^g \cap G^\tau = J^g \cap G^\tau$  which is  $\tau$ -invariant, we have  $\chi \circ \tau = \chi$ , thus  $\chi^2 = \chi(\chi \circ \tau) = 1$ . When  $\mathrm{char}(R) = l > 0$  the same argument in Proposition 1.6.26 can be used directly.

Finally using (3) and the distinction of the simple type, we have  $\mathrm{Hom}_{J^g \cap G^\tau}(\rho^g, \chi) \neq 0$ . Then the proof of (4) are the same of that in subsection 1.6.6, once we replace  $\gamma$  there with our  $\gamma$  here.  $\square$

<sup>6</sup>Noting that  $J^{1g} \cap G^\tau = J^1 \cap G^\delta$  and  $J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau = J^{1\gamma} \cap G^\delta$ , thus  $\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \mathrm{Hom}_{J^1 \cap G^\delta}(\eta, 1)$  and  $\mathrm{Hom}_{J^{1\varepsilon^{-1}\tau(g)} \cap G^\tau}(\eta^{\varepsilon^{-1}\tau(g)}, 1) = \mathrm{Hom}_{J^{1\gamma} \cap G^\delta}(\eta^\gamma, 1)$

**Corollary 1.9.6.** *For  $g \in G$  such that  $\mathrm{Hom}_{\mathbf{J}g \cap G^\tau}(\Lambda^g, 1) \neq 0$ , we may choose  $g$  up to changing the representative in the  $\mathbf{J}\text{-}G^\tau$  double coset, such that*

$$\gamma = \begin{cases} I_m & \text{or } \varpi_E I_m & \text{if } E/E_0 \text{ is unramified;} \\ I_m & \text{or } \mathrm{diag}(1, \dots, 1, \epsilon) & \text{or } \varpi_E J_{m/2} & \text{if } E/E_0 \text{ is ramified,} \end{cases}$$

as an element in  $\mathrm{GL}_m(E) \cong B^\times \hookrightarrow G$ , where  $\epsilon \in \mathfrak{o}_{E_0}^\times \setminus \mathrm{N}_{E/E_0}(\mathfrak{o}_E^\times)$

*Proof.* We have proved that  $\gamma = \tau_1(g)\varepsilon^{-1}g^{-1} \in B^\times \cap \mathbf{J} = E^\times \mathfrak{b}^\times$ . Changing  $g$  up to multiplying by an element in  $E^\times$  which doesn't change the double coset it represents, we may assume  $\gamma \in \mathfrak{b}^\times$  or  $\varpi_E \mathfrak{b}^\times$ . Using Proposition 1.2.2 and changing  $g$  up to multiplying by an element in  $\mathfrak{b}^\times$  on the left, we may assume that  $\gamma = \varpi_E^\alpha$ , and from the uniqueness we must have  $\varpi_E^\alpha = I_m$  or  $\varpi_E I_m$  when  $E/E_0$  is unramified, and  $\varpi_E^\alpha = I_m$  or  $\mathrm{diag}(1, \dots, 1, \epsilon)$  or  $\varpi_E J_{m/2}$  when  $E/E_0$  is totally ramified.  $\square$

Thus for  $g \in G$  as above, we get

$$\mathrm{Hom}_{\mathbf{J} \cap G^\delta}(\rho, \chi^{g^{-1}}) \cong \mathrm{Hom}_{\mathbf{J} \cap G^\delta}(\rho, \chi^{g^{-1}}) \cong \mathrm{Hom}_{\mathbf{J}g \cap G^\tau}(\rho^g, \chi) \neq 0.$$

Write  $\overline{H} = \mathbf{J} \cap G^\delta / \mathbf{J}^1 \cap G^\delta$  for the subgroup of  $\overline{G} = \mathrm{GL}_m(\mathfrak{l}) \cong \mathbf{J} / \mathbf{J}^1$ , which, from the expression of  $\gamma$  in Corollary 1.9.6, is either a unitary subgroup, or an orthogonal subgroup, or a symplectic subgroup of  $\overline{G}$ . Thus we have

$$\mathrm{Hom}_{\overline{H}}(\overline{\rho}, \overline{\chi'}) \neq 0,$$

where  $\overline{\rho}$  is a cuspidal representation of  $\overline{G}$  whose inflation is  $\rho|_{\mathbf{J}}$  and  $\overline{\chi'}$  is a quadratic character of  $\overline{H}$  whose inflation is  $\chi^{g^{-1}}|_{\mathbf{J} \cap G^\delta}$ .

When  $\overline{H}$  is unitary which also means that  $E/E_0$  is unramified, by Lemma 1.7.6 (or more precisely its argument)  $\overline{\chi'}$  can be extended to a quadratic character of  $\overline{G}$ . Thus  $\overline{\rho}\overline{\chi'}^{-1}$  as a cuspidal representation of  $\overline{G}$  is distinguished by  $\overline{H}$ , and thus it is  $\sigma$ -invariant by Proposition 1.9.2. Since  $p$  is odd, as a quadratic character  $\overline{\chi}$  is  $\sigma$ -invariant. Thus  $\overline{\rho}$  is also  $\sigma$ -invariant, or by Proposition 1.7.4,  $\overline{\rho}^{\tau_1} \cong \overline{\rho}^\vee$ . Thus both  $\kappa$  and  $\rho$  are  $\tau_1$ -selfdual, which means that  $\Lambda$  and  $\pi$  are  $\tau_1$ -selfdual. By Proposition 1.5.6,  $\pi$  is  $\sigma$ -invariant.

When  $\overline{H}$  is orthogonal which also means that  $E/E_0$  is totally ramified, comparing the central character as in §1.8.2 we have  $\overline{\rho}(-I_m) = \mathrm{id}$ . Thus  $\rho^{\tau_1}|_{\mathbf{J}} = \rho^{(t, -1)}|_{\mathbf{J}} \cong \rho|_{\mathbf{J}}$  by Proposition 1.7.4 and  $\rho(\tau_1(\varpi_E)) = \rho(-\varpi_E) = \rho(\varpi_E)$ , which means that  $\rho$  is  $\tau_1$ -selfdual, finishing the proof as above.

Finally by Proposition 1.8.4 and the fact that  $\mathrm{Sp}_m(\mathfrak{l})$  equals its derived subgroup, the case where  $\overline{H}$  is symplectic never occurs, which ends the proof of Theorem 1.9.1 when  $\pi$  is cuspidal.

### 1.9.3 The discrete series case

In the following two subsections we use the result in [FLO12] and our cuspidal result to finish the proof of Theorem 1.9.1. Since the characteristic 0 case has been proved in *ibid*, Theorem 6.1, we only consider the case where  $\mathrm{char}(R) = l > 0$ . In this subsection we assume that  $\pi$  is a discrete series representation, of which we give the definition below.

For  $\rho$  a cuspidal representation of  $\mathrm{GL}_{n'}(F)$ , we denote by  $f(\rho)$  the number of unramified characters  $\chi$  such that  $\rho\chi \cong \rho$  and we write  $q(\rho) = q^{f(\rho)}$ . We denote by  $o(\rho)$  the smallest positive integer such that  $l$  divides  $q(\rho)^{o(\rho)} - 1$ . We let  $e(\rho)$  be the smallest integer  $e'$  such that  $1 + q(\rho) + \dots + q(\rho)^{e'-1}$  is divided by  $l$ . Thus we always have  $e(\rho) \geq o(\rho)$  and the equality holds if and only if  $o(\rho) \neq 1$ . We write  $\Delta_{\rho, [a, b]}$  for the segment  $\{\rho\nu^a, \rho\nu^{a+1}, \dots, \rho\nu^b\}$ , where  $\nu(\cdot) = |\det(\cdot)|_F$ ,  $a, b \in \mathbb{Z}$  and  $k = b - a + 1$  is a positive integer. When  $k$  is smaller than  $e(\rho)$ , the normalized parabolic induction

$$\rho\nu^a \times \rho\nu^{a+1} \times \dots \times \rho\nu^b$$

has a unique non-degenerate irreducible quotient ( [MS14a], Lemma 7.14, Proposition 7.21.(3), Example 8.2.), which we denote by

$$\mathrm{St}(\Delta_{\rho,[a,b]}).$$

We call it a *discrete series representation*.

**Remark 1.9.7.** *Noting that in our definition a discrete series representation is always non-degenerate. One may also regard [MS14a], Définition 7.5.(2) as another possible definition for discrete series representations, which is a larger category than ours and includes degenerate representations.*

**Proposition 1.9.8.** *If  $kn' = n$  and  $\pi = \mathrm{St}(\Delta_{\rho,[a,b]})$  as above is distinguished by  $G^\tau$ , then both  $\mathrm{St}(\Delta_{\rho,[a,b]})$  and  $\rho$  are  $\sigma$ -invariant.*

*Proof.* Since  $\pi$  is distinguished by  $G^\tau$ , the parabolic induction  $\rho\nu^a \times \rho\nu^{a+1} \times \dots \times \rho\nu^b$  is also distinguished by  $G^\tau$ . By [FLO12], Lemma 6.10. (note that their argument also works for  $l$ -modular case), for  $M = \mathrm{GL}_{n'}(F) \times \dots \times \mathrm{GL}_{n'}(F)$  as a Levi subgroup of  $G$  and for each  $i \in \{0, 1, \dots, k-1\}$ , either there exists a unitary involution  $\tau'$  of  $\mathrm{GL}_{n'}(F)$  such that  $\rho\nu^{a+i}$  is distinguished by  $\mathrm{GL}_{n'}(F)^{\tau'}$ , or  $\rho\nu^{a+i}$  is isomorphic to  $\rho\nu^{a+i'}$  for another  $i' \in \{0, 1, \dots, k-1\}$ .

If for a certain  $i$  the first situation occurs, then using the theorem for the cuspidal case  $\rho\nu^i$  and  $\rho$  are  $\sigma$ -invariant [7], then  $\rho\nu^a \times \rho\nu^{a+1} \times \dots \times \rho\nu^b$  and  $\pi$  are  $\sigma$ -invariant which finishes the proof. Thus we assume that for each  $i$  we are in the second situation, and in particular we choose  $i$  to be the smallest integer in  $\{0, 1, \dots, k-1\}$  such that  $\rho^\sigma\nu^a \cong \rho\nu^{a+i}$ .

If  $i = 0$ , we finish the proof as above. In particular if  $q(\rho)$  equals 1 modulo  $l$ , we have  $\rho\nu \cong \rho$  which means that  $\rho^\sigma\nu^a \cong \rho\nu^{a+i} \cong \rho\nu^a$ , included in the  $i = 0$  case. So we assume that  $i > 0$  and  $l$  doesn't divide  $q(\rho) - 1$  indicating  $o(\rho) = e(\rho)$ . We have  $\rho\nu^{a+2i} \cong \rho^\sigma\nu^{a+i} \cong \rho\nu^a$  and thus  $o(\rho)$  divides  $2i$ . Since  $i \leq k < e(\rho) = o(\rho)$  from our assumption, we have  $2i = e(\rho) > k$ . Then  $\rho^\sigma\nu^{a+i-1} \cong \rho\nu^{a+i-1+i}$  and since  $\rho^\sigma\nu^{a+i-1}$  is isomorphic to  $\rho\nu^{a+j'}$  for a certain  $j' \in \{0, 1, \dots, k-1\}$ , we know that  $2i - 1 - j'$ , as an integer between  $2 - k$  and  $2i - 1 = e(\rho) - 1$ , is divided by  $e(\rho)$ , thus  $2i - 1 = j'$  which implies that  $2i - 1 \leq k - 1$ , or  $2i \leq k$ , contradictory! Thus we finish the proof.  $\square$

#### 1.9.4 The generic case

Finally we assume that  $\pi$  is generic and distinguished by  $G^\tau$ . Using [MS14a], Théorème 9.10, for  $i = 1, \dots, r$  there exist  $n_i, k_i \in \mathbb{N}$  and  $a_i, b_i \in \mathbb{Z}$  satisfying  $n_1k_1 + \dots + n_rk_r = n$  and  $b_i - a_i + 1 = k_i$ , and  $\rho_i$  cuspidal representations of  $\mathrm{GL}_{n_i}(F)$  satisfying  $k_i < e(\rho_i)$ , such that

$$\pi \cong \mathrm{St}(\Delta_{\rho_1,[a_1,b_1]}) \times \dots \times \mathrm{St}(\Delta_{\rho_r,[a_r,b_r]}),$$

where for  $i_1 \neq i_2$ , the corresponding segments  $\Delta_{\rho_{i_1},[a_{i_1},b_{i_1}]}$  and  $\Delta_{\rho_{i_2},[a_{i_2},b_{i_2}]}$  are not linked (lié in French) in the sense of *ibid*, Définition 7.3. We write  $\Delta_1, \dots, \Delta_r$  for those segments for short. For  $\Delta_1 + \dots + \Delta_r$  as a multisegment, we define its support to be  $\sum_{i=1}^r \sum_{j=a_i}^{b_i} \rho_i\nu^j$  as a multiset of cuspidal representations.

**Lemma 1.9.9.** *For each  $i$  there exist segments  $\Delta_{i_1}, \dots, \Delta_{i_{j_i}}$  with  $\Delta_i$  and  $\Delta_{i_1} + \dots + \Delta_{i_{j_i}}$  having the same support, such that for each corresponding representation  $\mathrm{St}(\Delta_{i_j})$ , either it is distinguished by a unitary subgroup, or there exist  $i' \in \{1, \dots, r\}$ ,  $j' \in \{1, \dots, j_{i'}\}$  such that  $\Delta_{i_j}^\sigma = \Delta_{i'j'}$  and  $\mathrm{St}(\Delta_{i'j'})^\sigma \cong \mathrm{St}(\Delta_{i_j})$ .*

*Proof.* The proof of the lemma is indicated in [FLO12], §6.2. By the geometric lemma, the restriction  $\pi|_{G^\tau} = \mathrm{St}(\Delta_{\rho_1,[a_1,b_1]}) \times \dots \times \mathrm{St}(\Delta_{\rho_r,[a_r,b_r]})|_{G^\tau}$  is written as a finite filtration of  $G^\tau$ -invariant subspaces. Since  $\pi$  is distinguished by  $G^\tau$ , there exists at least one subquotient in the filtration contributing to the

<sup>7</sup>As an unramified character  $\nu$  is always  $\sigma$ -invariant.

distinction. Thus by Lemma 6.4. of *loc. cit.*, for  $M = \mathrm{GL}_{n_1 k_1}(F) \times \dots \times \mathrm{GL}_{n_r k_r}(F)$  as a Levi subgroup of  $G$ , there exists a Levi subgroup  $L$  of  $M$ , such that the Jacquet module  $J_{L,M}(\mathrm{St}(\Delta_{\rho_1, [a_1, b_1]}) \otimes \dots \otimes \mathrm{St}(\Delta_{\rho_r, [a_r, b_r]}))$  as a representation of  $L$  is distinguished by a unitary subgroup of  $L$ . Using [MS14a], Proposition 7.16. to calculate the Jacquet module, we deduce that  $J_{L,M}(\mathrm{St}(\Delta_{\rho_1, [a_1, b_1]}) \otimes \dots \otimes \mathrm{St}(\Delta_{\rho_r, [a_r, b_r]}))$  must be of the form

$$\mathrm{St}(\Delta_{11}) \otimes \dots \otimes \mathrm{St}(\Delta_{1j_1}) \otimes \mathrm{St}(\Delta_{21}) \otimes \dots \otimes \mathrm{St}(\Delta_{2j_2}) \otimes \dots \otimes \mathrm{St}(\Delta_{r1}) \otimes \dots \otimes \mathrm{St}(\Delta_{rj_r}),$$

where  $\Delta_{i1}, \dots, \Delta_{ij_i}$  form a partition of  $\Delta_i$ . Finally by [FLO12], (6.5), each corresponding representation  $\mathrm{St}(\Delta_{ij})$  is either distinguished by a unitary subgroup, or there exist  $i' \in \{1, \dots, r\}$ ,  $j' \in \{1, \dots, j_{i'}\}$  such that  $\mathrm{St}(\Delta_{i'j'})^\sigma \cong \mathrm{St}(\Delta_{ij})$ . By considering the cuspidal support the latter case also implies that  $\Delta_{i'j'}^\sigma = \Delta_{ij}$ . □

**Lemma 1.9.10.** *For any multiset  $\mathfrak{s}$  of cuspidal representations, the multisegment  $\Delta_1 + \dots + \Delta_r$  such that*

- *its support is  $\mathfrak{s}$ ;*
- *its elements are not linked;*
- *$k_i < e(\rho_i)$  for each  $i$*

*is unique, if it exists.*

*Proof.* We choose  $\rho$  to be a cuspidal representation contained in  $\mathfrak{s}$ . We let  $k$  be the largest integer, such that there exist  $a, b \in \mathbb{Z}$  with  $\rho\nu^a, \rho\nu^{a+1}, \dots, \rho\nu^b$  in  $\mathfrak{s}$  and  $k = b - a + 1$ . From the non-linked condition,  $k < e(\rho)$  and the multisegment contains a segment with  $\Delta_{\rho, [a, b]}$  as its subsegment, thus it contains  $\Delta_{\rho, [a, b]}$  since  $k$  is the largest. Thus we finish the proof by eliminating  $\Delta_{\rho, [a, b]}$  and by induction on the cardinality of  $\mathfrak{s}$ . □

For  $\Delta_i$  as one segment given as above, if there exists one  $\Delta_{ij}$  as in the lemma such that  $\mathrm{St}(\Delta_{ij})$  is distinguished by a unitary subgroup, by Proposition 1.9.8 both  $\rho_i$  and  $\mathrm{St}(\Delta_i)$  are  $\sigma$ -invariant.

Now we consider those  $\Delta_i$  such that  $\rho_i^\sigma$  is not isomorphic to  $\rho_i$ . First we assume that  $\rho_i^\sigma$  is not isomorphic to  $\rho_i$  twisted by any power of  $\nu$ . We let  $\Delta_{i_1}, \dots, \Delta_{i_s}$  be those segments such that each  $\rho_{i_t}$  is isomorphic to  $\rho_i$  twisted by a certain power of  $\nu$ , correspondingly let  $\Delta_{i'_1}, \dots, \Delta_{i'_{s'}}$  be those segments such that each  $\rho_{i'_t}$  is isomorphic to  $\rho_i^\sigma$  twisted by a power of  $\nu$ . Thus using Lemma 1.9.9 every  $\Delta_{i_t}^\sigma$  equals a certain  $\Delta_{i'_t j'_t}$ . Thus  $\Delta_{i_1}^\sigma + \dots + \Delta_{i_s}^\sigma$  and  $\Delta_{i'_1} + \dots + \Delta_{i'_{s'}}$  have the same support and are equal to each other by Lemma 1.9.10. Thus  $\Delta_i^\sigma$  equals a certain  $\Delta_{i'}$  with  $i' \in \{1, \dots, r\}$  and  $\rho_i^\sigma$  is isomorphic to  $\rho_{i'}$  twisted by a power of  $\nu$ .

Now we assume that  $\rho_i^\sigma$  is isomorphic to  $\rho_i$  twisted by a certain power of  $\nu$ . As above we let  $\Delta_{i_1}, \dots, \Delta_{i_s}$  be those segments such that each  $\rho_{i_t}$  is isomorphic to  $\rho_i$  twisted by a power of  $\nu$ , still using Lemma 1.9.9 every  $\Delta_{i_t}^\sigma$  equals a certain  $\Delta_{i'_t j'_t}$ . Thus  $\Delta_{i_1}^\sigma + \dots + \Delta_{i_s}^\sigma$  and  $\Delta_{i_1} + \dots + \Delta_{i_s}$  have the same support and are equal to each other by Lemma 1.9.10. Thus  $\Delta_i^\sigma$  equals a certain  $\Delta_{i'}$  with  $i' \in \{1, \dots, r\}$ .

Thus we have proved that for each  $\Delta_i$  there exists  $i'$  such that  $\Delta_i^\sigma = \Delta_{i'}$  (it is possible that  $i' = i$ ), where  $i$  and  $i'$  range over  $\{1, \dots, r\}$ , thus

$$\pi^\sigma \cong \mathrm{St}(\Delta_1^\sigma) \times \dots \times \mathrm{St}(\Delta_r^\sigma) \cong \mathrm{St}(\Delta_1) \times \dots \times \mathrm{St}(\Delta_r) \cong \pi,$$

which finishes the proof.

**Remark 1.9.11.** *In general it should be interesting to know whether Theorem 1.9.1 and Proposition 1.9.2 are true or not for general irreducible  $l$ -modular representations.*

## 1.10 “ $l$ -modular” base change lift and applications

In this section, we study a “ $l$ -modular” version of cyclic base change lift via  $l$ -modular local Langlands correspondence developed by Vignéras. Here we use quotation mark to indicate that our map is not always compatible with the cyclic base change lift over  $\overline{\mathbb{Q}}_l$  if we consider the usual modulo  $l$  map  $r_l$ , instead we need to replace  $r_l$  by another modified modulo  $l$  map  $J_l$  whose definition will be given later. As an application, we will study all the  $\sigma$ -invariant cuspidal representations of  $G = \mathrm{GL}_n(F)$  over  $\overline{\mathbb{F}}_l$  having a  $\overline{\mathbb{Q}}_l$ -lift as a  $\sigma$ -invariant cuspidal representation of  $G$ .

### 1.10.1 $l$ -modular local Langlands correspondence

We briefly recall the  $l$ -modular local Langlands correspondence developed by Vignéras [Vig01]. Let  $F$  be a non-archimedean locally compact field and let  $R$  be  $\overline{\mathbb{Q}}_l$  or  $\overline{\mathbb{F}}_l$ . We write  $\mathcal{W}_F$  for the Weil group of  $F$ . By representations of  $\mathcal{W}_F$  over  $R$ , say Weil representations, we mean semisimple finite dimensional smooth representations. By a Weil-Deligne representation over  $R$  we mean a pair  $(\varrho, N)$ , where  $\varrho$  is a Weil representation of  $\mathcal{W}_F$  and  $N$  is a nilpotent linear transformation defined on the space of  $\varrho$ , such that

$$v_F(w)\varrho(w)N = N\varrho(w)$$

for any  $w \in \mathcal{W}_F$ , where  $v_F$  is given by the exact sequence

$$0 \rightarrow I_F \longrightarrow \mathcal{W}_F \xrightarrow{v_F} \mathbb{Z} \rightarrow 0$$

whose evaluation at the arithmetic Frobenius map is 1. We denote by  $\mathrm{Rep}_R(\mathcal{W}_F)$  and  $\mathrm{Rep}_R(WD_F)$  the equivalence classes of Weil representations and Weil-Deligne representations, by  $\mathrm{Irr}_R(\mathcal{W}_F)$  the equivalence classes of irreducible Weil representations, by  $\mathrm{Irr}_R(\mathcal{W}_F(n))$  the equivalence classes of irreducible Weil representations of dimension  $n$ , by  $\mathrm{Irr}_R(\mathrm{GL}_n(F))$  the equivalence classes of smooth irreducible representations of  $\mathrm{GL}_n(F)$ , by  $\mathrm{Rep}_R(WD_F(n))$  the equivalence classes of Weil-Deligne representations with the dimension of the corresponding Weil representation equaling  $n$  and by  $\mathrm{Scusp}_R(\mathrm{GL}_n(F))$  the equivalence classes of irreducible supercuspidal representations of  $\mathrm{GL}_n(F)$  over  $R$ . We identify  $\overline{\mathbb{Q}}_l$  with  $\mathbb{C}$  via an algebraic isomorphism,

**Theorem 1.10.1** (Laumon-Rapoport-Stuhler, Harris-Taylor, Henniart, Scholze, Vignéras [LRS93], [HT01], [Hen00], [Sch13], [Vig01]). (1) *The local Langlands correspondence is defined as a bijection*

$$\mathrm{LLC}_R : \mathrm{Scusp}_R(\mathrm{GL}_n(F)) \longrightarrow \mathrm{Irr}_R(\mathcal{W}_F(n))$$

*determined by certain desiderata.*

(2) *When  $R = \overline{\mathbb{Q}}_l$ , the map  $\mathrm{LLC}_{\overline{\mathbb{Q}}_l}$  can be extended to irreducible representations, which is a bijection*

$$\mathrm{LLC}_{\overline{\mathbb{Q}}_l} : \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(\mathrm{GL}_n(F)) \longrightarrow \mathrm{Rep}_{\overline{\mathbb{Q}}_l}(WD_F(n))$$

*determined by certain desiderata;*

(3)  $\mathrm{LLC}_{\overline{\mathbb{F}}_l}$  *is compatible with  $\mathrm{LLC}_{\overline{\mathbb{Q}}_l}$ , saying that for any  $\tilde{\pi}$  as an integral supercuspidal representation of  $\mathrm{GL}_n(F)$  over  $\overline{\mathbb{Q}}_l$  with the supercuspidal support of its modulo  $l$  reduction  $r_l(\tilde{\pi})$  denoted by  $\{\pi_1, \dots, \pi_s\}$  as a multiset,  $\mathrm{LLC}_{\overline{\mathbb{Q}}_l}(\tilde{\pi})$  is also integral and  $\bigoplus_{i=1}^s \mathrm{LLC}_{\overline{\mathbb{F}}_l}(\pi_i) = r_l(\mathrm{LLC}_{\overline{\mathbb{Q}}_l}(\tilde{\pi}))$ .*

**Remark 1.10.2.** *Our consideration of  $\mathrm{LLC}_{\overline{\mathbb{Q}}_l}$  and  $\mathrm{LLC}_{\overline{\mathbb{F}}_l}$  depends on the choice of the isomorphism  $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ . Actually it only depends on the choice of a certain square root of  $q$  in  $\overline{\mathbb{Q}}_l$  for  $q$  denoting the cardinality of the residue field of  $F$  (cf. [Dat07], §2.2).*

Recall that we have defined  $\nu = |\det(\cdot)|_F$  as a character of  $\mathrm{GL}_n(F)$  for any  $n$ . By abuse of notations we also regard  $\nu$  as a character of  $\mathcal{W}_F$  defined by  $\nu(w) = q^{-v_F(w)}$  for any  $w \in \mathcal{W}_F$ . By a *(super)cuspidal segment* of  $\mathrm{GL}_n(F)$ , we mean a multiset  $\Delta_{\rho,[a,b]} = \{\rho\nu^a, \rho\nu^{a+1}, \dots, \rho\nu^b\}$  with  $a \leq b$  integers and  $\rho$  a (super)cuspidal representation of  $\mathrm{GL}_{n'}(F)$  such that  $n'(b-a+1) = n$ . Similarly by a segment of  $\mathcal{W}_F(n)$ , we mean a multiset  $\Delta_{\varrho,[a,b]} = \{\varrho\nu^a, \varrho\nu^{a+1}, \dots, \varrho\nu^b\}$  with  $a \leq b$  integers and  $\varrho \in \mathrm{Irr}_R(\mathcal{W}_F)$  such that  $\dim(\varrho)(b-a+1) = n$ . In both cases we call  $n$  the length of the segment, and without imposing on the restriction on the length we call  $\Delta_{\rho,[a,b]}$  a (super)cuspidal segment of  $\mathrm{GL}(F)$ , and  $\Delta_{\varrho,[a,b]}$  a segment of  $\mathcal{W}_F$ . A multisegment of  $\mathrm{GL}(F)$  (resp.  $\mathcal{W}_F$ ) means a multiset composed of cuspidal segments of  $\mathrm{GL}(F)$  (resp. segments of  $\mathcal{W}_F$ ), whose length equals the sum of that of each segment. We denote by  $\mathrm{Mul}_R(\mathrm{GL}_n(F))$  the set of multisegments of  $\mathrm{GL}(F)$  of length  $n$  composed of **supercuspidal segments** and by  $\mathrm{Mul}_R(\mathcal{W}_F(n))$  the set of multisegments of  $\mathcal{W}_F$  of length  $n$ . The following proposition extend the local Langlands correspondence above to multisegments, whose proof is direct.

**Proposition 1.10.3.** *For each  $n$  the local Langlands correspondence as above extends to a bijection*

$$\mathrm{LLC}_R : \mathrm{Mul}_R(\mathrm{GL}_n(F)) \longrightarrow \mathrm{Mul}_R(\mathcal{W}_F(n)),$$

given by the relation

$$\mathrm{LLC}_R(\Delta_{\rho,[a,b]}) = \Delta_{\mathrm{LLC}_R(\rho),[a,b]}$$

for any  $a, b$  and  $\rho \in \mathrm{Scusp}_R(\mathrm{GL}_{n'}(F))$ , and

$$\mathrm{LLC}_R(\Delta_1 + \dots + \Delta_r) = \mathrm{LLC}_R(\Delta_1) + \dots + \mathrm{LLC}_R(\Delta_r)$$

for  $\Delta_i$  segments of  $\mathrm{GL}(F)$ .

We consider  $\mathrm{Mul}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathrm{GL}_n(F))$  as the subset of  $\mathrm{Mul}_{\overline{\mathbb{Q}_l}}(\mathrm{GL}_n(F))$  with each supercuspidal constituent integral, and for any  $n$  we define

$$r_l : \mathrm{Mul}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathrm{GL}_n(F)) \longrightarrow \mathrm{Mul}_{\overline{\mathbb{F}_l}}(\mathrm{GL}_n(F))$$

such that

$$r_l(\Delta_{\tilde{\rho},[a,b]}) = \sum_{\rho \in \mathrm{SC}(r_l(\tilde{\rho}))} \Delta_{\rho,[a,b]}$$

for any  $a, b$  and  $\tilde{\rho} \in \mathrm{Scusp}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathrm{GL}_{n'}(F))$  with  $\mathrm{SC}(r_l(\tilde{\rho}))$  denoting the supercuspidal support of  $r_l(\tilde{\rho})$ , and

$$r_l(\Delta_1 + \dots + \Delta_r) = r_l(\Delta_1) + \dots + r_l(\Delta_r)$$

for  $\Delta_i$  integral segments of  $\mathrm{GL}(F)$  over  $\overline{\mathbb{Q}_l}$ . Similarly we consider  $\mathrm{Mul}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathcal{W}_F(n))$  as the subset of  $\mathrm{Mul}_{\overline{\mathbb{Q}_l}}(\mathcal{W}_F(n))$  with each irreducible constituent integral, and for any  $n$  we define

$$r_l : \mathrm{Mul}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathcal{W}_F(n)) \longrightarrow \mathrm{Mul}_{\overline{\mathbb{F}_l}}(\mathcal{W}_F(n))$$

such that

$$r_l(\Delta_{\tilde{\varrho},[a,b]}) = \sum_{\varrho \in r_l(\tilde{\varrho})} \Delta_{\varrho,[a,b]}$$

for any  $a, b$  and  $\tilde{\varrho} \in \mathrm{Irr}_{\overline{\mathbb{Q}_l}}^{\mathrm{Int}}(\mathcal{W}_F(n'))$ , and

$$r_l(\Delta_1 + \dots + \Delta_r) = r_l(\Delta_1) + \dots + r_l(\Delta_r)$$

for  $\Delta_i$  integral segments of  $\mathcal{W}_F$  over  $\overline{\mathbb{Q}_l}$ .

**Corollary 1.10.4.**  $\text{LLC}_{\overline{\mathbb{Q}_l}}$  maps integral multisegments to integral multisegments and we have the following commutative diagram

$$\begin{array}{ccc} \text{Mul}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F)) & \xrightarrow{\text{LLC}_{\overline{\mathbb{Q}_l}}} & \text{Mul}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\mathcal{W}_F(n)) \\ r_l \downarrow & & \downarrow r_l \\ \text{Mul}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F)) & \xrightarrow{\text{LLC}_{\overline{\mathbb{F}_l}}} & \text{Mul}_{\overline{\mathbb{F}_l}}(\mathcal{W}_F(n)) \end{array}$$

*Proof.* The first part is obvious and the second part follows from Theorem [L.10.1](#)(3).  $\square$

Fix  $\psi_F : \mathfrak{o}_F \rightarrow \overline{\mathbb{Z}_l}^\times$  a non-trivial character and extend it to a character  $\psi_F : F \rightarrow R^\times$  by extension of scalar. For  $U_n$  the standard unipotent subgroup of  $\text{GL}_n(F)$  and  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s)$  a partition of  $n$ , we define a character of  $U_n$

$$\psi_\lambda(u) := \psi_F\left(\sum_i u_{i,i+1}\right),$$

where the sum ranges over  $i \neq \lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \lambda_2 + \dots + \lambda_s$ . We have a total order  $\lambda \geq \lambda'$  for partitions  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s)$  and  $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_{s'})$  of  $n$ , saying that either  $\lambda = \lambda'$ , or there exists  $k \in \{1, 2, \dots, s\}$  such that

$$\lambda_1 = \lambda'_1, \quad \lambda_2 = \lambda'_2, \quad \dots \quad \lambda_{k-1} = \lambda'_{k-1}, \quad \text{and} \quad \lambda_1 + \dots + \lambda_k > \lambda'_1 + \dots + \lambda'_k.$$

For  $\pi$  a smooth representation of  $\text{GL}_n(F)$  of finite length over  $R$ , we write  $\lambda_\pi$  for the largest partition  $\lambda$  such that

$$\text{Hom}_{R[U_n]}(\pi, \psi_\lambda) \neq 0,$$

called the derivative sequence of  $\pi$ . The dimension of the above  $R$ -vector space equals exactly the number of irreducible subquotients of  $\pi$  having the same derivative sequence. In particular, if the space is of dimension 1 as an  $R$ -vector space, we may associate to  $\pi$  with its unique irreducible subquotient  $\pi'$  such that  $\lambda_\pi = \lambda_{\pi'}$ , that is, amongst all the subquotients of  $\pi$ , the subquotient  $\pi'$  is unique having the highest derivative sequence.

For  $\Delta = \Delta_{\rho,[a,b]}$  with  $\rho$  a cuspidal representation of  $\text{GL}_{n'}(F)$ , when  $R = \overline{\mathbb{Q}_l}$  we define  $Z(\Delta)$  to be the subrepresentation of  $\rho\nu^a \times \dots \times \rho\nu^b$  which is unique up to isomorphism, such that its Jacquet module corresponding to the Levi subgroup  $\text{GL}_{n'}(F) \times \dots \times \text{GL}_{n'}(F)$  equals  $\rho\nu^a \otimes \dots \otimes \rho\nu^b$ . When  $R = \overline{\mathbb{F}_l}$  the above construction doesn't work due to the lack of uniqueness, but we have the following lemma:

**Lemma 1.10.5** ([\[Dat12\]](#), Proposition 2.2.3). *For  $\rho$  a cuspidal representation of  $\text{GL}_{n'}(F)$ , there exists an integral supercuspidal representation  $\tilde{\rho}$  of  $\text{GL}_{n'}(F)$  such that  $\rho = r_l(\tilde{\rho})$ . Moreover the representation  $r_l(Z(\Delta_{\tilde{\rho},[a,b]}))$  is irreducible and independent of the choice of  $\tilde{\rho}$ .*

Using this lemma we define  $Z(\Delta) := r_l(Z(\Delta_{\tilde{\rho},[a,b]}))$  for  $R = \overline{\mathbb{F}_l}$ . Moreover for  $\mathbf{m} = \Delta_1 + \dots + \Delta_r$  as a multisegment of  $\text{GL}_n(F)$ , the  $R$ -vector space

$$\text{Hom}_{R[U_n]}(Z(\Delta_1) \times \dots \times Z(\Delta_r), \psi_{\lambda_{Z(\Delta_1) \times \dots \times Z(\Delta_r)}})$$

is of dimension 1, and

$$Z(\mathbf{m}) = Z(\Delta_1, \dots, \Delta_r)$$

is defined to be the unique subquotient of  $Z(\Delta_1) \times \dots \times Z(\Delta_r)$  such that  $\lambda_{Z(\Delta_1) \times \dots \times Z(\Delta_r)} = \lambda_{Z(\mathbf{m})}$  (cf. [\[Dat12\]](#), §2.2.5). We have the following classification theorem for  $\text{Irr}_R(\text{GL}_n(F))$ .

**Proposition 1.10.6.** *We have the bijection*

$$\begin{aligned} \mathcal{Z}_R : \text{Mul}_R(\text{GL}_n(F)) &\longrightarrow \text{Irr}_R(\text{GL}_n(F)) \\ \mathfrak{m} &\longmapsto Z(\mathfrak{m}). \end{aligned}$$

Moreover when  $R = \overline{\mathbb{Q}_l}$  it maps integral multisegments to integral representations and vice versa.

*Proof.* The first part follows from [Vig98] Théorème, V.12. For the second part since the Jacquet functor and the parabolic induction maintain the property of being integral, and the subrepresentation of an integral representation is also integral ([MS14a], §1.2), we know that an irreducible representation is integral if and only if all the representations in its cuspidal support are integral, which finishes the proof.  $\square$

**Corollary 1.10.7.** *For  $\tilde{\pi}$  an irreducible integral representation of  $\text{GL}_n(F)$  over  $\overline{\mathbb{Q}_l}$ , we have  $\lambda_{\tilde{\pi}} = \lambda_{r_l(\tilde{\pi})}$  and  $\text{Hom}_{\overline{\mathbb{F}_l}[U_n]}(r_l(\tilde{\pi}), \psi_{\lambda_{r_l(\tilde{\pi})}}) \cong \overline{\mathbb{F}_l}$ , thus we may define a map*

$$\begin{aligned} J_l : \text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F)) &\longrightarrow \text{Irr}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F)) \\ \tilde{\pi} &\longmapsto J_l(\tilde{\pi}) \end{aligned}$$

with  $J_l(\tilde{\pi})$  denoting the unique irreducible subquotient of  $r_l(\tilde{\pi})$  such that  $\lambda_{\tilde{\pi}} = \lambda_{J_l(\tilde{\pi})}$ . Moreover  $J_l$  is surjective and we have the commutative diagram

$$\begin{array}{ccc} \text{Mul}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F)) & \xrightarrow{\mathcal{Z}_{\overline{\mathbb{Q}_l}}} & \text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F)) \\ r_l \downarrow & & \downarrow J_l \\ \text{Mul}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F)) & \xrightarrow{\mathcal{Z}_{\overline{\mathbb{F}_l}}} & \text{Irr}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F)) \end{array}$$

*Proof.* The first part follows from the fact that  $U_n$  is a pro- $p$ -group and  $p \neq l$ . The surjectivity of  $J_l$  follows from [Vig98], V.9.2. Now we prove the commutativity of the diagram. We choose  $\mathfrak{m} = \Delta_1 + \dots + \Delta_r \in \text{Mul}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F))$  and we write  $\tilde{\pi} = Z(\mathfrak{m})$  which is an irreducible representation. By definition  $J_l(\tilde{\pi})$  equals the unique subquotient of  $r_l(Z(\Delta_1)) \times \dots \times r_l(Z(\Delta_r))$  having the highest derivative sequence. By definition  $r_l(Z(\Delta_i)) = Z(\overline{\Delta}_i)$  for each  $i$ , where for  $\Delta_i = \Delta_{\tilde{\rho}_i, [a_i, b_i]}$ , we write  $\overline{\Delta}_i = \Delta_{\rho_i, [a_i, b_i]}$  as a segment over  $\overline{\mathbb{F}_l}$  with  $\rho_i = r_l(\tilde{\rho}_i)$  cuspidal (but not necessarily supercuspidal) over  $\overline{\mathbb{F}_l}$ . Then  $J_l(\tilde{\pi})$  is the unique subquotient of  $Z(\overline{\Delta}_1) \times Z(\overline{\Delta}_2) \times \dots \times Z(\overline{\Delta}_r)$  having the highest derivative sequence, which by definition equals  $Z(\sum_{i=1}^r \overline{\Delta}_i)$ . Finally using [MS14a], Théorème 9.36, we have  $Z(\sum_{i=1}^r \overline{\Delta}_i) = Z(r_l(\mathfrak{m}))$  which finishes the proof.  $\square$

For  $(\tilde{\varrho}, N) \in \text{Rep}_{\overline{\mathbb{Q}_l}}(WD_F(n))$ , we call it integral if  $\tilde{\varrho}$  is integral and  $N$  is realized from an element in  $\text{End}_{\overline{\mathbb{Z}_l}}(M)$  by extension of scalar with  $M$  denoting one  $\overline{\mathbb{Z}_l}$ -lattice corresponding to  $\tilde{\varrho}$ , and we define  $r_l(\tilde{\varrho}, N) := (r_l(\tilde{\varrho}), r_l(N))$ . For Weil-Deligne representations, we have the following classification theorem whose proof is clear by definition and the utilization of Jordan normal form.

**Proposition 1.10.8.** *We have a bijection*

$$\mathcal{G}_R : \text{Mul}_R(\mathcal{W}_F(n)) \longrightarrow \text{Rep}_R(WD_F(n))$$



determined by the relations

$$\mathcal{G}_R(\Delta_{\varrho,[a,b]}) := \left( \varrho\nu^a \oplus \varrho\nu^{a+1} \oplus \dots \oplus \varrho\nu^b, \begin{pmatrix} 0 & I_{n'} & 0 & \dots & 0 \\ \vdots & 0 & I_{n'} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & I_{n'} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \right)$$

for any  $a, b$  and  $\varrho \in \text{Irr}_R(\mathcal{W}_F(n'))$ , and

$$\mathcal{G}_R(\Delta_1 + \dots + \Delta_r) := \mathcal{G}_R(\Delta_1) \oplus \dots \oplus \mathcal{G}_R(\Delta_r)$$

for  $\Delta_i$  segments of  $\text{Irr}_R(\mathcal{W}_F)$ . Moreover if  $R = \overline{\mathbb{Q}_l}$  it maps integral multisegments to integral representations and vice versa.

**Corollary 1.10.9.** *We have the following commutative diagram:*

$$\begin{array}{ccc} \text{Mul}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\mathcal{W}_F(n)) & \xrightarrow{\mathcal{G}_{\overline{\mathbb{Q}_l}}} & \text{Rep}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(WD_F(n)) \\ r_l \downarrow & & \downarrow r_l \\ \text{Mul}_{\overline{\mathbb{F}_l}}(\mathcal{W}_F(n)) & \xrightarrow{\mathcal{G}_{\overline{\mathbb{F}_l}}} & \text{Rep}_{\overline{\mathbb{F}_l}}(WD_F(n)) \end{array}$$

*Proof.* By definition we only need to study a single segment, which is direct. □

Finally we define the so-called Zelevinsky correspondence which is analogous to the local Langlands correspondence and was indicated by Zelevinsky [\[Zel80\]](#). We define the map

$$\text{Zel}_R : \text{Irr}_R(\text{GL}_n(F)) \longrightarrow \text{Rep}_R(WD_F)$$

as the composition of  $\mathcal{Z}_R^{-1}$ ,  $\text{LLC}_R$  and  $\mathcal{G}_R$  which is a bijection. More precisely, for  $\pi = Z(\mathfrak{m})$  with  $\mathfrak{m} \in \text{Mul}_R(\text{GL}_n(F))$ , we define  $\text{Zel}_R(\pi) = \mathcal{G}_R(\text{LLC}_R(\mathfrak{m}))$ . In particular restricting to supercuspidal representations, the Zelevinsky correspondence coincides with the Langlands correspondence. We end this subsection with the  $l$ -modular property of Zelevinsky correspondence.

**Proposition 1.10.10.** *We have the following commutative diagram*

$$\begin{array}{ccc} \text{Irr}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(\text{GL}_n(F)) & \xrightarrow{\text{Zel}_{\overline{\mathbb{Q}_l}}} & \text{Rep}_{\overline{\mathbb{Q}_l}}^{\text{Int}}(WD_F) \\ J_l \downarrow & & \downarrow r_l \\ \text{Irr}_{\overline{\mathbb{F}_l}}(\text{GL}_n(F)) & \xrightarrow{\text{Zel}_{\overline{\mathbb{F}_l}}} & \text{Rep}_{\overline{\mathbb{F}_l}}(WD_F) \end{array}$$

*Proof.* We only need to combine the above three commutative diagrams together. □

### 1.10.2 $l$ -modular base change lift

In this subsection, we make use of the Zelevinsky correspondence defined above to construct a  $l$ -modular version of cyclic base change lift. To that end, we assume  $F/F_0$  to be a cyclic extension of degree  $r$  of non-archimedean locally compact fields of residue characteristic  $p \neq l$ . We fix  $\sigma \in \text{Gal}(F/F_0)$  a generator and  $\omega_{F/F_0}$  a character of  $F_0^\times$  with kernel  $N_{F/F_0}(F^\times)$  which is usually identified with a character of  $\text{GL}_n(F_0)$  by composing with the determinant map. We write  $\nu_0 = |\det(\cdot)|_{F_0}$  as a character of  $\text{GL}_n(F_0)$ , which is also regarded as a character of  $\mathcal{W}_{F_0}$  by abuse of notations. And for segments of  $\text{GL}(F_0)$  (resp.  $\text{GL}(F)$ ) and  $\mathcal{W}_{F_0}$  (resp.  $\mathcal{W}_F$ ), they are related to the character  $\nu_0$  (resp.  $\nu$ ). We fix an algebraic isomorphism  $\overline{\mathbb{Q}}_l \cong \mathbb{C}$  to identify this two fields. For  $\tilde{\rho}_0 \in \text{Scusp}_{\overline{\mathbb{Q}}_l}(\text{GL}_{n'}(F_0))$ , let  $c(\tilde{\rho}_0)$  be the cardinality of the set of isomorphism classes  $C(\tilde{\rho}_0) := \{\tilde{\rho}_0 \omega_{F/F_0}^k \mid k \in \mathbb{Z}\} / \cong$ . It is easy to see that  $c(\tilde{\rho}_0)$  divides  $r$ .

We also recall the Langlands classification, which says that for any  $\tilde{\pi}_0$  as an irreducible representation of  $\text{GL}_n(F_0)$  over  $\overline{\mathbb{Q}}_l$ , there exists a multisegment  $\mathbf{m}_0 = \Delta_1^0 + \dots + \Delta_k^0$  such that  $\tilde{\pi}_0 = L(\mathbf{m}_0)$ , where  $L(\mathbf{m}_0)$  denotes the unique irreducible quotient of  $\text{St}(\Delta_1^0) \times \dots \times \text{St}(\Delta_k^0)$  if  $\mathbf{m}_0$  is rearranged such that  $\Delta_i^0$  does not precede  $\Delta_j^0$  for  $i < j$  (see for example [\[MS13\]](#), Théorème 1.4).

**Theorem 1.10.11.** *The base change lift*

$$\text{BC}_{\overline{\mathbb{Q}}_l} : \text{Irr}_{\overline{\mathbb{Q}}_l}(\text{GL}_n(F_0)) \longrightarrow \text{Irr}_{\overline{\mathbb{Q}}_l}^{\sigma\text{-inv}}(\text{GL}_n(F))$$

satisfies and is determined by the following properties:

- For  $\tilde{\rho}_0 \in \text{Scusp}_{\overline{\mathbb{Q}}_l}(\text{GL}_{n'}(F_0))$ , the integer  $r/c(\tilde{\rho}_0)$  divides  $n'$ . And moreover there exists  $\tilde{\rho} \in \text{Scusp}_{\overline{\mathbb{Q}}_l}(\text{GL}_{n'/c(\tilde{\rho}_0)/r}(F))$  such that  $\tilde{\rho}^{\sigma^i} \cong \tilde{\rho}$  if and only if  $i$  is divided by  $r/c(\tilde{\rho}_0)$ , and for any  $a \leq b$  integers,

$$\text{BC}_{\overline{\mathbb{Q}}_l}(\text{St}(\Delta_{\tilde{\rho}_0, [a, b]})) = \text{St}(\Delta_{\tilde{\rho}, [a, b]}) \times \text{St}(\Delta_{\tilde{\rho}, [a, b]})^\sigma \times \dots \times \text{St}(\Delta_{\tilde{\rho}, [a, b]})^{\sigma^{r/c(\tilde{\rho}_0)-1}}. \quad (1.10.1)$$

Conversely let  $c$  be a positive integer dividing  $r$  with  $r/c$  dividing  $n'$ . For  $\tilde{\rho} \in \text{Scusp}_{\overline{\mathbb{Q}}_l}(\text{GL}_{n'/c/r}(F))$  such that  $\tilde{\rho}^{\sigma^i} \cong \tilde{\rho}$  if and only if  $i$  is divided by  $r/c$ , there exists  $\tilde{\rho}_0$  satisfying [\(1.10.1\)](#) and  $c(\tilde{\rho}_0) = c$ , and moreover

$$\text{BC}_{\overline{\mathbb{Q}}_l}^{-1}(\text{St}(\Delta_{\tilde{\rho}, [a, b]}) \times \text{St}(\Delta_{\tilde{\rho}, [a, b]})^\sigma \times \dots \times \text{St}(\Delta_{\tilde{\rho}, [a, b]})^{\sigma^{r/c-1}}) = \{\text{St}(\Delta_{\tilde{\rho}_0, [a, b]}) \mid \tilde{\rho}_0 \in C(\tilde{\rho}_0)\}.$$

- The base change lift is compatible with the Langlands quotient. That is, for  $\tilde{\pi}_0 = L(\mathbf{m}_0)$  as above, we have

$$\text{BC}_{\overline{\mathbb{Q}}_l}(L(\mathbf{m}_0)) = L(\text{BC}_{\overline{\mathbb{Q}}_l}(\mathbf{m}_0)),$$

where for any segment  $\Delta_{\tilde{\rho}_0, [a, b]}$  as above we define

$$\text{BC}_{\overline{\mathbb{Q}}_l}(\Delta_{\tilde{\rho}_0, [a, b]}) := \Delta_{\tilde{\rho}, [a, b]} + \Delta_{\tilde{\rho}^\sigma, [a, b]} + \dots + \Delta_{\tilde{\rho}^{\sigma^{r/c(\tilde{\rho}_0)-1}}, [a, b]}$$

as a multisegment of  $\text{GL}(F)$ , and for any multisegment  $\mathbf{m}_0 = \Delta_1^0 + \dots + \Delta_k^0$  we define

$$\text{BC}_{\overline{\mathbb{Q}}_l}(\mathbf{m}_0) := \sum_{i=1}^k \text{BC}_{\overline{\mathbb{Q}}_l}(\Delta_i^0).$$

In addition, it maps integral representations to integral representations.

*Proof.* All the properties are listed in [AC89], chapter 1, section 6 and [HL11], chapter 2, except the final one. Since base change lift transfers the information of central characters, it maps integral supercuspidal representations to integral supercuspidal representations. More precisely for  $\tilde{\rho}_0$  a supercuspidal representation of  $\mathrm{GL}_{n'}(F_0)$  and  $\mathrm{BC}_{\overline{\mathbb{Q}}_l}(\tilde{\rho}_0) = \tilde{\rho} \times \tilde{\rho}^\sigma \dots \times \tilde{\rho}^{\sigma^{r/c(\tilde{\rho}_0)-1}}$ , we have  $\omega_{\tilde{\rho}_0} = \prod_{i=0}^{r/c(\tilde{\rho}_0)-1} \omega_{\tilde{\rho}^{\sigma^i}} \circ N_{F/F_0}$ , where  $\omega_{\tilde{\rho}_0}$  and  $\omega_{\tilde{\rho}^{\sigma^i}}$  denote the central character of  $\tilde{\rho}_0$  and  $\tilde{\rho}^{\sigma^i}$  respectively. Thus  $\omega_{\tilde{\rho}_0}$  is integral if and only if  $\omega_{\tilde{\rho}}$  is integral, which means that  $\tilde{\rho}_0$  is integral if and only if  $\tilde{\rho}$  is integral by [Vig96], II.4.12, confirming the assertion above. Finally using the first two properties listed in the theorem it maps integral irreducible representations to integral irreducible representations.  $\square$

**Remark 1.10.12.** *Our consideration of base change lift over  $\overline{\mathbb{Q}}_l$  still depends on the choice of the isomorphism  $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ , or more precisely the choice of a certain square root of  $q$  in  $\overline{\mathbb{Q}}_l$  for  $q$  denoting the cardinality of the residue field of  $F_0$ . This is unlike the Jacquet-Langlands correspondence over  $\overline{\mathbb{Q}}_l$  which doesn't depend on the embedding since there the local Langlands correspondences (for  $\mathrm{GL}_n(F)$  and its inner form) are over the same base field, thus once we consider their composition and change the embedding, the changes of two local Langlands correspondences (as signs) compensate with each other. However for base change lift since the base fields (and in particular the cardinalities of residue fields) are not the same, this kind of cancelation doesn't happen in general (for example  $F/F_0$  is unramified and quadratic).*

One important property and motivation for the consideration of cyclic base change lift is that it satisfies the local Langlands functoriality. For  $(\varrho, N) \in \mathrm{Rep}_R(WD_{F_0})$ , since  $\mathcal{W}_F$  can be regarded as an open subgroup of  $\mathcal{W}_{F_0}$ , we define the restriction<sup>8</sup>

$$\begin{aligned} \mathrm{Res}_{F/F_0} : \mathrm{Rep}_R(WD_{F_0}) &\longrightarrow \mathrm{Rep}_R(WD_F) \\ (\varrho, N) &\longrightarrow ((\varrho|_{\mathcal{W}_F})^{ss}, N) \end{aligned}$$

**Proposition 1.10.13.** *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(\mathrm{GL}_n(F_0)) & \xrightarrow{\mathrm{LLC}_{\overline{\mathbb{Q}}_l}} & \mathrm{Rep}_{\overline{\mathbb{Q}}_l}(WD_{F_0}) \\ \mathrm{BC}_{\overline{\mathbb{Q}}_l} \downarrow & & \downarrow \mathrm{Res}_{F/F_0} \\ \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(\mathrm{GL}_n(F)) & \xrightarrow{\mathrm{LLC}_{\overline{\mathbb{Q}}_l}} & \mathrm{Rep}_{\overline{\mathbb{Q}}_l}(WD_F) \end{array}$$

*Proof.* The proof follows from the fact that the base change lift, the restriction  $\mathrm{Res}_{F/F_0}$  and the local Langlands correspondence transfer the information of  $L$ -factors and  $\varepsilon$ -factors, which determines the irreducible representations of general linear groups or Weil-Deligne representations. See [AC89], Proposition 6.9, [HL11], II.3.5 and [HT01].  $\square$

In addition, the restriction is compatible with the modulo  $l$  reduction, whose proof is direct:

**Proposition 1.10.14.** *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Rep}_{\overline{\mathbb{Q}}_l}^{\mathrm{Int}}(WD_{F_0}) & \xrightarrow{\mathrm{Res}_{F/F_0}} & \mathrm{Rep}_{\overline{\mathbb{Q}}_l}^{\mathrm{Int}}(WD_F) \\ r_l \downarrow & & \downarrow r_l \\ \mathrm{Rep}_{\overline{\mathbb{F}}_l}(WD_{F_0}) & \xrightarrow{\mathrm{Res}_{F/F_0}} & \mathrm{Rep}_{\overline{\mathbb{F}}_l}(WD_F) \end{array}$$

<sup>8</sup>The superscript “ss” denotes the semi-simplification.

For the future use, we need the functoriality with respect to the Zelevinsky correspondence. First we recall the Zelevinsky involution ( [Zel80], section 9), which is an involution between multisegments

$$T_{\text{Zel}} : \text{Mul}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F)) \longrightarrow \text{Mul}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F))$$

whose combinatorial definition is given by Mœglin-Waldspurger [MW86], leading to an involution between irreducible representations (still called the Zelevinsky involution)

$$T_{\text{Zel}} : \text{Irr}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F)) \longrightarrow \text{Irr}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F))$$

determined by the relation

$$T_{\text{Zel}}(Z(\mathbf{m})) = Z(T_{\text{Zel}}(\mathbf{m}))$$

for any multisegment  $\mathbf{m}$  over  $\overline{\mathbb{Q}_l}$ . Zelevinsky further conjectured that

$$\text{LLC}_{\overline{\mathbb{Q}_l}} \circ T_{\text{Zel}} = \text{Zel}_{\overline{\mathbb{Q}_l}}$$

which is known (For example see [MS13], Proposition A.7).

**Proposition 1.10.15.** *We have the following commutative diagram:*

$$\begin{array}{ccc} \text{Irr}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F_0)) & \xrightarrow{\text{Zel}_{\overline{\mathbb{Q}_l}}} & \text{Rep}_{\overline{\mathbb{Q}_l}}(WD_{F_0}) \\ \text{BC}_{\overline{\mathbb{Q}_l}} \downarrow & & \downarrow \text{Res}_{F/F_0} \\ \text{Irr}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F)) & \xrightarrow{\text{Zel}_{\overline{\mathbb{Q}_l}}} & \text{Rep}_{\overline{\mathbb{Q}_l}}(WD_F) \end{array}$$

*Proof.* By proposition [1.10.13] and the result mentioned above, we only need to prove that the base change lift is compatible with the Zelevinsky involution, that is, for any  $\tilde{\pi}_0 \in \text{Irr}_{\overline{\mathbb{Q}_l}}(\text{GL}_n(F_0))$  we have  $\text{BC}_{\overline{\mathbb{Q}_l}}(T_{\text{Zel}}(\tilde{\pi}_0)) = T_{\text{Zel}}(\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0))$ . Using Theorem [1.10.11], we have

$$\text{BC}_{\overline{\mathbb{Q}_l}}(L(\mathbf{m}_0)) = L(\mathbf{m})$$

for  $\mathbf{m} = \text{BC}_{\overline{\mathbb{Q}_l}}(\mathbf{m}_0)$ . Moreover we have

**Lemma 1.10.16.**  $T_{\text{Zel}}(\mathbf{m}) = \text{BC}_{\overline{\mathbb{Q}_l}}(T_{\text{Zel}}(\mathbf{m}_0))$ .

*Proof.* Using the algorithm given by Mœglin-Waldspurger in [MW86], we only need to consider the case where the elements in the support of  $\mathbf{m}_0$  belong to  $\{\tilde{\rho}_0 \nu_0^i | i \in \mathbb{Z}\}$ . We may write  $\mathbf{m} = \mathbf{m}_1 + \dots + \mathbf{m}_{r/c(\tilde{\rho}_0)}$ , where  $\mathbf{m}_i$  is the multisegment having the same combinatorial structure as  $\mathbf{m}_0$ , but with  $\tilde{\rho}_0 \nu_0^j$  in each segment replaced by  $\tilde{\rho}^{\sigma^{i-1}} \nu^j$ . Since the algorithm is independence of the supercuspidal representations in each segment but only the combinatorial structure, once we replace  $\mathbf{m}_0$  with  $T_{\text{Zel}}(\mathbf{m}_0)$ , each  $\mathbf{m}_i$  is replaced with  $T_{\text{Zel}}(\mathbf{m}_i)$ , which proves the lemma.  $\square$

By definition and [MS13], Proposition A.7, we have

$$T_{\text{Zel}}(L(\mathbf{m}_0)) = L(T_{\text{Zel}}(\mathbf{m}_0)) = Z(\mathbf{m}_0) \quad \text{and} \quad T_{\text{Zel}}(L(\mathbf{m})) = L(T_{\text{Zel}}(\mathbf{m})) = Z(\mathbf{m}).$$

Thus combining them together we have

$$T_{\text{Zel}}(\text{BC}_{\overline{\mathbb{Q}_l}}(\pi_0)) = T_{\text{Zel}}(\text{BC}_{\overline{\mathbb{Q}_l}}(L(\mathbf{m}_0))) = T_{\text{Zel}}(L(\mathbf{m})) = Z(\mathbf{m})$$

and

$$\begin{aligned} \text{BC}_{\overline{\mathbb{Q}_l}}(T_{\text{Zel}}(\pi_0)) &= \text{BC}_{\overline{\mathbb{Q}_l}}(T_{\text{Zel}}(L(\mathbf{m}_0))) = \text{BC}_{\overline{\mathbb{Q}_l}}(L(T_{\text{Zel}}(\mathbf{m}_0))) = L(\text{BC}_{\overline{\mathbb{Q}_l}}(T_{\text{Zel}}(\mathbf{m}_0))) = L(T_{\text{Zel}}(\mathbf{m})) \\ &= Z(\mathbf{m}). \end{aligned}$$

$\square$

Finally we may give our definition of  $l$ -modular base change lift. We consider the following diagram

$$\begin{array}{ccccc}
 \text{Rep}_{\mathbb{Q}_l}^{\text{Int}}(WD_{F_0}) & \xrightarrow{\text{Res}_{F/F_0}} & & \xrightarrow{\text{Res}_{F/F_0}} & \text{Rep}_{\mathbb{Q}_l}^{\text{Int}}(WD_F) \\
 & \searrow^{\text{Zel}_{\mathbb{Q}_l}} & & \nearrow^{\text{Zel}_{\mathbb{Q}_l}} & \\
 & & \text{Irr}_{\mathbb{Q}_l}^{\text{Int}}(\text{GL}_n(F_0)) & \xrightarrow{\text{BC}_{\mathbb{Q}_l}} & \text{Irr}_{\mathbb{Q}_l}^{\text{Int}, \sigma\text{-inv}}(\text{GL}_n(F)) \\
 & & \downarrow J_l & & \downarrow J_l \\
 & & \text{Irr}_{\mathbb{F}_l}(\text{GL}_n(F_0)) & \xrightarrow{\text{BC}_{\mathbb{F}_l}} & \text{Irr}_{\mathbb{F}_l}^{\sigma\text{-inv}}(\text{GL}_n(F)) \\
 & \swarrow^{\text{Zel}_{\mathbb{F}_l}} & & \searrow^{\text{Zel}_{\mathbb{F}_l}} & \\
 \text{Rep}_{\mathbb{F}_l}^{\text{Int}}(WD_{F_0}) & \xrightarrow{\text{Res}_{F/F_0}} & & \xrightarrow{\text{Res}_{F/F_0}} & \text{Rep}_{\mathbb{F}_l}^{\text{Int}}(WD_F)
 \end{array}
 \tag{I}$$

The diagram (I) is commutative by Proposition 1.10.15 and the diagram (II) and (III) are commutative by Proposition 1.10.10. In the diagram (IV) since  $\text{Zel}_{\mathbb{F}_l}$  is bijective, we may define the map  $\text{BC}_{\mathbb{F}_l}$  to make the diagram commutative. In this case from the commutativity of (I), (II), (III), (IV) and the outer diagram via Proposition 1.10.14, by diagram chasing the inner diagram (V) is also commutative. To sum up, we have proved the following theorem:

**Theorem 1.10.17.** *We may define the  $l$ -modular cyclic base change lift*

$$\text{BC}_{\mathbb{F}_l} : \text{Irr}_{\mathbb{F}_l}(\text{GL}_n(F_0)) \longrightarrow \text{Irr}_{\mathbb{F}_l}^{\sigma\text{-inv}}(\text{GL}_n(F))$$

which satisfies and is determined by the following commutative diagram

$$\begin{array}{ccc}
 \text{Irr}_{\mathbb{Q}_l}^{\text{Int}}(\text{GL}_n(F_0)) & \xrightarrow{\text{BC}_{\mathbb{Q}_l}} & \text{Irr}_{\mathbb{Q}_l}^{\text{Int}, \sigma\text{-inv}}(\text{GL}_n(F)) \\
 \downarrow J_l & & \downarrow J_l \\
 \text{Irr}_{\mathbb{F}_l}(\text{GL}_n(F_0)) & \xrightarrow{\text{BC}_{\mathbb{F}_l}} & \text{Irr}_{\mathbb{F}_l}^{\sigma\text{-inv}}(\text{GL}_n(F))
 \end{array}$$

**Remark 1.10.18.** *Similarly we have a finite version of  $l$ -modular base change lift. Let  $\mathfrak{l}/\mathfrak{l}_0$  be an extension of finite fields of characteristic  $p$  of degree  $r$ , then we may define the  $l$ -modular cyclic base change lift*

$$\text{bc}_{\mathbb{F}_l} : \text{Irr}_{\mathbb{F}_l}(\text{GL}_n(\mathfrak{l}_0)) \longrightarrow \text{Irr}_{\mathbb{F}_l}^{\sigma\text{-inv}}(\text{GL}_n(\mathfrak{l}))$$

which satisfies and is determined by the following commutative diagram

$$\begin{array}{ccc}
 \text{Irr}_{\mathbb{Q}_l}(\text{GL}_n(\mathfrak{l}_0)) & \xrightarrow{\text{bc}_{\mathbb{Q}_l}} & \text{Irr}_{\mathbb{Q}_l}^{\sigma\text{-inv}}(\text{GL}_n(\mathfrak{l})) \\
 \downarrow J_l & & \downarrow J_l \\
 \text{Irr}_{\mathbb{F}_l}(\text{GL}_n(\mathfrak{l}_0)) & \xrightarrow{\text{bc}_{\mathbb{F}_l}} & \text{Irr}_{\mathbb{F}_l}^{\sigma\text{-inv}}(\text{GL}_n(\mathfrak{l}))
 \end{array}$$

where  $\text{bc}_{\mathbb{Q}_l}$  denotes the Shintani base change lift as a bijection (cf. [Shi76]) and  $J_l$  is defined in the same manner as in the  $p$ -adic case.

To apply the above method, we use the Shintani base change lift and also its further development (for example [SZ05], Corollary 5.7.) in place of the theory of local base change lift of Arthur-Clozel,

and we use the classification theorem of Green ([Gre55], over  $\overline{\mathbb{Q}_l}$ ) and Dipper-James ([DJ86], over  $\overline{\mathbb{F}_l}$ ) of finite linear groups in place of the classification theorem for irreducible representations of  $p$ -adic linear groups and also the local Langlands correspondence. In this case the group  $\varprojlim_n \mathbb{F}_q^\times$  plays the role of the Weil group. Then we may simply imitate our proof above, for which we omit the detail.

### 1.10.3 Application

In this subsection, we assume that  $F/F_0$  is quadratic of residue characteristic  $p \neq 2$ . We are going to classify all the  $\sigma$ -invariant cuspidal representations  $\pi$  of  $\mathrm{GL}_n(F)$  over  $\overline{\mathbb{F}_l}$  coming from the modulo  $l$  reduction of a certain  $\sigma$ -invariant integral cuspidal representation  $\tilde{\pi}$  of  $\mathrm{GL}_n(F)$  over  $\overline{\mathbb{Q}_l}$ .

**Proposition 1.10.19** ([MS14a], section 6). *For  $\pi$  a cuspidal representation of  $\mathrm{GL}_n(F)$  over  $\overline{\mathbb{F}_l}$ , there exists a supercuspidal representation  $\rho$  of  $\mathrm{GL}_{n/k}(F)$  over  $\overline{\mathbb{F}_l}$  with  $k$  being 1 or  $e(\rho)l^s$  for  $s$  a non-negative integer, such that<sup>9</sup>  $\pi = Z(\rho, \rho\nu, \dots, \rho\nu^{k-1})$ .*

**Proposition 1.10.20.** *For  $\tilde{\rho}_0$  an integral cuspidal representation of  $\mathrm{GL}_{n'}(F_0)$  over  $\overline{\mathbb{Q}_l}$  such that  $\tilde{\rho} = \mathrm{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal,  $\tilde{\rho}$  is  $l$ -supercuspidal if and only if  $\tilde{\rho}_0$  is  $l$ -supercuspidal, where being  $l$ -supercuspidal means that the modulo  $l$  reduction is supercuspidal.*

*Proof.* If  $\tilde{\rho}_0$  is not  $l$ -supercuspidal, then  $r_l(\tilde{\rho}_0)$  is written as  $Z(\rho_0, \dots, \rho_0\nu_0^{k_0-1})$  with  $\rho_0$  supercuspidal for  $k_0 > 1$ . Let  $\tilde{\rho}'_0$  be a  $\overline{\mathbb{Q}_l}$ -lift of  $\rho_0$ , then by Theorem 1.10.17 we get

$$r_l(\tilde{\rho}) = \mathrm{BC}_{\overline{\mathbb{F}_l}}(r_l(\tilde{\rho}_0)) = \mathrm{BC}_{\overline{\mathbb{F}_l}}(J_l(Z(\tilde{\rho}'_0, \dots, \tilde{\rho}'_0\nu_0^{k_0-1}))) = J_l(\mathrm{BC}_{\overline{\mathbb{Q}_l}}(Z(\tilde{\rho}'_0, \dots, \tilde{\rho}'_0\nu_0^{k_0-1})))$$

which is not supercuspidal by Theorem 1.10.11 and direct calculation. Now we focus on the other direction, that is, we assume  $\tilde{\rho}_0$  to be  $l$ -supercuspidal and we prove that  $\tilde{\rho}$  is  $l$ -supercuspidal. We need the following lemma whose proof is a simple corollary of [Dat12], §2.2.4.

**Lemma 1.10.21.** *If  $l$  doesn't divide  $q(\tilde{\rho}) - 1$ , then  $\tilde{\rho}$  is  $l$ -supercuspidal.*

We first study the  $l \neq 2$  case. For  $\rho = r_l(\tilde{\rho})$ , using the functoriality of the local Langlands correspondence  $k = e(\rho)l^s$  equals the number of irreducible constituents of

$$r_l(\mathrm{LLC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho})) = \mathrm{Res}_{F/F_0}(r_l(\mathrm{LLC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0))),$$

where  $r_l(\mathrm{LLC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0))$  is irreducible since  $r_l(\tilde{\rho}_0)$  is supercuspidal. Since  $[\mathcal{W}_{F_0} : \mathcal{W}_F] = 2$ , we know that  $k$  is smaller than 2. If  $k = 1$ , then by definition  $\tilde{\rho}$  is  $l$ -supercuspidal. Otherwise we must have  $k = o(\rho) = e(\rho) = 2$  since  $l \neq 2$ . Thus  $l$  doesn't divide  $q(\tilde{\rho}) - 1 = q(\rho) - 1$ , meaning that  $\tilde{\rho}$  is  $l$ -supercuspidal by the above lemma.

We sketch the proof of the  $l = 2$  case using the explicit base change lift, which we refer to chapter 3, section 1, section 5 and section 6 for more information. Let  $E_0$  be the a parameter field of  $\tilde{\rho}_0$ , let  $d = [E_0 : F_0]$ , let  $n' = m'd$ , let  $\mathfrak{l}_0$  be the residue field of  $E_0$  and let  $\mathfrak{t}_0$  be the unramified extension of degree  $m'$  over  $\mathfrak{l}_0$ . Since  $\tilde{\rho}$  is also cuspidal,  $E = E_0 \otimes_{F_0} F$  is quadratic over  $E_0$  as a parameter field of  $\tilde{\rho}$ . Let  $\mathfrak{l}$  be the residue field of  $E$  and let  $\mathfrak{t} = \mathfrak{t}_0 \otimes_{\mathfrak{l}_0} \mathfrak{l}$ . Then there exists a  $\mathrm{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular character  $\xi_0$  of  $\mathfrak{t}_0^\times$  over  $\overline{\mathbb{Q}_l}$  characterizing the “level 0 part” of  $\tilde{\rho}_0$  (more precisely up to twisting a quadratic character) via the theory of Green. From our assumption  $\tilde{\rho}_0$  is  $l$ -supercuspidal, or equivalently the modulo  $l$  reduction of  $\xi_0$  as a character of  $\mathfrak{t}_0^\times$  over  $\overline{\mathbb{F}_l}$  is also  $\mathrm{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular. If  $E/E_0$  is unramified, then both  $\mathfrak{l}/\mathfrak{l}_0$  and  $\mathfrak{t}/\mathfrak{t}_0$  are quadratic. Using Theorem 3.6.2,  $\xi = \xi_0 \circ N_{\mathfrak{t}/\mathfrak{t}_0}$  is a  $\mathrm{Gal}(\mathfrak{t}/\mathfrak{l})$ -regular character of  $\mathfrak{t}^\times$  over  $\overline{\mathbb{Q}_l}$  characterizing the “level 0 part” of  $\tilde{\rho}$ . Moreover by direct verification the

<sup>9</sup>Here each  $\rho\nu^i$  is regarded as a segment and  $Z(\cdot)$  is defined in §1.10.1. In [MS14a] the notation  $\mathrm{St}(\rho, k)$  was used instead which is isomorphic to the representation  $Z(\rho, \rho\nu, \dots, \rho\nu^{k-1})$ .

modulo  $l$  reduction of  $\xi$  is  $\text{Gal}(\mathbf{t}/\mathbf{l})$ -regular. If  $E/E_0$  is totally ramified, then  $\mathbf{l} = \mathbf{l}_0$  and  $\mathbf{t} = \mathbf{t}_0$ . Still using Theorem 3.6.2,  $\xi = \xi_0^2$  is a  $\text{Gal}(\mathbf{t}_0/\mathbf{l}_0)$ -regular character of  $\mathbf{t}_0^\times$  over  $\overline{\mathbb{Q}_l}$  characterizing the “level 0 part” of  $\tilde{\rho}$ . Since  $l = 2$  and the modulo  $l$  reduction of  $\xi_0$  is  $\text{Gal}(\mathbf{t}_0/\mathbf{l}_0)$ -regular, the modulo  $l$  reduction of  $\xi$  is also  $\text{Gal}(\mathbf{t}_0/\mathbf{l}_0)$ -regular. Thus in both cases  $\tilde{\rho}$  is  $l$ -supercuspidal.  $\square$

**Lemma 1.10.22.** *For  $\rho$  a  $\sigma$ -invariant supercuspidal representation of  $\text{GL}_n(F)$  over  $\overline{\mathbb{F}_l}$ , there exists a supercuspidal representation  $\rho_0$  of  $\text{GL}_n(F_0)$  over  $\overline{\mathbb{F}_l}$  such that  $\text{BC}_{\overline{\mathbb{F}_l}}(\rho_0) = \rho$ . Furthermore,  $e(\rho_0)$  is independent of the choice of  $\rho_0$ .*

*Proof.* Using Theorem 0.2.3 we choose  $\tilde{\rho}$  to be a  $\sigma$ -invariant integral cuspidal representation of  $\text{GL}_n(F)$  over  $\overline{\mathbb{Q}_l}$  such that  $r_l(\tilde{\rho}) = \rho$ . Then by Theorem 1.10.11 we choose  $\tilde{\rho}_0$  to be an integral cuspidal representation of  $\text{GL}_n(F_0)$  over  $\overline{\mathbb{Q}_l}$  such that  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0) = \tilde{\rho}$ . If we write  $\rho_0 = r_l(\tilde{\rho}_0)$  which is cuspidal, by definition we have  $\text{BC}_{\overline{\mathbb{F}_l}}(\rho_0) = \rho$ . Using Proposition 1.10.20,  $\rho_0$  must be supercuspidal. Finally, by [BH03], Theorem A, and the fact that modulo  $l$  reduction maintains the endo-class, the endo-class of  $\rho$  equals the unique  $F/F_0$ -lift of that of  $\rho_0$ . Thus the endo-class of  $\rho_0$  is independent of the choice of  $\rho_0$ , and so is  $e(\rho_0)$  (cf. [MS14a], §5.2).  $\square$

**Lemma 1.10.23.** *Let  $\pi_0 = Z(\rho_0, \dots, \rho_0 \nu_0^{k_0-1})$  be a cuspidal representation of  $\text{GL}_n(F)$  over  $\overline{\mathbb{F}_l}$  with  $\rho_0$  supercuspidal, let  $T_0$  be the tame parameter field of  $\pi_0$  and  $\rho_0$ , let  $\tilde{\pi}_0$  be a  $\overline{\mathbb{Q}_l}$ -lift of  $\pi_0$ , and let  $\tilde{\rho}_0$  be a  $\overline{\mathbb{Q}_l}$ -lift of  $\rho_0$ . Assume that  $T = T_0 \otimes_{F_0} F$  is quadratic over  $T_0$ .*

- If  $T/T_0$  is unramified, then  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  being cuspidal implies that  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal, and the converse is true if and only if  $k_0$  is odd.
- If  $T/T_0$  is totally ramified and  $l \neq 2$ , and if  $\tilde{\rho}_0$  can be chosen such that  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal, then  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  is cuspidal.
- If  $T/T_0$  is totally ramified and  $l = 2$ , then  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal, and  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  is cuspidal if and only if  $k_0 = 1$ , or  $k_0 = 2$  and  $q(\rho_0) + 1$  is divided by 4.

*Proof.* If  $T/T_0$  is unramified, using Proposition 3.5.14,  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  is cuspidal if and only if  $n/[T_0 : F_0]$  is odd (noting that  $p \neq 2$ ). Similarly  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal if and only if  $n/k_0[T_0 : F_0]$  is odd. Thus the first claim is proved.

Now we assume that  $T/T_0$  is totally ramified. Let  $E_0$  be a parameter field of both  $\pi_0$  and  $\rho_0$ , let  $d = [E_0 : F_0]$ , let  $m = n/d$  which is a positive integer divided by  $k_0$ , let  $\mathbf{l}_0$  be the residue field of  $E_0$ , let  $\mathbf{t}_0/\mathbf{l}_0$  be an extension of degree  $m$  and let  $\mathbf{t}'_0/\mathbf{l}_0$  be its subextension of degree  $m/k_0$ . Recall that there exists a character  $\xi_0$  of  $\mathbf{t}_0^\times$  over  $\overline{\mathbb{Q}_l}$  which is  $\text{Gal}(\mathbf{t}_0/\mathbf{l}_0)$ -regular, characterizing the “level 0 part” of  $\tilde{\pi}_0$ , and moreover by *loc. cit.*  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  is cuspidal if and only if  $\xi_0^2$  is also  $\text{Gal}(\mathbf{t}_0/\mathbf{l}_0)$ -regular. Similarly, there exists a character  $\xi'_0$  of  $\mathbf{t}'_0^\times$  over  $\overline{\mathbb{Q}_l}$  which is  $\text{Gal}(\mathbf{t}'_0/\mathbf{l}_0)$ -regular, characterizing the “level 0 part” of  $\tilde{\rho}_0$ , and moreover  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal if and only if  $\xi_0'^2$  is also  $\text{Gal}(\mathbf{t}'_0/\mathbf{l}_0)$ -regular. Finally the modulo  $l$  reduction of  $\xi'_0$  is  $\text{Gal}(\mathbf{t}'_0/\mathbf{l}_0)$ -regular as a character over  $\overline{\mathbb{F}_l}$  and the modulo  $l$  reduction of  $\xi_0$  coincides with that of  $\xi'_0 \circ \text{N}_{\mathbf{t}_0/\mathbf{t}'_0}$  (see [DJ86] section 3).

Let  $s$  be the maximal integer such that  $l^s$  divides the cardinality of  $\mathbf{t}_0^\times$ . We write  $\xi_0 = (\xi'_0 \circ \text{N}_{\mathbf{t}_0/\mathbf{t}'_0}) \cdot \xi_1'$ , where  $\xi_1'$  is a character of  $\mathbf{t}_0^\times$  of order a power of  $l$  to be determined, such that the order of  $\xi_0$  is divided by  $l^s$ , and let  $\xi_1$  be a character of  $\mathbf{t}'_0^\times$  of order  $l^s$ . If  $l \neq 2$  and  $\xi_0'^2$  is  $\text{Gal}(\mathbf{t}'_0/\mathbf{l}_0)$ -regular, we have<sup>10</sup>  $\mathbf{l}_0[\xi_0'^2] \cong \mathbf{t}'_0$ . Thus

$$\mathbf{l}_0[\xi_0^2] = \mathbf{l}_0[(\xi'_0 \circ \text{N}_{\mathbf{t}_0/\mathbf{t}'_0})^2, \xi_1^2] \cong \mathbf{t}'_0[\xi_1^2],$$

<sup>10</sup>Here  $\xi_0'^2$  is identified with a root of unity in the algebraic closure of  $\mathbf{l}_0$  of the same order (where the choice of identification is not important), and thus  $\mathbf{l}_0[\xi_0'^2]/\mathbf{l}_0$  is a field extension.

and similarly since  $\xi_0$  is  $\text{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular, we have

$$\mathfrak{t}_0 \cong \mathfrak{l}_0[\xi_0] \cong \mathfrak{t}'_0[\xi_1].$$

Since  $l$  is odd and  $\xi_1$  is of order a power of  $l$ , we must have  $\mathfrak{t}'_0[\xi_1^2] = \mathfrak{t}'_0[\xi_1]$ , meaning that  $\xi_0^2$  is  $\text{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular and  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  is cuspidal. If  $l = 2$ , it is easy to see that  $\mathfrak{l}_0[\xi_0'] = \mathfrak{l}_0[\xi_0'^2]$ . Since  $\xi_0'$  is  $\text{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular,  $\xi_0'^2$  is  $\text{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular, meaning that  $\text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is cuspidal. Similarly we have

$$\mathfrak{t}_0 \cong \mathfrak{l}_0[\xi_0] \cong \mathfrak{t}'_0[\xi_1] \quad \text{and} \quad \mathfrak{l}_0[\xi_0^2] \cong \mathfrak{t}'_0[\xi_1^2].$$

Let  $Q_0$  be the cardinality of  $\mathfrak{t}'_0$  and recall that  $k_0 = [\mathfrak{t}_0 : \mathfrak{t}'_0]$ . An elementary argument shows that there exists  $\xi_0$  as above, such that the corresponding  $\xi_1$  satisfies  $\mathfrak{t}'_0[\xi_1^2] = \mathfrak{t}'_0[\xi_1]$ , if and only if  $k_0 = 1$ , or  $k_0 = 2$  and  $4 \nmid Q_0 - 1$ . Noting that  $q(\rho_0) = Q_0$ , we finish the proof.  $\square$

**Theorem 1.10.24.** *Let  $\pi$  be a  $\sigma$ -invariant cuspidal representation of  $\text{GL}_n(F)$ , let  $T$  be the tame parameter field of  $\pi$  and let  $T_0$  be the tame parameter field<sup>11</sup> of any supercuspidal representation of  $\text{GL}_n(F_0)$  over  $\overline{\mathbb{F}_l}$  whose base change is  $\pi$ . Write  $\pi = Z(\rho, \rho\nu, \dots, \rho\nu^{k-1})$  as in Proposition 1.10.19, for  $\rho$  a supercuspidal representation of  $\text{GL}_{n/k}(F)$ .*

(1) *Let  $\rho$  be  $\sigma$ -invariant and let  $\rho_0$  be any supercuspidal representation of  $\text{GL}_{n/k}(F_0)$  such that  $\text{BC}_{\overline{\mathbb{F}_l}}(\rho_0) = \rho$ , then  $\pi$  has a  $\sigma$ -invariant  $\overline{\mathbb{Q}_l}$ -lift if and only if*

- $k = 1$ , or  $k > 1$  and  $e(\rho_0) = e(\rho)$  in the case where  $l \neq 2$ ;
- $k = 1$  in the case where  $l = 2$  and  $T/T_0$  is unramified;
- $k = 1$ , or  $k = 2$  and  $q(\rho) + 1$  is divided by 4 in the case where  $l = 2$  and  $T/T_0$  is totally ramified.

(2) *If  $\rho$  is not  $\sigma$ -invariant, then  $\pi$  doesn't have a  $\sigma$ -invariant  $\overline{\mathbb{Q}_l}$ -lift if  $T/T_0$  is unramified, and conversely  $\pi$  always has a  $\sigma$ -invariant  $\overline{\mathbb{Q}_l}$ -lift if  $T/T_0$  is totally ramified;*

*Proof.* First we prove (1). To begin with we consider the case  $l \neq 2$ . We assume that  $\pi = r_l(\tilde{\pi})$  for  $\tilde{\pi}$  an integral  $\sigma$ -invariant cuspidal representation of  $\text{GL}_n(F)$  over  $\overline{\mathbb{Q}_l}$ , and we write  $\tilde{\pi} = \text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\pi}_0)$  for  $\tilde{\pi}_0$  an integral cuspidal representation of  $\text{GL}_n(F_0)$  over  $\overline{\mathbb{Q}_l}$ . We define  $\pi_0 = r_l(\tilde{\pi}_0)$  which is a cuspidal representation of  $\text{GL}_n(F_0)$  over  $\overline{\mathbb{F}_l}$ . By Proposition 1.10.19, we write  $\pi_0 = Z(\rho'_0, \rho'_0\nu_0, \dots, \rho'_0\nu_0^{k_0-1})$  with  $\rho'_0$  supercuspidal.

If  $T/T_0$  is unramified, by Lemma 1.10.23 we choose  $\tilde{\rho}_0$  to be an integral cuspidal representation of  $\text{GL}_{n/k_0}(F_0)$  lifting  $\rho'_0$ , such that  $\tilde{\rho} = \text{BC}_{\overline{\mathbb{Q}_l}}(\tilde{\rho}_0)$  is an integral cuspidal representation of  $\text{GL}_{n/k_0}(F)$ . Thus  $J_l(Z(\tilde{\rho}_0, \tilde{\rho}_0\nu_0, \dots, \tilde{\rho}_0\nu_0^{k_0-1})) = Z(\rho'_0, \rho'_0\nu_0, \dots, \rho'_0\nu_0^{k_0-1})$ . Using Theorem 1.10.17 and the fact that  $\text{BC}_{\overline{\mathbb{Q}_l}}(Z(\tilde{\rho}_0, \dots, \tilde{\rho}_0\nu_0^{k_0-1})) = Z(\tilde{\rho}, \dots, \tilde{\rho}\nu^{k_0-1})$ , for  $\rho' = r_l(\tilde{\rho})$  we have

$$Z(\rho, \dots, \rho\nu^{k-1}) = \pi = r_l(\tilde{\pi}) = J_l(Z(\tilde{\rho}, \dots, \tilde{\rho}\nu^{k_0-1})) = Z(\rho', \dots, \rho'\nu^{k_0-1}).$$

By Proposition 1.10.20  $\rho'$  is supercuspidal, thus we must have  $k_0 = k$  and we may assume  $\rho = \rho'$  without loss of generality (when  $k_0 = k = 1$  it is automatic). And since  $\text{BC}_{\overline{\mathbb{F}_l}}(\rho'_0) = \rho'$  we may also assume  $\rho_0 = \rho'_0$  without loss of generality. Thus if  $k_0 \neq 1$ , since  $\pi$  is cuspidal and  $\rho$  is supercuspidal, there exists a non-negative integer  $s$  such that  $e(\rho_0)l^{s_0} = k_0 = k = e(\rho)l^s$ . Since both  $e(\rho_0)$  and  $e(\rho)$  are either equal to  $l$  or prime to  $l$ , we must have  $s = s_0$  and  $e(\rho) = e(\rho_0)$ . If  $T/T_0$  is totally ramified, the condition  $e(\rho) = e(\rho_0)$  is automatic by direct calculation.

<sup>11</sup>By [BH03],  $T/T_0$  is a quadratic extension.



Conversely we show that the condition listed above is also sufficient. If  $k = 1$ , using Theorem 0.2.3 there exists a  $\sigma$ -invariant integral cuspidal representation  $\tilde{\rho}$  of  $\mathrm{GL}_n(F)$  over  $\overline{\mathbb{Q}}_l$  such that  $r_l(\tilde{\rho}) = \rho$ . If  $k > 1$  and there exists a supercuspidal representation  $\rho_0$  of  $\mathrm{GL}_{n/k}(F_0)$  such that  $\mathrm{BC}_{\overline{\mathbb{F}}_l}(\rho_0) = \rho$  and  $e(\rho_0) = e(\rho)$ , then  $Z(\rho_0, \dots, \rho_0 \nu_0^{k-1})$  is a cuspidal representation of  $\mathrm{GL}_n(F_0)$  over  $\overline{\mathbb{F}}_l$  which we denote by  $\pi_0$ . In particular when  $T/T_0$  is unramified we have  $q(\rho) = q(\rho_0)^2$ , thus  $e(\rho_0) = e(\rho)$  implies that  $o(\rho_0) = l$  is odd, meaning that  $k_0 = e(\rho_0)l^s$  is odd. Choose  $\tilde{\rho}_0$  to be a  $\overline{\mathbb{Q}}_l$ -lift of  $\rho_0$  and write  $\tilde{\rho} = \mathrm{BC}_{\overline{\mathbb{Q}}_l}(\tilde{\rho}_0)$ , then we have  $J_l(\tilde{\rho}) = \rho$  which is supercuspidal. Considering the supercuspidal support we know that  $\tilde{\rho}$  must be cuspidal. By Lemma 1.10.23, we choose  $\tilde{\pi}_0$  to be an integral cuspidal representation of  $\mathrm{GL}_n(F_0)$  over  $\overline{\mathbb{Q}}_l$  lifting  $\pi_0$ , such that  $\tilde{\pi} := \mathrm{BC}_{\overline{\mathbb{Q}}_l}(\tilde{\pi}_0)$  is cuspidal. By Theorem 1.10.17 we have  $r_l(\tilde{\pi}) = \pi$ .

Now we let  $l = 2$ . For  $\pi$  having a  $\sigma$ -invariant  $\overline{\mathbb{Q}}_l$ -lift, we define  $\tilde{\pi}$ ,  $\tilde{\pi}_0$ ,  $\pi_0$  and  $\rho'_0$  as in the  $l \neq 2$  case. When  $T/T_0$  unramified, since  $k_0$  is either 1 or even, using Lemma 1.10.23 we have  $k_0 = 1$ , saying that  $r_l(\tilde{\pi}_0) = \rho'_0$ . Using Proposition 1.10.20, we must have  $k = 1$ . When  $T/T_0$  is totally ramified, by Lemma 1.10.23, we have  $k_0 = 1$ , or  $k_0 = 2$  and  $q(\rho'_0) + 1$  is divided by 4. Using Proposition 1.10.20 and similar argument to the  $l \neq 2$  case we have  $k = k_0$  and  $q(\rho'_0) = q(\rho)$ . Conversely, if  $k = 1$ , using Theorem 0.2.3  $\pi = \rho$  has a  $\overline{\mathbb{Q}}_l$ -lift which is  $\sigma$ -invariant. If  $k = 2$ ,  $T/T_0$  is totally ramified and  $q(\rho_0) + 1 = q(\rho) + 1$  is not divided by 4, let  $\tilde{\pi}_0$  be an integral cuspidal representation of  $\mathrm{GL}_n(F_0)$  over  $\overline{\mathbb{Q}}_l$ , such that  $r_l(\tilde{\pi}_0) = Z(\rho_0, \rho_0 \nu_0)$ . By Lemma 1.10.23  $\tilde{\pi} = \mathrm{BC}_{\overline{\mathbb{Q}}_l}(\tilde{\pi}_0)$  is cuspidal. Thus by Theorem 1.10.17 we have  $r_l(\tilde{\pi}) = Z(\rho, \rho \nu) = \pi$ , finishing the proof of (1).

Finally we prove (2). Comparing the supercuspidal support, we must have  $o(\rho)$  is even and  $\rho^\sigma \cong \rho \nu^{o(\rho)/2}$ . In particular we have  $l \neq 2$ . When  $T/T_0$  is unramified if we assume on the contrary that there exists a  $\sigma$ -invariant  $\overline{\mathbb{Q}}_l$ -lift of  $\pi$ , then using the same argument as (1), there exists a  $\sigma$ -invariant supercuspidal representation  $\rho'$  of  $\mathrm{GL}_{n/k_0}(F)$ , such that

$$Z(\rho, \dots, \rho \nu^{k-1}) = \pi = Z(\rho', \dots, \rho' \nu^{k_0-1}).$$

Since both  $\rho$  and  $\rho'$  are supercuspidal,  $k = k_0$  and  $\rho$  is isomorphic to  $\rho'$  twisted by a power of  $\nu$ , which must be  $\sigma$ -invariant, contradictory! When  $T/T_0$  is totally ramified, we let  $\tilde{\rho}$  be a  $\overline{\mathbb{Q}}_l$ -lift of  $\rho$  as a cuspidal representation of  $\mathrm{GL}_{n/k}(F)$ . Then  $\tilde{\rho}^\sigma$  is not isomorphic to  $\tilde{\rho}$ . Using Theorem 1.10.11, there exists  $\tilde{\rho}_0$  as a supercuspidal representation of  $\mathrm{GL}_{2n/k}(F_0)$  over  $\overline{\mathbb{Q}}_l$ , such that  $\mathrm{BC}_{\overline{\mathbb{Q}}_l}(\tilde{\rho}_0) = \tilde{\rho} \times \tilde{\rho}^\sigma$ . We write  $\rho_0 = r_l(\tilde{\rho}_0)$  as a cuspidal representation of  $\mathrm{GL}_{2n/k}(F_0)$ . Using the same proof as the in first paragraph of Proposition 1.10.20,  $\rho_0$  is supercuspidal. Since by definition  $q(\rho_0) = q(\rho)^2$ , we have  $2o(\rho_0) = o(\rho)$ , thus  $k/2 = o(\rho_0)l^s$  and  $Z(\rho_0, \dots, \rho_0 \nu^{k/2})$  is cuspidal.

**Lemma 1.10.25.** *We may choose  $\tilde{\pi}_0$  to be a  $\overline{\mathbb{Q}}_l$ -lift of  $Z(\rho_0, \dots, \rho_0 \nu^{k/2})$  as a cuspidal representation of  $\mathrm{GL}_n(F_0)$ , such that  $\tilde{\pi} = \mathrm{BC}_{\overline{\mathbb{Q}}_l}(\tilde{\pi}_0)$  is cuspidal.*

*Proof.* For  $\mathfrak{l}_0$ ,  $\mathfrak{t}_0$  introduced as in the proof of Lemma 1.10.23 and for  $\xi_0$  as a  $\mathrm{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular character of  $\mathfrak{t}_0^\times$  characterizing the “level 0 part” of  $\tilde{\pi}_0$ , using Proposition 3.5.14,  $\tilde{\pi}$  is cuspidal if and only if  $\xi_0^2$  is  $\mathrm{Gal}(\mathfrak{t}_0/\mathfrak{l}_0)$ -regular. Let  $s'$  be the maximal positive integer such that  $l^{s'}$  divides the cardinality of  $\mathfrak{t}_0^\times$ , then it is elementary to prove that for  $\zeta'$  as a  $l^{s'}$ -th primitive root of unity,  $\mathfrak{l}_0[\zeta'] \cong \mathfrak{t}_0$ . Thus if we choose  $\xi_1$  to be a character of  $\mathfrak{t}_0^\times$  of order a power of  $l$ , such that the order of  $\xi_0 \xi_1$  is divided by  $l^{s'}$ , then we have

$$\mathfrak{t}_0 \cong \mathfrak{l}_0[\xi_0 \xi_1] \cong \mathfrak{l}_0[(\xi_0 \xi_1)^2].$$

Replacing  $\xi_0$  by  $\xi_0 \xi_1$ , the corresponding  $\tilde{\pi}$  is cuspidal. □

For such  $\tilde{\pi}_0$  and  $\tilde{\pi}$ , by Theorem 1.10.17 we get  $r_l(\tilde{\pi}) = \pi$ , which finishes the proof. □

Combining with [\[KM20\]](#), Theorem 3.4 and Theorem [\[0.2.1\]](#) for  $R = \overline{\mathbb{Q}_l}$ , we have

**Corollary 1.10.26.** *For  $\pi$  a cuspidal representation of  $G$  over  $\overline{\mathbb{F}_l}$  having a  $\sigma$ -invariant  $\overline{\mathbb{Q}_l}$ -lift, that is, satisfying the corresponding condition of Theorem [\[1.10.24\]](#), it is distinguished by  $G^\tau$  for any  $\tau$  as a unitary involution.*

**Remark 1.10.27.** *When  $T/T_0$  is totally ramified, Remark [\[1.8.6\]](#) indicates that all the  $\sigma$ -invariant cuspidal representations are distinguished, which is stronger than the corollary above. When  $T/T_0$  is unramified, it is interesting to know if the condition of being distinguished in the corollary is necessary or not, which is beyond the scope of the author.*



## Chapter 2

# Problem of distinction related to orthogonal subgroups of $\mathrm{GL}_n(F)$

### 2.1 Notation

#### 2.1.1 General notation

Let  $F$  be a non-archimedean locally compact field of residue characteristic  $p \neq 2$ . We write  $\mathfrak{o}_F$ ,  $\mathfrak{p}_F$ ,  $\mathfrak{k}$  for its ring of integers, the corresponding maximal ideal and its residue field respectively. We fix  $\psi_F : F \rightarrow \mathbb{C}^\times$  an additive character which is trivial on  $\mathfrak{p}_F$  but not on  $\mathfrak{o}_F$ .

Fix  $n$  a positive integer. We write  $G = \mathrm{GL}_n(F)$  as a locally profinite group. By *representations* of  $G$  and its closed subgroups, we always mean complex smooth representations. For a closed subgroup  $H$  of  $G$ , an element  $g \in G$  and a representation  $\pi$  of  $H$ , we write  $H^g := \{g^{-1}hg | h \in H\}$  as a subgroup of  $G$ , and  $\pi^g : g \mapsto \pi(ghg^{-1})$  as its representation. We write  $\pi^\vee$  for the contragredient of  $\pi$ . Given  $\tau$  a continuous involution of  $G$ , we write  $\pi^\tau$  for the representation  $\pi \circ \tau$  of  $\tau(H)$ . We say that  $\pi$  is  $\tau$ -selfdual if  $\tau(H) = H$  and  $\pi^\tau \cong \pi^\vee$ .

Given  $\pi$  a representation of  $H$  and  $\mu$  a representation of  $G^\tau \cap H$ , we say that  $\pi$  is  $\mu$ -*distinguished* if  $\mathrm{Hom}_{G^\tau \cap H}(\pi, \mu) \neq 0$ , where  $G^\tau$  denotes the subgroup of  $G$  consisting of the elements fixed by  $\tau$ . In particular, if  $\mu$  is the trivial character, we simply call  $\pi$   $G^\tau \cap H$ -*distinguished*.

#### 2.1.2 A brief recall of the simple type theory

In this subsection, we follow the introduction of the simple type theory given in chapter 1, section 3 summarizing results of [BK93], [BH96], [BH14b]. Since it seems redundant to repeat the same words again, we simply recall the necessary notation.

We write  $[\mathfrak{a}, \beta]$  for a *simple stratum* in  $M_n(F)$ , where  $\mathfrak{a}$  is a hereditary order in  $M_n(F)$  and  $\beta$  is an element in  $\mathrm{GL}_n(F)$  such that

- (1) the  $F$ -algebra  $E = F[\beta]$  is a field, where  $[E : F] = d$  and  $n = md$  for a positive integer  $m$ ;
- (2)  $E^\times$  normalizes  $\mathfrak{a}^\times$ .

We write  $B$  for the centralizer of  $E$  in  $M_n(F)$  identifying with  $M_m(E)$ , and  $\mathfrak{b} = \mathfrak{a} \cap B$  for the hereditary order in  $B$ . We denote by  $\mathfrak{p}_\mathfrak{a}$  (resp.  $\mathfrak{p}_\mathfrak{b}$ ) the Jacobson radical of  $\mathfrak{a}$  (resp.  $\mathfrak{b}$ ), and  $U^1(\mathfrak{a})$  (resp.  $U^1(\mathfrak{b})$ ) the compact open pro- $p$ -subgroup  $1 + \mathfrak{p}_\mathfrak{a}$  (resp.  $1 + \mathfrak{p}_\mathfrak{b}$ ) of  $\mathrm{GL}_n(F)$  (resp.  $B^\times$ ).

Associated to  $[\mathfrak{a}, \beta]$ , there are compact open subgroups

$$H^1(\mathfrak{a}, \beta) \subset J^1(\mathfrak{a}, \beta) \subset J(\mathfrak{a}, \beta)$$

of  $\mathfrak{a}^\times$ , and there is a finite set  $\mathcal{C}(\mathfrak{a}, \beta)$  of characters of  $H^1(\mathfrak{a}, \beta)$ , depending on the choice of  $\psi_F$ , called *simple characters*. We denote by  $\mathbf{J}(\mathfrak{a}, \beta)$  the subgroup of  $G$  generated by  $J(\mathfrak{a}, \beta)$  and the normalizer of

$\mathfrak{b}^\times$  in  $B^\times$  which is compact modulo the centre  $F^\times$ . We write  $\mathbf{J}, J, J^1, H^1$  for short for  $\mathbf{J}(\mathfrak{a}, \beta), J(\mathfrak{a}, \beta), J^1(\mathfrak{a}, \beta), H^1(\mathfrak{a}, \beta)$  respectively if  $\mathfrak{a}$  and  $\beta$  are clear to us. When  $\mathfrak{b}$  is a maximal order in  $B$ , we call the simple stratum  $[\mathfrak{a}, \beta]$  and the simple characters in  $\mathcal{C}(\mathfrak{a}, \beta)$  *maximal*. In this case  $\mathfrak{b}^\times/1 + \mathfrak{p}_{\mathfrak{b}} \cong \mathrm{GL}_m(\mathfrak{l})$ , where  $\mathfrak{l}$  is the residue field of  $E$ .

We denote by  $(\mathbf{J}, \Lambda)$  an *extended maximal simple type* (we always write *simple type* for short) in  $\mathrm{GL}_n(F)$ , which means that there are a maximal simple stratum  $[\mathfrak{a}, \beta]$  in  $M_n(F)$  and a maximal simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  such that  $\mathbf{J}(\mathfrak{a}, \beta) = \mathbf{J}$  and  $\theta$  is contained in the restriction of  $\Lambda$  to  $H^1(\mathfrak{a}, \beta)$ . We write  $\eta$  for the *Heisenberg representation* associated to  $\theta$ . For any representation  $\kappa$  of  $\mathbf{J}$  extending  $\eta$ , there is, up to isomorphism, a unique irreducible representation  $\rho$  of  $\mathbf{J}$  such that  $\Lambda \cong \kappa \otimes \rho$ , and moreover  $\rho|_J$  is the inflation of a supercuspidal representation of  $J/J^1 \cong \mathrm{GL}_m(\mathfrak{l})$ . For  $\pi$  a supercuspidal representation of  $G$ , there exists a unique  $G$ -conjugacy class of simple type  $(\mathbf{J}, \Lambda)$  such that  $\pi \cong \mathcal{C}\text{-Ind}_{\mathbf{J}}^G \Lambda$ , the compact induction of  $\Lambda$ .

For  $[\mathfrak{a}, \beta]$  a simple stratum in  $M_n(F)$  and  $[\mathfrak{a}', \beta']$  a simple stratum in  $M_{n'}(F)$  with  $n, n' \geq 1$ , and for a given  $F$ -algebra isomorphism  $\phi : F[\beta] \rightarrow F[\beta']$  such that  $\phi(\beta) = \beta'$ , we denote by

$$t_{\mathfrak{a}, \mathfrak{a}'}^{\beta, \beta'} : \mathcal{C}(\mathfrak{a}, \beta) \rightarrow \mathcal{C}(\mathfrak{a}', \beta')$$

the corresponding *transfer map*. We use capital Greek letter  $\Theta$  to denote the *endo-class* of a simple character  $\theta$  and  $\Theta_\pi$  to denote the endo-class of  $\pi$ , a supercuspidal representation of  $G$ . We write  $d = [F[\beta] : F]$  for the degree of  $\Theta$  which does not depend on the choice of  $[\mathfrak{a}, \beta]$  and  $\theta$ , but only on  $\Theta$  itself.

Let  $\Theta$  be as above and let  $T$  be its tame parameter field with respect to  $E/F$ , that is, the maximal tamely ramified subextension of  $E$  over  $F$ . Noting that  $T$  only depends on  $\Theta$  up to  $F$ -isomorphism, so it is also called the tame parameter field of  $\Theta$ . Let  $C \cong M_{n/t}(T)$  denote the centralizer of  $T$  in  $M_n(F)$ , where  $t = [T : F]$ . The intersection  $\mathfrak{c} = \mathfrak{a} \cap C$  is an order in  $C$ , which gives rise to a simple stratum  $[\mathfrak{c}, \beta]$ . The restriction of  $\theta$  to  $H^1(\mathfrak{c}, \beta)$ , denoted by  $\theta_T$  and called the *interior  $T/F$ -lift of  $\theta$* , is a simple character associated to the simple stratum  $[\mathfrak{c}, \beta]$ . If we change our choice of simple stratum  $[\mathfrak{a}, \beta]$  but fix  $T \hookrightarrow M_n(F)$  unchanged, then the map

$$\mathfrak{a} \mapsto \mathfrak{a} \cap C$$

is injective from the set of hereditary orders in  $M_n(F)$  normalized by  $T^\times$  to the set of hereditary orders in  $C$  (see [BH96], section 2). For  $[\mathfrak{a}, \beta_1], [\mathfrak{a}, \beta_2]$  two simple strata, and  $\theta_1 \in \mathcal{C}(\mathfrak{a}, \beta_1), \theta_2 \in \mathcal{C}(\mathfrak{a}, \beta_2)$  two simple characters, such that  $\theta_1$  and  $\theta_2$  have the same tame parameter field  $T$ , if

$$\mathcal{C}(\mathfrak{c}, \beta_1) = \mathcal{C}(\mathfrak{c}, \beta_2) \quad \text{and} \quad (\theta_1)_T = (\theta_2)_T,$$

then we have

$$\mathcal{C}(\mathfrak{a}, \beta_1) = \mathcal{C}(\mathfrak{a}, \beta_2) \quad \text{and} \quad \theta_1 = \theta_2$$

(see [BH96], Theorem 7.10, Theorem 7.15). In particular, when  $\beta_1 = \beta_2 = \beta$ , the interior  $T/F$ -lift is injective from  $\mathcal{C}(\mathfrak{a}, \beta)$  to  $\mathcal{C}(\mathfrak{c}, \beta)$ .

## 2.2 Symmetric matrices and orthogonal involutions

In this section, we recall some basic but important results about symmetric matrices and orthogonal involutions. Let  $E$  be a non-archimedean locally compact field of residue characteristic  $p \neq 2$ , let  $\varpi_E$  be a uniformizer of  $E$  and let  $m$  be a fixed positive integer.

## 2.2.1 Orbits of symmetric matrices, orthogonal involutions and orthogonal groups

Let  $\mathcal{S}$  denote the set of the symmetric matrices in  $\mathrm{GL}_m(E)$ , that is

$$\mathcal{S} := \{\varepsilon \in \mathrm{GL}_m(E) \mid {}^t\varepsilon = \varepsilon\}.$$

Especially, if we write

$$J_m := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathrm{M}_m(E),$$

then it is an element in  $\mathcal{S}$ .

We consider  $\mathrm{GL}_m(E)$ -action on  $\mathcal{S}$  as follows:

$$\varepsilon \cdot g := {}^t g \varepsilon g, \quad g \in \mathrm{GL}_m(E), \varepsilon \in \mathcal{S}.$$

We say that two elements in  $\mathcal{S}$  are *similar* if they are in the same  $\mathrm{GL}_m(E)$ -orbit. For  $\varepsilon \in \mathcal{S}$ , we denote by  $\mathrm{disc}_E(\varepsilon)$  its *discriminant*, saying the image of  $\det(\varepsilon)$  in  $E^\times/E^{\times 2}$  which is a group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . We denote by  $\square$

$$\mathrm{Hasse}_E(\varepsilon) = \prod_{i < j} \mathrm{Hil}_E(a_i, a_j) \in \{1, -1\}$$

its *Hasse invariant*, where  $\mathrm{diag}(a_1, \dots, a_m)$  denotes a diagonal matrix similar to  $\varepsilon$ , and

$$\mathrm{Hil}_E(a, b) = \begin{cases} 1, & \text{if } ax^2 + by^2 = 1 \text{ has a solution } (x, y) \in E \times E; \\ -1, & \text{otherwise.} \end{cases}$$

denotes the *Hilbert symbol* for  $a, b \in E^\times$ . Noting that the definition of  $\mathrm{Hasse}_E(\varepsilon)$  doesn't depend on the choice of  $\mathrm{diag}(a_1, \dots, a_m)$  similar to  $\varepsilon$  (see [O'M71](#), 63.13). When  $E$  is clear to us, we simply write  $\mathrm{disc}$ ,  $\mathrm{Hil}$  and  $\mathrm{Hasse}$  instead.

The following proposition characterizes all the  $\mathrm{GL}_m(E)$ -orbits in  $\mathcal{S}$ .

**Proposition 2.2.1** ([O'M71](#), Theorem 63.20). (1) When  $m = 1$ , there are four  $\mathrm{GL}_m(E)$ -orbits in  $\mathcal{S}$  represented by elements in  $E^\times/E^{\times 2}$ ;

(2) When  $m \geq 2$ , any two  $\mathrm{GL}_m(E)$ -orbits in  $\mathcal{S}$  are different if and only if their discriminants or Hasse invariants are different. Moreover,

- When  $m \geq 3$  there are eight  $\mathrm{GL}_m(E)$ -orbits;
- When  $m = 2$ , any  $\varepsilon \in \mathcal{S}$  with  $\mathrm{disc}(\varepsilon) = -1$  satisfies  $\mathrm{Hasse}(\varepsilon) = 1$ , and there are seven  $\mathrm{GL}_m(E)$ -orbits.

We may also consider the  $\mathrm{GL}_m(\mathfrak{o}_E)$ -orbits of  $\mathcal{S}$ . We consider  $\alpha = (\alpha_1, \dots, \alpha_r)$  of certain triples  $\alpha_i = (a_i, m_i, \varepsilon_i)$ , such that  $a_1 > \dots > a_r$  is a decreasing sequence of integers, and  $m_1, \dots, m_r$  are

<sup>1</sup>In [Hak13](#) Hakim used  $i \leq j$  instead of  $i < j$  in the product for the definition, however in the proof of various propositions (for example, Proposition 6.6. of *ibid.*) he indeed used the second definition ( $i < j$ ). This little inconsideration of course doesn't affect his results and proofs.

positive integers such that  $m_1 + \dots + m_r = m$ , and  $\epsilon_1, \dots, \epsilon_r$  are either 1 or  $\epsilon_0$ , where  $\epsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$  is fixed. For each  $\alpha = (\alpha_1, \dots, \alpha_r)$  as above, we introduce a symmetric matrix

$$\varpi_E^\alpha = \varpi_E^{\alpha_1} \oplus \dots \oplus \varpi_E^{\alpha_r},$$

where

$$\varpi_E^{\alpha_i} := \varpi_E^{\alpha_i} \text{diag}(1, \dots, 1, \epsilon_i) \in \text{GL}_{m_i}(E).$$

The following proposition studies all the  $\text{GL}_m(\mathfrak{o}_E)$ -orbits.

**Proposition 2.2.2** ([O'M71], §92). *Each  $\text{GL}_m(\mathfrak{o}_E)$ -orbit in  $\mathcal{S}$  contains exactly one representative of the form  $\varpi_E^\alpha$  defined as above.*

Now for  $\varepsilon \in \mathcal{S}$  a given symmetric matrix, we denote by

$$\tau_\varepsilon(x) := \varepsilon^{-1} x \varepsilon \quad \text{for any } x \in \text{GL}_m(E)$$

the orthogonal involution corresponding to  $\varepsilon$ . The group  $\text{GL}_m(E)$  acts on the set of orthogonal involutions by

$$g \cdot \tau_\varepsilon = \tau_{\varepsilon \cdot g} = \tau_{t g \varepsilon g}.$$

Given  $\varepsilon_1, \varepsilon_2$ , it is elementary to see that  $\tau_{\varepsilon_1} = \tau_{\varepsilon_2}$  if and only if  $\varepsilon_1 E^\times = \varepsilon_2 E^\times$ . Thus we build up a bijection between  $\mathcal{S}/E^\times$  and the set of orthogonal involutions, which is given by  $\varepsilon E^\times \mapsto \tau_\varepsilon$ . The following proposition studies the  $\text{GL}_m(E)$ -orbits of  $\mathcal{S}/E^\times$ , thus classifies all the  $\text{GL}_m(E)$ -orbits of orthogonal involutions.

**Proposition 2.2.3.** (1) *When  $m = 1$ , there is one  $\text{GL}_m(E)$ -orbit in  $\mathcal{S}/E^\times$ ;*

(2) *When  $m \geq 3$  is odd, there are two  $\text{GL}_m(E)$ -orbits in  $\mathcal{S}/E^\times$ . A representative in each orbit can be chosen to have any given discriminant, and two representatives with the same discriminant represent different orbits if and only if they have different Hasse invariants;*

(3) *When  $m = 2$ , there are four  $\text{GL}_m(E)$ -orbits in  $\mathcal{S}/E^\times$  determined by the discriminants;*

(4) *When  $m \geq 4$  is even, the discriminant leads to a map from  $(\mathcal{S}/E^\times)/\text{GL}_m(E)$  to  $E^\times/E^{\times 2}$  which is surjective. The fiber corresponding to  $(-1)^{m(m-1)/2}$ , the discriminant of  $J_m$ , is composed of two orbits distinguished exactly by the Hasse invariant, and the other three fibers are composed of exactly one orbit.*

*Proof.* The proof is a refinement of Proposition 2.2.1. For more detail, see [O'M71], §63. □

For  $\tau = \tau_\varepsilon$  an orthogonal involution, we denote by

$$\text{GL}_m(E)^\tau := \{x \in \text{GL}_m(E) \mid \tau(x) = x\}$$

the orthogonal group corresponding to  $\tau$ .

**Lemma 2.2.4.** *Let  $\tau_1$  and  $\tau_2$  be two orthogonal involutions such that  $\text{GL}_m(E)^{\tau_1} = \text{GL}_m(E)^{\tau_2}$ , then  $\tau_1 = \tau_2$ . As a result,  $\tau \mapsto \text{GL}_m(E)^\tau$  gives a bijection between  $\text{GL}_m(E)$ -orbits of orthogonal involutions and the set of  $\text{GL}_m(E)$ -conjugacy classes of orthogonal subgroups of  $\text{GL}_m(E)$ .*

*Proof.* For a proof, see [Hak13], Lemma 2.7. □

Combining Proposition 2.2.3 and Lemma 2.2.4, we get all the possible  $\text{GL}_m(E)$ -conjugacy classes of orthogonal groups.

**Proposition 2.2.5.** (1) When  $m = 1$ , there is only one orthogonal group  $\{1, -1\}$ ;

(2) When  $m \geq 3$  is odd, there are two  $\mathrm{GL}_m(E)$ -conjugacy classes of orthogonal groups, the one corresponding to the symmetric matrix  $J_m$  is split, and the other one is not quasisplit;

(3) When  $m = 2$ , there are four  $\mathrm{GL}_m(E)$ -conjugacy classes of orthogonal groups, the one corresponding to the symmetric matrix  $J_m$  is split, and the other three are quasisplit but not split;

(4) When  $m \geq 4$  is even, there are five  $\mathrm{GL}_m(E)$ -conjugacy classes of orthogonal groups. The one corresponding to the symmetric matrix  $J_m$  is split, and the one whose corresponding symmetric matrix is in the same fiber as  $J_m$  but not similar to  $J_m$ , as mentioned in Proposition [2.2.3](#), is not quasisplit, and the other three orthogonal groups are quasisplit but not split.

## 2.2.2 $\tau$ -split embedding

Now for  $E_m$  a field extension of degree  $m$  over  $E$  and  $\varepsilon \in \mathcal{S}$ , we say that an  $E$ -algebra embedding  $\iota : E_m \rightarrow \mathrm{M}_m(E)$  is  $\varepsilon$ -symmetric if its image consists of  $\varepsilon$ -symmetric matrices, or in other words,

$$\varepsilon^{-1} {}^t \iota(x) \varepsilon = \iota(x) \quad \text{for any } x \in E_m.$$

For  $\tau = \tau_\varepsilon$  an orthogonal involution, we say that  $E_m$  is  $\tau$ -split if there exists an embedding  $\iota$  as above such that it is  $\varepsilon$ -symmetric, or equivalently for any  $x \in E_m^\times$ , we have  $\tau(\iota(x)) = \iota(x)^{-1}$ . In particular, we get  $\tau(E_m^\times) = E_m^\times$ . We have the following important proposition which gives all the possible symmetric matrices via a given symmetric embedding:

**Proposition 2.2.6.** Let  $\tau = \tau_{\varepsilon_0}$  be a given orthogonal involution with  $\varepsilon_0 \in \mathcal{S}$  and let

$$\iota_0 : E_m \rightarrow \mathrm{M}_m(E)$$

be an  $\varepsilon_0$ -symmetric embedding. Then any symmetric matrix  $\varepsilon$  in  $\mathcal{S}$  such that there exists

$$\iota : E_m \rightarrow \mathrm{M}_m(E)$$

as an  $\varepsilon$ -symmetric embedding is similar to an element in  $\varepsilon_0 \iota_0(E_m^\times)$ .

*Proof.* We follow the proof of [Hak13](#), Proposition 4.3. For  $\varepsilon \in \mathcal{S}$  and corresponding  $\iota$  satisfying our condition, by the Skolem-Noether theorem, there exists  $g \in \mathrm{GL}_m(E)$  such that

$$\iota(x) = g^{-1} \iota_0(x) g$$

for any  $x \in E_m^\times$ . Then we have

$$\tau_0(\iota_0(x)) = \iota_0(x)^{-1} \quad \text{and} \quad \tau(\iota(x)) = \iota(x)^{-1},$$

thus

$$\tau(g)^{-1} \varepsilon^{-1} \varepsilon_0 \iota_0(x)^{-1} \varepsilon_0^{-1} \varepsilon \tau(g) = \tau(g)^{-1} \tau(\iota_0(x)) \tau(g) = \iota(x)^{-1} = g^{-1} \iota_0(x)^{-1} g,$$

which means that

$$\varepsilon_0^{-1} \varepsilon \tau(g) g^{-1} = \varepsilon_0^{-1} {}^t g^{-1} \varepsilon g^{-1}$$

commutes with any  $\iota_0(x) \in \iota_0(E_m^\times)$ . Thus  $\varepsilon_0^{-1} {}^t g^{-1} \varepsilon g^{-1} \in Z_{\mathrm{M}_m(E)}(\iota_0(E_m)) \setminus \{0\} = \iota_0(E_m^\times)$ , which means that  $\varepsilon$  is similar to an element in  $\varepsilon_0 \iota_0(E_m^\times)$ . □

In particular, we call an  $E$ -algebra embedding

$$\iota : E_m \rightarrow \mathrm{M}_m(E),$$

$J$ -symmetric if it is  $J_m$ -symmetric, omitting the size of matrices. The following proposition ensures the existence of  $J$ -symmetric embedding when  $E_m/E$  is tamely ramified.



**Proposition 2.2.7.** *When  $E_m/E$  is tamely ramified, there exists a  $J$ -symmetric embedding  $\iota$ .*

*Proof.* See for example [HL12], Proposition 5.15 or [Hak13], §4.2. □

**Remark 2.2.8.** *We don't know whether Proposition 2.2.7 is true or not when  $E_m/E$  is not necessarily tamely ramified.*

### 2.2.3 Calculation of Hilbert symbol and Hasse invariant in certain cases

In this subsection, we display elementary results for calculating Hilbert symbol and Hasse invariant.

**Lemma 2.2.9** ([HL12], Lemma 5.9). *If  $\varepsilon \in \mathrm{GL}_m(\mathfrak{o}_E) \cap \mathcal{S}$ , then  $\mathrm{Hasse}(\varepsilon) = 1$ .*

**Lemma 2.2.10.** *Let  $A \in \mathrm{M}_{n_1}(E)$  and  $B \in \mathrm{M}_{n_2}(E)$  be two symmetric matrices, then*

$$\mathrm{Hasse} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{Hasse}(A) \cdot \mathrm{Hasse}(B) \cdot \mathrm{Hil}(\det(A), \det(B)).$$

*Proof.* We assume that  $A$  is similar to  $\mathrm{diag}(a_1, \dots, a_{n_1})$  and  $B$  is similar to  $\mathrm{diag}(b_1, \dots, b_{n_2})$ , thus by definition

$$\begin{aligned} \mathrm{Hasse} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} &= \mathrm{Hasse}(\mathrm{diag}(a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2})) = \mathrm{Hasse}(A) \cdot \mathrm{Hasse}(B) \prod_{i,j=1}^{n_1, n_2} \mathrm{Hil}(a_i, b_j) \\ &= \mathrm{Hasse}(A) \cdot \mathrm{Hasse}(B) \cdot \mathrm{Hil}(\det(A), \det(B)). \end{aligned}$$

□

**Corollary 2.2.11.** *Let  $A_i \in \mathrm{M}_{n_i}(E)$  be symmetric matrices for  $i = 1, \dots, k$  such that for any  $1 \leq i < j \leq k$ , we have  $\mathrm{Hil}(\det(A_i), \det(A_j)) = 1$ . Then*

$$\mathrm{Hasse}(\mathrm{diag}(A_1, \dots, A_k)) = \prod_{i=1}^k \mathrm{Hasse}(A_i).$$

*Proof.* We use Lemma 2.2.10 for  $k - 1$  times to finish the proof. □

**Lemma 2.2.12.** *For  $\epsilon_1, \epsilon_2 \in \mathfrak{o}_E^\times$  and  $\varpi_E$  a uniformizer of  $E$ , we denote by  $\mathfrak{l}$  the residue field of  $E$ , and  $\bar{\epsilon}_1, \bar{\epsilon}_2$  the image of  $\epsilon_1, \epsilon_2$  in  $\mathfrak{l}$  respectively, then:*

(1)

$$\mathrm{Hil}(\varpi_E \epsilon_1, \varpi_E \epsilon_2) = \begin{cases} 1 & \text{if } -\bar{\epsilon}_1/\bar{\epsilon}_2 \in \mathfrak{l}^{\times 2}, \\ -1 & \text{otherwise.} \end{cases}$$

(2)

$$\mathrm{Hil}(\epsilon_1, \varpi_E \epsilon_2) = \begin{cases} 1 & \text{if } \bar{\epsilon}_1 \in \mathfrak{l}^{\times 2}, \\ -1 & \text{otherwise.} \end{cases}$$

*Proof.* For (1) we notice that

$$\mathrm{Hil}(\varpi_E \epsilon_1, \varpi_E \epsilon_2) = 1$$

if and only if

$$Z^2 + \epsilon_2/\epsilon_1 - \varpi_E C^2/\epsilon_1 = 0 \text{ has a solution for } Z \in \mathfrak{o}_E^\times \text{ and } C \in \mathfrak{o}_E.$$

Since if the equation  $\varpi_E \epsilon_1 X^2 + \varpi_E \epsilon_2 Y^2 = 1$  has a solution, comparing the order we must have  $X^{-1}, Y^{-1} \in \mathfrak{p}_E$  and  $X/Y \in \mathfrak{o}_E^\times$ . Thus we can change the variables  $Z = X/Y$  and  $C = \varpi_E^{-1} Y^{-1}$ . Using the Hensel lemma for the polynomial  $P(Z) = Z^2 + \epsilon_2/\epsilon_1 - \varpi_E C^2/\epsilon_1$  and the fact that  $p \neq 2$ , the condition above is true if and only if

$$\overline{Z}^2 = -\overline{\epsilon_2}/\overline{\epsilon_1} \text{ has a solution for } \overline{Z} \in \mathfrak{l}^\times,$$

which is equivalent to  $-\overline{\epsilon_1}/\overline{\epsilon_2} \in \mathfrak{l}^{\times 2}$ . Thus we finish the proof of (1), and the proof of (2) is similar.  $\square$

**Remark 2.2.13.** *In the latter sections, we mainly consider two cases:  $E = F$  or  $E/F$  is a field extension of degree  $d$  given by a certain simple stratum related to a given supercuspidal representation. In the former case, we have  $m = n$ ; In the latter case, we have  $m$  such that  $n = md$  with  $d = [E : F]$ . Moreover, we will simply write  $\det$ ,  $\text{disc}$  and  $\text{Hasse}$  for short when  $E = F$ .*

From now on until the end of this section, we assume  $E$  to be a tamely ramified extension of degree  $d = ef$  over  $F$ , where  $f$  denotes its residue class degree and  $e$  denotes its ramification index. Using Proposition 2.2.7, we fix a  $J$ -symmetric embedding  $E \hookrightarrow M_d(F)$ . We fix  $\epsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$  and  $\varpi_E$  a uniformizer of  $E$ , such that  $E^\times/E^{\times 2} = \{1, \epsilon_0, \varpi_E, \epsilon_0 \varpi_E\}$ . By Section 3 of [Hak13], we have three different cases:

**Proposition 2.2.14.** (1)  $N_{E/F}(E^\times)F^{\times 2}/F^{\times 2} = \{1\}$  if and only if  $E$  contains three quadratic subextensions over  $F$ , and exactly one of them is unramified. Thus both  $e$  and  $f$  are even;

(2)  $N_{E/F}(E^\times)F^{\times 2}/F^{\times 2}$  is of order 2 if and only if  $E$  contains exactly one quadratic subextension over  $F$ . Thus either  $e$  or  $f$  is even;

(3)  $N_{E/F}(E^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$  if and only if  $E$  contains no quadratic subextension over  $F$ . Thus  $d = ef$  is odd.

For case (1), we have the following lemma:

**Lemma 2.2.15.** *If  $N_{E/F}(E^\times)F^{\times 2}/F^{\times 2} = \{1\}$ , then we may further choose the uniformizer  $\varpi_E$  of  $E$ , such that*

$$\text{Hasse}(J_d \varpi_E) = 1 \quad \text{and} \quad \text{Hasse}(J_d \varpi_E \epsilon_0) = -1,$$

where  $J_d \varpi_E$  and  $J_d \varpi_E \epsilon_0$  are symmetric matrices in  $\text{GL}_d(F)$ .

*Proof.* We may use [Hak13], Proposition 6.6 directly.  $\square$

For case (2), first we assume that  $f$  is odd and  $e$  is even. We have:

**Lemma 2.2.16.** *For  $f$  odd and  $e$  even, we have  $\text{Hasse}(J_d \varpi_E) \neq \text{Hasse}(J_d \varpi_E \epsilon_0)$ .*

*Proof.* We use the proof of [Hak13], Proposition 6.6 directly, except that right now  $f$  is odd instead of being even. Our question reduces to calculate the following term

$$\text{Hasse}(\text{diag}(u_1, \dots, u_f, u_1 \varpi_F, \dots, u_f \varpi_F)) \quad (\text{with } u_1, \dots, u_f \in \mathfrak{o}_F^\times)$$

in the case where  $\prod_{i=1}^f u_i \in F^{\times 2}$  or  $\epsilon'_0 F^{\times 2}$  respectively with  $\epsilon'_0 \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$  fixed, and to show that they are different. From the calculation in *loc. cit.*, we have

$$\begin{aligned} \text{Hasse}(\text{diag}(u_1, \dots, u_f, u_1 \varpi_F, \dots, u_f \varpi_F)) &= \left( \prod_{i=1}^f \text{Hil}(u_i, \varpi_F) \right)^{2f-1} \cdot \text{Hil}(\varpi_F, \varpi_F)^{f(f-1)/2} \\ &= \text{Hil}\left(\prod_{i=1}^f u_i, \varpi_F\right) \cdot \text{Hil}(\varpi_F, \varpi_F)^{f(f-1)/2} \end{aligned}$$

Thus by Lemma 2.2.12(2), when  $\prod_{i=1}^f u_i \in F^{\times 2}$  or  $\epsilon'_0 F^{\times 2}$  respectively, the corresponding terms are different. □

**Corollary 2.2.17.** *Under the assumption of Lemma 2.2.16, the Hasse invariants*

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E)) \quad \text{and} \quad \text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0))$$

are different, where the two matrices are in  $M_m(M_d(F)) = M_{md}(F)$ .

*Proof.* We write

$$A = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E) \in M_{m-1}(M_d(F)) = M_{(m-1)d}(F),$$

then using Lemma 2.2.10, we have

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E)) = \text{Hasse}(A) \cdot \text{Hasse}(J_d \varpi_E) \cdot \text{Hil}(\det(A), \det(J_d \varpi_E))$$

and

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0)) = \text{Hasse}(A) \cdot \text{Hasse}(J_d \varpi_E \epsilon_0) \cdot \text{Hil}(\det(A), \det(J_d \varpi_E \epsilon_0)).$$

Thus using Lemma 2.2.16, we only need to show that

$$\text{Hil}(\det(A), \det(J_d \varpi_E)) = \text{Hil}(\det(A), \det(J_d \varpi_E \epsilon_0)),$$

which follows from the fact that  $\det(\epsilon_0) = N_{E/F}(\epsilon_0) \in F^{\times 2}$  when  $e$  is even. □

Now we assume that  $e$  is odd. First we consider the case where  $f$  is even. In this case,  $N_{E/F}(\epsilon_0) \notin F^{\times 2}$ . We choose  $\varpi'_E$  to be another uniformizer of  $E$  such that  $N_{E/F}(\varpi'_E) \in F^{\times 2}$ .

**Lemma 2.2.18.** *If  $e$  and  $m$  are odd and if  $f$  is even, then*

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi'_E)) = 1$$

and

$$\text{Hasse}(\text{diag}(J_d \varpi_E \epsilon_0, \dots, J_d \varpi_E \epsilon_0, J_d \varpi'_E \epsilon_0)) = -1,$$

where the two matrices are in  $M_m(M_d(F)) = M_{md}(F)$ .

*Proof.* To begin with, we state and proof the following general lemma which is useful not only in this proof, but in the latter sections.

**Lemma 2.2.19.** *Let  $E/L$  be a finite extension of non-archimedean locally compact fields of residue characteristic  $p \neq 2$  with odd degree, and let*

$$L^\times / L^{\times 2} \rightarrow E^\times / E^{\times 2}$$

be the homomorphism induced by the canonical embedding  $L \hookrightarrow E$ , then the homomorphism above induces two isomorphisms

$$L^\times / L^{\times 2} \cong E^\times / E^{\times 2} \quad \text{and} \quad \mathfrak{o}_L^\times / \mathfrak{o}_L^{\times 2} \cong \mathfrak{o}_E^\times / \mathfrak{o}_E^{\times 2}.$$

*Proof.* The embedding  $L \hookrightarrow E$  leads to the following embedding:

$$L^\times/E^{\times 2} \cap L^\times \hookrightarrow E^\times/E^{\times 2}.$$

First we have  $L^{\times 2} \subset E^{\times 2} \cap L^\times$ . And for  $x \in E^{\times 2} \cap L^\times$ , let  $x = y^2$  with  $y \in E^\times$ . Thus  $L[y]$  is a subextension of  $E$  over  $L$  which is of degree 1 or 2. Since  $[E : L]$  is odd, we must have  $L[y] = L$  and  $y \in L$ . So  $x \in L^{\times 2}$ , which means that  $E^{\times 2} \cap L^\times = L^{\times 2}$  since  $x$  is arbitrary. Thus the homomorphism in the lemma is injective, which is an isomorphism since  $[E^\times : E^{\times 2}] = [L^\times : L^{\times 2}] = 4$ .

Moreover, since  $|\mathfrak{o}_L^\times/\mathfrak{o}_L^{\times 2}| = |\mathfrak{o}_E^\times/\mathfrak{o}_E^{\times 2}| = 2$ , the isomorphism above also leads to an isomorphism

$$\mathfrak{o}_L^\times/\mathfrak{o}_L^{\times 2} \cong \mathfrak{o}_E^\times/\mathfrak{o}_E^{\times 2}.$$

□

Come back to the original proof. We write  $L$  for the maximal unramified subextension of  $E$  over  $F$ , then  $[L : F] = f$  and  $[E : L] = e$ . Since  $e$  is odd, by Lemma 2.2.19 we have an isomorphism

$$\mathfrak{o}_E^\times/\mathfrak{o}_E^{\times 2} \cong \mathfrak{o}_L^\times/\mathfrak{o}_L^{\times 2}.$$

Since the result doesn't depend on the choice of  $\varpi_E$ ,  $\varpi'_E$  and  $\epsilon_0$  as representatives in  $E^\times/E^{\times 2}$ , we may assume that  $\varpi_E^e = \varpi_L$  is a uniformizer in  $L$ , and  $\varpi'_E = \varpi'_L$  is a uniformizer in  $L$  such that  $N_{L/F}(\varpi'_L) \in F^{\times 2}$ , and  $\epsilon_0 \in \mathfrak{o}_L^\times \setminus \mathfrak{o}_L^{\times 2}$ . From the construction of the  $J$ -symmetric embedding in Proposition 2.2.7 (see the proof of Hak13, Proposition 6.6 for more details), we may write

$$J_d \varpi_E = \text{diag}(J_{(e-1)f}, J_f \varpi_L) \quad \text{and} \quad J_d \varpi'_E = \text{diag}(J_{e(f-1)}, J_f \varpi'_L)$$

and

$$J_d \varpi_E \epsilon_0 = \text{diag}(J_{(e-1)f} \epsilon_0, J_f \varpi_L \epsilon_0) \quad \text{and} \quad J_d \varpi'_E \epsilon_0 = \text{diag}(J_{e(f-1)} \epsilon_0, J_f \varpi'_L \epsilon_0).$$

Since  $\det(J_{(e-1)f}) \in \mathfrak{o}_F^\times$ , and since  $\det(\text{diag}(J_f \varpi_L, \dots, J_f \varpi_L, J_f \varpi'_L))$  is of even order in  $F^\times$ , using Lemma 2.2.9 and Corollary 2.2.11, we get

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi'_E)) = \text{Hasse}(\text{diag}(J_f \varpi_L, \dots, J_f \varpi_L, J_f \varpi'_L)), \quad (2.2.1)$$

where the matrix in the Hasse of the right hand side is of size  $fm$ . Similarly we have

$$\text{Hasse}(\text{diag}(J_d \varpi_E \epsilon_0, \dots, J_d \varpi_E \epsilon_0, J_d \varpi'_E \epsilon_0)) = \text{Hasse}(\text{diag}(J_f \varpi_L \epsilon_0, \dots, J_f \varpi_L \epsilon_0, J_f \varpi'_L \epsilon_0)), \quad (2.2.2)$$

where the matrix in the Hasse of the right hand side is also of size  $fm$ . Since  $L/F$  is unramified, we may write  $\varpi_L = \varpi_F v$  and  $\varpi'_L = \varpi_F v'$  with  $v, v' \in \mathfrak{o}_F^\times$ , thus the term in (2.2.1) equals

$$\text{Hasse}(\text{diag}(J_f v \varpi_F, \dots, J_f v \varpi_F, J_f v' \varpi_F)), \quad (2.2.3)$$

and the term in (2.2.2) equals

$$\text{Hasse}(\text{diag}(J_f v \epsilon_0 \varpi_F, \dots, J_f v \epsilon_0 \varpi_F, J_f v' \epsilon_0 \varpi_F)). \quad (2.2.4)$$

Since  $f$  is even,  $\det(J_f \varpi_F)$  and  $\det(J_f v' \varpi_F)$  are of even order in  $F^\times$ , thus by Lemma 2.2.11, (2.2.3) equals

$$\text{Hasse}(J_f v \varpi_F)^{m-1} \cdot \text{Hasse}(J_f v' \varpi_F) = \text{Hasse}(J_f v' \varpi_F)$$

and similarly (2.2.4) equals

$$\text{Hasse}(J_f v \epsilon_0 \varpi_F)^{m-1} \cdot \text{Hasse}(J_f v' \epsilon_0 \varpi_F) = \text{Hasse}(J_f v' \epsilon_0 \varpi_F).$$

We assume that  $J_f v'$  is similar to  $\text{diag}(1, \dots, 1, u_1)$  and  $J_f v' \epsilon_0$  is similar to  $\text{diag}(1, \dots, 1, u_2)$  with  $u_1, u_2 \in \mathfrak{o}_F^\times$ , then (2.2.3) equals

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, \varpi_F u_1)),$$

and (2.2.4) equals

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, \varpi_F u_2)).$$

By direct calculation, we get

$$\det(J_f v' \varpi_F) = (-1)^{f(f-1)/2} N_{L/F}(\varpi'_L) \in (-1)^{f(f-1)/2} F^{\times 2}$$

and

$$\det(J_f v' \epsilon_0 \varpi_F) = (-1)^{f(f-1)/2} N_{L/F}(\epsilon_0 \varpi'_L) \in (-1)^{f(f-1)/2} N_{L/F}(\epsilon_0) F^{\times 2},$$

where  $N_{L/F}(\epsilon_0) \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$ .

If  $-1 \in F^{\times 2}$  or if  $-1 \notin F^{\times 2}$  and  $4|f$ , then  $\det(J_f v') \in \mathfrak{o}_F^{\times 2}$  and  $\det(J_f v' \epsilon_0) \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$ . We may assume  $u_1 = 1$  and  $u_2 \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$ , where in the latter case we may further assume  $u_2 = -1$ . So by Lemma 2.2.12(1), when  $-1 \in F^{\times 2}$  we have

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, \varpi_F u_1)) = 1$$

and

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, \varpi_F u_2)) = (-1)^{f-1} = -1.$$

When  $-1 \notin F^{\times 2}$  and  $4 \nmid f$ , we have

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, \varpi_F u_1)) = (-1)^{f(f-1)/2} = 1,$$

and

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, -\varpi_F)) = (-1)^{(f-1)(f-2)/2} = -1.$$

If  $-1 \notin F^{\times 2}$  and  $4 \nmid f$ , then  $\det(J_f v') \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$  and  $\det(J_f v' \epsilon_0) \in \mathfrak{o}_F^{\times 2}$ . We may assume  $u_1 = -1$  and  $u_2 = 1$  and we have

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, -\varpi_F)) = (-1)^{(f-1)(f-2)/2} = 1$$

and

$$\text{Hasse}(\text{diag}(\varpi_F, \dots, \varpi_F, \varpi_F)) = (-1)^{f(f-1)/2} = -1.$$

Thus we finish the proof. □

Finally, we drop the assumption that  $f$  is even.

**Lemma 2.2.20.** *If  $e$  is odd,  $m$  is even and one of the three cases happens:*

- $2|d$ ;
- $2 \nmid d$  and  $4|m$ ;
- $2 \nmid d$ ,  $4 \nmid m$  and  $-1 \in F^{\times 2}$ ,

then  $\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0)) = -1$ , where the matrix in Hasse is in  $M_{md}(F)$ .

*Proof.* We write  $L$  for the maximal unramified extension of  $F$  contained in  $E$ , thus  $[L : F] = f$  and  $[E : L] = e$ . Since  $e$  is odd, by Lemma 2.2.19 we get

$$\mathfrak{o}_E^\times / \mathfrak{o}_E^{\times 2} \cong \mathfrak{o}_L^\times / \mathfrak{o}_L^{\times 2}.$$

Since the result doesn't depend on the choice of  $\varpi_E$  and  $\epsilon_0$  as representatives in  $E^\times / E^{\times 2}$ , we may choose  $\varpi_E$  as a uniformizer of  $E$  such that  $\varpi_E^e = \varpi_L$  is a uniformizer in  $L$ , and  $\epsilon_0 \in \mathfrak{o}_L^\times \setminus \mathfrak{o}_L^{\times 2}$ . As in Lemma 2.2.18, we may write

$$J_d \varpi_E = \text{diag}(J_{(e-1)f}, J_f \varpi_L) \quad \text{and} \quad J_d \varpi_E \epsilon_0 = \text{diag}(J_{e(f-1)} \epsilon_0, J_f \varpi_L \epsilon_0).$$

Thus by Corollary 2.2.11 and the fact that  $m$  is even, we get

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0)) = \text{Hasse}(\text{diag}(J_f \varpi_L, \dots, J_f \varpi_L, J_f \varpi_L \epsilon_0)), \quad (2.2.5)$$

where the last term in the Hasse is a matrix of size  $fm$ . Since  $L/F$  is unramified, we may write  $\varpi_L = \varpi_F v$  with  $v \in \mathfrak{o}_F^\times$ , thus the term in (2.2.5) equals

$$\text{Hasse}(\text{diag}(J_f v \varpi_F, \dots, J_f v \varpi_F, J_f v \varpi_F \epsilon_0)).$$

If we assume that  $J_f v$  is similar to  $\text{diag}(1, \dots, 1, u_1)$ , and  $J_f v \epsilon_0$  is similar to  $\text{diag}(1, \dots, 1, u_2)$ , then we get  $u_2/u_1 \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$ . Moreover we get

$$\text{Hasse}(\text{diag}(J_f v \varpi_F, \dots, J_f v \varpi_F, J_f v \varpi_F \epsilon_0)) = \text{Hasse}(\text{diag}(I_{m(f-1)} \varpi_F, u_1 \varpi_F, \dots, u_1 \varpi_F, u_2 \varpi_F)), \quad (2.2.6)$$

where the last diagonal matrix in Hasse is of size  $fm$ .

If  $-1 \in F^{\times 2}$ , we may choose either  $u_1 = 1$  and  $u_2 = \epsilon'_0$ , or  $u_1 = \epsilon'_0$  and  $u_2 = 1$  with  $\epsilon'_0 \in \mathfrak{o}_F^\times \setminus \mathfrak{o}_F^{\times 2}$ . Thus in the former case, by Lemma 2.2.12.(1) the (2.2.6) equals

$$\text{Hasse}(\text{diag}(I_{mf-1} \varpi_F, \varpi_F \epsilon'_0)) = (-1)^{mf-1} = -1,$$

and in the latter case, by Lemma 2.2.12.(1) the (2.2.6) equals

$$\text{Hasse}(\text{diag}(I_{mf-m+1} \varpi_F, I_{m-1} \varpi_F \epsilon'_0)) = (-1)^{(mf-m+1)(m-1)} = -1.$$

If  $-1 \notin F^{\times 2}$ , we may assume  $\epsilon'_0 = -1$ ,  $u_1$  equals 1 or  $-1$  and  $u_2 = -u_1$ , and for the two cases using Lemma 2.2.12.(1) the (2.2.6) equals

$$\text{Hasse}(\text{diag}(I_{fm-1} \varpi_F, -\varpi_F)) = (-1)^{(fm-1)(fm-2)/2} = -1$$

or

$$\text{Hasse}(\text{diag}(I_{(f-1)m+1} \varpi_F, -I_{m-1} \varpi_F)) = (-1)^{fm(fm-1)/2 - ((f-1)m+1)(m-1)} = -1,$$

where in both cases we use the fact that  $4|fm$  and  $2|m$ , thus we finish the proof.  $\square$

Finally we have the following lemma which completes Lemma 2.2.20.

**Lemma 2.2.21.** *If  $d$  is odd,  $m$  is even not divided by 4 and  $-1 \notin F^{\times 2}$ , then*

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E)) = -1,$$

where the matrix is in  $M_{md}(F)$ .

*Proof.* We may follow the same proof as Lemma 2.2.20, which finally shows that

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E)) = \text{Hasse}(I_{fm} \varpi_F).$$

Since  $-1 \notin F^{\times 2}$ , by Lemma 2.2.12.(1) the latter term equals  $(-1)^{fm(fm-1)/2}$ , which is  $-1$  since under our assumption  $fm \equiv 2 \pmod{4}$ . Thus we finish the proof.  $\square$

## 2.3 $\tau$ -selfdual type theorem

Let  $\pi$  be a supercuspidal representation of  $G$ . Let  $\tau = \tau_\varepsilon$  be the orthogonal involution corresponding to a symmetric matrix  $\varepsilon$ , such that for  $H = G^\tau$  as the orthogonal group corresponding to  $\tau$ , it satisfies the condition 2 of Theorem [0.3.1](#) with respect to  $\pi$ . For  $\mathfrak{a}$  an  $\mathfrak{o}_F$ -subalgebra of  $M_n(F)$ , we define

$$\tau(\mathfrak{a}) := \varepsilon^{-1} {}^t \mathfrak{a} \varepsilon$$

which is an  $\mathfrak{o}_F$ -subalgebra of  $M_n(F)$ . We say that  $\mathfrak{a}$  is  $\tau$ -stable if  $\tau(\mathfrak{a}) = \mathfrak{a}$ . For any  $g \in G$ , it is easy to show that  $\tau(\mathfrak{a}^g) = \tau(\mathfrak{a})^{\tau(g)}$ .

In this section, we follow the strategy in chapter 1, section 5 to prove the following theorem:

**Theorem 2.3.1.** *For  $\pi$  and  $\tau$  as above, there exists a maximal simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  contained in  $\pi$ , such that*

- (1)  $\tau(\mathfrak{a}) = \mathfrak{a}$  and  $\tau(H^1(\mathfrak{a}, \beta)) = H^1(\mathfrak{a}, \beta)$ ;
- (2)  $\theta \circ \tau = \theta^{-1}$ ;
- (3)  $\tau(\beta) = \beta^{-1}$ .

As a corollary of Theorem [2.3.1](#), we have the following  $\tau$ -selfdual type theorem.

**Theorem 2.3.2.** *For  $\pi$  and  $\tau$  as above, there exists a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  that compactly induces  $\pi$ .*

*Proof.* We only need to follow the proof of Theorem [1.5.3](#), with Theorem [1.5.2](#) replaced by Theorem [2.3.1](#). □

Now we state the following general theorem which implies Theorem [2.3.1](#)

**Theorem 2.3.3.** *Let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $M_n(F)$ , let  $T$  be the maximal tamely ramified subextension of  $E/F$ , let  $T_m$  be the unramified extension of degree  $m$  over  $T$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character. Let  $\tau$  be an orthogonal involution of  $G$  such that  $H = G^\tau$  satisfies the condition 2 of Theorem [0.3.1](#). Then there exist a maximal simple stratum  $[\mathfrak{a}', \beta']$  in  $M_n(F)$  and a simple character  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  such that*

- (1)  $\tau(\mathfrak{a}') = \mathfrak{a}'$  and  $\tau(H^1(\mathfrak{a}', \beta')) = H^1(\mathfrak{a}', \beta')$ ;
- (2)  $\theta'$  and  $\theta$  are in the same endo-class and  $\theta' \circ \tau = \theta'^{-1}$ ;
- (3)  $\tau(\beta') = \beta'^{-1}$ .

For  $\pi$  given as in Theorem [2.3.1](#), if we choose  $[\mathfrak{a}, \beta]$  to be a maximal simple stratum and  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  to be a simple character contained in  $\pi$ , then Theorem [2.3.3](#) implies Theorem [2.3.1](#). So from now on, we focus on the proof of Theorem [2.3.3](#). We write  $E = F[\beta]$ ,  $d = [E : F]$  and  $m = n/d$ . In the following subsections, we gradually consider the following three cases:  $E/F$  is maximal and totally wildly ramified,  $E/F$  is maximal and the general case.

To begin with, we state the following lemmas which will be useful in our future proof.

**Lemma 2.3.4.** *Let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $M_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , then for  $\tau$  as an orthogonal involution on  $G$ , the simple characters  $\theta \circ \tau$  and  $\theta^{-1}$  are in the same endo-class. In particular, if  $\tau(\mathfrak{a}) = \mathfrak{a}$ , then  $\theta \circ \tau$  is conjugate to  $\theta^{-1}$  by an element in  $U(\mathfrak{a})$ .*

*Proof.* We follow the same proof of Lemma [1.5.7](#), with  $\sigma$  in *loc. cit.* replaced by the trivial action. □

**Lemma 2.3.5.** *Let  $\tau = \tau_\varepsilon$  be the orthogonal involution on  $G$  corresponding to a symmetric matrix  $\varepsilon$ , let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $M_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character, such that*

$$\tau(\mathfrak{a}) = \mathfrak{a}, \quad \theta \circ \tau = \theta^{-1} \quad (\text{and } \tau(\beta) = \beta^{-1}).$$

*Then for  $\tau' = \tau_{\varepsilon'}$  as the orthogonal involution on  $G$  corresponding to the symmetric matrix  $\varepsilon' = {}^t g \varepsilon g$ , we have*

$$\tau'(\mathfrak{a}^g) = \mathfrak{a}^g, \quad \theta^g \circ \tau' = (\theta^g)^{-1} \quad (\text{and } \tau'(\beta^g) = (\beta^g)^{-1}).$$

*Proof.* Same proof as Lemma [1.5.8](#). □

**Lemma 2.3.6.** *Let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $M_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  such that  $\tau(\mathfrak{a}) = \mathfrak{a}$ ,  $\tau(H^1(\mathfrak{a}, \beta)) = H^1(\mathfrak{a}, \beta)$  and  $\theta \circ \tau = \theta^{-1}$ . Then there exists a simple stratum  $[\mathfrak{a}, \gamma]$  such that  $\theta \in \mathcal{C}(\mathfrak{a}, \gamma)$  and  $\tau(\gamma) = \gamma^{-1}$ .*

*Proof.* For  $\tau = \tau_\varepsilon$  with respect to a symmetric matrix  $\varepsilon$ , we define

$$\sigma_\varepsilon(x) := \varepsilon^{-1} {}^t x \varepsilon \quad \text{for any } x \in M_n(F)$$

as an anti-involution on  $M_n(F)$ . Then we may use the same argument in Proposition [1.5.19](#), with  $\sigma_t$  in *loc. cit.* replaced by  $\sigma_\varepsilon$ , and the original proof in [Ste01](#), Theorem 6.3 works. □

### 2.3.1 The maximal and totally wildly ramified case

In this subsection, we prove the following special case of Theorem [2.3.3](#).

**Proposition 2.3.7.** *Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $M_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character, where  $n = d$  and  $E/F$  is totally wildly ramified. Then for  $\tau = \tau_{I_n}$  the orthogonal involution on  $G$ , there exist a simple stratum  $[\mathfrak{a}', \beta']$  and a simple character  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  such that  $(\mathfrak{a}', \theta')$  is  $G$ -conjugate to  $(\mathfrak{a}, \theta)$  with the property  $\tau(\mathfrak{a}') = \mathfrak{a}'$  and  $\theta' \circ \tau = \theta'^{-1}$ . Moreover, we may further assume that  $\mathfrak{a}' \subset M_n(\mathfrak{o}_F)$ .*

*Proof.* We explain how the proof of Proposition [1.5.9](#) could be used directly in our case. First up to  $G$ -conjugacy, we may assume  $\mathfrak{a}$  to be the standard minimal order of  $M_n(F)$ . We have the following lemma corresponding to Lemma [1.5.11](#):

**Lemma 2.3.8.** *There exist  $g_1 \in \text{GL}_n(\mathfrak{o}_F)$  and  $a_1, \dots, a_n \in \mathfrak{o}_F^\times$  such that*

$$\tau(g_1)g_1^{-1} = A := \begin{pmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & \ddots & \ddots & a_2 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_{n-1} & \ddots & \ddots & 0 \\ a_n & 0 & \dots & 0 & 0 \end{pmatrix}.$$

*Moreover, if we define  $\mathfrak{a}'' := \mathfrak{a}^{g_1}$ , then we have  $\tau(\mathfrak{a}'') = \mathfrak{a}''$ .*

*Proof.* We choose  $a_1 = \dots = a_{(n-1)/2} = a_{(n+3)/2} = \dots = a_n = 1$ , and  $a_{(n+1)/2}$  equals 1 or  $-1$  to make sure that  $\det(A) = 1$ . Since the  $\mathfrak{o}_F$ -lattice of rank  $n$  equipped with a quadratic form corresponding to  $A$  is unimodular in the sense of [O'M71](#), §92, by §92:1 in *loc. cit.*, there exists  $g_1 \in \text{GL}_n(\mathfrak{o}_F)$  such that  ${}^t g_1^{-1} g_1^{-1} = A$ , or equivalently  $\tau(g_1)g_1^{-1} = A$ . Then we may use the same proof as that in Lemma [1.5.11](#) to obtain  $\tau(\mathfrak{a}'') = \mathfrak{a}''$ . □



By Lemma [2.3.8](#), we may choose  $g_1 \in \mathrm{GL}_n(\mathfrak{o}_F)$  such that  $\mathfrak{a}'' = \mathfrak{a}^{g_1}$  is  $\tau$ -invariant. Let  $M = \mathfrak{o}_F^\times \times \dots \times \mathfrak{o}_F^\times$  be the subgroup of  $\mathrm{GL}_n(\mathfrak{o}_F)$  via diagonal embedding, let  $M'' = M^{g_1}$  and  $U'' = U^{g_1} := U^1(\mathfrak{a})^{g_1}$ . Then using directly the proof of Proposition [1.5.9](#), with all the Galois involution in *loc. cit.* replaced by the trivial action, there exists  $x \in M''U''$  such that for  $\mathfrak{a}' = \mathfrak{a}''^x = \mathfrak{a}^{g_1x}$  and  $\theta' = \theta^{g_1x}$ , we have  $\tau(\mathfrak{a}') = \mathfrak{a}'$  and  $\theta' \circ \tau = \theta'^{-1}$ . Moreover since  $g_1x \in g_1M''U'' = MU^1g_1 \subset \mathrm{GL}_n(\mathfrak{o}_F)$  and  $\mathfrak{a} \subset \mathrm{M}_n(\mathfrak{o}_F)$ , we get  $\mathfrak{a}' = \mathfrak{a}^{g_1x} \subset \mathrm{M}_n(\mathfrak{o}_F)$ . □

### 2.3.2 The maximal case

In this subsection, we further use the result proved in [§2.3.1](#) to consider the following special case of Theorem [2.3.3](#).

**Proposition 2.3.9.** *Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $\mathrm{M}_n(F)$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character with  $n = d$ . Then for an orthogonal involution  $\tau = \tau_\varepsilon$  which is  $G$ -conjugate to  $\tau_{J_n}$ , there exist a simple stratum  $[\mathfrak{a}', \beta']$  and a simple character  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  such that  $(\mathfrak{a}', \theta')$  is  $G$ -conjugate to  $(\mathfrak{a}, \theta)$  with the property  $\tau(\mathfrak{a}') = \mathfrak{a}'$ ,  $\theta' \circ \tau = \theta'^{-1}$  and  $\tau(\beta') = \beta'^{-1}$ .*

**Remark 2.3.10.** *If we assume  $E/F$  to be totally wildly ramified, then by direct calculation and Lemma [2.2.9](#), we have*

$$\det(I_n) = \det(J_n) \text{ or } \det(-J_n) \quad \text{and} \quad \mathrm{Hasse}(I_n) = \mathrm{Hasse}(J_n) = \mathrm{Hasse}(-J_n) = 1.$$

Thus  $I_n$  is  $G$ -conjugate to  $J_n$  or  $-J_n$ , which means that  $\tau_{I_n}$  is  $G$ -conjugate to  $\tau_{J_n}$ . Choosing  $\varepsilon = I_n$ , Proposition [2.3.9](#) implies Proposition [2.3.7](#).

**Remark 2.3.11.** *Since  $\tau_{J_n}$  represents the split orthogonal group, it satisfies the condition of Theorem [2.3.3](#), which justifies that Proposition [2.3.9](#) is indeed a special case of Theorem [2.3.3](#).*

*Proof.* We write  $n = t(n/t)$  with  $t = [T : F]$  and  $n/t$  a power of  $p$  as an odd number, where  $T$  is the maximal tamely ramified subextension of  $E$  over  $F$ . We define

$$J_{t,n/t} := \mathrm{diag}(J_t, \dots, J_t)$$

as a matrix in  $\mathrm{M}_{n/t}(\mathrm{M}_t(F)) = \mathrm{M}_n(F)$ . Using Lemma [2.2.9](#), we have

$$\mathrm{Hasse}(J_{t,n/t}) = \mathrm{Hasse}(J_n) = \mathrm{Hasse}(-J_n) = 1.$$

Moreover by direct calculation we have

$$\det(J_{t,n/t}) = \det(J_n) \text{ or } \det(-J_n).$$

Thus using Proposition [2.2.1](#),  $J_{t,n/t}$  is similar to  $J_n$  or  $-J_n$ . Thus  $\tau_{J_{t,n/t}}$  is  $G$ -conjugate to  $\tau_{J_n}$  and  $\tau_\varepsilon$ . By Proposition [2.2.3](#), we may replace  $\varepsilon$  by multiplying an element in  $F^\times$  to make sure that  $\varepsilon$  is similar to  $J_{t,n/t}$ . Thus using Lemma [2.3.5](#), we only need to consider the case where  $\varepsilon = J_{t,n/t}$  and  $\tau = \tau_{J_{t,n/t}}$ . So from now on we assume  $\varepsilon = J_{t,n/t}$ .

Using Proposition [2.2.7](#), we may choose

$$\iota : T \hookrightarrow \mathrm{M}_t(F)$$

to be an  $F$ -algebra embedding which is  $J_t$ -symmetric. By abuse of notation, we consider the following embedding

$$\iota : \mathrm{M}_{n/t}(T) \hookrightarrow \mathrm{M}_{n/t}(\mathrm{M}_t(F)) = \mathrm{M}_n(F)$$

given by mapping each entry  $T$  to the corresponding  $M_t(F)$  via the original  $\iota$ . If we regard  $T$  as an  $F$ -subalgebra of  $M_{n/t}(T)$  given by the diagonal embedding, then  $\iota(T)^\times$  is fixed by  $\tau$ . By the Skolem-Noether theorem, we may choose  $g \in G$  such that  $\iota(T) = T^g$ . Thus using  $[\mathfrak{a}^g, \beta^g]$  and  $\theta^g$  to replace  $[\mathfrak{a}, \beta]$  and  $\theta$ , we may suppose  $\iota(T)$  to be the maximal tamely ramified extension with respect to  $E/F$ . Thus we identify  $T$  with  $\iota(T)$  and omit  $\iota$ .

Let  $C = M_{n/t}(T)$  denote the centralizer of  $T$  in  $M_n(F)$  and let  ${}^t c$  denote the transpose on  $C$ . For  $c = (c_{ij})_{ij} \in \mathrm{GL}_{n/t}(T)$ , we have

$$\tau(c) = (J_{t,n/t}^{-1} {}^t c J_{t,n/t})^{-1} = ((J_t^{-1} {}^t c_{ji} J_t)_{ij})^{-1} = ((c_{ji})_{ij})^{-1} = {}^t c c^{-1} = \tau'(c)$$

where we use the fact that  $\iota$  is  $J_t$ -symmetric and we write  $\tau'(x) = {}^t c x^{-1}$  for any  $x \in C^\times$ . Thus  $\tau'$  as the restriction of  $\tau$  to  $C^\times$  is the orthogonal involution  $\tau_{I_{n/t}}$  on  $C^\times = \mathrm{GL}_{n/t}(T)$ . As mentioned in §2.1.2, the intersection  $\mathfrak{c} = \mathfrak{a} \cap C$  gives rise to a simple stratum  $[\mathfrak{c}, \beta]$  and the restriction of  $\theta$  to  $H^1(\mathfrak{c}, \beta)$ , denoted by  $\theta_T$ , is the interior  $T/F$ -lift of  $\theta$ . Since  $E/T$  is totally wildly ramified, using Proposition 2.3.7 with  $G$ ,  $\theta$  and  $\tau$  replaced by  $C^\times$ ,  $\theta_T$  and  $\tau'$  respectively, there exists  $c \in C^\times$  such that  $\tau'(c^c) = c^c$  and  $\theta_T^c \circ \tau' = (\theta_T^c)^{-1}$ . As a corollary, we also have  $\tau'(H^1(\mathfrak{c}^c, \beta^c)) = H^1(\mathfrak{c}^c, \beta^c)$  and  $\mathcal{C}(\mathfrak{c}^c, -\beta^c) = \mathcal{C}(\mathfrak{c}^c, \tau'(\beta^c))$ .

By the injectivity of  $\mathfrak{a} \mapsto \mathfrak{a} \cap C$  between sets of hereditary orders mentioned in §2.1.2,  $\mathfrak{a}' := \mathfrak{a}^c$  is  $\tau$ -stable. Moreover if we write  $\theta' = \theta^c$  and  $T' = T^c$ , then from our construction of  $\tau$  and the definition of  $T'/F$ -lift, we know that

$$(\theta' \circ \tau)_{T'} = \theta' \circ \tau|_{H^1(\mathfrak{c}^c, \beta^c)} = \theta' \circ \tau'|_{H^1(\mathfrak{c}^c, \beta^c)} = \theta'_{T'} \circ \tau'$$

and

$$(\theta'^{-1})_{T'} = \theta'_{T'}^{-1}$$

are equal. Thus by the last paragraph of §2.1.2, the simple character  $\theta'$  satisfies the property  $\theta' \circ \tau = \theta'^{-1}$ .

Finally using Lemma 2.3.6 with  $\varepsilon = J_{t,n/t}$ , we may choose  $\beta'$  in the simple stratum such that  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  and  $\tau(\beta') = \beta'^{-1}$ , thus we finish the proof.  $\square$

Before we prove the general case, we state and prove the following important lemma which studies the set  $\varepsilon E'^\times$  consisting of symmetric matrices, where  $E' = F[\beta']$  with  $\beta'$  chosen as in Proposition 2.3.9.

**Lemma 2.3.12.** *We may choose  $[\mathfrak{a}', \beta']$  and  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$  satisfying the conclusion of Proposition 2.3.9 and  $T$  as a tame parameter field of  $\theta'$ , and we may fix  $\iota : T \hookrightarrow M_t(F)$  as a  $J$ -symmetric embedding given by Proposition 2.2.7, such that for any  $x \in E'^\times$ , there exists  $x_t \in T^\times$  such that  $\varepsilon x$  is similar to  $\mathrm{diag}(J_t \iota(x_t), \dots, J_t \iota(x_t))$ .*

*Proof.* First we assume  $\varepsilon = J_{t,d/t}$ . We recall that in the proof of Proposition 2.3.9, first we obtain a simple stratum  $[\mathfrak{a}', \beta]$  and a simple character  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta)$ , such that  $\tau(\mathfrak{a}') = \mathfrak{a}'$  and  $\theta' \circ \tau = \theta'^{-1}$ , then we use Lemma 2.3.6 to get  $\beta'$ . In this case we have  $\theta' \in \mathcal{C}(\mathfrak{a}', \beta) \cap \mathcal{C}(\mathfrak{a}', \beta')$ , thus  $J^1(\mathfrak{a}', \beta) = J^1(\mathfrak{a}', \beta')$  as the maximal pro- $p$ -subgroup of the normalizer of  $\theta$ . Moreover from our construction of  $[\mathfrak{a}', \beta]$ , for  $T$  the maximal tamely ramified subextension of  $E/F$  with  $E = F[\beta]$  and for  $\iota : T \hookrightarrow M_t(F)$  the chosen  $J$ -symmetric embedding, we have

$$T = \{\mathrm{diag}(\iota(x_t), \dots, \iota(x_t)) \in M_{d/t}(M_t(F)) = M_d(F)|x_t \in T\}.$$

Thus we get

$$\varepsilon T^\times = \{\mathrm{diag}(J_t \iota(x_t), \dots, J_t \iota(x_t)) \in M_{d/t}(M_t(F)) = M_d(F)|x_t \in T^\times\}.$$

We write  $T'$  for the maximal tamely ramified subextension of  $E'/F$  with  $E' = F[\beta']$ . By Lemma 2.2.19 with  $E = E'$  and  $L = T'$ , the embedding  $T' \hookrightarrow E'$  induces an isomorphism

$$T'^{\times}/T'^{\times 2} \cong E'^{\times}/E'^{\times 2}.$$

Thus for any  $x \in E'^{\times}$ , there exists  $y \in E'^{\times}$  such that  $xy^2 \in T'^{\times}$ . Thus

$$\varepsilon x = {}^t y^{-1} \varepsilon (xy^2) y^{-1},$$

where we use the fact that  $\varepsilon^{-1} {}^t y^{-1} \varepsilon = y^{-1}$ . Thus every element in  $\varepsilon E'^{\times}$  is similar to an element in  $\varepsilon T'^{\times}$ . Thus to finish the proof, we only need to show that any element in  $\varepsilon T'^{\times}$  is similar to an element in  $\varepsilon T^{\times}$ .

Using [BH14b], Proposition 2.6, there exists  $j \in J^1(\mathfrak{a}', \beta) = J^1(\mathfrak{a}', \beta')$  such that  $T' = T^j$ . For any  $x \in T^{\times}$ , we have  $j^{-1} x j \in T'^{\times}$ . Thus we get  $\tau(x) = x^{-1}$  and  $\tau(j^{-1} x j) = (j^{-1} x j)^{-1}$ , which implies that

$$k x k^{-1} = \tau(x^{-1}) = x,$$

where  $k := \tau(j) j^{-1} \in C \cap J^1(\mathfrak{a}', \beta) = J^1(\mathfrak{c}', \beta) \subset U^1(\mathfrak{c}')$  with  $C = Z_{M_d(F)}(T) = M_{d/t}(T)$ . Moreover we have

$$\varepsilon j^{-1} x j = (\varepsilon j^{-1} \varepsilon^{-1}) \varepsilon x j = {}^t \tau(j) \varepsilon x j = {}^t j {}^t k \varepsilon x j,$$

So we only need to show that  ${}^t k \varepsilon x$  is similar to  $\varepsilon x$ .

We denote by  $\tau'$  the restriction of  $\tau$  to  $C^{\times}$ , thus by definition  $\tau'(c) = {}^t c c^{-1}$  for any  $c \in C^{\times}$ , where  ${}^t c$  denotes the transpose on  $C$ . Since  $\tau(k)k = 1$ , we have  $\tau'(k)k = 1$ , or equivalently  ${}^t c k = k$ . Since  $\det_C(k) \in 1 + \mathfrak{p}_T \subset T^{\times 2}$  and  $\text{Hasse}_T(k) = 1$  by Proposition 2.3.7 and Lemma 2.2.9, by Proposition 2.2.1, there exists  $m \in C^{\times}$  such that

$${}^t c m m = {}^t c k \quad \text{or equivalently} \quad \tau(m)^{-1} m = \tau(k)^{-1},$$

where we denote by  $\det_C$  the determinant with respect to  $C = M_{d/t}(T)$  and by  $\text{Hasse}_T$  the Hasse invariant with respect to  $T$ . Thus

$${}^t k \varepsilon x = \varepsilon \tau(k)^{-1} x = \varepsilon \tau(m)^{-1} m x = {}^t m \varepsilon m x = {}^t m \varepsilon x m,$$

which means that  ${}^t k \varepsilon x$  is similar to  $\varepsilon x$ . So we finish the proof when  $\varepsilon = J_{t,d/t}$ .

For the general case, since  $\tau_{\varepsilon}$  and  $\tau_{J_{t,d/t}}$  are  $G$ -conjugate, we may choose  $\varepsilon$  up to multiplying an element in  $F^{\times}$ , such that  $\varepsilon = {}^t g J_{t,d/t} g$  with a certain  $g \in G$ . We assume that  $[\mathfrak{a}', \beta']$  and  $\theta'$  satisfy this lemma for  $\tau = \tau_{J_{t,d/t}}$ . We choose  $[\mathfrak{a}'', \beta''] := [\mathfrak{a}'^g, \beta'^g]$ ,  $\theta'' = \theta'^g$ , and by Lemma 2.3.5 we have

$$\tau_{\varepsilon}(\mathfrak{a}'') = \mathfrak{a}'', \quad \theta'' \circ \tau_{\varepsilon} = \theta''^{-1} \quad \text{and} \quad \tau_{\varepsilon}(\beta'') = \beta''^{-1}.$$

Moreover we have

$$\varepsilon E''^{\times} = {}^t g J_{t,d/t} g E'^{\times g} = {}^t g (J_{t,d/t} E'^{\times}) g,$$

which means that each element in  $\varepsilon E''^{\times}$  is similar to an element in  $J_{t,d/t} E'^{\times}$ . Thus  $[\mathfrak{a}'', \beta'']$ ,  $\theta''$  satisfy the condition of the lemma when  $\tau = \tau_{\varepsilon}$ . □

**Remark 2.3.13.** From the proof we may further observe that when  $\varepsilon = J_{t,d/t}$ , if we identify  $T$  with the maximal tamely ramified subextension of  $E'$  over  $F$  via an  $F$ -embedding, then  $x$  and  $x_t$  are in the same class of  $T^{\times}/T^{\times 2} \cong E'^{\times}/E'^{\times 2}$  given by Lemma 2.2.19 for  $E = E'$  and  $L = T$ .

Finally we state and prove the following corollary, saying the results for calculating Hasse invariant in §2.2.3 can be generalized to the case where  $E/F$  is not necessarily tamely ramified, once  $E$  is well-chosen.

**Corollary 2.3.14.** *For  $\varepsilon = J_d$  and  $[\mathfrak{a}', \beta']$ ,  $\theta'$  constructed in Lemma 2.3.12, the results in Lemma 2.2.15, Lemma 2.2.16, Corollary 2.2.17, Lemma 2.2.18, Lemma 2.2.20, Lemma 2.2.21 hold for  $E = E'$ .*

*Proof.* Since all the proofs are similar, we only prove Lemma 2.2.20 as an example.

First of all when  $d$  is even, by direct calculation and Lemma 2.2.9 we have  $\det(J_{t,d/t}) = \det(J_d)$  and  $\text{Hasse}(J_{t,d/t}) = \text{Hasse}(J_d) = 1$ . Thus  $J_{t,d/t}$  is similar to  $J_d$ . Using this fact and Remark 2.3.13, we deduce that when  $\varepsilon = J_d$ , we may assume  $x$  and  $x_t$  in the result of Lemma 2.3.12 to be in the same class of  $E'^{\times}/E'^{\times 2} \cong T^{\times}/T^{\times 2}$ , where we identify  $T$  with the maximal tamely ramified subextension of  $E'$  over  $F$  via an embedding. In particular, when  $x = \varpi_{E'}$  is a uniformizer of  $E'$ , we may assume  $x_t = \varpi_T$  to be a uniformizer of  $T$  in the same class as that of  $\varpi_{E'}$ , and when  $x = \varpi_{E'}\epsilon_0$  with  $\epsilon_0$  an element in  $\mathfrak{o}_{E'}^{\times} \setminus \mathfrak{o}_{E'}^{\times 2}$ , we may also assume  $x_t = \varpi_T\epsilon'_0$  with  $\epsilon'_0$  an element in  $\mathfrak{o}_T^{\times} \setminus \mathfrak{o}_T^{\times 2}$ . Thus using Lemma 2.3.12 for  $x = \varpi_{E'}$  and  $x = \varpi_{E'}\epsilon_0$ , we have

$$\begin{aligned} & \text{Hasse}(\text{diag}(J_d\varpi_{E'}, \dots, J_d\varpi_{E'}, J_d\varpi_{E'}\epsilon_0)) \\ &= \text{Hasse}(\text{diag}(\text{diag}(J_t\varpi_T, \dots, J_t\varpi_T), \dots, \text{diag}(J_t\varpi_T, \dots, J_t\varpi_T), \text{diag}(J_t\varpi_T\epsilon'_0, \dots, J_t\varpi_T\epsilon'_0))) \\ &= \text{Hasse}(\text{diag}(\text{diag}(J_t\varpi_T, \dots, J_t\varpi_T, J_t\varpi_T\epsilon'_0), \dots, \text{diag}(J_t\varpi_T, \dots, J_t\varpi_T, J_t\varpi_T\epsilon'_0))) \\ &= \text{Hasse}(\text{diag}(J_t\varpi_T, \dots, J_t\varpi_T, J_t\varpi_T\epsilon'_0))^{n/t} \\ &= \text{Hasse}(\text{diag}(J_t\varpi_T, \dots, J_t\varpi_T, J_t\varpi_T\epsilon'_0)), \end{aligned}$$

where the matrix in the third line is the direct sum of  $n/t$  copies of  $\text{diag}(J_t\varpi_T, \dots, J_t\varpi_T, J_t\varpi_T\epsilon'_0) \in M_{tm}(F)$ , and for the fourth line we use the fact that  $\det(\text{diag}(J_t\varpi_T, \dots, J_t\varpi_T, J_t\varpi_T\epsilon'_0))$  is of even order in  $F^{\times}$  and Corollary 2.2.11, and for the final line we use the fact that  $n/t$  is odd. Thus we may use the tamely ramified case to finish the proof.

When  $d$  is odd, if  $\det(J_{t,d/t}) = \det(J_d)$  we can still follow the proof above verbatim. If  $\det(J_{t,d/t}) = \det(-J_d)$ , we deduce that  $J_{t,d/t}$  is similar to  $-J_d$ . Thus following the above proof, when  $x = \varpi_{E'}$  (resp.  $\varpi_{E'}\epsilon_0$ ) we may choose  $x_t = -\varpi_T$  (resp.  $-\varpi_T\epsilon'_0$ ), where  $\varpi_{E'}$ ,  $\varpi_T$ ,  $\epsilon_0$ ,  $\epsilon'_0$  are defined as above. Thus for  $\varpi'_T = -\varpi_T$  as a uniformizer of  $T$  and using the same calculation, we have

$$\text{Hasse}(\text{diag}(J_d\varpi_{E'}, \dots, J_d\varpi_{E'}, J_d\varpi_{E'}\epsilon_0)) = \text{Hasse}(\text{diag}(J_t\varpi'_T, \dots, J_t\varpi'_T, J_t\varpi'_T\epsilon'_0)).$$

And still we use the tamely ramified case to finish the proof. □

### 2.3.3 The general case

In this subsection, we finish the proof of Theorem 2.3.3. For  $[\mathfrak{a}, \beta]$  and  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  given as in the theorem, we choose  $\beta_0 \in M_d(F)$  such that there exists an  $F$ -algebra isomorphism  $F[\beta_0] \rightarrow F[\beta]$  which maps  $\beta_0$  to  $\beta$ . Let  $\mathfrak{a}_0$  be the unique hereditary order of  $M_d(F)$  normalized by  $\beta_0$ . Thus  $[\mathfrak{a}_0, \beta_0]$  is a simple stratum of  $M_d(F)$  and we let  $\theta_0 = t_{\mathfrak{a}, \mathfrak{a}_0}^{\beta, \beta_0}(\theta)$  be the transfer of  $\theta$  as a simple character with respect to  $[\mathfrak{a}_0, \beta_0]$ . Using Proposition 2.3.9, for  $\tau_{J_d}$  the involution on  $\text{GL}_d(F)$ , there exist a simple stratum  $[\mathfrak{a}'_0, \beta'_0]$  and a simple character  $\theta'_0 \in \mathcal{C}(\mathfrak{a}'_0, \beta'_0)$  such that  $(\mathfrak{a}'_0, \theta'_0)$  is  $\text{GL}_d(F)$ -conjugate to  $(\mathfrak{a}_0, \theta_0)$  with the following property:

- (1)  $\tau_{J_d}(\mathfrak{a}'_0) = \mathfrak{a}'_0$  and  $\tau_{J_d}(H^1(\mathfrak{a}'_0, \beta'_0)) = H^1(\mathfrak{a}'_0, \beta'_0)$ ;
- (2)  $\theta'_0 \circ \tau_{J_d} = \theta'^{-1}_0$ ;

- (3)  $\tau_{J_d}(\beta'_0) = \beta'^{-1}_0$ ;  
(4) Corollary 2.3.14 holds.

Now we embed  $M_d(F)$  diagonally in  $M_n(F)$ , which gives an  $F$ -algebra homomorphism  $\iota' : F[\beta'_0] \hookrightarrow M_n(F)$ . We write  $\beta' = \iota'(\beta'_0) = \beta'_0 \otimes \dots \otimes \beta'_0$  and  $E' = F[\beta']$ . The centralizer of  $E'$  in  $M_n(F)$ , denoted by  $B'$ , is naturally identified with  $M_m(E')$ . Let  $\mathfrak{b}'$  be a maximal standard hereditary order in  $B'$  which may be identified with  $M_m(\mathfrak{o}_{E'})$ , and let  $\mathfrak{a}'$  be the unique hereditary order in  $M_n(F)$  normalized by  $E'^{\times}$  such that  $\mathfrak{a}' \cap B' = \mathfrak{b}'$ . Then we obtain a simple stratum  $[\mathfrak{a}', \beta']$  in  $M_n(F)$ . Let  $\theta' = t_{\mathfrak{a}'_0, \mathfrak{a}'_0}^{\beta'_0, \beta'_0}(\theta'_0) \in \mathcal{C}(\mathfrak{a}'_0, \beta'_0)$  be the transfer of  $\theta'_0$ .

We denote by  $T'$  the maximal tamely ramified subextension of  $E'/F$  and we denote by  $T'_m$  an unramified extension of degree  $m$  over  $T'$ . We denote by  $E'_m = T'_m E'$  an unramified extension of degree  $m$  over  $E'$ . Since  $E'/T'$  and  $E'_m/T'_m$  are totally wildly ramified, it is easy to check that

$$N_{T'/F}(T'^{\times})F^{\times 2}/F^{\times 2} = N_{E'/F}(E'^{\times})F^{\times 2}/F^{\times 2} \quad (2.3.1)$$

and

$$N_{T'_m/F}(T_m'^{\times})F^{\times 2}/F^{\times 2} = N_{E'_m/F}(E_m'^{\times})F^{\times 2}/F^{\times 2}. \quad (2.3.2)$$

The latter group is a subgroup of the former one, and both of them are subgroups of  $F^{\times}/F^{\times 2}$ , which is a group of order four.

We consider the following special orthogonal involutions  $\tau = \tau_{\varepsilon}$  such that

**Case (i)** If  $N_{T'_m/F}(T_m'^{\times})F^{\times 2}/F^{\times 2} = F^{\times}/F^{\times 2}$ , then  $\varepsilon = J_{d,m} = \text{diag}(J_d, \dots, J_d) \in M_m(M_d(F)) = M_n(F)$ ;

**Case (ii)** If  $N_{T'_m/F}(T_m'^{\times})F^{\times 2}/F^{\times 2}$  is a subgroup of  $F^{\times}/F^{\times 2}$  of order two, we consider the following two cases:

**(ii.a)** If  $2|m$ , then  $\varepsilon$  equals  $J_{d,m}$  or  $\text{diag}(J_d, \dots, J_d, J_d \epsilon)$ , where  $\epsilon \in \mathfrak{o}_{E'}^{\times}$ ;

**(ii.b)** If  $2 \nmid m$ , then  $\varepsilon$  equals  $J_{d,m}$  or  $\text{diag}(J_d \epsilon, \dots, J_d \epsilon)$ , where  $\epsilon \in E'^{\times}$ ;

**Case (iii)** If  $N_{T'_m/F}(T_m'^{\times})F^{\times 2}/F^{\times 2} = \{1\}$ , then  $\varepsilon$  equals  $J_{d,m}$  or  $\text{diag}(J_d \varpi_{E'}, \dots, J_d \varpi_{E'} \epsilon)$ , where  $\epsilon \in \mathfrak{o}_{E'}^{\times}$  and  $\varpi_{E'}$  is a certain uniformizer of  $E'$ . We distinguish the following two cases:

**(iii.a)**  $N_{T'/F}(T'^{\times})/F^{\times 2} = \{1\}$ ;

**(iii.b)**  $N_{T'/F}(T'^{\times})F^{\times 2}/F^{\times 2}$  is not trivial.

We want to check that for  $[\mathfrak{a}', \beta']$ ,  $\theta'$  and  $\tau = \tau_{\varepsilon}$  given as above, the conditions (1), (2) and (3) in Theorem 2.3.3 are satisfied. For each  $\varepsilon$  above, we may write  $\varepsilon = J_{d,m} a_{\varepsilon} \varepsilon_{E'}$ , where  $a_{\varepsilon} \in E'^{\times}$  and  $\varepsilon_{E'} = \text{diag}(1, \dots, 1, \epsilon) \in \text{GL}_m(E')$  with  $\epsilon \in \mathfrak{o}_{E'}^{\times}$ . Thus for  $x = (x_{ij})_{ij} \in \text{GL}_m(E')$ , we have

$$\begin{aligned} \tau(x) &= ((J_{d,m} a_{\varepsilon} \varepsilon_{E'})^{-1} {}^t((x_{ij})_{ij}) J_{d,m} a_{\varepsilon} \varepsilon_{E'})^{-1} = ((\varepsilon_{E'}^{-1} a_{\varepsilon}^{-1} ((J_d^{-1} {}^t x_{ji} J_d)_{ij}) a_{\varepsilon} \varepsilon_{E'})^{-1} \\ &= (\varepsilon_{E'}^{-1} a_{\varepsilon}^{-1} ((x_{ji})_{ij}) a_{\varepsilon} \varepsilon_{E'})^{-1} = (\varepsilon_{E'}^{-1} ({}^t x) \varepsilon_{E'})^{-1} = \tau'(x), \end{aligned} \quad (2.3.3)$$

where we write  ${}^t x$  for the transpose on  $\text{GL}_m(E')$  and  $\tau' := \tau_{\varepsilon_{E'}}$  for the orthogonal involution defined on  $\text{GL}_m(E')$  corresponding to  $\varepsilon_{E'}$ , and we use the fact that the embedding  $E' \hookrightarrow M_d(F)$  is  $J_d$ -symmetric and  $a_{\varepsilon}$  commutes with elements in  $\text{GL}_m(E')$ . Thus we proved that the restriction of  $\tau$  to  $\text{GL}_m(E')$  equals  $\tau'$  as an orthogonal involution on  $\text{GL}_m(E')$ . In particular, since  $\epsilon$  is an element in  $E'$ , we know that  $\varepsilon_{E'}$  commutes with elements in  $E'$  and we have  $\tau(\beta') = \beta'^{-1}$ . Thus condition (3) is verified.

Since  $\mathfrak{b}'$  is a maximal standard hereditary order in  $B'$  which may be identified with  $M_m(\mathfrak{o}_{E'})$ , it is  $\tau'$ -stable. Thus from our assumption of  $\tau$  and construction of  $\mathfrak{a}'$ , we deduce that  $\mathfrak{a}'$  is  $\tau$ -stable. By definition  $H^1(\mathfrak{a}', \beta')$  is  $\tau$ -stable, which means that condition (1) is verified.

Let  $M$  be the standard Levi subgroup of  $G$  isomorphic to  $\text{GL}_d(F) \times \dots \times \text{GL}_d(F)$ . Let  $P$  be the standard parabolic subgroup of  $G$  generated by  $M$  and upper triangular matrices, and let  $N$  be its unipotent radical. Let  $N^-$  be the unipotent radical of the parabolic subgroup opposite to  $P$  with

respect to  $M$ . By [SS08], Théorème 2.17, we have

$$H^1(\alpha', \beta') = (H^1(\alpha', \beta') \cap N^-) \cdot (H^1(\alpha', \beta') \cap M) \cdot (H^1(\alpha', \beta') \cap N), \quad (2.3.4)$$

$$H^1(\alpha', \beta') \cap M = H^1(\alpha'_0, \beta'_0) \times \dots \times H^1(\alpha'_0, \beta'_0). \quad (2.3.5)$$

By *loc. cit.*, the character  $\theta'$  is trivial on  $H^1(\alpha', \beta') \cap N^-$  and  $H^1(\alpha', \beta') \cap N$ , and the restriction of  $\theta'$  to  $H^1(\alpha', \beta') \cap M$  equals  $\theta'_0 \otimes \dots \otimes \theta'_0$ . We have

$$\theta' \circ \tau|_{H^1(\alpha', \beta') \cap N^-} = \theta' \circ \tau|_{H^1(\alpha', \beta') \cap N} = \theta'^{-1}|_{H^1(\alpha', \beta') \cap N^-} = \theta'^{-1}|_{H^1(\alpha', \beta') \cap N} = 1.$$

Moreover since  $\tau = \tau_\varepsilon$  with

$$\varepsilon = \text{diag}(J_d, \dots, J_d) \text{ or } \text{diag}(J_d \varepsilon, \dots, J_d \varepsilon) \text{ or } \text{diag}(J_d, \dots, J_d, J_d \varepsilon) \text{ or } \text{diag}(J_d \varpi_{E'}, \dots, J_d \varpi_{E'}, J_d \varpi_{E'} \varepsilon),$$

and since  $\varepsilon$  and  $\varpi_{E'}$  normalize  $\theta'_0$ , we have

$$\theta' \circ \tau|_{H^1(\alpha', \beta') \cap M} = \theta'_0 \circ \tau_{J_d} \otimes \dots \otimes \theta'_0 \circ \tau_{J_d} = \theta'^{-1} \otimes \dots \otimes \theta'^{-1} = \theta'^{-1}|_{H^1(\alpha', \beta') \cap M}.$$

Thus by equation (2.3.4), we have  $\theta' \circ \tau = \theta'^{-1}$ , which is the condition (2). Thus for those special orthogonal involutions, we finish the proof.

Finally we show that for a given orthogonal involution  $\tau$  and the corresponding orthogonal group  $H = G^\tau$  satisfying the condition of Theorem 2.3.3,  $\tau$  is conjugate to one of the orthogonal involutions mentioned in **Case (i)**, **(ii)** or **(iii)**. We consider them separately.

**Case (i)** By definition,

$$N_{E'_m/F}(E'_m{}^\times)F^{\times 2}/F^{\times 2} = N_{T'_m/F}(T'_m{}^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2} \quad (2.3.6)$$

then using Proposition 2.2.14 for  $E = T'_m$ , we deduce that  $[T'_m : F]$  is odd, thus  $n = [E'_m : F]$  is odd. By Lemma 2.2.9, we have

$$\text{Hasse}(J_n) = \text{Hasse}(-J_n) = \text{Hasse}(J_{d,m}) = 1.$$

And moreover

$$\det(J_{d,m}) = \det(J_n) \quad \text{or} \quad \det(J_{d,m}) = \det(-J_n).$$

So by Proposition 2.2.1,  $J_{d,m}$  is similar to  $J_n$  or  $-J_n$ , which means that  $\tau_{J_n}$  and  $\tau_{J_{d,m}}$  are in the same  $G$ -orbit.

**Case (ii)** By Lemma 2.2.9, we have

$$\text{Hasse}(J_n) = \text{Hasse}(J_{d,m}) = 1.$$

**(ii.a)** Since  $T'_m/T'$  is unramified and  $m$  is even, we get

$$N_{T'_m/F}(T'_m{}^\times)F^{\times 2}/F^{\times 2} = N_{T'/F}(\mathfrak{o}_{T'}^\times)F^{\times 2}/F^{\times 2} = N_{T'_m/F}(\mathfrak{o}_{T'_m}^\times)F^{\times 2}/F^{\times 2}.$$

Thus using equation (2.3.1) and (2.3.2) we know that

$$N_{E'_m/F}(\mathfrak{o}_{E'_m}^\times)F^{\times 2}/F^{\times 2} = N_{T'_m/F}(\mathfrak{o}_{T'_m}^\times)F^{\times 2}/F^{\times 2} = N_{E'/F}(\mathfrak{o}_{E'}^\times)F^{\times 2}/F^{\times 2} = N_{T'/F}(\mathfrak{o}_{T'}^\times)F^{\times 2}/F^{\times 2}$$

is a subgroup of  $F^\times/F^{\times 2}$  of order two. Thus there exists  $\varepsilon_0 \in \mathfrak{o}_{E'}^\times$  such that the image of  $N_{E'/F}(\varepsilon_0)$  in  $N_{E'/F}(\mathfrak{o}_{E'}^\times)F^{\times 2}/F^{\times 2}$  is nontrivial. From now on we fix one such  $\varepsilon_0$ .

**(ii.a.1)** If either of the three cases is true:

- $2|d$ ;
- $2 \nmid d$  and  $4|m$ ;
- $2 \nmid d$ ,  $4 \nmid m$  and  $-1 \in F^{\times 2}$ ,

then by direct calculation we get

$$1 = \text{disc}(J_{d,m}) = \text{disc}(J_n).$$

Thus by Proposition [2.2.1](#),  $J_{d,m}$  is in the same  $G$ -orbit as  $J_n$ , representing the  $G$ -conjugacy class of split orthogonal group. Moreover, we have

$$\det(J_d, \dots, J_d, J_d \epsilon_0) = N_{E'/F}(\epsilon_0)$$

which is non-trivial in  $F^\times/F^{\times 2}$ . Thus by Proposition [2.2.3](#) and Proposition [2.2.5](#), we know that  $\tau_\epsilon$  corresponds to the  $G$ -conjugacy class of orthogonal groups mentioned in Theorem [2.3.3](#), which is quasisplit but not split.

**(ii.a.2)** If  $2 \nmid d$ ,  $4 \nmid m$  and  $-1 \notin F^{\times 2}$ , we get

$$N_{T'_m/F}(T'_m{}^\times)F^{\times 2}/F^{\times 2} = \{1, -1\}.$$

By direct calculation we get

$$\det(\text{diag}(J_d, \dots, J_d, -J_d)) = \det(J_n) = -1$$

and

$$\det(J_{d,m}) = 1.$$

Thus if we further choose  $\epsilon = -1$  and  $\varepsilon = \text{diag}(J_d, \dots, J_d, -J_d)$ , then by Proposition [2.2.3](#) and Proposition [2.2.5](#),  $\tau_\varepsilon$  and  $\tau_{J_{d,m}}$  correspond to the two  $G$ -conjugacy classes of orthogonal groups respectively mentioned in Theorem [2.3.3](#), where the former class is split, and the latter class is quasisplit but not split.

**(ii.b)** Since  $m$  is odd, we deduce that

$$N_{T'_m/F}(T'_m{}^\times)F^{\times 2}/F^{\times 2} = N_{T'/F}(T'{}^\times)F^{\times 2}/F^{\times 2} = N_{E'/F}(E'{}^\times)F^{\times 2}/F^{\times 2}$$

and  $d$  is even by Proposition [2.2.14](#) with  $E = T'$ . We fix  $\epsilon \in N_{E'/F}(E'{}^\times)$  whose image in  $F^\times/F^{\times 2}$  is non-trivial. By direct calculation we get

$$\det(J_{d,m}) = (-1)^{md(d-1)/2} = (-1)^{md(n-1)/2} = \det(J_n).$$

Thus by Proposition [2.2.1](#),  $J_{d,m}$  is in the same  $G$ -orbit as  $J_n$ , representing the  $G$ -conjugacy class of split orthogonal group. Moreover, we have

$$\det(J_d \epsilon, \dots, J_d \epsilon) = N_{E'/F}(\epsilon)^m$$

which is non-trivial in  $F^\times/F^{\times 2}$ . Thus by Proposition [2.2.3](#) and Proposition [2.2.5](#),  $\tau_\epsilon$  correspond to the  $G$ -conjugacy class of orthogonal groups mentioned in Theorem [2.3.3](#), which is quasisplit but not split.

**Case (iii)** First of all since

$$N_{E'_m/F}(E'_m{}^\times)F^{\times 2}/F^{\times 2} = N_{T'_m/F}(T'_m{}^\times)F^{\times 2}/F^{\times 2} = \{1\},$$

by Proposition [2.2.14](#) with  $E = T'_m$  we know that  $4|[T'_m : F]$ . Thus  $4|[E'_m : F] = n$ . By direct calculation, we have

$$\det(J_{d,m}) = \det(J_n) = 1.$$

Moreover, by Lemma [2.2.9](#) we get

$$\text{Hasse}(J_{d,m}) = \text{Hasse}(J_n) = 1.$$

Thus by Proposition [2.2.1](#),  $J_{d,m}$  is in the same  $G$ -orbit as  $J_n$ , representing the  $G$ -conjugacy class of split orthogonal group. Thus we only need to show that for  $\varepsilon = \text{diag}(J_d \varpi_{E'}, \dots, J_d \varpi_{E'} \epsilon)$  with  $\varpi_{E'}$  and  $\epsilon \in \mathfrak{o}_E^\times$  well-chosen,  $\tau = \tau_\varepsilon$  corresponds to the non-quasisplit orthogonal group. By direct calculation, we have

$$\det(\varepsilon) = (-1)^{n(d-1)/2} N_{E'/F}(\varpi_{E'})^m N_{E'/F}(\epsilon) = N_{E'/F}(\varpi_{E'})^m N_{E'/F}(\epsilon).$$

(iii.a) Since

$$N_{E'/F}(E'^\times) F^{\times 2} / F^{\times 2} = N_{T'/F}(T'^\times) F^{\times 2} / F^{\times 2} = \{1\},$$

$\det(\varepsilon)$  is trivial as an element in  $F^\times / F^{\times 2}$ . Thus we only need to choose  $\epsilon$  such that  $\text{Hasse}(\varepsilon) = -1$ . By Lemma [2.2.15](#) and Corollary [2.3.14](#), we may choose  $\varpi_{E'}$  and  $\epsilon_0$  such that  $\text{Hasse}(J_d \varpi_{E'}) = 1$  and  $\text{Hasse}(J_d \varpi_{E'} \epsilon_0) = -1$ . Then using Corollary [2.2.11](#) and the fact that  $\det(J_d \varpi_{E'})$ ,  $\det(J_d \varpi_{E'} \epsilon_0) \in F^{\times 2}$ , we get

$$\text{Hasse}(\varepsilon) = \text{Hasse}(J_d \varpi_{E'})^{m-1} \text{Hasse}(J_d \varpi_{E'} \epsilon_0) = -1.$$

(iii.b) Since  $N_{E'/F}(E'^\times) F^{\times 2} / F^{\times 2}$  is not trivial and  $N_{E'_m/F}(E'_m{}^\times) F^{\times 2} / F^{\times 2}$  is trivial,  $m$  is even and there exists a uniformizer  $\varpi'_F$  of  $F$  such that

$$N_{E'/F}(E'^\times) F^{\times 2} / F^{\times 2} = \{1, \varpi'_F\}.$$

Thus

$$\det(\varepsilon) = N_{E'/F}(\varpi_{E'})^m N_{E'/F}(\epsilon) \equiv N_{E'/F}(\epsilon) \pmod{F^{\times 2}}.$$

Since  $N_{E'/F}(\epsilon) \in \mathfrak{o}_F^\times \cap N_{E'/F}(E'^\times)$ , its image in  $F^\times / F^{\times 2}$  is trivial, that is,  $\text{disc}(\varepsilon) = 1$ . So as in (iii.a), we only need to show that  $\text{Hasse}(\varepsilon) = -1$ . Fix  $\epsilon_0 \in \mathfrak{o}_{E'}^\times \setminus \mathfrak{o}_{E'}^{\times 2}$ , by Corollary [2.2.17](#) and Corollary [2.3.14](#), we may choose  $\epsilon$  equals 1 or  $\epsilon_0$ , such that

$$\text{Hasse}(\varepsilon) = \text{Hasse}(\text{diag}(J_d \varpi_{E'}, \dots, J_d \varpi_{E'}, J_d \varpi_{E'} \epsilon)) = -1.$$

So we finish the discussion for (iii.b).

Thus for  $H = G^\tau$  given as an orthogonal group in Theorem [2.3.3](#) with  $\tau = \tau_\varepsilon$ , we have shown that  $\tau$  is  $G$ -conjugate to one of the special orthogonal involutions mentioned in **Case (i)**, **(ii)** or **(iii)**. Furthermore, we may change  $\varepsilon$  up to multiplying by an element in  $E'^\times$  such that  $\varepsilon$  is similar to one of the special symmetric matrices mentioned in **Case (i)**, **(ii)** or **(iii)**. Using Lemma [2.3.5](#) and the special cases proved, we end the proof of Theorem [2.3.3](#).

**Remark 2.3.15.** In the proof of Theorem [2.3.3](#), we actually showed that for  $\tau$  as an involution in **Case (i)**, **(ii)** or **(iii)**, the choices of  $[\alpha', \beta']$  and  $\theta'$  are the same. Moreover,  $E = E' = F[\beta']$  satisfies Corollary [2.3.14](#).



## 2.4 Distinguished type theorem and the orbits of distinguished type

Let  $\pi$  be a supercuspidal representation of  $G$ , let  $T$  be a tame parameter field of  $\pi$  and let  $T_m$  be the unramified extension of degree  $m$  over  $T$ , where  $n = md$  is determined by  $\pi$  as before. From Theorem 2.3.1, Theorem 2.3.2, Theorem 2.3.3 and Remark 2.3.15, there exist a simple stratum  $[\mathfrak{a}, \beta]$ , a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  attached to  $\pi$  and a simple type  $(\mathbf{J}, \Lambda)$  containing  $\theta$  and compactly inducing  $\pi$  such that

$$(1) \tau_0(\mathfrak{a}) = \mathfrak{a} \text{ and } \tau_0(H^1(\mathfrak{a}, \beta)) = H^1(\mathfrak{a}, \beta);$$

$$(2) \theta \circ \tau_0 = \theta^{-1};$$

$$(3) \tau_0(\beta) = \beta^{-1};$$

$$(4) \tau_0(\mathbf{J}) = \mathbf{J} \text{ and } \Lambda^{\tau_0} = \Lambda^\vee;$$

$$(5) \text{Lemma 2.2.15, Lemma 2.2.16, Corollary 2.2.17, Lemma 2.2.18, Lemma 2.2.20, Lemma 2.2.21}$$

hold for  $E = F[\beta]$ .

Here we assume  $\tau_0 = \tau_{\varepsilon_0}$ , where  $\varepsilon_0$  is a symmetric matrix in  $M_n(F)$  as follows:

**Case (i)** If  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$ , then  $\varepsilon_0 = J_{d,m}$ .

**Case (ii)** If  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  is a subgroup of  $F^\times/F^{\times 2}$  of order two, we consider the following two cases:

**(ii.a)** If  $2|m$ , then  $\varepsilon_0$  equals  $J_{d,m}$  or  $\text{diag}(J_d, \dots, J_d, J_d\varepsilon_0)$ , where  $\varepsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$ ;

**(ii.b)** If  $2 \nmid m$ , then  $\varepsilon_0$  equals  $J_{d,m}$  or  $\text{diag}(J_d\varepsilon, \dots, J_d\varepsilon)$ , where  $\varepsilon$  is chosen to be either a uniformizer in  $E$  or an element in  $\mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$ , such that  $N_{E/F}(\varepsilon) \in N_{T_m/F}(T_m^\times) - F^{\times 2}$ .

**Case (iii)** If  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = \{1\}$ , we consider the following two cases:

**(iii.a)** If  $N_{T/F}(T^\times)F^{\times 2}/F^{\times 2} = \{1\}$ , then  $\varepsilon_0$  equals  $J_{d,m}$  or  $\text{diag}(J_d\varpi_E, \dots, J_d\varpi_E, J_d\varpi_E\varepsilon_0)$ , where  $\varepsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$  and  $\varpi_E$  is a uniformizer of  $E$  chosen by Lemma 2.2.15, such that  $\text{Hasse}(J_d\varpi_E) = 1$  and  $\text{Hasse}(J_d\varpi_E\varepsilon_0) = -1$ ;

**(iii.b)** If  $N_{T/F}(T^\times)F^{\times 2}/F^{\times 2}$  is not trivial, then  $\varepsilon_0$  equals  $J_{d,m}$  or  $\text{diag}(J_d\varpi_E, \dots, J_d\varpi_E, J_d\varpi_E\varepsilon_0)$  where  $\varepsilon_0 \in \mathfrak{o}_E^\times$  and  $\varpi_E$  is a certain uniformizer of  $E$ , such that  $\text{Hasse}(\text{diag}(J_d\varpi_E, \dots, J_d\varpi_E, J_d\varpi_E\varepsilon_0)) = -1$ .

Thus in different cases,  $G^{\tau_0}$  represents all possible  $G$ -conjugacy classes of orthogonal groups mentioned in Theorem 2.3.3 respectively.

From now on until the end of this section, we fix  $\varepsilon_0$ ,  $[\mathfrak{a}, \beta]$ ,  $\theta$  and  $(\mathbf{J}, \Lambda)$  as above. By (2.3.3) if we restrict  $\tau_0$  to  $B^\times = \text{GL}_m(E)$ , it becomes an orthogonal involution  $\tau_{\varepsilon_0 E}$  with respect to  $E$ , where  $\varepsilon_0 E$  equals  $I_m$  or  $\text{diag}(1, \dots, 1, \varepsilon_0)$  with  $\varepsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$ . We fix  $\varepsilon$  a symmetric matrix in  $\text{GL}_n(F)$  and  $\tau = \tau_\varepsilon$  an orthogonal involution on  $G$ . We write  $u = \varepsilon_0^{-1}\varepsilon$ , then by direct calculation we get

$$\tau(x) = u^{-1}\tau_0(x)u \quad \text{for any } x \in G$$

and

$$u\tau_0(u) = \varepsilon_0^{-1}\varepsilon\varepsilon_0^{-1}t\varepsilon_0{}^t\varepsilon^{-1}\varepsilon_0 = 1. \quad (2.4.1)$$

We write  $\gamma = u\tau(g)g^{-1}$ . We first state the following main theorem of this section:

**Theorem 2.4.1.** *For  $\pi$  a supercuspidal representation and  $G^\tau$  an orthogonal group of  $G$ , the representation  $\pi$  is distinguished by  $G^\tau$  if and only if there exists a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  of  $\pi$  such that  $\text{Hom}_{\mathbf{J} \cap G^\tau}(\Lambda, 1) \neq 0$ .*

The ‘‘if’’ part of this theorem is obvious, so we only need to prove the ‘‘only if’’ part of this theorem. We assume  $\pi$  to be distinguished by  $G^\tau$  and we choose  $(\mathbf{J}, \Lambda)$  to be  $\tau_0$ -selfdual as above. By direct calculation, we get

$$\tau(H^1) = \tau_0(H^1)^u = H^{1u}, \quad \theta^\tau \cong (\theta^{\tau_0})^u \cong (\theta^{-1})^u \quad \text{and} \quad \tau(\beta) = (\beta^{-1})^u, \quad (2.4.2)$$

and

$$\tau(\mathbf{J}) = \tau_0(\mathbf{J})^u = \mathbf{J}^u \quad \text{and} \quad \Lambda^\tau \cong (\Lambda^{\tau_0})^u \cong \Lambda^{\vee u}. \quad (2.4.3)$$

Using the Mackey formula and the Frobenius reciprocity, we have

$$0 \neq \text{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

The main step is to prove the following important theorem:

**Theorem 2.4.2.** *For  $g \in G$  such that  $\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ , we have  $\gamma = u\tau(g)g^{-1} \in \mathbf{J}$ .*

Thus for the simple type  $(\mathbf{J}^g, \Lambda^g)$ , we get

$$\tau(\mathbf{J}^g) = \tau_0(\mathbf{J})^{u\tau(g)} = \mathbf{J}^{\gamma g} = \mathbf{J}^g \quad \text{and} \quad (\Lambda^g)^\tau \cong (\Lambda^{\tau_0})^{u\tau(g)} \cong (\Lambda^\vee)^{\gamma g} \cong (\Lambda^g)^\vee,$$

where we use the fact that  $\gamma \in \mathbf{J}$  normalizes  $\mathbf{J}$  and  $\Lambda$ . Thus  $(\mathbf{J}^g, \Lambda^g)$  is what we want, which finishes the “only if” part of Theorem 2.4.1. So from now on, we focus on the proof of Theorem 2.4.2

### 2.4.1 Double cosets contributing to the distinction of $\theta$

In this subsection, we prove the following proposition:

**Proposition 2.4.3.** *For  $g \in G$ , the character  $\theta^g$  is trivial on  $H^{1g} \cap G^\tau$  if and only if  $\gamma \in JB^\times J$ .*

*Proof.* We follow the proof of [Séc19], Lemma 6.5. We choose  $\tau$ ,  $\chi$  and  $H$  in *loc. cit.* to be our  $\tau$ ,  $\theta$  and  $H^1$  respectively. We use the assumptions  $\tau(H^1) = H^{1u}$  and  $\theta \circ \tau = \theta^{-1u}$  to replace the original assumptions  $\tau(H) = H$  and  $\chi \circ \tau = \chi^{-1}$  respectively. And we use  $\gamma = u\tau(g)g^{-1}$  to replace  $\tau(g)g^{-1}$  in *loc. cit.* Finally we notice that  $\gamma$  intertwines  $\theta$  if and only if  $\gamma \in JB^\times J$ . With the replacements and remarks mentioned above, the original proof can be used directly.  $\square$

As a result, for  $g \in G$  such that  $\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \neq 0$ , restricting to  $H^{1g}$  we get  $\theta^g|_{H^{1g} \cap G^\tau} = 1$ , or equivalently  $\gamma \in JB^\times J$ .

### 2.4.2 The double coset lemma

In this section we prove the following double coset lemma:

**Lemma 2.4.4.** *Let  $g \in G$  and let  $\gamma = u\tau(g)g^{-1} \in JB^\times J$ . Then by changing  $g$  with another representative in  $JgG^\tau$ , we may assume  $\gamma \in B^\times$ .*

**Remark 2.4.5.** *By direct calculation, we get*

$$\gamma = u\tau(g)g^{-1} = \varepsilon_0^{-1} t g^{-1} \varepsilon g^{-1} = \tau_0(g) u g^{-1}, \quad (2.4.4)$$

and

$$\tau_0(\gamma)\gamma = g\tau_0(u)\tau_0(g)^{-1}\tau_0(g)ug^{-1} = g\tau_0(u)ug^{-1} = 1. \quad (2.4.5)$$

Since  $\tau_0(J) = J$ , if we change  $g$  with a new representative of  $JgG^\tau$ , the new  $\gamma$  belongs to the same  $J$ - $J$  double coset represented by the original  $\gamma$ , that is, the property  $\gamma \in JB^\times J$  doesn't depend on the choice of  $g$  in the  $J$ - $G^\tau$  double coset.

*Proof.* First of all, we need the following lemma:

**Lemma 2.4.6.** *There exists  $b \in B^\times$  such that  $\gamma \in JbJ$  and  $\tau_0(b)b = 1$ .*

*Proof.* Since  $\mathfrak{b}^\times$  is a maximal order of  $B^\times$ , using the Cartan decomposition for  $B^\times \cong \mathrm{GL}_m(E)$ , we may assume  $\gamma = xcy$  such that  $x, y \in J$  and

$$c = \mathrm{diag}(\varpi_E^{a_1} I_{m_1}, \dots, \varpi_E^{a_r} I_{m_r}), \quad (2.4.6)$$

where  $a_1 > \dots > a_r$  as integers and  $m_1 + \dots + m_r = m$ . By definition of  $\varepsilon_0$ , the restriction of  $\tau_0$  to  $B^\times$  is also an orthogonal involution  $\tau'_0$  defined by

$$\tau'_0(z) = \varepsilon_{0E}^{-1} {}^t z \varepsilon_{0E}^{-1} \quad \text{for any } z \in B^\times,$$

where  ${}^t E$  represents the transpose on  $\mathrm{GL}_m(E)$ . If we write  $b = c\varepsilon_{0E}$ , then by definition we get

$$\tau_0(b)b = \tau'_0(c\varepsilon_{0E})c\varepsilon_{0E} = \varepsilon_{0E}^{-1} {}^t c \varepsilon_{0E}^{-1} c \varepsilon_{0E} = \varepsilon_{0E}^{-1} {}^t c \varepsilon_{0E}^{-1} c \varepsilon_{0E} = \varepsilon_{0E}^{-1} {}^t c \varepsilon_{0E}^{-1} c \varepsilon_{0E} = \varepsilon_{0E}^{-1} \varepsilon_{0E} = 1.$$

So the choice of  $b$  satisfies our conditions. □

Now we write  $\gamma = x'bx$  with  $x, x' \in J$ ,  $b = c\varepsilon_{0E} \in B^\times$  and  $c$  as in (2.4.6). Replacing  $g$  by  $\tau_0(x')^{-1}g$  does not change the double coset  $JgG^\tau$  but changes  $\gamma$  into  $bx\tau_0(x')$ . So we may and will assume that  $\gamma = bx$  with  $x \in J$ .

Write  $K$  for the group  $J \cap J^b$ . Since  $\tau_0(b) = b^{-1}$  and  $\tau_0(J) = J$ , using (2.4.5) we have  $x \in J$  and  $bx b^{-1} = \gamma b^{-1} = \tau_0(\gamma^{-1})\tau_0(b) = \tau_0(x^{-1}) \in J$ , thus  $x \in K$ . Moreover, we have the following corollary of Lemma 2.4.6

**Corollary 2.4.7.** *The map  $\delta_b : k \mapsto b^{-1}\tau_0(k)b$  is an involution on  $K$ .*

For  $a_1 > \dots > a_r$  and  $m_1 + \dots + m_r = m$  as in (2.4.6), and  $M = \mathrm{GL}_{m_1 d}(F) \times \dots \times \mathrm{GL}_{m_r d}(F) \subseteq G$ , let  $P$  be the standard parabolic subgroup of  $G$  generated by  $M$  and upper triangular matrices. Let  $N$  and  $N^-$  be the unipotent radicals of  $P$  and its opposite parabolic subgroup with respect to  $M$ . By definition,  $b$  normalizes  $M$  and we have

$$K = (K \cap N^-) \cdot (K \cap M) \cdot (K \cap N).$$

We have similar properties for the subgroup  $V = K \cap B^\times = U \cap b^{-1}Ub$  of  $B^\times$ :

$$V = (V \cap N^-) \cdot (V \cap M) \cdot (V \cap N),$$

where  $U = U(\mathfrak{b})$  and  $U^1 = J^1 \cap B^\times = U^1(\mathfrak{b})$ . By definition,  $V$  is also fixed by  $\delta_b$ .

**Lemma 2.4.8.** *The subset*

$$K^1 = (K \cap N^-) \cdot (J^1 \cap M) \cdot (K \cap N)$$

*is a  $\delta_b$ -stable normal pro- $p$ -subgroup of  $K$ , and we have  $K = VK^1$ .*

*Proof.* The proof is the same as that in [Séc19], Lemma 6.10. □

**Lemma 2.4.9.** *For  $x \in K$  satisfying  $x\delta_x(y) = 1$ , there exist  $k \in K$  and  $v \in V$  such that*

- (1) *the element  $v$  is in  $\mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \dots \times \mathrm{GL}_{m_r}(\mathfrak{o}_E) \subseteq B^\times$  satisfying  $v\delta_b(v) = 1$ ;*
- (2)  *$\delta_b(k)xk^{-1} \in vK^1$ .*

*Proof.* We may follow the same proof as Lemma 1.6.9, by replacing  $\sigma$  and  $*$  in *loc. cit.* with trivial map and  $^t$ . Noting that instead of considering the three cases separately by using Lemma 1.6.10, Lemma 1.6.11 and Lemma 1.6.12 in *loc. cit.*, there is only one case to consider in our lemma and we only need to use Lemma 1.6.11 in *loc. cit.* to finish the proof.  $\square$

We finish the proof of Lemma 2.4.4. Applying Lemma 2.4.9 gives us  $k \in K$  and  $v \in V$  such that  $bv\tau_0(bv) = 1$  and  $\delta_b(k)xk^{-1} \in vK^1$ . Thus we have  $\tau_0(k)\gamma k^{-1} \in bvK^1$ . Therefore, replacing  $g$  by  $kg$  and  $b$  by  $bv$ , we may assume that  $\gamma$  is written as

$$\gamma = bx, \quad b\tau_0(b) = 1, \quad x \in K^1, \quad b \in \varpi_E^{a_1} \mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \dots \times \varpi_E^{a_r} \mathrm{GL}_{m_r}(\mathfrak{o}_E). \quad (2.4.7)$$

Furthermore, we have  $\delta_b(x)x = 1$ .

Since  $K^1$  is a  $\delta_b$ -stable pro- $p$ -group and  $p$  is odd, the first cohomology set of  $\delta_b$  in  $K^1$  is trivial. Thus  $x = \delta_b(y)y^{-1}$  for some  $y \in K^1$ , hence using (2.4.4) we have  $\gamma = \tau_0(g)ug^{-1} = \tau_0(y)by^{-1}$ . As a result, if we further use  $y^{-1}g$  to replace  $g$ , we get  $\gamma = b \in B^\times$ , which finishes the proof of Lemma 2.4.4.  $\square$

**Remark 2.4.10.** Noting that in [Séc19] and §1.6.1, the corresponding double coset lemma says that  $\gamma \in JB^\times J$  if and only if  $g \in JB^\times G^\tau$ . However in our case if we assume  $\varepsilon = \varepsilon_0$  and  $\gamma = \tau(g)g^{-1} \in JB^\times J$ , then it is possible that  $g$  is not in  $JB^\times G^\tau$ . We will discuss this new phenomenon and calculate all the possible  $J$ - $G^\tau$  cosets in §2.4.7.

### 2.4.3 Distinction of the Heisenberg representation

Let  $\eta$  be the Heisenberg representation of  $J^1$  associated to  $\theta$ , we have the following result as in [Séc19], Proposition 6.12 and Proposition 1.6.13.

**Proposition 2.4.11.** *Given  $g \in G$ , we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G^\tau}(\eta^g, 1) = \begin{cases} 1 & \text{if } \gamma = u\tau(g)g^{-1} \in JB^\times J, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* First we restrict  $\eta^g$  to  $H^{1g}$  which is isomorphic to  $\theta^{g(J^1:H^1)^{1/2}}$ . Using Proposition 2.4.3 when  $\gamma \notin JB^\times J$ , the dimension equals 0.

When  $\gamma \in JB^\times J$ , by Lemma 2.4.4 we may further assume  $\gamma \in B^\times$ . We denote

$$\delta(x) := (\tau(g)g^{-1})^{-1}\tau(x)\tau(g)g^{-1} \quad \text{for } x \in G$$

as an involution on  $G$ , then by definition and (2.4.5) we have

$$\mathrm{Hom}_{G^\tau}(\eta^g, 1) \cong \mathrm{Hom}_{G^\delta}(\eta, 1),$$

and

$$\gamma\delta(\gamma) = \gamma\gamma^{-1}\tau_0(\gamma)\gamma = 1. \quad (2.4.8)$$

Moreover, using (2.4.2) we have

$$\delta(H^1) = (\tau(g)g^{-1})^{-1}H^{1u}\tau(g)g^{-1} = H^{1\gamma} \quad \text{and} \quad \theta \circ \delta = (\theta^{-1})^{u\tau(g)g^{-1}} = (\theta^{-1})^\gamma. \quad (2.4.9)$$

So using Proposition 1.6.14, we finish the proof.  $\square$

### 2.4.4 Distinction of the extension of a Heisenberg representation

Let  $\kappa$  be an irreducible representation of  $\mathbf{J}$  extending  $\eta$ , then there exists a unique representation  $\rho$  of  $\mathbf{J}$  trivial on  $J^1$  up to isomorphism, such that  $\Lambda = \kappa \otimes \rho$ . First of all we have the following proposition:

**Proposition 2.4.12.** *Let  $g \in G$  such that  $\gamma \in JB^\times J$ .*

(1) *There is a unique character  $\chi$  of  $\mathbf{J}^g \cap G^\tau$  trivial on  $J^{1g} \cap G^\tau$  such that*

$$\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1}).$$

(2) *The canonical linear map*

$$\mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \otimes \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi) \rightarrow \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

*is an isomorphism.*

*Proof.* With the aid of Proposition 2.4.11, the proof is the same as that in [Séc19], Lemma 6.20.  $\square$

For  $g \in G$  such that  $\gamma = u\tau(g)g^{-1} = \tau_0(g)ug^{-1} \in JB^\times J$ , using  $u\tau(g) = \tau_0(g)u$  to replace  $g$ , we have

$$\tau_0(\tau_0(g)u)u(\tau_0(g)u)^{-1} = gu^{-1}\tau_0(g)^{-1} = (\tau_0(g)ug^{-1})^{-1} \in JB^\times J,$$

which means that we may consider  $u\tau(g)$  instead of  $g$  in Proposition 2.4.12. Thus there exists a unique character  $\chi'$  of  $\mathbf{J}^{u\tau(g)} \cap G^\tau$  trivial on  $J^{1u\tau(g)} \cap G^\tau$  such that

$$\mathrm{Hom}_{J^{1u\tau(g)} \cap G^\tau}(\eta^{u\tau(g)}, 1) \cong \mathrm{Hom}_{\mathbf{J}^{u\tau(g)} \cap G^\tau}(\kappa^{u\tau(g)}, \chi'^{-1}).$$

Moreover, we know that  $\tau(\mathbf{J}) = \mathbf{J}^u$ ,  $\tau(J) = J^u$ ,  $\tau(J^1) = J^{1u}$  and  $\tau(H^1) = H^{1u}$ , thus as in Lemma 1.4.2 and Lemma 1.6.15, it is easy to show that

$$\mathbf{J}^g \cap G^\tau = \mathbf{J}^{u\tau(g)} \cap G^\tau = J^g \cap G^\tau = J^{u\tau(g)} \cap G^\tau \quad (2.4.10)$$

As a result,  $\chi$  and  $\chi'$  are characters defined on the same group  $\mathbf{J}^g \cap G^\tau = \mathbf{J}^{u\tau(g)} \cap G^\tau$ .

**Proposition 2.4.13.** *For  $\chi$  and  $\chi'$  defined above as characters of  $\mathbf{J}^g \cap G^\tau = \mathbf{J}^{u\tau(g)} \cap G^\tau$ , we have  $\chi = \chi'$ .*

*Proof.* We write  $\delta(x) = (\tau(g)g^{-1})^{-1}\tau(x)\tau(g)g^{-1}$  for any  $x \in G$ . Using the basic results in simple type theory, we have  $\gamma = u\tau(g)g^{-1} \in I_G(\eta) = I_G(\kappa^0)$ , where  $\kappa^0 = \kappa|_J$  and  $I_G(\eta)$  (resp.  $I_G(\kappa^0)$ ) denotes the intertwining set of  $\eta$  (resp.  $\kappa^0$ ). Moreover we have

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{J \cap J^\gamma}(\kappa^{0\gamma}, \kappa^0)) = \dim_{\mathbb{C}}(\mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta)) = 1.$$

By direct calculation, we have  $J^1 \cap G^\delta = J^{1\gamma} \cap G^\delta$  as a subgroup of  $J^1 \cap J^{1\gamma}$  and  $H^1 \cap G^\delta = H^{1\gamma} \cap G^\delta$ . Using Proposition 1.6.20 for our  $\gamma$  and  $\delta$ , we have:

**Proposition 2.4.14.** *For a non-zero homomorphism  $\varphi \in \mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta) = \mathrm{Hom}_{J \cap J^\gamma}(\kappa^{0\gamma}, \kappa^0)$ , it naturally induces a  $\mathbb{C}$ -vector space isomorphism*

$$\begin{aligned} f_\varphi : \mathrm{Hom}_{J^1 \cap G^\delta}(\eta, 1) &\rightarrow \mathrm{Hom}_{J^{1\gamma} \cap G^\delta}(\eta^\gamma, 1), \\ \lambda &\mapsto \lambda \circ \varphi. \end{aligned}$$

Now we use Proposition [2.4.14](#) to finish the proof of Proposition [2.4.13](#). Using Proposition [2.4.11](#) for  $g$  and  $u\tau(g)$  respectively, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) = \dim_{\mathbb{C}} \mathrm{Hom}_{J^{1u\tau(g)} \cap G^\tau}(\eta^{u\tau(g)}, 1) = 1.$$

By Proposition [2.4.14](#), for  $0 \neq \varphi \in \mathrm{Hom}_{J^1 \cap J^{1\gamma}}(\eta^\gamma, \eta) = \mathrm{Hom}_{J^{1g} \cap J^{1u\tau(g)}}(\eta^g, \eta^{u\tau(g)})$ ,

$$\begin{aligned} f_\varphi : \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) &\rightarrow \mathrm{Hom}_{J^{1u\tau(g)} \cap G^\tau}(\eta^{u\tau(g)}, 1), \\ \lambda &\mapsto \lambda \circ \varphi, \end{aligned}$$

is bijective. If we choose

$$0 \neq \lambda \in \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \quad \text{and} \quad 0 \neq \lambda' := f_\varphi(\lambda) = \lambda \circ \varphi \in \mathrm{Hom}_{J^{1u\tau(g)} \cap G^\tau}(\eta^{u\tau(g)}, 1),$$

then for any  $v$  in the representation space of  $\eta$  and any  $x \in J^g \cap G^\tau = J^{u\tau(g)} \cap G^\tau$ , we have

$$\begin{aligned} \chi'(x)^{-1} \lambda'(v) &= \lambda'(\kappa^{0u\tau(g)}(x)v) && \text{(by Proposition [2.4.12](#) (1))} \\ &= \lambda(\varphi(\kappa^{0u\tau(g)}(x)v)) && \text{(by definition of } \lambda') \\ &= \lambda(\kappa^{0g}(x)\varphi(v)) && \text{(since } \varphi \in \mathrm{Hom}_{J^g \cap J^{u\tau(g)}}(\kappa^{0u\tau(g)}, \kappa^{0g})) \\ &= \chi(x)^{-1} \lambda(\varphi(v)) && \text{(by Proposition [2.4.12](#) (1))} \\ &= \chi(x)^{-1} \lambda'(v) && \text{(by definition of } \lambda'). \end{aligned}$$

Since  $v$  and  $x \in J^g \cap G^\tau = J^{u\tau(g)} \cap G^\tau$  are arbitrary, we have  $\chi'|_{J^{u\tau(g)} \cap G^\tau} = \chi|_{J^g \cap G^\tau}$ , which finishes the proof with the aid of [\(2.4.10\)](#).

### 2.4.5 Existence of a $\tau$ -selfdual extension of $\eta$

**Proposition 2.4.15.** *There is  $\kappa$  as an extension of  $\eta$  such that  $\kappa^{\tau_0^\vee} \cong \kappa$ .*

*Proof.* We refer to Proposition [1.6.24](#) for a proof. Noting that the restriction of  $\tau_0$  to  $\mathrm{GL}_m(\mathbf{l})$  becomes an orthogonal involution with respect to the symmetric matrix  $\overline{\varepsilon_{0E}} \in \mathrm{GL}_m(\mathbf{l})$ , where  $\overline{\varepsilon_{0E}}$  represents the image of  $\varepsilon_{0E}$  in  $\mathrm{GL}_m(\mathbf{l}) \cong \mathrm{GL}_m(\mathfrak{o}_E)/(1 + \mathrm{M}_m(\mathfrak{p}_E))$ , thus if we replace  $\sigma$  and  $\tau$  in the *loc. cit.* by the trivial action and  $\tau_0$ , then the same proof in the case where  $E/E_0$  is ramified in *loc. cit.* works for our proposition. □

From now on until the end of this section we fix  $\kappa$  as in Proposition [2.4.15](#). We have the following corollary:

**Corollary 2.4.16.** *The character  $\chi$  defined by Lemma [2.4.12](#) (1) is quadratic, that is,  $\chi^2 = 1$ .*

*Proof.* We have the following isomorphisms

$$\begin{aligned} \mathrm{Hom}_{J^{1u\tau(g)} \cap G^\tau}(\eta^{u\tau(g)}, 1) &\cong \mathrm{Hom}_{J^{1g} \cap G^\tau}(\eta^g, 1) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\kappa^g, \chi^{-1}) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\chi, \kappa^{g^\vee}) && \text{(by the duality of contragredient)} \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\kappa^{g^\vee}, \chi) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}(\kappa^{g^\vee} \circ \tau, \chi \circ \tau) \\ &\cong \mathrm{Hom}_{J^g \cap G^\tau}((\kappa^{\tau_0^\vee})^{u\tau(g)}, \chi \circ \tau) \\ &\cong \mathrm{Hom}_{J^{u\tau(g)} \cap G^\tau}(\kappa^{u\tau(g)}, \chi \circ \tau) && \text{(since } \kappa \text{ is } \tau_0\text{-selfdual)}. \end{aligned}$$

Using Proposition [2.4.13](#) and the uniqueness of  $\chi'$ , we have  $\chi \circ \tau = \chi^{-1}$ . Since  $\chi$  is defined on  $J^g \cap G^\tau = J^g \cap G^\tau$  which is  $\tau$ -invariant, we have  $\chi \circ \tau = \chi$ , thus  $\chi^2 = \chi(\chi \circ \tau) = 1$ . □

### 2.4.6 Proof of Theorem [2.4.2](#)

In this subsection, we finish the proof of Theorem [2.4.2](#). For  $g \in G$  given as in *loc. cit.*, by Lemma [2.4.4](#) and Cartan decomposition, we may replace  $g$  by another representative in the same  $J$ - $G^\tau$  double coset, such that

$$\gamma := u\tau(g)g^{-1} \in \varpi_E^{a_1} \mathrm{GL}_{m_1}(\mathfrak{o}_E) \times \dots \times \varpi_E^{a_r} \mathrm{GL}_{m_r}(\mathfrak{o}_E), \quad (2.4.11)$$

where  $a_i, m_i$  are defined as in Lemma [2.4.6](#). Thus there exists a unique standard hereditary order  $\mathfrak{b}_m \subseteq \mathfrak{b}$  such that

$$U^1(\mathfrak{b}_m) = (U \cap \delta(U^1))U^1 = (U \cap U^{1\gamma})U^1,$$

where we define  $U = U(\mathfrak{b})$ ,  $U^1 = U^1(\mathfrak{b})$  and  $\delta(x) = (\tau(g)g^{-1})^{-1}\tau(x)\tau(g)g^{-1}$  for any  $x \in G$  as an involution on  $G$ . First we have the following lemma whose proof is the same as that in [\[Séc19\]](#), Lemma 6.22, inspired by [\[HM08\]](#), Proposition 5.20:

**Lemma 2.4.17.** *We have  $U^1(\mathfrak{b}_m) = (U^1(\mathfrak{b}_m) \cap G^\delta)U^1$ .*

To finish the proof, it is enough to show that  $r = 1$  in [\(2.4.11\)](#). If not, we know that  $\mathfrak{b}_m$  by definition is a proper suborder of  $\mathfrak{b}$ . Furthermore,  $\overline{U^1(\mathfrak{b}_m)} := U^1(\mathfrak{b}_m)/U^1$  is a non-trivial unipotent subgroup of  $U/U^1 \cong \mathrm{GL}_m(\mathfrak{l})$ . Using Proposition [2.4.12](#)(2), we have

$$\mathrm{Hom}_{J \cap G^\delta}(\rho, \chi^{g^{-1}}) \cong \mathrm{Hom}_{J^g \cap G^\tau}(\rho^g, \chi) \neq 0.$$

Restricting to  $U^1(\mathfrak{b}_m) \cap G^\delta$ , we have

$$\mathrm{Hom}_{U^1(\mathfrak{b}_m) \cap G^\delta}(\rho, \chi^{g^{-1}}) \neq 0. \quad (2.4.12)$$

Using Lemma [2.4.17](#), we have the isomorphism

$$(U^1(\mathfrak{b}_m) \cap G^\delta)U^1/U^1 \cong U^1(\mathfrak{b}_m)/U^1.$$

We denote by  $\bar{\rho}$  the cuspidal representation of  $U^0/U^1 \cong \mathrm{GL}_m(\mathfrak{l})$  whose inflation is  $\rho|_{U^0}$ , and  $\overline{\chi^{g^{-1}}}$  the character of  $\overline{U^1(\mathfrak{b}_m)}$  whose inflation is  $\chi^{g^{-1}}$ . We consider the equation [\(2.4.12\)](#) modulo  $U^1$  and we have

$$\mathrm{Hom}_{\overline{U^1(\mathfrak{b}_m)}}(\bar{\rho}, \overline{\chi^{g^{-1}}}) \neq 0.$$

Since  $\chi^{g^{-1}}|_{J \cap G^\delta}$  is quadratic and  $\overline{U^1(\mathfrak{b}_m)}$  is a  $p$ -group with  $p \neq 2$ , we get  $\overline{\chi^{g^{-1}}}|_{\overline{U^1(\mathfrak{b}_m)}} = 1$ , thus

$$\mathrm{Hom}_{\overline{U^1(\mathfrak{b}_m)}}(\bar{\rho}, 1) \neq 0$$

which contradicts to the fact that  $\bar{\rho}$  is supercuspidal. So we finish the proof. □

### 2.4.7 Double cosets contributing to the distinction of $\pi$

In this subsection, we assume  $\varepsilon = \varepsilon_0$  and  $\tau = \tau_0$ . We want to study all the possible  $\mathbf{J}$ - $G^\tau$  double cosets contributing to the distinction of  $\pi$ . Precisely, we want to study those  $g \in G$  such that

$$\mathrm{Hom}_{\mathbf{J}g \cap G^\tau}(\Lambda^g, 1) \neq 0.$$

By Lemma 2.4.4, we may change  $g$  with another representative in  $\mathbf{J}gG^\tau$  to assume that  $\gamma = \tau(g)g^{-1} \in B^\times$ . Moreover, by Theorem 2.4.2 we get  $\gamma \in \mathbf{J}$ . As a result, we have

$$\gamma \in \mathbf{J} \cap B^\times = E^\times \mathfrak{b}^\times. \quad (2.4.13)$$

First by changing  $g$  up to multiplying an element in  $E^\times$  on the left, which doesn't change the double coset  $\mathbf{J}gG^\tau$ , we may assume  $\gamma \in \mathfrak{b}^\times$  or  $\varpi_E \mathfrak{b}^\times$ . Since  $\mathbf{J} \cap B^\times = \mathfrak{b}^\times = \mathrm{GL}_m(\mathfrak{o}_E)$ , using Proposition 2.2.2 we may change  $g$  up to multiplying an element in  $\mathfrak{b}^\times$  on the left, which doesn't change the double coset  $\mathbf{J}gG^\tau$ , such that

$$\gamma = I_m \text{ or } \mathrm{diag}(1, \dots, 1, \epsilon_0) \text{ or } \mathrm{diag}(\varpi_E, \dots, \varpi_E, \varpi_E) \text{ or } \mathrm{diag}(\varpi_E, \dots, \varpi_E, \varpi_E \epsilon_0). \quad (2.4.14)$$

By definition, we have

$$\mathrm{N}_{E/F}(\det_B(\gamma)) = \det(\gamma) \in F^{\times 2}, \quad (2.4.15)$$

where  $\det_B$  denotes the determinant on  $B^\times = \mathrm{GL}_m(E)$ . By studying different cases separately, we will give out all the possible double cosets of  $g$  satisfying the condition (2.4.14).

**Case (i)** If  $\mathrm{N}_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$ , then

$$\mathrm{N}_{E/F} : E^\times/E^{\times 2} \longrightarrow F^\times/F^{\times 2}$$

is bijective. Thus (2.4.15) shows that  $\det_B(\gamma) \equiv 1 \pmod{E^{\times 2}}$ . Thus from (2.4.14) and the fact that  $m$  is odd, we get  $\gamma = 1$ , which means that  $g \in G^\tau$ . Thus in this case there is only one double coset  $\mathbf{J}G^\tau$ .

**Case (ii)** If  $\mathrm{N}_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  is a subgroup of  $F^\times/F^{\times 2}$  of order two, we consider the following two cases:

**(ii.a)** If  $2|m$ , then from the same argument in §2.3.3 we have  $\mathrm{N}_{E_m/E}(E_m^\times)E^{\times 2}/E^{\times 2} = \{1, \epsilon_0\}$ , where  $\epsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$  as above. And moreover the ramification index of  $E/F$  is odd and  $\mathrm{N}_{E/F}(\epsilon_0) \notin F^{\times 2}$ . Using (2.4.14) and (2.4.15),  $\gamma$  equals  $I_m$  or  $\mathrm{diag}(\varpi_E, \dots, \varpi_E)$ .

**(ii.a.1)** We assume one of the three cases is true:

- $2|d$ ;
- $2 \nmid d$  and  $4|m$ ;
- $2 \nmid d$ ,  $4 \nmid m$  and  $-1 \in F^{\times 2}$ .

If  $\varepsilon = J_{d,m}$  and  $\varepsilon_{0E} = I_m$ , then in the case where  $\gamma = \tau(g)g^{-1} = I_m$ , we have  $g \in G^\tau$ . In the case where  $\gamma = \mathrm{diag}(\varpi_E, \dots, \varpi_E)$ , using Proposition 2.2.1 and the fact that

$$\det_B(\mathrm{diag}(\varpi_E, \dots, \varpi_E)) = \varpi_E^m \in E^{\times 2} \quad \text{and} \quad \mathrm{Hasse}_E(\mathrm{diag}(\varpi_E, \dots, \varpi_E)) = 1,$$

there exists  $g_1 \in B^\times$  such that  $\tau(g_1)g_1^{-1} = \mathrm{diag}(\varpi_E, \dots, \varpi_E)$ , where we denote by  $\mathrm{Hasse}_E$  the Hasse invariant for the symmetric matrices in  $B^\times = \mathrm{GL}_m(E)$  and we use Lemma 2.2.12 to calculate the Hasse invariant. Thus we have  $g \in \mathbf{J}g_1G^\tau$ . So there are two possible double cosets  $\mathbf{J}G^\tau$  and  $\mathbf{J}g_1G^\tau$ .



If  $\varepsilon = \text{diag}(J_d, \dots, J_d, J_d \epsilon_0)$  and  $\varepsilon_{0E} = \text{diag}(1, \dots, 1, \epsilon_0)$  with  $\epsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$ , then in the case where  $\gamma = I_m$ , we have  $g \in G^\tau$ . In the case where  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E)$ , by direct calculation we get

$${}^t g^{-1} \text{diag}(J_d, \dots, J_d, J_d \epsilon_0) g^{-1} = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0).$$

Using Lemma 2.2.9 we obtain  $\text{Hasse}({}^t g^{-1} \text{diag}(J_d, \dots, J_d, J_d \epsilon_0) g^{-1}) = 1$ . However by Lemma 2.2.20 and Corollary 2.3.14, we have  $\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0)) = -1$ , thus there doesn't exist any  $g \in G$  such that  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E)$ , so there is only one possible double coset  $\mathbf{J}G^\tau$ .

(ii.a.2) If  $2 \nmid d$ ,  $4 \nmid m$  and  $-1 \notin F^{\times 2}$ , then we may choose  $\epsilon_0 = -1 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$ .

If  $\varepsilon = \text{diag}(J_d, \dots, J_d, -J_d)$  and  $\varepsilon_{0E} = \text{diag}(1, \dots, 1, -1)$ , then in the case where  $\gamma = \tau(g)g^{-1} = I_m$ , we have  $g \in G^\tau$ . In the case where  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E)$ , using Proposition 2.2.1 and the fact that (by Lemma 2.2.12 for example)

$$\det_B(\text{diag}(\varpi_E, \dots, -\varpi_E)) = -\varpi_E^m \in \epsilon_0 E^{\times 2} \quad \text{and} \quad \text{Hasse}_E(\text{diag}(\varpi_E, \dots, -\varpi_E)) = 1,$$

there exists  $g_1 \in B^\times$  such that  ${}^t g_1^{-1} \varepsilon_{0E} g_1^{-1} = \text{diag}(\varpi_E, \dots, -\varpi_E)$ , or in other words  $\tau(g_1)g_1^{-1} = \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E)$ . Thus we have  $g \in \mathbf{J}g_1 G^\tau$ . So there are two possible double cosets  $\mathbf{J}G^\tau$  and  $\mathbf{J}g_1 G^\tau$ .

If  $\varepsilon = J_{d,m}$  and  $\varepsilon_{0E} = I_m$ , then in the case where  $\gamma = I_m$ , we have  $g \in G^\tau$ . In the case where  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E)$ , by direct calculation we get

$${}^t g^{-1} \text{diag}(J_d, \dots, J_d, J_d) g^{-1} = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E).$$

Using Lemma 2.2.9 we get  $\text{Hasse}({}^t g^{-1} \text{diag}(J_d, \dots, J_d, J_d) g^{-1}) = 1$ . Using Lemma 2.2.21 and Corollary 2.3.14 we have  $\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E)) = -1$ , thus there doesn't exist any  $g$  as above such that  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E)$ , so there is only one possible double coset  $\mathbf{J}G^\tau$ .

(ii.b) If  $2 \nmid m$ , then  $\varepsilon$  equals  $J_{d,m}$  or  $\text{diag}(J_d \epsilon, \dots, J_d \epsilon)$ , where  $\epsilon \in E^\times$ . In this case we have  $N_{E_m/F}(E_m^\times) F^{\times 2} / F^{\times 2} = N_{E/F}(E^\times) F^{\times 2} / F^{\times 2}$  and  $2 \mid d$ . Furthermore by Proposition 2.2.14, either the ramification index or the residue class degree of  $E/F$  is odd. We further consider the following two cases:

(ii.b.1) If the ramification index of  $E/F$  is odd, then  $\epsilon = \epsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$  such that  $N_{E/F}(\epsilon_0) \notin F^{\times 2}$ . By (2.4.14) and (2.4.15), we deduce that  $\gamma$  equals  $I_m$  or  $\text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E)$ , where  $\varpi'_E$  equals  $\varpi_E$  or  $\varpi_E \epsilon_0$  such that  $N_{E/F}(\varpi'_E) \in F^{\times 2}$ .

If  $\varepsilon = J_{d,m}$ , we have  $g \in G^\tau$  in the case where  $\gamma = I_m$ . In the case where  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E)$ , using Lemma 2.2.9, Lemma 2.2.18 and Corollary 2.3.14, we have

$$\det(J_{d,m} \text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E)) \in \det(J_{d,m}) F^{\times 2} \quad \text{and} \quad \text{Hasse}(J_{d,m} \text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E)) = 1,$$

thus by Proposition 2.2.1, there exists  $g_1 \in G$  such that

$${}^t g_1^{-1} J_{d,m} g_1^{-1} = J_{d,m} \text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E) = J_{d,m} \gamma,$$

or in other words  $\tau(g_1)g_1^{-1} = \gamma$ . Thus  $g \in g_1 G^\tau$ . So we get two double cosets  $\mathbf{J}G^\tau$  and  $\mathbf{J}g_1 G^\tau$ .

**Remark 2.4.18.** Since  $\det_B(\text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E)) = \varpi_E^m \varpi'_E \notin E^{\times 2}$ , it is impossible to choose  $g_1 \in B^\times$  such that  $\tau(g_1)g_1 = \gamma$ . Thus  $\mathbf{J}g_1 G^\tau$  is disjoint with  $\mathbf{J}B^\times G^\tau$ . Similar phenomena also occur in (ii.b.2) and (iii) below.

If  $\varepsilon = \text{diag}(J_d \epsilon_0, \dots, J_d \epsilon_0)$ , we get  $g \in G^\tau$  in the case where  $\gamma = I_m$ . In the case where  $\gamma = \text{diag}(\varpi_E, \dots, \varpi_E, \varpi'_E)$ , by direct calculation we have

$${}^t g^{-1} \text{diag}(J_d \epsilon_0, \dots, J_d \epsilon_0, J_d \epsilon_0) g^{-1} = \text{diag}(J_d \varpi_E \epsilon_0, \dots, J_d \varpi_E \epsilon_0, J_d \varpi'_E \epsilon_0). \quad (2.4.16)$$

By Lemma [2.2.9](#), we get

$$\text{Hasse}({}^t g^{-1} \text{diag}(J_d \epsilon_0, \dots, J_d \epsilon_0, J_d \epsilon_0) g^{-1}) = 1.$$

And by Lemma [2.2.18](#) and Corollary [2.3.14](#), we obtain  $\text{Hasse}(\text{diag}(J_d \varpi_E \epsilon_0, \dots, J_d \varpi_E \epsilon_0, J_d \varpi_E' \epsilon_0)) = -1$ , thus the condition [\(2.4.16\)](#) is never satisfied. Thus there is only one possible double coset  $\mathbf{J}G^\tau$ .

**(ii.b.2)** If the residue class degree of  $E/F$  is odd, then  $\epsilon = \varpi_E$  as a uniformizer of  $E$  such that  $N_{E/F}(\varpi_E) \notin F^{\times 2}$ . By [\(2.4.14\)](#) and [\(2.4.15\)](#), we get  $N_{E/F}(\det_B(\gamma)) \in F^{\times 2}$ , thus  $\det_B(\gamma)$  equals 1 or  $\epsilon_0$ , which means that  $\gamma$  equals  $I_m$  or  $\text{diag}(1, \dots, 1, \epsilon_0)$ .

If  $\epsilon = J_{d,m}$ , we have  $g \in G^\tau$  in the case where  $\gamma = I_m$ . In the case where  $\gamma = \text{diag}(1, \dots, 1, \epsilon_0)$ , using Lemma [2.2.9](#) we have

$$\det(J_{d,m} \text{diag}(1, \dots, 1, \epsilon_0)) \in \det(J_{d,m}) F^{\times 2} \quad \text{and} \quad \text{Hasse}(J_{d,m} \text{diag}(1, \dots, 1, \epsilon_0)) = \text{Hasse}(J_{d,m}) = 1,$$

thus by Proposition [2.2.1](#), there exists  $g_1 \in G$  such that

$${}^t g_1^{-1} J_{d,m} g_1^{-1} = J_{d,m} \text{diag}(1, \dots, 1, \epsilon_0),$$

or equivalently  $\tau(g_1)g_1^{-1} = \gamma$ . Thus  $g \in g_1 G^\tau$ . So we get two double cosets  $\mathbf{J}G^\tau$  and  $\mathbf{J}g_1 G^\tau$ .

If  $\epsilon = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E)$ , we get  $g \in G^\tau$  in the case where  $\gamma = I_m$ . In the case where  $\gamma = \text{diag}(1, \dots, 1, \epsilon_0)$ , by direct calculation we have

$${}^t g^{-1} \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E) g^{-1} = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0), \quad (2.4.17)$$

However by Corollary [2.2.17](#) and Corollary [2.3.14](#), this condition is never satisfied. Thus there is only one possible double coset  $\mathbf{J}G^\tau$ .

**Case (iii)** If  $N_{T_m/F}(T_m^\times) F^{\times 2} / F^{\times 2} = \{1\}$ , we consider the following two cases:

**(iii.a)** If  $N_{E/F}(E^\times) F^{\times 2} / F^{\times 2} = \{1\}$ , then  $\epsilon$  equals  $J_{d,m}$  or  $\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E \epsilon_0)$ , where  $\epsilon_0 \in \mathfrak{o}_E^\times \setminus \mathfrak{o}_E^{\times 2}$  and  $\varpi_E$  is a uniformizer of  $E$  satisfying Lemma [2.2.15](#) with  $E' = E$ .

If  $\epsilon = J_{d,m}$ , by [\(2.4.14\)](#) we have

$$\begin{aligned} {}^t g^{-1} J_{d,m} g^{-1} &= J_{d,m} \text{ or } \text{diag}(J_d, \dots, J_d, J_d \epsilon_0) \text{ or } \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E) \\ &\quad \text{or } \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0) \end{aligned} \quad (2.4.18)$$

Since the determinants of both sides of [\(2.4.18\)](#) are in  $F^{\times 2}$ , and by Lemma [2.2.9](#), Lemma [2.2.15](#) and Corollary [2.3.14](#), we have

$$\text{Hasse}(J_{d,m}) = \text{Hasse}(\text{diag}(J_d, \dots, J_d, J_d \epsilon_0)) = \text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E)) = 1,$$

and

$$\text{Hasse}(\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E, J_d \varpi_E \epsilon_0)) = -1,$$

then by Proposition [2.2.1](#) there exist  $g_0 = 1$ ,  $g_1$  and  $g_2$  which satisfy equation [\(2.4.18\)](#) with the first three terms on the right separately. Furthermore, equation [\(2.4.18\)](#) with the last term on the right is never satisfied. Thus there are exactly three double cosets  $\mathbf{J}G^\tau$ ,  $\mathbf{J}g_1 G^\tau$  and  $\mathbf{J}g_2 G^\tau$ .

If  $\epsilon = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E \epsilon_0)$ , then by [\(2.4.14\)](#) we have

$$\begin{aligned} {}^t g^{-1} \epsilon g^{-1} &= \epsilon I_m \text{ or } \epsilon \text{diag}(1, \dots, 1, \epsilon_0) \text{ or } \epsilon \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E) \\ &\quad \text{or } \epsilon \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E \epsilon_0) \end{aligned} \quad (2.4.19)$$

Since the determinants of both sides of (2.4.19) are in  $F^{\times 2}$ , and by Lemma 2.2.9, Lemma 2.2.15 and Corollary 2.3.14, we have

$$\begin{aligned} \text{Hasse}(\varepsilon \text{diag}(1, \dots, 1, \epsilon_0)) &= \text{Hasse}(\varepsilon \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E)) \\ &= \text{Hasse}(\varepsilon \text{diag}(\varpi_E, \dots, \varpi_E, \varpi_E \epsilon_0)) = 1, \end{aligned}$$

and

$$\text{Hasse}(\varepsilon) = -1.$$

Then equation (2.4.19) is never satisfied with the last three terms on the right, and  $g_0 = 1$  satisfies (2.4.19) with the first term on the right. Thus there is only one double coset  $\mathbf{J}G^\tau$ .

(iii.b) If  $N_{E/F}(E^\times)F^{\times 2}/F^{\times 2}$  is not trivial, then  $\varepsilon$  equals  $J_{d,m}$  or  $\text{diag}(J_d \varpi_E, \dots, J_d \varpi_E \epsilon_0)$ , where  $\epsilon_0 \in \mathfrak{o}_E^\times$  and  $\varpi_E$  is a uniformizer of  $E$ . Using the similar proof as (iii.a), with Lemma 2.2.15 replaced by Corollary 2.2.17, we can show that if  $\varepsilon = J_{d,m}$ , there are three double cosets  $\mathbf{J}g_0 G^\tau$ ,  $\mathbf{J}g_1 G^\tau$  and  $\mathbf{J}g_2 G^\tau$ , where  $g_0 = 1$ ,  $g_1$  and  $g_2$  are defined such that  $\tau(g_i)g_i^{-1}$  equal three of the four terms on the right side of equation (2.4.14). If  $\varepsilon = \text{diag}(J_d \varpi_E, \dots, J_d \varpi_E \epsilon_0)$ , then there is only one double coset  $\mathbf{J}G^\tau$ .

We sum up the main result of this subsection as the following proposition:

**Proposition 2.4.19. Case (i)** When  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = F^\times/F^{\times 2}$ , the only double coset contributing to the distinction is  $\mathbf{J}g_0 G^{\tau_0}$ , where we write  $g_0 = 1$  here and after to normalize the notation;

**Case (ii)** When  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2}$  is a subgroup of  $F^\times/F^{\times 2}$  of order 2, if  $G^{\tau_0}$  is quasisplit but not split, then the only double coset contributing to the distinction is  $\mathbf{J}g_0 G^{\tau_0}$ ; If  $G^{\tau_0}$  is split, then there are two different double cosets  $\mathbf{J}g_0 G^{\tau_0}$  and  $\mathbf{J}g_1 G^{\tau_0}$  contributing to the distinction, where  $\tau_0(g_1)g_1^{-1} \in B^\times$ ;

**Case (iii)** When  $N_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2} = \{1\}$ , if  $G^{\tau_0}$  is not quasisplit, then the only double coset contributing to the distinction is  $\mathbf{J}g_0 G^{\tau_0}$ ; If  $G^{\tau_0}$  is split, then there are three different double cosets  $\mathbf{J}g_0 G^{\tau_0}$ ,  $\mathbf{J}g_1 G^{\tau_0}$  and  $\mathbf{J}g_2 G^{\tau_0}$  contributing to the distinction, where  $\tau_0(g_1)g_1^{-1}, \tau_0(g_2)g_2^{-1} \in B^\times$ .

**Remark 2.4.20.** The above proposition doesn't guarantee that each of the double coset corresponds to a distinguished space, and it says nothing about the dimension. However in the next section we will find out that each double coset indeed contributes to the distinction and the corresponding dimension is one respectively.

**Remark 2.4.21.** We may also give out all the maximal simple characters contained in  $\pi$  that are  $\tau_0$ -selfdual. Let  $\theta$  be a fixed maximal simple character such that  $\theta \circ \tau_0 = \theta^{-1}$ . Any other maximal simple characters contained in  $\pi$  can be written as  $\theta^g$  with  $g \in G$ . Thus  $\theta^g$  is  $\tau_0$ -selfdual if and only if  $\gamma = \tau_0(g)g^{-1}$  normalizes  $\theta$ , that is,  $\gamma \in \mathbf{J}$ . Thus from the above argument,  $g$  is in the same  $\mathbf{J}G^{\tau_0}$  double coset as one of the  $g_i$  in Proposition 2.4.19. Thus one has a one-to-one correspondence between  $\mathbf{J}G^{\tau_0}$  double cosets in loc. cit. and  $G^{\tau_0}$ -orbits of  $\tau_0$ -selfdual maximal characters contained in  $\pi$ .

## 2.5 Proof of the main theorems

In this section, we finish the proof of our main theorem. Let  $\pi$  be a given supercuspidal representation of  $G$  and let  $\tau$  be a given orthogonal involution on  $G$ . First of all, if  $\pi$  is distinguished by  $G^\tau$ , then we restrict  $\pi$  to  $F^\times \cap G^\tau = \{1, -1\}$  which is contained in the centre of  $G$  and we get  $\omega_\pi(-1) = 1$ . So  $\omega_\pi(-1) = 1$  is indeed a necessary condition for  $\pi$  to be distinguished by  $G^\tau$ . So from now on we assume further that  $\pi$  satisfies this condition.

### 2.5.1 The finite field case

Let  $\mathbf{l}$  be a finite field of characteristic  $p \neq 2$  and let  $\bar{\mathbf{l}}$  be a fixed algebraic closure of  $\mathbf{l}$ . We denote by  $Fr \in \text{Gal}(\bar{\mathbf{l}}/\mathbf{l})$  the arithmetic Frobenius map, then we have  $\mathbf{H}^{Fr} := \mathbf{H}(\bar{\mathbf{l}})^{Fr} = \mathbf{H}(\mathbf{l})$  for any algebraic group  $\mathbf{H}$  over  $\mathbf{l}$ . For  $m \geq 1$ , let  $\mathbf{G} = \text{GL}_m$  be the reductive group over  $\mathbf{l}$ . Let  $\tau$  be an orthogonal involution of  $\mathbf{G}$  over  $\mathbf{l}$ , which means that the symmetric matrix corresponding to  $\tau$  is of coefficients in  $\mathbf{l}$ . Thus  $\tau$  commutes with  $Fr$  and  $\mathbf{G}^\tau$  denotes the corresponding orthogonal subgroup of  $\mathbf{G}$  as a reductive group over  $\mathbf{l}$ .

For  $\mathbf{S}$  a maximal torus of  $\mathbf{G}$  over  $\mathbf{l}$ , we write  $\mathbf{S}^\tau = \mathbf{S} \cap \mathbf{G}^\tau$ . We say that  $\mathbf{S}$  is  $\tau$ -stable if  $\tau(\mathbf{S}) = \mathbf{S}$ . Furthermore, we say that  $\mathbf{S}$  is  $\tau$ -split if  $\tau(x) = x^{-1}$  for any  $x \in \mathbf{S}(\mathbf{l})$ . We denote by  $\mathcal{T}$  the set of maximal tori  $\mathbf{S}$  of  $\mathbf{G}$ , such that there exists a Borel subgroup  $\mathbf{B}$  of  $\mathbf{G}$  over  $\mathbf{l}$  with the property  $\mathbf{S} = \mathbf{B} \cap \tau(\mathbf{B})$ . By [Vus74], Proposition 5, the  $\mathbf{G}^\tau$ -action on  $\mathcal{T}$  given by conjugation is transitive. Since this  $\mathbf{G}^\tau$ -action maintains the  $\tau$ -split property, and since  $\mathcal{T}$  contains a  $\tau$ -split torus<sup>2</sup>,  $\mathcal{T}$  consists of  $\tau$ -split tori.

As in [Lus90], section 2, for  $\mathbf{S}$  a  $\tau$ -stable maximal torus of  $\mathbf{G}$  over  $\mathbf{l}$ , we may define a character

$$\epsilon_{\mathbf{S}} : (\mathbf{S}^\tau)^{Fr} \rightarrow \{\pm 1\}.$$

given by

$$\epsilon_{\mathbf{S}}(t) = (-1)^{\text{rank}_{\mathbf{l}}(Z_{\mathbf{G}}((\mathbf{S}^\tau)^\circ)) + \text{rank}_{\mathbf{l}}(Z_{\mathbf{G}}((\mathbf{S}^\tau)^\circ) \cap Z_{\mathbf{G}}(t)^\circ)},$$

where  $(\cdot)^\circ$  denotes the connected component of an algebraic group,  $Z_{\mathbf{G}}(\cdot)$  denotes the centralizer of an element or an algebraic group in  $\mathbf{G}$  and  $\text{rank}_{\mathbf{l}}(\cdot)$  denotes the rank of an algebraic group over  $\mathbf{l}$ . For  $\mathbf{T}$  a maximal torus of  $\mathbf{G}$  over  $\mathbf{l}$ , we define

$$\Xi_{\mathbf{T}} = \{g \in \mathbf{G}(\bar{\mathbf{l}}) \mid g^{-1}\mathbf{T}g \text{ is } \tau\text{-stable}\}$$

as a variety over  $\bar{\mathbf{l}}$ , and we denote by  $\Xi_{\mathbf{T}}^{Fr} = \Xi_{\mathbf{T}} \cap \mathbf{G}^{Fr}$  its  $\mathbf{l}$ -points. By definition, it consists of  $\mathbf{T}^{Fr} \cdot (\mathbf{G}^\tau)^{Fr}$  double cosets. For  $\lambda$  a character of  $\mathbf{T}^{Fr}$  and  $\bar{\chi}$  a character of  $(\mathbf{G}^\tau)^{Fr}$ , we define

$$\Xi_{\mathbf{T}, \lambda, \bar{\chi}}^{Fr} = \{g \in \Xi_{\mathbf{T}}^{Fr} \mid \lambda(t) = \bar{\chi}(g^{-1}tg) \epsilon_{g^{-1}\mathbf{T}g}(g^{-1}tg) \text{ for any } t \in \mathbf{T}(\bar{\mathbf{l}}) \text{ such that } g^{-1}tg \in ((\mathbf{T}^\tau)^\tau)^{Fr}\}.$$

It is a subset of  $\Xi_{\mathbf{T}}^{Fr}$  which also consists of  $\mathbf{T}^{Fr} \cdot (\mathbf{G}^\tau)^{Fr}$  double cosets.

We choose  $\mathbf{l}_m/\mathbf{l}$  to be a finite extension of degree  $m$  and we fix an embedding  $\mathbf{l}_m \hookrightarrow \text{M}_m(\mathbf{l})$ . We assume  $\mathbf{T}$  to be *elliptic*, which means that  $\mathbf{T}(\mathbf{l}) = \mathbf{l}_m^\times$ . By [HM99], Lemma 2,  $\Xi_{\mathbf{T}}^{Fr}$  consists of a single  $\mathbf{T}^{Fr} \cdot (\mathbf{G}^\tau)^{Fr}$  double coset. Moreover, from its proof we know that for  $g \in \Xi_{\mathbf{T}}^{Fr}$ , we have  $g^{-1}\mathbf{T}g \in \mathcal{T}$ , thus  $g^{-1}\mathbf{T}g$  is  $\tau$ -split.

For any  $\bar{\rho}$  as a supercuspidal representation of  $\mathbf{G}(\mathbf{l})$ , by [DL76] we may associate it to a virtual character  $R_{\mathbf{T}, \lambda}$  as the trace of  $\bar{\rho}$ , where  $\lambda$  is a *non-singular* character of  $\mathbf{T}^{Fr}$ , that is,  $\lambda^{Fr^i} \neq \lambda$  for any  $i = 1, \dots, m-1$ . If

$$\text{Hom}_{\mathbf{G}^\tau(\mathbf{l})}(\bar{\rho}, \bar{\chi}) \neq 0,$$

then by [HL12], Theorem 3.11, we get  $\Xi_{\mathbf{T}, \lambda, \bar{\chi}}^{Fr} \neq 0$ , which means that  $\Xi_{\mathbf{T}, \lambda, \bar{\chi}}^{Fr} = \Xi_{\mathbf{T}}^{Fr}$  consists of a single  $\mathbf{T}^{Fr} \cdot (\mathbf{G}^\tau)^{Fr}$  double coset. Thus for  $h \in \Xi_{\mathbf{T}, \lambda, \bar{\chi}}^{Fr}$  we know that  $h^{-1}\mathbf{T}h$  is  $\tau$ -split, which also means that  $h^{-1}\mathbf{l}_m h$  is  $\tau$ -split. Thus we have proved the following proposition:

**Proposition 2.5.1.** *For  $\bar{\rho}$  a supercuspidal representation of  $\mathbf{G}(\mathbf{l})$  and  $\bar{\chi}$  a character of  $\mathbf{G}^\tau(\mathbf{l})$ , if*

$$\text{Hom}_{\mathbf{G}^\tau(\mathbf{l})}(\bar{\rho}, \bar{\chi}) \neq 0,$$

*then there exists a  $\tau$ -split embedding  $\mathbf{l}_m \hookrightarrow \text{M}_m(\mathbf{l})$ , where  $\mathbf{l}_m/\mathbf{l}$  is the field extension of degree  $m$ .*

<sup>2</sup>If  $\tau = \tau_{\bar{\epsilon}}$  corresponds to the diagonal symmetric matrix  $\bar{\epsilon} = \text{diag}(1, \dots, 1, \bar{\epsilon})$  with  $\bar{\epsilon}$  equalling 1 or  $\bar{\epsilon}_0 \in \mathbf{l}^\times \setminus \mathbf{l}^{\times 2}$ , then  $\mathbf{S}_0$  defined as the diagonal maximal torus is  $\tau$ -split which is contained in  $\mathcal{T}$ .

### 2.5.2 Orthogonal groups contributing to the distinction of $\pi$

In this subsection, we first assume that  $H = G^\tau$  satisfies the condition of Theorem 0.3.1. From the proof of Theorem 2.3.3,  $G^\tau$  is conjugate to  $G^{\tau_0}$  with  $\tau_0 = \tau_{\varepsilon_0}$  defined as in the beginning of §2.4. Since the property of distinction doesn't depend on the choice of the representative of a  $G$ -conjugacy class, we may suppose  $\tau = \tau_0$ .

We choose a  $\tau$ -selfdual simple type  $(\mathbf{J}, \Lambda)$  of  $\pi$  as in §2.4, then using the Mackey formula and the Frobenius reciprocity we get

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \prod_{g \in \mathbf{J} \backslash G / G^\tau} \mathrm{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1).$$

In §2.4.7, we studied all the possible double cosets that contribute to the distinction. By Proposition 2.4.19, we have

$$\mathrm{Hom}_{G^\tau}(\pi, 1) \cong \bigoplus_{g_i} \mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\Lambda^{g_i}, 1),$$

where  $g_i$  runs over a finite set of representatives, depending on **Case (i)**, **Case (ii)** or **Case (iii)** of *loc. cit.*

Moreover, we may write

$$\Lambda \cong \kappa \otimes \rho,$$

where by Proposition 2.4.15 we assume  $\kappa^{\tau^\vee} \cong \kappa$ , thus we also have  $\rho^{\tau^\vee} \cong \rho$ . By Proposition 2.4.12, we get

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\kappa^{g_i}, \chi_i^{-1}) = 1 \tag{2.5.1}$$

and

$$\mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\Lambda^{g_i}, 1) \cong \mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\kappa^{g_i}, \chi_i^{-1}) \otimes \mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\rho^{g_i}, \chi_i),$$

where  $\chi_i$  is a quadratic character of  $\mathbf{J}^{g_i} \cap G^\tau$ . Thus to finish the proof for  $\tau = \tau_0$ , we only need to calculate

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\rho^{g_i}, \chi_i).$$

We define  $\delta_i(x) = \gamma_i^{-1} \tau(x) \gamma_i$  for any  $x \in G$  with  $\gamma_i = \tau(g_i) g_i^{-1}$ , then by the exact definition of  $\tau$  and  $\delta_i$ , the restriction of  $\delta_i$  to  $\mathrm{GL}_m(\mathbf{l}) \cong J/J^1$  is an orthogonal involution, and we denote by  $\mathrm{GL}_m(\mathbf{l})^{\delta_i}$  the corresponding orthogonal group. So we have

$$\mathrm{Hom}_{\mathbf{J}^{g_i} \cap G^\tau}(\rho^{g_i}, \chi_i) \cong \mathrm{Hom}_{J \cap G^{\delta_i}}(\rho, \chi_i^{g_i^{-1}}) \cong \mathrm{Hom}_{\mathrm{GL}_m(\mathbf{l})^{\delta_i}}(\overline{\rho}, \overline{\chi_i^{g_i^{-1}}}),$$

where  $\overline{\rho}$  and  $\overline{\chi_i^{g_i^{-1}}}$  denote the representations of  $J/J^1$  and  $J \cap G^{\delta_i} / J^1 \cap G^{\delta_i}$  whose inflations equal  $\rho := \rho|_J$  and  $\chi_i^{g_i^{-1}}$  respectively. Using (2.5.1) we get  $\omega_\kappa(-1) = \overline{\chi_i^{g_i^{-1}}(-1)^{-1}}$ , where  $\omega_\kappa$  denotes the central character of  $\kappa$ . By [HL12], Proposition 6.7,  $\mathrm{Hom}_{\mathrm{GL}_m(\mathbf{l})^{\delta_i}}(\overline{\rho}, \overline{\chi_i^{g_i^{-1}}})$  is non-zero and of dimension 1 if and only if  $\omega_{\overline{\rho}}(-1) = \overline{\chi_i^{g_i^{-1}}(-1)}$ , or equivalently

$$\omega_\rho(-1) = \chi_i^{g_i^{-1}}(-1), \tag{2.5.2}$$

where  $\omega_{\overline{\rho}}$  and  $\omega_\rho$  denote the central character of  $\overline{\rho}$  and  $\rho$  respectively. If we denote by  $\omega_\Lambda$  and  $\omega_\pi$  the central character of  $\Lambda$  and  $\pi$  respectively, then we get

$$\omega_\pi(-1) = \omega_\Lambda(-1) = \omega_\kappa(-1) \omega_\rho(-1) = \chi_i^{g_i^{-1}}(-1)^{-1} \omega_\rho(-1), \tag{2.5.3}$$

Combining (2.5.2) with (2.5.3),  $\mathrm{Hom}_{\mathrm{GL}_m(\mathbf{l})^{\delta_i}}(\overline{\rho}, \overline{\chi_i^{g_i^{-1}}})$  is non-zero and of dimension 1 if and only if  $\omega_\pi(-1) = 1$ . Thus we proved the “if” part of Theorem 0.3.1 and Theorem 0.3.3.

### 2.5.3 Other orthogonal groups

In this subsection, we finish the proof of Theorem [0.3.1](#) by showing that if  $\pi$  is distinguished, then the corresponding orthogonal group must satisfy the condition of *loc. cit.*

Let  $\tau(x) = \varepsilon^{-1} {}^t x^{-1} \varepsilon$  for  $x \in G$  as an orthogonal involution and let  $G^\tau$  be the corresponding orthogonal group. We assume that  $\varepsilon_0 = J_{d,m}$  and we write  $\tau_0 = \tau_{\varepsilon_0}$ . We choose  $[\mathfrak{a}, \beta]$ ,  $\theta$  and  $(\mathbf{J}, \Lambda)$  as in [§2.4](#).

If  $\pi$  is distinguished by  $G^\tau$ , then by Theorem [2.4.1](#), Theorem [2.4.2](#) and Lemma [2.4.4](#), there exists  $g \in G$  with  $\gamma = u\tau(g)g^{-1} \in \mathbf{J} \cap B^\times = E^\times \mathfrak{b}^\times$ , such that  $(\mathbf{J}^g, \Lambda^g)$  is a simple type of  $\pi$  satisfying

$$\tau(\mathbf{J}^g) = \mathbf{J}^g, \quad (\Lambda^g)^{\tau^\vee} \cong \Lambda^g \quad \text{and} \quad \text{Hom}_{G^\tau}(\Lambda^g, 1) \neq 0.$$

Moreover from Proposition [2.4.12](#), if we write  $\Lambda = \kappa \otimes \rho$ , then there exists a character  $\chi$  of  $\mathbf{J}^g \cap G^\tau$  trivial on  $J^{1^g} \cap G^\tau$  such that

$$0 \neq \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\Lambda^g, 1) \cong \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1}) \otimes \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi),$$

where  $\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\kappa^g, \chi^{-1})$  is of dimension 1. Thus we get

$$\text{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi) = \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi) \neq 0.$$

If we define

$$\delta(x) = (\tau(g)g^{-1})^{-1} \tau(x) \tau(g)g^{-1} = \gamma^{-1} \varepsilon_0^{-1} {}^t x^{-1} \varepsilon_0 \gamma \quad \text{for any } x \in G$$

as an orthogonal involution of  $G$ , then we have

$$J = \delta(J), \quad J^1 = \delta(J^1), \quad J^g \cap G^\tau = (J \cap G^\delta)^g \quad \text{and} \quad J^{1^g} \cap G^\tau = (J^1 \cap G^\delta)^g.$$

By definition for  $x \in B^\times$ , we have  $\delta(x) = \gamma^{-1} {}^t_E x^{-1} \gamma$ , where  ${}^t_E$  denotes the transpose with respect to  $B \cong M_m(E)$ . Since  $\gamma \in E^\times \mathfrak{b}^\times$ , the restriction of  $\delta$  induces an orthogonal involution on  $\text{GL}_m(\mathfrak{l}) \cong J/J^1$ . Thus we have

$$\text{Hom}_{\text{GL}_m(\mathfrak{l})^\delta}(\bar{\rho}, \overline{\chi^{g^{-1}}}) = \text{Hom}_{J \cap G^\delta}(\rho, \chi^{g^{-1}}) = \text{Hom}_{J^g \cap G^\tau}(\rho^g, \chi) = \text{Hom}_{\mathbf{J}^g \cap G^\tau}(\rho^g, \chi) \neq 0,$$

where  $\bar{\rho}$  and  $\overline{\chi^{g^{-1}}}$  denote the representations of  $J/J^1$  and  $J \cap G^\delta / J^1 \cap G^\delta$  whose inflations equal  $\rho$  and  $\chi^{g^{-1}}$  respectively.

By Proposition [2.5.1](#), there exists  $\mathfrak{l}_m/\mathfrak{l}$  as a field extension of degree  $m$  and an embedding  $\mathfrak{l}_m \hookrightarrow M_m(\mathfrak{l})$  such that  $\mathfrak{l}_m$  is  $\delta$ -split. Using this embedding,  $\mathfrak{l}_m^\times$  can be regarded as a  $\delta$ -split subgroup of  $J$ . We denote by  $E_m = E[\mathfrak{l}_m^\times]$  the maximal unramified extension of degree  $m$  over  $E$  which is a  $\delta$ -split, thus  $E_m^g$  is a  $\tau$ -split which is  $F$ -isomorphic to  $E_m$ . In other words, there exists an  $F$ -embedding  $\iota : E_m \hookrightarrow M_n(F)$  which is  $\tau$ -split. We have proved Proposition [0.3.5](#).

Using the results in [§2.5.2](#) we know that  $\pi$  is distinguished by  $G^{\tau J_n}$ , thus we may in particular consider the argument above for  $\varepsilon = J_n$  and we deduce that  $E_m$  is  $\tau_{J_n}$ -split, that is, the condition of the following lemma is satisfied.

**Lemma 2.5.2** ([\[Hak13\]](#), Lemma 6.4). *Assume that there exists a  $J$ -symmetric embedding  $E_m \hookrightarrow M_n(F)$ . Then for  $Y_{E_m/F} = E_m^\times / (E_m^{\times 2} F^\times)$  and  $\mathcal{O}^{E_m}$  the set of  $E_m^\times$ -orbits of orthogonal involutions  $\tau$  such that  $E_m$  is  $\tau$ -split, the map*

$$\mu_{E_m/F} : Y_{E_m/F} \rightarrow \mathcal{O}^{E_m}$$

*which sends the coset of  $x \in E_m^\times$  to the orbit of  $\tau_{J_n x}$  is a bijection.*

In particular we have  $\tau_{J_n}, \tau \in \mathcal{O}^{E_m}$ . Since  $E_m/T_m$  is totally wildly ramified, as in Lemma [2.2.19](#) it is easy to see that  $E_m^\times/E_m^{\times 2}F^\times \cong T_m^\times/T_m^{\times 2}F^\times$ , and we denote by  $y_{T_m/F}$  the corresponding cardinality. Thus by [\[Hak13\]](#), Lemma 6.2,  $y_{T_m/F} - 1$  equals the number of quadratic extensions of  $F$  contained in  $T_m$ . Furthermore by [\[Hak13\]](#), Lemma 3.8 we have

$$y_{T_m/F} = \begin{cases} 1 & \text{Case (i),} \\ 2 & \text{Case (ii),} \\ 4 & \text{Case (iii).} \end{cases}$$

Thus in **Case (i)**, we have  $|\mathcal{O}^{E_m}| = 1$ , which means that  $\mathcal{O}^{E_m}$  consists of the  $E_m^\times$ -orbit represented by the split involution  $\tau_{J_n}$ , thus  $G^\tau$  is split. In **Case (ii)**, we have  $|\mathcal{O}^{E_m}| = 2$ . And by direct calculation,

$$\det(J_n E_m^\times)F^{\times 2}/F^{\times 2} = (-1)^{n(n-1)/2} \mathbb{N}_{E_m/F}(E_m^\times)F^{\times 2}/F^{\times 2} = (-1)^{n(n-1)/2} \mathbb{N}_{T_m/F}(T_m^\times)F^{\times 2}/F^{\times 2},$$

which is of order 2. Thus  $\mathcal{O}^{E_m}$  consists of two  $E_m^\times$ -orbits, one of which is split, and the other is quasisplit but not split with the determinants of its corresponding symmetric matrices contained in  $(-1)^{n(n-1)/2} \mathbb{N}_{T_m/F}(T_m^\times)F^{\times 2} \setminus (-1)^{n(n-1)/2} F^{\times 2}$ . Thus  $G^\tau$  is either split or quasisplit that satisfies the condition of Theorem [0.3.1](#). In **Case (iii)**,

$$\det(J_n E_m^\times)F^{\times 2}/F^{\times 2} = (-1)^{n(n-1)/2} \mathbb{N}_{E_m/F}(E_m^\times)F^{\times 2}/F^{\times 2} = \{1\}.$$

Thus by Proposition [2.2.5](#),  $G^\tau$  is either split or non-quasisplit. Combining these three cases together, we have shown that  $G^\tau$  must satisfy the condition of Theorem [0.3.1](#), which finishes the ‘‘only if’’ part of Theorem [0.3.1](#).

## Chapter 3

# Explicit base change lift and automorphic induction for supercuspidal representations

### 3.1 General notations

For  $F$  a non-archimedean locally compact field of residue characteristic  $p$ , we denote by  $\mathfrak{o}_F$  its ring of integers, by  $\mathfrak{p}_F$  the maximal ideal, by  $\mathbf{k}_F$  its residual field, by  $\boldsymbol{\mu}_F$  the group of roots of unity of order relatively prime to  $p$  in  $F^\times$ , by  $|\cdot|_F$  the corresponding discrete absolute value and by  $v_F$  the corresponding discrete valuation. We leave the notations  $A_F$  and  $C_F$  to stand for a certain  $F$ -algebra isomorphic to  $M_m(F)$  with the positive integer  $m$  to be specified, whose exact definitions will be given later according to the context, and we denote by  $\det_F$  the determinant map and by  $\mathrm{Tr}_{A_F}$  or  $\mathrm{Tr}_{C_F}$  the trace map. We leave the notation  $G_F$  to stand for  $A_F^\times$  or  $C_F^\times$  according to the context. For  $E/F$  a finite extension of non-archimedean locally compact fields, we denote by  $f(E/F)$  its residue class degree and by  $e(E/F)$  its ramification index, and by  $N_{E/F}$  and  $\mathrm{Tr}_{E/F}$  the norm and trace map respectively. Later without further mention, any finite separable extension  $E/F$  should be regarded as subextension in a fixed algebraic separable closure  $\overline{F}/F$ . We denote by  $\mathcal{W}_F$  the Weil group of  $F$  and by  $\mathcal{W}_E$  the Weil group of  $E$  as a subgroup of  $\mathcal{W}_F$ .

We regard  $\mathrm{GL}_n(F)$  as a locally profinite group endowed with  $p$ -adic topology. By representations of a locally profinite group we always mean smooth complex representations, and by characters we mean one dimensional smooth representations with complex values. We will use  $^\vee$  “check” to denote the contragredient of a smooth representation. For a character of  $F^\times$ , it is called unramified if its restriction to  $\mathfrak{o}_F^\times$  is trivial, and called tamely ramified if its restriction to  $1 + \mathfrak{p}_F$  is trivial.

We fix a prime number  $p$  and a non-archimedean locally compact field  $F_0$  of residue characteristic  $p$ , we fix once and for all an algebraic separable closure  $\overline{F}_0/F_0$ , and we write  $|\mathbf{k}_{F_0}| = q$ . We fix a character  $\psi_{F_0} : F_0 \rightarrow \mathbb{C}^\times$  of level 0, saying that it is trivial on  $\mathfrak{p}_{F_0}$  but not on  $\mathfrak{o}_{F_0}$ . For any  $F/F_0$  as a finite separable tamely ramified extension, we choose  $\psi_F = \psi_{F_0} \circ \mathrm{Tr}_{F/F_0}$  which is a character of  $F$  of level 0.

For  $G$  a group and  $\Sigma$  a subgroup of the group of automorphisms of  $G$ , we define the semi-direct product  $G \rtimes \Sigma$  as a group via the relations

$$(g, \sigma) \cdot (g', \sigma') = (g\sigma(g'), \sigma\sigma')$$

for any  $g, g' \in G$  and  $\sigma, \sigma' \in \Sigma$ .



## 3.2 Preliminaries for the simple type theory

In this section, we briefly recall the simple type theory built up by Bushnell and Kutzko, and further developed by Bushnell and Henniart in a series of articles. Our main reference will be [BK93], [BH96] and [BH14b].

### 3.2.1 Simple strata and simple characters

Let  $F$  be a non-archimedean locally compact field of residue characteristic  $p$  and let  $A = A_F = M_n(F)$  for  $n \geq 1$  fixed. We consider a *simple stratum*  $[\mathfrak{a}, m_\beta, 0, \beta]$  in  $A$  defined as in [BK93], section 1, where  $\mathfrak{a}$  is a hereditary in  $A$ , and  $\beta$  is an element in  $A^\times$  satisfying  $\beta\mathfrak{a} = \mathfrak{p}_\mathfrak{a}^{-m_\beta}$  for  $m_\beta$  a positive integer, such that  $E := F[\beta]$  is a field normalizing  $\mathfrak{a}$ . Since  $m_\beta$  is uniquely determined by  $\mathfrak{a}$  and  $\beta$ , we abbreviate the above notation to  $[\mathfrak{a}, \beta]$ . We denote by  $B$  the centralizer of  $E$  in  $A$  and we write  $\mathfrak{b} = \mathfrak{a} \cap B$  which is a hereditary order in  $B$ . We say that  $[\mathfrak{a}, \beta]$  is maximal if  $\mathfrak{b}$  is a maximal order in  $B$ . In particular for  $E'/E$  a finite extension with  $[E' : F] = n$ , we choose an  $F$ -algebra embedding  $E' \hookrightarrow A$  whose restriction to  $E$  is identity. Thus, the ideal chain of  $E'$  gives rise to a hereditary order in  $A$ , denoted by  $\mathfrak{a}_F(E')$ , and  $[\mathfrak{a}_F(E'), \beta]$  is a simple stratum in  $A$  which is maximal if and only if  $E'/E$  is unramified.

Associated with  $[\mathfrak{a}, \beta]$ , there is a chain of open compact subgroups (see [BK93], section 3)

$$H^1(\mathfrak{a}, \beta) \subset J^1(\mathfrak{a}, \beta) \subset J(\mathfrak{a}, \beta)$$

of  $\mathfrak{a}^\times$ , where  $J(\mathfrak{a}, \beta) = \mathfrak{b}^\times J^1(\mathfrak{a}, \beta)$ . We denote by  $\mathbf{J}(\mathfrak{a}, \beta)$  the subgroup of  $A^\times$  generated by  $J(\mathfrak{a}, \beta)$  and the normalizer of  $\mathfrak{b}^\times$  in  $B^\times$ , which is compact modulo  $F^\times$ . We may write  $H^1(\mathfrak{a}, \beta) = 1 + \mathfrak{h}^1(\mathfrak{a}, \beta)$  and  $J^1(\mathfrak{a}, \beta) = 1 + \mathfrak{j}^1(\mathfrak{a}, \beta)$  for  $\mathfrak{h}^1(\mathfrak{a}, \beta)$  and  $\mathfrak{j}^1(\mathfrak{a}, \beta)$  sub- $\mathfrak{o}_F$ -lattices of  $\mathfrak{p}_\mathfrak{a}$ .

We attach to  $[\mathfrak{a}, \beta]$  a finite set of characters of  $H^1(\mathfrak{a}, \beta)$  as in [BK93], §3.2, denoted by  $\mathcal{C}(\mathfrak{a}, \beta, \psi_F)$  and called *simple characters*, and we simply write  $\mathcal{C}(\mathfrak{a}, \beta)$  by omitting  $\psi_F$ . We use small Greek letter  $\theta$  (with additional superscripts and subscripts) to denote a simple character.

Given a simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta$  of  $H^1(\mathfrak{a}, \beta)$ , the normalizer of  $\theta$  equals  $\mathbf{J}(\mathfrak{a}, \beta)$ , whose unique maximal compact subgroup equals  $J(\mathfrak{a}, \beta)$ . The unique maximal normal pro- $p$ -subgroup of  $J(\mathfrak{a}, \beta)$  equals  $J^1(\mathfrak{a}, \beta)$ . Thus even if we change our choice of simple stratum  $[\mathfrak{a}, \beta]$ , the chain of subgroups

$$H^1(\mathfrak{a}, \beta) \subset J^1(\mathfrak{a}, \beta) \subset J(\mathfrak{a}, \beta) \subset \mathbf{J}(\mathfrak{a}, \beta)$$

is only determined by  $\theta$ . In particular,  $[\mathfrak{a}, \beta]$  is maximal if and only if  $J(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta) \cong \mathrm{GL}_m(\mathbf{k}_E)$ , which only depends on  $\theta$ . Thus we say that  $\theta$  is maximal if for one maximal simple stratum  $[\mathfrak{a}, \beta]$  we have  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ .

We explain our convention for the “null” case for a simple stratum  $[\mathfrak{a}, \beta]$ . In this case conventionally we write  $\beta = 0$ ,  $E = F$ ,  $A = B$ ,  $\mathfrak{a} = \mathfrak{b}$  and  $H^1(\mathfrak{a}, \beta) = J^1(\mathfrak{a}, \beta) = 1 + \mathfrak{p}_\mathfrak{a}$ . Moreover, the set  $\mathcal{C}(\mathfrak{a}, \beta)$  is a singleton consisting of the trivial character of  $H^1(\mathfrak{a}, \beta)$ . Later on our discussion will also include this case.

### 3.2.2 Endo-class and interior tame lifting

For  $[\mathfrak{a}, \beta]$  a simple stratum in  $M_n(F)$  and  $[\mathfrak{a}', \beta]$  a simple stratum in  $M_{n'}(F)$ , where we identify  $\beta$  with an element in both  $M_n(F)$  and  $M_{n'}(F)$  via certain  $F$ -algebra embeddings, we have a bijection (see [BK93], §3.6.)

$$t_{\mathfrak{a}, \mathfrak{a}'}^\beta : \mathcal{C}(\mathfrak{a}, \beta) \longrightarrow \mathcal{C}(\mathfrak{a}', \beta)$$

called the *transfer map*. Now let  $[\mathfrak{a}_1, \beta_1]$  and  $[\mathfrak{a}_2, \beta_2]$  be simple strata in  $M_{n_1}(F)$  and  $M_{n_2}(F)$  respectively. We say that a simple character  $\theta_1 \in \mathcal{C}(\mathfrak{a}_1, \beta_1)$  is *endo-equivalent* to another simple character  $\theta_2 \in \mathcal{C}(\mathfrak{a}_2, \beta_2)$ , if there exist simple strata  $[\mathfrak{a}', \beta_1]$  and  $[\mathfrak{a}', \beta_2]$  in  $M_{n'}(F)$  and simple characters

$\theta'_1 \in \mathcal{C}(\mathfrak{a}', \beta_1)$  and  $\theta'_2 \in \mathcal{C}(\mathfrak{a}', \beta_2)$ , such that  $\theta'_i = t_{\mathfrak{a}_i, \mathfrak{a}'}^{\beta_i}(\theta_i)$  for  $i = 1, 2$ , and  $\theta'_1$  is  $\mathrm{GL}_{n'}(F)$ -conjugate to  $\theta'_2$ . This definition does not depend on the choice of  $n'$ ,  $[\mathfrak{a}', \beta_1]$ ,  $[\mathfrak{a}', \beta_2]$  and the embeddings, which indeed gives an equivalence relation on the set of all the simple characters (see [BH96], section 8). We use the capital Greek letter  $\Theta$  (with superscripts and subscripts) to denote the corresponding equivalence class, called *endo-class*, of a certain simple character. We use  $\mathcal{E}(F)$  to denote the set of endo-classes of  $F$ . For a simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta$  in the endo-class  $\Theta$ , we say that  $\theta$  realizes  $\Theta$  and  $E = F[\beta]$  is a parameter field of  $\Theta$ . Such a parameter field is not unique, however its degree  $d = [E : F]$  and its maximal tamely ramified subextension  $T/F$  up to  $F$ -isomorphism are uniquely determined by  $\Theta$ . We call  $d$  the degree of  $\Theta$  and  $T/F$  the tame parameter field of  $\Theta$ .

Now we consider the *interior tame lifting* (see [BH96], section 7). Given a simple stratum  $[\mathfrak{a}, \beta]$  in  $A$ , let  $L/F$  be a tamely ramified subextension in  $A$  such that  $L[\beta]$  is a subfield of  $A$  normalizing  $\mathfrak{a}$ . We denote by  $A_L$  the centralizer of  $L$  in  $A$ , and we write  $\mathfrak{a}_L = \mathfrak{a} \cap A_L$  for the hereditary order in  $A_L$ . Thus  $[\mathfrak{a}_L, \beta]$  is a simple stratum in  $A_L$  and we have  $H^1(\mathfrak{a}_L, \beta) = A_L^\times \cap H^1(\mathfrak{a}, \beta)$ ,  $J^1(\mathfrak{a}_L, \beta) = A_L^\times \cap J^1(\mathfrak{a}, \beta)$ ,  $J(\mathfrak{a}_L, \beta) = A_L^\times \cap J(\mathfrak{a}, \beta)$  and  $\mathbf{J}(\mathfrak{a}_L, \beta) = A_L^\times \cap \mathbf{J}(\mathfrak{a}, \beta)$ . Moreover, given a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , we know that  $\theta_L := \theta|_{H^1(\mathfrak{a}_L, \beta)}$  is a simple character in  $\mathcal{C}(\mathfrak{a}_L, \beta)$ . We call such  $\theta_L$  the interior  $L/F$ -lift of  $\theta$ .

In general, given a tamely ramified extension  $L/F$ , there exists a surjection (see [BH14b], §2.3)

$$i_{L/F} : \mathcal{E}(L) \longrightarrow \mathcal{E}(F)$$

with finite fibers. For  $\Theta \in \mathcal{E}(F)$ , the elements in the finite set  $i_{L/F}^{-1}(\Theta)$  are called the  $L/F$ -lifts of  $\Theta$ . To find out all the fibers, we choose a simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  realizing  $\Theta$ . We have an  $F$ -algebra isomorphism  $F[\beta] \otimes_F L \cong \prod_{i=1}^k L_i$ , where  $L_i$  are fields over  $L$ . Then there is a canonical bijection

$$\begin{aligned} \{L_1, \dots, L_k\} &\longrightarrow i_{L/F}^{-1}(\Theta) \\ L_i &\longmapsto \Theta_i, \end{aligned}$$

such that  $L_i$  is a parameter field of  $\Theta_i$ . In particular, if  $\theta_L$  is the interior  $L/F$ -lift of  $\theta$ , then the endo-class of  $\theta_L$  is a  $L/F$ -lift of that of  $\theta$ .

Finally we state and proof two technical lemmas to end this subsection.

**Lemma 3.2.1.** *Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $A$ , let  $L/F$  be a tamely ramified subextension in  $A$  such that  $L[\beta]$  is a field normalizing  $\mathfrak{a}$ , and let  $L'$  be a subfield of  $L$  over  $F$ . Then for  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , we have  $(\theta_{L'})_L = \theta_L$ .*

*Proof.* By definition, we have  $\theta_{L'} = \theta|_{H^1(\mathfrak{a}_{L'}, \beta)}$  and  $\theta_L = \theta|_{H^1(\mathfrak{a}_L, \beta)}$ . Since  $A_L$  is also the centralizer of  $L$  in  $A_{L'}$ , considering the tamely ramified extension  $L/L'$  we have  $\mathfrak{a}_L = (\mathfrak{a}_{L'})_L$ ,  $[\mathfrak{a}_L, \beta] = [(\mathfrak{a}_{L'})_L, \beta]$ ,  $H^1(\mathfrak{a}_L, \beta) = A_L^\times \cap H^1(\mathfrak{a}_{L'}, \beta)$  and  $\theta_L = \theta_{L'}|_{H^1(\mathfrak{a}_L, \beta)}$ , which finishes the proof.  $\square$

**Lemma 3.2.2.** *Let  $[\mathfrak{a}, \beta]$  and  $[\tilde{\mathfrak{a}}, \beta]$  be two maximal simple strata in  $M_n(F)$  and  $M_{n'}(F)$  respectively and let  $L/F$  be a tamely ramified subextension in both  $M_n(F)$  and  $M_{n'}(F)$ , such that  $L[\beta]$  is a subfield of  $M_n(F)$  and  $M_{n'}(F)$  normalizing both  $\mathfrak{a}$  and  $\tilde{\mathfrak{a}}$ . Then for any  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , we have  $(t_{\mathfrak{a}, \tilde{\mathfrak{a}}}^\beta(\theta))_L = t_{\mathfrak{a}_L, \tilde{\mathfrak{a}}_L}^\beta(\theta_L)$  as a simple character in  $\mathcal{C}(\tilde{\mathfrak{a}}_L, \beta)$ .*

*Proof.* We write  $E' = L[\beta]$  as a field containing  $E$  and normalizing both  $\mathfrak{a}$  and  $\tilde{\mathfrak{a}}$ , and we write  $t = [L : F]$  and  $s = [E' : F]$ . We need to show that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{C}(\mathfrak{a}_L, \beta) & \xrightarrow{t_{\mathfrak{a}_L, \tilde{\mathfrak{a}}_L}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}_L, \beta) \\
\uparrow |_{H^1(\mathfrak{a}_L, \beta)} & & \uparrow |_{H^1(\tilde{\mathfrak{a}}_L, \beta)} \\
\mathcal{C}(\mathfrak{a}, \beta) & \xrightarrow{t_{\mathfrak{a}, \tilde{\mathfrak{a}}}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}, \beta)
\end{array} \tag{3.2.1}$$

First we assume that  $n = n'$ , thus both  $\mathfrak{a}$  and  $\tilde{\mathfrak{a}}$  are hereditary orders in  $M_n(F)$ . By [BK93], Theorem 3.6.1,  $\tilde{\theta} := t_{\mathfrak{a}, \tilde{\mathfrak{a}}}^\beta(\theta)$  is the unique simple character in  $\mathcal{C}(\tilde{\mathfrak{a}}, \beta)$  such that  $\theta|_{H^1(\mathfrak{a}, \beta) \cap H^1(\tilde{\mathfrak{a}}, \beta)} = \tilde{\theta}|_{H^1(\mathfrak{a}, \beta) \cap H^1(\tilde{\mathfrak{a}}, \beta)}$ . By definition,  $[\mathfrak{a}_L, \beta]$  and  $[\tilde{\mathfrak{a}}_L, \beta]$  are simple strata in  $M_{n/t}(L)$ , and  $\theta_L$  and  $\tilde{\theta}_L$  are the interior  $L/F$ -lifts of  $\theta$  and  $\tilde{\theta}$  respectively, and moreover

$$\begin{aligned}
\theta_L|_{H^1(\mathfrak{a}_L, \beta) \cap H^1(\tilde{\mathfrak{a}}_L, \beta)} &= \theta|_{\mathrm{GL}_{n/t}(L) \cap H^1(\mathfrak{a}, \beta) \cap H^1(\tilde{\mathfrak{a}}, \beta)} = \tilde{\theta}|_{\mathrm{GL}_{n/t}(L) \cap H^1(\mathfrak{a}, \beta) \cap H^1(\tilde{\mathfrak{a}}, \beta)} \\
&= \tilde{\theta}_L|_{H^1(\mathfrak{a}_L, \beta) \cap H^1(\tilde{\mathfrak{a}}_L, \beta)},
\end{aligned}$$

thus  $\tilde{\theta}_L = t_{\mathfrak{a}_L, \tilde{\mathfrak{a}}_L}^\beta(\theta_L)$ , which finishes the proof of the case  $n = n'$ .

Now we consider the case  $n' = nk$  for  $k$  a positive integer. By [BK93], §1.2, since  $E'$  normalizes  $\mathfrak{a}$ , there exist an  $E'$ -vector space  $V$  of dimension  $n/s$  and an  $\mathfrak{o}_{E'}$ -lattice chain  $\mathcal{L} = \{L_i\}$  in  $V$ , such that  $\mathrm{End}_F(V) \cong M_n(F)$  induces  $\mathrm{End}_{\mathfrak{o}_F}^0(\mathcal{L}) = \mathfrak{a}$ , where  $\mathrm{End}_{\mathfrak{o}_F}^0(\mathcal{L})$  denotes the ring of endomorphisms of  $\mathcal{L}$  as in *loc. cit.* We write  $\tilde{V} = V \oplus \dots \oplus V$  of  $k$ -copies and  $\tilde{\mathcal{L}} = \mathcal{L} \oplus \dots \oplus \mathcal{L}$  of  $k$ -copies as a  $\mathfrak{o}_{E'}$ -lattice chain of  $\tilde{V}$ , and we have  $\mathrm{End}_{\mathfrak{o}_F}^0(\tilde{\mathcal{L}}) \cong \tilde{\mathfrak{a}}$  given by the isomorphism  $\mathrm{End}_F(\tilde{V}) \cong M_{n'}(F)$ , which is naturally induced by  $\mathrm{End}_F(V) \cong M_n(F)$  given above. We further assume  $\mathrm{End}_{\mathfrak{o}_F}^0(\tilde{\mathcal{L}}) = \tilde{\mathfrak{a}}$ . To simplify the notation, we write

$$G = \mathrm{GL}_F(V') \quad \text{and} \quad M = \mathrm{GL}_F(V) \times \dots \times \mathrm{GL}_F(V)$$

of  $k$ -copies which is actually a Levi subgroup of  $G$ . Thus by [BK93], Proposition 7.1.19, we have

$$H^1(\tilde{\mathfrak{a}}, \beta) \cap M = H^1(\mathfrak{a}, \beta) \times \dots \times H^1(\mathfrak{a}, \beta)$$

of  $k$ -copies and for  $\tilde{\theta} := t_{\mathfrak{a}, \tilde{\mathfrak{a}}}^\beta(\theta)$  we have

$$\tilde{\theta}|_{H^1(\tilde{\mathfrak{a}}, \beta) \cap M} = \theta \otimes \dots \otimes \theta.$$

Similarly we may consider interior  $L/F$ -lifts. From our construction,  $\mathfrak{a}_L = \mathrm{End}_{\mathfrak{o}_L}^0(\mathcal{L})$  and  $\tilde{\mathfrak{a}}_L = \mathrm{End}_{\mathfrak{o}_L}^0(\tilde{\mathcal{L}})$  are hereditary orders in  $\mathrm{End}_L(V)$  and  $\mathrm{End}_L(\tilde{V})$ , and  $[\mathfrak{a}_L, \beta]$  and  $[\tilde{\mathfrak{a}}_L, \beta]$  are corresponding simple strata respectively. We write

$$G_L = \mathrm{GL}_L(\tilde{V}) \quad \text{and} \quad M_L = \mathrm{GL}_L(V) \times \dots \times \mathrm{GL}_L(V)$$

as a Levi subgroup of  $G_L$ , and if we denote by  $\theta_L$  and  $\tilde{\theta}_L$  the interior  $L/F$ -lifts of  $\theta$  and  $\tilde{\theta}$  respectively, by definition we have

$$\tilde{\theta}_L|_{H^1(\tilde{\mathfrak{a}}_L, \beta) \cap M_L} = \tilde{\theta}|_{H^1(\tilde{\mathfrak{a}}, \beta) \cap M_L} = \theta|_{H^1(\mathfrak{a}, \beta) \cap \mathrm{GL}_L(V)} \otimes \dots \otimes \theta|_{H^1(\mathfrak{a}, \beta) \cap \mathrm{GL}_L(V)} = \theta_L \otimes \dots \otimes \theta_L.$$

Using [BK93], Proposition 7.1.19 again, we must have  $\tilde{\theta}_L = t_{\mathfrak{a}_L, \tilde{\mathfrak{a}}_L}^\beta(\theta_L)$ .

Finally we consider the general case. We choose a simple stratum  $[\mathfrak{a}', \beta]$  in  $M_{nn'/s}(F)$ , such that  $\mathfrak{a}'$  is determined by the  $\mathfrak{o}_{E'}$ -lattice chain as the direct sum of  $n'/s$ -copies of the lattice chain corresponding

to  $\mathfrak{a}$ . Similarly we choose a simple stratum  $[\tilde{\mathfrak{a}}', \beta]$  in  $M_{nn'/s}(F)$ , such that  $\tilde{\mathfrak{a}}'$  is determined by the  $\mathfrak{o}_{E'}$ -lattice chain as the direct sum of  $n/s$ -copies of the lattice chain corresponding to  $\tilde{\mathfrak{a}}$ . Thus the following two diagrams are commutative:

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{a}_L, \beta) & \xrightarrow{t_{\mathfrak{a}_L, \mathfrak{a}'_L}^\beta} & \mathcal{C}(\mathfrak{a}'_L, \beta) \\ \uparrow |_{H^1(\mathfrak{a}_L, \beta)} & & \uparrow |_{H^1(\mathfrak{a}'_L, \beta)} \\ \mathcal{C}(\mathfrak{a}, \beta) & \xrightarrow{t_{\mathfrak{a}, \mathfrak{a}'}^\beta} & \mathcal{C}(\mathfrak{a}', \beta) \end{array} \quad \begin{array}{ccc} \mathcal{C}(\tilde{\mathfrak{a}}_L, \beta) & \xrightarrow{t_{\tilde{\mathfrak{a}}_L, \tilde{\mathfrak{a}}'_L}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}'_L, \beta) \\ \uparrow |_{H^1(\tilde{\mathfrak{a}}_L, \beta)} & & \uparrow |_{H^1(\tilde{\mathfrak{a}}'_L, \beta)} \\ \mathcal{C}(\tilde{\mathfrak{a}}, \beta) & \xrightarrow{t_{\tilde{\mathfrak{a}}, \tilde{\mathfrak{a}}'}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}', \beta) \end{array}$$

Since both  $[\mathfrak{a}', \beta]$  and  $[\tilde{\mathfrak{a}}', \beta]$  are simple strata in  $M_{nn'/s}(F)$ , using the previous case the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{a}'_L, \beta) & \xrightarrow{t_{\mathfrak{a}'_L, \tilde{\mathfrak{a}}'_L}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}'_L, \beta) \\ \uparrow |_{H^1(\mathfrak{a}'_L, \beta)} & & \uparrow |_{H^1(\tilde{\mathfrak{a}}'_L, \beta)} \\ \mathcal{C}(\mathfrak{a}', \beta) & \xrightarrow{t_{\mathfrak{a}', \tilde{\mathfrak{a}}'}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}', \beta) \end{array}$$

Combining these three diagrams together we get [\(3.2.1\)](#), which finishes the proof.  $\square$

### 3.2.3 Full Heisenberg representation

Let  $[\mathfrak{a}, \beta]$  be a simple stratum in  $A$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  be a simple character. We denote by  $I(\theta)$  the set of elements of  $A^\times$  which intertwine  $\theta$ . There exists a unique irreducible representation  $\eta$  of  $J^1(\mathfrak{a}, \beta)$  containing  $\theta$ , called the *Heisenberg representation* of  $\theta$ . We further consider a representation  $\kappa$  of  $\mathbf{J}(\mathfrak{a}, \beta)$  as an extension of  $\eta$ . Such a representation is called a *full Heisenberg representation* of  $\theta$  if every element in  $I(\theta)$  also intertwines  $\kappa$ , whose existence is guaranteed by [\[BH14b\]](#), §3.2. In particular  $\kappa := \kappa|_{J(\mathfrak{a}, \beta)}$  is a so-called  $\beta$ -extension.

### 3.2.4 Extended maximal simple type and supercuspidal representation

We fix a maximal simple stratum  $[\mathfrak{a}, \beta]$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$  with  $E = F[\beta]$ , we let  $T$  be the maximal tamely ramified subextension of  $E$  over  $F$ , and we write  $d = [E : F]$  and  $n = md$ . We consider an *extended maximal simple type*<sup>1</sup>  $(\mathbf{J}, \Lambda)$  of  $\theta$  in the sense of [\[BK93\]](#), section 6, such that  $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$  and  $\Lambda$  is a representation of  $\mathbf{J}$  whose restriction to  $H^1(\mathfrak{a}, \beta)$  is a multiple of  $\theta$ . We write  $\mathcal{T}(\theta)$  for the set of isomorphism classes of representations  $\Lambda$  of  $\mathbf{J}$ , such that  $(\mathbf{J}, \Lambda)$  is an extended maximal simple type of  $\theta$ . We denote by  $\mathcal{T}_m(E)$  the set of isomorphism classes of representations  $\rho$  of  $E^\times \mathrm{GL}_m(\mathfrak{o}_E)$  trivial on  $1 + M_m(\mathfrak{p}_E)$ , such that  $\rho|_{\mathrm{GL}_m(\mathfrak{o}_E)}$  is the inflation of a supercuspidal representation of  $\mathrm{GL}_m(\mathfrak{k}_E) \cong \mathrm{GL}_m(\mathfrak{o}_E)/(1 + M_m(\mathfrak{p}_E))$ . Via the natural isomorphism  $\mathbf{J}/J^1 \cong E^\times \mathrm{GL}_m(\mathfrak{o}_E)/(1 + M_m(\mathfrak{p}_E))$ , we will also regard elements in  $\mathcal{T}_m(E)$  as representations of  $\mathbf{J}$  trivial on  $J^1 = J^1(\mathfrak{a}, \beta)$  without further mention.

We denote by  $\Theta$  the endo-class of  $\theta$  and by  $\mathcal{A}_m^0(F, \Theta)$  the set of isomorphism classes of supercuspidal representations of  $\mathrm{GL}_{m \deg(\Theta)}(F)$  whose endo-class is  $\Theta$ .

**Proposition 3.2.3** ([\[BH14b\]](#), §3.6, [\[BK93\]](#), §6.2, §8.4.). (1) *Given any full Heisenberg representation  $\kappa$  of  $\theta$ , we have the following bijection*

$$\mathcal{T}_m(E) \longrightarrow \mathcal{T}(\theta), \quad \rho \longmapsto \kappa \otimes \rho.$$

<sup>1</sup>Later we will always write *simple type* for short.

(2) We have the following bijection

$$\mathcal{T}(\theta) \longrightarrow \mathcal{A}_m^0(F, \Theta), \quad \Lambda \longmapsto \text{ind}_{\mathbf{J}}^G \Lambda,$$

where  $\text{ind}_{\mathbf{J}}^G$  represents the compact induction.

Finally, we denote by  $X_1(T_m)$  (resp.  $X_1(E_m)$ ) the set of tamely ramified characters of  $T_m^\times$  (resp.  $E_m^\times$ ) endowed with a  $\Delta$ -action, where  $T_m$  (resp.  $E_m$ ) is the unramified extension of degree  $m$  over  $T$  (resp.  $E$ ) and  $\Delta = \text{Gal}(E_m/E) \cong \text{Gal}(T_m/T)$ . Since  $E_m/T_m$  is wildly ramified, it is easy to see that

$$X_1(T_m) \longrightarrow X_1(E_m), \quad \xi \longmapsto \xi_E := \xi \circ \mathbf{N}_{E_m/T_m} \quad (3.2.2)$$

is a bijection. We say that  $\xi$  is  $\Delta$ -regular if  $\xi^\delta$  is not equal to  $\xi$  for any non-trivial  $\delta \in \Delta$  and we have a similar definition for  $\xi_E$  to be  $\Delta$ -regular. We write  $X_1(T_m)^{\Delta\text{-reg}}$  ( $X_1(E_m)^{\Delta\text{-reg}}$ ) for the set of  $\Delta$ -regular tamely ramified characters of  $T_m^\times$  (resp.  $E_m^\times$ ), then it is easy to see that (3.2.2) induces a bijection between these two sets. We write  $\Delta \backslash X_1(T_m)^{\Delta\text{-reg}}$  (resp.  $\Delta \backslash X_1(E_m)^{\Delta\text{-reg}}$ ) for the corresponding quotient with respect to the  $\Delta$ -action.

**Proposition 3.2.4** ([BH14b], §3.5, Proposition). (1) For  $\xi_E \in X_1(E_m)^{\Delta\text{-reg}}$ , there exists a unique representation  $\rho_{\xi_E} \in \mathcal{T}_m(E)$  such that

$$\text{tr}(\rho_{\xi_E})(z\zeta) = (-1)^{m-1} \xi_E(z) \sum_{\delta \in \Delta} \xi_E^\delta(\zeta)$$

for any  $z \in E^\times$  and any  $\Delta$ -regular element  $\zeta \in \mu_{E_m}$ ;

(2) The representation  $\rho_{\xi_E}$  depends only on the  $\Delta$ -orbit of  $\xi_E$ , which induces the following bijection

$$\Delta \backslash X_1(E_m)^{\Delta\text{-reg}} \longrightarrow \mathcal{T}_m(E), \quad \xi_E \longmapsto \rho_{\xi_E};$$

(3) Combining (2) with (3.2.2), we have the following bijection

$$\Delta \backslash X_1(T_m)^{\Delta\text{-reg}} \longrightarrow \mathcal{T}_m(E), \quad \xi \longmapsto \rho_{\xi_E}, \quad (3.2.3)$$

and we write  $\rho_\xi := \rho_{\xi_E}$  for short.

As a corollary of the above two propositions, we have:

**Corollary 3.2.5.** If  $m = 1$ , then for  $\kappa$  a fixed full Heisenberg representation of  $\theta$  and any simple type  $\Lambda$  containing  $\theta$ , there exists a unique tamely ramified character  $\xi$  of  $T_m^\times = T^\times$ , such that  $\Lambda \cong \kappa \cdot (\xi \circ \mathbf{N}_{E/T})$ , where  $\xi \circ \mathbf{N}_{E/T}$  is regarded as a character of  $\mathbf{J}$  via the canonical isomorphism  $\mathbf{J}/J^1 \cong E^\times/(1 + \mathfrak{p}_E)$ .

Finally for  $\pi$  a supercuspidal representation of  $\text{GL}_n(F)$  containing  $\theta$  and  $\kappa$  a  $\beta$ -extension of  $\theta$ , the finite dimensional vector space  $\text{Hom}_{J^1}(\kappa, \pi)$  is endowed with a  $J/J^1$ -action given by

$$g \cdot f := \pi(g^{-1}) \circ f \circ \kappa(g), \quad g \in J, f \in \text{Hom}_{J^1}(\kappa, \pi),$$

thus is realized as a representation of  $\text{GL}_m(\mathbf{k}_E) \cong J/J^1$  denoted by  $K_\kappa(\pi)$ , which is necessarily irreducible and supercuspidal (cf. [SZ99]). If we choose a full Heisenberg representation  $\kappa$  extending  $\kappa$ , then we get a  $\Delta$ -regular character  $\xi_E$  of  $E_m^\times$  which is trivial on  $1 + \mathfrak{p}_{E_m}$  by the propositions above. In particular we get a  $\text{Gal}(\mathbf{k}_{E_m}/\mathbf{k}_E)$ -regular character  $\xi$  of  $\mathbf{k}_{E_m}^\times$  whose inflation equals  $\xi_E|_{\mathfrak{o}_{E_m}^\times}$  via the isomorphism  $\mathbf{k}_{E_m}^\times \cong \mathfrak{o}_{E_m}^\times/(1 + \mathfrak{p}_{E_m})$ . It is easily seen that the cuspidal representation of  $\text{GL}_m(\mathbf{k}_E)$  related to  $\xi$  via Green's theory [Gre55] and  $K_\kappa(\pi)$  coincide.

### 3.3 Symplectic signs

In this section, we sum up basic results corresponding to symplectic signs, which are useful when comparing the trace of two (full) Heisenberg representations. The main reference will be [BF83], [BH10].

Fix  $p$  a prime number and let  $\Gamma$  be a finite cyclic group of order relatively prime to  $p$ . We call  $(V, h)$  a symplectic  $\mathbb{F}_p[\Gamma]$ -module if

(1)  $V$  is a finite  $\mathbb{F}_p[\Gamma]$ -module;

(2)  $h$  is a non-degenerate, alternating, bilinear form  $V \times V \rightarrow \mathbb{F}_p$ , such that  $h(\gamma v_1, \gamma v_2) = h(v_1, v_2)$  for any  $v_1, v_2 \in V$  and  $\gamma \in \Gamma$ .

For two symplectic  $\mathbb{F}_p[\Gamma]$ -modules  $(V_1, h_1)$  and  $(V_2, h_2)$ , we may consider their direct sum  $(V_1 \oplus V_2, h_1 \oplus h_2)$  which is also a symplectic  $\mathbb{F}_p[\Gamma]$ -module, where

$$(h_1 \oplus h_2)((v_1, v_2), (v'_1, v'_2)) := h_1(v_1, v'_1) \cdot h_2(v_2, v'_2), \quad (v_1, v_2), (v'_1, v'_2) \in V_1 \oplus V_2.$$

Usually we omit the corresponding symplectic form and write  $V_1$ ,  $V_2$  and  $V = V_1 \oplus V_2$  instead. In general, every  $\mathbb{F}_p[\Gamma]$ -module can be written as direct sum of indecomposable  $\mathbb{F}_p[\Gamma]$ -modules, and indecomposable  $\mathbb{F}_p[\Gamma]$ -modules are classified into two basic types.

Given a symplectic  $\mathbb{F}_p[\Gamma]$ -module  $V$ , we may define a sign  $t_\Gamma^0(V) \in \{\pm 1\}$  and a quadratic character  $t_\Gamma^1(V) : \Gamma \rightarrow \{\pm 1\}$ . Here it is unnecessary to recall the exact definition, which the interested readers may find in [BH10]. We recall the following useful properties instead:

**Proposition 3.3.1.** (1) If  $V = V_1 \oplus V_2$  as symplectic  $\mathbb{F}_p[\Gamma]$ -modules, then  $t_\Gamma^0(V) = t_\Gamma^0(V_1) \cdot t_\Gamma^0(V_2)$  and  $t_\Gamma^1(V) = t_\Gamma^1(V_1) \cdot t_\Gamma^1(V_2)$ ;

(2) For any  $\gamma$  as a generator of  $\Gamma$ , the sign  $t_\Gamma(V) := t_\Gamma^0(V) \cdot t_\Gamma^1(V)(\gamma)$  doesn't depend on the choice of  $\gamma$ ;

(3) For  $\Sigma$  a subgroup of  $\Gamma$  acting trivially on  $V$  and  $\gamma$  a generator of  $\Gamma$ , we have  $t_\Gamma^0(V) = t_{\Gamma/\Sigma}^0(V)$  and  $t_\Gamma^1(V)(\gamma) = t_{\Gamma/\Sigma}^1(V)(\gamma)$ ;

(4) For  $\Delta$  a subgroup of  $\Gamma$  such that  $V^\Delta = V^\Gamma$  and  $\delta$  a generator of  $\Delta$ , we have  $t_\Gamma^0(V) \cdot t_\Gamma^1(V)(\delta) = t_\Delta^0(V) \cdot t_\Delta^1(V)(\delta)$ .

*Proof.* Properties (1)(2)(3) follow from direct definition, property (4) is [BH10], Proposition 5.  $\square$

We call  $t_\Gamma^0(V)$ ,  $t_\Gamma^1(V)$  and  $t_\Gamma(V)$  in the proposition symplectic signs, although  $t_\Gamma^1(V)$  is indeed a quadratic character. Now we focus on concrete examples. Let  $A = M_n(F)$ , let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $A$  and let  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ . For  $g_1, g_2 \in J^1(\mathfrak{a}, \beta)$ , the pairing

$$\langle g_1, g_2 \rangle \mapsto \theta(g_1 g_2 g_1^{-1} g_2^{-1})$$

induces a non-degenerate alternating bilinear form

$$J^1(\mathfrak{a}, \beta)/H^1(\mathfrak{a}, \beta) \times J^1(\mathfrak{a}, \beta)/H^1(\mathfrak{a}, \beta) \rightarrow \mathbb{F}_p,$$

which endows  $\mathcal{V} = J^1(\mathfrak{a}, \beta)/H^1(\mathfrak{a}, \beta)$  with a finite dimensional symplectic space structure ([BK93], Theorem 3.4.1). Moreover if we write  $g_1 = 1+x$  and  $g_2 = 1+y$  for  $x, y \in \mathfrak{j}^1(\mathfrak{a}, \beta)$ , then  $\theta(g_1 g_2 g_1^{-1} g_2^{-1}) = \psi_F(\text{Tr}_A(\beta(xy - yx)))$  ([BH05c], 6.1.1 Proposition). In other words, the symplectic structure on  $\mathcal{V} = J^1(\mathfrak{a}, \beta)/H^1(\mathfrak{a}, \beta) \cong \mathfrak{j}^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)$  can also be given by

$$\langle x, y \rangle \mapsto \psi_F(\text{Tr}_A(\beta(xy - yx))). \quad (3.3.1)$$

Now we denote by  $E_m$  the maximal unramified extension of degree  $m$  over  $E = F[\beta]$  and we fix an  $E$ -embedding  $E_m \hookrightarrow B$  such that  $\mathfrak{o}_{E_m}^\times$  is contained in  $\mathrm{GL}_m(\mathfrak{o}_E)$ , where  $n = md$  and  $B = M_m(E)$ . Let  $T_m$  be the maximal tamely ramified subextension of  $E_m$  over  $F$ . As a result,  $E_m^\times$  is contained in  $\mathbf{J}(\mathfrak{a}, \beta)$  and thus normalizes  $H^1(\mathfrak{a}, \beta)$ ,  $J^1(\mathfrak{a}, \beta)$ ,  $\mathcal{V}$  and  $\theta$ . For any uniformizer  $\varpi_F$  of  $F$ , the quotient  $E_m^\times / \langle \varpi_F \rangle$  acts on  $\mathcal{V}$  via conjugacy, which maintains the symplectic structure. Thus for any  $\Gamma$  as a cyclic subgroup of  $E_m^\times / \langle \varpi_F \rangle$ , we may endow  $\mathcal{V}$  with a  $\mathbb{F}_p[\Gamma]$  symplectic module structure.

**Example 3.3.2.** (1) For  $\mu_{T_m}$  being regarded as a subgroup of  $E_m^\times / \langle \varpi_F \rangle$ , we endow  $\mathcal{V}$  with  $\mathbb{F}_p[\mu_{T_m}]$  symplectic module structure and we have symplectic signs  $\epsilon_F^0(\mu_{T_m}) := t_{\mu_{T_m}}^0(\mathcal{V})$ ,  $\epsilon_F^1(\mu_{T_m}) := t_{\mu_{T_m}}^1(\mathcal{V})$  and  $\epsilon_F(\mu_{T_m}) := t_{\mu_{T_m}}(\mathcal{V})$ ;

(2) Let  $\varpi_T$  be a uniformizer of  $T$  and  $T_m$ , such that  $\varpi_F = \varpi_T^{e(T_m/F)} \zeta$  for  $\zeta$  a root of unity in  $T^\times$ . Thus the subgroup of  $E_m^\times / \langle \varpi_F \rangle$ , generated by  $\varpi_T$  and denoted by  $\langle \varpi_T \rangle / \langle \varpi_F \rangle$ , is cyclic of order relatively prime to  $p$ . Then we endow  $\mathcal{V}$  with  $\mathbb{F}_p[\langle \varpi_T \rangle / \langle \varpi_F \rangle]$  symplectic module structure and we have symplectic signs  $\epsilon_F^0(\varpi_T) := t_{\langle \varpi_T \rangle / \langle \varpi_F \rangle}^0(\mathcal{V})$ ,  $\epsilon_F^1(\varpi_T) := t_{\langle \varpi_T \rangle / \langle \varpi_F \rangle}^1(\mathcal{V})$  and  $\epsilon_F(\varpi_T) := t_{\langle \varpi_T \rangle / \langle \varpi_F \rangle}(\mathcal{V})$ ;

(3) Under the notation of (2) if we further assume  $\zeta = 1$ , then we actually have  $\langle \varpi_T \rangle / \langle \varpi_F \rangle \cong T^\times / F^\times \mathfrak{o}_T^\times \cong T_m^\times / F^\times \mathfrak{o}_{T_m}^\times$ . In this particular case, we use  $\epsilon_{T/F}^0$ ,  $\epsilon_{T/F}^1$  and  $\epsilon_{T/F}$  to denote the symplectic signs  $\epsilon_F^0(\varpi_T)$ ,  $\epsilon_F^1(\varpi_T)$  and  $\epsilon_F(\varpi_T)$  instead respectively, which are independent of the choice of  $\varpi_T$ .

Moreover, for  $L/F$  as a subextension of  $T_m/F$ , we write  $A_L$  for the centralizer of  $L$  in  $A$  and  $\mathfrak{a}_L = A_L \cap \mathfrak{a}$  as a hereditary order in  $A_L$ . Thus  $[\mathfrak{a}_L, \beta]$  is a simple stratum in  $A_L$  and  $\theta_L = \theta|_{H^1(\mathfrak{a}_L, \beta)}$  is the interior  $T/F$ -lift of  $\theta$ . Thus  $\mathcal{V}_L = J^1(\mathfrak{a}_L, \beta) / H^1(\mathfrak{a}_L, \beta)$  is a  $\mathbb{F}_p[\Gamma_L]$  symplectic module, where the symplectic structure is given by  $\theta_L$  and  $\Gamma_L$  is a cyclic subgroup of  $E_m^\times / \langle \varpi_L \rangle$  with  $\varpi_L$  a uniformizer of  $L$ .

**Example 3.3.3.** (1) We have symplectic signs  $\epsilon_L^0(\mu_{T_m}) := t_{\mu_{T_m}}^0(\mathcal{V}_L)$ ,  $\epsilon_L^1(\mu_{T_m}) := t_{\mu_{T_m}}^1(\mathcal{V}_L)$  and  $\epsilon_L(\mu_{T_m}) := t_{\mu_{T_m}}(\mathcal{V}_L)$ ;

(2) For  $\varpi_T$  a uniformizer of  $T$  and  $T_m$  such that  $\varpi_L = \varpi_T^{e(T_m/L)} \zeta_L$  for  $\zeta_L$  a root of unity in  $T^\times$ , we have symplectic signs  $\epsilon_L^0(\varpi_T) := t_{\langle \varpi_T \rangle / \langle \varpi_L \rangle}^0(\mathcal{V}_L)$ ,  $\epsilon_L^1(\varpi_T) := t_{\langle \varpi_T \rangle / \langle \varpi_L \rangle}^1(\mathcal{V}_L)$  and  $\epsilon_L(\varpi_T) := t_{\langle \varpi_T \rangle / \langle \varpi_L \rangle}(\mathcal{V}_L)$ ;

(3) If we further assume  $\zeta_L = 1$ , we may use  $\epsilon_{T/L}^0$ ,  $\epsilon_{T/L}^1$  and  $\epsilon_{T/L}$  to denote the symplectic signs  $\epsilon_L^0(\varpi_T)$ ,  $\epsilon_L^1(\varpi_T)$  and  $\epsilon_L(\varpi_T)$  respectively instead, which are independent of the choice of  $\varpi_T$ .

**Remark 3.3.4.** As indicated in [BH14b], section 5, the symplectic signs in the above examples are independent of the choice of the simple stratum  $[\mathfrak{a}, \beta]$  and the embedding  $E_m \hookrightarrow B$ , but only depend on  $\theta$  and thus only depend on the endo-class of  $\theta$ , since any such two maximal simple characters in the same endo-class are conjugate.

## 3.4 Cyclic base change and automorphic induction

In this section, we recall the definition and basic properties of cyclic base change and automorphic induction.

### 3.4.1 Cyclic base change

In this subsection, we sum up the results in [AC89], [HL11] to give a brief introduction of cyclic base change. Let  $F/F'$  be a finite cyclic extension of non-archimedean locally compact fields of degree  $r$ , let  $\Sigma = \mathrm{Gal}(F/F')$  and fix  $\sigma \in \Sigma$  a generator. For  $\pi$  an irreducible representation of  $\mathrm{GL}_n(F)$ , it is called  $\sigma$ -invariant if  $\pi^\sigma \cong \pi$ , or equivalently there exists an intertwining operator  $I_\sigma$  such that  $I_\sigma^r = \mathrm{id}$  and  $\pi(\sigma(g)) \circ I_\sigma = I_\sigma \circ \pi(g)$  for any  $g \in \mathrm{GL}_n(F)$ . Once  $I_\sigma$  is chosen, we may also regard  $\pi$  as an

irreducible representation of  $G_F \rtimes \Sigma$  defined by  $\pi(g, \sigma) = \pi(g) \circ I_\sigma$  acting on the same representation space. For  $\pi_0$  an irreducible representation of  $\mathrm{GL}_n(F')$ , we call  $\pi$  the *Shintani base change lift* of  $\pi_0$ , if there exist  $I_\sigma$  as above and a non-zero complex number  $c(I_\sigma)$  depending on  $I_\sigma$  such that

$$\mathrm{tr}(\pi)(g, \sigma) = c(I_\sigma) \cdot \mathrm{tr}(\pi_0)(g_0)$$

for any  $g \in \mathrm{GL}_n(F)$  such that  $g_0 := N_{F/F'}(g) = g\sigma(g)\dots\sigma^{r-1}(g)$  is a semisimple regular element in  $\mathrm{GL}_n(F')$ , where  $\mathrm{tr}(\pi)$  and  $\mathrm{tr}(\pi_0)$  denote the Harish-Chandra characters. We remark that such  $\pi$ , once exists, is uniquely determined by  $\pi_0$ . We denote by  $\mathrm{Irr}(\mathrm{GL}_n(F'))$  the set of equivalence classes of irreducible representations of  $\mathrm{GL}_n(F')$  and by  $\mathrm{Irr}^{\sigma\text{-inv}}(\mathrm{GL}_n(F))$  the set of equivalence classes of irreducible representations of  $\mathrm{GL}_n(F)$  that are  $\sigma$ -invariant.

**Theorem 3.4.1** ( [AC89], chapter I, section 6, [HL11], chapter II). *We have the map*

$$\begin{aligned} \mathrm{BC}_{F/F'} : \mathrm{Irr}(\mathrm{GL}_n(F')) &\longrightarrow \mathrm{Irr}^{\sigma\text{-inv}}(\mathrm{GL}_n(F)) \\ \pi_0 &\longmapsto \pi \end{aligned}$$

such that  $\pi_0$  is essentially tempered if and only if  $\pi$  is essentially tempered. In this case  $\pi$  is the Shintani base change lift of  $\pi_0$ , and their central characters  $\omega_{\pi_0}$  and  $\omega_\pi$  satisfy the relation

$$\omega_\pi(z) = \omega_{\pi_0}(N_{F/F'}(z))$$

for any  $z \in F^\times$ ;

In particular, we are interested in the supercuspidal case. We fix  $\omega_{F/F'}$  a character of  $F'^\times$  whose kernel is  $N_{F/F'}(F^\times)$ .

**Proposition 3.4.2** ( [AC89], chapter I, section 6, [HL11], chapter II). (1) *For  $\pi_0$  a supercuspidal representation of  $\mathrm{GL}_n(F')$ , let  $c(\pi_0)$  be the cardinality of the set of isomorphism classes*

$$C(\pi_0) := \{\pi_0 \omega_{F/F'}^k | k \in \mathbb{Z}\} / \cong$$

dividing  $r$ . Then  $r/c(\pi_0)$  divides  $n$  and there exists a supercuspidal representation  $\pi'$  of  $\mathrm{GL}_{nc(\pi_0)/r}(F)$  such that  $\pi'^{\sigma^i} \cong \pi'$  if and only if  $r/c(\pi_0)$  divides  $i$ , and

$$\mathrm{BC}_{F/F'}(\pi_0) = \pi' \times \pi'^{\sigma} \times \dots \times \pi'^{\sigma^{r/c(\pi_0)-1}}. \quad (3.4.1)$$

(2) *Conversely for  $c$  a positive integer dividing  $r$  such that  $r/c$  divides  $n$ , and for  $\pi'$  as a supercuspidal representation of  $\mathrm{GL}_{nc/r}(F)$ , such that  $\pi'^{\sigma^i} \cong \pi'$  if and only if  $r/c$  divides  $i$ , there exists  $\pi_0$  as a supercuspidal representation of  $\mathrm{GL}_n(F')$  satisfying (3.4.1),  $c(\pi_0) = c$  and*

$$\mathrm{BC}_{F/F'}^{-1}(\pi' \times \pi'^{\sigma} \times \dots \times \pi'^{\sigma^{r/c-1}}) = \{\pi'_0 | \pi'_0 \in C(\pi_0)\}.$$

**Remark 3.4.3.** *Since we may decompose  $F/F'$  into a sequence of subextensions  $F' \subsetneq F_1 \subsetneq \dots \subsetneq F_k = F$ , such that  $F_{i+1}/F_i$  and  $F_1/F'$  are cyclic of a prime degree, in practice we only need to focus on two special cases in the proposition above: either  $c(\pi_0) = r$  or  $c(\pi_0) = 1$ . In the former case  $\mathrm{BC}_{F/F'}(\pi_0)$  is a supercuspidal representation of  $\mathrm{GL}_n(F)$ , and in the latter case it equals the parabolic induction  $\pi' \times \pi'^{\sigma} \times \dots \times \pi'^{\sigma^{r-1}}$  with  $\pi'$  a supercuspidal representation of  $\mathrm{GL}_{n/r}(F)$ .*

### 3.4.2 Cyclic automorphic induction

In this subsection, we sum up the results in [HH95], [HL10], [BH10], [HL11] to give a brief introduction of cyclic automorphic induction.



### 3.4.2.1

Let  $K/F$  be a finite cyclic extension of non-archimedean locally compact fields of degree  $l$ . We fix a certain  $F$ -algebra embedding  $K \hookrightarrow M_n(F)$  to identify  $K$  with an  $F$ -subalgebra of  $M_n(F)$ , thus  $\mathrm{GL}_{n/l}(K)$  is regarded as the centralizer of  $K^\times$  in  $\mathrm{GL}_n(F)$ . Moreover, we fix the following data:

- (1) a generator  $\sigma_0$  of  $\Sigma_0 = \mathrm{Gal}(K/F)$ ;
- (2) a character  $\varkappa_{K/F}$  of  $F^\times$  with kernel  $N_{K/F}(K^\times)$ ;
- (3) an element  $e_l \in K^\times$  such that  $\sigma_0(e_l) = (-1)^{n(l-1)/l} e_l$ .

We call such a triple  $(\sigma_0, \varkappa_{K/F}, e_l)$  a *transfer system* for  $K/F$  in relative dimension  $n/l$ .

For  $g, g'$  two elements in  $\mathrm{GL}_{n/l}(K)$  with eigenvalues  $g_1, \dots, g_{n/l}$  and  $g'_1, \dots, g'_{n/l}$  in an algebraic closure of  $K$ , we define

$$\mathfrak{r}_{K/F}(g, g') = \prod_{i,j=1}^{n/l} (g_i - g'_j) \in K,$$

and moreover for  $g$  a semisimple regular element in  $\mathrm{GL}_{n/l}(K)$ , we define

$$\tilde{\Delta}_{K/F}(g) = \prod_{0 \leq i < j \leq l-1} \mathfrak{r}_{K/F}(\sigma_0^i(g), \sigma_0^j(g)).$$

By definition, we have  $e_l \tilde{\Delta}_{K/F}(g), \tilde{\Delta}_{K/F}(g)^2 \in F^\times$ . We further define

$$\begin{aligned} \Delta_{K/F}^1(g) &= |\tilde{\Delta}_{K/F}(g)|_F^{1/2} |\det(g)|_F^{n(l-1)/2l}, \\ \Delta_{K/F}^2(g) &= \varkappa_{K/F}(e_l \tilde{\Delta}_{K/F}(g)), \\ \delta_{K/F}(g) &= \Delta_{K/F}^2(g) / \Delta_{K/F}^1(g). \end{aligned}$$

These definitions, depending on the embedding and the choice of transfer factors, will be the starting point of the cyclic automorphic induction.

### 3.4.2.2

Let  $\pi$  be an irreducible representation of  $\mathrm{GL}_n(F)$  such that  $\pi \varkappa_{K/F} \cong \pi$ , where we identify  $\varkappa_{K/F}$  with  $\varkappa_{K/F} \circ \det_F$  as a character of  $\mathrm{GL}_n(F)$ . Equivalently, there exists a  $\varkappa_{K/F}$ -intertwining operator  $\Phi$  such that

$$\Phi \circ (\pi(g) \varkappa_{K/F}(g)) = \pi(g) \circ \Phi, \quad g \in \mathrm{GL}_n(F),$$

which determines  $\Phi$  up to a scalar. We define the  $\varkappa_{K/F}$ -trace of  $\pi$  as a distribution of the space of smooth compactly supported functions on  $\mathrm{GL}_n(F)$ :

$$\mathrm{tr}^{\varkappa_{K/F}}(\pi) : f \longmapsto \mathrm{tr}(\Phi \circ \pi(f)), \quad f \in C_c^\infty(\mathrm{GL}_n(F)),$$

which can also be realized as a locally constant function defined on the set of semisimple regular elements of  $\mathrm{GL}_n(F)$ . Similarly for  $\tau$  as an irreducible representation of  $\mathrm{GL}_{n/l}(K)$ , we have the ordinary trace

$$\mathrm{tr}(\tau) : f \longmapsto \mathrm{tr}(\tau(f)), \quad f \in C_c^\infty(\mathrm{GL}_{n/l}(K))$$

which can be realized as a locally constant function defined on the set of semisimple regular elements of  $\mathrm{GL}_{n/l}(K)$ . We call such  $\pi$  a  $\varkappa_{K/F}$ -lift of  $\tau$ , if a certain relation between  $\varkappa_{K/F}$ -trace of  $\pi$  and the trace of  $\tau$  is satisfied for all semisimple regular elements of  $\mathrm{GL}_K(K)$ , which we shall not recall here (see [HH95], §3.11). Since such a relation, if exists, uniquely determines  $\pi$ , in this case we also call  $\pi$

the automorphic induction of  $\tau$  and we write  $\pi = A_{K/F}(\tau)$ . In particular, for any elliptic elements  $g$  in  $\mathrm{GL}_{n/l}(K)$ , the above omitted relation is easy to describe and can be written as follows:

$$\mathrm{tr}^{\varkappa_{K/F}}(\pi)(g) = \delta_{K/F}(g) \cdot c(\tau, \Phi) \sum_{i=0}^{l-1} \mathrm{tr}(\tau)(\sigma_0^i(g)), \quad (3.4.2)$$

where  $c(\tau, \Phi)$  is a non-zero complex number independent of  $g$ . For supercuspidal representations, we sum up the following result:

**Proposition 3.4.4** ([BH05c], §1.2.). (1) *When  $\pi$  is supercuspidal satisfying  $\pi \varkappa_{K/F} \cong \pi$ , there exists a supercuspidal representation  $\tau$  such that  $A_{K/F}(\tau) = \pi$ . Each  $\tau$  is  $\Sigma_0$ -regular, saying that  $\tau^{\sigma_0^i}$  is not isomorphic to  $\tau$  for  $i = 1, 2, \dots, l-1$ , and its  $\Sigma_0$ -orbit is uniquely determined by  $\pi$ .*

(2) *Conversely, for any supercuspidal representation  $\tau$  that is  $\Sigma_0$ -regular, its automorphic induction  $\pi = A_{K/F}(\tau)$  is well defined and supercuspidal satisfying  $\pi \varkappa_{K/F} \cong \pi$ .*

(3) *For (1) and (2), (3.4.2) is enough to determine  $\pi$  via the  $\Sigma_0$ -orbit of  $\tau$ , and vice versa.*

### 3.4.2.3

We further assume that  $\pi = A_{K/F}(\tau)$  is generic. We let  $(U, \vartheta)$  be a Whittaker pair, where  $U$  is the unipotent radical of a Borel subgroup of  $\mathrm{GL}_n(F)$ , and  $\vartheta$  is a nondegenerate character of  $U$ . Being generic means that the vector space  $\mathrm{Hom}_U(\pi, \vartheta)$  is of dimension 1, thus we choose  $\lambda$  to be a non-zero vector in this space. Since the restriction of  $\varkappa_{K/F}$  to  $U$  is trivial, we have  $\mathrm{Hom}_U(\pi, \vartheta) = \mathrm{Hom}_U(\pi \varkappa_{K/F}, \vartheta)$ .

For  $\Phi$  an intertwining operator as before,  $\lambda \circ \Phi$  is another non-zero vector in  $\mathrm{Hom}_U(\pi \varkappa_{K/F}, \vartheta)$ , thus is proportional to  $\lambda$ . We may change  $\Phi$  up to a scalar such that  $\lambda \circ \Phi = \lambda$  and we denote by  $\Phi(\pi, \vartheta)$  this special intertwining operator. We write  $c^{K/F}(\vartheta) = c(\tau, \Phi(\pi, \vartheta))$  for the constant determined by (3.4.2), which is independent of  $\tau$  and  $\pi$  by the main result of [HL10].

In particular, for  $\psi$  an additive character of  $F$  of level 0, we may consider the special Whittaker pair  $(U_0, \vartheta_\psi)$ , where  $U_0$  is the unipotent radical of the upper triangular Borel subgroup, and  $\vartheta_\psi(u) := \psi(\sum_{i=1}^{n-1} u_{ii+1})$  for  $u = (u_{ij}) \in U_0$ . In this case we write  $c_n^{K/F}(\psi) = c^{K/F}(\vartheta_\psi)$  for the corresponding constant. We may also compare two different constants with respect to two Whittaker pairs: for any  $g \in \mathrm{GL}_n(F)$  such that  $(U, \vartheta) = (U_0^g, \vartheta_\psi^g)$ , we have (see [HL10], §3.3.)

$$c^{K/F}(\vartheta) = \varkappa_{K/F}(\det_F(g))^{-1} \cdot c_n^{K/F}(\psi), \quad (3.4.3)$$

where  $U_0^g := g^{-1}U_0g$  and  $\vartheta_\psi^g(u) := \vartheta_\psi(gug^{-1})$  for any  $u \in U_0^g$ .

### 3.4.2.4

We further assume  $\tau$  to be supercuspidal and  $\pi = A_{K/F}(\tau)$ . Let  $[\mathfrak{a}, \beta]$  be a maximal simple stratum in  $M_n(F)$  and let  $\theta$  be a simple character contained in  $\pi$ . Let  $E_m$  be an unramified extension of  $E = F[\beta]$  of degree  $m$  such that  $E_m^\times$  normalizes  $\mathfrak{a}$ , where  $n = md$  and  $d = [E : F]$ . We further choose  $K/F$  to be a cyclic subextension of  $E_m/F$  of degree  $l$ . We choose  $(\mathbf{J}(\mathfrak{a}, \beta), \Lambda)$  to be an extended maximal simple type containing  $\theta$  and compactly inducing  $\pi$ .

We fix  $V$  a vector space of dimension  $n$  over  $F$  and an isomorphism  $\mathrm{End}_F(V) \cong M_n(F)$ . By an  $F$ -flag  $\mathcal{F}$  of  $V$ , we mean a sequence

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n = V,$$

where  $V_i$  is a subspace of  $V$  of dimension  $i$  for  $i = 1, 2, \dots, n$ . Considering the stabilizer, we get a unipotent subgroup  $U_{\mathcal{F}}$  of  $\mathrm{GL}_n(F)$ . We have the following Uniform Induction Theorem:

**Theorem 3.4.5.** (1) *There exists a Whittaker pair  $(U, \vartheta)$  of  $\mathrm{GL}_n(F)$ , unique up to  $\mathbf{J}(\mathfrak{a}, \beta)$ -conjugacy, such that  $\Lambda|_{\mathbf{J}(\mathfrak{a}, \beta) \cap U}$  contains  $\vartheta|_{\mathbf{J}(\mathfrak{a}, \beta) \cap U}$ . As a result,*

- $\Phi_\pi^\theta := \Phi(\pi, \vartheta)$  is a  $\varkappa_{K/F}$ -intertwining operator which acts trivially on the  $\theta$ -isotypic subspace of the representation space of  $\pi$ ;
- (3.4.2) is satisfied for all  $\tau$  supercuspidal with  $\pi = A_{K/F}(\tau)$  supercuspidal containing  $\theta$ , and all elliptic elements  $g \in \mathrm{GL}_{n/l}(K)$ , where  $\Phi = \Phi_\pi^\theta$ . We set  $c_\theta^{K/F} = c(\tau, \Phi_\pi^\theta)$  for later use.

(2) *There exist an  $F$ -flag  $\mathcal{F}$ , the corresponding unipotent subgroup  $U = U_{\mathcal{F}}$  and a nondegenerate character  $\vartheta$  of  $U$ , such that*

$$\theta|_{U \cap H^1(\mathfrak{a}, \beta)} = \vartheta|_{U \cap H^1(\mathfrak{a}, \beta)}.$$

(3) *When  $E/F$  is of degree  $n$ , the Whittaker pair in (2) satisfies the result in (1).*

*Proof.* (1) follows from [BH14b], §1.3, §1.5, and (2) follows from [PS08], Theorem 3.3. To prove (3), since  $U$  is a pro- $p$ -group and  $\mathbf{J}(\mathfrak{a}, \beta) = E^\times J^1(\mathfrak{a}, \beta)$ , we have

$$U \cap \mathbf{J}(\mathfrak{a}, \beta) = U \cap J(\mathfrak{a}, \beta) = U \cap J^1(\mathfrak{a}, \beta).$$

Thus we only need to prove

$$\mathrm{Hom}_{U \cap J^1(\mathfrak{a}, \beta)}(\eta, \vartheta) \neq 0$$

for  $\eta$  the Heisenberg representation of  $\theta$ . Since we have

$$\eta^{\oplus (J^1(\mathfrak{a}, \beta): H^1(\mathfrak{a}, \beta))^{1/2}} \cong \mathrm{Ind}_{H^1(\mathfrak{a}, \beta)}^{J^1(\mathfrak{a}, \beta)} \theta,$$

we only need to prove that

$$\mathrm{Hom}_{U \cap J^1(\mathfrak{a}, \beta)}(\mathrm{Ind}_{H^1(\mathfrak{a}, \beta)}^{J^1(\mathfrak{a}, \beta)} \theta, \vartheta) \neq 0$$

or by the Mackey formula and the Frobenius reciprocity

$$\bigoplus_{H^1(\mathfrak{a}, \beta) \backslash J^1(\mathfrak{a}, \beta) / U \cap J^1(\mathfrak{a}, \beta)} \mathrm{Hom}_{U \cap H^1(\mathfrak{a}, \beta)}(\theta, \vartheta) \neq 0,$$

which follows from (2). □

### 3.4.3 Functorial property

One of the most important motivations for the base change and automorphic induction above is that they satisfy the functoriality of local Langlands correspondence for general linear groups. Let  $K/F$  be a finite separable extension of non-archimedean locally compact fields. We denote by  $\mathcal{W}_F$  the Weil group with respect to  $F$  and  $\mathcal{W}_K$  the Weil group with respect to  $K$  identifying with a subgroup of  $\mathcal{W}_F$ . For  $n$  a positive integer, we denote by  $\mathcal{A}_n^0(F)$  (resp.  $\mathcal{A}_n^0(K)$ ) the set of isomorphism classes of supercuspidal representations of  $\mathrm{GL}_n(F)$  (resp.  $\mathrm{GL}_n(K)$ ), and by  $\mathcal{G}_n^0(F)$  (resp.  $\mathcal{G}_n^0(K)$ ) the set of isomorphism classes of irreducible representations of  $\mathcal{W}_F$  (resp.  $\mathcal{W}_K$ ) of dimension  $n$ . Then we have the following local Langlands correspondence.

**Theorem 3.4.6** ([LRS93], [HT01], [Hen00], [Sch13]). *For  $F$  a non-archimedean locally compact field, the local Langlands correspondence is a bijection<sup>2</sup>*

$$\mathrm{LLC}_F : \mathcal{G}_n^0(F) \longrightarrow \mathcal{A}_n^0(F)$$

*satisfying certain desiderata.*

<sup>2</sup>We notice that the definition of  $\mathrm{LLC}_F$  here is actually the inverse of that considered in the introduction. This minor modification is more compatible with the notations in the results of Bushnell-Henniart.

For  $\rho \in \mathcal{G}_n^0(F)$  and  $\pi_\rho = \text{LLC}_F(\rho) \in \mathcal{A}_n^0(F)$ , we let  $\text{Res}_{K/F}(\rho) := \rho|_{\mathcal{W}_K} = \rho_1 \oplus \dots \oplus \rho_k$ , where  $\rho_i \in \mathcal{G}_{n_i}^0(K)$  for  $i = 1, \dots, k$  with  $n_1 + \dots + n_k = n$ . Then  $\pi_{\rho_i} := \text{LLC}_K(\rho_i)$  is a supercuspidal representation of  $\text{GL}_{n_i}(K)$ . The following proposition is known as the functoriality for the base change map:

**Proposition 3.4.7** ( [AC89], chapter I, section 6, [HL11], chapter II). (1) *The parabolic induction  $\pi_{\rho_1} \times \dots \times \pi_{\rho_k}$  is an irreducible representation of  $\text{GL}_n(K)$ , which we denote by  $\text{BC}_{K/F}(\pi_\rho)$ ;*  
 (2) *When  $K/F$  is cyclic, the definition of  $\text{BC}_{K/F}(\pi_\rho)$  coincide with that in §3.4.1.*

Similarly for  $\rho' \in \mathcal{G}_n^0(K)$  and  $\pi_{\rho'} = \text{LLC}_K(\rho') \in \mathcal{A}_n^0(K)$ , we let  $\text{Ind}_{K/F}(\rho') := \text{Ind}_{\mathcal{W}_K}^{\mathcal{W}_F} \rho' = \rho'_1 \oplus \dots \oplus \rho'_{k'}$ , where  $\rho'_i \in \mathcal{G}_{n'_i}^0(F)$  for  $i = 1, \dots, k'$  with  $n'_1 + \dots + n'_{k'} = n|\mathcal{W}_F/\mathcal{W}_K|$ . Then  $\pi_{\rho'_i} := \text{LLC}_F(\rho'_i)$  is a supercuspidal representation of  $\text{GL}_{n'_i}(F)$ . The following proposition is known as the functoriality for the automorphic induction:

**Proposition 3.4.8** ( [HH95]). (1) *The parabolic induction  $\pi_{\rho'_1} \times \dots \times \pi_{\rho'_{k'}}$  is an irreducible representation of  $\text{GL}_{n|\mathcal{W}_F/\mathcal{W}_K|}(F)$ , which we denote by  $A_{K/F}(\pi_{\rho'})$ ;*  
 (2) *When  $K/F$  is cyclic, the definition of  $A_{K/F}(\pi_{\rho'})$  coincides with that in §3.4.2.*

**Proposition 3.4.9.** *Let  $K/F$  and  $F/F'$  be finite separable extensions of non-archimedean locally compact fields.*

(1) *For  $\pi_{F'} \in \mathcal{A}_n^0(F')$  such that  $\text{BC}_{K/F'}(\pi_{F'})$  is supercuspidal, we have*

$$\text{BC}_{K/F'}(\pi_{F'}) = \text{BC}_{K/F}(\text{BC}_{F/F'}(\pi_{F'}));$$

(2) *For  $\pi_K \in \mathcal{A}_n^0(K)$  such that  $A_{K/F'}(\pi_K)$  is supercuspidal, we have*

$$A_{K/F'}(\pi_K) = A_{F/F'}(A_{K/F}(\pi_K));$$

(3) *If moreover  $K'$  is a subfield of  $K$  over  $F'$  such that  $K'F = K$  and  $K' \cap F = F'$ , and if  $\pi_{K'} \in \mathcal{A}_n^0(K')$  such that  $A_{K/F}(\text{BC}_{K/K'}(\pi_{K'}))$  is supercuspidal, then*

$$A_{K/F}(\text{BC}_{K/K'}(\pi_{K'})) = \text{BC}_{F/F'}(A_{K'/F'}(\pi_{K'})).$$

*Proof.* It follows from the equalities  $\text{Res}_{K/F'} = \text{Res}_{K/F} \circ \text{Res}_{F/F'}$ ,  $\text{Ind}_{K/F'} = \text{Ind}_{F/F'} \circ \text{Ind}_{K/F}$  and  $\text{Ind}_{K/F} \circ \text{Res}_{K/K'} = \text{Res}_{F/F'} \circ \text{Ind}_{K'/F'}$ . □

### 3.5 Basic classification

Let  $F/F_0$  be a tamely ramified cyclic extension of non-archimedean locally compact fields of degree  $r$  of residue characteristic  $p$ . We fix  $\omega_{F/F_0}$  a primitive character of the cyclic group  $F_0^\times/\mathcal{N}_{F/F_0}(F^\times)$ , and by abuse of notation we identify it with a character of  $\text{GL}_n(F_0)$  by composing with the determinant map. We write  $\Sigma = \text{Gal}(F/F_0)$  and we fix  $\sigma \in \Sigma$  a generator.

For  $\pi_0$  a supercuspidal representation of  $\text{GL}_n(F_0)$ , we denote by  $\pi$  the base change of  $\pi_0$  as an irreducible representation of  $\text{GL}_n(F)$ . Our aim is to give an explicit construction of  $\pi$  via  $\pi_0$  using the simple type theory. As mentioned in Remark 3.4.3 essentially we only need to focus on two cases: either  $\pi$  is supercuspidal, or  $\pi$  is the parabolic induction of  $r$  supercuspidal representations of  $\text{GL}_{n/r}(F)$ .

Let  $[\mathfrak{a}_0, \beta]$  be a maximal simple stratum in  $M_n(F_0)$  and let  $E_0 = F_0[\beta]$ . We write  $d = [E_0 : F_0]$  and  $n = md$ . We write  $E_{0,m}$  for the unramified extension of degree  $m$  over  $E_0$ , we fix an embedding  $E_{0,m} \hookrightarrow M_n(F_0)$  whose restriction to  $E_0$  is identity, and we define  $A_{F_0} := \text{End}_{F_0}(E_{0,m}) \cong M_n(F_0)$  as

the endomorphism ring of  $F$ -vector space  $E_{0,m}$  and  $G_{F_0} = A_{F_0}^\times \cong \mathrm{GL}_n(F_0)$ . Thus up to  $G_{F_0}$ -conjugacy, we may and will assume  $\mathfrak{a}_0 = \mathfrak{a}_{F_0}(E_{0,m})$ . We choose  $\theta_0 \in \mathcal{C}(\mathfrak{a}_0, \beta)$  to be a simple character contained in  $\pi_0$ , we denote by  $\eta_0$  the Heisenberg representations of  $\theta_0$ , and we choose  $\kappa_0$  to be a full Heisenberg representation extending  $\eta_0$ . Thus there is a unique representation  $\rho_0$  of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  trivial on  $\mathbf{J}^1(\mathfrak{a}_0, \beta)$  up to isomorphism, such that  $\Lambda_0 = \kappa_0 \otimes \rho_0$  compactly induces  $\pi_0$ . Here  $\rho_0|_{\mathbf{J}(\mathfrak{a}_0, \beta)}$  is the inflation of a supercuspidal representation  $\rho_0$  of  $\mathrm{GL}_m(\mathbf{k}_{E_0}) \cong \mathbf{J}(\mathfrak{a}_0, \beta)/\mathbf{J}^1(\mathfrak{a}_0, \beta)$ .

### 3.5.1 Supercuspidal case

In this subsection we assume  $\pi$  to be supercuspidal. We first restate the following corollary of Proposition 3.4.2

**Proposition 3.5.1.**  *$\pi$  is supercuspidal if and only if  $\pi_0 \omega_{F/F_0}^i \not\cong \pi_0$  for any  $i = 1, 2, \dots, r-1$ .*

Since  $\pi_0 \cong \mathrm{ind}_{\mathbf{J}(\mathfrak{a}_0, \beta)}^{G_{F_0}} \Lambda_0$ , for  $i = 1, \dots, r-1$  we have

$$\pi_0 \omega_{F/F_0}^i \not\cong \pi_0 \iff \Lambda_0 \omega_{F/F_0}^i \not\cong \Lambda_0 \iff \rho_0 \omega_{F/F_0}^i \not\cong \rho_0 \quad (3.5.1)$$

Thus in particular we have

$$\omega_{F/F_0}^i|_{\mathbf{J}(\mathfrak{a}_0, \beta)} = \omega_{F/F_0}^i|_{\mathrm{N}_{E_0/F_0}(E_0^\times)^m \mathrm{N}_{E_0/F_0}(\mathfrak{o}_{E_0}^\times)} \neq 1, \quad i = 1, \dots, r-1 \quad (3.5.2)$$

Since  $\mathrm{N}_{E_0/F_0}(E_0^\times)^m \mathrm{N}_{E_0/F_0}(\mathfrak{o}_{E_0}^\times) \subset \mathrm{N}_{E_0/F_0}(E_0^\times)$ , finally we have

$$\omega_{F/F_0}^i|_{\mathrm{N}_{E_0/F_0}(E_0^\times)} \neq 1, \quad i = 1, \dots, r-1 \quad (3.5.3)$$

**Proposition 3.5.2.** (3.5.3) is true if and only if for any non-trivial subextension  $F'/F_0$  of  $F/F_0$ , the field  $F'$  is not isomorphic to any subfield of  $E_0$  over  $F_0$ . Thus  $E = F[\beta] = F \otimes_{F_0} E_0$  is a field and  $E/E_0$  is of degree  $r$ .

*Proof.* (3.5.3) is true if and only if  $\mathrm{N}_{E_0/F_0}(E_0^\times)$  is not contained in the kernel of  $\omega_{F/F_0}^i$  for each  $i$ , thus  $\mathrm{N}_{E_0/F_0}(E_0^\times) \not\subseteq \mathrm{N}_{F'/F_0}(F'^\times)$  for any  $F'/F_0$  as a non-trivial subextension of  $F/F_0$ . By the local class field theory, it is equivalent to the fact that each  $F'$  is not isomorphic to a subfield of  $E_0$ .  $\square$

We write  $T_0$  (resp.  $T$ ) for the maximal tamely ramified extension of  $E_0$  (resp.  $E$ ) over  $F_0$  (resp.  $F$ ), and  $e_0$  (resp.  $e$ ) for the ramification index and  $f_0$  (resp.  $f$ ) for the residue class degree of  $T_0/F_0$  (resp.  $T/F$ ), and we have  $d = [E_0 : F_0] = [E : F]$ .

**Proposition 3.5.3.** (3.5.2) implies that:

- If  $F/F_0$  is unramified, then  $(r, m f_0) = 1$  and  $E/E_0$  is unramified;
- If  $F/F_0$  and  $E/E_0$  are totally ramified, then  $(r, e(E_0/F_0)) = 1$ ;
- If  $F/F_0$  is totally ramified and  $E/E_0$  is unramified, then  $r|e_0$  and  $(r, m) = 1$ .

*Proof.* We first consider the case where  $r$  is a prime number. By (3.5.2), we know that

$$\mathrm{N}_{E_0/F_0}(E_0^\times)^m \mathrm{N}_{E_0/F_0}(\mathfrak{o}_{E_0}^\times) \not\subseteq \mathrm{N}_{F/F_0}(F^\times), \quad (3.5.4)$$

which implies that

$$\mathrm{N}_{T_0/F_0}(T_0^\times)^m \mathrm{N}_{T_0/F_0}(\mathfrak{o}_{T_0}^\times) \not\subseteq \mathrm{N}_{F/F_0}(F^\times). \quad (3.5.5)$$

Choose  $\varpi_{F_0}$  to be a uniformizer of  $F_0$  and  $\varpi_{T_0}$  to be a uniformizer of  $T_0$ , such that  $\varpi_{T_0}^{e_0} = \varpi_{F_0} \zeta_{T_0}$  for a certain  $\zeta_{T_0} \in \boldsymbol{\mu}_{T_0}$ . Thus for  $F'_0$  denoting the maximal unramified subextension of  $T_0$  over  $F_0$ , the element  $\zeta'_0 := N_{F'_0/F_0}(\zeta_{T_0})$  is in  $\boldsymbol{\mu}_{F_0}$  and we have  $N_{T_0/F_0}(\varpi_{T_0}) = \varpi_{F_0}^{f_0} \zeta'_0$ . Moreover

$$N_{T_0/F_0}(T_0^\times)^m N_{T_0/F_0}(\mathfrak{o}_{T_0}^\times) = \langle \varpi_{F_0}^{mf_0} \zeta'^m \rangle N_{T_0/F_0}(\mathfrak{o}_{T_0}^\times) = \langle \varpi_{F_0}^{mf_0} \zeta'^m \rangle \mathbf{k}_{F_0}^{\times e_0} (1 + \mathfrak{p}_{F_0}).$$

If  $F/F_0$  is unramified, we have

$$N_{F/F_0}(F^\times) = \langle \varpi_{F_0}^r \rangle \mathfrak{o}_{F_0}^\times.$$

Thus (3.5.5) is equivalent to  $(r, mf_0) = 1$ , and moreover  $T/T_0$  and  $E/E_0$  are unramified. If  $F/F_0$  is totally and tamely ramified, we have  $(r, p) = 1$ . Choose  $\varpi_F$  to be a uniformizer of  $F$  such that  $\varpi_F^r = \varpi_{F_0} \zeta_0$  for a certain  $\zeta_0 \in \mathbf{k}_{F_0}^\times$ , and we have

$$N_{F/F_0}(F^\times) = \langle \varpi_{F_0} \zeta_0 \rangle N_{F/F_0}(\mathfrak{o}_F^\times) = \langle \varpi_{F_0} \zeta_0 \rangle \mathbf{k}_{F_0}^{\times r} (1 + \mathfrak{p}_{F_0}).$$

To ensure (3.5.5), either of the conditions is true:

- $(r, e_0) = 1$ ;
- $r|e_0$  and  $\zeta_0^{f_0} (\zeta_0^{mf_0})^{-1} \notin \mathbf{k}_{F_0}^r$ .

In the first case  $T/T_0$  is totally ramified, since  $r$  divides the ramification index of  $T/T_0$ . Thus  $E/E_0$  is also totally ramified. In the second case we must have  $(r, m) = 1$ . Moreover  $X^r = \varpi_{F_0} \zeta_{T_0}$  has a solution in  $T_0$  and  $X^r = \varpi_{F_0} \zeta_0$  has a solution in  $F$ , so we have

$$T \cong T_0[X]/(X^r - \varpi_{F_0} \zeta_0) \cong T_0[X]/(X^r - \zeta_0 \zeta_{T_0}^{-1}).$$

As a result  $T/T_0$  is unramified, thus  $E/E_0$  is also unramified.

In general, we choose  $F'/F_0$  to be a subextension of  $F/F_0$  of prime degree and we write  $E' = F'E_0$ . Then the proposition follows if we consider all such  $F'$  using the previous result. □

**Remark 3.5.4.** *As already indicated in the proof, if  $F/F_0$  is of prime degree, then the proposition classifies all the possibilities. In general, we may consider extensions  $F_0 \subset F_1 \subset F_2 \subset F$  and  $E_1 = E_0 F_1$  and  $E_2 = E_0 F_2$ , such that  $F_1/F_0$  is unramified,  $F_2/F_1$  is totally ramified and  $E_2/E_1$  is unramified, and both  $F/F_2$  and  $E/E_2$  are totally ramified. So essentially we only need to study the three cases listed in the proposition.*

**Corollary 3.5.5.** *In the first and second cases of Proposition 3.5.3, we have  $e = e_0$  and  $f = f_0$ .*

Now we assume that  $F/F_0$  and  $E/E_0$  are totally ramified.

**Lemma 3.5.6.** (3.5.1) implies that  $\boldsymbol{\rho}_0|_{J(\mathfrak{a}_0, \beta)} \cdot \omega_{F/F_0}^i \not\cong \boldsymbol{\rho}_0|_{J(\mathfrak{a}_0, \beta)}$  for  $i = 1, 2, \dots, r-1$ .

*Proof.* We choose  $\varpi_E$  to be a uniformizer of  $E$ . Since  $E/E_0$  is totally ramified,  $\varpi_{E_0} = N_{E/E_0}(\varpi_E)$  is a uniformizer of  $E_0$ . Thus by definition  $\det_{F_0}(\varpi_{E_0}) \in N_{F/F_0}(F^\times)$ , meaning that  $\omega_{F/F_0}(\varpi_{E_0}) = 1$ . Since  $\mathbf{J}(\mathfrak{a}_0, \beta) = \langle \varpi_{E_0} \rangle J(\mathfrak{a}_0, \beta)$ , we finish the proof. □

Since  $F/F_0$  is tamely ramified,  $\omega_{F/F_0}$  is trivial on  $1 + \mathfrak{p}_{F_0}$ . We denote by  $\omega$  the character of  $\mathbf{k}_{E_0}^\times$  whose inflation equals  $\omega_{F/F_0} \circ N_{E_0/F_0}|_{\mathfrak{o}_{E_0}^\times}$ . Moreover, since  $\omega_{F/F_0} \circ \det_{F_0}|_{\mathrm{GL}_m(\mathfrak{o}_{E_0})} = \omega_{F/F_0} \circ N_{E_0/F_0} \circ$

$\det_{E_0} |_{\mathrm{GL}_m(\mathfrak{o}_{E_0})}$ , we know that  $\omega_{F/F_0} \circ \det_{F_0} |_{J(\mathfrak{a}_0, \beta)}$  is the inflation of  $\omega \circ \det_{\mathbf{k}_{E_0}}$ . By abuse of notation, we identify  $\omega$  with  $\omega \circ \det_{\mathbf{k}_{E_0}}$  and we have

$$\rho_0 \cdot \omega^i \not\cong \rho_0 \quad (3.5.6)$$

for  $i = 1, 2, \dots, r-1$ .

By definition  $\mathbf{k}_{E_0, m}$  is the field of degree  $m$  over  $\mathbf{k}_{E_0}$ . We write  $\xi_0$  for a  $\mathrm{Gal}(\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0})$ -regular character of  $\mathbf{k}_{E_0, m}^\times$  corresponding to  $\rho_0$  via the theory of Green [Gre55]. Thus (3.5.6) implies that for any  $k \in \{0, 1, 2, \dots, m-1\}$  and any  $i \in \{1, 2, \dots, r-1\}$ ,

$$\xi_0 \cdot (\omega \circ N_{\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0}})^i \neq \xi_0^{Q^k}, \quad (3.5.7)$$

where  $Q$  denotes the cardinality of  $\mathbf{k}_{E_0}$ .

**Lemma 3.5.7.** *The character  $\omega \circ N_{\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0}}$  of  $\mathbf{k}_{E_0, m}^\times$  is of order  $r$ .*

*Proof.* By definition and direct calculation, if we write  $\omega'$  for the character of  $\mathbf{k}_{F_0}^\times \cong \mathfrak{o}_{F_0}^\times / 1 + \mathfrak{p}_{F_0}$  whose inflation equals  $\omega_{F/F_0} |_{\mathfrak{o}_{F_0}^\times}$ , then  $\omega \circ N_{\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0}} = (\omega' \circ N_{\mathbf{k}_{E_0, m}/\mathbf{k}_{F_0}})^{e(E_0/F_0)}$ . By definition  $\omega'$  has kernel  $\mathbf{k}_0^{\times r}$ , thus it is a character of order  $r$ . Using the fact that  $(r, e(E_0/F_0)) = 1$ , the character  $(\omega' \circ N_{\mathbf{k}_{E_0, m}/\mathbf{k}_{F_0}})^{e(E_0/F_0)}$  is also of order  $r$ . □

**Corollary 3.5.8.** *The character  $\xi := \xi_0^r$  of  $\mathbf{k}_{E_0, m}^\times$  is  $\mathrm{Gal}(\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0})$ -regular, thus it corresponds to a supercuspidal representation of  $\mathrm{GL}_m(\mathbf{k}_{E_0})$ .*

*Proof.* We assume on the contrary that there exists  $k \in \{1, \dots, m-1\}$  such that  $\xi^{Q^k} = \xi$ , which also means that  $\xi_0^{rQ^k} = \xi_0^r$ . As a result,  $\xi_0^{Q^k}$  equals  $\xi_0$  multiplying by a character of  $\mathbf{k}_{E_0, m}^\times$  of order dividing  $r$ , which is of the form  $(\omega \circ N_{\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0}})^i$  by Lemma 3.5.7, contradicting to (3.5.7). Thus  $\xi$  is  $\mathrm{Gal}(\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0})$ -regular. □

### 3.5.2 Non-supercuspidal case

Now we consider the following non-supercuspidal case, where there exists  $\pi'$  as a supercuspidal representation of  $\mathrm{GL}_{n/r}(F)$  such that

$$\pi \cong \pi' \times \pi'^{\sigma} \times \dots \times \pi'^{\sigma^{r-1}}.$$

Since  $\pi'$  is  $\sigma$ -regular, by [BH03], (5.1.2), we have  $A_{F/F_0}(\pi') = \pi_0$  and  $A_{F/F_0}^{-1}(\pi_0) = \{\pi', \pi'^{\sigma}, \dots, \pi'^{\sigma^{r-1}}\}$ , where  $A_{F/F_0}$  denotes the automorphic induction.

By Proposition 3.4.2, we have

$$\rho_0 \omega_{F/F_0} \cong \rho_0 \quad (3.5.8)$$

Restricting to  $J(\mathfrak{a}_0, \beta)$ , we have

$$\rho_0 \omega \cong \rho_0, \quad (3.5.9)$$

where  $\rho_0$  and  $\omega$  are defined as in the last subsection.

We focus on two special cases: either  $F$  is isomorphic to a subfield of  $E_0$ , or any non-trivial subextension  $F'/F_0$  of  $F/F_0$  is not isomorphic to a subfield of  $E_0$ . In particular if  $r$  is a prime number, then either of the two cases happens.

For the former case, by [BH03], Theorem B, the endo-class of  $\pi'$  is one of the  $F/F_0$ -lifts of the endo-class of  $\pi_0$ , which is of degree  $d/r$ . We say that  $\pi_0$  is the *interior automorphic induction* of  $\pi'$ .

For the latter case,  $E := E_0 \otimes_{F_0} F$  is a field of degree  $r$  over  $E_0$ . Thus  $\sigma$  can also be regarded as a generator of  $\text{Gal}(E/E_0)$ . Still by [BH03], Theorem B, the endo-class of  $\pi'$  is the unique  $F/F_0$ -lift of the endo-class of  $\pi_0$  which is of degree  $d$ . As a corollary we have  $d|(n/r)$ , or equivalently  $r|m$ . We say that  $\pi_0$  is the *exterior automorphic induction* of  $\pi'$ , which we study more precisely.

**Proposition 3.5.9.** *For  $F/F_0$  unramified, the field extension  $E/E_0$  is also unramified, and we have  $(r, f_0) = 1$ .*

*Proof.* Since  $F$  is not isomorphic to a subfield of  $E_0$ , we must have  $(r, f_0) = 1$ . Thus  $r$  divides the residue class degree of  $E/E_0$ , which means that  $E/E_0$  is also unramified.  $\square$

**Proposition 3.5.10.** *For  $F/F_0$  totally ramified,  $(r, e_0) = 1$  if and only if  $E/E_0$  is totally ramified, and  $r|e_0$  if and only if  $E/E_0$  is unramified.*

*Proof.* If  $(r, e_0) = 1$ , then  $r$  divides the ramification index of  $E/E_0$ , which means that  $E/E_0$  is totally ramified. If  $r|e_0$ , we may choose  $\varpi_{F_0}$  as a uniformizer of  $F_0$  and  $\zeta_{T_0} \in \mu_{T_0}$ , such that  $X^r = \varpi_{F_0} \zeta_{T_0}$  has a solution in  $E_0$ . Moreover, we may choose  $\varpi_F$  to be a uniformizer of  $F$  such that  $\varpi_F^r = \varpi_{F_0} \zeta_0$ , where  $\zeta_0 \in \mathbf{k}_{F_0}^\times$ . Thus  $E \cong E_0[X]/(X^r - \varpi_{F_0} \zeta_0) \cong E_0[X]/(X^r - \zeta_0 \zeta_{T_0}^{-1})$ , which implies that  $E/E_0$  is unramified.

For the other direction of the first equivalence if  $r$  is a prime number the proof is finished. In general we consider a certain subextension  $F'/F_0$  of  $F/F_0$  of degree  $l$  as a prime number. If  $E/E_0$  is totally ramified, then  $(l, e_0) = 1$  from the prime case. Since  $F'/F_0$  is arbitrary we must have  $(r, e_0) = 1$ . To finish the second equivalence if  $E/E_0$  is unramified, since  $r$  divides the ramification index of  $E_0/F_0$ , we must have  $r|e_0$ .  $\square$

**Proposition 3.5.11.** *For  $F/F_0$  and  $E/E_0$  totally ramified and for  $\xi_0$  a regular character of  $\mathbf{k}_{E_0, m}^\times$  corresponding to  $\rho_0$ , the character  $\xi_0^r$  is not  $\text{Gal}(\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0})$ -regular.*

*Proof.* By [3.5.9], we have  $\xi_0 \omega = \xi_0^{Q^k}$  for a certain  $k \in \{1, \dots, m-1\}$ . Since  $\omega$  is a character of order  $r$ , we have  $\xi_0^{r(Q^k-1)} = 1$ , meaning that  $\xi_0^r$  is not  $\text{Gal}(\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0})$ -regular.  $\square$

### 3.5.3 A brief summary

The following corollary gives a partial criterion for  $\pi$  being supercuspidal, whose proof is the combination of Proposition [3.5.3], Corollary [3.5.8], Proposition [3.5.9], Proposition [3.5.10] and Proposition [3.5.11].

**Corollary 3.5.12.** *For  $\pi_0$  and  $\pi = \text{BC}_{F/F_0}(\pi_0)$  as above such that  $E = E_0 \otimes_{F_0} F$  is a field of degree  $r$  over  $E$ ,*

- *if  $F/F_0$  is unramified, or if  $F/F_0$  is totally ramified but  $E/E_0$  is unramified, then  $\pi$  is supercuspidal if and only if  $(r, m) = 1$ . Moreover for the  $F/F_0$  unramified case we have  $(r, f_0) = 1$ ;*
- *if both  $F/F_0$  and  $E/E_0$  are totally ramified, then  $(r, e_0) = 1$ . In this case  $\pi$  is supercuspidal if and only if  $\xi_0^r$  is  $\text{Gal}(\mathbf{k}_{E_0, m}/\mathbf{k}_{E_0})$ -regular.*

**Remark 3.5.13.** *In the corollary, it is possible in the  $F/F_0$  totally ramified and  $E/E_0$  unramified case that  $r$  is NOT relatively prime to  $f_0$ . For example we choose  $r = 2$  and we assume  $F/F_0$  to be totally ramified. And we may choose  $\pi_0$  such that  $e_0 = f_0 = 2$ ,  $E_0 = T_0$  and  $E_0/F_0$  is not cyclic.*



Using the basic argument in the local class field theory, there exists a unique quadratic subextension of degree 2 of  $E_0/F_0$  which is unramified. In this case  $E = E_0 \otimes_{F_0} F$  must be a field, otherwise  $F$  is isomorphic to a subfield of  $E_0$  which is impossible. And  $E/E_0$  must be unramified since  $r|e_0$ . However  $r = f_0 = 2$ .

Finally when  $r$  is a prime number, our discussion above is actually exhaustive. Precisely we have the following proposition:

**Proposition 3.5.14.** *Assume  $r$  prime and let  $\pi_0$  be a supercuspidal representation of  $\mathrm{GL}_n(F_0)$ .*

1. *If  $F$  is isomorphic to a subfield of  $E_0$ , then  $\pi_0$  is the interior automorphic induction of a supercuspidal representation of  $\mathrm{GL}_{n/r}(F)$ .*
2. *If  $F$  is not isomorphic to a subfield of  $E_0$ , then  $E = F \otimes_{F_0} E_0$  is a field and moreover*
  - *if  $F/F_0$  is unramified, then  $E/E_0$  is unramified and  $(r, f_0) = 1$ . If  $F/F_0$  is totally ramified and  $E/E_0$  is unramified, then  $r|e_0$ . Moreover for these two cases  $(r, m) = 1$  if and only if  $\mathrm{BC}_{F/F_0}(\pi_0)$  is supercuspidal, otherwise  $r$  divides  $m$  and  $\pi_0$  is the exterior automorphic induction of a supercuspidal representation of  $\mathrm{GL}_{n/r}(F)$ .*
  - *if both  $F/F_0$  and  $E/E_0$  are totally ramified, then  $\mathrm{BC}_{F/F_0}(\pi_0)$  is supercuspidal if and only if  $\xi_0^r$  is a  $\mathrm{Gal}(\mathbf{k}_{E_{0,m}}/\mathbf{k}_{E_0})$ -regular character of  $\mathbf{k}_{E_{0,m}}^\times$ , otherwise  $r$  divides  $m$  and  $\pi_0$  is the exterior automorphic induction of a supercuspidal representation of  $\mathrm{GL}_{n/r}(F)$ . Moreover  $(r, e_0) = 1$ .*

## 3.6 Statement of the main theorems

In this section we state the main theorems providing an explicit construction of tamely ramified cyclic base change and automorphic inductions, whose proof will be given in §3.8. Let  $F/F_0$ ,  $\pi_0$ ,  $\pi$ ,  $[\mathfrak{a}_0, \beta]$ ,  $\theta_0$ ,  $\eta_0$ ,  $E_0$ ,  $E_{0,m}$ ,  $T_0$  be as in the last section and let  $T_{0,m}$  be the maximal tamely ramified subextension of  $E_{0,m}$  over  $F_0$ , and let  $\theta_{T_{0,m}}$  be the interior  $T_{0,m}/F_0$ -lift of  $\theta_0$ . We fix a uniformizer  $\varpi_{F_0}$  of  $F_0$ . We denote by  $C_{T_{0,m}}(\varpi_{F_0})$  the subgroup of  $T_{0,m}^\times$  containing  $\varpi_{F_0}$ , such that  $C_{T_{0,m}}(\varpi_{F_0}) \times U_{T_{0,m}}^1 \rightarrow T_{0,m}^\times$  is bijective. By [BH14b], §5.6, Lemma 2, we refine our choice of the full Heisenberg representation  $\kappa_0$  extending  $\eta_0$  such that

$$C_{T_{0,m}}(\varpi_{F_0}) \subset \mathrm{Ker}(\det(\kappa_0)) \quad \text{and} \quad \varpi_{F_0} \in \mathrm{Ker}(\kappa_0). \quad (3.6.1)$$

Thus we get  $\rho_0$  as the unique representation of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  trivial on  $J^1(\mathfrak{a}_0, \beta)$  up to isomorphism, such that  $\Lambda_0 = \kappa_0 \otimes \rho_0$  compactly induces  $\pi_0$ . Moreover by (3.2.3), there exists a  $\Delta_0$ -regular tamely ramified character  $\xi_0$  of  $T_{0,m}^\times$  unique up to  $\Delta_0$ -action such that  $\rho_0 = \rho_{\xi_0}$ , where  $\Delta_0 = \mathrm{Gal}(T_{0,m}/T_0)$ . We denote by  $\kappa_0$  the restriction of  $\kappa_0$  to  $\mathbf{J}(\mathfrak{a}_0, \beta)$  as a  $\beta$ -extension of  $\theta_0$ .

### 3.6.1 Base change in supercuspidal case

First we assume  $\pi$  to be supercuspidal. Using the results in §3.5,  $E = F \otimes_{F_0} E_0 \cong F[\beta]$  is a field of degree  $r$  over  $E_0$ , and  $E_m = F \otimes_{F_0} E_{0,m}$  is a field of degree  $r$  over  $E_{0,m}$ , and we define  $T = T_0 F$  and  $T_m = T_{0,m} F$ . We write  $\tilde{A}_{F_0} := \mathrm{End}_{F_0}(E_m) \cong \mathrm{M}_{nr}(F_0)$  for the endomorphism ring of  $E_m$  as an  $F$ -vector space and we denote by  $\tilde{\mathfrak{a}}_0 = \mathfrak{a}_{F_0}(E_m)$  the hereditary order in  $\tilde{A}_{F_0}$  given by the ideal chain of  $E_m$ , which gives a simple stratum  $[\tilde{\mathfrak{a}}_0, \beta]$  in  $\tilde{A}_{F_0}$ . We write  $\tilde{\theta}_0$  for the simple character of  $H^1(\tilde{\mathfrak{a}}_0, \beta)$  as the transfer of  $\theta_0$ . Since  $F/F_0$  is tamely ramified, we may consider the interior  $F/F_0$ -lift of  $\theta_0$  as follows: First let  $A_F$  be the centralizer of  $F$  in  $\tilde{A}_{F_0}$ , which is isomorphic to  $\mathrm{M}_n(F)$ . We

write  $\mathfrak{a} := \tilde{\mathfrak{a}}_0 \cap A_F$ , which by definition is identified with  $\mathfrak{a}_F(E_m)$  as a hereditary order in  $A_F$ . Thus we get a simple stratum  $[\mathfrak{a}, \beta]$  in  $A_F$ , and for  $G_F := A_F^\times$ , we have  $H^1(\mathfrak{a}, \beta) = H^1(\tilde{\mathfrak{a}}_0, \beta) \cap G_F$  and  $\theta_b = \tilde{\theta}_0|_{H^1(\mathfrak{a}, \beta)}$  serving as the interior  $F/F_0$ -lift of  $\tilde{\theta}_0$ . As a result, the endo-class of  $\theta_b$  is the  $F/F_0$ -lift of the endo-class of  $\theta_0$ . Using [BH03], Theorem A, the supercuspidal representation  $\pi$  of  $G_F$  contains  $\theta_b$ . The Galois action  $\sigma \in \text{Gal}(F/F_0) \cong \text{Gal}(E/E_0)$  naturally induces an action on  $A_F$ , and by definition  $\mathfrak{a}$  and  $H^1(\mathfrak{a}, \beta)$  are  $\sigma$ -invariant,  $\sigma(\beta) = \beta$  and  $\theta_b \circ \sigma = \theta_b$  (see [BH96], §11.3, Remark). We sum up the discussion above as the following proposition:

**Proposition 3.6.1.** *For  $\pi$  a supercuspidal representation of  $\text{GL}_n(F)$  as the base change of a supercuspidal representation of  $\text{GL}_n(F_0)$ , there are a simple stratum  $[\mathfrak{a}, \beta]$  in  $M_n(F)$  and a simple character  $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ , such that:*

- (1)  $\sigma(\mathfrak{a}) = \mathfrak{a}$  and  $\sigma(H^1(\mathfrak{a}, \beta)) = H^1(\mathfrak{a}, \beta)$ ;
- (2)  $\theta \circ \sigma = \theta$ ;
- (3)  $\sigma(\beta) = \beta$ .

We denote by  $\eta_b$  the Heisenberg representation of  $\theta_b$ , then  $\eta_b \circ \sigma \cong \eta_b$ . For  $\varpi_F$  a uniformizer of  $F$ , we denote by  $C_{T_m}(\varpi_F)$  the subgroup of  $T_m^\times$  containing  $\varpi_F$  such that  $C_{T_m}(\varpi_F) \times U_{T_m}^1 \rightarrow T_m^\times$  is a bijection, where  $T_m$  is the maximal tamely ramified subextension of  $E_m$  over  $F$ . In the next section, we will first choose  $\varpi_F$ , and then describe a unique way to construct a  $\sigma$ -invariant full Heisenberg representation  $\kappa_b$  extending  $\eta_b$ , such that

$$C_{T_m}(\varpi_F) \subset \text{Ker}(\det(\kappa_b)) \quad \text{and} \quad \varpi_F \in \text{Ker}(\kappa_b). \quad (3.6.2)$$

Right now we just assume the existence of such  $\kappa_b$ .

By Proposition 3.2.3, there is a unique  $\sigma$ -invariant representation  $\rho_b$  of  $J(\mathfrak{a}, \beta)$  trivial on  $J^1(\mathfrak{a}, \beta)$  up to isomorphism, such that  $\Lambda_b = \kappa_b \otimes \rho_b$  compactly induces  $\pi$ . And by 3.2.3, we may choose  $\xi_b$  to be a  $\Delta$ -regular tamely ramified character of  $T_m^\times$ , unique up to  $\Delta$ -action, such that  $\rho_b = \rho_{\xi_b}$ , where  $\Delta = \text{Gal}(T_m/T) \cong \text{Gal}(T_{0,m}/T_0) = \Delta_0$ . One interesting question is to compare  $\xi_0$  with  $\xi_b$ , which equivalently gives a direct construction of an extended maximal simple type of  $\pi$  via that of  $\pi_0$ .

**Theorem 3.6.2.** (1) *There exists a tamely ramified character  ${}_b\phi_{\theta_0}^{F/F_0}$  of  $T_m^\times$  depending only on the endo-class of  $\theta_{T_{0,m}}$ , such that  $\xi_b \cdot {}_b\phi_{\theta_0}^{F/F_0}$  and  $\xi_0 \circ N_{T_m/T_{0,m}}$  are in the same  $\Delta$ -orbit.*

- (2)  ${}_b\phi_{\theta_0}^{F/F_0}|_{\mathfrak{o}_{T_m}^\times}$  is a quadratic character.

Using (2), there exists a quadratic character  ${}_b\varphi_{\theta_0}^{F/F_0}$  of  $\mathfrak{o}_E^\times$  trivial on  $1 + \mathfrak{p}_E$  such that  ${}_b\varphi_{\theta_0}^{F/F_0} \circ N_{E_m/E} = {}_b\phi_{\theta_0}^{F/F_0}|_{\mathfrak{o}_{T_m}^\times} \circ N_{E_m/T_m}$ . The supercuspidal representation  $K_{\kappa_0}(\pi_0)$  of  $\text{GL}_m(\mathbf{k}_{E_0})$  is given by the regular character  $\xi_0$  of  $\mathbf{k}_{E_{0,m}}^\times$  whose inflation equals  $(\xi_0 \circ N_{E_{0,m}/T_{0,m}})|_{\mathfrak{o}_{E_{0,m}}^\times}$ . Moreover for  $\kappa_b = \kappa_b|_{J(\mathfrak{a}, \beta)}$  and  $\kappa'_b := \kappa_b \cdot ({}_b\varphi_{\theta_0}^{F/F_0} \circ \det_E)$ , the supercuspidal representation  $K_{\kappa'_b}(\pi)$  is given by the regular character  $\xi'_b$  of  $\mathbf{k}_{E_m}^\times$  whose inflation equals  $({}_b\varphi_{\theta_0}^{F/F_0} \circ N_{E_m/E}) \cdot (\xi_b \circ N_{E_m/T_m})|_{\mathfrak{o}_{E_m}^\times}$ , and by definition  $\xi'_b$  and  $(\xi_0 \circ N_{\mathbf{k}_{E_m}/\mathbf{k}_{E_{0,m}}})^{e(E/E_0)}$  are in the same  $\text{Gal}(\mathbf{k}_{E_m}/\mathbf{k}_{E_{0,m}})$ -orbit. In particular:

**Corollary 3.6.3.** *For  $[\mathfrak{a}_0, \beta]$ ,  $[\mathfrak{a}, \beta]$ ,  $\theta_0$ ,  $\theta_b$  as above such that  $E/E_0$  is unramified, there exist a  $\beta$ -extension  $\kappa_0$  of  $\theta_0$  and a  $\beta$ -extension  $\kappa'_b$  of  $\theta_b$  such that*

$$K_{\kappa'_b}(\text{BC}_{F/F_0}(\pi_0)) = \text{BC}_{\mathbf{k}_E/\mathbf{k}_{E_0}}(K_{\kappa_0}(\pi_0))$$

for any  $\pi_0$  supercuspidal containing  $\theta_0$  whose base change  $\text{BC}_{F/F_0}(\pi_0)$  is also supercuspidal, where  $\text{BC}_{\mathbf{k}_E/\mathbf{k}_{E_0}}$  denotes the Shintani base change given in [Shi76].

### 3.6.2 Interior automorphic induction

In this subsection, we assume that  $F$  is isomorphic to a subfield of  $E_0$  over  $F_0$  and we identify  $F$  with a subfield of  $E_0$  via a certain  $F_0$ -embedding. We denote by  $C_F$  the centralizer of  $F$  in  $A_{F_0}$  and we identify  $\mathrm{GL}_{m/r}(F)$  with  $G_F := C_F^\times$ . Thus there exists a supercuspidal representation  $\pi'$  of  $G_F$  such that  $A_{F/F_0}(\pi') = \pi_0$ . By [BH03], Theorem A, the endo-classes of  $\pi'^{\sigma^i}$  run over all the  $F/F_0$ -lifts of that of  $\theta_0$  when  $i = 0, 1, \dots, r-1$ . Changing  $\pi'$  by its twist with a certain  $\sigma^i$  if necessary, we further assume that the endo-class of  $\pi'$  is that of the interior  $F/F_0$ -lift of  $\theta_0$ .

By definition  $\mathfrak{c} = \mathfrak{a}_0 \cap C_F$  is a hereditary order and  $[\mathfrak{c}, \beta]$  is a simple stratum in  $C_F$ . We also have  $H^1(\mathfrak{c}, \beta) = H^1(\mathfrak{a}_0, \beta) \cap G_F$  and  $\theta_a := \theta_0|_{H^1(\mathfrak{c}, \beta)}$  serving as the interior  $F/F_0$ -lift of  $\theta_0$ . Thus  $\pi'$  contains  $\theta_a$ . We denote by  $\eta_a$  the Heisenberg representation of  $\theta_a$ . In the next section, we will construct a full Heisenberg representation  $\kappa_a$  extending  $\eta_a$ , which in particular satisfies

$$C_{T_{0,m}}(\varpi_F) \subset \mathrm{Ker}(\det(\kappa_a)) \quad \text{and} \quad \varpi_F \in \mathrm{Ker}(\kappa_a), \quad (3.6.3)$$

where  $\varpi_F$  will be chosen later as a uniformizer of  $F$ . Thus there exists  $\rho_a$  as a representation of  $\mathbf{J}(\mathfrak{c}, \beta)$  trivial on  $J^1(\mathfrak{c}, \beta)$  unique up to isomorphism, such that  $\Lambda_a = \kappa_a \otimes \rho_a$  compactly induces  $\pi'$ . Here  $\mathbf{J}(\mathfrak{c}, \beta)/J^1(\mathfrak{c}, \beta) \cong \mathbf{J}(\mathfrak{a}_0, \beta)/J^1(\mathfrak{a}_0, \beta) \cong E_0^\times \mathrm{GL}_m(\mathfrak{o}_{E_0})$ , and in this sense we identify the representations of  $\mathbf{J}(\mathfrak{c}, \beta)$  trivial on  $J^1(\mathfrak{c}, \beta)$  with the representations of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  trivial on  $J^1(\mathfrak{a}_0, \beta)$ . We choose  $\xi_a$  to be a  $\Delta_0$ -regular tamely ramified character of  $T_{0,m}^\times$  corresponding to  $\rho_a$  via (3.2.3). We denote by  $\kappa_a$  the restriction of  $\kappa_a$  to  $J(\mathfrak{c}, \beta)$ .

**Theorem 3.6.4.** (1) *There exists a tamely ramified character  ${}_a\phi_{\theta_0}^{F/F_0}$  of  $T_{0,m}^\times$  depending only on the endo-class of  $\theta_{T_{0,m}}$ , such that  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0$  are in the same  $\Delta_0$ -orbit.*

(2) *The restriction  ${}_a\phi_{\theta_0}^{F/F_0}|_{\mathfrak{o}_{T_{0,m}}^\times}$  is quadratic. Thus there exists a quadratic character  ${}_a\varphi_{\theta_0}^{F/F_0}$  of  $\mathfrak{o}_{E_0}^\times$  trivial on  $1 + \mathfrak{p}_{E_0}$  determined by  ${}_a\varphi_{\theta_0}^{F/F_0} \circ \mathrm{N}_{E_{0,m}/E_0} = {}_a\phi_{\theta_0}^{F/F_0} \circ \mathrm{N}_{E_{0,m}/T_{0,m}}|_{\mathfrak{o}_{E_{0,m}}^\times}$ , and moreover  $K_{\kappa'_a}(\pi')$  is isomorphic to  $K_{\kappa_0}(\pi_0)$  as a supercuspidal representation of  $\mathrm{GL}_m(\mathbf{k}_{E_0})$ , where  $\kappa'_a := \kappa_a \cdot ({}_a\varphi_{\theta_0}^{F/F_0} \circ \det_{E_0})$ .*

**Remark 3.6.5.** *In the theorem we may also consider all the representations of the form  $\pi'^{\sigma^i}$ . What we need to do is to replace the original  $F_0$ -embedding  $F \hookrightarrow E_0$  with its composition with  $\sigma^i$ . Thus the notations  $C_F$ ,  $\mathfrak{c}$ ,  $H^1(\mathfrak{c}, \beta)$ ,  $J^1(\mathfrak{c}, \beta)$ ,  $\mathbf{J}(\mathfrak{c}, \beta)$  and  $\Delta_0$  remain unchanged, but we replace  $\theta_a$ ,  $\eta_a$ ,  $\kappa_a$ ,  $\rho_a$ ,  $\Lambda_a$  and  $\xi_a$  with  $\theta_a^{\sigma^i}$ ,  $\eta_a^{\sigma^i}$ ,  $\kappa_a^{\sigma^i}$ ,  $\rho_a^{\sigma^i}$ ,  $\Lambda_a^{\sigma^i}$  and  $\xi_a^{\sigma^i}$  respectively to obtain the corresponding theorem.*

### 3.6.3 Exterior automorphic induction

We now consider exterior automorphic induction in this subsection. More precisely, we assume that any non-trivial subextension of  $F$  over  $F_0$  is not isomorphic to any subfield of  $E_0$ , and there exists a supercuspidal representation  $\pi'$  of  $\mathrm{GL}_{n/r}(F)$  such that  $A_{F/F_0}(\pi') = \pi_0$ . In this case we write  $E = E_0 \otimes_{F_0} F = F[\beta]$  as a field extension of degree  $r$  over  $E_0$ . We further consider the following two cases:

#### 3.6.3.1

Suppose that  $E/E_0$  is unramified. Since  $r$  divides  $m$ , we identify  $E$  with the subfield of  $E_{0,m}$  which is unramified of degree  $r$  over  $E_0$ . Since  $F/F_0$  is tamely ramified, for  $C_F$  the centralizer of  $F$  in  $A_{F_0}$ , and

<sup>3</sup>Indeed  $G_F$  may represent  $A_F^\times$  (for the base change case) or  $C_F^\times$  (for the automorphic induction case) by abuse of notations, however since we are in different situations, we hope this abbreviation won't cause any trouble.

$\mathfrak{c} = \mathfrak{a}_0 \cap C_F$ , we know that  $[\mathfrak{c}, \beta]$  is a simple stratum in  $C_F$ . Moreover we have  $H^1(\mathfrak{c}, \beta) = H^1(\mathfrak{a}_0, \beta) \cap C_F$  and  $\theta_a = \theta_0|_{H^1(\mathfrak{c}, \beta)}$  as the interior  $F/F_0$ -lift of  $\theta_0$ . Using [BH03], Theorem B,  $\pi'$  as a supercuspidal representation of  $G_F \cong \mathrm{GL}_{n/r}(F)$  contains  $\theta_a$ .

We denote by  $T_{m/r}$  the unramified extension of degree  $m/r$  over  $T$  which is contained in  $E_{0,m}$ , thus by definition  $T_{m/r} = T_{0,m}$ . For  $\varpi_F$  a uniformizer of  $F$ , we define  $C_{T_{m/r}}(\varpi_F)$  as before. In the next section we will first specify  $\varpi_F$ , and then give a direct construction of a full Heisenberg extension  $\kappa_a$  of  $\mathbf{J}(\mathfrak{c}, \beta)$  extending  $\eta_a$  which satisfies

$$\varpi_F \in \mathrm{Ker}(\kappa_a) \quad \text{and} \quad C_{T_{m/r}}(\varpi_F) \subset \mathrm{Ker}(\det(\kappa_a)), \quad (3.6.4)$$

and we denote by  $\kappa_a$  the restriction of  $\kappa_a$  to  $\mathbf{J}(\mathfrak{c}, \beta)$ . Corresponding to  $\pi'$ , we define  $\rho_a$  and  $\xi_a$  as in the interior case, but here  $\Delta = \mathrm{Gal}(T_{m/r}/T)$  is a subgroup of  $\Delta_0$ .

**Theorem 3.6.6.** (1) *There exists a tamely ramified character  ${}_a\phi_{\theta_0}^{F/F_0}$  of  $T_{0,m}^\times$  depending only on the endo-class of  $\theta_{T_{0,m}}$ , such that  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0$  are in the same  $\Delta_0$ -orbit.*

(2) *The restriction  ${}_a\phi_{\theta_0}^{F/F_0}|_{\mathfrak{o}_{T_{0,m}}^\times}$  is quadratic.*

Thus there exists a quadratic character  ${}_a\varphi_{\theta_0}^{F/F_0}$  of  $\mathfrak{o}_E^\times$  trivial on  $1 + \mathfrak{p}_E$  determined by  ${}_a\varphi_{\theta_0}^{F/F_0} \circ N_{E_{0,m}/E} = {}_a\phi_{\theta_0}^{F/F_0} \circ N_{E_{0,m}/T_{0,m}}|_{\mathfrak{o}_{E_{0,m}}^\times}$ . In this case the supercuspidal representation  $K_{\kappa_0}(\pi_0)$  is given by the regular character  $\xi_0$  of  $\mathbf{k}_{E_{0,m}}^\times$  whose inflation equals  $(\xi_0 \circ N_{E_{0,m}/T_{0,m}})|_{\mathfrak{o}_{E_{0,m}}^\times}$ , and for  $\kappa'_a := \kappa_a \cdot ({}_a\varphi_{\theta_0}^{F/F_0} \circ \det_E)$ , the supercuspidal representation  $K_{\kappa'_a}(\pi')$  is given by the regular character  $\xi_a$  of  $\mathbf{k}_{E_{m/r}}^\times$  whose inflation equals  $(\xi_a \circ N_{E_{m/r}/T_{m/r}})|_{\mathfrak{o}_{E_{m/r}}^\times}$ , thus  $\xi_0$  and  $\xi_a$  as characters of  $\mathbf{k}_{E_{0,m}}^\times = \mathbf{k}_{E_{m/r}}^\times$  are in the same  $\mathrm{Gal}(\mathbf{k}_{E_{0,m}}/\mathbf{k}_{E_0})$ -orbit. Moreover, we may also change  $\pi'$  above by  $\pi'^{\sigma^i}$  with  $i = 0, 1, \dots, r-1$ . Then for different  $i$  the corresponding  $\pi'^{\sigma^i}$  are inequivalent, thus the corresponding  $\xi_a$  are in different  $\Delta$ -orbits, and the corresponding  $\xi_a$  are in different  $\mathrm{Gal}(\mathbf{k}_{E_{m/r}}/\mathbf{k}_E)$ -orbits. As a result when  $i$  varies,  $\xi_a$  ranges over exactly all the  $\mathrm{Gal}(\mathbf{k}_{E_{m/r}}/\mathbf{k}_E)$ -orbits which are contained in the  $\mathrm{Gal}(\mathbf{k}_{E_{0,m}}/\mathbf{k}_{E_0})$ -orbit of  $\xi_0$ . In other words, we have proved:

**Corollary 3.6.7.** *The parabolic induction  $K_{\kappa'_a}(\pi') \times K_{\kappa'_a}(\pi'^{\sigma}) \times \dots \times K_{\kappa'_a}(\pi'^{\sigma^{r-1}})$  is isomorphic to the Shintani base change of  $K_{\kappa_0}(\pi_0)$  related to  $\mathbf{k}_E/\mathbf{k}_{E_0}$ .*

### 3.6.3.2

When  $E/E_0$  is totally ramified, we write  $E_m = E_{0,m} \otimes_{E_0} E$  for the unramified extension of degree  $m$  over  $E$  and  $E_{m/r}$  for its subextension of degree  $m/r$  over  $E$ . We denote by  $\tilde{C}_{F_0} = \mathrm{End}_{F_0}(E_{m/r})$  the endomorphism ring of  $F_0$ -vector space  $E_{m/r}$ . Then  $\tilde{\mathfrak{c}}_0 := \mathfrak{a}_{F_0}(E_{m/r})$  is a hereditary order and  $[\tilde{\mathfrak{c}}_0, \beta]$  is a simple stratum in  $\tilde{C}_{F_0}$ . We define  $\tilde{\theta}'_0 = t_{\mathfrak{a}_0, \tilde{\mathfrak{c}}_0}^\beta(\theta_0)$  the transfer of  $\theta_0$  as a simple character of  $H^1(\tilde{\mathfrak{c}}_0, \beta)$ . We write  $C_F \cong M_{n/r}(F)$  for the centralizer of  $F$  in  $\tilde{C}_{F_0}$ . Since  $F/F_0$  is tamely ramified,  $\mathfrak{c} = \tilde{\mathfrak{c}}_0 \cap C_F$  is a hereditary order in  $C_F$ , and  $[\mathfrak{c}, \beta]$  is a simple stratum in  $C_F$ , and  $H^1(\mathfrak{c}, \beta) = H^1(\tilde{\mathfrak{c}}_0, \beta) \cap C_F$  and  $\theta_a = \tilde{\theta}'_0|_{H^1(\mathfrak{c}, \beta)}$  is the interior  $F/F_0$ -lift.

Using [BH03], Theorem B, the supercuspidal representation  $\pi'$  of  $G_F$  contains  $\theta_a$ . We denote by  $T_{m/r}$  the unramified extension of degree  $m/r$  over  $T$  which is contained in  $E_{m/r}$  and we define  $C_{T_{m/r}}(\varpi_F)$  as before. In the next section, we will specify our choice of  $\varpi_F$  and will construct a full Heisenberg extension  $\kappa_a$  of  $\mathbf{J}(\mathfrak{c}, \beta)$  extending  $\eta_a$  which satisfies (3.6.4). Corresponding to  $\pi'$ , we define  $\kappa_a$ ,  $\rho_a$  and  $\xi_a$  as in the  $E/E_0$  unramified case, but here  $\Delta = \mathrm{Gal}(T_{m/r}/T)$ . We write  $\Delta' := \mathrm{Gal}(T_m/T) \cong \mathrm{Gal}(T_{0,m}/T_0) = \Delta_0$ .

**Theorem 3.6.8.** (1) *There exists a tamely ramified character  ${}_a\phi_{\theta_0}^{F/F_0}$  of  $T_m^\times$  depending only on the endo-class of  $\theta_{T_0,m}$ , such that  $(\xi_a \circ N_{T_m/T_{m/r}}) \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0 \circ N_{T_m/T_0,m}$  are in the same  $\Delta'$ -orbit.*

(2) *The restriction  ${}_a\phi_{\theta_0}^{F/F_0}|_{\mathfrak{o}_{T_m}^\times}$  is quadratic. Thus there exists a quadratic character  ${}_a\varphi_{\theta_0}^{F/F_0}$  of  $\mathfrak{o}_E^\times$  trivial on  $1 + \mathfrak{p}_E$  determined by  ${}_a\varphi_{\theta_0}^{F/F_0} \circ N_{E_m/E} = {}_a\phi_{\theta_0}^{F/F_0} \circ N_{E_m/T_m}|_{\mathfrak{o}_{E_m}^\times}$ , such that for  $K_{\kappa'_a}(\pi')$  the supercuspidal representation of  $\mathrm{GL}_{m/r}(\mathbf{k}_E)$  given by a regular character  $\xi'$ , and  $K_{\kappa_0}(\pi_0)$  the supercuspidal representation of  $\mathrm{GL}_m(\mathbf{k}_{E_0})$  given by a regular character  $\xi_0$ , the characters  $\xi_0^r$  and  $\xi' \circ N_{\mathbf{k}_{E_0,m}/\mathbf{k}_{E_m/r}}$  are in the same  $\mathrm{Gal}(\mathbf{k}_{E_0,m}/\mathbf{k}_{E_0})$ -orbit, where  $\kappa'_a := \kappa_a \cdot ({}_a\varphi_{\theta_0}^{F/F_0} \circ \det_E)$ .*

### 3.6.4 base change in non-supercuspidal case

Finally we finish the discussion for base change. We assume that there exists a supercuspidal representation  $\pi'$  of  $\mathrm{GL}_{n/r}(F)$  such that  $\pi = \mathrm{BC}_{F/F_0}(\pi_0) \cong \pi' \times \pi'^\sigma \times \dots \times \pi'^{\sigma^{r-1}}$ . Since  $\pi'$  is  $\sigma$ -regular, by [BH03], (5.1.2), we have  $A_{F/F_0}(\pi') = \pi$  and  $A_{F/F_0}^{-1}(\pi) = \{\pi', \pi'^\sigma, \dots, \pi'^{\sigma^{r-1}}\}$ . Thus the result in the last two subsections are enough to study  $\pi$ .

We give a ‘‘base change’’ version of Corollary 3.6.7 to end this section. We assume  $E/E_0$  to be unramified of degree  $r$  and we write  $F \otimes_{F_0} E_{0,m} = \prod_{i=1}^r E_i$ , where each  $E_i$  is isomorphic to  $E_{0,m}$ . We denote by  $\tilde{\mathfrak{a}}_0$  the hereditary order over  $F_0$  defined by the ideal chain  $\{\prod_{i=1}^r \mathfrak{p}_{E_i}^k | k \in \mathbb{Z}\}$ , thus  $[\tilde{\mathfrak{a}}_0, \beta]$  is a simple stratum in  $\tilde{A}_{F_0} := \mathrm{End}_{F_0}(F \otimes_{F_0} E_{0,m}) \cong M_{mr}(F_0)$ . Just as in the first paragraph of §3.6.1, we define  $\tilde{\theta}_0$ ,  $A_F$ ,  $\mathfrak{a}$ ,  $[\mathfrak{a}, \beta]$  and  $\theta_b := \tilde{\theta}_0|_{H^1(\mathfrak{a}, \beta)}$ . Moreover the flag

$$E_1 \subset E_1 \oplus E_2 \subset \dots \subset E_1 \oplus \dots \oplus E_r = F \otimes_{F_0} E_{0,m}$$

gives a parabolic subgroup  $P_b$  of  $A_F^\times$  with the corresponding Levi subgroup denoted by  $M_b$  and unipotent radical denoted by  $U_b$ . We write  $\mathfrak{a}_i = \mathfrak{a}_F(E_i)$  for the hereditary order and  $[\mathfrak{a}_i, \beta]$  the corresponding simple stratum in  $A_{F,i} := \mathrm{End}_F(E_i)$ , thus by definition  $M_b = A_{F,1}^\times \times \dots \times A_{F,r}^\times$ . We have

$$H^1(\mathfrak{a}, \beta) \cap M_b = H^1(\mathfrak{a}_1, \beta) \times \dots \times H^1(\mathfrak{a}_r, \beta)$$

and

$$\theta_b|_{H^1(\mathfrak{a}_b, \beta) \cap M_b} = \theta_1 \otimes \dots \otimes \theta_r,$$

where  $\theta_i$  is the corresponding simple character of  $H^1(\mathfrak{a}_i, \beta)$ . By identifying each  $E_i$  with  $E_{0,m}$ , we identify  $A_{F,i}$  with  $C_F$ ,  $[\mathfrak{a}_i, \beta]$  with  $[\mathfrak{c}, \beta]$  and  $\theta_i$  with  $\theta_a$  respectively. Via this identification, we further denote by  $\eta_i$  the Heisenberg representation and  $\kappa_i$  the  $\beta$ -extension corresponding to  $\eta_a$  and  $\kappa_a$  respectively, and we regard  $\pi'^{\sigma^{i-1}}$  as a supercuspidal representation of  $A_{F,i}^\times$ . We denote by  $\eta_b$  the Heisenberg extension of  $\theta_b$  and we may choose  $\kappa_b$  to be a  $\beta$ -extension extending  $\eta_b$ , such that

$$K_{\kappa_b} \circ \mathrm{Ind}_{P_b}^{A_F^\times} = \mathrm{Ind}_{\overline{P_b}}^{\mathrm{GL}_m(\mathbf{k}_E)} \circ (K_{\kappa_1} \otimes \dots \otimes K_{\kappa_r}),$$

as functors from the category of finite length smooth representations of  $M_b$  to that of  $\mathrm{GL}_m(\mathbf{l})$ , where  $\overline{P_b}$  is the corresponding parabolic subgroup of  $\mathrm{GL}_m(\mathbf{k}_E)$  (see [MS14b], section 5 for more details), thus in particular we have

$$K_{\kappa_b}(\pi' \times \dots \times \pi'^{\sigma^{r-1}}) = K_{\kappa_1}(\pi') \times K_{\kappa_2}(\pi'^\sigma) \times \dots \times K_{\kappa_r}(\pi'^{\sigma^{r-1}}).$$

By Corollary 3.6.7, we finally have

$$K_{\kappa_b}(\pi' \times \dots \times \pi'^{\sigma^{r-1}}) = \mathrm{BC}_{\mathbf{k}_E/\mathbf{k}_{E_0}}(K_{\kappa'_0}(\pi_0)).$$

We record this result as the following corollary:

**Corollary 3.6.9.** *For  $n = md$  with  $m$  divisible by  $r$  and  $[\mathfrak{a}_0, \beta]$ ,  $[\mathfrak{a}, \beta]$ ,  $\theta_0$ ,  $\theta_b$  as above, there exist a  $\beta$ -extension  $\kappa_0$  of  $\theta_0$  and a  $\beta$ -extension  $\kappa_b$  of  $\theta_b$ , such that*

$$K_{\kappa_\alpha}(\mathrm{BC}_{F/F_0}(\pi_0)) = \mathrm{BC}_{\mathbf{k}_E/\mathbf{k}_{E_0}}(K_{\kappa_0}(\pi_0))$$

for any supercuspidal representation  $\pi_0$  of  $\mathrm{GL}_n(F_0)$  containing  $\theta_0$ .

### 3.7 A precise construction of the full Heisenberg representation

In the previous section, we stated several results related to the base change lift and automorphic induction. More precisely, for  $F/F_0$  a cyclic tamely ramified extension of degree  $r$ , we considered  $\pi_0$  as a supercuspidal representation of  $\mathrm{GL}_n(F_0)$  such that either its base change  $\mathrm{BC}_{F/F_0}(\pi_0)$  as a representation of  $\mathrm{GL}_n(F)$  is supercuspidal, or there exists a supercuspidal representation  $\pi'$  of  $\mathrm{GL}_{n/r}(F)$  such that  $A_{F/F_0}(\pi') = \pi_0$ . We fixed a simple stratum  $[\mathfrak{a}_0, \beta]$  and a character  $\theta_0$  related to  $\pi_0$ , and we fixed a certain full Heisenberg representation  $\kappa_0$  of  $\theta_0$ , which gives a unique representation  $\rho_0$  and the corresponding  $\Delta_0$ -regular orbit represented by  $\xi_0$ . In the base change case, we constructed a  $\sigma$ -invariant simple stratum  $[\mathfrak{a}, \beta]$  and a  $\sigma$ -invariant simple character  $\theta_b$  related to  $\pi$  via that of  $\pi_0$ . In this section, we will give an explicit construction of a  $\sigma$ -invariant full Heisenberg representation  $\kappa_b$  of  $\theta_b$ . Similarly in the automorphic induction case, we constructed a simple stratum  $[\mathfrak{c}, \beta]$  and a simple character  $\theta_a$  related to  $\pi$  via that of  $\pi_0$ . Still in this section, we will give an explicit construction of a full Heisenberg representation  $\kappa_a$  of  $\theta_a$ .

#### 3.7.1 Several results of Bushnell-Henniart

In this subsection, we recall and reformulate several known results in a series of articles of Bushnell-Henniart.

##### 3.7.1.1

First of all, we recall the result related to the base change in the wild case. Let  $[\mathfrak{a}_0, \beta]$ ,  $[\mathfrak{a}, \beta]$ ,  $\theta_0$ ,  $\theta_b$  be as in §3.6.1, and we assume further that  $n = d = p^s$  for a certain  $s \in \mathbb{N}$  and  $E_0/F_0$  is totally wildly ramified. By definition,  $\pi'_0 := \mathrm{ind}_{\mathbf{J}(\mathfrak{a}_0, \beta)}^{G_{F_0}} \kappa_0$  is a supercuspidal representation of  $G_{F_0}$ . We fix a uniformizer  $\varpi_{F_0}$  of  $F_0$  and we choose  $\varpi_F$  to be a uniformizer of  $F$  such that  $N_{F/F_0}(\varpi_F) \in \langle \varpi_{F_0} \rangle \mu_{F_0}$ . By Corollary 3.5.12,  $\pi' := \mathrm{BC}_{F/F_0}(\pi'_0)$  is a supercuspidal representation of  $G_F$ .

**Theorem 3.7.1** ( [BH96], Proposition 14.10, [BH99], Theorem 1.8, Proposition 1.8.). (1) *There is a unique  $\sigma$ -invariant full Heisenberg representation  $\kappa_b$  of  $\theta_b$  such that  $\pi' \cong \mathrm{ind}_{\mathbf{J}(\mathfrak{a}, \beta)}^{G_F} \kappa_b$ ;*

(2) *When either  $r$  or  $v_{E_0}(\beta) = -m_\beta$  is relatively prime to  $p$ , the full Heisenberg representation  $\kappa_b$  can be constructed explicitly.*

**Corollary 3.7.2.** *For  $\varpi_F$  a uniformizer of  $F$  and  $\varpi_{F_0}$  a uniformizer of  $F_0$  such that  $N_{F/F_0}(\varpi_F) \in \langle \varpi_{F_0} \rangle \mu_{F_0}$ , the map  $\mathbf{b}_{F/F_0} : \kappa_0 \mapsto \kappa_b$  is a bijection between full Heisenberg representations of  $\theta_0$  and that of  $\theta_b$  satisfying respectively*

$$\langle \varpi_{F_0} \rangle \mu_{F_0} \subset \mathrm{Ker}(\kappa_0) \quad \text{and} \quad \langle \varpi_F \rangle \mu_F \subset \mathrm{Ker}(\kappa_b) \quad (3.7.1)$$

*Proof.* By Theorem 3.4.1 we have  $\omega_{\kappa_0} \circ N_{F/F_0} = \omega_{\kappa_b}$  with  $\omega_{\kappa_0}$  and  $\omega_{\kappa_b}$  denoting the central characters, thus

$$N_{F/F_0}(\langle \varpi_F \rangle \mu_F) \subset \mathrm{Ker}(\kappa_0) \quad \text{if and only if} \quad \langle \varpi_F \rangle \mu_F \subset \mathrm{Ker}(\kappa_b).$$

Since  $N_{F/F_0}(\langle \varpi_F \rangle \mu_F) \subset \langle \varpi_{F_0} \rangle \mu_{F_0}$ , the map  $\mathbf{b}_{F/F_0}$  must be surjective. Moreover, by [BH14b], §5.6., Lemma 2, both sets of the full Heisenberg representations satisfying the condition of the statement are of cardinality  $p^s$ , thus  $\mathbf{b}_{F/F_0}$  is actually a bijection.  $\square$

Let  $\kappa_0$  and  $\kappa_b$  be two full Heisenberg representations satisfying Theorem 3.7.1 (1) and (3.7.1). For any supercuspidal representation  $\pi_0$  of  $G_{F_0}$  containing  $\theta_0$ , by Corollary 3.2.5, there exists a unique tamely ramified character  $\xi_0$  of  $F_0^\times$ , such that  $\pi_0 \cong \text{ind}_{\mathbf{J}(\mathfrak{a}_0, \beta)}^{G_{F_0}}(\kappa_0 \cdot (\xi_0 \circ N_{E_0/F_0}))$ . We write  $\pi = \text{BC}_{F/F_0}(\pi_0)$  which is a supercuspidal representation of  $G_F$  containing  $\theta_b$ . Still by *loc. cit.* there exists a unique tamely ramified character  $\xi_b$  of  $F^\times$ , such that  $\pi \cong \text{ind}_{\mathbf{J}(\mathfrak{a}, \beta)}^{G_F}(\kappa_b \cdot (\xi_b \circ N_{E/F}))$ . Here  $\xi_0 \circ N_{E_0/F_0}$  and  $\xi_b \circ N_{E/F}$  are regarded as characters of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  and  $\mathbf{J}(\mathfrak{a}, \beta)$  respectively via the isomorphisms  $\mathbf{J}(\mathfrak{a}_0, \beta)/J^1(\mathfrak{a}_0, \beta) \cong E_0^\times/1 + \mathfrak{p}_{E_0}$  and  $\mathbf{J}(\mathfrak{a}, \beta)/J^1(\mathfrak{a}, \beta) \cong E^\times/1 + \mathfrak{p}_E$ .

**Proposition 3.7.3.** *We have  $\xi_b = \xi_0 \circ N_{F/F_0}$  in the setting above.*

*Proof.* Comparing the central character of  $\pi_0$  and  $\pi$  and using Theorem 3.7.1, we get  $\xi_b^n = (\xi_0 \circ N_{F/F_0})^n$ , which means that  $\xi_b$  and  $\xi_0 \circ N_{F/F_0}$  coincide on  $(1 + \mathfrak{p}_F)F^{\times n}$ . To finish the proof, we only need to show that  $\xi_b(\varpi_F) = \xi_0(N_{F/F_0}(\varpi_F))$ . We fix a uniformizer  $\varpi_E$  of  $E$  such that  $\varpi_E^n \in \varpi_F(1 + \mathfrak{p}_F)$ . We fix an integer  $k$  and we choose  $h_0 \in J^1(\mathfrak{a}_0, \beta)$  such that  $h'_0 := N_{E/E_0}(\varpi_E^k)h_0$  is an elliptic element in  $G_{F_0}$ , and moreover  $\text{tr}(\pi'_0)(h'_0) \neq 0$  (Using, for example, [BH14b], §9.5, Linear Independence Lemma). Since it is easy to see that  $\det_{F_0}(h'_0) \in N_{F/F_0}(F^\times)$ , we may choose  $h'$  to be an element in  $F[h'_0]^\times \subset G_F$  such that the norm  $N_{F[h'_0]/F_0[h'_0]}(h') = h'\sigma(h')\dots\sigma^{r-1}(h')$  equals  $h'_0$ . By direct calculation we have  $v_F(\det_F(h')) = k$ .

Since  $\kappa_b$  is  $\sigma$ -invariant, we may choose a certain extension to regard  $\kappa_b$  as a representation of  $\mathbf{J}(\mathfrak{a}, \beta) \rtimes \Sigma$ , which naturally extends  $\pi'$  as a representation of  $G_F \rtimes \Sigma$  via compact induction. Similarly since  $\xi_b$  is a  $\sigma$ -invariant character, we realize  $\kappa_b \cdot (\xi_b \circ N_{E/F})$  as a representation of  $\mathbf{J}(\mathfrak{a}, \beta) \rtimes \Sigma$  and  $\pi$  as a representation of  $G_F \rtimes \Sigma$ . Using the trace formula for the cyclic base change, we have

$$\text{tr}(\pi')(h', \sigma) = c' \cdot \text{tr}(\pi'_0)(h'_0) \quad \text{and} \quad \text{tr}(\pi)(h', \sigma) = c \cdot \text{tr}(\pi_0)(h'_0),$$

where  $c'$  and  $c$  are non-zero constants depending on the choice of the extension above.

**Lemma 3.7.4.** *We have*

$$\text{tr}(\pi_0)(h'_0) = \text{tr}(\pi'_0)(h'_0) \cdot \xi_0(N_{F/F_0}(\varpi_F^k)) \quad \text{and} \quad \text{tr}(\pi)(h', \sigma) = \text{tr}(\pi')(h', \sigma) \cdot \xi_b(\varpi_F^k).$$

*Proof.* For the first equation, using the Mackey formula<sup>4</sup> we have

$$\text{tr}(\pi'_0)(h'_0) = \sum_{\substack{g_0 \in \mathbf{J}(\mathfrak{a}_0, \beta) \setminus G_{F_0}, \\ g_0^{-1}h'_0g_0 \in \mathbf{J}(\mathfrak{a}_0, \beta)}} \text{tr}(\kappa_0)(g_0^{-1}h'_0g_0)$$

and

$$\text{tr}(\pi_0)(h'_0) = \sum_{\substack{g_0 \in \mathbf{J}(\mathfrak{a}_0, \beta) \setminus G_F, \\ g_0^{-1}h'_0g_0 \in \mathbf{J}(\mathfrak{a}_0, \beta)}} \text{tr}(\kappa_0)(g_0^{-1}h'_0g_0) \cdot (\xi_0 \circ N_{E_0/F_0})(g_0^{-1}h'_0g_0)$$

Since  $\det_{F_0}(g_0^{-1}h'_0g_0) = \det_{F_0}(h'_0)$ , it is easy to see that the images of  $g_0^{-1}h'_0g_0 \in \mathbf{J}(\mathfrak{a}_0, \beta)$  and  $h'_0$  in  $\mathbf{J}(\mathfrak{a}_0, \beta)/J^1(\mathfrak{a}_0, \beta) \cong E_0^\times/(1 + \mathfrak{p}_{E_0})$  are identical. Thus

$$(\xi_0 \circ N_{E_0/F_0})(g_0^{-1}h'_0g_0) = (\xi_0 \circ N_{E_0/F_0})(h'_0) = \xi_0(N_{F/F_0}(\varpi_F^k)),$$

<sup>4</sup>Note that there are only finite many non-zero terms in the sum of the right hand side, see for example [BH11], Lemma 1.2.

finishing the first equation. For the second equation, we have

$$\mathrm{tr}(\pi')(h', \sigma) = \sum_{\substack{g \in \mathbf{J}(\mathbf{a}, \beta) \backslash G_F, \\ g^{-1}h'\sigma(g) \in \mathbf{J}(\mathbf{a}, \beta)}} \mathrm{tr}(\kappa_b)(g^{-1}h'\sigma(g), \sigma)$$

and

$$\mathrm{tr}(\pi)(h', \sigma) = \sum_{\substack{g \in \mathbf{J}(\mathbf{a}, \beta) \backslash G_F, \\ g^{-1}h'\sigma(g) \in \mathbf{J}(\mathbf{a}, \beta)}} \mathrm{tr}(\kappa_b)(g^{-1}h'\sigma(g), \sigma) \cdot (\xi_b \circ N_{E/F})(g^{-1}h'\sigma(g))$$

Since  $\xi_b$  is  $\sigma$ -invariant, there exists a character  $\xi'_0$  of  $F_0^\times$  such that  $\xi_b = \xi'_0 \circ N_{F/F_0}$ . For  $g \in G_F$  such that  $g^{-1}h'\sigma(g) \in \mathbf{J}(\mathbf{a}, \beta)$  and  $N_{F/F_0}(h) := h'\sigma(h')\dots\sigma^{r-1}(h')$ , we have

$$(\xi_b \circ N_{E/F})(g^{-1}h'\sigma(g)) = (\xi'_0 \circ N_{E_0/F_0})(g^{-1}N_{F/F_0}(h')g) = (\xi'_0 \circ N_{E_0/F_0})(N_{F/F_0}(h')) = (\xi_b \circ N_{E/F})(h'),$$

where the equation in the middle follows from the fact that  $\det_F(g^{-1}N_{F/F_0}(h')g) = \det_F(N_{F/F_0}(h'))$ , and thus  $g^{-1}N_{F/F_0}(h')g$  and  $N_{F/F_0}(h')$  are identical in  $\mathbf{J}(\mathbf{a}, \beta)/\mathbf{J}^1(\mathbf{a}, \beta) \cong E^\times/(1 + \mathfrak{p}_E)$ . Thus

$$(\xi_b \circ N_{E/F})(g^{-1}h'\sigma(g)) = (\xi_b \circ N_{E/F})(h') = \xi_b(\varpi_F^k),$$

finishing the proof of the second equation. □

Using this lemma and the above equations, we get

$$0 \neq c' \cdot \mathrm{tr}(\pi'_0)(h'_0) \cdot \xi_b(\varpi_F^k) = c \cdot \mathrm{tr}(\pi'_0)(h'_0) \cdot \xi_0(N_{F/F_0}(\varpi_F^k)).$$

Since  $k$  is arbitrary, we must have  $c = c'$  and  $\xi_b(\varpi_F) = \xi_0(N_{F/F_0}(\varpi_F))$ , finishing the proof of the proposition. □

### 3.7.1.2

Now we consider unramified automorphic induction. Let  $[\mathbf{a}_0, \beta]$  and  $\theta_0$  be as before, let  $K_0$  be the maximal unramified subextension of  $E_{0,m}$  over  $F_0$ . We denote by  $A_{K_0}$  the centralizer of  $K_0$  in  $A_{F_0}$  and we write  $G_{K_0} := A_{K_0}^\times$ , thus  $\mathfrak{a}_{K_0} = \mathfrak{a}_0 \cap A_{K_0}$  is a hereditary order, and  $[\mathfrak{a}_{K_0}, \beta]$  is a simple stratum in  $A_{K_0}$ . Moreover  $H^1(\mathfrak{a}_{K_0}, \beta) = H^1(\mathfrak{a}_0, \beta) \cap G_{K_0}$  and  $\theta_{K_0} = \theta_0|_{H^1(\mathfrak{a}_{K_0}, \beta)}$  is the interior  $K_0/F_0$ -lift of  $\theta_0$ . We fix a uniformizer  $\varpi_{F_0}$  as in [§3.6.1](#) and we choose a full Heisenberg  $\kappa_0$  representation of  $\theta_0$  satisfying [\(3.6.1\)](#). The group  $E_0^\times \mathbf{J}^1(\mathfrak{a}_0, \beta)/\langle \varpi_{F_0} \rangle$  has a unique pro- $p$ -subgroup, and we denote by  ${}_p\mathbf{J}(\mathfrak{a}_0, \beta)(\varpi_{F_0})$  its inverse image in  $\mathbf{J}(\mathfrak{a}_0, \beta)$ . We have  ${}_p\mathbf{J}(\mathfrak{a}_{K_0}, \beta)(\varpi_{F_0}) = {}_p\mathbf{J}(\mathfrak{a}_0, \beta)(\varpi_{F_0}) \cap G_{K_0}$  which is a subgroup of  $\mathbf{J}(\mathfrak{a}_{K_0}, \beta) = \mathbf{J}(\mathfrak{a}_0, \beta) \cap G_{K_0}$ .

**Proposition 3.7.5** ([\[BH14b\]](#), §5.6, Lemma 3, Lemma 4 and Proposition). *There exists a unique  $\Delta_0$ -invariant full Heisenberg representation  $\kappa_{K_0}$  of  $\mathbf{J}(\mathfrak{a}_{K_0}, \beta)$  of  $\theta_{K_0}$  satisfying*

$$C_{T_{0,m}}(\varpi_{F_0}) \subset \mathrm{Ker}(\det(\kappa_{K_0})) \quad \text{and} \quad \varpi_{F_0} \in \mathrm{Ker}(\kappa_{K_0}), \quad (3.7.2)$$

such that for any  $h \in {}_p\mathbf{J}(\mathfrak{a}_{K_0}, \beta)(\varpi_{F_0})$  and any  $\zeta_0 \in \mu_{K_0}$  which is  $\mathrm{Gal}(K_0/F_0)$ -regular,

$$\mathrm{tr}(\kappa_{K_0})(h) = \epsilon_{F_0}^0(\mu_{K_0}) \cdot \epsilon_{F_0}^1(\mu_{K_0})(\zeta_0) \cdot \mathrm{tr}(\kappa_0)(\zeta_0 h).$$

(cf. [Example 3.3.2](#)) Moreover the map  $\mathbf{l}_{K_0/F_0}^{ur} : \kappa_0 \mapsto \kappa_{K_0}$  given above induces a bijection from the set of full Heisenberg extensions of  $\theta_0$  satisfying [\(3.6.1\)](#) to the set of full Heisenberg extensions of  $\theta_{K_0}$  satisfying [\(3.7.2\)](#).



Using [BH14b], §9.1, Proposition, the following unramified automorphic induction is bijective:

$$A_{K_0/F_0} : \Delta_0 \backslash \mathcal{A}_1^0(K_0, \Theta_{K_0})^{\Delta_0\text{-reg}} \longrightarrow \mathcal{A}_m^0(F_0, \Theta_0),$$

where  $\Theta_{K_0}$  denotes the endo-class of  $\theta_{K_0}$ , and  $\Delta_0 \backslash \mathcal{A}_1^0(K_0, \Theta_{K_0})^{\Delta_0\text{-reg}}$  denotes the  $\Delta_0$ -orbits of  $\Delta_0$ -regular representations in  $\mathcal{A}_1^0(K_0, \Theta_{K_0})$ . Thus we may choose  $\pi_{K_0}$  to be a supercuspidal representation of  $G_{K_0}$  containing  $\theta_{K_0}$ , such that  $\pi_0 = A_{K_0/F_0}(\pi_{K_0})$ . Let  $(\mathbf{J}(\mathfrak{a}_{K_0}, \beta), \Lambda_{K_0})$  be an extended maximal simple type compactly inducing  $\pi_{K_0}$ . By Corollary 3.2.5, there exists a unique tamely ramified character  $\xi_{K_0}$  of  $T_{0,m}^\times$ , such that  $\Lambda_{K_0} = \kappa_{K_0} \cdot (\xi_{K_0} \circ N_{E_{0,m}/T_{0,m}})$ .

**Theorem 3.7.6** ([BH14b], §9.1, Unramified Induction Theorem, §10.7, Corollary). *There exists a  $\Delta_0$ -fixed tamely ramified character  $\mu_{\theta_0}^{K_0/F_0}$  of  $T_{0,m}^\times$  uniquely determined by the following equations (cf. Example 3.3.2, 3.3.3):*

$$(1) \mu_{\theta_0}^{K_0/F_0}|_{\mu_{T_{0,m}}} = \epsilon_{F_0}^1(\mu_{T_{0,m}});$$

(2)  $\mu_{\theta_0}^{K_0/F_0}(\varpi_{F_0}) = \varkappa_{K_0/F_0}(\varpi_{F_0})^{n([K_0:F_0]-1)/2} = (-1)^{[E_{0,m}:K_0]([K_0:F_0]-1)}$ , where  $\varkappa_{K_0/F_0}$  is a primitive character of  $F_0^\times/N_{K_0/F_0}(K_0^\times)$ ;

(3) For  $\varpi_{T_0}$  a uniformizer of  $T_0$  lying in  $C_{T_0}(\varpi_{F_0})$  and  $L_0 = F_0[\varpi_{T_0}]$ ,

$$\mu_{\theta_0}^{K_0/F_0}(\varpi_{T_0})^{[E_0:T_0]} = \mathbf{d}' \cdot \epsilon_{K_0}(\varpi_{T_0}) \cdot \epsilon_{F_0}(\varpi_{T_0}) \cdot \epsilon_{L_0}^0(\mu_{T_{0,m}}) \cdot \epsilon_{F_0}^0(\mu_{T_{0,m}}),$$

where  $\mathbf{d}' = \varkappa_{K_0/F_0}(\varpi_{F_0})^{n([K_0:F_0]-1)[E_0:T_0]/2} = (-1)^{[E_{0,m}:K_0]([K_0:F_0]-1)[E_0:T_0]}$ .

More importantly for any  $\xi_0$  and  $\xi_{K_0}$  as above,  $\xi_0$  is  $\Delta_0$ -conjugate to  $\xi_{K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$ .

**Remark 3.7.7.** In the original statement of [BH14b], §10.7, Corollary, the sign  $\mathbf{d}'$  is not explicitly given, which is actually a simple corollary of *ibid.*, Transfer Lemma (Using the exact value of  $\mathbf{d}$  given in the last paragraph of its proof), (10.4.9), (10.5.1) and (10.6.3).

### 3.7.1.3

Finally we consider the automorphic induction in the maximal totally ramified case, that is, we assume that  $n = d$  and  $E_0/F_0$  is totally ramified. Thus we have  $m = 1$ ,  $E_0 = E_{0,m}$  and  $T_0 = T_{0,m}$ . Let  $A_{T_0}$  be the centralizer of  $T_0$  in  $A_{F_0}$  and let  $G_{T_0} = A_{T_0}^\times$ . Similarly we define  $\mathfrak{a}_{T_0}$ ,  $[\mathfrak{a}_{T_0}, \beta]$ ,  $H^1(\mathfrak{a}_{T_0}, \beta)$  and  $\theta_{T_0} = \theta_0|_{H^1(\mathfrak{a}_{T_0}, \beta)}$ , where  $\theta_{T_0}$  is the  $T_0/F_0$ -lift of  $\theta_0$ .

**Proposition 3.7.8** ([BH14b], §5.3, Proposition, §5.6, Lemma 2). *For  $\kappa_0$  a full Heisenberg representation of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  of  $\theta_0$ , there exists a unique full Heisenberg representation  $\kappa_{T_0}$  of  $\mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  of  $\theta_{T_0}$ , such that (cf. Example 3.3.2)*

$$\mathrm{tr}(\kappa_{T_0})(x) = \epsilon_{T_0/F_0} \cdot \mathrm{tr}(\kappa_0)(x)$$

for any  $x \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  such that  $v_{T_0}(\det_{T_0}(x))$  is relatively prime to  $t = [T_0 : F_0]$ . The map  $\mathbf{l}_{T_0/F_0}^{\mathrm{tr}} : \kappa_0 \mapsto \kappa_{T_0}$  given above is a bijection between full Heisenberg representations of  $\theta_0$  and that of  $\theta_{T_0}$ . Moreover,

$$C_{T_0}(\varpi_{F_0}) \subset \mathrm{Ker}(\det(\kappa_0)) \quad \text{and} \quad \varpi_{F_0} \in \mathrm{Ker}(\kappa_0)$$

if and only if

$$C_{T_0}(\varpi_{F_0}) \subset \mathrm{Ker}(\kappa_{T_0}).$$

**Remark 3.7.9.** In the maximal totally ramified case, by definition  $(\mathbf{J}(\mathfrak{a}_0, \beta), \kappa_0)$  is actually an extended maximal simple type containing  $\theta_0$ , and every extended maximal simple type containing  $\theta_0$  is of this form. Similarly,  $(\mathbf{J}(\mathfrak{a}_{T_0}, \beta), \kappa_{T_0})$  ranges over all the extended maximal simple type containing  $\theta_{T_0}$  when  $\kappa_{T_0}$  ranges over all the full Heisenberg representations of  $\theta_{T_0}$ .

Since  $T_0/F_0$  is not always cyclic (not even Galois), we cannot use the construction of Henniart-Herb [HH95] to define the automorphic induction  $A_{T_0/F_0}$ . However in §4.3. we gave the definition from the Galois side instead. We denote by  $\mathcal{W}_{F_0}$  the Weil group of  $F_0$  and by  $\mathcal{W}_{T_0}$  the Weil group of  $T_0$  regarded as a subgroup of  $\mathcal{W}_{F_0}$ . Since  $T_0/F_0$  is tamely ramified, we denote by  $\mathcal{P}_{F_0} = \mathcal{P}_{T_0}$  the wild inertia subgroup of  $\mathcal{W}_{F_0}$  (resp.  $\mathcal{W}_{T_0}$ ) and we denote by  $\widehat{\mathcal{P}}_{F_0} = \widehat{\mathcal{P}}_{T_0}$  the set of equivalence classes of smooth irreducible representations of  $\mathcal{P}_{F_0} = \mathcal{P}_{T_0}$ , which naturally endows with a  $\mathcal{W}_{F_0}$  (resp.  $\mathcal{W}_{T_0}$ )-action. By [BH14b], §6.1, Ramification Theorem, there exists a unique  $\alpha \in \widehat{\mathcal{P}}_{T_0}$  such that  $\mathcal{W}_{T_0} \cdot \alpha = \alpha$  and the following bijection is induced from the ordinary local Langlands correspondence:

$$\text{LLC}_{T_0} : \mathcal{G}_1^0(T_0, \alpha) \longrightarrow \mathcal{A}_1^0(T_0, \Theta_{T_0}),$$

where  $\mathcal{G}_1^0(T_0, \alpha)$  denotes the set of irreducible representations of  $\mathcal{W}_{T_0}$  whose restriction to  $\mathcal{P}_{T_0}$  contains  $\alpha$  with multiplicity 1. Since  $\alpha$  can be naturally regarded as an element in  $\widehat{\mathcal{P}}_{F_0}$ , thus as an orbit in  $\mathcal{W}_{F_0} \backslash \widehat{\mathcal{P}}_{F_0}$ , by *ibid.*, §6.1, Ramification Theorem and §6.2, Proposition, the following bijection is induced from the ordinary local Langlands correspondence:

$$\text{LLC}_{F_0} : \mathcal{G}_1^0(F_0, \alpha) \longrightarrow \mathcal{A}_1^0(F_0, \Theta_0),$$

where  $\mathcal{G}_1^0(F_0, \alpha)$  denotes the set of irreducible representations of  $\mathcal{W}_{F_0}$  whose restriction to  $\mathcal{P}_{F_0}$  overlaps with the orbit  $\alpha \in \mathcal{W}_{F_0} \backslash \widehat{\mathcal{P}}_{F_0}$  with multiplicity 1. By *ibid.*, §6.3, Tame Parameter Theorem and §1.5, Proposition, we have the following bijection:

$$\text{Ind}_{T_0/F_0} : \mathcal{G}_1^0(T_0, \alpha) \longrightarrow \mathcal{G}_1^0(F_0, \alpha).$$

Combining these together, the automorphic induction  $A_{T_0/F_0}$  leads to a bijection

$$A_{T_0/F_0} : \mathcal{A}_1^0(T_0, \Theta_{T_0}) \longrightarrow \mathcal{A}_1^0(F_0, \Theta_0),$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_1^0(T_0, \alpha) & \xrightarrow{\text{LLC}_{T_0}} & \mathcal{A}_1^0(T_0, \Theta_{T_0}) \\ \text{Ind}_{T_0/F_0} \downarrow & & \downarrow A_{T_0/F_0} \\ \mathcal{G}_1^0(F_0, \alpha) & \xrightarrow{\text{LLC}_{F_0}} & \mathcal{A}_1^0(F_0, \Theta_0) \end{array}$$

We fix  $\kappa_0$  as a full Heisenberg representation of  $\theta_0$  and we let  $\kappa_{T_0}$  be the full Heisenberg representation of  $\theta_{T_0}$  given by Proposition 3.7.8. Let  $\pi_0$  be a supercuspidal representation of  $G_{F_0}$  containing  $\theta_0$ , or in other words,  $\pi_0 \in \mathcal{A}_1^0(F_0, \Theta_0)$ . By Corollary 3.2.5, there exists a unique tamely ramified character  $\xi_0$  of  $T_0^\times$  such that

$$\text{ind}_{\mathbf{J}(a_0, \beta)}^{G_{F_0}}(\kappa_0 \cdot (\xi_0 \circ N_{E_0/T_0})) \cong \pi_0.$$

We choose  $\pi_{T_0}$  to be the supercuspidal representation of  $G_{T_0}$  containing  $\theta_{T_0}$  such that  $A_{T_0/F_0}(\pi_{T_0}) = \pi_0$ . Still by Corollary 3.2.5, there exists a unique tamely ramified character  $\xi_{T_0}$  of  $T_0^\times$  such that

$$\text{ind}_{\mathbf{J}(a_{T_0}, \beta)}^{G_{T_0}}(\kappa_{T_0} \cdot (\xi_{T_0} \circ N_{E_0/T_0})) \cong \pi_{T_0}.$$

**Proposition 3.7.10** ([BH14b], §8.2, Corollary). *There exists a unique tamely ramified character  $\mu_{\theta_0}^{T_0/F_0}$  of  $T_0^\times$  depending only on the endo-class of  $\theta_{T_0}$ , such that  $\xi_0 = \xi_{T_0} \cdot \mu_{\theta_0}^{T_0/F_0}$ .*

We further let  $L_0$  be a subfield of  $T_0$  over  $F_0$ , let  $A_{L_0}$  be the centralizer of  $L_0$  in  $A_{F_0}$  and let  $G_{L_0} = A_{L_0}^\times$ . We define  $\mathfrak{a}_{L_0}$ ,  $[\mathfrak{a}_{L_0}, \beta]$ ,  $H^1(\mathfrak{a}_{L_0}, \beta)$  and  $\theta_{L_0} = \theta_0|_{H^1(\mathfrak{a}_{L_0}, \beta)}$  in the similar way. By definition  $\theta_{T_0} = \theta_{L_0}|_{H^1(\mathfrak{a}_{T_0}, \beta)}$ , and by Proposition 3.7.8 with  $F_0$  replaced by  $L_0$ , there exists a full Heisenberg  $\kappa_{L_0}$  of  $\theta_{L_0}$ , such that

$$\mathrm{tr}(\kappa_{T_0})(x) = \epsilon_{T_0/L_0} \cdot \mathrm{tr}(\kappa_{L_0})(x)$$

for any  $x \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  such that  $v_{T_0}(\det_{T_0}(x))$  is relatively prime to  $[T_0 : L_0]$ . Thus we have the following corollary of Proposition 3.7.8:

**Corollary 3.7.11.** *For  $\kappa_0$  as a full Heisenberg representation of  $\mathbf{J}(\mathfrak{a}_0, \beta)$  containing  $\theta_0$ , there exists a unique Heisenberg representation  $\kappa_{L_0}$  of  $\mathbf{J}(\mathfrak{a}_{L_0}, \beta)$  containing  $\theta_{L_0}$ , such that*

$$\epsilon_{T_0/L_0} \cdot \mathrm{tr}(\kappa_{L_0})(x) = \epsilon_{T_0/F_0} \cdot \mathrm{tr}(\kappa_0)(x)$$

for all  $x \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  such that  $v_{T_0}(\det_{T_0}(x))$  is relative to  $[T_0 : F_0]$ , and the map  $\mathfrak{a}_{L_0/F_0}^{\mathrm{tr}} : \kappa_0 \mapsto \kappa_{L_0}$  given above is a bijection between full Heisenberg representations containing  $\theta_0$  and those containing  $\theta_{L_0}$ . Moreover for  $\varpi_{L_0}$  a uniformizer of  $L_0$  such that  $C_{T_0}(\varpi_{F_0}) = C_{T_0}(\varpi_{L_0})$ ,

$$C_{T_0}(\varpi_{F_0}) \subset \mathrm{Ker}(\det(\kappa_0)) \quad \text{and} \quad \varpi_{F_0} \in \mathrm{Ker}(\kappa_0)$$

if and only if

$$C_{T_0}(\varpi_{L_0}) \subset \mathrm{Ker}(\det(\kappa_{L_0})) \quad \text{and} \quad \varpi_{L_0} \in \mathrm{Ker}(\kappa_{L_0}).$$

With  $F_0$  replaced by  $L_0$ , we similarly consider the bijection  $A_{T_0/L_0} : \mathcal{A}_1^0(T_0, \Theta_{T_0}) \rightarrow \mathcal{A}_1^0(L_0, \Theta_{L_0})$ . Using Proposition 3.4.9 we have  $A_{T_0/F_0} = A_{L_0/F_0} \circ A_{T_0/L_0}$  as a bijection from  $\mathcal{A}_1^0(T_0, \Theta_{T_0})$  to  $\mathcal{A}_1^0(F_0, \Theta_0)$ . We denote by  $\pi_{L_0}$  the supercuspidal representation of  $G_{L_0}$  containing  $\theta_{L_0}$ , such that  $A_{L_0/F_0}(\pi_{L_0}) = \pi_0$ . Thus we also have  $A_{T_0/L_0}(\pi_{T_0}) = \pi_{L_0}$ . By Corollary 3.2.5, there exists a unique tamely ramified character  $\xi_{L_0}$  of  $T_0^\times$  such that

$$\mathrm{ind}_{\mathbf{J}(\mathfrak{a}_{L_0}, \beta)}^{G_{L_0}}(\kappa_{L_0} \cdot (\xi_{L_0} \circ N_{E_0/T_0})) \cong \pi_{L_0}.$$

Using Proposition 3.7.10 with  $F_0$  replaced by  $L_0$ , we may define a tamely ramified character  $\mu_{\theta_{L_0}}^{T_0/L_0}$  of  $T_0^\times$ . When  $L_0/F_0$  is cyclic of degree  $l$ , we fix a related transfer system  $(\sigma_0, \varkappa_{L_0/F_0}, e_l)$ , and we consider  $\tilde{\Delta}_{L_0/F_0}$ ,  $\Delta_{L_0/F_0}^1$ ,  $\Delta_{L_0/F_0}^2$  and  $\delta_{L_0/F_0}$  as in 3.4.2.

**Proposition 3.7.12.** *For  $L_0/F_0$  cyclic and of prime degree  $l$ , there exists a unique tamely ramified character  $\chi_{\theta_0}^{L_0/F_0}$  of  $T_0^\times$  such that*

$$\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0)) = \epsilon_{T_0/L_0} \cdot \epsilon_{T_0/F_0} \cdot c_{\theta_0}^{L_0/F_0} \cdot \delta_{L_0/F_0}(h_0) \tag{3.7.3}$$

for all  $h_0 \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  with  $v_{T_0}(\det_{T_0}(h_0))$  relatively prime to  $n$ , where  $c_{\theta_0}^{L_0/F_0}$  is the constant occurring Theorem 3.4.5. Moreover we have

$$\xi_0 = \xi_{L_0} \cdot \chi_{\theta_0}^{L_0/F_0} \quad \text{and} \quad \mu_{\theta_0}^{T_0/F_0} = \mu_{\theta_{L_0}}^{T_0/L_0} \cdot \chi_{\theta_0}^{L_0/F_0}.$$

*Proof.* This proposition is morally [BH14b], §8.9, Corollary, except that in *loc. cit.* they assume  $l$  to be the largest prime divisor of  $|\Gamma_0|$ , where  $\Gamma_0 = \mathrm{Aut}(T_0/F_0)$ . We explain how this assumption can be weakened to our settings.

As indicated in *ibid.*, §8.6, for any  $\gamma \in \Gamma_0$  we may choose  $g_\gamma$  contained in the normalizer of  $T_0^\times$  in  $G_{F_0}$ , such that  $g_\gamma^{-1}xg_\gamma = x^\gamma$  for every  $x \in T_0$  and  $\det(g_\gamma)$  equals 1 or  $-1$ . Thus

$$\varkappa : \gamma \mapsto \varkappa_{L_0/F_0}(\det(g_\gamma))$$

is a character of  $\Gamma_0$  of order dividing 2. By definition,  $\varkappa$  is trivial except when

$$l = 2, \quad q \equiv 3 \pmod{4} \quad \text{and} \quad [T_0 : F_0] \equiv |\Gamma_0| \equiv 2 \pmod{4}. \quad (3.7.4)$$

Using the same argument of *ibid.* (8.7.3), we may prove that

$$\epsilon_{T_0/F_0} \sum_{\gamma \in \Gamma_0} \chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0^\gamma)) \cdot \text{tr}(\pi_{T_0})(h_0^\gamma) \cdot \varkappa(\gamma) = \epsilon_{T_0/L_0} \cdot c_{\theta_0}^{L_0/F_0} \cdot \delta_{L_0/F_0}(h_0) \sum_{\gamma \in \Gamma_0} \text{tr}(\pi_{T_0})(h_0^\gamma)$$

for any elliptic  $h_0 \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  such that  $v_{T_0}(\det_{T_0}(h_0))$  is relatively prime to  $n$ . When the condition (3.7.4) fails, as in *ibid.* §8.7. we deduce that  $\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0)) = \chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0^\gamma))$ , and  $\chi_{\theta_0}^{L_0/F_0}|_{\mu_{T_0}}$  and  $\varkappa$  are trivial. When the condition (3.7.4) holds,  $\varkappa_{T_0/F_0}(-1) = -1$  and  $\varkappa$  is a character of order 2. By *ibid.* §8.8, Lemma 1 and Lemma 2, we deduce that  $\chi_{\theta_0}^{L_0/F_0}|_{\mu_{T_0}}$  is of order 2 and  $\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0)) = \chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0^\gamma)) \cdot \varkappa(\gamma)$ . Thus in both cases we have

$$\epsilon_{T_0/F_0} \cdot \chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0)) \sum_{\gamma \in \Gamma_0} \text{tr}(\pi_{T_0})(h_0^\gamma) = \epsilon_{T_0/L_0} \cdot c_{\theta_0}^{L_0/F_0} \cdot \delta_{L_0/F_0}(h_0) \sum_{\gamma \in \Gamma_0} \text{tr}(\pi_{T_0})(h_0^\gamma).$$

Thus for those  $h_0$  satisfying  $\sum_{\gamma \in \Gamma_0} \text{tr}(\pi_{T_0})(h_0^\gamma) \neq 0$ , we have

$$\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0)) = \epsilon_{T_0/F_0} \cdot \epsilon_{T_0/L_0} \cdot c_{\theta_0}^{L_0/F_0} \cdot \delta_{L_0/F_0}(h_0). \quad (3.7.5)$$

Finally for any  $h'_0 \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  with  $v_{T_0}(\det_{T_0}(h_0))$  relatively prime to  $n$ , by *ibid.* §8.1. Corollary we may choose  $h_0 \in \mathbf{J}(\mathfrak{a}_{T_0}, \beta)$  elliptic, such that  $\sum_{\gamma \in \Gamma_0} \text{tr}(\pi_{T_0})(h_0^\gamma) \neq 0$  and  $v_{T_0}(\det_{T_0}(h_0)) = v_{T_0}(\det_{T_0}(h'_0))$ . Using *ibid.*, §8.5 Transfer Lemma, the last paragraph of §8.7 and §8.8 Lemma 1, we have

$$\delta_{L_0/F_0}(h_0)/\delta_{L_0/F_0}(h'_0) = \chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0))/\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h'_0)).$$

Combining with (3.7.5) we have

$$\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h'_0)) = \epsilon_{T_0/F_0} \cdot \epsilon_{T_0/L_0} \cdot c_{\theta_0}^{L_0/F_0} \cdot \delta_{L_0/F_0}(h'_0),$$

which finishes the proof.  $\square$

**Corollary 3.7.13.** *When  $l$  is odd, for  $p \neq 2$  the character  $\chi_{\theta_0}^{L_0/F_0}$  defined above is trivial, and for  $p = 2$ , the character  $\chi_{\theta_0}^{L_0/F_0}$  is unramified such that  $\chi_{\theta_0}^{L_0/F_0}(\varpi_{T_0}^{[E_0:T_0]}) = 1$ .*

*Proof.* First we have the following lemma:

**Lemma 3.7.14.** *For  $l$  odd,  $\varpi_{E_0}$  a uniformizer of  $E_0$  such that  $\varpi_{E_0}^n/\varpi_{F_0} \in 1 + \mathfrak{p}_{F_0}$  and  $n$  an integer relatively prime to  $p$ , the value  $\delta_{L_0/F_0}(\varpi_{E_0}^a)$  is independent of  $a$ .*

*Proof.* Let  $E'_0/F_0$  be a Galois extension containing  $E_0$  and all the  $n$ -th roots of unity. Using the calculation in [BH05a], Lemma 4.5, we may prove that

$$\tilde{\Delta}_{L_0/F_0}(\varpi_{E_0}^a) = \varpi_{F_0}^{an(l-1)/2l} \cdot \epsilon \cdot \Pi \cdot u_0,$$

where  $\Pi$  is the product of certain differences of two different  $n$ -th roots of unity as an element in  $E'_0$  independent of  $a$ , whose exact expression is not important here,  $\epsilon$  is a sign and  $u_0 \in 1 + \mathfrak{p}_{E'_0}$ . Thus  $e_l \cdot \Pi \cdot u_0$  is an element in  $F^\times$ . By definition  $\Delta_{L_0/F_0}^1(\varpi_{E_0}^a) = 1$  and

$$\Delta_{L_0/F_0}^2(\varpi_{E_0}^a) = \varkappa_{L_0/F_0}(\varpi_{F_0}^{an(l-1)/2l} \cdot \epsilon) \cdot \varkappa_{L_0/F_0}(e_l \cdot \Pi \cdot u_0).$$

Since  $\varpi_{F_0} \in N_{T_0/F_0}(T_0^\times)$  and  $l$  is odd,  $\varkappa_{T_0/F_0}(\varpi_{F_0}^{an(l-1)/2l} \cdot \epsilon) = \varkappa_{T_0/F_0}(\epsilon) = 1$ . Moreover since  $\varkappa_{L_0/F_0}$  is tamely ramified,  $\varkappa_{L_0/F_0}(e_l \cdot \Pi \cdot u_0)$  is independent of  $u_0$ , and thus independent of  $a$ . Thus

$$\delta_{L_0/F_0}(\varpi_{E_0}^a) = \varkappa_{L_0/F_0}(e_l \cdot \Pi \cdot u_0)$$

is independent of  $a$ . □

As a by-product of the proof of Proposition 3.7.12, the character  $\chi_{\theta_0}^{L_0/F_0}$  is unramified when  $l$  is odd. If  $p \neq 2$ , we choose  $h_1 = \varpi_{E_0}$  and  $h_2 = \varpi_{E_0}^2$ , then using the above lemma and Proposition 3.7.12, we obtain  $\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_2 h_1^{-1})) = 1$ , meaning that  $\chi_{\theta_0}^{L_0/F_0}$  is trivial on a uniformizer of  $T_0$ , thus it is trivial. For  $p = 2$  and  $[E_0 : T_0] = 2^s$ , the case  $s = 0$  is trivial. And for  $s \geq 1$ , we choose  $h_{-1} = \varpi_{E_0}^{-1}$  and  $h_1 = \varpi_{E_0}$ , thus using the same argument we get  $\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_1 h_{-1}^{-1})) = 1$ , which implies that  $\chi_{\theta_0}^{L_0/F_0}(\varpi_{T_0}^2) = 1$ . □

When  $l = 2$  correspondingly we have:

**Corollary 3.7.15.** *For  $l = 2$ , the character  $\chi_{\theta_0}^{L_0/F_0}|_{\mu_{T_0}}$  is of order 2.*

*Proof.* It follows from [BH14b], §8.8, Lemma 1. □

**Corollary 3.7.16.** *For the character  $\mu_{\theta_0}^{T_0/F_0}$  of  $T_0^\times$  in Proposition 3.7.10, its restriction to  $\mu_{T_0}$  is quadratic.*

*Proof.* We may choose a sequence of field extensions  $F_0 \subset F_1 \subset \dots \subset F_k \subset T_0$ , such that  $F_i/F_{i-1}$  is cyclic and of prime order, and  $\text{Aut}(T_0/F_k) = \{1\}$ . Thus by Proposition 3.7.12

$$\mu_{\theta_0}^{T_0/F_0} = \mu_{\theta_{F_k}}^{T_0/F_k} \cdot \prod_{i=1}^k \chi_{\theta_{F_{i-1}}}^{F_i/F_{i-1}},$$

where  $\theta_{F_0} := \theta_0$  and  $\theta_{F_i}$  denotes the interior  $F_i/F_0$ -lift of  $\theta_0$ . By [BH14b], §8.3, Remark,  $\mu_{\theta_{F_k}}^{T_0/F_k}$  is unramified, and combining with Corollary 3.7.13 and Corollary 3.7.15 we finish the proof. □

### 3.7.2 Construction in the interior automorphic induction case

In this subsection, we follow the same setting as §3.6.2. We give an exact construction of the full Heisenberg representation  $\kappa_a$ .

#### 3.7.2.1

When  $F/F_0$  is unramified,  $F$  is also a subfield of  $K_0$  and  $T_0$ . We choose  $\varpi_F = \varpi_{F_0}$  to be a uniformizer of  $F$ . We denote by  $A_{K_0}$  the centralizer of  $K_0$  in  $A_{F_0}$  and we write  $\mathfrak{a}_{K_0} = \mathfrak{a}_0 \cap A_{K_0}$  as a hereditary order in  $A_{K_0}$ , thus  $[\mathfrak{a}_{K_0}, \beta]$  is a simple stratum in  $A_{K_0}$ . Moreover  $H^1(\mathfrak{a}_{K_0}, \beta) = H^1(\mathfrak{a}_0, \beta) \cap G_{K_0}$  and  $\theta_{K_0} = \theta_0|_{H^1(\mathfrak{a}_{K_0}, \beta)}$  is the interior  $K_0/F_0$ -lift of  $\theta_0$ . Using Proposition 3.7.5, there exists a unique  $\Delta_0$ -invariant full Heisenberg representation  $\kappa_{K_0}$  of  $\theta_{K_0}$ , such that  $\mathbf{l}_{K_0/F_0}^{ur}(\kappa_0) = \kappa_{K_0}$ . Moreover  $A_{K_0}$  is also the centralizer of  $K_0$  in  $C_F$ . By definition,  $\mathfrak{a}_{K_0} = \mathfrak{c} \cap A_{K_0}$ ,  $H^1(\mathfrak{a}_{K_0}, \beta) = H^1(\mathfrak{c}, \beta) \cap G_{K_0}$  and  $\theta_{K_0}$  is the interior  $K_0/F$ -lift of  $\theta_a$ . Thus replacing  $F_0$  by  $F$  in Proposition 3.7.5, there exists a unique

$\kappa_a$  as a full Heisenberg extension of  $\theta_a$ , such that  $\mathbf{l}_{K_0/F}^{ur}(\kappa_a) = \kappa_{K_0}$ . This gives the construction of  $\kappa_a$  in the unramified case. For ease of reference, we write  $\mathbf{a}_{F/F_0}^{ur} : \kappa_0 \mapsto \kappa_a$  for the above correspondence which is a bijection between those  $\kappa_0$  satisfying (3.6.1) and those  $\kappa_a$  satisfying (3.6.3).

$$\begin{array}{ccc}
 & \kappa_{K_0} & K_0 \\
 & \downarrow (\mathbf{l}_{K_0/F}^{ur})^{-1} & \downarrow mf_0/r \\
 & \kappa_a & F \\
 & \uparrow \mathbf{a}_{F/F_0}^{ur} & \uparrow r \\
 & \kappa_0 & F_0
 \end{array}$$

### 3.7.2.2

When  $F/F_0$  is totally ramified, we choose  $\varpi_F$  to be any uniformizer of  $F$ , such that  $\varpi_F^r/\varpi_{F_0}$  is in  $\mu_F$ , and in particular  $C_{T_{0,m}}(\varpi_{F_0}) = C_{T_{0,m}}(\varpi_F)$ . We denote by  $K$  the maximal unramified subfield of  $T_{0,m}$  over  $F$ , thus  $K/K_0$  is a totally tamely ramified extension of degree  $r$ . As in the previous case, we define  $\mathfrak{a}_{K_0}$ ,  $[\mathfrak{a}_{K_0}, \beta]$ ,  $H^1(\mathfrak{a}_{K_0}, \beta)$  and  $\theta_{K_0}$ . Similarly we write  $C_K$  for the centralizer of  $K$  in  $C_F$  and we write  $\mathfrak{c}_K = \mathfrak{c} \cap C_K$ , thus  $[\mathfrak{c}_K, \beta]$  is a simple stratum in  $C_K$ . Moreover  $H^1(\mathfrak{c}_K, \beta) = H^1(\mathfrak{c}, \beta) \cap G_K$  and  $\theta_K = \theta_a|_{H^1(\mathfrak{c}_K, \beta)}$  is the interior  $K/F$ -lift of  $\theta_a$ . By definition we also have  $\theta_K = \theta_{K_0}|_{H^1(\mathfrak{c}_K, \beta)}$ , which means that  $\theta_K$  is the interior  $K/K_0$ -lift of  $\theta_{K_0}$ . By Proposition 3.7.5 as above, there exists a unique  $\Delta_0$ -invariant full Heisenberg representation  $\kappa_{K_0}$  of  $\theta_{K_0}$  such that  $\mathbf{l}_{K_0/F_0}^{ur}(\kappa_0) = \kappa_{K_0}$ , which in particular satisfies (3.7.2). Since  $E_{0,m} = K_0[\beta]$  is a totally ramified extension of degree  $n/[K_0 : F_0]$  over  $K_0$ , by Corollary 3.7.11 with  $L_0 = K$  and  $F_0 = K_0$ , there exists a full Heisenberg extension  $\kappa_K$  of  $\mathbf{J}(\mathfrak{c}_K, \beta)$  containing  $\theta_K$  such that  $\mathbf{a}_{K/K_0}^{tr}(\kappa_{K_0}) = \kappa_K$ , and in particular we have

$$C_{T_{0,m}}(\varpi_F) \subset \text{Ker}(\det(\kappa_K)) \quad \text{and} \quad \varpi_F \in \text{Ker}(\kappa_K). \quad (3.7.6)$$

Using Proposition 3.7.5 for  $F_0 = F$  and  $K_0 = K$ , there exists a unique full Heisenberg representation  $\kappa_a$  of  $\theta_a$  satisfying equation (3.6.3), such that  $\mathbf{l}_{K/F}^{ur}(\kappa_a) = \kappa_K$ . This gives the construction of  $\kappa_a$  in the totally ramified case. For ease of reference, we write  $\mathbf{a}_{F/F_0}^{tr} : \kappa_0 \mapsto \kappa_a$  for the above correspondence, which is actually in accordance with the notation in Corollary 3.7.11. Such  $\mathbf{a}_{F/F_0}^{tr}$  is a bijection between those  $\kappa_0$  satisfying (3.6.1) and those  $\kappa_a$  satisfying (3.6.3).

$$\begin{array}{ccc}
 & \mathbf{a}_{K/K_0}^{tr} & \\
 \kappa_K & \longleftarrow & \kappa_{K_0} \\
 & \uparrow \mathbf{l}_{K_0/F_0}^{ur} & \\
 (\mathbf{l}_{K/F}^{ur})^{-1} \downarrow & & \uparrow \\
 \kappa_a & \longleftarrow & \kappa_0 \\
 & \mathbf{a}_{F/F_0}^{tr} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 K & \xrightarrow{r} & K_0 \\
 mf \downarrow & & \downarrow mf \\
 F & \xrightarrow{r} & F_0
 \end{array}$$

### 3.7.2.3

In general, we let  $F_1/F_0$  be the maximal unramified subextension of  $F/F_0$ . We choose  $\varpi_{F_1} = \varpi_{F_0}$  as a uniformizer of  $F_1$  and  $F_0$ , and  $\varpi_F$  as a uniformizer of  $F$  such that  $\varpi_F^{[F:F_1]}/\varpi_{F_0}$  is in  $\mu_F$ . Thus combining the above two cases together, we obtain  $\mathbf{a}_{F/F_0} = \mathbf{a}_{F/F_1}^{tr} \circ \mathbf{a}_{F_1/F_0}^{ur}$  as a bijection between those  $\kappa_0$  of  $\theta_0$  satisfying (3.6.1) and those  $\kappa_a$  of  $\theta_a$  satisfying (3.6.3).

## 3.7.3 Construction in the supercuspidal base change case

In this subsection, we follow the same setting as §3.6.1. Our aim is to construct the full Heisenberg extension  $\kappa_b$  of  $\mathbf{J}(\mathfrak{a}, \beta)$  of  $\theta_b$ .

**Lemma 3.7.17.** *We may choose  $\varpi_F$  as a uniformizer of  $F$ , and  $\varpi_{T_0}$  and  $\varpi_T$  as uniformizers of  $T_0$  and  $T$  respectively satisfying  $\varpi_{F_0} \in \mu_{T_0}\langle\varpi_{T_0}\rangle$ ,  $\varpi_F \in \mu_T\langle\varpi_T\rangle$  and  $N_{T/T_0}(\varpi_T) \in \mu_{T_0}\langle\varpi_{T_0}\rangle$ .*

*Proof.* We consider a sequence of field extensions  $F_0 \subset F_1 \subset F_2 \subset F$ , such that for  $E_1 = F_1E_0$  and  $E_2 = F_2E_0$ , both  $E_1/E_0$  and  $F_1/F_0$  are unramified,  $E_2/E_1$  is unramified and  $F_2/F_1$  is totally ramified, and both  $E/E_2$  and  $F/F_2$  are totally ramified. We write  $T_1 = F_1T_0$  and  $T_2 = F_2T_0$  and we have  $e_0 = e(T_0/F_0) = e(T_1/F_1)$ , and  $e = e(T_2/F_2) = e(T/F) = e_0/[F_2 : F_1]$ .

Using Hensel lemma, we choose  $\varpi_{T_0}$  to be a uniformizer of  $T_0$  such that  $\varpi_{F_0} = \varpi_{T_0}^{e_0}\zeta_{T_0}$  for a certain  $\zeta_{T_0} \in \mu_{T_0}$ , and  $\varpi_{F_2}$  to be a uniformizer of  $F_2$  such that  $\varpi_{F_0} = \varpi_{F_2}^{[F_2:F_1]}\zeta_{F_2}$  for a certain  $\zeta_{F_2} \in \mu_{F_0}$ , and finally  $\varpi_{T_2}$  to be a uniformizer of  $T_2$  such that  $\varpi_{F_2} = \varpi_{T_2}^e\zeta_{T_2}$  for a certain  $\zeta_{T_2} \in \mu_{T_2}$ . By definition,  $(\varpi_{T_2}/\varpi_{T_0})^{e_0} = \zeta_{T_0}/\zeta_{F_2}\zeta_{T_2}^{[F_2:F_1]}$  which is a root of unity in  $\mu_{T_2}$ , thus  $\varpi_{T_2}/\varpi_{T_0}$  must be a root of unity in  $\mu_{T_2}$  which we denote by  $\zeta'_{T_2}$ . Still using Hensel lemma, we choose  $\varpi_F$  to be a uniformizer of  $F$  such that  $\varpi_{F_2} = \varpi_F^{[F:F_2]}\zeta_F$  for a certain  $\zeta_F \in \mu_F$ , and  $\varpi_T$  to be a uniformizer of  $T$  such that  $\varpi_F = \varpi_T^e\zeta_T$  for a certain  $\zeta_T \in \mu_T$ . Thus by definition,  $(\varpi_T^{[F:F_2]}/\varpi_{T_2})^e = \zeta_{T_2}/\zeta_F\zeta_T^{[F:F_2]}$  which is a root of unity in  $\mu_T$ . Thus  $\varpi_T^{[F:F_2]}/\varpi_{T_2}$  must be a root of unity in  $\mu_T = \mu_{T_2}$  which we denote by  $\zeta'_T$ . By definition we get  $\varpi_{F_0} \in \mu_{T_0}\langle\varpi_{T_0}\rangle$ ,  $\varpi_F \in \mu_T\langle\varpi_T\rangle$ , and moreover

$$N_{T/T_0}(\varpi_T) = N_{T_2/T_0}(N_{T/T_2}(\varpi_T)) = N_{T_2/T_0}(\varpi_{T_2}\zeta'_T) = N_{T_2/T_0}(\zeta'_T\zeta'_{T_2})\varpi_{T_0}^{[F_2:F_0]} \in \mu_{T_0}\langle\varpi_{T_0}\rangle$$

□

We choose  $\varpi_F$ ,  $\varpi_{T_0}$  and  $\varpi_T$  as in the above lemma. For  $K_0$  the maximal unramified subextension of  $E_{0,m}$  over  $F_0$ , let  $A_{K_0}$ ,  $\mathfrak{a}_{K_0}$ ,  $[\mathfrak{a}_{K_0}, \beta]$ ,  $H^1(\mathfrak{a}_{K_0}, \beta)$  and  $\theta_{K_0}$  be as in the last subsection. Using Proposition 3.7.5 there exists a full Heisenberg representation  $\kappa_{K_0} = \mathbf{l}_{K_0/F_0}^{ur}(\kappa_0)$  of  $\theta_{K_0}$  satisfying (3.7.2). For the simple stratum  $[\mathfrak{a}_{K_0}, \beta]$  and simple character  $\theta_{K_0}$ , we are in the totally ramified maximal case, that is,  $E_{0,m} = K_0[\beta]$  is totally ramified over  $K_0$  of degree  $n/[K_0 : F_0]$ . Let  $A_{T_{0,m}}$  be the centralizer of  $T_{0,m}$  in  $A_{F_0}$  and let  $\mathfrak{a}_{T_{0,m}} = \mathfrak{a}_{K_0} \cap A_{T_{0,m}}$ . Thus  $[\mathfrak{a}_{T_{0,m}}, \beta]$  is a simple stratum in  $A_{T_{0,m}}$  and  $H^1(\mathfrak{a}_{T_{0,m}}, \beta) = H^1(\mathfrak{a}_{K_0}, \beta) \cap G_{T_{0,m}}$ . Moreover since  $T_{0,m}/K_0$  is tamely ramified,  $\theta_{T_{0,m}} = \theta_{K_0}|_{H^1(\mathfrak{a}_{T_{0,m}}, \beta)}$  is the interior  $T_{0,m}/K_0$ -lift of  $\theta_{K_0}$ . Using Proposition 3.7.8 for  $T_0 = T_{0,m}$  and  $F_0 = K_0$ , there exists a unique full Heisenberg representation  $\kappa_{T_{0,m}} = \mathbf{l}_{T_{0,m}/K_0}^{tr}(\kappa_{K_0})$  of  $\theta_{T_{0,m}}$  satisfying

$$\mu_{T_{0,m}}\langle\varpi_{T_0}\rangle = C_{T_{0,m}}(\varpi_{F_0}) \subset \text{Ker}(\kappa_{T_{0,m}}) \quad (3.7.7)$$

The composition of the two maps given above

$$\kappa_0 \xrightarrow{\mathbf{l}_{K_0/F_0}^{ur}} \kappa_{K_0} \xrightarrow{\mathbf{l}_{T_{0,m}/K_0}^{tr}} \kappa_{T_{0,m}}$$

is a bijection between those  $\kappa_0$  satisfying (3.6.1) and those  $\kappa_{T_{0,m}}$  satisfying (3.7.7).

The above argument also works for  $[\mathfrak{a}, \beta]$  and  $\theta_b$ . Let  $A_K$  be the centralizer of  $K$  in  $A_F$ . We similarly define  $\mathfrak{a}_K$ ,  $[\mathfrak{a}_K, \beta]$ ,  $H^1(\mathfrak{a}_K, \beta)$  and  $\theta_K = \theta_b|_{H^1(\mathfrak{a}_K, \beta)}$ . For any full Heisenberg representation  $\kappa_b$  of  $\mathbf{J}(\mathfrak{a}, \beta)$  of  $\theta_b$  satisfying (3.6.2), by Proposition 3.7.5 with  $F_0 = F$  and  $K_0 = K$ , there exists a unique full Heisenberg representation  $\kappa_K = \mathbf{l}_{K/F}^{ur}(\kappa_b)$  of  $\theta_K$ , which satisfies (3.7.6). For  $[\mathfrak{a}_K, \beta]$  and  $\theta_K$ , we are also in the totally ramified maximal case. We similarly define  $A_{T_m}$ ,  $\mathfrak{a}_{T_m}$ ,  $[\mathfrak{a}_{T_m}, \beta]$ ,  $H^1(\mathfrak{a}_{T_m}, \beta)$  and  $\theta_{T_m} = \theta_K|_{H^1(\mathfrak{a}_{T_m}, \beta)}$ . Thus by Proposition 3.7.8 with  $F_0 = K$  and  $T_0 = T_m$ , there exists a unique full Heisenberg representation  $\kappa_{T_m} = \mathbf{l}_{T_m/K}^{tr}(\kappa_K)$  of  $\theta_{T_m}$  satisfying

$$\mu_{T_m}\langle\varpi_T\rangle = C_{T_m}(\varpi_F) \subset \text{Ker}(\kappa_{T_m}). \quad (3.7.8)$$

The composition of the two maps given above

$$\kappa_b \xrightarrow{l_{K/F}^{ur}} \kappa_K \xrightarrow{l_{T_m/K}^{tr}} \kappa_{T_m} \quad (3.7.9)$$

is a bijection between those  $\kappa_b$  satisfying (3.6.1) and those  $\kappa_{T_m}$  satisfying (3.7.8). To connect the two sides, we need the following lemma:

**Lemma 3.7.18.** *We have  $\theta_{T_m} = (\theta_{T_0,m})_b$ , where  $(\theta_{T_0,m})_b$  is the simple character in  $\mathcal{C}(\mathfrak{a}_{T_m}, \beta)$  constructed by the procedure in §3.6.1. In particular, the endo-class of  $\theta_{T_m}$  is the  $T_m/T_0,m$ -lift of that of  $\theta_{T_0,m}$ .*

*Proof.* We write  $\tilde{A}_{F_0} = \text{End}_{F_0}(E_m)$  and  $\tilde{\mathfrak{a}}_0 = \mathfrak{a}_{F_0}(E_m)$  as a hereditary order in  $\tilde{A}_{F_0}$ . From the construction, we have

$$\theta_{T_m} = (\theta_b)_{T_m} = ((t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0))_F)_{T_m}.$$

Using Lemma 3.2.1 for the tamely ramified field extensions  $T_m/F/F_0$  and  $T_m/T_0,m/F_0$ , we have

$$\theta_{T_m} = ((t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0))_F)_{T_m} = (t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0))_{T_m} = ((t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0))_{T_0,m})_{T_m}.$$

On the other hand, we write  $\tilde{A}_{T_0,m} = \text{End}_{T_0,m}(E_m)$  and  $\tilde{\mathfrak{a}}_{T_0,m} = \mathfrak{a}_{T_0,m}(E_m)$  as a hereditary order in  $\tilde{A}_{T_0,m}$ . From the construction, we have

$$(\theta_{T_0,m})_b = (t_{\mathfrak{a}_{T_0,m}, \tilde{\mathfrak{a}}_{T_0,m}}^\beta(\theta_{T_0,m}))_{T_m}.$$

Using Lemma 3.2.2 for  $F = F_0$ ,  $L = T_0,m$ ,  $\mathfrak{a} = \mathfrak{a}_0$  and  $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_0$ , combining with the fact that  $(\mathfrak{a}_0)_{T_0,m} = \mathfrak{a}_{T_0,m}$  and  $(\tilde{\mathfrak{a}}_0)_{T_0,m} = \tilde{\mathfrak{a}}_{T_0,m}$ , we get

$$t_{\mathfrak{a}_{T_0,m}, \tilde{\mathfrak{a}}_{T_0,m}}^\beta(\theta_{T_0,m}) = (t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0))_{T_0,m}.$$

Combining these equations together, we finally get  $(\theta_{T_0,m})_b = \theta_{T_m}$ . □

**Remark 3.7.19.** *To sum up, we actually proved that each block in the following diagram is commutative:*

$$\begin{array}{ccccc} \mathcal{C}(\mathfrak{a}_0, \beta) & \xrightarrow{t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}_0, \beta) & \xrightarrow{|_{H^1(\mathfrak{a}, \beta)}} & \mathcal{C}(\mathfrak{a}, \beta) \\ |_{H^1(\mathfrak{a}_{T_0,m}, \beta)} \downarrow & & |_{H^1(\tilde{\mathfrak{a}}_{T_0,m}, \beta)} \downarrow & & |_{H^1(\mathfrak{a}_{T_m}, \beta)} \downarrow \\ \mathcal{C}(\mathfrak{a}_{T_0,m}, \beta) & \xrightarrow{t_{\mathfrak{a}_{T_0,m}, \tilde{\mathfrak{a}}_{T_0,m}}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}_{T_0,m}, \beta) & \xrightarrow{|_{H^1(\mathfrak{a}_{T_m}, \beta)}} & \mathcal{C}(\mathfrak{a}_{T_m}, \beta) \end{array}$$

Combining this lemma with Theorem 3.7.1 for the extension  $T_m/T_0,m$ , we choose  $\kappa_{T_m}$  to be the unique full Heisenberg extension extending  $\theta_{T_m}$ , such that

$$\text{BC}_{T_m/T_0,m}(\text{ind}_{\mathbf{J}(\mathfrak{a}_{T_0,m}, \beta)}^{G_{T_0,m}}(\kappa_{T_0,m})) = \text{ind}_{\mathbf{J}(\mathfrak{a}_{T_m}, \beta)}^{G_{T_m}}(\kappa_{T_m}).$$

By Corollary 3.7.2, such  $\kappa_{T_m}$  satisfies (3.7.8). Thus by (3.7.9) we get the corresponding full Heisenberg representation  $\kappa_b$  of  $\theta_b$  which satisfies (3.6.2). The map  $\kappa_0 \mapsto \kappa_b$  given above is a bijection between



those  $\kappa_0$  satisfying (3.6.1) and those  $\kappa_b$  satisfying (3.6.2), which we denote by  $\mathbf{b}_{F/F_0}$  for ease of reference.

$$\begin{array}{ccc}
 & \kappa_{T_m} \xleftarrow{\mathbf{b}_{T_m/T_{0,m}}} \kappa_{T_{0,m}} & T_m \xrightarrow{r} T_{0,m} \\
 (\mathbf{l}_{T_m/K}^{tr})^{-1} \downarrow & & \uparrow \mathbf{l}_{T_{0,m}/K_0}^{tr} \quad e \downarrow \quad \downarrow e_0 \\
 \kappa_K & & K \quad K_0 \\
 (\mathbf{l}_{K/F}^{ur})^{-1} \downarrow & & \uparrow \mathbf{l}_{K_0/F_0}^{ur} \quad mf \downarrow \quad \downarrow mf_0 \\
 \kappa_b \xleftarrow{\mathbf{b}_{F/F_0}} \kappa_0 & & F \xrightarrow{r} F_0
 \end{array}$$

### 3.7.4 Construction in the exterior automorphic induction case

In this subsection, we follow the same setting as §3.6.3. We will construct the full Heisenberg extension  $\kappa_a$  in the exterior automorphic induction case.

#### 3.7.4.1

First we consider the case where  $E/E_0$  is unramified and we follow the setting of 3.6.3.1. We write  $K_0$  for the maximal unramified subfield of  $T_{0,m}$  over  $F_0$  and we denote by  $A_{K_0}$  the centralizer of  $K_0$  in  $A_{F_0}$ . As before we define  $\mathfrak{a}_{K_0}$ ,  $[\mathfrak{a}_{K_0}, \beta]$ ,  $H^1(\mathfrak{a}_{K_0}, \beta)$  and  $\theta_{K_0} = \theta_0|_{H^1(\mathfrak{a}_{K_0}, \beta)}$ . Similarly we write  $K$  for the maximal unramified subfield of  $T_{m/r}$  over  $F$ . Thus  $K_0$  is a subfield of  $K$  by noting that  $T_{0,m} = T_{m/r}$ , and in particular  $K_0 = K$  if and only if  $F/F_0$  is unramified. Regarding  $K/F_0$  as a tamely ramified subextension of  $T_{0,m}/F_0$ , we may denote by  $A_K$  the centralizer of  $K$  in  $A_F$  and we may similarly define  $\mathfrak{a}_K$ ,  $[\mathfrak{a}_K, \beta]$ ,  $H^1(\mathfrak{a}_K, \beta)$  and  $\theta_K = \theta_0|_{H^1(\mathfrak{a}_K, \beta)}$ . Thus by definition  $\theta_{K_0}|_{H^1(\mathfrak{a}_K, \beta)} = \theta_K$ , meaning that  $\theta_K$  is the interior  $K/K_0$ -lift of  $\theta_{K_0}$ .

On the other hand, we denote by  $C_K$  the centralizer of  $K$  in  $C_F$ . Thus  $\mathfrak{c}_K = \mathfrak{c}_F \cap C_K$  is a hereditary order, and  $[\mathfrak{c}_K, \beta]$  is a simple stratum in  $C_K$ . By direct definition, we actually have  $A_K = C_K = \text{End}_K(E_{0,m})$ ,  $\mathfrak{a}_K = \mathfrak{c}_K = \mathfrak{a}_K(E_{0,m})$ . Using Lemma 3.2.1, we have  $\theta_K = \theta_a|_{H^1(\mathfrak{c}_K, \beta)}$  as the  $K/F$ -interior lift of  $\theta_a$ .

We choose  $\varpi_F$  to be a uniformizer of  $F$ , such that  $\varpi_F^{e(F/F_0)}/\varpi_{F_0}$  is in  $\mu_F$ . In particular we have  $C_{T_{0,m}}(\varpi_{F_0}) = C_{T_{m/r}}(\varpi_F)$ . For  $\kappa_0$  a full Heisenberg representation of  $\theta_0$  satisfying (3.6.1), by Proposition 3.7.5, there exists a full Heisenberg representation  $\kappa_{K_0} = \mathbf{l}_{K_0/F_0}^{ur}(\kappa_0)$  of  $\theta_{K_0}$  satisfying (3.7.2). Then using Corollary 3.7.11 with  $F_0 = K_0$  and  $L_0 = K$ , we get a full Heisenberg representation  $\kappa_K = \mathbf{a}_{K/K_0}^{tr}(\kappa_{K_0})$  of  $\theta_K$  satisfying

$$C_{T_{m/r}}(\varpi_F) \subset \text{Ker}(\det(\kappa_K)) \quad \text{and} \quad \varpi_F \in \text{Ker}(\kappa_K). \quad (3.7.10)$$

Finally using Proposition 3.7.5 again, there exists a full Heisenberg representation  $\kappa_a = (\mathbf{l}_{K/F}^{ur})^{-1}(\kappa_K)$  satisfying (3.6.4). Thus we obtain a bijection from those  $\kappa_0$  satisfying (3.6.1) to those  $\kappa_a$  satisfying (3.6.4), which is denoted by  ${}^e\mathbf{a}_{F/F_0}^{ur}$ .

$$\begin{array}{ccc}
 & \kappa_K \xleftarrow{\mathbf{a}_{K/K_0}^{tr}} \kappa_{K_0} & K \xrightarrow{f/f_0} K_0 \\
 (\mathbf{l}_{K/F}^{ur})^{-1} \downarrow & & \uparrow \mathbf{l}_{K_0/F_0}^{ur} \quad mf/r \downarrow \quad \downarrow mf_0 \\
 \kappa_a \xleftarrow{{}^e\mathbf{a}_{F/F_0}^{ur}} \kappa_0 & & F \xrightarrow{r} F_0
 \end{array}$$

#### 3.7.4.2

Then we consider the case where both  $E/E_0$  and  $F/F_0$  are totally ramified. Using Lemma 3.7.17 (which also works for our settings here), we choose  $\varpi_F$  as a uniformizer of  $F$ , and  $\varpi_{T_0}$  and  $\varpi_T$  as uniformizers

of  $T_0$  and  $T$  respectively satisfying  $\varpi_{F_0} \in \boldsymbol{\mu}_{T_0} \langle \varpi_{T_0} \rangle$ ,  $\varpi_F \in \boldsymbol{\mu}_T \langle \varpi_T \rangle$  and  $N_{T/T_0}(\varpi_T) \in \boldsymbol{\mu}_{T_0} \langle \varpi_{T_0} \rangle$ . Let  $K_0$  be the maximal unramified subfield of  $T_{0,m}$  over  $F_0$  and let  $K$  be the maximal unramified subfield of  $T_{m/r}$  over  $F$ . Then for  $K' = KK_0$  as a subfield of  $E_m$ , both  $K'/K_0$  and  $K'/K$  are cyclic extensions of degree  $r$ . The former is totally ramified, and the latter is unramified.

Let  $[\mathfrak{a}_0, \beta]$ ,  $\theta_0$  be given as before. Still we write  $A_{K_0}$  for the centralizer of  $K_0$  in  $A_{F_0}$ , and we define  $\mathfrak{a}_{K_0}$ ,  $[\mathfrak{a}_{K_0}, \beta]$ ,  $H^1(\mathfrak{a}_{K_0}, \beta)$  and  $\theta_{K_0} = \theta_0|_{H^1(\mathfrak{a}_{K_0}, \beta)}$  as before. We write  $\tilde{A}_{K_0} = \text{End}_{K_0}(E_m)$  and  $\tilde{\mathfrak{a}}_{K_0} = \mathfrak{a}_{K_0}(E_m)$  its hereditary order. Thus  $[\tilde{\mathfrak{a}}_{K_0}, \beta]$  is a simple stratum in  $\tilde{A}_{K_0}$  and we denote by  $\tilde{\theta}_{K_0}$  the transfer of  $\theta_{K_0}$  which is a simple character of  $H^1(\tilde{\mathfrak{a}}_{K_0}, \beta)$ . Furthermore we write  $A_{K'}$  for the centralizer of  $K'$  in  $\tilde{A}_{K_0}$  and  $\mathfrak{a}_{K'} = \tilde{\mathfrak{a}}_{K_0} \cap A_{K'}$  for its hereditary order. Thus  $[\mathfrak{a}_{K'}, \beta]$  is a simple stratum in  $A_{K'}$  and  $H^1(\mathfrak{a}_{K'}, \beta) = H^1(\tilde{\mathfrak{a}}_{K_0}, \beta) \cap G_{K'}$ . Moreover  $\tilde{\theta}_{K_0}|_{H^1(\mathfrak{a}_{K'}, \beta)}$  is the interior  $K'/K_0$ -lift of  $\tilde{\theta}_{K_0}$ .

On the other hand, let  $C_F$ ,  $\mathfrak{c}$ ,  $[\mathfrak{c}, \beta]$ ,  $\theta_a$  be defined as in §3.6.3 in the case where  $E/E_0$  is totally ramified. We write  $C_K$  for the centralizer of  $K$  in  $C_F$  and we may similarly define  $\mathfrak{c}_K$ ,  $[\mathfrak{c}_K, \beta]$ ,  $H^1(\mathfrak{c}_K, \beta)$  and  $\theta_K = \theta_a|_{H^1(\mathfrak{c}_K, \beta)}$ . We write  $\tilde{C}_K = \text{End}_K(E_m)$  and  $\tilde{\mathfrak{c}}_K = \mathfrak{a}_K(E_m)$  its hereditary order. Let  $\tilde{\theta}_K$  be the transfer of  $\theta_K$  which is a simple character of  $H^1(\tilde{\mathfrak{c}}_K, \beta)$ . Furthermore we write  $C_{K'}$  for the centralizer of  $K'$  in  $\tilde{C}_K$  and  $\mathfrak{c}_{K'} = \tilde{\mathfrak{c}}_K \cap C_{K'}$  for its hereditary order. By definition we actually have  $C_{K'} = A_{K'} = \text{End}_{K'}(E_m)$ ,  $\mathfrak{c}_{K'} = \mathfrak{a}_{K'} = \mathfrak{a}_{K'}(E_m)$  and  $[\mathfrak{c}_{K'}, \beta] = [\mathfrak{a}_{K'}, \beta]$  as a simple stratum in  $A_{K'}$ . Then  $\tilde{\theta}_K|_{H^1(\mathfrak{c}_{K'}, \beta)}$  is a simple character which is the interior  $K'/K$ -lift of  $\tilde{\theta}_K$ .

**Lemma 3.7.20.** *We have  $\tilde{\theta}_{K_0}|_{H^1(\mathfrak{a}_{K'}, \beta)} = \tilde{\theta}_K|_{H^1(\mathfrak{c}_{K'}, \beta)}$ , which we denote by  $\theta_{K'}$  as a simple character of  $H^1(\mathfrak{a}_{K'}, \beta) = H^1(\mathfrak{c}_{K'}, \beta)$ .*

*Proof.* From the construction, we have

$$\tilde{\theta}_{K_0}|_{H^1(\mathfrak{a}_{K'}, \beta)} = t_{\mathfrak{a}_{K_0}, \tilde{\mathfrak{a}}_{K_0}}^\beta(\theta_{K_0})|_{H^1(\mathfrak{a}_{K'}, \beta)} = (t_{\mathfrak{a}_{K_0}, \tilde{\mathfrak{a}}_{K_0}}^\beta(\theta_{K_0}))_{K'}.$$

Using Lemma 3.2.2 for  $F = F_0$ ,  $L = K_0$ ,  $\mathfrak{a} = \mathfrak{a}_0$  and  $\tilde{\mathfrak{a}} = \tilde{\mathfrak{a}}_0 := \mathfrak{a}_{F_0}(E_m)$ , and noting that  $(\mathfrak{a}_0)_{K_0} = \mathfrak{a}_{K_0}$  and  $(\tilde{\mathfrak{a}}_0)_{K_0} = \tilde{\mathfrak{a}}_{K_0} := \mathfrak{a}_{K_0}(E_m)$ , we get

$$t_{\mathfrak{a}_{K_0}, \tilde{\mathfrak{a}}_{K_0}}^\beta(\theta_{K_0}) = (t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0))_{K_0}.$$

Recall that  $\tilde{\mathfrak{c}}_0 := \mathfrak{a}_{F_0}(E_{m/r})$  is a hereditary order in  $\tilde{C}_{F_0} := \text{End}_{F_0}(E_{m/r})$ . Using the transitivity of the transfer map, we have

$$t_{\tilde{\mathfrak{c}}_0, \tilde{\mathfrak{a}}_0}^\beta(t_{\mathfrak{a}_0, \tilde{\mathfrak{c}}_0}^\beta(\theta_0)) = t_{\mathfrak{a}_0, \tilde{\mathfrak{a}}_0}^\beta(\theta_0) =: \tilde{\theta}_0$$

as a simple character in  $\mathcal{C}(\tilde{\mathfrak{a}}_0, \beta)$ . Using Lemma 3.2.1 for the tamely ramified extensions  $K'/K_0/F_0$  and  $K'/K/F_0$ , we get

$$((\tilde{\theta}_0)_{K_0})_{K'} = (\tilde{\theta}_0)_{K'} = ((\tilde{\theta}_0)_K)_{K'}.$$

Combining these equations together we get

$$\tilde{\theta}_{K_0}|_{H^1(\mathfrak{a}_{K'}, \beta)} = ((t_{\tilde{\mathfrak{c}}_0, \tilde{\mathfrak{a}}_0}^\beta(\tilde{\theta}'_0))_K)_{K'},$$

where by definition  $\tilde{\theta}'_0 := t_{\mathfrak{a}_0, \tilde{\mathfrak{c}}_0}^\beta(\theta_0)$  is a simple character in  $\mathcal{C}(\tilde{\mathfrak{c}}_0, \beta)$ . We use Lemma 3.2.2 for another time with  $F = F_0$ ,  $L = K$ ,  $\mathfrak{a} = \tilde{\mathfrak{c}}_0$  and  $\tilde{\mathfrak{a}}_0$ , and noting that by definition  $(\tilde{\mathfrak{c}}_0)_K = \mathfrak{c}_K := \mathfrak{a}_K(E_{m/r})$  and  $(\tilde{\mathfrak{a}}_0)_K = \tilde{\mathfrak{c}}_K := \mathfrak{a}_K(E_m)$ , we get

$$(t_{\tilde{\mathfrak{c}}_0, \tilde{\mathfrak{a}}_0}^\beta(\tilde{\theta}'_0))_K = t_{\mathfrak{c}_K, \tilde{\mathfrak{c}}_K}^\beta((\tilde{\theta}'_0)_K),$$

thus

$$\tilde{\theta}_{K_0}|_{H^1(\mathfrak{a}_{K'}, \beta)} = (t_{\mathfrak{c}_K, \tilde{\mathfrak{c}}_K}^\beta((t_{\mathfrak{a}_0, \tilde{\mathfrak{c}}_0}^\beta(\theta_0))_K))_{K'}.$$

On the other hand, by definition we have

$$\tilde{\theta}_K|_{H^1(\mathfrak{c}_{K'},\beta)} = (t_{\mathfrak{c}_K,\tilde{\mathfrak{c}}_K}^\beta((\theta_a)_K))_{K'} = (t_{\mathfrak{c}_K,\tilde{\mathfrak{c}}_K}^\beta(((t_{\mathfrak{a}_0,\tilde{\mathfrak{c}}_0}^\beta(\theta_0))_F)_K))_{K'}.$$

Using Lemma 3.2.1 with the tamely ramified extensions  $K/F/F_0$ , we get

$$(t_{\mathfrak{a}_0,\tilde{\mathfrak{c}}_0}^\beta(\theta_0))_K = ((t_{\mathfrak{a}_0,\tilde{\mathfrak{c}}_0}^\beta(\theta_0))_F)_K$$

which finally shows that

$$\tilde{\theta}_{K_0}|_{H^1(\mathfrak{a}_{K'},\beta)} = \tilde{\theta}_K|_{H^1(\mathfrak{c}_{K'},\beta)}.$$

□

**Remark 3.7.21.** *In the above proof, we actually showed that each block in the following diagram is commutative:*

$$\begin{array}{ccccccc} \mathcal{C}(\mathfrak{a}_0, \beta) & \xrightarrow{t_{\mathfrak{a}_0,\tilde{\mathfrak{c}}_0}^\beta} & \mathcal{C}(\tilde{\mathfrak{c}}_0, \beta) & \xrightarrow{|_{H^1(\mathfrak{c},\beta)}} & \mathcal{C}(\mathfrak{c}, \beta) & \xrightarrow{|_{H^1(\mathfrak{c}_K,\beta)}} & \mathcal{C}(\mathfrak{c}_K, \beta) \\ & \searrow^{t_{\mathfrak{a}_0,\tilde{\mathfrak{a}}_0}^\beta} & \downarrow^{t_{\tilde{\mathfrak{c}}_0,\tilde{\mathfrak{a}}_0}^\beta} & & & & \downarrow^{t_{\mathfrak{c}_K,\tilde{\mathfrak{c}}_K}^\beta} \\ & & \mathcal{C}(\tilde{\mathfrak{a}}_0, \beta) & \xrightarrow{|_{H^1(\tilde{\mathfrak{c}}_K,\beta)}} & \mathcal{C}(\tilde{\mathfrak{c}}_K, \beta) & & \downarrow^{t_{\mathfrak{c}_K,\tilde{\mathfrak{c}}_K}^\beta} \\ & & \downarrow^{t_{\mathfrak{a}_0,\tilde{\mathfrak{a}}_0}^\beta} & & & & \downarrow^{t_{\mathfrak{c}_K,\tilde{\mathfrak{c}}_K}^\beta} \\ & & \mathcal{C}(\tilde{\mathfrak{a}}_{K_0}, \beta) & \xrightarrow{|_{H^1(\tilde{\mathfrak{c}}_{K_0},\beta)}} & \mathcal{C}(\tilde{\mathfrak{c}}_{K_0}, \beta) & & \downarrow^{t_{\mathfrak{c}_{K_0},\tilde{\mathfrak{c}}_{K_0}^\beta} \\ & & \downarrow^{t_{\mathfrak{a}_{K_0},\tilde{\mathfrak{a}}_{K_0}^\beta} & & & & \downarrow^{t_{\mathfrak{c}_{K_0},\tilde{\mathfrak{c}}_{K_0}^\beta} \\ \mathcal{C}(\mathfrak{a}_{K_0}, \beta) & \xrightarrow{t_{\mathfrak{a}_{K_0},\tilde{\mathfrak{a}}_{K_0}^\beta} & \mathcal{C}(\tilde{\mathfrak{a}}_{K_0}, \beta) & \xrightarrow{|_{H^1(\mathfrak{c}_{K'},\beta)}} & \mathcal{C}(\mathfrak{c}_{K'}, \beta) & & \downarrow^{t_{\mathfrak{c}_{K'},\tilde{\mathfrak{c}}_{K'}^\beta} \end{array}$$

Given  $\kappa_0$  a full Heisenberg representation of  $\theta_0$  satisfying (3.6.1), by Proposition 3.7.5 we obtain a  $\Delta_0$ -invariant full Heisenberg representation  $\kappa_{K_0} = \mathbf{l}_{K_0/F_0}^{ur}(\kappa_0)$ , where  $\Delta_0 = \text{Gal}(T_{0,m}/T_0)$ . Then using §3.7.3 for the unramified extension  $K'/K_0$ , we obtain a full Heisenberg representation  $\kappa_{K'} = \mathbf{b}_{K'/K_0}(\kappa_{K_0})$  of  $\theta_{K'}$ . In particular, such  $\kappa_{K'}$  satisfies

$$C_{T_m}(\varpi_F) \subset \text{Ker}(\det(\kappa_{K'})) \quad \text{and} \quad \varpi_F \in \text{Ker}(\kappa_{K'}) \quad (3.7.11)$$

We use §3.7.3 again for the unramified extension  $K'/K$  to get a unique full Heisenberg representation of  $\theta_K$ , such that  $\mathbf{b}_{K'/K}(\kappa_K) = \kappa_{K'}$ . Finally using Proposition 3.7.5 for the unramified extension  $K/F$ , there exists a unique full Heisenberg representation  $\kappa_a$  of  $\theta_a$  satisfying (3.6.3) such that  $\mathbf{l}_{K/F}^{ur}(\kappa_a) = \kappa_K$ . We denote by  ${}^e\mathbf{a}_{F/F_0}^{tr}$  the above map from those full Heisenberg representations  $\kappa_0$  satisfying (3.6.1) to those full Heisenberg representations  $\kappa_a$  satisfying (3.6.3). From the construction it is a bijection.

$$\begin{array}{ccc} & \mathbf{b}_{K'/K_0} & \\ & \kappa_{K'} \longleftarrow \kappa_{K_0} & \\ & \downarrow^{(\mathbf{b}_{K'/K})^{-1}} & \uparrow^{\mathbf{l}_{K_0/F_0}^{ur}} \\ & \kappa_K & \\ & \downarrow^{(\mathbf{l}_{K/F}^{ur})^{-1}} & \\ & \kappa_a \longleftarrow \kappa_0 & \\ & \downarrow^{e\mathbf{a}_{F/F_0}^{tr}} & \end{array} \quad \begin{array}{ccc} K' & \xrightarrow{r} & K_0 \\ r \downarrow & & \downarrow^{mf_0} \\ K & & F \\ mf/r \downarrow & & \downarrow^r \\ F & \xrightarrow{r} & F_0 \end{array}$$

**Remark 3.7.22.** *In defining  $\mathbf{a}_{F/F_0}^{ur}$ ,  $\mathbf{a}_{F/F_0}^{tr}$ ,  $\mathbf{a}_{F/F_0}$ ,  $\mathbf{b}_{F/F_0}$ ,  ${}^e\mathbf{a}_{F/F_0}^{ur}$  and  ${}^e\mathbf{a}_{F/F_0}^{tr}$ , we always precise the uniformizers  $\varpi_{F_0}$  and  $\varpi_F$  at the beginning, however the definitions of those maps do not rely on our choice of the uniformizers. It is because those maps are combinations of the bijections between the full Heisenberg representations considered in Theorem 3.7.1, Proposition 3.7.5 and Proposition 3.7.8, whose definitions are independent of the choice of the uniformizers.*

## 3.8 Proof of the main theorems

In this section, we prove the results mentioned in section [3.6](#).

### 3.8.1 Interior automorphic induction

We follow the setting of [3.6.2](#) and [3.7.2](#), and we will prove Theorem [3.6.4](#).

#### 3.8.1.1

When  $F/F_0$  is unramified, by [3.7.1.2](#) we choose  $\pi_{K_0}$  to be a supercuspidal representation of  $G_{K_0}$  containing  $\theta_{K_0}$ , such that  $A_{K_0/F}(\pi_{K_0}) = \pi'$ . Since  $A_{F/F_0} \circ A_{K_0/F} = A_{K_0/F_0}$  by Proposition [3.4.9](#)(2), we also have  $A_{K_0/F_0}(\pi_{K_0}) = \pi_0$ . Moreover by Corollary [3.2.5](#), there exists a unique  $\Delta_0$ -regular tamely ramified character  $\xi_{K_0}$  of  $T_{0,m}^\times$ , such that

$$\pi_{K_0} \cong \text{ind}_{\mathbf{J}(\alpha_{K_0}, \beta)}^{G_{K_0}} (\kappa_{K_0} \cdot (\xi_{K_0} \circ N_{E_{0,m}/T_{0,m}})).$$

Using Theorem [3.7.6](#) for  $K_0/F_0$  and  $K_0/F$ , both  $\xi_0 \cdot (\mu_{\theta_0}^{K_0/F_0})^{-1}$  and  $\xi_a \cdot (\mu_{\theta_a}^{K_0/F})^{-1}$  are in the same  $\Delta_0$ -orbit as  $\xi_{K_0}$ . Thus

$${}_a\phi_{\theta_0}^{F/F_0} := (\mu_{\theta_a}^{K_0/F})^{-1} \cdot \mu_{\theta_0}^{K_0/F_0}$$

as a tamely ramified character of  $T_{0,m}^\times$  satisfies the condition (1) of the theorem.

#### 3.8.1.2

When  $F/F_0$  is totally ramified, we choose  $\pi_{K_0}$  to be a supercuspidal representation of  $G_{K_0}$  such that  $A_{K_0/F_0}(\pi_{K_0}) = \pi_0$ . Since  $K/K_0$  is totally ramified, by [3.7.1.3](#) we choose  $\pi_K$  to be a supercuspidal representation of  $G_K$  containing  $\theta_K$ , such that  $A_{K/K_0}(\pi_K) = \pi_{K_0}$ . Since

$$\pi_0 = A_{K_0/F_0}(A_{K/K_0}(\pi_K)) = A_{K/F_0}(\pi_K) = A_{F/F_0}(A_{K/F}(\pi_K))$$

by Proposition [3.4.9](#)(2),  $A_{K/F}(\pi_K)$  equals  $\pi'^{\sigma^i}$  for a certain  $i$ . Moreover, the endo-class of  $A_{K/F}(\pi_K)$  must be that of  $\theta_a$ , thus we have  $A_{K/F}(\pi_K) = \pi'$ . By Corollary [3.2.5](#), there exist unique tamely ramified characters  $\xi_{K_0}$  and  $\xi_K$  of  $T_{0,m}^\times$ , such that respectively

$$\pi_{K_0} \cong \text{ind}_{\mathbf{J}(\alpha_{K_0}, \beta)}^{G_{K_0}} (\kappa_{K_0} \cdot (\xi_{K_0} \circ N_{E_{0,m}/T_{0,m}})) \quad \text{and} \quad \pi_K \cong \text{ind}_{\mathbf{J}(\alpha_K, \beta)}^{G_K} (\kappa_K \cdot (\xi_K \circ N_{E_{0,m}/T_{0,m}})).$$

Using Theorem [3.7.6](#) for  $K_0/F_0$  and  $K/F$ , we deduce that  $\xi_0$  (resp.  $\xi_a$ ) is in the same  $\Delta_0$ -orbit as  $\xi_{K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$  (resp.  $\xi_K \cdot \mu_{\theta_a}^{K/F}$ ). Moreover by Proposition [3.7.10](#) for  $T_{0,m}/K_0$  and  $T_{0,m}/K$ , we have  $\xi_{K_0} = \xi_K \cdot \mu_{\theta_{K_0}}^{T_{0,m}/K_0} \cdot (\mu_{\theta_K}^{T_{0,m}/K})^{-1}$ . Thus

$${}_a\phi_{\theta_0}^{F/F_0} := (\mu_{\theta_a}^{K/F})^{-1} \cdot (\mu_{\theta_K}^{T_{0,m}/K})^{-1} \cdot \mu_{\theta_{K_0}}^{T_{0,m}/K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$$

as a tamely ramified character of  $T_{0,m}^\times$  satisfies the condition (1) of the theorem.

#### 3.8.1.3

In general, we consider the extensions  $F/F_1/F_0$ , such that  $F_1/F_0$  is unramified, and  $F/F_1$  is totally ramified. In this case, the character

$${}_a\phi_{\theta_0}^{F/F_0} := {}_a\phi_{\theta_{F_1}}^{F/F_1} \cdot {}_a\phi_{\theta_0}^{F_1/F_0}$$

satisfies the statement (1), where  $\theta_{F_1}$  denotes the interior  $F_1/F_0$ -lift of  $\theta_0$ . Moreover the statement (2) follows from Theorem [3.7.6](#) and Corollary [3.7.16](#).

### 3.8.2 Base change in supercuspidal case

We follow the setting of §3.6.1 and §3.7.3 and we will prove Theorem 3.6.2. By §3.7.1.2 and §3.7.1.3 we choose  $\pi_{K_0}$  to be a  $\Delta_0$ -regular supercuspidal representation of  $G_{K_0}$  containing  $\theta_{K_0}$  such that  $A_{K_0/F_0}(\pi_{K_0}) = \pi_0$ , and then  $\pi_{T_{0,m}}$  to be a supercuspidal representation of  $G_{T_{0,m}}$  containing  $\theta_{T_{0,m}}$  such that  $A_{T_{0,m}/K_0}(\pi_{T_{0,m}}) = \pi_{K_0}$ , where  $A_{T_{0,m}/K_0}$  is defined as in §3.4.3 since  $T_{0,m}/K_0$  is not necessarily cyclic. We write  $\pi_{T_m} = \text{BC}_{T_m/T_{0,m}}(\pi_{T_{0,m}})$  for the base change of  $\pi_{T_{0,m}}$  which is a supercuspidal representation of  $G_{T_m}$  by Corollary 3.5.12 and we write  $\pi_K = A_{T_m/K}(\pi_{T_m})$  for the automorphic induction of  $\pi_{T_m}$  defined as in §3.4.3 as a representation of  $G_K$ . Using Proposition 3.4.9 (3), we have  $\pi_K = \text{BC}_{K/K_0}(\pi_{K_0})$  which is supercuspidal by Corollary 3.5.12. Still using Proposition 3.4.9 (3) we have  $\pi = A_{K/F}(\pi_K)$ .

Now it is not difficult to study the relation between  $\xi_0$  and  $\xi_b$ . By Corollary 3.2.5 we choose  $\xi_{K_0}$  and  $\xi_K$  to be the unique tamely ramified characters of  $T_{0,m}^\times$  and  $T_m^\times$  respectively, such that

$$\pi_{K_0} \cong \text{ind}_{\mathbf{J}(\mathfrak{a}_{K_0}, \beta)}^{G_{K_0}}(\kappa_{K_0} \cdot (\xi_{K_0} \circ N_{E_{0,m}/T_{0,m}})) \quad \text{and} \quad \pi_K \cong \text{ind}_{\mathbf{J}(\mathfrak{a}_K, \beta)}^{G_K}(\kappa_K \cdot (\xi_K \circ N_{E_m/T_m})).$$

And similarly we choose  $\xi_{T_{0,m}}$  and  $\xi_{T_m}$  to be the unique tamely ramified characters of  $T_{0,m}^\times$  and  $T_m^\times$  respectively such that

$$\pi_{T_{0,m}} \cong \text{ind}_{\mathbf{J}(\mathfrak{a}_{T_{0,m}}, \beta)}^{G_{T_{0,m}}}(\kappa_{T_{0,m}} \cdot (\xi_{T_{0,m}} \circ N_{E_{0,m}/T_{0,m}})) \quad \text{and} \quad \pi_{T_m} \cong \text{ind}_{\mathbf{J}(\mathfrak{a}_{T_m}, \beta)}^{G_{T_m}}(\kappa_{T_m} \cdot (\xi_{T_m} \circ N_{E_m/T_m})).$$

Thus by Theorem 3.7.6,  $\xi_0$  equals  $\xi_{K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$  up to a  $\Delta_0$ -action and  $\xi_b$  equals  $\xi_K \cdot \mu_{\theta_b}^{K/F}$  up to a  $\Delta$ -action. Moreover by Proposition 3.7.10,  $\xi_{K_0} = \xi_{T_{0,m}} \cdot \mu_{\theta_{K_0}}^{T_{0,m}/K_0}$  and  $\xi_K = \xi_{T_m} \cdot \mu_{\theta_K}^{T_m/K}$ . Finally by Proposition 3.7.3,  $\xi_{T_{0,m}} \circ N_{T_m/T_{0,m}} = \xi_{T_m}$ . Combining these together, if we write

$${}_b\phi_{\theta_0}^{F/F_0} = (\mu_{\theta_b}^{K/F} \cdot \mu_{\theta_K}^{T_m/K})^{-1} \cdot (\mu_{\theta_0}^{K_0/F_0} \cdot \mu_{\theta_{K_0}}^{T_{0,m}/K_0}) \circ N_{T_m/T_{0,m}},$$

then  $\xi_b \cdot {}_b\phi_{\theta_0}^{F/F_0}$  and  $\xi_0 \circ N_{T_m/T_{0,m}}$  are in the same  $\Delta$ -orbit, finishing the proof of statement (1) in this case. The statement (2) follows from Theorem 3.7.6 and Corollary 3.7.16.

### 3.8.3 Exterior automorphic induction

Finally we follow the setting of §3.6.3 and §3.7.4, and we will prove Theorem 3.6.6 and Theorem 3.6.8.

#### 3.8.3.1

First we consider the case where  $E/E_0$  is unramified. Let  $\pi_K$  be a supercuspidal representation of  $G_K$  containing  $\theta_K$  such that  $A_{K/F}(\pi_K) = \pi'$ . Let  $\pi_{K_0} = A_{K/K_0}(\pi_K)$  which is a supercuspidal representation of  $G_{K_0}$  containing  $\theta_{K_0}$  by Proposition 3.4.4 and [BH03], Theorem B. By Proposition 3.4.9 (2), we have

$$\pi_0 = A_{F/F_0}(A_{K/F}(\pi_K)) = A_{K_0/F_0}(A_{K/K_0}(\pi_K)) = A_{K_0/F_0}(\pi_{K_0}).$$

By Corollary 3.2.5, we choose  $\xi_{K_0}$  (resp.  $\xi_K$ ) to be a tamely ramified character of  $T_{0,m}^\times = T_{m/r}^\times$  such that

$$\pi_{K_0} \cong \text{ind}_{\mathbf{J}(\mathfrak{a}_{K_0}, \beta)}^{G_{K_0}}(\kappa_{K_0} \cdot (\xi_{K_0} \circ N_{E_{0,m}/T_{0,m}})) \quad (\text{resp. } \pi_K \cong \text{ind}_{\mathbf{J}(\mathfrak{c}_K, \beta)}^{G_K}(\kappa_K \cdot (\xi_K \circ N_{E_{m/r}/T_{m/r}}))).$$

By Theorem 3.7.6 for  $K_0/F_0$  and  $K/F$ , we have a tamely ramified character  $\mu_{\theta_0}^{K_0/F_0}$  (resp.  $\mu_{\theta_a}^{K/F}$ ) of  $T_{0,m}^\times = T_{m/r}^\times$  such that  $\xi_{K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$  (resp.  $\xi_K \cdot \mu_{\theta_a}^{K/F}$ ) and  $\xi_0$  (resp.  $\xi_a$ ) are in the same  $\Delta_0$  (resp.

$\Delta$ )-orbit. Furthermore using Theorem 3.6.4 for  $F_0 = K_0$  and  $F = K$ , the characters  $\xi_K \cdot {}_a\phi_{\theta_{K_0}}^{K/K_0}$  and  $\xi_{K_0}$  are in the same  $\Delta_0$ -orbit. Thus if we define

$${}_a\phi_{\theta_0}^{F/F_0} := (\mu_{\theta_a}^{K/F})^{-1} \cdot {}_a\phi_{\theta_{K_0}}^{K/K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$$

as a tamely ramified character of  $T_{0,m}^\times$ , we conclude that  $\xi_a \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0$  are in the same  $\Delta_0$ -orbit. Finally using Theorem 3.7.6 and Corollary 3.7.16, we finish the proof of Theorem 3.6.6.

### 3.8.3.2

Then we assume that both  $F/F_0$  and  $E/E_0$  are totally ramified and we will prove Theorem 3.6.8. By §3.7.1.2 we choose  $\pi_K$  to be a supercuspidal representation of  $G_K = C_K^\times$  containing  $\theta_K$  such that  $A_{K/F}(\pi_K) = \pi'$ . By Corollary 3.5.12  $\pi_{K'} := \text{BC}_{K'/K}(\pi_K)$  is a supercuspidal representation of  $G_{K'} = C_{K'}^\times = A_{K'}^\times$ , containing  $\theta_{K'}$ .

**Lemma 3.8.1.** *There exists a supercuspidal representation  $\pi_{K_0}$  of  $G_{K_0}$  containing  $\theta_{K_0}$ , such that  $A_{K_0/F_0}(\pi_{K_0}) = \pi_0$  and  $\text{BC}_{K'/K_0}(\pi_{K_0}) = \pi_{K'}$ .*

*Proof.* We consider the equation

$$A_{F/F_0}(A_{K/F}(A_{K'/K}(\pi_{K'}))) = A_{K_0/F_0}(A_{K'/K_0}(\pi_{K'}))$$

which follows from [BH03], Lemma 6.2.(2). We choose  $\varkappa_{K'/K_0}$  as a character of  $K_0^\times$  whose kernel is  $\text{N}_{K'/K_0}(K'^\times)$ , and  $\varkappa_{K'/K}$  as a character of  $K^\times$  whose kernel is  $\text{N}_{K'/K}(K'^\times)$ , and  $\varkappa_{F/F_0}$  the character of  $F_0^\times$  satisfying  $\varkappa_{F/F_0} \circ \text{N}_{K_0/F_0} = \varkappa_{K'/K_0}$  whose kernel is  $\text{N}_{F/F_0}(F^\times)$ . Using *ibid.*, Theorem B and (5.1.2), there exists a supercuspidal representation  $\pi'_{K_0}$  of  $G_{K_0}$  containing  $\theta_{K_0}$ , such that

$$A_{K'/K_0}(\pi_{K'}) = \pi'_{K_0} \times \pi'_{K_0} \cdot \varkappa_{K'/K_0} \times \dots \times \pi'_{K_0} \cdot \varkappa_{K'/K_0}^{r-1}.$$

Let  $\pi'_0 := A_{K_0/F_0}(\pi'_{K_0})$  which is a supercuspidal representation of  $G_{F_0}$ , thus by *ibid.* (5.1.1) and the fact that automorphic induction commutes with parabolic induction for generic representations (see for example [HH95], section 5), we get

$$A_{K_0/F_0}(A_{K'/K_0}(\pi_{K'})) = \pi'_0 \times \pi'_0 \cdot \varkappa_{F/F_0} \times \dots \times \pi'_0 \cdot \varkappa_{F/F_0}^{r-1}$$

as an irreducible representation of  $G_{F_0}$ . On the other hand using [BH03] again, we have

$$A_{K'/K}(\pi_{K'}) = \pi_K \times \pi_K \cdot \varkappa_{K'/K} \times \dots \times \pi_K \cdot \varkappa_{K'/K}^{r-1},$$

thus

$$A_{K_0/F_0}(A_{K'/K_0}(\pi_{K'})) = A_{F/F_0}(A_{K/F}(A_{K'/K}(\pi_{K'}))) = \pi_0 \times \pi_0^c,$$

where  $\pi_0 = A_{F/F_0}(A_{K/F}(\pi_K))$  from the construction and  $\pi_0^c := A_{F/F_0}(A_{K/F}(\pi_K \cdot \varkappa_{K'/K} \times \dots \times \pi_K \cdot \varkappa_{K'/K}^{r-1}))$ . Since the factorization of an irreducible generic representation as parabolic induction of discrete series representations is unique, we get  $\pi_0 \cong \pi'_0 \cdot \varkappa_{F/F_0}^i$  for a certain  $i \in \{0, 1, \dots, r-1\}$ . Let  $\pi_{K_0} := \pi'_{K_0} \cdot \varkappa_{K'/K_0}^i$  be as a supercuspidal representation of  $G_{K_0}$ , then  $A_{K_0/F_0}(\pi_{K_0}) = \pi_0$ . Finally by [BH03], (5.1.2) and (5.1.3), we have  $\text{BC}_{K'/K_0}(\pi_{K_0}) = \pi_{K'}$ .  $\square$

Choose  $\pi_{K_0}$  as in the lemma. By Corollary 3.2.5, we choose  $\xi_{K_0}$  (resp.  $\xi_K$ ) to be the unique tamely unramified character of  $T_{0,m}^\times$  (resp.  $T_{m/r}^\times$ ) such that

$$\pi_{K_0} \cong \text{ind}_{\mathbf{J}(a_{K_0}, \beta)}^{G_{K_0}}(\kappa_{K_0} \cdot (\xi_{K_0} \circ N_{E_{0,m}/T_{0,m}})) \quad (\text{resp. } \pi_K \cong \text{ind}_{\mathbf{J}(c_K, \beta)}^{G_K}(\kappa_K \cdot (\xi_K \circ N_{E_{m/r}/T_{m/r}})))$$

and we choose  $\xi_{K'}$  to be the unique tamely unramified character of  $T_m^\times$  such that

$$\pi_{K'} \cong \text{ind}_{\mathbf{J}(c_{K'}, \beta)}^{G_{K'}}(\kappa_{K'} \cdot (\xi_{K'} \circ N_{E_m/T_m})).$$

Using Theorem 3.7.6, we have a tamely ramified character  $\mu_{\theta_0}^{K_0/F_0}$  (resp.  $\mu_{\theta_a}^{K/F}$ ) of  $T_{0,m}^\times$  (resp.  $T_{m/r}^\times$ ) such that  $\xi_0$  (resp.  $\xi_a$ ) and  $\xi_{K_0} \cdot \mu_{\theta_0}^{K_0/F_0}$  (resp.  $\xi_K \cdot \mu_{\theta_a}^{K/F}$ ) are in the same  $\Delta_0$  (resp.  $\Delta$ )-orbit. Since  $\pi_{K'} = \text{BC}_{K'/K_0}(\pi_{K_0}) = \text{BC}_{K'/K}(\pi_K)$ , by Theorem 3.6.2 there exists a tamely ramified character  ${}_b\phi_{\theta_{K_0}}^{K'/K_0}$  (resp.  ${}_b\phi_{\theta_K}^{K'/K}$ ) of  $T_m^\times$  such that  $\xi_{K'} \cdot {}_b\phi_{\theta_{K_0}}^{K'/K_0}$  (resp.  $\xi_{K'} \cdot {}_b\phi_{\theta_K}^{K'/K}$ ) equals  $\xi_{K_0} \circ N_{T_m/T_{0,m}}$  (resp.  $\xi_K \circ N_{T_m/T_{m/r}}$ ). Thus if we write

$${}_a\phi_{\theta_0}^{F/F_0} := (\mu_{\theta_a}^{K/F} \circ N_{T_m/T_{m/r}})^{-1} \cdot ({}_b\phi_{\theta_K}^{K'/K})^{-1} \cdot {}_b\phi_{\theta_{K_0}}^{K'/K_0} \cdot (\mu_{\theta_0}^{K_0/F_0} \circ N_{T_m/T_{0,m}})$$

as a tamely ramified character of  $T_m^\times$ , then  $(\xi_a \circ N_{T_m/T_{m/r}}) \cdot {}_a\phi_{\theta_0}^{F/F_0}$  and  $\xi_0 \circ N_{T_m/T_{0,m}}$  are in the same  $\Delta' = \text{Gal}(T_m/T)$ -orbit, finishing the proof of Theorem 3.6.8 (1), and the statement (2) follows from Theorem 3.7.6 and Corollary 3.7.16.

### 3.9 Calculation of ${}_b\phi_{\theta_0}^{F/F_0}$ in the $F/F_0$ unramified case

In the previous section, we proved the main theorems of §3.6. However, it should be more interesting to calculate the corresponding characters, which in practice gives the explicit construction of base change and automorphic induction<sup>5</sup>. In this section, we focus on the calculation of  ${}_b\phi_{\theta_0}^{F/F_0}$  when  $F/F_0$  is unramified.

The motivation for concentrating on this special case is two-fold. On the one hand, the values of this character are expected to be simple, since the symplectic signs in the definition compensate with each others, which is not the case when  $F/F_0$  is not unramified or when we consider the automorphic induction instead of base change, and which can also be expected from the known results of Bushnell-Henniart in the essentially tame case [BH05a], [BH05b], [BH10]. On the other hand, this special case is quite useful. For example, it will help us to update the values of the character related to the Comparison Theorem in [BH14b], or more concretely the character  $\mu_{\theta_0}^{T_0/F_0}$  in Proposition 3.7.10, which will be explained in the next section. Here comes the main theorem:

**Theorem 3.9.1.** *When  $F/F_0$  is unramified, the character  ${}_b\phi_{\theta_0}^{F/F_0}$  in Theorem 3.6.2 is unramified, and moreover  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = (-1)^{(t-1)(r-1)}$ , where  $t = [T_{0,m} : K_0]$  and  $[E_0 : T_0] = p^s$ .*

**Remark 3.9.2.** *In particular when  $E_0/F_0$  is totally and tamely ramified, we return to the known result [BH05b], Corollary 2.3.*

From now on until the end of this section, we will follow the settings of §3.6.1, §3.7.3 and §3.8.2, and we always assume that  $F/F_0$  is unramified.

<sup>5</sup>More precisely, we still need to admit the explicit construction of the full Heisenberg representation  $\kappa_b$  in Theorem 3.7.1 as a black box. Or in other words, we assume that the explicit construction of base change map in the totally wildly ramified case is fully understood.

### 3.9.1 Reduction to the maximal totally ramified case

In this subsection, we will show that Theorem [3.9.1](#) can be reduced to the maximal and totally ramified case.

#### 3.9.1.1

**Proposition 3.9.3.**  $\mu_{\theta_0}^{K_0/F_0} \circ N_{T_m/T_{0,m}} = \mu_{\theta_b}^{K/F}$  as tamely ramified characters of  $T_m^\times$ , where  $\mu_{\theta_0}^{K_0/F_0}$  and  $\mu_{\theta_b}^{K/F}$  are defined as in Theorem [3.7.6](#).

*Proof.* We use the exact formula of the two characters given in Theorem [3.7.6](#). By Proposition [3.5.3](#),  $r$  is relatively prime to  $mf_0$ , thus by definition  $K/K_0$  is unramified of degree  $r$ , and  $[K : F] = [K_0 : F_0] = mf_0$ . Moreover since  $[E_m : F] = [E_{0,m} : F_0] = n$ , we have  $[E_m : K] = [E_{0,m} : K_0]$ . Thus

$$\mu_{\theta_0}^{K_0/F_0}(\varpi_{F_0}) = (-1)^{[E_{0,m}:K_0]([K_0:F_0]-1)} = (-1)^{[E_m:K]([K:F]-1)} = \mu_{\theta_b}^{K/F}(\varpi_{F_0})$$

and

$$\mathbf{d}'_F := (-1)^{[E_{0,m}:K_0]([K_0:F_0]-1)p^s} = (-1)^{[E_m:K]([K:F]-1)p^s} =: \mathbf{d}'_{F_0}.$$

If  $r$  is odd, then

$$(-1)^{r[E_{0,m}:K_0]([K_0:F_0]-1)} = (-1)^{[E_m:K]([K:F]-1)}$$

and

$$(-1)^{r[E_{0,m}:K_0]([K_0:F_0]-1)p^s} = (-1)^{[E_m:K]([K:F]-1)p^s}.$$

If  $r$  is even, then  $[K_0 : F_0] = [K : F]$  is odd and thus

$$(-1)^{r[E_{0,m}:K_0]([K_0:F_0]-1)} = (-1)^{[E_m:K]([K:F]-1)} = 1$$

and

$$(-1)^{r[E_{0,m}:K_0]([K_0:F_0]-1)p^s} = (-1)^{[E_m:K]([K:F]-1)p^s} = 1.$$

Therefore we have

$$\mu_{\theta_0}^{K_0/F_0}(N_{T_m/T_{0,m}}(\varpi_{F_0})) = \mu_{\theta_0}^{K_0/F_0}(\varpi_{F_0})^r = \mu_{\theta_b}^{K/F}(\varpi_{F_0}) \quad \text{and} \quad \mathbf{d}'_F = (\mathbf{d}'_{F_0})^r.$$

To finish the proof, it is enough to prove the following proposition:

**Proposition 3.9.4.** (1)  $\epsilon_F^1(\boldsymbol{\mu}_{T_m}) = \epsilon_{F_0}^1(\boldsymbol{\mu}_{T_{0,m}}) \circ N_{T_m/T_{0,m}}$  as quadratic characters of  $\boldsymbol{\mu}_{T_m}$ ;

(2)  $\epsilon_F^0(\boldsymbol{\mu}_{T_m}) = \epsilon_{F_0}^0(\boldsymbol{\mu}_{T_{0,m}})^r$  and  $\epsilon_L^0(\boldsymbol{\mu}_{T_m}) = \epsilon_{L_0}^0(\boldsymbol{\mu}_{T_{0,m}})^r$ , where  $L_0 = F_0[\varpi_{T_0}]$  for  $\varpi_{T_0}$  a uniformizer of  $T_0$  lying in  $C_{T_0}(\varpi_{F_0})$  and  $L = F[\varpi_{T_0}]$ ;

(3)  $\epsilon_F(\varpi_{T_0}) = \epsilon_{F_0}(\varpi_{T_0})^r$  and  $\epsilon_L(\varpi_{T_0}) = \epsilon_{L_0}(\varpi_{T_0})^r$ .

Recall that all the symplectic signs in this proposition have been defined in Example [3.3.2](#) and [3.3.3](#). Once this proposition is proved, we further have

$$\mu_{\theta_0}^{K_0/F_0} \circ N_{T_m/T_{0,m}}|_{\boldsymbol{\mu}_{T_m}} = \mu_{\theta_b}^{K/F}|_{\boldsymbol{\mu}_{T_m}}$$

and

$$\mu_{\theta_0}^{K_0/F_0}(N_{T_m/T_{0,m}}(\varpi_{T_0})^{p^s}) = \mu_{\theta_b}^{K/F}(\varpi_{T_0}^{p^s}),$$

finishing the proof of Proposition [3.9.3](#). □



## 3.9.1.2

Thus we only need to prove Proposition 3.9.4. We refer to §3.2.1 for the definition of  $j^1(\mathfrak{a}, \beta)$ ,  $j^1(\mathfrak{a}_0, \beta)$ ,  $\mathfrak{h}^1(\mathfrak{a}, \beta)$ ,  $\mathfrak{h}^1(\mathfrak{a}_0, \beta)$  in the lemma below.

**Lemma 3.9.5** ([BH96], Proposition 11.14). *We have the isomorphisms  $j^1(\mathfrak{a}, \beta) \cong j^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{o}_{F_0}} \mathfrak{o}_F$ ,  $\mathfrak{h}^1(\mathfrak{a}, \beta) \cong \mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{o}_{F_0}} \mathfrak{o}_F$  and  $j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta) \cong j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{k}_{F_0}} \mathfrak{k}_F$ .*

We choose  $\{v_1, \dots, v_r\}$  to be a basis of  $\mathfrak{o}_F$  as a  $\mathfrak{o}_{F_0}$ -lattice, such that  $\text{Tr}_{F/F_0}(v_i v_j) = \delta_{ij}$  for  $1 \leq i, j \leq r$ , where  $\delta_{ij}$  equals 1 for  $i = j$  or 0 otherwise, which is possible for unramified extension  $F/F_0$ . Thus using the above lemma we have

$$j^1(\mathfrak{a}, \beta) \cong \bigoplus_{i=1}^r j^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{o}_{F_0}} v_i \quad \text{and} \quad \mathfrak{h}^1(\mathfrak{a}, \beta) \cong \bigoplus_{i=1}^r \mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{o}_{F_0}} v_i \quad (3.9.1)$$

as  $\mathfrak{o}_{F_0}$ -modules. Moreover, for any  $x_i, x_j \in j^1(\mathfrak{a}_0, \beta)$ , we have

$$\begin{aligned} & \psi_F(\text{Tr}_{A_F}(\beta((x_i \otimes v_i) \cdot (x_j \otimes v_j) - (x_j \otimes v_j) \cdot (x_i \otimes v_i)))) = \\ & \psi_{F_0}(\text{Tr}_{A_{F_0}}(\beta(x_i x_j \otimes \text{Tr}_{F/F_0}(v_i v_j) - x_j x_i \otimes \text{Tr}_{F/F_0}(v_j v_i)))) = \delta_{ij} \cdot \psi_{F_0}(\text{Tr}_{A_{F_0}}(\beta(x_i x_j - x_j x_i))). \end{aligned}$$

Combining with (3.3.1) we proved that:

**Lemma 3.9.6.** (3.9.1) gives an isomorphism of symplectic spaces over  $\mathbb{F}_p$ :

$$j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta) \cong \bigoplus_{i=1}^r j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{k}_{F_0}} v_i, \quad (3.9.2)$$

where the right hand side is the orthogonal direct sum of  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{o}_{F_0}} v_i$  identifying with  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta)$  as symplectic spaces via  $x \otimes v_i \mapsto x$ .

We study a finite cyclic group  $\Gamma$  acting on both sides of (3.9.2). First we consider  $\Gamma = \langle \varpi_{T_0} \rangle / \langle \varpi_{F_0} \rangle$ , where  $\varpi_{T_0}$  is a uniformizer of  $T_0$  lying in  $C_{T_0}(\varpi_{F_0})$  and the action on the left hand side is given by conjugation. We regard  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{k}_{F_0}} v_i$  as a symplectic  $\mathbb{F}_p[\Gamma]$ -module with the  $\Gamma$ -action giving by conjugation on the first coordinate of the tensor product, which is naturally isomorphic to the symplectic  $\mathbb{F}_p[\Gamma]$ -module  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta)$ . We define the  $\Gamma$ -action on the right hand side of (3.9.2) by acting on each summand  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathfrak{k}_{F_0}} v_i$  together. Thus comparing the symplectic signs of both sides, we get  $\epsilon_F(\varpi_{T_0}) = \epsilon_{F_0}(\varpi_{T_0})^r$ , finishing the proof of the first statement of Proposition 3.9.4(3).

We fix  $\zeta_0$  a given generator of  $\mu_{T_0, m}$  and  $\zeta$  a generator of  $\mu_{T_m}$  such that  $N_{T_m/T_0, m}(\zeta) = \zeta_0$ .

**Lemma 3.9.7.** For  $\zeta' \in \mu_F \cong \mathfrak{k}_F^\times$  a generator,  $x_0 := \zeta_0 \otimes \zeta'$  being regarded as an element in  $\mu_{T_m} \cong \mathfrak{k}_{T_m}^\times$  via the isomorphism  $\mathfrak{k}_{T_0, m} \otimes_{\mathfrak{k}_{F_0}} \mathfrak{k}_F \cong \mathfrak{k}_{T_m}$  satisfies  $x_0 = \zeta^u$  for an odd  $u$ .

*Proof.* Recall that  $q$  is the cardinality of  $\mathfrak{k}_{F_0}$ . Since  $\zeta_0$  is a generator of  $\mu_{T_0, m}$ , its order is exactly  $q^{mf_0} - 1$ . Similarly the order of  $\zeta'$  is  $q^r - 1$ , and thus the order of  $x_0$  is the least common divisor of  $q^{mf_0} - 1$  and  $q^r - 1$ , denoted by  $[q^{mf_0} - 1, q^r - 1]$ . To finish the proof, we only need to show that  $[q^{mf_0} - 1, q^r - 1]$  and  $q^{mf_0 r} - 1$  have the same maximal divisor as a power of 2, which is an easy exercise by noting that  $(mf_0, r) = 1$ . □

Now we let  $\Gamma = \langle x_0 \rangle$  be as a subgroup of  $\mu_{T_m}$  for  $x_0$  as in the lemma above, and we consider its action on (3.9.2). Still on the left hand side the  $\Gamma$ -action is given by conjugation, and on the right hand side it is given by  $\Gamma$  conjugation on each summand  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i$  together. Comparing the symplectic signs we have:

$$t_\Gamma^0(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)) = \prod_{i=1}^r t_\Gamma^0(j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i) \quad (3.9.3)$$

and

$$t_\Gamma^1(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)) = \prod_{i=1}^r t_\Gamma^1(j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i) \quad (3.9.4)$$

Since  $\zeta_0$  is a generator of  $\mathbf{k}_{T_0, m}^\times$ , it is easily seen that

$$\mathbf{k}_F[x_0] = \mathbf{k}_F[\zeta_0] = \mathbf{k}_{T_m} = \mathbf{k}_F[\zeta_0^2] = \mathbf{k}_F[x_0^2],$$

which implies that for  $\Gamma' := \langle x_0^2 \rangle$  as a subgroup of  $\Gamma$ ,

$$(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))^{\Gamma'} = (j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))^\Gamma = (j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))^{\mu_{T_m}}$$

Thus by Proposition 3.3.1(4) and the equation  $x_0 = \zeta^u$  with  $u$  being odd, we get

$$t_\Gamma^0(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)) \cdot t_\Gamma^1(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))(x_0) = \epsilon_F^0(\mu_{T_m}) \cdot \epsilon_F^1(\mu_{T_m})(x_0) = \epsilon_F^0(\mu_{T_m}) \cdot \epsilon_F^1(\mu_{T_m})(\zeta)$$

and

$$\begin{aligned} t_\Gamma^0(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)) \cdot t_\Gamma^1(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))(x_0^2) &= t_{\Gamma'}^0(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)) \cdot t_{\Gamma'}^1(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))(x_0^2) \\ &= \epsilon_F^0(\mu_{T_m}) \cdot \epsilon_F^1(\mu_{T_m})(x_0^2) = \epsilon_F^0(\mu_{T_m}) \cdot \epsilon_F^1(\mu_{T_m})(\zeta^2), \end{aligned}$$

which imply that

$$t_\Gamma^0(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta)) = \epsilon_F^0(\mu_{T_m}) \quad \text{and} \quad t_\Gamma^1(j^1(\mathfrak{a}, \beta)/\mathfrak{h}^1(\mathfrak{a}, \beta))(x_0) = \epsilon_F^1(\mu_{T_m})(\zeta). \quad (3.9.5)$$

Moreover  $\Xi := \langle x_0^{\mu_{T_0, m}^1} \rangle = \langle 1 \otimes \zeta^{\mu_{T_0, m}^1} \rangle$  as a subgroup of  $\Gamma$  acts trivially on  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i$ . And by definition, the symplectic  $\mathbb{F}_p[\Gamma/\Xi]$ -module  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i$  is identified with the symplectic  $\mathbb{F}_p[\mu_{T_0, m}]$ -module  $j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta)$  via the group isomorphism  $\Gamma/\Xi \rightarrow \mu_{T_0, m}$ ,  $x_0 \mapsto \zeta_0$ . Thus by Proposition 3.3.1(3), we have

$$\begin{aligned} t_\Gamma^0(j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i) &= \epsilon_{F_0}^0(\mu_{T_0, m}) \\ t_\Gamma^1(j^1(\mathfrak{a}_0, \beta)/\mathfrak{h}^1(\mathfrak{a}_0, \beta) \otimes_{\mathbf{k}_{F_0}} v_i)(x_0) &= \epsilon_{F_0}^1(\mu_{T_0, m})(\zeta_0). \end{aligned} \quad (3.9.6)$$

Combining (3.9.3), (3.9.4), (3.9.5), (3.9.6) together, we have

$$\epsilon_F^0(\mu_{T_m}) = \epsilon_{F_0}^0(\mu_{T_0, m})^r \quad \text{and} \quad \epsilon_F^1(\mu_{T_m})(\zeta) = \epsilon_{F_0}^1(\mu_{T_0, m})(\zeta_0^r).$$

So the first part of Proposition 3.9.4(2) is proved, and noting that  $\zeta$  generates  $\mu_{T_m}$  and  $\zeta_0 = N_{T_m/T_0, m}(\zeta)$ , we further have

$$\epsilon_F^1(\mu_{T_m}) = (\epsilon_{F_0}^1(\mu_{T_0, m}) \circ N_{T_m/T_0, m})^r.$$

When  $r$  is odd, we have

$$\epsilon_F^1(\mu_{T_m}) = (\epsilon_{F_0}^1(\mu_{T_0, m}) \circ N_{T_m/T_0, m})^r = \epsilon_{F_0}^1(\mu_{T_0, m}) \circ N_{T_m/T_0, m}.$$

When  $r$  is even,  $\epsilon_F^1(\boldsymbol{\mu}_{T_m}) = (\epsilon_{F_0}^1(\boldsymbol{\mu}_{T_0,m}) \circ N_{T_m/T_0,m})^r = 1$  and  $[\mathbf{k}_{T_0,m} : \mathbf{k}_{F_0}] = [K_0 : F_0]$  is relatively prime to  $r$  which is odd. Thus  $|\boldsymbol{\mu}_{T_0,m}/\boldsymbol{\mu}_{F_0}| = |\mathbf{k}_{T_0,m}^\times/\mathbf{k}_{F_0}^\times|$  is odd. Using Proposition 3.3.1(3), we have

$$\epsilon_{F_0}^1(\boldsymbol{\mu}_{T_0,m})(\zeta_0) = t_{\boldsymbol{\mu}_{T_0,m}/\boldsymbol{\mu}_{F_0}}^1(j^1(\mathbf{a}_0, \beta)/\mathfrak{h}^1(\mathbf{a}_0, \beta))(\zeta_0) = 1,$$

implying that

$$\epsilon_F^1(\boldsymbol{\mu}_{T_m}) = \epsilon_{F_0}^1(\boldsymbol{\mu}_{T_0,m}) \circ N_{T_m/T_0,m} = 1.$$

Thus Proposition 3.9.4(1) is proved no matter  $r$  is odd or even. Finally, for all the symplectic signs with respect to  $L_0$  and  $L$ , we use  $L_0, [\mathbf{a}_{L_0}, \beta], \theta_{L_0}, L, [\mathbf{a}_L, \beta], \theta_L$  to replace  $F_0, [\mathbf{a}_0, \beta], \theta_0, F, [\mathbf{a}, \beta], \theta_b$  respectively and the rest of the statements in Proposition 3.9.4 can be proved using the same argument as above.

### 3.9.1.3

We finish this subsection by explaining that to deal with Theorem 3.9.1 we only need to deal with the maximal and totally ramified case. We consider  $\pi_{K_0}, \pi_K$  exactly as in §3.8.2 and the corresponding character  ${}_b\phi_{\theta_{K_0}}^{K/K_0}$ . Thus from the construction of  ${}_b\phi_{\theta_{K_0}}^{K/K_0}$  and  ${}_b\phi_{\theta_0}^{F/F_0}$  there and Proposition 3.9.3, we have

$${}_b\phi_{\theta_0}^{F/F_0} = (\mu_{\theta_b}^{K/F})^{-1} \cdot {}_b\phi_{\theta_{K_0}}^{K/K_0} \cdot (\mu_{\theta_0}^{K_0/F_0} \circ N_{T_m/T_0,m}) = {}_b\phi_{\theta_{K_0}}^{K/K_0}.$$

So we only need to consider  $\pi_{K_0}$  and  $\pi_K = \text{BC}_{K/K_0}(\pi_{K_0})$  instead. Then  $K_0[\beta] = E_{0,m}$  is totally ramified over  $K_0$ , thus we reduce the theorem to the maximal and totally ramified case.

## 3.9.2 A special case of Theorem 3.9.1

From the argument of the previous subsection, **from now on until the end of this section we may and will assume that  $m = 1$  and  $E_0/F_0$  is totally ramified in the statement of Theorem 3.9.1.** In this subsection, we expect the following result is true:

**Conjecture 3.9.8.** *If  $F/F_0$  is unramified,  $t = [T_0 : F_0]$  is odd and  $(r, t) = 1$ , then  ${}_b\phi_{\theta_0}^{F/F_0}$  is a trivial character of  $T^\times$ .*

However due to the incompleteness of a key step which we will explain later, we could only prove the following special case:

**Proposition 3.9.9.** *If  $F/F_0$  is unramified,  $t = [T_0 : F_0]$  is odd and  $(r, t) = 1$ , then  ${}_b\phi_{\theta_0}^{F/F_0}$  is an unramified character of  $T^\times$ , and moreover  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = 1$ , where  $p^s = [E_0 : T_0]$ .*

### 3.9.2.1

We first recall our notations in §3.7.3 and §3.8.2 in the maximal and totally ramified case. Let  $\theta_0$  and  $\theta_b$  be the simple characters contained in  $\pi_0$  and  $\pi$  respectively as before, let  $\theta_{T_0}$  be the interior  $T_0/F_0$ -lift of  $\theta_0$  and let  $\theta_T$  be the interior  $T/F$ -lift of  $\theta_b$ . For  $\boldsymbol{\kappa}_0$  a full Heisenberg representation of  $\theta_0$  satisfying (3.6.1), we use Proposition 3.7.8 to construct the full Heisenberg representation  $\boldsymbol{\kappa}_{T_0} = \mathbf{l}_{T_0/F_0}^{tr}(\boldsymbol{\kappa}_0)$  of  $\theta_{T_0}$ . And then we use Theorem 3.7.1 to construct the full Heisenberg representation  $\boldsymbol{\kappa}_T = \mathbf{b}_{T/T_0}(\boldsymbol{\kappa}_{T_0})$  of  $\theta_T$ . Finally using Proposition 3.7.8 again we get the full Heisenberg representation  $\boldsymbol{\kappa}_b = (\mathbf{l}_{T/F}^{tr})^{-1}(\boldsymbol{\kappa}_T)$  of  $\theta_b$ . In particular,  $\boldsymbol{\kappa}_b$  and  $\boldsymbol{\kappa}_T$  are  $\sigma$ -invariant.

To calculate  ${}_b\phi_{\theta_0}^{F/F_0}$ , we only need to choose a special supercuspidal representation  $\pi_0$ . Thus in particular we assume  $\pi_0 \cong \text{ind}_{\mathbf{J}(\mathbf{a}_0, \beta)}^{G_{F_0}} \boldsymbol{\kappa}_0$ , or in other words, we assume  $\boldsymbol{\xi}_0 = 1$ . Thus by Theorem 3.6.2,

we have  $\pi \cong \text{ind}_{\mathbf{J}(\mathbf{a}, \beta)}^{G_F} \kappa'_b$  with  $\kappa'_b := \kappa_b \cdot ({}_b\phi_{\theta_0}^{F/F_0})^{-1}$ , where by abuse of notations we identify  ${}_b\phi_{\theta_0}^{F/F_0}$  with the corresponding character of  $\mathbf{J}(\mathbf{a}, \beta)/J^1(\mathbf{a}, \beta)$  as in Corollary 3.2.5. We write  $\pi_{T_0} := \text{ind}_{\mathbf{J}(\mathbf{a}_{T_0}, \beta)}^{G_{T_0}} \kappa_{T_0}$  as a supercuspidal representation of  $G_{T_0} = A_{T_0}^\times$  and  $\pi_T := \text{ind}_{\mathbf{J}(\mathbf{a}_T, \beta)}^{G_T} \kappa_T$  as a supercuspidal representation of  $G_T = A_T^\times$ , so we further have  $\text{BC}_{T/T_0}(\pi_{T_0}) = \pi_T$ . For  $\mathbf{J}(\mathbf{a}_0, \beta)$ ,  $\mathbf{J}(\mathbf{a}_{T_0}, \beta)$ ,  $\mathbf{J}(\mathbf{a}, \beta)$ ,  $\mathbf{J}(\mathbf{a}_T, \beta)$ , we will write  $\mathbf{J}_{F_0}$ ,  $\mathbf{J}_{T_0}$ ,  $\mathbf{J}_F$ ,  $\mathbf{J}_T$  for short respectively. Similar abbreviation for groups as  $J_{F_0}^1$  and  $H_{F_0}^1$  works without further mention.

By construction the central characters of  $\kappa_0$  and  $\kappa_b$  are trivial, thus the central character of  $\pi_0$  is trivial. Using the formula for the central characters of base change, the central character of  $\pi$  is also trivial. From the construction in the maximal case, every extended maximal simple type containing  $\theta_b$  and having trivial central character equals  $\kappa_b$  twisted by a character of  $\mathbf{J}_F/F^\times J_F^1$ , which is a cyclic group of order dividing  $n$  (see for example [BK93], Proposition 6.1.2). Thus:

**Proposition 3.9.10.**  ${}_b\phi_{\theta_0}^{F/F_0}$  is an unramified character of order dividing  $n$ .

### 3.9.2.2

Since  $\kappa_b$  is  $\sigma$ -invariant, we fix a representation  $\tilde{\kappa}_b$  of  $\mathbf{J}_F \rtimes \Sigma$  such that  $\tilde{\kappa}_b|_{\mathbf{J}_F} = \kappa_b$ , where  $\Sigma = \text{Gal}(F/F_0)$ . We have the following proposition as a generalization of Proposition 3.7.8

**Proposition 3.9.11.** *There exists a representation  $\tilde{\kappa}_T$  of  $\mathbf{J}_T \rtimes \Sigma$  extending  $\kappa_T$ , such that*

$$\text{tr}(\tilde{\kappa}_T)(g, \sigma^i) = \epsilon_{T/F} \cdot \text{tr}(\tilde{\kappa}_b)(g, \sigma^i)$$

(cf. Example 3.3.2) for all  $g \in \mathbf{J}_T$  such that  $(v_T(\det_T(g)), t) = 1$  and all integers  $i$ .

*Proof.* We follow exactly the same proof of [BH14b], §5.3, Proposition. We write  ${}_p\mathbf{J}_F = {}_p\mathbf{J}_F(\varpi_{F_0})$  for the subgroup of  $\mathbf{J}_F$  such that  ${}_p\mathbf{J}_F/\langle \varpi_{F_0} \rangle$  is the Sylow pro- $p$ -subgroup of  $\mathbf{J}_F/\langle \varpi_{F_0} \rangle$ , and we write  ${}_p\mathbf{J}_T = {}_p\mathbf{J}_F \cap \mathbf{J}_T$ . Since  $C_F(\varpi_{F_0}) \subset F^\times \subset \text{Ker}(\kappa_b)$ , we may view  $\tilde{\kappa}_b$  as the inflation of a representation of  $C_T(\varpi_{F_0})/C_F(\varpi_{F_0}) \rtimes ({}_p\mathbf{J}_F/\langle \varpi_{F_0} \rangle \rtimes \Sigma)$ . The group  $C_T(\varpi_{F_0})/C_F(\varpi_{F_0}) \cong T^\times/\mathfrak{o}_T^\times F^\times$  is of order  $t$ , which is relatively prime to the order of  ${}_p\mathbf{J}_F/\langle \varpi_{F_0} \rangle \rtimes \Sigma$ , thus using the Glauberman correspondence, we get a representation  $\tilde{\kappa}_T$  of  ${}_p\mathbf{J}_T/\langle \varpi_{F_0} \rangle \rtimes \Sigma$  such that

$$\text{tr}(\tilde{\kappa}_T)(x, \sigma^i) = \epsilon \cdot \text{tr}(\tilde{\kappa}_b)(\zeta \cdot (x, \sigma^i))$$

for any  $x \in {}_p\mathbf{J}_T/\langle \varpi_{F_0} \rangle$ , any integer  $i$  and any generator  $\zeta$  of  $C_T(\varpi_{F_0})/C_F(\varpi_{F_0})$ . Moreover if we restrict both sides to  ${}_p\mathbf{J}_T/\langle \varpi_{F_0} \rangle \times \{1\}$ , we come back to the trace formula in *loc. cit.*, which implies that  $\tilde{\kappa}_T|_{{}_p\mathbf{J}_T} = \kappa_T|_{{}_p\mathbf{J}_T}$  and  $\epsilon = \epsilon_{T/F}$ . We extend  $\tilde{\kappa}_T$  trivially to a representation of  $C_T(\varpi_{F_0})/C_F(\varpi_{F_0}) \times ({}_p\mathbf{J}_T/\langle \varpi_{F_0} \rangle \rtimes \Sigma)$  and inflate it to a representation of  $C_T(\varpi_{F_0}) \cdot ({}_p\mathbf{J}_T/\langle \varpi_{F_0} \rangle \rtimes \Sigma) = \mathbf{J}_T \rtimes \Sigma$ , still denoted by  $\tilde{\kappa}_T$ . This representation satisfies the condition of the proposition.  $\square$

Since  ${}_b\phi_{\theta_0}^{F/F_0}$  is  $\sigma$ -invariant and has been identified with a character of  $\mathbf{J}_F/J_F^1$  as we explained before, it extends to a character of  $\mathbf{J}_F \rtimes \Sigma$  whose restriction to  $\Sigma$  is trivial. Thus  $\tilde{\kappa}'_b := \tilde{\kappa}_b \cdot ({}_b\phi_{\theta_0}^{F/F_0})^{-1}$  is a representation of  $\mathbf{J}_F \rtimes \Sigma$  extending  $\kappa'_b$ . Let  $\tilde{\pi} = \text{ind}_{\mathbf{J}_F \rtimes \Sigma}^{G_F \rtimes \Sigma} \tilde{\kappa}'_b$  be a representation of  $G_F \rtimes \Sigma$  and let  $\tilde{\pi}_T = \text{ind}_{\mathbf{J}_T \rtimes \Sigma}^{G_T \rtimes \Sigma} \tilde{\kappa}_T$  be a representation of  $G_T \rtimes \Sigma$ . Thus  $\tilde{\pi}$  extends  $\pi$  and  $\tilde{\pi}_T$  extends  $\pi_T$ .

### 3.9.2.3

We denote by  $N_{T_0}$  (resp.  $N_T$ ) the normalizer of  $T_0^\times$  (resp.  $T^\times$ ) in  $G_{F_0}$  (resp.  $G_F$ ), and we write  $\Gamma_0 = \text{Aut}(T_0/F_0)$  (resp.  $\Gamma = \text{Aut}(T/F)$ ). Thus by definition we have  $\Gamma_0 \cong N_{T_0}/G_{T_0}$  (resp.  $\Gamma \cong N_T/G_T$ ). The following conjecture is expected to be true in general, although we cannot give a proof.

**Conjecture 3.9.12.** *Let  $a$  be an integer relatively prime to  $t$ . There exists an elliptic regular element  $h_0 \in \mathbf{J}_{T_0}$  such that*

- $v_{T_0}(\det_{T_0}(h_0)) = v_{F_0}(\det_{F_0}(h_0)) = a$ ;
- $F_0[h_0]/F_0$  is totally ramified with  $T_0/F_0$  its maximal tamely ramified subextension;
- $\mathrm{tr}(\pi_0)(h_0) = \epsilon_{T_0/F_0} \sum_{\gamma \in \Gamma_0} \mathrm{tr}(\pi_{T_0})(h_0^\gamma) \neq 0$  (cf. Example [3.3.2](#)).

Instead we will use and prove the following special case:

**Proposition 3.9.13.** *If moreover  $a$  is either relatively prime to  $n$ , or it is divisible by  $p^s$ , then the conjecture above is true.*

*Proof.* When  $a$  is relatively prime to  $n$ , it is derived from [\[BH14b\]](#), §8.1. When  $a = a'p^s$  for an integer  $a'$ , we explain how the similar idea works. Using the principal theorem (which is due to Howe [\[How77\]](#) when  $\mathrm{char}(F_0) = 0$ ) of [\[BHL10\]](#), for any elliptic element  $h'_0 \in G_{T_0}$  sufficiently close to 1, the trace  $\mathrm{tr}(\pi_0)(h'_0)$  is a non-zero constant  $C$ . We choose one such  $h'_0$  such that  $F_0[h'_0]/F_0$  is totally ramified of degree  $n$  and  $h'_0 \in J^1(\mathfrak{a}, \beta)$ , which is possible since  $J^1(\mathfrak{a}, \beta)$  is open. We write  $h_0 = \varpi_{T_0}^{a'} h'_0$  as an element in  $\mathbf{J}_{T_0}$ . By definition  $T_0$  is contained in the centralizer of  $F_0[h_0]$ , which is  $F_0[h_0]$  itself, thus  $F_0[h_0]/F_0$  is a totally ramified field extension of degree  $n$  with  $T_0/F_0$  its maximal tamely ramified subextension, and  $v_{T_0}(\det_{T_0}(h_0)) = a'p^s = a$ . Moreover

$$\epsilon_{T_0/F_0} \sum_{\gamma \in \Gamma_0} \mathrm{tr}(\pi_{T_0})(h_0^\gamma) = \epsilon_{T_0/F_0} \cdot C \cdot |\Gamma_0| \neq 0,$$

and we only need to show that it equals  $\mathrm{tr}(\pi_0)(h_0)$ . This follows from [\[BH14b\]](#), §8.1, Proposition, whose proof can be adapted here, since in *loc. cit.* the condition  $(n, a) = 1$  is only needed to prove Lemma 1 there, which is actually our second condition for  $h_0$  in our proposition and has been verified.  $\square$

We choose  $b$  to be an integer relatively prime to  $t$ , such that  $a = br$  satisfies the condition of Proposition [3.9.13](#). We fix  $h_0 \in \mathbf{J}_{T_0}$  in *loc. cit.*, thus  $\det_{F_0}(h_0) \in N_{F/F_0}(F^\times)$  since  $F/F_0$  is unramified of degree  $r$ . Using [\[AC89\]](#), Lemma 1.4, there exists  $h \in G_T$  such that  $N_{F/F_0}(h) = N_{T/T_0}(h) = h_0$ .

**Proposition 3.9.14.** *For  $h \in G_T$  such that  $N_{F/F_0}(h) = h_0$ , we have*

$$\mathrm{tr}(\tilde{\pi})(h, \sigma) = \epsilon_{T/F} \cdot {}_b\phi_{\theta_0}^{F/F_0}(\det_T(h))^{-1} \sum_{\gamma \in \Gamma_0} \mathrm{tr}(\tilde{\pi}_T)(h^\gamma, \sigma),$$

where  $\Gamma_0$  is naturally identified with a subgroup of  $\Gamma = \mathrm{Aut}(T/F)$ .

*Proof.* Using the Mackey formula, we have

$$\mathrm{tr}(\tilde{\pi})(h, \sigma) = \sum_{g \in G_F / \mathbf{J}_F, g^{-1}h\sigma(g) \in \mathbf{J}_F} \mathrm{tr}(\tilde{\kappa}'_b)(g^{-1}h\sigma(g), \sigma).$$

For  $g^{-1}h\sigma(g) \in \mathbf{J}_F$ , we further have  $N_{F/F_0}(g^{-1}h\sigma(g)) = g^{-1}h_0g \in \mathbf{J}_F$ .

**Lemma 3.9.15.** *For  $g \in G_F$  such that  $g^{-1}h_0g \in \mathbf{J}_F$ , there exists  $y \in N_T$  such that  $y\mathbf{J}_T = g\mathbf{J}_F \cap N_T$ , thus  $\gamma_y(t) := y^{-1}ty$  for any  $t \in T$  is an element in  $\Gamma$ . Moreover if  $g^{-1}h\sigma(g) \in \mathbf{J}_F$ , we further have  $\gamma_y \in \Gamma_0$ .*

*Proof.* As indicated in the proof of [BH14b], §8.1, Lemma 2, we may use *ibid.* §2.6, Conjugacy Lemma and §5.5, Lemma to choose  $y \in g\mathbf{J}_F$  such that  $y^{-1}h_0y \in \mathbf{J}_T$ . By definition,  $T$  is contained in the centralizer of  $y^{-1}h_0y$  in  $A_F$ , which is  $F[y^{-1}h_0y]$ . Thus both  $yTy^{-1}$  and  $T$  are contained in  $F[h_0]$  and tamely ramified over  $F$ , implying that  $y^{-1}Ty = T$  and  $y \in N_T$  and proving the first statement.

Since  $h \in G_T$ , the conjugation by  $y^{-1}\sigma(y)$  restricting to  $T$  equals the conjugation by  $y^{-1}h\sigma(y)$  restricting to  $T$ , which gives an action in  $\Gamma$ . Since  $y^{-1}h\sigma(y) \in \mathbf{J}_F = E^\times J_F^1$ , where  $E^\times$  is commutative with  $T^\times$  and  $J_F^1$  is a pro- $p$ -group with  $p$  relatively prime to  $|\Gamma|$ , the above two conjugations restricting to  $T$  give the trivial action in  $\Gamma$ . Thus

$$x = (y^{-1}\sigma(y))^{-1}ty^{-1}\sigma(y) = \sigma \circ \gamma_y \circ \sigma^{-1} \circ \gamma_y^{-1}(x) \quad (3.9.7)$$

for all  $x \in T$  by direct calculation. In particular, if we consider  $\varpi_{T_0}$  as a uniformizer of  $T_0$  such that  $\varpi_{T_0}^t = \varpi_{F_0}$  is a uniformizer of  $F_0$ , then  $\gamma_y(\varpi_{T_0}) = \zeta\varpi_{T_0}$  for a certain  $\zeta \in \mu_F$ . Choose  $x = \varpi_{T_0}$  in (3.9.7), we get  $\sigma(\zeta)\zeta^{-1}\varpi_{T_0} = \varpi_{T_0}$ , which means that  $\zeta \in \mu_{F_0}$ . Since  $T_0 = F_0[\varpi_{T_0}]$ , the restriction of  $\gamma_y$  to  $T_0$  is in  $\Gamma_0 = \text{Aut}(T_0/F_0)$  which finishes the proof.  $\square$

Using this lemma and Proposition 3.9.11, we further have

$$\begin{aligned} \text{tr}(\tilde{\pi})(h, \sigma) &= \sum_{y \in N_T/\mathbf{J}_T, y^{-1}h\sigma(y) \in \mathbf{J}_T} \text{tr}(\tilde{\kappa}'_b)(y^{-1}h\sigma(y), \sigma) \\ &= \sum_{\gamma \in \Gamma_0} \sum_{g \in G_T/\mathbf{J}_T, g^{-1}h^\gamma\sigma(g) \in \mathbf{J}_T} \text{tr}(\tilde{\kappa}'_b)(g^{-1}h^\gamma\sigma(g), \sigma) \\ &= \sum_{\gamma \in \Gamma_0} \sum_{g \in G_T/\mathbf{J}_T, g^{-1}h^\gamma\sigma(g) \in \mathbf{J}_T} \epsilon_{T/F} \cdot ({}_b\phi_{\theta_0}^{F/F_0}(\det_T(g^{-1}h^\gamma\sigma(g))))^{-1} \cdot \text{tr}(\tilde{\kappa}_T)(g^{-1}h^\gamma\sigma(g), \sigma) \end{aligned}$$

**Lemma 3.9.16.** *For  $g \in G_T$  such that  $g^{-1}h^\gamma\sigma(g) \in \mathbf{J}_T$ , we have  ${}_b\phi_{\theta_0}^{F/F_0}(\det_T(g^{-1}h^\gamma\sigma(g))) = {}_b\phi_{\theta_0}^{F/F_0}(\det_T(h^\gamma))$ .*

*Proof.* First we have  $N_{T/T_0}(g^{-1}h^\gamma\sigma(g)) = g^{-1}N_{T/T_0}(h^\gamma)g \in \mathbf{J}_T$ , as a result it is easy to see that  $N_{T/T_0}(g^{-1}h^\gamma\sigma(g)) \equiv N_{T/T_0}(h^\gamma) \pmod{J_T^1}$ . We write  $h^\gamma \equiv \varpi_{E_0}^k \zeta \pmod{J_T^1}$  for  $k \in \mathbb{N}$  and  $\zeta \in \mu_E$ , thus by direct calculation we have  $N_{T/T_0}(h^\gamma) \equiv N_{E/E_0}(\varpi_{E_0}^k \zeta) \pmod{J_T^1}$ . Similarly if we write  $g^{-1}h^\gamma\sigma(g) \equiv \varpi_{E_0}^{k'} \zeta' \pmod{J_T^1}$  for  $k' \in \mathbb{N}$  and  $\zeta' \in \mu_E$ , then we have  $N_{T/T_0}(g^{-1}h^\gamma\sigma(g)) \equiv N_{E/E_0}(\varpi_{E_0}^{k'} \zeta') \pmod{J_T^1}$ . As a result, we must have  $k = k'$  and  $N_{E/E_0}(\zeta) = N_{E/E_0}(\zeta')$ . Since  ${}_b\phi_{\theta_0}^{F/F_0}$  is unramified, we have

$${}_b\phi_{\theta_0}^{F/F_0}(\det_T(g^{-1}h^\gamma\sigma(g))) = {}_b\phi_{\theta_0}^{F/F_0}(N_{E/T}(\varpi_{E_0}^k \zeta')) = {}_b\phi_{\theta_0}^{F/F_0}(N_{E/T}(\varpi_{E_0}^k \zeta)) = {}_b\phi_{\theta_0}^{F/F_0}(\det_T(h^\gamma)).$$

$\square$

**Lemma 3.9.17.** *We have  $({}_b\phi_{\theta_0}^{F/F_0})^\gamma = {}_b\phi_{\theta_0}^{F/F_0}$  as characters of  $T^\times$  for any  $\gamma \in \Gamma$ .*

*Proof.* Since the action  $\gamma$  maps a certain uniformizer  $\varpi_T$  to its multiplication with a certain root of unity, we finish the proof by using the fact that  ${}_b\phi_{\theta_0}^{F/F_0}$  is unramified.  $\square$

Using these two lemmas and the Mackey formula again, we finish the calculation:

$$\begin{aligned} \mathrm{tr}(\tilde{\pi})(h, \sigma) &= \epsilon_{T/F} \cdot ({}_b\phi_{\theta_0}^{F/F_0}(\det_T(h)))^{-1} \sum_{\gamma \in \Gamma_0} \sum_{g \in G_T / \mathbf{J}_T, g^{-1}h^\gamma \sigma(g) \in \mathbf{J}_T} \mathrm{tr}(\tilde{\kappa}_T)(g^{-1}h^\gamma \sigma(g), \sigma) \\ &= \epsilon_{T/F} \cdot ({}_b\phi_{\theta_0}^{F/F_0}(\det_T(h)))^{-1} \sum_{\gamma \in \Gamma_0} \mathrm{tr}(\tilde{\pi}_T)(h^\gamma, \sigma) \end{aligned}$$

□

### 3.9.2.4

We choose  $h_0$  and  $h$  as before. For  $\pi_0$  and  $\pi = \mathrm{BC}_{F/F_0}(\pi_0)$ , using the trace formula of base change, there exists a non-zero constant  $c$  independent of the choice of  $h_0$  and  $h$ , such that

$$\mathrm{tr}(\tilde{\pi})(h, \sigma) = c \cdot \mathrm{tr}(\pi_0)(h_0). \quad (3.9.8)$$

Similarly for  $\pi_{T_0}$  and  $\pi_T = \mathrm{BC}_{T/T_0}(\pi_{T_0})$ , using the trace formula of base change there exists a non-zero constant  $c'$ , independent of the choice of  $h_0$  and  $h$ , such that

$$\mathrm{tr}(\tilde{\pi}_T)(h, \sigma) = c' \cdot \mathrm{tr}(\pi_{T_0})(h_0).$$

For  $\gamma \in \Gamma_0 \subset \Gamma$ , using  $h_0^\gamma$  and  $h^\gamma$  to replace  $h_0$  and  $h$  respectively, we further have:

$$\mathrm{tr}(\tilde{\pi}_T)(h^\gamma, \sigma) = c' \cdot \mathrm{tr}(\pi_{T_0})(h_0^\gamma). \quad (3.9.9)$$

Combining Proposition [3.9.13](#), Proposition [3.9.14](#), [\(3.9.8\)](#), [\(3.9.9\)](#) together, we get:

$$c \cdot \epsilon_{T_0/F_0} \cdot \mathrm{tr}(\pi_0)(h_0) = \epsilon_{T/F} \cdot ({}_b\phi_{\theta_0}^{F/F_0}(\det_T(h)))^{-1} \cdot c' \cdot \mathrm{tr}(\pi_0)(h_0),$$

which implies that

$${}_b\phi_{\theta_0}^{F/F_0}(\det_T(h)) = \epsilon_{T/F} \cdot \epsilon_{T_0/F_0} \cdot c' \cdot c^{-1}, \quad (3.9.10)$$

where  $v_T(\det_T(h)) = v_F(\det_F(h)) = b$  is relatively prime to  $t$ . Since  $t$  is odd, for  $b = p^s$  and  $b = 2p^s$ , which in particular guarantee the condition of Proposition [3.9.13](#) for  $a$ , there exist  $h = h_1$  and  $h = h_2$  respectively such that [\(3.9.10\)](#) is satisfied. As a result we get

$${}_b\phi_{\theta_0}^{F/F_0}(\det_T(h_2/h_1)) = 1 \quad \text{with} \quad v_T(\det_T(h_2/h_1)) = p^s,$$

concluding that  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = 1$  and finishing the proof of Proposition [3.9.9](#).

**Remark 3.9.18.** If Conjecture [3.9.12](#) is proved, we may use it to replace Proposition [3.9.13](#), and then we may choose  $b = 1$  and  $2$  in the argument instead to prove Conjecture [3.9.8](#) with the same proof.

**Remark 3.9.19.** Moreover the assumption  $t$  being odd in Conjecture [3.9.8](#) Proposition [3.9.9](#) can be dropped. When  $t$  is even, which implies that  $r$  is odd since  $(r, t) = 1$ , using Proposition [3.9.4](#) (3) we have  $\epsilon_{T/F} = \epsilon_{T_0/F_0}$ . Moreover we may use the Whittaker model to regularize our choice of  $\tilde{\kappa}_b$  and  $\tilde{\pi}$  to make sure that  $c = 1$  (see [\[AC89\]](#), §1.2.). Then we can show that  $\tilde{\kappa}_T$  and  $\tilde{\pi}_T$  are also “regularized” by the Whittaker model as extensions of  $\kappa_T$  and  $\pi_T$  respectively, meaning that  $c' = 1$ . Thus the right hand side of [\(3.9.10\)](#) is 1, and the introduction of  $h_2$  is avoided whose existence relies on the fact that  $(2, t) = 1$ .

### 3.9.3 A reductive procedure when $\Gamma_0$ is non-trivial

In this subsection, we consider the case where  $\Gamma_0 = \text{Aut}(T_0/F_0)$  is non-trivial. We fix a prime number  $l$  dividing  $|\Gamma_0|$ . We choose  $L_0$  to be the unique cyclic subextension of  $T_0$  of degree  $l$  over  $F_0$  and we write  $L = L_0F$  which is a subfield of  $T$  of degree  $l$  over  $F$ . We write  $\theta_{L_0}$  for the interior  $L_0/F_0$ -lift of  $\theta_0$ . We want to compare  ${}_b\phi_{\theta_0}^{F/F_0}$  with  ${}_b\phi_{\theta_{L_0}}^{L/L_0}$ . More concretely, we will prove the following proposition, which permits us to reduce the degree  $t = [T_0 : F_0]$  to  $t/l = [T_0 : L_0]$  by replacing  $F_0$  with  $L_0$ .

**Proposition 3.9.20.**  ${}_b\phi_{\theta_0}^{F/F_0} \cdot ({}_b\phi_{\theta_{L_0}}^{L/L_0})^{-1}$  is an unramified character of  $T^\times$ , and moreover

$${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}) \cdot ({}_b\phi_{\theta_{L_0}}^{L/L_0}(\varpi_{T_0}))^{-1} = (-1)^{(r-1)(l-1)n/l}.$$

#### 3.9.3.1

We choose  $\pi_{T_0}$  and  $\pi_T$  as in §3.8.2 (here  $T_0 = T_{0,m}$  and  $T = T_m$ ). We choose  $\pi_{L_0}$  to be the supercuspidal representation  $\pi_{L_0} = A_{T_0/L_0}(\pi_{T_0})$  of  $G_{L_0} := A_{L_0}^\times$ , where  $A_{L_0}$  is the centralizer of  $L_0$  in  $A_{F_0}$ . Thus  $\pi_{L_0}$  contains  $\theta_{L_0}$ . Using Proposition 3.4.9(3) and Corollary 3.5.12,

$$\pi_L = \text{BC}_{L/L_0}(\pi_{L_0}) = A_{T/L}(\pi_T)$$

is a supercuspidal representation of  $G_L := A_L^\times$ , where  $A_L$  is the centralizer of  $L$  in  $A_F$ . Again using Proposition 3.4.9(3), we have

$$\pi = \text{BC}_{F/F_0}(A_{L_0/F_0}(\pi_{L_0})) = A_{L/F}(\pi_L).$$

We write  $\theta_L$  for the interior  $L/F$ -lift of  $\theta_b$ . From the exact formula in §3.8.2, we have

$${}_b\phi_{\theta_0}^{F/F_0} = (\mu_{\theta_b}^{T/F})^{-1} \cdot (\mu_{\theta_0}^{T_0/F_0} \circ \text{N}_{T/T_0}) \quad \text{and} \quad {}_b\phi_{\theta_{L_0}}^{L/L_0} = (\mu_{\theta_L}^{T/L})^{-1} \cdot (\mu_{\theta_{L_0}}^{T_0/L_0} \circ \text{N}_{T/T_0}).$$

Using Proposition 3.7.12, we finally have

$${}_b\phi_{\theta_0}^{F/F_0} \cdot ({}_b\phi_{\theta_{L_0}}^{L/L_0})^{-1} = (\chi_{\theta_b}^{L/F})^{-1} \cdot (\chi_{\theta_0}^{L_0/F_0} \circ \text{N}_{T/T_0}), \quad (3.9.11)$$

where for  $L_0/F_0$  we fix a transfer system  $(\sigma_0, \varkappa_{L_0/F_0}, e_l)$  and for  $L/F$  we fix the transfer system  $(\sigma_0, \varkappa_{L/F}, e_l)$ , with  $\sigma_0$  a generator of  $\text{Gal}(L/F) \cong \text{Gal}(L_0/F_0)$ , and  $\varkappa_{L/F} := \varkappa_{L_0/F_0} \circ \text{N}_{F/F_0}$  and  $e_l \in L_0^\times$  such that  $\sigma_0(e_l) = (-1)^{(l-1)n/l}$ . Moreover  $\chi_{\theta_0}^{L_0/F_0}$  and  $\chi_{\theta_b}^{L/F}$  are determined by the formula (3.7.3).

#### 3.9.3.2

We fix a regular elliptic element  $h_0 \in \mathbf{J}_{T_0}$  such that  $v_{T_0}(\det_{T_0}(h_0)) = 1$ . We want to show that

$$\chi_{\theta_b}^{L/F}(\det_T(h_0)) = (\chi_{\theta_0}^{L_0/F_0}(\det_{T_0}(h_0)))^r \cdot (-1)^{(r-1)(l-1)n/l}. \quad (3.9.12)$$

Combining this equation with (3.9.11), we have

$${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}) \cdot ({}_b\phi_{\theta_{L_0}}^{L/L_0}(\varpi_{T_0}))^{-1} = {}_b\phi_{\theta_0}^{F/F_0}(\det_T(h_0)) \cdot ({}_b\phi_{\theta_{L_0}}^{L/L_0}(\det_T(h_0)))^{-1} = (-1)^{(r-1)(l-1)n/l},$$

finishing the proof of Proposition 3.9.20. So we focus on the proof of (3.9.12).



**3.9.3.3**

Using (3.7.3), we have

$$\chi_{\theta_b}^{L/F}(\det_T(h_0)) = \epsilon_{T/L} \cdot \epsilon_{T/F} \cdot c_{\theta_b}^{L/F} \cdot \delta_{L/F}(h_0)$$

and

$$\chi_{\theta_b}^{L_0/F_0}(\det_{T_0}(h_0)) = \epsilon_{T_0/L_0} \cdot \epsilon_{T_0/F_0} \cdot c_{\theta_b}^{L_0/F_0} \cdot \delta_{L_0/F_0}(h_0).$$

Using Proposition 3.9.4(3) (also with  $F$  and  $F_0$  replaced by  $L$  and  $L_0$ ), we have

$$\epsilon_{T/L} = \epsilon_{T_0/L_0}^r \quad \text{and} \quad \epsilon_{T/F} = \epsilon_{T_0/F_0}^r.$$

By direct calculation we have (cf. §3.4.2)

$$\tilde{\Delta}_{L/F}(h_0) = \tilde{\Delta}_{L_0/F_0}(h_0) \quad \text{and} \quad e_l \tilde{\Delta}_{L/F}(h_0) = e_l \tilde{\Delta}_{L_0/F_0}(h_0) \in F_0^\times.$$

Since  $|\cdot|_F = |\cdot|_{F_0}^r$ , we get  $\Delta_{L/F}^1(h_0) = \Delta_{L_0/F_0}^1(h_0)^r$ . And since  $\varkappa_{L/F} := \varkappa_{L_0/F_0} \circ N_{F/F_0}$ , by direct calculation we get

$$\Delta_{L/F}^2(h_0) = \varkappa_{L/F}(e_l \tilde{\Delta}_{L/F}(h_0)) = \varkappa_{L_0/F_0}(N_{F/F_0}(e_l \tilde{\Delta}_{L_0/F_0}(h_0))) = \Delta_{L_0/F_0}^2(h_0)^r.$$

Thus finally

$$\delta_{L/F}(h_0) = \Delta_{L/F}^2(h_0) / \Delta_{L/F}^1(h_0) = \Delta_{L_0/F_0}^2(h_0)^r / \Delta_{L_0/F_0}^1(h_0)^r = \delta_{L_0/F_0}(h_0)^r.$$

Thus to finish the proof, we only need to show that  $c_{\theta_b}^{L/F} = (-1)^{(r-1)(l-1)n/l} \cdot (c_{\theta_b}^{L_0/F_0})^r$

**3.9.3.4**

For a given  $F_0$ -algebra isomorphism  $A_{F_0} \cong \text{End}_{F_0}(E_0)$  and the induced isomorphism  $A_F \cong F \otimes_{F_0} A_{F_0} \cong \text{End}_F(F \otimes_{F_0} E_0) \cong \text{End}_F(E)$ , we consider the flag

$$\mathcal{F} : \quad 0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n = E_0$$

of the  $F_0$ -vector space  $E_0$ , also being regarded as a flag of the  $F$ -vector space  $E$  by extension of scalars, and then we get the corresponding unipotent subgroup  $U(F_0)$  of  $G_{F_0}$  and the unipotent subgroup  $U(F)$  of  $G_F$ . Moreover by Theorem 3.4.5(2), there exists a non-degenerate character  $\vartheta_0$  of  $U(F_0)$ , such that  $(U(F_0), \vartheta_0)$  is a Whittaker pair of  $G_{F_0}$  satisfying

$$\theta_0|_{H^1(\mathfrak{a}_0, \beta) \cap U(F_0)} = \vartheta_0|_{H^1(\mathfrak{a}_0, \beta) \cap U(F_0)}.$$

By extension of scalars we get a Whittaker pair  $(U(F), \vartheta)$  of  $G_F$ . To be more precise, the character  $\vartheta$  is realized as follows: The norm map induces the following group homomorphism

$$N_{F/F_0} : U(F)/U_{\text{der}}(F) \longrightarrow U(F_0)/U_{\text{der}}(F_0),$$

where  $U_{\text{der}}$  denotes the derived subgroup of  $U$ , thus

$$\vartheta(u) = \vartheta_0(N_{F/F_0}(u))$$

for any  $u \in U(F)$ , where  $u$  and  $N_{F/F_0}(u)$  should be regarded as in  $U(F)/U_{\text{der}}(F)$  and  $U(F_0)/U_{\text{der}}(F_0)$  respectively. Then by definition and [BH96], Theorem 12.6. the following condition is satisfied:

$$\theta_b|_{H^1(\mathfrak{a}, \beta) \cap U(F)} = \vartheta|_{H^1(\mathfrak{a}, \beta) \cap U(F)}.$$

By Theorem [3.4.5](#)(3), we have

$$c_{\theta_b}^{L/F} = c^{L/F}(\vartheta) \quad \text{and} \quad c_{\theta_0}^{L_0/F_0} = c^{L_0/F_0}(\vartheta_0).$$

Moreover, we choose a certain element  $g_0 \in G_{F_0}$  such that  $(U(F_0), \vartheta_0) = (U_0(F_0)^{g_0}, \vartheta_{\psi_{F_0}}^{g_0})$ , where  $U_0(F_0)$  is the upper triangular unipotent radical of  $G_{F_0}$ . Here  $g_0$  can be realized as the base change matrix, which is invariant up to extension of scalar. In other words, we also have  $(U(F), \vartheta) = (U_0(F)^{g_0}, \vartheta_{\psi_F}^{g_0})$ , where  $U_0(F)$  is the upper triangular unipotent radical of  $G_F$ . By [3.4.3](#), we have

$$c^{L_0/F_0}(\vartheta_0) = c_n^{L_0/F_0}(\psi_{F_0}) \cdot (\varkappa_{L_0/F_0}(\det_{F_0}(g_0)))^{-1}$$

and

$$c^{L/F}(\vartheta) = c_n^{L/F}(\psi_F) \cdot (\varkappa_{L/F}(\det_F(g_0)))^{-1} = c_n^{L/F}(\psi_F) \cdot (\varkappa_{L_0/F_0}(\det_{F_0}(g_0))^r)^{-1}.$$

Thus we actually proved that

$$c_{\theta_b}^{L/F} / (c_{\theta_0}^{L_0/F_0})^r = c_n^{L/F}(\psi_F) / c_n^{L_0/F_0}(\psi_{F_0})^r,$$

so we only need to calculate the latter term.

Finally we also point out that this ratio is independent of the choice of the embedding  $L_0 \hookrightarrow A_{F_0}$ , since by the Skolem-Noether Theorem, any two embeddings are conjugate by a certain  $g' \in G_{F_0}$ . Thus when we change one embedding to the other, the above ratio varies by multiplying  $\varkappa_{L_0/F_0}(\det_{F_0}(g'))^r / \varkappa_{L/F}(\det_F(g')) = 1$  ([HL10](#), §3.3.) and remains unchanged. Similarly using *ibid.* §3.2. this ratio is also independent of the choice of the transfer system  $(\sigma_0, \varkappa_{L_0/F_0}, e_l)$ .

### 3.9.3.5

We first calculate  $c_n^{L/F}(\psi_F) / c_n^{L_0/F_0}(\psi_{F_0})^r$  in a special case. We temporarily assume  $E_0 = L_0$  and  $E = L$ . In this case the supercuspidal representation  $\pi_0$  and  $\pi$  are essentially tame, saying that  $E_0/F_0$  and  $E/F$  are tamely ramified. Moreover,  $\pi_{L_0}$  (resp.  $\pi_L$ ) are  $\Gamma_0$ -regular (resp.  $\Gamma$ -regular) characters of  $G_{L_0} = L_0^\times$  (resp.  $G_L = L^\times$ ) for  $\Gamma_0 = \text{Gal}(L_0/F_0)$  (resp.  $\Gamma = \text{Gal}(L/F)$ ). Moreover  $\pi_L = \pi_{L_0} \circ N_{L/L_0}$ .

Using [BH05b](#) Corollary 2.3, we get  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}) = (-1)^{(r-1)(l-1)}$  with  $\varpi_{T_0}$  a uniformizer of  $T_0$  and  ${}_b\phi_{\theta_{L_0}}^{L/L_0}$  is trivial (noting that in the essentially tame case, our character  ${}_b\phi_{\theta_0}^{F/F_0}$  coincides with the character  $\nu$  in *loc. cit.*). If we choose the transfer system for  $L_0/F_0$  and  $L/F$  to be  $(\sigma_0, \varkappa_{L_0/F_0}, e'_l)$  and  $(\sigma_0, \varkappa_{L/F}, e'_l)$  respectively, where  $\sigma_0, \varkappa_{L_0/F_0}, \varkappa_{L/F}$  are as before, and  $e'_l$  is an element in  $L_0^\times$  such that  $\sigma_0(e'_l) = (-1)^{l-1}$ . Thus using the previous argument, we have

$$(-1)^{(r-1)(l-1)} = {}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}) / {}_b\phi_{\theta_{L_0}}^{L/L_0}(\varpi_{T_0}) = c_l^{L/F}(\psi_F) / c_l^{L_0/F_0}(\psi_{F_0})^r \quad (3.9.13)$$

### 3.9.3.6

Finally we come back to the general case. We let  $\xi_{L_0}$  be a  $\Gamma_0$ -regular character of  $L_0^\times$ , thus  $\pi'_0 = A_{L_0/F_0}(\xi_{L_0})$  is a supercuspidal representation of  $\text{Aut}_{F_0}(L_0) \cong \text{GL}_l(F_0)$ . Since the automorphic induction maintains the parabolic induction, for  $\tau_0 := \xi_{L_0} \times \dots \times \xi_{L_0}$  as a generic representation of  $G_{L_0} \cong \text{GL}_{n/l}(L_0)$ , the automorphic induction  $A_{L/F}(\tau_0)$  is exactly the parabolic induction  $\bar{\pi}_0 := \pi'_0 \times \dots \times \pi'_0$  as a generic representation of  $G_{F_0} \cong \text{GL}_n(F_0)$ . For  $\tau_0$  and  $\bar{\pi}_0$ , using [HL10](#), Proposition 3.7, we have

$$c_n^{L_0/F_0}(\psi_{F_0}) = \zeta_0 \cdot c_l^{L_0/F_0}(\psi_{F_0})^{n/l}, \quad (3.9.14)$$

where  $\zeta_0 = \varkappa_{L_0/F_0}(\zeta_0)$  and  $\zeta_0 := (-1)^{n(n/l-1)(l-1)/4} \cdot (e'_l)^{n/l} / e_l$  is an element in  $F_0^\times$ .

Correspondingly, we consider the  $L/F$ -side and we have

$$c_n^{L/F}(\psi_F) = \zeta \cdot c_l^{L/F}(\psi_F)^{n/l}, \quad (3.9.15)$$

where  $\zeta = \varkappa_{L/F}(\varsigma_0)$ . Since

$$\zeta = \varkappa_{L/F}(\varsigma_0) = \varkappa_{L_0/F_0}(\varsigma_0)^r = \zeta_0^r,$$

using (3.9.13), (3.9.14), (3.9.15), we finally have

$$c_n^{L/F}(\psi_F)/c_n^{L_0/F_0}(\psi_{F_0})^r = (-1)^{(r-1)(l-1)n/l},$$

finishing the proof of Proposition 3.9.20. We note the following interesting by-product as a corollary for ease of reference:

**Corollary 3.9.21.** *Let  $F, F_0$  be as before, let  $L_0/F_0$  be a cyclic totally ramified extension of degree  $l$  and let  $L = FL_0$ . For  $n$  divisible by  $l$ , we fix an  $F_0$ -embedding  $L_0 \hookrightarrow M_n(F_0)$  which induces an  $F$ -embedding  $L \hookrightarrow M_n(F)$  by extension of scalars, which respectively induces an embedding  $\mathrm{GL}_{n/l}(L_0) \hookrightarrow \mathrm{GL}_n(F_0)$  and an embedding  $\mathrm{GL}_{n/l}(L) \hookrightarrow \mathrm{GL}_n(F)$  by taking the centralizer, and we fix a transfer system  $(\sigma_0, \varkappa_{L_0/F_0}, e_l)$  for  $L_0/F_0$  and the transfer system  $(\sigma_0, \varkappa_{L/F}, e_l)$  for  $L/F$  with  $\varkappa_{L_0/F_0} \circ \mathbb{N}_{F/F_0} = \varkappa_{L/F}$ , then we have*

$$c_n^{L/F}(\psi_F)/c_n^{L_0/F_0}(\psi_{F_0})^r = (-1)^{(r-1)(l-1)n/l}.$$

### 3.9.4 The end of the proof

Finally we finish the proof of Theorem 3.9.1. As before we assume that  $m = 1$  and  $E_0/F_0$  is totally ramified.

We initiate by induction on  $t = [T_0 : F_0]$ . When  $t = 1$  we are in the wildly ramified case, and by definition  ${}_B\phi_{\theta_0}^{F/F_0}$  is trivial. We assume the theorem is true for  $[T_0 : F_0]$  smaller than  $t$ , and we will prove the theorem for  $[T_0 : F_0] = t > 1$ .

#### 3.9.4.1

First we assume that  $\Gamma_0$  is non-trivial. We choose a prime number  $l$  dividing  $|\Gamma_0|$ , we let  $L_0/F_0$  be the unique cyclic subsection of  $T_0$  over  $F_0$  of degree  $l$  and we let  $L = FL_0$ . We define  $\pi_{L_0}$  and  $\pi_L$  as before. Since  $[T_0 : L_0] = t/l < t$ , using the induction hypothesis the unramified character  ${}_B\phi_{\theta_{L_0}}^{L/L_0}$  satisfies

$${}_B\phi_{\theta_{L_0}}^{L/L_0}(\varpi_{T_0}^{p^s}) = (-1)^{(t/l-1)(r-1)}.$$

Combining with Proposition 3.9.20, the unramified character  ${}_B\phi_{\theta_0}^{F/F_0}$  satisfies

$${}_B\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = (-1)^{(t/l-1)(r-1)+p^s(r-1)(l-1)n/l}.$$

If  $p^s$  is odd we have

$$(-1)^{(t/l-1)(r-1)+p^s(r-1)(l-1)n/l} = (-1)^{(t/l-1)(r-1)+(r-1)(l-1)t/l} = (-1)^{(r-1)(t-1)},$$

and if  $p^s$  is even both  $t - 1$  and  $t/l - 1$  are even, thus

$$(-1)^{(t/l-1)(r-1)+p^s(r-1)(l-1)n/l} = (-1)^{(r-1)(t-1)} = 1,$$

which finishes the proof.

### 3.9.4.2

Now we assume that  $\Gamma_0$  is trivial, then  $t = [T_0 : F_0]$  is odd. We choose  $l$  to be the minimal prime divisor of  $t$ . We let  $F'_0/F_0$  be the unramified extension of degree  $l - 1$ , and we let  $F'/F'_0$  be the unramified extension of degree  $r$ , such that  $F'/F$  is unramified of degree  $l - 1$ . We further write  $T'_0 = T_0F'_0$  and  $T' = TF'$ .

$$\begin{array}{ccc} F' & \xrightarrow{r} & F'_0 \\ l-1 \downarrow & & \downarrow l-1 \\ F & \xrightarrow{r} & F_0 \end{array} \quad \begin{array}{ccc} T' & \xrightarrow{r} & T'_0 \\ l-1 \downarrow & & \downarrow l-1 \\ T & \xrightarrow{r} & T_0 \end{array}$$

We consider the base change lifts for unramified extensions  $F'/F$ ,  $F/F_0$ ,  $F'/F'_0$ ,  $F'_0/F_0$  and  $F'/F_0$ . Using the exact construction in §3.8.2, we have:

$${}_b\phi_{\theta_0}^{F'/F_0} = {}_b\phi_{\theta_{F'_0}}^{F'/F'_0} \cdot ({}_b\phi_{\theta_0}^{F'_0/F_0} \circ N_{T'/T'_0}) = {}_b\phi_{\theta_b}^{F'/F} \cdot ({}_b\phi_{\theta_0}^{F/F_0} \circ N_{T'/T}),$$

where  $\theta_{F'_0}$  denotes the simple character constructed in §3.6.1 ( $\theta_b$  in *loc. cit.*) with  $F$  replaced by  $F'_0$ . Since  $l$  is minimal,  $l - 1$  is relatively prime to  $t$ . Thus using Proposition 3.9.9 we have  ${}_b\phi_{\theta_b}^{F'/F}(\varpi_{T_0}^{p^s}) = {}_b\phi_{\theta_0}^{F'_0/F_0}(\varpi_{T_0}^{p^s}) = 1$ , and we obtain

$${}_b\phi_{\theta_{F'_0}}^{F'/F'_0}(\varpi_{T_0}^{p^s}) = {}_b\phi_{\theta_0}^{F/F_0}(N_{T'/T}(\varpi_{T_0}^{p^s})). \quad (3.9.16)$$

Since  $l$  divides  $q^{l-1} - 1$ , which also means that  $l$  divides  $|\Gamma'_0|$  for  $\Gamma'_0 = \text{Aut}(T'_0/F'_0)$ , we may use the above result to show that  ${}_b\phi_{\theta_{F'_0}}^{F'/F'_0}(\varpi_{T_0}^{p^s}) = (-1)^{(t-1)(r-1)n/t} = 1$  (since  $t - 1$  is even). By (3.9.16) we have  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{(l-1)p^s}) = 1$ .

Since  ${}_b\phi_{\theta_0}^{F/F_0}$  is an unramified character of order dividing  $n = p^s t$ , where  $t$  is relatively prime to  $l - 1$ , we finally have  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = 1$  which finishes the proof.

**Remark 3.9.22.** *In general our method here is not enough to give a full characterization of  ${}_b\phi_{\theta_0}^{F/F_0}$ , or equivalently to calculate  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0})$ . Indeed in the above argument, it is possible that  $l - 1$  is divisible by  $p^{s'}$  for a certain  $s'$ . Thus even if we have proved Conjecture 3.9.8 and have got the equation*

$${}_b\phi_{\theta_{F'_0}}^{F'/F'_0} = {}_b\phi_{\theta_0}^{F/F_0} \circ N_{T'/T}$$

in place of (3.9.16), it only provides information of  ${}_b\phi_{\theta_0}^{F/F_0}$  evaluating at  $\varpi_{T_0}^{p^{s'}}$  instead of  $\varpi_{T_0}$  itself.

## 3.10 Contribution to the calculation of the character $\mu_{\theta_0}^{T_0/F_0}$ .

In this section as before we let  $\theta_0$  be a simple character of degree  $d$  with respect to a simple stratum  $[\mathfrak{a}_0, \beta]$  in  $M_n(F_0)$ , let  $\theta_{T_0}$  be the interior  $T_0/F_0$ -lift of  $\theta_0$ . We further assume that  $n = d$  and  $E_0/F_0$  is totally ramified. Our aim is to update the information of the character  $\mu_{\theta_0}^{T_0/F_0}$  given by Proposition 3.7.10, which is closely related to the explicit Langlands correspondence in the totally ramified case<sup>6</sup>, however not well understood from the  $\text{GL}_n$  side, see [BH14b], §8.9. for a brief summary.

<sup>6</sup>Reducing from the general case to the totally ramified case is fully understood, which is Theorem 3.7.6

To do that, we choose  $\pi_0 = \text{ind}_{\mathbf{J}_{F_0}}^{G_{F_0}} \kappa_0$  as a supercuspidal representation of  $G_{F_0}$  and we write  $\pi_{T_0} = \text{ind}_{\mathbf{J}_{T_0}}^{G_{T_0}} (\kappa_{T_0} \cdot (\mu_{\theta_0}^{T_0/F_0})^{-1})$  which is the supercuspidal representation of  $G_{T_0} \cong \text{GL}_{p^s}(T_0)$  satisfying  $A_{T_0/F_0}(\pi_{T_0}) = \pi_0$ , where  $\kappa_0$  is a full Heisenberg representation of  $\theta_0$  satisfying (3.6.1), and  $\kappa_{T_0}$  is a full Heisenberg representation of  $\theta_{T_0}$ , which is uniquely determined by Proposition 3.7.8

Later we will focus on the case where  $t$  is odd. Indeed, in general we choose a sequence of field extensions

$$F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k \subsetneq T_0$$

such that  $F_i/F_{i-1}$  is cyclic of prime degree, and  $\text{Aut}(T_0/F_k)$  is trivial. For  $\theta_0$  the simple character contained in  $\pi_0$  as before, let  $\theta_{F_i}$  be the interior  $F_i/F_0$ -lift of  $\theta_0$ . Thus using Proposition 3.7.12, we get

$$\mu_{\theta_0}^{T_0/F_0} = \mu_{\theta_{F_k}}^{T_0/F_k} \cdot \prod_{i=1}^k \chi_{\theta_{F_{i-1}}}^{F_i/F_{i-1}}.$$

Those characters  $\chi_{\theta_{F_{i-1}}}^{F_i/F_{i-1}}$  are completely characterized by the formula given in *loc. cit.*, albeit not always having a simple expression, which can be regarded as “known”. As a result, essentially we only need to study the character  $\mu_{\theta_{F_k}}^{T_0/F_k}$ , whence  $[T_0 : F_k]$  is odd.

### 3.10.1 Evaluating at $\varpi_{T_0}^{p^s}$

Comparing the central character of  $\pi_0$  and  $\pi_{T_0}$ , we have

$$\begin{aligned} 1 = \omega_{\pi_0} &= \det(\text{LLC}_{F_0}^{-1}(\pi_0)) = \det(\text{Ind}_{T_0/F_0}(\text{LLC}_{T_0}^{-1}(\pi_{T_0}))) \\ &= (d_{T_0/F_0})^t \cdot \det(\text{LLC}_{T_0}^{-1}(\pi_{T_0})) \\ &= (d_{T_0/F_0})^t \cdot \omega_{\pi_{T_0}}|_{F_0^\times} = (d_{T_0/F_0})^t \cdot ((\mu_{\theta_0}^{T_0/F_0})^{p^s}|_{F_0^\times})^{-1}, \end{aligned}$$

or to sum up

$$(\mu_{\theta_0}^{T_0/F_0})^{p^s}|_{F_0^\times} = (d_{T_0/F_0})^t,$$

where  $d_{T_0/F_0}$  denotes the character  $\det(\text{Ind}_{T_0/F_0} \mathbf{1}_{\mathcal{W}_{T_0}})$  of  $F_0^\times$ , and from the first line to the second line we use [BH06], Proposition 29.2. If  $t$  is odd,  $d_{T_0/F_0}$  is exactly the unramified quadratic character with the value on a uniformizer equalling to the Jacobi symbol  $(\frac{q}{t})$ . As a result,

**Proposition 3.10.1.** *When  $t$  is odd,  $\mu_{\theta_0}^{T_0/F_0}$  is an unramified character of  $F_0^\times$  order  $2n$ .*

Moreover we have the following proposition:

**Proposition 3.10.2.** *When  $t$  is odd, we have  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}^{p^s}) = (\frac{q}{t})$ .*

*Proof.* We prove this proposition by induction on  $t$ . We first consider the case where  $\Gamma_0 = \text{Aut}(T_0/F_0)$  is non-trivial. We choose  $l$  to be a prime divisor of  $|\Gamma_0|$  and we let  $L_0/F_0$  be the cyclic subextension of  $T_0/F_0$  of degree  $l$ . As before we have  $\mu_{\theta_0}^{T_0/F_0} = \chi_{\theta_0}^{L_0/F_0} \cdot \mu_{\theta_{L_0}}^{T_0/L_0}$  and using Corollary 3.7.13 we know that  $\chi_{\theta_0}^{L_0/F_0}(\varpi_{T_0}^{p^s}) = 1$ . Thus using the induction hypothesis for  $T_0/L_0$ , we get  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}^{p^s}) = \mu_{\theta_{M_0}}^{T_0/L_0}(\varpi_{T_0}^{p^s}) = (\frac{q}{t/l}) = (\frac{q}{t})$ , since  $l|q-1$  and thus  $(\frac{q}{l}) = 1$ .

Now we assume that  $\Gamma_0$  is trivial. We let  $l$  be the minimal prime divisor of  $t$ . We choose  $F/F_0$  to be an unramified extension of degree  $l-1$  and we write  $T = T_0F$  which is a totally tamely ramified extension of degree  $t$  over  $F$ . We write  $\pi_T = \text{BC}_{T/T_0}(\pi_{T_0})$  as a supercuspidal representation of  $G_T$  and  $\pi = \text{BC}_{F/F_0}(\pi_0)$  as a supercuspidal representation of  $G_F$ . Thus using Proposition 3.4.9(3) we have

$A_{T/F}(\pi_T) = \pi$ . Using Theorem 3.6.2 and the exact definition of the corresponding character, we get  ${}_b\phi_{\theta_0}^{F/F_0} = (\mu_{\theta_b}^{T/F})^{-1} \cdot (\mu_{\theta_0}^{T_0/F_0} \circ N_{T/T_0})$ . Since  $(l-1, t) = 1$  and  $t$  is odd, by Proposition 3.9.9 we have  ${}_b\phi_{\theta_0}^{F/F_0}(\varpi_{T_0}^{p^s}) = 1$ , thus  $\mu_{\theta_b}^{T/F}(\varpi_{T_0}^{p^s}) = \mu_{\theta_0}^{T_0/F_0}(N_{T/T_0}(\varpi_{T_0}^{p^s}))$ . Since  $l|q^{l-1} - 1$  which also means that  $l||\Gamma|$  for  $\Gamma = \text{Aut}(T/F)$ , using the previous case we know that  $\mu_{\theta_b}^{T/F}(\varpi_{T_0}^{p^s}) = (\frac{q^{l-1}}{t})$ , which also means that  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}^{(l-1)p^s}) = (\frac{q}{t})^{l-1}$ . Since  $(\mu_{\theta_0}^{T_0/F_0})^{p^s}$  is a character of order  $2t$  and  $l-1$  is relatively prime to  $t$ , we must have  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}^{p^s})^2 = (\frac{q}{t})^2$ . Finally since  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}^{p^s})^t = \mu_{\theta_0}^{T_0/F_0}(\varpi_{F_0}^{p^s}) = (\frac{q}{t}) = (\frac{q}{t})^t$ , we have  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}^{p^s}) = (\frac{q}{t})$ .  $\square$

### 3.10.2 Epsilon factors

For  $\pi'_0$  (resp.  $\pi'_{T_0}$ ) a supercuspidal representation of  $G_{F_0}$  (resp.  $G_{T_0}$ ), we denote by  $\varepsilon(\pi'_0) := \varepsilon(\pi'_0, 1/2, \psi_{F_0})$  (resp.  $\varepsilon(\pi'_{T_0}) := \varepsilon(\pi'_{T_0}, 1/2, \psi_{T_0})$ ) the corresponding epsilon factors evaluating at  $1/2$ .

**Lemma 3.10.3.** *We have*

$$\varepsilon(\pi_0) = \varepsilon(\pi_{T_0}) \cdot \lambda_{T_0/F_0}^t,$$

where  $\lambda_{T_0/F_0} := \lambda_{T_0/F_0}(\psi_{F_0})$  denotes the Langlands constant.

*Proof.* It follows from [BH06], (30.4.2) and the fact that the local Langlands correspondence maintains epsilon factors.  $\square$

Now we define  $\pi'_{T_0} = \text{ind}_{\mathbf{J}_{T_0}}^{G_{T_0}}(\kappa_{T_0})$ , then

**Lemma 3.10.4** ([BH99], Theorem 1.4.).  $\varepsilon(\pi'_{T_0}) \cdot \mu_{\theta_0}^{T_0/F_0}(N_{E_0/T_0}(\beta)) = \varepsilon(\pi_{T_0})$ .

We write  $m_\beta = -v_{E_0}(\beta)$ , which by definition is a positive integer. Thus  $v_{T_0}(N_{E_0/T_0}(\beta)) = -m_\beta$ . Combining with the above two lemmas, we get

$$\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0})^{m_\beta} = \varepsilon(\pi'_{T_0}) \cdot \lambda_{T_0/F_0}^t / \varepsilon(\pi_0) = (\frac{q}{t}) \cdot \varepsilon(\pi'_{T_0}) / \varepsilon(\pi_0), \quad (3.10.1)$$

where we use the fact that  $t$  is odd and [BH05b], Lemma 1.5.(2) for the last equation.

### 3.10.3 A more detailed discussion for supercuspidal representations of Carayol type

In this subsection we assume that  $t$  is odd and  $m_\beta$  is relatively prime to  $p$  and we call the supercuspidal representation  $\pi_0$  with one corresponding simple stratum satisfying the latter condition of *Carayol type*. In this case Proposition 3.10.2 and (3.10.1) together give a full characterization of  $\mu_{\theta_0}^{T_0/F_0}$ . However the right hand side of equation (3.10.1) is quite vague, thus we will provide more details.

We denote by  $\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)$  (resp.  $\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)$ ) the group of fixed points of  $\beta$  (with  $\beta$  acting by conjugation) in  $\mathcal{V}_{F_0} := J^1(\mathfrak{a}_0, \beta)/H^1(\mathfrak{a}_0, \beta)$  (resp.  $\mathcal{V}_{T_0} := J^1(\mathfrak{a}_{T_0}, \beta)/H^1(\mathfrak{a}_{T_0}, \beta)$ ) and we define  $\psi_{A_{F_0}} := \psi_{F_0} \circ \text{Tr}_{A_{F_0}}$  (resp.  $\psi_{A_{T_0}} := \psi_{T_0} \circ \text{Tr}_{A_{T_0}}$ ).

**Proposition 3.10.5.** *We assume that  $E_0 \neq T_0$ .*

- When  $m_\beta$  is odd, we have

$$\varepsilon(\pi_0) = \text{tr}(\kappa_0^\vee)(\beta) \cdot \psi_{A_{F_0}}(\beta) / |\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)|^{1/2} \quad \text{and} \quad \varepsilon(\pi'_{T_0}) = \text{tr}(\kappa_{T_0}^\vee)(\beta) \cdot \psi_{A_{T_0}}(\beta) / |\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)|^{1/2},$$

- When  $m_\beta$  is even, there exist  $h_0 \in 1 + \mathfrak{p}_{\mathfrak{a}_0}^{m_\beta/2}$  and  $h'_0 \in 1 + \mathfrak{p}_{\mathfrak{a}_{T_0}}^{m_\beta/2}$ , such that

$$\mathrm{tr}(\kappa_0^\vee)(\beta h_0) \neq 0 \quad \text{and} \quad \mathrm{tr}(\kappa_{T_0}^\vee)(\beta h'_0) \neq 0,$$

and for any such  $h_0, h'_0$ , we have

$$\varepsilon(\pi_0) = \mathrm{tr}(\kappa_0^\vee)(\beta h_0) \cdot \psi_{A_{F_0}}(\beta h_0) / |\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)|^{1/2}$$

and

$$\varepsilon(\pi'_{T_0}) = \mathrm{tr}(\kappa_{T_0}^\vee)(\beta h'_0) \cdot \psi_{A_{T_0}}(\beta h'_0) / |\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)|^{1/2}.$$

*Proof.* The case where  $m_\beta$  is odd follows directly from [BH99], §6.1, Lemma 2 and §6.3, Proposition 1. The case where  $m_\beta$  is even follows from *ibid.* §6.1, Lemma 2 and §6.3, Proposition 2, and we only need to prove that the condition of §6.3, Proposition 2 is satisfied. For  $[\mathfrak{a}_0, m_\beta, m_\beta - 1, \alpha]$  a simple stratum equivalent to  $[\mathfrak{a}_0, m_\beta, m_\beta - 1, \beta]$ , we need to prove that  $F_0[\alpha]/F_0$  is not tamely ramified. By [BK93], Theorem 2.4.1, we know that  $e(F_0[\alpha]/F_0)$  divides  $e(E_0/F_0)$  and  $f(F_0[\alpha]/F_0) = 1$ , and moreover  $v_{F_0[\alpha]}(\alpha)/e(F_0[\alpha]/F_0) = v_{E_0}(\beta)/e(E_0/F_0) = -m_\beta/e(E_0/F_0)$ , thus  $e(F_0[\alpha]/F_0)$  is divisible by  $p^s$ , otherwise  $p$  divides  $m_\beta$ , contradictory! Thus  $F_0[\alpha]/F_0$  is not tamely ramified since  $E_0 \neq T_0$  and  $s \geq 1$ . For  $[\mathfrak{a}_{T_0}, m_\beta, m_\beta - 1, \alpha']$  a simple stratum equivalent to  $[\mathfrak{a}_{T_0}, m_\beta, m_\beta - 1, \beta]$ , similarly we prove that  $T_0[\alpha']/T_0$  is not tamely ramified, verifying the condition of §6.3, Proposition 2.  $\square$

**Corollary 3.10.6.** *Let  $k$  be an integer such that  $kp^s - m_\beta$  is relatively prime to  $t$ .*

- When  $m_\beta$  is odd, then

$$\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0})^{m_\beta} = \left(\frac{q}{t}\right) \cdot \frac{|\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)|^{1/2}}{|\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)|^{1/2}} \cdot \frac{\mathrm{tr}(\kappa_0^\vee)(\beta \varpi_{T_0}^k)}{\mathrm{tr}(\kappa_0^\vee)(\beta)}.$$

- When  $m_\beta$  is even, then for  $h_0$  and  $h'_0$  as in the proposition,

$$\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0})^{m_\beta} = \left(\frac{q}{t}\right) \cdot \frac{|\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)|^{1/2}}{|\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)|^{1/2}} \cdot \frac{\mathrm{tr}(\kappa_0^\vee)(\beta h'_0 \varpi_{T_0}^k)}{\mathrm{tr}(\kappa_0^\vee)(\beta h_0)} \cdot \frac{\psi_{A_{F_0}}(\beta h'_0)}{\psi_{A_{F_0}}(\beta h_0)}.$$

*Proof.* We only prove the second case, whence the first case is similar and simpler. By construction we know that  $\psi_{A_{T_0}}(\beta h'_0) = \psi_{A_{F_0}}(\beta h'_0)$ . Since  $\varpi_{T_0} \in C_{T_0}(\varpi_{F_0}) \subset \mathrm{Ker}(\kappa_{T_0}^\vee)$ , we have  $\kappa_{T_0}^\vee(\beta h'_0) = \kappa_{T_0}^\vee(\beta h'_0 \varpi_{T_0}^k)$ . Since  $v_{T_0}(\det_{T_0}(\beta h'_0 \varpi_{T_0}^k)) = kp^s - m_\beta$  is relatively prime to  $t$ , using Proposition 3.7.8 with  $\kappa_0$  and  $\kappa_{T_0}$  replaced by  $\kappa_0^\vee$  and  $\kappa_{T_0}^\vee$ , we get  $\mathrm{tr}(\kappa_{T_0}^\vee)(\beta h'_0) = \mathrm{tr}(\kappa_0^\vee)(\beta h'_0 \varpi_{T_0}^k)$ . Thus the proof follows from (3.10.1) and the above proposition.  $\square$

Unfortunately the author doesn't know how to proceed to simplify the result in the corollary. For example, the right hand side of the two equations should be a  $2n$ -th root of unity, which seems not clear from the expression itself. Instead we consider the following special case to end our discussion.

**Corollary 3.10.7.** *If  $m_\beta$  is relatively prime to  $n$ , then  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0}) = \left(\frac{q}{t}\right)$ .*

*Proof.* We may choose  $k = 0$ , and  $h_0 = h'_0$  if  $m_\beta$  is even, then we get

$$\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0})^{m_\beta} = \left(\frac{q}{t}\right) \cdot \frac{|\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)|^{1/2}}{|\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)|^{1/2}}.$$

Since  $\mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0})^{m_\beta}$  is a root of unity, we must have

$$|\mathcal{Z}_{\mathcal{V}_{F_0}}(\beta)|^{1/2} = |\mathcal{Z}_{\mathcal{V}_{T_0}}(\beta)|^{1/2} \quad \text{and} \quad \mu_{\theta_0}^{T_0/F_0}(\varpi_{T_0})^{m_\beta} = \left(\frac{q}{t}\right).$$

Combining with Proposition [3.10.2](#) we finish the proof. □





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**Titre :** Représentations supercuspidales de  $GL(n)$  sur un corps local non archimédien : distinction par un sous-groupe unitaire ou orthogonal, changement de base et induction automorphe.

**Mots clés :** groupes réductifs  $p$ -adiques, théorie des représentations, correspondance de Langlands locale, functorialité de Langlands locale, problème de la distinction, représentations modulaires, théorie des types simples

**Résumé :** Dans cette thèse, nous considérons quelques exemples concrets de la relation entre la correspondance de Langlands locale, sa functorialité et le problème de la distinction. Soit  $F/F_0$  une extension cyclique finie de corps localement compacts non-archimédiens de caractéristique résiduelle  $p$  et soit  $R$  un corps algébriquement clos de caractéristique  $l \neq p$ . Dans la première partie, nous supposons que  $F/F_0$  est quadratique et  $p \neq 2$ , et nous étudions les représentations irréductibles de  $GL_n(F)$  sur  $R$  distinguées par un sous-groupe unitaire. Nous résolvons complètement le problème pour les représentations supercuspidales et obtenons des résultats partiels pour les représentations génériques. Nous définissons également une version  $l$ -modulaire du changement de base cyclique. Dans la deuxième partie, nous supposons  $F = F_0$  et  $p \neq 2$ , et nous caractérisons complètement les représentations supercuspidales complexes de  $GL_n(F)$  distinguées par un sous-groupe orthogonal. Dans la dernière partie pour  $F/F_0$  modérément ramifiée, nous étudions le changement de base et l'induction automorphe pour des représentations supercuspidales complexes via la théorie des types simples.

**Title :** Supercuspidal representations of  $GL(n)$  over a non-archimedean local field : distinction by a unitary or orthogonal subgroup, base change and automorphic induction

**Keywords :**  $p$ -adic reductive groups, representation theory, local Langlands correspondence, local Langlands functoriality, problem of distinction, modular representations, simple type theory

**Abstract :** In this thesis, we consider several concrete examples of the relation among the local Langlands correspondence, its functoriality and the problem of distinction. Let  $F/F_0$  be a finite cyclic extension of non-archimedean locally compact fields of residue characteristic  $p$  and let  $R$  be an algebraically closed field of characteristic  $l \neq p$ . In the first part, we assume  $F/F_0$  to be quadratic and  $p \neq 2$ , and we study the irreducible representations of  $GL_n(F)$  over  $R$  distinguished by a unitary subgroup. We completely solve the problem for supercuspidal representations and get partial results for generic representations. Meanwhile we also define an  $l$ -modular version of the cyclic base change lift. In the second part, we assume  $F = F_0$  and  $p \neq 2$ , and we fully characterize the complex supercuspidal representations of  $GL_n(F)$  distinguished by an orthogonal subgroup. In the final part for  $F/F_0$  tamely ramified, we study the base change lift and the automorphic induction for complex supercuspidal representations via the simple type theory.