



Modelling and stability analysis of flexible robots : a distributed parameter port-Hamiltonian approach

Andrea Mattioni

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Andrea Mattioni. Modelling and stability analysis of flexible robots : a distributed parameter port-Hamiltonian approach. Automatic. Université Bourgogne Franche-Comté, 2021. English. NNT : 2021UBFCD030 . tel-03454705

HAL Id: tel-03454705

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THÈSE DE DOCTORAT DE L'ÉTABLISSEMENT
UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ
PRÉPARÉ À L'UNIVERSITÉ DE BOURGOGNE
FRANCHE-COMTÉ ET À L'UNIVERSITÉ DE
FRANCHE-COMTÉ

ÉCOLE DOCTORALE N° 37
SCIENCES POUR L'INGÉNIEUR ET MICROTECHNIQUES

Doctorat de Automatique (UBFC)

PAR

Andrea MATTIONI

**Modélisation et analyse de stabilité des
robots flexibles: une approche
Hamiltonienne à portes à paramètres
distribués**

Thèse présentée et soutenue à Besançon, le 23 Avril 2021

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DOCTORAL THESIS FROM THE UNIVERSITY OF
BOURGOGNE FRANCHE-COMTÉ PREPARED AT THE
UNIVERSITY OF FRANCHE-COMTÉ AND AT THE
UNIVERSITY OF BOURGOGNE FRANCHE-COMTÉ

DOCTORAL SCHOOL N° 37
ENGINEERING SCIENCES AND MICRO-TECHNOLOGIES

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Notation

SYMBOLS

Symbol	Description	Page
\mathbb{R}	Real numbers	10
\mathbb{N}	Natural Numbers: $\{1, 2, 3, \dots\}$	10
\mathbb{N}_0	Natural numbers with 0: $\mathbb{N} \cup \{0\}$	10
$\mathbb{R}^{n \times m}$	Set of n by m matrices with entries in \mathbb{R}	10
$\sigma(M)$	Eigenvalues set of the matrix M	10
$\frac{d}{d\xi}$ or $\frac{\partial}{\partial \xi}$ or f'	Spatial derivative of the function f	17
$\frac{d}{dt}$ or $\frac{\partial}{\partial t}$ or \dot{f}	Time derivative of the function f	12
f_+	Dini time derivative of a function f	14
df	Fréchet derivative of a function f	13
$d_x f$	Fréchet derivative of a function f with respect to the variable x	29
$C([a, b], \mathbb{R}^n)$	Space of continuous functions	10
$C^p([a, b], \mathbb{R}^n)$	Space of p times derivable functions	10
$L_p([a, b], \mathbb{R}^n)$	Vector space of p integrable functions	10
$H^p([a, b], \mathbb{R}^n)$	Sobolev space of order p	10
$\mathcal{L}(X, Y)$	Space of bounded liner operators from X to Y	11
$\mathbf{D}(A)$	Domain of the operator A	11
$\text{ran}(A)$	Range of the operator A	11
$\rho(A)$	Resolvent set of the operator A	11
$R(\lambda, A)$	Resolvent operator	11
A^*	Adjoint operator	11
$\langle \cdot, \cdot \rangle$	Inner product in the space X	10
$\ \cdot\ _X$	Norm in the space X	10
$\sup(M)$	Supremum of a set S	12
$\inf(M)$	Infinum of a set S	12
\limsup	Limit superior	14

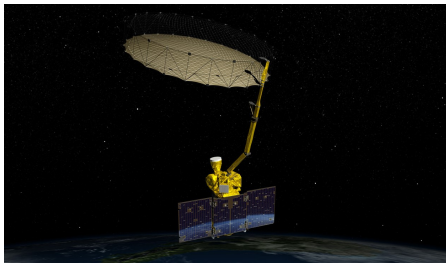
ABBREVIATIONS

Abbreviation	Description
pH	port-Hamiltonian
dpH	distributed port-Hamiltonian
m-pH	mixed port-Hamiltonian
FAPh	Functional analytic port-Hamiltonian
1-D	One dimensional
ODE	Ordinary Differential Equations
PDE	Partial Differential Equations
m-PDE-ODE	mixed PDE and ODE
C_0 -semigroup	Strongly continuous semigroup

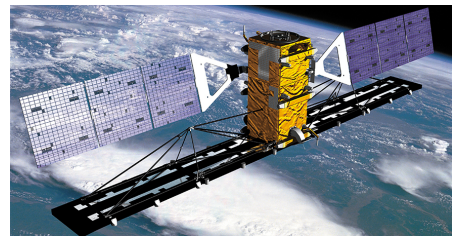
General Introduction

Since the late 60', the problem of modelling and control of flexible structures has received a lot of attention [1]. At that time, the research was motivated by the employment of large and light structures especially for space engineering applications. The large size and reduced mass density materials have low natural frequencies associated to them[2]. Low natural frequency modes may interfere with the attitude manoeuvring control's bandwidth, degrading its dynamic behaviour. Solar panels as well as big antennas and in general big appendices (see for example: the SMAP mission [3], the RADARSAT-2 mission [4] or the SMOS mission [5] in Figure 1) can be a source of flexible vibrations for a satellite during the attitude manoeuvre [6].

Nowadays, thanks to the impressive technological evolution of the last 50 years, flexible structures have found a large range of possible applications, for example in industrial robotics. The performances of robot manipulators can be improved by to increasing the payload-to-arms weight ratio diminishing the overall robot's weight. In this manner, it is possible to get a faster motion while keeping the same actuator. However, a faster motion with lightweight arms leads to the appearance of flexible deformations, that have to be attenuated if we do not want to decrease the performances [7]. Moreover, real robots frequently work on applications where they enter in contact with other objects or with the external environment. This is the case for flexible space robots



(a) SMAP mission satellite.



(b) RADARSAT-2 mission satellite.



(c) RADARSAT-2 mission satellite.

Figure 1: Space mission satellites pictures.

for debris removal [8], or minimally invasive surgery devices that get in contact with the patient's body [9]. The transition between an unconstrained to a constrained state excites the flexible proper modes of vibration of the structure and consequently, this transition phase is crucial for the system's stability. Along with flexible robots, wind turbine towers can also encounter flexible deformation problems. Wind turbines are large and heavy structures located in offshore or high altitude areas. These locations offer stronger winds for the creation of electric energy, but at the same time can cause the mechanical vibration of the tower, degrading this way the overall energy transformation performances [10, 11]. Besides, structural flexibility can also be encountered in microscale applications including micro-electromechanical systems, micro-assembly, biological systems etc. In these applications, compliant mechanisms can be preferred to rigid ones because of their miniaturization capability and often simpler fabrication and integration [12]. Possible examples of these class of systems are microgripper [13] or parallel continuous robots for precise positioning [14]. To fully exploit the potential advantages offered by lightweight flexible structures, one must derive a suitable control action able to effectively suppress their vibrational behaviour.

Strongly motivated by the previously mentioned studies, this thesis deals with the modelling and control of mechanisms that contain flexible deformable parts. Before introducing the core subject of the thesis, we provide some general notions and terminology about the tool that we use to study the modelling and control problem of flexible structures: *dynamical equations* (or *evolution equations*). Similarly to scale models [15], the study of dynamical equations is an engineering tool that allows predicting the behaviour of a specific process, without having it at our disposal. The stability analysis of a process modelled by dynamical equations studies its solutions' asymptotic behaviour and therefore if they evolve according to some desired specifications. Dynamical equations are differential equations that can be used to describe the time evolution of the process. We consider the solution $x = x(t)$ of the following general class of differential equations

$$\frac{dx(t)}{dt} = F(x(t)) \quad (1)$$

with initial data

$$x(0) = x_0. \quad (2)$$

The variable x belongs to a Hilbert space H , while F is a mapping from H to itself and is usually referred to as a *vector field*.

In the case where the variable x belongs to a n -dimensional Euclidean space $X = \mathbb{R}^n$, equation (1) consists of a set of Ordinary Differential Equations (ODE). In the case where X is a space of functions mapping from an initial domain Ω to a Euclidean space \mathbb{R}^n , equation (1) consists of a set of Partial Differential Equations (PDE). In the case where X consists of a union between a Euclidean space and a functional space, then (1) is referred to as a mixed ODE-PDE (m-ODE-PDE). Since (1) is an autonomous system, it is referred to as *unforced state equation*.

In case we add to equation (1) an input $u(t)$ and an output $y(t)$, the system can interact

with the external environment, and we call it *open systems*

$$\begin{aligned}\frac{dx(t)}{dt} &= F(x(t), u(t)) \\ y(t) &= h(x(t)).\end{aligned}\tag{3}$$

The input and output $u(t)$ and $y(t)$ belong to the input and output Hilbert spaces U and Y , respectively. Now the *vector field* F maps from $H \times U$ to X , and the function h maps from X to Y and is referred to as *output map*. The *control design* consists in finding the input $u(t)$ such that the state $x(t)$ behaves in a desired manner. The input $u(t)$ can be given as a direct function of time (*feed-forward*), as a function of the state (*state feed-back*) or as a function of the output (*output feed-back*). Once the control law is applied, the closed-loop system takes the form of equation (1), and we are interested in the behaviour of the *closed-loop* system state $x(t)$ when $t \rightarrow \infty$. Apart from linear-time invariant ODE and few other exceptions, normally we do not have at our disposal the explicit solution $x(t)$. Therefore, system theory and the study of evolution equations give us some tools to analyse the so-called *asymptotic behaviour* of the system, without explicitly knowing the solution $x(t)$.

Throughout the rest of the thesis, we try to answer the following *general questions*:

1. How can we obtain a dynamical model of a mechanism containing flexible parts?
2. How can we develop a control law able to move the mechanism with the desired behaviour?
3. How can we simulate the obtained model of the system and the closed-loop behaviour with the proposed control laws?
4. What is the asymptotic behaviour of a flexible mechanism when it enters in collision with an external object?

A possible way of proceeding is finding different methodologies that are optimal to answer all these questions separately. A certain methodology can be perfect to answer a specific question, but at the same time inappropriate for another one. However, in this thesis, we do not want to deal with these questions in a case-by-case manner. To answer these questions in a unified way, we decided to make use of the port-Hamiltonian (pH) framework. The pH framework is indeed dedicated to the modelling, analysis, design and control of multi-physical dynamical systems. A comprehensive guide to pH systems and their applications can be found in [16]. From an appropriate choice of the state variables together with the physical laws, a pH system provides a structure to the vector field and the output mapping of the general open systems introduced in equation (3), it gives advantages in the analysis while maintaining a general form that includes a very wide range of physical processes [17]. In the last two decades, the pH system theory has been extended from lumped parameter (ODE dynamical equations) systems to distributed parameter (PDE dynamical equations) systems, starting with the theory developed in [18]. At the current stage, there exists three different distributed pH (dpH)

formulations: the Stokes-Dirac structure formulation [18, 19], the jet-bundle formulation [20] and the *functional analytic* pH (FApH) formulation. Between the different distributed pH (dpH) formulations we decided to use the FApH approach that has its roots in the seminal work [21], that has been extended in the PhD thesis [22, 23] and in the monograph [24]. In these works, the differential equation (1) is restricted to be a dpH linear equation in the abstract differential form

$$\frac{\partial x}{\partial t}(\xi, t) = \sum_{k=0}^N P_k \frac{\partial^k}{\partial \xi^k} (\mathcal{H}x)(\xi, t) \quad (4)$$

with suitable boundary conditions. When the input $u(t)$ and output $y(t)$ are selected among the boundary variables (and therefore they are acting at the boundaries of the considered spatial domain), the system is called a *boundary control system*, and it is a subclass of the general class of open systems (3). The energy of the systems described by the dpH equation (4) is defined as $E = \frac{1}{2} \langle x, \mathcal{H}x \rangle$ and the norm of the considered state space X corresponds to the energy itself. The structure of the system together with the considered norm allows concluding about well-posedness (existence, uniqueness and smoothness of solutions) with a similar procedures used to show that the system is passive with respect to its energy. Well-posedness and stabilization problems for dpH systems have been studied in the case of static feedback [25], when they are connected with a dynamic linear system [26, 27] or with some classes of dynamic non-linear systems [28].

The first task that we are asked to accomplish when we want to analyse a physical process, is associate to this process an evolution equation in the form of (3). This procedure is normally referred to as *modelling*. Moving mechanisms usually contain flexible and rigid parts, hence both distributed and lumped parameter phenomena are present. The most common way of deriving equations of motion is through the use of the Hamilton principle: roughly speaking, the dynamic equations are found computing the partial derivatives of the Lagrangian (the difference between the kinetic and potential energy). Since distributed phenomena are present in the considered system, the Lagrangian depends on functions other than normal lumped states. A possibility is to find an approximated version of the Lagrangian, where the distributed variables are substituted by a set of lumped variables: the obtained model will be composed of only ODE, [29, 30]. An alternative is to maintain the distributed parameter nature of the system and obtain an m-ODE-PDE model [31], with the control input in the set of ODE. Different research articles considered the pH formulation of flexible mechanisms; some of them being simulation-oriented [32, 33], while some others control-oriented [34, 35]. Because of its passivity, the pH formulation of physical models helps in finding a control law that steers the system's solution to the desired equilibrium point.

The fundamental theory used to study systems' asymptotic stability has been established by the mathematician Alexandr M. Lyapunov in his famous doctoral thesis of 1892 *The general problem of stability of motion*. The theory has been applied to control theory only in the 1930s for nonlinear finite dimensional systems [36], but nowadays it became indispensable also for infinite dimensional systems as well as for discrete time systems. Roughly speaking, Lyapunov stability theory is based on finding a positive definite functional that is non-increasing along the system solutions. This theory is strictly

related to passivity concepts; in fact the energy functional is frequently selected as Lyapunov function candidate to prove the stability of a system. Besides, the FApH approach uses passivity concepts as a starting point for studying the well-posedness of systems. Hence, the FApH approach allows to create a bridge between abstract concepts like the existence of solutions and Lyapunov's stability of physical systems. A lot of research papers deal with the stability analysis of flexible mechanisms modelled by m-ODE-PDE systems on a case-by-case basis. To mention a few, in [37] is studied the stabilisation problem of a satellite with flexible appendices; in [11] is proposed a control strategy to stabilise the wind turbine tower's vibrations, while in [38] different control laws are proposed to stabilise the torsional vibrations of a drilling system. Since the subject is rather technical, a lot of works in the literature focus on the study of simple m-ODE-PDE mechanisms without a direct industrial application. In fact, several research articles focus on the stabilization of the Euler-Bernoulli beam [39, 40, 41, 42, 43], of the Timoshenko beam [44, 45, 46] or of the wave equation [47, 48, 49, 50] with dynamic boundary conditions. The idea underlying the use of the FApH approach is to understand and unify the case-to-case results in a more abstract level, by using the intrinsic properties of physical systems. For instance, in [51] the control by interconnection strategy has been extended to m-PDE-ODE pH (m-pH) systems, while in [28] some stability conditions are given for a pH system interconnected with a set of nonlinear ODE. In such way, a given control strategy can be applied to a large range of practical applications without the requirement of testing closed-loop stability at every change of the operating conditions. A favourable condition for controlling m-pH systems is to have direct access to the dpH boundary inputs. In this way, dissipation can be directly added in the boundary conditions such to obtain exponential stability of the closed-loop system [27, 28]. However, the considered class of m-PDE-ODE models of flexible mechanisms have control inputs only in the set of ODE. We call this issue *integral obstacle*.

Numerical simulations of closed-loop systems are essential when the stability result does not provide any information about the rate of convergence to the equilibrium point. Simulations of flexible mechanisms described by m-PDE-ODE dynamic equations require the PDE's spatial discretization. The discretization procedure consists of finding a set of ODE approximating the set of PDE. Among all the existing standard methods, there is a particular class that allows to approximate a dpH with a finite dimensional pH system. An exhaustive guide on discretization of dpH can be found in [52].

Considering the fourth general question, while there are many studies on the control of flexible manipulators in impact scenario using finite dimensional models [53, 54, 55], very few have discussed the collision issue using infinite-dimensional models [56]. A finite dimensional analysis provides a good approximation of the flexible phenomena in case of unconstrained conditions, but it can bring misleading results in the presence of impacts, where a large bandwidth of frequencies are excited. The dynamical model of a colliding flexible beam is expected to have instant changes in impact times. Therefore the model combines behaviours that are typical of continuous-time dynamical systems with behaviours that are typical of discrete-time dynamical systems. This definition perfectly fits into the class of Hybrid dynamical systems. The stability, as well as the control design theory of finite dimensional hybrid systems, have been extensively studied in the past 30 years, and a general introduction to this subject can be found in [57].

On the other side, only a few results have been established for infinite dimensional hybrid systems. In [58] are presented some general results on Lagrange, asymptotic and exponential stability (in all their variations) for the class of hybrid infinite dimensional systems, that do not require the determination of a Lyapunov function, as well as results that do involve Lyapunov functions. In [59] some conditions for obtaining exponential stability are given for a subclass of hybrid systems, namely switched operator systems. Other characterizations of exponentially stable switched operator equations can be found in [60, 61].

In the following paragraph, we give an overview of the contents of the four chapters that compose the thesis.

Before starting to deal with the main contributions of the thesis, we first recall in Chapter I some concepts about system and operator theory together with the prerequisites on dpH systems.

Chapter II is divided in two sections. In Section II.1 we propose a mathematically rigorous procedure to derive the models for mechanisms with possible flexible components, providing several examples from easy to more complex ones. The procedure is based on the general Least Action Principle that, in the considered examples, leads to m-ODE-PDE systems in a special boundary controlled m-pH format. Next, we analyse the passivity with respect to the internal energy of the considered class of m-pH systems. In Section II.2 we detail how to obtain a finite dimensional approximation of the Timoshenko beam equation and of the wave equation based on a structure preserving finite element discretization introduced in [62]. The finite dimension approximation is used throughout the thesis only for simulation purposes: all the successive analysis are carried on the m-pH systems.

Chapter III is divided in two main sections. In Section III.1 we detail the needed prerequisites on analysis of semilinear equations, *i.e.* the needed assumptions to obtain existence and uniqueness of solution as well as Lyapunov stability theory for semilinear equations. Next, we introduce a result that connects the concept of approximate observability to the zero solution of a general linear operator with admissible constant output. In the second Section III.2 we list four different possible control design methods applicable to the introduced class of boundary-controlled m-pH systems. Before introducing the control laws we give some sufficient conditions to obtain an approximately observable dpH system. This, together with the latter theorem, allows to find the largest invariant set and to conclude about stability for a class of m-pH systems in closed-loop with different control laws. Therefore we show under which conditions the m-pH systems are asymptotically stable in closed-loop with a classical PD and a non-linear passive control law extracted from previous works [42, 63]. Next, we propose two different control laws containing a so-called *strong dissipation* feedback: this term allows to overcome the *integral obstacle* and to insert dissipation on the PDE boundary. The first control law is used only to stabilize the system and leads to exponential stability, while the second also allows to position the system in the desired configuration but leads only to asymptotic stability. All the different control laws are clarified with the help of applicative examples on the models derived in Section II.1 of Chapter II.

Chapter IV is dedicated to the well-posedness and the stability analysis of a rotative flexible beam that undergoes an external impact. We begin by introducing the prerequisites on the analysis of switching infinite dimensional systems, including the extension to distributed parameter systems of the Lyapunov stability theorem extracted from [64]. The presentation holds for a general class of operator systems, therefore in Section IV.2 we first analyse a rotating rigid beam in an impact scenario. In this manner, we introduce all the ideas and procedures that are further investigated in Section IV.3, where we study a flexible rotating beam entering in contact with the external environment.

The concluding Chapter V provides final remarks of this thesis and several interesting perspectives on future works.

PUBLICATIONS ISSUED FROM THIS WORK

1. A. Mattioni, Y. Wu and Y. Le Gorrec. Infinite dimensional model of a double flexible-link manipulator: The port-Hamiltonian approach. *Applied Mathematical Modelling*, 83: 59-75, 2020.

A model of the double flexible manipulator in the FApH formalism has been derived. The obtained model has been discretized to obtain a finite dimensional approximation. Numerical simulations are shown to validate the obtained model.

2. A. Mattioni, J. Toledo and Y. Le Gorrec. Observer based nonlinear control of a rotating flexible beam. *IFAC-PapersOnLine*, 53(2):7479-7484, 2020.

A rotating flexible beam is shown to be asymptotically stable in closed-loop with an observer based nonlinear control. Numerical simulation are given to show the effectiveness of the proposed control law in comparison with a standard PD control.

3. A. Mattioni, Y. Wu, Y. Le Gorrec and H. Zwart. Stabilisation of a rotating beam clamped on a moving inertia with strong dissipation feedback. *59th IEEE Conference on Decision and Control (CDC)*, Jeju Island, South Korea, pp. 5056-5061, 2020.

A rotating flexible beam clamped on a moving inertia has been proved to be asymptotically stable in closed-loop with a *strong dissipation* feedback control. Numerical simulation are given to show the effectiveness of the proposed control law.

4. A. Mattioni, Y. Wu and Y. Le Gorrec. Modelling, control and stability analysis of flexible rotating beam's impacts during contact scenario. *American Control Conference*, New Orleans, United States, 2021.

The model of a rotating flexible beam impacting the external environment has

been showed and the equilibrium position of the system in closed-loop with a PD control law has been computed. Lyapunov stability has been assessed and numerical simulation have been given to validate the obtained results.

Chapter I

Background on operator theory and port-Hamiltonian systems

I.1	Prerequisites on functional analysis	10
I.2	Infinite dimensional port-Hamiltonian Systems	17
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I.1 PREREQUISITES ON FUNCTIONAL ANALYSIS

Preparing to the main part of this thesis we first recall some notions and results on functional analysis and theory on PDE [65, 66]. Meanwhile we fix the notation that will be used throughout this thesis. \mathbb{R} and \mathbb{R}_+ denote the spaces of real and positive real numbers, respectively. With $\mathbb{N} = \{1, 2, \dots\}$ we denote the natural numbers starting from 1, *i.e.* excluding 0. If we want to include 0 we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. With $\mathbb{R}^{n \times m}$ we denote the set of n by m matrices with entries in \mathbb{R} . We denote by $\sigma(A)$ the set of eigenvalues of the matrix A . We say that a symmetric matrix M is semi-positive (or positive) and we denote it by $M \geq 0$ ($M > 0$) if all its eigenvalues are non-negative $\lambda \geq 0$ (positive $\lambda > 0$) for $\lambda \in \sigma(M)$. A matrix M_1 is said to be greater equal (greater) than a matrix M_2 , and it will be denoted by $M_1 \geq M_2$ ($M_1 > M_2$), if the matrix $M_1 - M_2$ is semi-positive (positive) definite. Unless stated otherwise, we consider the finite dimensional space \mathbb{R}^n equipped with the Euclidean norm

$$\|x\| := \sqrt{\sum_{i=1}^n |x_i|^2}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \quad (\text{I.1})$$

and associated standard inner product

$$\langle x_1, x_2 \rangle_{\mathbb{R}^n} = x_1^T x_2 = \sum_{i=1}^n x_{1,i} x_{2,i}, \quad x_1, x_2 \in \mathbb{R}^n. \quad (\text{I.2})$$

Let $[a, b]$ with $a < b$ be an interval of \mathbb{R} , the *Lebesgue space* $L_p([a, b], \mathbb{R}^n)$ is the family of all the equivalence classes of L_p -integrable functions $f : [a, b] \mapsto \mathbb{R}^n$ for which the integral

$$\int_a^b |f(\xi)|^p d\xi < \infty \quad (\text{I.3})$$

with respect to the Lebesgue measure is finite. Two functions are said to be equivalent if they coincide on $[a, b] \setminus N$, where N is a set of Lebesgue measure zero. Note that the space $L_2([a, b], \mathbb{R}^n)$ equipped with the standard norm

$$\|x\|_{L_2} := \sqrt{\int_a^b |x|^2 d\xi}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in L_2([a, b], \mathbb{R}^n) \quad (\text{I.4})$$

is an Hilbert space. The inner product associated to the norm (I.4) is defined as

$$\langle x_1, x_2 \rangle_{L_2} = \int_a^b x_1^T x_2 d\xi. \quad (\text{I.5})$$

By $C([a, b], \mathbb{R}^n)$ we denote the space of continuous functions $f : [a, b] \mapsto \mathbb{R}^n$, while by $C^p([a, b], \mathbb{R}^n)$ with $p \in \mathbb{N}$ the space for which the derivative $\frac{d^p f}{d\xi^p}$ exists and lies in $C([a, b], \mathbb{R}^n)$. Also, we denote by $H^p([a, b], \mathbb{R}^n)$ the standard Sobolev space of order

$m \in \mathbb{N}$, consisting of m times derivable functions f with square integrable derivatives $\frac{d^m f}{d\xi^m}$, equipped with the norm

$$\|x\|_{H^p} := \sqrt{\int_a^b |x|^2 + \sum_{i=1}^p \left| \frac{d^i x}{d\xi^i} \right|^2 d\xi} \quad (\text{I.6})$$

and associated inner product

$$\langle x_1, x_2 \rangle_{H^p} = \int_a^b x_1^T x_2 + \sum_{i=1}^p \frac{d^i x_1}{d\xi^i}^T \frac{d^i x_2}{d\xi^i} d\xi. \quad (\text{I.7})$$

Let $W^{k,p}([a,b], \mathbb{R}^n)$ be the Sobolev space consisting of all functions $f : [a,b] \mapsto \mathbb{R}^n$ whose first k derivatives are functions in L_p . We say that a vector space X is compactly embedded in Y , and we denote it by $X \hookrightarrow Y$ if there exists an injective function $f : X \mapsto Y$ that is compact, *i.e.* every bounded sequence in X is mapped by f into a bounded sequence in Y with a subsequence that is converging in the norm of Y . The Rellich-Kondrachov Theorem establishes a compact embedding relation between the Sobolev spaces $W^{j,p}([a,b], \mathbb{R}^n)$ and $W^{i,n}([a,b], \mathbb{R}^n)$. In particular $W^{j,p}([a,b], \mathbb{R}^n)$ is compactly embedded in $W^{i,n}([a,b], \mathbb{R}^n)$ if $j > i$ and $j - \frac{1}{p} > i - \frac{1}{n}$.

Remark 1. The embeddings $H^p([a,b], \mathbb{R}^n) \hookrightarrow L_2([a,b], \mathbb{R}^n)$ are compact for every $p \in \mathbb{N}$.

If X and Y are real Banach spaces, $\mathcal{L}(X, Y)$ denotes the Banach space of all the linear and bounded operators $f : X \mapsto Y$. For $X = Y$ we simply write $\mathcal{L}(X) = \mathcal{L}(X, X)$. We denote by X a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle_X$ and norm $\|\cdot\|_X = \sqrt{\langle \cdot, \cdot \rangle_X}$. In general, we denote with $A : \mathbf{D}(A) \subset X \mapsto Y$ a closed (not necessarily bounded) operator that maps from a subspace $\mathbf{D}(A)$ of X (the domain of A) to Y . We denote by $\text{ran}(A)$ (the *range* of A) the image of $\mathbf{D}(A)$ under A . For the case $X = Y$ we say that $\lambda \in \mathbb{C}$ is in the *resolvent-set* $\rho(A)$ of A if $(\lambda I - A)^{-1}$ exists and is a bounded linear operator on a dense domain of X . For any $\lambda \in \rho(A)$ the *resolvent operator*

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \mapsto \mathbf{D}(A). \quad (\text{I.8})$$

Definition I.1.1. Let A be a linear operator on a Hilbert space X . Assume that the domain of A is dense in X . Then the *adjoint operator* $A^* : D(A^*) \subset X \mapsto X$ of A is defined as follows. The domain $D(A^*)$ of A^* consists of all $y \in X$ such that there exists a $y^* \in X$ satisfying

$$\langle Ax, y \rangle = \langle x, y^* \rangle \quad \text{for all } x \in \mathbf{D}(A). \quad (\text{I.9})$$

For each such $y \in D(A^*)$ the *adjoint operator* A^* is then defined in terms of y^* by

$$A^* y = y^*. \quad (\text{I.10})$$

■

An operator A is said to be *skew adjoint* if $A^* = -A$, while it is said *self adjoint* if $A^* = A$. In what follows we recall the semigroup theory in case of strongly continuous semigroup on a Hilbert space, and for doing that we need the norm's definition of a linear bounded operator.

Definition I.1.2. Let T be a bounded linear operator from $D(T) \subset X$ to Y . We define its norm $\|T\|$, by

$$\|T\| = \sup_{\substack{x \in D(T), \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}. \quad (\text{I.11})$$

■

Then, we can introduce the strongly continuous semigroup definition.

Definition I.1.3 (Strongly continuous semigroup). Let X be a Hilbert space. $(T(t))_{t \geq 0}$ is called a *Strongly continuous semigroup* (or in short C_0 -semigroup) if the following hold:

1. For all $t \geq 0$, $T(t)$ is a bounded linear operator on X , i.e. $T(t) \in \mathcal{L}(X)$;
2. $T(0) = I$;
3. $T(t + \tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;
4. For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \rightarrow 0$, i.e. $t \mapsto T(t)$ is strongly continuous at zero.

■

The semigroup concept is the generalization to abstract spaces of the exponential solution e^{Mt} of a linear matrix differential equation $\frac{dx}{dt} = Mx$, therefore it shares many properties with these exponentials. In the reminder of the manuscript we will also refer to the time derivative of x with the symbols $\frac{\partial x}{\partial t}$ and \dot{x} .

Theorem I.1.1. A strongly continuous semigroup $(T(t))_{t \geq 0}$ on the Hilbert space X has the following properties:

1. $\|T(t)\|$ is bounded on every finite sub-interval of $[0, \infty)$;
2. The mapping $t \mapsto T(t)$ is strongly continuous on the interval $[0, \infty)$;
3. For all $x \in X$ we have that $\frac{1}{t} \int_0^t T(s)x ds \rightarrow x$ as $t \rightarrow 0$;
4. If $w_0 = \inf_{t > 0} (\frac{1}{t} \log \|T(t)\|)$, then $w_0 = \lim_{t \rightarrow \infty} (\frac{1}{t} \log \|T(t)\|) < \infty$;
5. For every $w > w_0$, there exists a constant M_w such that for every $t \geq 0$ we have $\|T(t)\| \leq M_w e^{wt}$.

Proof. See Theorem 5.1.5 in [24].

□

In a similar way a matrix M can be associated to an exponential solution e^{Mt} we define an operator A associated to a C_0 -semigroup $(T(t))_{t \geq 0}$.

Definition I.1.4 (Infinitesimal generator). Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on the Hilbert space X . If the following limit exists

$$\lim_{t \rightarrow 0} \frac{T(t)x_0 - x_0}{t}, \quad (\text{I.12})$$

then we say that x_0 is an element of the domain of A , shortly $x_0 \in \mathbf{D}(A)$, and we define Ax_0 as

$$Ax_0 = \lim_{t \rightarrow 0} \frac{T(t)x_0 - x_0}{t}. \quad (\text{I.13})$$

We call A the *Infinitesimal generator* of the strongly continuous semigroup $(T(t))_{t \geq 0}$. ■

From point 5 of Theorem I.1.1, we know that every C_0 -semigroup satisfies $\|T(t)\| \leq Me^{wt}$ for some M and w . In the following definition we will introduce a subclass of C_0 -semigroup operators satisfying the property $\|T(t)\| \leq 1$.

Definition I.1.5. Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup on the Hilbert space X . Then the semigroup $(T(t))_{t \geq 0}$ is called a *contraction semigroup*, if $\|T(t)\| \leq 1$ for every $t \geq 0$. ■

Next we study the characterization of the infinitesimal generators of contraction C_0 -semigroup. We first introduce the definition of dissipative operator.

Definition I.1.6. A linear operator $A : \mathbf{D}(A) \subset X \mapsto X$, where X is an Hilbert space, is called *dissipative*, if

$$\langle Ax, x \rangle \leq 0, \quad x \in \mathbf{D}(A). \quad (\text{I.14})$$

■

The following theorem gives a necessary and sufficient condition for an operator A to generate a contraction C_0 -semigroup.

Theorem I.1.2 (Lumer-Phillips Theorem). *Let A be a linear operator with domain $\mathbf{D}(A)$ on a Hilbert space X . Then A is the infinitesimal generator of a contraction C_0 -semigroup $(T(t))_{t \geq 0}$ on X if and only if A is dissipative and $\text{ran}(I - A) = X$.*

Proof. See Theorem 6.1.7 of [24]. □

Next we introduce some concepts on differentiation theory that will be useful for computing the systems passivity property and for Lyapunov stability theory in Chapter III. We start with the concept of Frechet derivative

Definition I.1.7 (Fréchet derivative). Consider the mapping f from the Banach space X to the Banach space Y . Given $x, h \in X$, $h \neq 0$, if a linear bounded operator $df(x)$ exists such that

$$\lim_{\|h\|_X \rightarrow 0} \frac{\|f(x+h) - f(x) - df(x)h\|_Y}{\|h\|_X} = 0, \quad (\text{I.15})$$

then f is Fréchet differentiable at x , and $df(x)$ is said to be the Fréchet derivative at x . ■

Derivatives are usually computed to study the functions behaviour. In the case of study of *functionals* (the case where Y corresponds to the real numbers \mathbb{R}), the derivative's zeros intuitively correspond to the points in which the functional does not change. The next theorem makes the link between the minimums and the maximums of a functional and its Fréchet derivative.

Theorem I.1.3. *Let O be an open set of the Banach space X . If the mapping $f : O \mapsto \mathbb{R}$ has a minimum or a maximum at $x \in O$, and $df(x)$ exists, then $df(x) = 0$.*

Proof. See Theorem A.5.32 of [65]. □

Lyapunov's stability theory makes use of non-increasing functionals along solutions of an evolution equation to conclude about the solutions' asymptotic behaviour. To properly define the functional's time derivative, and to connect it with its non-increasing property, we need the concept of Dini derivative.

Definition I.1.8. Let f be a continuous functional from $[a, b] \subset \mathbb{R}$ to \mathbb{R} . The *Dini derivative* of f is defined as

$$\dot{f}_+(t) = \limsup_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}. \quad (\text{I.16})$$

■

With the normal concept of derivative it does not hold in general that if a functional f has a non-positive derivative, then f is non-increasing. A counter example of this is the Cantor functional (see [67] page 144-145). Conversely, the Dini derivative allows to establish a relation between the non-increasing property of a functional and its derivative.

Lemma I.1.1. *Assume that $f : [a, b] \mapsto \mathbb{R}$ is continuous. The following are equivalent:*

1. *The functional f is non-increasing, i.e. $f(t_2) \leq f(t_1)$ when $t_2 \geq t_1$;*
2. *The Dini derivative of f is non-positive, i.e. $\dot{f}_+(t) \leq 0$, $t \in (a, b)$.*

Proof. See Lemma A.5.44 of [65]. □

In most practical cases it is very hard to calculate the limit in (I.16), however functionals used for Lyapunov stability theory are usually Fréchet differentiable. In these cases, the Dini derivative can be computed making use of the Fréchet derivative. On this purpose we cite Lemma 11.2.5 of [65], that allows to compute the time derivative of a functional along the solutions of a semilinear equation.

Lemma I.1.2. *Let X be an Hilbert space and consider the semilinear equation*

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad t \geq 0, \quad x(0) = x_0 \quad (\text{I.17})$$

where A is the infinitesimal generator of the C_0 -semigroup $(T(t))_{t \geq 0}$ on the space X , and $f : X \mapsto X$ is locally Lipschitz continuous. Furthermore let $g : X \mapsto [0, \infty)$ be a continuous mapping. Then, if g is Fréchet differentiable, then for $z_0 \in \mathbf{D}(A)$, $g(x(t, z_0))$ is differentiable and

$$\dot{g}_+(x_0) = \left. \frac{dg(x(t, x_0))}{dt} \right|_{t=0} = dg(x_0)(Ax_0 + f(x_0)), \quad (\text{I.18})$$

where dg denotes the Fréchet derivative of g .

Proof. See Lemma 11.2.5 of [65]. □

We now introduce the concept of solution of the following semilinear equation

$$\dot{x} = Ax + f(x) \quad (\text{I.19})$$

where x is the state variable belonging to the Hilbert space X , A is the generator of a C_0 -semigroup on the space X and $f : X \mapsto X$ is a nonlinear mapping satisfying a Lipschitz continuity condition. For a complete discussion about existence and uniqueness of solution of the above equation, we refer the reader to [65]. In the following we only list the results that are employed in in Chapters III and IV. The definition of Lipschitz continuity is crucial in determining the existence of solutions for equation (I.19).

Definition I.1.9. A mapping f from X to X is *locally Lipschitz continuous* if for every $r > 0$ there exists an $L(r)$ such that for all $x_1, x_2 \in X$ satisfying $\|x_1\|, \|x_2\| \leq r$ there holds

$$\|f(x_1) - f(x_2)\| \leq L(r)\|x_1 - x_2\|. \quad (\text{I.20})$$

Furthermore, if $L(r)$ can be chosen independently of r , the mapping f is called *globally Lipschitz continuous*. ■

This is a rather simple condition to check and if the considered nonlinear function is obtained through the composition of simpler functions, it is easy to estimate if it is globally or at least locally Lipschitz continuous. From an intuitive point of view, a Lipschitz continuous function can be bounded by a linear function. So it is easy to check that linear and piecewise continuous linear functions are globally Lipschitz continuous. Polynomial functions are in general locally Lipschitz continuous, but discontinuous functions are not globally nor locally Lipschitz continuous.

Before stating the existence of solution theorem we clarify what do we mean by classical solution of (I.19).

Definition I.1.10. The function $x(t)$ is a *classical solution* of (I.19) on $[0, \tau)$ if $x(t) \in C^1([0, \tau); X)$, $x(t) \in \mathbf{D}(A)$ for all $t \in [0, \tau)$ and $x(t)$ satisfies (I.19) for all $t \in [0, \tau)$.

The function $x(t)$ is a *global classical solution* if $x(t)$ is a classical solution on $[0, \tau)$ for every $\tau > 0$. ■

This definition suggests that any possible $C^1([0, \tau], X)$ function such that if differentiated by time returns the right hand side of (I.19) can be called a classical solution. Nevertheless, Lemma 5.1.2 of [65] clarifies that for any initial condition $x_0 \in \mathbf{D}(A)$, the classical solution $x(t)$ of (I.19) is continuous and

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds, \quad t \in [0, \tau]. \quad (\text{I.21})$$

This equation tells us that the solution's behaviour is determined by the C_0 -semigroup generated by the linear operator A . However, (I.21) can be defined for initial conditions that do not necessarily belong to the domain of A , which naturally leads to the following definition.

Definition I.1.11. If the function $x(t)$ in (I.21) defines a continuous function, then it is called the *mild solution* of (I.19). ■

The definition of the mild solution is not an explicit formula, but another equation in the unknown state $x(t)$. Therefore, we do not know a priori this solution exists. In the following theorem, we define the conditions for the existence of a mild solution for equation (I.19) and its relation with the classical solution.

Theorem I.1.4. *Let A be the infinitesimal generator of the C_0 -semigroup $T(t)$ on the Hilbert space X and consider the following semilinear differential equation:*

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad t \geq 0 \quad x(0) = x_0. \quad (\text{I.22})$$

If $f : X \mapsto X$ is locally Lipschitz continuous, then there exists a $t_{\max} > 0$ such that the differential equation (I.22) has a unique mild solution on $[0, t_{\max})$ with the following properties:

1. *For $0 \leq t < t_{\max}$ the solution depends continuously on the initial condition, uniformly on any bounded interval $[0, \tau] \subset [0, t_{\max})$.*
2. *If $x_0 \in \mathbf{D}(A)$, then the mild solution is actually a classical solution on $[0, t_{\max})$.*

Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \rightarrow t_{\max}} \|x(t)\| = \infty. \quad (\text{I.23})$$

If the mapping f is globally Lipschitz continuous, then $t_{\max} = \infty$.

Proof. See theorem 11.1.5 in [65]. □

In case of open systems, *i.e* systems able to interact with the environment or with external systems, a very close concept to dissipative operator is the system's passivity notion. Consider the general nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \quad (\text{I.24})$$

where x is the state of the system and belong to an appropriate Hilbert space X , $u \in U$ is the input belonging to the input Hilbert space U , y is the output of the system belonging to the output Hilbert space Y and $f : X \times U \mapsto X$, $h : X \times U \mapsto Y$ are the vector field and the output mapping, respectively. Similarly for the case of semilinear differential equations, a function $x(t)$ is considered a classical solution of (I.24) if $x(t) \in C^1([0, \tau]; X)$ and $x(t)$ satisfies (I.24) for all $t \in [0, \tau)$. The function $x(t)$ is a global classical solution if it $x(t)$ is a classical solution on $[0, \tau)$ for every $\tau > 0$. We now define the passivity concept for a general nonlinear system (I.24)

Definition I.1.12. Consider the functional $E : X \mapsto \mathbb{R}$ depending on solutions $x(t)$ of system (I.24). The system (I.24) is said to be *passive* with respect to the storage function E if

$$\dot{E}_+ \leq y^T u. \quad (\text{I.25})$$

The time derivative in the previous definition is the Dini derivative of Definition I.1.8, and is intended as "along the solutions" of system (I.24).

I.2 INFINITE DIMENSIONAL PORT-HAMILTONIAN SYSTEMS

In this thesis we consider a special class of evolution equations, *i.e.* the class of 1-D dpH systems

$$\frac{\partial z}{\partial t}(\xi, t) = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}(\xi)z(\xi, t)) + P_0 \mathcal{H}(\xi)z(\xi, t) \quad (\text{I.26})$$

where $P_1 \in \mathbb{R}^{n \times n}$ is invertible and symmetric, $P_0 \in \mathbb{R}^{n \times n}$ is skew-symmetric and $\mathcal{H} \in C^1([0, L]; \mathbb{R}^{n \times n})$, $\mathcal{H}(\xi)$ is symmetric for all $\xi \in [0, L]$ and $mI \leq \mathcal{H}(\xi) \leq MI$ for all $\xi \in [0, L]$ and some $M, m > 0$ independent of ξ . The symbol $\frac{\partial f}{\partial \xi}$ denotes the 1-D spatial derivative of a function f , that in the reminder of the thesis will also be denoted with f' . In the following we will not write explicitly the space dependency of the parameters' density matrix \mathcal{H} , but it will be considered space varying if not differently mentioned. We consider the operator

$$\mathcal{J}z = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0 \mathcal{H}z \quad (\text{I.27})$$

on the state space $Z = L_2([0, L], \mathbb{R}^n)$ and with domain

$$\mathbf{D}(\mathcal{J}) = \{z \in Z \mid \mathcal{H}z \in H^1([0, L], \mathbb{R}^n)\}. \quad (\text{I.28})$$

The state space Z is equipped with the norm $\|z\|_Z = \sqrt{\langle z, \mathcal{H}z \rangle_{L_2}}$ associated with the inner product $\langle z_1, z_2 \rangle_Z = \langle z_1, \mathcal{H}z_2 \rangle_{L_2}$. The energy of the system (I.26) is defined as half of the defined squared norm

$$E = \frac{1}{2} \langle z, z \rangle_Z = \frac{1}{2} \langle z, \mathcal{H}z \rangle_{L_2}. \quad (\text{I.29})$$

Next, we add the possibility of the system of interacting with the external environment. In this thesis we consider that the distributed parameters systems interact with the external world through the boundary of their spatial domain. Therefore we introduce the boundary flow and effort defined as a parametrization of the boundary variables.

Definition I.2.1. Consider equation (I.26), then the *boundary effort* and *boundary flow* are defined as

$$\begin{bmatrix} f_{\partial}(t) \\ e_{\partial}(t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} \mathcal{H}z(L,t) \\ \mathcal{H}z(0,t) \end{bmatrix}. \quad (\text{I.30})$$

■

The boundary input u_z and output y_z are selected as a linear combination of the above defined boundary flow and effort. From a computational point of view, the boundary flow and effort parametrization is crucial for relating the energy variation to the defined input and output.

Theorem I.2.1. Let z be a classical solution of the pH system (I.26) with energy (I.29). Then the following balance equation holds:

$$\dot{E}_+(t) = f_{\partial}(t)^T e_{\partial}(t). \quad (\text{I.31})$$

Proof. We first compute the Fréchet derivative of the energy (I.29) as the linear term on h of the following difference

$$\begin{aligned} E(z+h) - E(z) &= \frac{1}{2} \langle z+h, z+h \rangle_Z - \frac{1}{2} \langle z, z \rangle_Z \\ &= \frac{1}{2} \langle z, z+h \rangle_Z + \frac{1}{2} \langle h, z+h \rangle_Z - \frac{1}{2} \langle z, z \rangle_Z \\ &= \frac{1}{2} \langle z, h \rangle_Z + \frac{1}{2} \langle h, z \rangle_Z + \frac{1}{2} \langle h, h \rangle_Z \\ &= \langle z, h \rangle_Z + \frac{1}{2} \langle h, h \rangle_Z. \end{aligned} \quad (\text{I.32})$$

Therefore, we obtain that $dEh = \langle z, h \rangle_Z$. Next, we compute the Dini derivative of Definition I.1.8 using Lemma I.1.2 the energy definition (I.29) and the operator (I.27),

$$\begin{aligned} \dot{E}_+ &= \langle z, \mathcal{J}z \rangle_Z \\ &= \langle z, \mathcal{H} \mathcal{J}z \rangle_{L_2} \\ &= \int_0^L (\mathcal{H}z)(\xi, t)^T P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z)(\xi, t) + (\mathcal{H}z)(\xi, t)^T P_0 (\mathcal{H}z)(\xi, t) d\xi. \end{aligned} \quad (\text{I.33})$$

Using the fact that P_0 is skew-symmetric, we write the expression as

$$\begin{aligned} \dot{E}_+ &= \int_0^L (\mathcal{H}z)(\xi, t)^T P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z)(\xi, t) d\xi \\ &= \frac{1}{2} \int_0^L \frac{\partial}{\partial \xi} [(\mathcal{H}z)(\xi, t)^T P_1 (\mathcal{H}z)(\xi, t)] d\xi \\ &= \frac{1}{2} [(\mathcal{H}z)(\xi, t)^T P_1 (\mathcal{H}z)(\xi, t)]_0^L \\ &= \frac{1}{4} [(\mathcal{H}z)(L, t)^T (\mathcal{H}z)(0, t)^T] \begin{bmatrix} P_1 & I \\ -P_1 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}z)(L, t) \\ (\mathcal{H}z)(0, t) \end{bmatrix}. \end{aligned} \quad (\text{I.34})$$

Finally, using the boundary effort and flow definition (I.30), we obtain

$$\begin{aligned}\dot{E}_+ &= \frac{1}{2} [f_\partial^T \ e_\partial^T] \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ &= f_\partial^T e_\partial.\end{aligned}\tag{I.35}$$

□

Roughly speaking, the input imposes a certain time-varying value at the boundary, the distributed state of the system evolves according to its describing PDE, an output that depends on the boundary value of the state is returned. From an application point of view, the input can be an actual actuation or can be the effect of some other external dynamical systems on the PDE, while the output can be the measured variables on the system or its effect on another external dynamical system. Throughout the rest of the thesis, and according to what we present in the modelling Chapter II, we consider that a part of the input are set equal to zero whereas the other part can be used for control. The input are defined by an operator acting on the distributed parameter state. Thus, we consider the following input operators:

$$\begin{aligned}\mathcal{B}_1(\mathcal{H}z)(t) &= W_{B,1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = u_z(t) \\ \mathcal{B}_2(\mathcal{H}z)(t) &= W_{B,2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0.\end{aligned}\tag{I.36}$$

The output operators are split accordingly:

$$\begin{aligned}\mathcal{C}_1(\mathcal{H}z)(t) &= W_{C,1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = y_z(t) \\ \mathcal{C}_2(\mathcal{H}z)(t) &= W_{C,2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \tilde{y}_z(t),\end{aligned}\tag{I.37}$$

such that $W_{B,1}, W_{C,1} \in \mathbb{R}^{m \times 2n}$ and $W_{B,2}, W_{C,2} \in \mathbb{R}^{(n-m) \times 2n}$. Note that the output $y_z(t)$ is selected to be the power-conjugated of $u_z(t)$, and therefore they are of the same dimensions. We define the complete input and output operators as the composition of the previously defined operators

$$\begin{aligned}\mathcal{B}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{B}_1(\mathcal{H}z) \\ \mathcal{B}_2(\mathcal{H}z) \end{bmatrix} = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ \mathcal{C}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{C}_1(\mathcal{H}z) \\ \mathcal{C}_2(\mathcal{H}z) \end{bmatrix} = \begin{bmatrix} W_{C,1} \\ W_{C,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}.\end{aligned}\tag{I.38}$$

We assume that for a 1-D dpH system, the following conditions holds.

Assumption 1.

1. The matrix $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible;

2. The matrices W_B and W_C are selected such that

$$\begin{bmatrix} W_B \Sigma W_B^T & W_B \Sigma W_C^T \\ W_C \Sigma W_B^T & W_C \Sigma W_C^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (\text{I.39})$$

It follows that the system

$$\begin{aligned} \dot{z} &= P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) = \mathcal{J}z \\ \mathcal{B}_1(\mathcal{H}z) &= u_z \\ \mathcal{C}_1(\mathcal{H}z) &= y_z \end{aligned} \quad (\text{I.40})$$

with domain

$$\mathbf{D}(\mathcal{J}) = \{z \in Z \mid \mathcal{H}z \in H^1([a, b], \mathbb{R}^n), \mathcal{B}_2 z = 0\} \quad (\text{I.41})$$

is a boundary control system (see Theorem 11.3.2 [24]). This means that for $u_z \in C^2([0, \infty), \mathbb{R}^m)$, $u_z(0) = \mathcal{B}_1 \mathcal{H}z(0)$ and $\mathcal{H}z(0) \in \mathbf{D}(\mathcal{J})$ the system (I.40)-(I.41) has a unique classical solution as defined in Definition I.1.10. This class of boundary control systems endows a large variety of systems ranging from mechanical to electric and diffusion processes all sharing the property of conserving the internal energy. In the next proposition, the defined operator \mathcal{J} is shown to be skew-adjoint in the defined state space Z , a property that is related to conservation of energy.

Proposition I.2.1. *Consider the operator (I.40) defined on the state space Z with homogeneous boundary conditions, i.e. $\mathcal{B}_2 z = 0$. Then, the operator \mathcal{J} is a skew-adjoint operator, i.e. $\mathcal{J} = -\mathcal{J}^*$ and $\mathbf{D}(\mathcal{J}) = \mathbf{D}(\mathcal{J}^*)$.*

Proof. See Theorem 2.24 of [22] considering the operator with homogeneous boundary conditions. \square

In the following lemmas we show some properties of the operator \mathcal{J} that will be useful later, during the control law design phase.

Lemma I.2.1. *Under Assumption I, for the boundary control system (I.40)-(I.41) with input and output defined in (I.36)-(I.37) with internal energy (I.29), the following properties hold:*

- i) $\dot{E}_+(t) = u_z^T y_z$;
- ii) $\langle \mathcal{J}z, z \rangle_Z = \langle z, \mathcal{J}z \rangle_Z = u_z^T y_z$.

Proof. 1) By Theorem I.2.1, we have that

$$\begin{aligned} \dot{E}_+ &= \frac{1}{2} [f_\partial^T \ e_\partial^T] \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ &= \frac{1}{2} [\mathcal{B}(\mathcal{H}z)^T \ \mathcal{C}(\mathcal{H}z)^T] [W_B^T \ W_C^T]^{-1} \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{B}(\mathcal{H}z) \\ \mathcal{C}(\mathcal{H}z) \end{bmatrix}. \end{aligned} \quad (\text{I.42})$$

Since for A and B invertible it holds $A^{-T}B^{-1}A^{-1} = (ABA^T)^{-1}$, using point 1 and 2 of Assumption 1 we obtain that

$$\begin{aligned}\dot{E}_+ &= \frac{1}{2}[\mathcal{B}(\mathcal{H}z)^T \mathcal{C}(\mathcal{H}z)^T] \left(\begin{bmatrix} W_B \\ W_C \end{bmatrix} \Sigma [W_B^T \ W_C^T] \right)^{-1} \begin{bmatrix} \mathcal{B}(\mathcal{H}z) \\ \mathcal{C}(\mathcal{H}z) \end{bmatrix} \\ &= \frac{1}{2}[\mathcal{B}(\mathcal{H}z)^T \mathcal{C}(\mathcal{H}z)^T] \Sigma \begin{bmatrix} \mathcal{B}(\mathcal{H}z) \\ \mathcal{C}(\mathcal{H}z) \end{bmatrix} = u_z^T y_z.\end{aligned}\quad (\text{I.43})$$

2) We notice that

$$\begin{aligned}\langle \mathcal{J}z, z \rangle_Z &= \langle \mathcal{J}z, \mathcal{H}z \rangle_{L_2} \\ &= \int_0^L \left(P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z)(\xi, t) \right)^T \mathcal{H}z(\xi, t) + (P_0(\mathcal{H}z)(\xi, t))^T \mathcal{H}z(\xi, t) d\xi \\ &= \int_0^L \left(\frac{\partial}{\partial \xi} (\mathcal{H}z)(\xi, t) \right)^T P_1(\mathcal{H}z)(\xi, t) - (\mathcal{H}z)(\xi, t)^T P_0(\mathcal{H}z)(\xi, t) d\xi \\ &= \int_0^L \left(\frac{\partial}{\partial \xi} (\mathcal{H}z)(\xi, t) \right)^T P_1(\mathcal{H}z)(\xi, t) d\xi \\ &= \frac{1}{2} \int_0^L \frac{\partial}{\partial \xi} ((\mathcal{H}z)(\xi, t)^T P_1(\mathcal{H}z)(\xi, t)) d\xi\end{aligned}\quad (\text{I.44})$$

and comparing it with what obtained in the proof of Theorem I.2.1, we can conclude $\langle \mathcal{J}z, z \rangle_Z = \langle z, \mathcal{J}z \rangle_Z$. \square

Lemma I.2.2. *Under Assumption 1, the operator \mathcal{J} defined in (I.40) with domain*

$$\mathbf{D}(\mathcal{J}) = \{z \in Z \mid \mathcal{H}z \in H^1([a, b], \mathbb{R}^n), \mathcal{B}z = 0\} \quad (\text{I.45})$$

generates a contraction C_0 -semigroup in the space Z and has a compact resolvent.

Proof. See Theorem 2.28 in [22]. \square

I.3 MIXED PORT-HAMILTONIAN SYSTEMS

In the previous section we introduced the class of 1-D dpH systems, while in this section we present some basic results for a class of systems composed by the boundary interconnection of a 1-D dpH system with a finite dimensional system. We refer to this class of PDE-ODE systems as m-pH (mixed port-Hamiltonian) systems. In Chapter II we show that some physical applications of flexible mechanisms can be cast in this class of m-pH systems. Consider the finite dimensional linear system defined as

$$\begin{cases} \dot{v} = (J - R)Qv + \tilde{g}u_v + gu \\ y = g^T Qv. \end{cases} \quad (\text{I.46})$$

where $v \in \mathbb{R}^{2m}$ is the finite dimensional state, u_v is the vector of restoring efforts coming from the dpH system, u corresponds to the vector containing the external control input and y is its power conjugated output. Let $J \in \mathbb{R}^{2m \times 2m}$ be a skew-adjoint matrix, let

$R \in \mathbb{R}^{2m \times 2m}$ be semi-positive definite and self-adjoint $R^T = R \geq 0$, let Q positive definite and self-adjoint $Q^T = Q > 0$, and let $\tilde{g}, g \in \mathbb{R}^{2m \times m}$. We underline that the defined finite dimensional system is not a proper pH system (see the definition in [17]) because there is no defined conjugated output to u_v . This will lead to a non power-preserving interconnection between the dpH system and the finite dimensional system. Therefore, even though the nomenclature “m-pH” may seem incorrect, we think it remains appropriate because of the pH structure of both distributed and finite dimensional systems. As depicted in Figure I.1, we interconnect the defined finite dimensional pH system (I.46) with the boundary control system (I.40)-(I.41) with the following interconnection law

$$u_z = y - Sy_z \quad u_v = -y_z \quad (\text{I.47})$$

where $S \in \mathbb{R}^{m \times m}$ is the direct feed-through matrix and is semi-positive definite $S \geq 0$. Let $x = [z \ v]^T$ be the total state with $x \in X = L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$, then the m-pH system is described by the following equations

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \mathcal{J} & 0 \\ -\tilde{g}\mathcal{C}_1\mathcal{H} & (J-R)Q \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ g \end{bmatrix} u(t) \\ &= Ax(t) + Bu(t) \\ y(t) &= g^T Qv \end{aligned} \quad (\text{I.48})$$

with domain of the operator A defined by

$$\mathbf{D}(A) = \{x \in X \mid \mathcal{H}z \in H^1([0, L], \mathbb{R}^n), \mathcal{B}_1(\mathcal{H}z) = g^T Qv, -S\mathcal{C}_1(\mathcal{H}z), \mathcal{B}_2(\mathcal{H}z) = 0\}. \quad (\text{I.49})$$

We highlight that in the considered class of systems, the external control input does not have direct access on the boundary conditions of the distributed parameter system. This property does not have to be seen as a limitation but rather a characteristic close to reality (as it will be shown in the modelling example in Chapter II) that has to be taken into consideration during the control law design. Next we generalise Lemma I.2.2, for the case of m-pH systems.

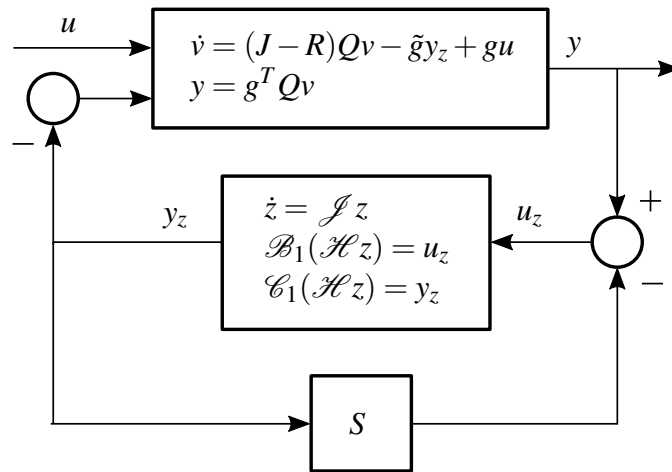


Figure I.1: Interconnection between the dpH system and the finite dimensional system.

Theorem I.3.1. *If there exists an inner product with which the operator $A : \mathbf{D}(A) \subseteq X \mapsto X$*

$$Ax = \begin{bmatrix} \mathcal{J} & 0 \\ -\tilde{g}\mathcal{C}_1\mathcal{H} & (J-R)Q \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} \quad (\text{I.50})$$

$$\mathbf{D}(A) = \{x \in X \mid \mathcal{H}z \in H^1([0, L], \mathbb{R}^n), \mathcal{B}_1(\mathcal{H}z) = g^T Qv \\ -S\mathcal{C}_1(\mathcal{H}z), \mathcal{B}_2(\mathcal{H}z) = 0\}$$

is dissipative $\langle Ax, x \rangle_X \leq 0$, and g, \tilde{g}, S are such that

$$g^T Q \tilde{g} > 0 \quad \text{or} \quad S > 0, \quad (\text{I.51})$$

then A generates a contraction C_0 -semigroup $(T(t))_{t \geq 0}$ on X . Moreover, in this case the operator A has a compact resolvent.

Proof. By the Lumer-Phillips Theorem we only have to show that $\text{ran}(\lambda I - A) = X$. The range condition consists in finding for a certain $\lambda > 0$, $(z, v) \in \mathbf{D}(A)$ such that

$$\lambda \begin{bmatrix} z \\ v \end{bmatrix} - A \begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} f_z \\ f_v \end{bmatrix}, \quad \begin{bmatrix} f_z \\ f_v \end{bmatrix} \in X. \quad (\text{I.52})$$

Writing the former equation in all its components

$$\begin{cases} (\lambda I - \mathcal{J})z = f_z \\ (\lambda I - (J - R)Q)v + \tilde{g}\mathcal{C}_1(\mathcal{H}z) = f_v \\ \mathcal{B}_1(\mathcal{H}z) = g^T Qv - S\mathcal{C}_1(\mathcal{H}z) \\ \mathcal{B}_2(\mathcal{H}z) = 0 \end{cases} \quad (\text{I.53})$$

and taking into account that $(\lambda I - (J - R)Q)^{-1}$ exists for $\lambda > 0$ since $J = -J^T$ and $R \geq 0$, we solve v in the second equation and substitute it in the third one. The problem becomes finding z such that $\mathcal{H}z \in H^1([0, L], \mathbb{R}^n)$ and

$$\begin{cases} (\lambda I - \mathcal{J})z = f_z \\ \mathcal{B}_1(\mathcal{H}z) + (g^T Q(\lambda I - (J - R)Q)^{-1} \tilde{g} + S)\mathcal{C}_1(\mathcal{H}z) = \tilde{f}_v \\ \mathcal{B}_2(\mathcal{H}z) = 0 \end{cases} \quad (\text{I.54})$$

where $\tilde{f}_v = g^T Q(\lambda I - (J - R)Q)^{-1} f_v$. Next, we define

$$Y = \begin{bmatrix} g^T Q(\lambda I - (J - R)Q)^{-1} \tilde{g} + S & 0 \\ 0 & 0 \end{bmatrix}$$

such to rewrite the problem as

$$\begin{cases} (\lambda I - \mathcal{J})z = f_z \\ (\mathcal{B} + Y\mathcal{C})(\mathcal{H}z) = \tilde{f}'_v \end{cases} \quad (\text{I.55})$$

with $\tilde{f}'_v = \begin{bmatrix} \tilde{f}_v \\ 0 \end{bmatrix}$. Using (I.30) and (I.38), we rewrite the second equation in (I.55) as

$$\frac{1}{\sqrt{2}} \begin{bmatrix} I & Y \end{bmatrix} \begin{bmatrix} W_B \\ W_C \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}z)(a) \\ (\mathcal{H}z)(b) \end{bmatrix} = \tilde{f}'_v. \quad (\text{I.56})$$

Since $[I \ Y]$ is right invertible and since $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ and $\begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}$ are invertible, we can find a z_p such that $\mathcal{H}z_p \in H^1([a, b], \mathbb{R}^n)$ and (I.56) is satisfied for $z = z_p$. We define the new variable $z_n = z - z_p$ to obtain

$$\begin{cases} (\lambda I - \mathcal{J})z_n = f_z - (\lambda I - \mathcal{J})z_p \\ \mathcal{B}_{cl}(\mathcal{H}z_n) = (\mathcal{B} + Y\mathcal{C})(\mathcal{H}z_n) = 0. \end{cases} \quad (\text{I.57})$$

From Theorem 3.3.6 of [23], the operator $J_{cl} = \mathcal{J}|_{D(J_{cl})}$ generates a contraction C_0 -semigroup on Z if J_{cl} is dissipative, with $D(J_{cl}) = \{z \in Z \mid \mathcal{H}z \in H^1([a, b], \mathbb{R}^n), \mathcal{B}_{cl}z = 0\}$. For every $z_n \in D(J_{cl})$, we take $v = (\lambda I - (J - R)Q)^{-1} \tilde{g}\mathcal{C}_1(\mathcal{H}z_n)$, so that

$$\begin{aligned} (\mathcal{B}_1 + S\mathcal{C}_1)(\mathcal{H}z_n) + g^T Qv &= 0 \\ \mathcal{B}_2(\mathcal{H}z_n) &= 0 \end{aligned} \quad (\text{I.58})$$

and hence $(z_n, v) \in \mathbf{D}(A)$. Finally, we use Lemma I.2.1 to write

$$\begin{aligned} \langle J_{cl}z_n, z_n \rangle_Z &= \langle \mathcal{J}z_n, z_n \rangle_Z \\ &= \langle \mathcal{B}_1(\mathcal{H}z_n), \mathcal{C}_1(\mathcal{H}z_n) \rangle_{\mathbb{R}^m} \\ &= \langle -(g^T Q(\lambda I - (J - R)Q)^{-1} \tilde{g} + S)\mathcal{C}_1(\mathcal{H}z_n), \mathcal{C}_1(\mathcal{H}z_n) \rangle_{\mathbb{R}^m} \\ &\leq -\langle g^T Q(\lambda I - (J - R)Q)^{-1} \tilde{g}\mathcal{C}_1(\mathcal{H}z_n), \mathcal{C}_1(\mathcal{H}z_n) \rangle_{\mathbb{R}^m} \\ &\quad - \langle S\mathcal{C}_1(\mathcal{H}z_n), \mathcal{C}_1(\mathcal{H}z_n) \rangle_{\mathbb{R}^m}. \end{aligned} \quad (\text{I.59})$$

Using the assumption $g^T Q\tilde{g} > 0$ and for λ large enough, it is true that

$$\langle J_{cl}z_n, z_n \rangle_Z \leq 0. \quad (\text{I.60})$$

On the other hand, using the other assumption $S > 0$, for λ large enough and defining μ the smallest eigenvalue of S it is true that

$$\langle J_{cl}z_n, z_n \rangle_Z \leq \frac{1}{2}\mu \|\mathcal{C}_1(\mathcal{H}z_n)\|^2 - \mu \|\mathcal{C}_1(\mathcal{H}z_n)\|^2 < 0 \quad (\text{I.61})$$

and thus, for any of the two assumptions in (I.51), J_{cl} generates a contraction C_0 -semigroup. Consequently, the resolvent operator $(\lambda I - \mathcal{J})^{-1}$ exists, and the unique solution of (I.57) is given by

$$z_n = (\lambda I - \mathcal{J})^{-1}(f_z - (\lambda I - \mathcal{J})z_p). \quad (\text{I.62})$$

Therefore the choice

$$\begin{aligned} z &= z_n + z_p \\ v &= (\lambda I - (J - R)Q)^{-1}(f_v + \tilde{g}\mathcal{C}_1(\mathcal{H}z)) \end{aligned} \quad (\text{I.63})$$

defines an element $(z, v) \in \mathbf{D}(A)$ for which the range condition is fulfilled, and from the Lumer-Phillips's theorem we conclude that the operator A generates a contraction C_0 -semigroup in the state space X .

Next we show that the resolvent operator is compact, *i.e.* that maps bounded sequences into bounded sequences with a convergent subsequence. We define the sequence

$$\{w_n\} = (\lambda I - A)^{-1}\{x_n\} \quad (\text{I.64})$$

with $\lambda > 0$ and $\{x_n\}$ a bounded sequence. The sequences are defined such that $\{w_n\} = [\{w_{n,1}\} \{w_{n,2}\}]^T \in H^1([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ and $\{x_n\} = [\{x_{n,1}\} \{x_{n,2}\}]^T \in X$. By Assumption 1 and Theorem III.2.9, A generates a contraction C_0 -semigroup in $L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the weighted norm (III.119). Since A generates a bounded C_0 -semigroup in X , from the Hille-Yoshida Theorem [65, Theorem 2.1.15] it follows that $\|(\lambda I - A)^{-1}\| < \frac{C}{c\lambda}$. This implies that $\{w_n\}$ is bounded in X , i.e. $\|w_{n,1}\|_{L_2}, \|w_{n,2}\|_{\mathbb{R}^{2m}} < \infty$. The Bolzano-Weierstrass Theorem implies that, since the sequence $\{w_{n,2}\} \in \mathbb{R}^{2m}$ is bounded, it also has a converging subsequence in \mathbb{R}^{2m} . We compute the H^1 norm of $w_{n,1}$

$$\|w_{n,1}\|_{H^1}^2 = \left\| \frac{\partial}{\partial \xi} w_{n,1} \right\|_{L_2}^2 + \|w_{n,1}\|_{L_2}^2. \quad (\text{I.65})$$

Using the operator \mathcal{J} definition and (I.64) we obtain

$$\begin{aligned} \left\| \frac{\partial}{\partial \xi} w_{n,1} \right\|_{L_2}^2 &= \left\| \mathcal{H}^{-1} P_1^{-1} \mathcal{J} w_{n,1} - \mathcal{H}^{-1} \frac{\partial \mathcal{H}}{\partial \xi} w_{n,1} - \mathcal{H} P_1^{-1} P_0 w_{n,1} \right\|_{L_2}^2 \\ &\leq 2 \left\| \mathcal{H}^{-1} \left(\mathcal{H}^{-1} (P_1^{-1} \lambda - P_1 P_0 \mathcal{H} - \frac{\partial \mathcal{H}}{\partial \xi}) w_{n,1} \right) \right\|_{L_2}^2 \\ &\quad + 2 \left\| \mathcal{H}^{-1} P_1^{-1} x_{n,1} \right\|_{L_2}^2 < \infty. \end{aligned} \quad (\text{I.66})$$

thus, $\{w_{n,1}\}$ is a bounded sequence in H^1 and from the Sobolev embedding Theorem, $\{w_{n,1}\}$ has a converging subsequence in L_2 . Therefore, A has a compact resolvent. \square

Related results to the theorem that we just presented can be found in Theorem 5.8 in [22] or Theorem 5.1.1 in [23]. The proposed theorem differs from Theorem 5.8 in [22] for two main reasons: first, we consider that part of the boundary input of the dpH system can be homogeneous and second we consider a linear finite dimensional system that has a specific structure but without assuming that all its poles have negative real part. Then, it also differs from Theorem 5.1.1 in [23] because we consider the possibility that part of the boundary input can be set equal to zero.

In case $\tilde{g} = g$ the m-pH system become of the form depicted in Figure I.2, and of equations

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \mathcal{J} & 0 \\ -g\mathcal{C}_1\mathcal{H} & (J-R)Q \end{bmatrix} \begin{bmatrix} z(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix} u(t) \\ &= Ax(t) + Bu(t) \\ y(t) &= g^T Qv \end{aligned} \quad (\text{I.67})$$

with domain of the operator A defined by

$$\mathbf{D}(A) = \{x \in X \mid \mathcal{H}z \in H^1([0, L], \mathbb{R}^n), \mathcal{B}_1(\mathcal{H}z) = g^T Qv - S\mathcal{C}_1(\mathcal{H}z), \mathcal{B}_2(\mathcal{H}z) = 0\}. \quad (\text{I.68})$$

The dissipativity of A follows directly from Theorem I.2.1.

Corollary I.3.1. *Under Assumption 1, the operator $A : \mathbf{D}(A) \subseteq X \mapsto X$*

$$Ax = \begin{bmatrix} \mathcal{J} & 0 \\ -g\mathcal{C}_1\mathcal{H} & (J-R)Q \end{bmatrix} \begin{bmatrix} z \\ v \end{bmatrix} \quad (\text{I.69})$$

$$\mathbf{D}(A) = \{x \in X \mid \mathcal{H}z \in H^1([0, L], \mathbb{R}^n), \mathcal{B}_1(\mathcal{H}z) = g^T Qv - S\mathcal{C}_1(\mathcal{H}z), \mathcal{B}_2(\mathcal{H}z) = 0\}$$

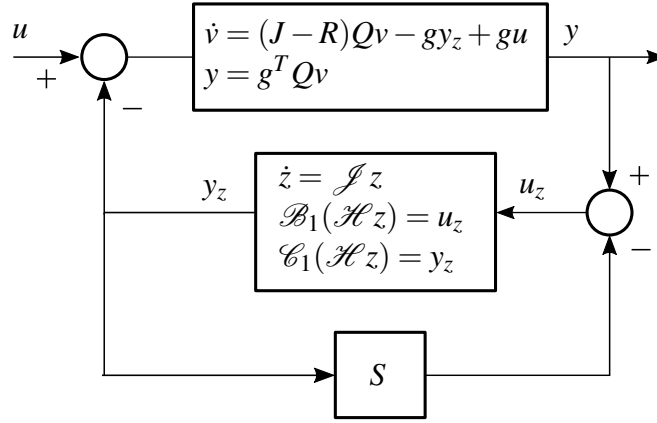


Figure I.2: Interconnection between the dpH system and the finite dimensional system.

generates a contraction C_0 -semigroup $(T(t))_{t \geq 0}$ on X equipped with the energy norm

$$\|x\|_X = \langle z, \mathcal{H}z \rangle_{L_2} + v^T Qv. \quad (\text{I.70})$$

Moreover, the operator A has a compact resolvent

Proof. According to Lemma I.2.1 and Assumption 1, the dissipativity condition writes

$$\begin{aligned} \langle Ax, x \rangle_X &= \langle \mathcal{J}z, \mathcal{H}z \rangle_{L_2} + (-g\mathcal{C}_1(\mathcal{H}z) + (J-R)Qv)^T Qv \\ &= \mathcal{C}_1(\mathcal{H}z)^T \mathcal{B}_1(\mathcal{H}z) - \mathcal{C}_1(\mathcal{H}z)g^T Qv - (Qv)^T RQv \\ &= \mathcal{C}_1(\mathcal{H}z)g^T Qv - \mathcal{C}_1(\mathcal{H}z)^T S\mathcal{C}_1(\mathcal{H}z) - \mathcal{C}_1(\mathcal{H}z)g^T Qv - (Qv)^T RQv \\ &= -\mathcal{C}_1(\mathcal{H}z)^T S\mathcal{C}_1(\mathcal{H}z) - (Qv)^T RQv \leq 0. \end{aligned} \quad (\text{I.71})$$

Then, since $g^T Qg > 0$, according to Theorem I.3.1 we conclude that the operator A generates a contraction C_0 -semigroup in X and has a compact resolvent. \square

Remark 2. When the input-output matrices g, \tilde{g} are such that $g^T Q\tilde{g} > 0$, the direct feed-through matrix S is not needed to be positive definite but only semi-positive definite. This allows Theorem I.3.1 and Corollary I.3.1 to include the cases in which no direct feed-through is present in the system when $g^T Q\tilde{g} > 0$, as well as the cases in which it is present without any requirement on g, \tilde{g} .

Chapter II

Modelling and discretization procedures for flexible mechanisms

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II.1 MODELLING OF M-PH MECHANICAL SYSTEMS

In this section we present how to derive the dynamical models for several different mechanisms containing flexible parts, firstly introducing a general modelling procedure and secondly writing the models as m-pH systems. Together with the model derivation, we precisely state the assumed working conditions of the considered systems such to define the range of validity of the derived models. Next, we show how to choose the state variables such to be able to rewrite the system as a pH system. The models of the considered class are composed by the interconnection of a 1-D distributed parameter part with lumped parameter one, where the control action is applied to the lumped parameter one. As we will see in the following, this is what happens when we consider flexible robots actuated by motors with an inertia that can not be neglected.

The models that we derive in this section are frequently encountered in the literature that deals with the study of flexible structures using the operator formalism. At the beginning of every example subsection, we list some possible applications that the proposed m-pH model can fit.

II.1.1 General procedure

The starting point to derive the equations of motion of a system is defining the action functional

$$A = \int_{t_1}^{t_2} \{L + W_{nc}\} dt \quad (\text{II.1})$$

i.e. the integral between two time instants $t_2 > t_1 > 0$ of the sum between the Lagrangian L and the work of non-conservative forces W_{nc} . The Lagrangian is defined as the difference between the kinetic and potential energy $L = T - P$, while the work of non-conservative forces W_{nc} includes the work done by the inputs and the friction forces on the system. To derive the equations of motion of our systems we refer to the general Principle of Least Action. This principle states that the true evolution of a system minimizes the action functional. More precisely, the first order variation of the action functional from the system's true trajectories must be equal to zero. In the case of lumped parameter systems, the Lagrangian depends on lumped generalized coordinates and the first order variation of the action functional corresponds to the first order term in its Taylor's expansion. Therefore, it is necessary to compute the partial derivatives with respect to the generalized coordinates of the Action functional. Since we deal with distributed parameter systems defined on a 1-D spatial domain, the Lagrangian depends on distributed parameter variables, consequently, we cannot rely on the same derivative concept used for lumped parameter systems, but we use the Fréchet derivative introduced in Definition I.1.7 together with the characterization theorem of minima and maxima (Theorem I.1.3).

Consider the vector of lumped $q = [q_1 \ q_2 \ \cdots \ q_{N_q}]^T$ and the vector of distributed $w = [w_1 \ w_2 \ \cdots \ w_{N_d}]^T$ generalized coordinates. The generalized coordinates describes completely the system's configuration along time, and are defined such that $q(t) : \mathbb{R} \mapsto \mathbb{R}^{N_q}$ and the $w_i = [w_{i,1} \ w_{i,2} \ \cdots \ w_{i,n_{d,i}}]^T$ components are defined such that $w_i(\xi_i, t) :$

$[0, L_i] \mapsto \mathbb{R}^{n_{d,i}} \times \mathbb{R}$, where $[0, L_i] \subset \mathbb{R}^+$ are the 1 dimensional spatial domain intervals. Note that the total number of distributed parameter generalized coordinates is $N_w = \sum_{i=0}^{N_d} n_{d,i}$, therefore the generalized coordinates distributed vector is defined as $w : \mathbb{R} \times [0, L_1] \times \dots \times [0, L_{N_d}] \mapsto \mathbb{R}^{N_w}$. Let p_i be the degree of the bigger spatial derivative present in the potential energy for the i -th generalized distributed coordinate, for $i = \{1, \dots, N_w\}$.

The Lagrangian is defined as the difference between the kinetic E_k and potential E_p energy

$$L(q, \dot{q}, w, \dot{w}, w'_1, \dots, w_1^{(p_1)}, \dots, w'_{N_w}, \dots, w_{N_w}^{(p_{N_w})}) = E_k(q, \dot{q}, w, \dot{w}) - E_p(q, w, w'_1, \dots, w_1^{(p_1)}, \dots, w'_{N_w}, \dots, w_{N_w}^{(p_{N_w})}), \quad (\text{II.2})$$

where $w'_j \equiv \frac{\partial w_j}{\partial \xi_i}$ represent the spatial derivative of the w_j $j \in \{1, \dots, N_w\}$ state with respect to its own spatial variable ξ_i $i \in \{1, \dots, N_d\}$. The derivation of dissipative system's equations using the principle of least action is still a subject under research, because of the difficulties in finding a good expression for the work of dissipative forces and on generalizing the Principle of least action. For more details about this subject, we refer to [68]. It is for this reason that in the following we will consider the work of not conservative forces as only composed by the work done by the inputs on the system.

We assume that the Action functional is Fréchet differentiable with respect to all the functions that compose it, in the sense of Definition I.1.7. We denote with $d_\alpha A$ the Fréchet derivative of A with respect to the coordinate α and with h_α the corresponding variation function (that in the literature are usually referred to as *virtual displacements*). We define the Action functional first order variation as

$$dAh = d_q Ah_q + d_{\dot{q}} Ah_{\dot{q}} + d_w Ah_w + d_{\dot{w}} Ah_{\dot{w}} + d_{w'_1} Ah_{w'_1} + \dots + d_{w_1^{(p_1)}} Ah_{w_1^{(p_1)}} + \dots + d_{w'_{N_w}} Ah_{w'_{N_w}} + \dots + d_{w_{N_w}^{(p_{N_w})}} Ah_{w_{N_w}^{(p_{N_w})}} \quad (\text{II.3})$$

for any variation function $h_q, h_{\dot{q}}, h_w, h_{\dot{w}}, h_{w'_i}, \dots, h_{w_i^{(p_i)}}$, $i = \{1, \dots, N_w\}$.

Remark 3.

1. It is important to note that the functions h_α are variations of the corresponding generalized coordinates α , therefore all the constraints on the boundary values of α should be respected also by h_α .
2. Each variation function h_α should belong to the same functions space as the corresponding generalized coordinate α .
3. As in item 1, the variational functions h_α should correspond to the variation of the corresponding generalized coordinates. α Therefore, to be consistent with physics, the variation of the derivative (temporal or spatial) of a generalized coordinate correspond to the derivative (temporal or spatial) of the variation of the generalized coordinate:

$$h_{\dot{w}} = \frac{\partial}{\partial t} h_w, \quad h_{w'} = \frac{\partial}{\partial z} h_w, \quad \dots, \quad h_{w^{(j)}} = \frac{\partial^j}{\partial z^j} h_w. \quad (\text{II.4})$$

Assuming that both the Lagrangian and the work of non-conservative forces are Fréchet differentiable for all the functions, we can write

$$dAh = \int_{t_1}^{t_2} \{d_q Lh_q + d_{\dot{q}} Lh_{\dot{q}} + d_w Lh_w + d_{\dot{w}} Lh_{\dot{w}} + d_{w'_1} Lh_{w'_1} + \dots + d_{w_1^{(p_1)}} Lh_{w_1^{(p_1)}} + \dots + d_{w'_{N_w}} Lh_{w'_{N_w}} + \dots + d_{w_{N_w}^{(p_{N_w})}} Lh_{w_{N_w}^{(p_{N_w})}} + d_q W_{nc} h_q\} dt. \quad (\text{II.5})$$

The principle of least action, together with Theorem I.1.3, states that the true evolution of a system described by the action functional A , satisfies

$$dAh = 0, \quad (\text{II.6})$$

for any h , where according to the principle of least action, we assume $h_q(t_1) = h_q(t_2) = h_w(t_1) = h_w(t_2) = 0$.

II.1.2 Examples

In what follows we show the modelling procedure to obtain the functional analytic model of four different mechanical mechanisms, starting from the general principle of least action stated in the previous section.

II.1.2.a Vibrating string with a tip mass

Several mechanical applications show wave propagation behaviours in their principal dynamics. An example is an overhead crane, which is frequently employed in industrial applications to move loads connected to it through a cable [47]. This type of mechanism can also be used in offshore engineering applications where the precise positioning of a load on the seabed is required [69]. Wave propagation's dynamics are also encountered in drilling systems, where the drill string is subjected to torsional distributed deformations and the bit has to be controlled to the desired speed [70]. In all these applications, the actuator inertia is not negligible, therefore the input acts on dynamic boundary conditions.

In this subsection, we use the modelling procedure introduced earlier to derive the model of a string with a tip mass, on which the previously mentioned examples can be cast. We consider a lossless vibrating string with varying parameters clamped at one side and a tip mass constrained to move in the vertical direction as depicted in Figure II.1.

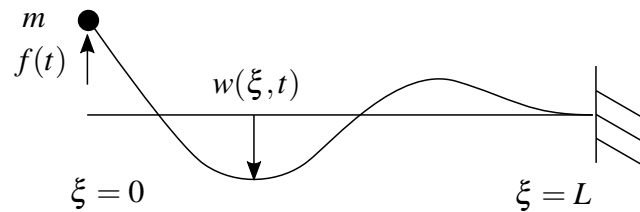


Figure II.1: Vibrating string with tip mass.

The system is considered without gravity: the string deformation evolves in a plane parallel to the ground. If gravity needs to be taken into account, its contribution has to be

added to the potential energy and the model derivation procedure does not change. The term $\xi \in [0, L]$ represents the spatial coordinate, while $w(t, \xi)$ is the deformation of the string at a point ξ and time t . Consider $\rho(\xi)$ and $T(\xi)$ the spatial depending density and tension, respectively. The kinetic E_k and potential E_p energies of the system write

$$E_k(t) = \frac{1}{2} \int_0^L \rho(\xi) \frac{\partial w}{\partial t}(\xi, t)^2 d\xi + \frac{1}{2} m \dot{w}(0, t)^2 \quad E_p(t) = \frac{1}{2} \int_0^L T(\xi) \frac{\partial w}{\partial \xi}(\xi, t)^2 d\xi, \quad (\text{II.7})$$

where since the string is clamped at its right side, the deformation function $w(\xi, t)$ is constrained to be $w(L, t) = 0$. We consider a force input $f(t)$ acting on the mass at $w(0, t)$. Since this input is the only non-conservative force, the work of non-conservative forces writes $W_{nc} = f(t)w(0, t)$. For this example we detail all the tedious Fréchet derivative computations that, for seek of brevity, is skipped in all the next examples. To be consistent with the notation of equation (II.5), we note that $q = w(0, t)$, while no new definition is needed for the distributed variable $w(\xi, t)$. We first note that $d_q L = d_w L = d_{w(i)} = 0$ for $i \geq 2$. Then, to find $d_{\dot{q}} L = d_{\dot{q}} E_k$ we begin by computing

$$\begin{aligned} E_k(\dot{q} + h_{\dot{q}}, \dot{w}) - E_k(\dot{q}, \dot{w}) &= \frac{1}{2} m (\dot{q} + h_{\dot{q}})^2 - \frac{1}{2} m \dot{q}^2 \\ &= m \dot{q} h_{\dot{q}} + \frac{1}{2} m h_{\dot{q}}^2, \end{aligned} \quad (\text{II.8})$$

thus we obtain $d_{\dot{q}} L h_{\dot{q}} = m \dot{q} h_{\dot{q}}$. To find $d_{\dot{w}} L = d_{\dot{w}} E_k$, we compute

$$\begin{aligned} E_k(\dot{q}, \dot{w} + h_{\dot{w}}) - E_k(\dot{q}, \dot{w}) &= \frac{1}{2} \int_0^L \rho \left(\frac{\partial w}{\partial t} + h_{\dot{w}} \right)^2 - \rho \frac{\partial w^2}{\partial t} d\xi \\ &= \int_0^L \rho \frac{\partial w}{\partial t} h_{\dot{w}} + \frac{1}{2} \rho h_{\dot{w}}^2 d\xi \end{aligned} \quad (\text{II.9})$$

thus we obtain $d_{\dot{w}} L h_{\dot{w}} = \int_0^L \rho \frac{\partial w}{\partial t} h_{\dot{w}} d\xi$. Finally, to find $d_{w'} L h_{w'} = -d_{w'} P h_{w'}$ we compute

$$\begin{aligned} E_p(w' + h_{w'}) - E_p(w') &= \frac{1}{2} \int_0^L T \left(\frac{\partial w}{\partial \xi} + h_{w'} \right)^2 - T \frac{\partial w^2}{\partial \xi} d\xi \\ &= \int_0^L T \frac{\partial w}{\partial \xi} h_{w'} + \frac{1}{2} T h_{w'}^2 d\xi \end{aligned} \quad (\text{II.10})$$

such to obtain $d_{w'} L h_{w'} = - \int_0^L T \frac{\partial w}{\partial \xi} h_{w'} d\xi$. The Fréchet derivative of the work of the nonconservative force writes $d_q W_{nc} h_q = f(t) h_q$. Therefore, equation (II.6) for the vibrating string with a tip mass writes

$$\begin{aligned} \int_{t_1}^{t_2} \left\{ \int_0^L \left(\rho \frac{\partial w}{\partial t} h_{\dot{w}} - T \frac{\partial w}{\partial \xi} h_{w'} \right) d\xi + m \dot{w}(0, t) h_{\dot{q}} + f h_q \right\} dt &= 0 \\ \int_{t_1}^{t_2} \left\{ \int_0^L \left(\rho \frac{\partial w}{\partial t} \frac{\partial}{\partial t} (h_w) - T \frac{\partial w}{\partial \xi} \frac{\partial}{\partial \xi} (h_w) \right) d\xi + m \dot{w}(0, t) \frac{d}{dt} (h_q) + f h_q \right\} dt &= 0. \end{aligned} \quad (\text{II.11})$$

Using integration by parts, we obtain

$$\begin{aligned} & \int_0^L \left[\rho \frac{\partial w}{\partial t} h_w \right]_{t_1}^{t_2} d\xi + [m\dot{w}(0,t)h_q]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left\{ - \left[T \frac{\partial w}{\partial \xi} h_w \right]_0^L \right. \\ & \left. + \int_0^L \left[-\frac{\partial}{\partial t} \left(\rho \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial \xi} \left(T \frac{\partial w}{\partial \xi} \right) \right] h_w d\xi + (f(t) - m\ddot{w}(0,t))h_q \right\} dt = 0 \end{aligned} \quad (\text{II.12})$$

and by the fact that the variation functions are null at t_1 and t_2 we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \int_0^L \left[-\frac{\partial}{\partial t} \left(\rho \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial \xi} \left(T \frac{\partial^2 w}{\partial \xi^2} \right) \right] h_w d\xi \right. \\ & \left. - \left[T \frac{\partial w}{\partial \xi} h_w \right]_0^L + (f - m\ddot{w}(0,t))h_q \right\} dt = 0. \end{aligned} \quad (\text{II.13})$$

According to item 1 of Remark 3, to be consistent with the variables' definition $w(0,t) = q$, we have that $h_w(0,t) = h_q(t)$.

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \int_0^L \left[-\frac{\partial}{\partial t} \left(\rho \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial \xi} \left(T \frac{\partial w}{\partial \xi} \right) \right] h_w d\xi \right. \\ & \left. + T(L) \frac{\partial w}{\partial \xi}(L,t) h_w(L,t) + \left(T(0) \frac{\partial w}{\partial \xi}(0,t) - m\ddot{w}(0,t) + f \right) h_q \right\} dt = 0. \end{aligned} \quad (\text{II.14})$$

According to item 1 of Remark 3, the boundary clamping condition at the $\xi = L$ side $w(L,t) = 0$ means that $h_w(L,t) = 0$, and therefore the former equation transforms to

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \int_0^L \left[-\frac{\partial}{\partial t} \left(\rho \frac{\partial w}{\partial t} \right) + \frac{\partial}{\partial \xi} \left(T \frac{\partial w}{\partial \xi} \right) \right] h_w d\xi \right. \\ & \left. + \left(T(0) \frac{\partial w}{\partial \xi}(0,t) - m\ddot{w}(0,t) + f \right) h_q \right\} dt = 0. \end{aligned} \quad (\text{II.15})$$

Since the variation functions h_w, h_q are different from zero, the only possibility of having this equation satisfied is to set the expressions multiplying the variation functions equal to zero. From the last two equations we can extract the dynamic equations of the vibrating string with the tip mass, together with its boundary conditions

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\rho(\xi) \frac{\partial w}{\partial t}(\xi,t) \right) = \frac{\partial}{\partial \xi} \left(T(\xi) \frac{\partial w}{\partial \xi}(\xi,t) \right) \\ & m\ddot{w}(0,t) = T(0) \frac{\partial w}{\partial \xi}(0,t) + f(t) \\ & w(L,t) = 0. \end{aligned} \quad (\text{II.16})$$

At this point we define the energy variable $z = [z_1 \ z_2]^T$ and p

$$z_1(\xi,t) = \rho(\xi) \frac{\partial w}{\partial t}(\xi,t), \quad z_2(\xi,t) = \frac{\partial w}{\partial \xi}(\xi,t), \quad p(t) = m \frac{dw}{dt}(0,t), \quad (\text{II.17})$$

and we define the input output operators for the string equation

$$\begin{aligned} \mathcal{B}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{B}_1(\mathcal{H}z) \\ \mathcal{B}_2(\mathcal{H}z) \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho(0)} z_1(0,t) \\ \frac{1}{\rho(L)} z_1(L,t) \end{bmatrix}, \\ \mathcal{C}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{C}_1(\mathcal{H}z) \\ \mathcal{C}_2(\mathcal{H}z) \end{bmatrix} = \begin{bmatrix} -T(0) z_2(0,t) \\ T(L) z_2(L,t) \end{bmatrix} \end{aligned} \quad (\text{II.18})$$

with energy density defined as

$$\mathcal{H}(\xi) = \begin{bmatrix} \frac{1}{\rho(\xi)} & 0 \\ 0 & T(\xi) \end{bmatrix}. \quad (\text{II.19})$$

The input operators are selected to be the input velocities at both sides of the spatial domain. This choice is motivated by the fact that we have to impose the clamping boundary condition at the $\xi = L$ side of the spatial domain, while the velocity at the $\xi = 0$ side is imposed as a consequence of the mass dynamics. Then, equations (II.16) with the new variable set writes

$$\begin{aligned} \dot{z} &= P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z) = \mathcal{J}z \\ \dot{p} &= -\mathcal{C}_1(\mathcal{H}z) + f(t), \end{aligned} \quad (\text{II.20})$$

with matrix P_1 defined as

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (\text{II.21})$$

and domain of the \mathcal{J} operator

$$\begin{aligned} \mathbf{D}(\mathcal{J}) &= \{z \in L_2([0, L], \mathbb{R}^2) \mid (\mathcal{H}z) \in H^1([0, L], \mathbb{R}^2), \\ &\quad \mathcal{B}_1(\mathcal{H}z) = \frac{1}{m}p, \mathcal{B}_2(\mathcal{H}z) = 0\}. \end{aligned} \quad (\text{II.22})$$

The energy of the vibrating string with a tip mass can be expressed as

$$E(z, p) = \frac{1}{2} \langle z, \mathcal{H}z \rangle_{L_2} + \frac{1}{2m} p^2. \quad (\text{II.23})$$

II.1.2.b Flexible rotating beam

In this subsection, we model a flexible beam clamped on a rotating hub. The majority of publications studying flexible rotating beams use the Euler-Bernoulli model to describe the flexible dynamics [71, 39, 40]. In this section we use the Timoshenko beam assumptions instead of the Euler-Bernoulli ones, obtaining a more detailed dynamic content in the models. In fact, the Euler Bernoulli model can be derived from the Timoshenko beam one, imposing a zero shear deformation [72]. Therefore, the possible applications of the proposed model are the same as the ones implementing Euler-Bernoulli equations. However, in this section we assume a slowly rotating operating condition (assumptions at page 34) [73] to obtain a linear model. We refer to Section V.2.1 for the study of the nonlinear model obtained without simplifying assumptions. The proposed model can be used in a lot of different applications ranging from flexible robotics [42, 44] to satellites with flexible appendices in attitude manoeuvring [63] or structural vibrations of wind turbines [74] (considering a translating instead of a rotating flexible beam with a slight change of boundary conditions).

We consider a frictionless flexible beam connected to a rotating hub as depicted in Figure II.2. The system is considered in absence of gravity: the flexible beam is rotating on a plane parallel to the ground. The rotor angle $\theta(t)$ is a real function of time, while J represents the rotary inertia of the hub to which the beam is connected. The term $\xi \in [0, L]$

identifies the spatial coordinate, while $w(\xi, t)$ and $\phi(\xi, t)$ represents the deflection and the relative rotation of a beam cross section in the rotating frame at position ξ and time t , respectively. $E(\xi), I(\xi)$ are the space depending Young's modulus and moment of inertia of the beam's cross section, respectively. The beam's cross section is assumed to be rectangular, hence its inertia is defined to be $I(\xi) = \frac{L_w(\xi)^3 L_t(\xi)}{12}$, where $L_w(\xi)$ and $L_t(\xi)$ are the width and the thickness of the beam, respectively. $\rho(\xi), I_\rho(\xi)$ are the space dependent density and mass moment of inertia of the beam's cross section, respectively. The mass moment of inertia of the cross section is defined as $I_\rho(\xi) = I(\xi)\rho(\xi)$. $K(\xi)$ is defined as $K(\xi) = kG(\xi)A(\xi)$, where k is a constant depending on the shape of the cross section ($k = 5/6$ for rectangular cross sections), $G(\xi)$ is the Shear modulus and $A(\xi)$ is the cross sectional area. Since the beam is clamped at the $\xi = 0$ side to the rotating hub, we set the reference deformation

$$w(0, t) = \phi(0, t) = 0. \quad (\text{II.24})$$

We write the kinetic energy of the rotating flexible beam as

$$E_k = \frac{1}{2} \int_0^L \{ \rho(\xi) v(\xi, t)^T v(\xi, t) + I_\rho(\xi) r(\xi, t)^2 \} d\xi + \frac{1}{2} J \dot{\theta}^2 \quad (\text{II.25})$$

where $v(\xi, t)$ and $r(\xi, t)$ represent the linear and angular velocity of the beam's cross section at position ξ and time t . To compute the linear velocity of every beam's cross section we begin by defining the position $p(\xi, t)$ of every point of the beam with respect to the fixed frame F_0

$$p(\xi, t) = \begin{bmatrix} \xi \cos(\theta) - w(\xi, t) \sin(\theta) \\ \xi \sin(\theta) + w(\xi, t) \cos(\theta) \end{bmatrix}. \quad (\text{II.26})$$

Differentiating by time we obtain the velocity of every beam's cross section point,

$$v(\xi, t) = \frac{dp}{dt}(\xi, t) = \begin{bmatrix} -\xi \dot{\theta} \sin(\theta) - \dot{w}(\xi, t) \sin(\theta) - w(\xi, t) \dot{\theta} \cos(\theta) \\ \xi \dot{\theta} \cos(\theta) + \dot{w}(\xi, t) \cos(\theta) - w(\xi, t) \dot{\theta} \sin(\theta) \end{bmatrix} \quad (\text{II.27})$$

and its square writes

$$v(\xi, t)^T v(\xi, t) = (\xi \dot{\theta}(t) + \dot{w}(\xi, t))^2 + (w(\xi, t) \dot{\theta}(t))^2. \quad (\text{II.28})$$

We make the following assumptions for the derivation of a linear model

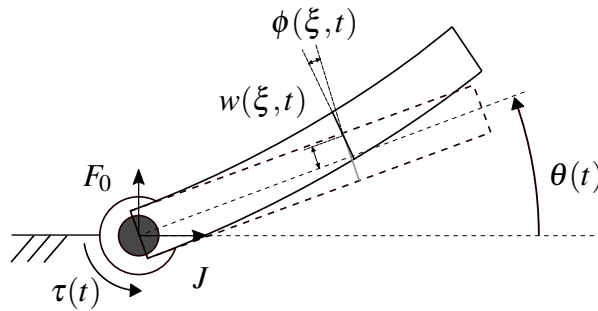


Figure II.2: Rotating flexible Timoshenko's beam.

1. The deformation $w(\xi, t)$ and the angular velocity $\dot{\theta}(t)$ are small in such a way that we can neglect the product $w(\xi, t)\dot{\theta}(t) \approx 0$.
2. The Timoshenko's assumptions hold for the flexible deformations.

Therefore, the square of the linear velocity can be approximated with

$$v(\xi, t)^T v(\xi, t) = (\xi \dot{\theta}(t) + \dot{w}(\xi, t))^2. \quad (\text{II.29})$$

We point out that this type of approximation is standard for the modelling procedure of flexible rotating beams, see for example [31, 44].

The angular velocity of the cross sections along the beam in the fixed frame F_0 is simply the sum between the relative cross section and the hub's angular velocity

$$r(\xi, t) = \dot{\phi}(\xi, t) + \dot{\theta}(t). \quad (\text{II.30})$$

Therefore, the kinetic energy of the system under consideration can be approximated by

$$E_k = \frac{1}{2} \int_0^L \{ \rho(\xi) (\xi \dot{\theta}(t) + \dot{w}(\xi, t))^2 + I_\rho(\xi) (\dot{\phi}(\xi, t) + \dot{\theta}(t))^2 \} d\xi + \frac{1}{2} J \dot{\theta}^2, \quad (\text{II.31})$$

the potential energy, without considering the gravity, follows directly from the Timoshenko assumptions for the beam's flexibility [75]

$$E_p = \int_0^L \left\{ K(\xi) \left(\frac{\partial w}{\partial \xi}(\xi, t) - \phi(\xi, t) \right)^2 + EI(\xi) \frac{\partial \phi}{\partial \xi}(\xi, t)^2 \right\} d\xi, \quad (\text{II.32})$$

while the work of non-conservative forces is only composed by the one done by the input torque acting on the hub $W_{nc} = \tau(t)\theta(t)$. Without expliciting the passages, we compute

$$\begin{aligned} d_{\dot{\theta}} L h_{\dot{\theta}} &= \left(\int_0^L \{ \rho \xi (\dot{w} + \xi \dot{\theta}) + I_\rho (\dot{\phi} + \dot{\theta}) \} d\xi + J \dot{\theta} \right) h_{\dot{\theta}} \\ d_{\dot{w}} L h_{\dot{w}} &= \int_0^L \rho (\dot{w} + \xi \dot{\theta}) h_{\dot{w}} d\xi \\ d_{\dot{\phi}} L h_{\dot{\phi}} &= \int_0^L I_\rho (\dot{\phi} + \dot{\theta}) h_{\dot{\phi}} d\xi \\ d_{\phi} L h_{\phi} &= + \int_0^L K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{\phi} d\xi \\ d_{w'} L h_{w'} &= - \int_0^L K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{w'} d\xi \\ d_{\phi'} L h_{\phi'} &= - \int_0^L EI \frac{\partial \phi}{\partial \xi} h_{\phi'} d\xi \\ d_{\theta} W_{nc} h_{\theta} &= \tau h_{\theta}. \end{aligned} \quad (\text{II.33})$$

Therefore, equation (II.6) for a rotating flexible beam connected to a rotating hub writes

$$\begin{aligned} \int_{t_1}^{t_2} \left\{ \int_0^L \rho (\dot{w} + \xi \dot{\theta}) h_{\dot{w}} + I_\rho (\dot{\phi} + \dot{\theta}) h_{\dot{\phi}} + K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{\phi} - K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{w'} \right. \\ \left. - EI \frac{\partial \phi}{\partial \xi} h_{\phi'} d\xi + \left(\int_0^L \rho \xi (\dot{w} + \xi \dot{\theta}) + I_\rho (\dot{\phi} + \dot{\theta}) d\xi + J \dot{\theta} \right) h_{\dot{\theta}} + \tau h_{\theta} \right\} dt = 0. \end{aligned} \quad (\text{II.34})$$

Using integration by parts, the assumption of null variation in the initial and final time together with $h_\phi = \frac{\partial}{\partial t} h_\phi$, $h_w = \frac{\partial}{\partial t} h_w$, $h_{\phi'} = \frac{\partial}{\partial \xi} h_\phi$ and $h_{w'} = \frac{\partial}{\partial \xi} h_w$, we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \int_0^L \left\{ \left[\frac{\partial}{\partial \xi} K \left(\frac{\partial w}{\partial \xi} - \phi \right) - \frac{\partial}{\partial t} \rho (\dot{w} + \xi \dot{\theta}) \right] h_w + \left[K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right. \right. \right. \\ & + \left. \left. - \frac{\partial}{\partial t} I_\rho (\dot{\phi} + \dot{\theta}) + \frac{\partial}{\partial \xi} EI \frac{\partial \phi}{\partial \xi} \right] h_\phi \right\} d\xi - \left[K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_w \right]_0^L - \left[EI \frac{\partial \phi}{\partial \xi} h_\phi \right]_0^L \\ & \left. - \left[\frac{\partial}{\partial t} \int_0^L \{ \rho \xi (\dot{w} + \xi \dot{\theta}) + I_\rho (\dot{\phi} + \dot{\theta}) \} d\xi + \frac{d}{dt} J \dot{\theta} - \tau \right] h_\theta \right\} dt = 0. \end{aligned} \quad (\text{II.35})$$

From which we can extract the following equations and boundary conditions

$$\begin{aligned} \frac{\partial}{\partial t} (\rho (\dot{w} + \xi \dot{\theta})) &= \frac{\partial}{\partial \xi} \left(K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right) \\ \frac{\partial}{\partial t} (I_\rho (\dot{\phi} + \dot{\theta})) &= \frac{\partial}{\partial \xi} \left(EI \frac{\partial \phi}{\partial \xi} \right) + K \left(\frac{\partial w}{\partial \xi} - \phi \right) \\ \frac{d}{dt} (J \dot{\theta}) &= - \frac{\partial}{\partial t} \int_0^L \{ \rho \xi (\dot{w} + \xi \dot{\theta}) + I_\rho (\dot{\phi} + \dot{\theta}) \} d\xi + \tau \\ K \left(\frac{\partial w}{\partial \xi} (L, t) - \phi (L, t) \right) &= 0, \quad EI \frac{\partial \phi}{\partial \xi} (L, t) = 0. \end{aligned} \quad (\text{II.36})$$

Before defining the energy variables we rewrite the third of the latter set of equations, substituting the two terms inside the integral with the first two of (II.36)

$$\begin{aligned} \frac{d}{dt} J \dot{\theta} &= - \int_0^L \left\{ \xi \frac{\partial}{\partial t} (\rho (\dot{w} + \xi \dot{\theta})) + \frac{\partial}{\partial t} (I_\rho (\dot{\phi} + \dot{\theta})) \right\} d\xi + \tau \\ &= - \int_0^L \left\{ \xi \frac{\partial}{\partial \xi} \left(K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right) + \frac{\partial}{\partial \xi} \left(EI \frac{\partial \phi}{\partial \xi} \right) + K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right\} d\xi + \tau \\ &= - \left[\xi K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right]_0^L - \int_0^L \left\{ \frac{\partial}{\partial \xi} \left(EI \frac{\partial \phi}{\partial \xi} \right) \right\} d\xi + \tau \\ &= - \left[\xi K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right]_0^L - \left[EI \frac{\partial \phi}{\partial \xi} \right]_0^L + \tau, \end{aligned} \quad (\text{II.37})$$

that using the boundary conditions in (II.36), transforms into

$$\frac{d}{dt} J \dot{\theta} = EI \frac{\partial \phi}{\partial \xi} (0, t) + \tau. \quad (\text{II.38})$$

Now, we can define the energy variables

$$z_1 = \rho (\dot{w} + \xi \dot{\theta}) \quad z_2 = I_\rho (\dot{\phi} + \dot{\theta}) \quad z_3 = \frac{\partial w}{\partial \xi} - \phi \quad z_4 = \frac{\partial \phi}{\partial \xi} \quad p = J \dot{\theta} \quad (\text{II.39})$$

and $z = [z_1 \ z_2 \ z_3 \ z_4]^T$ as the state related to the flexible beam. We define the input output operators such that

$$\mathcal{B}(\mathcal{H}z) = \begin{bmatrix} \frac{1}{I_\rho(0)} z_2(0, t) \\ \frac{1}{\rho(0)} z_1(0, t) \\ K(L) z_3(L, t) \\ EI(L) z_4(L, t) \end{bmatrix} \quad \mathcal{C}(\mathcal{H}z) = \begin{bmatrix} -EI(0) z_4(0, t) \\ -K(0) z_3(0, t) \\ \frac{1}{\rho(L)} z_1(L, t) \\ \frac{1}{I_\rho(L)} z_2(L, t) \end{bmatrix} \quad (\text{II.40})$$

and we split each of them in two different operators, such to obtain

$$\begin{aligned}\mathcal{B}_1(\mathcal{H}z) &= \frac{1}{I_p(0)}z_2(0,t) & \mathcal{C}_1(\mathcal{H}z) &= -EI(0)z_4(0,t) \\ \mathcal{B}_2(\mathcal{H}z) &= \begin{bmatrix} \frac{1}{\rho(0)}z_1(0,t) \\ K(L)z_3(L,t) \\ EI(L)z_4(L,t) \end{bmatrix} & \mathcal{C}_2(\mathcal{H}z) &= \begin{bmatrix} -K(0)z_3(0,t) \\ \frac{1}{\rho(L)}z_1(L,t) \\ \frac{1}{I_p(L)}z_2(L,t) \end{bmatrix}\end{aligned}\quad (\text{II.41})$$

where

$$\mathcal{H}(\xi) = \begin{bmatrix} \rho^{-1}(\xi) & 0 & 0 & 0 \\ 0 & I_p^{-1}(\xi) & 0 & 0 \\ 0 & 0 & K(\xi) & 0 \\ 0 & 0 & 0 & EI(\xi) \end{bmatrix}. \quad (\text{II.42})$$

Using the state variable definition (II.39) together with the original system's boundary conditions (II.24), we obtain

$$\begin{aligned}\frac{1}{I_p(0)}z_2(0,t) &= \dot{w}(0,t) = 0 \\ \frac{1}{\rho(0)}z_1(0,t) &= \dot{\phi}(0,t) + \dot{\theta}(t) = \frac{1}{J}p.\end{aligned}\quad (\text{II.43})$$

Therefore, we are able to characterize the system's boundary conditions : $\mathcal{B}_1(\mathcal{H}z) = -\frac{1}{J}p$, $\mathcal{B}_2(\mathcal{H}z) = 0$. Finally, we can rewrite (II.36) with the modified third equation (II.38) in the energy variables as a m-pH system

$$\begin{aligned}\dot{z} &= P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) = \mathcal{J}z \\ \dot{p} &= -\mathcal{C}_1(\mathcal{H}z) + \tau,\end{aligned}\quad (\text{II.44})$$

with matrices defined as

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{II.45})$$

and domain of the \mathcal{J} operator

$$\begin{aligned}\mathbf{D}(\mathcal{J}) &= \{z \in L_2([0,L], \mathbb{R}^4) \mid (\mathcal{H}z) \in H^1([0,L], \mathbb{R}^2), \\ &\quad \mathcal{B}_1(\mathcal{H}z) = \frac{1}{J}p, \mathcal{B}_2(\mathcal{H}z) = 0\}.\end{aligned}\quad (\text{II.46})$$

The energy of the rotating flexible beam can be expressed as

$$E(z, p) = \frac{1}{2} \langle z, \mathcal{H}z \rangle_{L_2} + \frac{1}{2J} p^2. \quad (\text{II.47})$$

II.1.2.c Rotating translating flexible beam

Consider a rotating a translating flexible beam as depicted in Figure II.3. The beam is assumed to move on a plane parallel to the ground, therefore gravity is not considered. The variable and parameters are the same as the one introduced in the previous Section II.1.2.b. Since the beam is also allowed to translate we consider m

the mass of the hub, while $s(t)$ is the hub's linear displacement. The deformations are considered null at the clamped side, *i.e.* the $\xi = 0$ side:

$$w(0, t) = \phi(0, t) = 0. \quad (\text{II.48})$$

Similarly as before, the kinetic energy of the rotating-translating flexible beam takes the form

$$E_k = \frac{1}{2} \int_0^L \{ \rho(\xi) v(\xi, t)^T v(\xi, t) + I_\rho(\xi) r(\xi, t)^2 \} d\xi + \frac{1}{2} J \dot{\theta}(t)^2 + \frac{1}{2} m \dot{s}(t)^2, \quad (\text{II.49})$$

where $v(\xi, t)$ and $r(\xi, t)$ represent the linear and angular velocity of the beam's cross section at position ξ and time t . To compute the linear velocity, we define every cross section beam's position with respect to the fixed frame F_0

$$p(\xi, t) = \begin{bmatrix} \xi \cos(\theta) - w(\xi, t) \sin(\theta) \\ \xi \sin(\theta) + w(\xi, t) \cos(\theta) + s(t) \end{bmatrix}. \quad (\text{II.50})$$

The velocity is obtained differentiating by time the former quantity

$$v(\xi, t) = \frac{dp}{dt}(\xi, t) = \begin{bmatrix} -\xi \dot{\theta} \sin(\theta) - \dot{w}(\xi, t) \sin(\theta) - w(\xi, t) \dot{\theta} \cos(\theta) \\ \xi \dot{\theta} \cos(\theta) + \dot{w}(\xi, t) \cos(\theta) - w(\xi, t) \dot{\theta} \sin(\theta) + \dot{s}(t) \end{bmatrix}. \quad (\text{II.51})$$

In this example, we make the following assumptions for the derivation of a linear model

1. The deformation $w(\xi, t)$ and the angular velocity $\dot{\theta}(t)$ are small in such a way that we can neglect the product $w(\xi, t) \dot{\theta}(t) \approx 0$.
2. The control strategy is such that $\theta(t) \approx 0$. We also assume that the initial condition is near to zero $\theta(0) \approx 0$, to be able to state that $\theta(t) \approx 0 \quad \forall t > 0$.
3. The Timoshenko's assumptions hold for the flexible deformations.

Taking into consideration the previous assumptions, the square of the velocity $v(\xi, t)$ can be written as

$$v(\xi, t)^T v(\xi, t) = (\dot{w}(\xi, t) + \xi \dot{\theta}(t) + \dot{s}(t))^2. \quad (\text{II.52})$$

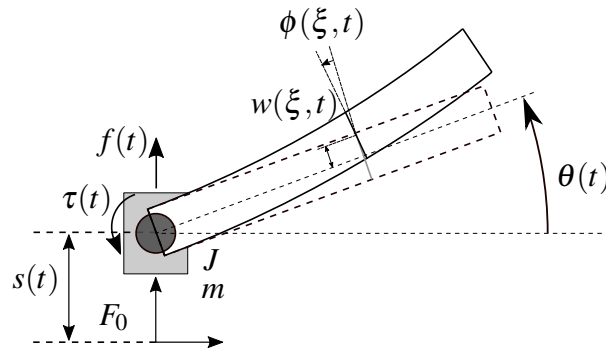


Figure II.3: Translating Rotating flexible Timoshenko's beam.

The angular velocity $r(\xi, t)$ remains as in (II.30), thus the kinetic energy of the system under consideration writes

$$E_k = \frac{1}{2} \int_0^L \left\{ \rho(\xi) (\dot{w}(\xi, t) + \xi \dot{\theta}(t) + s(t))^2 + I_\rho(\xi) (\dot{\phi}(\xi, t) + \dot{\theta}(t))^2 \right\} d\xi + \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m \dot{s}(t)^2, \quad (\text{II.53})$$

the potential energy, following directly from the Timoshenko's assumptions, remains equal to (II.32), while the work of the non-conservative forces has to take into account the force other than the torque input $W_{nc} = f(t)s(t) + \tau(t)\theta(t)$. The necessary Fréchet derivatives to find the equations of motions are obtained as follows

$$\begin{aligned} d_{\dot{\theta}} L h_{\dot{\theta}} &= \left(\int_0^L \left\{ \rho \xi (\dot{w} + \xi \dot{\theta} + s) + I_\rho (\dot{\phi} + \dot{\theta}) \right\} d\xi + J \dot{\theta} \right) h_{\dot{\theta}} \\ d_{\dot{s}} L h_{\dot{s}} &= \left(\int_0^L \left\{ \rho (\dot{w} + \xi \dot{\theta} + s) \right\} d\xi + m \dot{s} \right) h_{\dot{s}} \\ d_{\dot{w}} L h_{\dot{w}} &= \int_0^L \rho (\dot{w} + \xi \dot{\theta} + s) h_{\dot{w}} d\xi \\ d_{\dot{\phi}} L h_{\dot{\phi}} &= \int_0^L I_\rho (\dot{\phi} + \dot{\theta}) h_{\dot{\phi}} d\xi \\ d_{\phi} L h_{\phi} &= \int_0^L K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{\phi} d\xi \\ d_{w'} L h_{w'} &= - \int_0^L K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{w'} d\xi \\ d_{\phi'} L h_{\phi'} &= - \int_0^L EI \frac{\partial \phi}{\partial \xi} h_{\phi'} d\xi \\ d_{\theta} W_{nc} h_{\theta} &= \tau h_{\theta} \quad d_s W_{nc} h_s = \tau h_s. \end{aligned} \quad (\text{II.54})$$

Hence, equation (II.6) for a rotating translating flexible beam writes

$$\begin{aligned} \int_{t_1}^{t_2} \left\{ \int_0^L \left\{ \rho (\dot{w} + \xi \dot{\theta} + s) h_{\dot{w}} + I_\rho (\dot{\phi} + \dot{\theta}) h_{\dot{\phi}} + K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{\phi} \right. \right. \\ \left. \left. - K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{w'} - EI \frac{\partial \phi}{\partial \xi} h_{\phi'} \right\} d\xi + \tau h_{\theta} \right. \\ \left. + \left(\int_0^L \left\{ \rho \xi (\dot{w} + \xi \dot{\theta} + s) + I_\rho (\dot{\phi} + \dot{\theta}) \right\} d\xi + J \dot{\theta} \right) h_{\dot{\theta}} \right. \\ \left. + \left(\int_0^L \rho (\dot{w} + \xi \dot{\theta} + s) d\xi + m \dot{s} \right) h_{\dot{s}} + f h_s \right\} dt = 0. \end{aligned} \quad (\text{II.55})$$

After some very similar passages as the one did in the previous section, the equations of motion together with the boundary conditions are found as follows

$$\begin{aligned} \frac{\partial}{\partial t} (\rho (\dot{w} + \xi \dot{\theta} + s)) &= \frac{\partial}{\partial \xi} \left(K \left(\frac{\partial w}{\partial \xi} - \phi \right) \right) \\ \frac{\partial}{\partial t} (I_\rho (\dot{\phi} + \dot{\theta})) &= \frac{\partial}{\partial \xi} \left(EI \frac{\partial \phi}{\partial \xi} \right) + K \left(\frac{\partial w}{\partial \xi} - \phi \right) \\ \frac{d}{dt} (m \dot{s}) &= K(0) \left(\frac{\partial w}{\partial \xi}(0, t) - \phi(0) \right) + f \\ \frac{d}{dt} (J \dot{\theta}) &= EI(0) \frac{\partial \phi}{\partial \xi}(0, t) + \tau \\ K \left(\frac{\partial w}{\partial \xi}(L, t) - \phi(L, t) \right) &= 0, \quad EI \frac{\partial \phi}{\partial \xi}(L, t) = 0. \end{aligned} \quad (\text{II.56})$$

The energy variables are defined by

$$\begin{aligned} z_1 &= \rho(\dot{w} + \xi \dot{\theta} + \dot{s}) & z_2 &= I_\rho(\dot{\phi} + \dot{\theta}) & z_3 &= \frac{\partial w}{\partial \xi} - \phi & z_4 &= \frac{\partial \phi}{\partial \xi} \\ p_1 &= m\dot{s} & p_2 &= J\dot{\theta} \end{aligned} \quad (\text{II.57})$$

and $z = [z_1 \ z_2 \ z_3 \ z_4]^T$ is the flexible beam's state, and $p = [p_1 \ p_2]^T$ is the state related to the hub's motion. The input output operators are defined as in (II.40), but in this case, the input output operators are split as follows

$$\begin{aligned} \mathcal{B}_1(\mathcal{H}z) &= \begin{bmatrix} \frac{1}{\rho(0)} z_1(0, t) \\ \frac{1}{I_\rho(0)} z_2(0, t) \end{bmatrix} & \mathcal{C}_1(\mathcal{H}z) &= - \begin{bmatrix} K(0) z_3(0, t) \\ EI(0) z_4(0, t) \end{bmatrix} \\ \mathcal{B}_2(\mathcal{H}z) &= \begin{bmatrix} K(L) z_3(L, t) \\ EI(L) z_4(L, t) \end{bmatrix} & \mathcal{C}_2(\mathcal{H}z) &= \begin{bmatrix} \frac{1}{\rho(L)} z_1(L, t) \\ \frac{1}{I_\rho(L)} z_2(L, t) \end{bmatrix} \end{aligned} \quad (\text{II.58})$$

where $\mathcal{H}(\xi)$ is defined as in II.42. Equations of motion (II.56) in the energy variables coordinates, transform into a m-pH system

$$\begin{aligned} \dot{z} &= P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) = \mathcal{J}z \\ \dot{p} &= -\mathcal{C}_1(\mathcal{H}z) + u(t), \end{aligned} \quad (\text{II.59})$$

with $u(t) = [f(t) \ \tau(t)]^T$ and P_0, P_1 defined as in (II.45). The domain of the \mathcal{J} operator is defined as

$$\mathbf{D}(\mathcal{J}) = \{z \in L_2([0, L], \mathbb{R}^4) \mid z \in H^1([0, L], \mathbb{R}^2), \mathcal{B}_1(\mathcal{H}z) = Qp, \mathcal{B}_2(\mathcal{H}z) = 0\} \quad (\text{II.60})$$

where $Q = \text{diag}[m^{-1}, J^{-1}]$. The energy of the overall mechanism can be expressed as:

$$E = \frac{1}{2} \langle z, \mathcal{H}z \rangle_{L_2} + \frac{1}{2} p^T Q p. \quad (\text{II.61})$$

II.1.2.d Double flexible manipulator

In this section we consider a double flexible-link manipulator as depicted in Figure II.4. A possible application of the model proposed in this section is a flexible gripper as the one described in [45]. The system is composed by two flexible links connected with actuated revolute joints, i.e. motors. The motor fixed to the ground has only the shaft moving, while the other has both the stator and the shaft participating to the motion.

$\theta_1(t)$ and $\theta_2(t)$ represent the absolute angular rotation of the first and second rotor, respectively. With $\xi_1 \in [0, L_1]$ and $\xi_2 \in [0, L_2]$ we identify the spatial coordinates along the beams. The deflection of the two beams with respect to their own axis ξ_1 and ξ_2 , has been denoted with $w_1(\xi_1, t)$ and $w_2(\xi_2, t)$, while with $\phi_1(\xi_1, t)$ and $\phi_2(\xi_2, t)$ have been defined the relative (with respect to their own frame) rotation of the beam cross section. The beams are supposed to have a constant rectangular cross section width $L_{w,i}$, thickness $L_{t,i}$ and area $A_{s,i} = L_{w,i} L_{t,i}$, $i = \{1, 2\}$. All the physical parameters of the system are positive real, and their meaning are given as follows: $I_{h,1}, I_{h,21}, I_{h,22}, I_{h,3}, m_{h,2}, m_{h,3}$

represent respectively the rotary inertia of the shaft of the first motor, of the stator of the second motor, of the shaft of the second motor and of the payload; $m_{h,2}, m_{h,3}$ represent respectively the mass of the second motor and of the tip payload at the end of the second link. The beam's related parameters are defined as in section II.1.2.b for each flexible link. In accordance with the previous examples, we perform the following assumptions to obtain linear PDEs describing the beam's deformation dynamics

1. The deformations $w_1(\xi_1, t)$ and $w_2(\xi_2, t)$ and the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ are small in such a way that we can neglect the products $w_1(\xi_1, t)\dot{\theta}_1, w_2(\xi_2, t)\dot{\theta}_2 \approx 0$.
2. The Timoshenko's assumptions hold for the flexible deformation of each link.

The deformations of each beam are considered null at $\xi_1, \xi_2 = 0$ side:

$$w_1(0, t) = w_2(0, t) = \phi_1(0, t) = \phi_2(0, t) = 0. \quad (\text{II.62})$$

The kinetic energy E_k is composed by the one related to the distributed flexible parts and the one related to the lumped parameter system, *i.e.*:

$$\begin{aligned} E_k = & \frac{1}{2} \int_0^{L_1} \{ \rho_1(\xi_1) v_1(\xi_1, t)^T v_1(\xi_1, t) + I_{\rho 1}(\xi_1) r_1(\xi_1, t)^2 \} d\xi_1 \\ & + \frac{1}{2} \int_0^{L_2} \{ \rho_2(\xi_2) v_2(\xi_2, t)^T v_2(\xi_2, t) + I_{\rho 2}(\xi_2) r_2(\xi_2, t)^2 \} d\xi_2 \\ & + \frac{1}{2} I_{h,1} \dot{\theta}_1^2 + \frac{1}{2} I_{h,21} r_{h,21}(t)^2 + \frac{1}{2} m_{h,2} v_{h,2}(t)^T v_{h,2}(t) + \frac{1}{2} I_{h,22} \dot{\theta}_2^2 \\ & + \frac{1}{2} m_{h,3} v_{h,3}(t)^T v_{h,3}(t) + \frac{1}{2} I_{h,31} r_{h,31}(t)^2 \end{aligned} \quad (\text{II.63})$$

where $v_1(\xi_1, t)$ and $v_2(\xi_2, t)$ represent the linear velocities of the first and second beam, respectively, while $r_1(\xi_1, t)$ and $r_2(\xi_2, t)$ represent the angular velocities of the first and second beam, respectively. $r_{h,21}(t) = r_1(L_1, t)$ is the angular velocity of the stator of the second hub; $v_{h,2}(t) = v_1(L_1, t)$ is the linear velocity of the second hub; $v_{h,3}(t) = v_2(L_2, t)$ is the linear velocity of the end effector and $r_{h,31}(t) = r_2(L_2, t)$ its angular velocity. From now on we shall not explicit the time and space dependency of the variables, unless it

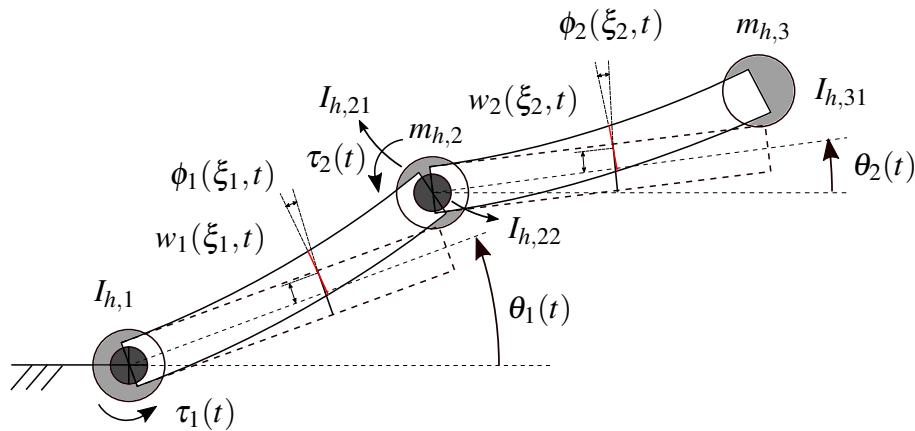


Figure II.4: Double flexible rotating Timoshenko's beam manipulator.

is not clear from the context. Similarly to Section II.1.2.b, and taking into account the assumptions at page 41, we obtain the linear and angular velocities v_1 , r_1 as follows

$$\begin{aligned} v_1 &= \begin{bmatrix} -\xi_1 \dot{\theta}_1 \sin(\theta_1) - \dot{w}_1 \sin(\theta_1) \\ \xi_1 \dot{\theta}_1 \cos(\theta_1) + \dot{w}_1 \cos(\theta_1) \end{bmatrix} \\ r_1 &= \dot{\phi}_1 + \dot{\theta}_1, \end{aligned} \quad (\text{II.64})$$

and consequently we obtain

$$\begin{aligned} v_{h,2} &= \begin{bmatrix} -L_1 \dot{\theta}_1 \sin(\theta_1) - \dot{w}_1(L_1) \sin(\theta_1) \\ L_1 \dot{\theta}_1 \cos(\theta_1) + \dot{w}_1 \cos(\theta_1) \end{bmatrix} \\ r_{h,21} &= \dot{\phi}_1(L_1) + \dot{\theta}_1. \end{aligned} \quad (\text{II.65})$$

The position of every point of the second beam is expressed by

$$p_2 = \begin{bmatrix} L_1 \cos(\theta_1) - w_1(L_1) \sin(\theta_1) + \xi_2 \cos(\theta_2) - w_2 \sin(\theta_2) \\ L_1 \sin(\theta_1) + w_1(L_1) \cos(\theta_1) + \xi_2 \sin(\theta_2) + w_2 \cos(\theta_2) \end{bmatrix}, \quad (\text{II.66})$$

and differentiating with respect to time we obtain

$$v_2 = \frac{dp_2}{dt} = \begin{bmatrix} L_1 \sin(\theta_1) \dot{\theta}_1 - \dot{w}_1(L_1) \sin(\theta_1) - w_1(L_1) \dot{\theta}_1 \cos(\theta_1) \\ -\xi_2 \dot{\theta}_2 \sin(\theta_2) - \dot{w}_2 \sin(\theta_2) - w_2 \dot{\theta}_2 \cos(\theta_2) \\ L_1 \cos(\theta_1) \dot{\theta}_1 + \dot{w}_1(L_1) \cos(\theta_1) - w_1(L_1) \dot{\theta}_1 \sin(\theta_1) \\ +\xi_2 \dot{\theta}_2 \cos(\theta_2) + \dot{w}_2 \cos(\theta_2) - w_2 \dot{\theta}_2 \sin(\theta_2) \end{bmatrix}. \quad (\text{II.67})$$

Using the assumptions at page 41, the linear velocity of each point of the second beam can be approximated as:

$$v_2 \approx \begin{bmatrix} -L_1 \sin(\theta_1) \dot{\theta}_1 - \dot{w}_1(L_1) \sin(\theta_1) - \xi_2 \dot{\theta}_2 \sin(\theta_2) - \dot{w}_2 \sin(\theta_2) \\ L_1 \cos(\theta_1) \dot{\theta}_1 + \dot{w}_1(L_1) \cos(\theta_1) + \xi_2 \dot{\theta}_2 \cos(\theta_2) + \dot{w}_2 \cos(\theta_2) \end{bmatrix}, \quad (\text{II.68})$$

while the angular velocity of every point of the second beam writes as

$$r_2 = \dot{\phi}_2 + \dot{\theta}_2. \quad (\text{II.69})$$

From the last two quantities we can derive the linear and angular velocities of the end effector

$$\begin{aligned} v_{h,3} &= \begin{bmatrix} -L_1 \sin(\theta_1) \dot{\theta}_1 - \dot{w}_1(L_1) \sin(\theta_1) - L_2 \dot{\theta}_2 \sin(\theta_2) - \dot{w}_2 \sin(\theta_2) \\ L_1 \cos(\theta_1) \dot{\theta}_1 + \dot{w}_1(L_1) \cos(\theta_1) + L_2 \dot{\theta}_2 \cos(\theta_2) + \dot{w}_2 \cos(\theta_2) \end{bmatrix} \\ r_{h,31} &= \dot{\phi}_2(L_2) + \dot{\theta}_2. \end{aligned} \quad (\text{II.70})$$

Substituting equations (II.64)-(II.65) and (II.68)-(II.70) in (II.63), we obtain the kinetic energy of the mechanism

$$\begin{aligned}
 E_k = & \frac{1}{2} \int_0^{L_1} \left\{ \rho_1 (\dot{w}_1 + \xi_1 \dot{\theta}_1)^2 + I_{\rho_1} (\dot{\phi}_1 + \dot{\theta}_1)^2 \right\} d\xi_1 + \frac{1}{2} I_{h,1} \dot{\theta}_1^2 \\
 & + \frac{1}{2} I_{h,21} (\dot{\phi}(L_1) + \dot{\theta}_1)^2 + \frac{1}{2} (m_{h,2} + m_2 + m_{h,3}) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)^2 \\
 & + \frac{1}{2} \int_0^{L_2} \left\{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2)^2 + I_{\rho_2} (\dot{\phi}_2 + \dot{\theta}_2)^2 \right\} d\xi_2 + \frac{1}{2} I_{h,22} \dot{\theta}_2^2 \\
 & + \frac{1}{2} m_{h,3} (\dot{w}_2(L_2) + L_2 \dot{\theta}_2)^2 + m_{h,3} \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) (\dot{w}_2(L_2) \\
 & + L_2 \dot{\theta}_2) + (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \cos(\theta_2 - \theta_1) \int_0^{L_2} \left\{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \right\} d\xi_2 \\
 & + \frac{1}{2} I_{h,31} (\dot{\phi}_2(L_2) + \dot{\theta}_2)^2
 \end{aligned} \tag{II.71}$$

with $m_2 = \int_0^{L_2} \rho_2 d\xi_2$. The potential energy follows directly from the Timoshenko's beam assumptions

$$\begin{aligned}
 E_p = & \frac{1}{2} \int_0^{L_1} \left[K_1 \left(\frac{\partial w_1}{\partial \xi_1} - \phi_1 \right)^2 + EI_1 \left(\frac{\partial \phi_1}{\partial \xi_1} \right)^2 \right] d\xi_1 \\
 & + \frac{1}{2} \int_0^{L_2} \left[K_2 \left(\frac{\partial w_2}{\partial \xi_2} - \phi_2 \right)^2 + EI_2 \left(\frac{\partial \phi_2}{\partial \xi_2} \right)^2 \right] d\xi_2,
 \end{aligned} \tag{II.72}$$

while the work of non-conservative forces, that also in this case corresponds to the input torques, can be written as $W_{nc} = \tau_1 \theta_1 + \tau_2 \theta_2$. In the following we list the Fréchet derivatives of the lagrangian $L = E_k - E_p$ with respect to all the generalized variables that compose them

$$\begin{aligned}
 d_{\dot{\theta}_1} L h_{\dot{\theta}_1} = & \left(I_{h,1} \dot{\theta}_1 + \int_0^{L_1} \left\{ \rho_1 \xi_1 (\dot{w}_1 + \xi_1 \dot{\theta}_1) + I_{\rho_1} (\dot{\phi}_1 + \dot{\theta}_1) \right\} d\xi_1 \right. \\
 & + L_1 \cos(\theta_2 - \theta_1) \left[\int_0^{L_2} \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) d\xi_2 \right. \\
 & + m_{h,3} (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) \left. \right] + I_{h,21} (\dot{\phi}_1(L_1) + \dot{\theta}_1) \\
 & + L_1 (m_2 + m_{h,2} + m_{h,3}) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \left. \right) h_{\dot{\theta}_1} \\
 d_{\dot{\theta}_2} L h_{\dot{\theta}_2} = & \left(I_{h,22} \dot{\theta}_2 + \int_0^{L_2} \left\{ \rho_2 \xi_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) + I_{\rho_2} (\dot{\phi}_2 + \dot{\theta}_2) \right\} d\xi_2 \right. \\
 & + m_{h,3} (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) + m_{h,3} L_2 \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1, t) \\
 & + L_1 \dot{\theta}_1) + \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \int_0^{L_2} \rho_2 \xi_2 d\xi_2 \\
 & + \frac{1}{2} I_{h,31} (\dot{\phi}_2(L_2) + \dot{\theta}_2) \left. \right) h_{\dot{\theta}_2} \\
 d_{\dot{w}_1} L h_{\dot{w}_1} = & \int_0^{L_1} \rho_1 (\dot{w}_1 + \xi_1 \dot{\theta}_1) h_{\dot{w}_1} d\xi_1 \\
 d_{\dot{w}_2} L h_{\dot{w}_2} = & \int_0^{L_2} \rho_2 [(\dot{w}_2 + \xi_2 \dot{\theta}_2) + \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)] h_{\dot{w}_2} d\xi_2 \\
 d_{\dot{\phi}_1} L h_{\dot{\phi}_1} = & \int_0^{L_1} I_{\rho_1} (\dot{\phi}_1 + \dot{\theta}_1) h_{\dot{\phi}_1} d\xi_1
 \end{aligned}$$

$$\begin{aligned}
 d_{\dot{\phi}_2} L h_{\dot{\phi}_2} &= \int_0^{L_2} I_{\rho 2} (\dot{\phi}_2 + \dot{\theta}_2) h_{\dot{\phi}_2} d\xi_2 \\
 d_{\dot{\phi}_1} L h_{\dot{\phi}_1} &= \int_0^{L_1} K_1 \left(\frac{\partial w_1}{\partial \xi_1} - \phi_1 \right) h_{\dot{\phi}_1} d\xi_1 \\
 d_{\dot{\phi}_2} L h_{\dot{\phi}_2} &= \int_0^{L_2} K_2 \left(\frac{\partial w_2}{\partial \xi_2} - \phi_2 \right) h_{\dot{\phi}_2} d\xi_2 \\
 d_{\theta_1} L h_{\theta_1} &= \left[m_{h,3} \sin(\theta_2 - \theta_1) (\dot{w}_1(L_1, t) + L_1 \dot{\theta}_1) (\dot{w}_2(L_2, t) + L_2 \dot{\theta}_2) \right. \\
 &\quad \left. + \sin(\theta_2 - \theta_1) (\dot{w}_1(L_1, t) + L_1 \dot{\theta}_1) \int_0^{L_2} \{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \} d\xi_2 \right] h_{\theta_1} \\
 d_{\theta_2} L h_{\theta_2} &= \left[-m_{h,3} \sin(\theta_2 - \theta_1) (\dot{w}_1(L_1, t) + L_1 \dot{\theta}_1) (\dot{w}_2(L_2, t) + L_2 \dot{\theta}_2) \right. \\
 &\quad \left. - \sin(\theta_2 - \theta_1) (\dot{w}_1(L_1, t) + L_1 \dot{\theta}_1) \int_0^{L_2} \{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \} d\xi_2 \right] h_{\theta_2} \\
 d_{w'_1} L h_{w'_1} &= - \int_0^{L_1} K_1 \left(\frac{\partial w_1}{\partial \xi_1} - \phi_1 \right) h_{w'_1} d\xi_1 \\
 d_{w'_2} L h_{w'_2} &= - \int_0^{L_2} K_2 \left(\frac{\partial w_2}{\partial \xi_2} - \phi_2 \right) h_{w'_2} d\xi_2 \\
 d_{\phi'_1} L h_{\phi'_1} &= - \int_0^{L_1} EI_1 \frac{\partial \phi_1}{\partial \xi_1} h_{\phi'_1} d\xi_1 \\
 d_{\phi'_2} L h_{\phi'_2} &= - \int_0^{L_2} EI_2 \frac{\partial \phi_2}{\partial \xi_2} h_{\phi'_2} d\xi_2 \\
 d_{\dot{w}_1(L_1)} L h_{\dot{w}_1(L_1)} &= \left[(m_{h,2} + m_2 + m_{h,3}) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) + m_{h,3} \cos(\theta_2 - \theta_1) (\dot{w}_2(L_2) \right. \\
 &\quad \left. + L_2 \dot{\theta}_2) + \cos(\theta_2 - \theta_1) \int_0^{L_2} \{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \} d\xi_2 \right] h_{\dot{w}_1(L_1)} \\
 d_{\dot{w}_2(L_2)} L h_{\dot{w}_2(L_2)} &= m_{h,3} (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) h_{\dot{w}_2(L_2)} \\
 d_{\dot{\phi}_1(L_1)} L h_{\dot{\phi}_1(L_1)} &= I_{h,21} (\dot{\phi}_1(L_1) + \dot{\theta}_1) h_{\dot{\phi}_1(L_1)} \\
 d_{\dot{\phi}_2(L_2)} L h_{\dot{\phi}_2(L_2)} &= I_{h,31} (\dot{\phi}_2(L_2) + \dot{\theta}_2) h_{\dot{\phi}_2(L_2)}.
 \end{aligned}$$

We substitute the Fréchet derivative of the previous equation in equation (II.6), and after applying integration by parts with respect to time and space as done in the previous examples, we obtain the dynamic equations of the double flexible manipulator. In this example we omit the tedious integration by parts procedure, and we directly show the resulting equations.

The ordinary differential equation governing the dynamics of the Shaft of the first motor θ_1 writes

$$\begin{aligned}
 &\frac{d}{dt} (I_{h,1} \dot{\theta}_1) + \frac{\partial}{\partial t} \int_0^{L_1} \{ \rho_1 \xi_1 (\dot{w}_1 + \xi_1 \dot{\theta}_1) + I_{\rho 1} (\dot{\phi}_1 + \dot{\theta}_1) \} d\xi_1 \\
 &+ \frac{d}{dt} (L_1 (m_{h,2} + m_2 + m_3) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)) + \frac{d}{dt} (I_{h,21} (\dot{\phi}_1(L_1) + \dot{\theta}_1)) \\
 &+ \frac{\partial}{\partial t} L_1 \cos(\theta_2 - \theta_1) \left(\int_0^{L_2} \{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \} d\xi_2 + m_{h,3} (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) \right) = \\
 &(\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \sin(\theta_2 - \theta_1) \left(\int_0^{L_2} \{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \} d\xi_2 \right. \\
 &\left. + m_{h,3} (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) \right) + \tau_1 - \tau_2,
 \end{aligned} \tag{II.73}$$

while the ordinary differential equation governing the dynamics of the Shaft of the second motor θ_2 is given as follows:

$$\begin{aligned} & \frac{d}{dt} (I_{h,21} \dot{\theta}_2) + \frac{\partial}{\partial t} \int_0^{L_2} \{ \rho_2 \xi_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) + I_{\rho_2} (\dot{\phi}_2 + \dot{\theta}_2) \\ & + \frac{\partial}{\partial t} (\rho_2 \xi_2 (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \cos(\theta_2 - \theta_1)) \} d\xi_2 + \frac{d}{dt} (I_{h,3} (\dot{\phi}_2(L_1) + L_1 \dot{\theta}_1)) \\ & \frac{d}{dt} (L_2 m_{h,3} ((\dot{w}_2(L_2) + L_2 \dot{\theta}_2) + \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1))) \\ & = \tau_2 - \sin(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \int_0^{L_2} \{ \rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) \} d\xi_2. \end{aligned} \quad (\text{II.74})$$

The set of PDE describing the absolute movement and the elastic deformations of the first flexible beam writes:

$$\begin{cases} \frac{\partial}{\partial t} (\rho_1 (\dot{w}_1 + \xi_1 \dot{\theta}_1)) = \frac{\partial}{\partial \xi_1} \left(K_1 \left(\frac{\partial w_1}{\partial \xi_1} - \phi_1 \right) \right) \\ \frac{\partial}{\partial t} (I_{\rho_1} (\dot{\phi}_1 + \dot{\theta}_1)) = \frac{\partial}{\partial \xi_1} \left(EI_1 \frac{\partial \phi_1}{\partial \xi_1} \right) + K_1 \left(\frac{\partial w_1}{\partial \xi_1} - \phi_1 \right). \end{cases} \quad (\text{II.75})$$

The two above PDE describe the translational and the rotational dynamics of every cross section of the first beam, respectively. The first two associated boundary conditions corresponds to the fixed deformation reference $w(0,t) = \phi(0,t) = 0$, while the last two are ODE describing the translational and rotational dynamics at the $\xi_1 = L_1$ beam's side, and they write as

$$\begin{aligned} & K_1 \left(\frac{\partial w_1}{\partial \xi_1}(L_1) - \phi(L_1) \right) + \frac{\partial}{\partial t} \int_0^{L_2} \{ \cos(\theta_2 - \theta_1) \rho_2 ((\dot{w}_2) + \xi_2 \dot{\theta}_2) \} d\xi_2 \\ & + \frac{d}{dt} (m_{h,3} ((\dot{w}_1(L_1) + L_1 \dot{\theta}_1) + \cos(\theta_2 - \theta_1) (\dot{w}_2(L_2) + L_2 \dot{\theta}_2))) \\ & + \frac{d}{dt} ((m_{h,2} + m_2) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)) = 0, \end{aligned} \quad (\text{II.76})$$

$$EI_1 \frac{\partial \phi_1}{\partial \xi_1}(L_1) + \tau_2 + \frac{d}{dt} I_{h,21} (\dot{\theta}_1 + \dot{\phi}_1(L_1)) = 0. \quad (\text{II.77})$$

The set of PDE describing the absolute movement and the elastic deformation of the second flexible beam writes

$$\begin{cases} \frac{\partial}{\partial t} (\rho_2 (\dot{w}_2 + \xi_2 \dot{\theta}_2) + \rho_2 (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \cos(\theta_2 - \theta_1)) = \frac{\partial}{\partial \xi_2} \left(K_2 \left(\frac{\partial w_2}{\partial \xi_2} - \phi_2 \right) \right) \\ \frac{\partial}{\partial t} (I_{\rho_2} (\dot{\phi}_2 + \dot{\theta}_2)) = \frac{\partial}{\partial \xi_2} \left(EI_2 \frac{\partial \phi_2}{\partial \xi_2} \right) + K_2 \left(\frac{\partial w_2}{\partial \xi_2} - \phi_2 \right) \end{cases} \quad (\text{II.78})$$

with boundary conditions $w_2(0,t) = \phi(0,t) = 0$, and

$$\begin{aligned} & K_2 \left(\frac{\partial w_2}{\partial \xi_2}(L_2,t) - \phi(L_2,t) \right) + \frac{d}{dt} (m_{h,3} ((L_2 \dot{\theta}_2 + \dot{w}_2(L_2)) \\ & + \cos(\theta_2 - \theta_1) (L_1 \dot{\theta}_1 + \dot{w}_1(L_1)))) = 0 \end{aligned} \quad (\text{II.79})$$

$$EI_2 \frac{\partial \phi_2}{\partial \xi_2}(L_2) + \frac{\partial}{\partial t} I_{h,31} (\dot{\theta}_2 + \dot{\phi}_2(L_2)) = 0. \quad (\text{II.80})$$

The energy variables of the infinite dimensional systems (II.75) (II.78), are defined as

$$\begin{aligned}
 z_{1,1} &= \rho_1 \left(\frac{\partial w_1}{\partial t} + z_1 \dot{\theta}_1 \right) & z_{2,1} &= \rho_2 \left(\left(\frac{\partial w_2}{\partial t} + z_2 \dot{\theta}_2 \right) + \right. \\
 & & & \left. (L_1 \dot{\theta}_1 + \dot{w}_1(L_1)) \cos(\theta_2 - \theta_1) \right) \\
 z_{1,2} &= I_\rho \left(\frac{\partial \phi}{\partial t} + \dot{\theta}_1 \right) & z_{2,2} &= I_{\rho_2} \left(\frac{\partial \phi}{\partial t} + \dot{\theta}_2 \right) \\
 z_{1,3} &= \frac{\partial w_1}{\partial z_1} - \phi_1 & z_{2,3} &= \frac{\partial w_2}{\partial z_2} - \phi_2 \\
 z_{1,4} &= \frac{\partial \phi_1}{\partial z} & z_{2,4} &= \frac{\partial \phi_2}{\partial z_2},
 \end{aligned} \tag{II.81}$$

where $z_1 = [z_{1,1} \ z_{1,2} \ z_{1,3} \ z_{1,4}]^T$ and $z_2 = [z_{2,1} \ z_{2,2} \ z_{2,3} \ z_{2,4}]^T$ are the states of the first and second flexible beam, respectively. The input output operators for both flexible beams are defined as

$$\begin{aligned}
 \mathcal{B}_{1,1}(\mathcal{H}_1 z_1) &= \begin{bmatrix} I_\rho^{-1} z_{1,2}(0) \\ \rho_i^{-1} z_{1,1}(L) \\ I_\rho^{-1} z_{1,2}(L) \end{bmatrix} & \mathcal{C}_{1,1}(\mathcal{H}_1 z_1) &= \begin{bmatrix} -EI_1 z_{1,4}(0) \\ K_1 z_{1,3}(L) \\ EI_1 z_{1,4}(L) \end{bmatrix} \\
 \mathcal{B}_{1,2}(\mathcal{H}_1 z_1) &= \rho_i^{-1} z_{1,1}(0) & \mathcal{C}_{1,2}(\mathcal{H}_1 z_1) &= -K_1 z_{1,3}(0) \\
 \mathcal{B}_{2,1}(\mathcal{H}_2 z_2) &= \begin{bmatrix} \rho_i^{-1} z_{1,1}(0) \\ I_\rho^{-1} z_{1,2}(0) \\ \rho_i^{-1} z_{1,1}(L) \\ I_\rho^{-1} z_{1,2}(L) \end{bmatrix} & \mathcal{C}_{2,1}(\mathcal{H}_2 z_2) &= \begin{bmatrix} -K_1 z_{1,3}(0) \\ -EI_1 z_{1,4}(0) \\ K_1 z_{1,3}(L) \\ EI_1 z_{1,4}(L) \end{bmatrix}
 \end{aligned} \tag{II.82}$$

where $\mathcal{H}_i(\xi_i) = \text{diag}[\rho_i^{-1}(\xi_i) \ I_{\rho,i}^{-1}(\xi_i) \ K_i(\xi_i) \ EI_i(\xi_i)]$. Equations (II.75) and (II.78) in the new variables (II.81) transforms into two boundary control pH Systems

$$\begin{aligned}
 \dot{z}_i &= \frac{\partial}{\partial \xi_i} P_1(\mathcal{H}_i z_i) + P_0(\mathcal{H}_i z_i) = \mathcal{J}_{i,b} z_i \quad i = \{1, 2\} \\
 \mathcal{B}_{i,1} z_i &= u_{z,i}(t) \quad \mathcal{C}_{i,1} z_i = y_{z,i}(t) \\
 \mathcal{B}_{i,2} z_i &= 0
 \end{aligned} \tag{II.83}$$

with matrices P_0 and P_1 defined as in (II.45). The energy of each flexible beam can be expressed as

$$E_{b,i} = \frac{1}{2} \int_0^{L_i} z_i^T \mathcal{H}_i z_i d\xi_i = \int_0^{L_i} H_{b,i} d\xi_i \tag{II.84}$$

where $H_{b,i}$ represent the energy densities.

The next step is to rewrite the obtained set ODE in the pH format thanks to an appropriate change of variables. Before doing so, we substitute the infinite dimension energy variables (II.81) computed at $z_1 = L_1$ in the payload equations (II.79)-(II.80)

$$\frac{d}{dt} (m_{h,3} [(\dot{w}_2(L_2) + L_2 \dot{\theta}_2) + \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)]) = -K_2 z_{2,3}(L_2), \tag{II.85}$$

$$\frac{d}{dt} I_{h,3} (\dot{\phi}_2(L_2) + \dot{\theta}_2) = -EI z_{2,4}(L_2). \tag{II.86}$$

After several developments, and using equation (II.85), the ODE describing the boundary translational dynamics of the first beam (II.76) writes

$$\begin{aligned} \frac{d}{dt} (m_I (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)) &= -K_1 z_{1,1}(L_1) + \cos(\theta_2 - \theta_1) K_2 z_{2,1}(0) \\ &+ (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2 - \theta_1) \int_0^{L_2} \{ \rho_2 [(\dot{w}_2 + \xi_2 \dot{\theta}_2) + \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)] \} d\xi_2 \\ &+ m_{h,3} (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_2 - \theta_1) [(\dot{w}_2(L_2) + L_2 \dot{\theta}_2) + \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)]. \end{aligned} \quad (\text{II.87})$$

The *virtual mass* term $m_I : \mathbb{R}^2 \mapsto \mathbb{R}^+$ depends on the angle configuration of the manipulator, and it is defined as follows:

$$m_I(\theta_1, \theta_2) = m_{h,2} + (m_2 + m_{h,3}) \sin^2(\theta_2 - \theta_1) > 0.$$

Similarly, substituting the infinite dimensional energy variables (II.81) in the boundary rotational dynamics of equation (II.77) leads to

$$\frac{d}{dt} I_{h,21} (\dot{\phi}_1(L_1) + \dot{\theta}_1) = -EI_1 z_{1,2}(L_1) - \tau_2.$$

Using (II.81), (II.83) and (II.87), (II.86), the dynamic equations of the two motors' shaft are derived from (II.73) and (II.74):

$$\begin{aligned} \frac{d}{dt} (I_{h,1} \dot{\theta}_1) &= EI_1 z_{1,2}(0) + \int_0^{L_2} \rho_2 (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \sin(\theta_2 - \theta_1) (\dot{w}_2 + \xi_2 \dot{\theta}_2) d\xi_2 \\ &+ m_{h,3} (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \sin(\theta_2 - \theta_1) (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) + \tau_1 \end{aligned} \quad (\text{II.88})$$

$$\begin{aligned} \frac{d}{dt} (I_{h,21} \dot{\theta}_2) &= EI_2 z_{2,4}(0) - \int_0^{L_2} \rho_2 (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \sin(\theta_2 - \theta_1) (\dot{w}_2 + \xi_2 \dot{\theta}_2) d\xi_2 \\ &- m_{h,3} (\dot{w}_1(L_1) + L_1 \dot{\theta}_1) \sin(\theta_2 - \theta_1) (\dot{w}_2(L_2) + L_2 \dot{\theta}_2) + \tau_2. \end{aligned} \quad (\text{II.89})$$

To define the pH representation of the above boundary dynamics, the energy states of the set of boundary dynamic equations (II.85)-(II.89) are defined as follows:

$$\begin{aligned} p_1 &= I_{h,1} \dot{\theta}_1, & p_2 &= m_I (\dot{w}_1(L_1) + L_1 \dot{\theta}_1), \\ p_3 &= I_{h,21} (\dot{\phi}_1(L_1) + \dot{\theta}_1), & p_4 &= I_{h,22} \dot{\theta}_2, \\ p_5 &= m_{h,3} [(\dot{w}_2(L_2) + L_2 \dot{\theta}_2) \\ &+ \cos(\theta_2 - \theta_1) (\dot{w}_1(L_1) + L_1 \dot{\theta}_1)] & p_6 &= I_{h,3} (\dot{\phi}_2(L_2) + \dot{\theta}_2) \\ q_1 &= \theta_1, & q_2 &= \theta_2. \end{aligned} \quad (\text{II.90})$$

The state of the ODE set is defined as $x_r = [p_1 \ p_2 \ p_3 \ p_4 \ p_5 \ p_6 \ q_1 \ q_2]^T \in X_r$. To write the pH formulation of the ODE set, the boundary conditions terms inside the ODE are considered as inputs, that are in turn related to the boundary outputs of the PDE. Thus, the input of the finite dimensional system is divided in three vectors. The first input vector collects the two torques applied in the first and second joints, the other two input vectors are used for the interconnection with the first and second set of PDE, respectively:

$$u_r = \begin{bmatrix} u_{r1} \\ u_{r2} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \quad u_1 = \begin{bmatrix} u_{r3} \\ u_{r4} \\ u_{r5} \end{bmatrix} = \begin{bmatrix} EI_1 \varepsilon_{1,r}(0) \\ K_1 \varepsilon_{1,t}(L_1) \\ EI_1 \varepsilon_{1,r}(L_1) \end{bmatrix}, \quad u_2 = \begin{bmatrix} u_{r6} \\ u_{r7} \\ u_{r8} \\ u_{r9} \end{bmatrix} = \begin{bmatrix} K_2 \varepsilon_{2,t}(0) \\ EI_2 \varepsilon_{2,r}(0) \\ K_2 \varepsilon_{2,t}(L_2) \\ EI_2 \varepsilon_{2,r}(L_2) \end{bmatrix}. \quad (\text{II.91})$$

The set of nonlinear ODE (II.85)-(II.89) can be written in the pH format as:

$$\begin{cases} \dot{x}_r = J_r(x_r, x_b) dH_{x_r} + g_r u_r + g_1 u_1 + g_2(x_r) u_2 \\ y_r = g_r^T dH_{x_r} \\ y_1 = g_1^T dH_{x_r} \\ y_2 = g_2(x_r)^T dH_{x_r}, \end{cases} \quad (\text{II.92})$$

where the skew-symmetric interconnection matrix $J_r(x_r, z_2) = -J_r(x_r, z_2)^T$ is defined as

$$J_r = \begin{bmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & -1 & 0 \\ -\alpha & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\text{II.93})$$

where the function $\alpha : \mathbb{R}^3 \times L_2([0, L_2], \mathbb{R}^1) \mapsto \mathbb{R}$ is given as:

$$\alpha(q_1, q_2, p_5, z_{2,1}) = \sin(q_2 - q_1) \left(\int_0^{L_2} z_{2,1} d\xi_2 + p_5 \right), \quad (\text{II.94})$$

while the input matrices are defined as

$$g_r = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ +\cos(q_2 - q_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\text{II.95})$$

dH_{x_r} represents the vector composed of all the Fréchet derivatives with respect to all the components of x_r of the energy

$$H_r = \frac{1}{2} x_r^T Q_r(x_r) x_r, \quad (\text{II.96})$$

where $Q_r = \text{diag} \left[I_{h,1}^{-1} m_l(q_1, q_2)^{-1} I_{h,21}^{-1} I_{h,22}^{-1} m_{h,3}^{-1} I_{h,3}^{-1} 0 0 \right]$ is the energy matrix related to the finite dimensional part. According to the states (II.81)-(II.90), and the input output definitions (II.91)-(II.92), the interconnection relations between the boundary control systems and the nonlinear set of ODE are defined as

$$\begin{aligned} u_{z,1} &= y_1 & u_{z,2} &= y_2 \\ u_1 &= -y_{z,1} & u_2 &= -y_{z,2} \end{aligned} \quad (\text{II.97})$$

According to the latter interconnection relations, the global system can be represented by the following equations

$$\begin{aligned}\dot{z}_1 &= P_1 \frac{\partial}{\partial \xi_1}(\mathcal{H}_1 z_1) + P_0(\mathcal{H}_1 z_1) = \mathcal{J}_1 z_1 \\ \dot{z}_2 &= P_1 \frac{\partial}{\partial \xi_2}(\mathcal{H}_2 z_2) + P_0(\mathcal{H}_2 z_2) = \mathcal{J}_2 z_2 \\ \dot{v} &= J_r dH_v - g_1 \mathcal{C}_{1,1}(\mathcal{H}_1 z_1) - g_2 \mathcal{C}_{2,1}(\mathcal{H}_2 z_2) + g_r u\end{aligned}\quad (\text{II.98})$$

and the global energy of the double flexible manipulator can be written as

$$E = \frac{1}{2} \int_0^{L_1} z_1^T \mathcal{H}_1 z_1 d\xi_1 + \frac{1}{2} \int_0^{L_2} z_2^T \mathcal{H}_2 z_2 d\xi_2 + \frac{1}{2} x_r^T Q_r(x_r) x_r. \quad (\text{II.99})$$

The operators \mathcal{J}_1 and \mathcal{J}_2 in (II.98) are defined with domain

$$\begin{aligned}\mathbf{D}(\mathcal{J}_1) &= \{z_1 \in Z \mid \mathcal{H}_1 z_1 \in H^1([0, L_1], \mathbb{R}^4), \mathcal{B}_{1,1}(\mathcal{H}_1 z_1) = g_1^T dH_v, \mathcal{B}_{1,2}(\mathcal{H}_1 z_1) = 0\} \\ \mathbf{D}(\mathcal{J}_2) &= \{z_2 \in Z \mid \mathcal{H}_2 z_2 \in H^1([0, L_2], \mathbb{R}^4), \mathcal{B}_{2,1}(\mathcal{H}_2 z_2) = g_2^T dH_v\}.\end{aligned}\quad (\text{II.100})$$

It is worth to remark that differently from the previous models, where both the describing ODE and PDE were linear, the model of the double flexible manipulator derived in this section has a nonlinear set of ODE. Moreover, the boundary conditions of both set of PDE depend nonlinearly from the finite dimensional states $\mathcal{B}_{1,1}(\mathcal{H}_1 z_1) = g_1^T dH_v$, $\mathcal{B}_{2,1}(\mathcal{H}_2 z_2) = g_2^T dH_v$. This implies that nonlinearities are present in the domain of both the PDE. The system described by equations (II.98)-(II.100) with homogeneous input $u = 0$, can be written as a semilinear operator equation with state $x = [z_1^T \ z_2^T \ v^T]^T \in X = Z \times Z \times \mathbb{R}^8$

$$\begin{aligned}\dot{x} &= \begin{bmatrix} \mathcal{J}_1 & 0 & 0 \\ 0 & \mathcal{J}_2 & 0 \\ -g_1 \mathcal{C}_{1,1} \mathcal{H}_1 & -g_2 \mathcal{C}_{2,1} \mathcal{H}_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ J_r dH_v \end{bmatrix} \\ &= Ax + f(x),\end{aligned}\quad (\text{II.101})$$

and domain

$$\mathbf{D}(A) = \{x \in X \mid \mathcal{H}_1 z_1, \mathcal{H}_2 z_2 \in H^1, \mathcal{B}_{1,1}(\mathcal{H}_1 z_1) = g_1^T dH_v, \mathcal{B}_{1,2}(\mathcal{H}_1 z_1) = 0, \mathcal{B}_{2,1}(\mathcal{H}_2 z_2) = g_2^T dH_v\}.\quad (\text{II.102})$$

The existence of solutions of this semilinear equations, in the sense of Definition 1.1.10 is not investigated in this manuscript.

II.1.3 Passivity of general mixed pH systems

In this section we define a general dynamic system able to cope with all the previous systems and we discuss about its passivity properties. The system's passivity is a key property in the control design and can help in defining a control law capable to guarantee the closed-loop stability.

Let k the number of 1-D dpH boundary control systems

$$\begin{cases} \dot{z}_1 = P_1 \frac{\partial}{\partial \xi_1}(\mathcal{H}_1 z_1) + P_0(\mathcal{H}_1 z_1) = \tilde{\mathcal{J}}_1(\mathcal{H}_1 z_1) = \mathcal{J}_1 z_1 \\ \vdots \\ \dot{z}_k = P_1 \frac{\partial}{\partial \xi_k}(\mathcal{H}_k z_k) + P_0(\mathcal{H}_k z_k) = \tilde{\mathcal{J}}_k(\mathcal{H}_k z_k) = \mathcal{J}_k z_k \end{cases}\quad (\text{II.103})$$

where $z_i \in L_2([a_i, b_i], \mathbb{R}^{n_i}) \forall i = \{1, 2, \dots, k\}$ are the associated states, and $\xi_i \in [a_i, b_i]$ are the spatial variables. Let $\mathcal{H}_i \in L^\infty([a_i, b_i], \mathbb{R}^{n_i \times n_i})$ such that $\mathcal{H}_i(\xi_i)^T = \mathcal{H}_i(\xi_i)$, $m_i I \leq \mathcal{H}_i \leq M_i I$ for almost every $\xi_i \in [a_i, b_i]$ and constants $M_i > m_i > 0$ independent of ξ_i for $i = \{1, \dots, k\}$.

Remark 4. In this section we define the operator $\tilde{\mathcal{J}}_k$ that is basically the same operator as \mathcal{J} but it does not include the energy density \mathcal{H} :

$$\tilde{\mathcal{J}} z = P_1 \frac{\partial}{\partial \xi} z + P_1 z. \quad (\text{II.104})$$

with domain

$$\mathbf{D}(\tilde{\mathcal{J}}) = \{z \in Z \mid z \in H^1, \mathcal{B}_1(\mathcal{H}z) = u_z, \mathcal{B}_2(\mathcal{H}z) = 0\}. \quad (\text{II.105})$$

and boundary operators defined in (II.106). Note that $\tilde{\mathcal{J}}_k$ is skew-adjoint and item 1 of Lemma I.2.1 holds in the L_2 space equipped with the standard norm (and not in the state space Z as for \mathcal{J}). We define this slightly different version of \mathcal{J} to highlight the general m-pH system relation with the Fréchet derivatives of its global energy.

Each boundary operator is compound by two different components, and is formulated in the boundary effort and flow variables of Definition I.2.1

$$\begin{aligned} \mathcal{B}_i(\mathcal{H}_i z_i) &= \begin{bmatrix} \mathcal{B}_{i,1}(\mathcal{H}_i z_i) \\ \mathcal{B}_{i,2}(\mathcal{H}_i z_i) \end{bmatrix} = \begin{bmatrix} W_{B1,i} \\ W_{B2,i} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ \mathcal{C}_i(\mathcal{H}_i z_i) &= \begin{bmatrix} \mathcal{C}_{i,1}(\mathcal{H}_i z_i) \\ \mathcal{C}_{i,2}(\mathcal{H}_i z_i) \end{bmatrix} = \begin{bmatrix} W_{C1,i} \\ W_{C2,i} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \end{aligned} \quad (\text{II.106})$$

where $W_{B1,i}, W_{C1,i} \in \mathbb{R}^{m_i \times 2n_i}$ and $W_{B2,i}, W_{C2,i} \in \mathbb{R}^{(n_i - m_i) \times 2n_i}$ are appropriate matrices such that the following assumption holds through.

Assumption 2. The matrices $W_B = \begin{bmatrix} W_{B1,i} \\ W_{B2,i} \end{bmatrix}$ are $n_i \times 2n_i$ full rank matrices for all $i \in \{1, 2, \dots, k\}$.

Then, we can define the input output operators related to each dpH boundary control system as

$$\begin{aligned} u_{z,i} &= \mathcal{B}_{i,1}(\mathcal{H}_i z_i) \quad 0 = \mathcal{B}_{i,2}(\mathcal{H}_i z_i) \\ y_{z,i} &= \mathcal{C}_{i,1}(\mathcal{H}_i z_i). \end{aligned} \quad (\text{II.107})$$

The energy related to each pH boundary control system is defined as

$$E_i = \frac{1}{2} \langle z_i, \mathcal{H}_i z_i \rangle_{L_2} = \frac{1}{2} \int_{a_i}^{b_i} z_i^T \mathcal{H}_i z_i d\xi_i = \int_{a_i}^{b_i} H_i d\xi_i \quad (\text{II.108})$$

where H_i is the energy density of the i -th pH boundary control system. We denote with dH_{i,z_i} the Fréchet derivative of the energy density H_i with respect to the state z_i .

Corollary II.1.1. The Fréchet derivatives vector dH_{i,z_i} can be computed as

$$dH_{i,z_i}^T = \mathcal{H}_i z_i. \quad (\text{II.109})$$

Proof. We start by computing

$$\begin{aligned} H_i(z_i + h_{z_i}) &= \frac{1}{2}(z_i + h_{z_i})^T \mathcal{H}_i(z_i + h_{z_i}) \\ &= \frac{1}{2}z_i^T \mathcal{H}_i z_i + \frac{1}{2}z_i^T \mathcal{H}_i h_{z_i} + \frac{1}{2}h_{z_i}^T \mathcal{H}_i z_i + \frac{1}{2}h_{z_i}^T \mathcal{H}_i h_{z_i} \\ &= \frac{1}{2}z_i^T \mathcal{H}_i z_i + z_i^T \mathcal{H}_i h_{z_i} + \frac{1}{2}h_{z_i}^T \mathcal{H}_i h_{z_i}. \end{aligned} \quad (\text{II.110})$$

Then, from the Fréchet derivative's Definition (I.1.7), we know that dH_{z_i} corresponds to the first order coefficient in h_{z_i} of

$$H_i(z_i + h_{z_i}) - H_i(z_i) = z_i^T \mathcal{H}_i h_{z_i} + \frac{1}{2}h_{z_i}^T \mathcal{H}_i h_{z_i}, \quad (\text{II.111})$$

therefore we obtain (II.109). \square

Using equality (II.109), we rewrite the set of boundary control pH systems (II.103) in the following form

$$\begin{cases} \dot{z}_1 = P_1 \frac{\partial}{\partial \xi_1} dH_{z_1}^T + P_0 dH_{z_1}^T = \bar{\mathcal{J}}_1 dH_{z_1}^T \\ \vdots \\ \dot{z}_k = P_1 \frac{\partial}{\partial \xi_k} dH_{z_k}^T + P_0 dH_{z_k}^T = \bar{\mathcal{J}}_k dH_{z_k}^T. \end{cases} \quad (\text{II.112})$$

Consider a nonlinear input-state-output finite dimensional pH system defined as

$$\begin{cases} \dot{v} = J(z, v) dH_v^T + g_v(z, v)u + g_1(z, v)u_{v,1} + \cdots + g_k(z, v)u_{v,k} \\ y = g_v(z, v)^T dH_v^T \\ y_{v,1} = g_1(z, v)^T dH_v^T \\ \vdots \\ y_{v,k} = g_k(z, v)^T dH_v^T \end{cases} \quad (\text{II.113})$$

where $v \in \mathbb{R}^{n_v}$ is the finite dimensional state, $z = [z_1 \cdots z_k]^T$ is the vector containing all the infinite dimensional states, $u \in \mathbb{R}^{m_u}$ is the vector of external inputs and $u_i \in \mathbb{R}^{m_i}$ are the inputs used to interconnect the finite dimensional system with the infinite dimensional systems. $J(z, v) = -J(z, v)^T$ is the skew symmetric interconnection matrix, $g_v(z, v)$ is a state dependent $n_v \times m_u$ input matrix and $g_i(z, v)$ are the $n_v \times m_i$ state dependent input interconnection matrices. H_v represents the finite dimensional energy, while dH_v is its Fréchet derivative with respect to the finite dimensional state variable v . Let the systems (II.112) and (II.113) be interconnected by the following power preserving interconnections for $i = \{1, \dots, k\}$

$$u_{v,i} = -y_{z,i} \quad u_{z,i} = y_{v,i}. \quad (\text{II.114})$$

Then, the interconnected system can be written in the following form

$$\begin{cases} \dot{x} = \bar{\mathcal{J}} dH^T + g(z, v)u \\ y = g(z, v)^T dH^T \end{cases} \quad (\text{II.115})$$

with state $x = [z_1 \cdots z_k v]^T \in X = L_2([a_1, b_1], \mathcal{R}^{n_1}) \times \cdots \times L_2([a_k, b_k], \mathcal{R}^{n_k}) \times \mathbb{R}^{n_v}$, operator $\bar{\mathcal{J}}$, vector dH and input matrix $g(z, v)$ defined as

$$\bar{\mathcal{J}} = \begin{bmatrix} \bar{\mathcal{J}}_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{\mathcal{J}}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{\mathcal{J}}_k & 0 \\ -g_1(z, v)\mathcal{C}_{1,1} & -g_2(z, v)\mathcal{C}_{2,1} & \cdots & -g_k(z, v)\mathcal{C}_{k,1} & J(z, v) \end{bmatrix} \quad (\text{II.116})$$

$$dH^T = \begin{bmatrix} dH_{1,z_1}^T \\ dH_{2,z_2}^T \\ \vdots \\ dH_{k,z_k}^T \\ dH_v^T \end{bmatrix} \quad g(z, v) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_v(z, v) \end{bmatrix}$$

with operator's domain

$$\mathbf{D}(\bar{\mathcal{J}}) = \{x \in X \mid \mathcal{H}_i z_i \in H^1([a_i, b_i], \mathbb{R}^{n_i}), \mathcal{B}_{i,1}(\mathcal{H}_i z_i) = g_i(z, v)^T dH_v^T, \mathcal{B}_{i,2}(\mathcal{H}_i z_i) = 0 \ \forall i \in \{1, \dots, k\}\}. \quad (\text{II.117})$$

Remark 5. We make the following observations regarding this system.

1. The defined operator $\bar{\mathcal{J}}$ is a nonlinear operator because it contains the state dependent input matrices and interconnection matrix. Moreover, the nonlinearities are present also at the boundaries of the spatial domains, as can be seen in the domain Definition (II.117).
2. The vector dH is defined as the composition of different Fréchet derivatives, but is not a Fréchet derivative itself. This because it is not possible to define a common function H whose Fréchet derivative corresponds to the definition of dH .

We define the energy of the overall interconnected system

$$E = \sum_{i=1}^k E_i + H_v \quad (\text{II.118})$$

and we assume that it is Fréchet differentiable.

Proposition II.1.1. *Consider the system (II.115) with operator's domain defined in (II.117). Then, this system is passive with respect to the energy functional (II.118), i.e.:*

$$\dot{E}_+(x_0) = y^T u \quad \forall x_0 \in X. \quad (\text{II.119})$$

Proof. Consider the Dini time derivative of Definition I.1.8 evaluated in x_0 , that thanks to Lemma I.1.2 we know that $\dot{E}_+ = dE(\bar{\mathcal{J}} dH + gu)$, where dE is the Fréchet derivative of E . We start by explicit the Fréchet derivative of E

$$dE = [dE_{z_1}^T \ dE_{z_2}^T \ \cdots \ dE_{z_k}^T \ dH_v^T], \quad (\text{II.120})$$

where after some simple computations we can find that

$$dE_{z_i} h_{z_i} = \int_{a_i}^{b_i} (\mathcal{H}_i z_i)^T h_{z_i} d\xi_i \quad \forall i \in \{1, 2, \dots, k\}. \quad (\text{II.121})$$

Then, we can compute

$$\begin{aligned} \dot{E}_+ = & \sum_{i=1}^k \left[\int_{a_i}^{b_i} (\mathcal{H}_i z_i)^T \bar{\mathcal{J}}_i(\mathcal{H}_i z_i) d\xi_i - dH_v g_i(z, v) \mathcal{C}_i(\mathcal{H}_i z_i) \right] \\ & + dH_v (J(z, v) dH_v + g_v(z, v) u) \end{aligned} \quad (\text{II.122})$$

that using Lemma I.2.1 ($\langle \mathcal{H}_i z_i, \bar{\mathcal{J}}_i \mathcal{H}_i z_i \rangle_{L_2} = \langle z_i, \mathcal{J} z_i \rangle_X$) and the skew symmetry of $J(z, v)$, can be rewritten as

$$\begin{aligned} \dot{E}_+ = & \sum_{i=1}^k \left[y_{z,i}^T u_{z,i} - \mathcal{C}_i(\mathcal{H}_i z_i)^T g_i(z, v)^T dH_v^T \right] + y^T u \\ = & \sum_{i=1}^k \left[y_{z,i}^T u_{z,i} - y_{z,i}^T u_{z,i} \right] + y^T u \\ = & y^T u. \end{aligned} \quad (\text{II.123})$$

□

Remark 6. All the previously presented mechanical models fit in the framework of system (II.115)-(II.117). Therefore, all of them are passive with respect to their proper global energy functional obtained by the sum between the energy associated to every single infinite dimensional pH system and the energy associated to the interconnected finite dimensional pH system.

II.2 DISCRETIZATION OF FLEXIBLE STRUCTURES

Numerical simulations of dynamical systems are used for mainly two reasons: compare the numerical simulations to real system data such to verify the validity of a proposed mathematical model and analyse the system time-behaviour when subjected to initial conditions and/or external input. Even if the model's validity is a fundamental step in systems' study with dynamical equations, it is not be discussed in this Thesis. In Chapter III and IV, numerical simulations are used to extract additional information about the system's behaviour when they are not available from the theoretical results. To be able to numerically simulate the system's time responses, the first step is to approximate the (possibly) present PDE with a set of ODE. This procedure is normally known as *discretization*. In the context of pH systems, several methods allow obtaining a finite dimensional pH system as the discretization of a dpH system. In [76] the author proposed a finite difference discretization scheme for dpH systems up to the 2 dimensional case. A structure-preserving partitioned finite element method has been proposed in [77] for the discretization of a 2D wave equation. In the following, we use the mixed finite element method proposed in [62] to obtain the finite dimensional approximation of the Timoshenko's beam equations and of the wave equations in all the input-output

configurations obtained in the previous examples. Always in [62], it has been shown that the discretization error (*i.e.* the difference between the exact solution and the one obtained through simulation of the discretized equations) of the proposed discretization scheme is of order $1/n^2$. This means that the larger the number of discretizing elements, the smaller the error of the obtained simulations with respect to the exact solution.

Let consider the state space $Z = L_2([0, L], \mathbb{R}^{n_q}) \times L_2([0, L], \mathbb{R}^{n_p})$ where $n_p + n_q = n$, and an element of the state space is denoted by $z = [q \ p]^T$. We define the flow and effort spaces as:

$$\mathcal{F} = L_2([0, L], \mathbb{R}^{n_q}) \times L_2([0, L], \mathbb{R}^{n_p}) \quad \mathcal{E} = H^1([0, L], \mathbb{R}^{n_q}) \times H^1([0, L], \mathbb{R}^{n_p}) \quad (\text{II.124})$$

with elements belonging to these space defined by $f = [f^q \ f^p]^T$ and $e = [e^q \ e^p]^T$, respectively. The energy associated to the 1-D dpH is defined as

$$E(t) = \frac{1}{2} \int_a^b z(\xi, t)^T \mathcal{H}(\xi) z(\xi, t) d\xi = \int_a^b H_z(\xi, t) d\xi \quad (\text{II.125})$$

where $\mathcal{H} = \text{diag}(\mathcal{H}_q, \mathcal{H}_p)$ is the parameters' density, and $H_z = H_q + H_p$ the energy's density. The effort and flow variables are defined with respect to the state of the system through the following constitutive relations

$$\begin{aligned} f^q(\xi, t) &= \frac{\partial q}{\partial t}(\xi, t) & f^p(\xi, t) &= \frac{\partial p}{\partial t}(\xi, t) \\ e^q(\xi, t) &= dH_{q,q}(\xi, t) & e^p(\xi, t) &= dH_{p,p}(\xi, t) \end{aligned} \quad (\text{II.126})$$

that are equivalent to

$$f(\xi, t) = \frac{\partial z}{\partial t}(\xi, t) \quad e(\xi, t) = dH_z(\xi, t) \quad (\text{II.127})$$

where $dH_{\alpha,\beta}$ stands for the Fréchet derivative of dH_α with respect to the state β .

Definition II.2.1. We define the boundary flow f_B and effort e_B in the sense of [62] as the e^p and e^q efforts computed at the boundaries of the spatial domain

$$\begin{aligned} f_B &= \begin{bmatrix} f_{Ba} \\ f_{Bb} \end{bmatrix} = \begin{bmatrix} e^p(a, t) \\ e^p(b, t) \end{bmatrix} = \begin{bmatrix} dH_{p,p}(a, t) \\ dH_{p,p}(b, t) \end{bmatrix} \\ e_B &= \begin{bmatrix} e_{Ba} \\ e_{Bb} \end{bmatrix} = \begin{bmatrix} e^q(a, t) \\ e^q(b, t) \end{bmatrix} = \begin{bmatrix} dH_{q,q}(a, t) \\ dH_{q,q}(b, t) \end{bmatrix}. \end{aligned} \quad (\text{II.128})$$

■

Remark 7. In the remainder of this chapter, we refer to the *boundary flow and effort* as the one defined in the sense of [62] in the previous Definition.

According to the effort and flow definition (II.127) and Corollary II.1.1, the 1-D dpH equations in the flow and effort variables writes

$$f = P_1 \frac{\partial}{\partial \xi} e + P_0 e. \quad (\text{II.129})$$

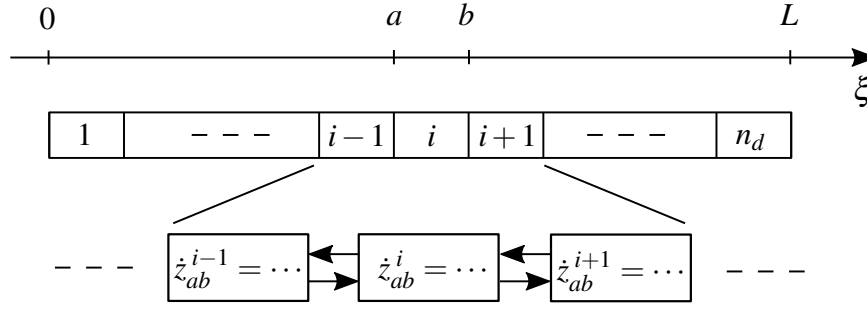


Figure II.5: Representation of a 1D PDE discretization.

Let n_d the number of discretizing element. In the following we first find the finite dimensional model of a single discretized element, then we interconnect all the elements to obtain the overall system as illustrated in Figure II.5. The flux f and the effort e are approximated in an interval $\xi \in [a, b]$, $a < b$, $b - a = Ln_d^{-1}$, $a, b \in [0, L]$ as follows:

$$f(\xi, t) = f_{ab}^z(t) w_{ab}(\xi) \quad e(\xi, t) = e_a(t) w_a(\xi) + e_b(t) w_b(\xi), \quad (\text{II.130})$$

where the function $f_{ab}^z(t) = [f_{ab}^p(t) \ f_{ab}^q(t)]^T$, identifies the approximated flow variable in the $[a, b]$ interval, while

$$e_a = \begin{bmatrix} e_a^p \\ e_a^q \end{bmatrix} = \begin{bmatrix} f_{Ba} \\ e_{Ba} \end{bmatrix} \quad e_b = \begin{bmatrix} e_b^p \\ e_b^q \end{bmatrix} = \begin{bmatrix} f_{Bb} \\ e_{Bb} \end{bmatrix} \quad (\text{II.131})$$

are the efforts computed at the $[a, b]$ sides, that themselves correspond to the flow and boundary efforts. According to Assumption 1 and 2 of [62], the base function $w_{ab}(z)$ should satisfy $\int_a^b w_{ab}(z) dz = 1$, while $w_a(z)$, $w_b(z)$ should satisfy $w_a(a) = 1$, $w_a(b) = 0$, $w_b(a) = 0$ and $w_b(b) = 1$. Hence, the base functions can be chosen as:

$$w_a(\xi) = -\frac{z}{b-a} + \frac{b}{b-a}, \quad w_b(\xi) = \frac{z}{b-a} - \frac{a}{b-a}, \quad w_{ab}(\xi) = \frac{1}{b-a}. \quad (\text{II.132})$$

We define the effort state in the $[a, b]$ interval as the mean value of the efforts at the $[a, b]$ boundaries:

$$e_{ab}^z(t) = \frac{1}{2}(e_a(t) + e_b(t)) \quad (\text{II.133})$$

where $e_{ab}^z(t) = [e_{ab}^p(t) \ e_{ab}^q(t)]^T$. Substituting the flow and effort approximation (II.131) and using the boundary effort and flow definitions (II.130) in equation (II.129) and (II.133), it is possible to get the following relation for both the Timoshenko's beam and the wave equation

$$\underbrace{\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & W_1 & W_2 \\ 0 & 0 & M & M \\ 0 & 0 & 0 & 0 \end{bmatrix}}_W \underbrace{\begin{bmatrix} f_{ab}^p \\ f_{ab}^q \\ f_{Ba} \\ f_{Bb} \end{bmatrix}}_{f_{ab}} + \underbrace{\begin{bmatrix} 0 & 0 & F_1 & F_2 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & M & M \end{bmatrix}}_F \underbrace{\begin{bmatrix} e_{ab}^p \\ e_{ab}^q \\ e_{Ba} \\ e_{Bb} \end{bmatrix}}_{e_{ab}} = 0 \quad (\text{II.134})$$

where for the wave equation the matrices are defined as

$$W_1 = 1 \quad W_2 = -1 \quad F_1 = 1 \quad F_2 = -1 \quad M = -\frac{1}{2}, \quad (\text{II.135})$$

while for the Timoshenko's beam equation

$$\begin{aligned} W_1 &= \begin{bmatrix} 1 & -\frac{b-a}{2} \\ 0 & 1 \end{bmatrix} & W_2 &= \begin{bmatrix} -1 & -\frac{b-a}{2} \\ 0 & -1 \end{bmatrix} & F_1 &= \begin{bmatrix} 1 & 0 \\ -\frac{b-a}{2} & 1 \end{bmatrix} \\ F_2 &= \begin{bmatrix} -1 & 0 \\ -\frac{b-a}{2} & -1 \end{bmatrix} & M &= \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}. \end{aligned} \quad (\text{II.136})$$

To write the explicit input state output equations of a single discretized model, it is necessary to split the energy in n_d components. To do so, we define the energy in the $[a, b]$ interval as

$$E_{ab} = \int_a^b z(\xi, t)^T \mathcal{H}(\xi) z(\xi, t) d\xi. \quad (\text{II.137})$$

As we did for the flow and effort variables, we approximate the state variables such that

$$z(\xi, t) = z_{ab}(t) w_{ab}(\xi), \quad (\text{II.138})$$

with $z_{ab} = [q_{ab}^T p_{ab}^T]^T$. We substitute the approximated state variable and the base function's definition in (II.137) to get

$$\begin{aligned} E_{ab} &= \int_a^b z_{ab}(t)^T w_{ab}(\xi)^2 \mathcal{H}(\xi) z_{ab}(t) d\xi \\ &= z_{ab}(t)^T \int_a^b w_{ab}(\xi)^2 \mathcal{H}(\xi) d\xi z_{ab}(t) \\ &= z_{ab}(t)^T \int_a^b \frac{1}{(b-a)^2} \mathcal{H}(\xi) d\xi z_{ab}(t). \end{aligned} \quad (\text{II.139})$$

We define the energy matrix in the $\xi \in [a, b]$ interval

$$H_{ab} = \int_a^b \frac{1}{(b-a)^2} \mathcal{H}(\xi) d\xi, \quad (\text{II.140})$$

such to write the energy as $E_{ab}(t) = z_{ab}(t)^T H_{ab} z_{ab}(t)$, where $H_{ab} = \text{diag}(H_{ab}^q, H_{ab}^p)$. We now derive the constitutive relation of the single discretized element starting from the ones of the original distributed parameter system (II.127). Substituting the flow approximation of (II.130) and the state approximation (II.138) in the flow constitutive relation of (II.127), we obtain

$$f_{ab}^z(t) = \frac{dz_{ab}}{dt}(t). \quad (\text{II.141})$$

In a similar way, we substitute the effort approximation of (II.130) in the effort constitutive relation of (II.127) to get

$$e_a(t) w_a(\xi) + e_b(t) w_b(\xi) = dH_z(\xi, t), \quad (\text{II.142})$$

that using the base function definition (II.132), the energy definition (II.125), Corollary (II.1.1), the state approximation (II.138) and integrating over the $[a, b]$ interval transforms into

$$\begin{aligned} e_a(t) \int_a^b -\frac{z}{b-a} + \frac{b}{b-a} d\xi + e_b(t) \int_a^b +\frac{z}{b-a} - \frac{a}{b-a} d\xi &= \int_a^b \frac{1}{b-a} \mathcal{H}(\xi) z_{ab}(t) d\xi \\ \frac{b-a}{2} (e_a(t) + e_b(t)) &= \int_a^b \frac{1}{b-a} \mathcal{H}(\xi) z_{ab}(t) d\xi. \end{aligned} \quad (\text{II.143})$$

Finally, we use the effort state e_{ab}^z definition (II.133) together with (II.140) in the the former equation to obtain

$$e_{ab}^z(t) = H_{ab} z_{ab}(t). \quad (\text{II.144})$$

We now define the input of a discretizing element such to impose the flow at the a -side and the effort at the b -side of the element; consequently, the output is composed by the effort at the a -side and the flow at the b -side:

$$u_{ab} = \begin{bmatrix} u_{ab,1} \\ u_{ab,2} \end{bmatrix} = \begin{bmatrix} -f_{Ba} \\ e_{Bb} \end{bmatrix} \quad y_{ab} = \begin{bmatrix} y_{ab,1} \\ y_{ab,2} \end{bmatrix} = \begin{bmatrix} e_{Ba} \\ f_{Bb} \end{bmatrix}. \quad (\text{II.145})$$

Substituting the constitutive relations (II.141) and (II.144), together with the input output definition (II.145) into equation (II.134), it is possible to obtain the following input-state-output representation of a single element dynamics :

$$\begin{aligned} \dot{z}_{ab} &= J_{ab} H_{ab} z_{ab} + g_{ab} u_{ab} \\ y_{ab} &= g_{ab,1} H_{ab} z_{ab} + D u_{ab} \end{aligned} \quad (\text{II.146})$$

with matrices defined as

$$\begin{aligned} J_{ab} &= \begin{bmatrix} 0 & W_2 M^{-1} \\ F_1 M^{-1} & 0 \end{bmatrix} & g_{ab} &= \begin{bmatrix} W_1 - W_2 & 0 \\ 0 & F_1 - F_2 \end{bmatrix} \\ g_{ab,1} &= \begin{bmatrix} -M^{-1} & 0 \\ 0 & -M^{-1} \end{bmatrix} & D &= \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \end{aligned} \quad (\text{II.147})$$

Remark 8. Let define the space

$$\mathbf{D} = \{f_{ab}, e_{ab} \in \mathbb{R}^{2n} \mid W f_{ab} + F e_{ab} = 0\}. \quad (\text{II.148})$$

According to [78], it is possible to prove that for both the Timoshenko's beam and the wave equations, \mathbf{D} is a Dirac structure. This implies that the interconnection relation (II.134) together with the consitutive relations (II.141) and (II.144) form a pH system (see [78, 79]). Therefore we have the following properties for the matrices defined in (II.147): $J_{ab} = -J_{ab}^T$, $g_{ab,1} = g_{ab,1}^T$, $D = -D^T$. In particular, from now on, we consider

$$g_{ab} = g_{ab,1} = \begin{bmatrix} -M^{-1} & 0 \\ 0 & -M^{-1} \end{bmatrix}. \quad (\text{II.149})$$

According to the previous remark, the pH dynamic equations for an element that takes as input the flow at the a -side and the effort at the b -side is defined by the following equation:

$$\begin{aligned} \dot{z}_{ab} &= J_{ab} H_{ab} z_{ab} + g_{ab} u_{ab} \\ y_{ab} &= g_{ab}^T H_{ab} z_{ab} + D u_{ab}. \end{aligned} \quad (\text{II.150})$$

Interconnecting the n_d elements in which the beam has been divided, we obtain a discretized beam model that takes as input the same as a single element: flow at the $\xi = 0$ side and effort at the $\xi = L$ side. As we can see from the beam's boundaries defined in examples of Sections II.1.2.a and II.1.2.d, there exists distribute parameters models that take as input the flow (velocity) at both sides of the beam. To obtain the discretized model of the beam with this type of boundary conditions it is necessary to define a new element that takes as input the flows at both sides of the $[a, b]$ discretizing interval. This element, which is referred to as elastic element, is then connected at the b -side of the last interconnected element such to convert the right effort input into a flow input.

The elastic element is derived setting in (II.134) both p -flow and p -effort equal to zero $f_{ab}^p = e_{ab}^p = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ I & W_1 & W_2 \\ 0 & M & M \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{ab}^q \\ f_{Ba} \\ f_{Bb} \end{bmatrix} + \begin{bmatrix} 0 & F_1 & F_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & M & M \end{bmatrix} \begin{bmatrix} e_{ab}^q \\ e_{Ba} \\ e_{Bb} \end{bmatrix} = 0. \quad (\text{II.151})$$

We now define the input and output of the elastic element

$$u_{ab}^e = \begin{bmatrix} u_{ab,1}^e \\ u_{ab,2}^e \end{bmatrix} = \begin{bmatrix} -f_{Ba} \\ f_{Bb} \end{bmatrix} \quad y_{ab}^e = \begin{bmatrix} y_{ab,1}^e \\ y_{ab,2}^e \end{bmatrix} = \begin{bmatrix} e_{Ba} \\ e_{Bb} \end{bmatrix} \quad (\text{II.152})$$

while its energy is defined as

$$H_{ab,e} = \frac{1}{2} q_{ab}^T H_{ab}^q q_{ab}. \quad (\text{II.153})$$

Using the constitutive relations (II.141) and (II.144) together with the input output definition (II.152) and corollary (II.1.1) in (II.151) we obtain the dynamic equations of the elastic element

$$\begin{aligned} \dot{q}_{ab} &= g_{ab}^e u_{ab}^e \\ y_{ab}^e &= g_{ab,1}^e H_{ab}^q q_{ab} \end{aligned} \quad (\text{II.154})$$

with matrices defined as

$$g_{ab}^e = \begin{bmatrix} -W_1 & W_2 \end{bmatrix} \quad g_{ab,1}^e = \begin{bmatrix} (F_2^{-1} F_1 - I)^{-1} M^{-1} \\ (F_1^{-1} F_2 - I)^{-1} M^{-1} \end{bmatrix}. \quad (\text{II.155})$$

Using the same considerations as in Remark 8, we obtain the dynamic equations of the elastic element in the pH form

$$\begin{aligned} \dot{q}_{ab} &= g_{ab}^e u_{ab}^e \\ y_{ab}^e &= (g_{ab}^e)^T H_{ab}^q q_{ab}. \end{aligned} \quad (\text{II.156})$$

Now that we derived the dynamic equations of the normal and elastic discretized elements, we define the following interconnection relations

$$\begin{cases} u_{ab,1}^{i+1} = -y_{ab,2}^i \\ u_{ab,2}^i = y_{ab,1}^{i+1}. \end{cases} \quad (\text{II.157})$$

Here we define the discretized model of the system with flow input at the $\xi = 0$ side and effort input at the $\xi = L$ side, that is obtained interconnecting n_d elements (II.150) using the interconnection relations (II.157). Since we are considering mechanical systems, imposing a flow input at one side of the boundary domain physically means imposing its velocity. Assuming the knowledge of the initial position of this boundary, imposing its velocity is equivalent to imposing its position. For example, if we have to discretize a Timoshenko beam clamped at one side of the spatial domain, the flow corresponding to this side has to be imposed equal to zero. Similarly, an effort input corresponds to a force. Therefore, if we have to simulate a Timoshenko beam with zero force and momentum at one side of the spatial domain, the effort corresponding to this side has to be set equal to zero.

Definition II.2.2 (Flow-effort input discretized system). Let $z_d \in \mathbb{R}^{n_d}$ the state composed by the composition of the n_d discretizing elements states

$$z_d = \begin{bmatrix} q_{1,ab} \\ \vdots \\ q_{n_d,ab} \\ p_{1,ab} \\ \vdots \\ p_{n_d,ab} \end{bmatrix}, \quad (\text{II.158})$$

then the dynamic equations of a discretized Timoshenko's beam or wave equation with flow input at the $\xi = 0$ side and effort input at the $\xi = L$ side writes

$$\begin{cases} \dot{z}_d = J_{fe} H_{fe} z_d + g_{fe} u_{fe} \\ y_{fe} = g_{fe}^T H_{fe} x + D_{fe} u_{fe} \end{cases} \quad (\text{II.159})$$

with matrices defined as

$$J_{fe} = \begin{bmatrix} 0 & S_{fe} \\ -S_{fe}^T & 0 \end{bmatrix}, \quad D_{fe} = \begin{bmatrix} 0 & (-1)^{n_d} \\ (-1)^{n_d-1} & 0 \end{bmatrix}$$

$$S_{fe} = \begin{bmatrix} W_2 M^{-1} & 0 & \dots & 0 \\ -M^{-2} & W_2 M^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n_d-1} M^{-2} & (-1)^{n_d-2} M^{-2} & \dots & W_2 M^{-1} \end{bmatrix} \quad (\text{II.160})$$

$$g_{fe} = \begin{bmatrix} -M^{-1} & 0 \\ \vdots & \vdots \\ (-1)^{n_d} M^{-1} & 0 \\ 0 & (-1)^{n_d} M^{-1} \\ \vdots & \vdots \\ 0 & -M^{-1} \end{bmatrix} \quad (\text{II.161})$$

where $J_{fe} \in \mathbb{R}^{n_d \times n_d}$, $S_{fe} \in \mathbb{R}^{\frac{n_d}{2} \times \frac{n_d}{2}}$, $g_{fe} \in \mathbb{R}^{n_d \times n}$, energy defined as

$$E = \frac{1}{2} z_d^T H_{fe} z_d, \quad H_{fe} = \begin{bmatrix} H_{fe,q} & 0 \\ 0 & H_{fe,p} \end{bmatrix} \in \mathbb{R}^{n_d \times n_d} \quad (\text{II.162})$$

and input output corresponding to

$$u_{fe} = \begin{bmatrix} f_{B0} \\ e_{BL} \end{bmatrix} = \begin{bmatrix} H_p p(0, t) \\ H_q q(L, t) \end{bmatrix} \quad y_{fe} = \begin{bmatrix} -e_{B0} \\ f_{BL} \end{bmatrix} = \begin{bmatrix} -H_q q(0, t) \\ H_p p(L, t) \end{bmatrix}. \quad (\text{II.163})$$

■

Here we define the discretized model of the system with flow input at $\xi = 0$ and $\xi = L$, that is obtained by interconnecting n_d elements (II.150) and one last elastic element (II.159) using the interconnection relations defined in (II.157).

Definition II.2.3 (Flow-Flow input discretized system). Let $z_d \in \mathbb{R}^{n \cdot n_d + \frac{n}{2}}$ the state composed by the composition of all the discretizing elements states

$$z_d = \begin{bmatrix} q_{1,ab} \\ \vdots \\ q_{n_d,ab} \\ q_{n_d+1,ab} \\ p_{1,ab} \\ \vdots \\ q_{n_d,ab} \end{bmatrix}, \quad (\text{II.164})$$

then the dynamic equations of a discretized Timoshenko's beam or wave equation with flow input at the $\xi = 0$ side and effort input at the $\xi = L$ side writes

$$\begin{cases} \dot{z}_d = J_{ff} H_{ff} z_d + g_{fe} u_{ff} \\ y_{ff} = g_{ff}^T H_{ff} x \end{cases} \quad (\text{II.165})$$

with matrices defined as

$$J_{ff} = \begin{bmatrix} 0 & S_{ff} \\ -S_{ff}^T & 0 \end{bmatrix} \quad S_{ff} = \begin{bmatrix} W_2 M^{-1} & 0 & \cdots & 0 \\ -M^{-2} & W_2 M^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n_d-1} M^{-2} & (-1)^{n_d-2} M^{-2} & \cdots & W_2 M^{-1} \\ (-1)^{n_d} W_1 M^{-1} & (-1)^{n_d-1} W_1 M^{-1} & \cdots & -W_1 M^{-1} \end{bmatrix} \quad (\text{II.166})$$

$$g_{ff} = \begin{bmatrix} -M^{-1} & 0 \\ \vdots & \vdots \\ (-1)^{n_d} M^{-1} & 0 \\ (-1)^{n_d+1} W_1 M^{-1} & W_2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \quad (\text{II.167})$$

where $J_{ff} \in \mathbb{R}^{(n \cdot n_d + \frac{n}{2}) \times (n \cdot n_d + \frac{n}{2})}$, $S_{ff} \in \mathbb{R}^{\frac{n}{2}(n_d+1) \times \frac{n}{2}n_d}$, $g_{ff} \in \mathbb{R}^{(n \cdot n_d + \frac{n}{2}) \times n}$, energy defined as

$$E = \frac{1}{2} z_d^T H_{ff} z_d, \quad H_{ff} = \begin{bmatrix} H_{ff,q} & 0 \\ 0 & H_{ff,p} \end{bmatrix} \quad (\text{II.168})$$

with $H_{ff,q} \in \mathbb{R}^{\frac{n}{2}(n_d+1) \times \frac{n}{2}(n_d+1)}$, $H_{ff,p} \in \mathbb{R}^{\frac{n}{2}n_d \times \frac{n}{2}n_d}$, and input output corresponding to

$$u_{ff} = \begin{bmatrix} f_{B0} \\ f_{BL} \end{bmatrix} = \begin{bmatrix} H_{ff,p} p(0,t) \\ H_{ff,p} p(L,t) \end{bmatrix} \quad y_{ff} = \begin{bmatrix} -e_{B0} \\ e_{BL} \end{bmatrix} = \begin{bmatrix} -H_{ff,q} q(0,t) \\ H_{ff,q} q(L,t) \end{bmatrix}. \quad (\text{II.169})$$

■

II.3 SUMMARY

In this Chapter we have presented the modelling procedure for a flexible mechanism containing deformable parts, such as strings or flexible beams. In Table II.1 we give a small summary of what we have presented in this chapter. In particular, in Section II.1.1 we have introduced an adaptation for flexible mechanisms of the Principle of Least Action. In Section II.1.2 we have given some examples such to understand how to use the Principle to extract the systems equations of motion starting from the knowledge

(1)	<p>Determine the Lagrangian and the work of non-conservative forces of the system:</p> <p>a) The <i>kinetic energy</i> depends on the mechanism kinematic structure.</p> <p>b) The <i>potential energy</i> depends on the assumptions made on the flexible part(s).</p>
(2)	Calculate the Fréchet derivatives of the Lagrangian with respect to the variables that compose it.
(3)	<p>a) Apply integration by parts such to obtain the <i>variation functions</i> with respect to the same variable.</p> <p>b) Extract the <i>equations of motion</i> and the <i>boundary conditions</i>.</p>
(4)	<p>Select the energy state variables such to express:</p> <p>a) the PDE as a dpH system;</p> <p>b) the ODE as a finite pH system;</p> <p>c) the energy density as a quadratic form of the dpH's state;</p> <p>d) the lumped parameter energy as a quadratic form of the finite pH's state.</p>
(5)	According to Section II.1.3, the obtained model is passive with respect to its total energy.
(6)	<p>Approximate the dpH system(s) with a finite dimensional pH system(s)</p> <ul style="list-style-type: none"> • If one side of the spatial domain is velocity controlled and the other one is force controlled, use the discretization model of Definition II.2.2. • If both sides of the spatial domain are velocity controller, use the discretization model of Definition II.2.3.

Table II.1: Algorithm for the FApH model derivation of a mechanism containing flexible part(s).

of the Lagrangian and the work of the non-conservative forces. Once the Lagrangian and the work of non-conservative forces are determined for a given system (item (1)), one can compute the Action functional and its derivatives with respect to the generalize variables that compose it (item (2)). Then, integration by parts should be applied to the Action's Fréchet derivatives such to be able to extract the equations of motions and the boundary conditions of the PDE (items (3)). In order to obtain a model in the pH format, an appropriate state selection should be made, *i.e.* one should select the energy variables (item (4)). According to Section II.1.3, the obtained model is passive with respect to its total energy (item (5)). Finally, a discretization algorithm is presented in Section II.2, *i.e.* we show how to obtain a set of ODE approximating the set of PDE present in the model (item (6)).

Chapter III

Control Design

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III.1 PRELIMINARIES ON ANALYSIS OF SEMILINEAR EQUATIONS

Mechanisms that integrate flexible parts naturally suffer from vibrations caused by flexible deformations. These type of undesired flexible phenomena can appear in different areas ranging from spatial application [1], to micro-manipulations [13]. To suppress the elastic vibrations, different technological solutions are possible. The first solution is to select more rigid materials for the mechanism parts that we do not want to deform. Most of the time, rigid materials are heavier than elastic ones, and because of the project specifications, this could not be a feasible solution. A second solution is to use actuators directly on the deformable element, i.e. distributed in-domain actuators (see for example [80] for piezoelectric or [81] for IPMC distributed actuators). This is a very interesting and performing solution that will not be discussed in this manuscript. Instead, we assume that the actuators that can deal with the flexible deformations are the same as the ones used to move the mechanism, i.e. angular and/or linear motors. In this chapter, we study how to integrate classical control actions with terms that include information related to flexible deformations. This information is usually coming from specific sensors such as strain gauge or laser sensors able to measure deformation as well as strain or velocities of the elastic beams.

Before investigating the asymptotic behaviour, the first problem is to understand if the closed-loop system is well-posed, in the sense that whether the modelling equations does admit a unique solution that depends continuously on the initial conditions. This first step, even if quite technical, is of crucial importance for a very simple reason: all the subsequent stability analysis assumes the existence and uniqueness of the solution. In other words, the stability analysis aims to find some solution's properties even if the solution is not known explicitly. But of course, a requirement is that this solution does exist and is unique. To study this we need to introduce a few but keys concepts and results that will be used in the remainder of this chapter and also in Chapter IV, where we will study a flexible rotating beam in a collision scenario.

III.1.1 Stability theory for semilinear equations

We consider a semilinear operator equation of the form

$$\dot{x} = Ax + f(x) \quad (\text{III.1})$$

where x is the state of the system and belongs to an Hilbert space X , A is a (not necessary bounded) dense operator in X with domain $\mathbf{D}(A)$ that generates a C_0 -semigroup in X and $f(x)$ is a locally Lipschitz continuous nonlinear mapping $f : X \mapsto X$.

Remark 9. Taking in consideration all the assumptions made on the semilinear equation (III.1), according to Theorem I.1.4 in Section I.1, we can conclude that there exist a unique local solution $x(t)$ on $[0, t_{max})$ of (III.1) with the property that if $t_{max} < \infty$, then

$$\lim_{t \rightarrow t_{max}} \|x(t)\| = \infty. \quad (\text{III.2})$$

In case of a semilinear equation (III.1) with an unbounded operator A , the explicit solution is often not available, nevertheless we are interested in its behaviour over time. When controlling a system, the control objective consists in general to maintain its trajectories around a certain equilibrium point, or better, that the distance between the trajectories and the equilibrium point decreases along time. Next, we define what we mean by equilibrium point for (III.1).

Definition III.1.1. The element $x_{eq} \in \mathbf{D}(A)$ is an *equilibrium point* of (III.1) when $Ax_{eq} + f(x_{eq}) = 0$. ■

In general, we do not have at our disposal explicit solutions, therefore we need to define different stability concepts and the respective tools to check them, to study the class of m-pH system considered in this manuscript. In case of linear finite dimensional systems, all the stability definitions are equivalent, therefore only one concept of stability is needed. In other words, if we know that the solution of a linear finite dimensional system asymptotically converges to the equilibrium point, we know that it does so with an exponential trend with respect to time. This is in general not true for infinite dimensional systems or even for nonlinear finite dimensional systems. This is why in the following definitions we introduce different stability concepts.

Definition III.1.2. The equilibrium point $x_{eq} = 0$ of (III.1) is said to be *Lyapunov stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\|x_0\| \leq \delta$ the solution satisfies $\|x(t)\| \leq \varepsilon$ for all $t \geq 0$.

The equilibrium point $x_{eq} = 0$ is said to be *asymptotically stable* if for every $x_0 \in X$ the corresponding solution $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

The equilibrium point $x_{eq} = 0$ is said to be *exponentially stable* if for every $x_0 \in X$ there exists a $w > 0$ such that $e^{wt}\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$. ■

There exist (especially for linear distributed parameter systems) other stability concepts as for example polynomial stability or weak stability, that allow to diversify even more the asymptotic behaviour of solutions. In this thesis we will focus on the stability concepts defined in the previous Definition III.1.2. We refer the interested reader to [82].

Remark 10. For the class of considered semilinear equations (III.1), we can assume without loss of generality that $x_{eq} = 0$. Let x_{eq} be an equilibrium position of (III.1), and define the new state $\bar{x}(t) = x(t) - x_{eq}$. We obtain the equation in the new shifted variable

$$\begin{aligned}\dot{\bar{x}}(t) &= A\bar{x}(t) + f(\bar{x}(t) + x_{eq}) + Ax_{eq} \\ &= A\bar{x}(t) + \tilde{f}(\bar{x}(t))\end{aligned}\tag{III.3}$$

where $\tilde{f}(\bar{x}(t)) = f(\bar{x}(t) + x_{eq}) + Ax_{eq}$. Next, it is possible to see that $\bar{x}_{eq} = 0$ and the new equation is written in the same semilinear equation format. It is possible to prove that $\bar{x}(t)$ is a solution of (III.3) if and only if $x(t)$ is a solution of (III.1). Moreover, the origin is an exponentially and asymptotically stable equilibrium point of (III.3) if and only if x_{eq} is an exponentially and asymptotically stable equilibrium point of (III.1) (see exercise 11.7 in [65]). In the case of general semilinear differential equation with multiple equilibrium positions, the fact of translating the state with respect to a particular equilibrium position allows to study the stability of that particular equilibrium position.

Lyapunov stability theory is a basic tool to study the stability of a wide spectrum of equations ranging from nonlinear to distributed parameter equations. Roughly speaking, this theory relies on the existence of a semi-positive definite functional (a Lyapunov functional), that needs to be non-increasing along the system's trajectories. The non-increasing condition of the Lyapunov functional is checked on the time derivative of this functional along the trajectories generated by the considered equation. The time derivative “along trajectories” is understood in the following manner: the time derivative of the Lyapunov functional in a point corresponds to the variation of the Lyapunov functional related to the generated trajectories when this point corresponds to the initial condition by the considered dynamic equations.

To define a Lyapunov functional we make use of the Dini derivative concept introduced in Definition I.1.8, and Lemma I.1.1 (page 14) that links the time derivative's sign to the functional non-increasing behaviour.

Definition III.1.3. Suppose that, for every $x_0 \in X$, the semilinear differential equation (III.1) possesses a local mild solution $x(t)$. A continuous functional $V : X \mapsto [0, \infty)$, is a *Lyapunov functional* for (III.1) on X if $V(x(t, x_0))$ is Dini differentiable at $t = 0$ for all $x_0 \in X$ and there holds

$$\dot{V}_+(x_0) := \limsup_{t \rightarrow 0} \frac{V(x(t, x_0)) - V(x_0)}{t} \leq 0. \quad (\text{III.4})$$

■

Since in most practical cases, the limit (III.4) it is not easy to compute, we rely on Lemma I.1.2 (page 14) to establish the relation between the Dini time derivative and the Fréchet derivative. In fact, if V is Fréchet differentiable, then for $z \in \mathbf{D}(A)$, $V(z(t, z_0))$ is Dini differentiable and

$$\dot{V}_+(x_0) := dV(x_0)(Ax_0 + f(x_0)) \quad (\text{III.5})$$

where dV is the Fréchet derivative of V . Therefore, we cite Theorem 11.2.4 from [65] that allows to conclude about boundedness of solutions (Lyapunov stability), using the concept of Lyapunov functional introduced in Definition III.1.3.

Theorem III.1.1. Let A be the infinitesimal generator of the C_0 -semigroup $T(t)$ on the Hilbert space X and let $f : X \mapsto X$ be locally Lipschitz continuous. If there exists a Lyapunov functional V for the semilinear differential equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad t \geq 0, \quad x(0) = x_0 \quad (\text{III.6})$$

with the property that $V(x) \rightarrow \infty$ whenever $\|x\| \rightarrow \infty$, then (III.6) has a bounded global mild solution for all initial conditions $x_0 \in X$. Moreover, if $x_0 \in \mathbf{D}(A)$ then the solution is a global classical solution.

To conclude about the other types of stability we need to ask a little bit more to the Lyapunov functional and to its time derivative. First of all we need the Lyapunov functional to be strictly increasing, then it is possible to prove that

- if $\dot{V}_+ \leq -\alpha(\|x\|)$ with α strictly increasing and such that $\alpha(0) = 0$, then the origin is an asymptotically stable equilibrium point.
- if $\dot{V}_+ \leq -\kappa V$ with $\kappa > 0$, then the origin is an exponentially stable equilibrium point.

Unfortunately, these types of inequalities are very difficult to obtain for a general class of systems, and needs to be studied and found on a case by case basis (see for example [74]). On the other hand, we will make use of another theorem that allows concluding about asymptotic stability without asking the strictly negative definitiveness of the Lyapunov functional's time derivative, *i.e.* the LaSalle's invariance principle. To state the LaSalle's invariance principle, we first have to introduce the concept of invariant set and convergence to a set.

Definition III.1.4. For $x_0 \in X$ suppose that (III.6) has a global mild solution $x(t, x_0)$; $t \geq 0$. A set $W \in X$ is an *invariant set* with respect to (III.6) if for every $x_0 \in W$ the solution $x(t, x_0)$ is in W for all $t \geq 0$. ■

Definition III.1.5. Let \mathbf{S} be a closed subset of the Hilbert space X and let $\{x_n\}$ be a sequence in X . Then x_n *converges to* \mathbf{S} if the following holds

$$\liminf_{n \rightarrow \infty} \inf_{s \in \mathbf{S}} \|x_n - s\| = 0. \quad (\text{III.7})$$

The continuous function $g(t)$ converges to \mathbf{S} when for every sequence t_n with $t_n \rightarrow \infty$, the sequence $g(t_n)$ converges to \mathbf{S} . ■

We now define the concept of trajectory set, and what we mean by pre-compact trajectories set.

Definition III.1.6. The *trajectories set* $\{x(t, x_0) \mid t \geq 0\}$ is said to be *pre-compact* if its closure $\overline{\{x(t, x_0) \mid t \geq 0\}}$ is compact.

We are now in position to state the LaSalle's invariance principle, that gives the conditions for which the solution $x(t, x_0)$ converges to a specific invariant set.

Theorem III.1.2 (LaSalle's invariance principle). *For $x_0 \in X$ suppose that (III.6) has a global mild solution $x(t, x_0)$; $t \geq 0$ and pre-compact trajectories set $\{x(t, x_0) \mid t \geq 0\}$. Let V be a Lyapunov functional for (III.6) and define*

$$\mathbf{S}_0 := \{x \in X \mid \dot{V}_+(x) = 0\}. \quad (\text{III.8})$$

Let \mathbf{S} be the largest invariant subset of \mathbf{S}_0 , then $x(t; x_0)$ converges to \mathbf{S} as $t \rightarrow \infty$.

This theorem is very similar to its finite dimensional counterpart, but the only, and indeed important, difference is the fact that the requirement on the pre-compactness of the trajectories set is replaced with the one on boundedness of solutions. This is because in case the state belongs to a finite dimensional space, boundedness of solutions directly implies pre-compactness of the trajectories set (Heine-Borel or Bolzano-Weierstrass

theorem). This direct implication is not true for infinite dimensional systems, see for instance the not-asymptotically stable transport equation with non-compact trajectories set but with Lyapunov function non-increasing along the system trajectories in [83].

Checking pre-compactness of the trajectories set or equivalently pre-compactness of state trajectories) is usually one of the most difficult tasks on proving stability using the LaSalle's invariance principle. In the following Corollary we derive the conditions to conclude about pre-compactness of trajectories generated by (III.6), as a trivial expansion of Theorem 3 in [84].

Corollary III.1.1. *Consider the semilinear differential equation (III.6) where A generates a bounded semigroup on X , and A has a compact resolvent.*

Suppose that for a $x_0 \in X$, the solution is such that $f(x(t, x_0)) \in L^p((0, \infty); U)$ for some $p \in [1, \infty)$, then the trajectories set $\{x(t, x_0) \mid t \geq 0\}$ is bounded and pre-compact.

Proof. The result is obtained taking $B = I$ in Theorem 3 of [84], and from the fact that the identity operator is a bounded linear operator and therefore infinite-time L^p -admissible for $T(t)$ for all $p \in [1, \infty)$. \square

For more details about infinite-time admissibility of (especially unbounded) input and output operators, we refer the interested reader to the comprehensive survey [85], as well as the book [86].

Another possibility to conclude about asymptotic stability for semilinear equations is the use of the combination of two results: the first one is due to Benchimol [87], while the second to Oosteeven [88] (Lemma 2.1.3 and 2.2.6). All these results make use of the approximate observability and controllability concepts for which we refer to Chapter 6 of [65]. To be able to define these concepts, consider the linear system $\Sigma(A, B, C, D)$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y &= Cx(t) + Du(t) \end{aligned} \quad (\text{III.9})$$

where A is the infinitesimal generator of a C_0 -semigroup $T(t)$ on a Hilbert space X , B is the input operator from the input space U to X , C is output operator from X to the output space Y and D is the feed-through operator from U to Y .

Definition III.1.7.

- (a) The system $\Sigma(A, B, -, -)$ is *approximately controllable* on $[0, \tau]$ if given an arbitrary $\varepsilon > 0$ it is possible to steer from the origin to within a distance ε from all points in the state space at time τ , i.e., if

$$\overline{\text{rank} \left(\int_0^\tau T(t - \tau) Bu(\tau) d\tau \right)} = X. \quad (\text{III.10})$$

- (b) The system $\Sigma(A, -, C, -)$ is *approximately observable* on $[0, \tau]$ (for some finite $\tau > 0$) if knowledge of the output in $L_2([0, \tau], Y)$ determines the initial state uniquely, i.e.

$$\ker(CT(t)x_0) = \{0\}, \quad (\text{III.11})$$

or equivalently that

$$CT(t)x_0 = 0 \text{ in } [0, \tau] \Rightarrow z = 0. \quad (\text{III.12})$$

Now we can introduce the two joint Oosteven's lemmas, making use of the approximate observability and controllability concepts, that allow to conclude about asymptotic stability for semilinear equations.

Theorem III.1.3. *Let X, U be Hilbert spaces, $B \in \mathcal{L}(U, X)$ and A the infinitesimal generator of a contraction C_0 -semigroup. Assume that A has a compact resolvent, and that the state linear system*

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= B^*x\end{aligned}\tag{III.13}$$

is approximately observable or approximately controllable on infinite time. Then

- *for all $\kappa > 0$, the operator $A - \kappa BB^*$ generates an asymptotically stable semigroup $T_{-\kappa BB^*}(t)$, i.e. $\lim_{t \rightarrow \infty} \|T_{-\kappa BB^*}(t)x_0\| = 0$;*
- *the closed-loop system $\Sigma(A - \kappa BB^*, B, B^*, 0)$ is input stable, meaning that for $u \in L_2((0, \infty); U)$ and $k < \infty$*

$$\left\| \int_0^\infty T_{-\kappa BB^*}(s)Bu(s)ds \right\|^2 \leq k \|u\|_{L_2((0, \infty); U)}^2;\tag{III.14}$$

- *for all $u \in L_2((0, \infty); U)$ we have*

$$\int_0^t T_{-\kappa BB^*}(t-s)Bu(s)ds \rightarrow 0 \quad \text{as } t \rightarrow \infty.\tag{III.15}$$

Theorems (III.1.2) and (III.1.3) both allow to conclude about asymptotic stability. Between the different formulations of the LaSalle's invariance principle, we decided to use the one for the semilinear operator equations of the form (III.6). Another formulation, considering a more general class of equations, can be found in [89]. While the LaSalle's invariance principle potentially allows determining asymptotic stability for a larger class of system, the Oosteven's theorem is restricted to semilinear equations in the form of (III.13). An important advantage of Theorem III.1.3 is that, differently from the LaSalle's invariance principle, pre-compactness of trajectories set does not have to be checked. Nevertheless, approximate observability or approximate controllability are needed to know that the state trajectories converge to the origin. We now introduce the concept of admissible output operator [90]

Definition III.1.8. Consider the linear system $\Sigma(A, -, C, -)$. The operator $C : X \mapsto Y$ is called an *admissible observation* operator for $T(t)$ if the estimate

$$\int_0^\tau \|CT(t)x_0\|^2 dt \leq k(\tau) \|x_0\|^2\tag{III.16}$$

holds for every $\tau > 0$ and for every $x_0 \in \mathbf{D}(A)$.

If $C \in \mathcal{L}(X, Y)$ then obviously it is admissible. In the following theorem, we make the connection between the approximate observability concept and the fact that the only solution of a homogeneous boundary control problem with constant output and zero as the only equilibrium position is the zero solution.

Lemma III.1.1. *For the linear system $\Sigma(A, -, C, -)$ the following statements are equivalent*

1. *If $y(t)$ is constant, then $x_0 = 0$.*
2. *$\Sigma(A, -, C, -)$ is approximately observable and its only equilibrium point is the origin.*

Proof. 1) \Rightarrow 2) Let x_0 be such that the output $y(t)$ of $\Sigma(A, -, C, -)$ is identically zero. By 1) we see that $x_0 = 0$, and thus $\Sigma(A, -, C, -)$ is approximately observable.

Let x_{eq} be an equilibrium solution, then $0 = Ax_{eq}$, and the corresponding output (corresponding to $x(t) \equiv x_{eq}$) is $y(t) = Cx_{eq}$. This is constant, and so by 1) $x_{eq} = 0$.

2) \Rightarrow 1) Let $y(t)$ be a constant output of $\Sigma(A, C)$ and let $x(t)$ be the corresponding state trajectories. Define $y_\Delta(t) := y(t + t_1) - y(t) = 0$, $t_1 > 0$. The corresponding state trajectories is $x_\Delta(t) = x(t + t_1) - x(t)$. By approximate observability we have that $x_\Delta(0) = 0$. Thus $x(t_1) - x(0) = 0$. Since t_1 was arbitrary, we have that $x(t) \equiv x(0)$, and thus $x(0)$ is an equilibrium solution. By assumption we conclude that $x(0) = x_0 = 0$. \square

This lemma will be applied to the infinite dimensional part of the obtained closed-loop system in order to find the large invariant subspace of the space for which the Lyapunov functional's derivative is zero.

III.2 CONTROL DESIGN FOR A M-PH CLASS OF SYSTEMS

In this section we would like to design different stabilizing control laws for a class of m-pH systems. Let $z \in L_2([0, L], \mathbb{R}^n)$, $p \in \mathbb{R}^m$ and consider the following m-pH system, as depicted in Figure III.1

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z) + P_0(\mathcal{H}z) \\ -y_z + u \end{bmatrix} \\ y &= M^{-1}p \quad u_z = y \end{aligned} \quad (III.17)$$

where $\mathcal{H} \in C^1([0, L]; \mathbb{R}^{n \times n})$, $\mathcal{H}(\xi)$ is self adjoint for all $\xi \in [0, L]$ and $cI \leq \mathcal{H}(\xi) \leq CI$ for all $\xi \in [0, L]$ and some $C, c > 0$ independent of ξ , $P_1 \in \mathbb{R}^{n \times n}$ is invertible and self adjoint, $P_0 \in \mathbb{R}^{n \times n}$ is skew adjoint, and with input output operators of the infinite dimensional part defined as

$$\begin{aligned} u_z = \mathcal{B}_1(\mathcal{H}z) &= W_{B,1} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} & \mathcal{B}_2(\mathcal{H}z) &= W_{B,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ y_z = \mathcal{C}_1(\mathcal{H}z) &= W_{C,1} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} & \mathcal{C}_2(\mathcal{H}z) &= W_{C,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \end{aligned} \quad (III.18)$$

such that $\text{rank}(W_{B,1}) = \text{rank}(W_{C,1}) = m$, and $\text{rank}(W_{B,2}) = \text{rank}(W_{C,2}) = n - m$ and $\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$ defined in Definition I.2.1 (page 18). Note that the output $y_z(t)$ has the same dimension as the input $u_z(t)$. We define the complete input and output operators as the composition

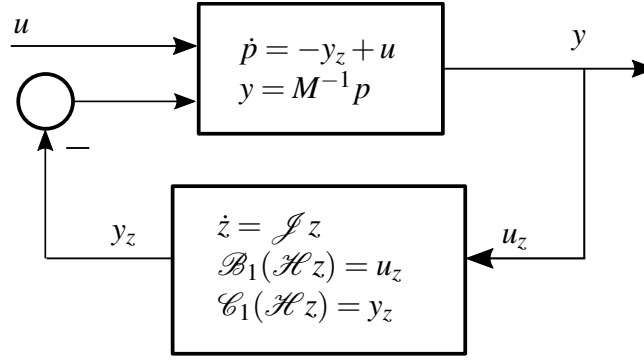


Figure III.1: Considered class of m-pH systems.

of the previously defined operators

$$\begin{aligned} \mathcal{B}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{B}_1(\mathcal{H}z) \\ \mathcal{B}_2(\mathcal{H}z) \end{bmatrix} = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ \mathcal{C}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{C}_1(\mathcal{H}z) \\ \mathcal{C}_2(\mathcal{H}z) \end{bmatrix} = \begin{bmatrix} W_{C,1} \\ W_{C,2} \end{bmatrix} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \end{aligned} \quad (\text{III.19})$$

with boundary flow and effort defined in Definition 1.2.1.

Remark 11. The output operator $\mathcal{C}(\mathcal{H}z)$ is a point evaluation of the state $z(\xi, t)$, therefore it is a bounded operator from the state space Z to the output space \mathbb{R}^n , hence an *admissible observation* operator according to Definition (III.1.8).

We now recall the conditions for the m-pH system (III.17)-(III.19) of Assumption 1 in Chapter I, that will be assumed throughout the rest of this chapter:

1. The matrix $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible;
2. W_B and W_C are such that

$$\begin{bmatrix} W_B \Sigma W_B^T & W_B \Sigma W_C^T \\ W_C \Sigma W_B^T & W_C \Sigma W_C^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (\text{III.20})$$

Note that the class of system defined by (III.17)-(III.19) corresponds to the one obtained by the physical examples of section II.1.2.a-II.1.2.c in Chapter II. In the following we propose different control laws and we analyse the closed-loop stability when applied to the aforementioned class of systems.

In Lemma III.1.1 we have shown the relation between the approximate observability property and the fact that a system admits zero as only solution. In the following theorem we give the conditions to obtain approximate observability for the class of 1-D dpH systems present in (III.17)-(III.19).

Theorem III.2.1. *Consider the equation*

$$\dot{z} = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) = \mathcal{J}z \quad (\text{III.21})$$

and boundary input/output operators (III.18)-(III.19), with homogeneous boundary conditions $\mathcal{B}(\mathcal{H}z) = 0$. Assume that the input/output of the system are chosen such that

$$\begin{aligned} \|\mathcal{H}z(0,t)\|^2 &\leq \|u_z(t)\|^2 + \|y_z(t)\|^2 \\ &\text{or} \\ \|\mathcal{H}z(L,t)\|^2 &\leq \|u_z(t)\|^2 + \|y_z(t)\|^2 \end{aligned} \quad (\text{III.22})$$

then the system is approximately observable with respect to the $y_z = \mathcal{C}_1(\mathcal{H}z)$ output.

Proof. By Proposition I.2.1 we know that \mathcal{J} is a skew-adjoint, therefore its eigenvalues belong to the imaginary axis. Operator (III.21) generates a unitary group if and only if it is a skew-adjoint operator (see Theorem 2.32 in [89]) and by Theorem I.2.2 its resolvent is compact, then we know by Theorem A.4.19 [65], that its eigenvectors forms an orthonormal basis. Since an orthonormal basis is a special case of a Riesz-Basis, operator (III.21) is a Riesz-spectral operator. Consequently, using theorem 6.3.6 of [65], to check that the system is approximately observable we have to show that there exists no eigenvector in the kernel of \mathcal{C}_1 . To show this, assume by contradiction that there exists an eigenvector v such that $\mathcal{C}_1 v = 0$. We now consider the first inequality of (III.22) to hold. Using the homogeneous boundary conditions together with $y_z = \mathcal{C}_1 v = 0$ in the first inequality of (III.22), we obtain $\mathcal{H}z(0,t) = 0$. We integrate both side of the eigenvalue problem's equation (obtained imposing $z = v(\xi)e^{i\omega t}$ in (III.21), where v is the eigenvector and $i\omega$ the corresponding eigenvalue)

$$i\omega v(\xi) = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}v)(\xi) + P_0(\mathcal{H}v)(\xi) \quad (\text{III.23})$$

to obtain

$$\begin{aligned} i\omega \int_0^s v(\xi) d\xi &= P_1 \int_0^s \frac{\partial}{\partial \xi}(\mathcal{H}v)(\xi) d\xi + \int_0^s P_0(\mathcal{H}v)(\xi) d\xi \\ \int_0^s (w\mathcal{H}^{-1}(\xi) - P_0)(\mathcal{H}v)(\xi) d\xi &= P_1[(\mathcal{H}v)(s) - (\mathcal{H}v)(0)] \\ \int_0^s (w\mathcal{H}^{-1} - P_0)\mathcal{H}v(\xi) d\xi &= P_1(\mathcal{H}v)(s) \end{aligned} \quad (\text{III.24})$$

With P_1 being full rank, the former equation is equivalent to

$$(\mathcal{H}v)(s) = P_1^{-1} \int_0^s g(\xi)(\mathcal{H}v)(\xi) d\xi, \quad (\text{III.25})$$

with $g(\xi) = (i\omega\mathcal{H}^{-1}(\xi) - P_0)$. From the former equation we get

$$\begin{aligned} \|(\mathcal{H}v)(s)\| &\leq \|P_1^{-1}\| \int_0^s \|g(\xi)\mathcal{H}v(\xi)\| d\xi \\ &\leq \|P_1^{-1}\| \int_0^s \|g(\xi)\| \cdot \|\mathcal{H}v(\xi)\| d\xi \\ &\leq K \int_0^s \|\mathcal{H}v(\xi)\| d\xi \end{aligned}$$

since both P_1^{-1} and $\|g(\xi)\|$ are bounded from above. Using the integral form of the Gronwall's Lemma we obtain that

$$\|(\mathcal{H}v)(s)\| \leq 0, \quad (\text{III.26})$$

that implies $(\mathcal{H}v)(s) \equiv 0$, which since $\mathcal{H}(\xi) \geq mI$ with $m > 0$ implies $v(s) \equiv 0$, that is a contradiction to the fact that v is an eigenvector. If the second inequality of (III.22) holds instead of the first one, we obtain $\mathcal{H}z(L, t) = 0$. Therefore, we integrate (III.23) from s to L such that

$$i\omega \int_s^L v(\xi) d\xi = P_1 \int_s^L \frac{\partial}{\partial \xi} (\mathcal{H}v)(\xi) d\xi + \int_s^L P_0 (\mathcal{H}v)(\xi) d\xi. \quad (\text{III.27})$$

The rest of the proof follows in a similar manner as before. \square

An immediate consequence is that a 1-D dpH system with a sufficient number of inputs and constant boundary output admits zero as only solution if the origin is its only equilibrium point.

Corollary III.2.1. *Consider the dpH system (III.21) with boundary input/output and operators defined in (III.18)-(III.19). Assume that the input/output of the system are selected such to fulfil (III.22), then if $y_z(t)$ is constant and the origin is the only equilibrium point, then $z(t) \equiv 0$ for all $t \geq 0$.*

Proof. It is a direct consequence of Lemma III.1.1 and Theorem III.2.1. \square

In the following, we will present different control laws applicable to the considered class of m-pH systems. At the end of each subsection, we present an application of the proposed control law for the models derived in Chapter II. The examples are endowed with numerical simulations that have been performed using Matlab[®].

III.2.1 Proportional Derivative control of a m-pH system

In this section, we consider the stabilization of a m-pH system with a PD control law. This corresponds to the classical feedback used to place the mechanism in the desired configuration, and in the following sections will be embodied with other terms that deal with the faster stabilization of the flexible deformation. We define the new variable q as the time integral of the momenta p multiplying the inverse of the mass matrix

$$q(t) = \int_0^t M^{-1} p(\tau) d\tau. \quad (\text{III.28})$$

Since the quantity $M^{-1}p$ physically corresponds to a velocity, its time integral corresponds to a position configuration in case the modelled system corresponds to a mechanical mechanism. We define the PD control law

$$u = -Kq - R_p M^{-1} p \quad (\text{III.29})$$

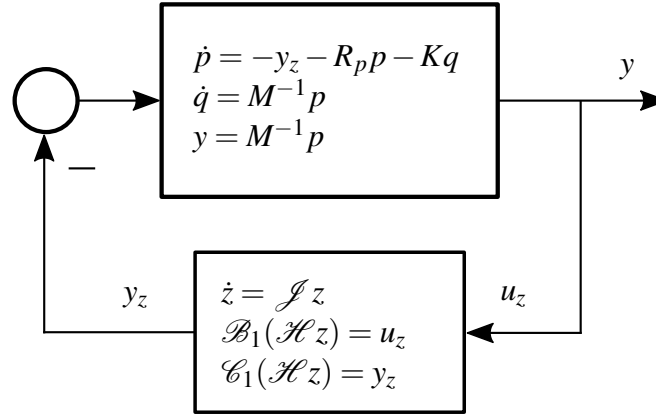


Figure III.2: m-pH system in closed-loop with a PD control law.

where the control matrices $K, R_p \in \mathbb{R}^{m \times m}$ are diagonal and positive definite, *i.e.* $K = \text{diag}[k_1 \ k_2 \ \dots \ k_m] > 0$ and $R_p = \text{diag}[r_{p,1} \ r_{p,2} \ \dots \ r_{p,m}] > 0$. This control law falls in the category of *output feedback*: the output of the considered system (III.17), is integrated to obtain the state q , and a linear composition of the states p and q is re-injected as feedback. Then the closed-loop system depicted in Figure III.2, with extended state $x = \begin{bmatrix} z \\ p \\ q \end{bmatrix}$, can be written as a linear operator equation

$$\begin{aligned} \dot{x} = Ax &= \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H} z) + P_0(\mathcal{H} z) \\ -\mathcal{C}_1(\mathcal{H} z) - Kq - R_p M^{-1} p \\ M^{-1} p \end{bmatrix} \\ y = Cx &= M^{-1} p \end{aligned} \quad (\text{III.30})$$

with state space $X = L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$, operator domain

$$\mathbf{D}(A) = \{x \in X \mid \mathcal{H} z \in H^1([0, L], \mathbb{R}^n), \mathcal{B}_1(\mathcal{H} z) = M^{-1} p, \mathcal{B}_2(\mathcal{H} z) = 0\}, \quad (\text{III.31})$$

and boundary operator defined as in (III.18). We define the inner product

$$\langle x_1, x_2 \rangle_X = \langle z_1, \mathcal{H} z_2 \rangle_{L_2} + p^T M^{-1} p + q^T K q \quad (\text{III.32})$$

such that to equip the state space with the associated norm $\|x\|_X = \sqrt{\langle x, x \rangle_X}$. We now show that the closed-loop operator A generates a C_0 -semigroup of contractions in the space X .

Theorem III.2.2. *Under Assumption 1, the closed-loop operator A (III.30)-(III.31) generates a contraction C_0 -semigroup in the space X and has a compact resolvent.*

Proof. We notice that system (III.30)-(III.31) can be rewritten in the same form as equation (I.69) in Corollary I.3.1 defining

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad R = \begin{bmatrix} -R_p & 0 \\ 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} M^{-1} & 0 \\ 0 & K \end{bmatrix} \quad g = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad S = 0. \quad (\text{III.33})$$

Therefore, according to Corollary I.3.1 we can conclude that the operator A generates a contraction C_0 -semigroup in the state space X and has a compact resolvent. \square

In the following theorem we employ two different methods to prove the asymptotic stability of the origin of the closed-loop operator: one using the LaSalle's invariance principle, while the other using the Oostveen's Theorem.

Theorem III.2.3. *Consider the system (III.30)-(III.31). Assume that the distributed parameter part of the system with homogeneous boundary conditions is approximately observable with respect to the output y_z , and $z_{eq} = 0$ is its only equilibrium point. Then, under Assumption 1 the equilibrium point $x_{eq} = 0$ of (III.30) is asymptotically stable.*

Proof. (With LaSalle's invariance principle). Since the resolvent of A is compact by Theorem I.3.1 and the nonlinearity is identically zero, we trivially obtain that the nonlinearity is p -integrable in infinite time, therefore the trajectories set is pre-compact in X by Corollary III.1.1. Define the following candidate Lyapunov functional

$$V = \frac{1}{2} \langle x, x \rangle, \quad (\text{III.34})$$

computing the time derivative according to (III.5) and using Theorem I.2.1 and Assumption 1, we obtain

$$\begin{aligned} \dot{V}_+(x) &= \langle z, \mathcal{J}z \rangle_{L_2} + p^T M^{-1} (-\mathcal{C}_1(\mathcal{H}z) - Kq - R_p M^{-1} p) + q^T K(M^{-1} p) \\ &= -(M^{-1} p)^T R_p (M^{-1} p) \leq 0. \end{aligned} \quad (\text{III.35})$$

We now want to find the Largest invariant set of $\{x \in X \mid \dot{V}_+(x) = 0\}$, that since $M, R_p > 0$, is equivalent to the set $\{x \in X \mid p = 0\}$. Therefore, we replace $p = 0$ in (III.30) to obtain

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ y_z = \mathcal{C}_1(\mathcal{H}z) = -Kq \end{cases} \quad (\text{III.36})$$

with boundary conditions $\mathcal{B}_1(\mathcal{H}z) = \mathcal{B}_2(\mathcal{H}z) = 0$, and constant q . Hence, the problem of finding the largest invariant set consists in finding the solution of (III.36). Because of Lemma III.1.1, since (III.36) is approximately observable by assumption, its only solution is $z = 0$. Then, the largest invariant set consists of the set containing only the origin. Finally by the LaSalle's invariant principle we can conclude that the origin is asymptotically stable. \square

Proof. (With Oostveen's Theorem). Firstly, we note that approximate observability of system (III.30) follows directly from approximate observability of its distributed parameter part. Next, we define

$$B = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \quad B^* = [0 \quad I \quad 0] \quad \kappa = R_p M^{-1}. \quad (\text{III.37})$$

where B^* is the adjoint of B . Hence, we define the weighted input-output matrices as $\tilde{B} = B\sqrt{\kappa}$ and $\tilde{B}^* = \sqrt{\kappa}B^*$. Then, the closed-loop system (III.30) can be rewritten as $\dot{x} = \tilde{A} + \kappa\tilde{B}\tilde{B}^*$. The operator \tilde{A} is the same as the operator A , but without dissipation, therefore under Assumption 1 \tilde{A} generates a contraction C_0 -semigroup and has a

Table III.1: Simulation parameters; PD controller example

Name	Variable	Value
Beam's Length	L	1 m
Beam's Width	L_w	0.1 m
Beam's Thickness	L_t	0.02 m
Density	ρ	$1.9\frac{\text{kg}}{\text{m}^3}$
Young's modulus	E	$8 \times 10^8\frac{\text{N}}{\text{m}^2}$
Bulk's modulus	K	$1.7 \times 10^9\frac{\text{N}}{\text{m}^2}$
Hub's inertia	J	$1\text{ kg} \cdot \text{m}^2$

compact resolvent (see Theorem III.2.2 with $R_p = 0$). Then by the Oosteven's Theorem III.1.3, we can conclude that the origin is an asymptotically stable equilibrium point. \square

Example 1 (Rotating flexible beam with PD controller). We consider a rotating flexible beam in closed-loop with a PD control law (III.30), with matrices and operators defined in Section II.1.2.b of Chapter II. We consider the control problem of stabilising a rotating flexible beam in a desired orientation $q = q_{eq}$, $p = 0$ and zero deformation $z = 0$. Defining $\tilde{q} = q - q_{eq}$ we retrieve the closed-loop formulation given in (III.30). Therefore we study, without loss of generality, the origin stabilization problem as depicted in Figure III.3. The physical parameters are listed in Table III.1, where the ones related to the flexible beam refers to a Polyethylene HDPE material. According to [73], we know that the model of a rotating flexible beam (II.44)-(II.46) is approximately controllable with respect to the input τ for time t large enough. Approximate controllability with respect to an input map B implies approximate observability with respect to its dual output $y = B^*x$ defined in (III.30) (see Lemma 6.2.14 of [65]). Approximate observability of (II.44)-(II.46) with respect to y implies approximate observability of the infinite dimensional part with respect to y_z . Then, it remains to show that the origin is the only

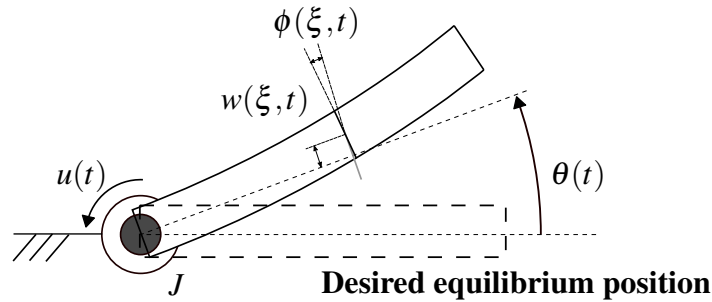


Figure III.3: Rotating flexible beam with origin equilibrium point.

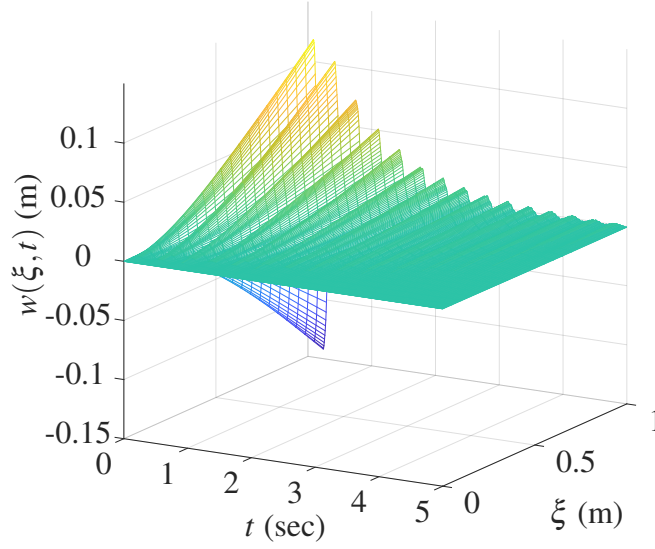


Figure III.4: Beam's deformation along time with PD control action.

equilibrium position of the distributed parameter part

$$\begin{aligned}
 \frac{\partial}{\partial \xi} K z_3(\xi, t) &= 0 \\
 \frac{\partial}{\partial \xi} E I z_4(\xi, t) + K z_3(\xi, t) &= 0 \\
 \frac{\partial}{\partial \xi} \frac{1}{\rho} z_1(\xi, t) - \frac{1}{I_p} z_2(\xi, t) &= 0 \\
 \frac{\partial}{\partial \xi} \frac{1}{I_p} z_2(\xi, t) &= 0
 \end{aligned} \tag{III.38}$$

with homogeneous boundary conditions $\frac{1}{\rho} z_1(0, t) = \frac{1}{I_p} z_2(0, t) = K z_3(L, t) = E I z_4(L, t) = 0$. The first and last equations with the respective boundary conditions return $K z_3(\xi, t) = \frac{1}{I_p} z_2(\xi, t) = 0$, that since $K(\xi), \frac{1}{I_p}(\xi) > 0$ for all $\xi \in [0, L]$ implies $z_3(\xi, t) = z_2(\xi, t) = 0$. Then, from the second and third equations we can also conclude $z_1(\xi, t) = z_4(\xi, t) = 0$, and therefore $z_{eq} = 0$. Using Proposition III.2.3 we can conclude that $x_{eq} = 0$ is an asymptotically stable equilibrium point of (III.30) with matrices defined in Section II.1.2.b of Chapter II.

To perform numerical simulations, we spatially discretize the infinite dimensional part of the system with the discretization method illustrated in in Section II.2 of Chapter II (page 53). The finite dimensional approximation of the closed-loop operator (III.30)-(III.31) writes

$$\begin{bmatrix} \dot{z}_d \\ \dot{p} \\ \dot{q} \end{bmatrix} = \left(\begin{bmatrix} J_{fe} & \bar{g}_{fe} & 0 \\ -\bar{g}_{fe}^T & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & r_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} H_{fe} & 0 & 0 \\ 0 & J^{-1} & 0 \\ 0 & 0 & k_1 \end{bmatrix} \begin{bmatrix} z_d \\ p \\ q \end{bmatrix} \tag{III.39}$$

with $z_d \in \mathbb{R}^{4n_d}$ and $n_d = 50$ the number of discretizing elements. The matrix J_{fe} is defined in (II.160), \bar{g}_{fe} is the vector corresponding to the 2nd column of g_{fe} defined in (II.161) and H_{fe} in (II.162). The control parameters are selected such that $k_1 = 500$ and

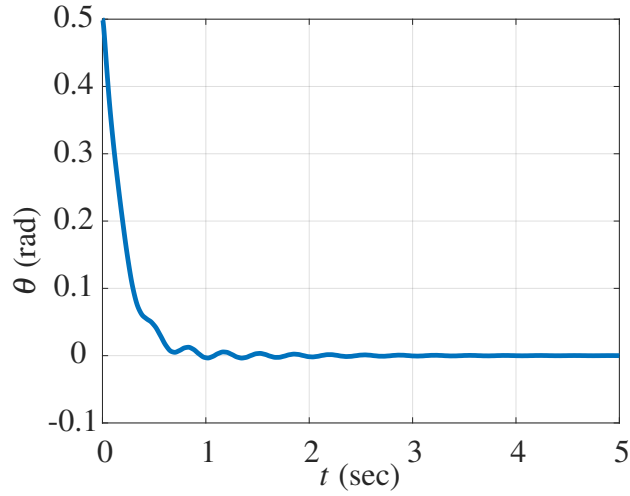


Figure III.5: Hub's angle along time with PD control action.

$r_p = 100$. In Figure III.4 and III.5 are shown the beam deformation along time, and the hub's angle trajectories along time, respectively. The considered initial conditions are taken such that $z_0 = 0$, $p_0 = 0$, $q_0 = 0.5$. In Figure III.6 we show the applied control law, highlighting the frequency content with a focus between second 1 and 2.

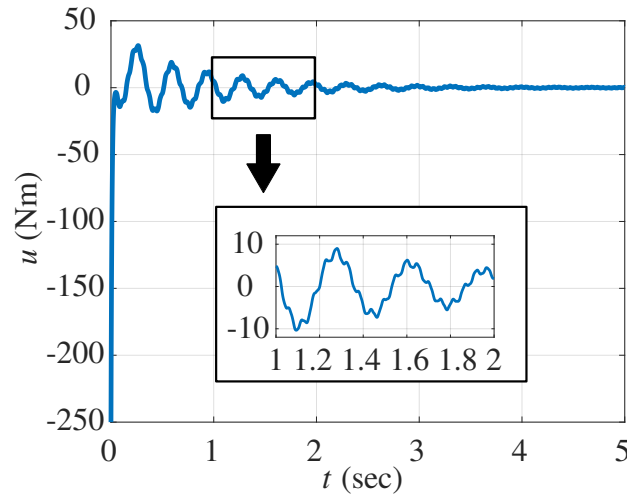


Figure III.6: Applied PD control law along time.

Remark 12. The model in [73] considers a flexible beam connected to the rotating inertia at a distance r from the center of rotation. In this thesis we implicitly assumed that the flexible beam is connected at the motor's rotor center of rotation. Therefore, the condition to ensure approximate controllability (page 447 of [73]) is automatically guaranteed. In case of real implementation, this condition must be carefully checked to avoid the incapability of observing some particular vibration mode.

III.2.2 Nonlinear passive control of a m-pH system

To be more effective in damping the vibration caused by flexibility, we want to add some information coming from the flexible deformation to the control action. In this section, we use the control law that has been proposed in [91] for the considered class of m-pH systems. In this work, the authors concluded about Lyapunov stability for a rotating flexible beam embodying the Euler-Bernoulli equations. In [42], a generalization of the same control law has been shown to asymptotically stabilize the same infinite dimensional model of the rotating flexible beam. In this section we study the same control law in closed-loop with the m-pH systems (III.17), for which we first conclude about well-posedness and then about asymptotic stability. We Consider a nonlinear control system given by the following equations

$$\begin{cases} \dot{x}_c = -R_c Q_c x_c + \Lambda(z) u_c \\ y_c = \Lambda^T(z) Q_c x_c \end{cases} \quad (\text{III.40})$$

where $x_c \in \mathbb{R}^m$ is the state vector, $Q_c = Q_c^T > 0$ the energy matrix, $R_c = R_c^T > 0$ the dissipation matrix. $\Lambda(z) = \text{diag}([\lambda_1(z) \cdots \lambda_m(z)])$ is the input-output matrix, composed by linear functions $\lambda_i : Z \mapsto \mathbb{R}$. The functions can represent, among others, the measurement of a boundary variable as well as a deformation at some point of the spatial domain: both can be represented as a linear function of the distributed parameter state. Therefore, the functions $\lambda_i(z)$ could represent a sensor's measurement. Their definition as general functions of the distributed parameter state has been employed to let open the possibility of finding a function that allows better performances in closed-loop. Consequently, since the control law depends on the total state, it can be classed as *state feedback*.

The dynamic controller introduced in (III.40) is passive with respect to its internal energy $E_c = \frac{1}{2} x_c^T Q_c x_c$. In fact,

$$\begin{aligned} \dot{E}_{c+} &= x_c^T Q_c (-R_c Q_c x_c + \Lambda(z) u_c) \\ &= -(Q_c x_c)^T R_c (Q_c x_c) + (\Lambda^T(z) Q_c x_c)^T u_c \\ &\leq y_c^T u_c. \end{aligned} \quad (\text{III.41})$$

We connect the controller input to the system's output $u_c = y$, and we define the following control law

$$u = -Kq - R_p M^{-1} p - y_c \quad (\text{III.42})$$

where the variable q is defined such that $\dot{q} = M^{-1} p$. The control matrices $K, R_p \in \mathbb{R}^{m \times m}$ are diagonal and positive definite, i.e. $K_q = \text{diag}[k_{q,1} \ k_{q,2} \ \dots \ k_{q,m}]$ and $K_p = \text{diag}[k_{p,1} \ k_{p,2} \ \dots \ k_{p,m}]$. We define the variable $v = \begin{bmatrix} p \\ q \\ x_c \end{bmatrix}$ and the closed-loop system obtained by applying (III.42) to system (III.17), as shown in Figure III.7, can be defined by a set of semi-linear equations

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \frac{\partial}{\partial \xi} P_1(\mathcal{H}z) + P_0(\mathcal{H}z) \\ -g\mathcal{C}_1(\mathcal{H}z) + (J - R)Qv \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ g & \tilde{g} \end{bmatrix} \begin{bmatrix} -\Lambda(z)\tilde{g}^T Qv \\ \Lambda(z)g^T Qv \end{bmatrix} \\ &= Ax + f(x), \end{aligned} \quad (\text{III.43})$$

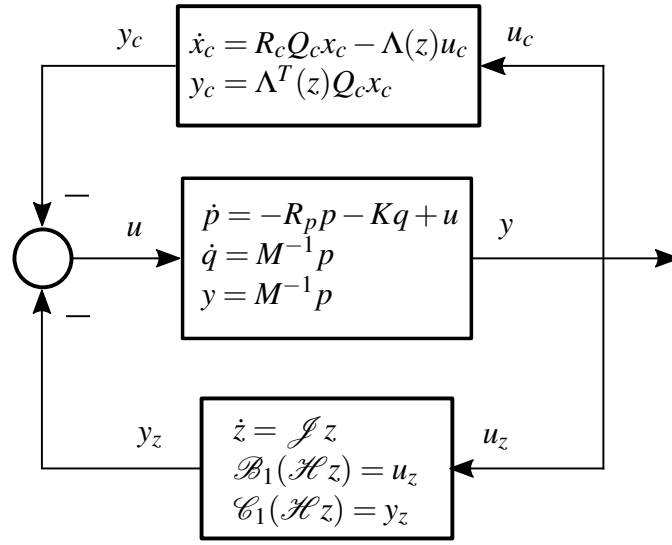


Figure III.7: m-pH system in closed-loop with a non-linear passive control law.

with state $x = \begin{bmatrix} z \\ v \end{bmatrix} \in X = L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{3m}$, and finite dimensional matrices

$$J = \begin{bmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} R_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_c \end{bmatrix} \quad Q = \begin{bmatrix} M^{-1} & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & Q_c \end{bmatrix} \quad g = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \quad \tilde{g} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \quad (\text{III.44})$$

and inner product on the state space defined as

$$\langle x_1, x_2 \rangle_X = \langle z_1, \mathcal{H} z_2 \rangle_{L_2} + v_1^T Q v_2. \quad (\text{III.45})$$

The linear operator in (III.43) is defined with domain

$$\mathbf{D}(A) = \{x \in X \mid z \in H^1([a, b], \mathbb{R}^n, \mathcal{B}_1(\mathcal{H} z) = g^T Q v, \mathcal{B}_2(\mathcal{H} z) = 0\}. \quad (\text{III.46})$$

In the following theorem we show that the operator A generates a contraction C_0 -semigroup in the defined state space X .

Theorem III.2.4. *Under Assumption 1 the operator A with domain $\mathbf{D}(A)$ generates a contraction C_0 -semigroup in the space X and has a compact resolvent.*

Proof. Since the operator A in (III.43) with domain (III.46) is in the same format as system (I.69) in Corollary I.3.1, according to Corollary I.3.1 we can conclude that the operator A generates a contraction C_0 -semigroup in the state space X and has a compact resolvent. \square

We are now in position to conclude about the existence of a unique bounded solution of the closed-loop system (III.43).

Theorem III.2.5. *Under Assumption 1, system (III.43) with operator domain $\mathbf{D}(A)$ (III.46) possesses for every initial condition $x_0 \in X$ a unique mild solution that is bounded. Furthermore, if $x_0 \in \mathbf{D}(A)$ the solution is classical.*

Proof. The proof is divided in two parts: in the first part we show that the semilinear equation (III.43) with operator domain $\mathbf{D}(A)$ (III.46) possesses a unique local mild solution, while in the second part we conclude that this solution is global.

By Theorem I.1.4, for the existence of a local mild solution we have to show that the function $f(x)$ is locally Lipschitz continuous. The Lipschitz continuity of $f(x)$ is equivalent to the Lipschitz continuity of $f_1 = \Lambda(z)\tilde{g}^T Qv$ and $f_2 = \Lambda(z)g^T Qv$. We start by the Lipschitz continuity of $f_1(x)$

$$\begin{aligned} \|f_1(x_2) - f_1(x_1)\| &= \|\Lambda(z_1)\tilde{g}^T Qv_1 - \Lambda(z_2)\tilde{g}^T Qv_2\| \\ &= \|\Lambda(z_1)\tilde{g}^T Q(v_1 - v_2) + (\Lambda(z_1) - \Lambda(z_2))\tilde{g}^T Qv_2\| \\ &\leq \|\Lambda(z_1)\tilde{g}^T Q(v_1 - v_2)\| + \|(\Lambda(z_1) - \Lambda(z_2))\tilde{g}^T Qv_2\|. \end{aligned} \quad (\text{III.47})$$

Assuming $\|x_1\|, \|x_2\| < r$, from the linearity of $\Lambda(z)$ we get $\|\Lambda(z)\| \leq M_\Lambda \|z\|$, and therefore we obtain

$$\|f_1(x_2) - f_1(x_1)\| \leq M_\Lambda r \lambda_Q \|v_1 - v_2\| + M_\Lambda \|z_1 - z_2\| \lambda_Q \|v_2\| \quad (\text{III.48})$$

where λ_{Q_c} is the biggest eigenvalue of Q_c . Since $\|v_1 - v_2\| \leq \|x_2 - x_1\|$ and $\|z_2 - z_1\| \leq \|x_2 - x_1\|$, we conclude

$$\|f_1(x_2) - f_1(x_1)\| \leq M_\Lambda \lambda_{Q_c} r \|x_2 - x_1\|. \quad (\text{III.49})$$

The Lipschitz continuity of $f_2(x)$ follows from very similar arguments.

Next, because of Theorem III.1.1, we remain to show that there exists a functional $V : X \mapsto \mathbb{R}^+$ such that $V(0) = 0$ and with time derivative $\dot{V}_+(x) \leq 0$, $\forall x_0 \in \mathbf{D}(A)$ and that the nonlinearity $f(x)$ is locally Lipschitz continuous. We begin by taking $V(x) = \frac{1}{2} \langle x, x \rangle_X$ as candidate Lyapunov functional, and we use (III.5), Assumption 1 and Theorem I.2.1 to write its time derivative along the system's solutions

$$\begin{aligned} \dot{V}_+(x) &= dV(x)(Ax + Bf(x)) \\ &= \langle z, \mathcal{J}z \rangle_Z + v^T Q(-g\mathcal{C}_1(\mathcal{H}z) + (J - R)Qv - g\Lambda(z)\tilde{g}Qv + \tilde{g}\Lambda(z)g^T Qv) \\ &= u_z^T y_z - v^T Qg\mathcal{C}_1(\mathcal{H}z) - v^T QRQv - v^T Qg\Lambda(z)\tilde{g}^T Qv + v^T Q\tilde{g}\Lambda(z)g^T Qv \\ &= u_z^T y_z - u_z^T y_z - (M^{-1}p)^T R_p(M^{-1}p) - (Q_c x_c)^T R_c(Q_c x_c) \end{aligned} \quad (\text{III.50})$$

therefore we obtain

$$\dot{V}_+(x) = -(M^{-1}p)^T R_p(M^{-1}p) - (Q_c x_c)^T R_c(Q_c x_c) \quad (\text{III.51})$$

and since R_p, R_c are semi-positive definite matrices, we can conclude that $\dot{V}_+(x) \leq 0$ and therefore the closed-loop system (III.43) admits a unique bounded mild solution. \square

Before stating the Theorem about the asymptotic stability of the closed-loop system, we need to show that the function $f(x)$ is square integrable in infinite time along the solutions of system (III.43).

Lemma III.2.1. *Under Assumption 1, the function $f(x)$ is square integrable in infinite time along solutions of system (III.43).*

Proof. Starting from (III.51), we use Lemma 11.2.5 of [65] to obtain

$$V(x(t)) = V(x_0) - \int_0^t (M^{-1}p(\tau))^T R_p (M^{-1}p(\tau)) - (Q_c x_c(\tau))^T R_c (Q_c x_c(\tau)) d\tau \quad (\text{III.52})$$

and since $V(x)$ is bounded from below and M, Q_c are strictly positive definite, the last equation implies $p, x_c \in L_2([0, \infty], \mathbb{R}^m)$. To show that $f(x)$ is square integrable, we begin by the function $f_1(x) = \Lambda(z)\tilde{g}^T Qv$:

$$\int_0^\infty \|f_1(x(t))\|^2 dt = \int_0^\infty \|\Lambda(z(t))\tilde{g}^T Qv(t)\|^2 dt. \quad (\text{III.53})$$

Use the linearity of $\Lambda(z)$, $\|\tilde{g}\| = 1$ and the boundedness of trajectories $\|x(t)\| \leq r$, to obtain

$$\begin{aligned} \int_0^\infty \|f_1(x(t))\|^2 dt &\leq M_\Lambda^2 \lambda_{Q_c}^2 \int_0^\infty \|z(t)\|^2 \|x_c(t)\|^2 dt \\ &\leq M_\Lambda^2 r^2 \lambda_{Q_c}^2 \int_0^\infty \|x_c(t)\|^2 dt. \end{aligned} \quad (\text{III.54})$$

Since x_c is square integrable on infinite time, it follows

$$\int_0^\infty \|f_1(x(t))\|^2 dt \leq \infty. \quad (\text{III.55})$$

To show the square integrability of $f_2(x)$ compute

$$\int_0^\infty \|f_2(x(t))\|^2 dt = \int_0^\infty \|\Lambda(z)g^T Qv\|^2 dt. \quad (\text{III.56})$$

With some very similar arguments used for $f_1(x)$, we obtain

$$\int_0^\infty \|f_2(x(t))\|^2 dt \leq M_\Lambda^2 r^2 \int_0^\infty \|M^{-1}p\|^2 dt \quad (\text{III.57})$$

and because of the square integrability of p we conclude that $f_2(x)$ is square integrable in infinite time. \square

Finally, we show that if the closed-loop system has the origin as only equilibrium point, then the origin is an asymptotically stable equilibrium point. As in the previous section we show the result using two different but equivalent proofs, *i.e.* firstly with the LaSalle's invariance principle and secondly with Oostveen's Theorem.

Theorem III.2.6. *Consider the closed-loop system (III.43). Under Assumption 1, if the distributed parameter part of the system with homogeneous boundary conditions is approximately observable with respect to the output y_z and $z_{eq} = 0$ is its only equilibrium point, then the origin is an asymptotically stable equilibrium.*

Proof. (With LaSalle's invariance principle). Since by Lemma III.2.1 we have that $f(x) \in L_2([0, \infty], \mathbb{R}^{2m})$, and by Theorem III.2.4 the resolvent of operator A is compact, we can conclude by Corollary III.1.1 that the trajectories set is pre-compact.

Consider the Lyapunov functional of Theorem III.2.5. We want to find the largest invariant set of $\{x \in X \mid \dot{V}_+(x) = 0\}$, that since $M, R_p, Q_c, R_c > 0$, is equivalent to the set

$\{x \in X \mid p, x_c = 0\}$ (see equation (III.51)). Therefore, we replace $p, x_c = 0$ in (III.43) to obtain

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ y_z = \mathcal{C}_1(\mathcal{H}z) = -Kq \end{cases} \quad (\text{III.58})$$

with boundary conditions $\mathcal{B}_1(\mathcal{H}z) = \mathcal{B}_2(\mathcal{H}z) = 0$, and constant q . Hence, the problem of finding the largest invariant set consists on finding the solution of (III.58). Because of Lemma III.1.1, since (III.58) is approximately observable by assumption, its only solution is $z = 0$. Then, the largest invariant set consists of the set containing only the origin. Finally by the LaSalle's invariant principle we can conclude that the origin is asymptotically stable. \square

Proof. (With Oostveen's Theorem). It is easy to see that approximate observability of the infinite dimensional part implies approximate observability of system (III.43) with respect to the output $y = B^*x$ where

$$B^* = \begin{bmatrix} 0 & M^{-1} & 0 \\ 0 & 0 & Q_c \end{bmatrix}. \quad (\text{III.59})$$

Define the weighted input-output matrices as $\tilde{B} = B\sqrt{\kappa}$ and $\tilde{B}^* = \sqrt{\kappa}B^*$, with

$$\kappa = \begin{bmatrix} R_p & 0 \\ 0 & R_c \end{bmatrix} > 0. \quad (\text{III.60})$$

Then, the closed-loop system (III.30) can be rewritten as $\dot{x} = A' + \kappa\tilde{B}\tilde{B}^*$. The operator A' is the same as the operator A , but without dissipation, therefore A' generates a contraction C_0 -semigroup and has a compact resolvent by Corollary I.3.1 (see Theorem III.2.4 with $R_p, R_c = 0$). Then by the Oostveen's theorem III.1.3, we can conclude that the origin is asymptotically stable. \square

Example 2 (Rotating flexible beam with non-linear passive controller). In this example we consider the stabilisation problem of a rotating flexible beam, as depicted in Figure III.3 of Example 1, but in this case using the nonlinear passive control law studied in this chapter. We refer to the model derived in Section II.1.2.b of Chapter II for the rotating flexible beam. The rotating flexible beam model in closed-loop with the control law (III.42) writes as a semi-linear operator equation (III.43) with domain (III.46). The model's physical parameters are listed in Table III.1. With the same arguments used in Example 1, it is possible to show that the distributed parameter part of the considered system is approximately observable with respect to the output y_z and that its only equilibrium point is $z_{eq} = 0$. Using Theorem III.2.6 we can conclude that $x_{eq} = 0$ is an asymptotically stable equilibrium point of (III.43) with matrices defined in Section II.1.2.b of Chapter II.

In this example we select the function $\lambda(z)$ of the nonlinear controller such that

$$\lambda(z) = \int_0^L \left\{ z_3(\xi, t) + \int_0^\xi z_4(\zeta, t) d\zeta \right\} d\xi. \quad (\text{III.61})$$

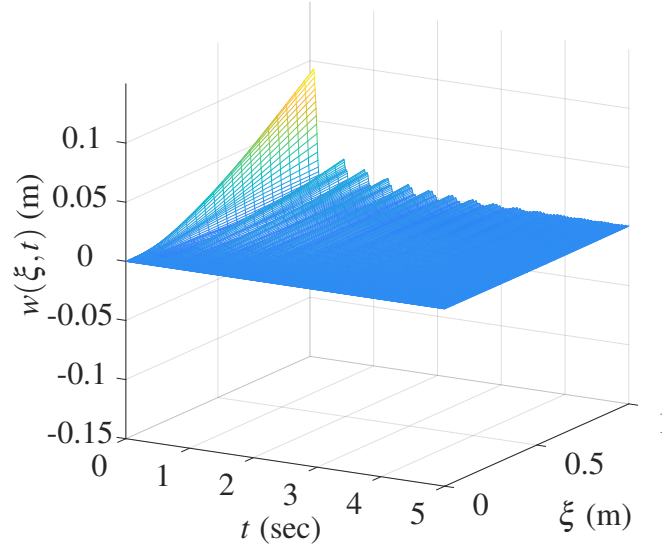


Figure III.8: Beam's deformation along time with passive non linear control action.

With this selection we impose the function $\lambda(z)$ to be equal to the deformation at the end of the beam $w(L, t)$. To simulate the closed-loop system, we spatially discretize the distributed parameter part of the system to obtain according to Section II.2 of Chapter II

$$\begin{aligned} \begin{bmatrix} \dot{z}_d \\ \dot{p} \\ \dot{q} \\ \dot{x}_c \end{bmatrix} &= \left(\begin{bmatrix} J_{fe} & \bar{g}_{fe} & 0 & 0 \\ -g_{fe}^T & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & r_p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_c \end{bmatrix} \right) \begin{bmatrix} H_{fe} & 0 & 0 & 0 \\ 0 & J^{-1} & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & k_c \end{bmatrix} \begin{bmatrix} z \\ p \\ q \\ x_c \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ \lambda_d(z_d)k_c x_c \\ 0 \\ -\frac{\lambda_d(z_d)}{J}p \end{bmatrix} \end{aligned} \quad (\text{III.62})$$

where λ_d corresponds to the discretized version of the λ function, $z_d \in \mathbb{R}^{4n_d}$ is the discretized distributed parameter state where n_d is the number of discretizing elements. The matrix J_{fe} is defined in (II.160), \bar{g}_{fe} is the vector corresponding to the 2nd column of g_{fe} defined in (II.161) and H_{fe} in (II.162). The control parameters are selected such that $k_1 = 500$, $c = 100$, $k_c = 3 \times 10^5$ and $r_c = 3 \times 10^{-8}$. We selected on purpose the parameters regarding the PD part of the control law equal to the ones selected in Example 1 on the previous section, such to be able to compare the obtained results. In Figure III.8 and III.9 are shown the beam deformation along time, and the hub's angle trajectories along time, respectively. Compared to the results obtained in Example 1, we can observe that the overall deformation is reduced and converge faster to the origin. The considered initial conditions are taken such that $z_b = 0$, $p = 0$, $q = 0.5$. In Figure III.10 we show the applied control law, highlighting the frequency content with a focus between second 1 and 2. We can observe that the control input in Figure III.10 has a

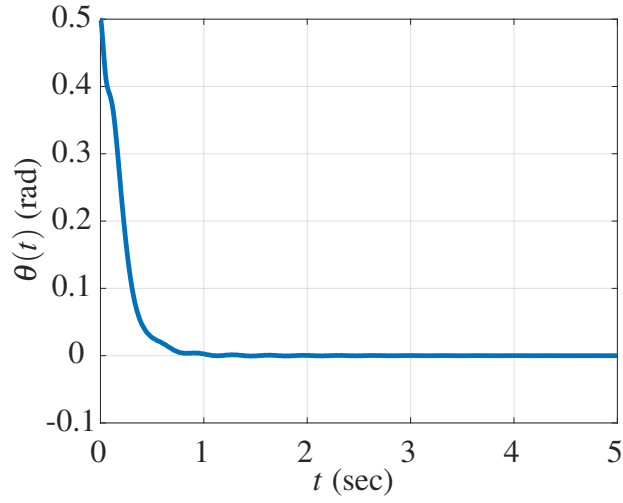


Figure III.9: Hub's angle along time with passive nonlinear control action.

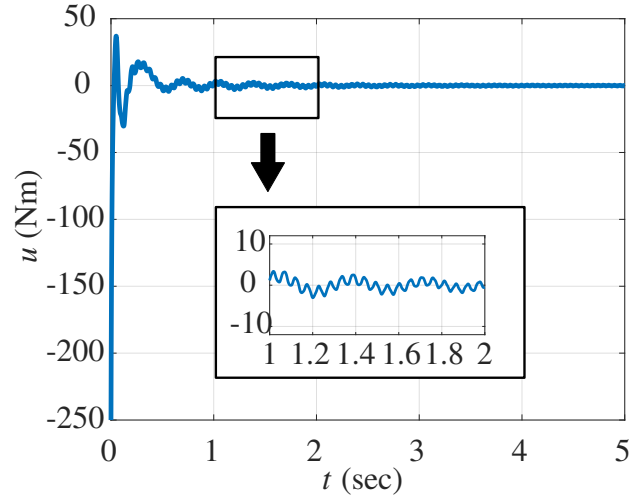


Figure III.10: Applied passive non linear control law along time.

frequency content similar to the control input obtained with the system in closed-loop with a PD control law. Therefore we can conclude that the nonlinear passive control law shown in this Section is feasible, in terms of actuator response velocity, as a PD control law.

III.2.3 Strong dissipation control of a m-pH system

It has been proven that linear operator equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu & x(0) &= x_0 \\ y &= Cx \end{aligned} \tag{III.63}$$

with A generator of a bounded group (i.e. $\sup_{t \in \mathbb{R}} \|T(t)\| < \infty$) on a infinite dimensional state space X , and input matrix $B \in \mathcal{L}(\mathbb{R}^n, X)$, are not exponentially stabilizable with classical bounded linear feedback $u = -Fx$ with $F \in \mathcal{L}(X, \mathbb{R}^n)$ (see in Lemma 8.4.1 of [65]). The systems represented by equation (III.17) can be written in the form of (III.63). This is therefore a subclass of the previously defined class of systems, and hence not exponentially stabilizable with classical bounded linear feedback. In this section we propose an (unbounded) linear feedback that allows to exponentially stabilize the considered class of system:

$$u(t) = -R_p M^{-1} p - R_p M^{-1} K_p \mathcal{C}_1(\mathcal{H}z) - K_p \frac{d}{dt}(\mathcal{C}_1(\mathcal{H}z)) \quad (\text{III.64})$$

where $R_p = \text{diag}([r_{p,1} \dots r_{p,m}]) \in \mathbb{R}^{m \times m}$, $K_p = \text{diag}([k_{p,1} \dots k_{p,m}]) \in \mathbb{R}^{m \times m}$ and the last term is known in the literature of stabilization of mixed PDEs-ODEs systems as *strong dissipation feedback*. The proposed linear feedback is unbounded because it contains a time derivative of the state: even if the state is bounded, there is no *a priori* assurance that also the control law is bounded. The control law (III.64) uses the output $\mathcal{C}_1(\mathcal{H}z)$ of the distributed parameter part together with the standard output $y = M^{-1}p$ of the system. Since all the variables needed to the application of the control law can be obtained through the use of sensors, we can class the control law (III.64) as *output feedback*. This type of control law has also been used in the stabilization of flexible beams, and it is normally referred to as *strain rate feedback* [92]. In case of application on moving flexible beams, this control input can be computed calculating an approximated and filtrated version of the time derivative of the strain measurement as explained in [41]. Applying the control law (III.64) to system (III.17), we obtain the closed-loop system depicted in Figure III.11 of equations

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ \dot{p} = -\mathcal{C}_1(\mathcal{H}z) - R_p M^{-1}(p + K_p \mathcal{C}_1(\mathcal{H}z)) - K_p \frac{d}{dt}(\mathcal{C}_1(\mathcal{H}z)). \end{cases} \quad (\text{III.65})$$

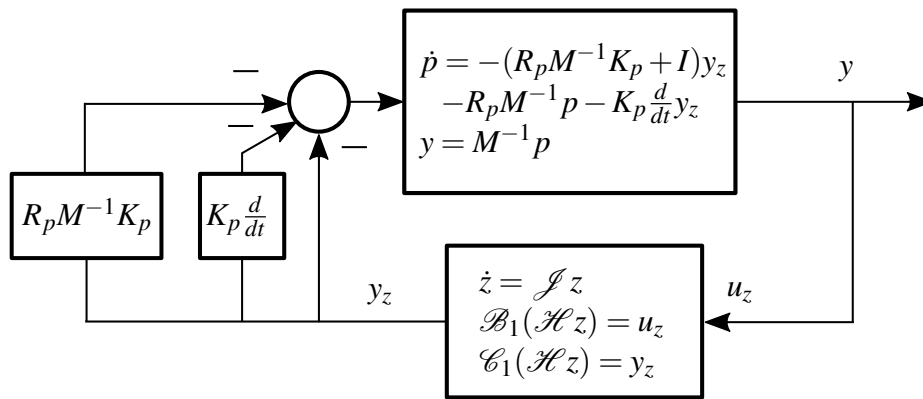


Figure III.11: m-pH system in closed-loop with a *strong feedback* control law.

To analyse the resulting closed-loop system we perform the change of variable $\eta = p + K_p \mathcal{C}_1(\mathcal{H}z)$. Therefore, we define $x = \begin{bmatrix} z \\ \eta \end{bmatrix} \in X = L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ such to write the system as a linear operator equation

$$\dot{x} = Ax = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ -\mathcal{C}_1(\mathcal{H}z) - R_p M^{-1} \eta \end{bmatrix} \quad (\text{III.66})$$

with domain defined as

$$\mathbf{D}(A) = \left\{ x \in X \mid z \in H^1([a, b], \mathbb{R}^n), \right. \\ \left. \mathcal{B}_1(\mathcal{H}z) = M^{-1}(\eta - K_p \mathcal{C}_1(\mathcal{H}z)), \mathcal{B}_2(\mathcal{H}z) = 0 \right\}. \quad (\text{III.67})$$

We define the inner product

$$\langle x_1, x_2 \rangle_X = \langle z_1, \mathcal{H}z_2 \rangle_{L_2} + \eta_1^T M^{-1} \eta_2 \quad (\text{III.68})$$

and we equip the state space X with the associated norm $\|x\| = \sqrt{\langle x, x \rangle_X}$. Since the made change of variables is bounded and invertible, studying the stability of (III.66) is equivalent to studying the stability of (III.65).

To conclude about exponential stability of the closed-loop system (III.66)-(III.67), throughout this section we assume the following assumption on the number of actuated inputs for the infinite dimensional part.

Assumption 3. The m input/output of the system are chosen such that

$$\begin{aligned} \|\mathcal{H}z(0, t)\|^2 &\leq \|u_z(t)\|^2 + \|y_z(t)\|^2 \\ &\quad \text{or} \\ \|\mathcal{H}z(L, t)\|^2 &\leq \|u_z(t)\|^2 + \|y_z(t)\|^2. \end{aligned} \quad (\text{III.69})$$

In practice, these inequalities require that we have all the boundary controls at least at one side of the spatial domain. We want to remark that these inequalities are the same required also in Theorem III.2.1, to conclude about approximate observability of a 1-D dpH system. Moreover, these same inequalities are required in [28] in order to have a 1-D dpH system that is exponentially stabilizable.

We start by showing that the closed-loop operator generates a contraction C_0 -semigroup in the defined state space X .

Theorem III.2.7. Under Assumption 1, system (III.66) with domain defined by (III.67) generates a contraction C_0 -semigroup in the state space X and has a compact resolvent.

Proof. Thanks to Theorem I.3.1 we only have to show that the operator A with domain $\mathbf{D}(A)$ is dissipative in the norm associated to the state space X . Therefore, knowing that $y_z = \mathcal{C}_1(\mathcal{H}z)$, we compute

$$\begin{aligned} \langle Ax, x \rangle &= \langle \mathcal{I}z, z \rangle_Z + (-\mathcal{C}_1(\mathcal{H}z) - R_p M^{-1} \eta)^T M^{-1} \eta \\ &= y_z^T u_z - \mathcal{C}_1(\mathcal{H}z)^T M^{-1} \eta - (M^{-1} \eta)^T R_p (M^{-1} \eta) \\ &= y_z^T M^{-1} (\eta - K_p y_z) - y_z^T M^{-1} \eta - (M^{-1} \eta)^T R_p (M^{-1} \eta) \\ &= -y_z^T K_p M^{-1} y_z - (M^{-1} \eta)^T R_p (M^{-1} \eta) \leq 0. \end{aligned} \quad (\text{III.70})$$

This shows that the operator A is dissipative in X and therefore that generates a contraction C_0 -semigroup in X and has a compact resolvent. \square

In the following lemmas we show some inequalities that will be necessary for the exponential stability proof of the closed-loop operator.

Lemma III.2.2. *Let $x(\zeta, t)$ be a solution generated by (III.66)-(III.67). Under Assumption 3 there exists a constant $\alpha > 0$ such that the state trajectories satisfies*

$$\begin{aligned} \alpha (||\mathcal{H}z(0, t)||^2 + ||\eta||^2) &\leq y_z^T K_p M^{-1} y_z + (M^{-1} \eta)^T R_p (M^{-1} \eta) \\ &\quad \text{or} \\ \alpha (||\mathcal{H}z(L, t)||^2 + ||\eta||^2) &\leq y_z^T K_p M^{-1} y_z + (M^{-1} \eta)^T R_p (M^{-1} \eta). \end{aligned} \quad (\text{III.71})$$

Proof. Use equation (III.69) and (III.67) to write

$$\begin{aligned} ||\mathcal{H}z(0, t)||^2 + ||\eta(t)||^2 &\leq ||u_z||^2 + ||y_z||^2 + ||\eta||^2 \\ &= ||M^{-1}(\eta + K_p y_z)||^2 + y_z^T y_z + \eta^T M^{-1} \eta \end{aligned} \quad (\text{III.72})$$

that since we are considering the norm associated to the inner product (III.68)

$$\begin{aligned} ||\mathcal{H}z(0, t)||^2 + ||\eta(t)||^2 &\leq (\eta + K_p y_z)^T M^{-2} (\eta + K_p y_z) + y_z^T y_z \\ &\quad + \eta^T M^{-1} \eta \\ &= y_z^T (K_p^2 M^{-2} + I) y_z + \eta^T (M^{-2} + M^{-1}) \eta \\ &\quad + 2\eta M^{-2} K_p y_z. \end{aligned} \quad (\text{III.73})$$

Then, we use the inequality

$$2\eta M^{-2} K_p y_z \leq y_z^T K_p^2 M^{-2} y_z + \eta^T M^{-2} \eta \quad (\text{III.74})$$

together with the fact that M and K_p are diagonal matrices, to write

$$\begin{aligned} ||\mathcal{H}z(0, t)||^2 + ||\eta(t)||^2 &\leq y_z^T (2K_p^2 M^{-2} + I) y_z + \eta^T (2M^{-2} + M^{-1}) \eta \\ &= y_z^T K_p (2K_p M^{-1} + K_p^{-1} M) M^{-1} y_z \\ &\quad + (M^{-1} \eta)^T R_p R_p^{-1} (2I + M) (M^{-1} \eta) \\ &\leq \gamma_1 y_z^T K_p M^{-1} y_z + \gamma_2 (M^{-1} \eta)^T R_p (M^{-1} \eta) \\ &\leq \max\{\gamma_1, \gamma_2\} (y_z^T K_p M^{-1} y_z + (M^{-1} \eta)^T R_p (M^{-1} \eta)) \end{aligned} \quad (\text{III.75})$$

where γ_1 and γ_2 are the biggest eigenvalues of $2K_p M^{-1} + K_p^{-1} M$ and $R_p^{-1} (2I + M)$, respectively. Finally define

$$\alpha = \frac{1}{\max\{\gamma_1, \gamma_2\}} \quad (\text{III.76})$$

such that equation (III.71) follows. \square

Lemma III.2.3. *Let $x(\zeta, t)$ be a solution generated by the closed-loop system (III.66) - (III.67), then under Assumption 1 and 3 the functional*

$$V(x) = \frac{1}{2} \langle x, x \rangle_X = \frac{1}{2} \int_0^L z(\xi, t)^T \mathcal{H}z(\xi, t) d\xi + \frac{1}{2} \eta^T M^{-1} \eta \quad (\text{III.77})$$

is a Lyapunov functional and satisfies for $t > 2\gamma L$, where γ is such that $P_1^{-1} + \gamma \mathcal{H}(\xi)$ and $-P_1^{-1} + \gamma \mathcal{H}(\xi)$ are positive definite,

$$c_1(t) V(x(t)) \leq \int_0^t ||\mathcal{H}z(0, \tau)||^2 d\tau + \int_0^t ||\eta(\tau)||^2 d\tau \quad (\text{III.78})$$

where $c_1(t) = \frac{2(t-2\gamma L)}{\beta_1}$, $\beta_1 = \max\{L, 1\}$ or

$$c_2(t)V(x(t)) \leq \int_0^t \|\mathcal{H}z(L, \tau)\|^2 d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \quad (\text{III.79})$$

where $c_2(t) = \frac{2(t-2\gamma L)}{\beta_2}$, $\beta_2 = \max\{Le^{\kappa L}, 1\}$ and κ is such that $\mathcal{H}(\xi)P_0^T P_1^{-1} + P_1^{-1}P_0\mathcal{H}(\xi) + \frac{\partial \mathcal{H}}{\partial \xi}(\xi) \leq \kappa \mathcal{H}(\xi)$.

Proof. We define the function $F : [0, L] \mapsto \mathbb{R}$ by

$$F(\xi) = \int_{\gamma\xi}^{t-\gamma\xi} z^T(\xi, \tau) \mathcal{H}(\xi) z(\xi, \tau) d\tau, \quad \xi \in [0, L], \quad (\text{III.80})$$

where we assume that $\gamma > 0$ and $t > 2\gamma L$. Differentiating the function F with respect to ξ gives

$$\begin{aligned} \frac{dF}{d\xi}(\xi) &= \int_{\gamma\xi}^{t-\gamma\xi} z^T(\xi, \tau) \frac{\partial}{\partial \xi}(\mathcal{H}(\xi) z(\xi, \tau)) d\tau \\ &\quad + \int_{\gamma\xi}^{t-\gamma\xi} \left(\frac{\partial}{\partial \xi} z(\xi, \tau) \right)^T \mathcal{H}(\xi) z(\xi, \tau) d\tau \\ &\quad - \gamma z^T(\xi, t - \gamma\xi) \mathcal{H}(\xi) z(\xi, t - \gamma\xi) - \gamma z^T(\xi, \gamma\xi) \mathcal{H}(\xi) z(\xi, \gamma\xi). \end{aligned} \quad (\text{III.81})$$

After some similar passages as in the proof of Lemma 9.1.2 in [24] we get

$$\begin{aligned} \frac{dF}{d\xi}(\xi) &= - \int_{\gamma\xi}^{t-\gamma\xi} z^T(\xi, \tau) \left(\mathcal{H}(\xi) P_0^T P_1^{-1} + P_1^{-1} P_0 \mathcal{H}(\xi) + \frac{\partial \mathcal{H}}{\partial \xi}(\xi) \right) z(\xi, \tau) d\tau \\ &\quad - z^T(\xi, t - \gamma\xi) (-P_1^{-1} + \gamma \mathcal{H}(\xi)) z(\xi, t - \gamma\xi) \\ &\quad - z^T(\xi, \gamma\xi) (P_1^{-1} + \gamma \mathcal{H}(\xi)) z(\xi, \gamma\xi). \end{aligned} \quad (\text{III.82})$$

Now we select γ large enough, such that $P_1^{-1} + \gamma \mathcal{H}$ and $-P_1^{-1} + \gamma \mathcal{H}$ are positive definite, such to obtain

$$\frac{dF}{d\xi}(\xi) \leq - \int_{\gamma\xi}^{t-\gamma\xi} z^T(\xi, \tau) \left(\mathcal{H}(\xi) P_0^T P_1^{-1} + P_1^{-1} P_0 \mathcal{H}(\xi) + \frac{\partial \mathcal{H}}{\partial \xi}(\xi) \right) z(\xi, \tau) d\tau. \quad (\text{III.83})$$

Since P_1 and P_0 are constant matrices and $\frac{\partial \mathcal{H}}{\partial \xi}(\xi)$ is bounded, there exists a constant $\kappa > 0$ such that for all $\xi \in [0, L]$ we have

$$\mathcal{H}(\xi) P_0^T P_1^{-1} + P_1^{-1} P_0 \mathcal{H}(\xi) + \frac{\partial \mathcal{H}}{\partial \xi}(\xi) \geq \kappa_1 \mathcal{H}(\xi) \quad (\text{III.84})$$

and therefore we obtain

$$\frac{dF}{d\xi}(\xi) \leq -\kappa \int_{\gamma\xi}^{t-\gamma\xi} z^T(\xi, \tau) \mathcal{H}(\xi) z(\xi, \tau) d\tau = -\kappa_1 F(\xi). \quad (\text{III.85})$$

The former inequality implies

$$F(\xi) \leq e^{-\kappa\xi} F(0) \quad \text{for } \xi \in [0, L], \quad (\text{III.86})$$

that in turn means $F(\xi) \leq F(0)$. In order to obtain the second inequality (III.79) we select

$$F(\xi) = \int_{\gamma(L-\xi)}^{\tau-\gamma(L-\xi)} z(\xi, t)^T \mathcal{H} z(\xi, t) dt. \quad (\text{III.87})$$

instead of (III.80) and following the same passages as in Lemma 9.1.2 of [24], it is possible to obtain that $F(\xi) \leq e^{\kappa L} F(0)$ where κ is such that

$$\mathcal{H}(\xi) P_0^T P_1^{-1} + P_1^{-1} P_0 \mathcal{H}(\xi) + \frac{\partial \mathcal{H}}{\partial \xi}(\xi) \leq \kappa \mathcal{H}(\xi). \quad (\text{III.88})$$

We now compute the time derivative of (III.77) using equation (III.5), Assumption 1 and the A dissipativity

$$\dot{V}_+(x) = \langle x, Ax \rangle_X \leq 0. \quad (\text{III.89})$$

Therefore, the functional V is a Lyapunov functional. For the rest of the proof we proceed similarly to the proof of Lemma 4.1 in [27].

Using the fact that the Lyapunov functional (III.77) is non-increasing along the system's trajectories, it holds

$$\begin{aligned} \int_{\gamma L}^{t-\gamma L} V(x(\tau)) d\tau &\geq V(x(t-\gamma L)) \int_{\gamma L}^{t-\gamma L} 1 d\tau \\ &= (t-2\gamma L) V(x(t-\gamma L)). \end{aligned} \quad (\text{III.90})$$

We use again the non-increasing property of the Lyapunov functional to write

$$\begin{aligned} &2(t-2\gamma L) V(x(t)) \\ &\leq 2(t-2\gamma L) V(x(t-\gamma L)) \\ &\leq 2 \int_{\gamma L}^{t-\gamma L} V(x(\tau)) d\tau \\ &= \int_0^L \int_{\gamma L}^{t-\gamma L} z^T(\xi, \tau) \mathcal{H} z(\xi, \tau) d\tau d\xi + \int_{\gamma L}^{t-\gamma L} \|\eta(\tau)\|^2 d\tau \\ &\leq \int_0^L \int_{\gamma \xi}^{t-\gamma \xi} z^T(\xi, \tau) \mathcal{H} z(\xi, \tau) d\tau d\xi + \int_0^t \|\eta(\tau)\|^2 d\tau \end{aligned} \quad (\text{III.91})$$

where, for the last inequality, we have increased the integration time. Use definition (III.80) and increase once again the integration interval of the second term to obtain

$$\begin{aligned} 2(t-2\gamma L) V(x(t)) &\leq \int_0^L F(\xi) d\xi + \int_0^t \|\eta(\tau)\|^2 d\tau \\ &\leq LF(0) + \int_0^t \|\eta(\tau)\|^2 d\tau \\ &= L \int_0^t z(0, \tau)^T \mathcal{H} z(0, \tau) d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \\ &\leq \beta \left(\int_0^t \|\mathcal{H} z(0, \tau)\|^2 d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \right) \end{aligned} \quad (\text{III.92})$$

where $\beta = \max\{L, 1\}$. Hence, we obtain

$$\frac{2(t-2\gamma L)}{\beta} V(x(t)) \leq \int_0^t \|\mathcal{H} z(0, \tau)\|^2 d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \quad (\text{III.93})$$

that shows inequality (III.78) with $c(\tau) = \frac{2(t-2\gamma L)}{\beta}$. The other inequality is obtained using $F(\xi) \leq F(L)e^{\kappa L}$ instead of $F(\xi) \leq F(0)$ in (III.92)

$$\begin{aligned}
 2(t-2\gamma L)V(x(t)) &\leq \int_0^L F(\xi)d\xi + \int_0^t \|\eta(\tau)\|^2 d\tau \\
 &\leq LF(0)e^{\kappa L} + \int_0^t \|\eta(\tau)\|^2 d\tau \\
 &= Le^{\kappa L} \int_0^t z(0, \tau)^T \mathcal{H} z(0, \tau) d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \\
 &\leq \beta_2 \left(\int_0^t \|\mathcal{H} z(0, \tau)\|^2 d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \right)
 \end{aligned} \tag{III.94}$$

where $\beta_2 = \max\{Le^{\kappa L}, 1\}$. □

Now we are in position to state the theorem on exponential stability of the origin of the closed-loop operator.

Theorem III.2.8. *Under Assumption 1 and Assumption 3, the origin of the closed-loop system described by equations (III.66)-(III.67) is exponentially stable.*

Proof. We use equations (III.70) and (III.89) to obtain

$$\dot{V}_+(x) = -y_z^T K_p M^{-1} y_z - (M^{-1} \eta)^T R (M^{-1} \eta). \tag{III.95}$$

We use inequality (III.71) in the above equation

$$\dot{V}_+(x) \leq -\alpha (\|\mathcal{H} z(0, t)\|^2 + \|\eta\|^2) \tag{III.96}$$

and integrating in time between 0 and t both sides of the above equation and using inequality (III.78), we obtain

$$\begin{aligned}
 V(x(t)) - V(x(0)) &\leq -\alpha \left(\int_0^t \|\mathcal{H} z(0, \tau)\|^2 d\tau + \int_0^t \|\eta(\tau)\|^2 d\tau \right) \\
 &\leq -\alpha c(t) V(x(t))
 \end{aligned} \tag{III.97}$$

which implies,

$$V(x(t)) \leq \frac{1}{1 + \alpha c(t)} V(x(0)). \tag{III.98}$$

It is possible to see that the Lyapunov functional V is equivalent to the square norm of the state, hence the above equation implies

$$\|x(t)\|^2 \leq \frac{1}{1 + \alpha c(t)} \|x(0)\|^2. \tag{III.99}$$

Let $T(t)$ be the semigroup generated by the operator A . From the latter equation we obtain for all $x_0 \in X$

$$\|T(t)x_0\|^2 \leq \frac{1}{1 + \alpha c(t)} \|x_0\|^2 \tag{III.100}$$

Table III.2: Simulation parameters; strong dissipation example

Name	Variable	Value
String's Length	L	1 m
Density	ρ	1 $\frac{kg}{m}$
Tension	T	1
Tip's mass	m	1 kg

that is equivalent to

$$\|T(t)\|^2 \leq \frac{1}{1 + \alpha c(t)}. \quad (\text{III.101})$$

Since $c(t)$ is a positive function such that $c(t) \rightarrow \infty$ for $t \rightarrow \infty$, there exists a $t^* > 0$ such that $\|T(t)\| < 1$ for all $t > t^*$. Consequently $w_0 = \inf_{t>0} \left(\frac{1}{t} \log \|T(t)\| \right) < 0$ and by Theorem I.1.1 we can conclude that there exist constants $M_w > 0$ and $w < 0$ such that $\|T(t)\| \leq M_w e^{wt}$ for all $t \geq 0$. \square

Remark 13. The previous theorem states that the norm of the state $\|x(t)\|$ can be bounded by an exponential, *i.e.*

$$\|x(t)\| \leq M_w e^{wt} \|x_0\| \quad (\text{III.102})$$

with $w < 0$ and in general $M_w \leq 0$. The constant M_w is not a priori known and is in general strictly greater than zero.

In the following example we show how to apply the strong dissipation feedback control law to an applicative example, and how it is possible to compute the exponential bound of the system's state norm along time.

Example 3 (Vibrating string with tip mass and Strong dissipation control). We study the control problem of the stabilisation of a clamped string with tip mass on the other side using a force applied on the tip mass. For this problem we use the model derived in Section II.1.2.a of Chapter II. The control objective is to stabilise the system in the zero state $p = 0$, $z = 0$, as depicted in Figure III.12. Consider the closed-loop operator (III.66)-(III.67) with matrices and input-output operators corresponding to the vibrating string with a tip mass introduced in Section II.1.2.a of Chapter II. The considered physical parameters are listed in Table III.2, while the control parameters are selected such that $k_p = 0.7$, $r_p = 5$. We notice that

$$\begin{aligned} \|\mathcal{H}z(0,t)\|^2 &= \left\| \frac{1}{\rho(0)} z_1(0,t) \right\|^2 + \|T(0)z_2(0,t)\|^2 \\ &= \|u_z(t)\|^2 + \|y_z(t)\|^2 \end{aligned} \quad (\text{III.103})$$

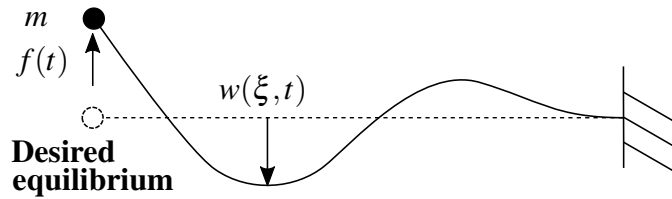


Figure III.12: String with a tip mass with origin equilibrium point.

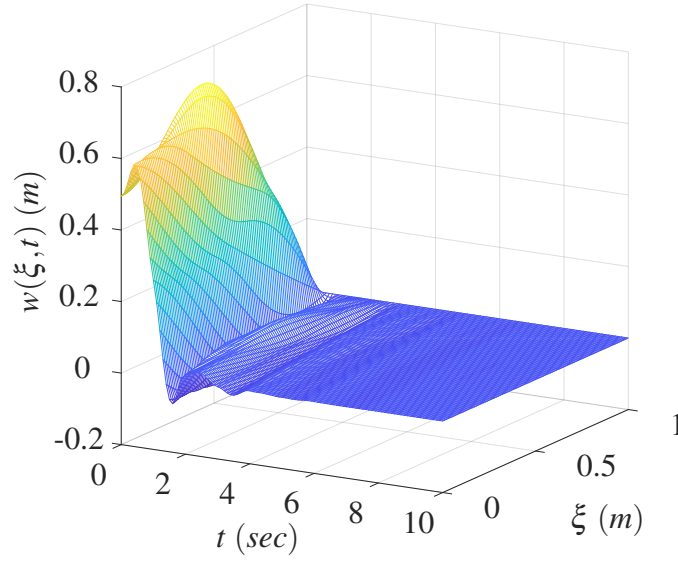


Figure III.13: String deformation along time.

and therefore, according to Theorem III.2.8, the origin of the string equation with a tip mass (II.20) in closed-loop with a strong dissipation feedback (III.64) is exponentially stable.

Therefore, we can compute all the parameters necessary to find the exponential bound of the state's norm. In particular, according to Lemma III.2.3 we can compute that for a wave equation with varying parameters

$$\gamma > \max_{\xi \in [0, L]} \sqrt{\frac{\rho(\xi)}{T(\xi)}}. \quad (\text{III.104})$$

Then, $\gamma_1 = 2k_p m^{-2} + k_p^{-1}$ $\gamma_2 = r_p^{-1}(2m^{-1} + m)$. Considering the control and the system's parameters in Table III.2, we find

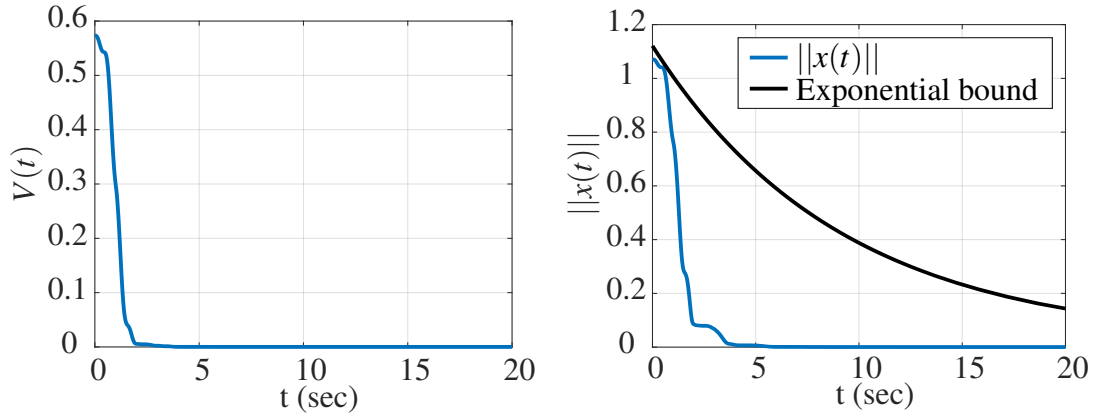
$$\gamma = 1 \quad \beta = 1 \quad \gamma_1 = 2.83 \quad \gamma_2 = 0.60 \quad (\text{III.105})$$

and consequently $\alpha = 0.35$.

Using the methodology introduced in Section II.2 of Chapter II, the spatial discretization of the distributed parameter part of the system writes

$$\begin{bmatrix} \dot{z}_d \\ \dot{\eta} \end{bmatrix} = \left(\begin{bmatrix} J_{ff} & \bar{g}_{ff} \\ -\bar{g}_{ff} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & r_p \end{bmatrix} \right) \begin{bmatrix} H_{ff} & 0 \\ 0 & m^{-1} \end{bmatrix} \begin{bmatrix} z_d \\ \eta \end{bmatrix} \quad (\text{III.106})$$

where $z_d \in \mathbb{R}^{2n_d}$ is the distributed parameter state with n_d the number of discretizing elements. The matrix J_{ff} is defined in (II.166), while \bar{g}_{ff} is the vector corresponding to the 1st column of g_{ff} defined in (II.167) and H_{ff} in (II.168). The initial conditions are set as following $z_1(\xi, 0) = 0$, $z_2(\xi, 0) = \sin(2\pi \frac{\xi}{L}) - \xi + 1$, $\eta = 0$. In Figure III.13



(a) Lyapunov functional evolution along time. (b) State's norm bounded by an exponential.

Figure III.14: Lyapunov functional and state's norm.

and III.14a are shown the string deformation and the Energy behaviour along time, respectively. In Figure III.14b we plot the state's norm together with the exponential bound assured by Theorem III.2.8. As highlighted in Remark 13, the system's norm is bounded by an exponential $\|x(t)\| \leq M_w e^{w_0 t} \|x_0\|$ for $t \geq 0$. The exponential bound has been computed as $w_0 = \inf_{t>0} \left(\frac{1}{t} \log \|T(t)\| \right)$ through Matlab[®] numerical methods, using the C_0 -semigroup norm defined in (III.101), and it results to be equal to $w_0 = -0.1099$. It is worth to remark that the exponential bound in Figure III.14b has been plotted fixing the parameter $M_w = 1.03$, that has been decided only after the state's norm decay was obtained. This to say that in case the parameter M_w is needed for any set of initial conditions, a more sharper analysis has to be carried.

The control law introduced in this section allows to stabilize the variables z and p without any control on the position of the tip mass. If the control objective is to stabilize the system in a desired position, we are forced to add in the control law a term proportional to the position, that will be investigated in the next section.

III.2.4 Strong dissipation plus position control of a m-pH system

In this section we add to the control law (III.64) the term corresponding to a position control proportional to the state variable q defined in (III.28). The variable q , in case of control of mechanical mechanisms, corresponds to a position configuration. Adding this variable in the control-loop means that the controller will be able to steer the mechanism to a desired configuration, as it was the case for the PD control in Section III.2.1 and the nonlinear passive controller in Section III.2.2. In this purpose, we propose the following control law

$$u = -R_p M^{-1} p - K q + (I - R_p M^{-1} K_p) \mathcal{C}_1(\mathcal{H}z) - K_p \frac{d}{dt}(\mathcal{C}_1(\mathcal{H}z)), \quad (\text{III.107})$$

where $R_p = \text{diag}([r_{p,1} \dots r_{p,m}])$, $K = \text{diag}([k_1 \dots k_m])$, $K_p = \text{diag}([k_{p,1} \dots k_{p,m}]) \in \mathbb{R}^{m \times m}$, where $r_{p,i}$, k_i , $k_{p,i} > 0$ for $i \in \{1, \dots, m\}$. A very similar control law has been obtained in [93] using a basketstepping control design. From an intuitive point of view,

the first two terms corresponds to a PD controller, the last term corresponds to a *strong dissipation* feedback, while the third term modifies the gain of the restoring force, relating it to the dissipation matrix R_p and the *strong dissipation* matrix K_p . For the same reasons explained in the the previous Section III.2.3, we can consider the control law (III.107) as an *output feedback*. The closed-loop system obtained by applying (III.107) to (III.17) is depicted in Figure III.15 and writes

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ \dot{x}_f = (J - R)Qx_f - g_1 R_p M^{-1} K_p \mathcal{C}_1(\mathcal{H}z) - g_1 K_p \frac{d}{dt}(\mathcal{C}_1(\mathcal{H}z)), \end{cases} \quad (\text{III.108})$$

where the finite dimensional state is $x_f = [\frac{p}{q}]$, and the matrices are defined as

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad R = \begin{bmatrix} R_p & 0 \\ 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} M^{-1} & 0 \\ 0 & K \end{bmatrix} \quad g_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (\text{III.109})$$

To analyse the obtained closed-loop system we perform the change of variables $\eta = p + K_p \mathcal{C}_1 z$, such to rewrite the system as

$$\begin{cases} \dot{z} = P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ \dot{v} = (J - R)Qv + g_2 M^{-1} K_p \mathcal{C}_1(\mathcal{H}z), \end{cases} \quad (\text{III.110})$$

where, $g_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $v = [\frac{\eta}{q}] \in \mathbb{R}^{2m}$. This system can be written as a linear operator equation of the form

$$\dot{x} = Ax = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ -g_2 M^{-1} K_p \mathcal{C}_1(\mathcal{H}z) + (J - R)Qv \end{bmatrix} \quad (\text{III.111})$$

with domain defined as

$$\mathbf{D}(A) = \left\{ x \in L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m} \mid \mathcal{H}z \in H^1([a, b], \mathbb{R}^n), \right. \\ \left. \mathcal{B}_1(\mathcal{H}z) = M^{-1}(\eta - K_p \mathcal{C}_1 z), \mathcal{B}_2(\mathcal{H}z) = 0 \right\} \quad (\text{III.112})$$

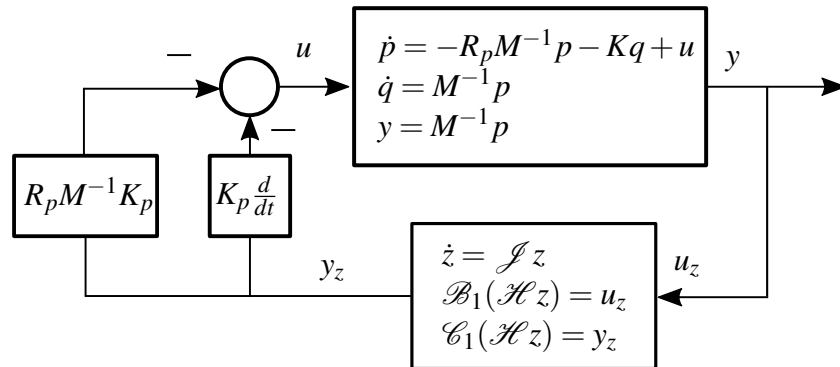


Figure III.15: m-pH system in closed-loop with a *strong dissipation* feedback plus position control law.

and state defined as $x = \begin{bmatrix} z \\ v \end{bmatrix}$. The closed-loop operator is defined as the non-power preserving interconnection between an infinite and a finite dimensional linear pH systems. Since the interconnection is not power preserving, it is not possible to show the contraction C_0 -semigroup generation in $L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the energy norm, as in classical interconnected m-pH systems [22]. Hence, in the next theorem we show that the closed-loop operator generates a contraction C_0 -semigroup in $L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with a special (energy-like) weighted norm.

Theorem III.2.9. *Under Assumption 1, there exists a weighted $L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ space such that the closed-loop operator (III.111) with domain defined by (III.112) generates a contraction C_0 -semigroup on this space, provided that $r_i^2 \geq m_i k_i$ for all $i \in \{1, \dots, m\}$. Moreover the operator A has a compact resolvent.*

Proof. Using Theorem I.3.1, we only have to show that there exists a space on which the operator A is dissipative. We define a new space $\Gamma = L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ with inner product

$$\langle x_1, x_2 \rangle_\Gamma = \langle z_1, z_2 \rangle_Z + v_1^T M_v v_2 \quad (\text{III.113})$$

where

$$M_v = \begin{bmatrix} K^{-1} M^{-1} R_p K_p^{-1} & K_p^{-1} \\ K_p^{-1} & 2KMR_p^{-1} K_p^{-1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \quad (\text{III.114})$$

To check the positive definitiveness of M_v , we use the Schur complements. In fact, since all the matrices in (III.114) are strictly positive definite, A_{22} and $A_{11} - A_{12}A_{22}^{-1}A_{21}$ are strictly positive definite matrices, from which it follows the positive definitiveness of M_v . We now check the dissipativity of the operator A in the new space Γ , considering Lemma I.2.1 and taking into account that $\mathcal{C}_1 z = y_z$,

$$\begin{aligned} \langle Ax, x \rangle_\Gamma &= \langle \mathcal{J} z, z \rangle_Z + ((J - R)Qv - g_2 M^{-1} K_p y_z)^T M_v v \\ &= u_z^T y_z - \eta^T K^{-1} M^{-2} R_p^2 K_p^{-1} \eta - 2\eta^T M^{-1} R_p K_p^{-1} q + \eta^T K_p^{-1} M^{-1} \eta \\ &\quad - q^T K_p K q + 2q^T K R_p^{-1} K_p^{-1} \eta - \eta^T M^{-1} y_z - 2q^T K R_p^{-1} y_z \\ &= -y_z K_p M^{-1} y_z - \eta^T K^{-1} M^{-2} R_p^2 K_p^{-1} \eta - 2\eta^T M^{-1} R_p K_p^{-1} q \\ &\quad + \eta^T K_p^{-1} M^{-1} \eta - q^T K_p K q + 2q^T K R_p^{-1} K_p^{-1} \eta - 2q^T K R_p^{-1} y_z. \end{aligned} \quad (\text{III.115})$$

Since all the matrices are diagonal, the previous inequality can be rewritten as

$$\begin{aligned} \langle Ax, x \rangle_\Gamma &= -y_z K_p M^{-1} y_z - \eta^T (K^{-1} M^{-2} R_p^2 K_p^{-1} - K_p^{-1} M^{-1}) \eta \\ &\quad 2\eta^T (M^{-1} R_p K_p^{-1} - K R_p^{-1} K_p^{-1}) q - 2q^T K R_p^{-1} y_z - q^T K_p^{-1} K q \\ &= -y_z K_p M^{-1} y_z - \eta^T K^{-1} M^{-2} R_p^2 K_p^{-1} (I - KMR_p^{-2}) \eta \\ &\quad 2\eta^T K_p^{-1} R_p M^{-1} (I - KMR_p^{-2}) q - 2q^T K R_p^{-1} y_z - q^T K_p^{-1} K q. \end{aligned} \quad (\text{III.116})$$

We define $\Lambda = KMR_p^{-2}$ and we rewrite the former equality as

$$\begin{aligned}
 \langle Ax, x \rangle_\Gamma &= -y_z K_p M^{-1} y_z - 2q^T K R_p^{-1} y_z - q^T (K_p^{-1} K - M^{-1} K_p^{-1} (I - \Lambda) \Lambda R_p^2) q \\
 &\quad - (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q)^T M^{-1} K_p^{-1} (I - \Lambda) (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q) \\
 &= -y_z K_p M^{-1} y_z - 2q^T K R_p^{-1} y_z - q^T K^2 M R_p^{-2} K_p^{-1} q \\
 &\quad - (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q)^T M^{-1} K_p^{-1} (I - \Lambda) (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q) \\
 &= -(K_p^{\frac{1}{2}} y_z + K M R_p^{-1} K_p^{-\frac{1}{2}} q)^T M^{-1} (K_p^{\frac{1}{2}} y_z + K M R_p^{-1} K_p^{-\frac{1}{2}} q) \\
 &\quad - (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q)^T M^{-1} K_p^{-1} (I - \Lambda) (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q).
 \end{aligned} \tag{III.117}$$

The assumption that $r_i^2 > m_i k_i$ for all $i \in \{1, \dots, m\}$ implies that the matrix $(I - \Lambda)$ is strictly positive definite. Therefore we obtain that $\langle Ax, x \rangle_\Gamma \leq 0$, and by means of Theorem I.3.1 we can conclude that the operator A generates a contraction C_0 -semigroup in the space Γ and has a compact resolvent. \square

It is well known that if an operator is the infinitesimal generator of a C_0 -semigroup in a space equipped with a certain norm, then it generates a C_0 -semigroup in all the spaces equipped with equivalent norms. In the next corollary we show that the norm defined through (III.113) is equivalent to the standard norm in $L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$. Hence, this directly implies that the closed-loop operator (III.111)-(III.112) generates a C_0 -semigroup in $L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the standard norm.

Corollary III.2.2. *Under Assumption 1, the closed-loop operator (III.111)-(III.112) generates a C_0 -semigroup in $L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the standard norm*

$$||x|| = \sqrt{\langle z, z \rangle_{L_2} + v^T v}. \tag{III.118}$$

Proof. It is sufficient to show that the norm associated to the inner product (III.113)

$$||x||_\Gamma = \sqrt{\langle z, \mathcal{H}z \rangle_{L_2} + v^T M_v v}, \tag{III.119}$$

is equivalent to the standard norm, i.e. that there exist $C > c \in \mathbb{R}^+$ such that

$$c||x|| \leq ||x||_\Gamma \leq C||x||. \tag{III.120}$$

The first inequality of (III.120) can be rewritten as

$$c\sqrt{\langle z, z \rangle_{L_2} + v^T v} \leq \sqrt{\langle z, \mathcal{H}z \rangle_{L_2} + v^T M_v v}, \tag{III.121}$$

that is equivalent to

$$c^2 (\langle z, z \rangle_{L_2} + v^T v) \leq \langle z, \mathcal{H}z \rangle_{L_2} + v^T M_v v. \tag{III.122}$$

The last inequality is fulfilled if

$$\langle z, (c^2 I - \mathcal{H})z \rangle \leq 0, \quad v^T (c^2 I - M_v)v \leq 0. \tag{III.123}$$

Since \mathcal{H} and M_v are strictly positive definite, it exists a constant $c \in \mathbb{R}$ such that both inequalities hold. The second inequality in (III.120) can be rewritten as

$$\sqrt{\langle z, \mathcal{H}z \rangle_{L_2} + v^T M_v v} \leq C \sqrt{\langle z, z \rangle_{L_2} + v^T v}, \quad (\text{III.124})$$

and it holds if

$$\langle z, (\mathcal{H} - C^2 I)z \rangle \leq 0, \quad v^T (M_v - C^2 I)v \leq 0. \quad (\text{III.125})$$

Since the entries of both \mathcal{H} and M_v are always finite, it exists a $C \in \mathbb{R}$ such that both these inequalities are fulfilled. We therefore conclude that (III.119) is equivalent to the standard norm in $L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$. As a consequence, since the closed-loop operator (III.111)-(III.112) generates a contraction C_0 -semigroup in $L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the norm (III.119), it also generates a C_0 -semigroup in the same space equipped with the standard norm. \square

Since by Theorem III.2.9 the operator A generates a contraction C_0 -semigroup $T(t)$ in $L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the norm (III.119), we have that

$$\|T(t)\|_{\Gamma} \leq 1. \quad (\text{III.126})$$

Using (III.120), it is possible to obtain the bound of the C_0 -semigroup generated by the operator A in $L_2([0, L], \mathbb{R}^n) \times \mathbb{R}^{2m}$ equipped with the standard norm (III.118)

$$\|T(t)\| \leq \frac{C}{c}. \quad (\text{III.127})$$

The asymptotic stability of the system described by equation (III.111) - (III.112) is equivalent to show the asymptotic stability of system (III.110). To show asymptotic stability we consider the state space $X = L_2([a, b], \mathbb{R}^n) \times \mathbb{R}^{2m}$ with inner product $\langle x_1, x_2 \rangle_X = \langle z_1, z_2 \rangle_{L_2} + v_1^T v_2$ and associated norm (III.118). We now prove that with the proper choice of control parameters the closed-loop system is asymptotically stable.

Theorem III.2.10. *Consider the closed-loop system (III.111)-(III.112). Assume that the distributed parameter part of the system with homogeneous boundary conditions is approximately observable with respect to the output y_z and $z_{eq} = 0$ is its only equilibrium point. Under Assumption 1, if the control gains $k_{p,i}, r_i, k_i$ with $i = \{1, \dots, m\}$ are chosen such that*

$$r_i^2 > 2m_i k_i, \quad k_{p,i} > 0 \quad (\text{III.128})$$

then the origin $x_{eq} = 0$ is an asymptotically stable equilibrium.

Proof. We define the candidate Lyapunov functional

$$V(x) = \frac{1}{2} \langle x, x \rangle_{\Gamma} \quad (\text{III.129})$$

with inner-product defined in (III.113). The time derivative of the Lyapunov functional can be computed as

$$\begin{aligned} \dot{V}_+(x) &= \langle x, Ax \rangle_{\Gamma} \\ &= \langle z, \mathcal{J}z \rangle_Z + v^T M_v ((J - R)Qv - g_2 M^{-1} K_p y_z) \end{aligned} \quad (\text{III.130})$$

that, because of Lemma I.2.1 and equations (III.115)-(III.117), can be rewritten as

$$\begin{aligned}\dot{V}_+(x) &= \langle \mathcal{J}z, z \rangle_Z + ((J-R)Qv - g_2 M^{-1} K_p y_z)^T M_v v \\ &= -(K_p^{\frac{1}{2}} y_z + KMR_p^{-1} K_p^{-\frac{1}{2}} q)^T M^{-1} (K_p^{\frac{1}{2}} y_z + KMR_p^{-1} K_p^{-\frac{1}{2}} q) \\ &\quad - (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q)^T M^{-1} K_p^{-1} (I - \Lambda) (\Lambda^{-\frac{1}{2}} \eta + R_p \Lambda^{\frac{1}{2}} q)\end{aligned}\quad (\text{III.131})$$

where $\Lambda = KMR_p^{-2}$. To use the LaSalle's invariance principle, we show that the largest invariant subset \mathbf{S} of $\mathbf{S}_0 = \{x_0 \in X \mid \dot{V}(x) = 0\}$ consists of only the origin of the state space. To do so, we characterize the set for which the Lyapunov functional's time derivative is equal to zero:

$$\begin{aligned}\mathbf{S}_0 &= \{x_0 \in X \mid \eta = -KMR_p^{-1} q, y_z = -KMR_p^{-1} K_p^{-1} q\} \\ &= \{x_0 \in X \mid \eta = -KMR_p^{-1} q, y_z = K_p^{-1} \eta\}.\end{aligned}\quad (\text{III.132})$$

Then, substitute the former relations in the closed-loop dynamic (III.111)-(III.112) to obtain

$$\begin{cases} \dot{z} = \mathcal{J}z \\ \dot{\eta} = 0 \\ \dot{q} = 0 \end{cases}\quad (\text{III.133})$$

with domain

$$\mathbf{D}(\mathcal{J}) = \{x \in X \mid z \in H^1([a, b], \mathbb{R}^n), \mathcal{B}(\mathcal{H}z) = 0\}, \quad (\text{III.134})$$

$\mathcal{C}_2 z(\mathcal{H}z)(\xi, t) = \tilde{y}(t)$, and the other part of the output

$$\mathcal{C}_1 z(\mathcal{H}z)(\xi, t) = y_z(t) = -KMR_p^{-1} K_p^{-1} q(t). \quad (\text{III.135})$$

System (III.133) implies that η and q must be constant along time, i.e. $\eta(t) = \eta^*$ $q(t) = q^*$. Hence, $x_0 \in E$ should verify

$$\begin{aligned}\dot{z}(t) &= \mathcal{J}z(t) \\ \mathcal{B}(\mathcal{H}z) &= 0 \quad \mathcal{C}_1(\mathcal{H}z) = -KMR_p^{-1} K_p^{-1} q^*.\end{aligned}\quad (\text{III.136})$$

Using the approximate observability of the infinite dimensional part of system together with Lemma III.1.1, we know that the only solution of (III.136) is $z = 0$, which in turn implies $y_z = 0$ and consequently $q^* = \eta^* = 0$ using the relations in (III.132). Thus the largest invariant set $\mathbf{S} \subset \mathbf{S}_0$ corresponds to $\mathbf{S} = \{0\}$. Since by Theorem I.3.1 we know that the resolvent of A is compact, we apply Corollary III.1.1 to obtain that the trajectories set is pre-compact in the space X . Then we can conclude by the LaSalle's invariance principle that the solution converges asymptotically to the origin. \square

Example 4 (Rotating translating flexible beam with strong dissipation plus position control). Consider the operator (III.111)-(III.112) corresponding to the translating rotating flexible beam model of Section II.1.2.c in Chapter II in closed-loop with the control law (III.107). The control objective consists on stabilising the system in the zero state $z = 0$, $p = q = 0$, as depicted in Figure III.16. The considered physical parameters are listed

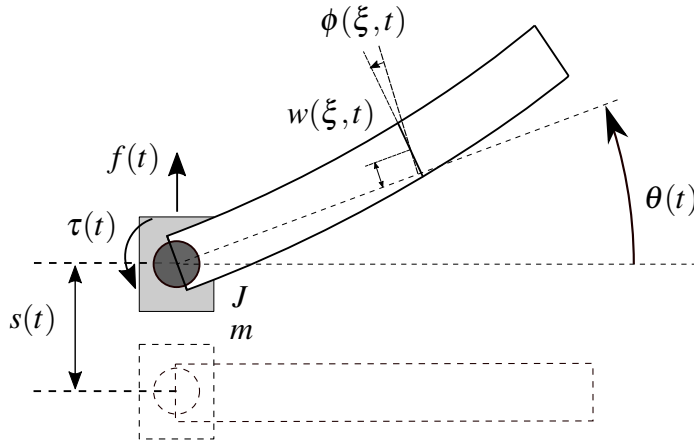


Figure III.16: Rotating translating beam with desired equilibrium point.

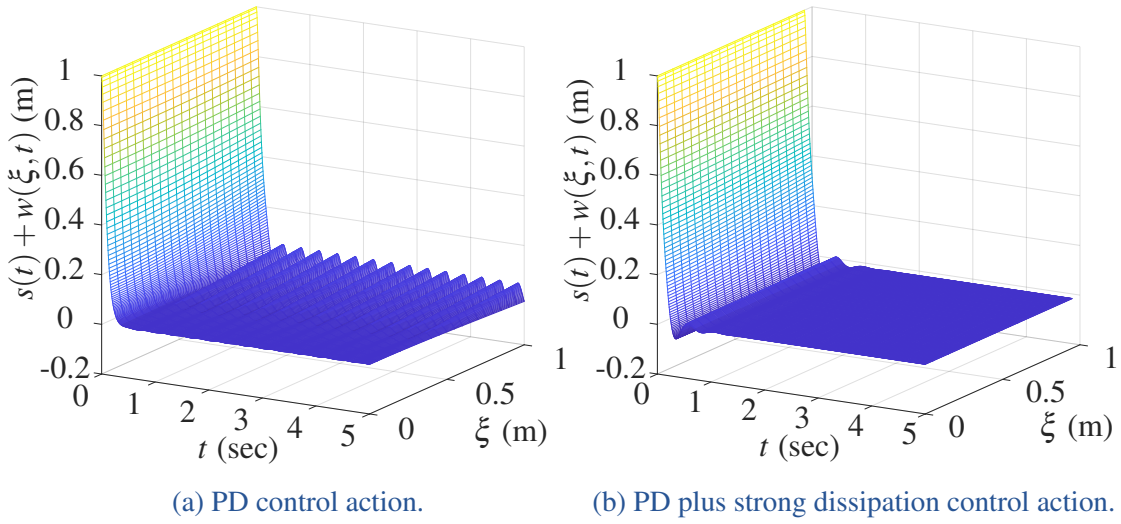


Figure III.17: Beam's Deformation plus displacement along time.

in Table III.1, together with the hub's mass $m = 1$ kg. We begin by computing

$$\begin{aligned}
 \|\mathcal{H}z(0, t)\|^2 &= \left\| \frac{1}{\rho(0)} z_1(0, t) \right\|^2 + \left\| \frac{1}{I_p(0)} z_2(0, t) \right\|^2 + \|K(0) z_3(0, t)\|^2 \\
 &\quad + \|EI(0) z_4(0, t)\|^2 \\
 &= \|u_z\|^2 + \|y_z\|^2
 \end{aligned} \tag{III.137}$$

that because of Theorem III.2.1, implies that the distributed parameter part of the system is approximately observable. For the same arguments as in Examples 1 it is possible to show that $z_{eq} = 0$ is the only equilibrium position of the distributed parameter part of the system. We select the control parameters in a way that the inequalities in (III.128) are respected: $r_{p,1} = r_{p,2} = 50$, $k_1 = k_2 = 500$, $k_{p,1} = k_{p,2} = 2$. Then, to perform the numerical simulation, we apply the discretization procedure introduced in Section II.2

of Chapter II to obtain

$$\begin{bmatrix} \dot{z}_d \\ \dot{\eta} \\ \dot{q} \end{bmatrix} = \left(\begin{bmatrix} J_{fe} & \bar{g}_{fe} & 0 \\ 0 & 0 & -I \\ -M^{-1}K_p\bar{g}_{fe} & I & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} H_{fe} & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} z_d \\ \eta \\ q \end{bmatrix} \quad (\text{III.138})$$

where $z_d \in \mathbb{R}^{4n_d}$ is the discretized distributed parameter state where $n_d = 20$ is the number of discretizing elements. The matrix J_{fe} is defined in (II.160), \bar{g}_{fe} corresponds to the first and second column of g_{fe} defined in (II.161) and H_{fe} is defined in (II.162). The initial conditions are set to be $z_0 = 0$, $p = 0$, $q_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Figure III.17 compares the displacement plus deformation behaviour along time in case the system is controlled with a PD or with a PD plus strong dissipation control action, respectively. We can appreciate that in case of a strong dissipation control law, the vibration are suppressed much faster than with a PD control law. Figure III.18 shows the closed-loop energy evolution in case PD control and PD plus strong dissipation control. We can observe that the closed-loop energy in case of application of PD plus strong dissipation control is not decreasing along the system's trajectories. This is the reason why it can not be used as Lyapunov functional to show the asymptotic stability of the closed-loop system. We remark that in this example we stabilize the system to the origin, but if we want to stabilize it around a different configuration s^*, θ^* , it suffices to define a translated position variable $q = \begin{bmatrix} s-s^* \\ \theta-\theta^* \end{bmatrix}$.

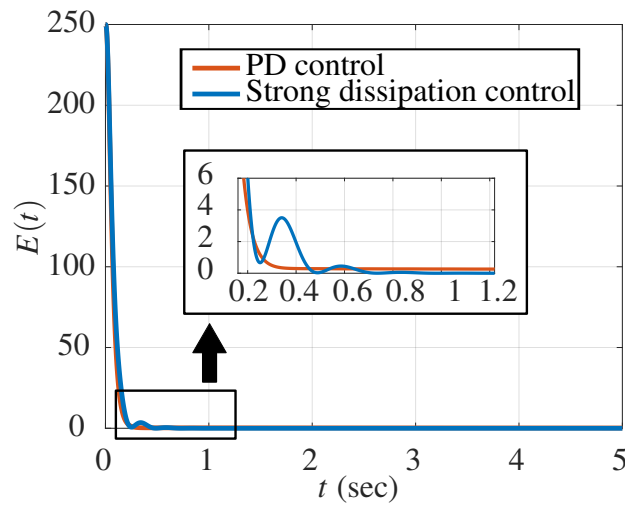


Figure III.18: closed-loop energy evolution in time.

III.3 SUMMARY

In this section we have first introduced the tools, based on Lyapunov Theory, for analysing the stability of semilinear equations. Then, we have defined the class of m-pH systems for which control laws are designed in the reminder of the chapter. This class of systems is general enough to include three out of the four model examples shown in Chapter II. In the main part of the Chapter we design four different control law for the stabilization of the defined m-pH system. In Table III.3 we briefly resume the characteristics of the control laws studied in this chapter.

Control law	Well-posedness	Asymptotic stability	Exponential stability
Proportional Derivative (PD)	✓	✓	
PD + nonlinear passive	✓	✓	
Strong Dissipation	✓	✓	✓
Strong Dissipation + position control	✓	✓	

Table III.3: Summary of the results obtained for the proposed control laws.

Chapter IV

Stability analysis of a rotating robot arm impacts during contact scenarios

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IV.1 PRELIMINARIES ON STABILITY OF SWITCHED SYSTEMS

A lot of critical tasks in robotics involve the contact between the manipulator and an external object or the environment, as space robots for debris removal [8], as well as minimally invasive surgery systems that make contact inside the patient body [9]. In this chapter, we propose a procedure that allows defining a well-posed closed-loop impact model for which we can conclude about its solution's asymptotic behaviour. With “impact” we mean the situation in which a robot (or in general a moving inertia) is first moving freely and then enters in contact with an external object. The impact scenario is studied for a rotating beam that is, on a first instance, considered rigid and then flexible. In the case of a rigid beam, the model can be written as a finite dimensional system, while in the case of a flexible beam its model should be of distributed parameter type. In both cases, the dynamical model of a colliding beam is expected to have instant changes in impact times. Therefore the model combines behaviours that are typical of continuous-time dynamical systems with behaviours that are typical of discrete-time dynamical systems. This definition perfectly fits into the class of Hybrid dynamical systems [57]. In this section, we provide the necessary background concerning dynamical systems determined by switching operator equations, in a way that both finite and infinite dimensional systems can be studied with the same approach. Another general framework for the study of discontinuous dynamical infinite dimensional systems has been proposed in [58], where Lyapunov and non-Lyapunov methods are employed to study the asymptotic behaviour of such systems. In the specific case of robots undergoing to impact with the external environment, a lot of efforts have been devoted to the rigid links' case [53, 54, 55], while very few have discussed the collision issue using infinite dimensional models [56].

Consider the general operator equation

$$\dot{x}(t) = f(x(t), m), \quad (\text{IV.1})$$

where $x \in X$ is the *continuous state* and belongs to an appropriate Hilbert space, and $m \in M = \{0, \dots, N\} \subset \mathbb{N}_0$ is the *discrete state*. The couple defined as the composition of the continuous and discrete state (x, m) is called *hybrid state*. If for each continuous state $x \in X$, only one discrete state $m \in M$ is possible, then the system is called a switching system. If in the same continuous state $x \in X$ different discrete states are possible, then the system is called an Hybrid system. Here, we consider switched systems, hence we partition the continuous state space in N disjoint regions

$$\Omega_1 \dots \Omega_N \subset X \quad (\text{IV.2})$$

where $\Omega_i \cup \dots \cup \Omega_N = X$ and $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$.

The i -th (in order of activation) discrete state m_i depends on the continuous state x , i.e. $m_i = \gamma(x)$ where $\gamma: X \mapsto M$ is the *discrete transition function*. Consider a family of linear operators $\mathbb{A} = \{A_m, m \in M\}$ defined on a common domain $\mathbf{D}(A_\kappa) = \mathbf{D}(A_\gamma)$ for

$\kappa, \gamma \in M$ and a family of functions $\mathbb{F} = \{f_m, m \in M\}$. The considered switched operator system is given by

$$\dot{x}(t) = A_{\gamma(x)}x(t) + f_{\gamma(x)}(x(t)). \quad (\text{IV.3})$$

We remark that the index of $A_{\gamma(x)}$ corresponds to the discrete transition function's value in the continuous state x . In the remainder of the chapter we will drop the notation about the index's dependency on the state x . The continuous state evolution of (IV.3) can be described as: starting at (x_0, m_0) at time t_0 , the continuous trajectory evolves according to $\dot{x} = A_{m_0}x + f_{m_0}(x)$. Let us assume that at time t_1 , x reaches a value x_1 that triggers a discrete change from m_0 to m_1 ; then the process evolves according to $\dot{x} = A_{m_1}x + f_{m_1}(x)$. Here, we consider hybrid systems with the continuous state that does not change during switching and therefore the hybrid state (x, m_i) becomes (x, m_j) . The changes of discrete state happen at the so called *switching sets*

$$D_{i,j} = \{x \in X \mid m_j = \gamma(x, m_i)\}. \quad (\text{IV.4})$$

We define a *switching sequence* anchored to a certain initial state

$$\{S_n(x_0)\} = (m_0, t_0), (m_1, t_1), \dots, (m_n, t_n), \dots \quad (\text{IV.5})$$

The switching sequence along (IV.3) describes completely the trajectory of the system according to the following rule: (m_i, t_i) means that the system evolves according to $\dot{x}(t) = A_{m_i}x + f_{m_i}(x)$ for $t_i \leq t \leq t_{i+1}$. We can take projections of the switching sequence, and they are defined as

$$\begin{aligned} \Pi_1(S_n(x_0)) &= m_0, m_1, \dots, m_n, \dots \\ \Pi_2(S_n(x_0)) &= t_0, t_1, \dots, t_n, \dots \end{aligned} \quad (\text{IV.6})$$

We denote by $S(x_0)|_m$ the endpoints of times for which the system m is active. Finally, let $\mathcal{E}(T) = t_0, t_2, t_4, \dots$ denote the even sequence and $\mathcal{O}(T) = t_1, t_3, t_5, \dots$ the odd sequence of $T : t_0, t_1, t_2, t_3, \dots$. We propose the equilibrium position definition given in [58, page 1278].

Definition IV.1.1. An hybrid state (x_{eq}, m_{eq}) is said to be an *hybrid equilibrium* of (IV.1) if the trajectory generated by the initial condition (x_{eq}, m_{eq}) is such that $x(t) = x_{eq}$ for all $t \geq 0$. ■

The hybrid equilibrium points may be obtained by finding the states satisfying

$$A_m x + f_m(x) = 0 \quad \forall m \in M. \quad (\text{IV.7})$$

All the continuous states satisfying (IV.7) are not hybrid equilibria because there may be not possible hybrid states. For example one solution of (IV.7) (x_{eq}, m_i) may not be possible, in the sense that x_{eq} is not contained in the region of the state space that is associated with the discrete state m_i . Without loss of generality the origin is assumed to be a continuous equilibrium of (IV.3) for which stability is investigated. In fact we can always translate the state with respect to one equilibrium position of which we want to investigate the stability. Therefore, since in general different semilinear equations have

different equilibrium positions, only one semilinear equation will have one equilibrium at the origin while all the other semilinear equations will have the equilibria different from the origin.

Now, we can define a single candidate Lyapunov functional V_m for a certain system's dynamic $A_mx + f_m(x)$.

Definition IV.1.2. Suppose that, for every $x_0 \in X$, the switched semilinear differential equation (IV.3) possesses a local mild solution $x(t)$. A continuous functional $V_m : X \mapsto [0, \infty)$ is a *Lyapunov functional* for $A_mx + f_m(x)$ and the trajectory $x(t)$ if:

- $V_m(x(t))$ is Dini differentiable;
- $\dot{V}_{m,+}(x_0) := \limsup_{t \rightarrow 0} \frac{V_m(x(t, x_0)) - V_m(x_0)}{t} \leq 0 \quad \forall x_0 \in \Omega_m$. ■

We introduce in the next Lemma an easy way to compute the Dini derivative. Note that for a functional V_m to be considered as a Lyapunov functional for A_mx , it is necessary that $\dot{V}_{m,+}(x_0)$ is non positive only in the region Ω_m , but $\dot{V}_{m,+}(x_0)$ can be computed in the whole state space X .

Lemma IV.1.1. *If the functional V_m is Fréchet differentiable, then for $x_0 \in \Omega_\kappa \cap D(A_\kappa)$, $\kappa \in M$, $V_m(x(t))$ is differentiable for $t = 0$ and*

$$\dot{V}_{m,t}(x_0) = \left. \frac{dV_m(x(t, x_0))}{dt} \right|_{t=0} = dV_m(x_0)(A_\kappa x_0 + f_\kappa(x_0)) \quad (\text{IV.8})$$

where dV_m denotes the Fréchet derivative of V_m .

Proof. Divide the state space in the different subspaces Ω_κ . Then, the time derivative equality in each Ω_κ is shown to hold as in Lemma 11.2.5 of [65]. □

In the previous lemma, we gave the formula for computing the time derivative of the Lyapunov functional V_m in any subspace Ω_κ . This means that it is possible to compute the Lyapunov functional time derivative in its own region ($m = \kappa$) or in a different region ($m \neq \kappa$), only substituting the vector field on which we want to compute a Lyapunov functional V_m time derivative in a certain region. At this point we are in position to state the bounded trajectory theorem for switched linear operator systems, that is an adaptation of Theorem 2.3 in [64] for a general class of switching linear operator systems.

Theorem IV.1.1. *Let assume that there exists a unique local mild solution of (IV.3). If there exist Lyapunov functionals V_m for every $A_mx + f_m(x)$ with the property that $\forall x \in \Omega_m$ $\alpha(\|x\|) \leq V_m(x) \leq \beta(\|x\|)$, where $\alpha, \beta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ are such that $\alpha(0) = \beta(0) = 0$ and*

$$\lim_{\|x\| \rightarrow \infty} \alpha(\|x\|) = \lim_{\|x\| \rightarrow \infty} \beta(\|x\|) = \infty, \quad (\text{IV.9})$$

that are non increasing in $\mathcal{E}(\mathcal{S}(x_0)|_m) \forall m \in M$, then (IV.3) is Lyapunov stable in the sense of Definition III.1.2 and has a globally bounded mild solution for every initial condition $x_0 \in X$.

Proof (for $M = \{0, 1\}$). From the assumption of local mild solution's existence, we know that for any x_0 there exists a t_{max} such that (IV.3) possess a mild solution on $[0, t_{max})$ and $t_{max} < \infty$ only if $\|x(t)\|$ diverges when $t \rightarrow t_{max}$.

Let $\varepsilon > 0$ be arbitrary. Let $m_m(\alpha) = \min\{V_m(x_0) \mid \|x_0\| = \alpha\}$. Pick $r_m < \varepsilon$ such that $\forall x_0 \in B(r_m) = \{x_0 \in X \mid \|x_0\| \leq r_m\}$, $V_m(x_0) < m_m(\varepsilon)$ for all $m \in M$. Let $r = \min(r_m)$. With this choice, if we select $\|x_0\| \leq r$, the evolution of the trajectory with either vector field $A_mx + f_m(x)$ is such that $\|x(t)\| \leq \varepsilon \quad \forall t \in [0, t_0]$.

Now pick $\rho_m < r$ such that $\forall x_0 \in B(\rho_m) = \{x_0 \in X \mid \|x_0\| \leq \rho_m\}$, $V_m(x_0) < m_m(r)$ for all $m \in M$. Set $\rho = \min(\rho_m)$. Thus, if we choose $\|x_0\|_X \leq \rho$ the trajectory's evolution with either vector field $A_mx + f_m(x)$ is such that $\|x(t)\| \leq r \quad \forall t \in [0, t_0]$.

Therefore, select $\|x_0\|_X \leq \rho$ such that at the first transition at time t_1 we have that $\|x(t_1)\| \leq r$ and at the second transition at time t_2 we have $\|x(t_2)\| \leq \varepsilon$. Then, because of the “switching-in” condition

$$V_m(x(t_2)) \leq V_m(x(t_0)) \leq r. \quad (\text{IV.10})$$

This procedure can be repeated to the infinite to conclude that for every ε there exists a $\rho > 0$ such that whenever $\|x_0\| < \rho$, the solution satisfies

$$\|x(t)\| \leq \varepsilon \quad \forall t \in \mathbb{R}^+. \quad (\text{IV.11})$$

Hence, the system (IV.3) is Lyapunov stable. Since the Lyapunov functions are bounded from above and from below by functions α and β that fulfil $\alpha(0) = \beta(0) = 0$ and the limit (IV.9), since ε is arbitrary we can take it bigger to augment arbitrarily r and consequently ρ . Therefore, we can bring ε to the infinite such to augment arbitrary ρ and to allow any initial condition $x_0 \in X$. Since the trajectory remains bounded the solution does not diverge and then $t_{max} = \infty$, i.e. (IV.3) has global mild solution for all $x_0 \in X$.

The case in which $M = \{0, 1, \dots, N\}$ can be shown creating $N + 1$ concentric balls and proceeding the same way as the case in which $M = \{0, 1\}$ (note that in the case $M = \{0, 1\}$ the first ball has radius ε , the second r , and the third ρ). \square

The non-increasing condition of V_m in $\mathcal{E}(S(x_0)|_m)$ concerns the value of each functional V_m each time is “switched in”. It means that the value of V_m at switching points should be smaller than that of the previous time it has become active or “switched in”.

IV.2 RIGID ROBOT ARM IMPACT

In this section we consider a rotating rigid beam that enters in contact with a compliant surface as shown in Figure IV.1. The control objective is to enter and maintain the contact with an external surface while, assuming the knowledge of the surface's stiffness, applying the desired contact force. Many studies have been devoted to investigating the stability of this problem with the use of a range of possible control laws (see for example [94, 95, 96] etc.). Therefore, the goal of this section is not to extend these

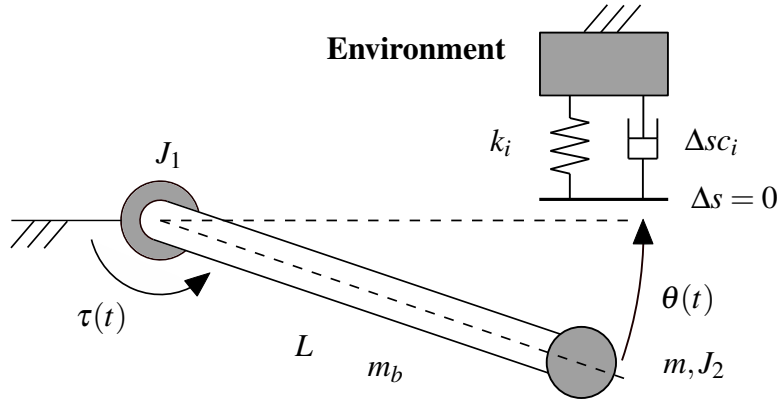


Figure IV.1: Rotating rigid beam's impact with the external environment.

studies, but rather to point out the problems in defining a well-posed closed-loop system and in studying its asymptotic behaviour. Furthermore, the study of the rigid beam case allows smoothing the transition to the flexible beam case that will be discussed in Section IV.3.

If friction and gravity are neglected the equivalent mechanical system is that of a rotating inertia J moving under the action of a control torque input $\tau(t)$. To be consistent with the next section, where we analyse a rotating flexible beam, we consider a link composed by a rotating hub of rotating inertia J_1 , a rigid beam of mass m_b and length L and a load of mass m and rotating inertia J_2 . Therefore the equivalent inertia can be computed as $J = J_1 + \frac{1}{3}m_b L^2 + mL^2 + J_2$. The continuous state is represented by $w = (\dot{\theta}, \theta) \in \mathbb{R}^2$, while the discrete state $m \in M = \{0, 1\}$ corresponds to the non-contact ($m = 0$) and contact ($m = 1$) scenario. The discrete transitions are determined by the function $\gamma: \mathbb{R}^2 \mapsto \{0, 1\}$, shown in figure IV.2 and defined as

$$\gamma = \begin{cases} 1 & \text{if } (\theta, \dot{\theta}) \in \{\theta > 0\} \cup \{\theta = 0, \dot{\theta} \geq 0\} \\ 0 & \text{Otherwise.} \end{cases} \quad (\text{IV.12})$$

For each $w \in \mathbb{R}^2$ only one $m \in M$ is allowed, therefore the considered system is a switching system and we can partition the state space in 2 disjoint regions

$$\begin{aligned} \Omega_1 &= \{(\theta, \dot{\theta}) \in \mathbb{R}^2 \mid \theta > 0\} \cup \{(\theta, \dot{\theta}) \in \mathbb{R}^2 \mid \theta = 0, \dot{\theta} \geq 0\} \\ \Omega_0 &= \{(\theta, \dot{\theta}) \in \mathbb{R}^2 \mid \theta < 0\} \cup \{(\theta, \dot{\theta}) \in \mathbb{R}^2 \mid \theta = 0, \dot{\theta} < 0\}. \end{aligned} \quad (\text{IV.13})$$

Note that the point $(0, 0)$ belongs to the contact region.

A common task between the rigid and the flexible beam's impact modelling procedure is to determine a suitable model of the impacting surface. A first approach is to consider that the surface's deformation is "sufficiently small" during the contact phases so that it remains rigid in a global observation scale [97]. A second approach is to consider a compliant surface, in case the surface's deformations are "sufficiently big" in a global observation scale [94]. In the remainder of the chapter, we consider a compliant impacting surface. According to [98] the compliant surface can be considered as a mass-less system composed of a spring and a damper with coefficients depending on

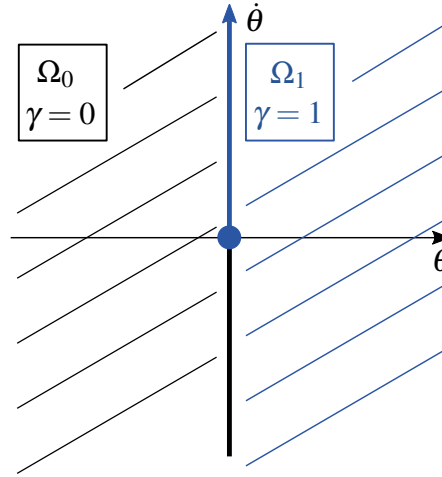


Figure IV.2: Contact and non-contact subspace representations.

the surface's deformation. In this work, the authors confirm with physical experiments that the energy loss of a point mass undergoing to impact is proportional to the one dissipated by a damper with a coefficient depending on the deformation of the impacting surface. Therefore, the resulting impact dissipation force is time continuous during impacts, a behaviour that is consistent with the continuous nature of physical quantities in classical continuum mechanics. In line with the previous work on impact scenario (see for example [95] and [94]), we consider a constant spring coefficient and a damping coefficient depending linearly on the deformation. Assuming that Δs corresponds to the surface's deformation, the impact force is given by

$$\lambda = -k_i \Delta s - c_i \Delta s \dot{\Delta s}. \quad (\text{IV.14})$$

The equation of motion in contact and non-contact scenario writes

$$J\ddot{\theta} = \gamma(\theta, \dot{\theta})L\lambda + \tau. \quad (\text{IV.15})$$

Since the surface's deformation is proportional to the rotation angle $\Delta s = L\theta$, we substitute the impact force (IV.14) in the former equation to obtain

$$J\ddot{\theta} = -\gamma(\theta, \dot{\theta})(\tilde{k}_i \theta + \tilde{c}_i \theta \dot{\theta}) + \tau \quad (\text{IV.16})$$

where $\tilde{k}_i = L^2 k_i$ and $\tilde{c}_i = L^3 c_i$. We select a proportional derivative control input

$$\tau = -k(\theta - \theta^o) - c\dot{\theta} \quad (\text{IV.17})$$

where $k, c > 0$ are the control gains and the value θ^o is selected such that to obtain the desired contact force $f_d = k_i \Delta s_d = k_i L \theta_d$ (proportional to the desired surface deformation Δs_d)

$$\theta^o = \frac{k + \tilde{k}_i}{Lk\tilde{k}_i} f_d = \frac{k + \tilde{k}_i}{k} \theta_d. \quad (\text{IV.18})$$

We define the continuous state $w = [\dot{\theta} \ \theta]$, and we apply the control law (IV.17) to the plant to obtain the semi-linear equation

$$\begin{aligned} \dot{w} &= \begin{bmatrix} -\frac{c}{J}\dot{\theta} - \frac{k}{J}\theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \frac{k}{J}\theta^o - \frac{\gamma(w)}{J}(\tilde{k}_i\theta + \tilde{c}_i\theta\dot{\theta}) \\ 0 \end{bmatrix} \\ &= Aw + f(w) \end{aligned} \quad (\text{IV.19})$$

for which we can define the following switching sets

$$\begin{aligned} D_{0,1} &= \{(\theta, \dot{\theta}) \in \mathbb{R}^2 \mid \theta = 0, \dot{\theta} \geq 0\} \\ D_{1,0} &= \{(\theta, \dot{\theta}) \in \mathbb{R}^2 \mid \theta = 0, \dot{\theta} < 0\}. \end{aligned} \quad (\text{IV.20})$$

IV.2.1 Existence and uniqueness of solutions

In this section we show that the defined semi-linear equation (IV.19) possesses a unique solution. To do so, we rely on the results presented in Section I.1 of Chapter I. We use these results instead of classical theory on Hybrid systems (see for example [57]) such to unify this analysis with the flexible beam's one, that will be presented in the next section.

Lemma IV.2.1. *The system defined by equations (IV.19) possesses for every initial condition $(\theta_0, \dot{\theta}_0) \in \mathbb{R}^2$ a unique continuous solution.*

Proof. Equipping the space \mathbb{R}^2 with the norm $\|w\| = \sqrt{J\dot{\theta}^2 + k\theta^2}$ one can see that the operator A generates a contraction C_0 -semigroup in this space, and therefore a C_0 -semigroup in \mathbb{R}^2 equipped with the standard norm. Hence, applying Theorem I.1.4 (page 16) we only have to show that the function $f(w)$ is locally Lipschitz continuous. Therefore we compute

$$\begin{aligned} \|f(w_2) - f(w_1)\| &= \|\gamma(w_2)(-\tilde{k}_i\theta_2 - \tilde{c}_i\theta_2\dot{\theta}_2) - \gamma(w_1)(-\tilde{k}_i\theta_1 - \tilde{c}_i\theta_1\dot{\theta}_1)\| \\ &\leq \tilde{k}_i\|\gamma(w_1)\theta_1 - \gamma(w_2)\theta_2\| + \tilde{c}_i\|\gamma(w_1)\theta_1\dot{\theta}_1 - \gamma(w_2)\theta_2\dot{\theta}_2\|. \end{aligned} \quad (\text{IV.21})$$

The first term in the former inequality corresponds to the function

$$\|\gamma(w_1)\theta_1 - \gamma(w_2)\theta_2\| = \begin{cases} 0 & \text{if } \gamma(w_1) = 0 \text{ and } \gamma(w_2) = 0 \\ \|\theta_1\| & \text{if } \gamma(w_1) = 1 \text{ and } \gamma(w_2) = 0 \\ \|\theta_2\| & \text{if } \gamma(w_1) = 0 \text{ and } \gamma(w_2) = 1 \\ \|\theta_1 - \theta_2\| & \text{if } \gamma(w_1) = 1 \text{ and } \gamma(w_2) = 1 \end{cases} \quad (\text{IV.22})$$

from which we can deduce that

$$\|\gamma(w_1)\theta_1 - \gamma(w_2)\theta_2\| \leq \|\theta_1 - \theta_2\| \leq \|w_1 - w_2\| = \|w_2 - w_1\| \quad (\text{IV.23})$$

since:

- For $\gamma(w_1) = 0$ and $\gamma(w_2) = 0$ it is true that $0 \leq \|\theta_1 - \theta_2\|$.
- For $\gamma(w_1) = 1$ and $\gamma(w_2) = 0$ it is true that $\|\theta_1\| \leq \|\theta_1 - \theta_2\|$ because we are adding a number of the same sign of θ_1 .

- For $\gamma(w_1) = 0$ and $\gamma(w_2) = 1$ it is true that $\|-\theta_2\| \leq \|\theta_1 - \theta_2\|$ because we are adding a number of the same sign of $-\theta_2$.

For the second term in inequality (IV.21) we compute

$$\begin{aligned} \|\gamma(w_1)\theta_1\dot{\theta}_1 - \gamma(w_2)\theta_2\dot{\theta}_2\| &= \|\gamma(w_1)\theta_1\dot{\theta}_1 + \gamma(w_1)\theta_1\dot{\theta}_2 - \gamma(w_1)\theta_1\dot{\theta}_2 - \gamma(w_2)\theta_2\dot{\theta}_2\| \\ &\leq \|\gamma(w_1)\theta_1(\dot{\theta}_1 - \dot{\theta}_2)\| + \|(\gamma(w_1)\theta_1 - \gamma(w_2)\theta_2)\dot{\theta}_2\| \\ &\leq \|\theta_1\| \cdot \|(\dot{\theta}_1 - \dot{\theta}_2)\| + \|\dot{\theta}_2\| \cdot \|(\gamma(w_1)\theta_1 - \gamma(w_2)\theta_2)\| \end{aligned} \quad (\text{IV.24})$$

that considering $\|w_1\|, \|w_2\| \leq r$ and according to (IV.23) becomes

$$\|\gamma(w_1)\theta_1\dot{\theta}_1 - \gamma(w_2)\theta_2\dot{\theta}_2\| \leq 2r\|w_2 - w_1\|. \quad (\text{IV.25})$$

Therefore we can conclude that $\|f(w_2) - f(w_1)\| \leq L(r)\|w_2 - w_1\|$ with $L(r) = \tilde{k}_i + 2\tilde{c}_i r$, that concludes the proof. \square

IV.2.2 Stability analysis

We now rewrite the closed-loop equation (IV.19) such to highlight the equilibrium position of the contact and non-contact equations

$$\begin{cases} J\ddot{\theta} = -k(\theta - \theta^o) - c\dot{\theta} & \text{if } \gamma = 0 \\ J\ddot{\theta} = -(k + \tilde{k}_i)(\theta - \theta_d) - (c + \tilde{c}_i\theta)\dot{\theta} & \text{if } \gamma = 1. \end{cases} \quad (\text{IV.26})$$

We perform the change of variables $x = [p \ q]^T$

$$q = \theta - \theta_d \quad p = J\dot{\theta} \quad (\text{IV.27})$$

such that the discrete transition function γ in the new variables transforms into

$$\gamma = \begin{cases} 1 & \text{if } x \in \{q > -\theta_d\} \cup \{q = -\theta_d, p \geq 0\} \\ 0 & \text{Otherwise} \end{cases} \quad (\text{IV.28})$$

with switching sets,

$$\begin{aligned} D_{0,1} &= \{x \in \mathbb{R}^2 \mid q = -\theta_d, p \geq 0\} \\ D_{1,0} &= \{x \in \mathbb{R}^2 \mid q = -\theta_d, p < 0\} \end{aligned} \quad (\text{IV.29})$$

and regions,

$$\begin{aligned} \Omega_1 &= \{x \in \mathbb{R}^2 \mid q > -\theta_d\} \cup \{x \in \mathbb{R}^2 \mid q = -\theta_d, p \geq 0\} \\ \Omega_0 &= \{x \in \mathbb{R}^2 \mid q < -\theta_d\} \cup \{x \in \mathbb{R}^2 \mid q = -\theta_d, p < 0\}. \end{aligned} \quad (\text{IV.30})$$

The closed-loop equation (IV.19) in the new variables (IV.27) writes as a switching operator equation

$$\dot{x}(t) = A_\gamma x(t) + f_\gamma(x(t)) \quad (\text{IV.31})$$

with

$$\begin{aligned} A_0 x &= \begin{bmatrix} -k(q - \theta^0 + \theta_d) - \frac{c}{J}p \\ \frac{p}{J} \end{bmatrix} & f_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ A_1 x &= \begin{bmatrix} -(k + \tilde{k}_i)q - \frac{c + \tilde{c}_i\theta_d}{J}p \\ \frac{p}{J} \end{bmatrix} & f_1 &= \begin{bmatrix} -\frac{\tilde{c}_i}{J}qp \\ 0 \end{bmatrix}. \end{aligned} \quad (\text{IV.32})$$

We notice that the equilibrium state of the first operator is $x_{eq,0} = [\theta^o - \theta_d]$ while the equilibrium state of the second is $x_{eq,1} = [0]$. According to Definition (IV.1.1), since the system cannot be initialized in $x_{eq,0}$ with active operator A_0x , the only possible hybrid equilibrium is $(x_{eq}, m_{eq}) = (x_{eq,1}, 1)$. In the next Theorem we show that x_{eq} is a Lyapunov stable equilibrium position for (IV.31).

Theorem IV.2.1. *The system described by the switching semi-linear operator equation (IV.31) with operators defined in (IV.32) and discrete transition function (IV.28) is Lyapunov stable and has a globally bounded solution for every initial condition $x_0 \in \mathbb{R}^2$.*

Proof. We define the following non-negative Lyapunov functional candidates for each of the system's vector fields

$$\begin{aligned} V_0 &= \frac{1}{2J}p^2 + \frac{1}{2}k(q - \theta^o + \theta_d)^2 \\ V_1 &= \frac{1}{2J}p^2 + \frac{1}{2}(k + \tilde{k}_i)q^2. \end{aligned} \quad (IV.33)$$

We compute the time derivative along the respective vector field to obtain

$$\begin{aligned} \dot{V}_{0,+}(x) &= dV_0(A_0x + f_0) = -\frac{c}{J^2}p^2 \\ \dot{V}_{1,+}(x) &= dV_1(A_1x + f_1) = -\frac{c + \tilde{c}_i(q + \theta_d)}{J^2}p^2 \end{aligned} \quad (IV.34)$$

Since the time derivatives are non-positive in the respective region of the state space, the functional defined in (IV.33) are Lyapunov functionals of the respective vector field according to Definition (IV.1.2). Let t_0 be the initial time. Assume that the initial condition is taken such that $x_0 \in \Omega_0$. Then, the time series $\Pi_2(\{S_n(\bar{x}_0)\})$ has even components $\mathcal{E}(\Pi_2(\{S_n(\bar{x}_0)\}))$ that correspond to the non-contact “switching-in” conditions, while has odd components $\mathcal{O}(\Pi_2(\{S_n(\bar{x}_0)\}))$ that correspond to the contact ones. Consider $[t_{i-1}, t_i)$ the non-contact time intervals, while $[t_i, t_{i+1})$ a contact's one with $i \in \{1, 3, 5, \dots\}$. Because of the first equation in (IV.34) we have that

$$V_0(x(t_{i-1})) \geq V_0(x(t_i)) \quad V_0(x(t_{i+1})) \geq V_0(x(t_{i+2})) \quad (IV.35)$$

that implies

$$\begin{aligned} \frac{1}{2J}p(t_{i-1})^2 \geq \frac{1}{2J}p(t_i)^2 &\Rightarrow |p(t_{i-1})| \geq |p(t_i)|, \\ \frac{1}{2J}p(t_{i+1})^2 \geq \frac{1}{2J}p(t_{i+2})^2 &\Rightarrow |p(t_{i+1})| \geq |p(t_{i+2})|. \end{aligned} \quad (IV.36)$$

We use the second equation in (IV.34) to obtain

$$V_1(x(t_i)) \geq V_1(x(t_{i+1})) \quad (IV.37)$$

that, similarly to before implies

$$|p(t_i)| \geq |p(t_{i+1})|. \quad (IV.38)$$

Combining (IV.36) with (IV.38) we obtain

$$|p(t_{i-1})| \geq |p(t_{i+1})| \quad |p(t_i)| \geq |p(t_{i+2})| \quad (IV.39)$$

Table IV.1: Simulation parameters; rigid beam impact

Name	Variable	Value
Beam's Length	L	1 m
Beam's mass	m_b	16 kg
Hub's inertia	J_1	$1\text{ kg}\cdot\text{m}^2$
Load's mass	m	1 kg
Load's inertia	J_2	$1\text{ kg}\cdot\text{m}^2$

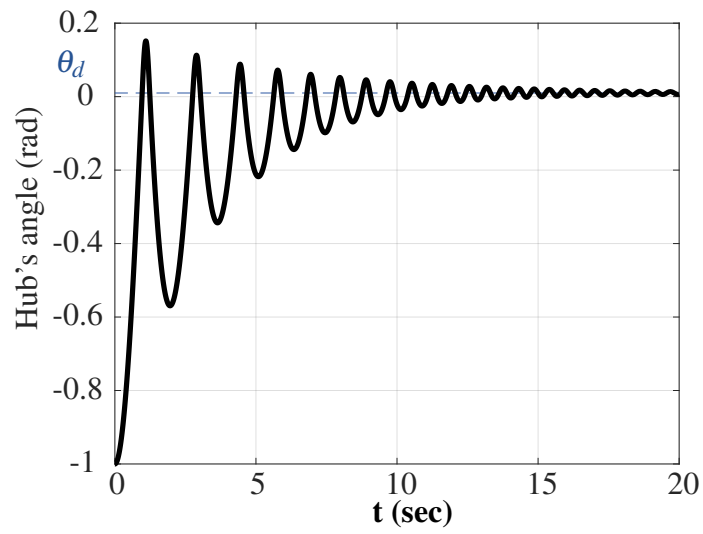
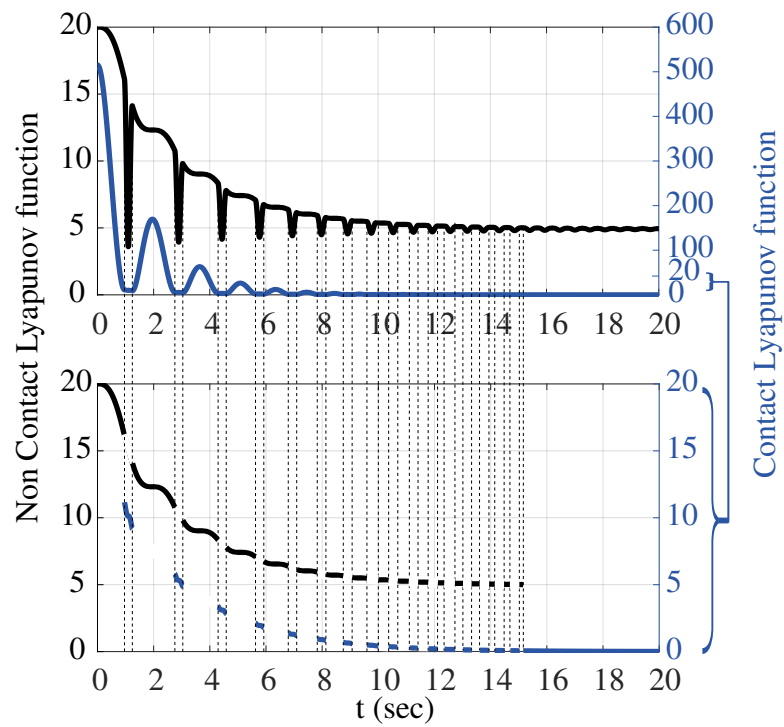
that respectively implies

$$V_0(x(t_{i-1})) \geq V_0(x(t_{i+1})) \quad V_1(x(t_i)) \geq V_1(x(t_{i+2})) \quad \text{for } i \in \{1, 3, 5, \dots\}. \quad (\text{IV.40})$$

The last inequalities show that the Lyapunov functionals are non-increasing in their “switching-in” times. The case in which $x_0 \in \Omega_1$ can be developed in the same manner as before, but with $[t_{i-1}, t_i)$ and $[t_{i+1}, t_{i+2})$ corresponding to the contact time interval, while $[t_i, t_{i+1})$ to the non-contact time interval for $i \in \{1, 3, 5, \dots\}$. According to Theorem IV.1.1 we can conclude that the system is Lyapunov stable and that the solutions are globally bounded for every $x_0 \in X$. \square

IV.2.3 Numerical Simulations

In this section we numerically simulate the closed-loop system with a PD control law of a rigid rotating link that enters in contact with the external environment, described by equation (IV.19). The system's parameters are listed in Table IV.1. The impact model parameters are set as $k_i = 1000$ and $c_i = 30$. We select the control parameters such that they are at least one order of magnitude smaller than the impact model parameters $k = 10$, $c = 3$. In this way, we can clearly appreciate the change of dynamics due to the impact with the external surface. Next, we set the control parameter $\theta^o = 1$ such that the desired equilibrium position is in the contact set Ω_1 . From (IV.2), we can compute the angle equilibrium $\theta_d = 0.0099$ rad. The simulations are performed with Matlab[®] using the “ode23t” time integration algorithm, while the initial conditions are set equal to $\theta(0) = -1\text{ rad}$ and $\dot{\theta}(0) = 0\text{ rad/sec}$. Figure IV.3 shows the time evolution of the angle $\theta(t)$. We remark that since the environment has a high stiffness coefficient ($k_i = 1000$), the angle's change of direction occurs in a small period of time. We notice that the angle converges to the desired equilibrium orientation θ_d , even though asymptotic stability has not been shown in this study. Figure IV.4 shows in the first plot the Lyapunov functionals (IV.33) along with solutions in the entire simulation time interval without considering the activation and deactivation time periods. In the second plot, the two Lyapunov functions are shown only in their respective activation time intervals. From this second plot, it is possible to appreciate both the non-increasing behaviour of the Lyapunov functionals in their respective activation time intervals, together with the respect of the “Switching-in” condition.

Figure IV.3: Angle evolution along time $\theta(t)$.Figure IV.4: Contact V_1 and non-contact V_0 Lyapunov functions behaviour along time for a rigid rotating beam.

IV.3 FLEXIBLE ROBOT ARM IMPACT

In this section we consider a system composed by a rotating flexible beam that enters in contact with the external environment as depicted in Figure IV.5. According to

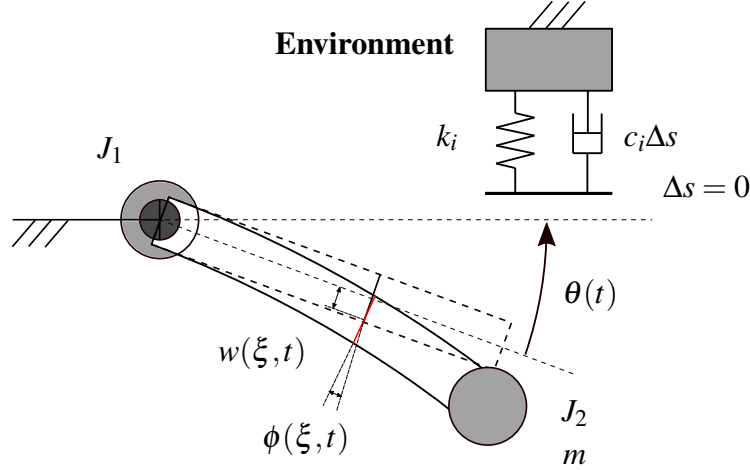


Figure IV.5: Rotating flexible beam's impact with the external environment.

the previous section the compliant surface is considered as a massless spring damper system, and the impact force considering $\Delta s = L\theta + w(L, t)$ writes

$$\lambda = -k_i(L\theta + w(L, t)) - c_i(L\theta + w(L, t))(L\dot{\theta} + \dot{w}(L, t)). \quad (\text{IV.41})$$

Note that the quantity $L\theta + w(L, t)$ corresponds to the distance of the end-effector from the external environment when is negative, and to the environment's deformation when it is positive. The elastic contribution of the impact force can be counted in the potential energy, while the dissipation in the work of the non-conservative forces. Therefore the kinetic H_k and potential energy H_p , using Timoshenko's assumptions, write

$$\begin{aligned} H_k &= \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}\int_0^L \left\{ \rho \left(\frac{\partial w}{\partial t} + \xi\dot{\theta} \right)^2 + I_\rho \left(\frac{\partial \phi}{\partial t} + \dot{\theta} \right)^2 \right\} d\xi \\ &\quad + \frac{1}{2}m(L\dot{\theta} + \dot{w}(L, t))^2 + \frac{1}{2}J_2(\dot{\theta} + \dot{\phi}(L, t))^2 \\ H_p &= \frac{1}{2}\int_0^L \left\{ K \left(\frac{\partial w}{\partial \xi} - \phi \right)^2 + EI \left(\frac{\partial \phi}{\partial \xi} \right)^2 \right\} d\xi \\ &\quad + \frac{1}{2}k_i\gamma(L\theta + w(L, t))(L\theta + w(L, t))^2 \end{aligned} \quad (\text{IV.42})$$

where γ is the discrete transition function and is defined as

$$\gamma = \begin{cases} 0 & \text{if } (L\theta + w(L, t) < 0) \text{ or } (L\theta + w(L, t) = 0) \text{ and } (\dot{\theta} + \dot{\phi}(L, t) < 0) \\ 1 & \text{if } (L\theta + w(L, t) > 0) \text{ or } (L\theta + w(L, t) = 0) \text{ and } (\dot{\theta} + \dot{\phi}(L, t) \geq 0) \end{cases} \quad (\text{IV.43})$$

and the work of non-conservative forces' Frechèt derivative equals

$$dW_{nc}h = \tau h_\theta - c_i\gamma(L\theta + w(L, t))(L\dot{\theta} + \dot{w}(L, t))h_{L\theta + w(L, t)}. \quad (\text{IV.44})$$

Using a similar modelling procedure as in Chapter II, we define the energy variables

$$\begin{aligned} z_1 &= \rho(\dot{w} + \xi \dot{\theta}) & z_2 &= I_\rho(\dot{\phi} + \dot{\theta}) & z_3 &= \frac{\partial w}{\partial \xi} - \phi & z_4 &= \frac{\partial \phi}{\partial \xi} \\ p_1 &= J_1 \dot{\theta} & p_2 &= m(L\dot{\theta} + \dot{w}(L, t)) & p_3 &= J_2(\dot{\theta} + \dot{\phi}(L, t)) \\ q_1 &= \theta & q_2 &= L\theta + w(L, t) & q_3 &= \theta + \phi(L, t) \end{aligned} \quad (\text{IV.45})$$

where $z = [z_1 \ z_2 \ z_3 \ z_4]^T$ is the state related to the flexible beam, $p = [p_1 \ p_2 \ p_3]^T$ and $q = [q_1 \ q_2 \ q_3]^T$ are the finite dimensional states related to the lumped inertias' dynamics. We define the input output operators of the infinite dimensional part of the system as

$$\begin{aligned} \mathcal{B}_1(\mathcal{H}z) &= W_{B,1} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I_\rho^{-1} z_2(0, t) \\ \rho^{-1} z_1(L, t) \\ I_\rho^{-1} z_2(L, t) \end{bmatrix} = u_z \\ \mathcal{B}_2(\mathcal{H}z) &= W_{B,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \rho^{-1} z_1(0, t) \\ \mathcal{C}_1(\mathcal{H}z) &= \tilde{W}_{C,1} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} -EI z_4(0, t) \\ K z_3(L, t) \\ EI z_4(L, t) \end{bmatrix} = y_z \\ \mathcal{C}_2(\mathcal{H}z) &= \tilde{W}_{C,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = -K z_3(0, t) \end{aligned} \quad (\text{IV.46})$$

where $W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ and $\tilde{W} = \begin{bmatrix} W_{C,1} \\ W_{C,2} \end{bmatrix}$ are such that:

1. The matrix $\begin{bmatrix} W_B \\ W_C \end{bmatrix}$ is invertible;
2. The matrices W_B and W_C are such that

$$\begin{bmatrix} W_B \Sigma W_B^T & W_B \Sigma W_C^T \\ W_C \Sigma W_B^T & W_C \Sigma W_C^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (\text{IV.47})$$

The total boundary input-output operators are defined as

$$\begin{aligned} \mathcal{B}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{B}_1(\mathcal{H}z) \\ \mathcal{B}_2(\mathcal{H}z) \end{bmatrix} = W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \\ \mathcal{C}(\mathcal{H}z) &= \begin{bmatrix} \mathcal{C}_1(\mathcal{H}z) \\ \mathcal{C}_2(\mathcal{H}z) \end{bmatrix} = W_C \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}. \end{aligned} \quad (\text{IV.48})$$

The equations in the new variables write

$$\begin{aligned} \dot{z} &= P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ \dot{p} &= -\mathcal{C}_1(\mathcal{H}z) + g_i \gamma \lambda + g \tau \\ \dot{q} &= M^{-1} p \end{aligned} \quad (\text{IV.49})$$

where the matrices P_1 , P_0 , \mathcal{H} are defined as in (II.42) and (II.45), while the remaining matrices write

$$M = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_2 \end{bmatrix} \quad g_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{IV.50})$$

We rewrite the discrete transition function γ as depending on the new state variables

$$\gamma(p, q) = \begin{cases} 0 & \text{if } (p, q) \in \{(p, q) \in \mathbb{R}^2 \mid q_2 < 0\} \cup \{(p, q) \in \mathbb{R}^2 \mid q_2 = 0, p_2 < 0\} \\ 1 & \text{if } (p, q) \in \{(p, q) \in \mathbb{R}^2 \mid q_2 > 0\} \cup \{(p, q) \in \mathbb{R}^2 \mid q_2 = 0, p_2 \geq 0\} \end{cases} \quad (\text{IV.51})$$

together with the impact function

$$\lambda = -k_i q_2 - \frac{c_i}{m} q_2 p_2. \quad (\text{IV.52})$$

With and abuse of notation, in the remainder we refer to the function $\gamma(p, q)$ as $\gamma(x)$. We use the original boundary conditions (IV.46) together with the state variables definition (IV.45) to derive that $u_z = M^{-1}p$, while the remaining boundary input is set equal to zero, i.e. $\mathcal{B}_2(\mathcal{H}z) = 0$. In this study we analyse the flexible rotating link in closed-loop with a PD controller

$$\tau(t) = -k(\theta(t) - \theta^o) - c\dot{\theta}(t) \quad (\text{IV.53})$$

with $k, c, \theta^o > 0$. We are interested in this simple control law because the aim of this section is not to obtain a certain desired performance of the closed-loop system, but rather set up a analysis procedure that can be employed in case of different control laws application.

Defining the new error state $\tilde{q}_1 = \theta - \theta^o$, we can write the closed-loop equations in the following semi-linear operator form

$$\begin{aligned} \dot{x} &= \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) \\ +EIz_4(0, t) - k\tilde{q}_1 - \frac{c}{J_1}p_1 \\ -Kz_3(L, t) - k_i q_2 \\ -EIz_4(L, t) - kq_3 \\ \frac{1}{J_1}p_1 \\ \frac{1}{m}p_2 \\ \frac{1}{J_2}p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ k_i \gamma(-x)q_2 - \frac{c_i}{m} \gamma(x)q_2 p_2 \\ kq_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= Ax + f(x) \end{aligned} \quad (\text{IV.54})$$

where $x = [z^T \ p^T \ \tilde{q}_1 \ q_2 \ q_3]^T \in X = L_2([0, L], \mathbb{R}^4) \times \mathbb{R}^6$ is the state of the system and the domain of the linear operator A is defined as

$$\mathbf{D}(A) = \{x \in X \mid z \in H^1([0, L], \mathbb{R}^4), \mathcal{B}_1(\mathcal{H}z) = M^{-1}p, \mathcal{B}_2(\mathcal{H}z) = 0\}. \quad (\text{IV.55})$$

The inner product in the state space is defined for $x_1, x_2 \in X$ as

$$\begin{aligned} \langle x_1, x_2 \rangle_X &= \langle z_1, \mathcal{H}z_2 \rangle_{L_2} + \frac{1}{J_1} p_{1,1} p_{1,2} + \frac{1}{m} p_{2,1} p_{2,2} + \frac{1}{J_2} p_{3,1} p_{3,2} \\ &\quad + k\tilde{q}_{1,1} \tilde{q}_{1,2} + k_i q_{2,1} q_{2,2} + kq_{3,1} q_{3,2} \end{aligned} \quad (\text{IV.56})$$

with associated norm $\|x\|_X = \sqrt{\langle x, x \rangle_X}$.

IV.3.1 Existence of solution and hybrid equilibrium

In this section we study the existence and uniqueness of a local mild solution for the defined semilinear equation (IV.54) modelling the flexible arm entering in contact with an external environment. Even if in the nonlinear function $f(x)$ is present the indicator function modelling the change of dynamic due to contact, we exploit the assumption that the damping coefficient depends on the deformation to be able to conclude that the function $f(x)$ is locally Lipschitz continuous. This property of the nonlinear function $f(x)$ permits the use of Theorem I.1.4 to conclude about existence of a local mild solution.

Theorem IV.3.1. *System (IV.54) with operator domain defined in $\mathbf{D}(A)$ (IV.55) possesses for every initial condition $x_0 \in X$ a unique mild solution on $[0, t_{max})$, where if $t_{max} < \infty$, then*

$$\lim_{t \rightarrow t_{max}} \|x(t)\| = \infty. \quad (\text{IV.57})$$

Furthermore, if $x_0 \in \mathbf{D}(A)$ the solution is classical.

Proof. By Theorem I.1.4 (page 16) we have to show that the operator A generates a contraction C_0 -semigroup and that the function $f(x)$ is locally Lipschitz continuous. By Theorem I.3.1, A generates a contraction C_0 -semigroup if it is dissipative on X . Therefore, using Lemma I.2.1 together with the inner product definition (IV.56) we compute

$$\begin{aligned} \langle Ax, x \rangle_X &= \langle \mathcal{J}z, \mathcal{H}z \rangle_{L_2} - \frac{1}{J_1} (+EIz_4(0, t) - k\tilde{q}_1 - \frac{c}{J_1}p_1)p_1 \\ &\quad + \frac{1}{m}(-Kz_3(L, t) - k_iq_2)p_2 + \frac{1}{J_2}(-EIz_4(L, t) - kq_3)p_3 \\ &\quad + \frac{k}{J_1}p_1\tilde{q}_1 + \frac{k_i}{m}p_2q_2 + \frac{k}{J_2}p_3q_3 \\ &= y_z^T u_z - y_z^T M^{-1}p - \frac{c}{J_1}p_1^2 \\ &= -\frac{c}{J_1}p_1^2 \leq 0 \end{aligned} \quad (\text{IV.58})$$

hence A generates a contraction C_0 -semigroup on X . We now show that the function $f(x)$ is locally Lipschitz continuous. First, we remark that the Lipschitz continuity of $f(x)$ is equivalent to the Lipschitz continuity of $k_i\gamma(-x)q_2 - \frac{c_i}{m}\gamma(x)q_2p_2$. Therefore we compute

$$\begin{aligned} \|f(x_2) - f(x_1)\| &= \|k_i\gamma(-x_2)q_{2,2} - \frac{c_i}{m}\gamma(x_2)q_{2,2}p_{2,2} - k_i\gamma(-x_1)q_{2,1} \\ &\quad + \frac{c_i}{m}\gamma(x_1)q_{2,1}p_{2,1}\| \\ &\leq k_i\|\gamma(-x_2)q_{2,2} - \gamma(-x_1)q_{2,1}\| \\ &\quad + \frac{c_i}{m}\|\gamma(x_1)q_{2,1}p_{2,1} - \gamma(x_2)q_{2,2}p_{2,2}\|. \end{aligned} \quad (\text{IV.59})$$

The first term in the former inequality corresponds to

$$\|\gamma(-x_2)q_{2,2} - \gamma(-x_1)q_{2,1}\| = \begin{cases} 0 & \text{if } \gamma(-x_2) = 0 \text{ and } \gamma(-x_1) = 0 \\ \|q_{2,2}\| & \text{if } \gamma(-x_2) = 1 \text{ and } \gamma(-x_1) = 0 \\ \|-q_{2,1}\| & \text{if } \gamma(-x_2) = 0 \text{ and } \gamma(-x_1) = 1 \\ \|q_{2,2} - q_{2,1}\| & \text{if } \gamma(-x_2) = 1 \text{ and } \gamma(-x_1) = 1 \end{cases} \quad (\text{IV.60})$$

and therefore we can conclude that

$$\|\gamma(-x_2)q_{2,2} - \gamma(-x_1)q_{2,1}\| \leq \|q_{2,2} - q_{2,1}\| \leq \|x_2 - x_1\| \quad (\text{IV.61})$$

using the same arguments after equation (IV.23) in Lemma IV.2.1. The second term in (IV.59) can be bounded by the following expression

$$\begin{aligned} \|\gamma(x_1)q_{2,1}p_{2,1} - \gamma(x_2)q_{2,2}p_{2,2}\| &= \|\gamma(x_1)q_{2,1}p_{2,1} + \gamma(x_1)q_{2,1}p_{2,2} \\ &\quad - \gamma(x_1)q_{2,1}p_{2,2} - \gamma(x_2)q_{2,2}p_{2,2}\| \\ &\leq \|\gamma(x_1)q_{2,1}(p_{2,1} - p_{2,2})\| \\ &\quad + \|p_{2,2}(\gamma(x_1)q_{2,1} - \gamma(x_2)q_{2,2})\| \\ &\leq \|q_{2,1}\| \cdot \|p_{2,1} - p_{2,2}\| + \\ &\quad \|p_{2,2}\| \cdot \|\gamma(x_1)q_{2,1} - \gamma(x_2)q_{2,2}\| \end{aligned} \quad (\text{IV.62})$$

that considering $\|x_1\|, \|x_2\| \leq r$ and according to (IV.61) becomes

$$\|\gamma(x_1)q_{2,1}p_{2,1} - \gamma(x_2)q_{2,2}p_{2,2}\| \leq 2r\|x_2 - x_1\|. \quad (\text{IV.63})$$

Therefore we can conclude that $\|f(x_2) - f(x_1)\| \leq L(r)\|x_2 - x_1\|$ with $L(r) = k_i + 2\frac{c_i r}{m}$. Therefore, according to Theorem I.1.4, we obtain the statement of the theorem. \square

We are now interested in studying the hybrid equilibrium of the semilinear equation (IV.54). We first note that the equilibrium position for the non contact condition corresponds to a state that is in the contact region, thus it is not a possible hybrid state (note that the equations in the non-contact region $\gamma(x) = 0$ results in a rotating flexible beam with a rotating and translating inertia in the free side for which we refer to Example 1 in Section III.2 of Chapter III). Consequently we investigate the equilibrium position for the equations in the contact region. Studying $Ax + f(x) = 0$ with $\gamma(x) = 1$, in the original coordinates with $q_1 = \theta$, $q_1^o = \theta^o$, results in equations

$$\begin{cases} P_1 \frac{\partial}{\partial \xi}(\mathcal{H}z) + P_0(\mathcal{H}z) = 0 \\ EIz_4(0, t) - k(q_1 - q_1^o) - \frac{c}{J_1}p_1 = 0 \\ -Kz_3(L, t) - k_i q_2 - \frac{c_i}{m}q_2 p_2 = 0 \\ EIz_4(L, t) = 0 \\ \frac{1}{J_1}p_1 = 0 \\ \frac{1}{m}p_2 = 0 \\ \frac{1}{J_2}p_3 = 0 \end{cases} \quad (\text{IV.64})$$

with boundary conditions

$$\begin{aligned} \frac{1}{\rho}z_1(0, t) &= 0 & \frac{1}{I_\rho}z_2(0, t) &= \frac{p_1}{J_1} \\ \frac{1}{\rho}z_1(L, t) &= \frac{p_2}{m} & \frac{1}{I_\rho}z_2(L, t) &= \frac{p_3}{J_2}. \end{aligned} \quad (\text{IV.65})$$

Since $J_1, m, J_2 > 0$, we have that $p_1 = p_2 = p_3 = 0$, and consequently the set of equations transforms into

$$\begin{cases} P_1 \frac{\partial}{\partial z}(\mathcal{H}z) + P_0(\mathcal{H}z) = 0 \\ EIz_4(0, t) = k(q_1 - q_1^o) \\ Kz_3(L, t) = -k_i q_2 \\ EIz_4(L, t) = 0 \end{cases} \quad (\text{IV.66})$$

with homogeneous boundary conditions

$$\frac{1}{\rho}z_1(0,t) = \frac{1}{I_p}z_2(0,t) = \frac{1}{\rho}z_1(L,t) = \frac{1}{I_p}z_2(L,t) = 0. \quad (\text{IV.67})$$

Take the first differential set of equations and write it in the extended form

$$\begin{aligned} \frac{\partial}{\partial \xi} K z_3(\xi, t) &= 0 \\ \frac{\partial}{\partial \xi} E I z_4(\xi, t) + K z_3(\xi, t) &= 0 \\ \frac{\partial}{\partial \xi} \frac{1}{\rho} z_1(\xi, t) - \frac{1}{I_p} z_2(\xi, t) &= 0 \\ \frac{\partial}{\partial \xi} \frac{1}{I_p} z_2(\xi, t) &= 0 \end{aligned} \quad (\text{IV.68})$$

with the same boundary conditions as before. The last two equations of (IV.68) with boundary conditions (IV.67) result into $z_1(\xi, t) = z_2(\xi, t) = 0$. From the first equation of (IV.68) and the boundary condition $z_3(L, t) = -\frac{k_i}{K}q_2$ we obtain

$$z_3(\xi, t) = -\frac{k_i}{K}q_2. \quad (\text{IV.69})$$

From the second equation of (IV.68) together with the first boundary condition in (IV.66) we obtain

$$z_4(\xi, t) = +\frac{k}{EI}(q_1 - q_1^o) + \frac{k_i}{EI}q_2\xi. \quad (\text{IV.70})$$

Using $z_4(L, t) = 0$ we obtain

$$Lk_iq_2 = k(q_1^o - q_1). \quad (\text{IV.71})$$

Now, from the state variable definition (IV.45) we compute

$$\begin{aligned} w(\xi, t) &= + \int_0^\xi \left\{ z_3(\zeta, t) + \int_0^\zeta \{ z_4(\alpha, t) \} d\alpha \right\} d\zeta \\ &= - \int_0^\xi \left\{ \frac{k_iq_2}{K} + \int_0^\zeta \left\{ \frac{k(q_1 - q_1^o)}{EI} + \frac{k_iq_2\alpha}{EI} \right\} d\alpha \right\} d\zeta \\ &= -\frac{k_iq_2}{K}\xi - \frac{k(q_1^o - q_1)}{2EI}\xi^2 + \frac{k_iq_2}{6EI}\xi^3 \end{aligned} \quad (\text{IV.72})$$

and because of (IV.71), we can write

$$w(\xi, t) = -\frac{k_iq_2}{K}\xi - \frac{k_iq_2L}{2EI}\xi^2 + \frac{k_iq_2}{6EI}\xi^3 \quad (\text{IV.73})$$

that computed at the $\xi = L$ boundary gives

$$w(L, t) = -\frac{k_iq_2}{k}L - \frac{k_iq_2}{3EI}L^3. \quad (\text{IV.74})$$

Using the q_1 and q_2 definitions of (IV.45), we know that

$$q_2 = Lq_1 + w(L, t) \quad (\text{IV.75})$$

then, substitute the $w(L, t)$ definition in the former equation to obtain

$$q_2 = Aq_1, \quad A = \frac{L}{1 + k_i \left(\frac{L}{K} + \frac{L^3}{3EI} \right)}. \quad (\text{IV.76})$$

Defining $B = \frac{k}{k_i LA + k}$ and substituting the former equation in (IV.71) we get

$$q_1^* = Bq_1^o, \quad q_2^* = ABq_1^o. \quad (\text{IV.77})$$

Then we are able to find the equilibrium positions for all the states as a function of the control parameter q_1^o

$$z_3^*(\xi) = -\frac{k_i A}{K} B q_1^o, \quad z_4^*(\xi) = \frac{k_i A}{EI} B (\xi - L) q_1^o. \quad (\text{IV.78})$$

We can now define the equilibrium vectors $z^* = [0 \ 0 \ z_3^* \ z_4^*]^T$, $p^* = [0 \ 0 \ 0]^T$, $q^* = [q_1^* \ q_2^* \ 0]^T$. The equilibrium state is defined as $x^* = \begin{bmatrix} z^* \\ p^* \\ q^* \end{bmatrix}$, while the hybrid equilibrium write $(x^*, 1)$. Hence, we define the new state as a shift of the state x with respect to the equilibrium position

$$\bar{x} = x - x^*. \quad (\text{IV.79})$$

The equations in the new state can be rewritten as

$$\begin{cases} \dot{\bar{z}}_1(\xi, t) = \frac{\partial}{\partial \xi} K(\bar{z}_3(\xi, t) + z_3^*(\xi)) \\ \dot{\bar{z}}_2(\xi, t) = \frac{\partial}{\partial \xi} EI(\bar{z}_4(\xi, t) + z_4^*(\xi)) + K(\bar{z}_3(\xi, t) + z_3^*(\xi)) \\ \dot{\bar{z}}_3(\xi, t) = \frac{\partial}{\partial \xi} \frac{1}{\rho} \bar{z}_1(\xi, t) - \frac{1}{I_\rho} \bar{z}_2(\xi, t) \\ \dot{\bar{z}}_4(\xi, t) = \frac{\partial}{\partial \xi} \frac{1}{I_\rho} \bar{z}_2(\xi, t) \\ \dot{\bar{p}}_1(t) = +EI(\bar{z}_4(0, t) + z_4^*(0)) - k(\bar{q}_1(t) + q_1^* - q_1^o) - \frac{c}{J_1} \bar{p}_1(t) \\ \dot{\bar{p}}_2(t) = -K(\bar{z}_3(L, t) + z_3^*(L)) - k_i \gamma(\bar{q}_2(t) + q_2^*)(\bar{q}_2(t) + q_2^*) \\ \quad - \frac{c_i}{m} \gamma(\bar{q}_2(t) + q_2^*)(\bar{q}_2(t) + q_2^*) \bar{p}_3(t) \\ \dot{\bar{p}}_3(t) = -EI(\bar{z}_4(L, t) + z_4^*(L)). \end{cases} \quad (\text{IV.80})$$

Using the equilibrium definitions (IV.78), the first two equations can be rewritten in the classical form

$$\begin{cases} \dot{\bar{z}}_1(\xi, t) = \frac{\partial}{\partial \xi} K \bar{z}_3(\xi, t) \\ \dot{\bar{z}}_2(\xi, t) = \frac{\partial}{\partial \xi} EI \bar{z}_4(\xi, t) + K \bar{z}_3(\xi, t) \end{cases} \quad (\text{IV.81})$$

allowing the pH representation for the infinite dimensional part of the system. Then, we can define the closed-loop system as described by a semi-linear switched operator equation

$$\dot{\bar{x}} = A_\gamma \bar{x} + f_\gamma \bar{x} \quad (\text{IV.82})$$

with non-contact and contact operators defined, respectively

$$A_0 \bar{x} = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H} \bar{z}) + P_0(\mathcal{H} \bar{z}) \\ EI \bar{z}_4(0, t) - k \bar{q}_1 - \frac{c}{J_1} \bar{p}_1 \\ -K \bar{z}_3(L, t) + A k_i B q_1^0 \\ -EI \bar{z}_4(L, t) \\ \frac{1}{J_1} \bar{p}_1 \\ \frac{1}{m} \bar{p}_2 \\ \frac{1}{J_2} \bar{p}_3 \end{bmatrix}, \quad (\text{IV.83})$$

$$A_1 \bar{x} + f_1(\bar{x}) = \begin{bmatrix} P_1 \frac{\partial}{\partial \xi}(\mathcal{H} \bar{z}) + P_0(\mathcal{H} \bar{z}) \\ EI \bar{z}_4(0, t) - k \bar{q}_1 - \frac{c}{J_1} \bar{p}_1 \\ -K \bar{z}_3(L, t) - k_i \bar{q}_2 \\ -EI \bar{z}_4(L, t) \\ \frac{1}{J_1} \bar{p}_1 \\ \frac{1}{m} \bar{p}_2 \\ \frac{1}{J_2} \bar{p}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{c_i}{m}(\bar{q}_2 + q_2^*) \bar{p}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{IV.84})$$

with domains

$$\mathbf{D}(A_1) = \mathbf{D}(A_2) = \{ \bar{x} \in X \mid \bar{z} \in H^1([0, L], \mathbb{R}^4), \mathcal{B}_1(\mathcal{H} \bar{z}) = M^{-1} \bar{p}, \mathcal{B}_2(\mathcal{H} \bar{z}) = 0 \}, \quad (\text{IV.85})$$

switching sets

$$\begin{aligned} D_{0,1} &= \{ \bar{x} \in X \mid \bar{q}_2 = -q_2^*, \bar{p}_2 \geq 0 \} \\ D_{1,0} &= \{ \bar{x} \in X \mid \bar{q}_2 = -q_2^*, \bar{p}_2 < 0 \}, \end{aligned} \quad (\text{IV.86})$$

and non-contact and contact regions defined, respectively, as

$$\begin{aligned} \Omega_0 &= \{ \bar{x} \in X \mid \bar{q}_2 < -q_2^* \} \cup \{ \bar{x} \in X \mid \bar{q}_2 = -q_2^*, \bar{p}_2 < 0 \} \\ \Omega_1 &= \{ \bar{x} \in X \mid \bar{q}_2 > -q_2^* \} \cup \{ \bar{x} \in X \mid \bar{q}_2 = -q_2^*, \bar{p}_2 \geq 0 \}. \end{aligned} \quad (\text{IV.87})$$

The infinite dimensional inputs and outputs are defined through the boundary operators defined in equation (IV.46)

$$u_{\bar{z}} = \mathcal{B}_1(\mathcal{H} \bar{z}) \quad y_{\bar{z}} = \mathcal{C}_1(\mathcal{H} \bar{z}), \quad (\text{IV.88})$$

while the discrete transition function in the new variables writes

$$\gamma(\bar{x}) = \begin{cases} 0 & \text{if } \bar{x} \in \{ \bar{x} \in X \mid \bar{q}_2 < -q_2^* \} \cup \{ \bar{x} \in X \mid \bar{q}_2 = -q_2^*, \bar{p}_2 < 0 \} \\ 1 & \text{if } \bar{x} \in \{ \bar{x} \in X \mid \bar{q}_2 > -q_2^* \} \cup \{ \bar{x} \in X \mid \bar{q}_2 = -q_2^*, \bar{p}_2 \geq 0 \}. \end{cases} \quad (\text{IV.89})$$

Remark 14. The representation as a switched semilinear system (IV.83)-(IV.84) has been obtained after a state translation (with respect to the equilibrium position) and the substitution of the indicator values (1 or 0) in the corresponding subspace (contact or non-contact) of the state space in the original semilinear equation (IV.54). Hence, existence and uniqueness of solution of (IV.54) implies that the switched semilinear system (IV.83)-(IV.84) has a unique mild and classical solution in the sense of Definitions I.1.10 and I.1.11, respectively. Because of the Lipschitz continuity of the studied

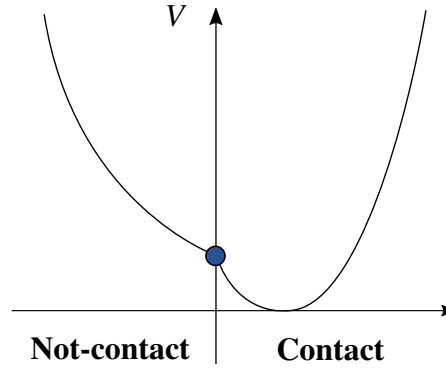


Figure IV.6: Non-contact and contact energy functionals combination to obtain a single candidate Lyapunov functional.

nonlinear function in (IV.54), we decided to use the solution concept of Definitions I.1.10 and I.1.11, without introducing any additional concept for the switched system's solution.

Remark 15. Both the contact and non-contact operators are defined on a common domain $\mathbf{D}(A_0) = \mathbf{D}(A_1)$ that is the same as the domain (IV.55) of the operator in the semilinear equation (IV.54). Therefore, since in Theorem IV.3.1 we concluded about existence of a unique classical solution of (IV.54)-(IV.55) where $x(t) \in \mathbf{D}(A)$ and $C^1([0, t], X)$ for all $t \geq 0$, it follows that $\bar{x}(t) \in \mathbf{D}(A_0) \in \mathbf{D}(A_1)$ for all $t \geq 0$. Moreover, we have that at any switching time t^* we have that the solution is time differentiable because it is obtained as a simple translation of the solution of (IV.54).

IV.3.2 Stability analysis

The model of the rotating flexible beam entering in impact with an external environment in closed-loop with a PD control law has been written as a semilinear differential equation (IV.54). Since in Theorem IV.3.1 we have shown that (IV.54) possesses a unique mild solution for all $x_0 \in X$, one can in principle use the stability theory of Section III.1.1 to conclude about its asymptotic behaviour. In this Chapter we decided to approach this problem from the switching systems perspective for one principal reason: consistently with the rest of the manuscript, we are interested in the use of energy functionals as candidate Lyapunov functionals. The energy functionals for the considered system are two: the energy in the non-contact scenario and the energy in the contact scenario. A first possibility is to combine the two Energy functionals in a single one as shown in Figure IV.6. This would lead to time differentiation problems during the switching times, and would force to define a new concept of time derivative (different from the Dini derivative of Definition I.1.8). Hence we decided to use Theorem IV.1.1, that makes use of multiple Lyapunov functionals, to be able to use the non-contact energy during non-contact phases and the contact one during contact phases.

Theorem IV.3.2. Consider the switched semilinear system (IV.82) with operators and nonlinear functions defined in (IV.83)-(IV.84) and discrete transition function (IV.89) with positive definite control parameters, i.e. $k, c, q_1^0 > 0$. Then the system is Lyapunov stable and has a globally bounded mild solution for every initial condition $\bar{x}_0 \in X$.

Proof. Since system (IV.82) corresponds to system (IV.54) with translated variables, thanks to Theorem IV.3.1, we know that it admits a unique solution that exists until an escape time t_{max} that is reached if and only if the solution diverges.

We define the following Lyapunov functional candidates for the non-contact and contact vector field, respectively

$$V_0 = \frac{1}{2} \int_0^L \left(\frac{1}{\rho} \bar{z}_1^2 + \frac{1}{I_\rho} \bar{z}_2^2 + K(\bar{z}_3 + z_3^*)^2 + EI(\bar{z}_4 + z_4^*)^2 \right) d\xi + \frac{1}{2J_1} \bar{p}_1^2 + \frac{1}{2m} \bar{p}_2^2 + \frac{1}{2J_2} \bar{p}_3^2 + \frac{1}{2} k(\bar{q}_1 - (1-B)q_1^0)^2 \quad (IV.90)$$

$$V_1 = \frac{1}{2} \langle \bar{z}, \mathcal{H} \bar{z} \rangle_{L_2} + \frac{1}{2J_1} \bar{p}_1^2 + \frac{1}{2m} \bar{p}_2^2 + \frac{1}{2J_2} \bar{p}_3^2 + \frac{1}{2} k \bar{q}_1^2 + \frac{1}{2} k_i \bar{q}_2^2. \quad (IV.91)$$

To compute the time derivative of V_0 along the system trajectories, we first put it in the following form

$$V_0 = \frac{1}{2} \langle \bar{z}, \mathcal{H} \bar{z} \rangle_{L_2} + \frac{1}{2J_1} \bar{p}_1^2 + \frac{1}{2m} \bar{p}_2^2 + \frac{1}{2J_2} \bar{p}_3^2 + \frac{1}{2} k(\bar{q}_1 - (1-B)q_1^0)^2 + W_z \quad (IV.92)$$

with

$$W_z = \frac{1}{2} \int_0^L (2Kz_3^* \bar{z}_3 + Kz_3^{*2} + 2EIz_4^* \bar{z}_4 + EIz_4^{*2}) d\xi. \quad (IV.93)$$

According to Remark 15, since $\bar{x} \in \mathbf{D}(A_0) = \mathbf{D}(A_1)$, we can compute the time derivative of V_0 and V_1 according to Lemma IV.1.1 for all $t \geq 0$, even during switching times. We now compute the time derivative of V_0 along the solutions in Ω_0

$$\begin{aligned} \dot{V}_{0,+}(\bar{x}) &= dV_{0,A_0} \bar{x} \\ &= \langle \bar{z}, \mathcal{J} \bar{z} \rangle_Z + \frac{1}{J_1} \bar{p}_1 (EI \bar{z}_4(0, t) - k \bar{q}_1 - \frac{c}{J_1} \bar{p}_1) \\ &\quad + \frac{1}{m} \bar{p}_2 (-K \bar{z}_3(L, t) + A k_i B q_1^0) - \frac{1}{J_2} \bar{p}_3 EI \bar{z}_4(L, t) \\ &\quad + \frac{k}{J_1} (\bar{q}_1 - (1-B)q_1^0) \bar{p}_1 + dW_z A_0 \bar{x} \end{aligned} \quad (IV.94)$$

where the last term can be computed as

$$\begin{aligned} dW_z A_0 \bar{x} &= \int_0^L \left(Kz_3^* \left(\frac{\partial}{\partial \xi} \frac{1}{\rho} \bar{z}_1 - \frac{1}{I_\rho} \bar{z}_2 \right) + EIz_4^* \frac{\partial}{\partial \xi} \frac{1}{I_\rho} \bar{z}_2 \right) d\xi \\ &= \int_0^L -k_i AB q_1^0 \left(\frac{\partial}{\partial \xi} \frac{1}{\rho} \bar{z}_1 - \frac{1}{I_\rho} \bar{z}_2 \right) + k_i AB q_1^0 (\xi - L) \frac{\partial}{\partial \xi} \frac{1}{I_\rho} \bar{z}_2 d\xi \\ &= \int_0^L \left(k_i AB q_1^0 \frac{\partial}{\partial \xi} \left(\frac{\xi}{I_\rho} \bar{z}_2 \right) - k_i AB q_1^0 \frac{\partial}{\partial \xi} \frac{1}{\rho} \bar{z}_1 - k_i AB q_1^0 \frac{\partial}{\partial \xi} \frac{L}{I_\rho} \bar{z}_2 \right) d\xi \\ &= k_i AB q_1^0 \left[\frac{\xi}{I_\rho} \bar{z}_2 - \frac{1}{\rho} \bar{z}_1 - \frac{L}{I_\rho} \bar{z}_2 \right]_0^L \\ &= k_i AB q_1^0 \left(-\frac{1}{\rho} \bar{z}_1(L, t) + \frac{L}{I_\rho} \bar{z}_2(0, t) \right). \end{aligned} \quad (IV.95)$$

We substitute the last expression in (IV.94), and using the state variables definition (IV.79) as well as Theorem I.2.1 and definitions of u_z, y_z in (IV.88), we obtain

$$\begin{aligned} \dot{V}_{0,+}(\bar{x}) = & u_z^T y_z + \frac{1}{J_1} \bar{p}_1 (EI \bar{z}_4(0, t) - k \bar{q}_1 - \frac{c}{J_1} \bar{p}_1) \\ & + \frac{1}{m} \bar{p}_2 (-K \bar{z}_3(L, t) + A k_i B q_1^0) - \frac{1}{J_2} \bar{p}_3 EI \bar{z}_4(L, t) \\ & + \frac{k}{J_1} (\bar{q}_1 - (1 - B) q_1^0) \bar{p}_1 + k_i A B q_1^0 \left(-\frac{1}{\rho} \bar{z}_1(L, t) + \frac{L}{I_p} \bar{z}_2(0, t) \right) \\ = & \mathcal{B}_1(\mathcal{H} \bar{z})^T \mathcal{C}_1(\mathcal{H} \bar{z}) - \mathcal{B}_1(\mathcal{H} \bar{z})^T \mathcal{C}_1(\mathcal{H} \bar{z}) \\ & + \frac{1}{J_1} \bar{p}_1 \left(-k \bar{q}_1 - \frac{c}{J_1} \bar{p}_1 \right) + \frac{1}{m} \bar{p}_2 A k_i B q_1^0 + \frac{k}{J_1} (\bar{q}_1 - (1 - B) q_1^0) \bar{p}_1 \\ & + k_i A B q_1^0 \left(-\frac{1}{\rho} \bar{z}_1(L, t) + \frac{L}{I_p} \bar{z}_2(0, t) \right). \end{aligned} \quad (\text{IV.96})$$

From the definition of B , we have $k(1 - B) = k_i A B L$. After some simplifications, the Lyapunov's derivative writes

$$\begin{aligned} \dot{V}_{0,+}(\bar{x}) = & -\frac{c}{J_1^2} \bar{p}_1^2 + \frac{k_i A B q_1^0}{m} \bar{p}_2 - \frac{k_i A B q_1^0 L}{J_1} \bar{p}_1 - \frac{k_i A B q_1^0}{\rho} \bar{z}_1(L, t) + \frac{k_i A B q_1^0 L}{I_p} \bar{z}_2(0, t) \\ = & -\frac{c}{J_1^2} \bar{p}_1^2 \leq 0. \end{aligned} \quad (\text{IV.97})$$

Since the contact operator (IV.84) and the contact Lyapunov function (IV.91) are in the classical form, for $\bar{x} \in \Omega_1$ we obtain

$$\dot{V}_{1,+}(\bar{x}) = -\frac{c}{J_1^2} \bar{p}_1^2 - \frac{c_i(\bar{q}_2 + q_2^*)}{m} \bar{p}_2^2 \quad (\text{IV.98})$$

and since in Ω_1 it holds that $\bar{q}_2 \geq -q_2^*$, we have

$$\dot{V}_{1,+}(\bar{x}) \leq -\frac{c}{J_1^2} \bar{p}_1^2. \quad (\text{IV.99})$$

Now we study the “switching in” behaviour of the two Lyapunov functionals V_0 and V_1 . Consider t_0 to be the initial time. Assume that the initial condition $\bar{x}(0) = \bar{x}_0$ is taken such that $\bar{x}_0 \in \Omega_0$. Since the asymptotically stable equilibrium position of the non-contact equations is in the Contact region (See the rotating flexible beam controlled with a PD control law in Example 1 in Section III.2 of Chapter III), there is always a contact phase after a non-contact one. Then, the time series $\Pi_2(\{S_n(\bar{x}_0)\})$ has even components $\mathcal{E}(\Pi_2(\{S_n(\bar{x}_0)\}))$ that corresponds to the non-contact “switching-in” conditions, while has odd components $\mathcal{O}(\Pi_2(\{S_n(\bar{x}_0)\}))$ that corresponds to the contact ones. According to Theorem IV.1.1, it remains to show that $V_0(t)$ is non-increasing in $t \in \mathcal{E}(\Pi_2(\{S_n(\bar{x}_0)\}))$ and that $V_1(t)$ is non-increasing in $t \in \mathcal{O}(\Pi_2(\{S_n(\bar{x}_0)\}))$. From (IV.97) and (IV.99), we directly obtain that $V_0(t_i) \leq V_0(t_{i-1})$ and $V_1(t_{i+1}) \leq V_1(t_i)$ for $i \in \{1, 3, 5, \dots\}$.

Now we show that $V_1(t_{i+2}) \leq V_1(t_{i+1})$. Note that the time interval $[t_{i+1}, t_{i+2})$, for $i \in \{1, 3, 5, \dots\}$ and the assumed initial condition, corresponds to a non-contact phase. So, our aim is to study the behaviour of the contact Lyapunov functional at the beginning and at the end of a non-contact phase. To do so, we first note that

$$V_0 = V_1 + W_z + W_q - \frac{1}{2} k_i \bar{q}_2^2 \quad (\text{IV.100})$$

where W_z is defined in (IV.93) and

$$W_q = \frac{1}{2}k(1-B)^2q_1^{o2} - k(1-B)q_1^o\bar{q}_1. \quad (\text{IV.101})$$

In light of (IV.100), inequality $V_0(t_{i+2}) \leq V_0(t_{i+1})$ for $i \in \{1, 3, 5, \dots\}$ can be rewritten as

$$V_1(t_{i+2}) + W_z(t_{i+2}) + W_q(t_{i+2}) - \frac{1}{2}k_i\bar{q}_2^2(t_{i+2}) \leq \begin{aligned} & V_1(t_{i+1}) + W_z(t_{i+1}) \\ & + W_q(t_{i+1}) - \frac{1}{2}k_i\bar{q}_2^2(t_{i+1}) \end{aligned} \quad (\text{IV.102})$$

that since $\bar{q}_2(t_{j-1}) = \bar{q}_2(t_j)$ for $j \in \mathbb{N}$ we obtain

$$\begin{aligned} V_1(t_{i+2}) + W_z(t_{i+2}) + W_q(t_{i+2}) &\leq V_1(t_{i+1}) + W_z(t_{i+1}) + W_q(t_{i+1}) \\ V_1(t_{i+2}) &\leq V_1(t_{i+1}) + W_z(t_{i+1}) - W_z(t_{i+2}) + W_q(t_{i+1}) - W_q(t_{i+2}). \end{aligned} \quad (\text{IV.103})$$

At this point we compute the time derivative of the quantity $W_z + W_q$ along the non-contact vector field $A_0\bar{x}$

$$\begin{aligned} \frac{d}{dt}(W_z + W_q) &= dW_z A_0\bar{x} + dW_q A_0\bar{x} \\ &= k_i AB q_1^o \left(-\frac{1}{\rho} \bar{z}_1(L, t) + \frac{L}{I_\rho} \bar{z}_2(0, t) \right) - \frac{k(1-B)q_1^o}{I_\rho} \bar{z}_2(0, t) \end{aligned} \quad (\text{IV.104})$$

and since $k(1-B) = k_iABL$, we obtain

$$\frac{d}{dt}(W_z + W_q) = -\frac{k_i AB q_1^o}{\rho} \bar{z}_1(L, t). \quad (\text{IV.105})$$

We compute

$$\int_{t_{i+1}}^{t_{i+2}} \frac{d}{dt}(W_z + W_q) dt = W_z(t_{i+2}) - W_z(t_{i+1}) + W_q(t_{i+2}) - W_q(t_{i+1}) \quad (\text{IV.106})$$

that, using the $\bar{z}_1 = z_1$ definition (IV.45) and the fact that at switching times $\bar{q}_2(t) = -q_2^* \Rightarrow L\theta(t) + w(L, t) = 0$ for $t \in \Pi_2(\{S_n(\bar{x}_0)\})$, $n \geq 1$, corresponds to

$$\begin{aligned} \int_{t_{i+1}}^{t_{i+2}} \frac{d}{dt}(W_z + W_q) dt &= - \int_{t_{i+1}}^{t_{i+2}} \frac{k_i AB q_1^o}{\rho} \bar{z}_1(L, t) dt \\ &= - \frac{k_i AB q_1^o}{\rho} \int_{t_{i+1}}^{t_{i+2}} L\dot{\theta}(t) + \dot{w}(L, t) dt \\ &= - \frac{k_i AB q_1^o}{\rho} (L\theta(t_{i+2}) + w(L, t_{i+2}) - L\theta(t_{i+1}) - w(L, t_{i+1})) \\ &= 0. \end{aligned} \quad (\text{IV.107})$$

From the last two equations we conclude that

$$W_z(t_{i+2}) - W_z(t_{i+1}) + W_q(t_{i+2}) - W_q(t_{i+1}) = 0 \quad (\text{IV.108})$$

that, from (IV.103), in turn implies

$$V_1(t_{i+2}) \leq V_1(t_{i+1}) \Rightarrow V_1(t_{i+2}) \leq V_1(t_{i+1}) \leq V_1(t_i) \Rightarrow V_1(t_{i+2}) \leq V_1(t_i) \quad (\text{IV.109})$$

for $i \in \{1, 3, 5, \dots\}$, that means that $V_1(t)$ is non increasing in $t \in \mathcal{O}(\Pi_2(\{S_n(\bar{x}_0)\}))$. The next step is showing that $V_0(t)$ is non increasing in $t \in \mathcal{E}(\Pi_2(\{S_n(\bar{x}_0)\}))$. Using relation (IV.100) together with the fact that $V_1(t_{i+1}) \leq V_1(t_i)$ for $i \in \{1, 3, 5, \dots\}$ we obtain

$$V_{nc}(t_{i+1}) \leq V_{nc}(t_i) + W_z(t_{i+1}) - W_z(t_i) + W_q(t_{i+1}) - W_q(t_i) \quad (\text{IV.110})$$

and with the same procedure as before we obtain that $V_{nc}(t)$ is non increasing in $t \in \mathcal{E}(\Pi_2(\{S_n(\bar{x}_0)\}))$. The case in which $x_0 \in \Omega_1$ can be developed in the same manner but with $[t_{i-1}, t_i)$ corresponding to the contact time interval, and $[t_i, t_{i+1})$ a non-contact time interval for $i \in \{1, 3, 5, \dots\}$. According to Theorem IV.1.1 we can conclude that the system is Lyapunov stable and has a globally bounded mild solution for every initial condition $x_0 \in X$. \square

IV.3.3 Numerical simulations

To perform the numerical simulations, a finite dimensional discretization of the infinite dimensional system has been considered. In particular, it has been used the discretization procedure described in Section II.2 of Chapter II, that allows to spatially approximate the resulting linear PDE with a linear pH system of dimension depending on the number of discretizing elements (in the shown simulations, the flexible beam has been divided into 150 elements). In particular, since the flexible beam model has velocity inputs at both sides of the spatial domain (see equation (IV.46)), its finite dimensional model corresponds to a *Flow-Flow input discretized system* introduced in Definition II.2.3 with n_d discretizing elements and state $z_d \in \mathbb{R}^{4n_d}$. Therefore the discretized version of the closed-loop system (IV.54) has a state defined by $x_d = \begin{bmatrix} z_d \\ p \\ q \end{bmatrix}$, and writes

$$\begin{bmatrix} \dot{z}_d \\ \dot{p} \\ \dot{q} \end{bmatrix} = \left(\begin{bmatrix} J_{ff} & \bar{g}_{ff} & 0 \\ -\bar{g}_{ff}^T & 0 & -I \\ 0 & I & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} H_{ff} & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} z_d \\ p \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ g_1 \\ 0 \end{bmatrix} f_d \quad (\text{IV.111})$$

with J_{ff} defined in (II.166), \bar{g}_{ff} defined as the 2nd-4th columns of matrix g_{ff} defined in (II.167) and

$$R_p = \begin{bmatrix} -\frac{c}{J_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad K = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{IV.112})$$

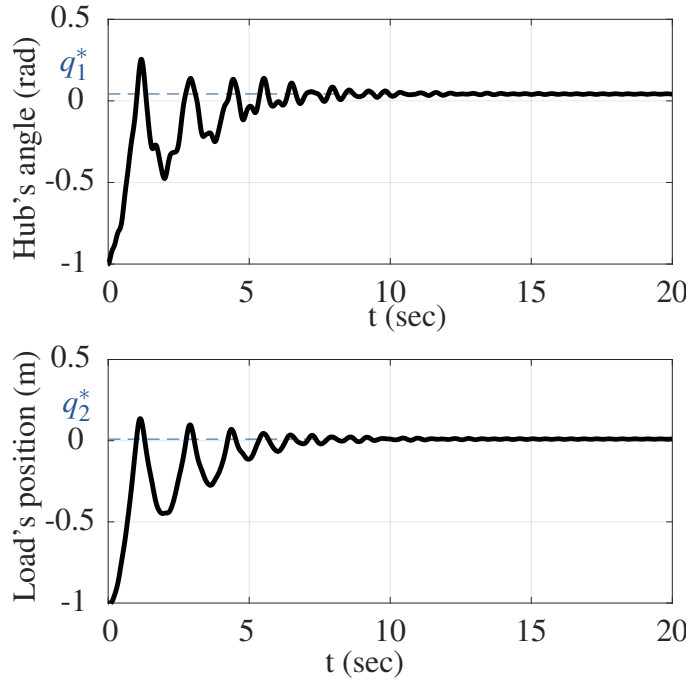
$$f_i = -k_i \gamma(x_d) q_2 - \frac{c_i}{m} \gamma(x_d) q_2 p_2.$$

Where the impact's model parameters are set equal to $k_i = 1000$ and $c_i = 30$. The simulations have been made in the Matlab[®] environment using the “ode23t” time integration algorithm, and the set of parameters used for simulations are listed in Table IV.2.

As in the rigid robot arm simulations, we select the controller parameters at least of one order of magnitude smaller than the impact model parameters $k = 10$, $c = 3$. In

Table IV.2: Simulation parameters; flexible beam impact

Name	Variable	Value
Beam's Length	L	1 m
Beam's Width	L_w	0.1 m
Beam's Thickness	L_t	0.02 m
Density	ρ	$16\frac{\text{kg}}{\text{m}^3}$
Young's modulus	E	$2 \times 10^9\frac{\text{N}}{\text{m}^2}$
Bulk's modulus	K	$6.85 \times 10^8\frac{\text{N}}{\text{m}^2}$
Hub's inertia	J_1	$1\text{ kg} \cdot \text{m}^2$
Load's mass	m	1 kg
Load's inertia	J_2	$1\text{ kg} \cdot \text{m}^2$

Figure IV.7: Hub's angle evolution along time $q_1(t)$ and Load's position evolution along time $q_2(t)$.

this way, we can clearly appreciate the change of dynamics due to the impact with the external surface. Next, we set the control parameter $\theta^o = 1$ such that the desired equilibrium position is in the contact set Ω_1 . In accordance with Section IV.3.1, it is possible to compute the equilibrium configuration of the system: $q_1^* = 0.0424\text{ rad}$, $q_2^* = 0.0096\text{ m}$, $\epsilon_t^*(\xi) = -1.3981 \times 10^{-8}$ and $\epsilon_r(\xi) = 0.0985(\xi - L)$. To perform numerical simulations, the beam's states as well as the finite dimensional momentum states are initialized to zero $x_b(0, \xi) = 0$, $p_1(0) = p_2(0) = p_3(0)$. The initial hub's angle has been initialized to $\theta(0) = q_1(0) = -1\text{ rad}$, accordingly to the load's initial position $q_2(0) = L\theta(0) + w(0, L) = -1\text{ m}$. Figure IV.7 shows the evolution in time of the hub's

angle and of the load position. It is important to note that the contact occurs when

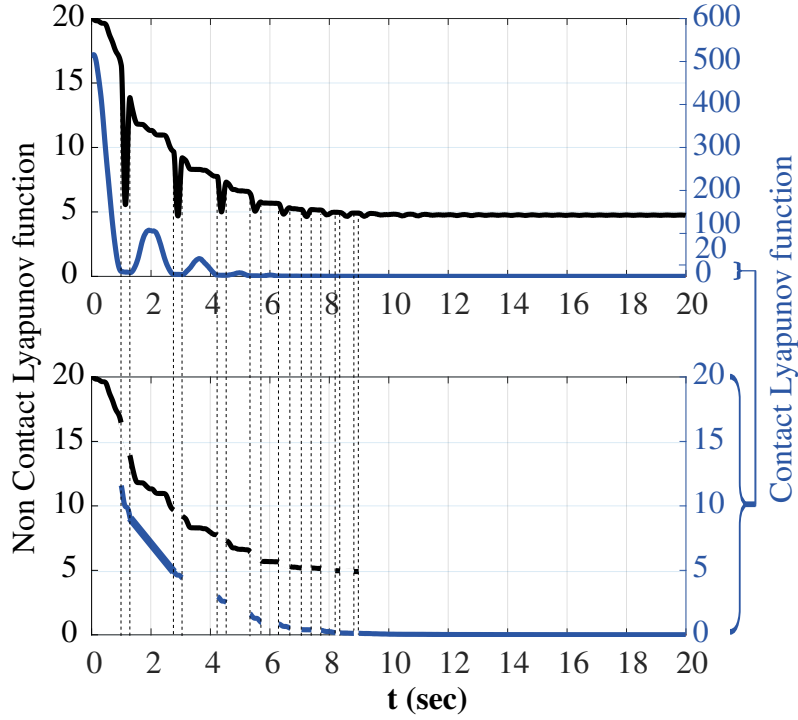


Figure IV.8: Contact V_1 and non-contact V_0 Lyapunov functions behaviour along time for a flexible rotating beam.

$q_2(t) \geq 0$, and in fact when it dynamically reaches this value, the q_2 variable is rejected back because of the spring force of the impact model. It is possible to appreciate that both angles asymptotically stabilize to the computed equilibrium positions. Figure IV.8 shows firstly the Lyapunov functionals (IV.91)-(IV.92) behaviour along with the solution in the entire simulation time interval without distinguishing between the active or non active time intervals, and secondly their behaviour during the respective activation time intervals. It is possible to appreciate that both the selected Lyapunov functionals are non-increasing in their activation phases and that the “Switching in” conditions are met. The time scaling in the second image in Figure IV.8 comes from the important values of $\|\bar{q}_2\|$ and thus of the term $\frac{1}{2}k_i\bar{q}_2^2$ in the contact Lyapunov functional (IV.92) during the non-contact time periods.

IV.4 SUMMARY

In this chapter, we have proposed a new framework for testing the Lyapunov stability of switching operator systems. Then, we derived the model of a rotating rigid link that enters in contact with the external (soft) environment. Using the Theoretical results presented at the beginning of the chapter, we show that the model of the rotating flexible link in closed-loop with a PD control law is Lyapunov stable. Next, we derive

the infinite dimensional model of a rotating flexible link entering in contact with the external environment. We apply a proportional derivative control law and we re-write the closed-loop model as a switching operator system. Finally, using the theoretical results presented at the beginning of the chapter, we show that also the model of the flexible rotating link is Lyapunov stable.

Chapter V

Conclusions and perspectives

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V.1 CONCLUSIONS

This thesis studies the modelling, the control design and the stability analysis of mechanisms including flexible parts. The port-Hamiltonian formulation has been used to model the mechanism containing flexible parts. This framework provides powerful analysis methods for proving well-posedness and allows the use of Lyapunov stability theory for studying the systems asymptotic behaviour. We enlightened that energy is crucial in all the different stages of the mechanisms' analysis: equations of motion are derived starting from the Lagrangian, control laws are designed using passivity concepts very similar to the ones used to prove the existence and uniqueness of solutions. Another key point of this framework is that allows dealing with a class of systems instead of studying a problem on a case-by-case basis. This working procedure allows understanding the underlying structure of mechanisms with flexible parts, and to conclude about results exploitable on a wide range of different applications. Moreover, the Lyapunov stability methods have been extended to conclude about stability in case of the presence of a specific class of non-linearities.

This thesis is mainly intended to offer an alternative way to the case-by-case methodology for studying flexible mechanisms, in such a way to exploit the similarities between models and to conclude general results. Nevertheless, the study of a specific problem allows to deeply understand a specific application and to get to conclusions that are not possible to generalise. Therefore we hope that the proposed material can provide different tools, procedures and a general understanding of ideas from which the reader can draw for solving specific application problems. In the following, we list the main contributions of this thesis.

1. In Chapter II, it has been described how to obtain a mixed ODE-PDE model for different flexible mechanisms starting from the very general principle of least action. We have seen how to write the models in the port-Hamiltonian framework: the PDE were written as open distributed 1-D port-Hamiltonian systems interacting at the boundary of the spatial domain with the ODE, rewritten as finite dimensional port-Hamiltonian systems, describing the boundary dynamics. The infinite and finite dimensional parts of these systems have been power preserving interconnected, and a general pH representation, enclosing all the different examples, has been found. Next, this general representation has been shown to be passive with respect to its internal energy. Furthermore, a structure preserving mixed finite element discretization method has been employed to discretize the 1-D pH system with any type of boundary conditions.
2. In Chapter III we first recall some basic notions on well-posedness and stability of semilinear operator equations. Then, we gave some necessary conditions to conclude about approximate observability of 1-D distributed port-Hamiltonian systems. This, together with another result linking approximate observability and the zero solution in case of constant output, are shown to be the key ingredients for proving asymptotic stability with the use of the La Salle invariance principle.

Next, we have proposed four control laws for a class of m-pH systems. Three of them can position the system in the desired equilibrium and asymptotically stabilize the closed-loop system, while the other one can exponentially stabilise the closed-loop (but not to position the system in the desired equilibrium). The proposed asymptotically stabilising control laws are as follows: a classic Proportional Derivative (PD) control, a PD control with a dynamic nonlinear term able to reduce flexible vibrations, a PD control plus strong dissipative feedback that can insert some boundary dissipation even if the control input does not directly act on the boundary conditions but only on the ODE. The exponentially stabilising control corresponds to a derivative plus strong dissipative feedback control. The well-posedness and stability results for each control law are accompanied by an applicative example on a flexible mechanism and the related numerical simulations.

3. In Chapter IV we have provided a framework to study the asymptotic behaviour of switched semilinear equations based on the extension of a result valid for nonlinear finite dimensional systems. This framework is based on the existence of multiple Lyapunov functionals and therefore fits perfectly with the used port-Hamiltonian formulation. Next, this framework has been used to study the Lyapunov stability of a rigid and flexible rotating beam that comes into contact with an external surface. Numerical simulations for both the rigid and the flexible rotating beam in the impact scenario have been provided to validate the theoretical results.

V.2 RECOMMENDATIONS FOR FUTURE WORK

In the following we list some ideas for some possible extensions of the results proposed in this thesis.

V.2.1 Nonlinear modelling of flexible mechanisms

In Chapter II, the models describing the flexible rotating and translating beams are obtained assuming that $w(\xi, t)\dot{\theta} \approx 0$ (see Assumptions II.1.2.b and II.1.2.c, at pages 34 and 38, respectively). It would be desirable to obtain the exact model without neglecting the former term, and to understand if the developed control laws in Chapter III still stabilise the system. In the following we present some preliminary ideas regarding this subject, applied on the rotating flexible beam.

Consider the kinetic energy of the rotating flexible beam in (II.25), with square distributed linear velocity (II.28) and angular velocity (II.30). Therefore, the Lagrangian writes

$$L = \frac{1}{2} \int_0^L \rho(\xi) (\xi \dot{\theta}(t) + \dot{w}(\xi, t))^2 + (w(\xi, t) \dot{\theta}(t))^2 + I_p(\xi) (\dot{\phi}(\xi, t) + \dot{\theta}(t))^2 d\xi + \frac{1}{2} J \dot{\theta}^2 - \int_0^L K(\xi) \left(\frac{\partial w}{\partial \xi}(\xi, t) - \phi(\xi, t) \right)^2 + EI(\xi) \frac{\partial^2 \phi}{\partial \xi^2}(\xi, t)^2 d\xi. \quad (\text{V.1})$$

Similarly as done in Chapter (II) we can compute the Lagrangian Fréchet derivatives

$$\begin{aligned}
d_{\dot{\theta}} L h_{\dot{\theta}} &= \left(\int_0^L \rho \xi (\dot{w} + \xi \dot{\theta}) + \rho w^2 \dot{\theta} + I_{\rho} (\dot{\phi} + \dot{\theta}) d\xi + J \dot{\theta} \right) h_{\dot{\theta}} \\
d_{\dot{w}} L h_{\dot{w}} &= \int_0^L \rho (\dot{w} + \xi \dot{\theta}) h_{\dot{w}} d\xi \\
d_w L h_w &= \int_0^L \rho w \dot{\theta}^2 d\xi \\
d_{\dot{\phi}} L h_{\dot{\phi}} &= \int_0^L I_{\rho} (\dot{\phi} + \dot{\theta}) h_{\dot{\phi}} d\xi \\
d_{\phi} L h_{\phi} &= - \int_0^L K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{\phi} d\xi \\
d_{w'} L h_{w'} &= - \int_0^L K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_{w'} d\xi \\
d_{\phi'} L h_{\phi'} &= - \int_0^L EI \frac{\partial \phi}{\partial \xi} h_{\phi'} d\xi \\
d_{\theta} W_{nc} h_{\theta} &= \tau h_{\theta}.
\end{aligned} \tag{V.2}$$

After integration by parts, equation (II.6) for a rotating flexible beam writes

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_0^L \left[\frac{\partial}{\partial \xi} K \left(\frac{\partial w}{\partial \xi} - \phi \right) + \rho w \dot{\theta}^2 - \frac{\partial}{\partial t} \rho (\dot{w} + \xi \dot{\theta}) \right] h_w d\xi - [K \left(\frac{\partial w}{\partial \xi} - \phi \right) h_w]_0^L \\
& + \int_0^L \left[K \left(\frac{\partial w}{\partial \xi} - \phi \right) + \frac{\partial}{\partial \xi} (EI \frac{\partial \phi}{\partial \xi}) - \frac{\partial}{\partial t} I_{\rho} (\dot{\phi} + \dot{\theta}) \right] h_{\phi} d\xi - [EI \frac{\partial \phi}{\partial \xi} h_{\phi}]_0^L \\
& - \left(\frac{\partial}{\partial t} \int_0^L \rho \xi (\dot{w} + \xi \dot{\theta}) + \rho w^2 \dot{\theta} + I_{\rho} (\dot{\phi} + \dot{\theta}) d\xi + \frac{d}{dt} J \dot{\theta} - \tau \right) h_{\theta} dt = 0
\end{aligned} \tag{V.3}$$

from which we can extract the following equations and boundary conditions

$$\begin{aligned}
\frac{\partial}{\partial t} \rho (\dot{w} + \xi \dot{\theta}) &= \frac{\partial}{\partial \xi} K \left(\frac{\partial w}{\partial \xi} - \phi \right) + \rho w \dot{\theta}^2 \\
\frac{\partial}{\partial t} I_{\rho} (\dot{\phi} + \dot{\theta}) &= \frac{\partial}{\partial \xi} EI \frac{\partial \phi}{\partial \xi} + K \left(\frac{\partial w}{\partial \xi} - \phi \right) \\
\frac{\partial}{\partial t} \left(\int_0^L \rho w^2 d\xi \dot{\theta} + J \dot{\theta} \right) &= - \frac{\partial}{\partial t} \int_0^L \rho \xi (\dot{w} + \xi \dot{\theta}) + I_{\rho} (\dot{\phi} + \dot{\theta}) d\xi + \tau \\
K \left(\frac{\partial w}{\partial \xi} (L, t) - \phi(L, t) \right) &= 0, \quad EI \frac{\partial \phi}{\partial \xi} (L, t) = 0.
\end{aligned} \tag{V.4}$$

Before defining the energy variables we rewrite the third of the latter set of equations, substituting the two terms inside the integral with the first two of (V.4)

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\int_0^L \rho w^2 d\xi \dot{\theta} + J \dot{\theta} \right) &= - \int_0^L \xi \frac{\partial}{\partial t} \rho (\dot{w} + \xi \dot{\theta}) + \frac{\partial}{\partial t} I_{\rho} (\dot{\phi} + \dot{\theta}) d\xi + \tau \\
&= - \int_0^L \xi \left[\frac{\partial}{\partial \xi} K \left(\frac{\partial w}{\partial \xi} - \phi \right) + \rho w \dot{\theta}^2 \right] + \frac{\partial}{\partial \xi} EI \frac{\partial \phi}{\partial \xi} \\
&\quad + K \left(\frac{\partial w}{\partial \xi} - \phi \right) - \xi \rho w \dot{\theta}^2 d\xi + \tau \\
&= - [\xi K \left(\frac{\partial w}{\partial \xi} - \phi \right)]_0^L - \int_0^L \frac{\partial}{\partial \xi} EI \frac{\partial \phi}{\partial \xi} d\xi - \int_0^L \xi \rho w \dot{\theta}^2 d\xi + \tau \\
&= - [\xi K \left(\frac{\partial w}{\partial \xi} - \phi \right)]_0^L - [EI \frac{\partial \phi}{\partial \xi}]_0^L - \int_0^L \xi \rho w \dot{\theta}^2 d\xi + \tau,
\end{aligned} \tag{V.5}$$

that using the boundary conditions in (II.36), transforms into

$$\frac{\partial}{\partial t} \left(\int_0^L \rho w^2 d\xi \dot{\theta} + J \dot{\theta} \right) = EI \frac{\partial \phi}{\partial \xi} (0, t) - \int_0^L \xi \rho w \dot{\theta}^2 d\xi + \tau. \tag{V.6}$$

Now, we can define the energy variables

$$\begin{aligned}
z_1 &= \rho (\dot{w} + \xi \dot{\theta}) & z_2 &= I_{\rho} (\dot{\phi} + \dot{\theta}) & z_3 &= \frac{\partial w}{\partial \xi} - \phi & z_4 &= \frac{\partial \phi}{\partial \xi} \\
p &= m(z) \dot{\theta}
\end{aligned} \tag{V.7}$$

where $m(z)$ is the “virtual mass” depending on the distributed parameter state and is defined as

$$m(z) = \int_0^L \rho w(z)^2 d\xi + J \quad (\text{V.8})$$

and the deformation w can be expressed as linear function of the new variables

$$w(z) = \int_0^\xi z_3 + \int_0^\xi z_4 ds d\xi. \quad (\text{V.9})$$

We define the input-output operators as in (II.40) while the density energy matrix as in (II.42). Then, we can rewrite (V.4) with the modified third equation (V.6) in the energy variables

$$\begin{aligned} \dot{z} &= P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z) + P_0 (\mathcal{H}z) + f(z, p) = \mathcal{J}z + f(z, p) \\ \dot{p} &= -\mathcal{C}_1 (\mathcal{H}z) - \int_0^L \xi \rho \frac{w(z)}{m(z)^2} p^2 d\xi + \tau, \end{aligned} \quad (\text{V.10})$$

with

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad f(z, p) = \begin{bmatrix} \rho \frac{w(z)}{m(z)^2} p^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{V.11})$$

and domain of the \mathcal{J} operator

$$\begin{aligned} \mathbf{D}(\mathcal{J}) &= \{z \in L_2([0, L], \mathbb{R}^4) \mid (\mathcal{H}z) \in H^1([0, L], \mathbb{R}^2), \\ &\quad \mathcal{B}_1(\mathcal{H}z) = \frac{1}{m(z)}p, \mathcal{B}_2(\mathcal{H}z) = 0\} \end{aligned} \quad (\text{V.12})$$

where the input-output operators are defined as

$$\begin{aligned} \mathcal{B}_1(\mathcal{H}z) &= \frac{1}{I_\rho(0)} z_2(0, t) & \mathcal{C}_1(\mathcal{H}z) &= -EI(0) z_4(0, t) \\ \mathcal{B}_2(\mathcal{H}z) &= \begin{bmatrix} \frac{1}{\rho(0)} z_1(0, t) \\ K(L) z_3(L, t) \\ EI(L) z_4(L, t) \end{bmatrix} & \mathcal{C}_2(\mathcal{H}z) &= \begin{bmatrix} -K(0) z_3(0, t) \\ \frac{1}{\rho(L)} z_1(L, t) \\ \frac{1}{I_\rho(L)} z_2(L, t) \end{bmatrix}. \end{aligned} \quad (\text{V.13})$$

Then, we define the total energy

$$E = \frac{1}{2} \langle z, \mathcal{H}z \rangle_{L_2} + \frac{1}{2m(z)} p^2. \quad (\text{V.14})$$

Next we show that the system (V.10) is passive with respect to the input τ , output $y = \frac{1}{m(z)}p$ and energy E . So, we compute the energy's time derivative using the Dini derivative concept introduced Definition I.1.8 together with Lemma I.1.2 (page 14)

$$\begin{aligned}
\dot{E}_+ &= \langle z, \mathcal{H}(\mathcal{J}z + f(z, p)) \rangle_{L_2} + \frac{1}{2} \frac{d}{dt} \left(\frac{1}{m(z)} \right) p^2 + \frac{1}{m(z)} p (\mathcal{C}_1(\mathcal{H}z) \\
&\quad - \int_0^L \xi \rho \frac{w(z)}{m(z)^2} p^2 d\xi + \tau) \\
&= \langle z, \mathcal{H} \mathcal{J}z \rangle_{L_2} + \langle z, \mathcal{H} f(z, p) \rangle_{L_2} - \int_0^L \rho \frac{w(z)}{m(z)^2} \left(\int_0^\xi \dot{z}_3 + \int_0^\xi \dot{z}_4 ds d\xi \right) p^2 d\xi \\
&\quad + \frac{1}{m(z)} p \mathcal{C}_1(\mathcal{H}z) - \frac{1}{m(z)} \int_0^L \xi \rho \frac{w(z)}{m(z)^2} p^3 d\xi + y\tau \\
&= -\frac{1}{m(z)} p \mathcal{C}_1(\mathcal{H}z) + \int_0^L z_1 \frac{w(z)}{m(z)^2} p^2 d\xi - \int_0^L \rho \frac{w(z)}{m(z)^2} \left(\int_0^\xi \frac{\partial}{\partial \xi} \frac{1}{\rho} z_1 - \frac{1}{I_\rho} z_2 \right. \\
&\quad \left. + \int_0^\xi \frac{\partial}{\partial \xi} \frac{1}{I_\rho} z_2 ds d\xi \right) p^2 d\xi + \frac{1}{m(z)} p \mathcal{C}_1(\mathcal{H}z) - \frac{1}{m(z)} \int_0^L \xi \rho \frac{w(z)}{m(z)^2} p^3 d\xi + y\tau \\
&= \int_0^L z_1 \frac{w(z)}{m(z)^2} p^2 d\xi - \int_0^L \rho \frac{w(z)}{m(z)^2} \left(\int_0^\xi \frac{\partial}{\partial \xi} \frac{1}{\rho} z_1 - \frac{1}{I_\rho} z_2(0) d\xi \right) p^2 d\xi - \int_0^L \xi \rho \frac{w(z)}{m(z)^3} p^3 d\xi + y\tau \\
&= \int_0^L z_1 \frac{w(z)}{m(z)^2} p^2 d\xi - \int_0^L \frac{w(z)}{m(z)^2} (z_1 - z_1(0)) p^2 d\xi \\
&\quad + \int_0^L \rho \frac{w(z)}{m(z)^2} \xi \frac{1}{I_\rho} z_2(0) p^2 d\xi - \int_0^L \xi \rho \frac{w(z)}{m(z)^3} p^3 d\xi + y\tau.
\end{aligned} \tag{V.15}$$

Finally, since by the domain definition $\frac{1}{\rho} z_1(0) = 0$ and $\frac{1}{I_\rho} z_2(0) = \frac{1}{m(z)}p$ we obtain

$$\dot{E}_+(t) = y(t)\tau(t) \tag{V.16}$$

that indeed shows that the defined system is passive with respect to its internal energy. The system passivity is a good starting point for the analysis of a dynamical system. Therefore, different possible questions rise to continue investigating this research line:

1. Is it possible to define a general system able to enclose all the flexible mechanisms' models when no approximation is made during the modelling procedure?
2. Is the defined model (V.10)-(V.12) well-posed when interconnected in closed-loop with even a simple proportional derivative control law?
3. What would be the asymptotic behaviour of this type of system?

Already proving the well-posedness of the derived system would be a very difficult challenge because of the presence of a nonlinear function in the PDE and in the domain of the operator \mathcal{J} .

V.2.2 Asymptotic stability of the flexible rotating beam in impact scenario

A natural continuation of what we have presented in Chapter IV would be proving the asymptotic stability of the PD controlled flexible rotating beam in impact scenario. To do so, it would be first necessary to show that the solution of the controlled flexible rotating beam possesses a finite number of switches in a finite interval of time (the opposite is known as *Zeno behaviour*). A possibility can be exploiting the continuity of the infinite dimensional solution together with the fact that the equilibrium position is

away from the Switching set, from which we can deduce the non-Zeno behaviour of the state q_2 . Therefore, the asymptotic stability can be shown with a similar procedure as the one used in [56].

Moreover, the asymptotic behaviour of the flexible beam in impact scenario should be investigated in closed-loop with the different control laws proposed in Chapter III. Since all the stability analysis have been done using Lyapunov stability theory, the possibility of using the proposed Lyapunov functionals, with some slight modifications, in the stability proofs for the impact case should be investigated further.

This study on an application example can lead to the generalization to a class of semi-linear switching port-Hamiltonian systems that can enclose a large set of flexible mechanisms that enter in contact with an external object.

V.2.3 Further investigation on 1-D distributed port-Hamiltonian systems

As discussed in Chapter III, for obtaining approximate observability of the distributed port-Hamiltonian system, it is necessary to have control and observation on all the boundary variables at least at one side of the spatial domain

$$\begin{aligned} ||\mathcal{H}z(0,t)||^2 &\leq ||u_z(t)||^2 + ||y_z(t)||^2 \\ \text{or} \\ ||\mathcal{H}z(L,t)||^2 &\leq ||u_z(t)||^2 + ||y_z(t)||^2. \end{aligned} \tag{V.17}$$

If the former condition is not verified, it is necessary to rely on specific results on the consider distributed parameter equations (as it was the case in example 1 for the PD controlled rotating Timoshenko beam). A possible extension can be trying to conclude about approximate observability of distributed port-Hamiltonian systems weakening the assumption of total control and observation at one side of the spatial domain. To conclude about approximate observability in case only part of the controls are available at one side of the spatial domain (inequality (V.17) not verified), it maybe be necessary to consider only a subclass of the class of distributed pH systems considered in this manuscript putting further assumptions on the matrices that define the system

$$\dot{z} = P_1 \frac{\partial}{\partial \xi} (\mathcal{H}z) + P_0 (\mathcal{H}z). \tag{V.18}$$

Although nonlinearity in the connected set of ODE has been considered in this thesis and in a lot of different other works, a major challenge would be proving well-posedness and eventually stability considering a class of nonlinearities also in the distributed parameter equations. A motivation in this direction has been given in the previous Section V.2.1. In that section we pointed out the possible nonlinearity structure that can appear from the modelling of moving flexible structure, highlighting the passivity property that can be key in showing the existence and uniqueness of solutions.

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Titre : Modélisation et analyse de stabilité des robots flexibles: une approche Hamiltonienne à portes à paramètres distribués

Mots clés: Systèmes Hamiltoniens à ports, Synthèse du contrôleur, Systèmes à paramètres distribués, Modélisation

Résumé : L'objectif de cette thèse est de fournir un cadre mathématique permettant d'expliciter les modèles dynamiques d'une classe de mécanismes flexibles, de concevoir des lois de commandes adaptées et d'analyser le comportement asymptotique en boucle fermée qui en résulte. D'un point de vue mathématique, les parties flexibles sont décrites par des équations aux dérivées partielles (EDP), alors que la dynamique des parties rigides est décrite par des équations aux dérivées ordinaires (EDO). Par conséquent, le modèle global est décrit par un ensemble mixte de EDO-EDP (m-EDO-EDP), qui est étudié dans cette thèse en utilisant l'approche hamiltonienne à ports combinée à la théorie des semi-groupes. Tout d'abord, nous définissons une procédure rigoureuse basée sur le principe de moindre action afin d'établir le modèle des mécanismes

avec d'éventuels composants flexibles, en fournissant plusieurs exemples illustratifs. Les parties à paramètres distribués sont modélisées comme des systèmes de contrôle frontière unidimensionnels. Dans un second temps, différentes lois de commande stabilisantes sont synthétisées sur une classe de systèmes m-EDP-EDO linéaires. Les lois de commande proposées permettent d'atteindre une stabilité asymptotique ou exponentielle. Enfin, nous nous intéressons au problème de contact entre un bras rotatif et son environnement dans le cas où le système en rotation est considéré comme étant rigide ou flexible. Puisque ce système présente des changements instantanés dans les temps d'impact, nous étudions ce problème à l'aide de la théorie de commutation appliquée à des systèmes de dimensions infinie.

Title : Modelling and stability analysis of flexible robots: a distributed parameter port-Hamiltonian approach

Keywords : Port Hamiltonian systems, Control design, Distributed parameter systems, Modelling

Abstract : The objective of this thesis is to provide a mathematical framework that allows to explicit the dynamical model of a class of flexible mechanisms, to design their control law and to analyze the resulting closed loop asymptotic behaviour. From a mathematical point of view, the flexible parts are distributed parameter systems whose dynamics are described by Partial Differential Equations (PDE), while the dynamics of the rigid parts are described by Ordinary Differential Equation (ODE). Therefore, the total model is described by a mixed set of ODE-PDE (m-PDE-ODE). For studying these dynamic models, this thesis uses the port-Hamiltonian framework combined with the infinite-dimensional semigroup theory. First, we define a rigorous procedure based on the Least Action Principle for deriving the model of mechanisms with possible flexible

components, providing several illustrative examples. The general class of nonlinear systems enclosing all the proposed examples is shown to be passive with respect to its mechanical energy. In this class of systems, the distributed parameter parts are modelled as one dimensional boundary control systems. Second, we restrict ourselves to a linear class of m-ODE-PDE systems for which we propose different control laws. We show that the proposed control laws allow achieving asymptotic or exponential stability. Finally, a rotating arm that enters in contact with the external environment is studied in case the link is considered as being both rigid or flexible. Since this system exhibits instant changes in the impact times, we study this problem with the help of switching theory applied to infinite dimensional systems.