Hypersurfaces cubiques spéciales

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Special cubic fourfolds

by

Hanine AWADA

"It's fine to work on any problem, so long as it generates interesting mathematics along the way – even if you don't solve it at the end of the day."

Andrew Wiles

Résumé

Le problème de la rationalité des hypersurfaces cubiques lisses dans $\mathbb{P}^5$ est un des problèmes les plus mystérieux en géométrie algébrique. On s'attend à ce que la cubique générale soit non rationnelle. Mais pour l'instant uniquement des exemples sporadiques d'hypersurfaces cubiques rationnelles sont connues. Dans cette thèse, nous nous intéressons surtout aux hypersurfaces cubiques spéciales c'est-à-dire des hypersurfaces cubiques contenant une surface algébrique non homologue à une intersection complète. Elles forment une union infinie dénombrable de diviseurs $\mathcal{C}_d$ (appelés diviseurs de Hassett) dans l'espace de modules des hypersurfaces cubiques $\mathcal{C}$.

Dans un premier temps, nous étudions l'intersection des diviseurs de Hassett $\mathcal{C}_d$ dans $\mathcal{C}$. Nos résultats sont obtenus en se basant sur des calculs de réseaux. Plus précisément, nous considérons des diviseurs qui paramètrent des hypersurfaces cubiques birationnelles à des fibrations en surfaces sur $\mathbb{P}^2$. La rationalité des hypersurfaces cubiques en question est équivalente à celle des surfaces sur le corps de fonctions de $\mathbb{P}^2$ et souvent cela dépend de l'existence de sections rationnelles des fibrations sur $\mathbb{P}^2$ associées. L'exemple le plus facile est donné par une fibration en surfaces quadriques. En intersectant ces diviseurs avec d'autres (qui paramètrent des cubiques rationnelles), nous construisons alors de nouvelles classes d'exemples d'hypersurfaces cubiques qui: sont birationnelles à des fibrations en surfaces sur $\mathbb{P}^2$, sont rationnelles, mais dont la fibration associée n'admet pas de sections. Par la suite, nous montrons que l'intersection de jusqu'à 20 diviseurs $\mathcal{C}_{d_k}$ dans l'espace de modules $\mathcal{C}$ est non vide sous certaines conditions sur les discriminants d_k. En s'appuyant sur ce dernier résultat et en utilisant certaines propriétés de surfaces K3 de rang de Néron-Severi 19, nous produisons dans chaque diviseur de Hassett $\mathcal{C}_d$ une infinité de familles de dimension 1 d'hypersurfaces cubiques, telles que leurs motifs de Chow sont de dimension finie et de type abélien. Nous obtenons un résultat semblable aussi pour certaines variétés de Hyperkähler associées aux cubiques.

Ensuite, nous considérons les familles universelles des hypersurfaces au-dessus des diviseurs $\mathcal{C}_d$. Nous proposons deux méthodes différentes pour démontrer l'unirationalité de ces familles universelles pour $8 \leq d \leq 42$. Finalement, nous étudions une autre classe de variétés de Fano, les variétés de Gushel-Mukai de dimension 4, et nous développons une méthode générale pour démontrer l'unirationalité de certains espaces de modules des variétés de GM de dimension 4 avec m points marqués.

Abstract

The rationality problem of smooth cubic hypersurfaces of dimension four (cubic fourfolds for short) is one of the most challenging open problem in algebraic geometry. In this thesis, we are interested in special cubic fourfolds, that is cubic fourfolds containing a surface which is not homologous to a complete intersection. These cubic fourfolds are parametrized by a countable union of divisors $\mathcal{C}_d$ in the moduli space of cubic fourfolds $\mathcal{C}$.

First, we work on the theory of intersection of these divisors $\mathcal{C}_d$ in $\mathcal{C}$. More precisely, we consider some classes of cubic fourfolds that are birational to fibrations over $\mathbb{P}^2$, where the fibers are rational surfaces. The rationality of these fibered cubic fourfolds is strongly related to the rationality of these surfaces over the function field of $\mathbb{P}^2$ and to the existence of rational sections of the associated fibration. By intersecting the divisors parametrizing these fibered cubic fourfolds with other ones whose elements are known to be rational, via lattice theory, we provide explicit description of these intersections in terms of irreducible components and we exhibit new examples of rational fibered cubic fourfolds such that the associated fibration doesn't have sections. Furthermore, we extend this intersection by giving necessary condition for up to 20 divisors to intersect. We apply this construction to build a countable infinity of one dimensional families of cubic fourfolds with finite dimensional Chow motives of abelian type inside every divisor $\mathcal{C}_d$. This also implies abelianity and finite dimensionality of the motive of certain related Hyperkähler varieties.

Then, we consider *universal* cubic fourfolds over certain divisors $\mathcal{C}_d$ in $\mathcal{C}$. We propose two methods to prove their unirationality, for the divisors $\mathcal{C}_d$, in the range $8 \leq d \leq 42$. Finally, we study another family of Fano fourfolds, *Gushel-Mukai fourfolds*, and develop a general method to show the unirationality of the moduli spaces of m-pointed (cubic or Gushel-Mukai) fourfolds.

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Contents

Chapter 1

Introduction

Les hypersurfaces cubiques lisses complexes dans $\mathbb{P}^5$ sont parmi les objets à la fois les plus mystérieux et les plus étudiés en géométrie algébrique. Ceci est dû à la compréhension, seulement partielle, qu'on a de leur géométrie birationnelle. Toutefois, pour des objets similaires de dimension inférieure, ce problème est résolu. En effet, une courbe cubique lisse complexe (courbe de genre 1) est non rationnelle. Ainsi, depuis le $19^{ème}$ siècle, on sait qu'une surface cubique lisse sur $\mathbb{C}$ est rationnelle. De plus, en 1972, Clemens et Griffiths [23] ont prouvé que les hypersurfaces cubiques lisses complexes de dimension 3 sont non rationnelles en utilisant la Jacobienne intermédiaire. Par contre, pour des hypersurfaces cubiques lisses de dimension 4, on n'admet pas encore de réponses claires concernant leur géométrie birationnelle. On s'attend à ce que la plupart des hypersurfaces cubiques soient non rationnelles mais pour l'instant aucun exemple irrationnel n'est connu. Pourtant, au fil du temps, multiples exemples d'hypersurfaces cubiques rationnelles ont été construits (voir [19, 46, 105, 106]). En particulier, les études de l'espace de modules $\mathcal{C}$ des hypersurfaces cubiques lisses et de l'application de période [65, 68, 109] en parallèle avec certaines études à travers la théorie de Hodge [46–48] et la théorie de catégories dérivées [59, 60] ont permis de mieux comprendre ces hypersurfaces cubiques et d'avancer dans l'étude de leur rationalité. Selon la célèbre conjecture, soutenue par le travail de Hassett et de Kuznetsov, une hypersurface cubique est rationnelle si et seulement si elle est associée à une surface K3. En effet, Hassett s'est demandé si toutes les hypersurfaces cubiques rationnelles sont *spéciales* et ont des surfaces K3 Hodge-théoriquement associées. En plus, Kuznetsov a conjecturé que toute hypersurface cubique $X \subseteq \mathbb{P}^5$ est rationnelle si et seulement si sa composante de Kuznetsov $Ku(X)$ de la categorie derivée bornée $\mathrm{D}^b(X)$ est derivée équivalente à une

surface K3, où $Ku(X)$ est l'orthogonale à droite de $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$ (dans ce cas, on dit que $Ku(X)$ est *géométrique*). Récemment, en se basant sur les conditions de stabilité sur $Ku(X)$, Bayer-Lahoz-Macrì-Nuer-Perry-Stellari [17, Corollary 29.7] ont prouvé qu'une hypersurface cubique *spéciale* X a une surface K3 Hodge-théoriquement associée si et seulement si sa composante de Kuznetsov $Ku(X)$ est *géométrique*.

Cette thèse est dédiée à l'étude des hypersurfaces cubiques lisses de dimension 4 en utilisant la théorie de Hodge, la géométrie projective et la théorie des réseaux. Le texte est composé de trois parties bien differenciées:
La première partie est consacrée aux définitions introductives et à l'évocation de certaines notions de réseaux et de théorie de Hodge en relation avec les hypersurfaces cubiques, cruciales pour la suite. Dans la deuxième partie, nous nous intéressons à l'intersection des diviseurs de Hassett $\mathcal{C}_d$ dans l'espace de modules $\mathcal{C}$. Nos résultats sont obtenus en se basant sur des calculs de réseaux. En effet, dans un premier temps, nous nous intéressons aux diviseurs contenant des hypersurfaces cubiques fibrées en surfaces rationnelles tels que $\mathcal{C}_8$ et $\mathcal{C}_{18}$. La rationalité des hypersurfaces cubiques en question est fortement liée à celle des fibres sur le corps de fonctions de $\mathbb{P}^2$ et à l'existence des sections rationnelles des fibrations sur $\mathbb{P}^2$ associées. En intersectant ces diviseurs avec d'autres tels que tous les hypersurfaces cubiques contenues dans ces diviseurs sont rationnelles comme $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ et $\mathcal{C}_{38}$, nous produisons alors de nouveaux exemples d'hypersurfaces cubiques fibrées rationnelles tout en décrivant l'intersection en terme de composantes irréductibles. En utilisant Macaulay2 [39], ces exemples sont illustrés explicitement. Cette partie correspond à l'article [9].
Dans un second temps, nous nous intéressons à l'intersection de plus de deux diviseurs. Toujours en s'appuyant sur des calculs de réseaux mais cette fois-ci d'une nouvelle manière introduite dans [112], nous montrons que l'intersection jusqu'à 20 diviseurs $\mathcal{C}_{d_k}$ dans l'espace de modules $\mathcal{C}$ est non vide sous certaines conditions sur les discriminants d_k:

$$\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset, \text{ pour } 1 \leq n \leq 20.$$

Ainsi, nous trouvons des lieux, de dimension $20 - n$, paramétrant des hypersurfaces cubiques avec des surfaces K3 associées d'un rang arbitraire $n \in \{1, .., 20\}$. En s'appuyant sur ce dernier résultat et en utilisant certaines propriétés de surfaces K3 de rang 19, nous produisons une infinité de familles d'hypersurfaces cubiques de dimension 1, contenues dans chaque diviseur de Hassett $\mathcal{C}_d$, telles que leurs motifs de Chow sont de dimension finie et de

type abélien. Cet exemple s'ajoute à d'autres vérifiant la conjecture de Kimura-O'Sullivan selon laquelle tous les motifs de Chow des variétés projectives lisses sont de dimension finie. Il s'agit de l'article [11].
Dans la troisième partie, nous nous intéressons toujours à la géométrie birationnelle des hypersurfaces cubiques de dimension 4 mais d'une manière différente. En effet, nous définissons les familles universelles $\mathcal{C}_{d,m}$ au-dessus des diviseurs $\mathcal{C}_d$. Nous démontrons que ces familles universelles sont unirationnelles pour $8 \leq d \leq 42$, pour tout m. Ceci avec deux méthodes différentes. L'une basée sur les familles universelles des surfaces K3 et la relation entre certaines hypersurfaces cubiques de dimension 4 et les surfaces K3 associées. L'autre est basée sur la représentation des diviseurs de Hassett en tant que cubiques contenant une certaine surface rationnelle. Ces résultats se trouvent dans l'article [10]. Ainsi, nous étudions une autre classe de variétés de Fano: les variétés de Gushel-Mukai de dimension 4. Bien que ces objets diffèrent des hypersurfaces cubiques, ils présentent quelques similarités concernant leur géométrie birationnelle. De même, en utilisant la théorie de Hodge, nous décrivons quelques lieux paramétrant des familles de variétés de GM de dimension 4 dans leur espace de modules $\mathcal{M}^4_{GM}$. Nous définissons alors les familles universelles au-dessus de certains de ces lieux dans $\mathcal{M}^4_{GM}$. Finalement, nous développons une méthode générale pour démontrer l'unirationalité de certains espaces de modules des variétés de GM (mais aussi des hypersurfaces cubiques) de dimension 4 $m-$pointées. Il s'agit de l'article [12].

Vers la fin, se trouvent trois Annexes contenant des résultats concernant des objets utilisés durant cette thèse. La première est consacrée aux surfaces K3 et à quelques résultats concernants leur théorie de Hodge et leurs familles universelles. Dans la deuxième Annexe, nous introduisons les motifs de Chow et nous évoquons quelques résultats connus concernant les motifs de Chow des hypersurfaces cubiques de dimension 4 et ceux des surfaces K3. Dans l'Annexe C, nous regroupons certaines généralités sur le variétés de GM de dimension 4. Enfin, l'ensemble des notations utilisées tout au long de ce texte sont regroupées dans la partie Notation.

Nous allons maintenant donner quelques détails de plus sur les différents chapitres.

1.1 Hypersurfaces cubiques spéciales

1.1.1 Définitions

On dénote par $\mathcal{C}$ l'espace de modules des hypersurfaces cubiques complexes lisses de dimension 4.

Hassett a défini ce qu'on appelle une hypersurface cubique *spéciale* X dans $\mathbb{P}^5$, correspondant à l'hypersurface cubique contenant une surface algébrique $\mathbf{S}$ non homologue à une intersection complète *i.e* non homologue à un multiple de h^2, avec $h = c_1(\mathcal{O}_X(1))$ la classe d'une section d'hyperplan. On définit alors pour cette hypersurface cubique le sous-réseau saturé de $H^{2,2}(X, \mathbb{Z})$, $K_d := \mathbb{Z}h^2 + \mathbb{Z}\mathbf{S}$ avec d le determinant (ou discriminant) de la forme d'intersection sur K_d. Les hypersurfaces cubiques *spéciales* forment une infinité dénombrable de diviseurs irréductibles $\mathcal{C}_d$ dans $\mathcal{C}$, correspondant à celles admettant un sous-réseau K_d de discriminant d, avec d un entier tel que $d \geq 8$ et $d \equiv 0, 2 \ (mod\ 6)$. Ces diviseurs $\mathcal{C}_d$ sont appelés les diviseurs de *Noether-Lefschetz* ou même simplement diviseurs de Hassett.
Pour certaines petites valeurs de d, les diviseurs $\mathcal{C}_d$ sont definis explicitement en terme de familles de surfaces, non uniques, contenues dans des éléments généraux de ces diviseurs. Ainsi, $\mathcal{C}_8$ est le lieu des hypersurfaces cubiques contenant un plan [47], $\mathcal{C}_{14}$ est la clôture du lieu des hypersurfaces cubiques contenant une surface réglée rationnelle lisse de degré 4 ou même, d'une façon équivalente, de ceux qui contiennent une surface del Pezzo de degré 5 (voir [19, 21])... Selon une conjecture due à Harris-Hassett-Kuznetsov, il devrait exister une infinité dénombrable de diviseurs de Hasssett paramétrant les hypersurfaces cubiques rationnelles.

1.1.2 Association de surfaces K3

Hassett a prouvé ainsi que pour un nombre infini de diviseurs de Hassett, on peut associer à des hypersurfaces cubiques, des surfaces K3. Plus précisement, la structure de Hodge sur la cohomologie non spéciale $K_d^{\perp}$ d'une hypersurface cubique X dans $\mathcal{C}_d$ est isomorphe à la structure de Hodge de la cohomologie primitive $H^2_{prim}(S, \mathbb{Z})$ d'une surface K3 polarisée S de degré d si et seulement si d n'est pas divisible par 4, 9, ou tout nombre premier impair $p \equiv 2 \ (mod\ 3)$. Selon les conjectures, la rationalité des hypersurfaces cubiques spéciales est liée fortement à l'association de surfaces K3. Pour l'instant, toutes les hypersurfaces

cubiques dans $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$ et $\mathcal{C}_{42}$ sont prouvées rationnelles, ce qui s'adapte parfaitement à la conjecture [13, 21, 91–93].

En plus, Hassett a montré que la variété de Fano de droites $F(X)$ associée à une hypersurface cubique générique X dans $\mathcal{C}_d$ est isomorphe à $S^{[2]}$, le schéma de Hilbert des sous-schémas de longueur deux d'une surface K3 S lorsque $d = 2(n^2 + n + 1)$ pour $n \geq 2$ (voir [46]). Ce résultat sera ultérieurement plus exploré (voir [1, 36]). Notamment, Addington [1, Theorem 2] a prouvé que $F(X)$ est birationnelle à $S^{[2]}$ si et seulement si $X \in \mathcal{C}_d$, avec d de la forme $\frac{2n^2+2n+2}{a^2}$ pour un certain $n,\ a \in \mathbb{Z}$.

1.2 Intersections des diviseurs de Hassett

L'intersection des diviseurs $\mathcal{C}_d$ dans l'espace de modules $\mathcal{C}$ est considérée dans [8] en étudiant l'intersection de deux diviseurs $\mathcal{C}_8$ et $\mathcal{C}_{14}$. Les auteurs [8] ont alors fourni des exemples d'hypersurfaces cubiques rationnelles contenant un plan et une surface del Pezzo de degré 5. En se basant sur des calculs de réseaux (voir [86] ou [51, Chap. 14] pour plus de details sur la théorie des réseaux) et les travaux de Laza [65], Looijenga [68], Mayanskiy [71] et Voisin [109], ils ont décrit l'intersection en terme de composantes irréductibles non vides. En outre, Addington [3, Theorem 4.1] a montré que $\mathcal{C}_8$ intersecte tout diviseur de Hassett $\mathcal{C}_d$. Dernièrement, Yang and Yu [112] ont prouvé que l'intersection de n'importe quels deux diviseurs $\mathcal{C}_{d_1}$ et $\mathcal{C}_{d_2}$ dans $\mathcal{C}$ est non vide. Par conséquent, ils en ont déduit que tous les diviseurs de Hassett $\mathcal{C}_d$ contiennent l'union de trois sous-variétés de codimension deux dans $\mathcal{C}$ paramétrant des hypersurfaces cubiques rationnelles, correspondant aux intersections avec les diviseurs $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ et $\mathcal{C}_{38}$ [112, Theoreme 3.3]. Très récemment, ils ont démontré que l'intersection de tous les diviseurs de Hassett est non vide en utilisant les hypersurfaces cubiques polarisables (voir [113]).

1.2.1 Hypersurfaces cubiques fibrées rationnelles

Récemment, dans [9], on considère certaines classes d'hypersurfaces cubiques birationnelles à des fibrations en surfaces rationnelles sur $\mathbb{P}^2$ en les intersectant avec d'autres classes d'hypersurfaces cubiques rationnelles.

On considère d'abord les hypersurfaces cubiques dans $\mathcal{C}_8$ contenant un plan. La projection du plan établit une stucture de fibrations de surfaces quadriques sur $\mathbb{P}^2$. Hassett [47] a identifié un lieu dans $\mathcal{C}_8$ correspondant à des sous-variétés de codimension 2 dans $\mathcal{C}$ paramé-

trant les hypersurfaces cubiques rationnelles. Il a prouvé que la trivialité de l'*invariant de Clifford* des cubiques contenant un plan implique leur rationalité. On appelle "*l'invariant de Clifford*" la classe de Brauer $\alpha \in Br(S)$ d'une fibration en $\mathbb{P}^1$ au dessus d'une surface K3 S lisse de degré 2. Cette fibration en $\mathbb{P}^1$ est le schéma de Hilbert $\mathcal{H}(0,1)$ des droites dans les fibres de la fibration en quadriques. En effet, la trivialité de l'*invariant de Clifford* est équivalente à l'existence des sections rationnelles de la fibration. Dans [9], nous considérons des exemples d'hypersurfaces cubiques rationnelles contenant un plan tels que la fibration associée n'a pas de section rationnelle et donc avec la classe de Brauer α non triviale en étudiant l'intersection de $\mathcal{C}_8$ avec $\mathcal{C}_{26}$ et $\mathcal{C}_{38}$. Ainsi, nous décrivons explicitement les composantes irréductibles dans les intersections et nous nous assurons de la trivialité ou pas de la classe de Brauer pour chacune des composantes:

Théorème I-1: *L'intersection $\mathcal{C}_8 \cap \mathcal{C}_{26}$ dans l'espace de modules $\mathcal{C}$ a huit composantes irréductibles indexées par le discriminant $d_\tau \in \{29, 36, 48, 53, 61, 64, 68, 69\}$ de la forme d'intersection des éléments génériques dans chaque composante.*
Quatre de ces composantes contiennent des hypersufaces cubiques rationnelles telles que la fibration associée n'a pas de sections.

Théorème I-2: *L'intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$ dans l'espace de modules $\mathcal{C}$ a dix composantes irréductibles indexées par le discriminant $d_\tau \in \{36, 45, 61, 68, 80, 85, 93, 96, 100, 101\}$ de la forme d'intersection des éléments génériques dans chaque composante.*
Cinq de ces composantes contiennent des hypersufaces cubiques rationnelles telles que la fibration n'a pas de sections.

Dans [9], on étudie ainsi d'autres exemples d'hypersurfaces cubiques fibrées rationnelles. Ces hypersurfaces cubiques vivent dans $\mathcal{C}_{18}$. En effet, l'hypersurface cubique générique X dans $\mathcal{C}_{18}$ contient une surface réglée elliptique T de degré 6 (voir [4]). Le système linéaire de quadriques à travers T définit une application rationnelle $X \dashrightarrow \mathbb{P}^2$ dont les fibres sont des surfaces del Pezzo de degré 6. En particulier, X est birationnelle à une fibration en surfaces del Pezzo de degré 6. Pour une fibration "good[1]", on peut associer des classes de Brauer comme suit: on appelle $\mathcal{H}(0,3)$ et $\mathcal{H}(0,2)$ les schémas de Hilbert sur $\mathbb{P}^2$ paramétrant les

[1]La notion de fibration " du Val" en surfaces del Pezzo de degré six est une généralisation de la notion "good" [58].

courbes connexes de genre 0 avec un anticanonique de degré respectivement 3 et 2; $\mathcal{H}(0,3)$ est une fibration en $\mathbb{P}^2$ au-dessus d'une surface K3 S de degré 2 et $\mathcal{H}(0,2)$ est une fibration en $\mathbb{P}^1$ au-dessus d'un triple revêtement d'une surface non singulière Z sur $\mathbb{P}^2$; On considère alors les deux classes de Brauer $\beta \in Br(S)$ associée à $\mathcal{H}(0,3)$ au-dessus de S et $\gamma \in Br(Z)$, celle associée à $\mathcal{H}(0,2)$ au-dessus de Z sur $\mathbb{P}^2$. La trivialité des classes de Brauer β et γ est équivalente à l'existence d'une 2-section ou 3-section de la fibration sur $\mathbb{P}^2$ (voir [58]). Dans [4], les hypersurfaces cubiques dans $\mathcal{C}_{18}$ données sont rationnelles quand la fibration a une section rationnelle *i.e* quand les deux classes de Brauer sont triviales. En intersectant $\mathcal{C}_{18}$ avec $\mathcal{C}_{14}$ (et d'autres diviseurs tels que $\mathcal{C}_{26}$ et $\mathcal{C}_{38}$), nous montrons que cette condition n'est pas nécessaire pour la rationalité de la cubique dans $\mathcal{C}_{18}$, en fournissant des exemples d'hypersurfaces cubiques rationnelles telles que la fibration "good" en surfaces del Pezzo de degré 6 associée n'a pas de sections.

Théorème II: *Il existe des hypersurfaces cubiques rationnelles dans $\mathcal{C}_{18}$ fibrées en "good" surfaces del Pezzo de degré 6 telles que les classes de Brauer associées ne sont pas toutes les deux triviales.*

Ainsi, on donne des descriptions complètes des composantes irréductibles de l'intersection entre $\mathcal{C}_{18}$ et $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$. On s'abstient de les décrire ici mais elles se trouvent dans les Théorèmes 3.2.10, 3.2.11 et 3.2.12. Pour chaque composante, on étudie aussi la trivialité ou pas des deux classes de Brauer.

Des exemples explicites d'hypersurfaces cubiques rationnelles fibrées en surfaces quadriques ou surfaces del Pezzo de degré 6 sont ensuite construites à l'aide de Macaulay2 [39].

1.2.2 Familles d'hypersurfaces cubiques avec des motifs de Chow de dimension finie et de type abélien

Suite au résultat de Yang et Yu [112] confirmant l'intersection non vide de n'importe quels deux diviseurs de Hassett dans l'espace de modules $\mathcal{C}$, il semble naturel de se poser la question sur l'intersection de plusieurs diviseurs dans $\mathcal{C}$. En faisant des calculs de réseaux, dans [11], nous généralisons le résultat de [112]. En particulier, en fixant des conditions arithmétiques sur 20 discriminants $d_1, .., d_{20}$, nous montrons que jusqu'à vingt diviseurs

s'intersectent dans l'espace de modules $\mathcal{C}$. De plus, nous montrons qu'on peut construire des familles d'hypersurfaces cubiques, contenues dans ces intersections de diviseurs, associées à des surfaces K3 S de rang arbitraire $1 \leq \rho(S) \leq 20$.

Théorème III:

(i) *Pour* $3 \leq n \leq 20$,

$$\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset,$$

avec $d_k \geq 8$, $d_k \equiv 0, 2 \ (mod\ 6)$ *et* $d_3, .., d_n = 6\ d'$ *ou* $6\ d' + 2$, d' *un nombre carré.*

(ii) Il existe des familles d'hypersurfaces cubiques $\mathcal{F} \subseteq \bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$, *de dimension* $20 - n$, *telles que pour une hypersurface cubique générique* $X \in \mathcal{F}$, $rang(H^2(X, \Omega_X^2) \cap H^4(X, \mathbb{Z})) = n + 1$.

(iii) En plus, si au moins un des $\mathcal{C}_{d_k}$ *de l'intersection contient des hypersurfaces cubiques avec une surface K3 associée, alors les hypersurfaces cubiques de* $\mathcal{F}$ *ont une surface K3 associée de rang supérieur ou égal à* n.

Ainsi, en utilisant les résultats de Bolognesi et Pedrini [20] et de Bülles [22] sur la décomposition de Chow-Künneth du motif de Chow des hypersurfaces cubiques (voir Annexe B), nous arrivons à mettre en relation la dimension finie et l'abélianité du motif de Chow de $X \in \mathcal{F}$, $h(X)$, aux mêmes propriétés de celui de la surface K3 associée. Finalement, en combinant ces résultats avec certaines propriétés des surfaces K3 d'un rang $\rho \geq 19$ (voir [89]), nous obtenons le théorème suivant:

Théorème IV: *Chaque diviseur de Hassett contient une infinité dénombrable de familles de dimension 1 d'hypersurfaces cubiques telles que leurs motifs de Chow sont de dimension finie et de type abélien.*

Ce résultat se trouve dans [11]. Ainsi, nous considérons les variétés d'Hyperkähler suivantes, associées à une hypersurface cubique X contenue dans ces familles construites: $F(X)$

la variété de Fano de droites sur X et $L(X)$ la variété de dimension 8 construite à partir de l'espace de modules des courbes cubiques twistées sur X ne contenant pas un plan (*i.e* $X \notin \mathcal{C}_8$) (voir [67]). Nous montrons que ces variétés d'Hyperkähler associées à X ont des motifs de Chow finiment générés et abéliens aussi.

1.3 Familles universelles d'hypersurfaces cubiques

Les familles universelles sont bien exploitées dans le cadre des espaces de modules des courbes [45]. De même, dans le contexte des surfaces K3, des résultats multiples sont apparus concernant leurs familles universelles, notamment récemment dans [15, 34, 35, 70]. Nous avons alors étudié ces objets pour l'espace de modules des hypersurfaces cubiques de dimension 4. En effet, vu que les cubiques génériques dans les diviseurs $\mathcal{C}_d$ n'admettent pas des automorphismes projectifs (voir [37, Thm 3.8]), on peut alors définir des familles universelles d'hypersurfaces cubiques $\mathcal{C}_{d,1} \to \mathcal{C}_d$ au moins sur des ouverts de chaque diviseur de Hassett $\mathcal{C}_d$. Ces familles universelles correspondent aux espaces de modules des hypersurfaces cubiques 1-pointées. Nous montrons, dans [10], le suivant:

Théorème V: *Les familles universelles d'hypersurfaces cubiques $\mathcal{C}_{d,1}$ sont unirationnelles, pour $8 \leq d \leq 42$.*

Ainsi inductivement, nous déduisons que:

Théorème VI: *Les familles universelles d'hypersurfaces cubiques $\mathcal{C}_{d,m}$ sont unirationnelles pour $8 \leq d \leq 42$, pour tout m.*

De plus, en utilisant des résultats récents sur la non-négativité de la dimension de Kodaira des espaces de modules des surfaces K3 pointées (voir [42, 103]) et leur relation avec les hypersurfaces cubiques, on s'assure que pour une infinité de valeurs de d, les familles universelles $\mathcal{C}_{d,1}$ ne sont pas unirationnelles:

Proposition VII: *Les familles universelles des hypersurfaces cubiques $\mathcal{C}_{d,1}$ ne sont pas unirationnelles si:*

1. $d > 80,\ d \equiv 2\ (mod\ 6),\ 4 \not| d$ *et tel que pour tout nombre premier impair* $p,\ p|d$, *on a* $p \equiv 1\ (mod\ 3)$;

2. $d = 6n + 2,\ n > 13$ *et* $n \neq 15$;

3. $d = 6n$ *pour* $n > 16$ *et* $n \neq 18, 20, 22, 30$.

1.4 Variétés de Gushel-Mukai

Ensuite nous nous sommes intéressés à d'autres variétés de Fano, notamment les variétés lisses de Gushel-Mukai (variétés de GM dans la suite). D'après un résultat de Mukai [80], ces variétés Y de dimension n sont données par une intersection dimensionnellement transverse d'un cône sur la Grassmannienne avec un espace linéaire et une hypersurface quadrique:

$$Y = C\mathbb{G}(1,4) \cap \mathbb{P}(W) \cap Q,$$

avec W un sous-espace vectoriel de dimension n+5 et Q une hypersurface quadrique dans $\mathbb{P}(W)$. Elles sont de dimension au plus 6.

Nous abordons dans cette thèse plus précisément les variétés de GM lisses de dimension 4. Elles sont des variétés de Fano de degré 10 et d'indexe 2. Elles peuvent être réalisées comme indiqué précédemment par une intersection dimensionnellement transverse d'un cône $C\mathbb{G}(1,4) \subset \mathbb{P}^{10}$ sur la Grassmannienne $\mathbb{G}(1,4) \subset \mathbb{P}^9$ avec un sous-espace linéaire $\mathbb{P}^8 \subset \mathbb{P}^{10}$ et une hypersurface quadrique $Q \subset \mathbb{P}^{10}$. Ces variétés de GM de dimension 4 existent en deux types; soit elles sont isomorphes à un revêtement double d'une section linéaire de $\mathbb{G}(1,4)$ branchée sur une section quadrique (appelées de type Gushel), soit isomorphes à une section quadrique d'une section linéaire de $\mathbb{G}(1,4)$ (appelées ordinaires[2]). Ces dernières peuvent être vues comme des hypersurfaces quadriques lisses dans une variété de del Pezzo lisse de dimension 5, $\mathbb{Y}^5 := \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$. Elles sont paramétrées par un ouvert V dans l'espace projectif $\mathbb{P}(\mathcal{O}_{\mathbb{Y}^5}(2))$ de dimension 39. En effet, les variétés de GM ordinaires forment un ouvert dans $\mathcal{M}^4_{GM}$, l'espace de modules des variétés de GM de dimension 24. Cet ouvert est complémentaire à un fermé irréductible de codimension 2 dans $\mathcal{M}^4_{GM}$, lieu des GM de type Gushel.

[2]Celles pour lesquelles le sous-espace $\mathbb{P}^8 \subset \mathbb{P}^{10}$ ne contient pas le sommet du cône.

1.4.1 Théorie de Hodge des variétés de GM

Selon le point de vue de la géometrie birationnelle, les variétés de GM de dimension 4 se comportent largement de manière semblable aux hypersurfaces cubiques dans $\mathbb{P}^5$ et ont beaucoup de propriétés en commun. Elles sont encore une fois toutes unirationnelles, les exemples rationnels sont simples à construire, mais aucun exemple nonrationnel est connu. Encore, à travers l'étude de la théorie de Hodge, dans [26], les auteurs ont introduit le lieu de *Noether-Lefschetz* dans l'espace de modules $\mathcal{M}^4_{GM}$ correspondant à l'ensemble des variétés de GM de dimension 4 *spéciales i.e* contenant une surface telle que sa classe de cohomologie ne vient pas de la Grassmannienne $\mathbb{G}(1,4)$. D'une façon équivalente, on dit qu'une variété de GM de dimension 4, $Y \in \mathcal{M}^4_{GM}$ est *spéciale* si et seulement si $\text{rang}(H^{2,2}(Y) \cap H^4(Y,\mathbb{Z})) \geq 3$. Ces variétés sont paramétrées par une réunion infinie dénombrable d'hypersurfaces $\bigcup_d (\mathcal{M}^4_{GM})_d \subset \mathcal{M}^4_{GM}$, indexées par des entiers $d \geq 10$ avec $d \equiv 0, 2, 4 \ (mod\ 8)$. Si $d \equiv 0 \ (mod\ 4)$, l'hypersurface $(\mathcal{M}^4_{GM})_d$ est irréductible, alors que si $d \equiv 2 \ (mod\ 8)$, c'est la réunion de deux composantes irréductibles $(\mathcal{M}^4_{GM})'_d \cup (\mathcal{M}^4_{GM})''_d$. Pour une Y très générale dans $(\mathcal{M}^4_{GM})_d$, le réseau $H^{2,2}(Y) \cap H^4(Y,\mathbb{Z})$ est de rang 3 et de discriminant d.

Pour certaines valeurs de d, $Y \in (\mathcal{M}^4_{GM})_d$ a une surface K3 associée de degré d; pour d'autres, elles ont des hypersurfaces cubiques de dimension 4 associées de discriminant d (voir [26, Section 6.2, 6.3]). La notion d'association de surfaces K3 mène à une conjecture similaire à celle de Hassett et Kuznetsov pour la rationalité des hypersurfaces cubiques; Y est rationelle si et seulement si elle a une surface K3 associée [48, 59, 60, 93, 102]. Les premiers composantes irréductibles du lieu de *Noether-Lefschetz* dans $\mathcal{M}^4_{GM}$ sont: $(\mathcal{M}^4_{GM})'_{10}$, $(\mathcal{M}^4_{GM})''_{10}$, $(\mathcal{M}^4_{GM})_{20}$.. Toutes les variétés de GM dans ces composantes sont rationnelles (voir [26, 49, 90]).

1.4.2 Familles universelles de variétés de GM

Dans le même esprit que pour les hypersurfaces cubiques de dimension 4, nous considérons les variétés de GM de dimension 4 universelles au-dessus de certains diviseurs dans $\mathcal{M}^4_{GM}$. Notamment pour certaines familles nous prouvons que leur familles universelles sont unirationnelles. En effet, pour certaines familles de GM de dimension 4, des surfaces K3 apparaissent dans leurs constructions géométriques. C'est le cas par exemple de $(\mathcal{M}^4_{GM})_{20}$, lieu des variétés de GM contenant une surface rationnelle de degré 9 et de genre sectionnel

2. En ce basant sur leur construction géométrique, nous prouvons ainsi, dans [12], que:

Proposition VIII: *La famille universelle* $(\mathcal{M}^4_{GM})_{20,1}$ *est unirationnelle.*

Ainsi, nous nous sommes intéressés à $(\mathcal{M}^4_{GM})^{nod}_{20}$, un lieu contenu dans $(\mathcal{M}^4_{GM})_{20}$ et defini à partir de certaines surfaces K3 de genre 11. Nous démontrons alors cette fois-ci la rationalité de $(\mathcal{M}^4_{GM})^{nod}_{20}$ et de leur famille universelle $(\mathcal{M}^4_{GM})^{nod}_{20,1}$.

Finallement, nous proposons une méthode générale s'appliquant sur les hypersurfaces cubiques et les variétés de GM de dimension 4 pour démontrer l'unirationalité de certaines de ces familles universelles $m-$pointées au-dessus de leurs espaces de modules.

Théorème IX: *Les espaces de modules des familles* $m-$*pointées au-dessus des lieux suivants sont unirationnels:*

1. $\mathcal{C}_{14}$*: hypersurfaces cubiques contenant une surface del Pezzo de degré 5;*
2. $\mathcal{C}_{26}$*: hypersurfaces cubiques contenant une surface réglée de degré 7 avec 3 noeuds;*
3. $\mathcal{C}_{38}$*: hypersurfaces cubiques contenant une surface de Coble généralisée;*
4. $\mathcal{C}_{42}$*: hypersurfaces cubiques contenant une surface rationnelle de degré 9 et de genre sectionnel 2;*
5. $(\mathcal{M}^4_{GM})'_{10}$*: variétés de GM contenant une surface* τ*-quadrique;*
6. $(\mathcal{M}^4_{GM})''_{10}$*: variétés de GM contenant une surface del Pezzo de degré 5;*
7. $(\mathcal{M}^4_{GM})_{20}$*: variétés de GM contenant une surface rationnelle de degré 9 et de genre sectionnel 2;*

Chapter 2

General facts on cubic fourfolds

2.1 Introduction

Cubic fourfolds are smooth cubic hypersurfaces $X \subseteq \mathbb{P}^5$. Despite the simplicity of their definition, they are among the most mysterious objects in algebraic geometry. This is related to the difficulty to understand their birational geometry. It is expected that very general cubic fourfolds are nonrational but, until now, only examples of rational cubic fourfolds are known.

Over time, many techniques were deployed successfully to understand more the birational geometry of cubic fourfolds: Hassett [46, 47] studied cubic fourfolds via Hodge theory sense, Kuznetsov [59] developed the derived category theory of cubic fourfolds. Similar conjectures under both point of views were made toward the rationality of cubic fourfolds. Hassett wondered whether all rational cubic fourfolds were *special* and have Hodge-theoretically associated K3 surfaces (see [46–48]). In addition, Kuznetsov established a semiorthogonal decomposition of the bounded derived category of X,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle Ku(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle.$$

He conjectured that a smooth cubic fourfold X is birational to $\mathbb{P}^4$ if and only its Kuznetsov component $Ku(X)$ is derived equivalent to a K3 surface S *i.e.* $Ku(X) \cong \mathrm{D}^{\mathrm{b}}(S)$ for some K3 surface S. In this case, we say that $Ku(X)$ is *geometric* (see also [3]). Addington-Thomas [1, Theorem 1.1] showed that for a cubic fourfold X, if $Ku(X)$ is *geometric* then X is special and has a Hodge-theoretically associated K3 surface. Conversely, they proved that these sets

of cubics described by Hassett contain a Zariski open dense subset of cubics X admitting the derived equivalence (for which $Ku(X)$ is *geometric*) (see [52] for the twisted version). Recently, based on the stability conditions on $Ku(X)$, in [17, Corollary 29.7], Bayer-Lahoz-Macrì-Nuer-Perry-Stellari proved that a cubic fourfold X has a Hodge-theoretically associated K3 surface if and only if its Kuznetsov component $Ku(X)$ is *geometric*.

In the following we collect some generalities on cubic fourfolds and known results related to their Hodge and lattice theories.

2.1.1 Basic definitions

Throughout, we work over the complex field $\mathbb{C}$.

Definition 2.1.1. *Let X be a smooth projective variety of dimension n.*

We say that X is rational if there exists a birational map $\mathbb{P}^n \dashrightarrow X$, i.e. there exist $U \subseteq X$, $V \subseteq \mathbb{P}^n$ open subsets, such that U is isomorphic to V. This is equivalent to say that the function field $\mathbb{C}(X)$ is isomorphic to the extension $\mathbb{C}(x_1, .., x_n)$.

We say that X is unirational if there exists a dominant rational map $\mathbb{P}^N \dashrightarrow X$, for an arbitrary N. This is equivalent to have the function field of X, $\mathbb{C}(X)$, contained in a purely transcendental field extension of $\mathbb{C}$.

There are some birational invariants for varieties that are birationally equivalent. We mention for example the plurigenera $P_X^d = \dim H^0(X, K_X^d)$ for $d \geq 0$ with K_X the canonical bundle of X, the Kodaira dimension, the fundamental group $\pi_1(X)$, some Hodge numbers (as $h^{p,0} = H^0(X, \Omega^p)$ where Ω^p is the sheaf of $p-$forms)...

2.1.2 Moduli space of cubic fourfolds

Cubic hypersurfaces of dimension 4 are defined by an equation

$$\mathbf{P}(x_1, x_2, x_3, x_4, x_5, x_6) = 0$$

where $\mathbf{P} \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ is a degree three homogeneous polynomial.
Cubic hypersurfaces of dimension four are parametrized by $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3))) \cong \mathbb{P}^{55}$. The smooth ones form a Zariski open dense subset $\mathcal{U} \subset \mathbb{P}^{55}$.

The invariant theory of cubic hypersurfaces, with respect to the natural $PGL(6)$-action, is known by the work of Laza [65, 66].

Theorem 2.1.2. *[66, Thm. 1.1] A cubic hypersurface $X \subset \mathbb{P}^5$ is not GIT stable if and only if one of the following conditions holds:*

1. *X is singular along a curve C spanning a linear subspace of $\mathbb{P}^5$ of dimension at most 3;*
2. *X contains a singularity that deforms to a singularity of class P_8, X_9 or J_{10}.*

In particular, if X is a cubic hypersurface with isolated singularities, then X is stable if and only if X has at worst simple singularities.

Since we work with smooth cubic hypersurfaces in $\mathbb{P}^5$, the *GIT* quotient that we will consider is quiet different than the one constructed by Laza [66]. In fact, it is known that two cubic fourfolds are isomorphic if and only if they are projectively equivalent by the action of $PGL(6, \mathbb{C})$. As a result, the moduli space of cubic fourfolds $\mathcal{C}$ can be described as the *GIT* quotient

$$\mathcal{C} := \mathcal{U}//PGL(6, \mathbb{C}).$$

Actually, the moduli stack of cubic fourfolds is a Deligne-Mumford stack with a coarse moduli space $\mathcal{C}$, which is a quasi-projective variety of dimension $55 - 35 = 20$ (see [83, Chapter 4.2]).

2.2 Lattice and Hodge theory for cubic fourfolds

We recall that the Hodge decomposition holds for smooth projective varieties over $\mathbb{C}$ (for Kähler manifolds in general) as follows:

$$H^k(X, \mathbb{Z}) \otimes \mathbb{C} \cong \bigoplus_i H^{i,k-i}(X), \quad H^{i,k-i}(X) := H^{k-i}(X, \Omega_X^i)$$

with two structures on this decomposition:

1. a conjugation action with $H^{i,k-i} = \overline{H^{k-i,i}}$;
2. a non degenerate integral quadratic form on $H^k(X, \mathbb{Z})$.

In our case, the cohomology of a cubic fourfold X, $H^*(X,\mathbb{Z})$, is well understood and torsion-free (see [19]). The Hodge diamond of X has the following form:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

We focus on the middle cohomology of X, containing some nontrivial Hodge theoretic information. The Hodge-Riemann bilinear relations imply that $H^4(X,\mathbb{Z})$ is a unimodular lattice under the intersection form $(-,-)$ of signature (21,2). Let L be the *cohomology lattice* corresponding to $H^4(X,\mathbb{Z})$, and $L_{prim} = H^4_{prim}(X,\mathbb{Z}) := \langle h^2\rangle^\perp := \{v \in H^4(X,\mathbb{Z}) \mid (v,h^2) = 0\}$ the *primitive cohomology lattice*, where $h \in H^2(X,\mathbb{Z})$ is the *hyperplane class* defined by the embedding $X \subset \mathbb{P}^5$.
Note that L_{prim} is an even lattice (see [46, §2]).

Furthermore, using the Abel-Jacobi map that shows the correspondence between X and the Fano variety of lines $F(X)$ on the Hodge structure level (see [19] or §2.3.3 for more details about $F(X)$):

$$\theta : \ L \longrightarrow H^2(F(X),\mathbb{Z})\ (-1)^1$$

and the induced isomorphism:

$$L_{prim} \longrightarrow H^2(F(X),\mathbb{Z})_{prim}(-1)\ ,$$

Hassett [46, Proposition 2.1.2] computed explicitly the *cohomology lattice* for cubic fourfolds as follows:

$$\begin{aligned} L := H^4(X,\mathbb{Z}) \cong & \quad E_8^{\oplus 2} \bigoplus U^{\oplus 2} \bigoplus I_{3,0} \\ \cong & \quad (+1)^{\oplus 21} \bigoplus (-1)^{\oplus 2}, \end{aligned}$$

[1] $H^2(F(X),\mathbb{Z})$ is equipped with the *Beauville-Bogomolov* form.

and the *primitive cohomology lattice*

$$L_{prim} := H^4(X,\mathbb{Z})_{prim} := \langle h^2 \rangle^{\perp} \cong E_8^{\oplus 2} \bigoplus U^{\oplus 2} \bigoplus A_2,$$

where:
$I_{3,0}$ is the rank 3 lattice of signature (3,0) whose intersection matrix is the identity matrix of rank 3 and h the hyperplane class defined by

$$h^2 = (1,1,1) \in I_{3,0},$$

E_8 is the unimodular positive definite even rank 8 lattice associated to the corresponding Dynkin diagram defined by the following matrix:

$$E_8 := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

U is the hyperbolic plane defined by $U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the rank 2 lattice $A_2 := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Moreover, we denote by

$$A(X) = H^{2,2}(X) \cap H^4(X,\mathbb{Z}) = H^2(X,\Omega_X^2) \cap H^4(X,\mathbb{Z})$$

the lattice of integral middle Hodge classes of X; it is a positive definite lattice by the Riemann bilinear relations; and since the (integral) Hodge conjecture [2] holds for cubic fourfolds (see [111, Theorem 18] or [73, Corollary 0.3] for a recent proof), it coincides with $CH^2(X)$ the Chow group of codimension 2 cycles on X up to rational equivalence (see [24, §5]). Moreover, thanks to Voisin [109, Proposition 1], $A(X)$ has no roots (i.e., $\nexists\, v \in A(X)$

[2] The algebraic cohomology of a cubic fourfold is generated by the classes of algebraic cycles.

with $(v.v) = 2$). We denote by $d(A(X)) \in \mathbb{Z}$ its discriminant, the determinant of its Gram matrix.

These lattices will be useful for next Chapters.

Theorem 2.2.1 (Torelli theorem for cubic fourfolds, [68, 109])**.** *Let X, X' be cubic fourfolds. Suppose there is an isomorphism of Hodge structures $\psi : H^4(X, \mathbb{C}) \xrightarrow{\simeq} H^4(X', \mathbb{C})$, Then X and X' are isomorphic.*

2.3 Special cubic fourfolds

For a very general cubic fourfold X, any algebraic surface $\mathbf{S} \subset X$ is homologous to a multiple of h^2, so that $\mathrm{rk}(A(X)) = 1$ as follows

$$A(X) \cong \mathbb{Z}h^2 \text{ and } [\mathbf{S}] \equiv nh^2,\ n = \frac{deg(\mathbf{S})}{3}.$$

However, Hassett [46] defined *special* cubic fourfolds corresponding to cubic fourfolds containing an algebraic surface $\mathbf{S}$ which is not homologous to a complete intersection *i.e* the cohomology class of $\mathbf{S}$ is linearly independent of h^2. Then, as a consequence of the Hodge conjecture, one can say that a cubic fourfold X is *special* if and only if $A(X)$ has rank at least 2.

Definition 2.3.1. *A labelling K_d of a special cubic fourfold is a rank 2 saturated (i.e. the quotient group $A(X)/K_d$ is torsion free) sublattice such that*

$$h^2 \in K_d \subseteq A(X).$$

The discriminant d of a labelling $K_d := \langle h^2, \mathbf{S} \rangle$ with $\mathbf{S}$ the algebraic surface of degree $d_{\mathbf{S}}$, is the determinant of the intersection form on K_d:

$$K_d := \begin{array}{c|cc} & h^2 & \mathbf{S} \\ \hline h^2 & 3 & d_{\mathbf{S}} \\ \mathbf{S} & d_{\mathbf{S}} & (\mathbf{S},\mathbf{S}) \end{array},$$

such that the self-intersection is given by the highest Chern class of the normal bundle to $\mathbf{S}$ in X:

$$(\mathbf{S},\mathbf{S}) = c_2(\mathcal{N}_{\mathbf{S}/X}) = 6\mathbf{h}^2 + 3\mathbf{h}K_{\mathbf{S}} + K_{\mathbf{S}}^2 - \chi_{\mathbf{S}},$$

where $\mathbf{h} = h|_{\mathbf{S}}$, $K_{\mathbf{S}}$ is the canonical divisor and $\chi_{\mathbf{S}}$ denotes the Euler characteristic of $\mathbf{S}$.

A cubic fourfold X is called special (of discriminant d) if X is associated to a labelling K_d.

Theorem 2.3.2 ([46]). *Let $\mathcal{C}_d \subset \mathcal{C}$ denote the special cubic fourfolds of discriminant d. Then $\mathcal{C}_d$ is an irreducible nonempty divisor if and only if*

$$(*) \qquad d \geq 8 \text{ and } d \equiv 0, 2 \ (mod\ 6).$$

Special cubic fourfolds form a countably infinite union of irreducible divisors $\mathcal{C}_d \subset \mathcal{C}$, admitting labellings K_d. These divisors $\mathcal{C}_d$ are called *Noether-Lefschetz* divisors (or for short Hassett divisors).
Only few $\mathcal{C}_d$ can be defined explicitly in terms of particular surfaces, not unique, contained in a cubic fourfold $X \in \mathcal{C}_d$. In fact, Hassett provided first explicit descriptions of the generic members of the divisors $\mathcal{C}_d$ for $d \leq 20,\ d \neq 18$ (see [46, 47]). Later, Nuer [87] described explicitly the generic members of $\mathcal{C}_d$ for $12 \leq d \leq 38$ and $d = 42$ using the deformation theory of Hilbert schemes of flags, semicontinuity arguments and explicit Macaulay2 calculations. Recently, Russo and Staglianò [92, 93] gave an alternative descriptions in terms of surfaces contained in generic elements of $\mathcal{C}_d$ for $d = 14,\ 26,\ 38$ and 42. In the following, we will give more details on the descriptions of certain divisors $\mathcal{C}_d$ that will appear in next Chapters.

2.3.1 Examples of special cubic fourfolds

We describe in the following table some examples of divisors $\mathcal{C}_d$ in $\mathcal{C}$.

Divisor $\mathcal{C}_d \subset \mathcal{C}$	Descriptions	Labelling K_d
$\mathcal{C}_8$	$\mathcal{C}_8$ is usually described as the locus of cubic fourfolds containing a plane P.	$K_8 :=$ |	h^2	P | | h^2 | 3 | 1 | | P | 1 | 3 |
$\mathcal{C}_{12}$	$\mathcal{C}_{12}$ corresponds to cubic fourfolds containing a rational normal cubic scroll Σ_3.	$K_{12} :=$ |	h^2	Σ_3 | | h^2 | 3 | 3 | | Σ_3 | 3 | 7 |
$\mathcal{C}_{14}$	Cubic fourfolds containing quartic scrolls, quintic scrolls or quintic del Pezzo surfaces form the divisor $\mathcal{C}_{14}$ [19, 46]. Later, we will denote by Σ_4 the class of a rational quartic scroll or a degeneration of such a surface (see [21] or [91, §2] for new surfaces defining $\mathcal{C}_{14}$).	$K_{14} :=$ |	h^2	Σ_4 | | h^2 | 3 | 4 | | Σ_4 | 4 | 10 |
$\mathcal{C}_{18}$	A generic cubic fourfold in $\mathcal{C}_{18}$ contains an elliptic ruled surface T of degree 6.	$K_{18} :=$ |	h^2	T | | h^2 | 3 | 6 | | T | 6 | 18 |
$\mathcal{C}_{26}$	Cubic fourfolds in $\mathcal{C}_{26}$ contain S_{26}, a surface obtained as the projection of a smooth del Pezzo surface of degree 7 in $\mathbb{P}^7$ from a line intersecting the secant variety in one general point (see [91, §3]). $\mathcal{C}_{26}$ corresponds also to cubic fourfolds containing a 3−nodal septic scrolls (see [34]).	$K_{26} :=$ |	h^2	S_{26} | | h^2 | 3 | 7 | | S_{26} | 7 | 25 |
$\mathcal{C}_{38}$	Cubic fourfolds in $\mathcal{C}_{38}$ contain a "generalized" Coble surface S_{38}. It is a general degree 10 smooth surface of sectional genus 6 obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having 10 fixed triple points (see [91, §4]).	$K_{38} :=$ |	h^2	S_{38} | | h^2 | 3 | 10 | | S_{38} | 10 | 46 |

$\mathcal{C}_{42}$	Cubic fourfolds in $\mathcal{C}_{42}$ contain the rational surface S_{42} of degree 9 and sectional genus 2 with 5 nodes (see [93]). $\mathcal{C}_{42}$ corresponds also to cubic fourfolds containing a 8−nodal degree 9 scroll (see [35]).	$K_{42} :=$ |	h^2	S_{42} | | h^2 | 3 | 9 S_{42} | 9 | 41

As already mentioned, Nuer [87] gave alternative descriptions for generic cubics in $\mathcal{C}_d$ for $12 \leq d \leq 44$, $d \neq 42$ by means of certain rational surfaces. For $12 \leq d \leq 38$ these surfaces, with certain cohomological invariants (see [87, Table 2]), are obtained as blow-ups of $\mathbb{P}^2$ at p points in general positions, embedded into $\mathbb{P}^5$ via certain ample linear system. In particular, he considered an open subset $U_d \subset (\mathbb{P}^2)^p$ parametrizing generic p-tuples of the distinct point. Let $\mathbf{S}$ be the rational surface obtained as the blow-up of $\mathbb{P}^2$ along the p points $x_1, \ldots, x_p$, and let us embed it into $\mathbb{P}^5$ via the linear series $|aL - (E_1 + \cdots + E_i) - 2(E_{i+1} + \cdots + E_{i+j}) - 3(E_{i+j+1} + \cdots + E_p)|$, where E_m, for $m = 1, .., p$, are exceptional divisors and L the pull-back of the hyperplane class on $\mathbb{P}^2$; the parameters (a, i, j, p) of the linear system are those displayed in [87, Table 1]. They are chosen appropriately so that the surface image of the linear system is contained in $\mathbb{P}^5$ and the cubics in $\mathbb{P}H^0(\mathcal{I}_{\mathbf{S}/\mathbb{P}^5}(3))$ all belong to $\mathcal{C}_d$, for a fixed value of d. Moreover, for $d = 44$, Nuer showed that a generic element of $\mathcal{C}_{44}$ contains a Fano model embedded in an Enriques surface.

2.3.2 Associated K3 surfaces

Hassett gives also the connection between *special* cubic fourfolds and K3 surfaces on the Hodge structure level. He identified infinite number of divisors $\mathcal{C}_d$ with associated K3 surfaces (see Appendix A).

Theorem 2.3.3 ([46]). *For a special cubic fourfold $X \in \mathcal{C}_d$, there exists a polarized K3 surface S of degree d such that $K_d^{\perp} \subset L$ is Hodge-isometric to $H^2_{prim}(S, \mathbb{Z})(-1)$ if and only if*

$$(**) \qquad d \text{ is not divisible by 4, 9, or any odd prime number } p \equiv 2 \ (mod\ 3)\ ^{3}.$$

In fact, for those d as in Theorem 2.3.3, Hassett proved also that $\mathcal{C}_d$ and $\mathcal{F}_g$ are related, where $\mathcal{F}_g$ is the 19-dimensional moduli space of polarized K3 surfaces of genus $g = \frac{d+2}{2}$. In particular, he showed that there is a rational map:

[3] This condition is equivalent to $d|2n^2 + 2n + 2$, for some $n \in \mathbb{Z}$ $(**)$.

$$\mathcal{F}_{\frac{d+2}{2}} \dashrightarrow \mathcal{C}_d \tag{2.1}$$

which is a birational isomorphism for $d \equiv 2\ (mod\ 6)$ and a degree two cover for $d \equiv 0\ (mod\ 6)$.

It is now conjectured that the rationality of cubic fourfolds depends on existence of such an associated K3 surface. According to the conjecture, supported by Hassett and Kuznestov's works ([46], [47], [59]), a cubic fourfold is rational if and only if it has an associated K3 surface in this sense.

First values of discriminants d verifying $(**)$ are $d = 14, 26, 38, 42, 62....$ Currently, known examples of rational cubic fourfolds are $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$ and $\mathcal{C}_{42}$ but also a countable infinite union of divisors in $\mathcal{C}_8$, $\mathcal{C}_{18}$ and certain divisors in $\mathcal{C}_{20}$ (see [4, 21, 32, 46, 92, 93, 113]). In fact, rationality has been known classically for the first family $\mathcal{C}_{14}$, by the work of Fano [13] (see also [21]). Moreover, in two recent papers [91] and [93], a new way of showing the rationality of cubic fourfolds has been introduced based on the congruences of 5-secant conics for the first three cases $\mathcal{C}_{14}, \mathcal{C}_{26}$ and $\mathcal{C}_{38}$ (see [91]) and congruences of 8-secant twisted cubics for $\mathcal{C}_{42}$ (see [93]). In particular, in these papers the authors construct specific irreducible surfaces $S_d \subset \mathbb{P}^5$ contained in a general element of certain $\mathcal{C}_d$, for $d = 14,\ 26,\ 38$ or 42. These surfaces S_d admit a four dimensional family of $(3e-1)-$ secant curves of degree e parametrized by rational variety such that through any general point of $\mathbb{P}^5$ there passes a unique curve of the family. In this case, we say that these surfaces S_d admit congruences of $(3e-1)-$ secant curves of degree e. Then cubics containing S_d are birational to the base of the rational family (or to the rational parameter space of the congruence), and hence are rational as well. More details of these constructions are given in §4.4.1 (Chapter 4).
Clearly, for the rest of the divisors $\mathcal{C}_8$, $\mathcal{C}_{18}$ and $\mathcal{C}_{20}$, their discriminants d don't satisfy $(**)$. Although, the rational examples of cubics that they contain belong (most probably) to the intersection with certain divisors $\mathcal{C}_d$ with d satisfying $(**)$ (see [113], [9], [32]).

2.3.3 Fano variety of lines

For a cubic fourfold X, the Fano variety of lines $F(X) = \{l \in \mathbb{G}(1,5) |\ l \subseteq X\}$ is a four-dimensional smooth projective variety. It is an Hyperkähler variety, *i.e.* $F(X)$ is simply connected and $H^0(F(X), \Omega^2_{F(X)})$ is spanned by an everywhere non-degenerate form (see

[19], [53, Chapter 4] or [5, Section 1]). Hassett [46, Section 6] showed also that for a generic special cubic fourfold $X \in \mathcal{C}_{2(n^2+n+1)}$, for $n \geq 2$ an integer, there is an isomorphism

$$F(X) \cong S^{[2]} \tag{2.2}$$

between $F(X)$ and the Hilbert scheme of length two subschemes $S^{[2]}$ of the associated K3 surface S.
Moreover, Hassett showed that, if for the generic $X \in \mathcal{C}_d$, the Fano variety $F(X)$ is isomorphic to $S^{[2]}$ for some K3 surface S then

$$(***) \qquad d \text{ is of the form } \frac{2n^2+2n+2}{a^2} \text{ for some } n,\ a \in \mathbb{Z}\ .$$

Furthermore, Addington [1] proved that $X \in \mathcal{C}_d$ for some d satisfying $(***)$ if and only if $F(X)$ is birational to $S^{[2]}$ for some K3 surface S (see also [36]).

2.3.4 Unirationality of Hassett divisors $\mathcal{C}_d$

Questions about the unirationality of Hassett divisors $\mathcal{C}_d$ have been investigated in multiple works. It is natural to ask, on one hand, what is the smallest d for which $\mathcal{C}_d$ is not unirational and, on the other hand, is there a minimal d after which $\mathcal{C}_d$ is of general type.

Although the unirationality of $\mathcal{C}_d$ for $d = 8,\ 12,\ 14,\ 20$ was already known, using the description mentioned above (see §2.3.1), Nuer [87] proved the following:

Proposition 2.3.4. *([87, Corollary 4.1, 4.2]) For $12 \leq d \leq 44$ and $d \neq 42$, the moduli space $\mathcal{C}_d$ is unirational.*

In fact, the main theorem [87, Thm. 3.1] shows that there exists a vector bundle $V_d \to U_d$ such that the fiber over $(x_1, \ldots, x_p)$ is the space of global sections $H^0(\mathcal{I}_{\mathbf{S}|\mathbb{P}^5}(3))$. The associated projective bundle is clearly unirational. The natural classifying morphism $\mathbb{P}(V_d) \to \mathcal{C}_d$, obtained by the universal property of $\mathcal{C}_d$ is dominant and this implies the unirationality of $\mathcal{C}_d$.

Furthermore, due to Gritsenko-Hulek-Sankaran [42], multiple results were made concerning the non-negativity of the Kodaira dimension of the moduli space of polarized K3 surfaces of certain degree. Following Nuer [87] and using the connection between cubic fourfolds and K3 surfaces (see [46]), we write their "translation" in terms of cubic fourfolds via the rational map (2.1):

Proposition 2.3.5. *Let $d > 80$, $d \equiv 2 \ (mod\ 6)$, $4 \not| d$ such that for any odd prime p, $p|d$ implies $p \equiv 1 \ (mod\ 3)$. Then the Kodaira dimension of $\mathcal{C}_d$ is non-negative. If moreover $d > 122$, then $\mathcal{C}_d$ is of general type.*

Later on certain gaps were filled due to Tanimoto-Várilly-Alvarado [103] using automorphic form techniques:

Proposition 2.3.6. *([103, Theorem 1.1]) The divisor $\mathcal{C}_{6n+2}$ is of general type for $n > 18$ and $n \neq 20, 21, 25$ and has non-negative Kodaira dimension for $n > 13$ and $n \neq 15$. Moreover, $\mathcal{C}_{6n}$ is of general type for $n > 18$, $n \neq 20, 22, 23, 25, 30, 32$ and has non-negative Kodaira dimension for $n > 16$ and $n \neq 18, 20, 22, 30$.*

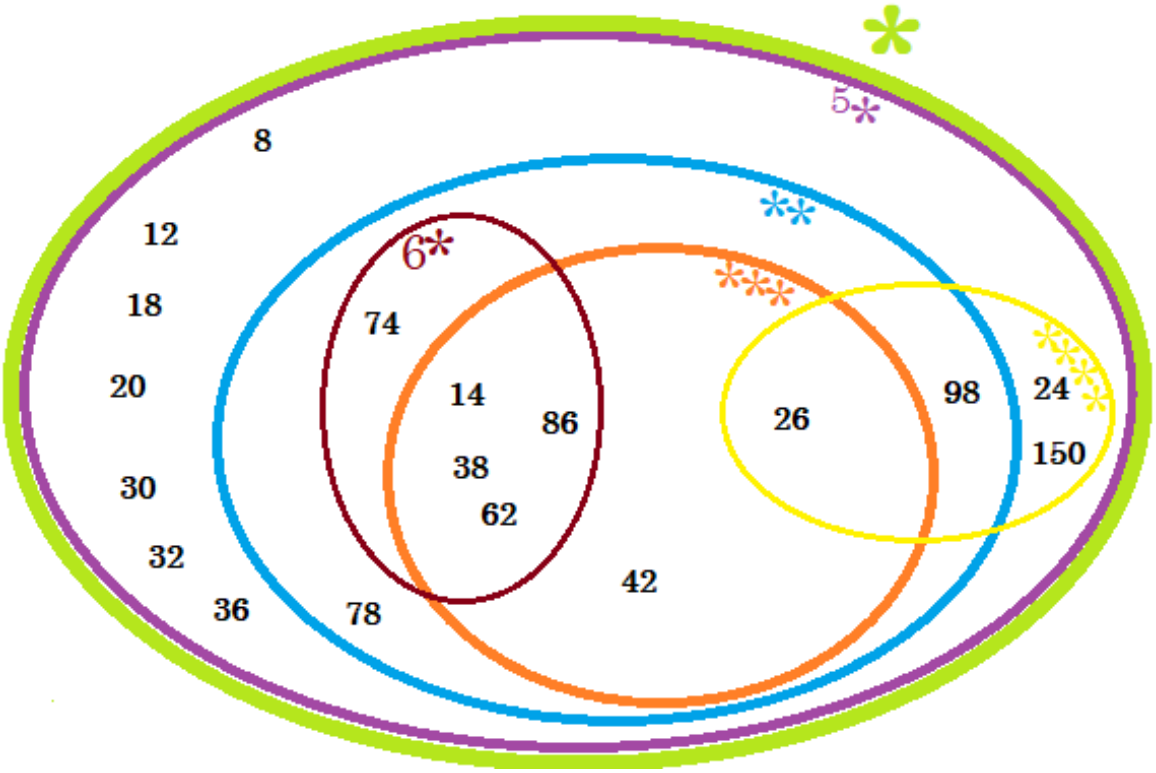
2.4 Arithmetic conditions on discriminants

Next we collect all arithmetic conditions on the discriminants d of Hassett divisors $\mathcal{C}_d$ that appear all along this manuscript. We define some elements that are not mentioned yet: for a cubic fourfold $X \in \mathcal{C}_d$, we denote by $S^{[4]}$ the Hilbert scheme of four points on a K3 surface S, $L(X)$ the 8fold associated to X (see Section 3.3.2), $t(X)$ (resp. $t_2(S)$) the transcendental motive of X (resp. of a K3 surface S) (see Appendix B). For more details, we refer to [1, 2, 52, 54].

Label	Condition on d	Comments for $X \in \mathcal{C}_d$
$(*)$	$d \geq 8$ and $d \equiv 0$ or 2 $(mod\ 6)$	Hassett divisors $\mathcal{C}_d$ are indexed by d satisfying $(*)$
$(**)$	$d \not| 4, 9$, or any odd prime number $p \equiv 2$ $(mod\ 3)$	$\iff H^2(S,\mathbb{Z})_{prim}(-1) \simeq K_d^{\perp}$ (Hodge isometry); $Ku(X) \cong \mathrm{D}^b(S)$
$(***)$	$d = \frac{2n^2+2n+2}{a^2}$, for some $n,\ a \in \mathbb{Z}$	$\iff F(X) \sim S^{[2]}$
$(****)$	$d = 6d'$ or $6d'+2$ with d' a square	This condition is introduced in Section 3.3 to prove the non-emptiness of the intersection of multiple divisors with discriminants satisfying this condition
$(5*)$	$\exists f,\ g \in \mathbb{Z}$ with $g\|(2n^2+2n+2),\ n \in \mathbb{Z}$ and $d = f^2 g$	$\implies t(X) \cong t_2(S)(-1)$
$(6*)$	$d = \frac{6n^2+6n+2}{a^2},\ \ n,\ a \in \mathbb{Z}$	$\iff L(X) \sim S^{[4]}$

One can notice that the condition $(***)$ is strictly stronger than $(**)$, which is in turn strictly stronger than $(5*)$. Of course, all values d, that we are interested in, verify the condition $(*)$. The conditions $(****)$ and $(6*)$ are not comparable with others. However, the sets of d satisfying $(****)$ and $(6*)$ intersect in a certain way with the sets of d satisfying other conditions. Next we try to schematize this situation. For the reader's convenience, some values of d were positioned inside the ellipses where they belong [4].

[4]The author is aware that this is not the most formal way to represent this idea but still think that this visualizes better the concept.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Chapter 3

Intersection of Hassett divisors $\mathcal{C}_d$ in the moduli space $\mathcal{C}$

3.1 Introduction

From discussions above, it is significant to study the intersection of Hassett divisors. Then one can understand more the locus of rational cubics inside $\mathcal{C}$, find new examples of rational cubics and much more... Intersections of Hassett divisors $\mathcal{C}_d$ were considered in multiple papers (see [3, 8, 32, 112, 113]). In particular, the intersection $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has been known nonempty for 20 years (see [47]). In [8], this intersection is also described explicitly. This allowed the authors to establish new examples of rational cubic fourfolds X containing a plane and a quintic del Pezzo surface. In fact, based on lattice computations and the works of Laza [65], Looijenga [68], Mayanskiy [71] and Voisin [109], they provide description of the intersection in terms of irreducible components and classify exactly which components are supported by cubic fourfolds and which ones are empty. Furthermore, Addington-Thomas [3, Theorem 4.1] proved that $\mathcal{C}_8$ intersects any Hassett divisor. Moreover in [21], the authors mentioned that $\mathcal{C}_{14}$ intersects many other Hassett divisors without specifying exactly which ones. Lately, Yang and Yu [112] proved that $\mathcal{C}_{14}$ intersects all divisors. Actually, they proved that the intersection of any two Hassett divisors is nonempty. As a consequence, they conclude that every Hassett divisor contains a union of three codimension two subvarieties in $\mathcal{C}$ parametrizing rational cubic fourfolds corresponding to the intersection with the divisors $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ (see [112, Theorem 3.3]). Actually they are four, since in the meanwhile

also $\mathcal{C}_{42}$ has been shown to parametrize rational cubic fourfolds (see [93]).

Recently, in [9], we consider some classes of cubic fourfolds birational to fibrations over $\mathbb{P}^2$, where the fibers are rational surfaces. This relates strongly the rationality of the cubics to the rationality of these surfaces over the function field of $\mathbb{P}^2$ and to the existence of rational sections of the fibration. By studying the intersection of these classes of cubics with others, well known to be rational (such as cubics in $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ (see [91])), we understand whether the existence of a rational section is also necessary for the rationality of the cubics.
We study first the divisor $\mathcal{C}_8$, whose general element is a cubic containing a plane. These cubics are birational to quadric fibrations over $\mathbb{P}^2$. Hassett [47] identified countably infinite divisors in $\mathcal{C}_8$ consisting of cubics containing a plane and proved that they are rational whenever the associated quadric surface bundle has a section. However, intersections of $\mathcal{C}_8$ with the divisors $\mathcal{C}_{26}$, $\mathcal{C}_{38}$ give us new classes of smooth rational cubic fourfolds whose associated quadric surface bundle doesn't have a rational section, hence not contained in the divisors of $\mathcal{C}_8$ described by Hassett. In addition, based on lattice computations, we describe explicitly the intersections by evoking their irreducible components and we provide conditions on the intersection pairing of generic elements for the (non)existence of sections of the associated quadric surface bundle. This is equivalent to the (non)triviality of the Brauer class $\alpha \in Br(S)$ associated to a degree 2 K3 surface S arising from a cubic fourfolds in $\mathcal{C}_8$ (see §3.2.1 for more details):

Theorem I-1: *The intersection $\mathcal{C}_8 \cap \mathcal{C}_{26}$ in the moduli space of cubic fourfolds has eight irreducible components indexed by the discriminant $d_\tau \in \{29, 36, 48, 53, 61, 64, 68, 69\}$ of the intersection pairing of generic elements inside each component.*
Four of these components contain rational cubic fourfolds whose quadric fibration has no section.

Theorem I-2: *The intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$ in the moduli space of cubic fourfolds has ten irreducible components indexed by the discriminant $d_\tau \in \{36, 45, 61, 68, 80, 85, 93, 96, 100, 101\}$ of the intersection pairing of generic elements inside each component.*
Five of these components contain rational cubic fourfolds whose quadric fibration has no

section.

In [9], we consider also the divisor $\mathcal{C}_{18}$, closure of the locus of cubic fourfolds containing an elliptic ruled surface. The linear system of quadrics through this surface defines a (possibly *good*) sextic del Pezzo fibration over $\mathbb{P}^2$ (see Section 3.2.2 for a complete definition of a "good" fibration). In particular, to a *good* fibration, we can associate two Brauer classes. Let us denote by $\mathcal{H}(0,3)$ and $\mathcal{H}(0,2)$ the relative Hilbert schemes over $\mathbb{P}^2$ parametrizing connected genus zero curves of respectively anticanonical degree three and two. In fact, $\mathcal{H}(0,3)$ is a $\mathbb{P}^2$-bundle over a smooth degree two K3 surface S and let $\beta \in Br(S)$ be the associated Brauer class. In addition, $\mathcal{H}(0,2)$ is a $\mathbb{P}^1$-bundle over a triple cover Z of $\mathbb{P}^2$ and we define the associated Brauer class $\gamma \in Br(Z)$. In [4], it is proved that cubics in $\mathcal{C}_{18}$ are rational whenever the fibration has rational section *i.e.* both Brauer classes β and γ are trivial. In [9], we show that this condition is not necessary for the rationality of a cubic fourfold in $\mathcal{C}_{18}$.

Theorem II: *There exist rational cubic fourfolds in $\mathcal{C}_{18}$ fibered in "good" sextic del Pezzo surfaces such that the associated Brauer classes are not both trivial.*

We proceed, once again, by intersecting $\mathcal{C}_{18}$ with the divisors $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ and providing explicit description of these intersections. These results are obtained by studying the intersection lattices of cubic fourfolds in these loci of codimension two in the moduli space, and by checking their non emptiness. We abstain from describing them here, they correspond to Theorems 3.2.10, 3.2.11 and 3.2.12. Moreover, for each irreducible component in each intersection, we also study the (non)triviality of the two Brauer classes and we exhibit new examples of rational cubic fourfolds whose associated *good* sextic del Pezzo fibration has no rational sections (see §3.2.2):

Explicit examples of these rational cubic fourfolds fibered both in quadric surfaces and sextic del Pezzo surfaces over $\mathbb{P}^2$ are provided in §3.2.3. The equations of the cubics themselves are constructed with Macaulay2 [39][1].

[1]The "goodness" condition on the del Pezzo fibration is quite difficult to obtain computationally.

Furthermore, generalizing the construction of Yang and Yu [112], we study in [11] to what extent one can intersect Hassett divisors $\mathcal{C}_d$ in $\mathcal{C}$. In particular, we exhibit arithmetic conditions on 20 indexes $d_1, \ldots, d_{20}$ that assure that the divisors $\mathcal{C}_{d_1}, \ldots, \mathcal{C}_{d_{20}}$ all intersect one another. Using lattice theoretical computations and a result of [112], we prove the following:

Theorem III:

(i) *For* $3 \leq n \leq 20$,

$$\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset,$$

with $d_k \geq 8$, $d_k \equiv 0, 2 \ (mod\ 6)$ *and* $d_3, .., d_n = 6d'$ *or* $6d'+2$, d' *a square.*[2]

(ii) There exist families of cubic fourfolds $\mathcal{F} \subseteq \bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$, *of dimension* $20-n$, *such that a generic cubic fourfold* $X \in \mathcal{F}$ *has* $\mathrm{rk}(A(X)) = n+1$.

(iii) In addition, if at least one of the divisors $\mathcal{C}_{d_k}$ *in the intersection parametrizes cubic fourfolds with an associated K3 surface (in other words,* d_k *satisfies the condition* $(**)$*), then cubic fourfolds in the families* $\mathcal{F}$ *have an associated K3 surface with a Néron-Severi group of rank greater than or equal to* n.

Based on these results, we construct new one dimensional families of cubic fourfolds with finite dimensional, abelian Chow motive (see Appendix B). In fact, few examples of cubic fourfolds belonging to the category generated by motives of abelian varieties $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$ are known, mainly due to the beautiful work of Laterveer [62–64]. However, one would be interested to understand how these cubics are positioned in the geography of the moduli space, that is whether they belong to any of the divisors $\mathcal{C}_d$. These one dimensional families of cubic fourfolds with finite dimensional, abelian motive that we construct, are quite ubiquitous in the moduli space, since any Hassett divisor contains some of them:

Theorem IV: *Every Hassett divisor* $\mathcal{C}_d$ *contains a countable infinity of one dimensional families of cubic fourfolds, whose Chow motive is finite dimensional and Abelian.*

[2] See Remark 3.3.10 for a comparison with the recent result of Yang and Yu [113].

This also implies abelianity and finite dimensionality of the motive of related Hyperkähler varieties, such as the Fano variety of lines and the LLSvS 8fold (see [67]).

3.2 Intersection of two Hassett divisors: rational fibered cubic fourfolds with nontrivial Brauer classes

3.2.1 Cubic fourfolds fibered in quadric surfaces over $\mathbb{P}^2$

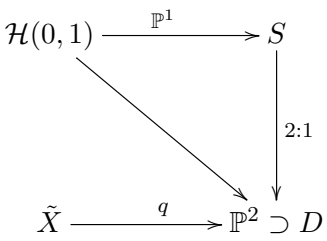
Hassett [47] identifies countably many divisors in $\mathcal{C}_8$ consisting of cubic fourfolds containing a plane parametrizing rational cubic fourfolds. Each of these loci is a codimension two subvariety in the moduli space of cubic fourfolds $\mathcal{C}$.

Now consider a cubic fourfold X in $\mathcal{C}_8$ containing a plane P. Let h denote the *hyperplane class* of X, and let Q denote the class of the *quadric surface* residual to P so that $h^2 = P+Q$. Let $\tilde{X}$ denote the blow-up of X along P. The projection from P resolves into a morphism

$$q\colon \tilde{X} \to \mathbb{P}^2.$$

The fibers of this morphism are quadric surfaces in the class Q. In particular, cubic fourfolds containing a plane are birational to quadric surface bundles over $\mathbb{P}^2$.

Recall that the *discriminant divisor* D is defined as the locus over which q fails to be smooth. We say that the plane P is *good* if the fibers of q have at most isolated singularities. This is equivalent to having $D \subset \mathbb{P}^2$ smooth or also to having X not containing another plane intersecting P (see [109, §1 Lemma 2], [7, §1.5]). In this case, the cover $S \to \mathbb{P}^2$ branched over D, a sextic curve, is a smooth K3 surface S of degree 2. If D is smooth, the relative Hilbert scheme of lines $\mathcal{H}(0,1)$ of the morphism q is an étale projective $\mathbb{P}^1$-bundle over S (see [20, §5]). To such an object we can associate a Brauer class $\alpha \in Br(S)$ which is trivial if and only if q has a rational section.



Theorem 3.2.1 ([47]). *If there is a class $C \in A(X)$ such that (C, Q) is odd then X is rational over $\mathbb{C}$.*

In particular, q has a rational section if and only if there exists an algebraic cycle $C \in A(X)$ such that $(C, Q) = 1$. In other words, the associated Brauer class α is trivial if and only if there exists an algebraic cycle $C \in A(X)$ such that $(C, Q) = 1$. According to Hassett, the triviality of the Brauer class implies the rationality of X over $\mathbb{C}$.

In the following, by lattice-theoretic calculations, we describe classes of rational cubic fourfolds containing a plane whose fibration in quadric surfaces has no section respectively in $\mathcal{C}_8 \cap \mathcal{C}_{26}$ and in $\mathcal{C}_8 \cap \mathcal{C}_{38}$, hence not contained in the class of cubics described by Hassett.

3.2.1.1 $\mathcal{C}_8 \cap \mathcal{C}_{26}$:

A smooth cubic fourfold X is in $\mathcal{C}_8$ or $\mathcal{C}_{26}$ if and only if $A(X)$ has primitive sublattice $K_8 := \langle h^2, P\rangle$ or $K_{26} := \langle h^2, S_{26}\rangle$, such that P is a plane contained in a general element of $\mathcal{C}_8$ and S_{26} is a surface contained in a general element of $\mathcal{C}_{26}$ (as defined in §2.3.1).

A cubic fourfold $X \in \mathcal{C}_8 \cap \mathcal{C}_{26}$ has a sublattice $\langle h^2, P, S_{26}\rangle \subset A(X)$ with the following Gram matrix

$$\begin{array}{c} \\ h^2 \\ P \\ S_{26} \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & P & S_{26} \end{array} \\ \begin{pmatrix} 3 & 1 & 7 \\ 1 & 3 & \tau \\ 7 & \tau & 25 \end{pmatrix} \end{array}$$

For some $\tau = (P, S_{26}) \in \mathbb{Z}$ depending on X. The values of τ may be restricted following some properties and works of Voisin [109] or Yang and Yu [112].

Denote by A_τ the lattice of rank 3 whose bilinear form has the previous Gram matrix. Let $\mathcal{C}_{A_\tau} \subset \mathcal{C}$ be the locus of smooth cubic fourfolds such that there is a primitive embedding $A_\tau \subset A(X)$ of lattices preserving h^2.

Proposition 3.2.2. *The irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{26}$ are the subvarieties of codimension two $\mathcal{C}_{A_\tau}$ for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$.*

Proof. By construction, $\mathcal{C}_8 \cap \mathcal{C}_{26} = \cup_{\tau\in\mathbb{Z}}\ \mathcal{C}_{A_\tau}$. First we determine which components of $\mathcal{C}_{A_\tau}$ are possibly nonempty. Since for $X \in \mathcal{C}_8 \cap \mathcal{C}_{26}$, $A(X)$ is a positive definite lattice, by Sylvester's criterion, the sublattice A_τ must have positive discriminant. As $d(A_\tau) = -3\tau^2 + 14\tau + 53$, the only values of τ making a positive discriminant are $\tau \in I = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$. Hence, $\mathcal{C}_8 \cap \mathcal{C}_{26} = \cup_{\tau\in I}\ \mathcal{C}_{A_\tau}$.

Then, we prove that $\mathcal{C}_{A_\tau}$ is empty for $\tau = -2, 7$ by finding primitive short roots (that is, primitive vectors of norm 2) in $A_{\tau,prim} = \langle h^2\rangle^\perp$. Indeed, the vectors $(1, -3, 0)$ and $(-3, 2, 1)$ form a basis for $A_{\tau,prim}$. For all $R_{a,b} \in A_{\tau,prim}$, $R_{a,b} = (a-3b, -3a+2b, b)$ for some $a, b \in \mathbb{Z}$, $\tau \in I$; for $\tau = -2$, we find primitive short roots $\pm R_{0,1}$; for $\tau = 7$, we find primitive short roots $\pm R_{1,1} = \pm(-2, -1, 1)$. Hence, by [109, §4 Proposition 1], $\mathcal{C}_{A_\tau}$ is empty for $\tau = -2, 7$. We are left with $\mathcal{C}_{A_\tau}$ possibly nonempty only for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$.

To check the (non)emptiness of these $\mathcal{C}_{A_\tau}$, we proceed as follows; for all $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$, A_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in A_\tau \subset A(X) \subset L$$

Furthermore, let $v = xh^2 + yP + zS_{26} \in A_\tau$ for $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 3y^2 + 25z^2 + 2xy + 14xz + 2\tau yz,\ \tau \in \{-1, 0, 1, 2, 3, 4, 5, 6\}$$

For these values of τ, there exists no $v \in A_\tau$ such that $(v, v) = 2$. Thus, by [112, Lemma 2.4] $\mathcal{C}_{A_\tau}$ is nonempty and has codimension two. The corresponding discriminants $d(A_\tau)$ are 36, 53, 64, 69, 68, 61, 48 and 29.

τ	-1	0	1	2	3	4	5	6
$d(A_\tau)$	36	53	64	69	68	61	48	29

We prove now that the $\mathcal{C}_{A_\tau}$ are irreducible. We first note that the rank of the lattice $A(X)$ is an upper-semicontinuous function on $\mathcal{C}$; the irreducible components of $\mathcal{C}_{A_\tau}$ correspond then to rank 3 overlattices B of A_τ (a finite index sublattice for some τ *i.e.* B/A_τ is a finite abelian group) which is primitively embedded into $H^4(X, \mathbb{Z})$. By standard lattice theory, we have that for an embedding $A_\tau \hookrightarrow B$ with finite index $[B : A_\tau] = |B/A_\tau|$, $|d(B)|.[B : A_\tau]^2 = |d(A_\tau)|$ (see [86] or [97]);
We will prove that no proper finite overlattices exist; for $\tau = 0, 2, 4, 6$ the discriminants of A_τ are squarefree, so there are no proper finite overlattices.

For the remaining cases, we can take h^2 and P as a part of a basis of the overlattice B. Let U be a vector that completes this to a basis such that $U = xh^2 + yP + zS_{26}$ with $x, y, z \in \mathbb{Q}$. Consider the natural embedding of A_τ in B that can be written as follows:

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \\ 0 & 0 & -1/z \end{pmatrix} \in M_{3,3}(\mathbb{Z}).$$

We can take $z = \frac{1}{n}$, for some $n \in \mathbb{Z}$ and $x' = nx, y' = yn \in \mathbb{Z}$; then $U = \frac{1}{n}(x'h^2 + y'P + S_{26})$. By adding multiples of h^2 and P, we may ensure that $0 \leq x', y' < n$.
Note that $n = [B : A_\tau] = [B_{prim} : A_{\tau,prim}]$, with B_{prim} the finite proper overlattice of $A_{\tau,prim}$ (this follows from standard lattice theory).

Computing intersections:

$$\begin{aligned} (U, h^2) &= \frac{1}{n}(3x' + y' + 7) = a_1, \\ (U, P) &= \frac{1}{n}(x' + 3y' + \tau) = b_1, \\ (U, U) &= \frac{1}{n^2}(3x'^2 + 3y'^2 + 14x' + 2\tau y' + 2x'y' + 25) = c_1. \end{aligned}$$

Hence, the Gram matrix of B is:

$$\begin{array}{c} \\ h^2 \\ P \\ U \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & P & U \end{array} \\ \begin{pmatrix} 3 & 1 & a_1 \\ 1 & 3 & b_1 \\ a_1 & b_1 & c_1 \end{pmatrix} \end{array}$$

Now we check each case separately for possible values of τ, n, x' and y':

1. $\tau = -1$: We see that n can be 2, 3 or 6.
   Remark that for all possible values of n, x' and y' other than ($n = 3$, $x' = 1$ and $y' = 2$), the Gram matrix of B is not in $M_{3,3}(\mathbb{Z})$. For ($n = 3$, $x' = 1$ and $y' = 2$), B has the following Gram matrix:

$$B := \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & 6 \end{pmatrix},$$

which has the short root $(-2, 0, 1)$. Then no such overlattices exist. Thus $\mathcal{C}_{A_{-1}}$ is irreducible.

2. $\tau = 1$: We see that n can be 2, 4 or 8.
   Observe that for all possible values of n, x' and y', the Gram matrix of B is not in $M_{3,3}(\mathbb{Z})$. Thus $\mathcal{C}_{A_1}$ is irreducible.

3. $\tau = 3$: n can be 2.
   We notice that for all possible values of x' and y', the Gram matrix of B is non-integral. Then no such overlattices exist. Thus $\mathcal{C}_{A_3}$ is irreducible.

4. $\tau = 5$: We observe that n can be 2 or 4.
   Remark that for $n = 4$, $|d(B)| = 3$ which is impossible by [113, Lemma 7.8].
   Otherwise, for all possible values of x' and y', the Gram matrix of B is non-integral. Then no such overlattices exist. Thus $\mathcal{C}_{A_5}$ is irreducible.

□

In the following, we address the question of the (non)triviality of the Brauer class.

Theorem 3.2.3. *Let X be a general cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{26}$ (so that $A(X)$ has rank 3) containing a good plane P. The Brauer class $\alpha \in Br(S)$ of X is trivial if and only if τ is even.*

Proof. If τ is even, then a cycle $S_{26} + 3P \in A(X)$ exists such that $(S_{26} + 3P, Q) = 1 - \tau \equiv 1$ (*mod* 2) (odd). Hence α is trivial by the application of the criterion of Hassett (see [47]).

If τ is odd, then A_τ has rank 3 and even discriminant, hence α is nontrivial (see [8, Proposition 2]). □

Corollary 3.2.4. *The four irreducible components $\mathcal{C}_{A_\tau}$ of $\mathcal{C}_8 \cap \mathcal{C}_{26}$ corresponding to $\tau = -1$, 1, 3, 5 contain examples of rational cubic fourfolds whose associated quadric surface bundles don't have a rational section.*

3.2.1.2 $\mathcal{C}_8 \cap \mathcal{C}_{38}$:

Using same method as before, we compute intersection between $\mathcal{C}_8$ and $\mathcal{C}_{38}$ this time.

A cubic fourfold X is in $\mathcal{C}_8$ or $\mathcal{C}_{38}$ if and only if $A(X)$ has primitive sublattice $K_8 := \langle h^2, P\rangle$ or $K_{38} := \langle h^2, S_{38}\rangle$, such that P and S_{38} are as defined in §2.3.1. Hence, X in $\mathcal{C}_8 \cap \mathcal{C}_{38}$ has a sublattice $\langle h^2, P, S_{38}\rangle \subset A(X)$ with Gram matrix:

$$\begin{array}{c} \\ h^2 \\ P \\ S_{38} \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & P & S_{38} \end{array} \\ \begin{pmatrix} 3 & 1 & 10 \\ 1 & 3 & \tau \\ 10 & \tau & 46 \end{pmatrix} \end{array}$$

For some $\tau \in \mathbb{Z}$ depending on X.

Proposition 3.2.5. *The irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{38}$ are the subvarieties of codimension two $\mathcal{C}_{A_\tau}$ for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$.*

Proof. As $d(A_\tau) = -3\tau^2 + 20\tau + 68$, the only values of τ inducing a positive discriminant are $\tau \in J = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
Let $A_{\tau,prim} = \{(x, y, z) \in \mathbb{Z}^3 \mid 3x+y+10z = 0\}$; indeed, the vectors $(1, -3, 0)$ and $(-3, -1, 1)$ form a basis for $A_{\tau,prim}$; for all $R_{a,b} \in A_{\tau,prim}$, $R_{a,b} = (a - 3b, -3a - b, b)$ for some $a, b \in \mathbb{Z}$, $\tau \in J$.
For $\tau = -2$, we find primitive short roots $\pm R_{-1,1} = \pm(-4, 2, 1)$. Hence, by [109, §4 Proposition 1], $\mathcal{C}_{A_{-2}}$ is empty ; We are left with $\mathcal{C}_{A_\tau}$ possibly nonempty only for $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

For all $\tau \in \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, A_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in A_\tau \subset A(X) \subset L$$

Furthermore, let $v = xh^2 + yP + zS_{38} \in A_\tau$, for $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 3y^2 + 46z^2 + 2xy + 20xz + 2\tau yz,$$

For $\tau = 9$, we have that $(-2h^2 - 2P + S_{38}, -2h^2 - 2P + S_{38}) = 2$. $\mathcal{C}_{A_9} \subset \mathcal{C}$ is then empty (see [112, Lemma 2.4]). For the rest of the values of τ, there exists no $v \in A_\tau$ such that $(v, v) = 2$, $\mathcal{C}_{A_\tau} \subset \mathcal{C}$ are nonempty and have codimension 2 (see [112, Lemma 2.4]). The corresponding discriminants $d(A_\tau)$ are 45, 68, 85, 96, 101, 100, 93, 80, 61 and 36.

τ	-1	0	1	2	3	4	5	6	7	8
$d(A_\tau)$	45	68	85	96	101	100	93	80	61	36

We prove now that $\mathcal{C}_{A_\tau}$ are irreducible. For $\tau = 1, 3, 5, 7$ the discriminants are squarefree, so there are no proper finite overlattices. For the remaining cases, we can take h^2 and P as a part of a basis of an overlattice B and we complete it with U' such that $U' = xh^2 + yP + zS_{38}$ with $x, y, z \in \mathbb{Q}$. As we did in the previous section, let $z = \frac{1}{n}$ for some $n \in \mathbb{Z}$ and $x' = nx, y' = yn \in \mathbb{Z}$. We can write $U' = \frac{1}{n}(x'h^2 + y'P + S_{38})$. By adding multiples of h^2 and P, we may ensure that $0 \leq x', y' < n$.

Computing intersections:

$$\begin{aligned}(U', h^2) &= \frac{1}{n}(3x' + y' + 10) = a_2,\\ (U', P) &= \frac{1}{n}(x' + 3y' + \tau) = b_2,\\ (U', U') &= \frac{1}{n^2}(3x'^2 + 3y'^2 + 20x' + 2\tau y' + 2x'y' + 46) = c_2.\end{aligned}$$

Hence, B has Gram matrix:

$$\begin{array}{c} \\ h^2 \\ P \\ U' \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & P & U' \end{array} \\ \begin{pmatrix} 3 & 1 & a_2 \\ 1 & 3 & b_2 \\ a_2 & b_2 & c_2 \end{pmatrix} \end{array}$$

Now we check each case separately for possible values of τ, n, x' and y'; We notice that for all cases other than $(\tau = -1, n = 3, x' = 1, y' = 2)$ and $(\tau = 8,\ n = 3,\ x' = 1,\ y' = 2)$ the Gram matrix of B is non-integral.

For $(\tau = -1,\ n = 3,\ x' = 1,\ y' = 2)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 1 & 5 \\ 1 & 3 & 2 \\ 5 & 2 & 9 \end{pmatrix}$$

which has primitive short roots $\pm(1, 0, -1)$.

For $(\tau = 8, n = 3, x' = 1, y' = 2)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 1 & 5 \\ 1 & 3 & 5 \\ 5 & 5 & 13 \end{pmatrix}$$

which has a primitive short root $(-2,-1,1)$.
We conclude that no such overlattices exist. Thus all components $\mathcal{C}_{A_\tau}$ are irreducible. □

Theorem 3.2.6. *Let X be a general cubic fourfold in $\mathcal{C}_8 \cap \mathcal{C}_{38}$ (so that $A(X)$ has rank 3) containing a good plane P. The Brauer class $\alpha \in Br(S)$ of X is trivial if and only if τ is odd.*

Proof. If τ is odd, then a cycle $S_{38}+5P \in A(X)$ exists such that $(S_{38}+5P, Q) = -\tau \equiv 1 \ (mod\ 2)$ (odd). Hence α is trivial by the application of the criterion (see [47]).

If τ is even, then A_τ has rank 3 and even discriminant, hence α is nontrivial (see [8, Proposition 2]). □

Corollary 3.2.7. *$\mathcal{C}_8 \cap \mathcal{C}_{38}$ has five smooth irreducible components, corresponding to $\tau = 0,2,4,6,8$, containing examples of rational cubic fourfolds whose associated quadric surface bundles don't have a rational section.*

In §3.2.3, we construct an explicit example of rational smooth cubic fourfold containing a *good* plane in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$, whose associated Brauer class $\alpha \in Br(S)$ is nontrivial.

3.2.2 Cubic fourfolds fibered in sextic del Pezzo surfaces over $\mathbb{P}^2$

In [4], other examples of rational fibered cubic fourfolds in $\mathcal{C}_{18}$, parametrized by a countably infinite union of codimension two subvarieties, have been studied. We recall that a generic cubic fourfold in $\mathcal{C}_{18}$ contains an elliptic ruled surface T of degree 6. Let $X \subset \mathcal{C}_{18}$ be a generic cubic fourfold containing the elliptic ruled surface T. We denote by $\tilde{X}$ the blow-up of X along T and let

$$\pi \colon \tilde{X} \to \mathbb{P}^2$$

be the map induced by the linear system of quadrics containing T. For generic X, the generic fiber of π is del Pezzo surface of degree 6 (see Appendix A.3 for a general definition of a del Pezzo surface).

Note that a singular sextic del Pezzo surface is a surface $\mathcal{P}$ with ADE singularities and ample anticanonical class such that $K_{\mathcal{P}}^2 = 6$. They are classified as follows (see [58, Section 2]):

• Type I: $\mathcal{P}$ has one A_1 singularity and is obtained by blowing up $\mathbb{P}^2$ in three collinear

points, and blowing down the proper transform of the line containing them;
• Type II: $\mathcal{P}$ has one A_1 singularity, obtained by blowing up $\mathbb{P}^2$ in two infinitely near points and a third point not on the line associated with the infinitely near points, then blowing down the proper transform of the first exceptional divisor over the infinitely near points;
• Type III: $\mathcal{P}$ has two A_1 singularities and is obtained by blowing up two infinitely near points and a third point all contained in a line, and blowing down the proper transforms of the first exceptional divisor over the infinitely near point and the line;
• Type IV: $\mathcal{P}$ has one A_2 singularity and is obtained by blowing up a curvilinear subscheme of length three not contained in a line and blowing down the first two exceptional divisors;
• Type V: $\mathcal{P}$ has an A_1 and an A_2 singularity and is obtained by blowing up a curvilinear subscheme contained in a line, and blowing down the proper transforms of the first two exceptional divisors and the line.

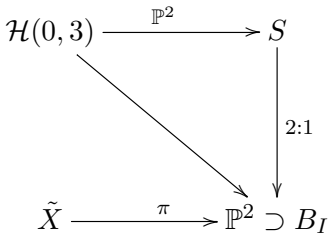
Type I and II occur in codimension one in the moduli stack of (singular) sextic del Pezzo surfaces. Type III and IV occur in codimension two; type V occurs in codimension three.

Definition 3.2.8. *[4, Definition 11] Let $\mathbb{P}$ be a smooth complex projective surface. A good del Pezzo fibration consists of a smooth fourfold G and a flat projective morphism $\pi\colon G \to \mathbb{P}$ such that the fibers are either smooth or singular sextic del Pezzo surfaces of Type I, II, III or IV. Let B_i denote the closure of the locus of Type i fibers in $\mathbb{P}$. B_i has the following properties*
• *B_I is a non-singular curve;*
• *B_{II} is a curve, non-singular away from B_{IV};*
• *B_{III} is finite and coincides with the intersection of B_I and B_{II}, which is tranverse;*
• *B_{IV} is finite and B_{II} has cusps at B_{IV};*
• *B_V is empty.*

The discriminant curve of π has two irreducible components, a smooth sextic curve B_I and a sextic curve B_{II} with 9 cusps (see [4], [58] or [20] *for more details*). For a *good* del Pezzo fibration π, we consider the following constructions:

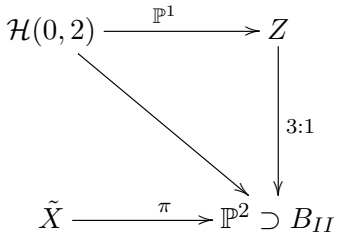
We need to consider surfaces inside two different Hilbert schemes of curves in the fibers of π. Let us denote by $\mathcal{H}(0,3)$ the relative Hilbert scheme of connected genus zero curves with anticanonical degree 3 contained in the fibers. There are two 2-dimensional families of such curves on a del Pezzo sextic. One is given by the strict transforms of the lines in $\mathbb{P}^2$ that do not pass through any of the three base points. The second is given by conics passing through

the three base points. Hence the Stein factorization yields an étale $\mathbb{P}^2$-bundle $\mathcal{H}(0,3)$ over a smooth degree two K3 surface S branched on B_I.



If the del Pezzo fibration has a rational section, then the projective bundle $\mathcal{H}(0,3) \to S$ has a rational section as well and the Brauer class $\beta \in Br(S)$ is trivial.

Let us consider now $\mathcal{H}(0,2) \to \mathbb{P}^2$ the relative Hilbert scheme of connected genus zero curves with anticanonical degree 2 contained in the fibers. The Stein factorization yields an étale $\mathbb{P}^1$-bundle $\mathcal{H}(0,2)$ over a non singular surface Z. In fact, associated to a *good* del Pezzo fibration of π, there is a non-singular triple cover $Z \to \mathbb{P}^2$ branched along a cuspidal sextic B_{II}. On every geometric fiber, the $\mathbb{P}^1$-bundle is given by the strict transform of the lines through each of the 3 blown-up points P_1, P_2, $P_3 \in \mathbb{P}^2$ of the corresponding del Pezzo of degree 6.



The Brauer class $\gamma \in Br(Z)$ of the $\mathbb{P}^1$-bundle $\mathcal{H}(0,2) \to Z$ is trivial whenever the del Pezzo fibration π has a rational section.

For more details on the Brauer classes of a sextic del Pezzo surface see [6] or [59]. For the rationality of the *good* fibration in sextic del Pezzo surfaces it is enough to have the two Brauer classes β and γ trivial (see [4, Proposition 8] or [58] *for more details*).

Furthermore, for a *good* sextic del Pezzo fibration $\pi\colon \tilde{X} \to \mathbb{P}^2$, we have that: (see [58, Proposition 5.20])

1. The Brauer class $\beta \in Br(S)$ is trivial if and only if π has a rational 2-multisection.

2. The Brauer class $\gamma \in Br(Z)$ is trivial if and only if π has a rational 3-multisection.

In terms of cycles of dimension 2, this is equivalent to the following:

Proposition 3.2.9. *Let F be the class of a fiber of the good sextic del Pezzo fibration $\pi\colon \tilde{X} \to \mathbb{P}^2$, we have that:*

1. *The Brauer class β is trivial if and only if $\exists W \in A(X)|\ (W,F) = 2$.*
2. *The Brauer class γ is trivial if and only if $\exists W \in A(X)|\ (W,F) = 3$.*

In the following, we describe irreducible components of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$, $\mathcal{C}_{18} \cap \mathcal{C}_{26}$ and $\mathcal{C}_{18} \cap \mathcal{C}_{38}$.

3.2.2.1 $\mathcal{C}_{18} \cap \mathcal{C}_{14}$:

Let X be a cubic fourfold in $\mathcal{C}_{18} \cap \mathcal{C}_{14}$; $A(X)$ has primitive sublattices $K_{18} := \langle h^2, T\rangle$ and $K_{14} := \langle h^2, \Sigma_4\rangle$, such that Σ_4 is the class of a rational quartic scroll or a degeneration of such a surface (see [21]), T is the class of an elliptic ruled surface (see §2.3.1). $X \in \mathcal{C}_{18} \cap \mathcal{C}_{14}$ has a sublattice $\langle h^2, \Sigma_4, T\rangle \subset A(X)$ with the following Gram matrix

$$\begin{array}{c} \\ h^2 \\ \Sigma_4 \\ T \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & \Sigma_4 & T \end{array} \\ \begin{pmatrix} 3 & 4 & 6 \\ 4 & 10 & \tau \\ 6 & \tau & 18 \end{pmatrix} \end{array}$$

For some $\tau \in \mathbb{Z}$ depending on X.

Denote by A_τ the lattice of rank 3 whose bilinear form has the previous Gram matrix. We recall that $\mathcal{C}_{A_\tau} \subset \mathcal{C}$ is the locus of smooth cubic fourfolds such that there is a primitive embedding $A_\tau \subset A(X)$ of lattices preserving h^2.

Theorem 3.2.10. *The intersection $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ has exactly five irreducible components $\mathcal{C}_{A_\tau}$ given by the following Gram matrix*

$$\begin{array}{c} \\ h^2 \\ \Sigma_4 \\ T \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & \Sigma_4 & T \end{array} \\ \begin{pmatrix} 3 & 4 & 6 \\ 4 & 10 & \tau \\ 6 & \tau & 18 \end{pmatrix} \end{array},$$

where $\tau \in \{4, 5, 6, 7, 8\}$.

Proof. Note that A_τ, defined as a sublattice of a positive definite lattice, must have positive discriminant by Sylvester's criterion. As $d(A_\tau) = -3(\tau^2 - 16\tau + 36)$, the only values of τ making a positive discriminant are $\tau \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$.
For these values of τ, A_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in A_\tau \subset A(X) \subset L$$

Furthermore, let $v = xh^2 + y\Sigma_4 + zT \in A_\tau$. For $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 10y^2 + 18z^2 + 8xy + 12xz + 2\tau yz.$$

For $\tau = 3$, $(-4h^2 + \Sigma_4 + T, -4h^2 + \Sigma_4 + T) = 2$; for $\tau = 13$, $(-\Sigma_4 + T, -\Sigma_4 + T) = 2$. Thus $\mathcal{C}_{A_3}$ and $\mathcal{C}_{A_{13}}$ are empty (see [112, Lemma 2.4]).
For all $\tau \in \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$, there exists no $v \in A_\tau$ such that $(v, v) = 2$. Hence, for these values of τ, $\mathcal{C}_{A_\tau} \subset \mathcal{C}$ is nonempty and has codimension 2 (see [112, Lemma 2.4]). The corresponding discriminants $d(A_\tau)$ are 36, 57, 72, 81, 84, 81, 72, 57, 36. Note that there are isometries between the lattices A_4, A_5, A_6, A_7 and A_9, A_{10}, A_{11}, A_{12} respectively (see [113, Remark 7.7]). Thus, next, we may only consider A_τ for $\tau = 4,\ 5,\ 6,\ 7,\ 8$.

We now prove that $\mathcal{C}_{A_\tau}$ is irreducible. For $\tau = 5$ the discriminant is squarefree, so there are no proper finite overlattices.
For the remaining cases, we can take h^2 and Σ_4 as a part of a basis of an overlattice B and complete it with V such that $V = xh^2 + y\Sigma_4 + zT$ with $x, y, z \in \mathbb{Q}$. As we did in previous section, we can take $z = \frac{1}{n}$ for some $n = [B : A_\tau] \in \mathbb{Z}$ and $x' = nx, y' = yn \in \mathbb{Z}$, we can write $V = \frac{1}{n}(x'h^2 + y'\Sigma_4 + T)$. By adding multiples of h^2 and Σ_4, we may ensure that $0 \leq x', y' < n$.

Computing intersections:

$$\begin{aligned}
(V, h^2) &= \frac{1}{n}(3x' + 4y' + 6) = a_3, \\
(V, \Sigma_4) &= \frac{1}{n}(4x' + 10y' + \tau) = b_3, \\
(V, V) &= \frac{1}{n^2}(3(x')^2 + 10(y')^2 + 12x' + 2\tau y' + 8x'y' + 18) = c_3.
\end{aligned}$$

Then B has the following Gram matrix:

$$\begin{array}{c} \\ h^2 \\ \Sigma_4 \\ V \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & \Sigma_4 & V \end{array} \\ \begin{pmatrix} 3 & 4 & a_3 \\ 4 & 3 & b_3 \\ a_3 & b_3 & c_3 \end{pmatrix} \end{array}$$

Now we check each case separately for possible values of τ, n, x' and y'.

For $(\tau = 4,\ n = 2,\ x' = 0,\ y' = 1)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 10 & 7 \\ 5 & 7 & 9 \end{pmatrix},$$

which has primitive short roots $\pm(1, 0, -1)$.

For $(\tau = 4,\ n = 6,\ x' = 2,\ y' = 3)$, the Gram matrix of B is:

$$\begin{pmatrix} 3 & 4 & 4 \\ 4 & 10 & 7 \\ 4 & 7 & 6 \end{pmatrix}.$$

B has primitive short roots such as $(-4, -2, 5)$.

For $(\tau = 4,\ n = 3,\ x' = 2,\ y' = 0)$, B has the following form

$$\begin{pmatrix} 3 & 4 & 4 \\ 4 & 10 & 4 \\ 4 & 4 & 6 \end{pmatrix},$$

which has primitive short roots $\pm(-2, 0, 1)$.

For $(\tau = 6,\ n = 2,\ x' = 0,\ y' = 1)$, the Gram matrix of B is

$$\begin{pmatrix} 3 & 4 & 5 \\ 4 & 10 & 8 \\ 5 & 8 & 10 \end{pmatrix},$$

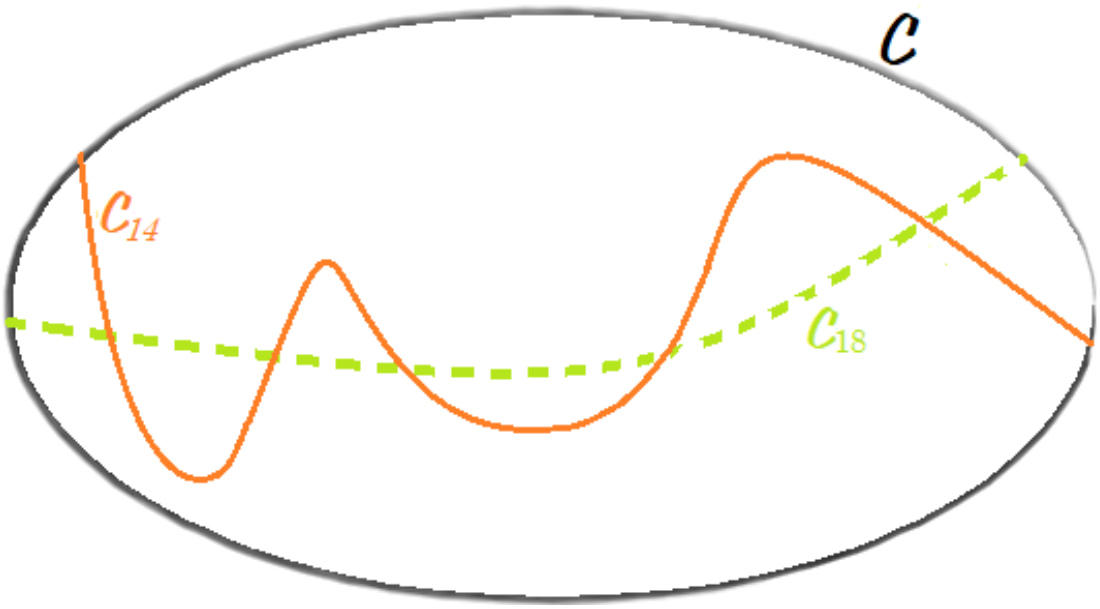
which has primitive short roots $\pm(-2, 0, 1)$.

Therefore, by Voisin [109, §4, Proposition 1], no overlattices exist in these cases.

For $(\tau = 6,\ n = 6)$, $d(B) = 2$ which is impossible by [113, Lemma 7.8].

For all other possible values τ, n, x' and y', the Gram matrices of B are non-integer. Thus all the components $\mathcal{C}_{A_\tau}$ are irreducible. □

Let F be the class of a fiber of a *good* sextic del Pezzo fibration $\pi : \tilde{X} \longrightarrow \mathbb{P}^2$, that is



$$F = 4h^2 - T.$$

For such a fibration, let us check the triviality (or not) of the associated Brauer classes for each component of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$.
Let $W_{a,b,c} \in A(X)$ be a cycle such that

$$W_{a,b,c} = ah^2 + b\Sigma_4 + cT, \text{ for } a, b, c \in \mathbb{Z}$$

We have that $(W_{a,b,c}, F) = 6a + (16 - \tau)b + 6c$. We will check for possible values of τ the triviality of β or γ for a *good* sextic del Pezzo fibration.

In the following table, we collect our results of the (non)triviality of the Brauer classes for the different components of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ depending on the existence of a cycle $W \in A(X)$ such that $(W, F) = 2$ or 3. This is true if we assume that the sextic del Pezzo fibration associated to the cubic fourfold is *good*.

τ	$(W_{a,b,c}, F) = 2$	$(W_{a,b,c}, F) = 3$	β		γ	
			triv.	nontriv.	triv.	nontriv.
4	$\emptyset$	$\emptyset$		×		×
5	$W_{0,4,-7}$	$W_{0,3,-5}$	×		×	
6	$W_{0,2,-3}$	$\emptyset$	×			×
7	$\emptyset$	$W_{0,1,-1}$		×	×	
8	$W_{0,1,-1}$	$\emptyset$	×			×

In §3.2.3, we give an explicit example of a rational cubic fourfold in $\mathcal{C}_{18}$ associated with a *good* sextic del Pezzo fibration and with one nontrivial Brauer class.

3.2.2.2 $\mathcal{C}_{18} \cap \mathcal{C}_{26}$:

Let X be a cubic fourfold in $\mathcal{C}_{18} \cap \mathcal{C}_{26}$; $A(X)$ has primitive sublattice $K_{18} := \langle h^2, T\rangle$ and $K_{26} := \langle h^2, S_{26}\rangle$, such that T and S_{26} as defined in §2.3.1.

Theorem 3.2.11. *The intersection $\mathcal{C}_{18} \cap \mathcal{C}_{26}$ has exactly seven irreducible components $\mathcal{C}_{A_\tau}$ given by the following Gram matrix*

$$\begin{array}{c} \\ h^2 \\ T \\ S_{26} \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & T & S_{26} \end{array} \\ \begin{pmatrix} 3 & 6 & 7 \\ 1 & 18 & \tau \\ 7 & \tau & 25 \end{pmatrix} \end{array},$$

where $\tau \in \{8, 9, 10, 11, 12, 13, 14\}$.

Proof. The cubic fourfold $X \in \mathcal{C}_{18} \cap \mathcal{C}_{26}$ has a sublattice $A_\tau := \langle h^2, T, S_{26}\rangle \subset A(X)$ with the Gram matrix:

$$\begin{array}{c} \\ h^2 \\ T \\ S_{26} \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & T & S_{26} \end{array} \\ \begin{pmatrix} 3 & 6 & 7 \\ 6 & 18 & \tau \\ 7 & \tau & 25 \end{pmatrix} \end{array}$$

For some $\tau \in \mathbb{Z}$ depending on X.

As $d(A_\tau) = -3(\tau^2 - 28\tau + 144)$, the only values making a positive discriminant are $\tau \in \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$.

For these values of τ, A_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in A_\tau \subset A(X) \subset L$$

Furthermore, let $v = xh^2 + yT + zS_{26} \in A_\tau$ $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 18y^2 + 25z^2 + 12xy + 14xz + 2\tau yz,$$

For $\tau = 7$, $(-5h^2+T+S_{26}, -5h^2+T+S_{26}) = 2$; for $\tau = 21$, $(h^2+T-S_{26}, h^2+T-S_{26}) = 2$. Thus $\mathcal{C}_{A_7}$ and $\mathcal{C}_{A_{21}}$ are empty (see [112, Lemma 2.4]).

For all $\tau \in \{8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$, there exists no $v \in A_\tau$ such that $(v, v) = 2$. Then for these values of τ, $\mathcal{C}_{A_\tau} \subset \mathcal{C}$ is nonempty and has codimension 2 (see [112, Lemma 2.4]). Note that there are isometries between the lattices A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13} and A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, A_{20} respectively (see [113, Remark 7.7]). Thus, next, we may consider only A_τ for $\tau = 8,\ 9,\ 10,\ 11,\ 12,\ 12,\ 13,\ 14$ of discriminant $d(A_\tau)$ respectively 48, 81, 108, 129, 144, 153, 156.

We now prove that $\mathcal{C}_{A_\tau}$ is irreducible. For $\tau = 11$ the discriminant is squarefree, so there are no proper finite overlattices. For the remaining cases, we can take h^2 and S_{18} as a part of a basis of an overlattice B and complete it with V' such that $V' = xh^2 + yT + zS_{26}$ with $x, y, z \in \mathbb{Q}$. Let $z = \frac{1}{n}$ for some $n \in \mathbb{Z}$ and $x' = nx, y' = yn \in \mathbb{Z}$; we can write $V' = \frac{1}{n}(x'h^2+y'T+S_{26})$. By adding multiples of h^2 and T, we may ensure that $0 \leq x', y' < n$. Recall that $n = [B : A_\tau] = [B_{prim} : A_{\tau,prim}]$, with B_{prim} the finite proper overlattice of $A_{\tau,prim}$.

Computing intersections:

$$\begin{aligned}
(V', h^2) &= \frac{1}{n}(3x' + 6y' + 7) = a_4, \\
(V', T) &= \frac{1}{n}(6x' + 18y' + \tau) = b_4 \\
(V', V') &= \frac{1}{n^2}(3x'^2 + 18y'^2 + 2\tau y' + 14x' + 12x'y' + 25) = c_4.
\end{aligned}$$

B has the following Gram matrix:

$$\begin{array}{c} \\ h^2 \\ T \\ V' \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & T & V' \end{array} \\ \begin{pmatrix} 3 & 6 & a_4 \\ 6 & 18 & b_4 \\ a_3 & b_3 & c_4 \end{pmatrix} \end{array}$$

Now we check each case separately for the possible values of τ, n, x' and y'. The Gram matrices of B are non-integer for all possible values of τ, n, x' and y' other than:
For $(\tau = 8,\ n = 2,\ x' = 1,\ y' = 1)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 16 \\ 8 & 16 & 22 \end{pmatrix}$$

which has primitive short roots $\pm(-2, 0, 1)$. This is impossible by Voisin [109, §4, Proposition 1].
For $n = 4$, $d(B) = 3 \equiv 3\ (mod\ 4)$ which is a contradiction by [113]. For $(\tau = 10,\ n = 2)$, $d(B) = 27 \equiv 3\ (mod\ 4)$ which is again impossible for the same reason.
For $(\tau = 12,\ n = 2,\ x' = 1,\ y' = 1)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 18 \\ 8 & 18 & 24 \end{pmatrix}.$$

The vectors $(-2, 1, 0)$ and $(-8, 0, 3)$ form a basis for B_{prim} which has the following Gram matrix:

$$\begin{pmatrix} 6 & 6 \\ 6 & 24 \end{pmatrix}.$$

B_{prim} has primitive short roots $\pm(\frac{-2}{3}, \frac{1}{6})$.
Again for $(\tau = 8$ but $\ n = 4,\ x' = 1,\ y' = 1)$, B with this Gram matrix

$$\begin{pmatrix} 3 & 6 & 4 \\ 6 & 18 & 9 \\ 4 & 9 & 6 \end{pmatrix}$$

has primitive short roots $\pm(-2, 0, 1)$.
Thus by Voisin [109, §4, Proposition 1], no overlattices exist in these cases.

For $(\tau = 14,\ n = 2)$, $d(B) = 39 \equiv 3\ (mod\ 4)$ which is not possible.
Thus all the components $\mathcal{C}_{A_\tau}$ are irreducible. □

For a *good* del Pezzo fibration π, let us check the triviality of the associated Brauer classes β and γ for each component of $\mathcal{C}_{18} \cap \mathcal{C}_{26}$.
For $W_{a,b,c}$ a cycle in $A(X)$ such that

$$W_{a,b,c} = ah^2 + bT + cS_{26}, \text{ for } a, b, c \in \mathbb{Z}$$

and F the class of a fiber of π, we have that $(W_{a,b,c}, F) = 6a + 6b + (28 - \tau)c$.

In the following table, we study the existence of some particular 2-cycles with specific intersections with the fibers and deduce the (non)triviality of the Brauer classes for the different components of $\mathcal{C}_{18} \cap \mathcal{C}_{26}$. This is true if we assume that the associated sextic del Pezzo fibration is *good* for a general cubic fourfold in the intersection.

τ	$(W_{a,b,c}, F) = 2$	$(W_{a,b,c}, F) = 3$	β		γ	
			triv.	nontriv.	triv.	nontriv.
8	$W_{0,7,-2}$	$\emptyset$	$\times$			$\times$
9	$W_{0,13,-4}$	$W_{0,10,-3}$	$\times$		$\times$	
10	$\emptyset$	$\emptyset$		$\times$		$\times$
11	$W_{0,6,-2}$	$W_{0,9,-3}$	$\times$		$\times$	
12	$W_{0,3,-1}$	$\emptyset$	$\times$			$\times$
13	$\emptyset$	$W_{0,3,-1}$		$\times$	$\times$	
14	$W_{0,5,-2}$	$\emptyset$	$\times$			$\times$

3.2.2.3 $\mathcal{C}_{18} \cap \mathcal{C}_{38}$:

Let X be a cubic fourfold in $\mathcal{C}_{18} \cap \mathcal{C}_{38}$; $A(X)$ has primitive sublattice $K_{18} := \langle h^2, T \rangle$ and $K_{38} := \langle h^2, S_{38} \rangle$, such that T is the class of an elliptic ruled surface and S_{38} is the general degree 10 smooth surface of sectional genus 6 contained in a general element of $\mathcal{C}_{38}$ (see §2.3.1). $X \in \mathcal{C}_{18} \cap \mathcal{C}_{38}$ has a sublattice $A_\tau := \langle h^2, T, S_{38} \rangle \subset A(X)$ with Gram matrix:

$$\begin{array}{c} \\ h^2 \\ T \\ S_{38} \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & T & S_{38} \end{array} \\ \begin{pmatrix} 3 & 6 & 10 \\ 6 & 18 & \tau \\ 10 & \tau & 46 \end{pmatrix} \end{array}$$

For some $\tau \in \mathbb{Z}$ depending on X.

Theorem 3.2.12. *The irreducible components of $\mathcal{C}_{18} \cap \mathcal{C}_{38}$ are the subvarieties of codimension two $\mathcal{C}_{A_\tau}$ given by rank 3 lattices represented by*

$$A_\tau := \begin{pmatrix} 3 & 6 & 10 \\ 6 & 18 & \tau \\ 10 & \tau & 46 \end{pmatrix} \text{ and } B_{16} := \begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 17 \\ 8 & 17 & 24 \end{pmatrix}$$

where $12 \leq \tau \leq 20$ and B_{16} is an overlattice of A_{16}.

Proof. As $d(A_\tau) = -3(\tau^2 - 40\tau + 324)$, the only values of τ making a positive discriminant are $\tau \in \{12, 13, ..., 28\}$.
For these values of τ, A_τ is a positive definite saturated sublattice of rank 3:

$$h^2 \in A_\tau \subset A(X) \subset L$$

Furthermore, let $v = xh^2 + yT + zS_{38} \in A_\tau$, for $x, y, z \in \mathbb{Z}$, we get

$$(v, v) = 3x^2 + 18y^2 + 46z^2 + 12xy + 20xz + 2\tau yz,$$

For these values of τ, there exists no $v \in A_\tau$ such that $(v, v) = 2$; thus $\mathcal{C}_{A_\tau} \subset \mathcal{C}$ is nonempty and has codimension 2 (see [112, Lemma 2.4]). Note that there are isometries between the lattices A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19} and A_{21}, A_{22}, A_{23}, A_{24}, A_{25}, A_{26}, A_{27}, A_{28} respectively (see [113, Remark 7.7]). Thus, next, we may consider only A_τ for $\tau = 12, 13, 14, 15, 16, 17, 18, 19, 20$. The corresponding discriminants $d(A_\tau)$ are 36, 81, 120, 153, 180, 201, 216, 225, 228. We now prove that $\mathcal{C}_{A_\tau}$ is irreducible. For $\tau = 17$ the discriminant is squarefree, so there are no proper finite overlattices. For the remaining cases, we can take h^2 and T as a part of a basis of an overlattice B and complete it with V'' such that $V'' = xh^2 + yT + zS_{38}$ with $x, y, z \in \mathbb{Q}$. Let $z = \frac{1}{n}$ for some $n = [B : A_\tau] \in \mathbb{Z}$ and $x' = nx, y' = yn \in \mathbb{Z}$, we can write $V'' = \frac{1}{n}(x'h^2 + y'T + S_{38})$. By adding multiples of h^2 and T, we may ensure that $0 \leq x', y' < n$.

Computing intersections:

$$\begin{aligned} (V'', h^2) &= \frac{1}{n}(3x' + 6y' + 10) = a_5, \\ (V'', T) &= \frac{1}{n}(6x' + 18y' + \tau) = b_5, \\ (V'', V'') &= \frac{1}{n^2}(3x'^2 + 18y'^2 + 2\tau y' + 20x' + 12x'y' + 46) = c_5. \end{aligned}$$

B has the following Gram matrix:

$$\begin{array}{c} \\ h^2 \\ T \\ V'' \end{array} \begin{array}{c} \begin{array}{ccc} h^2 & T & V'' \end{array} \\ \begin{pmatrix} 3 & 6 & a_5 \\ 6 & 18 & b_5 \\ a_5 & b_5 & c_5 \end{pmatrix} \end{array}$$

By checking each case separately for the possible values of τ, n, x' and y', one sees that all the Gram matrices of B are non-integer except for:

For $(\tau = 12,\ n = 2,\ x' = 0,\ y' = 1)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 15 \\ 8 & 15 & 22 \end{pmatrix},$$

which has primitive short roots $\pm(-2, 0, 1)$. This is impossible by Voisin [109, §4, Proposition 1].

For $(\tau = 14,\ n = 2)$, $d(B) = 30 \equiv 2\ (mod\ 4)$ which is impossible by [113, Lemma 7.8].

For $(\tau = 16,\ n = 2,\ x' = 0,\ y' = 1)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 17 \\ 8 & 17 & 24 \end{pmatrix}$$

of discriminant 45 that we will denoted by B_{16}. Hence $\mathcal{C}_{A_{16}}$ consists of 2 irreducible components.

For $(\tau = 18,\ n = 2)$, $d(B) = 54 \equiv 2\ (mod\ 4)$ which is impossible by [113, Lemma 7.8].

For $(\tau = 19,\ n = 5,\ x' = 3,\ y' = 1)$, B has the following Gram matrix

$$\begin{pmatrix} 3 & 6 & 5 \\ 6 & 18 & 11 \\ 5 & 11 & 9 \end{pmatrix},$$

which has primitive short roots $\pm(-1, 0, 1)$; For $(\tau = 20,\ n = 2,\ x' = 0,\ y' = 1)$, the Gram matrix of B is

$$\begin{pmatrix} 3 & 6 & 8 \\ 6 & 18 & 19 \\ 8 & 19 & 26 \end{pmatrix},$$

which has primitive short roots such as $(-2,0,5)$. Therefore, by Voisin [109, §4, Proposition 1], no overlattices exist in these cases as well.

□

Let us check the triviality of the Brauer classes β and γ associated to the *good* fibration π for each component of $\mathcal{C}_{18} \cap \mathcal{C}_{38}$.
For $W_{a,b,c}$ a cycle in $A(X)$ such that $W_{a,b,c} = ah^2 + bT + cS_{38}$, for $a, b, c \in \mathbb{Z}$ and F the class of a fiber of π, we have that

$$(W_{a,b,c}, F) = 6a + 6b + (40 - \tau)c.$$

In the following table, we study the (non)triviality of the Brauer classes for the different components of $\mathcal{C}_{18} \cap \mathcal{C}_{38}$ depending on the existence of a multisection of degree 2 or 3. This is true if we assume that the associated sextic del Pezzo fibration is *good*.

τ	$(W_{a,b,c}, F) = 2$	$(W_{a,b,c}, F) = 3$	β		γ	
			triv.	nontriv.	triv.	nontriv.
12	$W_{0,5,-1}$	$\emptyset$	×			×
13	$\emptyset$	$W_{0,5,-1}$		×	×	
14	$W_{0,9,-2}$	$\emptyset$	×			×
15	$W_{0,17,-4}$	$W_{0,13,-3}$	×		×	
16	$\emptyset$	$\emptyset$		×		×
16′	$\emptyset$	$\emptyset$		×		×
17	$W_{0,8,-2}$	$W_{0,12,-3}$	×		×	
18	$W_{0,4,-1}$	$\emptyset$	×			×
19	$\emptyset$	$W_{0,4,-1}$		×	×	
20	$W_{0,7,-2}$	$\emptyset$	×			×

3.2.3 Explicit examples of rational fibered cubic fourfolds

We shall give here explicit examples of rational fibered cubic fourfolds with nontrivial Brauer classes. All our computations have been done using Macaulay2 [39]. We work over the finite field F_{3331} but our equations hold over fields of characteristic zero.

Let $\mathbb{P}^5 = Proj(F_{3331}[x_0, ..., x_5])$ and $\mathbb{P}^2 = Proj(F_{3331}[t_0, ..., t_2])$.

We provide first an example of rational cubic fourfold in $\mathcal{C}_8$ containing a *good* plane with nontrivial Brauer class using §3.2.1. Let S_{38} be the smooth surface of degree 10 contained in a general element of $\mathcal{C}_{38}$ as defined in §2.3.1.
Let P be the plane whose ideal is generated by P_1, P_2 and P_3 as follows:

$$\begin{aligned} P_1 &= x_2 - 1277x_3 - 733x_4 + 1373x_5, \\ P_2 &= x_1 + 1280x_3 + 1650x_4 - 1339x_5, \\ P_3 &= x_0 - 954x_3 - 133x_4 - 1153x_5. \end{aligned}$$

Note that P and S_{38} intersect transversely in 2 points. They are contained in the cubic fourfold X cut out by

$$\begin{aligned} C' := {} & x_0^3 - 994x_0^2x_1 - 337x_0x_1^2 - 1402x_1^3 - 297x_0^2x_2 + 22x_0x_1x_2 \\ & - 295x_1^2x_2 - 1639x_0x_2^2 - 331x_1x_2^2 + 830x_2^3 - 182x_0^2x_3 - 717x_0x_1x_3 \\ & + 38x_1^2x_3 - 655x_0x_2x_3 + 1519x_1x_2x_3 + 1348x_2^2x_3 - 359x_0x_3^2 + 1500x_1x_3^2 \\ & + 850x_2x_3^2 + 270x_3^3 - 1479x_0^2x_4 + 906x_0x_1x_4 + 862x_1^2x_4 + 354x_0x_2x_4 \\ & - 1561x_1x_2x_4 - 1196x_2^2x_4 + 530x_0x_3x_4 - 690x_1x_3x_4 - 151x_2x_3x_4 \\ & + 1101x_3^2x_4 + 989x_0x_4^2 + 1045x_1x_4^2 + 966x_2x_4^2 + 1292x_3x_4^2 - 868x_4^3 \\ & - 1612x_0^2x_5 - 767x_0x_1x_5 + 1027x_1^2x_5 - 108x_0x_2x_5 + 533x_1x_2x_5 - 338x_2^2x_5 \\ & + 45x_0x_3x_5 + 269x_1x_3x_5 - 469x_2x_3x_5 + 946x_3^2x_5 + 686x_0x_4x_5 \\ & + 1070x_1x_4x_5 + 26x_2x_4x_5 - 741x_3x_4x_5 + 1403x_4^2x_5 + 663x_0x_5^2 + 1067x_1x_5^2 \\ & + 1609x_2x_5^2 - 1403x_3x_5^2. \end{aligned}$$

X is a smooth cubic hypersurface in $\mathbb{P}^5$ contained in the intersection $\mathcal{C}_8 \cap \mathcal{C}_{38}$. The discriminant divisor $D \in \mathbb{P}^2$ of the map $q\colon \tilde{X} \to \mathbb{P}^2$ is a smooth sextic curve defined as follows:

$$\begin{aligned} D : t_0^6 + 249t_0^5t_1 - 66t_0^4t_1^2 - 1000t_0^3t_1^3 - 306t_0^2t_1^4 - 862t_0t_1^5 - 408t_1^6 \\ - 588t_0^5t_2 + 138t_0^4t_1t_2 + 72t_0^3t_1^2t_2 - 1637t_0^2t_1^3t_2 - 1358t_0t_1^4t_2 \\ + 1364t_1^5t_2 - 528t_0^4t_2^2 + 994t_0^3t_1t_2^2 + 1595t_0^2t_1^2t_2^2 - 1525t_0t_1^3t_2^2 \\ - 510t_1^4t_2^2 - 857t_0^3t_2^3 + 1334t_0^2t_1t_2^3 + 1248t_0t_1^2t_2^3 + 1295t_1^3t_2^3 \\ - 668t_0^2t_2^4 + 936t_0t_1t_2^4 + 236t_1^2t_2^4 + 708t_0t_2^5 + 151t_1t_2^5 + 160t_2^6 = 0. \end{aligned}$$

Hence, X is a smooth rational cubic fourfold containing a *good* plane P inside the irreducible component of $\mathcal{C}_8 \cap \mathcal{C}_{38}$ indexed by $\tau = (P, S_{38}) = 2$. By Corollary 3.2.7, the quadric surface bundle associated to cubic fourfolds in this component has no rational section.

We exhibit now an example of rational cubic fourfold $X' \in \mathcal{C}_{18}$ for which the fibration in sextic del Pezzo surfaces is *good*, with a nontrivial Brauer class. This illustrates Theorem II in §3.1.

Using §3.2.2, we know the possible values of $\tau = (T, \Sigma_4)$ corresponding to the intersection pairing for the elliptic ruled surface T and the quartic rational normal scroll Σ_4 contained in a general element of respectively $\mathcal{C}_{18}$ and $\mathcal{C}_{14}$.
Note that $\mathcal{C}_{14}$ could be described as the closure of the locus of cubic fourfolds containing a quintic del Pezzo surface (see §2.3.1, [21]).
Let $\mathbf{D}$ be a quintic del Pezzo surface in $\mathbb{P}^5$ as defined in [21, Section 4]. We have that $\Sigma_4 = 3h^2 - \mathbf{D}$.

To construct T, we define three quadrics Q_1, Q_2 and Q_3 as follows:

$$\begin{aligned} Q_1 &= x_2x_3 - 967x_0x_4 + 622x_1x_4 - 1581x_2x_4 + 1403x_0x_5 - 122x_2x_5, \\ Q_2 &= x_1x_3 + 1069x_0x_4 + 1170x_1x_4 + 76x_2x_4 + 1072x_0x_5 - 881x_1x_5 + 237x_2x_5, \\ Q_3 &= x_0x_3 + 242x_1x_4 - 565x_2x_4 - 370x_0x_5 - 518x_1x_5 - 901x_2x_5. \end{aligned}$$

Each of these quadrics contains two disjoint planes:

$$\Pi_1 = \{x_0 = x_1 = x_2\} \text{ and } \Pi_2 = \{x_3 = x_4 = x_5\}.$$

T is obtained by saturating the ideal generated by Q_1, Q_2 and Q_3 with respect to the defining ideals of the planes Π_1 and Π_2. In other words, T is the residual intersection of the Q_i.

Since T and $\mathbf{D}$ intersect transversely in 12 points, we have that $\langle T, \Sigma_4\rangle = \langle T, 3h^2 - \mathbf{D}\rangle = 3\langle T, h^2\rangle - \langle T, \mathbf{D}\rangle = 6$.

The surfaces T and $\mathbf{D}$ are contained in the cubic fourfold X' cut out by

$$\begin{aligned} C'' := {} & x_0x_1x_3 - 242x_1^2x_3 + 354x_0x_2x_3 - 843x_1x_2x_3 - 552x_2^2x_3 \\ & + 462x_1x_3^2 - 983x_2x_3^2 + 606x_0^2x_4 + 141x_0x_1x_4 - 674x_0x_2x_4 + 1157x_1x_2x_4 \\ & + 275x_2^2x_4 - 462x_0x_3x_4 - 1645x_1x_3x_4 + 116x_2x_3x_4 \\ & + 1645x_0x_4^2 + 641x_2x_4^2 - 354x_0^2x_5 + 1098_0x_1x_5 - 1056x_1^2x_5 \\ & + 552x_0x_2x_5 - 463x_1x_2x_5 + 376x_0x_3x_5 + 1600x_1x_3x_5 \\ & - 1533x_2x_3x_5 - 1008x_0x_4x_5 - 742x_1x_4x_5 - 267x_2x_4x_5 + 1345x_0x_5^2 \\ & + 455x_1x_5^2 \end{aligned}$$

X' is a smooth irreducible subscheme of $\mathbb{P}^5$ of dimension 4 and degree 3.

Let us check that the sextic del Pezzo fibration associated to X' is *good*.

The discriminant locus of the map $\pi\colon \tilde{X}' \to \mathbb{P}^2$ is a reducible curve of degree 12 with two irreducible components:

$$\begin{aligned} B_I : {} & -60t_0^6 + 223t_0^5t_1 + 1421t_0^4t_1^2 + 69t_0^3t_1^3 + 61t_0^2t_1^4 + 311t_0t_1^5 + 1537t_1^6 \\ & + 781t_0^5t_2 + 117t_0^4t_1t_2 + 1452t_0^3t_1^2t_2 + 625t_0^2t_1^3t_2 + 144t_0t_1^4t_2 + 1560t_1^5t_2 \\ & - 203t_0^4t_2^2 - 314t_0^3t_1t_2^2 - 673t_0^2t_1^2t_2^2 - 239t_0t_1^3t_2^2 - 1007t_1^4t_2^2 - 1633t_0^3t_2^3 \\ & + 1643t_0^2t_1t_2^3 - 530t_0t_1^2t_2^3 + 71t_1^3t_2^3 + 849t_0^2t_2^4 - 1293t_0t_1t_2^4 + 1356t_1^2t_2^4 \\ & - 928t_0t_2^5 + 1055t_1t_2^5 + t_2^6 = 0 \end{aligned}$$

$$\begin{aligned}B_{II} : 1262t_0^6 &- 979t_0^5t_1 + 315t_0^4t_1^2 + 1534t_0^3t_1^3 + 1545t_0^2t_1^4 - 551t_0t_1^5\\ &- 451t_1^6 - 1475t_0^5t_2 - 812t_0^4t_1t_2 + 182t_0^3t_1^2t_2 - 205t_0^2t_1^3t_2\\ &+ 1579t_0t_1^4t_2 - 1528t_1^5t_2 - 21t_0^4t_2^2 + 495t_0^3t_1t_2^2 + 124t_0^2t_1^2t_2^2\\ &- 1633t_0t_1^3t_2^2 - 503t_1^4t_2^2 + 1452t_0^3t_2^3 - 995t_0^2t_1t_2^3 + 825t_0t_1^2t_2^3\\ &- 1491t_1^3t_2^3 + 278t_0^2t_2^4 + 381t_0t_1t_2^4 - 1183t_1^2t_2^4 + 1412t_0t_2^5 + 1530t_1t_2^5 + t_2^6 = 0\end{aligned}$$

B_I is a smooth irreducible subscheme of $\mathbb{P}^2$ of dimension 1 and degree 6, B_{II} has 9 cusps. They intersect transversely in 36 points. This proves that π is *good* (see §3.2.2).

Hence, X' is a rational smooth cubic fourfold in the irreducible component corresponding to $\tau = 6$ of $\mathcal{C}_{18} \cap \mathcal{C}_{14}$ with a *good* associated sextic del Pezzo fibration and a nontrivial Brauer class γ.

3.3 Intersection of more than two Hassett divisors

3.3.1 Rational cubic fourfolds with associated K3 surfaces of arbitrary rank

In this Section, we need to recall some lattice theory from §2.2.
Let E_8^1 and E_8^2 be the two copies of E_8, the unimodular positive definite even rank 8 lattice represented by its matrix in the basis $\langle t_k^i \rangle$, for $k = 1, .., 8$, $i = 1, 2$. In addition, let U^1 and U^2 be two copies of the hyperbolic plane U. We have that for $k = 1, 2$,

$$U^k \simeq \mathbb{Z}.e_1^k \bigoplus \mathbb{Z}.e_2^k$$

with the quadratic form given by

$$(e_1^k, e_1^k) = 0,\ (e_1^k, e_2^k) = 1,\ (e_2^k, e_2^k) = 0.$$

A_2 is given by the bilinear form $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the canonical basis $\langle a_1, a_2 \rangle$.

Using previous notations, we can denote now L and L_{prim} as follows:

$$L \simeq \quad E_8^1 \bigoplus E_8^2 \bigoplus U^1 \bigoplus U^2 \bigoplus I_{3,0},$$

$$L_{prim} \simeq \quad E_8^1 \bigoplus E_8^2 \bigoplus U^1 \bigoplus U^2 \bigoplus A_2.$$

Proposition 3.3.1. *[112, Theorem 3.1] Any two Hassett divisors intersect i.e $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \neq \emptyset$ for any integers d_1 and d_2 satisfying the condition $(*)$:*

$$d_1,\ d_2 \geq 8 \text{ and } d_1,\ d_2 \equiv 0 \text{ or } 2 \ (mod\ 6).$$

Remark 3.3.2. *If X is a cubic fourfold in $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$, then there exists infinitely many divisors $\mathcal{C}_d$ containing X. In fact, $A(X)$ is a rank 3 lattice generated by h^2 and both surfaces S_1 and S_2 defining respectively $\mathcal{C}_{d_1}$ and $\mathcal{C}_{d_2}$. For every vector u in the sublattice $\langle S_1, S_2 \rangle$, $u^\perp \subset A(X)$ is of rank 2. This defines a labelling (thus a divisor containing X), and moving u appropriately we obtain as many different labellings as we want.*

We denote by $(***)$ the following condition on the discriminant d:

$$(***) \qquad d = 6d' \text{ or } d = 6d' + 2, \text{ where } d' \text{ is a square.}$$

(Prime numbers appearing inside the previous formula are not necessary distinct)

Theorem 3.3.3 (Theorem III)**.** *(i) For $3 \leq n \leq 20$,*

$$\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset,$$

with $d_k \geq 8,\ d_k \equiv 0, 2\ (mod\ 6)$ and $d_3, .., d_n = 6d'$ or $6d' + 2$, d' a square.

(ii) There exist families of cubic fourfolds $\mathcal{F} \subseteq \bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$, of dimension $20 - n$, such that a generic cubic fourfold $X \in \mathcal{F}$ has $\mathrm{rk}(A(X)) = n + 1$.

(iii) In addition, if at least one of the divisors $\mathcal{C}_{d_k}$ in the intersection parametrizes cubic fourfolds with an associated K3 surface (in other words, d_k satisfies the condition $(**)$), then cubic fourfolds in the families $\mathcal{F}$ have an associated K3 surface with a Néron-Severi group of rank greater than or equal to n.

For the sake of simplicity, the proof of the Theorem will be divided into different steps. We mainly use the following Proposition:

Proposition 3.3.4. *[48, Proposition 12] Let M be a positive definite lattice of rank $rk(M) \geq 2$, that admits a saturated embedding*

$$h^2 \in M \subset L,$$

Let $\mathcal{C}_M \subset \mathcal{C}$ be the locus of cubic fourfolds X having algebraic classes with lattice structure

$$h^2 \in M \subset A(X) \subset L.$$

If $\mathcal{C}_M$ is non-empty, then it has codimension $rk(M) - 1$ and there exists $X \in \mathcal{C}_M$ with $A(X) = M$.

To prove the first point (i) of the Theorem, we proceed as following. First, in Lemma 3.3.5, we give a complete proof for the non emptiness of the intersection of any three divisors $\mathcal{C}_{d_1}, \mathcal{C}_{d_2}$ and $\mathcal{C}_{d_3}$ for d_1, d_2 and d_3 satisfying $(*)$ and d_3 satisfying $(****)$ as well. The proof is divided into 4 parts covering all possible cases, depending alternatively on the values of the discriminants. Then in Proposition 3.3.7, we prove that $\bigcap_{1\leq k\leq 20} \mathcal{C}_{d_k} \neq \emptyset$ for particular choices of the integers d_k. Finally in Theorem 3.3.8, we prove the full statement of (i). Moreover, the part (ii) of the Theorem 3.3.3 can be deduced from the proof of (i) and Proposition 3.3.4. In fact, in the proof of (i), in order to show the non emptiness, we construct families of cubic fourfolds in the intersection of Hassett divisors with lattice of algebraic cycles of a given rank. Then, applying Proposition 3.3.4, we deduce (ii). On the other hand, Theorem 3.3.3 (iii) can be concluded from Corollary 3.3.9.

Lemma 3.3.5. *We have $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \neq \emptyset$ for any integers d_1, d_2, d_3 satisfying $(*)$ such that d_3 satisfies $(****)$.*

Proof. The proof is divided in multiple cases depending on the values of the discriminants.

Case 1: if $d_1, d_2, d_3 \equiv 0 \ (mod\ 6)$ such that $d_1 = 6n_1, d_2 = 6n_2$ and $d_3 = 6n_3$, for $n_1, n_2 \geq 2$ and n_3 a square.
We consider the rank 4 lattice,

$M := \langle h^2, \alpha_1, \alpha_2, \alpha_3 \rangle$, with $\alpha_1 = e_1^1 + n_1 e_2^1, \alpha_2 = e_1^2 + n_2 e_2^2$ and $\alpha_3 = \sqrt{n_3} a_1$.
The Gram matrix of M with respect to this basis is:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2n_1 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 \\ 0 & 0 & 0 & 2n_3 \end{pmatrix}.$$

Therefore, M is a positive definite saturated sublattice of L. Moreover, for any nonzero $v = x_1 h^2 + x_2 \alpha_1 + x_3 \alpha_2 + x_4 \alpha_3$, where $x_1, .., x_4$ are integers not all zeros, we have

$$(v, v) = 3x_1^2 + 2n_1 x_2^2 + 2n_2 x_3^2 + 2n_3 x_4^2 \geq 3,$$

$\forall n_1, n_2 \geq 2, n_3$ as required.

We denote by $\mathcal{C}_M \subset \mathcal{C}$ the locus of smooth cubic fourfold such that there is a primitive embedding $M \subset A(X)$ preserving h^2. The locus $\mathcal{C}_M$ is nonempty of codimension 3 by [112, Proposition 2.3]. Let us consider now the sublattices

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$
$$K_{d_2} := \langle h^2, \alpha_2 \rangle \subset M$$
$$K_{d_3} := \langle h^2, \alpha_3 \rangle \subset M$$

of discriminant respectively d_1, d_2 and d_3. By [112, Lemma 2.4, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3}$ as required.

Case 2: if $d_1, d_2 \equiv 0 \ (mod\ 6), d_3 \equiv 2 \ (mod\ 6)$ such that $d_1 = 6n_1, d_2 = 6n_2$ and $d_3 = 6n_3 + 2$ for $n_1, n_2 \geq 2$, and n_3 a square.
This time we consider the positive definite saturated rank 4 sublattice of L:

$$M := \langle h^2, \alpha_1, \alpha_2, \alpha_3 + (0, 0, 1) \rangle.$$

The Gram matrix of M is:

$$\begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2n_1 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 \\ 1 & 0 & 0 & 2n_3 + 1 \end{pmatrix}.$$

Additionally, for any nonzero $v = x_1h^2 + x_2\alpha_1 + x_3\alpha_2 + x_4(\alpha_3 + (0,0,1))$, we have that:

$$\begin{aligned}(v,v) = &\quad 3x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + (2n_3+1)x_4^2 + 2x_1x_4 \\ = &\quad 2x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + 2n_3x_4^2 + (x_1+x_4)^2 \geq 3,\end{aligned}$$

$\forall n_1, n_2 \geq 2$ and n_3 a square. Then, applying [112, Proposition 2.3], we have that $\mathcal{C}_M$ is nonempty of codimension 3. Now we define the following saturated rank 2 sublattices of M:

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \text{ with Gram matrix } \begin{pmatrix} 3 & 0 \\ 0 & 2n_1 \end{pmatrix},$$

$$K_{d_2} := \langle h^2, \alpha_2 \rangle \text{ with Gram matrix } \begin{pmatrix} 3 & 0 \\ 0 & 2n_2 \end{pmatrix},$$

$$K_{d_3} := \langle h^2, \alpha_3 + (0,0,1) \rangle \text{ with Gram matrix } \begin{pmatrix} 3 & 1 \\ 1 & 2n_3+1 \end{pmatrix}.$$

Again, by [112, Lemma 2.4] and [112, Proposition 2.3], we have that $\mathcal{C}_M \subset \mathcal{C}_{K_{d_1}} = \mathcal{C}_{d_1}$, $\mathcal{C}_M \subset \mathcal{C}_{K_{d_2}} = \mathcal{C}_{d_2}$ and $\mathcal{C}_M \subset \mathcal{C}_{K_{d_3}} = \mathcal{C}_{d_3}$. Consequently, $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \neq \emptyset$ in this case as well.

Case 3: if $d_1 \equiv 0 \ (mod\ 6)$ and $d_2, d_3 \equiv 2 \ (mod\ 6)$ such that $d_1 = 6n_1, d_2 = 6n_2 + 2$ and $d_3 = 6n_3 + 2$ for $n_1 \geq 2, n_2 \geq 1$ and n_3 a square.
We consider the rank 4 lattice,

$$M := \langle h^2, \alpha_1, \alpha_2 + (0,1,0), \alpha_3 + (0,0,1) \rangle.$$

The Gram matrix of M is:

$$\begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 2n_1 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 \\ 1 & 0 & 0 & 2n_3+1 \end{pmatrix}.$$

M is a positive definite saturated sublattice of L. Moreover, for any nonzero $v = x_1h^2 + x_2\alpha_1 + x_3(\alpha_2 + (0,1,0)) + x_4(\alpha_3 + (0,0,1)) \in M$,

$$\begin{aligned}(v,v) = &\quad 3x_1^2 + 2n_1x_2^2 + (2n_2+1)x_3^2 + (2n_3+1)x_4^2 + 2x_1x_3 + 2x_1x_4\\ = &\quad x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + 2n_3x_4^2 + (x_1+x_3)^2 + (x_1+x_4)^2 \geq 3,\end{aligned}$$

$\forall n_1 \geq 2, n_2 \geq 1, n_3$ as required. Then $\mathcal{C}_M$ is nonempty of codimension 3 by [112, Proposition 2.3]. Now let us consider the saturated rank 2 sublattices of M:

$$K_{d_1} := \langle h^2, \alpha_1\rangle \text{ of Gram matrix } \begin{pmatrix} 3 & 0 \\ 0 & 2n_1 \end{pmatrix},$$

$$K_{d_2} := \langle h^2, \alpha_2 + (0,1,0)\rangle \text{ of Gram matrix } \begin{pmatrix} 3 & 1 \\ 1 & 2n_2+1 \end{pmatrix},$$

$$K_{d_3} := \langle h^2, \alpha_3 + (0,0,1)\rangle \text{ of Gram matrix } \begin{pmatrix} 3 & 1 \\ 1 & 2n_3+1 \end{pmatrix}.$$

It is straightforward to see that $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \neq \emptyset$ also in this case.

Case 4: if $d_1, d_2, d_3 \equiv 2 \ (mod\ 6)$ such that $d_1 = 6n_1 + 2, d_2 = 6n_2 + 2$ and $d_3 = 6n_3 + 2$, for $n_1, n_2 \geq 1$ and n_3 a square.
Let us consider the lattice M as follows

$$M := \langle h^2, \alpha_1 + (1,0,0), \alpha_2 + (0,1,0), \alpha_3 + (0,0,1)\rangle$$

The Gram matrix of M with respect to this basis is:

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 2n_1+1 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 \\ 1 & 0 & 0 & 2n_3+1 \end{pmatrix},$$

and M is a rank 4 positive definite saturated sublattice of L.
Furthermore, for any nonzero $v = x_1h^2 + x_2(\alpha_1 + (1,0,0)) + x_3(\alpha_2 + (0,1,0)) + x_4(\alpha_3 +$

$(0,0,1)) \in M$, we have that:

$$\begin{aligned}(v,v) = \; & 3x_1^2 + (2n_1+1)x_2^2 + (2n_2+1)x_3^2 + (2n_3+1)x_4^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4 \\ = \; & 2n_1x_2^2 + 2n_2x_3^2 + 2n_3x_4^2 + (x_1+x_2)^2(x_1+x_3)^2 + (x_1+x_4)^2 \geq 3,\end{aligned}$$

$\forall n_1, n_2 \geq 1, n_3$ as required. As usual, we consider rank 2 saturated sublattices of M:

$$\begin{aligned}K_{d_1} &:= \langle h^2, \alpha_1 + (1,0.0)\rangle, \\ K_{d_2} &:= \langle h^2, \alpha_2 + (0,1,0)\rangle, \\ K_{d_3} &:= \langle h^2, \alpha_3 + (0,0,1)\rangle.\end{aligned}$$

Applying [112, Lemma 2.4] and [112, Proposition 2.3], $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \neq \emptyset$ also in this case. □

Remark 3.3.6. *The condition required on the third discriminant is a sufficient condition not necessary. Yang and Yu [112, Theorem 3.3] proved, for example, that $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{14} \neq \emptyset$ for d_1 and d_2 satisfying $(*)$.*

Remark that by Proposition 3.3.4, the proof of Lemma 3.3.5 implies the existence of an explicit 17-dimensional family of cubic fourfolds in the intersection of the three divisors, generically with an algebraic cycles lattice of rank 4.

Proposition 3.3.7. $\bigcap_{k=1}^{20} \mathcal{C}_{d_k} \neq \emptyset$, $\forall d_k \equiv 0 \pmod 6$ *such that* $d_3, .., d_{20} = 6d'$, d' *a square.*

Proof. We will prove that $\forall n \geq 4$, $\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$.
Since for $n \leq 3$, this formula has already been proved (see Proposition 3.3.1 and Lemma 3.3.5), we proceed to consider the case $n = 4$.

For $n = 4$: $d_1 = 6n_1, d_2 = 6n_2, d_3 = 6n_3, d_4 = 6n_4$ for $n_1, n_2 \geq 2$ and n_3, n_4 squares.
Let us consider the rank 5 lattice,

$$M := \langle h^2, \alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle,$$

with $\alpha_1 = e_1^1 + n_1 e_2^1, \alpha_2 = e_1^2 + n_2 e_2^2$, $\alpha_3 = \sqrt{n_3}a_1$ and $\alpha_4 = \sqrt{n_4}a_2$.
The lattice M is a positive definite saturated sublattice of L with the following Gram matrix:

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 \\ 0 & 0 & 0 & 2n_3 & \sqrt{n_3 n_4} \\ 0 & 0 & 0 & \sqrt{n_3 n_4} & 2n_4 \end{pmatrix}.$$

Furthermore, for any nonzero $v = x_1 h^2 + x_2\alpha_1 + x_3\alpha_3 + x_4\alpha_3 + x_5\alpha_4 \in M$, we have that

$$(v,v) = 3x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 \geq 3,$$

$\forall n_1, n_2 \geq 2$, n_3, n_4 squares (thus $n_3, n_4 \geq 4$). The locus $\mathcal{C}_M$ is then nonempty of codimension 4. Now let us consider rank 2 saturated sublattices of M:

$$K_{d_i} := \langle h^2, \alpha_i \rangle, i : 1, .., 4$$

of discriminant respectively d_i. By [112, Lemma 2.4] and [112, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i.

Then the same method is applied to check the non emptiness of $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$, for $5 \leq n \leq 19$, hence we refrain to give details. Nevertheless, let us go through the final, $n = 20$, step.

For n=20: if $d_i = 6n_i$, $i = 1, .., 20$, with $n_1, n_2 \geq 2$ and $n_3, .., n_{20}$ squares.
We need to consider the rank 21 lattice M generated by:
$\langle h^2, \alpha_1, \alpha_2, \sqrt{n_3}a_1, \sqrt{n_4}a_2, \sqrt{n_5}t_1^1, \sqrt{n_6}t_3^1, \sqrt{n_7}t_6^1, \sqrt{n_8}t_1^2, \sqrt{n_9}t_3^2, \sqrt{n_{10}}t_6^2, \sqrt{n_{11}}t_2^1,$
$\sqrt{n_{12}}t_2^2, \sqrt{n_{13}}t_4^1, \sqrt{n_{14}}t_4^2, \sqrt{n_{15}}t_7^1, \sqrt{n_{16}}t_7^2, \sqrt{n_{17}}t_8^1, \sqrt{n_{18}}t_8^2, \sqrt{n_{19}}t_5^1, \sqrt{n_{20}}t_5^2\rangle$, with the following Gram matrix:

$$\left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right)$$

such that

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2n_3 & \sqrt{n_3n_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{n_3n_4} & 2n_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2n_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2n_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{10} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{n_5n_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{n_6n_{11}} & 0 & -\sqrt{n_6n_{13}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_6n_{19}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{n_7n_{15}} & 0 & 0 & 0 & -\sqrt{n_7n_{19}} & 0 \\ 0 & -\sqrt{n_8n_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{n_9n_{12}} & 0 & -\sqrt{n_9n_{14}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_9n_{20}} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{n_{10}n_{16}} & 0 & 0 & 0 & -\sqrt{n_{10}n_{20}} \end{pmatrix}$$

$$C = \begin{pmatrix} 2n_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2n_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2n_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2n_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2n_{15} & 0 & -\sqrt{n_{15}n_{17}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2n_{16} & 0 & -\sqrt{n_{16}n_{18}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{n_{15}n_{17}} & 0 & 2n_{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{n_{16}n_{18}} & 0 & 2n_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{19} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{20} \end{pmatrix}$$

The lattice $M \subset L$ is a positive definite saturated lattice containing h^2. Moreover, for all $v \in M$, we have that

$$(v, v) \neq 2,$$

for $n_1, n_2 \geq 2$, $n_3, .., n_{20}$ as required. Then, by [112, Propsition 2.3, Lemma 2.4], the locus $\mathcal{C}_M$ is nonempty of codimension 20.

Finally, we consider the rank 2 saturated sublattices $K_{d_i} \subset M$ generated by h^2 and another element from the basis of M: $\alpha_1, \alpha_2, \sqrt{n_3}a_1, \sqrt{n_4}a_2, \sqrt{n_5}t_1^1, \sqrt{n_6}t_3^1, \sqrt{n_7}t_6^1, \sqrt{n_8}t_1^2, \sqrt{n_9}t_3^2, \sqrt{n_{10}}t_6^2, \sqrt{n_{11}}t_2^1, \sqrt{n_{12}}t_2^2, \sqrt{n_{13}}t_4^1, \sqrt{n_{14}}t_4^2, \sqrt{n_{15}}t_7^1, \sqrt{n_{16}}t_7^2, \sqrt{n_{17}}t_8^1, \sqrt{n_{18}}t_8^2, \sqrt{n_{19}}t_5^1, \sqrt{n_{20}}t_5^2$. K_{d_i} are of discriminant d_i respectively. Applying [112, Propsition 2.3, Lemma 2.4] again, we obtain $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap .. \cap \mathcal{C}_{d_{20}}$ with the required conditions on d_i. □

As before, Proposition 3.3.4 implies that the proof of Proposition 3.3.7 yields families of dimension $20 - n$ of cubic fourfolds inside the intersection of the n different divisors. The generic rank of $A(X)$ for these cubics is $n + 1$.

The proof of the next theorem is again divided into multiple cases. In fact, Lemma 3.3.5 and Proposition 3.3.7 are particular cases of the next theorem, which generalises them. Next, we prove that the intersection of 20 divisors $\mathcal{C}_{d_i}$ is nonempty whenever $d_i \equiv 0, 2 \ (mod\ 6)$, for all i.

Theorem 3.3.8. *[Theorem 3.3.3 (i)]We have* $\bigcap_{k=1}^{20} \mathcal{C}_{d_k} \neq \emptyset$ *for* d_k *satisfying* $(*)$ *and* $d_3, .., d_{20} = 6d'$ *or* $6d' + 2$, *with* d' *a square.*

Proof. Let us consider $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$.

For $n = 2$: see [112, Theorem 3.1].

For $n = 3$: see Lemma 3.3.5.

For $1 \leq n \leq 20$: for $d_k \equiv 0 \ (mod\ 6)$, $d_3, .., d_{20} = 6d'$, see Proposition 3.3.7.

In the following, we will describe in details the case $n = 4$. For higher values of n, it will be clear that one needs to use exactly the same strategy, hence for the sake of shortness, we will omit most of them. Details are left to the interested readers.

For $n = 4$:

Case 1: if all discriminants $d_1,\ d_2,\ d_3,\ d_4 \equiv 0 \ (mod\ 6)$. This case is proved in Proposition 3.3.7.

Case 2: if $d_1 = 6n_1, d_2 = 6n_2, d_3 = 6n_3, d_4 = 6n_4 + 2$ such that $n_1, n_2 \geq 2$ and n_3, n_4 squares.

We consider the rank 5 lattice,

$M := \langle h^2, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + (0, 0, 1)\rangle$, with $\alpha_1 = e_1^1 + n_1 e_2^1, \alpha_2 = e_1^2 + n_2 e_2^2$, $\alpha_3 = \sqrt{n_3}a_1$ and $\alpha_4 = \sqrt{n_4}a_2$.

The Gram matrix of M with respect to this basis is:

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 1 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 \\ 0 & 0 & 0 & 2n_3 & \sqrt{n_3 n_4} \\ 1 & 0 & 0 & \sqrt{n_3 n_4} & 2n_4 + 1 \end{pmatrix},$$

and M is a positive definite saturated sublattice of L.

For all $v \in M$ $v = x_1 h^2 + x_2\alpha_1 + x_3\alpha_2 + x_4\alpha_3 + x_5(\alpha_4 + (0, 0, 1)) \in M$, we have that

$$(v, v) = 2x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1 + x_5)^2 \neq 2,$$

for all $n_1, n_2 \geq 2$, n_3, n_4 squares (thus $n_3, n_4 \geq 4$). Then $\mathcal{C}_M$ is nonempty of codimension 4. Moreover, we consider

$$K_{d_i} := \langle h^2, \alpha_i\rangle,\ i : 1, .., 4.$$

Hence, by [112, Lemma 2.4, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i in this case.

Case 3: if $d_1 = 6n_1, d_2 = 6n_2, d_3 = 6n_3 + 2, d_4 = 6n_4 + 2$ such that $n_1, n_2 \geq 2$ and n_3, n_4 squares.

We consider the rank 5 lattice,

$$M := \langle h^2, \alpha_1, \alpha_2, \alpha_3 + (0,1,0), \alpha_4 + (0,0,1) \rangle,$$

with $\alpha_1 = e_1^1 + n_1 e_2^1, \alpha_2 = e_1^2 + n_2 e_2^2$, $\alpha_3 = \sqrt{n_3} a_1$ and $\alpha_4 = \sqrt{n_4} a_2$.

The Gram matrix of M is

$$\begin{pmatrix} 3 & 0 & 0 & 1 & 1 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 \\ 1 & 0 & 0 & 2n_3 + 1 & \sqrt{n_3 n_4} \\ 1 & 0 & 0 & \sqrt{n_3 n_4} & 2n_4 + 1 \end{pmatrix},$$

and M is a positive definite saturated sublattice of L.

In addition, for all $v \in M$ $v = x_1 h^2 + x_2 \alpha_1 + x_3 \alpha_2 + x_4(\alpha_3 + (0,1,0)) + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) = x_1^2 + 2n_1 x_2^2 + 2n_2 x_3^2 + n_3 x_4^2 + n_4 x_5^2 + (\sqrt{n_3} x_4 + \sqrt{n_4} x_5)^2 + (x_1 + x_5)^2 + (x_1 + x_4)^2 \neq 2,$$

for all $n_1, n_2 \geq 2$, n_3, n_4 squares. Then $\mathcal{C}_M$ is nonempty of codimension 4 by [112, Proposition 2.3]. Moreover, we consider

$$K_{d_i} := \langle h^2, \alpha_i \rangle, \ i : 1, .., 4.$$

By [112, Lemma 2.4, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i in this case.

Case 4: if $d_1 = 6n_1, d_2 = 6n_2 + 2, d_3 = 6n_3 + 2, d_4 = 6n_4 + 2$ such that $n_1 \geq 2$, $n_2 \geq 1$ and n_3, n_4 squares.

We consider the rank 5 lattice,

$$M := \langle h^2, \alpha_1, \alpha_2 + (1,0,0), \alpha_3 + (0,1,0), \alpha_4 + (0,0,1) \rangle.$$

The Gram matrix of M is:

$$\begin{pmatrix} 3 & 0 & 1 & 1 & 1 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 & 0 \\ 1 & 0 & 0 & 2n_3+1 & \sqrt{n_3n_4} \\ 1 & 0 & 0 & \sqrt{n_3n_4} & 2n_4+1 \end{pmatrix}.$$

Hence, M is a positive definite saturated sublattice of L.
Furthermore, for all $v \in M$, $v = x_1h^2 + x_2\alpha_1 + x_3(\alpha_2 + (1,0,0)) + x_4(\alpha_3 + (0,1,0)) + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) =$$
$$2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1+x_5)^2 + (x_1+x_4)^2 + (x_1+x_3)^2 \neq 2,$$

for all $n_1 \geq 2$, $n_2 \geq 1$ and n_3, n_4 squares (thus $n_3, n_4 \geq 4$). Then $\mathcal{C}_M$ is nonempty of codimension 4. Moreover, we consider

$$K_{d_i} := \langle h^2, \alpha_i \rangle,\ i : 1, .., 4.$$

We have then that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i in this case.

Case 5: if $d_1 = 6n_1 + 2, d_2 = 6n_2 + 2, d_3 = 6n_3 + 2, d_4 = 6n_4 + 2$ such that $n_1, n_2 \geq 1$ and n_3, n_4 squares.
We consider the rank 5 lattice,

$$M := \langle h^2, \alpha_1 + (1,0,0), \alpha_2 + (1,0,0), \alpha_3 + (0,1,0), \alpha_4 + (0,0,1) \rangle$$

with the following Gram matrix:

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 2n_1+1 & 0 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 & 0 \\ 1 & 0 & 0 & 2n_3+1 & \sqrt{n_3n_4} \\ 1 & 0 & 0 & \sqrt{n_3n_4} & 2n_4+1 \end{pmatrix}.$$

The lattice M is a positive definite saturated sublattice of L.
For all $v \in M$, $v = x_1h^2 + x_2(\alpha_1 + (1,0,0)) + x_3(\alpha_2 + (1,0,0)) + x_4(\alpha_3 + (0,1,0)) + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) = 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1+x_5)^2 + (x_1+x_4)^2 + (x_1+x_3)^2 + (x_1+x_2)^2 - x_1^2 \neq 2,$$

for all $n_1, n_2 \geq 1$, n_3, n_4 squares. Then $\mathcal{C}_M$ is nonempty of codimension 4. We consider the rank 2 saturated sublattices

$$K_{d_i} := \langle h^2, \alpha_i \rangle,\ i : 1, .., 4.$$

Applying [112, Lemma 2.4], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i in this case.

For $5 \leq n \leq 19$: In the same way, one can continue to consider, for each n-intersection of Hassett divisors, certain positive definite saturated (n+1)-rank lattice E containing h^2 and check that for a nonzero $v \in E$, $(v.v) \geq 3$ or equivalently for all $v \in E$ $(v,v) \neq 2$. This allows us to conclude the non emptiness of the intersection (by [112, Lemma 2.4, Propositon 2.3]).

Finally, in order to study the intersection of 20 Hassett divisors $\mathcal{C}_{d_k}$ such that all $d_k \equiv 2\ (mod\ 6)$, We consider, for example, the rank 21 lattice M, generated by

$\langle h^2, \alpha_1 + (0,1,0), \alpha_2 + (1,0,0), \sqrt{n_3}a_1 + (1,0,0), \sqrt{n_4}a_2 + (0,1,0), \sqrt{n_5}t_1^1 + (0,0,1), \sqrt{n_6}t_3^1 + (0,0,1), \sqrt{n_7}t_6^1 + (1,0,0), \sqrt{n_8}t_1^2 + (0,1,0), \sqrt{n_9}t_3^2 + (0,0,1), \sqrt{n_{10}}t_6^2 + (0,0,1), \sqrt{n_{11}}t_2^1 + (0,0,1), \sqrt{n_{12}}t_2^2 + (0,0,1), \sqrt{n_{13}}t_4^1 + (0,0,1), \sqrt{n_{14}}t_4^2 + (1,0,0), \sqrt{n_{15}}t_7^1 + (0,1,0), \sqrt{n_{16}}t_7^2 + (0,1,0), \sqrt{n_{17}}t_8^1 + (0,1,0), \sqrt{n_{18}}t_8^2 + (0,1,0), \sqrt{n_{19}}t_5^1 + (0,1,0), \sqrt{n_{20}}t_5^2 + (0,1,0)\rangle$.

The Gram matrix of M is:

$$\left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right)$$

with

$$A = \begin{pmatrix}
3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2n_1+1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 2n_2+1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 2n_3+1 & \sqrt{n_3n_4} & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & \sqrt{n_3n_4} & 2n_4+1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 2n_5+1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 2n_6+1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 2n_7+1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2n_8+1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2n_9+1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2n_{10}+1
\end{pmatrix}$$

$$B = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1-\sqrt{n_5n_{11}} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sqrt{n_6n_{11}} & 1 & 1-\sqrt{n_6n_{13}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_6n_{19}} & 0 \\
0 & 0 & 0 & 1 & -\sqrt{n_7n_{15}} & 0 & 0 & 0 & -\sqrt{n_7n_{19}} & 0 \\
0 & -\sqrt{n_8n_{12}} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1-\sqrt{n_9n_{12}} & 1 & -\sqrt{n_9n_{14}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_9n_{20}} \\
1 & 1 & 1 & 0 & 0 & -\sqrt{n_{10}n_{16}} & 0 & 0 & 0 & -\sqrt{n_{10}n_{20}}
\end{pmatrix}$$

$$C = \begin{pmatrix}
2n_{11}+1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2n_{12}+1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2n_{13}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2n_{14}+1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2n_{15}+1 & 0 & -\sqrt{n_{15}n_{17}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2n_{16}+1 & 0 & -\sqrt{n_{16}n_{18}} & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{n_{15}n_{17}} & 0 & 2n_{17}+1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{n_{16}n_{18}} & 0 & 2n_{18}+1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{19}+1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{20}+1
\end{pmatrix}$$

M is a positive definite saturated sublattice of L. For all $v \in M, (v,v) \neq 2$, $\forall n_1, n_2 \geq 1$, n_k squares for $k = 3, .., 20$.

We consider rank 2 saturated sublattices $h^2 \in K_{d_i}$ of discriminant d_i as before.

Therefore, by [112, Lemma 2.4, Proposition 2.3], $\bigcap_{k=1}^{20} \mathcal{C}_{d_k} \neq \emptyset$ for all d_k satisfying $(*)$ and for $k \geq 3$, d_k satisfies also $(***)$. □

Let us denote by $\mathcal{F}$ the families of cubic fourfolds constructed in Theorem 3.3.8. By definition they are contained in the non empty intersection $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$, and all these cubics verify $M \subset A(X)$, for M the fixed rank $n+1$ lattice (see Proposition 3.3.4). The lattice M coincides with the lattice of algebraic classes $A(X)$, for a generic $X \in \mathcal{F}$. A straightforward result is the following:

Corollary 3.3.9. *Let $\mathcal{F} \subseteq \bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$ be the family of cubic fourfolds as constructed above and suppose that at least one of the divisors $\mathcal{C}_{d_k}$ parametrizes cubic fourfolds with an associated K3 surface. Then there exist cubic fourfolds in $\mathcal{F}$ with an associated K3 surface S such that* $\mathrm{rk}(NS(S)) = n$.

Proof. This is straightforward, since a generic cubic fourfold $X \in \mathcal{F} \subseteq \bigcap_{k=1}^{n} \mathcal{C}_{d_k}$ have $\mathrm{rk}(A(X)) = n+1$, hence the primitive cohomology of the associated K3 surface has rank $n-1$, and the Néron-Severi group has rank n. Note that the intersection of $\mathcal{F}$ with further Hassett divisors may represent cubic fourfolds whose associated K3 surface S has $\mathrm{rk}(NS(S)) \geq n$. □

Remark 3.3.10. *After our paper [11] came out, Yang and Yu [113] proved that the intersection of all Hassett divisors is nonempty by showing that the Fermat cubic fourfold is contained in every Hassett divisor, hence in the intersection of all divisors. This intersection is rational, of dimension at least* 13. *However, our construction allowed us to produce concrete examples of cubic fourfolds contained in certain families of arbitrary dimension, from 0 to 18, in $\mathcal{C}$. In addition, for generic cubic fourfolds X in these families, the lattice $A(X)$ is of any prescribed rank.*

3.3.2 Families of cubic fourfolds with finite dimensional Chow motives of abelian type

Let $\mathcal{M}_{rat}(\mathbb{C})$ be the (contravariant) category of Chow motives and let $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$ be the strictly full, thick, rigid, tensor subcategory of $\mathcal{M}_{rat}(\mathbb{C})$ generated by the motives of abelian varieties (see Appendix B for background material of motives).

Next, using the intersection theory developed in §3.3, we construct new one dimensional families of cubic fourfolds with finite dimensional, abelian motive (see [11]).

Theorem 3.3.11. *Every Hassett divisor $\mathcal{C}_d$ contains a one dimensional family of cubic fourfolds, whose Chow motive is finite dimensional and abelian.*

Proof. Let us consider $\mathcal{C}_d \subset \mathcal{C}$ any divisor of special cubic fourfolds. By Theorem 3.3.3, one can choose appropriately 17 divisors $\mathcal{C}_{d_1}, \dots \mathcal{C}_{d_{17}}$ and consider the intersection

$$\mathcal{C}_d \cap \mathcal{C}_{14} \cap (\bigcap_{k=1}^{17} \mathcal{C}_{d_k}),$$

which is nonempty.

In particular, by the proof of Theorem 3.3.3, one sees that a family of cubic fourfolds $\mathcal{F} \subseteq \mathcal{C}_d \cap \mathcal{C}_{14} \cap (\bigcap_{k=1}^{17} \mathcal{C}_{d_k})$ is a one dimensional algebraic subvariety of $\mathcal{C}_d$, that is, a family of cubic fourfolds of discriminant d. On the other hand, by construction, the family $\mathcal{F}$ is also contained in $\mathcal{C}_{14}$, hence all the cubics in $\mathcal{F}$ have an associated K3 surface. Moreover, by the results of Section 3.3 (see Corollary 3.3.9), we observe that cubic fourfolds in $\mathcal{F}$ have associated K3 surfaces with Néron-Severi rank $\rho(S) \geq 19$. More precisely: the generic cubic fourfold in $\mathcal{F}$ has an associated K3 surface with $\rho(S) = 19$; the intersection points of $\mathcal{F}$ with further Hassett divisors may represent cubic fourfolds whose associated K3 surface has $\rho(S) = 20$. By Theorem B.4 the Chow motives of these K3 surfaces are finite dimensional and Abelian. Now we need to evoke the isomorphism of Eq. B.2. The divisor $\mathcal{C}_{14}$ is among those whose cubic fourfolds have Chow motives that decomposes as follows

$$h(X) = \mathbb{1} \oplus \mathbb{L} \oplus (\mathbb{L}^2)^{\oplus \rho_2(X)} \oplus t_2(S)(-1) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4$$

where $t_2(S)$ is the transcendental part of the motive $h(S)$ of the associated K3 surface. Now, if the motive of the associated K3 surface is finite dimensional or Abelian, then also the motive of the cubic fourfold has the same property.

This means in turn that, by Theorem B.4, all the cubics in $\mathcal{F}$ have finite dimensional and abelian Chow motive, since the associated K3 surfaces have Néron-Severi group of rank bigger or equal to 19. □

Remark 3.3.12. *It is well-known that K3 surfaces S with Picard number $\rho(S) = \mathrm{rk}(NS(S)) \geq 19$ are very special, since these are the maximal possible ranks of the Néron-Severi group.*

They are called singular K3 surfaces and have some very particular features. On one hand, as seen before, their Chow motives are abelian and finite dimensional (see Appendix B). On the other hand, they admit a Shioda-Inose structure [99]. In particular, there is a birational map from the K3 surface to a Kummer surface (see [75], [94]).

Remark 3.3.13. *Let us point out that, since all the family $\mathcal{F}$ is contained in $\mathcal{C}_{14}$, then, by results of [21], all cubic fourfolds in $\mathcal{F}$ are rational.*

In the proof of Theorem 3.3.11, we chose $\mathcal{C}_{14}$ for simplicity, since it is the first of the series that has d verifying condition:

$$(5*) \qquad \exists f, g \in \mathbb{Z} \text{ with } g|(2n^2+2n+2)\ n \in \mathbb{N} \text{ and } d = f^2 g,$$

and where cubics have associated K3 surfaces (*i.e.* d verifies also the condition $(**)$). Any other divisor $\mathcal{C}_d$ obeying $(5*)$, such that cubics in $\mathcal{C}_d$ have associated K3 surfaces, would have worked. Hence we can say even more.

Corollary 3.3.14 (Theorem IV)**.** *Every Hassett divisor $\mathcal{C}_d$ contains a countable infinity of one dimensional families of cubic fourfolds, whose Chow motive is finite dimensional and Abelian.*

Proof. Just consider, in $(5*)$, $f = 1$ and $g = 2n^2 + 2n + 2$, for any $n \in \mathbb{N}$. This gives an infinite series of values s, such that cubics in $\mathcal{C}_s$ have associated K3 surfaces and verify the isomorphism (B.2). Hence, this construction

$$\mathcal{F} \subset \mathcal{C}_s \cap \mathcal{C}_d \cap (\bigcap_{k=1}^{17} \mathcal{C}_{d_k})$$

gives rise to an infinite one dimensional families of cubic fourfolds with Chow motives of abelian type and finite dimensional as it corresponds, in each case, to different locus of cubic fourfolds. For the reader's convenience one can consider the case for example where d and $d_k \equiv 0\ (mod\ 6)$ for $k = 1, .., 17$, with M the rank 20 lattice associated to $\mathcal{F}$. By fixing all divisors except for $\mathcal{C}_s$, the discriminant of M varies as well. In fact, in this particular case, $d(M) = s.d(M')$, with M' a fixed sublattice of rank 18 (see §3.3.1 for similar constructions). □

3.3.2.1 Density of cubic fourfolds with abelian motive

Theorem 3.3.15. *Let d be not divisible by 4, 9, or any odd prime number $p \equiv 2 \pmod 3$. Then there exists a countable, dense (in the complex topology) set of points in a non-empty Zariski open subset inside $\mathcal{C}_d$ such that the corresponding fourfolds have finite dimensional Chow motive.*

Proof. Let $X \in \mathcal{C}_d$, for d in the range of the claim here above. That is: X has one (or two, see [46]) associated polarized K3 surface S_X in $\mathcal{F}_{\frac{d+2}{2}}$. Then the map

$$\mathcal{F}_{\frac{d+2}{2}} \quad \to \quad \mathcal{C}_d; \tag{3.1}$$

$$S_X \quad \mapsto \quad X; \tag{3.2}$$

is rational and dominant. Hence, if d is in the range here above, there exists an open set $\mathcal{U}_d$ of $\mathcal{C}_d$ such that for every $X \in \mathcal{U}_d$ there exists a K3 surface S_X of degree d associated to X. We observe also that singular K3 surfaces form a (countable) subset of the moduli space $\mathcal{F}_{\frac{d+2}{2}}$ which is dense in the complex topology. The proof of this fact goes along the same lines as the proof of the density of all K3 surfaces in the period domain (see [44, Corollary VIII.8.5]). By the dominance of the map in (3.1), this directly implies the claim. □

3.3.2.2 Some remarks on Hyperkähler varieties

We mention here some consequences on Abelianity and finite dimensionality of the motive of some Hyperkähler varieties related to cubic fourfolds from the results of the preceding section.

Notably, we will consider $F(X)$ the Fano variety of lines and $L(X)$, the 8-fold constructed in [67] from the space of twisted cubic curves on a cubic fourfold not containing a plane. The 4-dimensional $F(X)$ is in general deformation equivalent to the Hilbert scheme $S^{[2]}$, with S a K3 surface, while $L(X)$ is deformation equivalent to $S^{[4]}$.

By [25, Section 6], the Hilbert scheme $S^{[n]}$ of a K3 surface S with finite dimensional (or abelian) motive has finite dimensional (or abelian) motive. Now, recall that our family $\mathcal{F}$ of cubic fourfolds from Theorem 3.3.11 entirely lies in $\mathcal{C}_{14}$. Moreover, for all cubics X in $\mathcal{C}_{14}$, we have a birationality $F(X) \sim S^{[2]}$ (see [1]), where S is the associated K3 surface. Hence,

since birational Hyperkähler varieties have isomorphic Chow motives, it is straightforward to check that we have the following.

Proposition 3.3.16. *All Hyperkähler fourfolds $F(X)$, $X \in \mathcal{F}$, have finitely generated and abelian Chow motive.*

Remark 3.3.17. *Once again, we can play the same game as before by taking $\mathcal{C}_{\frac{2n^2+2n+2}{a^2}}$, for $n, a \in \mathbb{Z}$, instead of $\mathcal{C}_{14}$. By [1, Thm. 2], having a d of this shape (i.e. d satisfying the condition $(***)$ (see §2.4)) is equivalent to having a birational equivalence between $F(X)$ and the Hilbert square of the associated K3 surface. Hence, we have a countably infinite set of families of Fano varieties $F(X)$ with finite dimensional and abelian motive, whose cubic fourfolds all lie in a fixed $\mathcal{C}_d$.*

On the other hand, let us now consider the Hyperkähler 8fold $L(X)$. In order to define properly $L(X)$ we need to assume that X does not contain a plane, i.e. $X \notin \mathcal{C}_8$. Then, the analogue of Proposition 3.3.16 is the following.

Proposition 3.3.18. *All Hyperkähler 8folds $L(X)$, $X \in \mathcal{F}$, have finitely generated and Abelian Chow motive.*

Proof. Let $X \in \mathcal{C}_d$ be a cubic fourfold not containing a plane, such that S is its associated K3 surface. In [85, Thm. 3] the authors show that the 8fold $L(X)$ is birational to $S^{[4]}$ if and only if

$$(6*) \qquad d = \frac{6n^2 + 6n + 2}{a^2}, \quad n,\ a \in \mathbb{Z}.$$

The first integer of the list is once again 14, hence for all the cubic fourfolds of our family $\mathcal{F}$ we have $L(X) \sim S^{[4]}$. Since birational Hyperkähler varieties have isomorphic Chow motives, the results from [25] complete the proof, and we have a one-dimensional family of $L(X)$ with finite dimensional and abelian motive for all $\mathcal{C}_d$. □

Remark 3.3.19. *By taking $\mathcal{C}_{14}$ in the proof of Proposition 3.3.16 and 3.3.18, we observe that this verifies both condition $(***)$ and $(6*)$. Then the intersection*

$$\mathcal{F} \subseteq \mathcal{C}_{14} \cap \Big(\bigcap_{k=1}^{18}\Big)\mathcal{C}_{d_k} \neq \emptyset. \tag{3.3}$$

defines a one dimensional family, inside any $\mathcal{C}_d$ of cubic fourfolds such that the corresponding Fano varieties and LLSvS 8folds have finite dimensional and abelian motive, too.

Chapter 4

Universal families of Fano fourfolds

4.1 Introduction

Universal families are well known and studied objects in the realm of moduli spaces of curves [33], where forgetful maps are the algebraic geometer's everyday tools. Also in the context of K3 surfaces several results about these objects have appeared in the literature, also very recently in [15, 34, 35, 70]. Probably because the birational geometry of a Fano fourfold, such as a cubic fourfold, is already so difficult to understand as we have seen in the previous chapters, the problem of studying the same issues for their universal families seems to have been overtaken. Thanks to the recent advances in the study of rationality of a Fano fourfold, such as a cubic fourfold but also a Gushel-Mukai fourfold (see Appendix C for some generalities on Gushel-Mukai fourfolds), it seems natural to ask what can be said in families, in particular over certain relevant loci in their moduli spaces. This is what we will talk about in this chapter.

First, we are interested in universal cubics fourfolds over certain divisors $\mathcal{C}_d$ in $\mathcal{C}$. In [10], we prove the following fact:

Theorem V: *The universal cubic fourfolds $\mathcal{C}_{d,1}$ are unirational in the range $8 \leq d \leq 42$.*

More precisely, we propose two different methods, since each one seems to have its own interest, concerning different aspects of the geometry of cubic fourfolds. On one hand, we use some recent results of Farkas-Verra [34, 35] about universal K3 surfaces and the relation

with cubic fourfolds given by associated K3 surfaces. On the other hand, we use the presentation of Hassett divisors as cubics containing certain rational surfaces, as done by Nuer [87] and Russo-Staglianò [91, 93], and Kollár's theorem [57] on unirationality of cubics. This second argument, applied inductively, also allows us to prove the following fact:

Theorem VI: *The universal families $\mathcal{C}_{d,m}$ are unirational for all values of $m \in \mathbb{N}^*$, $8 \leq d \leq 42$.*

Furthermore, by collecting some recent results of Gritsenko-Hulek-Sankaran [41, 42] and Várilly-Alvarado-Tanimoto [104] about the Kodaira dimension of moduli spaces of K3 surfaces and of the Hassett divisors $\mathcal{C}_d$, we were able to specify an infinite range of values of d where the universal families over these divisors $\mathcal{C}_d$ cannot be unirational. In fact, in [42], it is proven that the moduli space of polarized K3 surfaces of degree $2m$ has non-negative Kodaira dimension or is even of general type for an infinity of values (see Proposition 2.3.5 for details). Furthermore Várilly-Alvarado and Tanimoto [104] have started pursuing the project of filling certain gaps, obtaining another relevant range of values where the Kodaira dimension of $\mathcal{C}_d$ is positive (see Proposition 2.3.6). These results give us *negative* information about the divisors $\mathcal{C}_d$ where the universal cubic $\mathcal{C}_{d,1}$ can not be unirational.

Proposition VII: *The universal cubic fourfold $\mathcal{C}_{d,1}$ is not unirational if:*

1. *$d > 80$, $d \equiv 2 \ (mod\ 6)$, $4 \not| d$ and such that for any odd prime p, $p|d$ implies $p \equiv 1 \ (mod\ 3)$;*
2. *$d = 6n + 2$, $n > 13$ and $n \neq 15$;*
3. *$d = 6n$ for $n > 16$ and $n \neq 18, 20, 22, 30$.*

In this chapter, we are interested also in another class of Fano fourfold called the Gushel-Mukai fourfold (GM fourfold, for short). The interest in these fourfolds is related to their rationality problems and similarities with cubic fourfolds. In a fashion very similar to cubic fourfolds, all GM fourfolds are unirational, moreover some examples of rational GM fourfolds are not hard to construct. Nevertheless, one expects that a very general GM fourfold should be irrational, but no example has yet been proven to be irrational (see Appendix C).

We denote by $\mathcal{M}^4_{GM}$ the moduli space of GM fourfolds of dimension 24 (see [26]). Once again, similarly to the case of cubic fourfolds (see [46], [47]), via Hodge theory and the period map, Debarre, Iliev and Manivel [26] defined the *Hodge-special* GM fourfolds, that is, fourfolds that contain a surface whose cohomology class does not come from the tautological classes of the Grassmannian $\mathbb{G}(1,4)$. They showed that these families of fourfolds in $\mathcal{M}^4_{GM}$, called *Noether-Lefschetz loci*, correspond to a countable union of hypersurfaces $(\mathcal{M}^4_{GM})_d$ in $\mathcal{M}^4_{GM}$. In this chapter, we consider exclusively *ordinary* GM fourfolds. They can be viewed as smooth quadric hypersurfaces in a smooth del Pezzo fivefold $\mathbb{Y}^5 := \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$, thus parametrized by an open subset V in the 39-dimensional projective space $\mathbb{P}(\mathcal{O}_{\mathbb{Y}^5}(2))$; they correspond to the points of an open subset in $\mathcal{M}^4_{GM}$ viewed as the *GIT* quotient $V//PGL(9,\mathbb{C})$.

One can, in the same spirit of cubic fourfolds, consider universal fourfolds over some specific families of GM fourfolds. First, we notice that, when GM fourfolds are rational, a geometric construction is very often provided by means of K3 surfaces (for example seen as base loci of linear systems). The birational geometry of the universal GM families once again depends very much on the birational geometry of certain moduli spaces $\mathcal{F}_{g,n}$ of n-pointed K3 surfaces of genus g (see [70], [15]). Of course this also gives information about the birational geometry of the base of the family inside $\mathcal{M}^4_{GM}$. Notably we will consider the divisor $(\mathcal{M}^4_{GM})_{20}$ and using its geometric description given in [49, 101], we prove that:

Proposition VIII: *The universal family $(\mathcal{M}^4_{GM})_{20,1}$ is unirational.*

We restrict then our attention to a dimension one locus $(\mathcal{M}^4_{GM})^{nod}_{20}$ inside $(\mathcal{M}^4_{GM})_{20}$, defined via certain genus 11 K3 surfaces contained in a Noether-Lefschetz divisor. By describing the birational geometry of the Noether-Lefschetz divisor in the moduli of K3 surfaces, and exploiting the relation between these surfaces and the GM fourfolds, we prove that $(\mathcal{M}^4_{GM})^{nod}_{20}$ and the universal family $(\mathcal{M}^4_{GM})^{nod}_{20,1}$ above it are rational.

On the way to our Theorem, we also show the following result:

Proposition A.2.2: *$\mathcal{F}_{11,7}$, the moduli space of $7-$pointed K3 surfaces of genus 11, is unirational .*

Last, we propose a unified, general, abstract method to show the unirationality of the m-pointed universal (cubic and GM) fourfolds over their moduli spaces. This can be applied to any family $\mathcal{X}_{\mathcal{S}}$ of (cubic and GM) fourfolds that contains surfaces from a given family $\mathcal{S}$, under some hypotheses (see Remark 4.4.12) on $\mathcal{X}_{\mathcal{S}}$ and $\mathcal{S}$.

Theorem IX: *The universal m-pointed fourfold over the following irreducible codimension-one loci:*

$$\mathcal{C}_{14},\ \mathcal{C}_{26},\ \mathcal{C}_{38},\ \mathcal{C}_{42},\ (\mathcal{M}^4_{GM})'_{10},\ (\mathcal{M}^4_{GM})''_{10},\ (\mathcal{M}^4_{GM})_{20},$$

are unirational.

4.2 Unirationality of universal families of cubic fourfolds

4.2.1 Existence of universal cubic fourfolds

Similarly to the case of curves [33] and K3 surfaces [15, 34, 35, 70], we give the following definition of universal families of cubic fourfolds:

Definition 4.2.1. *By universal cubic fourfold* over a divisor $\mathcal{C}_d$ we mean the moduli space $\mathcal{C}_{d,1}$ of 1-pointed cubic fourfolds equipped with the forgetful map $\mathcal{C}_{d,1} \to \mathcal{C}_d$.

Despite the immediate existence of these universal families in the sense of stacks, it needs however a little bit of justification when working on coarse moduli spaces.

Proposition 4.2.2. *The generic cubic fourfold in any divisor $\mathcal{C}_d$ does not have projective automorphisms, hence a universal family of cubic fourfolds exists over an open subset of each divisor in the moduli space $\mathcal{C}$.*

Proof. By Theorem 2.1.2, since we are working with smooth cubic hypersurfaces, all points are stable in the *GIT* sense considered in §2.1.2. In addition, in [37, Thm 3.8] it is proven that the families of cubic fourfolds with a non-trivial projective automorphism have dimension at most 14. This means that these families cannot contain any divisor of the moduli space. Hence the generic element of any divisor has no non trivial automorphism and universal families then exists at least over an open subset of our divisors in the moduli space. □

4.2.2 Method 1: unirationality via universal K3 surfaces

4.2.2.1 Some properties of scrolls and associated K3 surfaces

By Section 4.2.1, there exists, at least over a dense subset of $\mathcal{C}_{26}$ and $\mathcal{C}_{42}$, a universal family of cubic fourfolds. For simplicity, and since we are however working in the birational category, we will still denote the two universal families by $\mathcal{C}_{26,1} \to \mathcal{C}_{26}$ and $\mathcal{C}_{42,1} \to \mathcal{C}_{42}$, without stressing the fact that they may not be defined everywhere.

Definition 4.2.3. *(see §2.3.1) We recall that:*

- $\mathcal{C}_{26} := \overline{\{\text{Cubic fourfolds containing a } 3-\text{nodal septic scroll}\}}$
- $\mathcal{C}_{42} := \overline{\{\text{Cubic fourfolds containing a } 8-\text{nodal degree 9 scroll}\}}$.

Let us recall here the construction of the scrolls defining these divisors (see Appendix A.3). We refer to [34, 35], for more details.

Let $\mathbb{F}_1 = Bl_w(\mathbb{P}^2)$ be a Hirzebruch surface, for $w \in \mathbb{P}^2$; we denote by l the class of a line and E the exceptional divisor.
A smooth septic scroll $\Sigma_7 := R_{3,4} \subseteq \mathbb{P}^8$ is the image of the linear system $\psi_{4l-3E} : \mathbb{F}_1 \hookrightarrow \mathbb{P}^8$. We denote by $\Gamma := \langle p_1, p_2, p_3 \rangle \in \mathbb{G}(2,8)$ the linear span of three general points in the secant variety $Sec(\Sigma_7) \subseteq \mathbb{P}^8$. The image Σ_7^3 of the projection with center Γ:

$$\pi_\Gamma : \Sigma_7 \longrightarrow \mathbb{P}^5$$

is a $3-$nodal septic scroll.

Again to construct an $8-$nodal nonic scroll, we start with a smooth degree 9 scroll $\Sigma_9 := R_{4,5} \subseteq \mathbb{P}^{10}$, image of the linear system $\psi_{5l-4E} : \mathbb{F}_1 \hookrightarrow \mathbb{P}^{10}$. We choose a general $4-$plane $\Omega \in \mathbb{G}(4,10)$ which is $8-$secant to the secant variety $Sec(\Sigma_9) \subseteq \mathbb{P}^{10}$. Assuming $\Omega \cap \Sigma_9 = \emptyset$, an $8-$nodal nonic scroll $\Sigma_9^8 \subseteq \mathbb{P}^5$ is the image of the restriction to Σ_9 of the projection with center Ω:

$$\pi_{\Omega|\Sigma_9} : \Sigma_9 \longrightarrow \mathbb{P}^5.$$

Furthermore, cubic fourfolds inside these two divisors have an associated K3 surface (actually cubics in $\mathcal{C}_{42}$ have two associated K3 surfaces (see [35, Section 2])). The associated K3 surfaces for cubics from $\mathcal{C}_{26}$ have genus 14 and degree 26, whereas for cubics from $\mathcal{C}_{42}$ they have genus 22 and degree 42.

By §2.3.2, for $\mathcal{C}_{26}$ and $\mathcal{C}_{42}$, we have the following rational maps

$$\begin{array}{ccc} \mathcal{F}_{14} & \overset{\sim}{\dashrightarrow} & \mathcal{C}_{26}; \\ \mathcal{F}_{22} & \overset{2:1}{\dashrightarrow} & \mathcal{C}_{42}. \end{array}$$

Moreover, for a general $X \in \mathcal{C}_{2(n^2+n+1)}$, $n \geq 2$ (remark that 26 and 42 all verify this equality), the isomorphism (2.2)

$$S^{[2]} \cong F(X) \hookrightarrow \mathbb{G}(1,5)$$

holds and the family of scrolls contained inside X is precisely parametrized by the associated K3 surface S. The construction, roughly speaking, goes as follows. For each $p \in S$, one defines a rational curve

$$\Delta_p := \{y \in S^{[2]} : \{p\} = Supp(y)\}.$$

The image of Δ_p inside $F(X)$ then defines a (possibly singular) scroll $R_p \subset X$.

Proposition 4.2.4. *[34, 35]*

1. *The family of septimic 3-nodal scrolls inside a generic cubic fourfold X in $\mathcal{C}_{26}$ is the genus 14 K3 surface associated to X.*

2. *The family of 8-nodal degree 9 scrolls inside a generic cubic fourfold X in $\mathcal{C}_{42}$ is the genus 22 K3 surface associated to X.*

Recall from Proposition A.2.1 that $\mathcal{F}_{14,1}$ and $\mathcal{F}_{22,1}$ are unirational.

Remark 4.2.5. *Given the result of [37], Proposition 4.2.2 is straightforward and very general. For $d = 26$, and 42, since the isomorphism (2.2) holds, one could also argue in a more "geometric" way as follows. Any automorphism $\phi \in Aut(X)$ of a smooth cubic fourfold X induces an automorphism of the Fano variety of lines $F(X)$. Then, via the isomorphism $F(X) \cong S^{[2]}$ with the symmetric square of the associated K3 surface, one can use the fact that the generic K3 surface has no non-trivial automorphism to conclude that ϕ is the identity. Remark also that, by Proposition 4.2.4, the fact that the associated K3 surface has no non-trivial automorphism implies that any $\phi \in Aut(X)$ should send each scroll onto itself.*

4.2.2.2 Unirationality of $\mathcal{C}_{26,1}$ and $\mathcal{C}_{42,1}$

We will show next that:

Theorem 4.2.6. *The universal cubic fourfolds $\mathcal{C}_{26,1}$ and $\mathcal{C}_{42,1}$ are unirational.*

The strategy will be the same for the two cases, hence we will summarize here below the properties, that hold for both divisors, that we will need. In order to keep the notation not too tedious, when we will say that a certain property holds for $\mathcal{C}_n$ we will assume $n = 26, 42$.

Let X be a generic smooth cubic fourfold in $\mathcal{C}_n$. We will denote by $K(X)$ the associated K3 surface and by $S(X)$ the family of scrolls (as defined in Definition 4.2.3) contained in $X \subset \mathbb{P}^5$. For $n = 26$ septic 3−nodal scrolls, for $n = 42$ 8−nodal degree 9 scrolls. One can rephrase Proposition 4.2.4 by saying that $K(X)$ is isomorphic to $S(X)$. More generally, we will denote by $\mathcal{S}_n$ the Hilbert scheme of scrolls contained in cubics in $\mathcal{C}_n$ (those appearing in Definition 4.2.3), and by S_n the $PGL(6,\mathbb{C})$-quotient of $\mathcal{S}_n$. These moduli quotients have been also considered in [34, 35, 61]. Taking example from these papers let us give the following definition.

Definition 4.2.7. *Let us denote by*

$$\mathcal{H}_n = \{(X, R) : R \subset X,\ [X] \in \mathcal{C}_n,\ R \in S(X)\}$$

the "nested" Hilbert scheme given by the couples (X cubic fourfold whose class lives in $\mathcal{C}_n$) + (R rational normal scroll in $S(X)$); and by $H_n := \mathcal{H}_n//PGL(6,\mathbb{C})$ the corresponding moduli quotient.

From [34, 35] we have the following result.

Proposition 4.2.8. *The universal K3 surfaces $\mathcal{F}_{\frac{n+2}{2},1}$ are birational to H_n.*

In particular, in [34, 35] the authors show that S_n is unirational and H_n is birational to a projective bundle over S_n, and hence unirational as well. We add just one more character to the plot, that is:

Definition 4.2.9. *We denote by $\mathcal{H}_{n,1}$ the Hilbert scheme of the triples (X, R, p) as follows:*

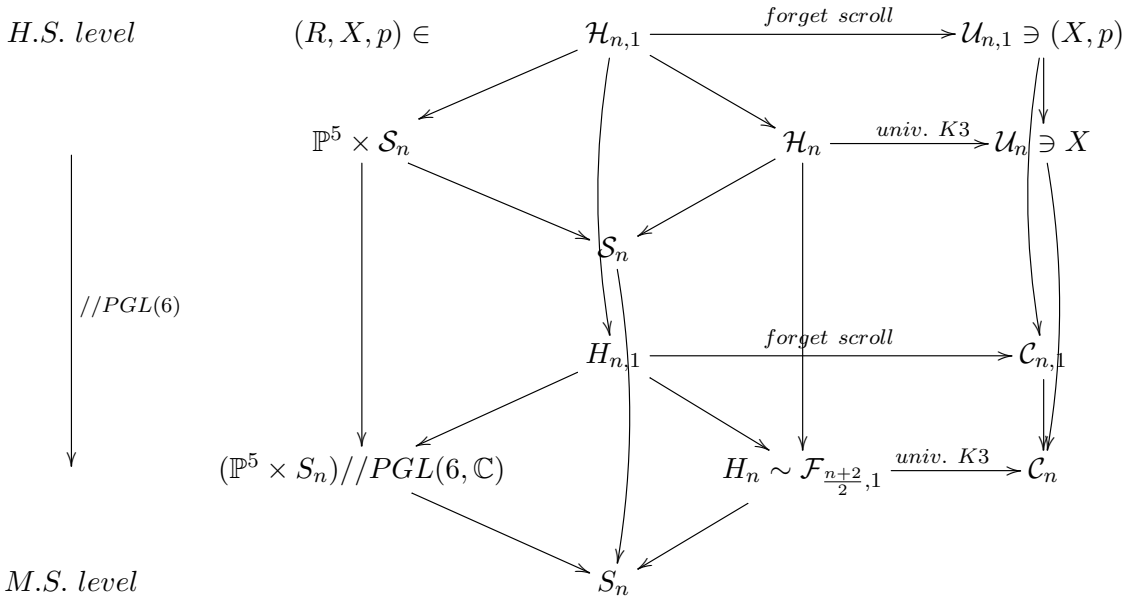
$$\mathcal{H}_{n,1} = \{(X, R, p)) : R \subset X,\ [X] \in \mathcal{C}_n,\ R \in S(X),\ p \in X\}.$$

We will denote by $H_{n,1}$ the corresponding moduli quotient by $PGL(6,\mathbb{C})$.

Of course, as a consequence of Proposition 4.2.2, this is generically a fibration in cubic hypersurfaces over $\mathcal{H}_n$. Finally we will denote by $\mathcal{U}_n$ the open subset complementary to the discriminant inside $|\mathcal{O}_{\mathbb{P}^5}(3)|$ (actually inside $\mathcal{U}$), and by $\mathcal{U}_{n,1}$ the universal family over $\mathcal{U}_n$.

Theorem 4.2.10. *The Hilbert scheme $\mathcal{H}_{n,1}$ is unirational.*

Proof. Let us try to give a global picture of the situation.



(4.1)

Here above all vertical arrows shall be intended as quotients by the $PGL(6)$-action. On the *upper* level of the diagram all spaces are Hilbert schemes, whereas on the *lower* they are moduli quotients (H.S and M.S. for short). By the unirationality of $\mathcal{S}_n$ we have the unirationality of $\mathbb{P}^5\times\mathcal{S}_n$. Then we observe that there is a natural forgetful map

$$\phi_n : \mathcal{H}_{n,1} \to \mathbb{P}^5\times\mathcal{S}_n; \tag{4.2}$$

$$(X,R,p) \mapsto (p,R). \tag{4.3}$$

Namely, the fiber over a given couple (p,R) is exactly the linear system $\mathbb{P}H^0(\mathbb{P}^5,\mathcal{I}_{R\cup p}(3))$ of cubics in $\mathbb{P}^5$ containing R and p. Over an open subset of $\mathbb{P}^5\times\mathcal{S}_n$ (i.e. the locus where

$p \notin R$) the rank does not change and it is 11, and 5 respectively for $d = 26$ and 42. This in turn implies that $\mathcal{H}_{n,1}$ is unirational. □

Proposition 4.2.11. *The universal cubic fourfold $\mathcal{C}_{n,1} \to \mathcal{C}_n$ is unirational.*

Proof. Recall from Proposition 4.2.2 that the generic cubic fourfold in the divisors $\mathcal{C}_n$ has no projective automorphism. Call $V \subset \mathcal{C}_n$ the dense locus where cubics have no automorphism. It is straightforward to see - and we have already implicitly used this in Diagram 4.1 - that, over V, the universal cubic has a natural quotient structure

$$\mathcal{C}_{n,1} = \mathcal{U}_{n,1}//PGL(6,\mathbb{C}). \tag{4.4}$$

Hence $\mathcal{C}_{n,1}$ is the natural moduli space for couples (X,p), up to the action of $PGL(6)$. The upshot is that there exist a natural rational surjective forgetful map (up to $PGL(6)$-action)

$$\varphi_R : H_{n,1} \to \mathcal{C}_{n,1}; \tag{4.5}$$
$$(X,R,p) \mapsto (X,p); \tag{4.6}$$

that forgets the scrolls contained in X. By Theorem 4.2.10, the variety $H_{n,1}$ is unirational, and it dominates $\mathcal{C}_{n,1}$, thus also $\mathcal{C}_{n,1}$ is unirational. □

Remark 4.2.12. *Using the same method and a result of Di Tullio [107], one could prove the unirationality of the universal family $\mathcal{C}_{14,1}$ over $\mathcal{C}_{14}$, the closure of the locus of cubic fourfolds containing a smooth quartic scroll. In fact, shortly after our paper [10] came out, Di Tullio [107] showed that the universal family $\mathcal{F}_{8,1}$ is rational. Recall that the isomorphism 2.1 holds for cubics in $\mathcal{C}_{14}$ and we have as well a birational map $\mathcal{F}_8 \dashrightarrow \mathcal{C}_{14}$. Nevertheless, in the next Section we obtain the same result by different methods.*

4.2.3 Method 2: unirationality through rational special surfaces

4.2.3.1 Special cubics in $\mathcal{C}_d$ in the range $8 \leq d \leq 38$

The goal of this Section is to prove the unirationality of the non-empty families of universal cubics $\mathcal{C}_{d,1}$ for $8 \leq d \leq 38$ by generalizing some results of Nuer, and using a

celebrated result of Kollár [57]. Then, by applying inductively the same argument, we will show that $\mathcal{C}_{d,m}$ - the universal cubic with m marked points - is also unirational in the same range. Let us recall shortly the results we need.

Extending a result of Segre that holds only over $\mathbb{Q}$, Kollár shows the following

Theorem 4.2.13. *[57] Let k be a field and $X \subset \mathbb{P}^{n+1}$ a smooth cubic hypersurface of dimension $n \geq 2$ over k. Then the following are equivalent:*

1. *X is unirational (over k);*
2. *X has a k-point.*

We are going to use this in a relative setting. In order to do this we recall some notations due to Nuer [87] (see §2.3.1). In [87], the author studies the birational geometry of the divisors $\mathcal{C}_d$, for $12 \leq d \leq 38$, via rational surfaces contained in cubic fourfolds. Let $U_d \subset (\mathbb{P}^2)^p$ be the open subset parametrizing generic p-tuples of distinct points such that the surface $\mathbf{S}$ obtained as the blow-up of $\mathbb{P}^2$ along the p points $x_1, \ldots, x_p$ and embed into $\mathbb{P}^5$ via a linear system has certain invariants (see [87, Table 2]). We denote by $V_d \to U_d$ the vector bundle such that the fiber over $(x_1, \ldots, x_p)$ is the space of global sections $H^0(\mathcal{I}_{\mathbf{S}|\mathbb{P}^5}(3))$.

Let us now denote by $\mathcal{X}_{d,1}$ the universal cubic over $\mathbb{P}(V_d)$, and by $\pi_{p+1} : (\mathbb{P}^2)^{p+1} \to (\mathbb{P}^2)^p$ the forgetful map that forgets the last point. Now we can claim :

Theorem 4.2.14. *The universal cubic fourfolds $\mathcal{C}_{d,1}$ are unirational if $12 \leq d \leq 38$.*

Proof. By pulling back through π, we have the following commutative diagram.

$$\begin{array}{ccccc} \pi^*\mathcal{X}_{d,1} & \longrightarrow & \mathcal{X}_{d,1} & \longrightarrow & \mathcal{C}_{d,1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}(\pi^*V_d) & \longrightarrow & \mathbb{P}(V_d) & \longrightarrow & \mathcal{C}_d \\ \downarrow & & \downarrow & & \\ (\mathbb{P}^2)^{p+1} & \xrightarrow{\pi} & U_d & & \end{array} \qquad (4.7)$$

with s a section of $\mathbb{P}(\pi^*V_d) \to (\mathbb{P}^2)^{p+1}$.

Actually the LHS column is only defined on $\pi^{-1}(U_d)$ but for simplicity we will not write this. Working in the birational category, this does not affect our results.

We observe that the pulled-back family $\pi^*\mathcal{X}_{d,1}$, seen as universal family over $\mathbb{P}(\pi^*V_d)$ has a tautological rational section s, that is the image in $\mathbf{S}$ of the $(p+1)^{th}$ point from $(\mathbb{P}^2)^{p+1}$. This means that $\pi^*\mathcal{X}_{d,1}$ has a rational point over the function field of $(\mathbb{P}^2)^{p+1}$. By Kollár's Theorem $\pi^*\mathcal{X}_{d,1}$ is then unirational over this field. $\pi^*\mathcal{X}_{d,1}$ dominates $\mathcal{X}_{d,1}$ and this in turn dominates $\mathcal{C}_{d,1}$, which is then unirational. □

Remark 4.2.15. *An easy adaptation of this argument allows us to show the unirationality of $\mathcal{C}_{8,1}$ as well. Of course, all the planes in $\mathbb{P}^5$ are projectively equivalent, and the linear system of cubics through a given 2-plane P is $|\mathcal{I}_{P/\mathbb{P}^5}(3)| \cong \mathbb{P}^{45}$. The plane P has rational points over $\mathbb{C}$ and the universal cubic $\mathcal{X}_{8,1} \subset \mathbb{P}^{45} \times \mathbb{P}^5$ contains the "constant" plane $\mathbb{P}^{45} \times P$. The variety $\mathcal{X}_{8,1}$ also dominates $\mathcal{C}_{8,1}$, by definition. Hence $\mathcal{X}_{8,1}$ has rational points over the function field of $\mathbb{P}^{45}$ and hence is unirational over this field. By the same argument as in Theorem 4.2.14, this implies the unirationality of $\mathcal{C}_{8,1}$.*

4.2.3.2 Special cubics in $\mathcal{C}_{42}$

The unirationality of universal cubics $\mathcal{C}_{42,1}$ goes along the same lines as in the previous section, but we need to extract a couple of quite subtle result from [93] about rational surfaces contained in cubics belonging to this divisor.

In [93], the authors construct a 48-dimensional unirational family $\widetilde{\mathcal{S}^{9,2}}$ of 5-nodal surfaces of degree 9 and sectional genus 2, such that the divisor $\mathcal{C}_{42}$ can be described as the locus of cubics containing surfaces from this family. Next, we reconstruct briefly their argument. A detailed construction of such families is mentioned later on in § 4.4.1.5.

More precisely (see [93, Rmk. 4.3]) they construct a family $\mathcal{W}$ of del Pezzo quintics with some special intersection theoretical properties. The construction of such surfaces depend on certain choices (points on $\mathbb{P}^2$, a line secant to a particular rational surface, etc.) which all depend on free, rational parameters. This implies that the family $\mathcal{W}$ is unirational. By construction, each del Pezzo quintic $D \in \mathcal{W}$ is contained in one smooth del Pezzo fivefold $\mathbb{Y}^5 \subset \mathbb{P}^8$, that is an hyperplane section of the Grassmannian $\mathbb{G}(1,4)$. The fivefold $\mathbb{Y}^5$ contains a rational, 3-dimensional, family of planes, whose members all have class $(2,2)$ inside the Chow ring $CH^\bullet(\mathbb{G}(1,4))$ of the Grassmannian. The projection with center any one of these

planes defines a birational morphism between any del Pezzo quintic from $\mathcal{W}$ and some 5-nodal $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}^{9,2}}$. As observed in [93], this in turn implies that $\widetilde{\mathcal{S}^{9,2}}$ is unirational. Let us denote by $\mathbb{S}$ the rational parameter space that dominates $\widetilde{\mathcal{S}^{9,2}}$. Other relevant work about $\mathcal{C}_{42}$ has been done in [61] and [49].

Remark 4.2.16. *We observe that the construction above implies that any $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}^{9,2}}$ is birational to a del Pezzo quintic over $\mathbb{C}(\mathbb{S})$, the function field of $\mathbb{S}$. This means that on the same field there exists a birational map $\mathbb{P}^2 \dashrightarrow \widetilde{\mathbf{S}}$, as we explain in the following Lemma.*

Lemma 4.2.17. *Let $\widetilde{\mathbf{S}}$ be any surface in $\widetilde{\mathcal{S}^{9,2}}$, then $\widetilde{\mathbf{S}}$ is rational over the function field of $\mathbb{S}$.*

Proof. As we have observed in Remark 4.2.16, $\widetilde{\mathbf{S}}$ is birational to a del Pezzo quintic, over $\mathbb{C}(\mathbb{S})$. By a well-known result of Enriques [31], a del Pezzo quintic is rational over any field. □

Proposition 4.2.18. *The universal cubic $\mathcal{C}_{42,1}$ is unirational.*

Proof. Let us denote by $\mathbb{P}(V_{42})$ the relative linear system over $\widetilde{\mathcal{S}^{9,2}}$ of cubics through $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}^{9,2}}$, and by $\mathcal{X}_{42,1}$ the universal family over $\mathbb{P}(V_{42})$, that dominates $\mathcal{C}_{42,1}$. Then the situation is the following.

$$\begin{array}{ccccccc}
\rho^*\pi^*\mathcal{X}_{42,1} & \longrightarrow & \pi^*\mathcal{X}_{42,1} & \longrightarrow & \mathcal{X}_{42,1} & \longrightarrow & \mathcal{C}_{42,1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{P}(\rho^*\pi^*V_{42}) & \longrightarrow & \mathbb{P}(\pi^*V_{42}) & \longrightarrow & \mathbb{P}(V_{42}) & \xrightarrow{\varphi} & \mathcal{C}_{42} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{P}^2 \times \mathbb{S} & \xrightarrow{\rho} & \mathbb{S} & \xrightarrow{\pi} & \widetilde{\mathcal{S}^{9,2}} & &
\end{array} \qquad (4.8)$$

(with s a section of the leftmost vertical arrow)

Here $\pi : \mathbb{S} \to \widetilde{\mathcal{S}^{9,2}}$ is the unirational parametrization described above, ρ the second projection and φ is the natural classifying map. Now, all cubics that are the fibers of the fibration $\mathcal{X}_{42,1} \to \mathbb{P}(V_{42})$ contain by construction a surface $\widetilde{\mathbf{S}} \in \widetilde{\mathcal{S}^{9,2}}$, and since these surfaces

are rational over $\mathbb{C}(\mathbb{S})$ (they are birational to del Pezzo quintics), they have rational points over the same field. Summing up, there exists a section $s : \mathbb{P}^2 \times \mathbb{S} \to \rho^*\pi^*\mathcal{X}_{42,1}$, that - thanks to Kollár's Theorem - makes $\rho^*\pi^*\mathcal{X}_{42,1}$ unirational over $\mathbb{P}^2 \times \mathbb{S}$ and hence implies the unirationality of $\mathcal{C}_{42,1}$. □

Remark 4.2.19. *As the reader may observe, the universal cubic $\mathcal{C}_{44,1}$ is missing from our description. Cubics in this divisor contain a Fano model of an Enriques surface [87, Thm. 3.2]. The divisor $\mathcal{C}_{44}$ is unirational, so in order to apply Kollár's Theorem and show that $\mathcal{C}_{44,1}$ is unirational, one should show that the generic Enriques surface has a rational point.*

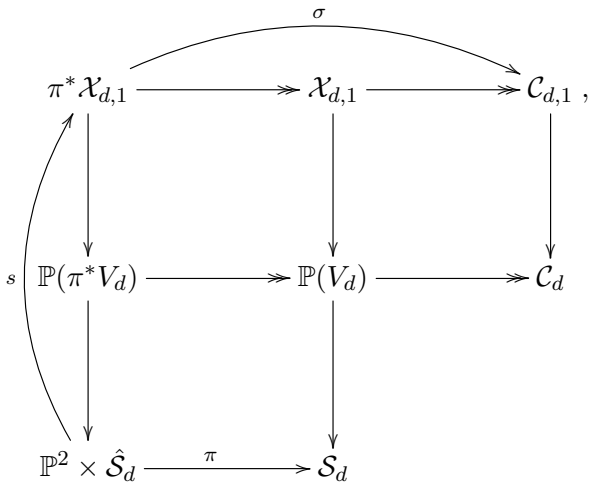
4.2.3.3 Unirationality of $\mathcal{C}_{d,m}$

In this section we are going to use the inductivity of the construction of $\mathcal{C}_{d,m}$ - the moduli space of cubic fourfolds in $\mathcal{C}_d$ with m marked points - and Kollár's theorem in order to show the unirationality of $\mathcal{C}_{d,m}$ for all m.

Recall in fact that $\mathcal{C}_{d,m}$ is just the universal cubic fourfold over $\mathcal{C}_{d,m-1}$. This allows us to use our machinery to prove inductively the following.

Theorem 4.2.20 (Theorem VI)**.** *The moduli spaces $\mathcal{C}_{d,m}$ are unirational for all m, if $8 \leq d \leq 42$.*

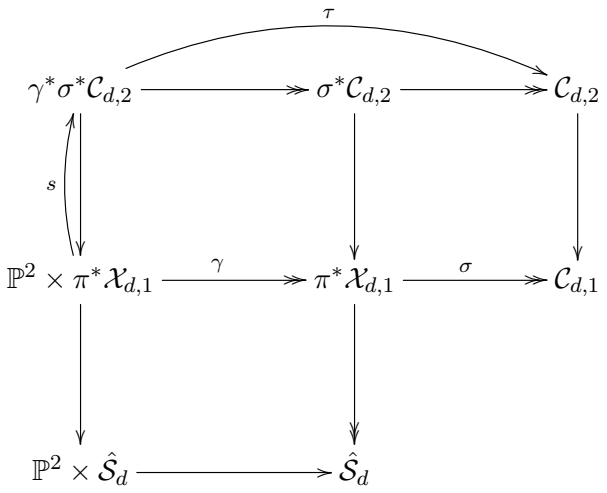
Proof. We will work inductively on m. For $m = 0$ (or respectively 1) the claim is true thanks to [87] and [93] (or resp. Sections 4.2.3.1 and 4.2.3.2). We will denote by $\mathcal{S}_d$ the unirational family of rational surfaces contained in cubics in $\mathcal{C}_d$, $\mathbb{P}(V_d) \to \mathcal{S}_d$ the relative ideal of cubic hypersurfaces through each surface and $\mathcal{X}_{d,1} \to \mathbb{P}(V_d)$ the universal cubic over the relative linear system. As seen in Theorem. 4.2.14 and Proposition 4.2.18, the variety $\mathcal{X}_{d,1}$ is unirational and dominates $\mathcal{C}_{d,1}$ through the natural classifying map of the coarse moduli space $\mathcal{C}_{d,1}$. The situation is described by the following diagram:



(4.9)

where $\mathbb{P}^2 \times \hat{\mathcal{S}}_d$ is the appropriate rational space that assures the existence of the section s. As seen in Theorems 4.2.14 and 4.2.18, this boils down to taking $\mathbb{P}^2 \times (\mathbb{P}^2)^p$ for $8 \leq d < 38$, and the rational parameter space $\mathbb{P}^2 \times \mathbb{S}$ for $d = 42$.

We remark that s is the section that makes $\pi^*\mathcal{X}_{d,1}$ unirational following Kollár's Theorem. Call σ the classifying map to $\mathcal{C}_{d,1}$. Then, we can add one point and consider the universal cubic $\mathcal{C}_{d,2} \to \mathcal{C}_{d,1}$, that fits in the following diagram



(4.10)

where γ is the second projection and σ the dominant map from Diagram 4.9. Recall that $\pi^*\mathcal{X}_{d,1}$ is unirational, and dominates $\hat{\mathcal{S}}_d$.

We observe that, exactly as we have done in Diag 4.9 for $\mathcal{C}_{d,1}$, since $\pi^*\mathcal{X}_{d,1}$ dominates $\hat{\mathcal{S}}_d$, we are able to define a natural rational section $s : \mathbb{P}^2 \times \pi^*\mathcal{X}_d \to \gamma^*\sigma^*\mathcal{C}_{d,2}$. Hence the universal cubic has rational points over $\mathbb{P}^2 \times \pi^*\mathcal{X}_{d,1}$ and this space is unirational. By Kollár's theorem and since $\gamma^*\sigma^*\mathcal{C}_{d,2}$ dominates $\mathcal{C}_{d,2}$, we get that $\mathcal{C}_{d,2}$ is unirational.

Now it is straightforward to see how to continue the argument by induction; we just draw the diagram for the following step for clarity. Recall that $\gamma^*\sigma^*\mathcal{C}_{d,2} = \tau^*\mathcal{C}_{d,2}$.

$$\begin{array}{ccccc}
 & & \overset{\omega}{\frown} & & \\
\lambda^*\tau^*\mathcal{C}_{d,3} & \twoheadrightarrow & \tau^*\mathcal{C}_{d,3} & \twoheadrightarrow & \mathcal{C}_{d,3} \\
\downarrow\uparrow s & & \downarrow & & \downarrow \\
\mathbb{P}^2 \times \tau^*\mathcal{C}_{d,2} & \xrightarrow{\lambda} & \tau^*\mathcal{C}_{d,2} & \xrightarrow{\tau} & \mathcal{C}_{d,2} \\
\downarrow & & \downarrow & & \\
\mathbb{P}^2 \times \hat{\mathcal{S}}_d & \longrightarrow & \hat{\mathcal{S}}_d & &
\end{array} \tag{4.11}$$

□

4.2.4 Some results of non-unirationality.

In this section using some recent results about the Kodaira dimension of moduli spaces of K3 surfaces and of Hassett divisors $\mathcal{C}_d$, we show that for an infinite range of values of d, the universal cubic fourfold over $\mathcal{C}_d$ can not be unirational. Let us first claim the following straightforward Lemma, in which we need to assume that d is even.

Lemma 4.2.21. *Suppose that the Kodaira dimension of $\mathcal{F}_{\frac{d+2}{2}}$ or of $\mathcal{C}_d$ is non-negative, then the universal cubic fourfold $\mathcal{C}_{d,1}$ is not unirational.*

Proof. The universal cubic $\mathcal{C}_{d,1}$ dominates $\mathcal{C}_d$ via the forgetful map, hence if $\mathcal{C}_{d,1}$ was unirational, then $\mathcal{C}_d$ would also be unirational and has negative Kodaira dimension. If $\mathcal{F}_{\frac{d+2}{2}}$ has non-negative Kodaira dimension, then via the birational map of (2.1) also $\mathcal{C}_d$ has, and $\mathcal{C}_d$ can not be unirational, as seen before. □

We will apply Lemma 4.2.21 to the propositions 2.3.5 and 2.3.6, due to Gritsenko-Hulek-Sankaran [42] and Tanimoto-Várilly-Alvarado [103]. We obtain the following Proposition:

Proposition 4.2.22 (Proposition VII)**.** *The universal cubic fourfold $\mathcal{C}_{d,1}$ is not unirational if:*

1. $d > 80$, $d \equiv 2 \ (mod\ 6)$, $4 \not| d$ *and such that for any odd prime* p, $p|d$ *implies* $p \equiv 1 \ (mod\ 3)$;
2. $d = 6n + 2$, $n > 13$ *and* $n \neq 15$;
3. $d = 6n$ *for* $n > 16$ *and* $n \neq 18, 20, 22, 30$.

4.3 Universal Gushel-Mukai fourfolds

4.3.1 Existence of universal families of GM fourfolds

Definition 4.3.1. *By the universal Gushel-Mukai fourfold over $(\mathcal{M}^4_{GM})_d$ we mean the moduli space $(\mathcal{M}^4_{GM})_{d,1}$ of $1-$pointed GM fourfolds.*

The construction of these universal GM fourfolds over the coarse moduli space $\mathcal{M}^4_{GM}$ needs more justification. The following Proposition is a direct consequence of [28, Proposition 3.21].

Proposition 4.3.2. *For a very general ordinary smooth Gushel-Mukai fourfold Y, the group of automorphisms $Aut(Y)$ is trivial.*

The upshot is the following:

Corollary 4.3.3. *The universal family of ordinary Gushel-Mukai fourfolds exists over an open subset of their coarse moduli space.*

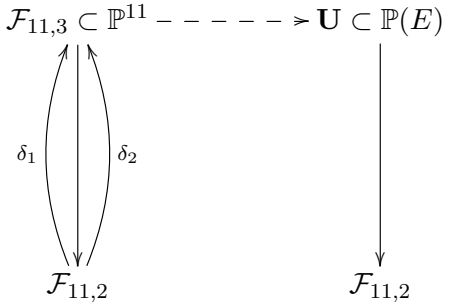
We recall, in this section, the explicit description of the divisor $(\mathcal{M}^4_{GM})_{20}$, corresponding to GM fourfolds containing a rational surface of degree 9 and genus 2 (see Appendix C). Then, using the unirationality of some moduli spaces of pointed K3 surfaces that appeared in its geometric construction, we prove the unirationality of universal families of these GM fourfolds.

4.3.2 Unirationality of a family of universal GM fourfolds

In this section we will consider a codimension 1 locus in $\mathcal{M}^4_{GM}$, $(\mathcal{M}^4_{GM})_{20}$. Let us recall shortly the geometric construction of a general fourfold belonging to this family (see [49, 93]). There exists a birational map $\mathbb{P}^4 \dashrightarrow Y$, defined by the linear system of hypersurfaces of degree 9 having double points along $U \subset \mathbb{P}^4$, a degree 10 surface which is a projection of a genus 11 K3 surface.
More precisely, we start from a K3 surface $Z \subset \mathbb{P}^{11}$ of degree 20 and sectional genus 11. We take two points $p, q \in Z$, and perform first a triple projection from p to $\mathbb{P}^5$, then a simple projection off q to $\mathbb{P}^4$. The image is the required surface U. With this in mind, one can prove the following result:

Proposition 4.3.4 (Proposition VIII)**.** *The universal family $(\mathcal{M}^4_{GM})_{20,1}$ of 1-pointed GM-fourfolds is unirational.*

Proof. The philosophy is to do the above rationality construction in families. We need then to consider the moduli space $\mathcal{F}_{11,2}$ of polarized K3 surface of genus 11 with two marked points. The moduli space $\mathcal{F}_{11,3}$ comes equipped with an embedding inside a $\mathbb{P}^{11}$-bundle over $\mathcal{F}_{11,2}$ and with two sections $\delta_1, \delta_2 : \mathcal{F}_{11,2} \to \mathcal{F}_{11,3}$. Performing a relative triple projection from the image of δ_1 and a simple one from the image of δ_2 we obtain a $\mathbb{P}^4$-bundle $\mathbb{P}(E)$ over $\mathcal{F}_{11,2}$ containing the family $\mathbf{U}$ of degree 10 surfaces.



Since $\mathcal{F}_{11,2}$ is unirational [15, Theorem 0.1], the projective bundle $\mathbb{P}(E)$ is unirational. The relative linear system of degree 9 hypersurfaces, with multiplicity two along $\mathbf{U}$ gives a rational dominant map between the $\mathbb{P}^4$-bundle $\mathbb{P}(E)$ and $(\mathcal{M}^4_{GM})_{20,1}$. □

Remark 4.3.5. *This implies obviously the unirationality of the actual divisor $(\mathcal{M}^4_{GM})_{20}$.*

4.3.2.1 A rational Noether-Lefschetz divisor of genus 11 K3 surfaces

In [49], Hoff and Staglianò also consider a codimension one subfamily of genus 11 K3 surfaces, that forms a Noether-Lefschetz divisor inside $\mathcal{F}_{11}$. This divisor seems particularly interesting under our point of view, since the wealth of geometry going on here allows us to strengthen our rationality results concerning the corresponding universal families of GM fourfolds related to these K3 surfaces. But let us give a couple more details about these surfaces.

We start from a smooth Fano threefold F of type $X_{22} \subset \mathbb{P}^{13}$. It is well known that the generic tangent hyperplane sections of F are one-nodal (a double point) K3 surfaces (see [80], [95]). The projection off the node of such a K3 surface gives a K3 surface in $\mathbb{P}^{11}$, of degree 20 and sectional genus 11, containing a further conic (the exceptional divisor over the node). In fact, such a construction gives a Noether-Lefschetz divisor inside the 19-dimensional moduli space of K3 surfaces of genus 11, and the intersection lattice of these surfaces contains a sublattice of type

$$\begin{pmatrix} 20 & 2 \\ 2 & -2 \end{pmatrix}.$$

Before studying the universal family of GM fourfolds obtained from these special K3 surfaces, we need to show some results on the birational geometry of their Noether-Lefschetz locus. We will denote by $\mathcal{V}_n^{nod}$ the moduli space of $n-$pointed one nodal K3 surfaces of sectional genus 12, obtained by cutting a X_{22}-type 3fold with tangent hyperplanes as above. The generic element of $\mathcal{V}_n^{nod}$ is represented by a vector $(F, p, H, q_1, \ldots, q_n)$, where F is a Fano threefold of type X_{22}, p is a point of F, H is a hyperplane tangent to F in p, and $q_1, \ldots, q_n$ are n points on the surface $S_H := F \cap H$. We will also denote by $\mathcal{X}_{22}$ the (rational, see [81]) moduli space of Fano threefolds of type X_{22}. All these Fano threefolds are rational and birational to each other, we will need to fix one $\widetilde{X}_{22} \in \mathcal{X}_{22}$.

Theorem 4.3.6. *The moduli space $\mathcal{V}_n^{nod}$ is rational if $n \leq 9$.*

Proof. Let us consider the rational map

$$\varphi : \mathcal{V}_n^{nod} \to \mathcal{X}_{22} \times \widetilde{X}_{22}^{n+1} \tag{4.12}$$

$$(F, p, H, q_1, \dots, q_n) \mapsto (F, p, q_1, \dots, q_n). \tag{4.13}$$

Remark that $\mathcal{X}_{22} \times \widetilde{X}_{22}^{n+1}$ is rational (and of dimension $3n+9$) since it is the product of rational varieties. Then, the fiber of φ over $(F, p, q_1, \dots, q_n)$ is exactly the linear system of hyperplanes in $\mathbb{P}^{13}$ that are tangent to F in p, and pass through $q_1, \dots, q_n$. This shows that $\mathcal{V}_n^{nod}$ is birational to a $\mathbb{P}^{9-n}-$projective bundle over $\mathcal{X}_{22} \times \widetilde{X}_{22}^{n+1}$, and hence is rational if $n \leq 9$. □

We recall that the projection off the node sends birationally $\mathcal{V}_n^{nod}$ onto a $(18+2n)$-dimensional Noether-Lefschetz locus inside $\mathcal{F}_{11,n}$. Let us denote by $(\mathcal{M}_{GM}^4)_{20}^{nod}$, the moduli space of GM fourfolds obtained from the Noether-Lefschetz K3 surfaces described above, and by $(\mathcal{M}_{GM}^4)_{20,1}^{nod}$ the universal family above, obtained by restricting the construction of Proposition 4.3.4. The moduli space $(\mathcal{M}_{GM}^4)_{20}^{nod}$ is of dimension 22, and is contained in $(\mathcal{M}_{GM}^4)_{20}$.

Corollary 4.3.7. *The universal family $(\mathcal{M}_{GM}^4)_{20,1}^{nod}$ is rational. The moduli space $(\mathcal{M}_{GM}^4)_{20}^{nod}$ is rational.*

Proof. The moduli space $\mathcal{V}_3^{nod}$ of nodal, 3-pointed K3 surfaces can be embedded in a $\mathbb{P}^{12}$-bundle, and endowed with two sections $\delta_1, \delta_2 : \mathcal{V}_2^{nod} \to \mathcal{V}_3^{nod}$, over $\mathcal{V}_2^{nod}$. Since we are working in the birational category, we can even consider (at least an open subset of) $\mathcal{V}_2^{nod}$ as contained in $\mathcal{F}_{11,2}$. Now, we project fiberwise off the node, obtaining a family of Noether-Lefschetz K3 surfaces in a $\mathbb{P}^{11}$-bundle, with two sections, over $\mathcal{V}_2^{nod}$. Then, as we did in Proposition 4.3.4, we perform the two projections off the sections and we obtain a $\mathbb{P}^4$-bundle over $\mathcal{V}_2^{nod}$, containing a family $\mathcal{T}$ of degree 10 surfaces. The moduli space $\mathcal{V}_2^{nod}$ is rational, hence the $\mathbb{P}^4$-bundle is rational as well. Then, by applying the relative linear system of degree 9 hypersurfaces through $\mathcal{T}$ as in Proposition 4.3.4, we obtain a rational family of GM fourfolds over $\mathcal{V}_2^{nod}$, hence $(\mathcal{M}_{GM}^4)_{20,1}^{nod}$ is rational. By construction $(\mathcal{M}_{GM}^4)_{20}^{nod}$ is birational to $\mathcal{V}_2^{nod}$ and hence rational. □

4.4 Unified method for the unirationality of universal Fano fourfolds

4.4.1 Explicit geometric descriptions of some Hassett divisors in the moduli space $\mathcal{C}$ of cubic fourfolds and in the moduli space $\mathcal{M}^4_{GM}$ of GM fourfolds

In this section, we shall recall some explicit descriptions of unirational irreducible families $\mathcal{S}$ in the Hilbert scheme of $\mathbb{P}^5$ (respectively in the Hilbert scheme of a fixed smooth del Pezzo fivevold $\mathbb{Y}^5 = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$) such that the closure of the locus of cubic fourfolds (resp., GM fourfolds) containing a surface of the family $\mathcal{S}$ describes a Noether-Lefschetz divisor in the corresponding moduli space. We shall focus on the fact that starting from a pair $(\mathbf{S}, X)$, where $\mathbf{S}$ is a general member of the family $\mathcal{S}$ and X is a general fourfold containing $\mathbf{S}$, we can build an explicit birational map $X \overset{\sim}{\dashrightarrow} \mathbb{P}^4$, defined over the same field of definition as $\mathbf{S}$ and X.

4.4.1.1 Cubic fourfolds containing a quintic del Pezzo surface

A quintic del Pezzo surface is the image of $\mathbb{P}^2$ via the linear system of cubic curves with 4 base points in general position. The cubic fourfolds containing a quintic del Pezzo surface describe the divisor $\mathcal{C}_{14} \subset \mathcal{C}$ (see §2.3.1).

Theorem 4.4.1 ([13, 21, 74]). *A quintic del Pezzo surface $\mathbf{S} \subset \mathbb{P}^5$ admits a* congruence of secant lines*: through the general point of $\mathbb{P}^5$ there passes a unique secant line to $\mathbf{S}$. The general line of this congruence can be realized as the general fiber of the dominant map*

$$\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|H^0(\mathcal{I}_{\mathbf{S}}(2))|$ of quadric hypersurfaces through $\mathbf{S}$.

If X is a general cubic fourfold containing $\mathbf{S}$, then the restriction of μ induces a birational map $\mu|_X : X \overset{\sim}{\dashrightarrow} \mathbb{P}^4$.

4.4.1.2 Cubic fourfolds containing a 3-nodal septic scroll

Let $\mathbf{S} \subset \mathbb{P}^5$ be the projection of a rational normal septic scroll $\Sigma_7 \subset \mathbb{P}^8$ from a plane spanned by three general points on the secant variety of Σ_7. Thus $\mathbf{S}$ is a rational septic

scroll having 3 non-normal nodes.
The family of rational 3-nodal septic scrolls, constructed as above, is irreducible, unirational and of dimension 44. Cubic fourfolds containing such a surface describe the divisor $\mathcal{C}_{26} \subset \mathcal{C}$ (see [33]).

Theorem 4.4.2 ([92, 93])**.** **S** *admits a* congruence of 5-secant conics*: through the general point of* $\mathbb{P}^5$ *there passes a unique conic curve which is* 5*-secant to* **S**. *The general conic curve of this congruence can be realized as the general fiber of the dominant map*

$$\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|H^0(\mathcal{I}^2_{\mathbf{S}}(5))|$ *of quintic hypersurfaces with double points along* **S**.

If X *is a general cubic fourfold containing* **S***, then the restriction of* μ *induces a birational map* $\mu|_X : X \xrightarrow{\sim} \mathbb{P}^4$.

4.4.1.3 Cubic fourfolds containing a "generalized" Coble surface

Cubic fourfolds containing the "generalized" Coble surface **S** describe the divisor $\mathcal{C}_{38} \subset \mathcal{C}$ (see §2.3.1). These surfaces form an irreducible unirational family $\mathcal{S}^{10,6} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ of dimension 47 (see [87]).

Theorem 4.4.3 ([91–93])**.** *A general surface* $[\mathbf{S}] \in \mathcal{S}^{10,6}$ *admits a* congruence of 5-secant conics*: through the general point of* $\mathbb{P}^5$ *there passes a unique conic curves which is* 5*-secant to* **S**. *The general conic curve of this congruence can be realized as the general fiber of the dominant map*

$$\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|H^0(\mathcal{I}^2_{\mathbf{S}}(5))|$ *of quintic hypersurfaces with double points along* **S**.

If X *is a general cubic fourfold containing* **S***, then the restriction of* μ *induces a birational map* $\mu|_X : X \xrightarrow{\sim} \mathbb{P}^4$

4.4.1.4 GM fourfolds of discriminant 10

4.4.1.4.1 τ-quadric surfaces A *τ-quadric* surface is a two-dimensional linear section of a Schubert variety $\Sigma_{1,1} \simeq \mathbb{G}(1,3) \subset \mathbb{G}(1,4)$ (see Appendix C). A standard parameter count (see [26, Proposition 7.4]) shows that the closure inside $\mathcal{M}^4_{GM}$ of the family of fourfolds containing a τ-quadric surface forms the divisor $(\mathcal{M}^4_{GM})'_{10} \subset \mathcal{M}^4_{GM}$, one of the two irreducible

components of the Noether-Lefschetz locus in $\mathcal{M}^4_{GM}$ parametrizing fourfolds of discriminant 10. In particular, since the family of τ-quadric surfaces in $\mathbb{G}(1,4)$ is unirational, we deduce that the divisor $(\mathcal{M}^4_{GM})'_{10}$ is also unirational.

Theorem 4.4.4 ([26]; see also [49])**.** *The projection of a general fourfold* $[X] \in (\mathcal{M}^4_{GM})'_{10}$ *containing a* τ*-quadric surface* $\mathbf{S}$*, from the linear span* $\langle \mathbf{S} \rangle \simeq \mathbb{P}^3$ *of* $\mathbf{S}$*, gives a birational map* $X \dashrightarrow \mathbb{P}^4$*.*

4.4.1.4.2 Quintic del Pezzo surfaces A quintic del Pezzo surface can be realized as a two-dimensional linear section of $\mathbb{G}(1,4)$ (see Appendix C). A standard parameter count (see [26, Proposition 7.7], and also [101]) shows that the closure inside $\mathcal{M}^4_{GM}$ of the family of fourfolds containing a quintic del Pezzo surface forms the divisor $(\mathcal{M}^4_{GM})''_{10} \subset \mathcal{M}^4_{GM}$; one of the two irreducible components of the Noether-Lefschetz locus in $\mathcal{M}$ of fourfolds of discriminant 10. In particular, since the family of quintic del Pezzo surfaces in $\mathbb{G}(1,4)$ is unirational, we deduce that the divisor $(\mathcal{M}^4_{GM})''_{10}$ is also unirational.

Theorem 4.4.5 ([90]; see also [26, 31, 98])**.** *The projection of a general fourfold* $[X] \in (\mathcal{M}^4_{GM})''_{10}$ *containing a quintic del Pezzo surface* $\mathbf{S}$*, from the linear span* $\langle \mathbf{S} \rangle \simeq \mathbb{P}^5$ *of* $\mathbf{S}$*, induces a dominant map* $X \dashrightarrow \mathbb{P}^2$ *whose generic fiber is a quintic del Pezzo surface. (In particular,* X *is rational. Indeed as mentioned before, from a classic result of Enriques, a quintic del Pezzo surface defined over an infinite field* K *is rational over* K*.)*

4.4.1.5 GM fourfolds of discriminant 20

Throughout this subsection, we continue to let $\mathbb{Y}^5 \subset \mathbb{P}^8$ denote a fixed del Pezzo fivefold. Recall first two well-known ways to parametrize $\mathbb{Y}^5$ over its field of definition.

1. If $P \subset \mathbb{Y}^5$ is a plane in $\mathbb{Y}^5$ with class $\sigma_{2,2}$ in $\mathbb{G}(1,4)$, then the projection of $\mathbb{Y}^5$ from P gives a birational map $\mathbb{Y}^5 \dashrightarrow \mathbb{P}^5$, whose inverse is defined by the linear system of quadrics through a rational normal cubic scroll $\Sigma_3 \subset \mathbb{P}^4 \subset \mathbb{P}^5$.

2. If $C \subset \mathbb{Y}^5$ is a conic such that its linear span P is not contained in $\mathbb{Y}^5$, then the projection of $\mathbb{Y}^5$ from P gives a birational map $\mathbb{Y}^5 \dashrightarrow \mathbb{P}^5$, whose inverse is defined by the linear system of cubics through a rational quartic scroll $\Theta_4 \subset \mathbb{P}^5$, obtained as a general projection of a rational normal threefold scroll in $\mathbb{P}^6$.

Let $\mathcal{S}^{9,2} \subset \mathrm{Hilb}_{\mathbb{Y}^5}$ be the 25-dimensional unirational family of smooth rational surfaces of degree 9 and genus 2. having class $6\sigma_{3,1} + 3\sigma_{2,2}$ in the Chow ring of $\mathbb{G}(1,4)$. Let $T \subset \mathbb{P}^6$ be the image of the plane via the linear system of quartic curves having 8 general base points $p_1, \ldots, p_8$. Thus T is a smooth rational surface of degree 8 and sectional genus 3 cut out by 7 quadrics. These 7 quadrics define a *special* Cremona transformation

$$\varphi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^6$$

of type $(2,4)$, which has been classically studied in [96] (see also [50]).

Let us recall a bit of geometry from the papers [50, 96]. The pencil of plane cubics through the 8 base points $p_1, \ldots, p_8$ yields a pencil of elliptic normal quartic curves on T passing through a special point $q \in T$, and the union of the linear spans of these curves gives a cone of vertex q over a Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Let $H \simeq \mathbb{P}^5 \subset \mathbb{P}^6$ be a general hyperplane in $\mathbb{P}^6$. The restriction of φ to H gives a birational map

$$\varphi|_H : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^6$$

onto a quartic hypersurface $Z \subset \mathbb{P}^6$, whose base locus, that is the intersection of T with the hyperplane H, is an octic curve $C \subset \mathbb{P}^5$ of arithmetic genus 3 contained in a Segre threefold $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The image $\overline{\varphi(\Sigma)}$ is a smooth quadric surface $Q \subset Z \subset \mathbb{P}^6$ (which is double as a component of the base locus of the inverse of $\varphi|_H$). The pullback via the restriction $\varphi|_H$ of a line in one of the two pencils of lines on Q is a $\mathbb{P}^2$ of the ruling of Σ, while the pullback of a general line in the other pencil of lines on Q is a smooth quintic del Pezzo surface containing C. In particular, the curve C is the base locus of a pencil $\{D_\lambda\}_\lambda$ of quintic del Pezzo surfaces contained in Σ and whose general member is smooth.

Everything we have said about C continues to hold true even if we take the hyperplane $H \subset \mathbb{P}^6$ to be general among the hyperplanes containing a general tangent plane to T. But in this case (and only in this case), the curve C has a node and it can be embedded in a rational quartic scroll $\Theta_4 \subset \mathbb{P}^5$ as the one considered above. Indeed, such a nodal curve C can be realized as a nodal projection of a smooth curve of degree 8 and genus 2 contained in a smooth rational normal quartic scroll threefold in $\mathbb{P}^6$; see [93, Section 4] for more details on this last step. Then the birational map $\mathbb{P}^5 \dashrightarrow \mathbb{Y}^5 \subset \mathbb{P}^8$ defined by the linear system of cubics through Θ_4 induces an isomorphism between a general quintic del Pezzo surface

of the pencil $\{D_\lambda\}_\lambda$ with a smooth rational surface $\mathbf{S} \subset \mathbb{Y}^5 \subset \mathbb{P}^8$ of degree 9 and sectional genus 2, cut out by 19 quadrics, and having class $6\sigma_{3,1} + 3\sigma_{2,2}$ in $\mathbb{G}(1,4)$.

Theorem 4.4.6 ([93]). *The surfaces $\mathbf{S} \subset \mathbb{Y}^5$ of degree 9 and sectional genus 2 form an irreducible unirational family $\mathcal{S}^{9,2} \subset \mathrm{Hilb}_{\mathbb{Y}^5}$.*

The closure of the family of quadric hypersurfaces in $\mathbb{Y}^5$ containing a surface of the family $\mathcal{S}^{9,2}$, after passing to the quotient modulo $PGL(9,\mathbb{C})$, describes the divisor $(\mathcal{M}^4_{GM})_{20} \subset \mathcal{M}^4_{GM}$, the irreducible component of the Noether-Lefschetz locus in $\mathcal{M}^4_{GM}$ of fourfolds of discriminant 20.

These GM fourfolds were actually already mentioned in the previous section.

Theorem 4.4.7 ([49]). *Let $\mathbf{S} \subset \mathbb{Y}^5$ be a surface corresponding to a general member of the family $\mathcal{S}^{9,2}$. Then $\mathbf{S}$ admits inside $\mathbb{Y}^5$ a* congruence of 3-secant conic curves*, that is, through the general point of $\mathbb{Y}^5$ there passes a unique conic which is 3-secant to $\mathbf{S}$ and is contained in $\mathbb{Y}^5$. The general conic curve of this congruence can be realized as the general fiber of the dominant map*

$$\mu : \mathbb{Y}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|H^0(\mathcal{I}^2_{\mathbf{S},\mathbb{Y}^5}(5))|$ of quintic hypersurfaces in $\mathbb{Y}^5$ with double points along $\mathbf{S}$.

If X is a general quadric hypersurface in $\mathbb{Y}^5$ containing $\mathbf{S}$, then the restriction of μ induces a birational map $\mu|_X : X \xrightarrow{\simeq} \mathbb{P}^4$.

4.4.1.6 Cubic fourfolds of discriminant 42

The 25-dimensional family $\mathcal{S}^{9,2} \subset \mathrm{Hilb}_{\mathbb{Y}^5}$ considered in the previous subsection and the 3-dimensional family of planes in $\mathbb{Y}^5$ with class $\sigma_{2,2}$ can be combined together to get a family of surfaces in $\mathbb{P}^5$ of dimension $48 = 25 + 3 - \dim \mathrm{Aut}(\mathbb{Y}^5) + \dim \mathrm{Aut}(\mathbb{P}^5)$. Indeed, let $\mathbf{S} \subset \mathbb{Y}^5$ be a surface corresponding to a general member of the family $\mathcal{S}^{9,2}$, and let $P \subset \mathbb{Y}^5$ be a general plane with class $\sigma_{2,2}$ in $\mathbb{G}(1,4)$. Then the projection of $\mathbf{S}$ from P gives a rational surface $\widetilde{\mathbf{S}} \subset \mathbb{P}^5$ of degree 9 and sectional genus 2, cut out by 9 cubics and having 5 non-normal nodes. Let us denote by $\widetilde{\mathcal{S}^{9,2}} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ the (closure of the) family of surfaces $\widetilde{\mathbf{S}}$ obtained as above.

Theorem 4.4.8 ([93]). *The family $\widetilde{\mathcal{S}^{9,2}} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ is irreducible and unirational of dimension 48. The cubic fourfolds containing a surface of the family $\widetilde{\mathcal{S}^{9,2}}$ describe the divisor $\mathcal{C}_{42} \subset \mathcal{C}$.*

Theorem 4.4.9 ([93]). *Let $\widetilde{\mathbf{S}} \subset \mathbb{P}^5$ be a surface corresponding to a general member of the family $\widetilde{\mathcal{S}^{9,2}}$. Then $\widetilde{\mathbf{S}}$ admits a congruence of 8-secant twisted cubic curves, that is, through the general point of $\mathbb{P}^5$ there passes a unique twisted cubic which is 8-secant to $\widetilde{S}$.*

The general cubic curve of this congruence can be realized as the general fiber of the dominant map

$$\mu : \mathbb{P}^5 \dashrightarrow W \subset \mathbb{P}^7$$

onto a smooth del Pezzo fourfold $W = \mathbb{G}(1,4) \cap \mathbb{P}^7$, defined by the linear system $|H^0(\mathcal{I}^3_{\widetilde{\mathbf{S}},\mathbb{P}^5}(8))|$ of octic hypersurfaces with triple points along $\widetilde{\mathbf{S}}$.

If X is a general cubic fourfold containing $\widetilde{\mathbf{S}}$, then the restriction of μ induces a birational map $\mu|_X : X \overset{\simeq}{\dashrightarrow} W$.

Remark 4.4.10. *It follows from well-known classic results that a del Pezzo fourfold $W = \mathbb{G}(1,4) \cap \mathbb{P}^7$ defined over an infinite field K is rational over K.*

4.4.2 Unirationality of moduli spaces of $m-$pointed fourfolds

The goal of this Section is to prove the following result.

Theorem 4.4.11 (Theorem IX). *The moduli spaces of $m-$pointed fourfolds over the following moduli loci are unirational:*

1. *$\mathcal{C}_{14}$: cubic fourfolds containing a quintic del Pezzo surface (§ 4.4.1.1);*
2. *$\mathcal{C}_{26}$: cubic fourfolds containing a 3-nodal septic scroll (§ 4.4.1.2);*
3. *$\mathcal{C}_{38}$: cubic fourfolds containing a generalized Coble surface (§ 4.4.1.3);*
4. *$\mathcal{C}_{42}$: cubic fourfolds containing a 5-nodal rational surface of degree 9 and sectional genus 2 (§ 4.4.1.6);*
5. *$(\mathcal{M}^4_{GM})'_{10}$: GM fourfolds containing a τ-quadric surface (§ 4.4.1.4.1);*
6. *$(\mathcal{M}^4_{GM})''_{10}$: GM fourfolds containing a quintic del Pezzo surface (§ 4.4.1.4.2);*

7. $(\mathcal{M}^4_{GM})_{20}$*: GM fourfolds containing a smooth rational surface of degree 9 and sectional genus 2 (§ 4.4.1.5);*

Remark 4.4.12. *Recall that we denoted by* $\mathcal{U} \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ *(respectively,* $\mathcal{V} \subset H^0(\mathcal{O}_{\mathbb{Y}^5}(2))$*) the open set parametrizing smooth cubic hypersurface in* $\mathbb{P}^5$ *(respectively, smooth quadric hypersurfaces in* $\mathbb{Y}^5$*). If* $\mathcal{S}$ *is a family of surfaces in the the Hilbert scheme* $\mathrm{Hilb}_{\mathbb{P}^5}$ *of* $\mathbb{P}^5$ *(respectively, in the Hilbert scheme* $\mathrm{Hilb}_{\mathbb{Y}^5}$ *of* $\mathbb{Y}^5$*), then we denote by* $\mathcal{X}_{\mathcal{S}}$ *the closure inside* $\mathcal{U}$ *(respectively, inside* $\mathcal{V}$*) of the family of fourfolds that contain some surface* $[\mathbf{S}] \in \mathcal{S}$*, and we let* $\widetilde{\mathcal{X}_{\mathcal{S}}} = \mathcal{X}_{\mathcal{S}}//\mathrm{Aut}(\mathbb{P}^5) \subseteq \mathcal{C}$ *(respectively,* $\widetilde{\mathcal{X}_{\mathcal{S}}} = \overline{\mathcal{X}_{\mathcal{S}}//\mathrm{Aut}(\mathbb{Y}^5)} \subseteq \mathcal{M}^4_{GM}$*). We observe that the families of fourfolds described in § 4.4.1 (and object of Theorem 4.4.11) all share the following properties:*

1. $\mathcal{S}$ *is irreducible and unirational; so that the same holds true for the corresponding family* $\mathcal{X}_{\mathcal{S}}$*, and hence for* $\widetilde{\mathcal{X}_{\mathcal{S}}}$*.*

2. *If* $(\mathbf{S}, X)$ *is a couple where* $\mathbf{S}$ *is a general member of the family* $\mathcal{S}$ *and* X *is a general fourfold containing* $\mathbf{S}$*, then we are able to build, starting from the pair* $(\mathbf{S}, X)$*, an explicit birational map* $\psi_{(\mathbf{S},X)} : \mathbb{P}^4 \dashrightarrow X$*, defined over the same field of definition as* $\mathbf{S}$ *and* X*.*

Remark 4.4.13. *The family* $\mathcal{X}_{\mathcal{S}}$ *carries the universal 1-pointed fourfold* $\mathcal{X}_{\mathcal{S},1} \to \mathcal{X}_{\mathcal{S}}$*. And inductively one can define a tower of maps*

$$\cdots \to \mathcal{X}_{\mathcal{S},m} \to \mathcal{X}_{\mathcal{S},m-1} \to \cdots \to \mathcal{X}_{\mathcal{S},1} \to \mathcal{X}_{\mathcal{S}}.$$

By quotienting out by the automorphisms of $\mathbb{P}^5$ or $\mathbb{Y}^5$, we can give straight away the following definition.

Definition 4.4.14. *We will denote by* $\widetilde{\mathcal{X}_{\mathcal{S},m}}$ *the moduli space of* $m-$*pointed (cubic or GM) fourfolds. It is the quotient by the respective group of automorphisms of the family* $\mathcal{X}_{\mathcal{S},m}$*.*

Of course there are forgetful maps

$$\widetilde{\mathcal{X}_{\mathcal{S},m}} \overset{\pi_m}{\to} \widetilde{\mathcal{X}_{\mathcal{S},m-1}} \overset{\pi_{m-1}}{\to} \widetilde{\mathcal{X}_{\mathcal{S},m-2}} \to \cdots,$$

with evident meaning.

Proof of Theorem 4.4.11. Let us consider the unirational family $\mathcal{S}$ of surfaces inside $\mathbb{P}^5$ (resp., $\mathbb{Y}^5$). Over the function field of $\mathcal{S}$, we can write the equations of the surface $\mathbf{S} \in \mathcal{S}$, and notably describe their ideal. Call n the dimension of $H^0(\mathbb{P}^5, \mathcal{I}_{\mathbf{S}}(3))$ (resp., $H^0(\mathbb{Y}^5, \mathcal{I}_{\mathbf{S}}(2))$). This is generically constant over $\mathcal{S}$. This means that the Hilbert scheme of couples $(\mathbf{S}, X)$, where X is a fourfold containing the surface $\mathbf{S} \in \mathcal{S}$, is birational to a $\mathbb{P}^{n-1}-$bundle over $\mathcal{S}$, and hence unirational. Let us denote it by $P_{(\mathbf{S},X)}$. This sits naturally inside the product $\mathcal{S} \times \mathcal{X}$, where $\mathcal{X}$ is the moduli space of cubic (resp., GM) fourfolds, and has two natural projections to the two components.

Now we observe that, by definition, over $P_{(\mathbf{S},X)}$ there is a natural *double* universal family. That is: since $P_{(\mathbf{S},X)}$ parametrizes the couples $(\mathbf{S}, X)$, then on one side we have the universal family $\mathcal{S}_1$ of 1-pointed surfaces - that is the universal family of surfaces over $\mathcal{S}$, pulled-back to $P_{(\mathbf{S},X)}$, on the other we have the universal family $\mathcal{X}_{\mathcal{S},1}$ of 1-pointed cubic (resp., GM) fourfolds with the forgetful map $\mathcal{X}_{\mathcal{S},1} \xrightarrow{\pi} P_{(\mathbf{S},X)}$. By definition there is a fiberwise inclusion:

$$\begin{array}{ccc} \mathcal{S}_1 & \hookrightarrow & \mathcal{X}_{\mathcal{S},1} \\ & \searrow \quad \swarrow & \\ & P_{(\mathbf{S},X)} & \end{array}$$

Now, by assumption (2) of our working hypotheses at the beginning of this section, we have that - over $P_{(\mathbf{S},X)}$ - we can define a relative linear system on $\mathcal{X}_{\mathcal{S},1}$ (with base locus supported on $\mathcal{S}_1$) that defines a birational map from $\mathcal{X}_{\mathcal{S},1}$ to a $\mathbb{P}^4$-bundle over $P_{(\mathbf{S},X)}$, that we denote $\mathbb{P}^4_{P_{(\mathbf{S},X)}}$. Now $\mathbb{P}^4_{P_{(\mathbf{S},X)}}$ is rational over $P_{(\mathbf{S},X)}$ and $P_{(\mathbf{S},X)}$ is unirational, hence the universal family $\mathcal{X}_{\mathcal{S},1}$ is unirational, since it is birational to $\mathbb{P}^4_{P_{(\mathbf{S},X)}}$.

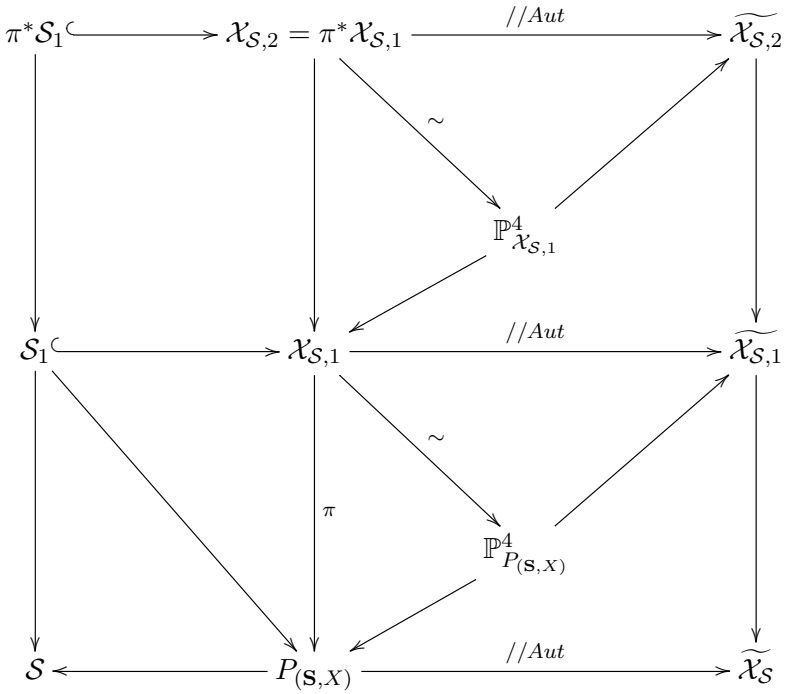
Exactly as one does for $\mathcal{X}_{\mathcal{S},1}$, we can construct a universal cubic (resp., GM) fourfold over $\mathcal{X}_{\mathcal{S},1}$, just by taking the pull-back $\pi^*\mathcal{X}_{\mathcal{S},1}$ over $\mathcal{X}_{\mathcal{S},1}$. We denote by $\mathcal{X}_{\mathcal{S},2}$ this family, and we observe that it tautologically contains $\pi^*\mathcal{S}_1$, as the following diagram follows.

$$\begin{array}{ccc} \pi^*\mathcal{S}_1 & \hookrightarrow & \mathcal{X}_{\mathcal{S},2} = \pi^*\mathcal{X}_{\mathcal{S},1} \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \hookrightarrow & \mathcal{X}_{\mathcal{S},1} \end{array}$$

Thus $\mathcal{X}_{\mathcal{S},2}$ has the same property (2) as $\mathcal{X}_{\mathcal{S},1}$ and one can define a relative linear system defining the birationality between $\mathcal{X}_{\mathcal{S},2}$ and a $\mathbb{P}^4$-bundle over $\mathcal{X}_{\mathcal{S},1}$ - that we denote by $\mathbb{P}^4_{\mathcal{X}_{\mathcal{S},1}}$.

By the same argument as above, since $\mathcal{X}_{\mathcal{S},1}$ is unirational, $\mathcal{X}_{\mathcal{S},2}$ is unirational as well. Then, inductively, the same argument shows the unirationality of the universal families $\mathcal{X}_{\mathcal{S},m}$, for all m.

Now the natural classifying maps, given by the quotient by the automorphisms groups, make the following diagram commute.



It is then clear that also the corresponding moduli space $\widetilde{\mathcal{X}_{\mathcal{S},m}}$, corresponding to the families $\mathcal{X}_{\mathcal{S},m}$ are unirational.

To conclude the proof we observe that, thanks to the properties (1) and (2), we can plug any one of the seven loci mentioned in the claim inside this construction, and get the result. □

Remark 4.4.15. *Several other special families of cubic (resp., GM) fourfolds verify the hypotheses required in this section. We nevertheless decided to concentrate on certain particular descriptions of codimension one loci.*

In fact we did not only choose some codimension 1 loci, but we also chose a particular description of them. For example, for cubic fourfolds in $\mathcal{C}_{14}$ we could have chosen quartic scrolls as surfaces defining the divisor. We remark however that in that case our argument

would not have worked since the quartic scroll defines a birational map to a 4-dimensional quadric, and a quadric bundle is not automatically rational over its base.

Appendix A

K3 Surfaces

We mention here some facts about K3 surfaces. For more details see [16, 51].

A.1 Basic facts on K3 surfaces

Definition A.1.1. *A K3 surface is a smooth compact complex surface S such that*

$$\Omega_S^2 \cong \mathcal{O}_S \text{ and } H^1(S, \mathcal{O}_S) = 0.$$

Every complex K3 surface is Kähler [100] and the existence of Kähler metric implies that the Hodge decomposition holds on its cohomology group:

$$H^n(X, \mathbb{C}) := H^n(X, \mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

such that

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

The Hodge diamond of a K3 surface S is the following

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

with $h^{p,q}(S) := dim H^q(S, \Omega_S^p)$.

As one can see the most interesting behaviour happens in the middle cohomology corresponding to $H^2(X,\mathbb{Z})$. This group with the cup-product pairing:

$$H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$$

forms a lattice of rank 22.

Proposition A.1.2. *For a complex K3 surface S,*

$$H^2(S,\mathbb{Z}) \cong U^{\oplus 3} \bigoplus E_8^{\oplus 2}(-1)$$

with E_8 the rank 8 lattice associated to the corresponding Dynkin diagram and U the hyperbolic plane.

Theorem A.1.3. *(Torelli Theorem [69]) Let S and S' be two complex K3 surfaces, then S and S' are biholomorphic if and only if $H^2(S,\mathbb{Z})$ and $H^2(S',\mathbb{Z})$ are Hodge isometric i.e there is a lattice isometry $H^2(S,\mathbb{Z}) \xrightarrow{\sim} H^2(S',\mathbb{Z})$ such that its complexification $H^2(S,\mathbb{C}) \to H^2(S',\mathbb{C})$ preserves the Hodge decomposition.*

There is two important sublattices of $H^2(S,\mathbb{Z})$. The first one is associated to the Néron-Severi group $NS(S)$. In fact, the Lefschetz theorem on (1,1)-classes yields an isomorphism

$$Pic(S) \cong NS(S) := H^{1,1}(S) \cap H^2(S,\mathbb{Z})$$

Hence the rank of the Néron-Severi group $NS(S)$ called the Picard number $\rho(S) = \mathrm{rk}(NS(S))$ is in between 1 and 20. The second sublattice that occurs also frequently is the transcendental lattice $T(S)$ corresponding to the smallest sublattice of $H^2(S,\mathbb{Z})$ whose complexification contains a generator σ of $H^{2,0}(S)$.

A.1.1 Polarized K3 surfaces

Definition A.1.4. *A polarized K3 surface of degree d is a pair $[S,H]$ where S is a K3 surface and $H \longrightarrow S$ an ample primitive line bundle with $H.H = d$.*

The geometry of these classes of K3 surfaces was studied in [18, 72]. Let $\mathcal{F}_g$ be the 19-dimensional quasi-projective (coarse) moduli space of polarized K3 surfaces of genus g (or degree $d = 2g-2$) parametrizing pairs $[S,H]$, where S is a K3 surface and $H \in Pic(S)$ is the polarization with $H^2 = d$ (see [14, 16, 44]). In the celebrated series of papers [76–79, 82],

Mukai described the construction of general polarized K3 surfaces of genus $g \leq 11$ as well as $g = 13, 16, 18, 20$. In particular for those values, $\mathcal{F}_g$ is unirational. In addition, using the rational map 2.1, Nuer [87] proved that $\mathcal{F}_8$, $\mathcal{F}_{14}$ and $\mathcal{F}_{20}$ are unirational based on the unirationality of $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$. Moreover, Farkas and Verra [34, 35] proved that $\mathcal{F}_{14}$ and $\mathcal{F}_{22}$ are unirational based on the unirationality of the universal K3 surfaces $\mathcal{F}_{14,1}$ and $\mathcal{F}_{22,1}$ (see next section). $\mathcal{F}_g$ in turn are known to be of general type for $g > 122$ by groundbreaking work of Gritsenko, Hulek and Sankaran [42] (see also [41, 110]).

A.2 Universal families of K3 surfaces

Universal families over K3 surfaces are well studied and several results about these objects have appeared recently in [34], [35], [15] and [70]. In the following, we mention some facts about the birational geometry of certain universal families of K3 surfaces.

Recall that for a generic K3 surface of $\mathcal{F}_g$ that has no automorphism, universal families exist:

$$f_g : \mathcal{F}_{g,1} \quad \to \quad \mathcal{F}_g,$$

at least over an open subset of each moduli space. This is enough while working in birational category. This has allowed the study of $\mathcal{F}_{14,1}$ and $\mathcal{F}_{22,1}$ by Farkas-Verra as mentioned before. They considered universal K3 surface as moduli spaces for couples (X, R), where X is a cubic fourfold, and R is a scroll - belonging to a given class of surfaces - contained in X.

Proposition A.2.1. *[34, 35, 107] The universal K3 surfaces $\mathcal{F}_{14,1}$[1] and $\mathcal{F}_{22,1}$ are unirational.*

Recently, Ma [70] studied certain $\mathcal{F}_{g,n}$, the moduli space of $n-$pointed K3 surfaces of genus g and their Kodaira dimension. Many results were made on the (uni)rationality of these moduli spaces. Though probably already known to the specialists, the following result does not seem to have appeared elsewhere, and is original.

Proposition A.2.2. *The moduli space $\mathcal{F}_{11,7}$ is unirational.*

Proof. By [15, Theorem 0.4], we have that $\mathcal{F}_{11,7}$ is birational to a $\mathbb{Z}_2^{\bigoplus 2}-$quotient of $\mathcal{M}_{9,9}$, the moduli space of curves of genus 9 with $9-$marked points. Since $\mathcal{M}_{9,9}$ is unirational (see

[1] The universal K3 surface $\mathcal{F}_{14,1}$ is actually rational.

[33, Proposition 5.7]), the quotient $\mathcal{M}_{9,9}/\mathbb{Z}_2^{\oplus 2}$ is unirational as well. Therefore, $\mathcal{F}_{11,7}$ is unirational. □

A.3 Some other surfaces

We define in this Section certain other surfaces that appeared frequently in the previous chapters.

A.3.1 Scrolls

Definition A.3.1. *A scroll is a ruled surface embedded in a projective space such that the fibers of the ruling are straight lines.*

To describe rational normal scrolls, an important family of scrolls, let us define the rational normal curve of certain degree d in $\mathbb{P}^d$, which is the image of the Veronese map of degree d given by:

$$\begin{array}{rccc} v_d: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^d \\ & [x:y] & \longmapsto & [x^d : x^{d-1}y : \ldots : xy^{d-1} : y^d]. \end{array}$$

To build a rational normal scroll $R_{a,b}$ in $\mathbb{P}^{a+b+1}$, we choose two complementary linear spaces of dimension a and b, and in each we put a rational normal curve, the image of the line under v_a and v_b respectively. The surface $R_{a,b}$ is constructed by taking the union of all of the lines joining $v_a(p)$ to $v_b(p)$ as p varies in $\mathbb{P}^1$.

Remark A.3.2. *For $1 \leq a \leq b$, $R_{a,b}$ is actually isomorphic to the Hirzebruch surface $\mathbb{F}_n$, for $n = b - a$, and of degree $a + b$.*

A.3.2 Del Pezzo surfaces

Definition A.3.3. *A del Pezzo surface is a complete (non-singular) surface with ample anticanonical bundle. Its degree d is the self intersection number (K, K) of its canonical class K.*

The possible degrees run between $d = 1$ and $d = 9$. Topologically, del Pezzo surfaces are determined by their degree except for $d = 8$. If d is different from 8, then a del Pezzo surface of degree d, is a generic blow up of the complex projective plane $\mathbb{P}^2$ in $9 - d$ points

in general positions *i.e* no 3 points lie on a line, no 6 points lie on a conic, and no 8 points lie on a singular cubic, with one of the points at the singularity. But if $d = 8$, there are 2 isomorphism types; one is a Hirzebruch surface given by the blow up of the projective plane at one point; and the other is the product of two projective lines $\mathbb{P}^1 \times \mathbb{P}^1$, which is the only del Pezzo surface that cannot be obtained by starting with the projective plane and blowing up points.

Appendix B

Chow motives of cubic fourfolds and K3 surfaces

Here we collect some facts and known results on the Chow groups and motives of K3 surfaces and cubic fourfolds. We also include some remarks on the notion of finite dimensionality of motives in the sense of Kimura and O'Sullivan (see [20, 84, 89]).

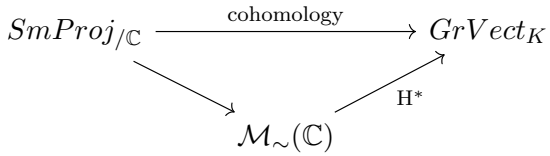
B.1 Introduction to the theory of motives

The theory of motives was created by Grothendieck in the 1960's. This comes from the intent to create a "universal cohomology theory" for several already known cohomology theories such as Betti cohomology, de Rham cohomology, étale cohomology, cristalline cohomology... Indeed, these cohomology theories are examples of what is called Weil cohomology theory with cohomology functor from the category $SmProj_{\mathbb{C}}$ of smooth projective varieties over $\mathbb{C}$ to the category $GrVect_K$ of graded $K-$vector spaces, for some field K of characteristic zero:

$$SmProj_{/\mathbb{C}} \longrightarrow GrVect_K.$$

The idea behind the theory of motives is that all Weil cohomology theories should factor through a "category of motives", i.e. any Weil cohomology theory can be expressed as the

following composition of functors:



where $\mathcal{M}_\sim(\mathbb{C})$ is the category of motives over $\mathbb{C}$ and H^* is called a realization functor. One can get then the different Weil cohomology theories via different realization functors from $\mathcal{M}_\sim(\mathbb{C})$ to $GrVect_K$.

B.2 Basic notions of Chow motives

Let $Z^i(X)$ be the group of algebraic cycles of codimension i on a variety X which is the free abelian group generated by irreducible closed subvarieties of codimension i in X. Two cycles z_1 and z_2 in $Z^i(X)$ are rationally equivalent if there exists an algebraic cycle $z \in Z^i(X \times \mathbb{P}^1)$ and $a,\ b \in \mathbb{P}^1$ such that z_1 is the fiber of z over a and z_2 is the fiber of z over b. The Chow group of cycles of codimension i on X, $CH^i(X)$, is $Z^i(X)$ modulo rational equivalence.
When working with rational coefficients instead of integral coefficients, we add the subscript $\mathbb{Q}$ to our objects.

Definition B.1. *A correspondence from X of dim d to another variety Y is a cycle in the product $X \times Y$. We will denote:*

$$Corr^i(X,Y) := CH^{d+i}(X \times Y)_{\mathbb{Q}}$$

Definition B.2. *The category of Chow motives $\mathcal{M}_{rat}(\mathbb{C})$ consists of triples (X,p,m) with X is smooth projective variety over $\mathbb{C}$, $p \in Corr^0_{rat}(X,X)$ is a projector and m an integer. The morphisms are as follows:*

$$Hom_{\mathcal{M}_{rat}}((X,p,m),(Y,q,n)) = q \circ Corr^{n-m}_{rat}(X,Y) \circ p$$

The composition is the composition of correspondences and $\mathcal{M}_{rat}(\mathbb{C})$ is an additive, pseudo-abelian and rigid category.

We have a contravariant functor that sends a smooth projective variety X to its motive $h(X)$ as follows:

$$\begin{aligned} h\colon SmProj_{/\mathbb{C}} &\to \mathcal{M}_{rat}(\mathbb{C}) \\ X &\mapsto h(X) = (X, \Delta_X, 0) \\ f : X \to Y &\mapsto h(f) = \Gamma_f^T : h(Y) \to h(X), \end{aligned}$$

with Δ_X the diagonal embedding $X \hookrightarrow X \times X$ and Γ_f^T is the transpose of the graph of f.

We define the standard motives: the unit motive which is the motive of a point $\mathbb{1} = h(point) := (Spec(\mathbb{C}), id, 0)$ and the Lefschetz motive $\mathbb{L} := (\mathbb{P}^1, \mathbb{P}^1 \times point, 0) \simeq (Spec(\mathbb{C}), id, -1)$.

B.3 Chow-Künneth decomposition of motives

We say that the motive of a smooth projective variety X of dimension d, $h(X) \in \mathcal{M}_{rat}(\mathbb{C})$ has a Chow-Künneth decomposition if there exist orthogonal projectors $\pi_i = \pi_i(X) \in Corr^0_{rat}(X, X)$ for $0 \leq i \leq 2d$ such that $\pi_0 + ... + \pi_{2d} = \Delta_X$ and there is a direct sum decomposition

$$h(X) = \bigoplus_{0 \leq i \leq 2d} h_i(X)$$

where $h_i(X) = (X, \pi_i, 0)$, such that for any $0 \leq i \leq 2d$, we have $H^*(h_i(X)) = H^i(X)$.

We recall here a result of [20] on the existence of a Chow-Künneth decomposition for the motive of a cubic fourfold $X \subset \mathbb{P}^5$. Namely we have

Proposition B.2. *For a cubic fourfold X,*

$$h(X) = \mathbb{1} \oplus \mathbb{L} \oplus (\mathbb{L}^2)^{\oplus \rho_2(X)} \oplus t(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4, \tag{B.1}$$

where $\rho_2(X) = \operatorname{rk} CH^2(X) \equiv rank H^{2,2}(X, \mathbb{Q})$, with $CH^2(X) \subset H^4(X, \mathbb{Z})$, and $t(X)$ is the transcendental motive of X, i.e. $H^*(t(X)) = H^4(t(X)) = T(X)_{\mathbb{Q}}$, with $T(X)$ the transcendental lattice of X.

Moreover, for $X \in \mathcal{C}_d$, Bülles [22] shows that for certain values of d, there exists a K3 surface S such that

$$t(X) \simeq t_2(S)(-1). \tag{B.2}$$

Here $t_2(S)$ is the transcendental motive of S , i.e.

$$h(S) = \mathbb{1} \oplus \mathbb{L}^{\oplus \rho(S)} \oplus t_2(S) \oplus \mathbb{L}^2,$$

where $\rho(S)$ is the Picard rank of the Néron-Severi group $NS(S)$ and $H^*(t_2(S)) = H^2(t_2(S)) = T(S)_{\mathbb{Q}}$.
More precisely, the isomorphism (B.2) holds whenever d satisfies the following numerical condition

$$(5*) \qquad \exists f, g \in \mathbb{Z} \text{ with } g|(2n^2+2n+2)\ n \in \mathbb{N} \text{ and } d = f^2 g.$$

Note that an isomorphism $t(X) \simeq t_2(S)(-1)$ can never hold if X is not special, i.e. if $\rho_2(X) = 1$, see [20, Prop.3.4].

Remark B.3.1. *The condition* $(5*)$ *(hence the isomorphism B.2) is likely to be necessary but not sufficient for the rationality of cubics in the divisors* $\mathcal{C}_d$ *(see § 2.4).*

B.4 Finite dimensional motives

Kimura [56] and O'Sullivan [88] introduced the notion of finite dimentionality in the context of motives.
Let Σ_n be the symmetric group of order n. For an object $M = h(X)$ in the category of Chow motives, we denote by $\bigwedge^n M$ the $n^{th}-$exterior power of M which is the image of the following projector:

$$\tfrac{1}{n!}\Sigma_{\sigma\in\Sigma_n} sign(\sigma)[\Gamma_\sigma] : M^{\otimes n} \to M^{\otimes n},$$

where Γ_σ is the natural endomorphism induced by σ on $M^{\otimes n}$.
Similarly, $S^n M$ the $n^{th}-$symmetric power is defined as the image of the projector

$$\tfrac{1}{n!}\Sigma_{\sigma\in\Sigma_n}[\Gamma_\sigma].$$

A motive M is evenly (respectively oddly) finite-dimensional if there exists an $n \in \mathbb{N}^*$ such that $\bigwedge^n M = 0$ (respectively $S^n M = 0$), its dimension is then defined as the maximal number n for which $\bigwedge^n M \neq 0$ (respectively $S^n M \neq 0$).
M is finite-dimensional if it can be decomposed into direct sum $M_+ \bigoplus M_-$, with M_+ evenly finite dimensional and M_- oddly finite dimensional. In this case, $\dim(M) = \dim(M_+) + \dim(M_-)$.

Remark B.4.1. *The unit motive* $\mathbb{1}$ *is evenly finite dimensional with dim* $\mathbb{1} = 1$*, since* $\bigwedge^2 \mathbb{1} = (Spec(\mathbb{C}) \times Spec(\mathbb{C}), id - id, 0) = 0$*. This is the case also for the Lefschetz motive* $\mathbb{L}$ *as well, for the same reason.*

Conjecture (Kimura and O'Sullivan): *All motives should be finite dimensional.*

All known examples of finite-dimensional motives belong to $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$, the category generated by the motives of abelian varieties ([108]). The following classes of smooth projective varieties are known to have motives belonging to $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$:

1. projective spaces, Grassmannian varieties, projective homogeneous varieties, toric varieties ;
2. smooth projective curves ;
3. Kummer K3 surfaces ;
4. Hilbert schemes of points on abelian surfaces;
5. Fermat hypersurfaces ;
6. Cubic 3-folds and their Fano surfaces of lines, see [38] and [30].

In addition, finite dimensionality of the motives of certain K3 surfaces has been addressed in [89] as follows

Theorem A.3. *Let* S *be a smooth complex projective K3 surface with* $\rho(S) = 19, 20$*. Then the motive* $h(S) \in \mathcal{M}_{rat}(\mathbb{C})$ *is finite dimensional and lies in the subcategory* $\mathcal{M}_{rat}^{Ab}(\mathbb{C})$*.*

Appendix C

Gushel-Mukai fourfolds

C.1 Generalities on GM fourfolds

A Gushel-Mukai fourfold (or GM fourfold for short) Y is a prime Fano fourfold of degree 10 and index 2 with $Pic(Y) = \mathbb{Z}H$ for an ample divisor H such that $H^4 = 10$ and $K_Y \cong -2H$. By a result of Mukai [80] (extending to all dimensions a result of Gushel proved in [43] only for dimension 3), a GM fourfold Y can be embedded in $\mathbb{P}^8$ as a complex smooth dimensionally tranverse intersection of a cone in $\mathbb{P}^{10}$ over the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with a linear space W isomorphic to $\mathbb{P}^8$ and a quadric hypersurface Q. GM fourfolds are isomorphic:

1. either to a quadric section of a linear section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ (this is the case when W does not contain the vertex of the cone over the Grassmannian $\mathbb{G}(1,4)$). These GM fourfolds are called *ordinary* GM foufolds.

2. or otherwise to a double cover of a linear section of $\mathbb{G}(1,4)$ branched along a quadric section (called *Gushel-type* fourfolds).

The moduli space of GM fourfolds, obtained as a *GIT* quotient, has dimension 24 and is denoted by $\mathcal{M}^4_{GM}$. The locus of *Gushel-type* fourfolds is an irreducible closed subset of codimension 2 in $\mathcal{M}^4_{GM}$ (see [26], [29, Theorem 5.15] or [27]).

C.2 *Hodge-special* GM fourfolds

Let $\rho : \mathcal{M}^4_{GM} \to \mathcal{D}$ be the period map from the moduli space onto the 20− dimensional quasi-projective period domain $\mathcal{D}$. Each GM fourfold Y determines a period point $\rho(Y) \in \mathcal{D}$. The map ρ is dominant with smooth fibers of dimension 4 (see [26] or [27]).

The Hodge diamond of Y is as follows (see [55, Lemma 4.1]):

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 22 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

In order to focus on the middle cohomology of Y, containing the most nontrivial Hodge theoretic information, we define a positive definite lattice $A(Y) = H^{2,2}(Y) \cap H^4(Y, \mathbb{Z}) \supseteq A(\mathbb{G}(1,4))$ containing the rank 2 lattice $\Gamma_2 := H^4(\mathbb{G}(1,4), \mathbb{Z})|_Y$ defined as follows

	$\sigma_{1,1\|Y}$	$\sigma_{2\|Y} - \sigma_{1,1\|Y}$
$\sigma_{1,1\|Y}$	2	0
$\sigma_{2\|Y} - \sigma_{1,1\|Y}$	0	2

Note that the Schubert cycles $\sigma_{1,1}$ and σ_2 form a basis for Γ_2 (see [40]).

Recall that Y is said to be *Hodge-special* if it contains a surface whose cohomology class does not come from the tautological classes of the Grassmannian $\mathbb{G}(1,4)$. Equivalently, Y is *Hodge-special* if the rank of $A(Y)$ is at least 3.

Definition C.2.1. *One can associate to a very general Hodge-special Gushel-Mukai fourfold a labelling of discriminant d which is a primitive positive-definite rank 3 sublattice $K_d \subset H^4(Y, \mathbb{Z})$ containing the rank 2 lattice Γ_2 and the smooth surface $\mathbf{S}$ such that $[\mathbf{S}] \notin A(\mathbb{G}(1,4))$ with the following intersection matrix of determinant d:*

$$\begin{pmatrix} 2 & 0 & b \\ 0 & 2 & a-b \\ b & a-b & \mathbf{S}_Y^2 \end{pmatrix}$$

in the basis $(\sigma_{1,1|Y}, \sigma_{2|Y} - \sigma_{1,1|Y}, [\mathbf{S}])$ *where* $[\mathbf{S}] = a\sigma_{3,1} + b\sigma_{2,2}$ *in terms of Schubert cycles in* $\mathbb{G}(1,4)$ *for some integers* a *and* b *(see [49], [101] for more details).*

These special fourfolds are parametrized by a countable union of hypersurfaces $(\mathcal{M}^4_{GM})_d \in \mathcal{M}^4_{GM}$ labelled by the integers $d \equiv 0, 2$ or 4 $(mod\ 8)$ (see [26, Lemma 6.1]). The hypersurface $(\mathcal{M}^4_{GM})_d$ is irreducible if $d \equiv 0$ $(mod\ 4)$ and has two irreducible components $(\mathcal{M}^4_{GM})'_d$ and $(\mathcal{M}^4_{GM})''_d$ if $d \equiv 2$ $(mod\ 8)$ (see [26, Corollary 6.3]). The same holds true for the hypersurfaces $\rho((\mathcal{M}^4_{GM})_d) = \mathcal{D}_d \subset \mathcal{D}$.

C.2.1 Examples of families of GM fourfolds

In [26, Sect. 7] and [49] several families of fourfolds in $\mathcal{M}^4_{GM}$ were studied and explicit descriptions of the general member of these families were given. More recently, Staglianò [101], inspired by the work of Nuer [87], gives an explicit description of the first irreducible components of the *Noether-Lefschetz* locus in terms of classes of rational smooth surfaces that the fourfolds have to contain. They are described as the closure in $\mathcal{M}^4_{GM}$ of families of fourfolds containing certain rational surfaces that can be embedded in a smooth del Pezzo fivefold. We regroup next certain families of GM fourfolds that are considered in this manuscript.

Surface $\mathbf{S} \subset \mathbb{Y}^5$	class in $\mathbb{G}(1,4)$	Codim in $\mathcal{M}^4_{GM}$	$(\mathcal{M}^4_{GM})_d$ in $\mathcal{M}^4_{GM}$	Intersection matrix of K_d in the basis $< \sigma_{1,1}, \sigma_2 - \sigma_{1,1}, [\mathbf{S}] >$
$\tau-$quadric surface	$\sigma_{3,1} + \sigma_{2,2}$	1	$(\mathcal{M}^4_{GM})'_{10}$	$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$
Quintic del Pezzo surface	$3\sigma_{3,1} + 2\sigma_{2,2}$	1	$(\mathcal{M}^4_{GM})''_{10}$	$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}$
Rational surface of degree 9 and genus 2	$6\sigma_{3,1} + 3\sigma_{2,2}$	1	$(\mathcal{M}^4_{GM})_{20}$	$\begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 3 \\ 3 & 3 & 14 \end{pmatrix}$

Notation

X a cubic fourfold in $\mathbb{P}^5$

$\mathcal{U} \subset \mathbb{P}^{55}$ the Zariski open parametrizing smooth cubic hypersurfaces in $\mathbb{P}^5$

$\mathcal{C}$ the moduli space of cubic fourfolds

Ω_X^p the sheaf of $p-$forms on X

$H^q(X, \Omega_X^p)$ the cohomology group of Ω_X^p

$h^{p,q} = dim\ H^q(X, \Omega_X^p)$ the Hodge number

$h \in H^2(X, \mathbb{Z})$ the hyperplane class

L the cohomology lattice of X corresponding to $H^4(X, \mathbb{Z})$

L_{prim} the primitive cohomology lattice of X

$F(X)$ the Fano variety of lines of X

$\theta :\ L \longrightarrow H^2(F(X), \mathbb{Z})\ (-1)$ the Abel-Jacobi map

$I_{3,0}$ the rank 3 lattice of signature (3,0) whose intersection matrix is the identity matrix of rank 3

E_8 the unimodular positive definite even rank 8 lattice associated to the corresponding Dynkin diagram defined by the following matrix:

$$E_8 := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

U the hyperbolic plane defined by $U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

A_2 the lattice $A_2 := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$A(X) = H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ the lattice of algebraic 2-cycles on X

$d(A(X))$ the discriminant of $A(X)$

$\mathcal{C}_d$ the divisors in $\mathcal{C}$ parametrizing special cubic fourfolds

K_d a labelling associated to $\mathcal{C}_d$

$K_d^{\perp}$ the non special cohomology lattice i.e the orthogonal complement of K_d in L

$\mathbf{S}$ an algebraic surface

$d_{\mathbf{S}}$ the degree of $\mathbf{S}$

$K_{\mathbf{S}}$ the canonical divisor of $\mathbf{S}$

$\chi_{\mathbf{S}}$ the Euler characteristic of $\mathbf{S}$

$\mathcal{N}_{\mathbf{S}/X}$ the normal bundle of $\mathbf{S}$

$L(X)$ the 8fold associated to X

$\mathrm{D}^{\mathrm{b}}(X)$ the bounded derived category of X

$Ku(X)$ the Kuznetsov component of X, the right orthogonal to $\mathcal{O}(X), \mathcal{O}_X(1), \mathcal{O}_X(2)$

S a K3 surface

$NS(S)$ the Néron-Severi group of S

$\rho(S)$ the rank of $NS(S)$

$T(S)$ the transcendental lattice of S

$S^{[2]}$ the Hilbert scheme of length two subschemes of S

$S^{[n]}$ the Hilbert scheme of n points on S

$Br(S)$ the Brauer group of S

$\alpha,\ \beta$ and γ the Brauer classes

$\mathcal{F}_g$ the moduli space of genus g K3 surfaces

$\mathcal{F}_{g,n}$ the moduli space $n-$pointed K3 surfaces of genus g

$\mathcal{C}_{d,1} \to \mathcal{C}_d$ a universal cubic fourfold

$\mathcal{C}_{d,n}$ the moduli space of $n-$marked cubic fourfolds

$CH^i(X)$ the Chow group of codimension i cycles of X

$\mathcal{M}_{rat}(\mathbb{C})$ the category of Chow motives

$\mathcal{M}_{rat}^{Ab}(\mathbb{C})$ the category generated by motives of abelian varieties

$h(X)$ the Chow motive of X

$t(X)$ the transcendental motive of X

$\mathbb{1} = h(point)$ the motive of a point

$\mathbb{L}$ the Lefschetz motive

$GrVect_K$ the category of graded $K-$vector spaces

$SmProj_{\mathbb{C}}$ the category of smooth projective varieties over $\mathbb{C}$

$\mathbb{G}(1,4)$ the Grassmannian of lines in $\mathbb{P}^4$

Y a Gushel-Mukai fourfold

$\mathcal{M}^4_{GM}$ the moduli space of Gushel-Mukai fourfolds

$\mathcal{D}$ the period domain associated to Gushel-Mukai fourfolds

$A(Y) = H^{2,2}(Y) \cap H^4(Y,\mathbb{Z}) \supseteq A(\mathbb{G}(1,4))$ the positive definite lattice associated to Y

$\Gamma_2 := H^4(\mathbb{G}(1,4),\mathbb{Z})|_Y$ the rank 2 lattice coming from $\mathbb{G}(1,4)$

$(\mathcal{M}^4_{GM})_d \subset \mathcal{M}^4_{GM}$ the hypersurfaces parametrizing special Gushel-Mukai fourfolds

$(\mathcal{M}^4_{GM})_{d,1}$ the moduli space of $1-$marked Gushel-Mukai fourfolds

$(\mathcal{M}^4_{GM})_{d,n}$ the moduli space of $n-$marked Gushel-Mukai fourfolds

$\mathcal{V}^{nod}_n$ the moduli space of $n-$pointed one nodal K3 surfaces of sectional genus 12

$\mathcal{M}^{4nod}_{W_{9,2}} \subset \mathcal{M}^4_{W_{9,2}} \subset \mathcal{M}^4_{GM}$ the moduli space of GM fourfolds obtained from the K3 surfaces in $\mathcal{V}^{nod}_n$

$\mathbb{Y}^5$ a del Pezzo fivefold

$\mathcal{S}$ a family of surfaces

$\mathcal{X}_{\mathcal{S}}$ a family of fourfolds that contains surfaces from $\mathcal{S}$

References

[1] N. Addington, "On two rationality conjectures for cubic fourfolds," *Math. Res. Lett.*, vol. 23, no. 1, pp. 1–13, 2016. [Online]. Available: https://doi.org/10.4310/MRL.2016.v23.n1.a1

[2] N. Addington and F. Giovenzana, "On the period of Lehn, Lehn, Sorger, and van Straten's symplectic eightfold," 2020.

[3] N. Addington and R. Thomas, "Hodge theory and derived categories of cubic fourfolds," *Duke Math. J.*, vol. 163, no. 10, pp. 1885–1927, 2014. [Online]. Available: https://doi.org/10.1215/00127094-2738639

[4] N. Addington, B. Hassett, Y. Tschinkel, and A. Várilly-Alvarado, "Cubic fourfolds fibered in sextic del Pezzo surfaces," *Amer. J. Math.*, vol. 141, no. 6, pp. 1479–1500, 2019. [Online]. Available: https://doi.org/10.1353/ajm.2019.0041

[5] A. B. Altman and S. L. Kleiman, "Foundations of the theory of Fano schemes," *Compositio Math.*, vol. 34, no. 1, pp. 3–47, 1977. [Online]. Available: http://www.numdam.org/item?id=CM_1977__34_1_3_0

[6] A. Auel and M. Bernardara, "Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields," *Proceedings of the London Mathematical Society*, vol. 117, no. 1, p. 1–64, Feb 2018. [Online]. Available: http://dx.doi.org/10.1112/plms.12119

[7] A. Auel, M. Bernardara, and M. Bolognesi, "Fibrations in complete intersections of quadrics, clifford algebras, derived categories, and rationality problems," *Journal de Mathématiques Pures et Appliquées*, vol. 102, no. 1, p. 249–291, Jul 2014. [Online]. Available: http://dx.doi.org/10.1016/j.matpur.2013.11.009

[8] A. Auel, M. Bernardara, M. Bolognesi, and A. Várilly-Alvarado, "Cubic fourfolds containing a plane and a quintic del Pezzo surface," *Algebr. Geom.*, vol. 1, no. 2, pp. 181–193, 2014. [Online]. Available: https://doi.org/10.14231/AG-2014-010

[9] H. Awada, "Rational fibered cubic fourfolds with nontrivial Brauer classes," 2019.

[10] H. Awada and M. Bolognesi, "Unirationality of certain universal families of cubic fourfolds," 2020, *to appear in the proceedings of "Rationality of algebraic varieties" (Schiermonnikoog, April 15-19, 2019).*

[11] H. Awada, M. Bolognesi, and C. Pedrini, "K3 surfaces and cubic fourfolds with abelian motive," 2021.

[12] H. Awada, M. Bolognesi, and G. Staglianò, "Some moduli of n-pointed Fano fourfolds," 2021.

[13] G. b. Fano, "Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4o ordine," *Commentarii Mathematici Helvetici*, vol. 15, pp. 71–80, 1942.

[14] W. L. Baily, Jr. and A. Borel, "Compactification of arithmetic quotients of bounded symmetric domains," *Ann. of Math. (2)*, vol. 84, pp. 442–528, 1966. [Online]. Available: https://doi.org/10.2307/1970457

[15] I. Barros, "Geometry of the moduli space of n-pointed K3 surfaces of genus 11," *Bulletin of the London Mathematical Society*, vol. 50, no. 6, p. 1071–1084, Sep 2018. [Online]. Available: http://dx.doi.org/10.1112/blms.12203

[16] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., ser. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2004, vol. 4. [Online]. Available: https://doi.org/10.1007/978-3-642-57739-0

[17] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, and P. Stellari, "Stability conditions in families," 2020.

[18] A. Beauville, *Complex algebraic surfaces*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1983, vol. 68, translated from the French by R. Barlow, N. I. Shepherd-Barron and M. Reid.

[19] A. Beauville and R. Donagi, "La variété des droites d'une hypersurface cubique de dimension 4," *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 301, no. 14, pp. 703–706, 1985.

[20] M. Bolognesi and C. Pedrini, "The transcendental motive of a cubic fourfold," *J. Pure Appl. Algebra*, vol. 224, no. 8, pp. 106 333, 16, 2020. [Online]. Available: https://doi.org/10.1016/j.jpaa.2020.106333

[21] M. Bolognesi, F. Russo, and G. Staglianò, "Some loci of rational cubic fourfolds," *Math. Ann.*, vol. 373, no. 1-2, pp. 165–190, 2019. [Online]. Available: https://doi.org/10.1007/s00208-018-1707-7

[22] T.-H. Bülles, "Motives of moduli spaces on K3 surfaces and of special cubic fourfolds," *Manuscripta Math.*, vol. 161, no. 1-2, pp. 109–124, 2020. [Online]. Available: https://doi.org/10.1007/s00229-018-1086-0

[23] C. H. Clemens and P. A. Griffiths, "The intermediate Jacobian of the cubic threefold," *Ann. of Math. (2)*, vol. 95, pp. 281–356, 1972. [Online]. Available: https://doi.org/10.2307/1970801

[24] J.-L. Colliot-Thélène, "Descente galoisienne sur le second groupe de Chow: mise au point et applications," *Doc. Math.*, no. Extra vol.: Alexander S. Merkurjev's sixtieth birthday, pp. 195–220, 2015.

[25] M. A. A. de Cataldo and L. Migliorini, "The Chow groups and the motive of the Hilbert scheme of points on a surface," *J. Algebra*, vol. 251, no. 2, pp. 824–848, 2002. [Online]. Available: https://doi.org/10.1006/jabr.2001.9105

[26] O. Debarre, A. Iliev, and L. Manivel, "Special prime Fano fourfolds of degree 10 and index 2," in *Recent advances in algebraic geometry*, ser. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2015, vol. 417, pp. 123–155.

[27] O. Debarre, "Gushel-Mukai varieties," 2020.

[28] O. Debarre and A. Kuznetsov, "Gushel-Mukai varieties: classification and birationalities," *Algebr. Geom.*, vol. 5, no. 1, pp. 15–76, 2018. [Online]. Available: https://doi.org/10.14231/ag-2018-002

[29] ——, "Gushel-Mukai varieties: moduli," *Internat. J. Math.*, vol. 31, no. 2, pp. 2 050 013, 59, 2020. [Online]. Available: https://doi.org/10.1142/S0129167X20500135

[30] H. Diaz, "The motive of the Fano surface of lines," *C. R. Math. Acad. Sci. Paris*, vol. 354, no. 9, pp. 925–930, 2016. [Online]. Available: https://doi.org/10.1016/j.crma.2016.07.003

[31] F. Enriques, "Sulle irrazionalità da cui può farsi dipendere la risoluzione d'un' equazione algebrica f(xys) = 0 con funzioni raziionali di due parametri," *Mathematische Annalen*, vol. 49, pp. 1–23, 1897. [Online]. Available: http://eudml.org/doc/157836

[32] Y.-W. Fan and K.-W. Lai, "New rational cubic fourfolds arising from Cremona transformations," 2020.

[33] G. Farkas and A. Verra, "The classification of universal Jacobians over the moduli space of curves," *Comment. Math. Helv.*, vol. 88, no. 3, pp. 587–611, 2013. [Online]. Available: https://doi.org/10.4171/CMH/297

[34] ——, "The universal $K3$ surface of genus 14 via cubic fourfolds," *J. Math. Pures Appl. (9)*, vol. 111, pp. 1–20, 2018. [Online]. Available: https://doi.org/10.1016/j.matpur.2017.07.014

[35] ——, "The unirationality of the moduli space of K3 surfaces of genus 22," 2019.

[36] S. Galkin and E. Shinder, "The Fano variety of lines and rationality problem for a cubic hypersurface," 2014.

[37] V. González-Aguilera and A. Liendo, "Automorphisms of prime order of smooth cubic n-folds," *Arch. Math. (Basel)*, vol. 97, no. 1, pp. 25–37, 2011. [Online]. Available: https://doi.org/10.1007/s00013-011-0247-0

[38] S. Gorchinskiy and V. Guletskiĭ, "Motives and representability of algebraic cycles on threefolds over a field," *J. Algebraic Geom.*, vol. 21, no. 2, pp. 347–373, 2012. [Online]. Available: https://doi.org/10.1090/S1056-3911-2011-00548-1

[39] D. R. Grayson and M. E. Stillman, "Macaulay2, a software system for research in algebraic geometry," Available at http://www.math.uiuc.edu/Macaulay2/.

[40] P. Griffiths and J. Harris, *Principles of algebraic geometry*, ser. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994, reprint of the 1978 original. [Online]. Available: https://doi.org/10.1002/9781118032527

[41] V. Gritsenko, K. Hulek, and G. K. Sankaran, "Moduli of K3 surfaces and irreducible symplectic manifolds," in *Handbook of moduli. Vol. I*, ser. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2013, vol. 24, pp. 459–526.

[42] V. A. Gritsenko, K. Hulek, and G. K. Sankaran, "The Kodaira dimension of the moduli of $K3$ surfaces," *Invent. Math.*, vol. 169, no. 3, pp. 519–567, 2007. [Online]. Available: https://doi.org/10.1007/s00222-007-0054-1

[43] N. P. Gushel', "On Fano Varieties of Genus 6," *Izvestiya: Mathematics*, vol. 21, no. 3, pp. 445–459, Jun. 1983.

[44] A. Harder and A. Thompson, "The geometry and moduli of K3 surfaces," in *Calabi-Yau varieties: arithmetic, geometry and physics*, ser. Fields Inst. Monogr. Fields Inst. Res. Math. Sci., Toronto, ON, 2015, vol. 34, pp. 3–43. [Online]. Available: https://doi.org/10.1007/978-1-4939-2830-9_1

[45] J. Harris, "An introduction to the moduli space of curves," in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, ser. Adv. Ser. Math. Phys. World Sci. Publishing, Singapore, 1987, vol. 1, pp. 285–312.

[46] B. Hassett, "Special cubic fourfolds," *Compositio Mathematica*, vol. 120, pp. 1–23, 01 2000.

[47] ——, "Some rational cubic fourfolds," *J. Algebraic Geom.*, vol. 8, no. 1, pp. 103–114, 1999.

[48] ——, "Cubic fourfolds, K3 surfaces, and rationality questions," in *Rationality problems in algebraic geometry*, ser. Lecture Notes in Math. Springer, Cham, 2016, vol. 2172, pp. 29–66.

[49] M. Hoff and G. Staglianò, "New examples of rational Gushel-Mukai fourfolds," *Mathematische Zeitschrift*, Feb 2020. [Online]. Available: http://dx.doi.org/10.1007/s00209-020-02498-5

[50] K. Hulek, S. Katz, and F.-O. Schreyer, "Cremona transformations and syzygies," *Math. Z.*, vol. 209, no. 1, pp. 419–443, 1992.

[51] D. Huybrechts, *Lectures on K3 surfaces*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016, vol. 158. [Online]. Available: https://doi.org/10.1017/CBO9781316594193

[52] ——, "The K3 category of a cubic fourfold," *Compositio Mathematica*, vol. 153, no. 3, p. 586–620, Mar 2017. [Online]. Available: http://dx.doi.org/10.1112/S0010437X16008137

[53] ——, "Compact hyperkähler manifolds," in *Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001)*, ser. Universitext. Springer, Berlin, 2003, pp. 161–225.

[54] ——, "Hodge theory of cubic fourfolds, their Fano varieties, and associated K3 categories," 2018.

[55] A. Iliev and L. Manivel, "Fano manifolds of degree ten and EPW sextics," *Ann. Sci. Éc. Norm. Supér. (4)*, vol. 44, no. 3, pp. 393–426, 2011. [Online]. Available: https://doi.org/10.24033/asens.2146

[56] S.-I. Kimura, "Chow groups are finite dimensional, in some sense," *Math. Ann.*, vol. 331, no. 1, pp. 173–201, 2005. [Online]. Available: https://doi.org/10.1007/s00208-004-0577-3

[57] J. Kollár, "Unirationality of cubic hypersurfaces," *J. Inst. Math. Jussieu*, vol. 1, no. 3, pp. 467–476, 2002. [Online]. Available: https://doi.org/10.1017/S1474748002000117

[58] A. Kuznetsov, "Derived Categories of Families of Sextic del Pezzo Surfaces," *International Mathematics Research Notices*, 06 2019, rnz081. [Online]. Available: https://doi.org/10.1093/imrn/rnz081

[59] ——, "Derived categories of cubic fourfolds," in *Cohomological and geometric approaches to rationality problems*, ser. Progr. Math. Birkhäuser Boston, Boston, MA, 2010, vol. 282, pp. 219–243. [Online]. Available: https://doi.org/10.1007/978-0-8176-4934-0_9

[60] ——, "Derived categories view on rationality problems," in *Rationality problems in algebraic geometry*, ser. Lecture Notes in Math. Springer, Cham, 2016, vol. 2172, pp. 67–104.

[61] K.-W. Lai, "New cubic fourfolds with odd-degree unirational parametrizations," *Algebra Number Theory*, vol. 11, no. 7, pp. 1597–1626, 2017. [Online]. Available: https://doi.org/10.2140/ant.2017.11.1597

[62] R. Laterveer, "A family of cubic fourfolds with finite-dimensional motive," *J. Math. Soc. Japan*, vol. 70, no. 4, pp. 1453–1473, 2018. [Online]. Available: https://doi.org/10.2969/jmsj/74497449

[63] ——, "A remark on the motive of the Fano variety of lines of a cubic," *Ann. Math. Qué.*, vol. 41, no. 1, pp. 141–154, 2017. [Online]. Available: https://doi.org/10.1007/s40316-016-0070-x

[64] ——, "Some cubics with finite-dimensional motive," *Bol. Soc. Mat. Mex. (3)*, vol. 24, no. 2, pp. 319–327, 2018. [Online]. Available: https://doi.org/10.1007/s40590-017-0173-7

[65] R. Laza, "The moduli space of cubic fourfolds via the period map," *Ann. of Math. (2)*, vol. 172, no. 1, pp. 673–711, 2010. [Online]. Available: https://doi.org/10.4007/annals.2010.172.673

[66] ——, "The moduli space of cubic fourfolds," *J. Algebraic Geom.*, vol. 18, no. 3, pp. 511–545, 2009. [Online]. Available: https://doi.org/10.1090/S1056-3911-08-00506-7

[67] C. Lehn, M. Lehn, C. Sorger, and D. van Straten, "Twisted cubics on cubic fourfolds," *J. Reine Angew. Math.*, vol. 731, pp. 87–128, 2017. [Online]. Available: https://doi.org/10.1515/crelle-2014-0144

[68] E. Looijenga, "The period map for cubic fourfolds," *Invent. Math.*, vol. 177, no. 1, pp. 213–233, 2009. [Online]. Available: https://doi.org/10.1007/s00222-009-0178-6

[69] E. Looijenga and C. Peters, "Torelli theorems for Kähler $K3$ surfaces," *Compositio Math.*, vol. 42, no. 2, pp. 145–186, 1980/81. [Online]. Available: http://www.numdam.org/item?id=CM_1980__42_2_145_0

[70] S. Ma, "Mukai models and Borcherds products," 2019.

[71] E. Mayanskiy, "Intersection lattices of cubic fourfolds," 2011.

[72] A. L. Mayer, "Families of $k-3$ surfaces," *Nagoya Math. J.*, vol. 48, pp. 1–17, 1972. [Online]. Available: https://projecteuclid.org:443/euclid.nmj/1118798783

[73] G. Mongardi and J. C. Ottem, "Curve classes on irreducible holomorphic symplectic varieties," *Commun. Contemp. Math.*, vol. 22, no. 7, pp. 1 950 078, 15, 2020. [Online]. Available: https://doi.org/10.1142/S0219199719500780

[74] U. Morin, "Sulla razionalità dell'ipersuperficie cubica dello spazio lineare S_5," *Rend. Semin. Mat. Univ. Padova*, vol. 11, pp. 108–112, 1940.

[75] D. R. Morrison, "On $K3$ surfaces with large Picard number," *Invent. Math.*, vol. 75, no. 1, pp. 105–121, 1984. [Online]. Available: https://doi.org/10.1007/BF01403093

[76] S. Mukai, "K3 surfaces of genus sixteen," in *Minimal models and extremal rays (Kyoto, 2011)*, ser. Adv. Stud. Pure Math. Math. Soc. Japan, [Tokyo], 2016, vol. 70, pp. 379–396. [Online]. Available: https://doi.org/10.2969/aspm/07010379

[77] ——, "Polarized $K3$ surfaces of genus thirteen," in *Moduli spaces and arithmetic geometry*, ser. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2006, vol. 45, pp. 315–326. [Online]. Available: https://doi.org/10.2969/aspm/04510315

[78] ——, "Curves and $K3$ surfaces of genus eleven," in *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*, ser. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1996, vol. 179, pp. 189–197.

[79] ——, "Polarized $K3$ surfaces of genus 18 and 20," in *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, ser. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1992, vol. 179, pp. 264–276. [Online]. Available: https://doi.org/10.1017/CBO9780511662652.019

[80] ——, "Biregular classification of Fano 3-folds and Fano manifolds of coindex 3," *Proc. Nat. Acad. Sci. U.S.A.*, vol. 86, no. 9, pp. 3000–3002, 1989. [Online]. Available: https://doi.org/10.1073/pnas.86.9.3000

[81] ——, "Fano 3-folds," in *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, ser. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1992, vol. 179, pp. 255–263. [Online]. Available: https://doi.org/10.1017/CBO9780511662652.018

[82] ——, "Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10," in *Algebraic geometry and commutative algebra, Vol. I.* Kinokuniya, Tokyo, 1988, pp. 357–377.

[83] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., ser. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, 1994, vol. 34. [Online]. Available: https://doi.org/10.1007/978-3-642-57916-5

[84] J. P. Murre, J. Nagel, and C. A. M. Peters, *Lectures on the theory of pure motives*, ser. University Lecture Series. American Mathematical Society, Providence, RI, 2013, vol. 61. [Online]. Available: https://doi.org/10.1090/ulect/061

[85] F. N.Addington, "On the period of Lehn, Lehn, Sorger, and van Straten's symplectic eightfold," *https://arxiv.org/pdf/2003.10984.pdf*, 2020.

[86] V. V. Nikulin, "Integral symmetric bilinear forms and some of their applications," *Mathematics of the USSR-Izvestiya*, vol. 14, no. 1, pp. 103–167, feb 1980. [Online]. Available: https://doi.org/10.1070%2Fim1980v014n01abeh001060

[87] H. Nuer, "Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces," *Algebr. Geom.*, vol. 4, no. 3, pp. 281–289, 2017. [Online]. Available: https://doi.org/10.14231/AG-2017-015

[88] P. O'Sullivan, "The structure of certain rigid tensor categories," *C. R. Math. Acad. Sci. Paris*, vol. 340, no. 8, pp. 557–562, 2005. [Online]. Available: https://doi.org/10.1016/j.crma.2005.03.018

[89] C. Pedrini, "On the finite dimensionality of a K3 surface," *Manuscripta Math.*, vol. 138, no. 1-2, pp. 59–72, 2012. [Online]. Available: https://doi.org/10.1007/s00229-011-0483-4

[90] L. Roth, "Algebraic varieties with canonical curve sections," *Ann. Mat. Pura Appl. (4)*, vol. 29, pp. 91–97, 1949. [Online]. Available: https://doi.org/10.1007/BF02413916

[91] F. Russo and G. Staglianò, "Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds," *Duke Math. J.*, vol. 168, no. 5, pp. 849–865, 2019. [Online]. Available: https://doi.org/10.1215/00127094-2018-0053

[92] F. Russo and G. Staglianò, "Explicit rationality of some cubic fourfolds," 2019.

[93] ——, "Trisecant flops, their associated k3 surfaces and the rationality of some fano fourfolds," 2019.

[94] A. Sarti, "Transcendental lattices of some $K3$-surfaces," *Math. Nachr.*, vol. 281, no. 7, pp. 1031–1046, 2008. [Online]. Available: https://doi.org/10.1002/mana.200510657

[95] F.-O. Schreyer, "Geometry and algebra of prime Fano 3-folds of genus 12," *Compositio Math.*, vol. 127, no. 3, pp. 297–319, 2001. [Online]. Available: https://doi.org/10.1023/A:1017529016445

[96] J. G. Semple and J. A. Tyrrell, "The $T_{2,4}$ of S_6 defined by a rational surface ${}^3F^8$," *Proc. Lond. Math. Soc.*, vol. s3-20, pp. 205–221, 1970.

[97] J.-P. Serre, *A course in arithmetic*. Springer-Verlag, New York-Heidelberg, 1973, translated from the French, Graduate Texts in Mathematics, No. 7.

[98] N. I. Shepherd-Barron, "The rationality of quintic Del Pezzo surfaces—a short proof," *Bull. London Math. Soc.*, vol. 24, no. 3, pp. 249–250, 1992. [Online]. Available: https://doi.org/10.1112/blms/24.3.249

[99] T. Shioda and H. Inose, "On singular $K3$ surfaces," in *Complex analysis and algebraic geometry*, 1977, pp. 119–136.

[100] Y. T. Siu, "Every $K3$ surface is Kähler," *Invent. Math.*, vol. 73, no. 1, pp. 139–150, 1983. [Online]. Available: https://doi.org/10.1007/BF01393829

[101] G. Staglianò, "On some families of Gushel-Mukai fourfolds," 2020.

[102] ——, "Some new rational Gushel fourfolds," 2020.

[103] S. Tanimoto and A. Várilly-Alvarado, "Kodaira dimension of moduli of special cubic fourfolds," *J. Reine Angew. Math.*, vol. 752, pp. 265–300, 2019. [Online]. Available: https://doi.org/10.1515/crelle-2016-0053

[104] ——, "Kodaira dimension of moduli of special cubic fourfolds," *J. Reine Angew. Math.*, vol. 752, pp. 265–300, 2019. [Online]. Available: https://doi.org/10.1515/crelle-2016-0053

[105] S. L. Tregub, "Three constructions of rationality of a cubic fourfold," *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, no. 3, pp. 8–14, 1984.

[106] ——, "Two remarks on four-dimensional cubics," *Uspekhi Mat. Nauk*, vol. 48, no. 2(290), pp. 201–202, 1993. [Online]. Available: https://doi.org/10.1070/RM1993v048n02ABEH003729

[107] D. D. Tullio, "Rationality of the universal K3 surface of genus 8," 2020.

[108] C. Vial, "Remarks on motives of abelian type," *Tohoku Math. J. (2)*, vol. 69, no. 2, pp. 195–220, 2017. [Online]. Available: https://doi.org/10.2748/tmj/1498269623

[109] C. Voisin, "Théorème de Torelli pour les cubiques de $\mathbb{P}^5$." *Inventiones mathematicae*, vol. 86, pp. 577–602, 1986. [Online]. Available: http://eudml.org/doc/143409

[110] ——, "Géométrie des espaces de modules de courbes et de surfaces $K3$ (d'après Gritsenko-Hulek-Sankaran, Farkas-Popa, Mukai, Verra, et al.)," 2008, no. 317, pp. Exp. No. 981, x, 467–490, séminaire Bourbaki. Vol. 2006/2007.

[111] ——, "Some aspects of the Hodge conjecture," *Japanese Journal of Mathematics*, vol. 2, pp. 261–296, 09 2007.

[112] S. Yang and X. Yu, "Rational cubic fourfolds in Hassett divisors," *C. R. Math. Acad. Sci. Paris*, vol. 358, no. 2, pp. 129–137, 2020. [Online]. Available: https://doi.org/10.5802/crmath.4

[113] ——, "On lattice polarizable cubic fourfolds," 2021.