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# Static replication of European options and dynamic replication of correlation swaps

Sébastien Bossu

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Sébastien Bossu. Static replication of European options and dynamic replication of correlation swaps. Business administration. Université Paris-Saclay, 2021. English. NNT : 2021UPASI014 . tel-03457424

**HAL Id: tel-03457424**

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La réplication statique des options  
européennes et la réplication dynamique  
des swaps de corrélation

*Static replication of European options and  
dynamic replication of correlation swaps*

**Thèse de doctorat de l'université Paris-Saclay**

École doctorale n°630, Droit Économie Management (DEM)

Spécialité de doctorat : Sciences de gestion

Unité de recherche : Réseaux Innovation Territoires & Mondialisation, 92330 Sceaux

Référent : Faculté de droit, économie et gestion

**Thèse présentée et soutenue à Paris-Saclay,  
le 24 septembre 2021, par**

**Sébastien Bossu**

**Composition du Jury**

<b>Stéphane Menozzi</b> Professeur, Université Paris-Saclay	Président
<b>Areski Cousin</b> Professeur, Université de Strasbourg	Rapporteur & Examineur
<b>Jean-Pierre Fouque</b> Professor, University of California, Santa Barbara	Rapporteur & Examineur
<b>Antoon Pelsser</b> Professor, University of Amsterdam & Maastricht University	Rapporteur & Examineur
<b>Bruno Dupire</b> Head of Quantitative Research, Bloomberg LP	Examineur
<b>Robert A. Jarrow</b> Professor, Cornell University	Examineur
<b>Adil Reghai</b> Responsable Recherche Quantitative Dérivés Actions et Matières premières, Natixis	Examineur

**Direction de la thèse**

<b>Stéphane Crépey</b> Professeur, Université Paris-Saclay	Directeur de thèse
<b>Peter Carr</b> Professor, New York University	Invité
<b>Andrew Papanicolaou</b> Assistant professor, North Carolina State University	Invité



# Acknowledgements

I am forever grateful to Peter Carr, Stéphane Crépey and Andrew Papanicolaou for their interest, availability and guidance throughout my research — I could not have dreamed of a better team of advisors. Many thanks to my committee: Stéphane Menozzi, Areski Cousin, Jean-Pierre Fouque, Antoon Pelsser, Bruno Dupire, Robert Jarrow, Adil Reghai. I also want to thank Michel Guillard for welcoming my thesis within his doctoral school. A special thank you to Robert Kohn at NYU Courant for his interest in my work on Radon transforms and for connecting me with experts in this highly specialized area of mathematics, and to Petter Kolm for his support and interest. Finally, this thesis would not have been possible without the loving support of many friends and family: *mille fois merci!*



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# Chapter 1

## Introduction

### 1.1 A brief history of quantitative finance

Quantitative finance, also known as financial mathematics, is a transdisciplinary branch of knowledge that sits at the intersection of economics, mathematics, management science, and decision science. Its main objective is to solve mathematical problems arising from finance, with a particular focus on applications to financial markets and derivative securities: options, futures, and complex transactions known as “structured products”. Other notable domains of application include investment management and risk management.

The foundational lineaments of quantitative finance may be traced back to 1900 with the publication of Louis Bachelier’s thesis under the title “Théorie de la Spéculation”. However, important concepts such as compound interest, discounting, arbitrage and utility theory were known well ahead. Bachelier introduced advanced probability concepts such as random walks and Brownian motions for the purpose of option pricing, which are still in use today.

Five decades later, modern portfolio theory was initiated by Harry Markowitz (1952) while Kenneth Arrow and Gérard Debreu (1954) formalized complete market economies using state-price securities to prove the existence of general economic equilibrium. During the 1960s, Paul Samuelson (1965) and others proposed option valuation formulas that were very close to those proved by Fisher Black and Myron Scholes (1973) and Robert Merton (1973). In 1997, Scholes and Merton received the economics prize in memory of Alfred Nobel for their work (Black died just a few months before the Nobel committee’s meeting).

Today, quantitative finance typically posits that the prices of certain financial assets such as stocks, bonds, or even a class of derivatives, are known, and proceeds to value other assets consistently with this information. This approach differs from the cornucopia of valuation methods, models and theories that are based on structural or

fundamental factors — for example: DCF<sup>1</sup> methods, CAPM<sup>2</sup> and other factor models of asset pricing, GARCH<sup>3</sup> and other related econometric models, the microeconomic theory of supply and demand, utility theory, game theory. The latter are often grouped as “P”-type approaches based on objective or historical factors, in contrast with the “Q”-type approach used in quantitative finance that takes dynamic hedging and arbitrage techniques into account.

The Q versus P terminology comes from financial probability theory whereby P is the objective probability measure satisfying the empirical principle that more risk is rewarded by higher expected returns; whereas Q is the “risk-neutral” probability measure under which investors expect no risk premium. The latter approach is particularly adapted to derivatives pricing without creating any arbitrage opportunity. Since the 1970s, the volume of derivatives transactions has soared a hundred times over and a myriad of new “exotic” contracts were created, necessitating ever more sophisticated Q-type models for their valuation.

In this context, a wide range of multi-asset options whose value depends on the prices of several underlying assets, have been designed and traded. While the pricing methodology of some contracts such as basket call options has become relatively consensual<sup>4</sup>, the valuation of many other multi-asset contracts, including best-of and worst-of options and dispersion options, requires complex and sophisticated probabilistic models, such as Gaussian and non-Gaussian copula, local volatility and correlation models, stochastic volatility and correlation, and their variants with jumps.

The main conclusion of this thesis is that this probabilistic sophistication may not be justified for a large class of multi-asset options that turn out to be decomposable as a static portfolio of basket calls whose valuation is simpler. This central theoretical result is demonstrated in Chapters 3 and 4 for dispersion options and all European multi-asset options with homogeneous payoff, including best-of and worst-of options. Further research will extend this result to all European multi-asset options regardless of homogeneity, and investigate the case of path-dependent options whose value depends on the time series of underlying asset prices.

## 1.2 Organization of this thesis and main results

Chapters 2 to 4 are directly related to the seminal work of Breeden and Litzenberger (1978) and Carr and Madan (1998) on the pricing and static replication of an exotic option with an ad hoc portfolio of simpler options. This line of work does not properly fall within the P-Q typology but rather into a third, “model-free” type valid for any choice of probability measure consistent with the no-arbitrage principle. Below is a summary of our main scientific contributions:

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<sup>1</sup>Discounted Cash Flow

<sup>2</sup>Capital Asset Pricing Model

<sup>3</sup>Generalized Auto-Regressive Conditional Heteroskedasticity

<sup>4</sup>To this end, large issuers contribute basket call prices to the private “IHS Markit Totem” database on a regular basis, and receive summary statistics

Ch.2: The **one-dimensional inverse problem** of replicating any option with a static portfolio of other options may be formulated as a **first-kind integral equation**. Subject to certain mathematical conditions, a solution exists, is unique, and may be calculated by means of spectral decomposition techniques. The truncated spectral decomposition possesses interesting properties for fast option pricing.

Ch.3: The **multidimensional inverse problem** of replicating a dispersion option with a static portfolio of basket calls may be reformulated as a **fractional integral equation**. Subject to regularity conditions, a solution exists, is unique and may be calculated as a fractional derivative of the dispersion payoff. In particular, we derived solutions for the dispersion call and put options.

Ch.4: The **multidimensional inverse problem** of replicating a European multi-asset option with homogeneous payoff by a static portfolio of basket calls may be converted to a **Radon transform inverse problem**. Subject to regularity conditions, a solution exists, is unique and may be calculated as an inverse Radon transform. In particular, we derived solutions for two-asset best-of and worst-of options. In addition, we derived a novel mathematical formula to calculate the inverse Radon transform, which we applied to generalize the Breeden and Litzenberger (1978) formula to the joint implied distribution across multiple assets.

Finally, Chapter 5 collates earlier publications that squarely fit within the Q category and explore the linkages between several sophisticated derivative transactions, namely: the variance swap, whose floating leg is the square of an asset's realized volatility; the correlation swap whose floating leg is the average realized correlation between the returns of multiple assets; and variance dispersion trades that play on the volatility gap between an index and its constituent assets. In particular, we proved that the correlation swap may be approximately replicated by dynamically trading zero-cost variance dispersion trades.

## 1.3 Connection with other theories and scholarly work in management science

### 1.3.1 Arbitrage pricing and replication theory

Chapters 2 to 4 are very much related to the fundamental work done by Arrow and Debreu (1954), Ross (1976) and of course Breeden and Litzenberger (1978), among others. In a finite-state economy with finitely many securities, Arrow and Debreu proved the existence in complete markets of portfolios replicating state-price securities that each pay off \$1 in a single state of the world and 0 otherwise. In this framework, any state-contingent asset with payoff vector  $\mathbf{f}$  may be replicated by a unique portfolio of the primitive securities in quantities  $\phi = \mathbf{A}^{-1}\mathbf{f}$ , where  $\mathbf{A}$  is the matrix of payoff vectors of the primitive securities. In the same finite framework Ross showed how an incomplete

market of primitive assets such as stocks and bonds can be completed by including call and put options written on these assets.

For the financial mathematician, a natural question here is whether these results may be extended to infinitely many states of the world. This question is not a purely gratuitous academic exercise: the issue of whether the world is finite or infinite remains open and the subject of debate among economists, philosophers of physics and mathematics (e.g., Finetti, 1970). In addition, it is worth noting that even if the world were finite, most scientists agree that the concept of infinity is fertile for modelling, approximating, simplifying, and generally enhancing our understanding of the finite world.

With regards to the subject at hand, theoretical studies on infinite-state complete market economies were led by Bewley (1972) in the countable case, and Green and Jarrow (1987), Mas-Colell (1986) and Nachman (1988) in the continuous case. It is in this context that Breeden and Litzenberger (1978) showed how the prices of Arrow-Debreu state-price securities<sup>5</sup> may be calculated from call option prices in a one-asset economy with continuous state space — a key result known in quantitative finance as the implied distribution. Twenty years later, Carr and Madan (1998) formally proved that virtually any single-asset contingent claim  $F$  may be replicated by a continuum of calls and puts in quantities  $\phi = F''$ . In parallel, many researchers have been investigating the case of incomplete markets with countably infinite or continuous state space — see Duffie (1996) for an excellent summary of their results and challenges. Jarrow (1988) and others have also investigated the relationship between APT and CAPM.

Chapter 2 of this thesis shows how the Breeden and Litzenberger and Carr and Madan results may be generalized to a continuum of arbitrary “replicant” options augmenting a one-asset economy with continuous state space, and analyzed and solved within the mathematical theory of integral equations. Remarkably, a necessary condition for market completeness is that the integral operator  $\mathcal{G}$  induced by the replicant options must not be compact. However, partial completeness may be achieved with a compact kernel and, for a potentially large class of contingent claims  $F$ , a unique replicating portfolio  $\phi = \mathcal{G}^{-1}F$  may then be calculated by spectral decomposition of  $\mathcal{G}$ .

Chapters 3 and 4 further expand this theoretical approach to multi-asset economies with continuous state space that are augmented by a continuum of vanilla basket calls<sup>6</sup>. Work in this direction was initiated by Baxter (1998), Lipton (2001) and Carr and Laurence (2011). We showed that completeness is achieved for a large class of multi-asset contingent claims whose payoff is either a radial or a multivariate homogeneous function. More importantly, for radial payoff we showed that the unique replicating portfolio up to first-order terms and a multiplicative constant is  $\phi(r) = \left(\frac{d}{dr^2}\right)^{\frac{n-1}{2}} r^n F''(r)$  (ch. 3, eq. 7). For homogeneous payoff, it is  $\phi = \mathcal{R}^{-1} \frac{\partial^2 F}{\partial k^2}$  (ch. 4, eq. 1b) where  $\mathcal{R}^{-1}$  is an inverse Radon transform, and in odd dimension<sup>7</sup> we showed that it may alternatively be calculated as  $\phi = \mathcal{R}_1^{(n-1)} \frac{\partial^2 F}{\partial k^2}$  where  $\mathcal{R}_1^{(n-1)}$  is a differentiation of the direct Radon transform (ch 4, eq. 7). Our novel formula for

<sup>5</sup>In continuous state space, Arrow-Debreu securities take the more elusive form of Dirac's delta functions.

<sup>6</sup>Ross (1976) uses the term “portfolio options” instead of “basket calls” which was coined by practitioners.

<sup>7</sup>In even dimension, the same formula applies followed by a Hilbert transform.

$\mathcal{R}^{-1}$  was established by calculating the fundamental solution for  $\mathcal{R}$ , which is the replicating portfolio for a multi-asset Arrow-Debreu security (multidimensional Dirac delta function). As an application, we obtained a generalization of the Breeden and Litzenberger formula for the joint implied distribution across multiple assets (ch. 4, eq. 13).

In addition to all these theoretical contributions, we also derived closed-form formulas for the replication of standard dispersion calls and puts (radial options), as well as two-asset best-of or worst-of calls and puts. Here, an important technical caveat is that these particular solutions  $\phi$  involve generalized functions, such as derivatives of Dirac's delta function or *pseudofunctions* subject to integral regularization.

### 1.3.2 Multi-asset option pricing

Early work in this area was led by Margrabe (1978) and Longstaff (1990) for exchange options, Stulz (1982), Johnson (1987) and Rubinstein (1991) for two-asset best-of and worst-of options, Kirk (1995) for spread options<sup>8</sup>. In terms of numerical methods, finite difference grids for partial differential equations (PDE), Monte-Carlo simulations and binomial trees were respectively introduced to price single-asset options by Brennan and Schwartz (1977), Boyle (1977) and Cox, Ross, and Rubinstein (1979). Boyle (1988), Boyle, Evnine, and Gibbs (1989), Kamrad and Ritchken (1991), Rubinstein (1994b) subsequently extended these methods to multiple underlying assets. Other popular numerical methods are quadrature techniques, fast fourier transforms (Carr and Madan, 1999), and recently machine learning (e.g. Lokeshwar, Bhardawaj, and Jain, 2019; E, Han, and Jentzen, 2017; Buehler et al., 2019).

Specifically, Boyle (1988) priced two-asset American worst-of puts using a trinomial tree together with approximations of the bivariate joint lognormal distribution of terminal asset prices. Kamrad and Ritchken (1991) developed a more general methodology for any number of state variables based on a multinomial tree. Rubinstein (1994b) proposed to price two-asset options using binomial pyramids, which he applied to best-of and worst-of options.

Many multi-asset option pricing articles have since been published. Today, the dominant numerical method to price multi-asset options is Monte-Carlo simulations, particularly in dimensions higher than 3 where PDE grid and binomial tree methods are stalled by the curse of dimensionality. Recently, machine learning techniques have gained in popularity for pricing, hedging and trading strategies involving multi-asset options.

### 1.3.3 Implied distribution and market anticipations

The seminal paper of Breeden and Litzenberger (1978) is cited by 2700 publications and has inspired thousands more. Notable applications include monetary policy (Malz, 1995; Bates, 1996; Malz, 1997; Söderlind and Svensson,

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<sup>8</sup>On commodities markets, spread options are also known as basis options

1997); estimation of risk aversion in utility theory (Bliss and Panigirtzoglou, 2004); implied binomial trees and numerical estimates of option-implied probabilities (Rubinstein, 1994a; Jackwerth and Rubinstein, 1996; Dupire, 1994; Derman and Kani, 1994). We refer the reader to the excellent article by Figlewski (2018) for a thorough review of scholarly work accomplished on this topic.

### 1.3.4 Portfolio theory and asset correlation

Chapter 5 is connected to modern portfolio theory spearheaded by Markowitz (1952) and enhanced by many other influential contributors, including Treynor (1962), Sharpe (1964), Mossin (1966), Lintner (1969), Jorion (1986), Michaud (1989), Black and Litterman (1992), to name only a few. Prior work on average financial correlation may be traced back to Aneja, Chandra, and Gunay (1989) who published an early version of the “shortcut formula” (1.5) as  $\bar{c} = (s_y^2 - N)/[N(N-1)]$  where  $s_y^2$  is the variance of a portfolio of  $N$  assets in quantities inversely proportional to their respective volatilities. In financial econometrics, Engle (2002) proposed a dynamic conditional correlation model for multivariate GARCH processes, while Solnik and Roulet (2000) developed a formula for average correlation that has an inverse relationship with cross-sectional dispersion.

### 1.3.5 Implied volatility skew and equity correlation

Chapter 5 is also connected to a large body of research on implied volatility skew in relation to equity correlation. On the modelling side, Heston (1993) is credited with the first successful option pricing model featuring a correlation parameter between the underlying asset and its instant volatility. The Heston model produces implied volatility skew when this parameter is negative, as empirically observed for equities. In empirical studies, Driessen, Maenhout, and Vilkov (2009) found evidence of high implied correlation versus realized correlation and investigated a correlation trading strategy aiming to monetize the correlation risk premium. The reader is invited to consult their list of references for further perspective on the scholarly relevance of this topic.

The following paragraphs describe my research interests, their motivation and challenges, together with my contributions and achievements.

## 1.4 Static option replication theory (chapters 2 to 4)

Breeden and Litzenberger (1978) and Carr and Madan (1998) have shown that, subject to mild differentiability conditions, any single-asset option with payoff function  $F(x)$  where  $x \geq 0$  is the terminal underlying asset price can

be replicated with a weighted sum of call options paying off  $\max(0, x - K)$  indexed by a continuum of exercise prices  $K \geq 0$ . Specifically, they indirectly solved the **integral equation**:

$$F(x) \stackrel{\text{a.e.}}{\equiv} \int_0^{\infty} \phi(K) \max(0, x - K) \, dK, \quad x \in \mathbb{R}_+. \quad (1.1)$$

Mathematically this means that ramp functions  $(x \mapsto \max(0, x - K))_{K \in \mathbb{R}_+}$  form a spanning set of the space of differentiable functions (possibly in a generalized sense), except perhaps for certain values of  $x$  whose set has zero measure. The solution turns out to be simply  $\phi(y) = F''(y)$  as recovered by differentiating the integral equation twice, or integrating by parts, or by means of Taylor's theorem with remainder in integral form.

### 1.4.1 Dimension 1

In my 2021 article co-authored with Peter and Andrew and published in *Quantitative Finance*, presented in Chapter 2 of this thesis, we show how the one-dimensional replication inverse problem (1.1) may be generalized to other integral kernels  $G(x, y)$ :

$$F(x) \stackrel{\text{a.e.}}{\equiv} \int_{-\infty}^{\infty} G(x, y) \phi(y) \, dy, \quad x \in \mathbb{R},$$

with particular focus on the “straddle kernel”  $G(x, y) := |x - y|$  which better lends itself to **spectral decomposition** as a symmetric and injective kernel. In  $L^2$  Hilbert space a solution  $\phi$  exists and the eigenbasis turns out to be a modified Fourier basis with interesting properties for fast numerical pricing, particularly in the context of large option books. Another benefit is that closed-form formulas for the basis options may be derived from the characteristic function of the underlying asset when the latter is known.

I also investigated the case when the integral kernel corresponds to butterflies  $G(x, y) := \max(0, c - |x - y|)$  where  $c > 0$  is a constant. This case turned out to be very technical with many block matrix calculations, but I ultimately succeeded to derive the eigenbasis.

### 1.4.2 Dimension $n$

In my latest two working papers presented in Chapters 3 and 4 of this thesis, I consider the following generalization to  $\mathbb{R}^n$ :

$$F(\mathbf{x}) \stackrel{\text{a.e.}}{\equiv} \int_{\mathbb{R}^n} \varphi(\mathbf{y}) \max(0, \mathbf{x} \cdot \mathbf{y} - k) \, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, \quad (1.2)$$

where  $k$  is an external parameter,  $F(\mathbf{x})$  is the payoff function of a multi-asset option,  $\max(0, \mathbf{x} \cdot \mathbf{y} - k)$  is a basket call option, i.e. a call option on the terminal value of a hypothetical portfolio of the assets with weights  $\mathbf{y} \in \mathbb{R}^n$ . This is a difficult inverse problem due to its multidimensional nature. A major consequence when a solution exists is that

the price of the multi-asset option is independent from any probabilistic model beyond the pricing of basket calls of arbitrary weights  $\mathbf{y}$ , for which there is industry consensus.

Various differentiation techniques allow to convert integral equation (1.2) to a **Radon transform inverse problem** of the form

$$f(\mathbf{x}) \stackrel{\text{a.e.}}{=} \int_{\mathbb{R}^n} \phi(\mathbf{y}) \delta(\mathbf{x} \cdot \mathbf{y} - k) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, \quad (1.3)$$

where  $\delta$  is Dirac's delta function and  $f, \phi$  are functions related to  $F, \varphi$ . General inverse formulas exist for the Radon transform (e.g. Rubin, 2015, pp 175–204), but they involve multidimensional Fourier transforms or spherical integrals that can be hard to solve. During my investigations, I found out that for a solution to exist at all we must often venture outside of the realm of absolutely integrable  $L^1$  functions posited by standard theory. For example, we may extend the solution space to generalized functions and pseudofunctions subject to integral regularizations such as Cauchy principal values or Hadamard finite part.

### Solution for radial functions

When the multi-asset option payoff is a radial function  $F(\mathbf{x}) := F_0(|\mathbf{x}|)$ , I showed how the replication problem (1.2) may be converted to Radon form (1.3) by radial differentiation with  $f(\mathbf{x}) = f_0(|\mathbf{x}|)$ ,  $f_0(s) := F_0''(s)s^2/k^2$ ,  $s := |\mathbf{x}|$ . The left-hand side being a radial function, the solution  $\phi$  can be shown to be radial as well and the Radon transform on the right-hand side collapses to a one-dimensional **fractional integral**. The solution  $\phi_0(|\mathbf{y}|)$  may then be recovered by fractional differentiation techniques as

$$\phi_0(r) = \begin{cases} \frac{1}{\pi^{(n-1)/2}} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} r^n F_0''(r), & n \text{ odd,} \\ \frac{2}{\pi^{n/2}} \frac{d}{dr^2} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} \left( \frac{d}{ds^2} \right)^{\frac{n-2}{2}} [s^n F_0''(s)] ds, & n \text{ even.} \end{cases} \quad (1.4)$$

The above solution is valid for a large class of well-behaved payoff functions  $F_0$  that cancel around the origin, including dispersion calls with payoff  $F_0(|\mathbf{x}|) = \max(0, |\mathbf{x}| - K)$ , but excluding dispersion puts with payoff  $F_0(|\mathbf{x}|) = \max(0, K - |\mathbf{x}|)$ . I overcame this limitation by isolating first-order terms which turn out to be replicable by a continuous portfolio of zero-strike basket calls with *normalized* basket weights, and obtained explicit solutions for the dispersion put.

## Solution for multi-asset options with absolutely homogeneous payoff

When the multi-asset payoff function  $F(x; k)$  is absolutely homogeneous and also depends on external parameter  $k$ , I showed that the solution  $\phi$  may be calculated by Fourier transform methods as

$$\phi(\mathbf{y}) = \left[ |t|^{n-1} \frac{\widehat{\partial^2 F}}{\partial k^2}(t\mathbf{y}; k) \right]^\wedge \Big|_{k=\lambda=1} (\lambda)$$

where the  $\wedge, \vee$  symbols respectively denote the one-dimensional direct and  $n$ -dimensional inverse Fourier transforms. I applied this formula to successfully derive replication identities for two-asset best-of call and put options, as well as worst-of. I also derived an analytical formula for the Radon transform inverse which I used to obtain an explicit formula for the implied joint distribution.

## 1.5 Fundamentals of correlation matrices and dynamic replication of correlation swaps with variance dispersion trades (Chapter 5)

### 1.5.1 Mean correlation

In various pieces of research (2004, 2005, 2014) collated in Chapter 5 of this thesis, I studied correlation matrices  $R = (\rho_{i,j})_{1 \leq i,j \leq n}$  and two associated entry-wise means frequently used in econometrics and finance, respectively the off-diagonal and full-matrix mean coefficient defined as

$$\begin{cases} \rho(x) := \frac{\|x\|_R^2 - |x|^2}{(x \cdot e)^2 - |x|^2}, & \cos(\widehat{x, e}) \neq 1/n; \\ \hat{\rho}(x) := \frac{\|x\|_R^2}{(x \cdot e)^2}, & x \perp e, \end{cases} \quad x \in \mathbb{R}^n,$$

where  $\|x\|_R := \sqrt{x \cdot Rx}$  is the quadratic norm induced by the correlation matrix  $R$ , and  $e := (1, \dots, 1)$  is the first diagonal vector of  $\mathbb{R}^n$ . A special case of interest in finance is when the weights  $x$  are market capitalizations multiplied by asset volatilities, giving the “shortcut formula”

$$\rho(x) = \frac{\sigma_P^2 - \sum_i w_i^2 \sigma_i^2}{(\sum_i w_i \sigma_i)^2 - \sum_i w_i^2 \sigma_i^2}, \quad x_i = w_i \sigma_i, \quad (1.5)$$

where  $\sigma_P$  is portfolio volatility and  $\sum_i w_i = 1$ . Remarkably, the computational complexity of the formula is only  $O(n)$  compared to the original definition of  $\rho(x)$  which is  $O(n^2)$ .

Mathematically, correlation matrices are characterized by being symmetric, positive-semidefinite with unit diagonal coefficients. In the context of finance and economics, they are often structured around one very large eigenvalue

with transversal eigenvector (“the market”), while all other eigenvalues are small (“random noise”) — e.g. Potters, Bouchaud, and Laloux (2005). During my initial empirical investigations I observed that, for large enough  $n \geq 20$  and non-pathological weights  $x$ , the difference between the two correlation means  $\rho$  and  $\hat{\rho}$  was small. I then showed that  $\rho(x) \sim \hat{\rho}(x)$  as  $n \rightarrow \infty$  by various methods. From a practitioner’s perspective, this property translates into the **correlation proxy formula** that I found in January 2004 as a young analyst at J.P. Morgan in London:

$$\rho \approx \frac{\sigma_P^2}{\bar{\sigma}^2}$$

where  $\sigma_P$  is portfolio volatility and  $\bar{\sigma} := \sum_i w_i \sigma_i$  is average volatility of constituents. Impact of this result on the financial industry can be observed by its subsequent dissemination in specialist literature<sup>9</sup>.

Mathematically, I found that mean correlation is straightforwardly related to **Rayleigh’s quotient**  $\mathfrak{R}(x) := \|x\|_R^2 / \|x\|^2$ . Specifically,

$$\hat{\rho}(x) = \frac{\frac{1}{n} \mathfrak{R}(x)}{\cos^2 \theta}, \quad \rho(x) = \frac{\frac{1}{n} \mathfrak{R}(x) - \frac{1}{n}}{\cos^2 \theta - \frac{1}{n}},$$

where  $\theta := \arccos \frac{x \cdot e}{|x||e|}$  is the angle in  $[0, \pi)$  between vectors  $x$  and  $e$ . Based on the above, I obtained that the speed of convergence of  $\rho$  to  $\hat{\rho}$  is at least  $O(1/n)$ .

In ongoing research I use spectral decomposition to investigate the difference  $\rho(x) - \hat{\rho}(x)$  which can be re-expressed as  $\hat{\rho}(x) - \hat{\rho}(y)$  for a suitable choice of  $y \in \mathbb{R}^n$ . Specifically, I proved that

$$\left| \hat{\rho}(x) - \hat{\rho}(y) - \frac{\lambda_n}{n} \left( \frac{\cos^2(\widehat{x, v_n})}{\cos^2(\widehat{x, e})} - \frac{\cos^2(\widehat{y, v_n})}{\cos^2(\widehat{y, e})} \right) \right| \leq \lambda_{n-1} |a| |b| \sin(\widehat{a, v_n}) \sin(\widehat{b, v_n})$$

where  $\lambda_1 \leq \dots \leq \lambda_n$  are the eigenvalues of  $R$  with associated orthonormal eigenvectors  $v_1, \dots, v_n$ , and  $a, b := x/x \cdot e \pm y/y \cdot e$ . I believe this inequality can be further improved.

## 1.5.2 Arbitrage pricing and dynamic replication of correlation swaps

In various industry papers and in my book (2005, 2007, 2014), I showed how *correlation swaps* may be dynamically replicated by continuously trading zero-cost *variance dispersion trades*. Specifically, a correlation swap is a forward contract on mean correlation with payoff (equal to profit or loss, p/l):

$$p/l_{\text{correl}} := \rho - K = \frac{\sum_{i < j} w_i w_j \rho_{i,j}}{\sum_{i < j} w_i w_j} - K$$

<sup>9</sup>Citation standards are relatively informal in the industry, and as a result my 2004 proxy formula is often stated without attribution, or attributed to other authors of posterior publications such as the 2005 marathon report on “Correlation vehicles” written by my J.P. Morgan colleagues Nick Granger and Peter Allen. As a matter of fact, there is no intellectual property on specific mathematical formulas or their derivations, which is probably for the better of research productivity.

where  $K \in \mathbb{R}$  is a fixed strike price,  $w \in (0, 1)^n$  is a vector of weights summing to 1, and  $\rho_{i,j}$  is Pearson's correlation coefficient of daily returns between any two underlying assets  $1 \leq i, j \leq n$ . The p/l on variance dispersion trades, on the other hand, is

$$p/l_{\text{vardisp}} = \sigma_P^2 - \rho' \times \sum_i w_i \sigma_i^2 - K_{\rho'} \quad (1.6)$$

where  $\sigma_P$  is portfolio volatility,  $\sum_i w_i \sigma_i^2$  is average constituent variance,  $\rho' > 0$  is a constant corresponding to a "leg ratio" and  $K_{\rho'} \in \mathbb{R}$  is a fixed residual cost. A zero-cost trade is thus obtained by choosing

$$\rho'^* = \frac{\sigma_P^{*2}}{\sum_i w_i \sigma_i^{*2}}$$

where  $\sigma^* := \sqrt{\mathbb{E}(\sigma^2)}$  denotes the fair price of variance. The above formula defines a *variance-based* alternative metric for mean correlation which satisfies  $0 \leq \rho' \leq \rho$  by Jensen's inequality. Substituting  $\sigma_P^2 = \rho' \times \sum_i w_i \sigma_i^2$  into equation (1.6) and factoring yields

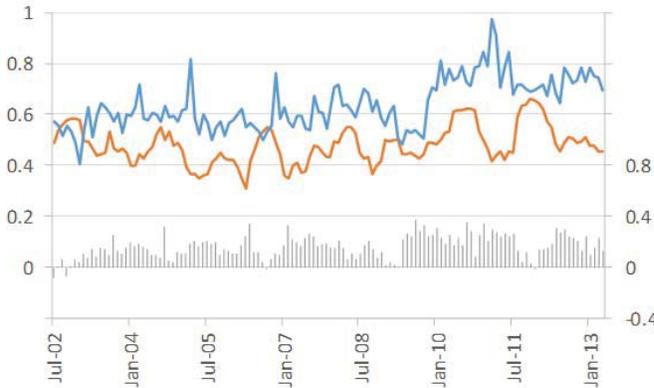
$$p/l_{\text{vardisp}} = (\rho' - \rho'^*) \times \sum_i w_i \sigma_i^2$$

which is positive (profit) if historical correlation  $\rho'$  is greater than the price of correlation  $\rho'^*$ , and negative otherwise (loss). As such, the p/l of a zero-cost variance dispersion trade is simply that of a correlation swap<sup>10</sup> multiplied by mean constituent variance  $\sum_i w_i \sigma_i^2$ . Figure 1.1 shows how the correlation alpha has been historically attractive and persistent for the Dow Jones EuroStoxx 50 and S&P 500 indices.

Based on this observation, I showed that a correlation swap can be dynamically replicated by continuous trading of zero-cost variance dispersions. In essence, my dynamic trading strategy cancels the exposure to mean constituent variance in order to isolate pure correlation exposure.

Figure 1.1: Historical correlation alpha. Data sources: OptionMetrics, Bloomberg, Ogee Group.

1.1a. 6-month variance-based implied correlation (blue) versus realized correlation 6 months later (orange), EuroStoxx 50, 2002–2013



1.1b. 12-month at-the-money implied correlation (blue) versus realized correlation 12 months later (orange), S&P 500, 2001–2017



<sup>10</sup>In practice, correlation swaps trade at a substantial discount against implied correlation, in large part because they are very difficult to replicate. Correlation arbitrageurs looking to sell correlation will typically prefer to trade variance dispersion.

## 1.6 Other research themes (not included in this thesis)

### 1.6.1 Delta of variance swaps

In a short working paper and conference presentation I investigated the “delta” of a variance swap, that is, the price sensitivity of variance against the underlying asset. By Carr and Madan (1998) theory, the variance swap has a “model-free” price based on vanilla calls and puts, and its delta may be calculated accordingly; but note however that the call and put delta calculation is model-dependent. Remarkably, the variance delta can be shown to vanish for a wide class of option pricing models, namely *scale-invariant models*. This counter-intuitive result that goes against the well-known empirical principle that “when stock prices go down, volatility goes up” suggests that scale-invariant models should not be used to calculate the delta of a variance swap.

### 1.6.2 Multidimensional scale invariance

In ongoing research I extended the concept of scale-invariance to a multidimensional setting, and made a minor correction to a theorem by Alexander and Nogueira (2007) for which I offered a new proof.

**Definition.**  $(\mathbf{X}_t)_{t \geq 0}$  is a *scale-invariant process* when its joint cumulative distribution function is homogeneous of degree zero, that is, for all  $\mathbf{a}, \mathbf{x} \in \mathbb{R}_+^n$  and  $\lambda, t > 0$ ,

$$\mathbb{P}(\mathbf{X}_t \leq \lambda \mathbf{x} \mid \mathbf{X}_0 = \lambda \mathbf{a}) = \mathbb{P}(\mathbf{X}_t \leq \mathbf{x} \mid \mathbf{X}_0 = \mathbf{a}).$$

**Theorem.** Given a European option with maturity  $T$  and homogeneous payoff  $G(\mathbf{X}_T, \mathbf{K})$  of degree<sup>11</sup>  $k \in \mathbb{R}$  with value  $g_t(\mathbf{a}, \mathbf{x}, \mathbf{K}) := \mathbb{E}[G(\mathbf{X}_T, \mathbf{K}) \mid \mathbf{X}_t = \mathbf{x}, \mathbf{X}_0 = \mathbf{a}]$ ,  $0 \leq t \leq T$ ,

1. If  $(\mathbf{X}_t)$  is scale-invariant, then the option value  $g_t(\mathbf{a}, \mathbf{x}, \mathbf{K})$  is also homogeneous of degree  $k$  for all  $0 \leq t \leq T$ .
2. Conversely, if the option value  $g_t(\mathbf{a}, \mathbf{x}, \mathbf{K})$  is homogeneous of degree  $k$  for all  $0 \leq t \leq T$  and the payoff function  $G(\mathbf{x}, \mathbf{K})$  induces an injective adjoint integral operator, i.e. the homogeneous integral equation of the first kind

$$0 = \int_{\mathbb{R}_+^n} G(\mathbf{x}, \mathbf{K}) \phi(\mathbf{x}) \, d\mathbf{x}, \quad \mathbf{K} \in \mathbb{R}_+^n,$$

only has the trivial solution  $\phi \equiv 0$ , then  $(\mathbf{X}_t)_{0 \leq t \leq T}$  is a scale-invariant process.

---

<sup>11</sup>That is, for all  $\lambda > 0$ ,  $G(\lambda \mathbf{X}_T, \lambda \mathbf{K}) = \lambda^k G(\mathbf{X}_T, \mathbf{K})$ .

### 1.6.3 Spreadsheet compiler

I created an algorithm to compile a spreadsheet into a procedural language such as C or VBA, which can be used to considerably speed up the Monte-Carlo method compared to iterated spreadsheet recalculation. A demo is available from [my company website](#). A conspicuous application in finance is the pricing of multi-asset options, but there are other possible applications<sup>12</sup>. The example below illustrates my procedure.

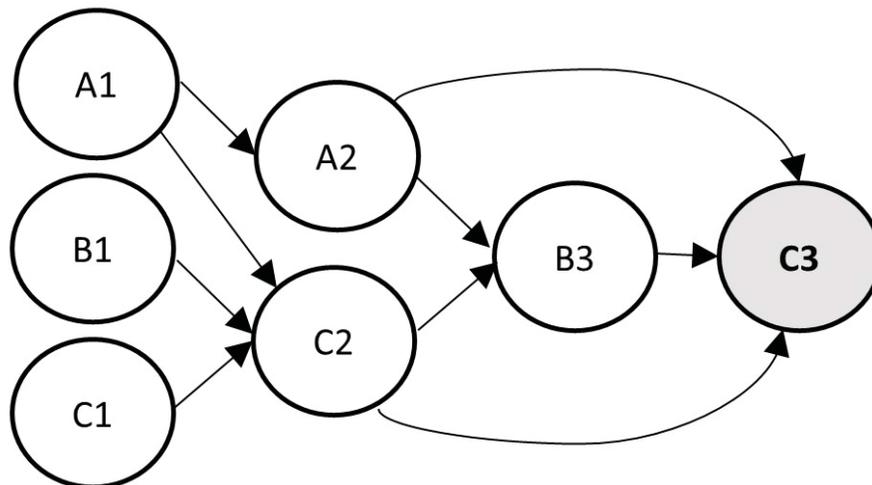
In the spreadsheet shown in Figure 1.2a, cell C4 depends on three parents cells B3, A2 and C2; cell B3 itself depends on A2 and C2; and so forth. This type of interdependence is best modelled by a **simple graph**, as shown in Figure 1.2b. With a recursive algorithm I assign a computation rank so as to translate the graph into a step-by-step procedural script. It is worth emphasizing that “real-life” spreadsheets can have very complex cell dependencies whose computation ranks cannot simply be read from left to right as in the example.

Figure 1.2: A sample spreadsheet and the graph of its cell dependencies

1.2a. Spreadsheet

	A	B	C
1	1	2	3
2	=A1		=A1*B1+C1
3		=(A2+C2)^2	
4			=B3/(A2+C2)

1.2b. Dependency graph



<sup>12</sup>Applications of graph theory in economic and social sciences include optimal transport costs and economic networks (e.g., Naimzada, Stefani, and Torriero, 2009).

# Bibliography

- Alexander, Carol and Leonardo M. Nogueira (2007). “Model-free hedge ratios and scale-invariant models”. In: *Journal of Banking & Finance* 31.6, pp. 1839–1861.
- Aneja, Yash P., Ramesh Chandra, and Erdal Gunay (1989). “A Portfolio Approach to Estimating the Average Correlation Coefficient for the Constant Correlation Model”. In: *The Journal of Finance* 44.5, pp. 1435–1438.
- Arrow, Kenneth J. and Gerard Debreu (1954). “Existence of an Equilibrium for a Competitive Economy”. In: *Econometrica* 22.3, pp. 265–290.
- Bachelier, Louis (1900). “Théorie de la Spéculation”. In: *Annales Scientifiques de L'Ecole Normale Supérieure* 17. Reprinted in P. H. Cootner (ed), 1964, *The Random Character of Stock Market Prices*, Cambridge, Mass. MIT Press, pp. 21–88.
- Bates, David S (1996). “Dollar jump fears, 1984–1992: distributional abnormalities implicit in currency futures options”. In: *Journal of international Money and Finance* 15.1, pp. 65–93.
- Baxter, Martin (1998). “Hedging in Financial Markets”. In: *ASTIN Bulletin* 28.1, pp. 5–16.
- Bewley, Truman F. (1972). “Existence of equilibria in economies with infinitely many commodities”. In: *Journal of Economic Theory* 4.3, pp. 514–540.
- Black, Fischer and Robert Litterman (1992). “Global Portfolio Optimization”. In: *Financial Analysts Journal* 48.5, pp. 28–43.
- Black, Fischer and Myron Scholes (1973). “The pricing of options and corporate liabilities”. In: *Journal of political economy* 81.3, p. 637.
- Bliss, Robert R and Nikolaos Panigirtzoglou (2004). “Option-implied risk aversion estimates”. In: *The Journal of Finance* 59.1, pp. 407–446.
- Bossu, Sébastien (2005). *Arbitrage pricing of equity correlation swaps*. Tech. rep. J.P. Morgan.
- (2007). *A New Approach For Modelling and Pricing Correlation Swaps*. Tech. rep. Dresdner Kleinwort.
- (2014). *Advanced Equity Derivatives. Volatility & Correlation*. John Wiley & Sons.
- Bossu, Sébastien, Peter Carr, and Andrew Papanicolaou (2021). “A functional analysis approach to the static replication of European options”. In: *Quantitative Finance* 21 (4), pp. 637–655.

- Bossu, Sébastien and Yi Gu (2004). *Fundamental relationship between an index's volatility and the correlation and average volatility of its components*. Tech. rep. J.P. Morgan.
- Boyle, Phelim P. (1977). "Options: A Monte Carlo approach". In: *Journal of Financial Economics* 4.3, pp. 323–338.
- (1988). "A Lattice Framework for Option Pricing with Two State Variables". In: *Journal of Financial and Quantitative Analysis* 23.1, pp. 1–12.
- Boyle, Phelim P., Jeremy Evnine, and Stephen Gibbs (1989). "Numerical Evaluation of Multivariate Contingent Claims". In: *Review of Financial Studies* 2.2, pp. 241–250.
- Breeden, Douglas T. and Robert H. Litzenberger (1978). "Prices of state-contingent claims implicit in option prices". In: *Journal of business*, pp. 621–651.
- Brennan, Michael J. and Eduardo S. Schwartz (1977). "The Valuation of American Put Options". In: *The Journal of Finance* 32.2, pp. 449–462.
- Buehler, H., L. Gonon, J. Teichmann, and B. Wood (2019). "Deep hedging". In: *Quantitative Finance* 19.8, pp. 1271–1291.
- Carr, Peter and Peter Laurence (2011). "Multi-asset Stochastic Local Variance Contracts". In: *Mathematical Finance* 21.1, pp. 21–52.
- Carr, Peter and Dilip Madan (1998). "Towards a theory of volatility trading". In: *Volatility: New estimation techniques for pricing derivatives*. Ed. by Robert A. Jarrow. Vol. 29. Risk books, pp. 417–427.
- (1999). "Option Valuation Using the Fast Fourier Transform". In: *Journal of Computational Finance* 2, pp. 61–73.
- Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979). "Option Pricing: A Simplified Approach". In: *Journal of Financial Economics*, pp. 229–263.
- Derman, Emanuel and Iraj Kani (Feb. 1994). "Riding on a Smile". In: *Risk Magazine* 7, pp. 139–145.
- Driessen, Joost, Pascal J Maenhout, and Grigory Vilkov (2009). "The Price of Correlation Risk: Evidence from Equity Options". In: *The Journal of Finance* 64.3, pp. 1377–1406.
- Duffie, Darrell (1996). "Incomplete security markets with infinitely many states: An introduction". In: *Journal of Mathematical Economics* 26.1. Special Issue on Equilibrium with Incomplete Markets and an Infinite State Space, pp. 1–8.
- Dupire, Bruno (Jan. 1994). "Pricing with a Smile". In: *Risk Magazine* 7, pp. 18–20.
- E, W., J. Han, and A. Jentzen (2017). "Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations". In: *Communications in Mathematics and Statistics* 5.4, pp. 370–398.
- Engle, Robert (2002). "Dynamic Conditional Correlation". In: *Journal of Business & Economic Statistics* 20.3, pp. 339–350.
- Figlewski, Stephen (2018). "Risk-Neutral Densities: A Review". In: *Annual Review of Financial Economics* 10.1, pp. 329–359.

- Finetti, Bruno de (1970). *Teoria delle probabilità*. English translation: Finetti, B. de, 1974, *Theory of probability*, John Wiley & Sons, New York. Einaudi.
- Green, Richard C. and Robert A. Jarrow (1987). "Spanning and completeness in markets with contingent claims". In: *Journal of Economic Theory* 41.1, pp. 202–210.
- Heston, Steven (1993). "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options". In: *Review of Financial Studies* 6, pp. 327–343.
- Jackwerth, Jens Carsten and Mark Rubinstein (1996). "Recovering Probability Distributions from Option Prices". In: *The Journal of Finance* 51.5, pp. 1611–1631.
- Jarrow, Robert A. (1988). "Preferences, Continuity, and the Arbitrage Pricing Theory". In: *The Review of Financial Studies* 1.2, pp. 159–172.
- Johnson, Herb (Sept. 1987). "Options on the Maximum or the Minimum of Several Assets". In: *Journal of Financial and Quantitative Analysis* 22.3, pp. 277–283.
- Jorion, Philippe (1986). "Bayes-Stein Estimation for Portfolio Analysis". In: *The Journal of Financial and Quantitative Analysis* 21.3, pp. 279–292.
- Kamrad, Bardia and Peter Ritchken (1991). "Multinomial Approximating Models for Options with k State Variables". In: *Management Science* 37.12, pp. 1640–1652.
- Kirk, E (1995). "Correlation in the Energy Markets". In: *Managing Energy Price Risk*. Risk Publications and Enron.
- Lintner, John (1969). "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets: A Reply". In: *The Review of Economics and Statistics* 51.2, pp. 222–224.
- Lipton, Alexander (2001). *Mathematical Methods For Foreign Exchange: A Financial Engineer's Approach*. World Scientific Publishing Company.
- Lokeshwar, Vikranth, Vikram Bhardawaj, and Shashi Jain (Nov. 2019). *Neural Network for Pricing and Universal Static Hedging of Contingent Claims*. Working paper. Indian Institute of Science, Bangalore, India. URL: <https://ssrn.com/abstract=3491209>.
- Longstaff, Francis A. (July 1990). "The valuation of options on yields". In: *Journal of Financial Economics* 26.1, pp. 97–121.
- Malz, Allan M. (July 1995). "Currency option markets and exchange rates: A case study of the U.S. dollar in March 1995". In: *Current Issues in Economics and Finance* 1.4.
- (1997). "Estimating the Probability Distribution of the Future Exchange Rate from Option Prices". In: *The Journal of Derivatives* 5.2, pp. 18–36.
- Margrabe, William (1978). "The Value of an Option to Exchange One Asset for Another". In: *Journal of Finance* 33.1, pp. 177–86.
- Markowitz, Harry (1952). "Portfolio selection". In: *The Journal of Finance* 7.1, pp. 77–91.

- Mas-Colell, Andreu (1986). "The Price Equilibrium Existence Problem in Topological Vector Lattices". In: *Econometrica* 54.5, pp. 1039–1053.
- Merton, Robert C. (1973). "Theory of Rational Option Pricing". In: *The Bell Journal of Economics and Management Science* 4.1, pp. 141–183.
- Michaud, Richard O. (1989). "The Markowitz Optimization Enigma: Is 'Optimized' Optimal?" In: *Financial Analysts Journal* 45.1, pp. 31–42.
- Mossin, Jan (1966). "Equilibrium in a Capital Asset Market". In: *Econometrica* 34.4, pp. 768–783.
- Nachman, David C. (1988). "Spanning and Completeness with Options". In: *Review of Financial Studies*, pp. 311–328.
- Naimzada, Ahmad K., Silvana Stefani, and Anna Torriero, eds. (2009). *Networks, Topology and Dynamics: Theory and Applications to Economics and Social Systems*. Lecture Notes in Economics and Mathematical Systems. Springer Berlin Heidelberg.
- Potters, Marc, Jean-Philippe Bouchaud, and Laurent Laloux (Sept. 2005). *Financial Applications of Random Matrix Theory: Old Laces and New Pieces*. Science & Finance (CFM) working paper archive. Science & Finance, Capital Fund Management.
- Ross, Stephen A. (1976). "Options and Efficiency". In: *The Quarterly Journal of Economics* 90.1, pp. 75–89.
- Rubin, Boris (2015). *Introduction to Radon Transforms*. Cambridge University Press.
- Rubinstein, Mark (1991). "Somewhere over the rainbow". In: *Risk* 4.11, pp. 61–63.
- (1994a). "Implied Binomial Trees". In: *The Journal of Finance* 49.3, pp. 771–818.
- (1994b). "Return to Oz". In: *Risk* 7, pp. 67–71.
- Samuelson, Paul A. (1965). "Rational Theory of Warrant Pricing". In: *Industrial Management Review* 6, pp. 13–31.
- Sharpe, William F. (1964). "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk". In: *The Journal of Finance* 19.3, pp. 425–442.
- Söderlind, Paul and Lars Svensson (1997). "New techniques to extract market expectations from financial instruments". In: *Journal of Monetary Economics* 40.2. Monetary Policy and Financial Markets, pp. 383–429.
- Solnik, Bruno and Jacques Roulet (2000). "Dispersion as Cross-Sectional Correlation". In: *Financial Analysts Journal* 56.1, pp. 54–61.
- Stulz, Rene M. (July 1982). "Options on the minimum or the maximum of two risky assets : Analysis and applications". In: *Journal of Financial Economics* 10.2, pp. 161–185.
- Treynor, Jack L. (1962). *Toward a Theory of Market Value of Risky Assets*. Unpublished manuscript.





## Chapter 2

# “A functional analysis approach to the replication of European options”, Quantitative Finance, 2021



Quantitative Finance



ISSN: (Print) (Online) Journal homepage: <https://www.tandfonline.com/loi/rqf20>

### A functional analysis approach to the static replication of European options

Sébastien Bossu , Peter Carr & Andrew Papanicolaou

To cite this article: Sébastien Bossu , Peter Carr & Andrew Papanicolaou (2020): A functional analysis approach to the static replication of European options, Quantitative Finance, DOI: [10.1080/14697688.2020.1810857](https://doi.org/10.1080/14697688.2020.1810857)

To link to this article: <https://doi.org/10.1080/14697688.2020.1810857>



Published online: 04 Nov 2020.



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# A functional analysis approach to the static replication of European options

SÉBASTIEN BOSSU\*, PETER CARR and ANDREW PAPANICOLAOU

New York University, New York City, USA

(Received 11 July 2019; accepted 31 July 2020; published online 4 November 2020)

The replication of any European contingent claim by a static portfolio of calls and puts with strikes forming a continuum, formally proven by Carr and Madan [Towards a theory of volatility trading. In *Volatility: New Estimation Techniques for Pricing Derivatives*, edited by R.A. Jarrow, Vol. 29, pp. 417–427, 1998 (Risk books)], is part of the more general theory of integral equations. We use spectral decomposition techniques to show that exact payoff replication may be achieved with a discrete portfolio of special options. We discuss applications for fast pricing of vanilla options that may be suitable for large option books or high frequency option trading, and for model pricing when the characteristic function of the underlying asset price is known.

**Keywords:** Derivatives; Options; Static replication; Payoff; Integral equation; Functional analysis; Spectral theorem; Breeden-Litzenberger formula; Implied distribution

## 1. Introduction

We consider the general problem of replicating a target European option<sup>†</sup> with a static portfolio of cash, the underlying asset and a selection of ‘replicant’ European options. Replication problems arise in many areas of finance, such as in asset pricing theory where an asset is replicated with a finite number of other assets (e.g. Černý 2016, ch. 1, 2) using the techniques of finite-dimensional linear algebra, or option pricing theory, where Carr and Madan (1998) formally proved that any European option may be replicated with a portfolio of cash, forward contracts, and European call and/or put options with a continuum of strike prices. A key consequence of payoff replication is that if the prices of the replicating options are known, then the price of the target European option is also known and enforced by no-arbitrage considerations.

Specifically, given a target European option’s payoff  $F(x)$  to be replicated, where  $x \in \mathcal{X} \subseteq \mathbb{R}_+$  is the terminal price of the option’s underlying asset, and a family of replicating European options’ payoffs  $G(x, y)$  indexed by  $y \in \mathcal{Y} \subseteq \mathbb{R}$ , we are looking for portfolio quantities or weights such that, for all  $x \in \mathcal{X}$ ,

$$F(x) = c + qx + \int_{y \in \mathcal{Y}} G(x, y) \phi(y) d\mu(y), \quad (1)$$

\*Corresponding author. Email: sbossu@nyu.edu

<sup>†</sup>We use the term ‘option’ to designate any derivative contract, also known as ‘contingent claim’, on a single underlying asset

where  $c$ ,  $q$  and  $\phi(y)$  are the respective quantities of cash, underlying asset and replicating option with index  $y$ , and  $\mu$  is an appropriate measure. In particular, if  $\mathcal{Y}$  is discrete and  $\mu$  is the counting measure, the above equation becomes  $F(x) = c + qx + \sum_{y \in \mathcal{Y}} G(x, y) \phi(y)$  or, with the more habitual subscript notation for discrete sums,

$$F(x) = c + qx + \sum_{n \in \mathcal{Y}} \phi_n G_n(x).$$

We will especially focus on the case where both variables  $x$ ,  $y$  belong to a continuous interval such as  $[a, b]$  or  $(a, b)$  where  $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$  may be infinite, and  $\mu$  is the Lebesgue measure, so that we may write equation (1) as

$$f(x) = \int_a^b G(x, y) \phi(y) dy, \quad (2)$$

where  $f(x) := F(x) - c - qx$  is the target payoff function  $F(x)$  up to affine terms. Observe that the second and higher derivatives of  $f$  and  $F$  coincide.

The origin of the Carr-Madan replication formula may be traced back to the seminal paper of Breeden and Litzenberger (1978) who showed that the terminal distribution of the underlying asset implicit in option prices, also known as the *implied distribution*, could be recovered by differentiating call prices twice with respect to the strike price. This elegant theoretical result allowed pricing any other European option payoff consistently with existing vanilla options.

However, it was not until the 1990s that practitioners and researchers became particularly interested in replication and hedging strategies for non-vanilla option payoffs, on the back of the expansion of option markets and the search for option contract innovation. Evidence of such interest can be found in the work of Dupire (1993), as well as Derman *et al.* (1994) who discuss especially static replication of barrier options.

Much research (e.g. Demeterfi *et al.* 1999) has been devoted to the static replication of the log-contract first introduced by Neuberger (1990), leading to the development of volatility and variance swap markets. In this context, Carr and Madan (1998) offered a general replication result that did not solely apply to the log-contract and was also probability- and model-free. To this day, option practitioners refer to the idea that any European option payoff can be replicated with a continuous portfolio of vanilla calls and puts as the ‘Carr–Madan result’. Its most visible impact may be seen in the new calculation methodology of the VIX (see The CBOE volatility index–VIX 2009), which was adopted in 2002 by the Chicago Board Options Exchange.

In other related literature, Carr and Wu (2013) consider the static hedging of a longer-dated vanilla option using a continuum of shorter-term options. Balder and Mahayni (2006) expand on this work and explore various discretization strategies when the strikes are pre-specified and the underlying price dynamics are unknown, and recently Wu and Zhu (2017) propose a model-free strategy of statically hedging options with nearby options in strike and maturity dimensions. Madan and Milne (1994) price options under a Gaussian measure using Hermite polynomials as a basis. Carton de Wiart and Dempster (2011) use wavelet theory for partial differential equations used in derivatives pricing. Papanicolaou (2018) expresses a consistency condition between SPX Stochastic Volatility and VIX Market Models as an integral equation and solves it using an eigen series decomposition. Di Tella *et al.* (2019) find a sparse set of tradeable assets for semi-static hedging under a variance-optimal loss criterion.

Our ambition for this paper is to show the relevance and usefulness of functional analysis tools and concepts in the context of payoff replication. We establish that perfect replication can be achieved with a discrete portfolio of special options forming an orthogonal eigensystem, rather than a continuous portfolio of vanilla options with overlapping payoffs. In practice, a satisfactory approximation may be achieved with a smaller number of these special options compared with integral discretization schemes, and for some target options including vanillas our approach is more accurate than existing Fourier series methods.

The remainder of our paper is organized as follows: In section 2, we show that the Carr–Madan result is part of the general theory of integral equations. In section 3, we present key results of the theory about the existence and uniqueness of solutions, with particular focus on spectral decomposition within Hilbert spaces. In section 4, we proceed with the spectral decomposition of the ‘straddle kernel’, and we interpret our results in terms of option replication in section 5. In section 6, we propose a numerical application for fast pricing of vanilla options. In section 7, we propose a theoretical application to derive pricing formulas when the characteristic function of the underlying asset price is known. Finally,

in section 8 we consider the case of the ‘butterfly kernel’ and derive equations for its eigensystem that may be solved numerically. Section 9 concludes.

## 2. Carr–Madan as part of the theory of integral equations

In functional analysis, equation (2) is known as a *Fredholm linear integral equation of the first kind*, and  $G(x, y)$  is called the *integral kernel* or, with slight abuse of terminology, the *integral operator*. A shorthand notation for the equation is often  $f = \langle G, \phi \rangle$  or simply  $f = G\phi$ . When  $f(x)$  is identically zero the equation is called *homogeneous*; otherwise it is called *inhomogeneous*. Many authors further categorize an integral equation as *singular* when it has a convergent improper integral, as in equation (2) when either bound  $a, b$  is infinite.

Many integral kernels that are relevant to finance vanish for  $y \geq x$  or  $y \leq x$ , in which case equation (2) respectively simplifies to

$$f(x) = \int_a^x G(x, y)\phi(y) dy, \quad \text{or} \quad f(x) = \int_x^b G(x, y)\phi(y) dy.$$

These equations are known as a *Volterra integral equations of the first kind* and they have special properties and methods (e.g. Polyanin and Manzhirov 2008, ch. 10, 11).

We will see in section 3 that solving equation (2) is considerably easier when the integral kernel  $G(x, y)$  is *symmetric* and *injective*, as defined later. Table 1 lists several examples of kernels that are relevant to quantitative finance and indicates whether they are symmetric and/or injective. Note that, to a degree, log contracts and options trade on derivatives markets as options, futures and swaps on VIX and realized variance. Note also that, thanks to the development of electronic option markets, many option strategies combining vanilla options, such as straddles or butterfly spreads, quote and trade directly on dedicated platforms usually known as *complex order books*.

### 2.1. Carr–Madan kernel

The kernel  $G(x, y) := (x - y)^+$  corresponds to the payoff replication problem with call options of various strike prices  $y \in \mathcal{Y}$ . When all strike prices form the continuum  $\mathcal{Y} = \mathbb{R}_+$ , the solution to equation (2) is then  $\phi(y) = f''(y)$  as shown by Carr and Madan (1998) using standard calculus techniques. In fact, this solution can be viewed as a corollary of Taylor’s theorem with remainder in integral form,

$$F(x) = F(0) + F'(0)x + \int_0^x (x - t)F''(t) dt.$$

Substituting  $(x - t)^+$  which is identically zero for  $t > x$  yields the Carr–Madan formula at origin:

$$F(x) = F(0) + F'(0)x + \int_0^\infty (x - t)^+ F''(t) dt.$$

Table 1. Examples of integral kernels<sup>†</sup>

Kernel	European option payoff kernels		
	$G(x, y), x, y > 0$	Symmetric	Injective
Forward contracts	$x - y$	No	No
Calls and puts	$(x - y)^+, (y - x)^+$	No	Yes
Straddles	$ x - y $	Yes	Yes
Powers of the above	$G(x, y)^c$		
Strangles	$( x - y  - c)^+$	Yes	Yes ( $1/c \in 2\mathbb{N}$ )
Butterfly spreads	$(c -  x - y )^+$	Yes	Yes
Binary options	$H(x - y), H(y - x)$	No	Yes
Risk reversals	$(x - y - c)^+ - (y - x - c)^+$	No	Yes ( $1/c \in 2\mathbb{N}$ )
Log contracts	$\ln x/y$	No	No
Log calls and puts	$(\ln x/y)^+, (\ln y/x)^+$	No	Yes
Kernel	Mathematical kernels		
	$G(x, y), x, y \in \mathbb{R}$	Symmetric	Injective
Power	$x^y, x, y > 0$	No	Yes
Gaussian	$\frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}$	Yes	Yes
Laplace transform	$e^{-xy}$	Yes	Yes
Fourier transform	$e^{-2i\pi xy}$	Yes	Yes

<sup>†</sup>  $c > 0$  is a constant parameter,  $H(\cdot)$  is Heaviside's step function, and  $i$  is the imaginary number.

The general Carr–Madan formula involves both call and put options whose strike prices are respectively above or below an arbitrary split level  $x_0 \geq 0$ :

$$F(x) = F(x_0) + F'(x_0)(x - x_0) + \int_0^{x_0} F''(y)(y - x)^+ dy + \int_{x_0}^{\infty} F''(y)(x - y)^+ dy. \quad (3)$$

Observe how the second term  $F'(x_0)(x - x_0)$  corresponds to a long or short position in forward contracts with delivery price  $x_0$ . A convenient choice for practical applications is to set  $x_0$  to the underlying asset's current forward price (respectively its current spot price), in which case all call and put options are out-of-the-money-forward (respectively out-of-the-money-spot).

The Carr–Madan formula (3) may be viewed as the solution  $\phi(y) = f''(y)$  to the integral equation (2) with target function  $f(x) := F(x) - F(x_0) - F'(x_0)(x - x_0)$  and *Carr–Madan kernel*

$$G(x, y; x_0) := (x - y)^+ H(y - x_0) + (y - x)^+ H(x_0 - y), \quad (4)$$

where  $H(\cdot)$  is Heaviside's step function. An alternative proof to Taylor's theorem is to carefully differentiate both sides of equation (2) twice, either with the help of Dirac's delta functions or by invoking Leibniz's integral rule.

## 2.2. Alternative expression

It is worth noting that the Carr–Madan kernel (4) may be rewritten as

$$G(x, y; x_0) = (x - y) [H(x - y) - H(x_0 - y)],$$

by substituting  $H(y - x_0) = 1 - H(x_0 - y)$  and then  $(x - y)^+ - (y - x)^+ = x - y$  into equation (4). Substituting the

above into (2), we obtain the Volterra equation of the first kind,

$$f(x) = \int_{x_0}^x (x - y)\phi(y) dy,$$

which is forward for  $x > x_0$  and backward for  $x < x_0$ .

## 2.3. Limitations

The Carr–Madan formula has two major limitations:

- (1) In practice, only a finite number of vanilla option strikes are available and the formula must be discretized accordingly. Hedging is imperfect and approximation errors get in the way.
- (2) In the theory of integral equations, the Carr–Madan kernel  $G(x, y; x_0)$  (equation (4)) is not symmetric and does not have an orthonormal decomposition.

In this paper we address the above limitations by substituting the 'better' *straddle kernel*  $G(x, y) := |x - y|$  which is symmetric and therefore admits an orthonormal decomposition. This kernel remains tractable in terms of practical applications as it corresponds to the family of all straddles with a continuum of strikes  $y \in \mathbb{R}_+$ . Moreover, the following identity shows that the straddle kernel has a one-to-one correspondence with the Carr–Madan kernel:

$$G(x, y; x_0) = \frac{|x - y|}{2} + \frac{x - y}{2} [H(y - x_0) - H(x_0 - y)].$$

This identity is straightforwardly established by substituting  $(\pm u)^+ = (|u| \pm u)/2$  into equation (4).

### 3. Existence and uniqueness of solutions

#### 3.1. Solving first-kind Fredholm equations

Early theory for integral equations was developed by Volterra (1896), Fredholm (1903), Hilbert (1904), Schmidt (1907), Riesz (1916). It turns out that first-kind Fredholm equations are very much related to second-kind equations,

$$f(x) = \lambda\phi(x) - \int_a^b G(x, y)\phi(y) dy,$$

where  $\lambda$  is a nonzero complex parameter<sup>†</sup>.

Much of the literature about integral equations is dedicated to the theoretical and numerical resolution of second-kind equations with a continuous kernel operating on continuous or square-integrable functions. Famously, the *Fredholm alternative* states that, for any  $\lambda \neq 0$ , either the homogeneous Fredholm integral equation of the second kind,

$$0 = \lambda\phi(x) - \int_a^b G(x, y)\phi(y) dy,$$

has a nontrivial solution  $\phi \neq 0$  and  $\lambda$  is called an *eigenvalue*, or the inhomogeneous equation,

$$f(x) = \lambda\phi(x) - \int_a^b G(x, y)\phi(y) dy,$$

always has a unique solution for any  $f(x)$  and  $\lambda$  is called a *regular value*. Note that when  $\lambda$  is an eigenvalue, the second-kind inhomogeneous equation has either no solution or infinitely many solutions.

First-kind equations can be significantly more challenging to solve. It is worth emphasizing that there may be no solution at all, and that the theory about the existence and uniqueness of solutions is very limited compared to the Fredholm alternative available for second-kind equations. Fundamentally, the difficulty for finding a solution results from the smoothing property of integration. To illustrate this point, consider a well-behaved continuous kernel  $G(x, y)$  and an input function  $\phi(y)$  that is only piecewise continuous. The resulting output  $\int_a^b G(x, y)\phi(y) dy$  will be smoother than  $\phi(y)$ . Therefore, if  $f(x)$  is a continuous target function, it is very possible that solutions  $\phi(y)$  are all discontinuous, and that no solution exists within the class of continuous functions (e.g. section 8 and figure 6).

This observation is relevant to our payoff replication problem wherein a continuous solution is neither required nor expected; in fact, we will be mostly interested in square-integrable solutions.

#### 3.2. Formal framework

Let  $E$  denote the infinite-dimensional vector space of the payoff functions under consideration, such as  $C([a, b])$  or

$L^2([a, b])$ . Define the linear operator:

$$\begin{aligned} \mathcal{G} : E &\rightarrow E \\ \phi &\mapsto \mathcal{G}\phi : x \mapsto \int_a^b G(x, y)\phi(y) dy. \end{aligned}$$

With these notations, the first-kind linear integral equation (2) may be written as  $\mathcal{G}\phi = f$ . The existence of solutions for all  $f \in E$  then corresponds to  $\mathcal{G}$  being a surjective operator, i.e.  $\mathcal{G}(E) = E$ , while the uniqueness of any solution corresponds to  $\mathcal{G}$  being an injective operator, i.e.  $\mathcal{G}^{-1}(0_E) = \{0_E\}$  where  $0_E$  is the null function of  $E$ .

A standard theoretical requirement is for  $\mathcal{G}$  to be a *compact operator* (see Kress 2014, pp. 25–6, for a formal definition). It turns out that compact operators are never surjective (Kress 2014, pp. 297–8), and thus there always are infinitely many target functions  $f \in E$  for which the first-kind equation has no solution at all. In contrast, the identity operator  $I : E \rightarrow E, \phi \mapsto \phi$  is trivially surjective and thus never compact (Kress 2014, p. 27), and it can be shown that the second-kind operator  $\lambda I - \mathcal{G}, \lambda \neq 0$  is surjective if and only if it is injective (Kress 2014, p. 38). Within this framework, the Fredholm alternative translates into a discussion whether  $\lambda I - \mathcal{G}$  is injective.

On the topic of eigenvalues, it is worth noting that three classic important properties from finite-dimensional linear algebra extend to infinite-dimensional Hilbert spaces  $E$ :

- (1) For a large class of integral operators, the series of eigenvalues  $\sum \lambda_n$  converges to the operators's trace  $\int_a^b G(x, x) dx$  (Lax 2002, p. 329).
- (2) *Perron-Frobenius theorem*: if the integral operator  $\mathcal{G}$  is positive<sup>‡</sup>, it has a positive eigenvalue which is the largest in absolute value among all eigenvalues, and its eigenfunction is positive (Lax 2002, p. 253).
- (3) *Mercer's theorem*: if the integral operator  $\mathcal{G}$  is symmetric and satisfies  $\int_a^b \int_a^b \phi(x)G(x, y)\phi(y) dx dy \geq 0$  then it is a positive-semidefinite operator and all its eigenvalues are nonnegative (Lax 2002, p. 343).

#### 3.3. Spectral decomposition of continuous symmetric kernels

When the vector space of payoff functions is the Hilbert space of square-integrable functions on a finite segment  $E = L^2([a, b])$ , the linear map  $\mathcal{G}$  corresponding to the square-integrable kernel  $G \in L^2([a, b] \times [a, b])$  is called a *Hilbert-Schmidt integral operator*. If the kernel  $G(x, y)$  is continuous, the operator  $\mathcal{G}$  is always compact and therefore never surjective, i.e. there always are target functions  $f \in L^2([a, b])$  for which the first-kind integral equation  $\mathcal{G}\phi = f$  has no solution at all.

By Hilbert-Schmidt theory, when the kernel  $G(x, y)$  is continuous and symmetric, all eigenvalues of  $\mathcal{G}$  are real and form a finite or countable subset of  $\mathbb{R}$  and there is an orthonormal system of eigenfunctions  $(\phi_n)$ . In practical applications, we can find all nonzero eigenvalues  $\lambda_n$  of  $\mathcal{G}$  and their associated eigenfunctions  $\phi_n$  by solving the homogeneous integral equation of the second kind  $(\lambda_n I - \mathcal{G})\phi_n \equiv 0$ , for which

<sup>†</sup> Observe that when  $\lambda = 0$  we have a first-kind equation.

<sup>‡</sup> Here, an operator is positive when the function  $\mathcal{G}\phi$  is positive for any nonnegative and nonnull function  $\phi$ .

numerous methods exist. Moreover, we have the spectral decomposition (Eidelman *et al.* 2004, p. 94),

$$G(x, y) = \sum_n \lambda_n \phi_n(x) \phi_n(y), \quad (5)$$

where the convergence of the series is understood in the sense of  $L^2([a, b] \times [a, b])$ . As a corollary,  $\sum_n \lambda_n^2 = \int_a^b \int_a^b G(x, y)^2 dx dy$ . Substituting the above spectral decomposition identity (5) into equation (2) we obtain that, when a solution  $\phi$  exists, the target function  $f$  is attained by a linear combination of all eigenfunctions  $\phi_n$ ,

$$f(x) = \sum_n \lambda_n \phi_n(x) \int_a^b \phi_n(y) \phi(y) dy.$$

The financial interpretation of the above equation is that the target option payoff  $F(x)$  discussed in section 1 is perfectly replicated by a combination of cash and underlying asset together with a *discrete* portfolio of independent ‘spectroreplicant’ options, i.e.

$$F(x) = c + qx + \sum_n w_n \phi_n(x), \quad (6)$$

where  $c, q$  are the quantities of cash and underlying asset, and  $w_n := \lambda_n \int_a^b \phi_n(y) \phi(y) dy$  is the weight or quantity of the  $n$ th spectroreplicant option paying off  $\phi_n(x)$ .

### 3.4. Unique square-integrable solution for continuous, symmetric and injective kernels

In some cases an explicit solution  $\phi(y)$  to a first-kind equation with symmetric kernel may be obtained using non-spectral techniques, such as the convolution method for difference kernels (e.g. Srivastava and Buschman 2013, ch. 3). However, many equations do not solve in this manner. Fortunately, theory provides for a criterion about the existence of a unique solution when the continuous and symmetric kernel  $G(x, y)$  induces an injective integral operator  $\mathcal{G}$  on the Hilbert space of square-integrable functions  $E = L^2([a, b])$  or  $E = L^2(a, b)$ .

Indeed, when  $\mathcal{G}$  is symmetric and injective the orthonormal eigensystem  $(\phi_n)$  is complete and therefore a basis of  $E$ , and all eigenvalues are real. Denoting  $f_n := \int_a^b f(x) \phi_n(x) dx$  the coordinates of any target function  $f \in E$  in the basis, it is then easy to see that the function

$$\phi(y) := \sum_n \frac{f_n}{\lambda_n} \phi_n(y)$$

is a well-defined element of  $E$  if and only if the series  $\sum f_n^2 / \lambda_n^2$  converges, in which case it is the unique solution to the first-kind integral equation  $f = \mathcal{G}\phi$ .

Note that if  $\mathcal{G}$  is symmetric but *not injective*, solutions exist if and only if the series  $\sum_{\lambda_n \neq 0} f_n^2 / \lambda_n^2$  converges. The solution set is then the affine space  $\hat{\phi} + \mathcal{G}^{-1}(0_E)$  where  $\hat{\phi} := \sum_{\lambda_n \neq 0} f_n \phi_n / \lambda_n$  is unique. In the context of payoff replication it is worth emphasizing that the nullspace portfolios  $\phi \in \mathcal{G}^{-1}(0_E)$  replicate the null payoff and thus always have

zero cost. As such, they do not change the economics of replicating the target payoff and may be ignored. For ease of exposition we only consider injective kernels.

## 4. Spectral decomposition of the straddle kernel

In this section and the following three, we focus on payoff replication with straddles as replicant options. The corresponding straddle kernel  $G(x, y) := |x - y|$ , where  $y$  is the strike price, is continuous and symmetric and thus admits a spectral decomposition over any finite segment  $[a, b]$ . Moreover, there must be at least one negative and one positive eigenvalue since the kernel trace vanishes:  $\int_a^b |x - x| dx = 0$ . In fact, since the straddle kernel induces a positive operator, it must have a positive eigenvalue which is the largest among all absolute eigenvalues.

For ease of exposure, and without loss of generality, we first derive the spectral decomposition of the straddle kernel on the unit interval  $[a, b] = [0, 1]$  with corresponding integral equation

$$f(x) = \int_0^1 |x - y| \phi(y) dy, \quad 0 \leq x \leq 1.$$

The decomposition for an arbitrary interval  $[a, b]$  is then straightforwardly obtained through the affine map  $x \mapsto a + (b - a)x$  and similarly for  $y$ . Note that differentiating the above integral equation twice against  $x$  yields the solution  $\phi(x) = \frac{1}{2} f''(x)$  which is unique<sup>†</sup>. In particular, the homogeneous equation only has the trivial solution and thus the kernel is injective. Furthermore, we can see that when  $f(x) \equiv 0$ , i.e. the target payoff function  $F(x)$  is purely affine, the integral equation only has the trivial solution.

To find the eigensystem we must solve the homogeneous second-kind equation

$$\lambda \phi(x) = \int_0^1 |x - y| \phi(y) dy, \quad (7)$$

for  $\lambda \neq 0$ . Again, differentiating twice against  $x$  yields that eigenfunctions must satisfy the homogeneous second-order linear differential equation

$$\lambda \phi''(x) = 2\phi(x), \quad 0 \leq x \leq 1,$$

whose general solution is of the form

$$\begin{cases} \phi(x) = \alpha e^{2\omega x} + \beta e^{-2\omega x} & \text{if } \lambda > 0, \\ \alpha \cos 2\omega x + \beta \sin 2\omega x & \text{if } \lambda < 0, \end{cases} \quad (8a) \quad (8b)$$

where  $\alpha, \beta$  are real coefficients and  $\omega := 1/\sqrt{2|\lambda|}$  is the semi-angular frequency associated with  $\lambda$ .

Following the notations of section 3.3 we index eigenelements by nonnegative integers  $n \in \mathbb{N}$  from largest to smallest

<sup>†</sup> For a formal proof of uniqueness, suppose that  $\tilde{\phi}$  is another solution; then  $\int_0^1 |x - y| (\phi(y) - \tilde{\phi}(y)) dy = 0$  and differentiating twice against  $x$  yields  $\tilde{\phi} \equiv \phi$ .

absolute eigenvalue  $|\lambda_n|$ . In the next section 4.1 we will see that there is only one positive eigenvalue  $\lambda_0$  which is the largest among all absolute eigenvalues.

#### 4.1. Eigenfunction associated with the positive eigenvalue

Substituting (8) into equation (7) and simplifying, the straddle integral operator maps an eigenfunction  $\phi_0$  with positive eigenvalue  $\lambda_0 > 0$  to

$$\begin{aligned} & \int_0^1 |x-y|\phi_0(y) dy \\ &= \lambda_0 \left[ \phi_0(x) + (\beta - \alpha e^{2\omega}) (1 + e^{-2\omega}) \omega x \right. \\ & \quad \left. - \frac{\alpha}{2} e^{2\omega} (1 - 2\omega + e^{-2\omega}) - \frac{\beta}{2} e^{-2\omega} (1 + 2\omega + e^{2\omega}) \right]. \end{aligned} \quad (9)$$

For the remainder terms which are affine in  $x$  to vanish we must have  $\beta = \alpha e^{2\omega}$ . After substitution into equation (9) and simplifications, we obtain that  $\omega$  must be the only fixed point of the hyperbolic cotangent  $\omega_0 \approx 1.19968$ ; equivalently, the only positive eigenvalue of the straddle kernel is

$$\lambda_0 = \frac{1}{2\omega_0^2} \approx 0.34741.$$

Finally, solving  $\int_0^1 \phi_0^2(y) dy = 1$  for  $\alpha$  we obtain the normalized eigenfunction

$$\begin{aligned} \phi_0(x) &= \frac{\sqrt{2}}{\cosh \omega_0} \cosh \omega_0 (1 - 2x) \\ &\approx 0.78126 \times \cosh[1.19968 \times (1 - 2x)], \end{aligned} \quad (10)$$

which is a positive function as expected from the Perron-Frobenius theorem.

#### 4.2. Eigenfunctions associated with negative eigenvalues

Substituting (8b) into equation (7) and simplifying through trigonometric identities, the straddle integral operator maps an eigenfunction  $\phi_n, n \geq 1$  with negative eigenvalue  $\lambda_n < 0$  to

$$\begin{aligned} & \int_0^1 |x-y|\phi_n(y) dy \\ &= \lambda_n \left[ \phi_n(x) + 2\omega \cos \omega (\alpha \sin \omega - \beta \cos \omega) x \right. \\ & \quad \left. + \left( \beta \omega - \frac{\alpha}{2} \right) \cos 2\omega - \left( \alpha \omega + \frac{\beta}{2} \right) \sin 2\omega - \frac{\alpha}{2} \right]. \end{aligned} \quad (11)$$

The remainder terms affine in  $x$  vanish when either

- (a)  $\beta = 0$  and  $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$ ; or
- (b)  $\beta = \alpha \tan \omega$ , where  $\omega \neq (\pi/2) + k\pi, k \in \mathbb{Z}$  satisfies  $\cos \omega + \omega \sin \omega = 0$ , i.e. it is an opposite fixed point of the cotangent function.

Solving  $\int_0^1 \phi_n^2(y) dy = 1$  for  $\alpha$  and simplifying through trigonometric identities, we obtain the alternating system of normalized eigenfunctions

$$\phi_n(x) = \begin{cases} \sqrt{2} \cos n\pi x & \text{if } n \geq 1 \text{ is odd,} \\ \frac{\sqrt{2}}{\cos \omega_n} \cos \omega_n (1 - 2x) & \text{if } n \geq 2 \text{ is even,} \end{cases} \quad (12)$$

where  $\omega_n$  is the only opposite fixed point of the cotangent function in the interval  $((n-1)\pi/2, n\pi/2)$  when  $n \geq 2$  is even. With the convention  $\omega_n := n\pi/2$  when  $n \geq 1$  is odd, the negative eigenvalues  $\lambda_n$  are indexed from largest to smallest in absolute value:

$$\lambda_n = -\frac{1}{2\omega_n^2}, \quad n \geq 1.$$

#### 4.3. Remarks about the straddle eigensystem

The straddle eigensystem derived in sections 4.1 and 4.2 may be viewed as a modified Fourier basis of the Hilbert space  $L^2([0, 1])$ , with the benefit that the basis is ordered by magnitude of eigenvalues. Additionally, we have the following properties:

- (a) The eigenfunctions  $\phi_n, n \geq 1$  take positive and negative values. This may have a numerical benefit when replicating a target payoff  $f(x)$  which is small in absolute value.
- (b) The eigensystem is consistent with the spectral decomposition of linear and symmetric Toeplitz matrices (Bünger 2014) which are a discrete version of the straddle kernel.
- (c) All eigenfunctions satisfy  $\phi_n(0) = \sqrt{2}$  and  $\phi_n(1) = (-1)^n \sqrt{2}$ .
- (d) Since the kernel trace vanishes, we have:

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda_{2k+2} &= \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} - \lambda_0 \\ &= \frac{2}{\pi^2} \frac{\pi^2}{8} - \lambda_0 \approx -0.09741. \end{aligned}$$

- (e) By definition, for each eigenfunction we have  $\phi_n'' = \frac{2}{\lambda_n} \phi_n = -4\omega_n^2 \phi_n$  for  $n \geq 1$  and  $\phi_0'' = 4\omega_0^2 \phi_0$ .
- (f) Asymptotically, when  $n \geq 2$  is even, we have  $\omega_n \sim n\pi/2$  as  $n \rightarrow \infty$ . Indeed, inverting the opposite fixed point equation  $\cot \omega_n = -\omega_n$  produces  $\omega_n = -\operatorname{arccot} \omega_n + (n\pi/2)$ , and the inverse cotangent function is bounded. Therefore, for large  $n$ , we have  $\omega_n = n\pi/2$  if  $n$  is odd and  $\omega_n \sim n\pi/2$  if  $n$  is even.

#### 4.4. Spectral decomposition on the unit interval

Substituting the normalized eigenfunction expressions of equations (10) and (12) into the spectral decomposition equation (5), and then simplifying, the spectral decomposition

Table 2. Top 20 eigenvalues and related coefficients of the straddle kernel

$n$	$\lambda_n (\times 10^{-3})$	$\omega_n$	$c_n$	Err. Norm
0	+ 347.4082690	1.199678640	+ 0.212046516	0.214416
1	- 202.6423673	1.570796327	- 0.405284735	0.070073
2	- 63.84909579	2.798386046	- 0.144005020	0.028871
3	- 22.51581859	4.712388980	- 0.045031637	0.018071
4	- 13.34411279	6.121250467	- 0.027400487	0.012186
5	- 8.105694691	7.853981634	- 0.016211389	0.009099
6	- 5.758866886	9.317866462	- 0.011650392	0.007045
7	- 4.135558516	10.99557429	- 0.008271117	0.005703
8	- 3.206946639	12.48645440	- 0.006455031	0.004716
9	- 2.501757621	14.13716694	- 0.005003515	0.003998
10	- 2.042994806	15.64412837	- 0.004102685	0.003437
11	- 1.674730308	17.27875959	- 0.003349461	0.003001
12	- 1.415208556	18.79640437	- 0.002838428	0.002646
13	- 1.199067262	20.42035225	- 0.002398135	0.002359
14	- 1.038184585	21.94561288	- 0.002080680	0.002118
15	- 0.900632744	23.56194490	- 0.001801265	0.001917
16	- 0.794086718	25.09291041	- 0.001590696	0.001745
17	- 0.701184662	26.70353756	- 0.001402369	0.001598
18	- 0.627008356	28.23893658	- 0.001255589	0.001470
19	- 0.561336198	29.84513021	- 0.001122672	0.001359

of the straddle kernel on the unit interval  $[0, 1]$  is

$$\begin{aligned}
 |x - y| &= c_0 \cosh \omega_0(1 - 2x) \cdot \cosh \omega_0(1 - 2y) \\
 &+ \sum_{k=0}^{\infty} c_{2k+1} \cos[(2k+1)\pi x] \cdot \cos[(2k+1)\pi y] \\
 &+ \sum_{k=0}^{\infty} c_{2k+2} \cos \omega_{2k+2}(1 - 2x) \cdot \cos \omega_{2k+2}(1 - 2y),
 \end{aligned} \tag{13}$$

where  $c_n$  are the scaling coefficients:

$$c_n := \begin{cases} 1/(\omega_0 \cosh \omega_0)^2 & \text{if } n = 0, \\ -4/(n\pi)^2 & \text{if } n \geq 1 \text{ is odd,} \\ -1/(\omega_n \cos \omega_n)^2 & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

In table 2, we report numerical estimates of  $\lambda_n, \omega_n, c_n$  together with the  $L^2$  norm of the running spectral decomposition error<sup>†</sup>  $|x - y| - \sum_{k=0}^{n-1} \lambda_k \phi_k(x) \phi_k(y)$ . Figure 1 illustrates the goodness of fit using 1, 2 and 6 eigenfunctions associated with top eigenvalues. As predicted by the rapidly decaying error norm, we can see that few eigenfunctions are needed to obtain a visually excellent fit.

#### 4.5. Spectral decomposition on a finite segment $[a, b]$

Using affine transformations, it is easy to show that an orthonormal eigensystem for the straddle kernel defined over

<sup>†</sup> In the orthonormal eigensystem the error norm is  $\|\sum_{k=n}^{\infty} \lambda_k \phi_k\| = \sqrt{\sum_{k=n}^{\infty} \lambda_k^2}$

an arbitrary finite segment  $[a, b]$  is simply

$$\left( \frac{1}{\sqrt{b-a}} \phi_n \left( \frac{x-a}{b-a} \right) \right), \quad a \leq x \leq b, n \geq 0,$$

with associated eigenvalues  $(b-a)^2 \lambda_n$ , where  $\phi_n, \lambda_n$  are defined in sections 4.1 and 4.2. The corresponding spectral decomposition is then given as

$$\begin{aligned}
 |x - y| &= (b-a)c_0 \cosh \omega_0 \left( 1 - 2 \frac{x-a}{b-a} \right) \\
 &\cdot \cosh \omega_0 \left( 1 - 2 \frac{y-a}{b-a} \right) \\
 &+ (b-a) \sum_{k=0}^{\infty} c_{2k+1} \cos \left[ (2k+1)\pi \frac{x-a}{b-a} \right] \\
 &\cdot \cos \left[ (2k+1)\pi \frac{y-a}{b-a} \right] \\
 &+ (b-a) \sum_{k=0}^{\infty} c_{2k+2} \cos \omega_{2k+2} \left( 1 - 2 \frac{x-a}{b-a} \right) \\
 &\cdot \cos \omega_{2k+2} \left( 1 - 2 \frac{y-a}{b-a} \right),
 \end{aligned}$$

where the coefficients  $c_n$  are given in equation (13).

#### 5. Consequences for option replication and pricing

Because equation (2) with straddle kernel has the unique solution  $\phi(y) = \frac{1}{2}f''(y) = \frac{1}{2}F''(y)$  when it exists, the weights of the spectrally replicant options in equation (6) may be further specified as

$$w_n = \frac{b-a}{2} \lambda_n \int_a^b \phi_n \left( \frac{x-a}{b-a} \right) F''(x) dx. \tag{14}$$

A proxy of order  $n \geq 1$  for the target payoff is then simply obtained by the truncation

$$\hat{F}_n(x) := c + qx + \sum_{k=0}^{n-1} w_k \phi_k(x)$$

and the  $L^2$  norm of the replication error is then given as

$$\|F - \hat{F}_n\|_2 := \left( \int_a^b [F(x) - \hat{F}_n(x)]^2 dx \right)^{1/2} = \sqrt{\sum_{k=n}^{\infty} w_k^2}.$$

To determine the cash and underlying asset quantities  $c, q$  we need two independent conditions. For instance, integrating the right-hand side of equation (1) by parts and evaluating at the boundaries  $x = a, b$  with straddle kernel  $G(x, y) := |x - y|$ ,

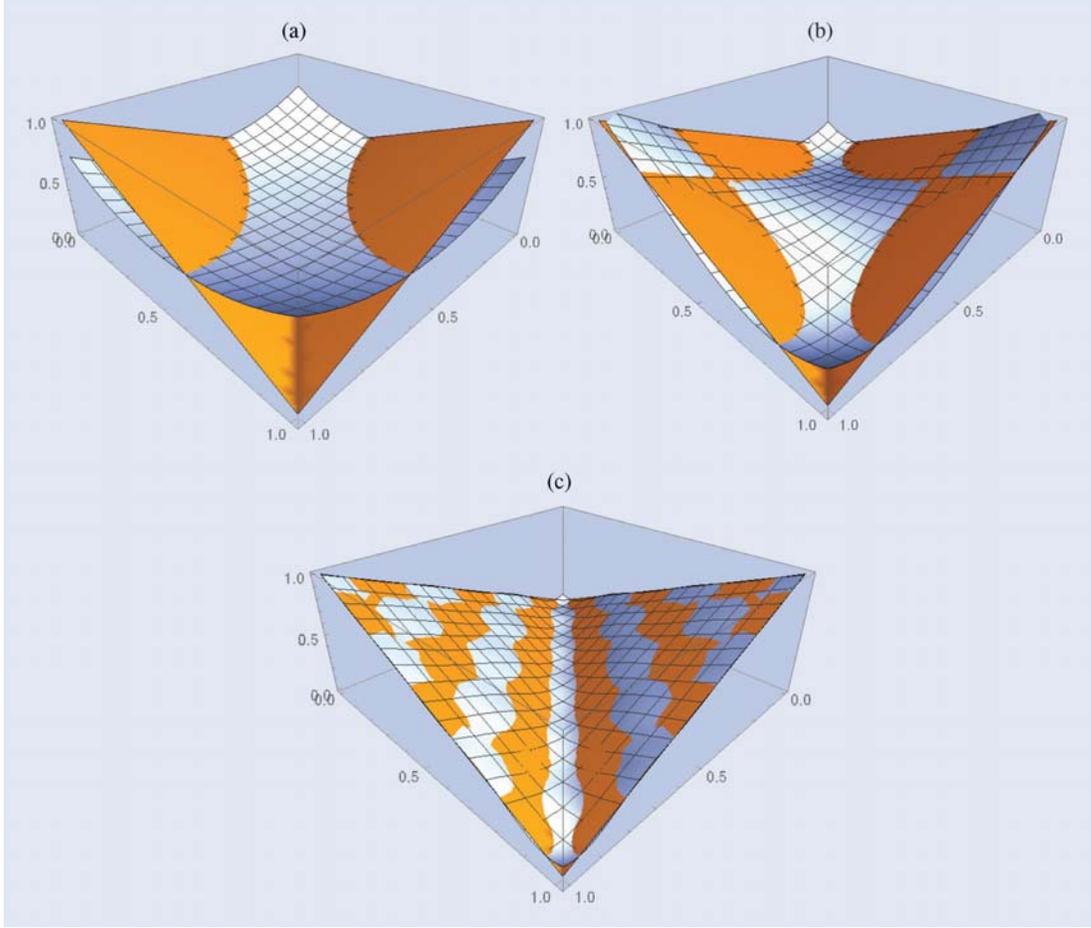


Figure 1. Straddle kernel fit with top eigenfunctions. (a) Top eigenfunction. (b) Top two eigenfunctions and (c) Top six eigenfunctions

we obtain

$$\begin{aligned}
 F(a) &= c + qa + \frac{1}{2} \int_a^b (y - a)F''(y) dy \\
 &= c + qa + \frac{1}{2} [(b - a)F'(b) - (F(b) - F(a))], \\
 F(b) &= c + qb + \frac{1}{2} \int_a^b (b - y)F''(y) dy \\
 &= c + qb - \frac{1}{2} [(b - a)F'(a) - (F(b) - F(a))].
 \end{aligned}$$

Solving for  $c, q$  yields

$$\begin{aligned}
 c &= \frac{1}{2} [F(a) + F(b) - aF'(a) - bF'(b)], \\
 q &= \frac{1}{2} [F'(a) + F'(b)].
 \end{aligned}$$

### 5.1. Benefits for option replication and hedging

Given that the spectrorreplicant options induced by the straddle kernel do not trade, the practical benefits of equation (6) in

terms of hedging are limited in this case. However, our general framework is not confined to the straddle kernel and leaves the door open to other symmetric kernels  $G(x, y)$  that might decompose into more practical spectrorreplicant options.

Nevertheless, in terms of approximation accuracy, we found that our spectrorreplication approach tends to perform better than a classical Fourier series decomposition such as the COS method of Fang and Oosterlee (2009), for at least two European target payoffs of high practical relevance: the log contract and vanilla calls. This is a useful property in view of the pricing applications discussed in the following sections. For the log contract  $F(x) := \ln x$ , figure 2(a) shows that the  $L^2$  norm of the error decays more rapidly in our method than the Fourier series method as the number of terms  $n$  grows, and is more accurate for  $n \geq 16$  terms. For vanilla calls  $F(x) := (x - K)^+$ , figure 2(b) shows that the error norm of our method is about 42% lower across all strikes with only  $n = 10$  terms, while table 3 below reports a consistently lower error norm at various truncation orders  $n$ .

We provide below some arguments analyzing the comparative performance of our spectrorreplication method against the Fourier COS series expansion over the unit interval  $[0, 1]$ . Let

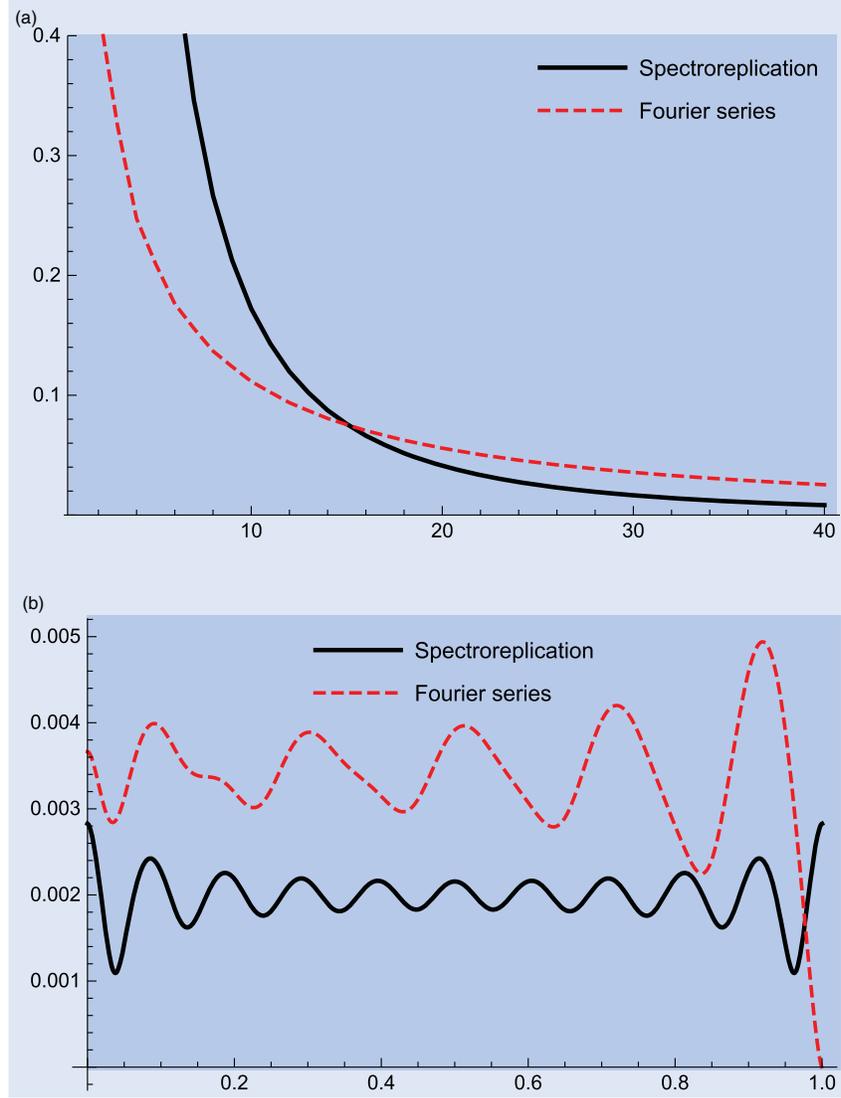


Figure 2. Comparison of proxy error norms between the spectroreplication and Fourier basis methods. (a) Error norm decay for the log contract as the truncation order  $n$  increases, using  $[a, b] = [0.01, 1.01]$  and (b) Error norm at order  $n = 10$  for vanilla calls as a function of strike  $0 \leq K \leq 1$ , using  $[a, b] = [0, 1]$ .

$\psi_n(x) := \cos(n\pi x)$  for  $n \geq 0$ , and denote the scalar product of two functions by  $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$ . The proxies at order  $n$  are

$$\hat{F}_n(x) := c + qx + \sum_{k=0}^{n-1} \frac{\lambda_k}{2} \langle F'', \phi_k \rangle \phi_k(x),$$

$$\check{F}_n(x) := \langle F, \psi_0 \rangle + 2 \sum_{k=1}^{n-1} \langle F, \psi_k \rangle \psi_k(x).$$

By orthogonality of Fourier basis functions we have  $\langle F, \check{F}_n \rangle = \|\check{F}_n\|^2$  and thus  $\|F - \check{F}_n\|^2 = \|F\|^2 - \langle F, \check{F}_n \rangle =$

$\langle F, F - \check{F}_n \rangle$ . Similarly,  $\|F - \hat{F}_n\|^2 = \langle F(x) - c - qx, F(x) - \hat{F}_n(x) \rangle$ , so that

$$\begin{aligned} \|F - \hat{F}_n\|^2 - \|F - \check{F}_n\|^2 &= \langle F(x) - c - qx, F(x) - \hat{F}_n(x) \rangle - \langle F, F - \check{F}_n \rangle \\ &= \langle F(x) - c - qx, \check{F}_n(x) - \hat{F}_n(x) \rangle \\ &= \langle c + qx, F(x) - \check{F}_n(x) \rangle \\ &= \langle F(x) - c - qx, \check{F}_n(x) - \hat{F}_n(x) \rangle \\ &= -q \langle x, F(x) - \check{F}_n(x) \rangle, \end{aligned} \quad (15)$$

Table 3. Comparison of proxy error norms for vanilla calls on the domain  $[a, b] = [0, 1]$ , averaged across all strikes  $0 \leq K \leq 1$  spaced 0.01 apart, at various truncation orders  $n$

$n$	Spectroreplication	Fourier series	Ratio
5	0.0061372	0.010485	0.59
10	0.0018960	0.003265	0.58
15	0.0010996	0.0020171	0.55
20	0.00071673	0.0011074	0.65
25	0.00051208	0.00061407	0.83
30	0.00039795	0.00052633	0.76
35	0.00031651	0.00041700	0.76
40	0.00026859	0.00031527	0.85

whose sign will depend on the particular choice of target payoff  $F$ . In Appendix we show that, for large enough  $n$ ,

$$\begin{aligned} \check{F}_n(x) - \hat{F}_n(x) &\approx \langle F, \psi_0 \rangle - c - qx \\ &\quad - \int_0^1 \left[ \frac{1}{3} + \frac{x^2 - (x+y) + y^2}{2} \right] F''(y) dy \\ &\quad + 2 \sum_{k=1}^{n-1} \frac{(-1)^k F'(1) - F'(0)}{(k\pi)^2} \psi_k(x). \end{aligned}$$

Substituting into equation (15) and rearranging we get  $\|F - \hat{F}_n\|^2 - \|F - \check{F}_n\|^2 \approx A + B_n$  where

$$\begin{aligned} A &:= \int_0^1 (F(x) - c - qx) (\langle F, \psi_0 \rangle - c - qx) dx \\ &\quad - \int_0^1 \int_0^1 (F(x) - c - qx) \\ &\quad \times \left[ \frac{1}{3} + \frac{x^2 - (x+y) + y^2}{2} \right] F''(y) dx dy, \\ B_n &:= 2 \sum_{k=1}^{n-1} \frac{(-1)^k F'(1) - F'(0)}{(k\pi)^2} \\ &\quad \times \int_0^1 (F(x) - c - qx) \psi_k(x) dx \\ &\quad - q \int_0^1 x (F(x) - \check{F}_n(x)) dx. \end{aligned}$$

Note that the approximate criterion  $A + B_n$  is easy to calculate for any  $n$  as it only involves the Fourier proxy  $\check{F}_n$  and basis functions  $\psi_k(x) := \cos(k\pi x)$  and does *not* require to calculate the spectroreplicant proxy  $\hat{F}_n(x)$ .

For the log-contract  $F(x) := \ln(x + 0.01)$  we find  $A \approx 199.63027$  and  $B_{30} \approx -199.63170$  giving  $\|F - \hat{F}_{30}\|^2 - \|F - \check{F}_{30}\|^2 \approx A + B_{30} \approx -0.00143$ . This further suggests that, for large enough  $n$ , our spectroreplication approach is more accurate than the Fourier cosine series.

For the vanilla call  $F(x) := (x - K)^+$  we find  $A = -(K^4/24) + (K^2/12) - \frac{1}{48}$  and

$$\begin{aligned} B_{10} &= -\frac{K^3}{12} + \frac{K^2}{8} - \frac{1}{24} + \frac{2.16387}{\pi^4} \\ &\quad + \frac{4}{\pi^4} \left[ \cos(K\pi) + \frac{\cos(3K\pi)}{81} + \frac{\cos(5K\pi)}{625} \right. \\ &\quad \left. + \frac{\cos(7K\pi)}{2401} + \frac{\cos(9K\pi)}{6561} \right] \\ &\quad - \frac{1}{\pi^4} \left[ \frac{\cos(2K\pi)}{8} + \frac{\cos(4K\pi)}{128} \right. \\ &\quad \left. + \frac{\cos(6K\pi)}{648} + \frac{\cos(8K\pi)}{2048} \right]. \end{aligned}$$

Figure 3 gives the signed square root of the relative squared error  $\|F - \hat{F}_{10}\|^2 - \|F - \check{F}_{10}\|^2$  versus its proxy  $A + B_{10}$  as functions of strike  $0 \leq K \leq 1$ . We can see that both measures are negative, suggesting again that the spectroreplication approach is more accurate than the Fourier cosine series.

## 5.2. Benefits for option pricing

When equation (6) holds and all relevant quantities converge in  $L^2$ , the price of the target option is simply given as

$$F = c + qX + \sum_{n=0}^{\infty} w_n \Phi_n, \quad (16)$$

where  $F, X$  are the respective prices of the target option and underlying asset,  $\Phi_n$  is the price of the  $n$ th spectroreplicant option, and all prices are forward (i.e. paid on the common maturity date.) The above pricing equation can be established using classical arbitrage arguments under the assumptions that short-selling and the instant trading of infinitely many securities are both feasible. In practice, just as with the Carr–Madan formula, the latter assumption is not realistic and must be mitigated by selecting a finite number of replicant options. For example, a proxy of order  $n \geq 1$  based on the largest absolute eigenvalues would be

$$\hat{F}_n := c + qX + \sum_{k=0}^{n-1} w_k \Phi_k. \quad (17)$$

The key benefit of equation (17) compared to a discretization of the Carr–Madan formula is that spectroreplicant options are orthogonal in the sense of the scalar product  $\langle f, g \rangle = \int f(x)g(x) dx$ . In contrast, the continuum of call and put replicants in Carr–Madan are very codependent due to their overlapping payoff functions. This suggests that, for non-pathological target payoff  $F(x)$ , a limited number  $n$  of spectroreplicant options is enough to achieve satisfactory pricing accuracy.

An obvious practical disadvantage of equation (17) is that the fair prices  $(\Phi_k)_{0 \leq k \leq n}$  of spectroreplicant options must be discovered by another method. In this regard, we may distinguish between:

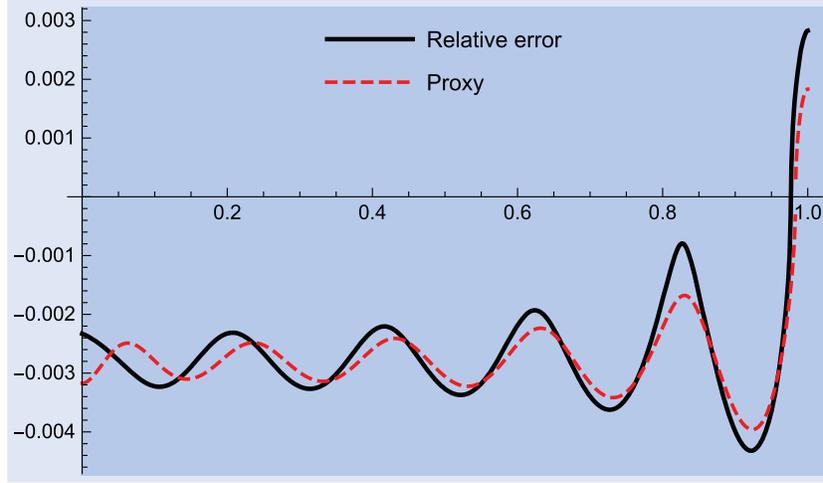


Figure 3. Comparison between exact and proxy relative squared error at order  $n = 10$  for the vanilla call  $F(x) := (x - K)^+$ , as functions of strike  $0 \leq K \leq 1$

- (1) A model-based option pricing method, as shown in section 7;
- (2) A model-free option pricing method, such as the Carr–Madan formula for option prices, either discretized along listed option strikes as shown in section 6 or using a numerical integration scheme together with the Black and Scholes (1973) formula and a model of the implied volatility smile.

Here, we must pause to dissipate any concern of circularity within the second approach: if we price spectrorreplicant options using the Carr–Madan formula, do we not end up where we started and effectively price the target option using the Carr–Madan kernel? This would be true if we used infinitely many spectrorreplicants, but the key benefit of equation (17) is that only a small number  $n$  of spectrorreplicants is required to achieve satisfactory accuracy. In a practical implementation, the fixed weights  $w_k$  and the fair spectrorreplicant prices  $\Phi_k$  need only be precomputed once; then, for each target option in the book with specific payoff  $F(x)$ , computing the proxy price  $\hat{F}_n$  only requires  $n + 3$  multiplications and additions.

As an illustration, consider a vanilla option market with 200 listed strikes, and an option book of 1000 exotic European options. Pricing each exotic option individually costs 200 operations using the discretized Carr–Madan formula, for a total of 200,000 operations for the entire book. In contrast, precomputing 20 spectrorreplicant option prices costs 4000 operations, and then pricing all exotic options in the book using equation (17) only costs 20,000 operations, for a grand total of 24,000 operations — an 88% gain in efficiency.

This gain of speed is likely to be very relevant for electronic market-making, risk management of large portfolios of options or high frequency option trading. Moreover, the computational cost of refreshing the prices  $\Phi_k$  throughout the trading day can be mitigated using Greek sensitivities.

## 6. Numerical application: fast vanilla option pricing

### 6.1. Proxy formula for vanilla option prices

For the vanilla call target payoff  $F(x) := (x - K)^+$  where  $K$  is the strike price, the second-order derivative is Dirac’s delta function  $F''(x) = \delta(x - K)$ ; substituting into equation (14) we obtain the proxy formula for the call price

$$\hat{c}_n(K) = \frac{X - K}{2} + \sum_{k=0}^{n-1} w_k(K) \Phi_k, \quad (18)$$

with weights

$$w_k(K) = \frac{b - a}{2} \lambda_k \phi_k \left( \frac{K - a}{b - a} \right).$$

Similarly, the put proxy formula is given as

$$\hat{p}_n(K) = \frac{K - X}{2} + \sum_{k=0}^{n-1} w_k(K) \Phi_k$$

with the same weights  $w_k(K)$ .

Table 4. Spectrorreplicant option prices for the S&P 500 option market as of 20 November 2018

$n$	$\Phi_n$	$n$	$\Phi_n$
0	0.9636258443	10	-0.2156408037
1	-1.010953323	11	0.2308846696
2	-0.08605859769	12	-0.1771790305
3	0.562828561	13	0.09907369714
4	-0.8733519895	14	-0.02579248218
5	0.682335075	15	-0.03158846323
6	-0.3531713684	16	0.05740449608
7	0.08321391368	17	-0.06505514540
8	0.1384173306	18	0.05272523143
9	-0.2156408037	19	-0.03139028403

### 6.2. Numerical results

We repriced 30-day out-of-the-money options on the S&P 500 index using the top 20 spectre-replicant options, based on sample bid and offer data as of 20 November 2018. We report the spectre-replicant option prices  $\Phi_k$  in table 4 obtained with a VIX-style discretization of the Carr–Madan formula (3). Then, we compute the proxy option prices for strike prices ranging from  $a = 1225$  to  $b = 3075$  using the formulas above.

In figure 4(a) we plot our results for listed strikes between 1225 and 3075 on a scale from 0 to 1, where 0 corresponds to the market bid and 1 corresponds to the market offer price. Remarkably, all but two proxy option prices lie within the bid-offer range.

A valuable additional benefit of the spectral decomposition method is to provide a natural ‘fit’ of the implied volatility smile for arbitrary strikes  $a \leq K \leq b$ . In figure 4(b), we show our results in the slightly extended range [1000, 3300]. We can see that the fit is visually pleasing and the extrapolated values on the left and right regions of the chart look plausible.

### 6.3. Arbitrage considerations

It is worth emphasizing that the proxy formula of equation (18) is not theoretically free of arbitrage due to the oscillatory nature of the spectre-replicant options. Indeed, the tails of the corresponding implied distribution  $h_{n+1}(K) := \hat{c}''_{n+1}(K)$  can become negative as shown in figure 5, indicating

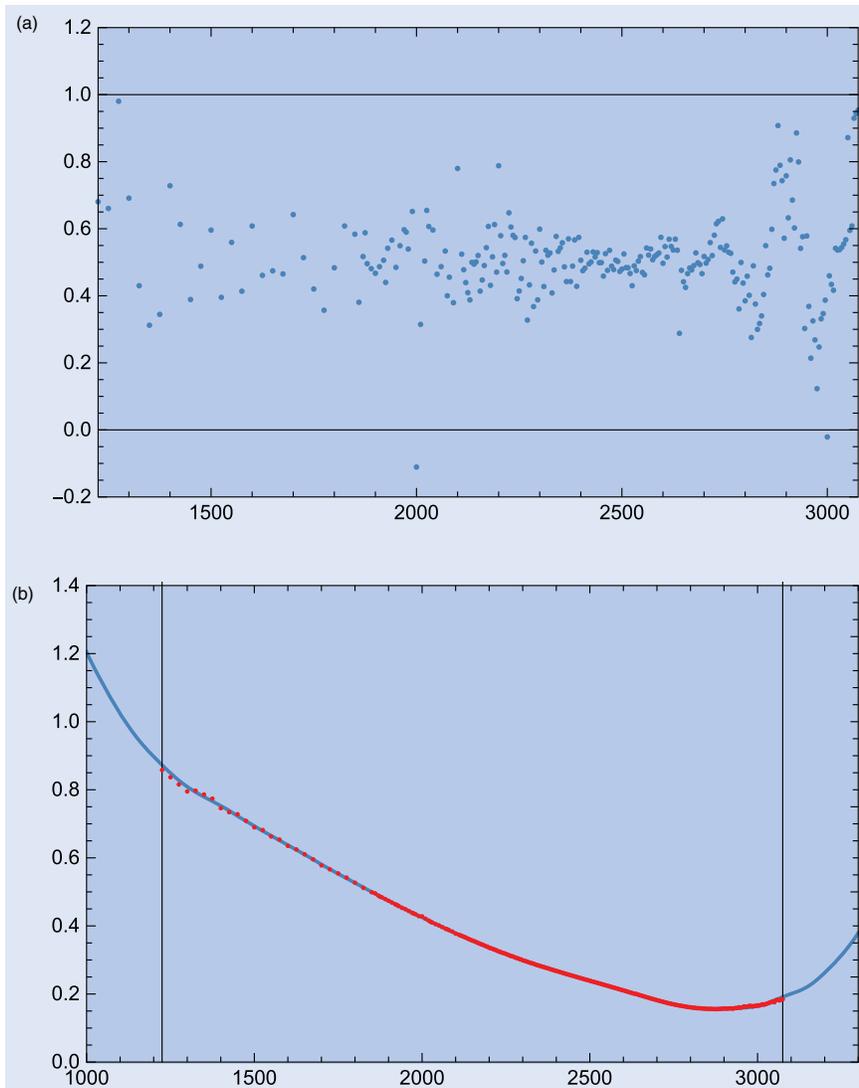


Figure 4. Proxy OTM option prices and corresponding proxy implied volatility smile. (a) Proxy prices on a market bid-offer 0-1 scale and (b) Proxy implied volatility smile

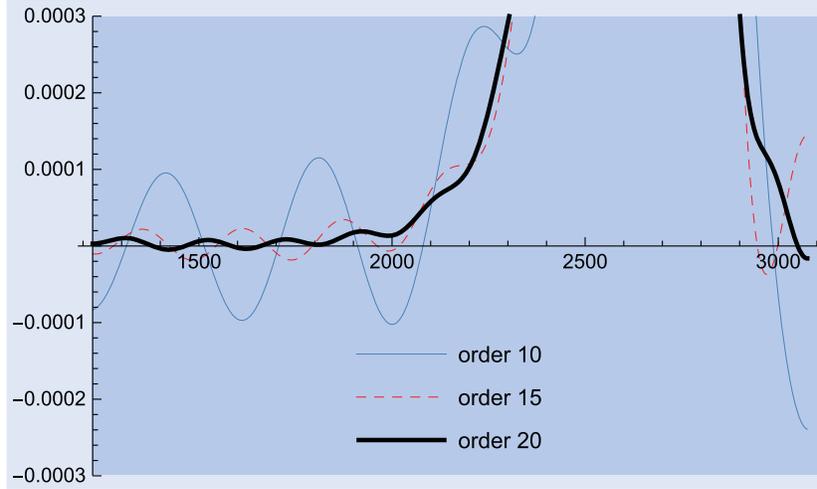


Figure 5. Implied distribution tails corresponding to proxy call prices

the theoretical existence of butterfly arbitrages. However, our empirical results shown in figure 4(a) suggest that such arbitrages are unlikely to have any practical relevance once bid-offer spreads are taken into account.

As expected, in the limit as  $n \rightarrow \infty$ , the proxy formula is arbitrage-free as long as all spectreptic prices  $\Phi_n$  are known and priced off a valid implied distribution  $h(K)$ . This can be verified by substituting  $\Phi_n = \int_a^b \phi_n(x-a)/(b-a)h(x) dx$  into (18) to get

$$\begin{aligned} c(K) &:= \frac{X-K}{2} + \sum_{n=0}^{\infty} w_n(K) \Phi_n \\ &= \frac{X-K}{2} + \int_a^b h(x) dx \sum_{n=0}^{\infty} w_n(K) \phi_n \left( \frac{x-a}{b-a} \right). \end{aligned}$$

Substituting into the above the expression for  $w_n(K)$ , recognizing the spectral decomposition (5) of the straddle kernel  $|x-K|$ , and differentiating both sides twice against  $K$  we recover the implied distribution  $c''(K) = h(K)$ . In addition, note that the truncated implied distribution  $h_{n+1}(K)$  has the simple expression

$$h_{n+1}(K) := \hat{c}_{n+1}''(K) = \sum_{k=0}^n \Phi_k \phi_k \left( \frac{K-a}{b-a} \right),$$

which is obtained by differentiating (18) twice against  $K$  using  $\phi_n''(y) = (2/\lambda_n)\phi_n(y)$ .

### 7. Theoretical application: formulas for spectreptic option prices when the characteristic function is known

Consider an option pricing model where the characteristic function  $\varphi(z) := \mathbb{E}(e^{izX_T})$ ,  $z \in \mathbb{C}$  of the terminal underlying

value  $X_T$  at time  $T$  is known in closed form. The characteristic function for  $\tilde{X}_T := (X_T - a)/(b - a)$  is then

$$\tilde{\varphi}(z) = e^{-i(a/(b-a))z} \varphi \left( \frac{z}{b-a} \right),$$

and we may recover pricing formulas for spectreptic options through the identities

$$\mathbb{E} \cosh \omega(1 - 2\tilde{X}_T) = \frac{1}{2} e^{\omega} [\tilde{\varphi}(2i\omega) + \tilde{\varphi}(-2i\omega)], \quad (19a)$$

$$\mathbb{E} \cos \omega \tilde{X}_T = \frac{1}{2} [\tilde{\varphi}(\omega) + \tilde{\varphi}(-\omega)], \quad (19b)$$

$$\mathbb{E} \cos \omega(1 - 2\tilde{X}_T) = \frac{1}{2} e^{i\omega} \tilde{\varphi}(-2\omega) + \frac{1}{2} e^{-i\omega} \tilde{\varphi}(2\omega). \quad (19c)$$

At this point, we are faced with two technical issues. In many classical pricing models such as Black and Scholes (1973), Heston (1993), Merton (1976), the price  $X_T$  belongs to the entire real half-line rather than a finite interval  $[a, b]$ . In addition, the characteristic function is only known for the *log-price*. The second issue may be resolved by rewriting the target payoff in log-price space as  $\bar{F}(x) := F(e^x)$ , so that  $x$  now corresponds to  $\ln X_T$ , without changing the kernel  $G(x, y) := |x - y|$  which now corresponds to log straddles. The coefficients  $w_n$  in equation (14) then become

$$\begin{aligned} w_n &= \frac{b-a}{2} \lambda_n \int_a^b \phi_n \left( \frac{x-a}{b-a} \right) \bar{F}''(x) dx \\ &= \frac{b-a}{2} \lambda_n \int_a^b \phi_n \left( \frac{x-a}{b-a} \right) [e^x F'(e^x) + e^{2x} F''(e^x)] dx \end{aligned}$$

and may be used in the pricing equations (16) and (17).

The finite domain issue is harder to address. Fang and Oosterlee (2009) propose a somewhat arbitrary formula (eq.(49)) for the range  $[a, b]$  based on the *cumulants* of  $X_T$ . We derive an alternative approach in the following paragraphs. Note that once a suitable range  $[a, b]$  has been found, both the spectreptic prices  $\Phi_n$  obtainable from equations (19) and the

resulting target option price  $F$  of equation (16) will have the required numerical accuracy for the chosen model.

Let  $\check{F}$  denote the model price of the target option, while  $F$  remains its price over the restricted domain  $[a, b]$  :

$$\begin{aligned}\check{F} &:= \mathbb{E}[F(X_T)] = \int_{-\infty}^{\infty} F(e^x)p(x) dx \\ &\approx \int_a^b F(e^x)p(x) dx =: F,\end{aligned}$$

where  $p(x)$  is the probability density function of  $\ln X_T$ . An upper bound for the absolute error is then

$$|\check{F} - F| \leq \int_{\mathbb{R} \setminus [a,b]} |F(e^x)| p(x) dx.$$

If the payoff function  $F(e^x)$  is bounded we have the trivial upper bound

$$|\check{F} - F| \leq [1 - (P(b) - P(a))] \sup_{x \in \mathbb{R} \setminus [a,b]} |F(e^x)|,$$

where  $P(x)$  is the cumulative distribution function of  $\ln X_T$ . Otherwise, if the payoff function is unbounded but square-integrable under measure  $P$ , we may apply Cauchy-Schwartz's inequality to the product  $F(e^x)\sqrt{p(x)} \times \sqrt{p(x)}$  to write

$$\begin{aligned}|\check{F} - F|^2 &\leq \int_{\mathbb{R} \setminus [a,b]} F(e^x)^2 p(x) dx \int_{\mathbb{R} \setminus [a,b]} p(x) dx \\ &\leq [1 - (P(b) - P(a))] \int_{\mathbb{R} \setminus [a,b]} F(e^x)^2 p(x) dx \\ &\leq [1 - (P(b) - P(a))] \mathbb{E}[F(X_T)^2],\end{aligned}$$

where the last bound above is loose. Typically the value of  $\mathbb{E}[F(X_T)^2]$  is not known but in practice we may use an estimate obtained from an analytically tractable model such as Black and Scholes (1973).

Finally, if the probability range  $P(b) - P(a)$  is not available in closed form, we may use the classical Bienaymé-Chebyshev's inequality

$$1 - (P(m_1 + L) - P(m_1 - L)) \leq \frac{m_2 - m_1^2}{L^2}$$

where  $L > 0$  is arbitrary and  $m_k := \mathbb{E}(\ln^k X_T)$ ,  $k \in \{1, 2\}$  are moments recoverable from the characteristic function. Alternatively, a tighter bound could be obtained from the formula (e.g. Durrett 2019, p.115)

$$1 - (P(L) - P(-L)) \leq \frac{L}{2} \int_{-2/L}^{2/L} [1 - \varphi(x)] dx.$$

As an illustration we show how  $[a, b]$  may be determined within the Black and Scholes (1973) model, for which the characteristic function of the log-price  $x \equiv \ln X_T$  is known to be

$$\varphi_{BS}(x) = e^{ix[\ln X_0 + (r - \sigma^2/2)T] - x^2 \sigma^2 T/2},$$

where  $X_0$  is the underlying asset spot price,  $r$  is the continuous interest rate and  $\sigma$  is the volatility parameter. The probability

range  $P(b) - P(a)$  is also well known in closed form as

$$\begin{aligned}P(b) - P(a) &= \mathbb{P}(a \leq \ln X_T \leq b) \\ &= N\left(\frac{b - m_1}{\sigma\sqrt{T}}\right) - N\left(\frac{a - m_1}{\sigma\sqrt{T}}\right),\end{aligned}$$

where  $m_1 = \ln X_0 + (r - \sigma^2/2)T$ . Choosing  $b = m_1 + L\sigma\sqrt{T}$ ,  $a = m_1 - L\sigma\sqrt{T}$ ,  $L > 0$ , the above simplifies to

$$P(b) - P(a) = N(L) - N(-L) = 1 - 2N(-L).$$

For a bounded target payoff such as a call spread with strikes spaced  $\Delta K$  apart, a given precision target  $\varepsilon$  is achieved by choosing  $L$  such that  $2N(-L)\Delta K \leq \varepsilon$ , i.e.  $L \geq N^{-1}(\varepsilon/2\Delta K)$ . Using  $X_0 = 1$ ,  $r = 0.05$ ,  $T = 1$ ,  $\sigma = 0.2$ ,  $\Delta K = 1$  and a target precision  $\varepsilon = 0.01$  we find  $L \approx 2.58$  giving  $[a, b] = [-0.4851, 0.5452]$ .

For an unbounded payoff such as the log contract  $F(X_T) := \ln(X_T) \equiv x$ , we have  $\mathbb{E}[F(X_T)^2] = \int_{-\infty}^{\infty} x^2 p(x) dx = m_2 = m_1^2 + \sigma^2 T$  and a given precision target  $\varepsilon$  is achieved by choosing  $L \geq -N^{-1}(\varepsilon^2/2m_2)$ . Using the same parameters as above we find  $L \approx 3.03$  and  $[a, b] = [-0.5760, 0.6361]$ .

## 8. Spectral decomposition of the butterfly kernel

To further underscore the generality of our approach, in this final section we consider the butterfly kernel

$$\begin{aligned}G(x, y; c) &:= (c - |x - y|)^+ \\ &= (x - y + c)^+ - 2(x - y)^+ + (x - y - c)^+\end{aligned}$$

for finite domain  $[a, b]$  and fixed call spread parameter  $0 < c \leq \frac{1}{2}(b - a)$ . As stated in table 1, this kernel is symmetric and injective. Indeed, we can write  $G(x, y; c) = cK(x - y)$  where  $K(z) := (1 - |z|/c)^+$ , and  $G$  is a positive-definite kernel if and only if  $K$  is a positive-definite function. By Bochner's theorem (Lax 2002, p. 144) a function is positive-definite if and only if it is the Fourier transform of a probability density, and it is easy to verify that

$$\hat{K}(u) := \frac{1}{c\pi} \cdot \frac{1 - \cos cu}{u^2}$$

is such a density. Indeed, by Fubini and then the property that the real number  $\phi(y)$  is equal to its conjugate, we may write

$$\begin{aligned}&\int_a^b \int_a^b \phi(x)K(x - y)\phi(y) dx dy \\ &= \int_a^b \int_a^b \phi(x)\phi(y) \int_{-\infty}^{\infty} e^{i(x-y)u} \hat{K}(u) du dx dy \\ &= \int_{-\infty}^{\infty} \hat{K}(u) du \int_a^b \phi(x)e^{iux} dx \int_a^b \overline{\phi(y)e^{iuy}} dy \geq 0,\end{aligned}$$

and equality implies  $\phi \equiv 0$ . Therefore, the butterfly kernel only has strictly positive eigenvalues and it is injective.

For ease of exposure, and without loss of generality, we assume  $[a, b] = [0, 1]$ ,  $0 < c \leq \frac{1}{3}$  as we did in section 4. Differentiating the integral equation (2) with butterfly kernel  $G(x, y; c)$  twice against  $x$  we obtain the linear recurrence equation for  $\phi$

$$f''(x) = \phi(x - c) - 2\phi(x) + \phi(x + c),$$

with the convention  $\phi(x) \equiv 0$  for  $x < 0$  or  $x > 1$ . When  $c = 1/N$ ,  $N \in \mathbb{N} \setminus \{0, 1, 2\}$ , the solution is

$$\begin{aligned} \phi(x) = & -\frac{N-n}{N+1} \sum_{k=0}^n (n+1-k)f''(x-kc) \\ & -\frac{n+1}{N+1} \sum_{k=1}^{N-n-1} (N-n-k)f''(x+kc), \\ & x = nc + r, \quad 0 \leq r < c, \quad n \in \{0, 1, \dots, N-1\}, \end{aligned} \quad (20)$$

wherein  $n$  is the Euclidean quotient of  $x$  by  $c$  with remainder  $r$  (i.e.  $x$  modulo  $c$ ). In particular, the homogeneous equation with  $f(x) \equiv 0$  only has the trivial solution  $\phi(x) \equiv 0$ , thereby confirming that the butterfly kernel is injective when  $c = 1/N$ .

It is worth noting that the solution (20) is typically discontinuous at every step  $c$ , and that the integral  $\int_0^1 G(x, y; c)\phi(y) dy$  matches  $f(x)$  up to affine terms. Figure 6 shows the solution obtained for  $F(x) = e^x$  and  $c = 1/6$ .

In the fashion of section 4, it is possible to identify the general form of eigenfunctions  $\phi(x)$  satisfying

$$\lambda\phi''(x) = \begin{cases} -2\phi(x) + \phi(x+c) & \text{for } 0 \leq x < c, \\ \phi(x-c) - 2\phi(x) + \phi(x+c) & \text{for } c \leq x \leq 1-c, \\ \phi(x-c) - 2\phi(x) & \text{for } 1-c < x \leq 1, \end{cases}$$

for an eigenvalue  $\lambda > 0$ . This may be done by splitting the domain  $[0, 1]$  at every step  $c$  and solving the system of second-order ordinary linear differential equations

$$\lambda\mathbf{u}''(x) = -\mathbf{A}\mathbf{u}(x), \quad 0 \leq x < c,$$

where  $\mathbf{u}(x) := (\phi(x), \phi(x+c), \dots, \phi(x+(N-1)c))^T$  is a vector of length  $N$  and  $\mathbf{A}$  is the familiar  $N \times N$  tridiagonal matrix

$$\mathbf{A} := \begin{bmatrix} 2 & -1 & & (0) \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ (0) & & -1 & 2 \end{bmatrix},$$

which is positive-definite with principal square root  $\mathbf{A}^{1/2}$ ; formulas for the spectral elements of  $\mathbf{A}$  can be found in e.g. Smith (1985), pp. 55, 154–156. The homogeneous second-kind integral equation with butterfly kernel may then be written in terms of  $\mathbf{u}$  as

$$\lambda\mathbf{u}(x) = \int_0^c \mathbf{G}(x, y)\mathbf{u}(y) dy \quad (21)$$

where  $\mathbf{G}(x, y) := ((c - |x - y + (n - p)c|)^+)_{0 \leq n, p \leq N-1}$  is a  $N \times N$  matrix defined for  $0 \leq x, y < c$ .

The general solution to the system of second-order ordinary linear differential equations  $\lambda\mathbf{u}'' = -\mathbf{A}\mathbf{u}$  is known to be

$$\mathbf{u}(x) = \cos\left(\frac{x}{\sqrt{\lambda}}\mathbf{A}^{1/2}\right)\mathbf{k}_1 + \sin\left(\frac{x}{\sqrt{\lambda}}\mathbf{A}^{1/2}\right)\mathbf{k}_2,$$

where  $\mathbf{k}_1, \mathbf{k}_2$  are two column vectors of  $N$  constant coefficients. Substituting into (21) and integrating by parts twice

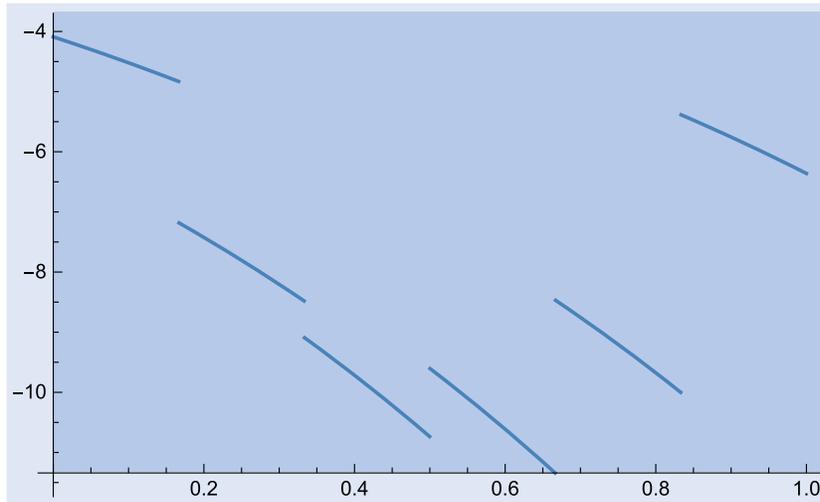


Figure 6. Solution to integral equation with butterfly kernel  $G(x, y; c)$ ,  $c = \frac{1}{6}$ , target function  $F(x) = e^x$ , domain  $[a, b] = [0, 1]$

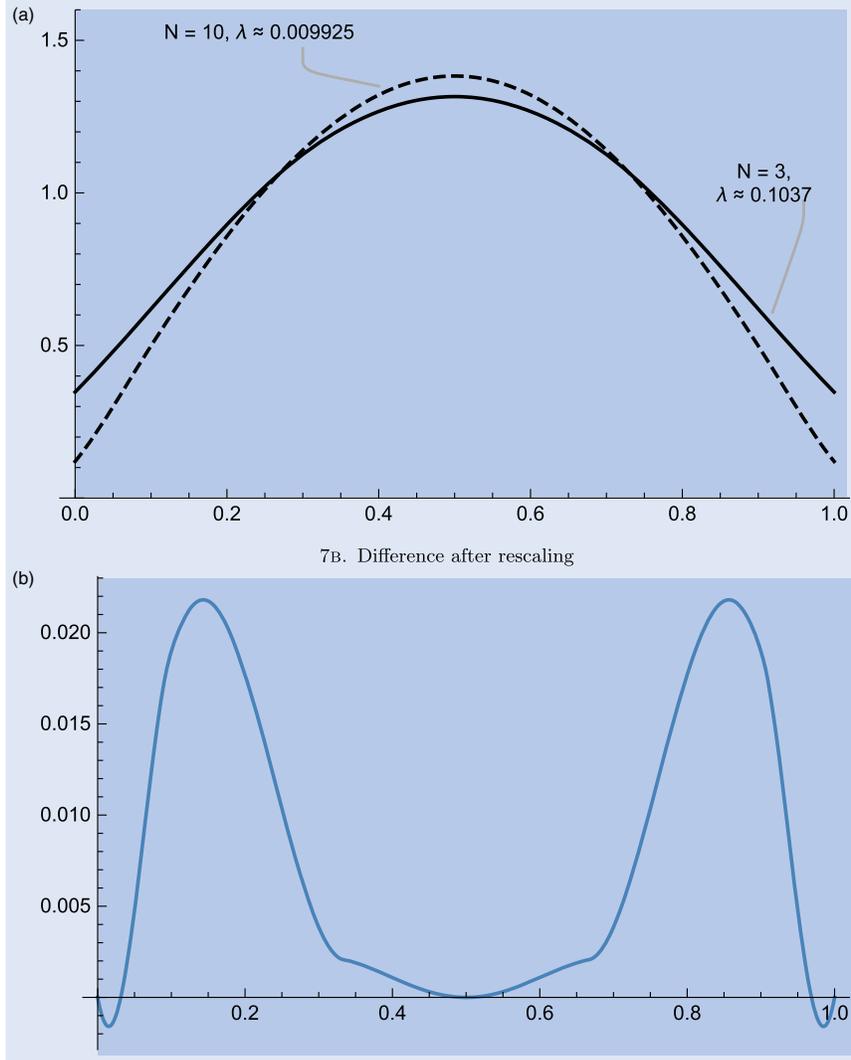


Figure 7. Comparison of top eigenfunctions of the butterfly kernel  $G(x, y; c)$ ,  $c = 1/N$  over the domain  $[a, b] = [0, 1]$ . (a) Top normalized eigenfunction for  $N = 3$  and  $N = 10$ . (b) Difference after rescaling

we obtain<sup>†</sup>

$$\lambda \mathbf{u}(x) = -\lambda \left[ \mathbf{G}(x, y) \mathbf{A}^{-1} \mathbf{u}'(y) - \mathbf{G}_y(x, y) \mathbf{A}^{-1} \mathbf{u}(y) \right]_{y=0}^{y=c^-} - \lambda \int_0^c \mathbf{G}_{yy}(x, y) \mathbf{A}^{-1} \mathbf{u}(y) dy,$$

where  $\mathbf{G}_y, \mathbf{G}_{yy}$  are the first- and second-order partial derivatives of  $\mathbf{G}(x, y)$  against  $y$ . Substituting the identities  $\mathbf{G}(x, c) = x\mathbf{I} + (c-x)\mathbf{L}$ ,  $\mathbf{G}(x, 0) = (c-x)\mathbf{I} + x\mathbf{L}^T$ ,  $\mathbf{G}_y(x, c) = -\mathbf{I} + \mathbf{L}$ ,  $\mathbf{G}_y(x, 0) = \mathbf{I} - \mathbf{L}^T$ , where  $\mathbf{L}$  is the lower shift matrix with ones on the subdiagonal, we may rewrite the bracket in the

<sup>†</sup> Observe that the matrix versions of  $\cos, \sin$  commute with any power of the argument matrix, and that  $-\lambda \mathbf{A}^{-1} \mathbf{u}'(x), -\lambda \mathbf{A}^{-1} \mathbf{u}(x)$  are respectively first- and second-order antiderivatives of  $\mathbf{u}(x)$ .

above equation as the block matrix expression

$$\begin{aligned} \mathbf{b}_\lambda(x) &:= [\mathbf{I} - \mathbf{L} \quad x(\mathbf{I} - \mathbf{L}) + c\mathbf{L}] \\ &\times \begin{bmatrix} \mathbf{C}_\lambda & \mathbf{S}_\lambda \\ -\frac{1}{\sqrt{\lambda}} \mathbf{A}^{1/2} \mathbf{S}_\lambda & \frac{1}{\sqrt{\lambda}} \mathbf{A}^{1/2} \mathbf{C}_\lambda \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} \mathbf{k}_1 \\ \mathbf{A}^{-1} \mathbf{k}_2 \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{I} - \mathbf{L}^T & \frac{x}{\sqrt{\lambda}} (\mathbf{I} - \mathbf{L}^T) \mathbf{A}^{1/2} - \frac{c}{\sqrt{\lambda}} \mathbf{A}^{1/2} \end{bmatrix} \\ &\times \begin{bmatrix} \mathbf{A}^{-1} \mathbf{k}_1 \\ \mathbf{A}^{-1} \mathbf{k}_2 \end{bmatrix} \end{aligned}$$

where  $\mathbf{C}_\lambda := \cos((c/\sqrt{\lambda})\mathbf{A}^{1/2})$  and  $\mathbf{S}_\lambda := \sin((c/\sqrt{\lambda})\mathbf{A}^{1/2})$ . The vector  $\mathbf{b}_\lambda(x)$  is affine in  $x$  and will vanish if and only if the intercept and slope vectors  $\mathbf{b}_\lambda(0), \mathbf{b}'_\lambda$  are zero, leading to

the homogeneous block matrix equation in  $\mathbf{k}_1, \mathbf{k}_2$

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \left( \begin{bmatrix} \mathbf{I} - \mathbf{L} & c\mathbf{L} \\ \mathbf{O} & \mathbf{I} - \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{C}_\lambda & \mathbf{S}_\lambda \\ -\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\mathbf{S}_\lambda & \frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\mathbf{C}_\lambda \end{bmatrix} + \begin{bmatrix} \mathbf{I} - \mathbf{L}^T & -\frac{c}{\sqrt{\lambda}}\mathbf{A}^{1/2} \\ \mathbf{O} & (\mathbf{I} - \mathbf{L}^T)\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2} \end{bmatrix} \right) \begin{bmatrix} \mathbf{A}^{-1}\mathbf{k}_1 \\ \mathbf{A}^{-1}\mathbf{k}_2 \end{bmatrix},$$

where  $\mathbf{0}$  is the null column vector of  $\mathbb{R}^N$  and  $\mathbf{O}$  is the null matrix of  $\mathbb{R}^{N \times N}$ . It is worth noting that solving the above equation is difficult: we need to find  $\lambda$  such that the  $2N \times 2N$  block matrix between parentheses is singular, and then find the corresponding nullspace to identify non-trivial solutions  $\mathbf{k}_1, \mathbf{k}_2$ . However, with some algebra we can simplify this problem for some eigenvalues  $\lambda$ , as detailed below.

Left-multiplying both sides of the previous equation by  $\begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \sqrt{\lambda}\mathbf{A}^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{I} - \mathbf{L} & c\mathbf{L} \\ \mathbf{O} & \mathbf{I} - \mathbf{L} \end{bmatrix}^{-1}$ , we obtain

$$\begin{aligned} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} &= \left( \begin{bmatrix} \mathbf{C}_\lambda & \mathbf{S}_\lambda \\ -\mathbf{S}_\lambda & \mathbf{C}_\lambda \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \sqrt{\lambda}\mathbf{A}^{-1/2} \end{bmatrix} \right. \\ &\times \begin{bmatrix} \mathbf{I} - \mathbf{L} & c\mathbf{L} \\ \mathbf{O} & \mathbf{I} - \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} - \mathbf{L}^T & -\frac{c}{\sqrt{\lambda}}\mathbf{A}^{1/2} \\ \mathbf{O} & (\mathbf{I} - \mathbf{L}^T)\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2} \end{bmatrix} \\ &\times \left. \begin{bmatrix} \mathbf{A}^{-1}\mathbf{k}_1 \\ \mathbf{A}^{-1}\mathbf{k}_2 \end{bmatrix} \right). \end{aligned} \quad (22)$$

It is easy to show that the second term above between parentheses simplifies to

$$\begin{bmatrix} \mathbf{e}\mathbf{v}^T - \mathbf{L}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{-1/2}(\mathbf{e}\mathbf{v}^T - \mathbf{L}^T)\mathbf{A}^{1/2} \end{bmatrix} - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{1/2}\mathbf{v} \end{bmatrix}^T,$$

where  $\mathbf{e} := (1, \dots, 1)^T$  is the first diagonal vector of  $\mathbb{R}^N$ ,  $\mathbf{v} := (1, 0, \dots, 0)^T$  is the first coordinate vector, and  $\mathbf{w} := (1, 2, \dots, N)^T$ . Equation (22) may thus be rewritten as

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \left( \mathbf{M}_\lambda - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{1/2}\mathbf{v} \end{bmatrix}^T \right) \begin{bmatrix} \mathbf{A}^{-1}\mathbf{k}_1 \\ \mathbf{A}^{-1}\mathbf{k}_2 \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{M}_\lambda &:= \begin{bmatrix} \mathbf{C}_\lambda & \mathbf{S}_\lambda \\ -\mathbf{S}_\lambda & \mathbf{C}_\lambda \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{e}\mathbf{v}^T - \mathbf{L}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{-1/2}(\mathbf{e}\mathbf{v}^T - \mathbf{L}^T)\mathbf{A}^{1/2} \end{bmatrix}. \end{aligned}$$

When  $\mathbf{M}_\lambda$  is invertible, the Sherman-Morrison formula (e.g. Golub and Loan 1996, p. 51) states that

$$\mathbf{M}_\lambda - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{1/2}\mathbf{v} \end{bmatrix}^T$$

is singular if and only if  $\lambda$  satisfies the scalar equation

$$1 - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{1/2}\mathbf{v} \end{bmatrix}^T \mathbf{M}_\lambda^{-1} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} = 0, \quad (23)$$

and in this case  $\mathbf{M}_\lambda^{-1} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix}$  is in the nullspace, giving a nontrivial solution

$$\begin{bmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{bmatrix} \mathbf{M}_\lambda^{-1} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix}.$$

In figure 7 we plot the top eigenfunction that we obtained by numerically solving equation (23) for  $N = 3$  and  $N = 10$  and then computing  $\mathbf{k}_1, \mathbf{k}_2$  as written above. As expected, the eigenfunctions are continuous and smooth. Note that there may be eigenvalues  $\lambda$  for which  $\mathbf{M}_\lambda$  is not invertible, in which case equation (23) cannot be relied upon.

## 9. Summary and conclusions

Integral equations and theory from functional analysis can be used to generalize the formula of Carr and Madan (1998) and allow from some improvements for the replication of European contingent claims. The results in this paper show how replication can be achieved with a discrete portfolio of special options forming an orthogonal eigensystem, which is easier to manage numerically when compared to a continuous portfolio of vanilla options. Our approach considers a general class of symmetric kernels, including the straddle and butterfly kernels, which have spectral decompositions allowing for replication with a discrete portfolio in the form of a series of special option payoffs. Truncation of this series provides a numerical method for fast pricing of vanilla options. An advantage of this method is that it may be possible to use a smaller number of special options rather than integral discretization schemes. For kernels having eigenfunctions given by the Fourier basis elements, the replicating series can take advantage of probability densities with explicit Fourier transforms. Overall, the results presented in this paper are part of a greater potential generalization, which in future work will extend to multivariate European payoffs and some families of exotic payoffs.

## Acknowledgements

The corresponding author thanks Stéphane Crépey and two anonymous referees for useful comments and suggestions, as well as Yucheng Wang for checking some results. All remaining errors are his.

## Disclosure statement

No potential conflict of interest was reported by the authors.

## References

- Bünger, F., Inverses, determinants, eigenvalues, and eigenvectors of real symmetric Toeplitz matrices with linearly increasing entries. *Linear Algebra Appl.*, 2014, **459**, 595–619.
- Balder, S. and Mahayni, A., Robust hedging with short-term options. *Wilmott Mag.*, 2006, **9**, 72–78.
- Black, F. and Scholes, M., The pricing of options and corporate liabilities. *J. Polit. Econ.*, 1973, **81**(3), 637.
- Breeden, D.T. and Litzenberger, R.H., Prices of state-contingent claims implicit in option prices. *J. Business*, 1978, **51**(4), 621–651.
- Carr, P. and Madan, D., Towards a theory of volatility trading. In *Volatility: New Estimation Techniques for Pricing Derivatives*, edited by R.A. Jarrow, Vol. 29, pp. 417–427, 1998 (Risk Books).
- Carr, P. and Wu, L., Static hedging of standard options. *J. Financ. Econometrics*, 2013, **12**(1), 3–46.
- Carton de Wiart, B. and Dempster, M.A.H., Wavelet optimized valuation of financial derivatives. *Int. J. Theor. Appl. Finance*, 2011, **14**, 1113–1137.
- Černý, A., *Mathematical Techniques in Finance. Tools for Incomplete Markets*, 2nd ed., 2016 (Princeton University Press: Princeton, NJ).
- Demeterfi, K., Derman, E., Kamal, M. and Zou, J., *More Than You Ever Wanted to Know About Volatility Swaps*. Goldman Sachs quantitative strategies research notes, 1999 (Goldman, Sachs & Co: New York).
- Derman, E., Ergener, D. and Kani, I., *Static Options Replication*. Goldman Sachs quantitative strategies research notes, 1994 (Goldman, Sachs & Co: New York).
- Di Tella, P., Haubold, M. and Keller-Ressel, M., Semistatic and sparse variance-optimal hedging. *Math. Finance*, forthcoming issue.
- Dupire, B., *Arbitrage Pricing With Stochastic Volatility*. Technical report, 1993 (Banque Paribas).
- Durrett, R., *Probability: Theory and Examples*. 5th ed, 2019.
- Eidelman, Y., Milman, V.D. and Tsolomitis, A., *Functional Analysis: An Introduction*, 2004 (American Mathematical Society: Providence, RI).
- Fang, F. and Oosterlee, C.W., A novel pricing method for European options based on Fourier-Cosine series expansions. *SIAM. J. Sci. Comput.*, 2009, **31**(2), 826–848.
- Fredholm, I., Sur une classe d'équations fonctionnelles. *Acta Mathematica*, 1903, **27**(1), 365–390.
- Golub, G.H. and Loan, C.F.V., *Matrix Computations*, 1996 (Johns Hopkins University Press: Baltimore, MD).
- Heston, S.L., A closed-Form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.*, 1993, **6**(2), 327–343.
- Hilbert, D., Über eine Anwendung der Integralgleichungen auf ein Problem der Funktionentheorie. In *Verhandlungen des dritten internationalen Mathematiker Kongresses*, edited by A. Krazer, pp. 233–240, 1904 (Leipzig: Teubner).
- Kress, R., *Linear Integral Equations*, 3rd ed. 2014 (Springer Verlag: New York).
- Lax, P.D., *Functional Analysis*, 2002 (Wiley-Interscience: New York).
- Madan, D. and Milne, F., Contingent claims valued and hedged by pricing and investing in a basis. *Math. Finance*, 1994, **4**(3), 223–245.
- Merton, R.C., Option pricing when underlying stock returns are discontinuous. *J. Financ. Econ.*, 1976, **3**, 125–144.
- Neuberger, A., Volatility trading. Working paper, London Business School, 1990.
- Papanicolaou, A., Consistent inter-model specification for time-homogeneous SPX stochastic volatility and VIX market models. Working paper, arXiv.org, 2018.
- Polyanin, A.D. and Manzhirov, A.V., *Handbook of Integral Equations*, 2nd ed., 2008 (Chapman and Hall/CRC: Boca Raton, FL).
- Riesz, F., Über lineare funktionalgleichungen. *Acta Math.*, 1916, **41**(1), 71–98.
- Schmidt, E., Zur Theorie der linearen und nichtlinearen Integralgleichungen. *Math. Ann.*, 1907, **63**(4), 433–476.
- Smith, G.D., *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, 3rd ed., 1985 (Clarendon Press/Oxford University Press: Oxford).
- Srivastava, H.M. and Buschman, R.G., *Theory and Applications of Convolution Integral Equations*, Rev., enl., updated ed. of *Convolution Integral Equations, with Special Function Kernels*, edited by M. Hazewinkel, Vol. 79, p. c1977, 2013 (Springer Science & Business Media).
- The CBOE volatility index–VIX. Technical report, Chicago Board Options Exchange, 2009.
- Volterra, V., Sulla inversione degli integrali definiti. In *Rendiconti della Classe di Scienze Fisiche, Matematiche e Naturali*, Vol. 5, pp. 177–185, 289–300, 1896 (Accademia Nazionale dei Lincei).
- Wu, L. and Zhu, J., Simple robust hedging with nearby contracts. *J. Financ. Econom.*, 2017, **15**(1), 1–35.
- Zagier, D., The dilogarithm function. In: *Frontiers in Number Theory, Physics, and Geometry II: On Conformal Field Theories, Discrete Groups and Renormalization*, edited by P.M. Pierre Cartier, B. Julia, and P. Vanhove. pp. 3–65, 2007 (Springer: Berlin, Heidelberg).

### Appendix. Derivation of proxy formula for the difference between the truncated Fourier series and spectroreplicant methods

Keeping the notations of section 5.1 with  $[a, b] = [0, 1]$ , a double integration by parts on  $\langle F, \psi_n \rangle$ ,  $n \geq 1$  yields

$$\langle F, \psi_n \rangle = -\frac{1}{(n\pi)^2} \langle F'', \psi_n \rangle - \frac{1}{(n\pi)^2} F'(0) + \frac{(-1)^n}{(n\pi)^2} F'(1),$$

so that

$$\begin{aligned} \check{F}_n(x) &= \langle F, \psi_0 \rangle - 2 \sum_{k=1}^{n-1} \frac{\langle F'', \psi_k \rangle}{(k\pi)^2} \psi_k(x) \\ &\quad + 2 \sum_{k=1}^{n-1} \left[ \frac{(-1)^k}{(k\pi)^2} F'(1) - \frac{F'(0)}{(k\pi)^2} \right] \psi_k(x). \end{aligned}$$

The above decomposition gives us an insight into the type of integral equation solved by a Fourier series. Specifically, we may reconstruct the corresponding integral kernel at order  $n \geq 1$  as

$$K_n(x, y) := - \sum_{k=1}^{n-1} \frac{1}{(k\pi)^2} [\sqrt{2}\psi_k(x)][\sqrt{2}\psi_k(y)],$$

so that

$$\begin{aligned} \check{F}_n(x) &= \langle F, \psi_0 \rangle + \int_0^1 K_n(x, y) F''(y) dy \\ &\quad + 2 \sum_{k=1}^{n-1} \left[ \frac{(-1)^k}{(k\pi)^2} F'(1) - \frac{F'(0)}{(k\pi)^2} \right] \psi_k(x). \end{aligned}$$

As  $n \rightarrow \infty$ , the Fourier expansion kernel  $K_n(x, y)$  converges pointwise to

$$K(x, y) := - \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} [\sqrt{2}\psi_k(x)][\sqrt{2}\psi_k(y)]$$

$$= -\frac{1}{2\pi^2} \left[ \text{Li}_2\left(e^{i\pi(x+y)}\right) + \text{Li}_2\left(e^{-i\pi(x+y)}\right) \right. \\ \left. + \text{Li}_2\left(e^{i\pi(x-y)}\right) + \text{Li}_2\left(e^{-i\pi(x-y)}\right) \right],$$

where  $\text{Li}_2(z) := \sum_{k=1}^{\infty} z^k/k^2$  is the complex dilogarithm function. From the identity  $\text{Li}_2(z) + \text{Li}_2(z^{-1}) = -(\pi^2/6) - \frac{1}{2} \ln^2(-z)$  (e.g. Zagier 2007, p.8) we further obtain

$$K(x, y) = \frac{1}{6} + \frac{1}{4\pi^2} \left[ \ln^2\left(-e^{i\pi(x+y)}\right) + \ln^2\left(-e^{i\pi(x-y)}\right) \right] \\ = \frac{1}{6} - \frac{1}{4} \left[ (x+y-1)^2 + \min\left((x-y-1)^2, (x-y+1)^2\right) \right] \\ = -\frac{1}{3} - \frac{1}{2} \left( x^2 - 2 \max(x, y) + y^2 \right) \\ = \frac{|x-y|}{2} - \left( \frac{1}{3} + \frac{x^2 - (x+y) + y^2}{2} \right).$$

The above identity establishes that the straddle kernel  $G(x, y) := |x - y|$  is embedded within the Fourier expansion kernel together with a residual kernel:

$$K \equiv \frac{1}{2}G + R, \quad R(x, y) := -\left( \frac{1}{3} + \frac{x^2 - (x+y) + y^2}{2} \right).$$

Writing  $K_n = \frac{1}{2}G_n + (K_n - \frac{1}{2}G_n)$  where  $G_n(x, y) := \sum_{k=0}^{n-1} \lambda_k \phi_k(x)\phi_k(y)$  is the truncated straddle kernel at order  $n$ , and noting that  $\int_0^1 G_n(x, y)(F''(y)/2) dy = \hat{F}_n(x) - c - qx$ , we have

$$\check{F}_n(x) = \hat{F}_n(x) + (\langle F, \psi_0 \rangle - c - qx) \\ + \int_0^1 \left[ K_n(x, y) - \frac{1}{2}G_n(x, y) \right] F''(y) dy \\ + 2 \sum_{k=1}^{n-1} \left[ \frac{(-1)^k}{(k\pi)^2} F'(1) - \frac{F'(0)}{(k\pi)^2} \right] \psi_k(x),$$

with  $K_n - \frac{1}{2}G_n \xrightarrow[n \rightarrow \infty]{} R$  pointwise, so that, for large  $n$ ,

$$\check{F}_n(x) - \hat{F}_n(x) \approx (\langle F, \psi_0 \rangle - c - qx) \\ + \int_0^1 R(x, y)F''(y) dy \\ + 2 \sum_{k=1}^{n-1} \left[ \frac{(-1)^k}{(k\pi)^2} F'(1) - \frac{F'(0)}{(k\pi)^2} \right] \psi_k(x).$$



## **Chapter 3**

# **“Static replication of European standard dispersion options”, Quantitative Finance, forthcoming**

Note: This version was accepted for publication conditionally upon minor revisions.

# STATIC REPLICATION OF EUROPEAN STANDARD DISPERSION OPTIONS

SÉBASTIEN BOSSU\*, PETER CARR†, AND ANDREW PAPANICOLAOU‡

ABSTRACT. The replication of any European contingent claim by a static portfolio of calls and puts with strikes forming a continuum, formally proven by Carr and Madan (1998), extends to “standard dispersion” options written on the Euclidean norm of a vector of  $n$  asset performances. With the help of integral equation techniques we derive replicating portfolios for calls, puts and indeed any claim contingent on standard dispersion using vanilla basket calls whose basket weights span an  $n$ -dimensional continuum. Consequently multi-asset standard dispersion options admit a model-free price enforced by arbitrage, just as single-asset European claims do.

## 1. INTRODUCTION

Over the past few decades, an array of derivative instruments and trading strategies have appeared where the payoff is based on some measure of statistical dispersion of one or more underlying assets. In the single-asset category, realized volatility and variance swaps appeared in the 1990s, then VIX futures and options in the 2000s as well as other volatility-related exotic options. In the multi-asset category, examples include vanilla price dispersion trades, realized variance dispersion trades, correlation swaps, or call and put options written on cross-sectional price dispersion<sup>1</sup> as illustrated in figure 1. Significant market activity for dispersion instruments can be observed in annual reports of many large quantitative hedge funds<sup>2</sup>. Accurate pricing and hedging of these instruments is notoriously more complex compared to other multi-asset options such as basket options (e.g. Brigo et al., 2004) or worst-of and best-of options.

In our preceding publication (2021) we considered the inverse problem of replicating a single-asset European option with cash, the asset and a “continuous portfolio” of arbitrary “replicant” options indexed by a single real variable such as a strike price. In this paper we extend our framework to the multi-asset class of “standard dispersion” options written on the Euclidean norm of a vector of  $n$  asset performances, which we seek to replicate with cash and a continuous portfolio of replicant basket calls indexed along  $n$  real variables corresponding to basket levers or weights.

Specifically, given a target payoff function  $F(s)$  written on standard dispersion  $s := \sqrt{\sum_i x_i^2}$  of  $n \geq 2$  asset performances  $x_1, x_2, \dots, x_n$ , we wish to find quantities  $\varphi(y_1, y_2, \dots, y_n)$  of vanilla basket

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*Date:* October 15, 2021.

\* NYU Courant. This corresponding author thanks Hao Lu and Yucheng Wang for their help with Appendices B and C. All remaining errors are his.

† NYU Tandon. ‡ North Carolina State University.

<sup>1</sup>In the financial industry, price dispersion is more commonly defined as mean absolute deviation corresponding to the “taxicab”  $\ell_1$ -norm, whereas our approach is based on the Euclidean  $\ell_2$ -norm for ease of mathematics. We do not discuss to what extent our  $\ell_2$  approach may approximate  $\ell_1$  instruments because an exact replication of the latter will be derived in a follow-up paper using different mathematical methods.

<sup>2</sup>For example: Infinity Q Alpha Fund SEC Form N-CSR 31 Aug. 2020, pp. 5, 8–, Assenagon Alpha Annual Report 31 Jan. 2020, p. 7.

FIGURE 1. Sample terms of an industry dispersion option. Source: large investment bank.

Interactive Bankers, N.A.

“Bankers you can talk to”

[Sample Term sheet](#)

### 3-year Dispersion Warrant on five shares in USD quanto

The following product is a warrant where the investor receives a Bonus linked to the performance of five stocks compared to the basket minus a Strike Level. The product has no capital protection at any time and there can be a partial or total loss of any capital invested. Investment is therefore highly speculative and should only be considered by investors who can afford to lose their entire investment amount.

<b>Issuer &amp; Guarantor</b>	Interactive Bankers, N.A. (credit rating Aa3, unsecured)																														
<b>Issue Type</b>	Warrant																														
<b>Issue Amount</b>	USD 3,000,000																														
<b>Number of Warrants</b>	3,000																														
<b>Notional Amount per Warrant (N)</b>	1 Warrant = USD 1,000																														
<b>Settlement Currency</b>	USD quanto																														
<b>Issue Price per Warrant</b>	USD 60																														
<b>Listing</b>	None																														
<b>Trade Date (T)</b>	[today]																														
<b>Strike Date</b>	T																														
<b>Issue Date</b>	T + 5 days																														
<b>Redemption Date</b>	T + 3 years																														
<b>Underlying Shares</b>	<table border="1"> <thead> <tr> <th>i</th> <th>Name</th> <th>Ticker</th> <th>Share<sup>i</sup><sub>initial</sub></th> <th>Weight w<sub>i</sub></th> </tr> </thead> <tbody> <tr> <td>1</td> <td>Apple</td> <td>AAPL</td> <td>[114]</td> <td>20%</td> </tr> <tr> <td>2</td> <td>Microsoft</td> <td>MSFT</td> <td>[210]</td> <td>20%</td> </tr> <tr> <td>3</td> <td>Airbus</td> <td>AIR</td> <td>[64]</td> <td>20%</td> </tr> <tr> <td>4</td> <td>Yamaha</td> <td>7951</td> <td>[5000]</td> <td>20%</td> </tr> <tr> <td>5</td> <td>Beyond Meat</td> <td>BYND</td> <td>[170]</td> <td>20%</td> </tr> </tbody> </table>	i	Name	Ticker	Share <sup>i</sup> <sub>initial</sub>	Weight w <sub>i</sub>	1	Apple	AAPL	[114]	20%	2	Microsoft	MSFT	[210]	20%	3	Airbus	AIR	[64]	20%	4	Yamaha	7951	[5000]	20%	5	Beyond Meat	BYND	[170]	20%
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<b>Settlement Amount</b>	On the Redemption Date, the Issuer will pay to the holder the following amount in U.S. dollars:																														
<b>Where</b>	$\text{Bonus} = \max(0\%, \text{Dispersion} - \text{Strike})$ $\text{Dispersion} = \sum_{i=1}^5 w_i \times \text{abs} \left( \frac{\text{Share}_{\text{final}}^i}{\text{Share}_{\text{initial}}^i} - \text{Basket}_{\text{final}} \right)$																														
<b>With</b>	Strike = 20% Basket <sub>initial</sub> = 1 $\text{Basket}_{\text{final}} = \sum_{i=1}^5 w_i \times \frac{\text{Share}_{\text{initial}}^i}{\text{Share}_{\text{initial}}^i}$ Share <sup>i</sup> <sub>initial</sub> with i from 1 to 5 is the official closing price of Underlying Share i on the Strike Date Share <sup>i</sup> <sub>final</sub> with i from 1 to 5 is the official closing price of Underlying Share i on the Redemption Date																														
<b>Business Day Convention</b>	Following Business Day																														
<b>Governing law</b>	U.S. law																														

calls<sup>3</sup> across all possible basket weights  $y_1, y_2, \dots, y_n$  that replicate the target payoff up to a fixed amount of cash  $c$ :

$$F\left(\sqrt{\sum_{i=1}^n x_i^2}\right) = c + \int \cdots \int \left(\sum_{i=1}^n x_i y_i - k\right)^+ \varphi(y_1, \dots, y_n) dy_1 \cdots dy_n,$$

where  $k > 0$  is a fixed moneyness parameter,  $t^+ := \max(0, t)$  denotes the positive part of a real number  $t$  and  $\int \cdots \int$  denotes a multiple integral over a suitable domain. For maximum generality we let all our variables  $x_i, y_i$  be positive or negative real numbers and we leave the definition of asset performance unspecified with the important caveat that the replicant option payoffs  $(\sum_i x_i y_i - k)^+$  are defined accordingly. A typical definition would be the gross returns to maturity or the price ratios of  $n$  underlying assets<sup>4</sup>.

Switching to vector notations, in the language of functional analysis<sup>5</sup> we want to solve the multidimensional integral equation of the first kind

$$F(|\mathbf{x}|) - c = \int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n, \quad (1)$$

for the unknown function  $\varphi(\mathbf{y})$  and constant  $c$ . Here,  $\mathbf{x} \cdot \mathbf{y} := \sum_i x_i y_i$  denotes the canonical dot product of Euclidean space  $\mathbb{R}^n$  with associated norm  $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$ , and  $(\mathbf{x} \cdot \mathbf{y} - k)^+$  is the known integral kernel. This inverse problem is mathematically nontrivial and may be viewed as a multidimensional generalization of the Breeden and Litzenberger (1978) and Carr and Madan (1998) inverse problems for a particular class of sophisticated, multi-asset options.

**1.1. Background and review.** Growing interest from practitioners and academics alike in dispersion trading strategies and options can be observed in e.g. Bossu (2005), Jacquier and Slaoui (2007), Driessen, Maenhout, and Vilkov (2009), Bouzoubaa and Osseiran (2010), Bossu (2014), Schofield (2017, pp. 475–477). Evidence of research interest in static option replication strategies can be found in the work of Dupire (1993), Derman, Ergener, and Kani (1994), Pelsser (2003), Baldeaux and Rutkowski (2010), to name just a few.

Interest in static replication is justified by the resulting model-free price for the target instrument, even if actual arbitrage enforcement could be difficult to implement due to liquidity and transaction cost issues. The most successful illustration of this approach is the decomposition of the log-contract into a continuous portfolio of out-of-the-money calls and puts on the S&P 500 index, whose discretization underpins the calculation of the VIX (see *The CBOE volatility index—VIX 2009*, for details). While the majority of such calls and puts are illiquid and do not trade very often, the VIX is widely regarded as an excellent, model-free gauge of aggregate implied volatility and estimate of the fair price of a variance swap.

Further practical motivations for decomposing a dispersion option as a sum of basket calls may include technical limitations of risk systems which are often designed to work with simpler instruments, in which case having a payoff equivalence can save a lot of time and reprogramming

<sup>3</sup>This includes basket call options on all  $n$  underlying assets, as well as any subset: single-asset calls (case where all weights  $y_i$  but one are zero), two-asset calls (case where all weights  $y_i$  but two are zero), and so forth.

<sup>4</sup>Another possible definition of asset performances could be the time series of  $n$  daily returns  $x_1, \dots, x_n$  with respect to a single asset, in which case  $s$  would be the asset's realized volatility. However, the corresponding replicant options would then be based on various weighted sums of daily returns resembling *cliquet options* which are less compelling than basket calls in terms of practical applications.

<sup>5</sup>We refer to our 2021 publication for further background in functional analysis.

costs; as well as specific hedging needs of large derivatives issuers to offload excess covariance or correlation risk accumulated by selling simpler multi-asset options<sup>6</sup>.

In other related literature, Baxter (1998, p. 13) mentions a generalization of the Breeden and Litzenberger formula to a vector of assets in  $\mathbb{R}^n$  based on Fourier transforms, while Lipton (2001, pp. 291–292) proposes a generalization of the Carr and Madan formula for two assets using Radon transforms. Expanding on the latter approach, Carr and Laurence (2011) derive a multi-asset version of the Dupire (1993) local volatility formula, while Austing (2011) uses standard calculus tools to replicate basket options using best-of and worst-of options. Recently, Pötz (2020) investigates efficient basket option pricing with Chebyshev quadrature techniques, and Cui and Xu (2021) derive a multi-asset extension of the Carr and Madan formula as multiple integrals of products of call options.

**1.2. Results and organization of this paper.** Our main contribution is to establish that any standard dispersion option with sufficiently regular payoff is replicated by a continuous portfolio of vanilla basket calls, and consequently admits a model-free arbitrage price so long as the prices of basket call options of arbitrary basket weights are known. We also provide closed-form solutions to replicate the dispersion call, zero-strike dispersion call, and dispersion put. To achieve this result, we relied on a fair amount of technical machinery presented in Appendix A, leveraging on existing fractional calculus techniques in relation to Radon transforms which we adapted to our needs. In addition, we overcame a substantial mathematical limitation that the payoff function satisfy  $F'(0) = 0$  by isolating the first-order term which we proved to be replicable with zero-strike basket calls.

The remainder of our paper is organized as follows: In section 2 we discuss the concept of constrained and unconstrained continuous portfolios of vanilla basket calls. In sections 3 and 4 we derive solutions for the replication of standard dispersion calls and puts. In section 5 we extend our results to arbitrary target payoff functions. In section 6 we consider a numerically tractable application for the “Mexican hat” dispersion straddle. In section 7 we show how the dispersion call decomposition may be expanded as continuous portfolios of various basket securities in finite quantities, before discussing the consequences of our results for the pricing of dispersion options in our concluding section 8.

## 2. CONTINUOUS PORTFOLIOS OF VANILLA BASKET CALLS

In this opening section, we discuss the financial interpretation of the multiple integral  $\int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) \, d\mathbf{y}$  to the right-hand side of integral equation (1) as a continuous portfolio of vanilla basket calls indexed by basket weights  $\mathbf{y}$ . For maximum generality the basket weights  $\mathbf{y}$  in equation (1) are unconstrained, contrary to industry practice where they typically sum to 1. As a result, the moneyness control parameter  $k$  is *not* interpreted as a direct strike price. When the sum of weights is positive, correspondence is easily obtained by simple standardization:

$$(\mathbf{x} \cdot \mathbf{y} - k)^+ = \left( \sum_{i=1}^n y_i \right) \left( \mathbf{x} \cdot \frac{\mathbf{y}}{\sum_i y_i} - \frac{k}{\sum_i y_i} \right)^+, \quad \sum_i y_i > 0,$$

which is a quantity  $\sum_i y_i$  of basket calls with standardized basket weights  $\mathbf{y}/\sum_i y_i$  summing to 1 and strike price  $k/\sum_i y_i$ . Another consequence of letting basket weights unconstrained within

<sup>6</sup>An interesting price property of dispersion calls and puts is that they are short correlation instruments which help issuers reduce their correlation risk exposure.

$\mathbb{R}^n$  is that some weights may be negative, resulting in a long-short basket<sup>7</sup> which is uncommon in the derivatives industry. However, a long-short basket may be viewed as a “spread” between two long-only baskets:

$$(\mathbf{x} \cdot \mathbf{y} - k)^+ = \left( \sum_{y_i > 0} x_i y_i - \sum_{y_i < 0} x_i |y_i| - k \right)^+,$$

in which case  $k$  is interpreted as a “residual” strike price. Such call and put options on the performance spread between two assets, also known *outperformance options*, are well understood by practitioners. Again, weights may be standardized to sum to 1 within each basket for better correspondence with industry practice.

It is possible to introduce constraints on basket weights as alternative formulations of our replication problem (1), at the greater risk of finding no solution. For example, a long-only constraint can be expressed as an integral over  $\mathbb{R}_+^n$  rather than  $\mathbb{R}^n$ . More complex types of constraints  $\mathbf{y} \in \mathcal{S} \subseteq \mathbb{R}^n$ , such as weights summing to 1, are best expressed as a *surface integral*

$$\int_{\mathcal{S}} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) \, d\mathbf{y},$$

where  $d\mathbf{y}$  now denotes the infinitesimal change in surface area. Two particularly important types of constraints encountered in this paper are

- *Unit sum of weights* corresponding to the hyperplane  $\mathcal{S} := \{\mathbf{y} \in \mathbb{R}^n : \sum_i y_i = 1\}$  with surface integral

$$\int_{\sum_i y_i = 1} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) \, d\mathbf{y} = \int_{\mathbf{y} \cdot \mathbf{e} = 1} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) \, d\mathbf{y} = \int_{\mathbb{R}^n} \delta(\mathbf{y} \cdot \mathbf{e} - 1) (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) \, d\mathbf{y},$$

where  $\mathbf{e} := (1, \dots, 1)$  is the first diagonal vector of  $\mathbb{R}^n$  and  $\delta$  is Dirac’s delta function. This type of surface integral is known as a *Radon transform* and may be financially interpreted as a *standardized continuous portfolio* of basket calls.

- *Unit sum of squares of weights*:  $\sum_i y_i^2 = 1$ , or  $|\mathbf{y}| = 1$  in vector notation. While this type of constraint is uncommon in the industry, it is well known to mathematicians as a surface integral over the unit hypersphere<sup>8</sup>

$$\int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u} - k)^+ \varphi(\mathbf{u}) \, d\mathbf{u} = \int_{\mathbb{S}^{n-1}} (\mathbf{x} \cdot \mathbf{u} - k)^+ \varphi(\mathbf{u}) \, d\mathbf{u},$$

where we use the letter  $\mathbf{u}$  instead of  $\mathbf{y}$  to emphasize it is a unit vector. From a quantitative finance perspective we may name the above a *normalized continuous portfolio* of basket calls.

<sup>7</sup>Short-only basket calls are also possible and better interpreted as long-only basket puts with negative moneyness parameter  $-k$ .

<sup>8</sup>The  $n$ -dimensional unit hypersphere, or simply unit sphere, is an object of algebraic dimension  $n$  as subset of vector space  $\mathbb{R}^n$ . In the academic literature, it is often denoted  $\mathbb{S}^{n-1}$  in reference to its *geometric* dimension  $n - 1$  as easily visualized for  $n = 2$  or  $3$ . Unlike the  $n$  exponent in  $\mathbb{R}^n$  which denotes the Cartesian product  $\mathbb{R} \times \dots \times \mathbb{R}$ , the  $n - 1$  superscript in  $\mathbb{S}^{n-1}$  does not appear to have a particular meaning and some authors indeed prefer the subscript notation  $\mathbb{S}_{n-1}$ .

## 3. REPLICATION OF STANDARD DISPERSION CALLS

Throughout this paper we handle the parameter  $k$ , which appears to the right-hand side of integral equation (1) but not to the left-hand side, as a *constant parameter* susceptible to appear in the solution  $\varphi(\mathbf{y})$ , which we denote  $\varphi(\mathbf{y}; k)$  every so often to emphasize its parametric dependence. In contrast, other authors tend to assume that  $\varphi$  is independent from  $k$ , which is more restrictive. We begin with a pair of technical definitions about the regularity of the payoff function  $F$  that can be replicated under our theory, and the regularity of the solution  $\varphi$ :

- Definition.** (D1) A payoff function  $F(s)$  is *sufficiently regular* if it is twice continuously differentiable at the origin and the associated function  $\phi(r)$  defined in formula (A.7) exists as a regular generalized function<sup>9</sup> with  $\phi(r) = O(r^3)$  as  $r \rightarrow 0$ .
- (D2) A solution  $\varphi(\mathbf{y})$  to the replication problem (1) is *regular* if there is a regular generalized function  $\phi$  such that  $\varphi(\mathbf{y}; k) = \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$  and  $\phi(r) = O(r^3)$  as  $r \rightarrow 0$ . Otherwise it is *singular*.

*Remark.* In odd dimension  $n$ , if  $F$  is bounded and piecewise-differentiable of order  $(n+3)/2$ , then  $\phi(r)$  exists as a regular generalized function.

Based on these definitions, the standard dispersion call with payoff  $F(|\mathbf{x}|) := (|\mathbf{x}| - K)^+$ ,  $K > 0$  qualifies as *sufficiently regular* and satisfies  $F'(0) = 0$ . Following proposition A.4, the replication problem (1) has a *regular solution*, as outlined below:

**Proposition 1.** The standard dispersion call is replicable with vanilla basket calls as per equation (1) with  $c = 0$  and

$$\varphi_C\left(\mathbf{y}; \frac{k}{K}\right) = \begin{cases} \frac{2}{\pi^{(n-1)/2}} \delta\left(\frac{n-1}{2}\right) \left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right), & n \text{ odd,} \\ (-1)^{n/2} \frac{2 \Gamma\left(\frac{n-1}{2}\right)}{\pi^{(n+1)/2}} \frac{H\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)}{\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)^{(n+1)/2}} \mathfrak{P}, & n \text{ even,} \end{cases} \quad (2)$$

where  $\delta$  is Dirac's delta function,  $H$  is Heaviside's step function,  $\Gamma$  is Euler's gamma function, and the pilcrow symbol  $\mathfrak{P}$  indicates a *pseudofunction* subject to Hadamard regularization (Kanwal, 2004, pp. 71–74).

*Remark.* The solution vanishes as  $K \rightarrow 0$  and thus cannot be used to replicate the zero-strike dispersion call with payoff  $|\mathbf{x}|$ , as predicted by proposition A.4.

<sup>9</sup>(See e.g. Kanwal, 2004, p.22) for a definition

*Proof.* Substituting  $F''(s) = \delta(s - K)$  into equation (A.7), then sifting and simplifying,

$$\begin{aligned} \phi_{\mathbb{C}}(r) &= \begin{cases} \frac{1}{\pi^{(n-1)/2}} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} r^n \delta(r - K), & n \text{ odd,} \\ \frac{2}{\pi^{n/2}} \left( \frac{d}{dr^2} \right)^{n/2} \int_0^r \frac{s^{n+1} \delta(s - K)}{\sqrt{r^2 - s^2}} ds, & n \text{ even;} \end{cases} \\ &= \begin{cases} \frac{K^n}{\pi^{(n-1)/2}} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} \delta(r - K), & n \text{ odd,} \\ \frac{2K^{n+1}}{\pi^{n/2}} \left( \frac{d}{dr^2} \right)^{n/2} \frac{H(r - K)}{\sqrt{r^2 - K^2}}, & n \text{ even.} \end{cases} \end{aligned}$$

Substituting  $H(r - K) = H(r^2 - K^2)$  together with its chain rule version  $\delta(r - K) = 2K\delta(r^2 - K^2)$  into the above expression,

$$\begin{aligned} \phi_{\mathbb{C}}(r) &= \begin{cases} \frac{2K^{n+1}}{\pi^{(n-1)/2}} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} \delta(r^2 - K^2), & n \text{ odd,} \\ \frac{2K^{n+1}}{\pi^{n/2}} \left( \frac{d}{dr^2} \right)^{n/2} \frac{H(r^2 - K^2)}{\sqrt{r^2 - K^2}}, & n \text{ even;} \end{cases} \\ &= \begin{cases} \frac{2K^{n+1}}{\pi^{(n-1)/2}} \delta^{\left(\frac{n-1}{2}\right)}(r^2 - K^2), & n \text{ odd,} \\ \frac{2K^{n+1}}{\pi^{n/2}} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \cdots \left( -\frac{n-1}{2} \right) \frac{H(r^2 - K^2)}{(r^2 - K^2)^{(n+1)/2}} \mathbf{1}, & n \text{ even.} \end{cases} \end{aligned}$$

Substituting  $k \mapsto k/r$ , dividing both sides by  $r^{n+1}$ , homogenizing the delta function and simplifying yields expression (2) as stated for  $\varphi(\mathbf{y}; k) = \phi(k/|\mathbf{y}|)/|\mathbf{y}|^{n+1}$ .  $\square$

**Corollary.** By the chain rule for the derivative of the delta function (Kanwal, 2004, p. 50) we have in dimensions 2 and 3:

$$\varphi_{\mathbb{C}}\left(\mathbf{y}; \frac{k}{K}\right) = \begin{cases} \frac{2}{\pi} \delta' \left( \frac{k^2}{K^2} - |\mathbf{y}|^2 \right) = -\frac{K^3/k^3}{2\pi} \delta \left( |\mathbf{y}| - \frac{k}{K} \right) - \frac{K^2/k^2}{2\pi} \delta' \left( |\mathbf{y}| - \frac{k}{K} \right), & n = 3, \\ -\frac{2}{\pi} \frac{H\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)}{\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)^{3/2}} \mathbf{1}, & n = 2. \end{cases} \quad (3)$$

*Remark.* We may validate the solution for  $n = 3$  by inserting it into equation (A.10) together with  $c = F(0) = 0$ ; substituting  $r \mapsto \sqrt{r}$  and simplifying; then integrating by parts and sifting to obtain

$$\begin{aligned} F(|\mathbf{x}|) &= 2 \int_0^\infty r^3 \delta'(k^2/K^2 - r^2) \frac{(|\mathbf{x}| - k/r)^{+2}}{|\mathbf{x}|} dr \\ &= \int_0^\infty \delta'(k^2/K^2 - r) \frac{(|\mathbf{x}| \sqrt{r} - k)^{+2}}{|\mathbf{x}|} dr \\ &= \int_0^\infty \delta(k^2/K^2 - r) \frac{2(r|\mathbf{x}| - k)^+}{2\sqrt{r}} dr = (|\mathbf{x}| - K)^+, \end{aligned}$$

as required.

The solutions of equations (2) and (3) call for several comments. At first glance, the formulas in odd and even dimensions may look very different; however, a careful examination of the proof

will reveal that they only differ by a half integral. The true nature of the solution is thus best investigated in odd dimension. In addition, as esoteric as Dirac's delta functions might perhaps seem, their presence is hardly surprising: in dimension 1, the Carr-Madan solution to replicate a call-type payoff with vanilla calls, say  $F(s) := (s^2 - K)^+$ , would be  $F''(s) = 2H(s^2 - K) + 8K\delta(s^2 - K) + 4s^2(s^2 - K)\delta'(s^2 - K)$  which involves the delta function and its derivative. This is due to the piecewise nature of the payoff function, as is often the case in finance.

The delta function in the solution  $\varphi_C$  is activated over the hypersphere of radius  $k/K$ . This tells us that the basket weights spanned by the replicating continuous portfolio are constrained to be normalized, as defined in section 2. Interpreting derivatives of the delta function is more subtle. Informally, they may be represented using finite differences:  $\delta'(t) = \lim_{\varepsilon \rightarrow 0} \frac{\delta(t+\varepsilon) - \delta(t-\varepsilon)}{\varepsilon}$ ,  $\delta''(t) = \lim_{\varepsilon \rightarrow 0} \frac{\delta(t+\varepsilon) - 2\delta(t) + \delta(t-\varepsilon)}{\varepsilon^2}$ , and so forth. Under this representation,  $\delta'$  translates into infinitely leveraged call spreads,  $\delta''$  translates into infinitely leveraged butterfly spreads, etc. Formally, integration by parts may be used to expand the solution into finite quantities of basket calls, binary basket calls and higher-order derivatives of basket calls, as explained in section 7.

#### 4. REPLICATION OF STANDARD DISPERSION PUTS

As noted in the remark to proposition 1, the zero-strike standard dispersion call with payoff  $|\mathbf{x}|$  does not admit a regular solution. By put-call parity, this issue applies to standard dispersion puts as well. Fortunately, this limitation may be circumvented by including *zero-strike* basket calls with payoff  $(\mathbf{x} \cdot \mathbf{y})^+$  in the replicant kernel, in which case we have the decompositions given below.

**Proposition 2.** The zero-strike standard dispersion call is replicated with an equally weighted normalized portfolio of zero-strike basket calls, as follows:

$$|\mathbf{x}| = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ d\mathbf{u}. \quad (4)$$

*Proof.* By slice integration (Rubin, 2015, p. 29),

$$\int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ d\mathbf{u} = |\mathbb{S}^{n-2}| |\mathbf{x}| \int_{-1}^1 t^+ (1-t^2)^{(n-3)/2} dt,$$

where  $|\mathbb{S}^{n-2}| = 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$  is the surface area of the  $(n-1)$ -dimensional unit sphere. Solving the integral, simplifying and rearranging yields the identity as stated.  $\square$

**Corollary.** By put-call parity, the standard dispersion put with payoff  $F(|\mathbf{x}|) := (K - |\mathbf{x}|)^+$  is replicated by a combination of cash, a short normalized continuous portfolio of zero-strike basket calls replicating the zero-strike dispersion call, and a long continuous portfolio of basket calls replicating the standard dispersion call:

$$(K - |\mathbf{x}|)^+ = K - \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ d\mathbf{u} + \int_{\mathbb{R}^n} \varphi_C\left(\mathbf{y}; \frac{k}{K}\right) (\mathbf{x} \cdot \mathbf{y} - k)^+ d\mathbf{y}, \quad (5)$$

where  $\varphi_C(\mathbf{y}; k/K)$  is given by formula (2).

## 5. GENERAL REPLICATION OF STANDARD DISPERSION OPTIONS

Having established that standard dispersion calls and puts are replicable with vanilla basket calls, it follows from the Carr and Madan formula that any standard dispersion option with well-behaved payoff  $F(|\mathbf{x}|)$  is replicable as well:

$$F(|\mathbf{x}|) = F(s_0) + F'(s_0)(|\mathbf{x}| - s_0) + \int_0^{s_0} F''(K)(K - |\mathbf{x}|)^+ dK + \int_{s_0}^{\infty} F''(K)(|\mathbf{x}| - K)^+ dK,$$

where  $s_0 \geq 0$  is an arbitrary split level. The following pair of theorems gives the general solution to replication problem (1) for any sufficiently regular payoff function.

**Theorem 1** (GENERAL DECOMPOSITION). Any standard dispersion option paying off  $F(|\mathbf{x}|)$ , where  $F$  is a sufficiently regular payoff function, is replicated with a combination of cash, a normalized continuous portfolio of zero-strike basket calls, and a continuous portfolio of positive-strike basket calls, as follows:

$$F(|\mathbf{x}|) = F(0) + F'(0) \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ d\mathbf{u} + \int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}} d\mathbf{y}, \quad (6)$$

where

$$\phi(r) = \begin{cases} \frac{1}{\pi^{(n-1)/2}} \left(\frac{d}{dr^2}\right)^{\frac{n-1}{2}} r^n F''(r), & n \text{ odd,} \\ \frac{2}{\pi^{n/2}} \frac{d}{dr^2} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} \left(\frac{d}{ds^2}\right)^{\frac{n-2}{2}} [s^n F''(s)] ds, & n \text{ even.} \end{cases} \quad (7)$$

*Proof.* Let  $F_1(s) := F(s) - F'(0)s$ . Then  $F_1$  is sufficiently regular with  $F_1(0) = F(0)$ ,  $F_1'(0) = 0$ , and  $F_1''$  coincides with  $F''$ . By proposition A.4, a regular solution  $\varphi$  exists for  $F_1$ . By proposition A.2 this solution is given as  $\varphi(\mathbf{y}; k) = \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$ , and we have

$$F_1(|\mathbf{x}|) = F(0) + \int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}; k) d\mathbf{y}.$$

Substituting  $F_1(|\mathbf{x}|) := F(|\mathbf{x}|) - F'(0)|\mathbf{x}|$ , then equation (4) and rearranging yields the decomposition (6) as stated.  $\square$

Mathematically, including zero-strike basket calls in the replicant kernel is equivalent to extending the solution space to *singular solutions* (see definition (D2)):

**Theorem 2** (GENERAL SOLUTION). Any standard dispersion option with payoff  $F(|\mathbf{x}|)$ , where  $F$  is sufficiently regular, is replicated with vanilla basket calls as per equation (1) with

$$\begin{cases} c = F(0), \\ \varphi_{\mathbb{F}}(\mathbf{y}; k) = \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}} + F'(0) \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}, \end{cases} \quad (8)$$

where  $\phi(r)$  is given by formula (7).

*Remark.* The singular term in  $\frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}$  may be viewed as a corrective term to inversion formula (A.7) when allowing for singular solutions. When  $F'(0) = 0$ , both formulas coincide and the solution is regular, as for the standard dispersion call.

*Proof.* Plugging the proposed solution (8) into the right-hand side of equation (1) and splitting the integral,

$$\int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) \, d\mathbf{y} = F(0) + F'(0) \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \int_{\mathbb{R}^n} \frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}} \, d\mathbf{y} + \int_{\mathbb{R}^n} \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}} \, d\mathbf{y}. \quad (9)$$

Switching to cylindrical coordinates in the first integral, simplifying, and homogenizing the delta function ; substituting  $r \mapsto k/r$ ; then sifting,

$$\begin{aligned} \int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}} \, d\mathbf{y} &= \int_0^\infty \delta\left(\frac{k}{r}\right) \frac{k}{r^2} \, dr \int_{|\mathbf{u}|=1} \left(\mathbf{x} \cdot \mathbf{u} - \frac{k}{r}\right)^+ \, d\mathbf{u} \\ &= \int_0^\infty \delta(r) \, dr \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u} - r)^+ \, d\mathbf{u} \\ &= \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ \, d\mathbf{u} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} |\mathbf{x}|, \end{aligned}$$

where we used identity (4) in the last step. Substituting the above into equation (9) and then decomposition (6) yields  $F(|\mathbf{x}|)$  as required.  $\square$

**Corollary.** The replication problem (1) admits singular solutions for the following dispersion options:

- (a) For the zero-strike standard dispersion call with payoff  $F(|\mathbf{x}|) := |\mathbf{x}|$ ,

$$\varphi_C(\mathbf{y}; \infty) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}.$$

- (b) For the standard dispersion put with payoff  $F(|\mathbf{x}|) := (K - |\mathbf{x}|)^+$ ,

$$\begin{cases} c = K, \\ \varphi_P(\mathbf{y}; k/K) = \varphi_C(\mathbf{y}; k/K) - \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}, \end{cases}$$

where  $\varphi_C(\mathbf{y}; k/K)$  is given by formula (2).

## 6. NUMERICAL APPLICATION: REPLICATION OF THE “MEXICAN HAT” DISPERSION STRADDLE

The “Mexican hat” dispersion straddle option with payoff  $F(|\mathbf{x}|) := 1 - e^{-|\mathbf{x}|^2}$  is a good example of a continuous and bounded payoff function for which the replication problem has a “nice”, numerically tractable continuous solution. The payoff function  $F(s) := 1 - e^{-s^2}$  is clearly sufficiently regular and satisfies  $F(0) = F'(0) = 0$ . Substituting  $F''(s) = 2(1 - 2s^2)e^{-s^2}$  into formula (7),

$$\phi(r) = \begin{cases} \frac{2}{\pi^{(n-1)/2}} \left(\frac{d}{dr^2}\right)^{\frac{n-1}{2}} e^{-r^2} (r^n - 2r^{n+2}), & n \text{ odd,} \\ \frac{4}{\pi^{n/2}} \frac{d}{dr^2} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} \left(\frac{d}{ds^2}\right)^{\frac{n-2}{2}} \left[ e^{-s^2} (s^n - 2s^{n+2}) \right] ds, & n \text{ even.} \end{cases}$$

For ease of exposure we merely proceed with the cases  $n = 2, 3$  whereby

$$\phi(r) = \begin{cases} \frac{2}{\pi} \frac{d}{dr^2} e^{-r^2} (r^3 - 2r^5), & n = 3, \\ \frac{4}{\pi} \frac{d}{dr^2} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} e^{-s^2} (s^2 - 2s^4) ds, & n = 2. \end{cases}$$

As shown in Appendix C, the integral above solves to  $r + r^3 - (1 + r^2 + 2r^4)\mathfrak{D}(r)$  wherein  $\mathfrak{D}(r) := e^{-r^2} \int_0^r e^{t^2} dt$  is Dawson's function. Substituting this expression together with  $\frac{d}{dr^2} = \frac{1}{2r} \frac{d}{dr}$  into the above equation,

$$\begin{aligned} \phi(r) &= \begin{cases} \frac{1}{r\pi} \frac{d}{dr} e^{-r^2} (r^3 - 2r^5), & n = 3, \\ \frac{4}{r\pi} \frac{d}{dr} [r + r^3 - (1 + r^2 + 2r^4)\mathfrak{D}(r)] & n = 2; \end{cases} \\ &= \begin{cases} \frac{r}{\pi} (3 - 12r^2 + 4r^4) e^{-r^2}, & n = 3, \\ \frac{4}{\pi} [r - r^3 + (2r^4 - 3r^2)\mathfrak{D}(r)], & n = 2. \end{cases} \end{aligned}$$

The solution  $\varphi$  to replication problem (1) is thus

$$\varphi(\mathbf{y}; k) = \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}} = \begin{cases} \frac{k}{\pi|\mathbf{y}|^5} \left( 3 - \frac{12k^2}{|\mathbf{y}|^2} + \frac{4k^4}{|\mathbf{y}|^4} \right) e^{-k^2/|\mathbf{y}|^2}, & n = 3, \\ \frac{4k}{\pi|\mathbf{y}|^4} \left( 1 - \frac{k^2}{|\mathbf{y}|^2} \right) - \frac{8k^2}{\pi|\mathbf{y}|^5} \left( \frac{3}{2} - \frac{k^2}{|\mathbf{y}|^2} \right) \mathfrak{D} \left( \frac{k}{|\mathbf{y}|} \right), & n = 2. \end{cases}$$

Figure 2 shows the payoff  $F$  and its replicating solution  $\phi$  for  $n = 2$  assets. For  $n = 3$ , the solution may be verified by inserting it into equation (A.10) to obtain

$$F(|\mathbf{x}|) = c + \int_0^\infty \frac{k}{r^2} \left( 3 - \frac{12k^2}{r^2} + \frac{4k^4}{r^4} \right) e^{-k^2/r^2} \frac{(|\mathbf{x}| - k/r)^{+2}}{|\mathbf{x}|} dr,$$

which solves to the target payoff  $1 - e^{-|\mathbf{x}|^2}$  as required after substituting  $r \mapsto k/r$  and  $c = F(0) = 0$ .

## 7. THEORETICAL APPLICATION: TRACTABLE EXPANSION OF THE DISPERSION CALL DECOMPOSITION

Solution formula (2) for replicating a dispersion call is mathematically correct but it involves generalized functions that present a singularity at  $|\mathbf{y}| = k/K$  implying infinite quantities of basket calls to buy or sell. This would typically not be an issue in theoretical pricing applications thanks to the dampening effect of the expectation operator, but it is an issue for discretization, numerical integration, and of course trading. There is a well-known parallel in the single-asset case whereby a binary option with payoff  $F(x) = H(x - K)$  may be represented as the limit-case of a levered call spread with strikes  $K - \varepsilon$  and  $K$ :

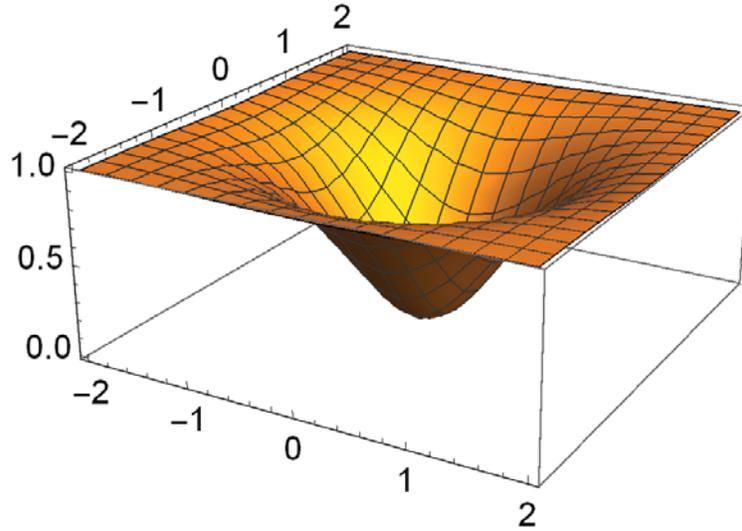
$$H(x - K) = \frac{d}{dx} (x - K)^+ = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [(x + \varepsilon - K)^+ - (x - K)^+].$$

An equivalent mathematical representation of the above is

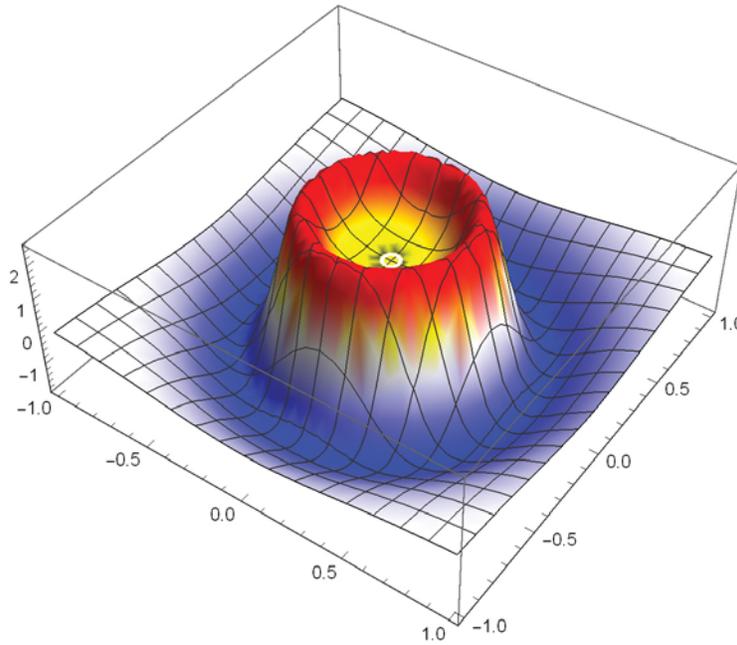
$$H(x - K) = \int_0^\infty \delta(\kappa - K) d(x - \kappa)^+ = \int_0^\infty \delta'(\kappa - K) (x - \kappa)^+ d\kappa,$$

FIGURE 2. Two-asset “Mexican hat” straddle dispersion payoff  $F(x_1, x_2) = 1 - e^{-x_1^2 - x_2^2}$  and its replicating solution  $\phi(y_1, y_2)$  as quantity of basket calls with monyness parameter  $k = 1$ .

2A. Payoff function



2B. Replicating solution



where the second expression stems from integration by parts and is consistent with the Carr and Madan formula at origin. Similarly, standard dispersion calls may be replicated with *ad hoc* continuous portfolios of vanilla basket calls, binary basket calls and so forth in finite quantities, as discussed below.

**7.1. Odd dimension.** We begin with the case  $n = 3$  before discussing the general case. Substituting solution (3) into equation (A.9) and switching the order of integration, then sifting, we get

$$\begin{aligned} (|\mathbf{x}| - K)^+ &= -\frac{K^2}{2\pi k^2} \int_{|\mathbf{u}|=1} \int_0^\infty r^2 \left[ \frac{K}{k} \delta\left(r - \frac{k}{K}\right) + \delta'\left(r - \frac{k}{K}\right) \right] (r\mathbf{x} \cdot \mathbf{u} - k)^+ dr d\mathbf{u} \\ &= -\frac{K^2}{2\pi k^2} \int_{|\mathbf{u}|=1} \left[ \frac{k}{K} \left(\frac{k}{K}\mathbf{x} \cdot \mathbf{u} - k\right)^+ + \int_0^\infty r^2 \delta'\left(r - \frac{k}{K}\right) (r\mathbf{x} \cdot \mathbf{u} - k)^+ dr \right] d\mathbf{u} \\ &= \int_{|\mathbf{u}|=1} \left[ \frac{1}{\pi} (\mathbf{x} \cdot \mathbf{u} - K)^+ - \frac{K}{2\pi} H(\mathbf{x} \cdot \mathbf{u} - K) \right] d\mathbf{u}, \quad n = 3, \end{aligned}$$

where we integrated by parts, sifted and simplified terms in the last step. Thus, in dimension  $n = 3$ , the standard dispersion call option with dispersion strike  $K$  is replicated by a normalized continuous portfolio of long basket calls in quantity  $1/\pi$  and short binary basket calls in quantity  $K/2\pi$ , with fixed moneyness parameter  $K$  and basket weights  $\mathbf{u} = (u_1, u_2, u_3)$  subject to the constraint  $u_1^2 + u_2^2 + u_3^2 = 1$ .

It is worth observing that the presence of binary options in the above decomposition provides some insight into the dynamic hedging challenges for dispersion calls: for every binary option near the money, its delta price sensitivity becomes very large and the delta-hedging strategy prescribed by standard option theory is not feasible. In practice this issue can be mitigated by replacing binary options with tight call spreads so as to obtain an “overhedge” for the issuer — see e.g. Demeterfi et al. (1999, pp. 37–39), Taleb (1997, pp. 286–290), Bossu (2014, pp. 1, 2, 37–39) for further details. The following proposition gives the general form of the expansion:

**Proposition 3.** In odd dimension  $n \geq 3$ , the standard dispersion call is replicated by a normalized continuous portfolio of vanilla basket calls and their payoff derivatives up to order  $(n - 1)/2$ , such as binary basket calls (step function), basket Arrow-Debreu securities<sup>10</sup> (delta function) and higher-order derivatives, as follows:

$$(|\mathbf{x}| - K)^+ = \frac{1}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} \left( \frac{d}{dr} \right)^{\frac{n-1}{2}} \left[ r^{\frac{n-2}{2}} (\sqrt{r}\mathbf{x} \cdot \mathbf{u} - K)^+ \right] \Big|_{r=1} d\mathbf{u}, \quad n \text{ odd},$$

where  $\left( \frac{d}{dr} \right)^{\frac{n-1}{2}} \left[ r^{\frac{n-2}{2}} (\sqrt{r}\mathbf{x} \cdot \mathbf{u} - K)^+ \right]$  may be further expanded using Leibniz’s product rule.

<sup>10</sup>Alternatively,  $\delta(\mathbf{x} \cdot \mathbf{u} - K)$  may be interpreted as the limit-case of a levered butterfly spread  $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} [(\mathbf{x} \cdot \mathbf{u} - K + \varepsilon)^+ - 2(\mathbf{x} \cdot \mathbf{u} - K)^+ + (\mathbf{x} \cdot \mathbf{u} - K - \varepsilon)^+]$ .

*Proof.* Substituting the solution formula (2) into equation (A.9) and switching the order of integration; then substituting  $r \mapsto \frac{k}{K}\sqrt{r}$  and simplifying,

$$\begin{aligned} (|\mathbf{x}| - K)^+ &= \frac{2}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} \int_0^\infty \delta\left(\frac{n-1}{2}\right) \left(\frac{k^2}{K^2} - r^2\right) r^{n-1} (r\mathbf{x} \cdot \mathbf{u} - k)^+ dr d\mathbf{u} \\ &= \frac{1}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} \int_0^\infty \delta\left(\frac{n-1}{2}\right) (1-r) r^{\frac{n-2}{2}} (\sqrt{r}\mathbf{x} \cdot \mathbf{u} - K)^+ dr d\mathbf{u}, \quad n \text{ odd.} \end{aligned}$$

Integrating by parts  $(n-1)/2$  times and sifting yields the decomposition as stated.  $\square$

**Corollary.** For  $n = 5$  the decomposition expands as

$$\begin{aligned} (|\mathbf{x}| - K)^+ &= \frac{1}{\pi^2} \int_{|\mathbf{u}|=1} \frac{d^2}{dr^2} \left[ r^{3/2} (\sqrt{r}\mathbf{x} \cdot \mathbf{u} - K)^+ \right] \Big|_{r=1} d\mathbf{u} \\ &= \int_{|\mathbf{u}|=1} \left[ \frac{2}{\pi^2} (\mathbf{x} \cdot \mathbf{u} - K)^+ + \frac{5K}{4\pi^2} H(\mathbf{x} \cdot \mathbf{u} - K) + \frac{1}{4\pi^2} \delta(\mathbf{x} \cdot \mathbf{u} - K) \right] d\mathbf{u}, \quad n = 5. \end{aligned}$$

**7.2. Even dimension.** In even dimension the standard dispersion call also decomposes into continuous portfolios of vanilla basket calls, binary basket calls and higher-order payoff derivatives, after *ad hoc* Hadamard regularization of the pseudofunction written in solution formula (2). We illustrate below how this is done in dimension  $n = 2$ .

**Proposition 4.** In dimension  $n = 2$ , the standard dispersion call is replicated by a normalized continuous portfolio of vanilla and binary basket calls together with a constrained portfolio of basket Arrow-Debreu securities, as follows:

$$\begin{aligned} (|\mathbf{x}| - K)^+ &= \frac{1}{2} \int_{|\mathbf{u}|=1} \left[ (\mathbf{x} \cdot \mathbf{u} - K)^+ + KH(\mathbf{x} \cdot \mathbf{u} - K) \right] d\mathbf{u} \\ &\quad - \frac{K^2}{\pi} \int_{|\mathbf{y}| \leq 1} \frac{\arcsin|\mathbf{y}|}{|\mathbf{y}|^3} \delta(\mathbf{x} \cdot \mathbf{y} - K) d\mathbf{y}, \quad n = 2. \end{aligned}$$

*Proof.* Substituting solution formula (3) into equation (A.9),

$$(|\mathbf{x}| - K)^+ = -\frac{2}{\pi} \int_0^\infty r \frac{H(k^2/K^2 - r^2)}{(k^2/K^2 - r^2)^{3/2}} \mathbf{1} dr \int_{|\mathbf{u}|=1} (r\mathbf{x} \cdot \mathbf{u} - k)^+ d\mathbf{u}, \quad n = 2,$$

subject to Hadamard regularization of the singularity at  $r = k/K$ . Substituting  $r \mapsto \frac{k}{K}\sqrt{1-r}$  and simplifying; regularizing; and then integrating by parts,

$$\begin{aligned} (|\mathbf{x}| - K)^+ &= -\frac{1}{\pi} \int_0^1 \frac{H(r)}{r^{3/2}} \mathbf{1} dr \int_{|\mathbf{u}|=1} (\sqrt{1-r}\mathbf{x} \cdot \mathbf{u} - K)^+ d\mathbf{u} \\ &= -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left[ \int_\varepsilon^1 \frac{dr}{r^{3/2}} \int_{|\mathbf{u}|=1} (\sqrt{1-r}\mathbf{x} \cdot \mathbf{u} - K)^+ d\mathbf{u} - \frac{2}{\sqrt{\varepsilon}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u} - K)^+ d\mathbf{u} \right] \\ &= \frac{1}{2\pi} \int_0^1 \frac{dr}{\sqrt{r(1-r)}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u}) H(\sqrt{1-r}\mathbf{x} \cdot \mathbf{u} - K) d\mathbf{u}, \quad n = 2, \end{aligned}$$

which is a convergent improper integral. Substituting  $r \mapsto 1 - r^2$  and then integrating by parts and sifting,

$$\begin{aligned} (|\mathbf{x}| - K)^+ &= \frac{1}{\pi} \int_0^1 \frac{dr}{\sqrt{1-r^2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u}) H(r\mathbf{x} \cdot \mathbf{u} - K) d\mathbf{u} \\ &= \frac{1}{2} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u}) H(\mathbf{x} \cdot \mathbf{u} - K) d\mathbf{u} \\ &\quad - \frac{K^2}{\pi} \int_0^1 \frac{\arcsin r}{r^2} dr \int_{|\mathbf{u}|=1} \delta(r\mathbf{x} \cdot \mathbf{u} - K) d\mathbf{u}, \quad n = 2. \end{aligned}$$

The second term above may be rewritten as the surface integral over the unit disk  $\{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\}$ ,

$$- \frac{K^2}{\pi} \int_{|\mathbf{y}| \leq 1} \frac{\arcsin|\mathbf{y}|}{|\mathbf{y}|^3} \delta(\mathbf{x} \cdot \mathbf{y} - K) d\mathbf{y}.$$

Substituting the above together with  $(\mathbf{x} \cdot \mathbf{u})H(\mathbf{x} \cdot \mathbf{u} - K) = (\mathbf{x} \cdot \mathbf{u} - K)^+ + KH(\mathbf{x} \cdot \mathbf{u} - K)^+$  into the prior expression, we obtain the decomposition as stated.  $\square$

## 8. CONSEQUENCES FOR ARBITRAGE PRICING AND CONCLUSIONS

It is standard industry practice to price a given European multi-asset option with an *ad hoc* model capturing the option's idiosyncratic risks in terms of dynamic hedging, together with empirical "street adjustments" compensating for certain unavoidable risks such as payoff discontinuities. In the early days, a wide range of multi-asset options would typically be priced using a multi-asset Black-Scholes or local volatility model with constant correlation (e.g. Bossu, 2014, pp. 82–84) — for instance: basket calls or puts, best-of and worst-of calls or puts, quanto options. Recently, the derivatives industry appears to have shifted toward local correlation and stochastic correlation models that better reflect complex joint dynamics between asset prices, particularly for best-of and worst-of options. Evidence of this shift can be found in the works of Langnau (2010), Reghai (2010), Austing (2011), among others.

Dispersion options are typically viewed as risky instruments to hedge that require a sophisticated pricing model, perhaps featuring stochastic volatility and correlation, and jumps. The replication results in this paper indicate that this view may not be entirely justified. Instead, the existence of a static replicating portfolio suggests standard dispersion options should be priced with the same model used for vanilla basket calls, under penalty of arbitrage. However, the presence of potentially discontinuous payoffs such as binary baskets calls in the replicating portfolio, as found for the dispersion call in section 7, together with the dynamic hedging challenges associated with negative basket weights, might still justify some street adjustments not accounted for by our theory.

Overall, the results presented in this paper constitute a first step toward extending the seminal work of Carr and Madan (1998) and Breeden and Litzenberger (1978) to the static replication and pricing of multi-asset options, leveraging on advanced mathematical tools and theory such as Radon transforms that have vast potential for further applications in quantitative finance and indeed other scientific fields.

## APPENDIX A. CLASSICAL SOLUTION FORMULAS

General references for this section are especially Rubin (2015, pp. 26–68, 127–143), as well as Deans (1983) and Natterer (2001).

## A.1. Conversion to one-dimensional fractional integral equation of the first kind.

**Proposition A.1.** If the target dispersion payoff function  $F(s)$  is twice differentiable (possibly in a generalized sense), the multidimensional inverse problem (1) of replicating a dispersion option with basket calls converts to the one-dimensional fractional integral equation of the first kind:

$$f(s) = \frac{2}{\Gamma[(n-1)/2]} \int_0^s r \phi(r) (s^2 - r^2)^{\frac{n-3}{2}} dr, \quad (\text{A.1})$$

where  $\Gamma$  is Euler's gamma function,  $f(s) := \frac{s^n F''(s)}{\pi^{(n-1)/2}}$  and  $\varphi(\mathbf{y}; k) \equiv \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$ .

*Proof.* Writing integral equation (1) in cylindrical coordinates  $\mathbf{x} \mapsto s\mathbf{v}$  where  $s := |\mathbf{x}| \geq 0$  is a nonnegative real number and  $\mathbf{v} := \mathbf{x}/|\mathbf{x}|$  is a unit vector of  $\mathbb{R}^n$  yields

$$F(s) - c = \int_{\mathbb{R}^n} (s\mathbf{v} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y}) d\mathbf{y}, \quad s \geq 0, |\mathbf{v}| = 1. \quad (\text{A.2})$$

Differentiating both sides twice against  $s$  and then sifting,

$$\begin{aligned} F''(s) &= \int_{\mathbb{R}^n} (\mathbf{v} \cdot \mathbf{y})^2 \delta(s\mathbf{v} \cdot \mathbf{y} - k) \varphi(\mathbf{y}) d\mathbf{y} \\ &= \frac{k^2}{s^2} \int_{\mathbb{R}^n} \delta(s\mathbf{v} \cdot \mathbf{y} - k) \varphi(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Multiplying both sides by  $s^2/k^2$  and switching back to Cartesian coordinates,

$$\frac{|\mathbf{x}|^2}{k^2} F''(|\mathbf{x}|) = \int_{\mathbb{R}^n} \delta(\mathbf{x} \cdot \mathbf{y} - k) \varphi(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n,$$

which is a Radon transform inverse problem of the target *radial function*  $s \mapsto F''(s) s^2/k^2$  with Cartesian parameters  $(\mathbf{x}, k) \in \mathbb{R}^{n+1}$ . Conversion of the transform with cylindrical parameters  $(\mathbf{x}, k) \in \mathbb{S}^{n-1} \times \mathbb{R}$  into a *modified Erdélyi-Kober fractional integral* is covered in Rubin (2015, pp. 140–142). In particular the solution  $\varphi$ , if it exists, is also radial, i.e.  $\varphi(\mathbf{y}) = \psi(|\mathbf{y}|)$  where  $\psi(r)$  is a function of a single variable. The case at hand with Cartesian parameters is straightforwardly adapted as follows. Rewriting the integral to the right-hand side of the above equation in cylindrical coordinates  $\mathbf{y} \mapsto r\mathbf{u}$ ,  $|\mathbf{u}| = 1$ ,

$$\frac{|\mathbf{x}|^2}{k^2} F''(|\mathbf{x}|) = \int_0^\infty r^{n-1} \psi(r) dr \int_{|\mathbf{u}|=1} \delta(r\mathbf{x} \cdot \mathbf{u} - k) d\mathbf{u}, \quad (\text{A.3})$$

where the inner integral is a surface integral over the  $n$ -dimensional unit sphere  $\mathbb{S}^{n-1} := \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\}$  introduced in section 2. By *slice integration* (Rubin, 2015, p. 29), this spherical

integral collapses to

$$\begin{aligned} \int_{|\mathbf{u}|=1} \delta(r\mathbf{x} \cdot \mathbf{u} - k) d\mathbf{u} &= |\mathbb{S}^{n-2}| \int_{-1}^1 \delta(r|\mathbf{x}|t - k) (1-t^2)^{\frac{n-3}{2}} dt \\ &= \frac{|\mathbb{S}^{n-2}|}{r|\mathbf{x}|^{n-2}} \left( |\mathbf{x}|^2 - \frac{k^2}{r^2} \right)^{\frac{n-3}{2}} H\left(r - \frac{k}{|\mathbf{x}|}\right), \end{aligned}$$

where  $|\mathbb{S}^{n-2}| = 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$  is the surface area of the  $(n-1)$ -dimensional unit sphere, and  $H$  is Heaviside's step function. Substituting into equation (A.3) and simplifying,

$$\frac{|\mathbf{x}|^2}{k^2} F''(|\mathbf{x}|) = \frac{|\mathbb{S}^{n-2}|}{|\mathbf{x}|^{n-2}} \int_{k/|\mathbf{x}|}^{\infty} r^n \psi(r) \left( |\mathbf{x}|^2 - \frac{k^2}{r^2} \right)^{\frac{n-3}{2}} \frac{dr}{r^2},$$

which, for fixed  $k > 0$ , is a radial equation as both sides are functions of  $|\mathbf{x}|$  only. Substituting  $r \mapsto k/r$  and  $|\mathbb{S}^{n-2}| = 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$ , simplifying and rearranging,

$$\frac{|\mathbf{x}|^n F''(|\mathbf{x}|)}{\pi^{(n-1)/2}} = \frac{2}{\Gamma[(n-1)/2]} \int_0^{|\mathbf{x}|} r \left(\frac{k}{r}\right)^{n+1} \psi\left(\frac{k}{r}\right) (|\mathbf{x}|^2 - r^2)^{\frac{n-3}{2}} dr. \quad (\text{A.4})$$

The above integral is a *left-sided* modified Erdélyi-Kober fractional integral of the function  $\phi(r) := (k/r)^{n+1} \psi(k/r)$ . Finally, substituting  $s := |\mathbf{x}|$ ,  $f(s) := s^n F''(s)/\pi^{(n-1)/2}$  and  $\phi(r)$  into the above and then simplifying yields the fractional integral equation (A.1) as stated.  $\square$

## A.2. Inversion formulas.

**Proposition A.2.** Provided that the following standard and fractional derivatives exist (possibly in a generalized sense), the solution to the fractional integral equation (A.1) is given as

$$\phi(r) = \begin{cases} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} f(r), & n \text{ odd,} \\ \frac{2}{\sqrt{\pi}} \frac{d}{dr^2} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} \left( \frac{d}{ds^2} \right)^{\frac{n-2}{2}} f(s) ds, & n \text{ even.} \end{cases} \quad (\text{A.5})$$

*Proof.* See Rubin (2015, pp. 65–68) for inversion of modified Erdélyi-Kober fractional integral operators with particular focus on *right-sided* operators in theorem 2.44, and pp. 142–143 for an application to the Radon transform of radial functions. The case of left-sided operators is similar and illustrated in appendix B.  $\square$

*Remark.* A variant of the above formula for  $n$  even has the differential operator taken out of the integral:

$$\phi(r) = \begin{cases} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} f(r), & n \text{ odd,} \\ \frac{2}{\sqrt{\pi}} \left( \frac{d}{dr^2} \right)^{n/2} \int_0^r \frac{sf(s)}{\sqrt{r^2 - s^2}} ds, & n \text{ even.} \end{cases} \quad (\text{A.6})$$

**Corollary.** The solution  $\varphi(\mathbf{y})$  to the replication problem (1), if it exists, is given as  $\varphi(\mathbf{y}; k) = \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$  with

$$\phi(r) = \begin{cases} \frac{1}{\pi^{(n-1)/2}} \left( \frac{d}{dr^2} \right)^{\frac{n-1}{2}} r^n F''(r), & n \text{ odd,} \\ \frac{2}{\pi^{n/2}} \frac{d}{dr^2} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} \left( \frac{d}{ds^2} \right)^{\frac{n-2}{2}} [s^n F''(s)] ds, & n \text{ even,} \end{cases} \quad (\text{A.7})$$

where the differential operator may be taken out of the integral as remarked above for  $n$  even.

*Proof.* Immediate from substituting  $f(s) := s^n F''(s)/\pi^{(n-1)/2}$  into equation (A.5).  $\square$

**A.3. Existence and uniqueness of solutions.** In function spaces of interest, fractional integrals are injective linear operators<sup>11</sup> and thus if a solution  $\phi$  to equation (A.1) exists, it is unique almost everywhere. For example, consider the left-sided half-integral operator  $J : \phi \mapsto J\phi$  where

$$J\phi(s) := \int_0^s \frac{\phi(r)}{\sqrt{s-r}} dr, s > 0.$$

Half-differentiating against  $s$ ,

$$J'J\phi(s) = \frac{d}{ds} \int_0^s \frac{J\phi(t)}{\sqrt{s-t}} dt = \frac{d}{ds} \int_0^s \int_0^t \frac{\phi(r)}{\sqrt{(s-t)(t-r)}} dr dt.$$

Provided the function space is compatible with integral reordering (such as the space of  $L^1$  functions),

$$J'J\phi(s) = \frac{d}{ds} \int_0^s \phi(r) \int_r^s \frac{1}{\sqrt{(s-t)(t-r)}} dr dt = \pi \frac{d}{ds} \int_0^s \phi(r) dr = \pi\phi(s).$$

Consequently, the nullspace of  $J$  is just the null function, and  $J$  is an injective operator. However, it is worth emphasizing that a solution  $\phi$  to equation (A.1) may not always exist, particularly if we impose smoothness or regularity requirements such as continuity over  $(0, \infty)$ , as shown below.

**Proposition A.3.** For  $n \geq 3$ , the left-sided modified Erdélyi-Kober fractional integral operator  $I : \phi \mapsto I\phi$  where

$$I\phi(s) := \frac{2}{\Gamma[(n-1)/2]} \int_0^s r\phi(r)(s^2 - r^2)^{(n-3)/2} dr, s > 0,$$

is an endomorphism of the space of continuous functions over  $(0, \infty)$ .

*Proof.* Substituting  $r \mapsto s\sqrt{r}$  and simplifying,

$$I\phi(s) = s^{n-3} \frac{1}{\Gamma[(n-1)/2]} \int_0^1 \phi(s\sqrt{r}) (1-r)^{\frac{n-3}{2}} dr, \quad (\text{A.8})$$

which is continuous in  $s$  for  $n \geq 3$  if  $\phi(r)$  is continuous in  $r$  (see also Luchko and Trujillo, 2007, th. 2.2).  $\square$

**Corollary.** (a) If the payoff function  $F$  is not twice continuously differentiable over  $(0, \infty)$ , then the replication problem (1) has no continuous solution  $\varphi$ .

(b) If  $\phi(r) = O(r^3)$  then  $F''$  is continuous at the origin.

<sup>11</sup>See e.g. Kress (2014, p.2) for a definition.

*Proof.* (a) Contraposition of proposition A.3 when  $I\phi(s) = f(s) := s^n F''(s)/\pi^{(n-1)/2}$ .  
 (b) Replace  $I\phi(s)$  with  $f(s) := s^n F''(s)/\pi^{(n-1)/2}$  into equation (A.8), divide both sides by  $s^n$  and let  $s \rightarrow 0$ . □

Here, it is worth emphasizing that many payoff functions that are relevant to finance, beginning with calls and puts, are not twice continuously differentiable over the entire domain  $(0, \infty)$ ; we must therefore look for solutions outside of classical theory such as generalized functions. Fortunately, the inversion formulas of Section A.2 are compatible with generalized functions and yield a solution for standard dispersion calls with payoff  $F(s) := (s - K)^+$  as shown in Section 3. However, the following proposition shows that the *zero-strike* call  $F(s) := s$  (and thus puts  $F(s) := (K - s)^+$  by reason of put-call parity) is not replicable in this manner. We resolve this impasse in Sections 4 and 5 by including singular generalized functions in the solution space.

**Proposition A.4.** Let  $F$  be a sufficiently regular payoff function. The replication problem (1) has a regular solution if and only if  $F'(0) = 0$ .

*Proof.* If  $\varphi(\mathbf{y})$  is a regular solution to the replication problem (1), we may differentiate both sides of equation (A.2) against  $s$  and let  $s \rightarrow 0$  to get  $F'(0) = 0$ . Conversely, assume that  $F'(0) = 0$ . The solution  $\phi$  given in (A.7) is known to solve equation (A.1); integrate the latter twice to retrace our steps and retrieve equation (A.2) up to a linear term  $\lambda s$  where  $\lambda = F'(0) = 0$ . Hence,  $\varphi(\mathbf{y}; k) = \phi(k/|\mathbf{y}|)/|\mathbf{y}|^{n+1}$  is a regular solution to the replication problem. □

**Corollary.** There is no regular solution to the replication problem (1) for the class of affine standard dispersion options with payoff  $F(|\mathbf{x}|) := c + \lambda|\mathbf{x}|$ ,  $\lambda \neq 0$ .

*Proof.* Immediate from  $F'(0) = \lambda \neq 0$ . Alternatively, substitute  $F''(s) = 0$  into formula (A.7) to obtain a degenerate  $\phi \equiv 0$ . □

**A.4. Another one-dimensional conversion.** In the spirit of our previous paper (2021), we present another conversion of the multidimensional integral equation (1) with basket call kernel  $(\mathbf{x} \cdot \mathbf{y} - k)^+$  to a one-dimensional equation with integral kernel  $G(|\mathbf{x}|, r)$  indexed by  $r \in (0, \infty)$ . This alternative expression can be handy to validate a solution  $\varphi$  obtained by the fractional calculus methods used earlier.

**Proposition A.5.** The replication problem (1) converts to the Fredholm integral equation of the first kind

$$F(s) - c = \int_0^\infty G(s, r)\psi(r) dr, \quad s \geq 0,$$

with integral kernel

$$G(s, r) = |\mathbb{S}^{n-2}| \left[ \frac{rs}{n-1} \left( r^2 - \frac{k^2}{s^2} \right)^{(n-1)/2} - \frac{k}{2} r^{n-1} \mathbf{B} \left( 1 - \frac{k^2}{r^2 s^2}, \frac{n-1}{2}, \frac{1}{2} \right) \right] H(rs - k),$$

where  $s := |\mathbf{x}|$ ,  $\varphi(\mathbf{y}) \equiv \psi(|\mathbf{y}|)$ , and  $\mathbf{B}(x; a, b) := \int_0^x t^{a-1}(1-t)^{b-1} dt$  is the incomplete beta function.

*Proof.* Switching to cylindrical coordinates  $\mathbf{y} \mapsto r\mathbf{u}, |\mathbf{u}| = 1$  and rearranging, equation (1) becomes

$$F(|\mathbf{x}|) - c = \int_0^\infty \psi(r) dr \int_{|\mathbf{u}|=1} r^{n-1} (r\mathbf{x} \cdot \mathbf{u} - k)^+ d\mathbf{u}. \quad (\text{A.9})$$

By slice integration the sphere integral collapses to

$$\int_{|\mathbf{u}|=1} r^{n-1} (r\mathbf{x} \cdot \mathbf{u} - k)^+ d\mathbf{u} = r^n |\mathbf{x}| |\mathbb{S}^{n-2}| \int_{-1}^1 \left(t - \frac{k}{r|\mathbf{x}|}\right)^+ (1-t^2)^{(n-3)/2} dt.$$

Denoting  $\alpha := k/(rs)$  for  $k < rs$ , splitting the integral at  $t = \alpha$ , applying the reverse chain rule to one split integral and substituting  $t \mapsto \sqrt{1-t}$  inside the other,

$$\begin{aligned} \int_{-1}^1 (t - \alpha)^+ (1-t^2)^{(n-3)/2} dt &= \int_\alpha^1 t (1-t^2)^{(n-3)/2} dt - \alpha \int_\alpha^1 (1-t^2)^{(n-3)/2} dt \\ &= \frac{1}{n-1} (1-\alpha^2)^{(n-1)/2} - \frac{\alpha}{2} \int_0^{1-\alpha^2} t^{(n-3)/2} (1-t)^{-1/2} dt \\ &= \frac{1}{n-1} (1-\alpha^2)^{(n-1)/2} - \frac{\alpha}{2} \text{B}(1-\alpha^2; \frac{n-1}{2}, \frac{1}{2}), \end{aligned}$$

where we recognized the incomplete beta function  $\text{B}(x; a, b)$  in the last step. Substituting  $\alpha := k/(rs)$ , multiplying both sides by  $r^n s |\mathbb{S}^{n-2}|$  and simplifying yields the formula for  $G(s, r)$  as stated.  $\square$

**Corollary.** For  $n = 3$  we have the simpler expression

$$G(s, r) = \frac{\pi r^3}{s} \left(s - \frac{k}{r}\right)^{+2}, \quad n = 3.$$

*Proof.* By slice integration, the sphere integral in equation (A.9) simplifies to

$$\int_{|\mathbf{u}|=1} r^{n-1} (r\mathbf{x} \cdot \mathbf{u} - k)^+ d\mathbf{u} = r^3 |\mathbf{x}| |\mathbb{S}^1| \int_{-1}^1 \left(t - \frac{k}{r|\mathbf{x}|}\right)^+ dt = 2\pi r^3 |\mathbf{x}| \frac{1}{2} \left(1 - \frac{k}{r|\mathbf{x}|}\right)^{+2},$$

which simplifies to  $\frac{\pi r^3}{s} \left(s - \frac{k}{r}\right)^{+2}$  as stated.  $\square$

In other words, a dispersion option payoff  $F(|\mathbf{x}|)$  on three underlying assets may be replicated with cash and a continuous portfolio of “smooth dispersion calls” indexed by  $r \in (0, \infty)$  as

$$F(|\mathbf{x}|) = c + \int_0^\infty \pi r^3 \psi(r) \frac{(|\mathbf{x}| - k/r)^{+2}}{|\mathbf{x}|} dr, \quad n = 3, \quad (\text{A.10})$$

provided that a solution  $\varphi(\mathbf{y}) \equiv \psi(|\mathbf{y}|)$  to integral equation (1) exists in the first place.

## APPENDIX B. INVERSION OF MODIFIED ERDÉLYI-KOBER FRACTIONAL INTEGRAL EQUATION

We show how the left-sided modified Erdélyi-Kober fractional integral equation

$$f(x) = 2 \int_0^x yg(y)(x^2 - y^2)^{\frac{n-3}{2}} dy, \quad x \geq 0, \quad (\text{B.1})$$

is solved for  $g(y)$  by repeated differentiation against  $x^2$ , together with further analysis when  $n$  is even. When  $n \geq 5$  is odd, the exponent  $m := \frac{n-3}{2}$  is a positive integer and we may differentiate both sides against  $x^2$  to obtain

$$\frac{d}{dx^2}[f(x)] = 2xg(x)(x^2 - x^2)^m \frac{dx}{dx^2} + 2m \int_0^x yg(y)(x^2 - y^2)^{m-1} dy,$$

where we used Leibniz's integral rule. Since  $m > 0$  when  $n \geq 5$  the first term vanishes, and we may iterate this process to write

$$\left(\frac{d}{dx^2}\right)^m [f(x)] = 2m! \int_0^x yg(y) dy,$$

which is also satisfied when  $n = 3, m = 0$  with the conventions  $(d/dx^2)^0 = \text{id}$  and  $0! = 1$ . Differentiating against  $x$ , dividing both sides by  $2m!x$  and substituting  $\frac{1}{2x} \frac{d}{dx} = \frac{d}{dx^2}$  we recover

$$\frac{1}{m!} \left(\frac{d}{dx^2}\right)^{m+1} [f(x)] = g(x), \quad n \text{ odd},$$

which solves the integral equation for  $n \geq 3$  odd. For  $n \geq 2$  even, the exponent  $\frac{n-3}{2}$  is now an integer and a half, and equation (B.1) is a proper fractional integral equation. Half-integrating both sides yields

$$\int_0^x f(s) \frac{s}{\sqrt{x^2 - s^2}} ds = 2 \int_0^x \frac{s}{\sqrt{x^2 - s^2}} \int_0^s yg(y)(s^2 - y^2)^{\frac{n-3}{2}} dy ds.$$

Switching the order of integration<sup>12</sup>, then substituting  $s \mapsto \sqrt{y^2 + (x^2 - y^2)s}$  and simplifying,

$$\begin{aligned} \int_0^x f(s) \frac{s}{\sqrt{x^2 - s^2}} ds &= 2 \int_0^x yg(y) \int_y^x s \frac{(s^2 - y^2)^{\frac{n-3}{2}}}{\sqrt{x^2 - s^2}} ds dy \\ &= \int_0^x yg(y)(x^2 - y^2)^{\frac{n-2}{2}} \int_0^1 s^{\frac{n-3}{2}} (1-s)^{-1/2} ds dy \\ &= \text{B}\left(\frac{n-1}{2}, \frac{1}{2}\right) \int_0^x yg(y)(x^2 - y^2)^{\frac{n-2}{2}} dy, \end{aligned}$$

where we recognized the inner integral as a beta function  $\text{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  in the last step. The exponent  $\frac{n-2}{2}$  in the above expression being an integer, we may repeatedly differentiate both sides against  $x^2$  as we did in odd dimension to obtain

$$\left(\frac{d}{dx^2}\right)^{n/2} \int_0^x f(s) \frac{s}{\sqrt{x^2 - s^2}} ds = \frac{n-2}{2}! \text{B}\left(\frac{n-1}{2}, \frac{1}{2}\right) g(x).$$

Simplifying and rearranging yields the solution

$$g(x) = \frac{1}{\Gamma[(n-1)/2]\sqrt{\pi}} \left(\frac{d}{dx^2}\right)^{n/2} \int_0^x f(s) \frac{s}{\sqrt{x^2 - s^2}} ds, \quad n \text{ even}.$$

<sup>12</sup>By Fubini's theorem this is licit if  $g$  is absolutely integrable, but not only.

APPENDIX C. CALCULATION OF THE SOLUTION FOR THE “MEXICAN HAT” PAYOFF IN  
DIMENSION  $n = 2$

Substituting  $s \mapsto r \sin \theta$  and simplifying, then substituting  $\sin^2 \theta = 1 - \cos^2 \theta$  and simplifying,

$$\begin{aligned} & \int_0^r \frac{s}{\sqrt{r^2 - s^2}} e^{-s^2} (s^2 - 2s^4) ds \\ &= \int_0^{\pi/2} r \sin \theta e^{-r^2 \sin^2 \theta} (r^2 \sin^2 \theta - 2r^4 \sin^4 \theta) d\theta \\ &= \int_0^{\pi/2} r \sin \theta e^{r^2 \cos^2 \theta - r^2} \left[ r^2 - r^2 \cos^2 \theta - 2(r^2 - r^2 \cos^2 \theta)^2 \right] d\theta. \end{aligned}$$

Substituting  $t = r \cos \theta$  and simplifying; expanding the square; separating terms; integrating by parts twice and simplifying,

$$\begin{aligned} \int_0^r \frac{s}{\sqrt{r^2 - s^2}} e^{-s^2} (s^2 - 2s^4) ds &= e^{-r^2} \int_0^r e^{t^2} [r^2 - t^2 - 2(r^2 - t^2)^2] dt \\ &= (r^2 - 2r^4) \mathfrak{D}(r) + e^{-r^2} \int_0^r 2te^{t^2} (2r^2t - t/2 - t^3) dt \\ &= r + r^3 - (1 + r^2 + 2r^4) \mathfrak{D}(r), \end{aligned}$$

where  $\mathfrak{D}(r) := e^{-r^2} \int_0^r e^{t^2} dt$ .

REFERENCES

- Austing, Peter (2011). “Repricing the Cross Smile: An Analytic Joint Density”. In: *Risk Magazine* (July).
- Baldeaux, Jan and Marek Rutkowski (2010). “Static Replication of Forward-Start Claims and Realized Variance Swaps”. In: *Applied Mathematical Finance* 17.2, pp. 99–131.
- Baxter, Martin (1998). “Hedging in Financial Markets”. In: *ASTIN Bulletin* 28.1, pp. 5–16.
- Bossu, Sébastien (2005). *Arbitrage pricing of equity correlation swaps*. Tech. rep. J.P. Morgan.
- (2014). *Advanced Equity Derivatives*. John Wiley & Sons.
- Bossu, Sébastien, Peter Carr, and Andrew Papanicolaou (2021). “A functional analysis approach to the static replication of European options”. In: *Quantitative Finance* 21 (4), pp. 637–655.
- Bouzoubaa, Mohamed and Adel Osseiran (2010). *Exotic options and hybrids : a guide to structuring, pricing and trading*. Wiley Finance series. John Wiley & Sons.
- Breeden, Douglas T. and Robert H. Litzenberger (1978). “Prices of state-contingent claims implicit in option prices”. In: *Journal of business*, pp. 621–651.
- Brigo, Damiano, Fabio Mercurio, Francesco Rapisarda, and Rita Scotti (2004). “Approximated moment-matching for basket-options pricing”. In: *Quantitative Finance* 4, pp. 1–16.
- Carr, Peter and Peter Laurence (2011). “Multi-asset Stochastic Local Variance Contracts”. In: *Mathematical Finance* 21.1, pp. 21–52.
- Carr, Peter and Dilip Madan (1998). “Towards a theory of volatility trading”. In: *Volatility: New estimation techniques for pricing derivatives*. Ed. by Robert A. Jarrow. Vol. 29. Risk books, pp. 417–427.
- Cui, Zhenyu and Yuewu Xu (2021). *A New Representation of the Risk-Neutral Distribution and Its Applications*. Working paper. Stevens Institute of Technology and Fordham University.

- Deans, Stanley R. (1983). *The Radon Transform and Some of Its Applications*. Wiley Interscience. John Wiley & Sons.
- Demeterfi, Kresimir, Emanuel Derman, Michael Kamal, and Joseph Zou (1999). *More than you ever wanted to know about volatility swaps*. Goldman Sachs quantitative strategies research notes. Goldman, Sachs & Co New York.
- Derman, Emanuel, Deniz Ergener, and Iraj Kani (1994). *Static options replication*. Goldman Sachs quantitative strategies research notes. Goldman, Sachs & Co New York.
- Driessen, Joost, Pascal J Maenhout, and Grigory Vilkov (2009). “The Price of Correlation Risk: Evidence from Equity Options”. In: *The Journal of Finance* 64.3, pp. 1377–1406.
- Dupire, Bruno (1993). *Arbitrage pricing with stochastic volatility*. Tech. rep. Banque Paribas.
- Jacquier, Antoine and Saad Slaoui (2007). *Variance Dispersion and Correlation Swaps*. Working paper.
- Kanwal, Ram P. (2004). *Generalized Functions: Theory and Applications*. 3rd ed. Birkhäuser Boston.
- Kress, Rainer (2014). *Linear integral equations*. 3rd ed. Vol. 82. Springer Verlag, New York.
- Langnau, Alex (2010). “A Dynamic Model for Correlation”. In: *Risk Magazine* (April), pp. 74–78.
- Lipton, Alexander (2001). *Mathematical Methods For Foreign Exchange: A Financial Engineer’s Approach*. World Scientific Publishing Company.
- Luchko, Yuri and Juan J. Trujillo (2007). “Caputo-type modification of the Erdélyi-Kober fractional derivative”. In: *Fract. Calc. Appl. Anal.* 10 (3), pp. 249–267.
- Natterer, Frank (2001). *The Mathematics of Computerized Tomography*. Society for Industrial and Applied Mathematics.
- Pelsser, Antoon (2003). “Pricing and hedging guaranteed annuity options via static option replication”. In: *Insurance: Mathematics and Economics* 33.2, pp. 283–296.
- Pötz, Christian (2020). “Function approximation for option pricing and risk management. Methods, theory and applications”. PhD thesis. School of Mathematical Sciences, Queen Mary University.
- Reghai, Adil (2010). “Breaking Correlation Breaks”. In: *Risk Magazine* (October), pp. 90–95.
- Rubin, Boris (2015). *Introduction to Radon Transforms*. Cambridge University Press.
- Schofield, Neil C. (2017). *Equity Derivatives Corporate and Institutional Applications*. Palgrave Macmillan UK.
- Taleb, Nassim Nicholas (1997). *Dynamic Hedging: Managing Vanilla and Exotic Options*. John Wiley & Sons.
- The CBOE volatility index—VIX* (2009). Tech. rep. Chicago Board Options Exchange.

## **Chapter 4**

# **Static replication of European multi-asset options with homogeneous payoff**

# STATIC REPLICATION OF EUROPEAN MULTI-ASSET OPTIONS WITH HOMOGENEOUS PAYOFF

SÉBASTIEN BOSSU\*

ABSTRACT. The replication of any European contingent claim by a static continuous portfolio of calls and puts, formally proven by Carr and Madan (1998), extends to multi-asset claims with absolutely homogeneous payoff. Using sophisticated tools from integral geometry, we show how such claims may be replicated with a continuum of vanilla basket calls and derive closed-form solutions to replicate two-asset best-of and worst-of options. We also derive a novel mathematical formula to invert the Radon transform which we apply to obtain a tractable expression of the joint implied distribution. Consequently, a large class of multi-asset options admit a model-free price enforced by arbitrage, just as single-asset European claims do.

## 1. INTRODUCTION

In dimension  $n \geq 2$  the general European payoff replication problem takes the following form: given a target multi-asset European payoff  $F(x_1, x_2, \dots, x_n)$ , where  $x_1, \dots, x_n$  are  $n$  asset performances, and given a family of “replicant” multi-asset European options with payoffs  $G(x_1, \dots, x_n, y_1, \dots, y_n)$  indexed along  $n$  continuums of variables  $y_1, y_2, \dots, y_n$  that typically correspond to asset levers or weights, find a static “continuous portfolio” of the basic options in quantities  $\phi(y_1, \dots, y_n)$  that replicate the target payoff:

$$F(x_1, \dots, x_n) = \int \cdots \int G(x_1, \dots, x_n, y_1, \dots, y_n) \phi(y_1, \dots, y_n) dy_1 \cdots dy_n,$$

where  $\int \cdots \int$  denotes a multiple integral against the index variables  $y_1, \dots, y_n$  over a suitable domain. For maximum generality we let all our variables  $x_i, y_i$  be positive or negative real numbers and we leave the definition of asset performance unspecified with the important caveat that the replicant option payoffs  $G(\cdots)$  are defined accordingly. A typical definition would be the gross returns to maturity or the price ratios of  $n$  underlying assets.

This paper covers replication with vanilla basket calls that pay off  $G(\cdots) := (\sum_{i=1}^n x_i y_i - k)^+$  where  $t^+ := \max(0, t)$  denotes the positive part of a real number  $t$  and  $k$  is a moneyness parameter. We further postulate that the target payoff  $F$  also depends on parameter  $k$  and write  $F(x_1, \dots, x_n; k)$ , and that it is absolutely homogeneous, that is,  $F(\lambda x_1, \dots, \lambda x_n; \lambda k) = |\lambda| F(x_1, \dots, x_n; k)$  for any real number  $\lambda$ . Switching to vector notations, in the language of functional analysis we want to solve the multidimensional integral equation of the first kind

$$F(\mathbf{x}; k) = \int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \phi(\mathbf{y}) d\mathbf{y}, \quad (\mathbf{x}, k) \in \mathbb{R}^{n+1}, \quad (1a)$$

for an unknown function  $\phi(\mathbf{y})$  that is independent from  $k$ . Here,  $\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^n x_i y_i$  denotes the canonical dot product of Euclidean space  $\mathbb{R}^n$  with associated norm  $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$ . This integral equation may be viewed

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*Date:* April 27, 2021.

\* NYU Courant. The author thanks Guillaume Bal and Paul Garrett for useful discussions about Radon transforms and integral regularizations. All remaining errors are his.

as a multi-asset generalization of the Breeden and Litzenberger (1978) and Carr and Madan (1998) inverse problems. When a solution  $\phi(\mathbf{y})$  exists, the target option admits a model-free arbitrage price so long as basket call prices of arbitrary basket weights  $\mathbf{y}$  are known.

Differentiating both sides twice against  $k$ , equation (1a) converts to the *Radon transform inverse* problem

$$f(\mathbf{x}; k) = \int_{\mathbb{R}^n} \delta(\mathbf{x} \cdot \mathbf{y} - k) \phi(\mathbf{y}) \, d\mathbf{y} =: \mathcal{R}\phi(\mathbf{x}, k), \quad (\mathbf{x}, k) \in \mathbb{R}^{n+1}, \quad (1b)$$

where  $f = \frac{\partial^2 F}{\partial k^2}$ ,  $\delta$  denotes Dirac's delta function, and  $\mathcal{R}\phi(\mathbf{x}, k)$  is the Radon transform of  $\phi$  with  $n + 1$  Cartesian parameters  $(\mathbf{x}, k) \in \mathbb{R}^{n+1}$ . From a quantitative finance perspective, the above integral equation may be interpreted as a replication problem of the ersatz payoff function  $f$  with basket Arrow-Debreu securities  $G(\mathbf{x}, \mathbf{y}) := \delta(\mathbf{x} \cdot \mathbf{y} - k)$ .

In Bossu, Carr, and Papanicolaou (2020) integral equation (1a) was solved for “standard dispersion options” whose payoff is a *radial function*  $F(\mathbf{x}) := F_0(|\mathbf{x}|)$ ,  $F_0 \in \mathbb{R}^{\mathbb{R}}$ , in which case the Radon transform inverse problem (1b) collapses to a one-dimensional fractional integral equation and solves as a fractional derivative of  $F_0$ . We refer to Sections 1 and 2 of this prior paper for a discussion of motivations behind the replication problem (1a), and we shall adopt the same notations.

Our main theoretical contribution is to show how multi-asset European options with absolutely homogeneous payoff can be replicated by a continuous portfolio of vanilla basket calls, and consequently admit a “model-free” arbitrage price so long as the prices of basket call options of arbitrary basket weights are known. We also derive a novel mathematical formula for the inverse Radon transform which we use to obtain a tractable expression of the joint implied distribution based on vanilla basket call prices. Finally, we apply our theory to obtain the first replication identities for best-of and worst-of options on two assets.

In related literature, Henkin and Shananin (1990) study a similar inverse problem arising from mathematical economics over the domain  $\mathbb{R}_+^n$ . Lipton (2001, pp. 291–292) proposes a generalization of the Carr and Madan formula for two assets using Radon transforms, while Carr and Laurence (2011) derive a multi-asset version of the Dupire (1993) local volatility formula. Recently, Cui and Xu (2021) use standard multivariate calculus techniques to derive a multi-asset version of the Carr and Madan formula as multiple integrals of products of call options.

The remainder of this paper is organized as follows: In Section 2 we review the Radon transform with Cartesian parameterization and its mathematical properties, and derive inverse formulas. In Section 3 we discuss how equations (1a) and (1b) are related and how the replication problem may be solved. In Section 4 we apply our framework for best-of and worst-of options in dimension  $n = 2$ . In Section 5 we present as theoretical application a new formula for the joint implied distribution, before discussing general consequences for arbitrage pricing in our concluding Section 6.

## 2. THE RADON TRANSFORM WITH CARTESIAN PARAMETERIZATION AND ITS INVERSE

General references for this section are especially Rubin (2015, pp. 26–68, 127–143), and Deans (1983), Natterer (2001). A fair body of theoretical and applied research has been published on the Radon transform and its inverse with particular focus on cylindrical parameters, that is for unit vectors  $\mathbf{x}$ . This section summarizes some key results from existing theory which we adapted to our Cartesian parameterization  $(\mathbf{x}, k) \in \mathbb{R}^{n+1}$ , together with some original theoretical extensions.

**2.1. Existence and elementary properties of the Radon transform.** A standard requirement for integral transforms is that  $\phi$  be absolutely or square integrable in the first place so that the transform  $\mathcal{R}\phi$  be well defined. If  $k \neq 0$ , the behavior of  $\phi$  at origin does not matter too much and it is enough to require

$$\int_{|\mathbf{y}|>a} \frac{|\phi(\mathbf{y})|}{|\mathbf{y}|} d\mathbf{y} < \infty$$

for any  $a > 0$  (see Rubin, 2015, pp. 132–133 and 145–147 for a detailed discussion of existence). Note that the Radon transform may also exist for other types of functions that do not meet the above criterion. If  $\phi$  is good enough for  $\mathcal{R}\phi$  to be well defined, the transform  $\mathcal{R}\phi(\mathbf{x}, k)$  satisfies:

(P1) Scaling property: by linear change of variable,

$$\mathcal{R}\phi(\lambda\mathbf{x}, k) = \frac{1}{|\lambda|^n} \mathcal{R}\left[\phi\left(\frac{\mathbf{y}}{\lambda}\right)\right](\mathbf{x}, k),$$

and more generally for any invertible matrix  $A$  with inverse transpose  $A^{-T}$ ,

$$\mathcal{R}\phi(A\mathbf{x}, k) = \frac{1}{|\det A|} \mathcal{R}[\phi(A^{-T}\mathbf{y})](\mathbf{x}, k).$$

(P2) Absolute homogeneity of degree -1: by the same property of Dirac's delta function,

$$\mathcal{R}\phi(\lambda\mathbf{x}, \lambda k) = \frac{1}{|\lambda|} \mathcal{R}\phi(\mathbf{x}, k).$$

Property (P2) yields the following alternative parameterizations:

- Cylindrical parameters:  $\mathcal{R}\phi(\mathbf{x}, k) = \frac{1}{|\mathbf{x}|} \mathcal{R}\phi\left(\frac{\mathbf{x}}{|\mathbf{x}|}, \frac{k}{|\mathbf{x}|}\right)$ . This is the classic parameterization used in integral geometry which we denote  $R\phi(\mathbf{u}, t)$  where  $(\mathbf{u}, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$ .
- Standard Cartesian parameters:  $\mathcal{R}\phi(\mathbf{x}, k) = \frac{1}{|k|} \mathcal{R}\phi\left(\frac{\mathbf{x}}{k}, 1\right)$ . We shall denote this parameterization  $\mathcal{R}_1\phi(\mathbf{x})$ .

Note that the operators  $R$  and  $\mathcal{R}_1$  are *not* absolutely homogeneous in their respective parameters; however  $R$  is even symmetric:  $R\phi(-\mathbf{u}, -t) = R\phi(\mathbf{u}, t)$ .

## 2.2. Fourier slice theorem. Inversion by Fourier method.

**2.2.1. Cylindrical parameters.** A well-known key result is that the  $n$ -dimensional Fourier transform may be decomposed as a Radon transform followed by a one-dimensional Fourier transform (e.g., Rubin, 2015, p. 157):

$$\hat{\phi}(\lambda\mathbf{u}) = [R\phi(\mathbf{u}, t)]^\wedge(\lambda), \quad (\mathbf{u}, \lambda) \in \mathbb{S}^{n-1} \times \mathbb{R}, \quad (2)$$

where  $\hat{\phi}$  is the  $n$ -dimensional Fourier transform of  $\mathbf{y} \mapsto \phi(\mathbf{y})$  and  $[g(t)]^\wedge(\cdot)$  denotes the one-dimensional Fourier transform of a function  $t \mapsto g(t)$ . Substituting  $\mathbf{u} = \mathbf{x}/|\mathbf{x}|, \lambda = |\mathbf{x}|$  into the above and taking  $n$ -dimensional inverse Fourier transforms of both sides gives the classic Radon transform inverse formula by Fourier method (Rubin, 2015, p. 176),

$$\phi(\mathbf{y}) = \left[ \left[ R\phi\left(\frac{\mathbf{x}}{|\mathbf{x}|}, t\right) \right]^\wedge(|\mathbf{x}|) \right]^\vee(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^n,$$

where  $[g(\mathbf{x})]^\vee(\cdot)$  denotes the  $n$ -dimensional inverse Fourier transform of a function  $\mathbf{x} \mapsto g(\mathbf{x})$ .

2.2.2. *Standard Cartesian parameters.* The Fourier slice theorem is straightforwardly converted to standard Cartesian parameterization by substituting  $R\phi(\mathbf{u}, t) = |t|^{-1}\mathcal{R}_1\phi(\mathbf{u}/t)$ , and  $\mathbf{u} = \mathbf{x}/|\mathbf{x}|, \lambda \mapsto \lambda|\mathbf{x}|$  into equation (2) to obtain

$$\hat{\phi}(\lambda\mathbf{x}) = \left[ \frac{1}{|t|} \mathcal{R}_1 \phi \left( \frac{\mathbf{x}}{t|\mathbf{x}|} \right) \right]^\wedge (\lambda|\mathbf{x}|) = \left[ \frac{1}{|t|} \mathcal{R}_1 \phi(\mathbf{x}/t) \right]^\wedge (\lambda), \quad (\mathbf{x}, \lambda) \in \mathbb{R}^{n+1}, \quad (3)$$

where we used the scaling property of the Fourier transform and simplified in the last step. Substituting  $\lambda = 1$ , taking  $n$ -dimensional inverse Fourier transforms of both sides and switching the order of Fourier transforms yields the inversion formula

$$\phi(\mathbf{y}) = \left[ \frac{1}{|t|} [\mathcal{R}_1 \phi(\mathbf{x}/t)]^\vee(\mathbf{y}) \right]^\wedge (1) = \left[ |t|^{n-1} \widetilde{\mathcal{R}_1 \phi}(t\mathbf{y}) \right]^\wedge (1), \quad \mathbf{y} \in \mathbb{R}^n, \quad (4)$$

where we used the scaling property of the  $n$ -dimensional inverse Fourier transform in the last step.

2.3. **Inversion by backprojection.** For completeness, as an alternative to the aforementioned Fourier method, we discuss a popular method of inversion known as *backprojection*.

2.3.1. *Cylindrical backprojection.* Subject to regularity conditions, the Radon transform  $R\phi(\mathbf{u}, t) \equiv h(\mathbf{u}, t)$  with cylindrical parameters  $(\mathbf{u}, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$  is inverted as (e.g. Natterer, 2001, ch. II, eq. (2.5))

$$\phi(\mathbf{y}) = \begin{cases} \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1}} \int_{|\mathbf{u}|=1} \mathcal{H}h^{(n-1)}(\mathbf{u}, \mathbf{u} \cdot \mathbf{y}) \, d\mathbf{u}, & n \text{ even,} \\ \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1}} \int_{|\mathbf{u}|=1} h^{(n-1)}(\mathbf{u}, \mathbf{u} \cdot \mathbf{y}) \, d\mathbf{u}, & n \text{ odd.} \end{cases} \quad (5)$$

where the  $(n-1)$ st derivative of the target function  $h(\mathbf{u}, t)$  is taken against the radial argument  $t$ , and  $\mathcal{H}$  denotes the Hilbert transform operator

$$\mathcal{H}g(\tau) \equiv \mathcal{H}_t[g(t)](\tau) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{\tau - t} \P dt$$

whose convergence is understood as Cauchy principal value, as we emphasized by the pircrow symbol  $\P$ . In integral geometry, integrals of the form  $\int_{|\mathbf{u}|=1} h(\mathbf{u}, \mathbf{u} \cdot \mathbf{y}) \, d\mathbf{u}$  are known as *dual Radon transform* (Rubin, 2015, pp. 138–140), *adjoint operator* or *backprojection operator*.

2.3.2. *Cartesian backprojection.* In the remainder of this section we derive a backprojection formula to invert the Radon transform with Cartesian parameters, which we did not see in our reference literature and present as a novel mathematical result. To do so, we propose to extend the concept of *fundamental solution* or *Green's function* to the Radon transform operator  $\mathcal{R}_1$ , which is a linear operator. In other words, we wish to solve the multidimensional integral equation

$$\delta(\mathbf{x} - \mathbf{K}) = \int_{\mathbb{R}^n} \delta(\mathbf{x} \cdot \mathbf{y} - 1) \varphi_{\mathbf{K}}(\mathbf{y}) \, d\mathbf{y} =: (\mathcal{R}_1 \varphi_{\mathbf{K}})(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

where  $\mathbf{K}$  is a constant vector and  $\varphi_{\mathbf{K}}$  is the fundamental solution. From a quantitative finance perspective, this corresponds to replicating the multi-asset Arrow-Debreu security  $\delta(\mathbf{x} - \mathbf{K})$  with a continuous portfolio of basket Arrow-Debreu securities  $\delta(\mathbf{x} \cdot \mathbf{y} - k)$ . With the fundamental solution in hand, we may formally

invert the Radon transform  $\mathcal{R}_1$  as  $\phi(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{K})\varphi_{\mathbf{K}}(\mathbf{y}) d\mathbf{K}$ , provided that this integral converge in some sense. As shown in Appendix A, the fundamental solution is given as

$$\varphi_{\mathbf{K}}(\mathbf{y}) = \frac{(-1)^{n-1}}{(2\pi)^{n-1}} \left( \mathcal{H}^{n-1} \delta^{(n-1)} \right) (1 - \mathbf{y} \cdot \mathbf{K}), \quad (6)$$

where  $\mathcal{H}^{n-1}$  denotes the composition of order  $n-1$  of the Hilbert transform. Substituting, commuting the Hilbert transform with the translation  $t \mapsto t - \mathbf{y} \cdot \mathbf{K}$ , then taking it outside of the integral,

$$\phi(\mathbf{y}) = \frac{(-1)^{n-1}}{(2\pi)^{n-1}} \mathcal{H}_t^{n-1} \left[ \int_{\mathbb{R}^n} f(\mathbf{K}) \delta^{(n-1)}(t - \mathbf{y} \cdot \mathbf{K}) d\mathbf{K} \right] (1).$$

Recognizing the partial derivative against  $t$  of the Radon transform  $\mathcal{R}f(\mathbf{y}, t)$  we obtain the remarkable inverse formula

$$\begin{aligned} \phi(\mathbf{y}) &= \frac{(-1)^{n-1}}{(2\pi)^{n-1}} \mathcal{H}_t^{n-1} \left[ \frac{\partial^{n-1}}{\partial t^{n-1}} \mathcal{R}f(\mathbf{y}, t) \right] (1) = \begin{cases} \frac{(-1)^{n/2}}{(2\pi)^{n-1}} \mathcal{H}_t \left[ \mathcal{R}^{(n-1)} f(\mathbf{y}, t) \right] (1), & n \text{ even,} \\ \frac{(-1)^{(n-1)/2}}{(2\pi)^{n-1}} \mathcal{R}_1^{(n-1)} f(\mathbf{y}), & n \text{ odd;} \end{cases} \quad (7) \\ &= \begin{cases} \frac{(-1)^{n/2}}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \frac{1}{1-t} \frac{\partial^{n-1}}{\partial t^{n-1}} \int_{\mathbb{R}^n} \delta(\mathbf{x} \cdot \mathbf{y} - t) f(\mathbf{x}) d\mathbf{x} dt, & n \text{ even,} \\ \frac{(-1)^{(n-1)/2}}{(2\pi)^{n-1}} \frac{\partial^{n-1}}{\partial t^{n-1}} \Big|_{t=1} \int_{\mathbb{R}^n} \delta(\mathbf{x} \cdot \mathbf{y} - t) f(\mathbf{x}) d\mathbf{x}, & n \text{ odd,} \end{cases} \end{aligned}$$

where we introduced the notations  $\mathcal{R}^{(n-1)} f(\mathbf{y}, t) := \frac{\partial^{n-1}}{\partial t^{n-1}} \mathcal{R}f(\mathbf{y}, t)$  and  $\mathcal{R}_1^{(n-1)} f(\mathbf{y}) := \mathcal{R}^{(n-1)} f(\mathbf{y}, 1)$ . Thus, in Cartesian parameterization the inverse Radon transform  $\mathcal{R}_1^{-1} f$  turns out to be straightforwardly related to the partial derivative of order  $n-1$  of the direct transform  $\mathcal{R}f$  against the second argument. Compared to the classic backprojection formula (5) under cylindrical parameterization, our method presents some important differences and advantages:

- Formula (7) fits the natural Cartesian coordinate system of the original integral equation (1b). No conversion from Cartesian to cylindrical coordinates is needed.
- In practice, spherical integrals can be hard to calculate. In contrast our formula (7) is based on the Radon transform of the target function  $f$  which may be easier to calculate as an integral over  $\mathbb{R}^n$ .
- The classic backprojection formula (5) requires high-order differentiation of the target function inside the spherical integral, which tends to exacerbate singularities. In our inversion formula (7), differentiation is performed after integration.

### 3. SOLVING THE REPLICATION PROBLEM

If a solution  $\phi$  to the replication problem (1a) exists, it must solve the Radon transform inverse problem (1b) with  $f = \frac{\partial^2 F}{\partial k^2}$ . Conversely, if  $\phi$  solves equation (1b), by double integration against  $k$  it also solves the replication problem (1a) up to an additive term  $h_1(\mathbf{x}) + k h_2(\mathbf{x})$ , where  $h_1, h_2$  are two functions that are independent from  $k$ . Asymptotic considerations yield

$$h_2(\mathbf{x}) = \lim_{k \rightarrow \infty} \frac{\partial F}{\partial k}(\mathbf{x}; k), \quad h_1(\mathbf{x}) = \lim_{k \rightarrow \infty} [F(\mathbf{x}; k) - k h_2(\mathbf{x})],$$

provided both limits exist and are finite. Depending on the target multi-asset payoff  $F$ , the additive term  $h_1(\mathbf{x}) + k h_2(\mathbf{x})$  may vanish and the replication problem (1a) is solved as is. If it does not vanish, it might still be replicable with a combination of cash, underlying assets and perhaps single-asset options. Otherwise,

other methods beyond the scope of this paper would need to be employed to further break it down into replicable securities.

Observe that if the payoff function  $F$  is absolutely homogeneous, then  $f = \frac{\partial^2 F}{\partial k^2}$  is absolutely homogeneous of degree -1, thereby satisfying property (P2) of the Radon transform operator  $\mathcal{R}$ . Homogenizing both sides of equation (1b) with respect to  $k \neq 0$  and simplifying, it is easily converted to standard Cartesian form

$$f(\mathbf{x}; 1) =: f_1(\mathbf{x}) = \mathcal{R}_1 \phi(\mathbf{x}) := \mathcal{R} \phi(\mathbf{x}, 1), \quad \mathbf{x} \in \mathbb{R}^n,$$

whose solution  $\phi = \mathcal{R}_1^{-1} f_1$  may be calculated by Fourier methods as discussed in Section 2.2.2, that is,

$$\phi(\mathbf{y}) = [ |t|^{n-1} \check{f}_1(t\mathbf{y}) ]^\wedge(1) = \left[ |t|^{n-1} \widetilde{\frac{\partial^2 F}{\partial k^2}}(t\mathbf{y}; k) \right]^\wedge \Big|_{k=\lambda=1} (\lambda)$$

wherein the  $^\wedge, ^\vee$  symbols respectively denote the one-dimensional direct and  $n$ -dimensional inverse Fourier transforms. Alternatively, our backprojection formula (7) may be used.

#### 4. APPLICATION: REPLICATION OF TWO-ASSET BEST-OF AND WORST-OF OPTIONS

Consider a best-of option payoff  $F(x_1, x_2; k) := F_0(x_1 \vee x_2; k)$  where  $F_0(s; k)$  is an arbitrary wrapper function of the best-of asset  $x_1 \vee x_2 := \max(x_1, x_2)$ . By the well-known even-odd decomposition  $x_1 \vee x_2 = \frac{x_1 + x_2}{2} + \frac{|x_1 - x_2|}{2}$ , the best-of asset  $x_1 \vee x_2$  is replicated with a 50-50 portfolio on the assets together with a long at-the-money straddle on the spread  $x_1 - x_2$ , also known as an exchange or Margrabe straddle.

It is worth emphasizing that the best-of function  $(x_1, x_2) \mapsto x_1 \vee x_2$  is *positively* homogeneous but not *absolutely* homogeneous. In what follows, we circumvent this issue by taking absolute values  $|x_1| \vee |x_2|$ . From a financial perspective, this does not pose any problem as long as  $x_1, x_2$  correspond to positive asset prices or price ratios.

**4.1. General solution for two-asset best-of options.** If the wrapper function  $F_0(s; k)$  is absolutely homogeneous in  $s, k$ , then the absolute best-of option payoff  $F(x_1, x_2; k) := F_0(|x_1| \vee |x_2|; k)$  is absolutely homogeneous in  $x_1, x_2, k$  and we may deploy solution formula (4) as

$$\phi_\vee(y_1, y_2) = \int_{-\infty}^{\infty} |t| e^{-it} \check{f}_\vee(ty_1, ty_2) dt \tag{8}$$

with

$$f_\vee(x_1, x_2) \equiv \mathcal{R}_1 \phi(x_1, x_2) := f_0(|x_1| \vee |x_2|), \quad f_0(s) := \frac{\partial^2 F_0}{\partial k^2}(s, 1).$$

Decomposing  $f_0(|x_1| \vee |x_2|) = f_0(|x_1|)H(|x_1| - |x_2|) + f_0(|x_2|)H(|x_1| - |x_2|)$ , it is enough to calculate the bidimensional inverse Fourier transform of the first term:

$$\begin{aligned}
 [f_0(|x_1|)H(|x_1| - |x_2|)]^\vee(y_1, y_2) &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} e^{i(x_1 y_1 + x_2 y_2)} f_0(|x_1|)H(|x_1| - |x_2|) dx_1 dx_2 \\
 &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{i x_1 y_1} f_0(|x_1|) \int_{-|x_1|}^{|x_1|} e^{i x_2 y_2} dx_2 dx_1 \\
 &= \frac{1}{2\pi^2 y_2} \int_{-\infty}^{\infty} e^{i x_1 y_1} f_0(|x_1|) \sin(|x_1| y_2) dx_1, \\
 &= \frac{1}{\pi^2 y_2} \int_0^{\infty} f_0(s) \cos(y_1 s) \sin(y_2 s) ds,
 \end{aligned}$$

which holds by continuity at  $y_2 = 0$  through  $\sin(s y_2)/y_2 \sim s$ . By symmetry,

$$\widetilde{f}_\vee(y_1, y_2) = \frac{1}{\pi^2} \int_0^{\infty} f_0(s) \left[ \frac{1}{y_2} \cos(y_1 s) \sin(y_2 s) + \frac{1}{y_1} \sin(y_1 s) \cos(y_2 s) \right] ds, \quad y_1, y_2 \neq 0.$$

Substituting into equation (8), simplifying and switching the order of integration,

$$\phi_\vee(y_1, y_2) = \frac{1}{\pi^2} \int_0^{\infty} f_0(s) \int_{-\infty}^{\infty} e^{-it(\operatorname{sgn} t)} \left[ \frac{1}{y_2} \cos(t y_1 s) \sin(t y_2 s) + \frac{1}{y_1} \sin(t y_1 s) \cos(t y_2 s) \right] ds, \quad y_1, y_2 \neq 0.$$

After calculations and simplifications of the inner direct Fourier transform with respect to  $t$ , we obtain that the solution for the best-of option is

$$\begin{aligned}
 \phi_\vee(y_1, y_2) &= -\frac{4}{\pi^2} \int_0^{\infty} \frac{s f_0(s)}{(1 - (y_1 - y_2)^2 s^2)(1 - (y_1 + y_2)^2 s^2)} \mathfrak{H} ds \\
 &= -\frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{s(f_0(s) - f_0(-s))}{(1 - (y_1 - y_2)^2 s^2)(1 - (y_1 + y_2)^2 s^2)} \mathfrak{H} ds.
 \end{aligned} \tag{9}$$

where the ptilde symbol  $\mathfrak{H}$  indicates the integral needs regularization. It is worth noting that, by partial fraction decomposition, the above integral may be decomposed into a sum of Hilbert transforms of the odd part of  $f_0$ , that is the function  $s \mapsto (f_0(s) - f_0(-s))$ .

**4.2. Replication of two-asset best-of call and put.** The payoff function of the absolute best-of call,  $F(x_1, x_2; k) := (|x_1| \vee |x_2| - |k|)^+$ , is absolutely homogeneous with second-order derivative

$$\frac{\partial^2 F_0}{\partial k^2}(|x_1| \vee |x_2|; k) = 2\delta(k)H(|x_1| \vee |x_2| - |k|) + \delta(|x_1| \vee |x_2| - |k|).$$

Evaluating at  $k = 1$ , we obtain  $f_0(s) := \frac{\partial^2 F_0}{\partial k^2}(s; 1) = \delta(s - 1)$ . Substituting into formula (9) and sifting, we obtain that the solution for the best-of call is

$$\phi_{\vee}^{\mathbb{C}}(y_1, y_2) = \frac{-4/\pi^2}{(1 - (y_1 - y_2)^2)(1 - (y_1 + y_2)^2)} \mathbb{1}, \quad (10)$$

as verified in Appendix B. It is worth emphasizing that this solution is a *pseudofunction* subject to regularization of its singularities along the diagonals  $y_1 - y_2 = \pm 1$  and antidiagonals  $y_1 + y_2 = \pm 1$ , as we indicated by the pilcrow symbol  $\mathbb{1}$ . The corresponding replication identities are then given as:

- For the best-of call,

$$(x_1 \vee x_2 - k)^+ = -\frac{4}{\pi^2} \iint_{-\infty}^{\infty} \frac{(x_1 y_1 + x_2 y_2 - k)^+}{(1 - (y_1 - y_2)^2)(1 - (y_1 + y_2)^2)} \mathbb{1} dy_1 dy_2, \quad x_1, x_2, k > 0; \quad (11)$$

- For the best-of put,

$$(k - x_1 \vee x_2)^+ = k - \frac{x_1 + x_2}{2} - \frac{|x_1 - x_2|}{2} + \frac{4}{\pi^2} \iint_{-\infty}^{\infty} \frac{(x_1 y_1 + x_2 y_2 - k)^+}{(1 - (y_1 - y_2)^2)(1 - (y_1 + y_2)^2)} \mathbb{1} dy_1 dy_2, \quad x_1, x_2, k > 0, \quad (12)$$

which stems from put-call parity together with even-odd decomposition of  $x_1 \vee x_2$ .

**4.3. Two-asset worst-of options.** The results in this section straightforwardly extend to worst-of options by means of the maximum-minimum identity

$$F_0(x_1 \wedge x_2; k) = F_0(x_1; k) + F_0(x_2; k) - F_0(x_1 \vee x_2; k).$$

where  $x_1 \wedge x_2 := \min(x_1, x_2)$ . For example, the worst-of call is replicated as

$$(x_1 \wedge x_2 - k)^+ = (x_1 - k)^+ + (x_2 - k)^+ + \frac{4}{\pi^2} \iint_{-\infty}^{\infty} \frac{(x_1 y_1 + x_2 y_2 - k)^+}{(1 - (y_1 - y_2)^2)(1 - (y_1 + y_2)^2)} \mathbb{1} dy_1 dy_2, \quad x_1, x_2, k > 0,$$

which is long vanilla calls on each single asset together with a long continuous portfolio of vanilla basket calls.

## 5. THEORETICAL APPLICATION: MULTI-ASSET GENERALIZATION OF BREEDEN-LITZENBERGER FORMULA

In the manner of Breeden and Litzenberger (1978), we may write the undiscounted price of a vanilla basket call option as the risk-neutral expectation of its payoff

$$c(\mathbf{w}, k) := \mathbb{E}[(\mathbf{w} \cdot \mathbf{S} - k)^+] = \int_{\mathbb{R}^n} (\mathbf{w} \cdot \mathbf{y} - k)^+ \pi(\mathbf{y}) d\mathbf{y},$$

where  $\mathbf{S}$  is the random vector of terminal underlying asset prices with risk-neutral density  $\pi(\mathbf{y}) := \mathbb{P}(\mathbf{S} = \mathbf{y})$ . Recognizing an integral equation of the form (1a), we may again differentiate both sides twice against  $k$  to convert this equation into Radon form (1b),

$$c_{kk}(\mathbf{w}, k) := \frac{\partial^2 c}{\partial k^2}(\mathbf{w}, k) = \int_{\mathbb{R}^n} \delta(\mathbf{w} \cdot \mathbf{y} - k) \pi(\mathbf{y}) d\mathbf{y} =: \mathcal{R}\pi(\mathbf{w}, k).$$

Here, it is worth noting that  $c_{kk}(\mathbf{w}, k) = \mathbb{E}[\delta(\mathbf{w} \cdot \mathbf{S} - k)]$  is absolutely homogeneous of order -1 as required by property (P2), even though  $c(\mathbf{w}, k)$  is merely positively homogeneous. The risk-neutral density implied by basket call prices is thus recovered as the inverse Radon transform of  $c_{kk}$ , that is,  $\pi(\mathbf{y}) = (\mathcal{R}^{-1}c_{kk})(\mathbf{y})$ , in agreement with Carr and Laurence (2011, p. 26) who proved this result under cylindrical parameterization. This result may be further explicated by substituting our Cartesian backprojection formula (7) to obtain

$$\pi(\mathbf{y}) := \mathbb{P}(\mathbf{S} = \mathbf{y}) = \begin{cases} \frac{(-1)^{n/2}}{(2\pi)^{n-1}} \left( \mathcal{H}_t \left[ \mathcal{R}^{(n-1)}c_{kk}(\mathbf{y}, t) \right] \right) (1), & n \text{ even,} \\ \frac{(-1)^{(n-1)/2}}{(2\pi)^{n-1}} \left( \mathcal{R}_1^{(n-1)}c_{kk} \right) (\mathbf{y}), & n \text{ odd.} \end{cases} \quad (13)$$

This new formula for the implied risk-neutral density makes the Carr and Laurence result more explicit, is more analytically tractable and also allows to combine all differentiation inside the Radon transform thanks to the linearity of all integro-differential operators involved:

$$\pi(\mathbf{y}) := \mathbb{P}(\mathbf{S} = \mathbf{y}) = \begin{cases} \frac{(-1)^{n/2}}{(2\pi)^{n-1}} \left( \mathcal{H}_t \left[ \mathcal{R} \frac{\partial^{n+1}c}{\partial k^{n+1}}(\mathbf{y}, t) \right] \right) (1), & n \text{ even,} \\ \frac{(-1)^{(n-1)/2}}{(2\pi)^{n-1}} \left( \mathcal{R}_1 \frac{\partial^{n+1}c}{\partial k^{n+1}} \right) (\mathbf{y}), & n \text{ odd.} \end{cases}$$

## 6. CONSEQUENCES FOR ARBITRAGE PRICING AND CONCLUSIONS

It is standard industry practice to price a given European multi-asset option with an *ad hoc* model capturing the option's idiosyncratic risks in terms of dynamic hedging, together with empirical "street adjustments" compensating for certain unavoidable risks such as payoff discontinuities. In the early days, a wide range of multi-asset options would typically be priced using a multi-asset Black-Scholes or local volatility model with constant correlation (e.g. Bossu, 2014, pp. 82–84) — for instance: basket calls or puts, best-of and worst-of calls or puts, quanto options. Recently, the derivatives industry appears to have shifted toward local correlation and stochastic correlation models that better reflect complex joint dynamics between asset prices, particularly for best-of and worst-of options. Evidence of this shift can be found in the works of Langnau (2010), Reghai (2010), Austing (2011), among others.

European multi-asset options are often viewed as risky instruments to hedge that need a sophisticated pricing model, perhaps featuring stochastic volatility and correlation, and jumps. The replication results in this paper indicate that this view may not be entirely justified. Instead, the existence of a static replicating portfolio suggests multi-asset options should be priced with the same model used for vanilla basket calls, under penalty of arbitrage. However, the potential presence of singularities corresponding to infinite quantities for some vanilla basket calls, as found for the best-of call in Section 4, together with the dynamic hedging challenges that may be associated with negative basket weights, might still justify some street adjustments not accounted for by our theory.

Overall, the results presented in this paper constitute a major step toward a comprehensive extension of the seminal work by Carr and Madan (1998) and Breeden and Litzenberger (1978) to the static replication and pricing of multi-asset options, leveraging on the advanced mathematical tools and theory of integral geometry and Radon transforms that have vast potential for further applications in quantitative finance and beyond. Future research in this direction is projected to cover all European multi-asset options with sufficiently regular payoff, with applications to include best-of and worst-of options on any number of assets,  $\ell_1$  dispersion options, correlation swaps and options, to name a few.

## APPENDIX A. DERIVATION OF THE FUNDAMENTAL SOLUTION FOR THE RADON TRANSFORM OPERATOR

By the Fourier slice theorem,

$$\widehat{\varphi_{\mathbf{K}}}(\lambda \mathbf{x}) = \left[ \frac{1}{|t|} (\mathcal{R}_1 \varphi_{\mathbf{K}})(\mathbf{x}/t) \right]^{\wedge}(\lambda) = \left[ \frac{1}{|t|} \delta(\mathbf{x}/t - \mathbf{K}) \right]^{\wedge}(\lambda) = [|t|^{n-1} \delta(\mathbf{x} - t\mathbf{K})]^{\wedge}(\lambda)$$

where we used the homogeneity of the delta function in the last step. Rescaling the left-hand side with respect to  $\lambda$  before taking the inverse multidimensional Fourier transform of both sides against  $\mathbf{x}$ , and switching the order of Fourier transforms to the right-hand side,

$$\begin{aligned} |\lambda|^{-n} \varphi_{\mathbf{K}}(\mathbf{y}/\lambda) &= \frac{1}{(2\pi)^n} \left[ |t|^{n-1} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\mathbf{y}} \delta(\mathbf{x} - t\mathbf{K}) \, d\mathbf{x} \right]^{\wedge}(\lambda) \\ &= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} |t|^{n-1} e^{-it(\lambda - \mathbf{K}\cdot\mathbf{y})} \, dt \\ &= \frac{1}{(2\pi)^n} [|t|^{n-1}]^{\wedge}(\lambda - \mathbf{K}\cdot\mathbf{y}). \end{aligned} \tag{14}$$

The Fourier transform of  $t \mapsto |t|^{n-1}$  is classically represented as

$$[|t|^{n-1}]^{\wedge}(\nu) = \begin{cases} 2\pi (-1)^{\frac{n-1}{2}} \delta^{(n-1)}(\nu), & n \text{ odd,} \\ 2 (-1)^{n/2} \frac{(n-1)!}{\nu^n} \mathcal{P}, & n \text{ even,} \end{cases}$$

wherein the pilcrow symbol  $\mathcal{P}$  indicates a *pseudofunction* subject to Hadamard regularization<sup>1</sup>. Substituting the above into equation (14) for  $\nu = \lambda - \mathbf{y}\cdot\mathbf{K}$ , homogenizing both sides against  $\lambda$  and simplifying yields the fundamental solution

$$\varphi_{\mathbf{K}}(\mathbf{y}) = \begin{cases} \frac{(-1)^{(n-1)/2}}{(2\pi)^{n-1}} \delta^{(n-1)}(1 - \mathbf{y}\cdot\mathbf{K}) & n \text{ odd,} \\ 2 \frac{(-1)^{n/2}}{(2\pi)^n} \frac{(n-1)!}{(1 - \mathbf{y}\cdot\mathbf{K})^n} \mathcal{P} & n \text{ even,} \end{cases}$$

which is singular over the hyperplane  $\mathbf{y}\cdot\mathbf{K} = 1$ . We may further unify the above expression by means of the Hilbert transform as

$$\varphi_{\mathbf{K}}(\mathbf{y}) = \frac{(-1)^{n-1}}{(2\pi)^{n-1}} \left( \mathcal{H}^{n-1} \delta^{(n-1)} \right) (1 - \mathbf{y}\cdot\mathbf{K}),$$

wherein  $\mathcal{H}^{n-1}$  is the composition of order  $n-1$  of the Hilbert transform.

## APPENDIX B. VERIFICATION OF SOLUTION FOR BEST-OF CALL

The Radon transform of solution (10) is

$$\mathcal{R}_1 \phi_{\vee}^{\mathcal{C}}(x_1, x_2) = -\frac{4}{\pi^2} \iint_{-\infty}^{\infty} \frac{\delta(x_1 y_1 + x_2 y_2 - 1)}{(1 - (y_1 - y_2)^2)(1 - (y_1 + y_2)^2)} \mathcal{P} \, dy_1 \, dy_2,$$

which is a singular integral needing regularization. As this can be a difficult endeavor when two variables are involved, we shall use the Fourier slice theorem whereby

$$\mathcal{R}_1 \phi(\mathbf{x}) = [\widehat{\phi}(\lambda \mathbf{x})]^{\vee}(1)$$

<sup>1</sup>In other words, the function  $\nu \mapsto 1/\nu^n$  for  $n$  even should be construed as a *tempered distribution* (a type of generalized function), just we have a generalized function in odd dimension. See Hadamard (1923, pp. 133–143) and Riesz (1938).

Partial fraction decomposition of  $\phi$  yields

$$\begin{aligned} \phi_{\vee}^C(y_1, y_2) = & -\frac{1}{2\pi^2} \left( \frac{1}{y_1} + \frac{1}{y_2} \right) \left( \frac{1}{1 - (y_1 + y_2)} - \frac{1}{1 + y_1 + y_2} \right) \\ & - \frac{1}{2\pi^2} \left( \frac{1}{y_1} - \frac{1}{y_2} \right) \left( \frac{1}{1 - (y_1 - y_2)} - \frac{1}{1 + y_1 - y_2} \right). \end{aligned}$$

By the convolution theorem,

$$\begin{aligned} \hat{\phi}_{\vee}^C = & -\frac{1}{2\pi^2} \frac{1}{(2\pi)^2} (-2i\pi^2 \delta(z_1) \operatorname{sgn} z_2 - 2i\pi^2 \delta(z_2) \operatorname{sgn} z_1) ** (4i\pi^2 \delta(z_1 - z_2) \cos z_1 \operatorname{sgn} z_1) \\ & - \frac{1}{2\pi^2} \frac{1}{(2\pi)^2} (2i\pi^2 \delta(z_1) \operatorname{sgn} z_2 - 2i\pi^2 \delta(z_2) \operatorname{sgn} z_1) ** (4i\pi^2 \delta(z_1 + z_2) \cos z_1 \operatorname{sgn} z_1) \end{aligned}$$

wherein the double asterisk  $**$  denotes bidimensional convolution. Convoluting and simplifying,

$$\hat{\phi}_{\vee}^C(x_1, x_2) = (\operatorname{sgn}(x_1 + x_2) - \operatorname{sgn}(x_1 - x_2)) \cos x_2 \operatorname{sgn} x_2 (\operatorname{sgn}(x_1 + x_2) + \operatorname{sgn}(x_1 - x_2)) \cos x_1 \operatorname{sgn} x_1$$

The inverse Fourier transform of  $\lambda \mapsto \hat{\phi}_{\vee}^C(\lambda x_1, \lambda x_2)$  is then

$$\begin{aligned} [\hat{\phi}_{\vee}^C]^{\vee}(t) = & (\operatorname{sgn}(x_1 + x_2) - \operatorname{sgn}(x_1 - x_2)) \operatorname{sgn} x_2 \overline{\cos(\lambda x_2)}(t) \\ & + (\operatorname{sgn}(x_1 + x_2) + \operatorname{sgn}(x_1 - x_2)) \operatorname{sgn} x_1 \overline{\cos(\lambda x_1)}(t) \\ = & \frac{1}{2} (\operatorname{sgn}(x_1 + x_2) - \operatorname{sgn}(x_1 - x_2)) \operatorname{sgn} x_2 (\delta(t - x_2) + \delta(t + x_2)) \\ & + \frac{1}{2} (\operatorname{sgn}(x_1 + x_2) + \operatorname{sgn}(x_1 - x_2)) \operatorname{sgn} x_1 (\delta(t - x_1) + \delta(t + x_1)) \end{aligned}$$

Substituting  $(\operatorname{sgn}(x_1 + x_2) + \operatorname{sgn}(x_1 - x_2)) \operatorname{sgn} x_1 = 2H(|x_2| - |x_1|)$ ,  $\delta(t - x_2) + \delta(t + x_2) = \delta(t - |x_2|)$  into the first term and similarly for the second term,

$$[\hat{\phi}(\lambda x_1, \lambda x_2)]^{\vee}(t) = \delta(t - |x_1|)H(|x_1| - |x_2|) + \delta(t - |x_2|)H(|x_2| - |x_1|) = \delta(t - |x_1| \vee |x_2|),$$

which evaluates to  $\delta(1 - |x_1| \vee |x_2|)$  as required for  $t = 1$ . Following Section 3, the best-of call is replicated as per equation (1a) up to an additive term  $h_1(x_1, x_2) + k h_2(x_1, x_2)$  with

$$\begin{cases} h_2(x_1, x_2) = (\operatorname{sgn} k) \lim_{k \rightarrow \infty} H(|x_1| \vee |x_2| - |k|) = 0 \\ h_1(x_1, x_2) = \lim_{k \rightarrow \infty} [(|x_1| \vee |x_2| - |k|)^+ - k h_2(x_1, x_2)] = 0, \end{cases}$$

which completes the proof that  $\hat{\phi}_{\vee}^C$  solves the best-of call replication problem.

## REFERENCES

- Austing, Peter (2011). “Repricing the Cross Smile: An Analytic Joint Density”. In: *Risk Magazine* (July).
- Bossu, Sébastien (2014). *Advanced Equity Derivatives*. John Wiley & Sons.
- Bossu, Sébastien, Peter Carr, and Andrew Papanicolaou (2020). *Static replication of European standard dispersion options*. Working Paper. NYU.
- Breeden, Douglas T. and Robert H. Litzenberger (1978). “Prices of state-contingent claims implicit in option prices”. In: *Journal of business*, pp. 621–651.
- Carr, Peter and Peter Laurence (2011). “Multi-asset Stochastic Local Variance Contracts”. In: *Mathematical Finance* 21.1, pp. 21–52.
- Carr, Peter and Dilip Madan (1998). “Towards a theory of volatility trading”. In: *Volatility: New estimation techniques for pricing derivatives*. Ed. by Robert A. Jarrow. Vol. 29. Risk books, pp. 417–427.

- Cui, Zhenyu and Yuewu Xu (2021). *A New Representation of the Risk-Neutral Distribution and Its Applications*. Working paper. Stevens Institute of Technology and Fordham University.
- Deans, Stanley R. (1983). *The Radon Transform and Some of Its Applications*. Wiley Interscience. John Wiley & Sons.
- Dupire, Bruno (1993). *Arbitrage pricing with stochastic volatility*. Tech. rep. Banque Paribas.
- Hadamard, Jacques (1923). *Lectures on Cauchy's problem in linear partial differential equations*. Yale University Press. URL: <https://openlibrary.org/books/OL14018648M>.
- Henkin, Gennadi Markovich and Aleksandr Alekseevich Shananin (1990). "Bernstein theorems and Radon transform, application to the theory of production functions". In: *Translations of Mathematical Monographs: Mathematical Problems of Tomography*. Ed. by Izrail M. Gelfand and Semen G. Gindikin. Vol. 81. American Mathematical Society, pp. 189–223.
- Langnau, Alex (2010). "A Dynamic Model for Correlation". In: *Risk Magazine* (April), pp. 74–78.
- Lipton, Alexander (2001). *Mathematical Methods For Foreign Exchange: A Financial Engineer's Approach*. World Scientific Publishing Company.
- Natterer, Frank (2001). *The Mathematics of Computerized Tomography*. Society for Industrial and Applied Mathematics.
- Reghai, Adil (2010). "Breaking Correlation Breaks". In: *Risk Magazine* (October), pp. 90–95.
- Riesz, Marcel (1938). "Intégrales de Riemann-Liouville et potentiels". In: *Acta Litt. Ac Sient. Univ. Hung. Francisco-Josephinae, Sec. Sci. Math.* 9:1–1, pp. 1–42.
- Rubin, Boris (2015). *Introduction to Radon Transforms*. Cambridge University Press.

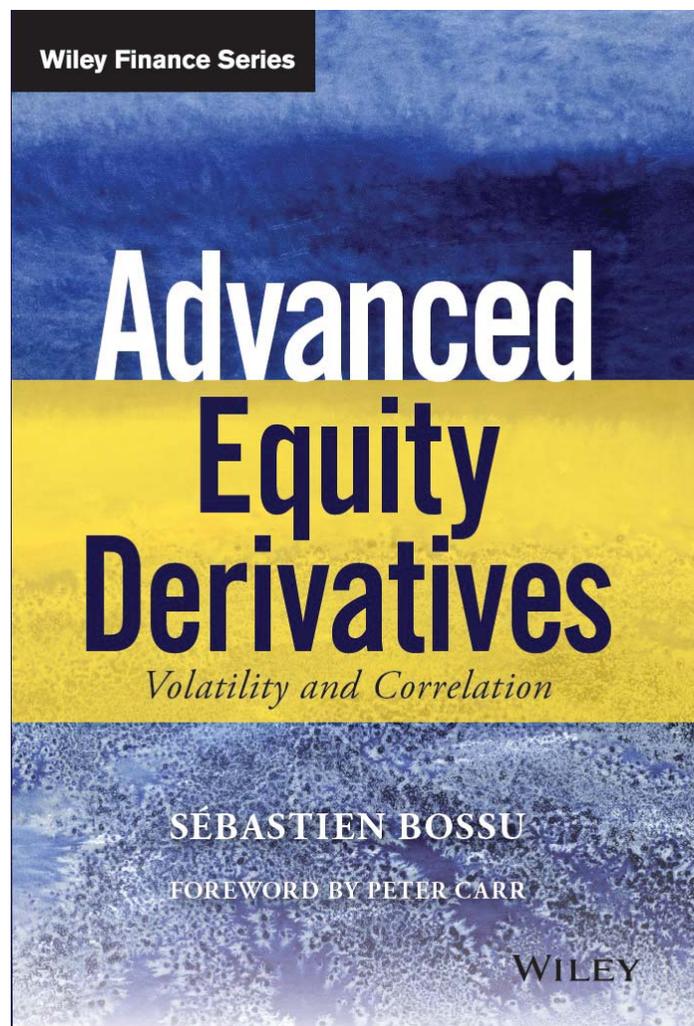




## Chapter 5

# Dynamic replication of correlation swaps

5.1 Excerpts from *Advanced Equity Derivatives*, Wiley, 2014



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# **Advanced Equity Derivatives**

*Volatility and Correlation*

SÉBASTIEN BOSSU

**WILEY**

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***Library of Congress Cataloging-in-Publication Data:***

Bossu, Sébastien.

Advanced equity derivatives : volatility and correlation / Sébastien Bossu.  
pages cm. – (Wiley finance series)

Includes bibliographical references and index.

ISBN 978-1-118-75096-4 (cloth); ISBN 978-1-118-77484-7 (ePDF);

ISBN 978-1-118-77471-7 (ePub)

1. Derivative securities. I. Title.

HG6024.A3B67 2014

332.64'57–dc23

2013046823

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

*“As for the expense,” gravely declared  
the deputy Haffner who never opened his mouth  
except on great occasions, “our children will pay for it,  
and nothing will be more just.”*

Emile Zola, *La Curée* (*The Kill*)



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## Foreword

I am pleased to introduce Sébastien Bossu's new book, *Advanced Equity Derivatives*, which is a great contribution to the literature in our field. Years of practical experience as an exotics structure, combined with strong theoretical skills, allowed Sébastien to write a condensed yet profound text on a variety of advanced topics: volatility derivatives and volatility trading, correlation modelling, dispersion trading, local and stochastic volatility models, to name just a few.

This book not only reviews the most important concepts and recent developments in option pricing and modelling, but also offers insightful explications of great relevance to researchers as well as traders. For instance, readers will find formulas to overhedge convex payoffs, the derivation of Feller conditions for the Heston model, or an exposition of the latest local correlation models to correctly price basket options.

Perhaps the most exciting aspect of this book is its treatment of the latest generation of equity derivatives, namely volatility and correlation derivatives. Readers will find a wealth of information on these new securities, including original analyses and models to approach their valuation. The chapters on correlation are particularly commendable, as they shed light on an otherwise still obscure area.

The content quality, selection of topics, and level of insight truly set this book apart. I have no doubt that equity derivatives practitioners around the world, be they traders, quants or investors, will find it extremely pertinent, and I wish this book every success.

Peter Carr

*Dr. Peter Carr has over 18 years of experience in the derivatives industry and is currently Global Head of Market Modeling at Morgan Stanley, as well as Executive Director of the Math Finance program at NYU's Courant Institute. He has over 70 publications in academic and industry-oriented journals and serves as an associate editor for eight journals related to mathematical finance. Dr. Carr is also the Treasurer of the Bachelier Finance Society, a trustee for the Museum of Mathematics in New York, and has received numerous awards, including Quant of the Year by Risk magazine in 2003, the ISA Medal for Science in 2008, and Financial Engineer of the Year in 2010.*



## Preface

In 2004, while working as an equity derivatives analyst at J.P. Morgan in London, I came upon an esoteric trade: someone was simultaneously selling correlation and buying it back for a (risky) profit using two different methods. I became obsessed with the rationale behind this trade, and, after writing down the math, I discovered with excitement that with some corrections this trade led to a pure dynamic arbitrage strategy—the kind you normally find only in textbooks.

I could see, however, that transaction costs and other market frictions made the strategy very hard to implement in practice, especially for price takers on the buy side. But the fact remained that correlation could be bought and sold at very different prices, and that didn't make sense to me. So I developed a simple “toy” model to see how this gap might be accounted for, and as I suspected I found that there should be little difference. What this meant is that one of the two correlation instruments involved in the trade, namely the correlation swap, was not priced at “fair value” according to my analysis.

Later on I refined my model, which I introduce in the last chapter of this book among other topics, and reached similar conclusions. I am very pleased that the topic of equity correlation has gained tremendous momentum since 2004, and it is one of this book's ambitions to introduce the work of others in this highly specialized field. I have no doubt that many new exciting results are yet to be discovered in the coming years.

I also wanted to cover other key advanced concepts in equity derivatives that are relevant to traders, quantitative analysts, and other professionals. Many of these concepts, such as implied distributions and local volatilities, are now well-known and established in the field, while others, such as local and stochastic correlation, lie at the forefront of current research.

To get the most out of this book, readers must already be familiar with the terminology and standard pricing theory of equity derivatives, which can be found in my textbook *Introduction to Equity Derivatives: Theory & Practice*, second edition, also published by John Wiley & Sons.

I relied on a fair amount of advanced mathematics, and therefore a graduate scientific education is a prerequisite here, especially for those readers who want to solve the problems included at the end of each chapter.

The book is made of nine chapters, which are meant to be read sequentially, starting with an exposition of the most widespread exotic derivatives and culminating with cutting-edge concepts on stochastic correlation, which are necessary to correctly price the next generation of equity derivatives such as correlation swaps.

Some simplifications, such as zero interest rates and dividends, were often necessary to avoid convoluted mathematical expressions. I strongly encourage readers to check the particular assumptions used for each formula before transposing it into another context.

I hope this book will prove insightful and useful to its target audience. I am always interested to hear feedback; please do not hesitate to contact me to share your thoughts.

## Acknowledgments

I would like to thank David Hait and his team at OptionMetrics for providing me with very useful option data. I am grateful to my team at Wiley—Bill Falloon, Meg Freeborn, and associates—for their guidance and professionalism throughout the publication process.

Many thanks also to a group of individuals who, directly or indirectly, made this book possible: Romain Barc, Martin Bertsch, Eynour Boutia, Peter Carr, Jose Casino, Mauro Cesa, Emanuel Derman, Jim Gatheral, Fabrice Rouah, Simone Russo, Roberto Silvotti, and Paul Wilmott.

Last, a special mention to John Lyttle at Ogee Group for his help on many figures and problem solutions.



# Exotic Derivatives

Strictly speaking, an exotic derivative is any derivative that is not a plain vanilla call or put. In this chapter we review the payoff and properties of the most widespread equity derivative exotics.

## 1-1 SINGLE-ASSET EXOTICS

### 1-1.1 Digital Options

A **European digital** or **binary option** pays off \$1 if the underlying asset price is above the strike  $K$  at maturity  $T$ , and 0 otherwise:

$$\text{Digital Payoff} = \begin{cases} 1 & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$$

In its American version, which is more uncommon, the option pays off \$1 *as soon as* the strike level is hit.

The Black-Scholes price formula for a digital option is simply:

$$e^{-rT}N(d_2) = e^{-rT}N\left(\frac{\ln(F/K) - \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)$$

where  $F$  is the forward price of  $S$  for maturity  $T$ ,  $r$  is the continuous interest rate, and  $\sigma$  is the volatility parameter. When there is an implied volatility smile this formula is inaccurate and a corrective term must be added (see Section 2-1.3).

Digital options are not easy to dynamically hedge because their delta can become very large near maturity. Exotic traders tend to overhedge them

with a tight call spread whose range may be determined according to several possible empirical rules, such as:

- *Daily volatility rule*: Set the range to match a typical stock price move over one day. For example, if the annual volatility of the underlying stock is 32% annually; that is,  $32\%/\sqrt{252} \approx 2\%$  daily, a digital option struck at \$100 would be overhedged with \$98–\$100 call spreads.
- *Normalized liquidity rule*: Set the range so that the quantity of call spreads is in line with the market liquidity of call spreads with 5% range. The quantity of call spreads is  $N/R$  where  $N$  is the quantity of digitals and  $R$  is the call spread range. If the tradable quantity of call spreads with range 5% is  $V$ , the normalized tradable quantity of call spreads with range  $R$  would be  $V \times R / 0.05$ . Solving for  $R$  gives  $R = \sqrt{0.05 \times \frac{N}{V}}$ . In practice  $V$  is either provided by the option trader or estimated using the daily trading volume of the stock.

### 1-1.2 Asian Options

In an **Asian call** or **put**, the final underlying asset price is replaced by an average:

$$\text{Asian Call Payoff} = \max(0, A_T - K)$$

$$\text{Asian Put Payoff} = \max(0, K - A_T)$$

where  $A_T = \frac{1}{n} \sum_{i=1}^n S_{t_i}$  for a set of pre-agreed fixing dates  $t_1 < t_2 < \dots < t_n \leq T$ . For example, a one-year at-the-money Asian call on the S&P 500 index with quarterly fixings pays off  $\max\left(0, \frac{S_{0.25} + S_{0.5} + S_{0.75} + S_1}{4} - S_0\right)$ , where  $S_0$  is the current spot price and  $S_{0.25}, \dots, S_1$  are the future spot prices observed every three months.

On occasion, the strike may also be replaced by an average, typically over a short initial observation period.

Fixed-strike Asian options are always cheaper than their European counterparts, because  $A_T$  is less volatile than  $S_T$ .

There is no closed-form Black-Scholes formula for arithmetic Asian options. However, for geometric Asian options where  $A_T = \exp\left[\frac{1}{T} \int_0^T \ln S_t dt\right]$ , the Black-Scholes formulas may be used with adjusted volatility  $\hat{\sigma} = \sigma/\sqrt{3}$  and dividend yield  $\hat{q} = \frac{1}{2} \left(r + q + \frac{\sigma^2}{6}\right)$ , as shown in Problem 1.3.

A common numerical approximation for the price of arithmetic Asian options is obtained by fitting a lognormal distribution to the actual risk-neutral moments of  $A_T$ .

### 1-1.3 Barrier Options

In a **barrier call** or **put**, the underlying asset price must hit, or never hit, a certain barrier level  $H$  before maturity:

- For a knock-in option, the underlying must hit the barrier, or else the option pays nothing.
- For a knock-out option, the underlying must *never* hit the barrier, or else the option pays nothing.

Barrier options are always cheaper than their European counterparts, because their payoff is subject to an additional constraint. On occasion, a fixed cash “rebate” is paid out if the barrier condition is not met.

Similar to digital options, barrier options are not easy to dynamically hedge: their delta can become very large near the barrier level. Exotic traders tend to overhedge them by shifting the barrier a little in their valuation model.

Continuously monitored barrier options have closed-form Black-Scholes formulas, which can be found, for instance, in Hull (2012). The preferred pricing approach is the local volatility model (see Chapter 4).

In practice the barrier is often monitored on a set of pre-agreed fixing dates  $t_1 < t_2 < \dots < t_n \leq T$ . Monte Carlo simulations are then commonly used for valuation.

Broadie, Glasserman, and Kou (1997) derived a nice result to switch between continuous and discrete barrier monitoring by shifting the barrier level  $H$  by a factor  $\exp(\pm\beta\sigma\sqrt{\Delta t})$  where  $\beta \approx 0.5826$ ,  $\sigma$  is the underlying volatility, and  $\Delta t$  is the time between two fixing dates.

### 1-1.4 Lookback Options

A **lookback call** or **put** is an option on the maximum or minimum price reached by the underlying asset until maturity:

$$\text{Lookback call payoff} = \max(0, \max_{0 \leq t \leq T} S_t - K);$$

$$\text{Lookback put payoff} = \max(0, K - \min_{0 \leq t \leq T} S_t).$$

Lookback options are always more expensive than their European counterparts: about twice as much when the strike is nearly at the money, as shown in Problem 1.5.

Continuously monitored lookback options have closed-form Black-Scholes formulas, which can be found, for instance, in Hull (2012). The preferred pricing approach is the local volatility model (see Chapter 4).

In practice the maximum or minimum is often monitored on a set of pre-agreed fixing dates  $t_1 < t_2 < \dots < t_n \leq T$ . Monte Carlo simulations are then commonly used for valuation.

### 1-1.5 Forward Start Options

In a **forward start option** the strike is determined as a percentage  $k$  of the spot price on a future start date  $t_0 > 0$ :

$$\text{Forward start call payoff} = \max(0, S_T - kS_{t_0});$$

$$\text{Forward start put payoff} = \max(0, kS_{t_0} - S_T).$$

At  $t = t_0$  a forward start option becomes a regular option. Note that the forward start feature is not specific to vanilla options and can be added to any exotic option that has a strike.

Forward start options have closed-form Black-Scholes formulas. The preferred pricing approach is to use a stochastic volatility model (see Chapter 4).

### 1-1.6 Cliquet Options

A **cliquet** or **ratchet option** consists of a series of consecutive forward start options, for example:

$$\text{Monthly cliquet option payoff} = \max \left[ 0, \sum_{i=1}^{12} \min \left( 5\%, \frac{S_{i/12}}{S_{(i-1)/12}} - 1 \right) \right]$$

where 5% is the local cap amount. In other words, this particular cliquet option pays off the greater of zero and the sum of monthly returns, each capped at 5%.

Cliquet options can be very difficult to value and especially hedge.

## 1-2 MULTI-ASSET EXOTICS

Multi-asset exotics are based on several underlying stocks or indices, and thus their fair value depends on the level of correlation between the underlying assets. They are typically priced on a Monte Carlo simulation engine with local volatilities (see Chapter 4 and Chapter 6, Section 6-5).

### 1-2.1 Spread Options

The payoff of a **spread option** is based on the difference in gross return between two underlying assets:

$$\text{Spread option payoff} = \max \left( 0, \frac{S_T^{(1)}}{S_0^{(1)}} - \frac{S_T^{(2)}}{S_0^{(2)}} - k \right)$$

where  $k$  is the residual strike level (in %). For example, a spread option on Apple Inc. vs Google Inc. with 5% strike pays off the outperformance of Apple over Google in excess of 5%: if Apple's return is 13% and Google's is 4%, the option pays off  $13\% - 4\% - 5\% = 4\%$ .

The value of a spread option is very sensitive to the level of correlation between the two assets. Specifically the option value increases as correlation decreases: the lower the correlation, the wider the two assets are expected to spread apart.

In practice hedging spread options can be difficult because the spread  $\frac{S_T^{(1)}}{S_0^{(1)}} - \frac{S_T^{(2)}}{S_0^{(2)}}$  is often nearly orthogonal to the basket  $\frac{1}{2} \left[ \frac{S_T^{(1)}}{S_0^{(1)}} + \frac{S_T^{(2)}}{S_0^{(2)}} \right] - 1$ .

When  $k = 0$  a spread option is also known as an exchange option. A closed-form Black-Scholes formula is then available which can be found, for instance, in Hull (2012).

## 1-2.2 Basket Options

A **basket call** or **put** is an option on the gross return of a portfolio of  $n$  underlying assets:

$$\text{Basket call payoff} = \max \left( 0, \sum_{i=1}^n w_i \frac{S_T^{(i)}}{S_0^{(i)}} - k \right);$$

$$\text{Basket put payoff} = \max \left( 0, k - \sum_{i=1}^n w_i \frac{S_T^{(i)}}{S_0^{(i)}} \right),$$

where the weights  $w_1, \dots, w_n$  sum to 100% and the strike  $k$  is expressed as a percentage (e.g., 100% for at the money).

### EXAMPLE

#### Equally-Weighted Stock Basket Call

Option seller: ABC Bank Co.  
 Notional amount: \$20,000,000  
 Issue date: [Today]  
 Maturity date: [Today + 3 years]

(Continued)

**EXAMPLE** *(Continued)*

## Equally-Weighted Stock Basket Call

Underlying stocks: IBM (IBM), Microsoft (MSFT), Google (GOOG)  
 Payoff:

$$\text{Notional} \times \max \left( 0, \frac{1}{3} \left( \frac{IBM_{\text{final}}}{IBM_{\text{initial}}} + \frac{MSFT_{\text{final}}}{MSFT_{\text{initial}}} + \frac{GOOG_{\text{final}}}{GOOG_{\text{initial}}} \right) - 1 \right)$$

Option price: 17.4%

The value of a basket option is sensitive to the level of pairwise correlations between the assets. The lower the correlation, the less volatile the portfolio and the cheaper the basket option.

Basket options do not have closed-form Black-Scholes formulas. A common approximation technique is to fit a lognormal distribution to the actual moments of the basket and then use formulas for the single-asset case.

**1-2.3 Worst-Of and Best-Of Options**

A **worst-of call** or **put** is an option on the lowest gross return between  $n$  underlying assets:

$$\text{Worst-of call payoff} = \max \left( 0, \min_{1 \leq i \leq n} \frac{S_T^{(i)}}{S_0^{(i)}} - k \right);$$

$$\text{Worst-of put payoff} = \max \left( 0, k - \min_{1 \leq i \leq n} \frac{S_T^{(i)}}{S_0^{(i)}} \right),$$

where the strike  $k$  is expressed as a percentage (e.g., 100% for at the money). For example, a worst-of at-the-money call on Apple, Google, and Microsoft pays off the worst stock return between the three companies, if positive.

Similarly, a **best-of call** or **put** is an option on the highest gross return between  $n$  underlying assets.

Worst-of calls and best-of puts are always cheaper than any of their single-asset European counterparts, while best-of calls and worst-of puts are always more expensive.

### 1-2.4 Quanto Options

The payoff of a **quanto option** is paid out in a different currency from the underlying assets, at a guaranteed exchange rate. For example, a call on the S&P 500 index quanto euro pays off  $\max(0, S_T - K)$  in euros instead of dollars, thereby guaranteeing an exchange rate of 1 euro per dollar.

The actual exchange rate between the asset currency and the quanto currency is in fact an implicit additional underlying asset. The value of quanto options is very sensitive to the correlation between the primary asset and the implicit exchange rate.

Quanto options are an example of hybrid exotic options involving different asset classes—here equity and foreign exchange.

In terms of pricing, the quanto feature is often approached using a technique called change of numeraire. In summary, this technique says that the risk-neutral dynamics of an asset quantoed in a different currency from its natural currency has the same volatility coefficient but an adjusted drift coefficient.

#### FOCUS ON CHANGE OF NUMERAIRE

This technique builds upon the concepts of change of measure and Girsanov's theorem, which are explained in Appendix 1.A.

Consider a world with two currencies, say dollars and euros, and a non-income-paying asset  $S$  with dollar price  $S^{\$}$  and euro price  $S^{\text{€}}$ . Denote  $X$  the exchange rate of one dollar into euros, so that  $S_t^{\text{€}} = S_t^{\$} X_t$ . Assume that  $S^{\$}$  and  $X$  both follow a geometric Brownian motion under the dollar risk-neutral measure  $\mathbb{Q}^{\$}$ , specifically:

$$\text{For } S^{\$}: dS_t^{\$}/S_t^{\$} = r_{\$} dt + \sigma dW_t$$

$$\text{For } X: dX_t/X_t = v dt + \eta dZ_t$$

where  $W, Z$  are standard Brownian motions under  $\mathbb{Q}^{\$}$  with correlation  $\rho$ ,  $r_{\$}$  is the dollar interest rate, and all other parameters are free.

Because the original Girsanov theorem applies to independent Brownian motions, we rewrite  $Z = \rho W + \bar{\rho} W^{\perp}$  where  $W^{\perp}$  is a standard Brownian motion under  $\mathbb{Q}^{\$}$  independent from  $W$  and  $\bar{\rho} = \sqrt{1 - \rho^2}$  is the orthogonal complement of  $\rho$ . The diffusion equation for  $X$  then becomes:

$$dX_t/X_t = v dt + \eta \rho dW_t + \eta \bar{\rho} dW_t^{\perp}$$

Applying the Ito-Doebelin theorem to the product  $S_t^\epsilon = S_t^\$ X_t$  we obtain after simplifying:

$$dS_t^\epsilon/S_t^\epsilon = (r_\$ + \nu + \rho\sigma\eta)dt + (\sigma + \rho\eta)dW_t + \eta\bar{\rho}dW_t^\perp \quad (1.1)$$

Because  $S^\epsilon$  is a euro tradable asset we must also have:

$$dS_t^\epsilon/S_t^\epsilon = r_\epsilon dt + (\sigma + \rho\eta)d\tilde{W}_t + \eta\bar{\rho}d\tilde{W}_t^\perp \quad (1.2)$$

where  $\tilde{W}, \tilde{W}^\perp$  are independent standard Brownian motions under the euro risk-neutral measure  $\mathbb{Q}^\epsilon$ . This is the diffusion equation of the *composite* asset S after conversion from dollars to euros.

The processes  $\tilde{W}, \tilde{W}^\perp$  are affine transformations of the original processes  $W, W^\perp$ ; specifically:

$$\begin{cases} \tilde{W}_t = W_t + \gamma_1 t \\ \tilde{W}_t^\perp = W_t^\perp + \gamma_2 t \end{cases}$$

where  $\gamma_1$  and  $\gamma_2$  are particular coefficients. Substituting into Equation (1.1) and connecting with Equation (1.2) we obtain that  $\gamma_1, \gamma_2$  must satisfy:

$$r_\$ + \nu + \rho\sigma\eta = r_\epsilon + \gamma_1(\sigma + \rho\eta) + \gamma_2\eta\bar{\rho}$$

In order to determine  $\gamma_1, \gamma_2$  uniquely, we need another equation. This is provided by the dynamics of  $X$ , which is a euro-tradable asset (it is the price in euros of \$1):

$$dX_t/X_t = (r_\epsilon - r_\$)dt + \eta\rho d\tilde{W}_t + \eta\bar{\rho}d\tilde{W}_t^\perp$$

Following the same reasoning we find that  $\gamma_1, \gamma_2$  must also satisfy:

$$\nu = r_\epsilon - r_\$ + \gamma_1\eta\rho + \gamma_2\eta\bar{\rho}$$

Solving for  $\gamma_1, \gamma_2$  we find:

$$\begin{cases} \gamma_1 = \eta\rho \\ \gamma_2 = \frac{\nu + r_\$ - r_\epsilon - \eta^2\rho^2}{\eta\bar{\rho}} \end{cases}$$

The dynamics of  $S^\$$  may thus be rewritten as:

$$dS_t^\$/S_t^\$ = r_\$ dt + \sigma dW_t = r_\$ dt + \sigma(d\tilde{W}_t - \gamma_1 dt) = (r_\$ - \rho\sigma\eta)dt + \sigma d\tilde{W}_t$$

This is the diffusion equation for  $S^{\$}$  quanto euro. In particular, the forward price of  $S^{\$}$  quanto euro is:

$$\mathbb{E}^{\mathbb{Q}^{\$}}(S_T^{\$}) = S_0^{\$} e^{(r_{\$} - \rho\sigma\eta)T}$$

### 1-3 STRUCTURED PRODUCTS

Structured products combine several securities together, especially exotic options. They are typically sold as equity-linked notes (ELN) or mutual funds to small investors as well as large institutions. These notes and funds are sometimes traded on exchanges.

#### EXAMPLE

##### Capital Guaranteed

##### Performance Note

Issuer: ABC Bank Co.

Notional amount: \$10,000,000

Issue date: [Today]

Maturity date: [Today + 5 years]

Underlying index: S&P 500 (SPX)

Payoff:

$$\text{Notional} \times \left[ 100\% + \text{Participation} \times \max \left( 0, \frac{SPX_{\text{final}}}{SPX_{\text{initial}}} - 1 \right) \right]$$

Participation: 50%

##### Reverse Convertible Note

Issuer: ABC Bank Co.

Notional amount: €2,000,000

Issue date: [Today]

Maturity date: [Today + 3 years]

Underlying stock: Kroger Co. (KR)

Payoff:

(a) If, between the start and maturity dates, Kroger Co. always trades above the Barrier level, Issuer will pay:

$$\text{Notional} \times \max \left( 115\%, \frac{S_{\text{final}}}{S_{\text{initial}}} \right)$$

(b) Otherwise, Issuer will pay:

$$\text{Notional} \times \frac{S_{\text{final}}}{S_{\text{initial}}}$$

Barrier level: 70%

In the Capital Guaranteed Performance Note, investors are guaranteed<sup>1</sup> to get their \$10 mn capital back after five years. This is much safer than a direct \$10 mn investment in the S&P 500 index, which could result in a loss. In exchange for this protection, investors receive a smaller share in the S&P 500 performance: 50% instead of 100%.

In the Reverse Convertible Note, investors may lose on their €2 mn capital if Kroger Co. ever trades below the 70% barrier, but never more than a direct investment in the stock (ignoring dividends). Otherwise, investors receive at least €2.3 mn after three years, and never less than a direct investment in the stock (again, ignoring dividends).

In some cases it is possible to break down a structured product into a portfolio of securities whose prices are known and find its value. In all other cases the payoff is typically programmed on a Monte Carlo simulation engine.

Multi-asset structured products significantly expand the payoff possibilities of exotic options. They allow investors to play on correlation and express complex investment views.

### EXAMPLE

#### Worst-Of Reverse Convertible Note Quanto CHF

Issuer: ABC Bank Co.

Notional amount: CHF 5,000,000

Issue date: [Today]

Maturity date: [Today + 3 years]

Underlying indexes: S&P 500 (SPX), EuroStoxx-50 (SX5E), Nikkei 225 (NKY)

Payoff:

- (a) If, between the start and maturity dates, all underlying indexes always trade above the Barrier level, Issuer will pay:

$$\text{Notional} \times \max \left( 120\%, \min \left( \frac{SPX_{\text{final}}}{SPX_{\text{initial}}}, \frac{SX5E_{\text{final}}}{SX5E_{\text{initial}}}, \frac{NKY_{\text{final}}}{NKY_{\text{initial}}} \right) \right)$$

(Continued)

<sup>1</sup>Provided the issuer does not go bankrupt.

**EXAMPLE** (Continued)

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Worst-Of Reverse Convertible Note Quanto CHF

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(b) Otherwise, Issuer will pay:

$$\text{Notional} \times \min \left( \frac{SPX_{\text{final}}}{SPX_{\text{initial}}}, \frac{SX5E_{\text{final}}}{SX5E_{\text{initial}}}, \frac{NKY_{\text{final}}}{NKY_{\text{initial}}} \right)$$

Barrier level: 50% of Initial Price

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Multi-asset structured product valuation is almost always done using Monte Carlo simulations. Hedging correlation risk is often difficult or expensive, and exotic trading desks tend to accumulate large exposures, which can cause significant losses during a market crash.

**REFERENCES**

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- Baxter, Martin, and Andrew Rennie. 1996. *Financial Calculus: An Introduction to Derivative Pricing*. New York: Cambridge University Press.
- Brodie, Mark, Paul Glasserman, and Steven Kou. 1997. "A Continuity Correction for Discrete Barrier Options." *Mathematical Finance* 7 (4): 325–348.
- Hull, John C. 2012. *Option, Futures, and Other Derivatives*, 8th ed. New York: Prentice Hall.

**PROBLEMS****1.1 "Free" Option**


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Consider a European call option on an underlying asset  $S$  with strike  $K$  and maturity  $T$  where "you only pay the premium if you win," that is, if  $S_T > K$ .

- Draw the diagram of the net P&L of this "free" option at maturity. Is it really "free"?
- Find a replicating portfolio for the "free" option using vanilla and exotic options.
- Calculate the fair value of the "free" option premium using the Black-Scholes model with 20% volatility,  $S_0 = K = \$100$ , one-year maturity, zero interest and dividend rates.

## 1.2 Autocallable

Consider an exotic option expiring in one, two, or three years on an underlying asset  $S$  with the following payoff mechanism:

- If after one year  $S_1 > S_0$  the option pays off  $1 + C$  and terminates;
- Else if after two years  $S_2 > S_0$  the option pays off  $1 + 2C$  and terminates;
- Else if after three years  $S_3 > 0.7 \times S_0$  the option pays off  $\max(1 + 3C, S_3/S_0)$ ;
- Otherwise, the option pays off  $S_3/S_0$ .

Assuming  $S_0 = \$100$ , zero interest and dividend rates, and 25% volatility, estimate the level of  $C$  so that the option is worth 1 using Monte Carlo simulations.

## 1.3 Geometric Asian Option

Consider a geometric Asian option on an underlying  $S$  with payoff  $f(A_T)$  where  $A_T = \exp\left(\frac{1}{T} \int_0^T \ln S_t dt\right)$ . Assume that  $S$  follows a geometric Brownian motion with parameters  $(r - q, \sigma)$  under the risk-neutral measure.

(a) Using the Ito-Doebelin theorem, show that

$$A_T = S_0 \exp\left(\frac{1}{2} \left(r - q - \frac{1}{2}\sigma^2\right) T + \frac{\sigma}{T} \int_0^T W_t dt\right)$$

(b) Using the Ito-Doebelin theorem, show that  $\int_0^T W_t dt = \int_0^T (T - t) dW_t$ . What is the distribution of this quantity?

(c) Show that  $A_T$  is lognormally distributed with parameters  $\left(\ln S_0 + \left(r - \hat{q} - \frac{1}{2}\hat{\sigma}^2\right) T, \hat{\sigma}\sqrt{T}\right)$  where  $\hat{\sigma} = \sigma/\sqrt{3}$  and  $\hat{q} = \frac{1}{2}\left(r + q + \frac{\sigma^2}{6}\right)$ .

## 1.4 Change of Measure

In the context of Appendix 1.A, verify that  $\mathbb{E}^{\mathbb{Q}}(S_T) = S_0 e^{rT}$  using the expression for  $d\mathbb{Q}/d\mathbb{P}$ .

## 1.5 At-the-Money Lookback Options

The Black-Scholes closed-form formula for an at-the-money lookback call is given as:

$$\text{Lookback}_0 = e^{-rT} S_0 (N(-\alpha_2) - 1) + S_0 N(\alpha_1) \left(1 + \frac{\sigma^2}{2r}\right) - e^{-rT} \frac{\sigma^2}{2r} S_0 N(\alpha_3)$$

where  $\alpha_{1,2} = \left(\frac{r}{\sigma} \pm \frac{1}{2}\sigma\right) \sqrt{T}$  and  $\alpha_3 = \left(-\frac{r}{\sigma} + \frac{1}{2}\sigma\right) \sqrt{T}$ .

Using a first-order Taylor expansion of the cumulative normal distribution  $N(\cdot)$  show that for reasonable rates and maturities we have the proxy:

$$\text{Lookback}_0 \approx \frac{4S_0\sigma\sqrt{T}}{\sqrt{2\pi}}$$

which is twice as much as the European call proxy:  $c_0 \approx \frac{2S_0\sigma\sqrt{T}}{\sqrt{2\pi}}$ .

### 1.6 Siegel's Paradox

*This problem is about foreign exchange rates and goes beyond the scope of equity derivatives.*

Consider two currencies, say dollars and euros, and suppose that their corresponding interest rates,  $r_{\$}$  and  $r_{\text{€}}$ , are constant. Let  $X$  be the euro–dollar exchange rate defined as the number of dollars per euro. The traditional risk-neutral process for  $X$  is thus:

$$dX_t = (r_{\$} - r_{\text{€}})X_t dt + \sigma X_t dW_t$$

where  $W$  is a standard Brownian motion.

- (a) Using the Ito-Doeblin theorem, show that the risk-neutral dynamics for the dollar-euro exchange rate, that is, the number  $1/X$  of euros per dollar, is:

$$d\frac{1}{X_t} = (r_{\text{€}} - r_{\$} + \sigma^2)\frac{1}{X_t} dt + \sigma\frac{1}{X_t} dW_t$$

- (b) Symmetry suggests that the drift of  $1/X$  should be  $r_{\text{€}} - r_{\$}$  instead—this is Siegel's paradox. Use your knowledge of quantos (see Section 1-2.4) to resolve the paradox.

## APPENDIX 1.A: CHANGE OF MEASURE AND GIRSANOV'S THEOREM

Recall that the Black-Scholes model assumes that the underlying asset price process follows a geometric Brownian motion:

$$dS_t/S_t = \mu dt + \sigma dW_t$$

where  $W$  is a standard Brownian motion under some objective probability measure  $\mathbb{P}$ ,  $\mu$  is the objective drift coefficient, and  $\sigma$  is the objective volatility coefficient.

However, the drift coefficient  $\mu$  disappears from option pricing equations as a result of delta-hedging, and option prices may equivalently be calculated as discounted expected payoffs under a special probability

measure  $\mathbb{Q}$  called risk-neutral. Under  $\mathbb{Q}$ , the underlying asset price process follows the geometric Brownian motion:

$$dS_t/S_t = rdt + \sigma dW'_t$$

where  $W'$  is a standard Brownian motion under  $\mathbb{Q}$ ,  $r$  is the continuous interest rate, and  $\sigma$  is the *same* volatility coefficient.

To understand how  $\mathbb{P}$  and  $\mathbb{Q}$  relate, consider the *undiscounted* expected payoff:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(f(S_T)) &= \int_0^{\infty} f(s) \mathbb{Q}\{S_T = s\} ds \\ &= \int_0^{\infty} f(s) \frac{\mathbb{Q}\{S_T = s\}}{\mathbb{P}\{S_T = s\}} \mathbb{P}\{S_T = s\} ds \end{aligned}$$

If we define the ratio of densities  $h(s) = \frac{\mathbb{Q}\{S_T=s\}}{\mathbb{P}\{S_T=s\}}$  then we can write:

$$\mathbb{E}^{\mathbb{Q}}(f(S_T)) = \int_0^{\infty} f(s) h(s) \mathbb{P}\{S_T = s\} ds = \mathbb{E}^{\mathbb{P}}(f(S_T) h(S_T)).$$

The change of measure from  $\mathbb{P}$  to  $\mathbb{Q}$  is thus equivalent to multiplying by the random variable  $h(S_T)$  called a Radon-Nikodym derivative and properly denoted  $\frac{d\mathbb{Q}}{d\mathbb{P}}$ . Girsanov's theorem states that  $\mathbb{Q}$  exists and is properly defined by a Radon-Nikodym derivative of the form:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\frac{r-\mu}{\sigma} W_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right)$$

Furthermore,  $W'_t = W_t + \frac{r-\mu}{\sigma} t$  is then a Brownian motion under  $\mathbb{Q}$ . Problem 1.4 verifies that  $\mathbb{E}^{\mathbb{Q}}(S_T) = S_0 e^{rT}$ .

For a rigorous yet accessible exposition of the change of measure technique and Girsanov's theorem we refer the reader to Baxter and Rennie (1996).

# Introducing Correlation

Correlation is almost as ubiquitous as volatility in quantitative finance. For example the downward-sloping volatility smile observed in equities may be explained by the negative correlation between stock prices and volatility. In this chapter we introduce various measures of correlation between assets, investigate their properties, and present simple multiasset extensions of the Black-Scholes and Local Volatility models.

## 6-1 MEASURING CORRELATION

Correlation is the degree to which two quantities are linearly associated. A correlation of +1 or -1 means that the linear relationship is perfect, while a correlation of 0 typically<sup>1</sup> indicates independence.

There are two kinds of correlation between two financial assets:

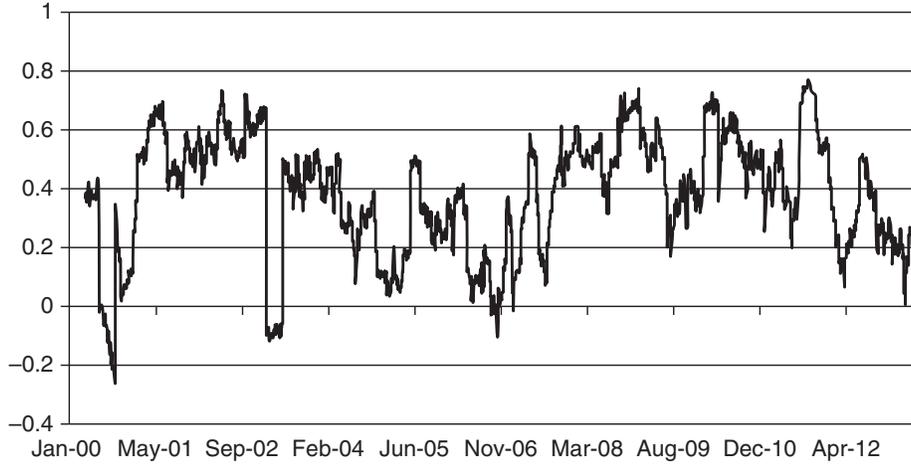
1. Historical correlation, based on historical returns;
2. Implied correlation, derived from option prices.

### 6-1.1 Historical Correlation

Historical correlation between two assets  $S^{(1)}$  and  $S^{(2)}$  is usually measured as the Pearson's correlation coefficient between their  $N$  historical returns observed at regular intervals:

$$\rho_{1,2}^{\dagger} = \frac{\text{Cov}_{1,2}^{\dagger}}{\sigma_1^{\dagger} \sigma_2^{\dagger}} = \frac{\sum_{i=1}^N (r_i^{(1)} - \bar{r}^{(1)}) (r_i^{(2)} - \bar{r}^{(2)})}{\sqrt{\sum_{i=1}^N (r_i^{(1)} - \bar{r}^{(1)})^2 \times \sum_{i=1}^N (r_i^{(2)} - \bar{r}^{(2)})^2}}$$

<sup>1</sup>Recall that if two random variables  $X, Y$  are independent their correlation must be zero; however, the converse is not necessarily true.



**FIGURE 6.1** Historical correlation of daily returns between Apple and Microsoft over a three-month rolling window since 2000.

where  $\text{Cov}_{1,2}^\dagger$  is historical covariance,  $\sigma^\dagger$ 's are historical standard deviations,  $r_i^{(j)}$  is the return on asset  $S^{(j)}$  for observation  $i$ , and  $\bar{r}^{(j)} = \frac{1}{N} \sum_{i=1}^N r_i^{(j)}$  is the mean return on asset  $S^{(j)}$ . Returns may be computed on an arithmetic or logarithmic basis; occasionally the mean returns are assumed to be zero.

Figure 6.1 shows the evolution of the historical correlation between Microsoft and Apple over a three-month rolling window since 2000. We can see that this correlation has varied quite significantly over time.

Note that using daily returns can produce misleading results for assets trading within different time zones; in this case it is preferable to estimate correlation using weekly returns. Figure 6.2 compares the two methods for the S&P 500 and Nikkei 225 indexes. We can see that the correlation observed on weekly returns is significantly higher.

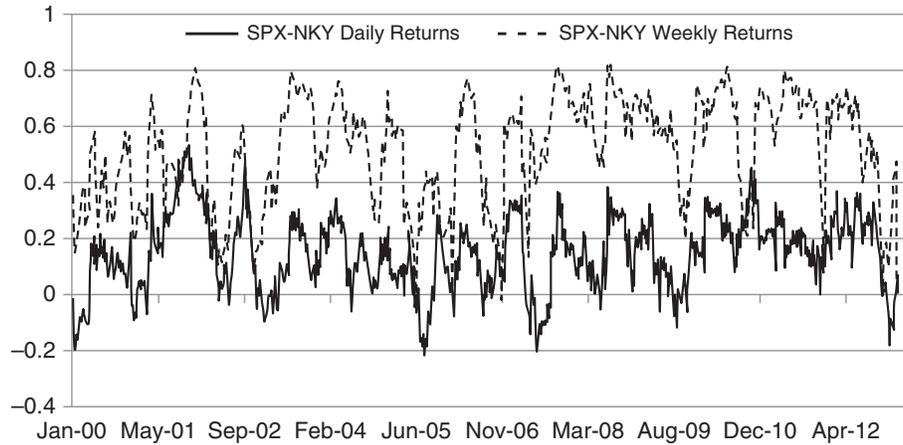
### 6-1.2 Implied Correlation

Implied correlation between two assets  $S^{(1)}$  and  $S^{(2)}$  is derived from an option price, such as a quote for an over-the-counter (OTC) basket option. Typically the quote is converted into an implied basket volatility  $\sigma_{\text{Basket}}^*$  from which implied correlation may be extracted through the formula:

$$\sigma_{\text{Basket}}^* = \sqrt{w_1^2 \sigma_1^{*2} + w_2^2 \sigma_2^{*2} + 2w_1 w_2 \sigma_1^* \sigma_2^* \rho_{1,2}^*}, \text{ that is,}$$

$$\rho_{1,2}^* = \frac{\sigma_{\text{Basket}}^{*2} - w_1^2 \sigma_1^{*2} - w_2^2 \sigma_2^{*2}}{2w_1 w_2 \sigma_1^* \sigma_2^*} = \frac{\sigma_{\text{Basket}}^{*2} - (w_1^2 \sigma_1^{*2} + w_2^2 \sigma_2^{*2})}{(w_1 \sigma_1^* + w_2 \sigma_2^*)^2 - (w_1^2 \sigma_1^{*2} + w_2^2 \sigma_2^{*2})}$$

where  $w_j$  is the weight on asset  $S^{(j)}$  and  $\sigma_j^*$  is the implied volatility of asset  $S^{(j)}$ .



**FIGURE 6.2** Historical correlation of daily and weekly returns between S&P 500 and Nikkei 225 over a three-month rolling window since 2000.

Conventionally all implied volatilities are for the same moneyness level  $k$  (strike over spot) and maturity  $T$ , and weights are equal.

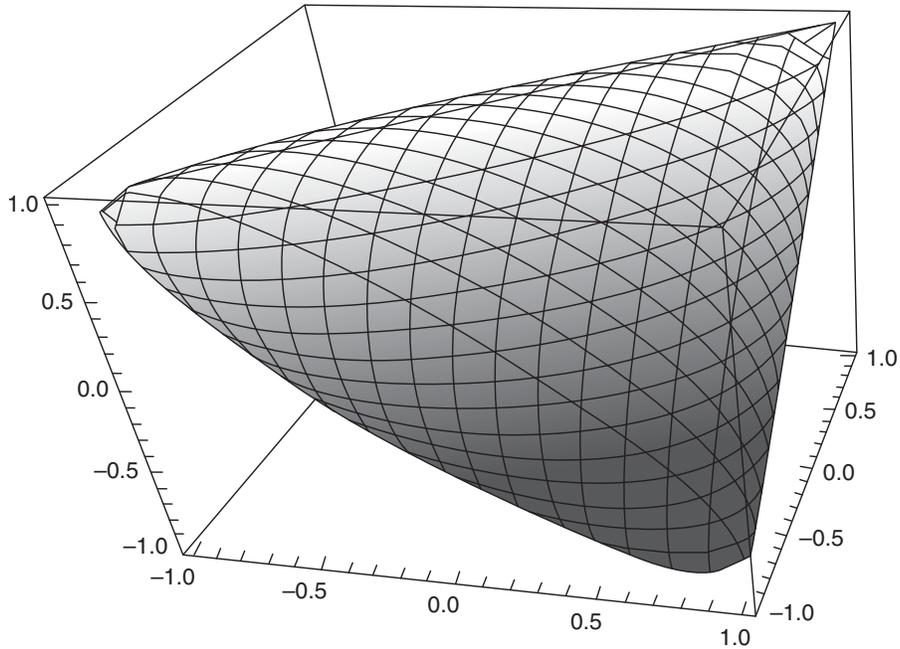
## 6-2 CORRELATION MATRICES

Very often we are interested in correlation for a selection of  $n \geq 2$  assets. This leads to a correlation matrix of the form:

$$R = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,n} \\ \rho_{2,1} & 1 & \rho_{2,3} & \cdots & \rho_{2,n} \\ \rho_{3,1} & \rho_{3,2} & 1 & \cdots & \rho_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1} & \rho_{n,2} & \rho_{n,3} & \cdots & 1 \end{pmatrix}$$

where  $\rho_{i,j}$  is the pairwise correlation coefficient between assets  $S^{(i)}$  and  $S^{(j)}$ , which may either be historical or implied. Note that  $R$  is symmetric because  $\rho_{i,j} = \rho_{j,i}$ .

Not every symmetric matrix with entries in  $[-1, 1]$  and a diagonal of 1's is a candidate for a correlation matrix  $R$ . This is because the correlation between assets  $S^{(i)}$  and  $S^{(j)}$  and assets  $S^{(j)}$  and  $S^{(k)}$  says something about the correlation between assets  $S^{(i)}$  and  $S^{(k)}$ —intuitively, if Microsoft and Apple are highly correlated, and Apple and IBM are also highly correlated, then Microsoft and IBM must also have some positive correlation.



**FIGURE 6.3** Envelope of admissible correlation values when  $n = 3$ .

Figure 6.3 shows the envelope of admissible correlation values when  $n = 3$ . We can see that certain regions, such as around the corner  $(-1, -1, -1)$  are not admissible.

Specifically, correlation matrices must be positive-semidefinite; that is, their eigenvalues must all be nonnegative. This property is always verified for historical correlation but not necessarily for implied correlation. Additionally the sum of all eigenvalues must equal the trace, that is,  $n$ .

A common fix for an indefinite candidate matrix  $M$  is to replace its negative eigenvalues with zeros and adjust its positive eigenvalues to maintain a sum of  $n$ :

$$R = \Omega D_{\text{adj}} \Omega^T$$

where  $\Omega$  is the orthogonal matrix of eigenvectors of  $M$  with eigenvalues  $(\lambda_1, \dots, \lambda_n)$  and  $D_{\text{adj}}$  is the diagonal matrix of adjusted eigenvalues with entries  $\lambda_i^{\text{adj}} = \frac{\lambda_i^+}{\sum_{j=1}^n \lambda_j^+}$ . Alternatively one may use the method proposed by Higham (2002).

In equities correlation matrices have other empirical properties. Plerou et al. (2002) and Potters, Bouchaud, and Laloux (2005) found for U.S. stocks

that the top eigenvalue typically dominates all the other ones. Furthermore, the corresponding eigenvector is more or less an equally weighted portfolio of all the stocks. This suggests that one factor (“the market”) strongly drives the behavior of each stock.

### 6-3 CORRELATION AVERAGE

To summarize the overall level of correlation across  $n$  assets, it is common practice to compute the average of the correlation matrix, excluding the diagonal of 1’s. The formula for a given weighting vector  $x$  is then:

$$\rho(x) = \frac{\sum_{i<j} x_i x_j \rho_{i,j}}{\sum_{i<j} x_i x_j} = \frac{x^T R x - x^T x}{(x^T e)^2 - x^T x} \quad (6.1)$$

where  $e$  is the vector of 1’s. In the main case of interest where all the weights are nonnegative we have:

$$-1 \leq -\frac{x^T x}{(x^T e)^2 - x^T x} \leq \rho(x) \leq (n-1) \frac{x^T x}{(x^T e)^2 - x^T x} \leq 1$$

but in general  $\rho(x)$  could lie outside of these bounds. In practice, when applying sensible weights to a large equity correlation matrix,  $\rho(x)$  can safely be assumed to be positive.

Common choices for  $x$  are:

- Equal weights:  $x = e$ . In this case the average correlation formula simplifies to:

$$\rho(e) = \frac{2}{n(n-1)} \sum_{i<j} \rho_{i,j}$$

and we have the bounds:

$$-\frac{1}{n-1} \leq \rho(e) \leq 1$$

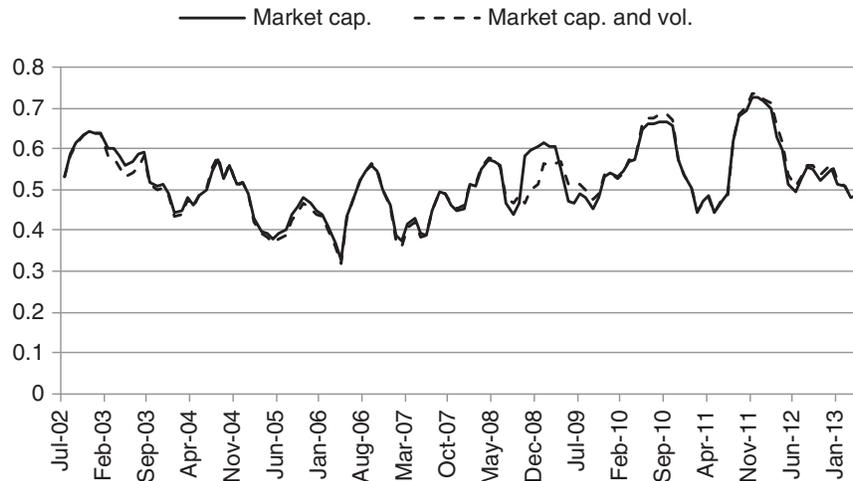
- Market capitalization weights:  $x = w$ . This is particularly relevant when the  $n$  stocks are the constituents of an equity index such as the S&P 500.

- Volatility and market capitalization weights:  $x = (w_1\sigma_1, \dots, w_n\sigma_n)^T$ , where  $\sigma$ 's may either be historical or implied volatilities. This case is particularly appealing because of the identity<sup>2</sup> or shortcut formula:

$$\rho \begin{pmatrix} w_1\sigma_1 \\ \vdots \\ w_n\sigma_n \end{pmatrix} = \frac{\sigma_{\text{Basket}}^2 - \sum_{i=1}^n w_i^2 \sigma_i^2}{\left( \sum_{i=1}^n w_i \sigma_i \right)^2 - \sum_{i=1}^n w_i^2 \sigma_i^2}$$

where  $\sigma_{\text{Basket}}$  is the volatility of the all-stock portfolio with weights  $w$ . Assimilating an equity index to a portfolio of stocks with fixed weights,<sup>3</sup> this formula allows us to compute the average implied correlation using only listed option prices.

In practice, for large baskets ( $n > 30$ ), these various choices for  $x$  tend to produce similar results within a few correlation points, as observed by Tierens and Anadu (2004) and illustrated in Figure 6.4.



**FIGURE 6.4** Realized average correlation for the EuroStoxx 50 index over a six-month rolling window using market capitalization weights and volatility and market capitalization weights.

<sup>2</sup>Note that the identity is exact for arithmetic returns but only approximate for logarithmic returns.

<sup>3</sup>Note that in reality equity index weights continuously change with stock prices. However, these variations tend to be limited, especially over short time horizons.

### 6-3.1 Correlation Proxy

Equation (6.1) is related to a mathematical quantity known as the Rayleigh quotient  $\mathfrak{R}(x) = \frac{x^T R x}{x^T x}$ ; specifically, dividing both numerator and denominator by  $n x^T x = (x^T x)(e^T e)$ :

$$\rho(x) = \frac{\frac{1}{n} \mathfrak{R}(x) - \frac{1}{n}}{\frac{(x^T e)^2}{(x^T x)(e^T e)} - \frac{1}{n}} = \frac{\frac{1}{n} \mathfrak{R}(x) - \frac{1}{n}}{\cos^2 \theta - \frac{1}{n}}$$

where  $\theta$  is the angle between vectors  $x$  and  $e$ .

As  $n \rightarrow \infty$  we have the proxy formula:

$$\rho(x) \sim \frac{\frac{1}{n} \mathfrak{R}(x)}{\cos^2 \theta} = \frac{x^T R x}{(x^T e)^2}$$

subject to certain technical conditions, which are met in practice. In particular, for volatility and market capitalization weights, the proxy formula equates the now well-known squared ratio of basket volatility to average stock volatility:

$$\rho(x) \sim \left( \frac{\sigma_{\text{Basket}}}{\sum_i w_i \sigma_i} \right)^2$$

#### FOCUS ON THE PROXY FORMULA

It is easy to establish the proxy formula when all correlation coefficients are positive (see, e.g., Bossu and Henrotte (2012)). However, when some correlation coefficients are negative we must use a more elaborate proof. Specifically, using the spectral decomposition of  $R$ , we may write:

$$\mathfrak{R}(x) = \sum_{i=1}^n \lambda_i \frac{(x^T v_i)^2}{(x^T x)(v_i^T v_i)}$$

where  $v$ 's form an orthogonal basis of eigenvectors and  $\lambda$ 's are their associated eigenvalues.

Therefore  $\mathfrak{R}(x) \geq \sum_{i=1}^n \lambda_i \min_{1 \leq j \leq n} \frac{(x^T v_j)^2}{(x^T x)(v_j^T v_j)} = n \min_{1 \leq j \leq n} \frac{(x^T v_j)^2}{(x^T x)(v_j^T v_j)}$  since all eigenvalues must sum to  $n$ . Assuming that  $x$  is never orthogonal to any eigenvector  $v_j$  (also in the limit) then  $\mathfrak{R}(x) \rightarrow \infty$  and thus  $\frac{1}{n}\mathfrak{R}(x) - \frac{1}{n} \sim \frac{1}{n}\mathfrak{R}(x)$ . Furthermore, if  $x$  is also never orthogonal to  $e$  (also in the limit) then  $\cos^2\theta - \frac{1}{n} \sim \cos^2\theta$ , which completes the proof that  $\rho(x) \sim \frac{\frac{1}{n}\mathfrak{R}(x)}{\cos^2\theta} = \frac{x^T R x}{(x^T e)^2}$ .

### 6-3.2 Some Properties of the Correlation Proxy

We now focus on some fundamental properties of the proxy formula  $\hat{\rho}(x) = \frac{x^T R x}{(x^T e)^2}$ . In what follows it is assumed that the eigenvalues of  $R$  are sorted by ascending order.

First, a property of the Rayleigh quotient is that it must be comprised between the top and bottom eigenvalues, which implies that:

$$0 \leq \frac{\lambda_1/n}{\cos^2\theta} \leq \hat{\rho}(x) \leq \frac{\lambda_n/n}{\cos^2\theta} \leq \frac{1}{\cos^2\theta}$$

Note that the lower bound  $\frac{\lambda_1/n}{\cos^2\theta}$  can be slightly improved in the unconstrained case (see Problem 6.1) and that tighter numerical bounds can be computed through quadratic optimization methods in the constrained case where  $x \geq 0$ .

Second, another quantity of interest is the distance between two average correlation measures  $\Delta = |\hat{\rho}(x) - \hat{\rho}(y)|$ . Restricting ourselves to vectors  $x$  and  $y$  such that  $x^T e = y^T e = 1$  we may rewrite without loss of generality:

$$\Delta = \left| x^T R x - y^T R y \right| = \left| (x + y)^T R (x - y) \right|$$

The Cauchy-Schwarz inequality then gives the general upper bound  $\Delta \leq \lambda_n \|x + y\| \|x - y\|$  but in practice it is not satisfactory. To find a better upper bound we must look at the spectral decomposition of  $R$ :

$$R = \sum_{i=1}^n \lambda_i \frac{v_i v_i^T}{v_i^T v_i}$$

where  $v_i$  is an eigenvector with associated eigenvalue  $\lambda_i$ . Thus, for any vectors  $a$  and  $b$ :

$$\frac{a^T R b}{\sqrt{a^T a} \sqrt{b^T b}} = \sum_{i=1}^n \lambda_i \frac{a^T v_i}{\sqrt{a^T a} \sqrt{v_i^T v_i}} \frac{v_i^T b}{\sqrt{v_i^T v_i} \sqrt{b^T b}} = \sum_{i=1}^n \lambda_i \cos(\widehat{a, v_i}) \cos(\widehat{v_i, b})$$

where  $\widehat{(u, v)}$  denotes the absolute angle in  $[0, \pi]$  between any two vectors  $u$  and  $v$ .

Recalling that the top eigenvalue of stock correlation matrices dominates all other eigenvalues, we are induced to split the sum accordingly:

$$\frac{a^T R b}{\sqrt{a^T a} \sqrt{b^T b}} = \sum_{i=1}^{n-1} \lambda_i \cos(\widehat{a, v_i}) \cos(\widehat{v_i, b}) + \lambda_n \cos(\widehat{a, v_n}) \cos(\widehat{v_n, b})$$

Furthermore  $\cos \alpha \cos \beta = \cos(\alpha + \beta) + \sin \alpha \sin \beta$ , so that:

$$\begin{aligned} \frac{a^T R b}{\sqrt{a^T a} \sqrt{b^T b}} &= \sum_{i=1}^{n-1} \lambda_i \cos(\widehat{a, v_i}) \cos(\widehat{v_i, b}) + \lambda_n \sin(\widehat{a, v_n}) \sin(\widehat{v_n, b}) \\ &\quad + \lambda_n \cos[(\widehat{a, v_n}) + (\widehat{v_n, b})] \end{aligned}$$

We now invoke the fifth property of the Euclidean metric<sup>4</sup> to get  $|\cos(\widehat{a, v_i})| \leq \sin(\widehat{a, v_n})$ ,  $|\cos(\widehat{v_i, b})| \leq \sin(\widehat{v_n, b})$  and for  $(\widehat{a, v_n}) + (\widehat{v_n, b}) \leq \frac{\pi}{2}$ :  $0 \leq \cos[(\widehat{a, v_n}) + (\widehat{v_n, b})] \leq \cos(\widehat{a, b})$ , so that:

$$\left| \frac{a^T R b}{\sqrt{a^T a} \sqrt{b^T b}} \right| \leq n \sin(\widehat{a, v_n}) \sin(\widehat{v_n, b}) + \lambda_n \cos(\widehat{a, b})$$

because the eigenvalues sum to  $n$ .

Taking  $a = x + y$ ,  $b = x - y$  and rearranging terms we get:

$$\Delta \leq n \|x + y\| \|x - y\| \left[ \sin(\widehat{x + y, v_n}) \sin(\widehat{v_n, x - y}) + \frac{\lambda_n}{n} \cos(\widehat{x + y, x - y}) \right]$$

<sup>4</sup>See Dattorro (2008) who cites Blumenthal (1933). See also Laurence et al. (2008) who cite De Finetti (1937).

In practice the quantity between brackets is usually small because  $x$ ,  $y$  are “close” to  $v_n$  and  $x + y$ ,  $x - y$  are nearly orthogonal.

### **FOCUS ON THE FIFTH PROPERTY**

The fifth property of the Euclidean metric is a triangle inequality for angles in three dimensions, which is surprisingly not documented in mainstream geometry textbooks. Specifically it states that for any three vectors  $u$ ,  $v$ , and  $w$  we have:

$$|\widehat{(u, v)} - \widehat{(v, w)}| \leq \widehat{(u, w)} \leq \widehat{(u, v)} + \widehat{(v, w)}$$

where all angles are measured between 0 and  $\pi$ . Taking cosines we equivalently have:

$$\cos[\widehat{(u, v)} + \widehat{(v, w)}] \leq \cos \widehat{(u, w)} \leq \cos[\widehat{(u, v)} - \widehat{(v, w)}].$$

As a corollary if e.g.  $\widehat{(v, w)} = \pi/2$  then  $|\cos \widehat{(u, w)}| \leq \cos\left(\frac{\pi}{2} - \widehat{(u, v)}\right) = \sin \widehat{(u, v)}$

## **6-4 BLACK-SCHOLES WITH CONSTANT CORRELATION**

Extending Black-Scholes to a basket of  $n$  underlying assets  $S^{(1)}, \dots, S^{(n)}$  with constant correlation is fairly straightforward, except perhaps notation-wise.

Given a vector of volatilities  $(\sigma_1, \dots, \sigma_n)$  and a correlation matrix  $(\rho_{i,j})$ , assume that the prices of the underlying assets follow  $n$  correlated geometric Brownian motions:

$$\begin{aligned} dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} dW_t^{(1)} \\ dS_t^{(2)} &= \mu_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} dW_t^{(2)} \\ &\vdots \\ dS_t^{(n)} &= \mu_n S_t^{(n)} dt + \sigma_n S_t^{(n)} dW_t^{(n)} \end{aligned}$$

where  $dW_t^{(i)}dW_t^{(j)} \equiv \rho_{i,j}dt$ . If the derivative's value only depends on time and the  $n$  spot prices, we have  $D_t = f(t, S_t^{(1)}, \dots, S_t^{(n)})$  and we can apply the multidimensional version of the Ito-Doebelin theorem to get:

$$\begin{aligned} dD_t = df &= \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial S^{(i)}} dS_t^{(i)} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial S^{(i)} \partial S^{(j)}} \sigma_i \sigma_j \rho_{i,j} S_t^{(i)} S_t^{(j)} dt \\ &= \frac{\partial f}{\partial t} dt + \nabla f^T dS_t + \frac{1}{2} dS_t^T \nabla^2 f dS_t \end{aligned}$$

where  $\nabla f$  and  $\nabla^2 f$  are the gradient and Hessian of  $f$ , respectively.

A portfolio long one unit of derivative and short  $\delta_i = \frac{\partial f}{\partial S^{(i)}}$  units of each asset  $S^{(i)}$  is then riskless, and by the same reasoning as in the single-asset case we obtain a multidimensional partial differential equation for  $f$  whose only parameters are the interest rate  $r$ , the volatility vector and the correlation matrix:

$$rf = \frac{\partial f}{\partial t} + r \sum_{i=1}^n \frac{\partial f}{\partial S^{(i)}} S_t^{(i)} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial S^{(i)} \partial S^{(j)}} \sigma_i \sigma_j \rho_{i,j} S_t^{(i)} S_t^{(j)}$$

Solving partial differential equations in high dimension is very hard mathematically and computationally. In practice, the numerical method of choice to implement the multiasset Black-Scholes model is Monte Carlo simulation under the risk-neutral measure. The Cholesky decomposition of the correlation matrix is then typically used to generate correlated Brownian motions from uncorrelated ones.

### FOCUS ON THE CHOLESKY DECOMPOSITION

The Cholesky decomposition of a symmetric, positive-definite matrix  $A$  is the lower triangular matrix  $C$  with strictly positive diagonal entries such that  $A = CC^T$ . It can be computed with a short algorithm of complexity  $O(n^3)$ .

The Cholesky decomposition  $C$  of a correlation matrix  $R$  may be used to generate correlated standard normals  $Y = XC^T$  from a sample  $X$  of uncorrelated ones with  $m$  rows and  $n$  columns. Indeed the covariance estimate for  $Y$  up to a multiplicative factor is:

$$Y^T Y = CX^T X C^T \approx CC^T = R$$

where we used  $X^T X \approx I$ , which is true for large  $m$ .

## 6-5 LOCAL VOLATILITY WITH CONSTANT CORRELATION

Another straightforward extension of a popular model is local volatility with constant correlation (LVCC). Keeping the notations of Section 6-4, this model assumes dynamics of the form:

$$\begin{aligned} dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + \sigma_1^{\text{loc}}(t, S_t^{(1)}) S_t^{(1)} dW_t^{(1)} \\ dS_t^{(2)} &= \mu_2 S_t^{(2)} dt + \sigma_2^{\text{loc}}(t, S_t^{(2)}) S_t^{(2)} dW_t^{(2)} \\ &\vdots \\ dS_t^{(n)} &= \mu_n S_t^{(n)} dt + \sigma_n^{\text{loc}}(t, S_t^{(n)}) S_t^{(n)} dW_t^{(n)} \end{aligned}$$

where  $\sigma_i^{\text{loc}}(t, S)$  is the local volatility function for asset  $S^{(i)}$  (see Chapter 4) and  $dW_t^{(i)} dW_t^{(j)} \equiv \rho_{ij} dt$  as before.

The same reasoning as in Section 6-4 then applies, with identical results after substituting local volatilities. Again Monte Carlo simulations are overwhelmingly preferred to other numerical methods such as multidimensional binomial trees or finite difference lattices.

Until recently the local volatility model with constant correlation was widely used to price a broad range of multiasset exotic options. In Chapter 8, we introduce the next generation of models where correlation is allowed to vary.

## REFERENCES

- Blumenthal, Leonard M. 1933. "On the four-point property." *Bulletin of the American Mathematical Society*, 39: 423–426.
- Bossu, Sébastien, and Philippe Henrotte. 2012. *An Introduction to Equity Derivatives: Theory and Practice*, 2nd ed. Chichester, UK: John Wiley & Sons.
- Dattorro, Jon. 2008. *Convex Optimization & Euclidean Distance Geometry*. Palo Alto, CA: Meboo Publishing USA.
- De Finetti, Bruno. 1937. "A proposito di correlazione." *Supplemento Statistico ai Nuovi problemi di Politica, Storia ed Economia*, 3: 41–57. English translation in Laurence et al. (2008).
- Higham, Nicholas J. 2002. "Computing the Nearest Correlation Matrix—A Problem from Finance." *IMA Journal of Numerical Analysis* 22: 329–343.
- Laurence, Peter, Tai-Ho Hwang, and Luca Barone. 2008. "Geometric Properties of Multivariate Correlation in de Finetti's Approach to Insurance Theory." *Electronic Journal for History of Probability and Statistics* 4(2).

- Plerou, Vasiliki, Parameswaran Gopikrishnan, Bernd Rosenow, Luis A. Nunes Amaral, Thomas Guhr, and H. Eugene Stanley. 2002. "Random Matrix Approach to Cross Correlations in Financial Data." *Physical Review E* 65: 1–18.
- Potters, Marc, Jean-Philippe Bouchaud, and Laurent Laloux. 2005. "Financial Applications of Random Matrix Theory: Old Laces and New Pieces." *Acta Physica Polonica B* (36): 2767–2784.
- Tierens, Ingrid, and Margaret Anadu. 2004. "Does It Matter Which Methodology You Use to Measure Average Correlation across Stocks?" Goldman Sachs Equity Derivatives Strategy report, April 2004.

## PROBLEMS

### 6.1 Lower Bound for Average Correlation

Let  $R$  be a  $n \times n$  correlation matrix. For any  $n \times n$  positive-definite matrix  $A$  define  $\hat{\rho}_A(x) = \frac{x^T A x}{(x^T e)^2}$  where  $e$  is the vector of 1's and  $x$  is an arbitrary vector which is nonorthogonal to  $e$ .

- Show that  $\hat{\rho}_R(e) \leq \frac{1}{n} \lambda_n$  where  $\lambda_n$  is the top eigenvalue of  $R$ .
- Show that  $\hat{\rho}_R(x) \geq [\hat{\rho}_{R^{-1}}(e)]^{-1}$ . *Hint: This can be formulated as a constrained optimization problem and solved with; for example, the Lagrangian method.*
- We want to approximate the distance  $d = \hat{\rho}_R(e) - [\hat{\rho}_{R^{-1}}(e)]^{-1}$  when  $R$  is an equity correlation matrix with top eigenvalue  $\lambda_n \gg \lambda_{n-1}$  and the corresponding top eigenvector  $v_n$  is an all-stock portfolio close to  $e/n$  (up to a scaling factor).
  - Show that  $d$  may be rewritten as  $d = \frac{1}{n}(A - H)$  where  $A$ ,  $H$  are respectively the arithmetic and harmonic weighted averages of the eigenvalues of  $R$ , with weights  $\alpha_i = \cos^2(\widehat{e, v_i})$ . *Hint: Use Parseval's identity to show that  $\sum_{i=1}^n \alpha_i = 1$ .*
  - Argue that  $d \approx \left[ \frac{1-\alpha_n}{n-1} \left( 1 - \frac{\lambda_n}{n} \right) + \alpha_n \frac{\lambda_n}{n} \right] - \frac{1}{\frac{1-\alpha_n}{n-1} \sum_{i=1}^{n-1} \frac{n}{\lambda_i} + \alpha_n \frac{n}{\lambda_n}}$

### 6.2 Geometric Basket Call

Consider a call option with payoff  $\max(0, b_T - k)$  on a geometric basket calculated as  $b_T = \prod_{i=1}^n \left( \frac{S_T^{(i)}}{S_0^{(i)}} \right)^{w_i}$  where  $S_t^{(i)}$  is the price of the underlying asset  $S^{(i)}$  at time  $t$  and the nonnegative basket weights  $(w_i)$  sum to 1.

- (a) In the Black-Scholes model with constant correlation, show that under the risk-neutral measure  $b_T$  is lognormally distributed and find the distribution parameters as functions of volatilities and correlations.
- (b) Find a closed-form formula for the price of the call.

### 6.3 Worst-Of Put Pricing

Using the Black-Scholes model with constant correlation and Monte Carlo simulations, calculate the price of a one-year at-the-money worst-of put option (see Section 1-2.3) on Apple, Microsoft, and Google, in accordance with the following parameters:

- Interest rate: 1%
- Dividend rates: Apple 3%, Microsoft 2.8%, Google 0%
- Volatilities: Apple 30%, Microsoft 26%, Google 23%
- Correlations: Apple-Google: 35%, Apple-Microsoft: 30%, Google-Microsoft: 50%

### 6.4 Continuously Monitored Correlation

Consider the LVCC model for two assets  $S^{(1)}$  and  $S^{(2)}$ . Define the continuously monitored realized correlation coefficient as:

$$c = \frac{\int_0^T \frac{dS_t^{(1)}}{S_t^{(1)}} \frac{dS_t^{(2)}}{S_t^{(2)}}}{\sqrt{\int_0^T \left[ \frac{dS_t^{(1)}}{S_t^{(1)}} \right]^2 \times \int_0^T \left[ \frac{dS_t^{(2)}}{S_t^{(2)}} \right]^2}}.$$

Show that  $c \leq \rho_{1,2}$

## Correlation Trading

With the development of multiasset exotic products it became possible, and at times necessary, to trade correlation more or less directly. The first correlation trades were actually dispersion trades where a long or short position on a multi-asset option is offset by a reverse position on single-asset options. Recently pure correlation trades appeared in the form of correlation swaps.

### 7-1 DISPERSION TRADING

The payoff of a dispersion trade is of the form:

$$\text{Basket Option Payoff} - \beta \times \sum_i \text{Weight}_i \times \text{Single Option Payoff}_i$$

where  $\beta$  is an arbitrary coefficient or leg ratio, which is typically determined so that the trade has zero initial cost, and all other notations are self-explanatory.

The intuition behind dispersion trades is that the basket option's leg provides exposure to volatility and correlation. To isolate the correlation exposure, it is necessary to hedge, if only approximately, the volatility exposure: this is precisely the purpose of the short single options' leg.

The two most popular types of dispersion trades are *vanilla dispersions*, based on vanilla options (typically straddles), and *variance dispersions*, based on variance swaps.

#### 7-1.1 Vanilla Dispersion Trades

The payoff formula for a vanilla dispersion trade on a selection of  $n$  stocks  $S^{(1)}, \dots, S^{(n)}$  with weights  $w_1, \dots, w_n$  is given as:

$$\left| \sum_{i=1}^n w_i \frac{S_T^{(i)}}{S_0^{(i)}} - k \right| - \beta \sum_{i=1}^n w_i \left| \frac{S_T^{(i)}}{S_0^{(i)}} - k \right|$$

## Stochastic Correlation

Stochastic correlation models may provide a more realistic approach to the pricing and hedging of certain types of exotic derivatives, such as worst-of and best-of options and correlation swaps and correlation options. In this chapter, we review various types of stochastic correlation models and propose a framework for the pricing of realized correlation derivatives that is consistent with variance swap markets.

### 9-1 STOCHASTIC SINGLE CORRELATION

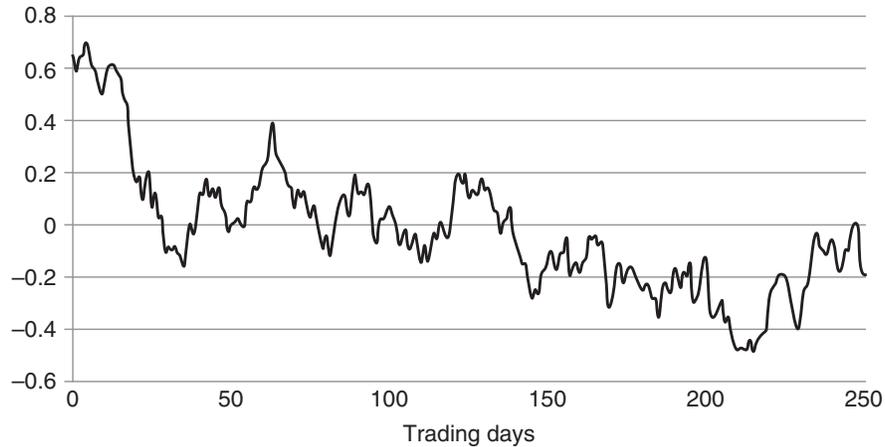
Consider the following general model framework for two assets  $S^{(1)}$  and  $S^{(2)}$ :

$$\begin{cases} dS_t^{(1)}/S_t^{(1)} = \mu_1(t, \dots)dt + \sigma_1(t, \dots)dW_t^{(1)} \\ dS_t^{(2)}/S_t^{(2)} = \mu_2(t, \dots)dt + \sigma_2(t, \dots)dW_t^{(2)} \\ (dW_t^{(1)})(dW_t^{(2)}) = \rho(t, \dots)dt \end{cases}$$

where  $\mu$ 's are instant drift coefficients,  $\sigma$ 's are instant volatility coefficients, and  $\rho$  is the instant correlation coefficient between the driving Brownian motions  $W$ 's. Here all the coefficients may be stochastic, and we focus on  $\rho$ .

There are some simple ways to make  $\rho$  stochastic and comprised between  $-1$  and  $1$ ; for example, take  $\rho_t = \sin(\alpha + \beta Z_t)$  where  $Z$  is an independent Brownian motion. The dynamics of  $d\rho_t$  may then be found by means of the Ito-Doebelin theorem. One issue with this approach is that the parameters may not be very intuitive.

A better approach is to specify diffusion dynamics for  $\rho$  and examine the Feller conditions at bounds  $-1$  and  $1$  (see Section 2-4.2.2). A popular process here is the affine Jacobi process, also known as a Wright-Fisher



**FIGURE 9.1** Sample path of an affine Jacobi process with parameters  $\rho_0 = 0.65$ ,  $\bar{\rho} = -0.1$ ,  $\kappa = 10.6$ ,  $\alpha = 1$ .

process, which is very similar to Heston's stochastic volatility process (see Section 2-4.2.2):

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \alpha\sqrt{1 - \rho_t^2}dZ_t$$

where  $\bar{\rho}$  is the long-term mean,  $\kappa$  is the mean reversion speed, and  $\alpha$  is the volatility of instant correlation. The Feller condition is then  $\frac{\alpha^2}{\kappa} - 1 < \bar{\rho} < 1 - \frac{\alpha^2}{\kappa}$ . A technical analysis of this type of process can be found in van Emmerich (2006).

Figure 9.1 shows the path obtained for an affine Jacobi process with parameters  $\rho_0 = 0.65$ ,  $\bar{\rho} = -0.1$ ,  $\kappa = 10.6$ ,  $\alpha = 1$ . Observe how all values are comprised between  $-1$  and  $1$ .

## 9-2 STOCHASTIC AVERAGE CORRELATION

We now shift our focus to average correlation measures  $\rho(x) = \frac{\sum_{i < j} x_i x_j \rho_{i,j}}{\sum_{i < j} x_i x_j}$  as introduced in Section 6-3. Because the correlation matrix  $R = (\rho_{i,j})_{1 \leq i, j \leq n}$  must be positive-definite at all times we cannot naively extend the single correlation case with, for instance,  $n(n - 1)/2$  affine Jacobi processes and take their average. Note that as a consequence of positive-definiteness  $\rho(x)$  is actually comprised between 0 and 1 for large  $n$ .

Before we go into further detail we must distinguish between non-tradable correlation, such as rolling historical or implied correlations, and

tradable correlation, such as the historical correlation observed over a fixed time period  $[0, T]$ :

- Nontradable average correlation can be modeled quite freely, using, for example, a standard Jacobi process between 0 and 1 or econometric processes such as Constant and Dynamic Conditional Correlation models (see, e.g., Engle (2009)).
- Tradable average correlation requires special consideration to be consistent with other related securities such as variance swaps.

The rest of this section is devoted to the study of tradable average correlation.

### 9-2.1 Tradable Average Correlation

Consider  $\hat{\rho} = \frac{\sigma_{\text{Basket}}^2}{\sum_{i=1}^n w_i \sigma_i^2}$  which was introduced in Section 7-1.2 and is related to the proxy formula  $\hat{\rho} = \left( \frac{\sigma_{\text{Basket}}}{\sum_{i=1}^n w_i \sigma_i} \right)^2$  introduced in Section 6-3.1. Because  $\hat{\rho}$  is the ratio of two tradable assets—namely, basket variance and average constituent variance—we can derive its dynamics from those of the two tradable assets. For example, suppose we have:

$$\begin{cases} dX_t/X_t = f_t(X_t, Y_t) dW_t \\ dY_t/Y_t = g_t(X_t, Y_t) dZ_t \\ (dW_t)(dZ_t) = h_t(X_t, Y_t) dt \end{cases}$$

where  $X_t$  is the price of basket variance at time  $t$ ,  $Y_t \geq X_t$  is the price of average constituent variance at time  $t$ , and the driving Brownian motions  $W, Z$  are taken under the forward-neutral measure.

Using the Ito-Doeblin theorem the resulting dynamics for  $\hat{\rho} = \frac{X}{Y}$  are then:

$$d\hat{\rho}_t/\hat{\rho}_t = (g_t^2 - f_t g_t h_t) dt + \sqrt{f_t^2 - 2f_t g_t h_t + g_t^2} dB_t \quad (9.1)$$

where  $B$  is another standard Brownian motion constructed from  $W$  and  $Z$ .

Note that, as the ratio of two prices,  $\hat{\rho}_t$  is *not* the price of correlation at time  $t$ , which is why the drift coefficient in Equation (9.1) is nonzero under the forward-neutral measure:

$$\hat{\rho}_t = \frac{X_t}{Y_t} = \frac{\mathbb{E}_t(X_T)}{\mathbb{E}_t(Y_T)} \neq \mathbb{E}_t\left(\frac{X_T}{Y_T}\right) = \mathbb{E}_t(\hat{\rho}_T)$$

Because  $\hat{\rho}$  is invariant when multiplying  $X$  and  $Y$  by the same scalar  $\lambda$ , we may further focus on one-dimensional reductions of the model (see Section 7-2.3) and assume that  $f, g, h$  are functions of  $X/Y$ :

$$\begin{cases} dX_t/X_t = f_t \left( \frac{X_t}{Y_t} \right) dW_t \\ dY_t/Y_t = g_t \left( \frac{X_t}{Y_t} \right) dZ_t \\ (dW_t)(dZ_t) = h_t \left( \frac{X_t}{Y_t} \right) dt \end{cases}$$

In this case Equation (9.1) becomes one-dimensional; that is, the drift and volatility coefficients depend only on time and  $\hat{\rho}_t$ . This makes the following Feller analysis considerably easier.

Omitting the time subscript for ease of exposure and using  $x$  to denote the state variable we may rewrite Equation (9.1) as:

$$dx = [g^2(x) - f(x)g(x)h(x)] x dt + x \sqrt{f^2(x) - 2f(x)g(x)h(x) + g^2(x)} dB \quad (9.2)$$

The Feller conditions at bounds 0 and 1 are then:

$$\begin{cases} \lim_{x \downarrow 0} \int_x^{x_0} s(y) dy = \infty \\ \lim_{x \uparrow 1} \int_{x_0}^x s(y) dy = \infty \\ s(y) = \exp \left( - \int_{y_0}^y \frac{2 [g^2(u) - f(u)g(u)h(u)]}{u[f^2(u) - 2f(u)g(u)h(u) + g^2(u)]} du \right) \end{cases}$$

Dividing both the numerator and denominator by  $g^2(u)$ , the integrand in  $s(y)$  may be rewritten as  $\frac{1}{u} \left[ 1 + \frac{1-p^2(u)}{p^2(u)-2p(u)h(u)+1} \right]$  with  $p(u) = \frac{f(u)}{g(u)}$ . Furthermore,

- As  $x \rightarrow 0$  a sufficient condition is that  $\lim_0 \frac{1-p^2}{p^2-2pb+1} = \ell \geq 0$  in which case we have  $s(y) \approx \exp \left( - \int_{y_0}^y \frac{1+\ell}{u} du \right) = \left( \frac{y_0}{y} \right)^{1+\ell}$  for  $y_0$  and  $y$  close to 0, and thus  $\lim_{x \downarrow 0} \int_x^{x_0} s(y) dy = \infty$ . A formal proof of sufficiency is proposed in Appendix 9.A.
- As  $x \rightarrow 1$  a necessary condition is that  $s(y) \rightarrow \infty$ , which in turn implies that  $\frac{1-p^2}{p^2-2pb+1}$  diverges (see Appendix 9.B for a formal proof). An analysis of this quantity over the domain  $p \geq 0$  and  $|h| \leq 1$  reveals that the only singularity is at (1, 1). Thus, as a corollary we have the weak necessary

condition  $f(u) \sim g(u)$  and  $h(u) \rightarrow 1$  as  $u \rightarrow 1$ . This configuration intuitively makes sense: if average correlation is close to 1, there is almost no diversification effect, and basket variance and average constituent variance become almost identical.

Additionally, we want  $f \geq g$  because basket variance is more volatile than average constituent variance, which unfortunately makes the sufficient condition stated above ineffective, since  $p \geq 1$ . We must keep all these properties in mind when researching suitable functions  $f$ ,  $g$ , and  $h$ .

### 9-2.2 The B-O Model

The following model, which we call the B-O model (for beta-omega), is a further step towards a suitable stochastic average correlation model:

$$\begin{cases} dX_t/X_t = 2\frac{T-t}{T} \left[ \omega + \beta \left( 1 - \frac{X_t}{Y_t} \right) \right] dW_t \\ dY_t/Y_t = 2\omega \frac{T-t}{T} dZ_t \\ (dW_t)(dZ_t) = \left[ \frac{X_t}{Y_t} + \frac{\omega}{\omega + \beta} \left( 1 - \frac{X_t}{Y_t} \right) \right] dt \end{cases} \quad (9.3)$$

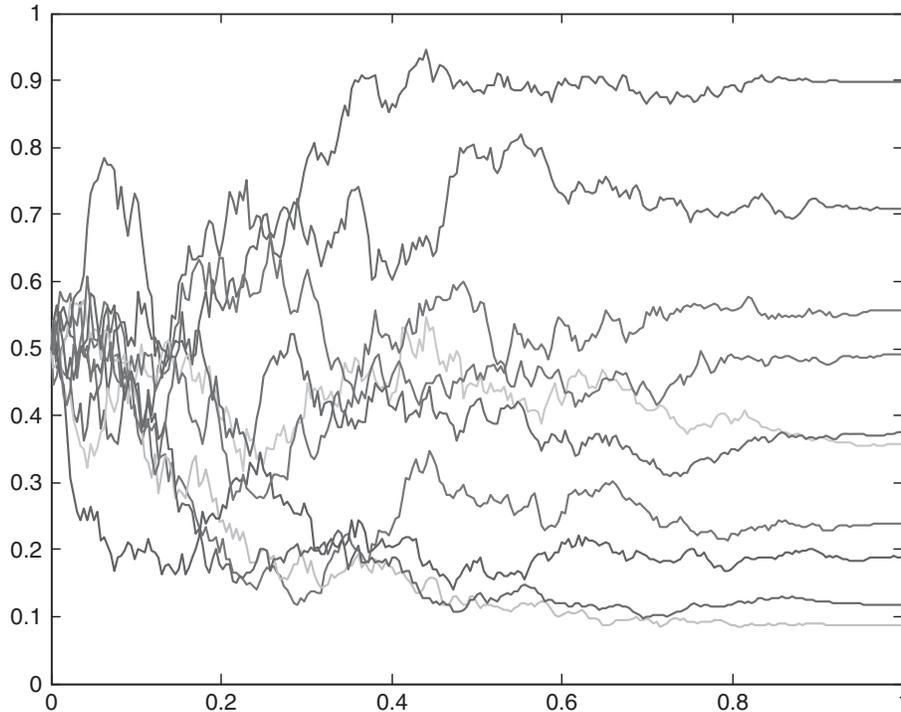
where  $\omega$  is the instant volatility of constituent volatility and  $\beta$  is the “additional” volatility of basket volatility.<sup>1</sup> The corresponding dynamics for the average correlation  $\hat{\rho} \equiv x$  are then given by Equation (9.2) using the functions:

$$\begin{cases} f_t(x) = 2\frac{T-t}{T}[\omega + \beta(1-x)] \\ g_t(x) = 2\omega \frac{T-t}{T} \\ h_t(x) = x + \frac{\omega}{\omega + \beta}(1-x) \end{cases}$$

Unfortunately, both lower and upper bounds  $[0,1]$  turn out to be attracting in the B-O model, making it unsuitable for extreme starting values  $\rho_0$  and long-term horizons  $T$ . However, empirical simulations exhibit plausible paths. Further research is needed here.

Figure 9.2 shows 10 sample paths obtained with parameters  $\omega = 70\%$ ,  $\beta = 40\%$  and  $\hat{\rho}_0 = 0.5$ . Remarkably enough, using Monte Carlo simulations the price of correlation  $\mathbb{E}(\hat{\rho}_T)$  in this model appears to be close to the initial value  $\hat{\rho}_0 = \frac{X_0}{Y_0} = \frac{\sigma_{\text{Basket}}^{*2}}{\sum_{i=1}^n w_i \sigma_i^{*2}}$ , also known as variance-implied correlation. This suggests that the fair strike of a correlation swap on  $\hat{\rho}_T$  should

<sup>1</sup>Note that  $dX_t/X_t = 2(\omega + \beta)\frac{T-t}{T}dW_t$  when  $\hat{\rho}_t = X_t/Y_t$  is equal to 0, and that  $dX_t/X_t = 2\omega\frac{T-t}{T}dW_t$  when  $\hat{\rho}_t$  is equal to 1.



**FIGURE 9.2** Ten sample paths using the B-O model with parameters  $\omega = 70\%$ ,  $\beta = 40\%$ , and  $\hat{\rho}_0 = 0.5$ .

be close to  $\hat{\rho}_0$ , and by extension a similar result should apply to standard correlation swaps.

### 9-3 STOCHASTIC CORRELATION MATRIX

A yet more ambitious endeavor is to devise a model for the evolution of the entire correlation matrix  $R_t = (\rho_{i,j}(t))_{1 \leq i,j \leq n}$  through time. As pointed out earlier, the difficulty here is to ensure that  $R_t$  is positive-definite at all times.

It is worth emphasizing that, when correlations are tradable, we should also ensure that the induced dynamics of average correlation  $\hat{\rho}_t$  be consistent with variance swaps under the forward-neutral measure.

As already pointed out in Section 6-2, equity correlation matrices have structure—namely, there is typically one large eigenvalue dominating all others, and the associated eigenvector corresponds to an all-stock portfolio. As such an equity correlation matrix cannot be viewed as any kind of random matrix.

Here we need to be more specific about the meaning of a (symmetric) random matrix. This concept was first introduced by Wishart (1928) in the form  $M = XX^T$  where  $X$  is an  $n \times n$  matrix of independent and identically distributed random variables; the special case where  $X$  is Gaussian deserves particular attention since it tends to the identity matrix as  $n \rightarrow \infty$ . Another approach is Wigner's, whereby  $M = \frac{1}{2}(X + X^T)$ ; a remarkable property is that the empirical distribution of ordered eigenvalues then follows the semi-circle law:

$$\frac{1}{n} \#\{i: \lambda_i \leq \lambda\} \xrightarrow{n \rightarrow \infty} \frac{1}{2\pi} \int_{-2}^{\lambda} \sqrt{4 - x^2} dx \quad (|\lambda| \leq 2)$$

### 9-3.1 Spectral Decomposition and the Common Factor Model

The empirical analysis of equity correlation matrices suggests that they may be viewed as the sum of a (truly) random matrix and an orthogonal projector onto the maximal eigenvector. Following the spectral theorem we may indeed write:

$$R = \left( \sum_{i=1}^{n-1} \lambda_i v_i v_i^T \right) + \lambda_n v_n v_n^T$$

where  $(v_1, \dots, v_n)$  is an orthonormal basis of eigenvectors with eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$ . The residual matrix  $\sum_{i=1}^{n-1} \lambda_i v_i v_i^T = R - \lambda_n v_n v_n^T$  may then be approximated by a Wishart-type matrix.

For large  $n$  we could ignore  $\sum_{i=1}^{n-1} \lambda_i v_i v_i^T$  altogether and write:

$$R \approx \hat{R} = (I - \lambda_n D) + \lambda_n v_n v_n^T = \begin{pmatrix} 1 & \lambda_n a_1 a_2 & \cdots & \lambda_n a_1 a_n \\ \lambda_n a_2 a_1 & 1 & & \lambda_n a_2 a_n \\ \vdots & & \ddots & \vdots \\ \lambda_n a_n a_1 & \lambda_n a_n a_2 & \cdots & 1 \end{pmatrix}$$

where  $a_1, \dots, a_n$  are the entries of the maximal eigenvector  $v_n$  and  $D = \text{diag}(a_1^2, \dots, a_n^2)$ . Note that  $\hat{R}$  has different eigenelements from  $R$ ; however,  $\lambda_n$

is related to average correlation because  $\rho(\hat{R}; v_n) = \frac{v_n^T \hat{R} v_n - 1}{(v_n^T e)^2 - 1} = \frac{\lambda_n}{n} \frac{1 - \sum_{i=1}^n a_i^4}{\frac{1}{n}(v_n^T e)^2 - \frac{1}{n}} \sim$

$\frac{\lambda_n/n}{\cos^2(\bar{v}_n, e)}$  as  $n \rightarrow \infty$ .

This approach corroborates Boortz's Common Factor Model (2008) whereby:

$$R_t = \begin{pmatrix} 1 & \xi_{t,1}\xi_{t,2} & \cdots & \xi_{t,1}\xi_{t,n} \\ \xi_{t,2}\xi_{t,1} & 1 & & \xi_{t,2}\xi_{t,n} \\ \vdots & & \ddots & \vdots \\ \xi_{t,n}\xi_{t,1} & \xi_{t,n}\xi_{t,2} & \cdots & 1 \end{pmatrix}$$

where  $(\xi_{t,1}, \dots, \xi_{t,n})$  is a vector of correlated stochastic processes in  $(-1, 1)$ , such as affine Jacobi processes. One issue with the Common Factor Model is that the (equally weighted) average realized correlation has a risk-neutral drift, which has no particular reason to fit in the framework of Section 9-2.1. In other words the Common Factor Model does not appear to be consistent with variance swap markets.

### 9-3.2 The $n \times n$ Fischer-Wright Model

Recent work by Ahdida and Alfonsi (2012) alternatively proposes the following stochastic process for the correlation matrix  $R_t$ , which is a generalization of the Jacobi process:

$$dR_t = \left[ \kappa (\bar{R} - R_t) + (\bar{R} - R_t)\kappa \right] dt + \sum_{i=1}^n \alpha_i \left( \sqrt{R_t - R_t E_{i,i} R_t} dW_t E_{i,i} + E_{i,i} dW_t^T \sqrt{R_t - R_t E_{i,i} R_t} \right)$$

where the matrix  $\bar{R}$  is the long-term correlation mean,  $\kappa = \text{diag}(\kappa_1, \dots, \kappa_n)$  is a diagonal matrix of mean-reversion speeds,  $\alpha = \text{diag}(\alpha_1, \dots, \alpha_n)$  is a diagonal matrix of volatility coefficients,  $E_{i,i} = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$  is the diagonal matrix with coefficient 1 at position  $(i,i)$  and 0 elsewhere,  $\sqrt{H}$  denotes the unique square root of a positive-semidefinite matrix  $H$ , and  $(W_t)$  is an  $n \times n$  matrix of independent standard Brownian motions.

Subject to the condition  $\kappa \bar{R} + \bar{R} \kappa - (n-2)\alpha^2$  being positive-semidefinite, the Ahdida-Alfonsi process is guaranteed to remain a valid correlation matrix through time; however, a corrected Euler scheme is required for simulation.

Unfortunately, Ahdida and Alfonsi have not studied the eigenlements of their respective correlation matrix processes. and it is difficult to tell how realistic their model is within the realm of equity correlation matrices. In particular, there is no guarantee that the induced dynamics of average correlation can be made consistent with realistic dynamics of basket variance and

average constituent variance in the fashion described early in the chapter. Further research is thus needed.

## REFERENCES

- Ahdida, Abdelkoddousse, and Aurélien Alfonsi. 2012. “A Mean-Reverting SDE on Correlation Matrices.” arXiv:1108.5264.
- Boortz, C. Kaya. 2008. “Modelling Correlation Risk.” Diplomarbeit preprint, Institut für Mathematik, Technische Universität Berlin & Quantitative Products Laboratory, Deutsche Bank AG.
- Engle, Robert. 2009. *Anticipating Correlations: A New Paradigm for Risk Management*. Princeton, NJ: Princeton University Press.
- van Emmerich, Cathrin. 2006. “Modelling Correlation as a Stochastic Process.” Bergische Universität Wuppertal. Preprint.
- Wishart, John. 1928. “The Generalised Product Moment Distribution in Samples from a Normal Multivariate Population.” *Biometrika* 20A (1–2): 32–52.

## PROBLEMS

### 9.1

Consider a stock  $S$ , which does not pay dividends, with dollar price  $S^{\$}$ , and let  $X$  be the exchange rate of one dollar into euros. Assume that  $S^{\$}$  and  $X$  both follow geometric Brownian motions under the dollar risk-neutral measure with joint dynamics:

$$\begin{cases} dS_t^{\$}/S_t^{\$} = r_{\$}dt + \sigma dW_t \\ dX_t/X_t = \nu dt + \eta dZ_t \end{cases}$$

where  $r_{\$}$  is the constant dollar interest rate,  $\sigma$ ,  $\nu$  and  $\eta$  are free constant parameters, and  $W$ ,  $Z$  are standard Brownian motions with stochastic correlation  $(dW_t)(dZ_t) \equiv \tilde{\rho}_t dt$ .

- (a) Show that the forward price of  $S$  quanto euro for maturity  $T$  is  $S_0^{\$} \mathbb{E} \left[ \exp \left( r_{\$} T - \sigma \eta \int_0^T \tilde{\rho}_t dt \right) \right]$ .
- (b) Assume that  $S_0^{\$} = \$100$ ,  $r_{\$} = 0$ ,  $\sigma = 25\%$ ,  $\eta = 10\%$ ,  $d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \alpha \sqrt{1 - \rho_t^2} dB_t$  with  $\rho_0 = -0.65$ ,  $\bar{\rho} = -0.2$ ,  $\kappa = 10.6$ ,  $\alpha = 1$ . Compute the one-year forward price of  $S$  quanto euro using Monte Carlo simulations over 252 trading days. *Answer: €100.60*

## 9.2

Consider the model for stochastic average correlation:

$$d\widehat{\rho}_t/\widehat{\rho}_t = \omega^2(1 - \widehat{\rho}_t)dt + \omega \frac{1 - \widehat{\rho}_t}{\sqrt{1 - \widehat{\rho}_t/2}} dB_t$$

- (a) Verify that the process remains within (0,1) and that the lower bound is non-attracting.
- (b) Define  $h(x) = \sqrt{x(2-x)}$ . Find  $f(x)$ ,  $g(x)$  such that  $\widehat{\rho} \equiv x$  satisfies Equation (9.2). *Hint: Show that  $\frac{1-pb}{p^2-2pb+1} = \frac{1-x/2}{1-x}$  where  $p = f/g$  and solve for  $p$ .*
- (c) Do you think that this model is suitable?

### APPENDIX 9.A: SUFFICIENT CONDITION FOR LOWER BOUND UNATTAINABILITY

Following the notations of Section 9-2.1, suppose that  $\lim_0 \frac{1-p^2}{p^2-2pb+1} = \ell \geq 0$ . By the definition of a limit this means that for arbitrary  $\varepsilon > 0$  there exists an  $\alpha > 0$  such that:

$$\text{for all } 0 \leq u \leq \alpha, \frac{1 - p^2(u)}{p^2(u) - 2p(u)h(u) + 1} \leq \ell + \varepsilon$$

Thus, for all  $0 < u \leq \alpha$ ,  $-\frac{1}{u} \left[ 1 + \frac{1-p^2(u)}{p^2(u)-2p(u)h(u)+1} \right] \geq -\frac{1+\ell+\varepsilon}{u}$ . By integration over  $[y_0, y] \subset [0, \alpha]$  we get:

$$\begin{aligned} -\int_{y_0}^y \frac{du}{u} \left[ 1 + \frac{1 - p^2(u)}{p^2(u) - 2p(u)h(u) + 1} \right] &\geq -\int_{y_0}^y \frac{1 + \ell + \varepsilon}{u} du \\ &= -(1 + \ell + \varepsilon) \ln \frac{y}{y_0}. \end{aligned}$$

Taking exponentials:

$$s(y) \geq \left( \frac{y_0}{y} \right)^{1+\ell+\varepsilon}$$

and thus  $\lim_{x \downarrow 0} \int_x^{x_0} s(y) dy = \infty$  since  $\int_0^{x_0} \frac{dy}{y^{1+\beta}}$  diverges for any  $\beta \geq 0$ .

### APPENDIX 9.B: NECESSARY CONDITION FOR UPPER BOUND UNATTAINABILITY

Suppose that  $\frac{1-p^2}{p^2-2pb+1}$  converges to a finite limit  $\ell$ . By the definition of a limit this means that for arbitrary  $\varepsilon > 0$  there exists an  $\alpha < 1$  such that:

$$\text{for all } \alpha \leq u \leq 1, \ell - \varepsilon \leq \frac{1 - p^2(u)}{p^2(u) - 2p(u)b(u) + 1} \leq \ell + \varepsilon$$

Thus, for all  $\alpha \leq u \leq 1$ ,  $\frac{1+\ell-\varepsilon}{u} \leq \frac{1}{u} \left[ 1 + \frac{1-p^2(u)}{p^2(u)-2p(u)b(u)+1} \right] \leq \frac{1+\ell+\varepsilon}{u}$ . By integration over  $[y_0, y] \subset [\alpha, 1]$  we get:

$$(1 + \ell - \varepsilon) \ln \frac{y}{y_0} \leq \int_{y_0}^y \frac{du}{u} \left[ 1 + \frac{1 - p^2(u)}{p^2(u) - 2p(u)b(u) + 1} \right] \leq (1 + \ell + \varepsilon) \ln \frac{y}{y_0}$$

Taking exponentials:

$$\left( \frac{y_0}{y} \right)^{1+\ell+\varepsilon} \leq s(y) \leq \left( \frac{y_0}{y} \right)^{1+\ell-\varepsilon}$$

and thus  $\lim_{x \uparrow 1} \int_{x_0}^x s(y) dy$  is finite since  $\int_{x_0}^1 \frac{dy}{y^\beta}$  converges for any  $\beta$ , thereby contradicting the requirement that  $\lim_{x \uparrow 1} \int_{x_0}^x s(y) dy = \infty$ .

## **Author's Note**

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## 5.2 “Arbitrage Pricing of Equity Correlation Swaps”, J.P. Morgan report, 2005

JULY 2005

### ARBITRAGE PRICING OF EQUITY CORRELATION SWAPS

Sebastien Bossu  
sebastien.bossu@jpmorgan.com  
Equity Derivatives Group

JPMorgan Securities Ltd. – London

EQUITY DERIVATIVES



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## Summary

In this report we propose a ‘toy model’ for pricing derivatives on the realized variance of an Asset, which we apply for pricing correlation swaps on the components of an equity index. We find that the fair strike of a correlation swap is approximately equal to a particular measure of implied correlation, and that the corresponding hedging strategy relies upon dynamic trading of variance dispersions.

I thank my colleague Manos Venardos for his contribution and comments. All errors are mine.

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## Introduction

Volatility and variance modeling has been an active research area within quantitative finance since the publication of the Black-Scholes model in 1973. Initially, research efforts have mostly focused on extending the Black-Scholes model for pricing calls and puts in the presence of implied volatility ‘smile’ (Hull-White 1987, Heston 1993, Dupire 1993a & 1993b, Derman-Kani 1994.) In the mid 1990’s, new instruments known as variance swaps also appeared on equities markets and made squared volatility a tradable asset (Neuberger 1990, Demeterfi-Derman 1999.) As variance became an asset class of its own, various forms of volatility derivatives have appeared, for example volatility swaps, forward contracts and options on the new CBOE Volatility Index (VIX.) The modeling of these new instruments is difficult because they overlap with certain exotic derivatives such as cliquet options which highly depend on the dynamics of the implied volatility surface.

In recent years the research on volatility and variance modeling has embraced the pricing and hedging of these volatility derivatives. Here we must distinguish between two types:

- Derivatives on realized volatility, where the payoff explicitly involves the historical volatility of the underlying Asset observed between the start and maturity dates, e.g. volatility swaps.
- Derivatives on implied volatility, where the payoff will be determined at maturity by the implied volatility surface of the underlying asset, e.g. forward-starting variance swaps, cliquet options, or options on the VIX.

It is important to notice that the first category can be seen as derivatives on a variance swap of same maturity. Leveraging on this observation and on earlier work by Dupire (1993b), Buehler (2004) models a continuous term structure of forward variance swaps, while Duanmu (2004), Potter (2004) and Carr-Sun (2005) model a fixed-term variance swap. All these approaches are based on dynamic hedging with one or several variance swap instruments.

The second category is beyond the scope of this report. We refer the interested reader to the work on the dynamics of the implied volatility surface carried out by Schonbucher (1998), Cont-Fonseca (2002), Brace et al. (2002).

Despite the development of exotic and hybrid markets which offer derivatives on several underlying assets, correlation modeling in the context of option pricing theory has been relatively under-investigated in the financial literature.

Correlation swaps appeared in the early 2000’s as a means to hedge the parametric risk exposure of exotic desks to changes in correlation. Exotic derivatives indeed frequently involve multiple assets, and their valuation requires a correlation matrix for input. Unlike volatility, whose implied levels have become observable due to the development of listed option markets, implied correlation coefficients are unobservable, which makes the pricing of correlation swaps a perfect example of ‘chicken-egg problem.’

In this report, we show how a correlation swap on an equity index can be viewed as a simple derivative on two types of tradable variance, and derive a closed-form formula for its arbitrage price relying upon dynamic trading of these instruments. For this purpose, we start by proposing a ‘toy model’ for tradable variance in Section 1 which we apply for pricing single volatility derivatives. In Section 2, we introduce a proxy for the payoff of correlation swaps that has the property of involving only tradable variance payoffs, and we extend the toy model for variance to derive the theoretical price of a correlation swap.

## 1. A toy model for tradable variance

Our purpose is to introduce a simplified model which can be used to price derivatives on realized volatility. We depart from the traditional stochastic volatility models such as Heston (1993) by modeling directly the fair price of a variance swap with the same maturity as the derivative. Here, the underlying tradable asset is the variance swap itself which, at any point in time, is a linear mixture between past realized variance and future implied variance.

This approach lacks the sophistication of other methods and does not address the issue of possible arbitrage with other derivatives instruments. But its simplicity allows us to find closed-form formulas based on a reduced number of intuitive parameters, so that everyone can form an opinion on the rationality of our results.

### The Model

In this section we limit our considerations to a market with two tradable assets: variance and cash. We follow in part the guidelines by Duanmu (2004) to introduce a simplified, ‘toy model’ for the variance asset which is a straightforward modification of the Black-Scholes model for asset prices. We make the usual economic assumptions of constant interest rate  $r$ , absence of arbitrage, infinite liquidity, unlimited short-selling, absence of transaction costs, and continuous flow of information. We have the usual set up of a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with Brownian filtration  $(\mathcal{F}_t)$  and an equivalent risk-neutral pricing measure  $\mathbb{Q}$ .

We further assume that only the variance swap is tradable, but not the Asset itself<sup>1</sup>. Let  $v_t(0, T)$  be the price at time  $t$  of the floating leg of a variance swap for the period  $[0, T]$  where  $T$  denotes the maturity or settlement date of the swap. From now on we use the reduced notation  $v_t$  and we use the terms ‘variance’ and ‘variance swap’ interchangeably.

We specify the dynamics of  $(v_t)$  through the following diffusion equation under the risk-neutral measure  $\mathbb{Q}$ :

$$dv_t = rv_t dt + 2\omega \frac{T-t}{T} v_t dW_t$$

where  $r$  and  $\omega$  are model parameters corresponding to the short-term interest rate and the volatility of volatility, and  $(W_t)$  is a standard Brownian motion under  $\mathbb{Q}$ .

Hence  $v_0$  is the price at inception of the variance swap which can be observed on the market or calculated using the replicating portfolio of puts and calls described in e.g. Demeterfi-Derman (1999); and  $v_T$  is the price of the same variance swap at maturity which coincides with the realized variance for the period  $[0, T]$ .

Our toy model for variance is thus a log-normal diffusion whose volatility parameter linearly collapses to zero between the start date and the maturity date. Note that by Ito-Doebelin this is equivalent to assume that volatility follows a log-normal diffusion with a time-dependent

volatility parameter  $\omega \frac{T-t}{T}$ .

### Comparison with stochastic volatility models

We now make the comparison with standard stochastic volatility models of the instantaneous asset variance  $(X_t)$ . The usual mean-reverting model is

<sup>1</sup> We make this assumption to avoid modeling the Asset price process itself, and escape the debate on model consistency with vanilla option prices. Clearly this is not a realistic assumption, hence the expression ‘toy model.’

$$dX_t = \lambda(\theta - X_t)dt + \xi X_t^a dW_t$$

where  $\lambda$ ,  $\theta$ ,  $\xi$ ,  $a$  are constant parameters. In this framework the price at time  $t$  of a variance swap over the period  $[0, T]$  is given by:

$$\begin{aligned} v_t &= e^{-r(T-t)} \frac{1}{T} \left[ \int_0^t X_s ds + E \left[ \int_t^T X_s ds \middle| F_t \right] \right] \\ &= e^{-r(T-t)} \frac{1}{T} \left[ \int_0^t X_s ds + \theta(T-t) + \frac{1}{\lambda} (1 - e^{-\lambda(T-t)}) (X_t - \theta) \right] \end{aligned}$$

This price is independent of the volatility of volatility specification controlled by the parameter  $\xi$ . Since the variance swap price is an affine function of the instantaneous variance, the dynamics of  $(v_t)$  are straightforwardly obtained:

$$dv_t = rv_t dt + e^{-r(T-t)} \frac{1}{\lambda T} (1 - e^{-\lambda(T-t)}) \xi X_t^a dW_t$$

We may now use the variance swap price expression to obtain the dynamics of  $(v_t)$  in terms of  $v_t$  only. When  $a = 1$  this simplifies to:

$$dv_t = rv_t dt + \xi \left( v_t - \frac{1}{T} e^{-r(T-t)} \left[ \int_0^t X_s ds + \theta(T-t) - \frac{1}{\lambda} (1 - e^{-\lambda(T-t)}) \theta \right] \right) dW_t$$

and we can see that the volatility factor between brackets converges to zero as we approach maturity.

In contrast to the toy model, the volatility specification of the variance swap in a stochastic volatility model is a power of the instantaneous variance, not the variance swap price. For short maturities the two models are comparable.

#### Terminal distribution

Using the Ito-Doebelin theorem, we can write the diffusion equation for  $\ln v$ :

$$d \ln(v_t) = \left[ r - 2\omega^2 \left( \frac{T-t}{T} \right)^2 \right] dt + 2\omega \frac{T-t}{T} dW_t$$

Thus, for all times  $0 < t < t' < T$ , we have:

$$v_{t'} = v_t \exp \left[ r(t'-t) - 2 \frac{\omega^2}{T^2} \int_t^{t'} (T-s)^2 ds + 2 \frac{\omega}{T} \int_t^{t'} (T-s) dW_s \right]$$

Calculating the first integral explicitly we obtain:

$$v_{t'} = v_t \exp \left[ r(t'-t) - \frac{2}{3} \omega^2 T \left( \left( \frac{T-t}{T} \right)^3 - \left( \frac{T-t'}{T} \right)^3 \right) + 2 \frac{\omega}{T} \int_t^{t'} (T-s) dW_s \right]$$

In particular, the terminal variance  $v_T$  has the following expression:

$$v_T = v_t \exp \left[ r(T-t) - \frac{2}{3} \omega^2 T \left( \frac{T-t}{T} \right)^3 + 2 \frac{\omega}{T} \int_t^T (T-s) dW_s \right] \quad (1)$$

Furthermore, the stochastic integral  $\int_t^T (T-s) dW_s$  has a normal distribution with zero mean and standard deviation  $\frac{(T-t)^{3/2}}{\sqrt{3}}$ . Thus,  $v_T$  has a conditional lognormal distribution with

mean  $\ln(v_t) + r(T-t) - \frac{2}{3} \omega^2 T \left( \frac{T-t}{T} \right)^3$  and standard deviation  $\frac{2}{\sqrt{3}} \omega \sqrt{T} \left( \frac{T-t}{T} \right)^{3/2}$ .

### Application: arbitrage pricing of volatility derivatives

As an example of an application of our toy model for variance, we derive the arbitrage price of a European contingent claim on realized volatility  $v_T$  at maturity. We denote  $f(v_T)$  the payoff and  $f_t = f(t, v_t)$  the F-adapted price process of such contingent claim.

Following the fundamental theorem of asset pricing, the price of the contingent claim equals the discounted conditional expectation of its payoff:

$$f_t = e^{-r(T-t)} E[f(v_T) | F_t]$$

We now proceed to derive closed-form formulas for two contingent claims of particular interest:

- A forward contract on realized volatility, whose payoff is the square root of variance:  
 $f(v_T) = \sqrt{v_T}$ ;
- A call option on realized variance struck at level  $K$ , whose payoff is:  
 $f(v_T) = \text{Max}(0, v_T - K)$ .

#### Forward contract on realized volatility

Taking the square root of (1) we can write for all  $0 < t < T$ :

$$f(v_T) = \sqrt{v_t} e^{-r(T-t)} \exp \left( -\frac{1}{3} \omega^2 T \left( \frac{T-t}{T} \right)^3 + \frac{\omega}{T} \int_t^T (T-s) dW_s \right)$$

Taking conditional expectations and discounting then yields:

$$f_t = \sqrt{v_t} e^{-r(T-t)} \exp \left( -\frac{1}{3} \omega^2 T \left( \frac{T-t}{T} \right)^3 \right) E \left[ \exp \left( \frac{\omega}{T} \int_t^T (T-s) dW_s \right) \middle| F_t \right] \quad (2)$$

But:

$$E \left[ \exp \left( \frac{\omega}{T} \int_t^T (T-s) dW_s \right) \right] = \exp \left( \frac{1}{2} \frac{\omega^2}{T^2} \int_t^T (T-s)^2 ds \right) = \exp \left( \frac{1}{6} \omega^2 T \left( \frac{T-t}{T} \right)^3 \right)$$

Substituting this result in (2), we obtain a closed-form formula for the price of the forward contract on volatility:

$$f_t = \sqrt{v_t} e^{-r(T-t)} \exp\left(-\frac{1}{6} \omega^2 T \left(\frac{T-t}{T}\right)^3\right)$$

In particular:

$$f_0 = \sqrt{v_0} \exp\left(-\frac{1}{2} rT - \frac{1}{6} \omega^2 T\right)$$

A corollary is that the convexity adjustment  $c$  between the fair strikes of newly issued variance and volatility swaps can be expressed as a function of volatility of volatility:

$$c = \sqrt{v_0} e^{rT} - f_0 e^{rT} = \sqrt{v_0} e^{rT} \left[1 - \exp\left(-\frac{1}{6} \omega^2 T\right)\right]$$

Which gives the rule of thumb:

$$c \approx \frac{1}{6} \sqrt{v_0} e^{rT} \omega^2 T$$

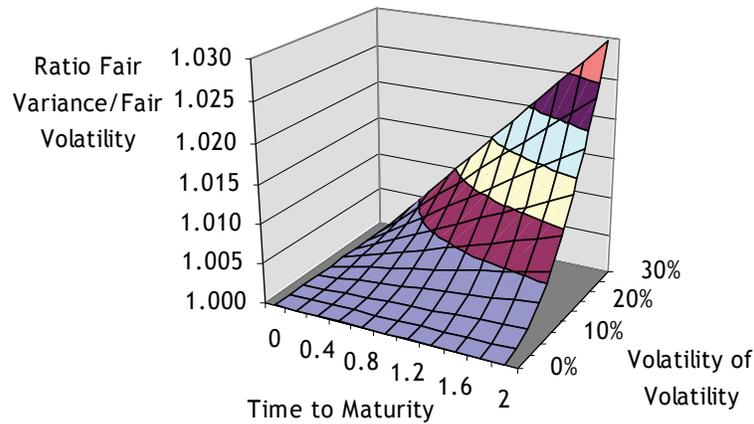
This result has some resemblance with Duanmu's who finds  $c = \sqrt{v_0} e^{rT} \left[1 - \exp\left(-\frac{1}{4} \xi^2\right)\right]$

with a time-dependent volatility of volatility  $\omega(T-t) = \frac{\xi}{\sqrt{T-t}}$ . However, we believe our

result is more consistent with the intuition that the longer the maturity, the higher the convexity effect. Exhibit 1.1 below shows how the convexity adjustment behaves as a function of volatility of volatility and maturity.

Exhibit 1.1

Convexity Adjustment between Variance and Volatility Swaps



Another point of interest is the corresponding dynamic hedging strategy for replicating volatility swaps using variance swaps. Following our approach, the quantity  $\delta_t$  of variance to hold at a given point in time  $t$  would be:

$$\delta_t = \frac{\partial f}{\partial v} = \frac{1}{2\sqrt{v_t}e^{r(T-t)}} \exp\left(-\frac{1}{6}\omega^2 T \left(\frac{T-t}{T}\right)^3\right)$$

It is worth noting that at time  $t = 0$  this delta is equal to:

$$\delta_0 = \frac{1}{2\sqrt{v_0}e^{rT}} \exp\left(-\frac{1}{6}\omega^2 T\right)$$

which is in line, modulus the convexity adjustment, with the market practice of calculating the notional of a newly issued variance swap according to the formula:

$$\text{Variance Swap Notional} = \frac{\text{Vega Notional}}{2 \times \text{Variance Swap Strike}}$$

#### Call option on realized variance

Because  $v_T$  has a lognormal distribution, the closed-form formula for a call on realized variance struck at level  $K$  is identical to the Black-Scholes formula for a call on a zero-dividend paying stock with a constant volatility parameter<sup>2</sup>  $\sigma = \frac{2}{\sqrt{3}}\omega \frac{T-t}{T}$ . Substitution yields:

$$f_t = v_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$

where  $N$  is the cumulative standard normal distribution and:

$$d_1 = \frac{\ln\sqrt{\frac{v_t e^{r(T-t)}}{K}} + \frac{1}{3}\omega^2 T \left(\frac{T-t}{T}\right)^3}{\frac{1}{\sqrt{3}}\omega\sqrt{T}\left(\frac{T-t}{T}\right)^{3/2}}, \quad d_2 = d_1 - \frac{2}{\sqrt{3}}\omega\sqrt{T}\left(\frac{T-t}{T}\right)^{3/2}.$$

Note that the quantity  $\sqrt{\frac{v_t e^{r(T-t)}}{K}}$  corresponds to the ratio of implied variance to the option's strike expressed in volatility points.

Exhibits 1.2 to 1.4 below show how the arbitrage price of a 3-year call on variance struck at  $20^2$  compares to the original Black-Scholes call formula at  $t = 0, 1$  and  $2$  years. To generate these graphs we assumed a volatility of volatility of 20% and a 0% interest rate. Option prices are expressed in percentage of the strike. We can see that the call on variance is worth more than Black-Scholes at  $t = 0$ , and less at  $t = 1$  and  $t = 2$ , which indicates a higher time decay or theta.

<sup>2</sup> By 'constant volatility parameter' we mean that at time  $t$  the call on realized variance has the same price as a call on an asset  $S$  whose price follows the diffusion  $dS_\tau = rS_\tau d\tau + \sigma S_\tau dW_\tau$  where  $\tau \in [t, \infty)$  is the time dimension of the diffusion and  $\sigma$  does not depend on  $\tau$ .

Exhibit 1.2

Call on Realized Variance: Toy Model versus Black-Scholes at  $t = 0$

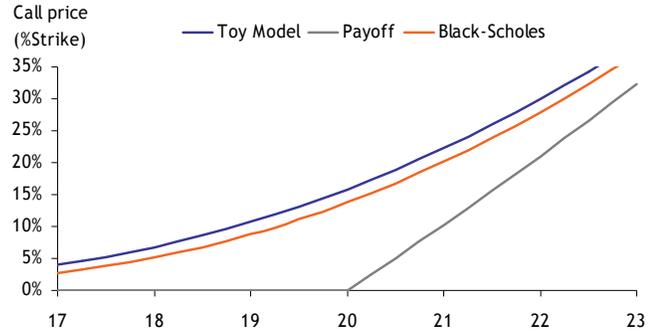


Exhibit 1.3

Call on Realized Variance: Toy Model versus Black-Scholes at  $t = 1$

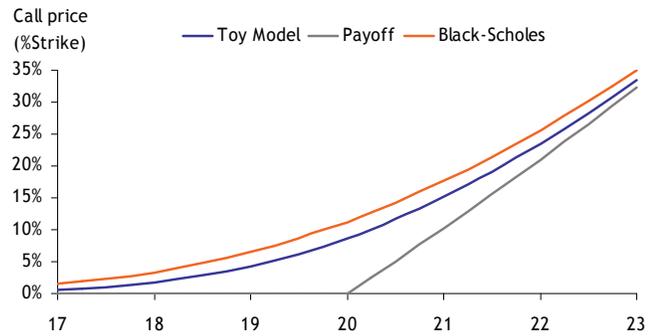
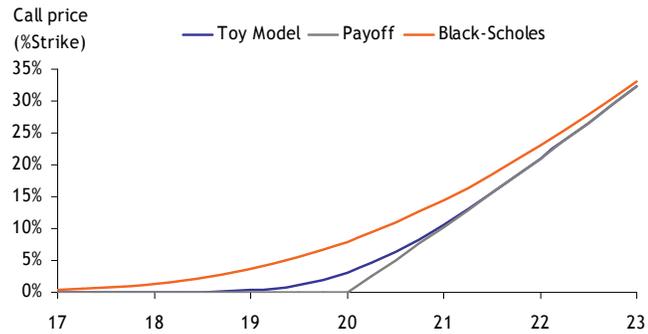


Exhibit 1.4

Call on Realized Variance: Toy Model versus Black-Scholes at  $t = 2$



## 2. Pricing and hedging of quasi-correlation claims

### Correlation Swaps

A correlation swap is a derivative instrument on a basket of  $n$  Stocks whose payoff is given as:

$$c_T = \frac{\sum_{1 \leq i < j \leq n} w_i w_j \rho_{i,j}}{\sum_{1 \leq i < j \leq n} w_i w_j} - K$$

where  $w$  is a vector of arbitrary non-negative weights summing to 1,  $\rho$  a positive-definite matrix of pair-wise correlation coefficients, and  $K$  a scalar called strike.

In practice the correlation coefficients are calculated using the canonical statistical formula on the time series of the Stocks' daily log-returns. The first term in the formula corresponds to the weighted average of the correlation matrix, excluding the diagonal of 1's. We call this quantity the realized average correlation between the  $n$  Stocks for the period  $[0, T]$ .

### Implied index correlation

In the case of equity indices, an implied average correlation measure can be backed out from implied volatilities:

$$\bar{\rho}_{Implied} = \frac{\sigma_{Index}^2 - \sum_{i=1}^n w_i^2 \sigma_i^2}{2 \sum_{1 \leq i < j \leq n} (w_i \sigma_i)(w_j \sigma_j)}$$

where  $n$  is the number of Stocks in the index,  $\sigma_{Index}$  is the implied volatility of the index,  $\sigma$  is the vector of implied volatilities and  $w$  is the vector of index weights.

This measure is justified by the well-known relationship between the variance of a portfolio and the covariance of its components, which is the founding block of portfolio theory (Markovitz 1952):

$$\sigma_{Portfolio}^2 = \sum_{i=1}^n w_i^2 \sigma_i^2 + 2 \sum_{1 \leq i < j \leq n} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

There are, however, some minor differences between an equity index and a portfolio of stocks. In a portfolio weights are fixed, whereas in an index they vary with stock prices. Additionally the formula above is only exact for standard returns ( $\frac{\Delta P}{P}$ ), not log-returns. In normal market conditions and over reasonable observation periods, these differences can be ignored.

### Implied correlation and 'fair' correlation

Intuitively, one would expect the 'fair value' of a correlation swap on an equity index to be related to the index implied correlation. However, in the absence of a replication strategy, the concept of 'fair value' is quite sloppy. This is complicated by the existence of implied volatility surfaces that translate into implied correlation surfaces: there is not a single measure of implied correlation.

Later on we establish the formal existence of a quasi-replication strategy for equity index correlation swaps relying upon dynamic trading of variance on the index and its components, and show that the ‘fair value’ of a correlation swap is roughly equal to a particular measure of implied correlation, after discounting. This dynamic replication strategy is more easily exposed using the rules of thumb which we introduce below.

### Correlation proxy

In 2004 several papers (Bossu-Gu, Tierens-Anadu, Statman-Scheid) have investigated the relationship between portfolio volatility and average correlation. The conclusion which can be drawn is that for a sufficient number of Stocks and in normal conditions<sup>3</sup> we have the rule of thumb:

$$\bar{\rho} \approx \left( \frac{\sigma_{Index}}{\sum_{i=1}^n w_i \sigma_i} \right)^2$$

where  $\bar{\rho}$  denotes either realized or implied average correlation,  $\sigma$  a vector of either realized or implied volatilities, and  $w$  a vector of components’ weights in the index.

In essence, average correlation is thus the squared ratio of index volatility to the average volatility of its components. We push this paradigm one level further by noticing that this proxy measure is conceptually close to the ratio of index variance to the average variance of its components:

$$\left( \frac{\sigma_{Index}}{\sum_{i=1}^n w_i \sigma_i} \right)^2 \approx \frac{\sigma_{Index}^2}{\sum_{i=1}^n w_i \sigma_i^2}$$

We call the quantity on the left-hand side the volatility-based correlation proxy and that on the right-hand side the variance-based proxy. In practice those two proxy measures typically differ by a few correlation points for the major equity indices, both for implied and historical data. It should also be noted that the variance-based proxy is always lower than or equal to the volatility-based one<sup>4</sup>.

Our motivation for introducing the variance-based proxy should be clear: in this form, average index correlation becomes the ratio of two tradable types of variances: index variance and average components’ variance. In fact these variances are frequently traded one against the other in the so-called variance dispersion trades, with the objective of taking advantage of the gap between implied correlation and realized correlation, as illustrated in Exhibit 2.1 on the Dow Jones EuroStoxx 50 index.

<sup>3</sup> Well-behaved weights and volatilities, actual correlation above 0.15.

<sup>4</sup> This property is a straightforward consequence of Jensen’s inequality:  $(\sum_i w_i \sigma_i)^2 \leq \sum_i w_i \sigma_i^2$ .

Exhibit 2.1

1-year Implied and Realized Correlation of the DJ EuroStoxx 50 index



Source: JPMorgan

**Quasi-correlation claims**

We call a quasi-correlation claim a variance derivative whose payoff is:

$$c_T = \frac{a_T}{b_T}$$

where  $a_T$  denotes index realized variance and  $b_T$  the average components' realized variance, defined as follows:

$$a_T = \frac{1}{T} [\ln S]_T$$

$$b_T = \frac{1}{T} \sum_{i=1}^n w_i [\ln S^i]_T$$

with  $S$  denoting the price process of the index,  $(S^1, \dots, S^n)$  the vector of price processes of the components, and  $[\cdot]$  the quadratic variation.

**Arbitrage Pricing**

We now extend our toy model to find the arbitrage price of a quasi-correlation claim. We consider a market of two tradable variance assets  $a$  and  $b$  and cash, and we make the same economic assumptions as in Section 1.

We specify the following dynamics for the  $F$ -adapted price processes  $a$  and  $b$  under a risk-neutral measure  $Q$ :

$$da_t = ra_t dt + 2\omega_a \frac{T-t}{T} a_t dW_t$$

$$db_t = rb_t dt + 2\omega_b \frac{T-t}{T} b_t \left( \chi dW_t + \sqrt{1-\chi^2} dZ_t \right)$$

where  $r$  is the short-term interest rate,  $\omega$ 's are volatility of volatility parameters for  $a$  and  $b$ ,  $\chi$  is the instant correlation parameter between  $a$  and  $b$ , and  $(W_t)$ ,  $(Z_t)$  are two independent Brownian motions under  $Q$ .

Denoting  $(c_t)$  the price process for the quasi-correlation claim, and applying the Ito-Doebelin theorem on  $\ln(a/b)$ , we find:

$$d \ln \frac{a_t}{b_t} = 2(\omega_b^2 - \omega_a^2) \left( \frac{T-t}{T} \right)^2 dt + 2(\omega_a + \omega_b \chi) \frac{T-t}{T} dW_t + 2\omega_b \sqrt{1-\chi^2} \frac{T-t}{T} dZ_t$$

Whence for all  $0 < t < T$ :

$$c_T = \frac{a_T}{b_T} = \frac{a_t}{b_t} \exp \left[ \frac{2}{3} (\omega_b^2 - \omega_a^2) T \left( \frac{T-t}{T} \right)^3 + 2 \frac{\omega_a - \omega_b \chi}{T} \int_t^T (T-s) dW_s - 2 \frac{\omega_b \sqrt{1-\chi^2}}{T} \int_t^T (T-s) dZ_s \right]$$

Taking conditional expectations under  $Q$  and discounting yields:

$$c_t = \frac{a_t}{b_t} \exp \left[ -r(T-t) + \frac{2}{3} \left[ \omega_b^2 - \omega_a^2 + (\omega_a - \omega_b \chi)^2 + \omega_b^2 (1-\chi^2) \right] T \left( \frac{T-t}{T} \right)^3 \right]$$

Expanding the squares and simplifying terms, we obtain:

$$c_t = \frac{a_t}{b_t} \exp \left[ -r(T-t) + \frac{4}{3} (\omega_b^2 - \omega_a \omega_b \chi) T \left( \frac{T-t}{T} \right)^3 \right]$$

In particular, at time  $t = 0$ , we have:

$$c_0 = \frac{a_0}{b_0} \exp \left[ -rT + \frac{4}{3} (\omega_b^2 - \omega_a \omega_b \chi) T \right]$$

Here, it is worth noting that if the volatility of volatility parameters are of the same order and the correlation of variances is high, we have  $c_0 \approx \frac{a_0}{b_0} e^{-rT}$ , which is nothing else but the discounted variance-based implied correlation proxy.

Exhibits 2.2 to 2.4 below show how the fair strikes of a 1-year quasi-correlation claim compare to the variance-based implied correlation, for various levels of volatility of volatility and variance correlation parameters. We can see that when  $\omega_a$  and  $\omega_b$  are close the ratio is close to 1. This suggests that our result is relatively model-independent in the sense that it does not heavily depend on the model parameters.

Exhibit 2.2

Ratio of Fair Quasi-Correlation to Variance-Based Implied Correlation for $\chi = 0.5$							
$\omega_a \backslash \omega_b$	0%	5%	10%	15%	20%	25%	30%
0%	1	1.003	1.013	1.030	1.055	1.087	1.127
5%	1	1.001	1.009	1.023	1.045	1.074	1.112
10%	1	0.999	1.004	1.016	1.035	1.062	1.096
15%	1	0.996	0.999	1.009	1.025	1.049	1.081
20%	1	0.994	0.994	1.002	1.016	1.037	1.065
25%	1	0.992	0.990	0.995	1.006	1.025	1.051
30%	1	0.989	0.985	0.988	0.997	1.013	1.036

Exhibit 2.3

Ratio of Fair Quasi-Correlation to Variance-Based Implied Correlation for $\chi = 1$							
$\omega_a \backslash \omega_b$	0%	5%	10%	15%	20%	25%	30%
0%	1	1.003	1.013	1.030	1.055	1.087	1.127
5%	1	1	1.007	1.020	1.041	1.069	1.105
10%	1	0.997	1	1.010	1.027	1.051	1.083
15%	1	0.993	0.993	1	1.013	1.034	1.062
20%	1	0.990	0.987	0.990	1	1.017	1.041
25%	1	0.987	0.980	0.980	0.987	1	1.020
30%	1	0.983	0.974	0.970	0.974	0.983	1

Exhibit 2.4

Ratio of Fair Quasi-Correlation to Variance-Based Implied Correlation for $\chi = 0$							
$\omega_a \backslash \omega_b$	0%	5%	10%	15%	20%	25%	30%
0%	1	1.003	1.013	1.030	1.055	1.087	1.127
5%	1	1.002	1.010	1.025	1.048	1.078	1.116
10%	1	1.000	1.007	1.020	1.041	1.069	1.105
15%	1	0.998	1.003	1.015	1.034	1.060	1.094
20%	1	0.997	1.000	1.010	1.027	1.051	1.083
25%	1	0.995	0.997	1.005	1.020	1.043	1.073
30%	1	0.993	0.993	1.000	1.013	1.034	1.062

### Dynamic Hedging Strategy

We now examine in further detail the hedging strategy for quasi-correlation claims. The hedging coefficients or deltas for the two variance assets are given as:

$$\delta_t^a = \frac{c_t}{a_t}$$

$$\delta_t^b = -\frac{c_t}{b_t}$$

In practice, this means that if we are short a claim we must hold a long position in index variance against a short position in average components' variance, in dynamic quantities. This type of spread trade is known as a variance dispersion. We must emphasize that here the weights between the two legs are not equal – in fact, the ratio of deltas is equal to the fair value of the claim:

$$\left| \frac{\delta_t^b}{\delta_t^a} \right| = \frac{a_t}{b_t} = c_t$$

In particular, at  $t = 0$ , this ratio is equal to the variance-based implied correlation proxy, and the initial delta-hedge is known as a correlation-weighted variance dispersion trade<sup>5</sup>.

Furthermore, the cost of setting up the delta-hedge is nil at all times:

$$\delta_t^a a_t + \delta_t^b b_t = \frac{c_t}{a_t} a_t - \frac{c_t}{b_t} b_t = 0$$

Thus, the hedging strategy is entirely self-funded.

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<sup>5</sup> For a detailed analysis of dispersion trading, please refer to our 2005 report *Correlation Vehicles*, JPMorgan European Equity Derivatives Strategy, N. Granger and P. Allen.

## Conclusion

Because standard correlation swaps have a payoff approximately equal to that of a quasi-correlation claim minus the strike, it follows that the hedging strategy for the latter is a quasi-replication strategy for the former in the sense that it replicates the payoff modulus the error of the correlation proxy. In other words, correlation swaps on an equity index should trade at a strike close to the variance-based implied correlation proxy. It should be pointed out that at the time of writing, over-the-counter transactions typically take place at a significantly lower strike, which may indicate the existence of dynamic arbitrage opportunities.

The implications are vast from both practical and theoretical standpoints. On the practical side, the identification of a quasi-replication strategy is a crucial step for the development of the correlation swap market. On the theoretical side, we see at least three research areas which should be affected by our results: the pricing and hedging of exotic derivatives on multiple equity assets (in particular the long-debated issue of correlation skew), the stochastic modeling of volatility and correlation, and the pricing and hedging of options on realized correlation as a branch of the pricing theory of derivatives on realized variance.

## References

- Brace, A., B. Goldys, J. van der Hoek and R. Womersley, 2002, Markovian Models in the Stochastic Implied Volatility Framework, Working paper.
- Buehler, H., 2004, Consistent Variance Curve Models, Deutsche Bank AG London, Working paper.
- Black, F., and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81, 637-654.
- Bossu, S. and Y. Gu, 2004, Fundamental relationship between an index's volatility and the average volatility and correlation of its components, JPMorgan Equity Derivatives, Working paper.
- Carr, P. and J. Sun, 2005, A New Approach for Option Pricing Under Stochastic Volatility, Bloomberg LP, Working paper.
- Demeterfi, K., E. Derman, M. Kamal, J. Zou, 1999, More Than You Ever Wanted to Know About Volatility Swaps, Goldman Sachs Quantitative Strategies Research Notes.
- Derman, E. and I. Kani, 1994, The Volatility Smile and Its Implied Tree, Goldman Sachs Quantitative Strategies Research Notes.
- Duanmu, Z., 2004, Rational Pricing of Options on Realized Volatility – The Black Scholes Way, Global Derivatives and Risk Management Conference, Madrid.
- Dupire, B., 1993a, Pricing and hedging with smiles, Paribas Capital Markets, Swaps and Options Research Team, Working paper.
- Dupire, B., 1993b, Arbitrage Pricing with Stochastic Volatility, Paribas Capital Markets, Swaps and Options Research Team, Working paper.
- Cont R., J. da Fonseca and V. Durrelman, 2002, Stochastic Models of Implied Volatility Surfaces, Economic Notes by Banca Monte dei Paschi di Siena SpA, vol. 21, no 2-2002, pp. 361-377.
- Heston, S. L., 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *The Review of Financial Studies*, 6, 2, 327-343.
- Hull, J. C. and A. White, 1987, The Pricing of Options on Assets with Stochastic Volatilities, *The Journal of Finance*, 42, 2, 281-300.
- Markowitz, H. M., 1952, Portfolio Selection, *The Journal of Finance*, 7 (1), 77-91.
- Neuberger, A. 1990, Volatility Trading, London Business School, Working paper.
- Potter, C. W., 2004, Complete Stochastic Volatility Models With Variance Swaps, Oxford University, Working paper.
- Schonbucher, P., 1998, A Market Model for Stochastic Implied Volatility, Department of Statistics, Bonn University, Working paper.
- Statman, M. and J. Scheid, 2004, Dispersion, Working paper. <http://ssrn.com/abstract=603682>
- Tierens, I. and M. Anadu, 2004, Does it matter which methodology you use to measure average correlation across stocks?, Goldman Sachs Equity Derivatives Strategy.



### 5.3 “Fundamental Relationship Between An Index’s Volatility And The Correlation And Average Volatility Of Its Components”, J.P. Morgan report, 2004

JUNE 2004<sup>1</sup>

#### FUNDAMENTAL RELATIONSHIP BETWEEN AN INDEX'S VOLATILITY AND THE CORRELATION AND AVERAGE VOLATILITY OF ITS COMPONENTS

Sebastien Bossu  
sebastien.bossu@jpmorgan.com

Yi Gu  
yi.gu@jpmorgan.com

Equity Derivatives Institutional Marketing  
JPMorgan Securities Ltd. - London

CONFIDENTIAL



<sup>1</sup>Original publication February 2004. All data and charts as of 25 April 2004.

## Overview

In this document, we derive and analyse the fundamental relationship between an index's volatility and the correlation and average volatility of its components:

$$\text{Index Volatility} \approx \sqrt{\text{Correlation}} \times (\text{Average Component Volatility}) .$$

This relationship holds in practice when correlation is above 0.15, and the number of components is above 20<sup>2</sup>.

- Section 1 reviews the definition of realised and implied correlation.
- Section 2 derives the relationship from fundamentals.
- Section 3 evaluates the strength of the relationship using statistical methods.
- Appendix A is a short reference of statistical formulas.

---

<sup>2</sup> Additionally, weights and components' volatilities must be reasonable to avoid concentration on specific stocks.

## 1. Realised and Implied Correlation

In this section, the definition of realised and implied correlation is reviewed.

### Realised Correlation

Realised correlation on an index is traditionally defined as the average of the realised correlation matrix between the index components, excluding the diagonal of 1's:

$$\rho_{Realised} = \frac{\sum_{i<j}^n w_i w_j \rho_{ij}}{\sum_{i<j}^n w_i w_j}$$

where n is the number of components, w's are the component weights, and  $\rho$ 's are the pairwise correlations:

$$\rho_{i,j} = \frac{COV_{i,j}}{\sigma_i \sigma_j}.$$

Hence:

$$\rho_{Realised} = \frac{\sum_{i<j}^n w_i w_j \frac{COV_{ij}}{\sigma_i \sigma_j}}{\sum_{i<j}^n w_i w_j}.$$

A slightly different way of defining realised correlation could be given as the ratio of average covariance to average 'cross-volatility':

$$\begin{aligned} \rho'_{Realised} &= \frac{\sum_{i<j} w_i w_j COV_{ij}}{\sum_{i<j} w_i w_j \sigma_i \sigma_j} \\ &= \frac{\sum_{i<j} w_i w_j \sigma_i \sigma_j \rho_{i,j}}{\sum_{i<j} w_i w_j \sigma_i \sigma_j} \end{aligned}$$

This is not the market practice, but would be more consistent with the way *implied correlation* is defined.

Exhibit 1 below shows the evolution of realised correlation over a one-year rolling window for two indices: EuroStoxx 50 and S&P 500.

Exhibit 1

## 1Y Realised Correlation—EuroStoxx 50 and S&amp;P 500



Source: JPMorgan—DataQuery.

### Implied Correlation

Implied Correlation is the correlation parameter extracted from market option prices on an index and its components:

$$\rho_{\text{Implied}} = \frac{\sigma_{\text{Index}}^2 - \sum_{i=1}^n w_i^2 \sigma_i^2}{2 \sum_{i < j} w_i w_j \sigma_i \sigma_j}$$

where  $n$  is the number of components,  $w$ 's are the component weights, and  $\sigma$ 's are implied volatilities.

This definition is derived from the well-known probability formula (see Appendix):

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i < j} \text{Cov}(X_i, X_j)$$

which translates as follows in portfolio theory:

$$\sigma_p^2 = \sum_{i=1}^n w_i^2 \sigma_i^2 + 2 \sum_{i < j} w_i w_j \sigma_i \sigma_j \rho_{i,j}$$

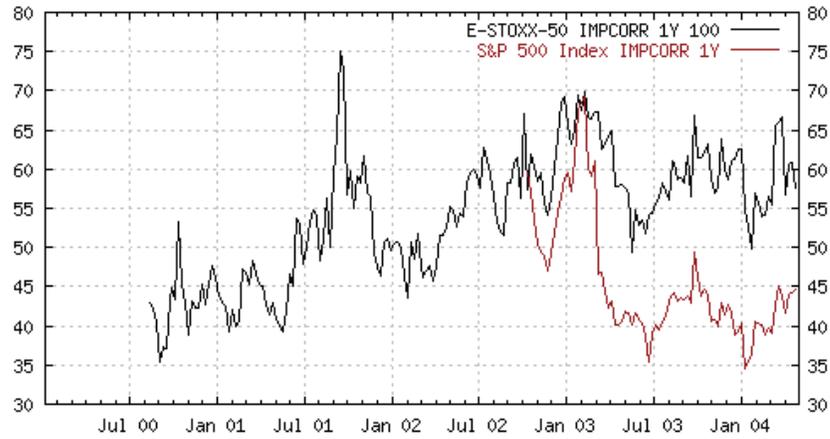
$\rho_{\text{Implied}}$  is thus closely connected to  $\rho'_{\text{Realised}}$ , and commonly interpreted as the market's expectation of the future realised correlation. However, the existence of a volatility surface (skew and term structure) translates into another surface for implied correlation levels. When it comes to determine the 'fair strike' of a correlation swap<sup>3</sup>, traders will have a look at both implied and realised levels.

<sup>3</sup> A correlation swap is an instrument which pays off the notional multiplied by the realised correlation between inception and maturity in exchange for a fixed amount.

Exhibit 2 below shows the evolution of ATM 1Y implied correlation for two indices: EuroStoxx 50 and S&P 500.

Exhibit 2

**1Y ATM Implied Correlation—EuroStoxx 50 and S&P 500**



Source: JPMorgan—DataQuery.

## 2. Fundamental relationship between an index's volatility and the correlation and volatility of its components

In this section, a Proxy for implied and realised correlation is derived from fundamentals, leading to the relationship.

### Correlation Proxy

The following mathematical equation holds for any given numbers  $x_1, \dots, x_n$ :

$$\left( \sum_{i=1}^n x_i \right)^2 = \sum_{i=1}^n x_i^2 + 2 \sum_{i<j} x_i x_j .$$

Whence:

$$\rho_{Implied} = \frac{\sigma_{Index}^2 - \sum_{i=1}^n w_i^2 \sigma_i^2}{\left( \sum_{i=1}^n w_i \sigma_i \right)^2 - \sum_{i=1}^n w_i^2 \sigma_i^2} .$$

For a sufficiently large number of components (in practice  $n > 20$ ) and well-behaved weights

and volatilities, the second term on both the numerator and denominator,  $\sum_{i=1}^n w_i^2 \sigma_i^2$ ,

becomes negligible. This is most straightforwardly observed when the components are equally weighted:

$$\sum_{i=1}^n \frac{1}{n^2} \sigma_i^2 \leq \frac{1}{n^2} \sum_{i=1}^n \sigma_{\max}^2 = \frac{\sigma_{\max}^2}{n} \xrightarrow{n \rightarrow +\infty} 0$$

The limiting case gives us the proxy:

$$\rho \approx \frac{\sigma_{Index}^2}{\sigma_{Average}^2}$$

where  $\sigma_{Average} = \sum_{i=1}^n w_i \sigma_i$  is the average implied volatility of the components. Note that this

proxy will become inaccurate if the true implied correlation becomes low (in practice  $< 0.15$ ). Since most indices select their components regionally or sectorially, this should rarely be observed in practice.

The same derivations hold for the second definition of realised correlation,  $\rho'_{Realised}$ , once it has been noticed that:

$$\rho'_{Realised} = \frac{\sigma_{Index}^2 - \sum_{i=1}^n w_i^2 \sigma_i^2}{2 \sum_{i<j} w_i w_j \sigma_i \sigma_j}$$

by backing out the numerator through the formula for the volatility of a portfolio and its components (see Section 1.)

## Interpretation

In essence, correlation is the squared ratio of index volatility to the average volatility of the components. In other words:

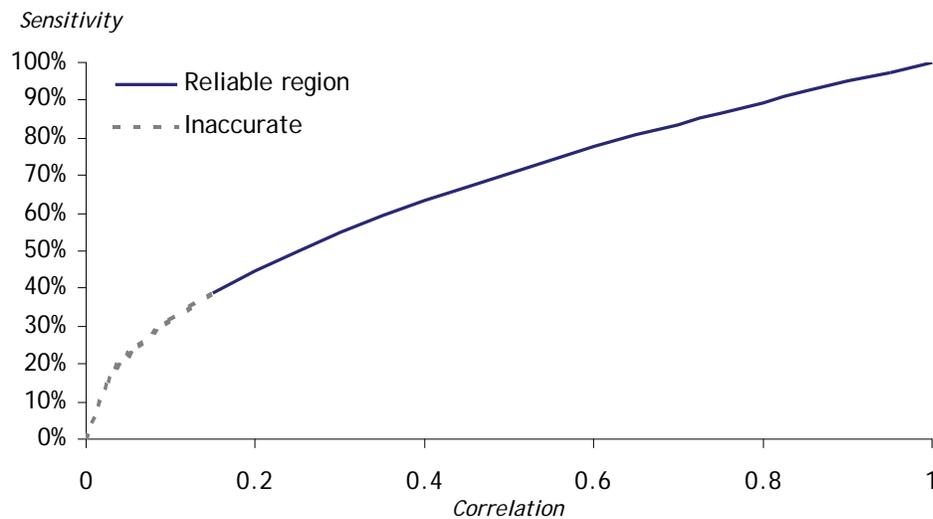
$$\text{Index Volatility} \approx \sqrt{\text{Correlation}} \times (\text{Average Component Volatility})$$

This relationship means that index volatility is less sensitive to changes in components' volatility when correlation is low. However, the relationship becomes weaker when correlation reaches the region 0.15-0.25; and becomes inaccurate below 0.15.

Exhibit 3 below shows the sensitivity of index volatility to average component volatility in function of correlation.

Exhibit 3

Sensitivity of index volatility to average component volatility.



### 3. Statistical Analysis

In this section, the strength of the relationship derived previously is evaluated using statistical methods on both implied and realised data.

New functions in DataQuery introduced in January 2004<sup>4</sup> give access to both implied and realised data for volatility, correlation and average volatility of the major indices.

#### Implied Data

Exhibits 4 and 5 below compare the index ATM implied volatility (IVOL) for both one-year and 3-month maturities against  $\sqrt{IMPCORR} \times AVGIVOL$  on the EuroStoxx 50 between 2000 and 2004. The fundamental relationship is so accurate in both cases that the two lines are almost indistinguishable.

This is confirmed by the regression results in Exhibit 6: the  $R^2$  is 0.99873.

#### Exhibit 4

1Y ATM implied volatility of EuroStoxx50 vs. fundamentals

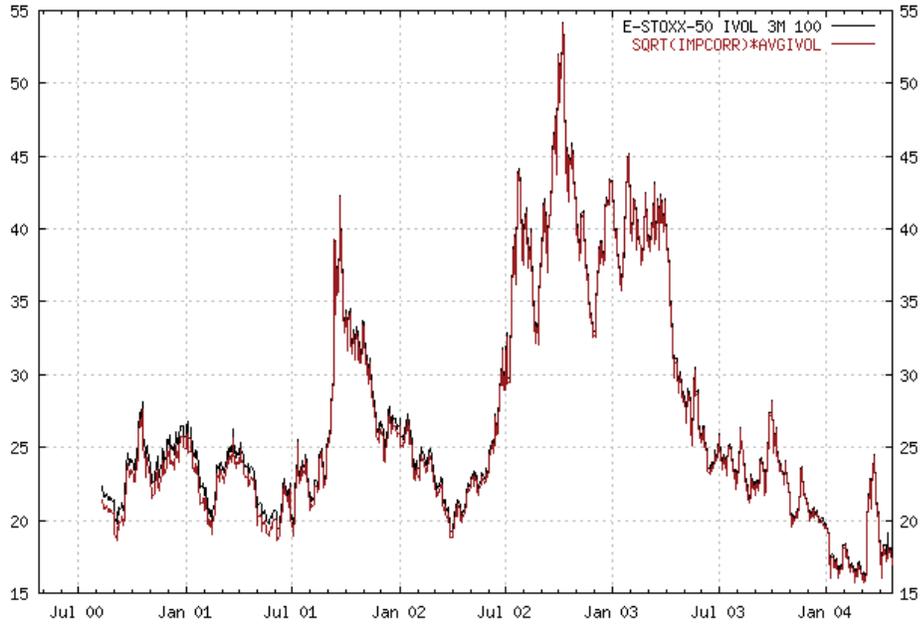


Source: JPMorgan—DataQuery.

<sup>4</sup> *Equity Derivatives in DataQuery*, EDG Strategy London, 26 January 2004.

Exhibit 5

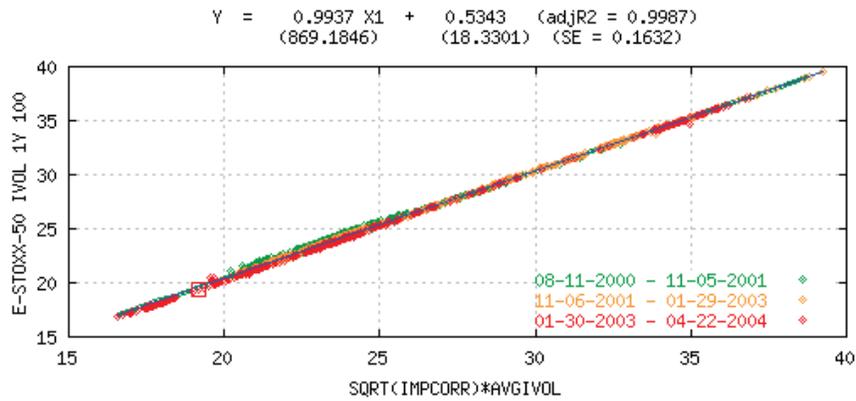
3M ATM implied volatility of EuroStoxx50 vs. fundamentals



Source: JPMorgan—DataQuery.

Exhibit 6

1Y ATM implied volatility of EuroStoxx50 vs. fundamentals—Regression



Source: JPMorgan—DataQuery.

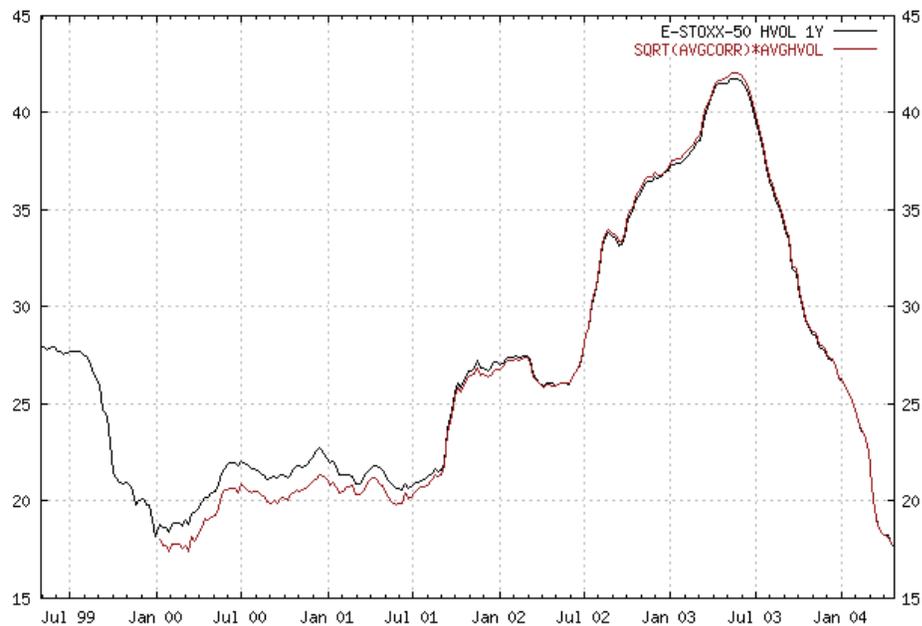
## Realised Data

The same methodology is used for realised data. Exhibits 7 and 8 below compare the index realised volatility (HVOL) for both one-year and 3-month rolling windows against  $\sqrt{AVGCORR} \times AVGHVOL$  on the EuroStoxx 50 between 2000 and 2004.

The relationship is also very strong (see the regression results in Exhibit 9, with a  $R^2$  of 0.9975), but slightly less so for the period starting January 2000 and ending August 2001. Interestingly, this period coincides with a low realised correlation (see Exhibit 1 in Section 1), where the fundamental relationship is expected to be weaker.

### Exhibit 7

#### 1Y realised volatility of EuroStoxx50 vs. fundamentals



Source: JPMorgan—DataQuery.

Exhibit 8

3M realised volatility of EuroStoxx50 vs. fundamentals

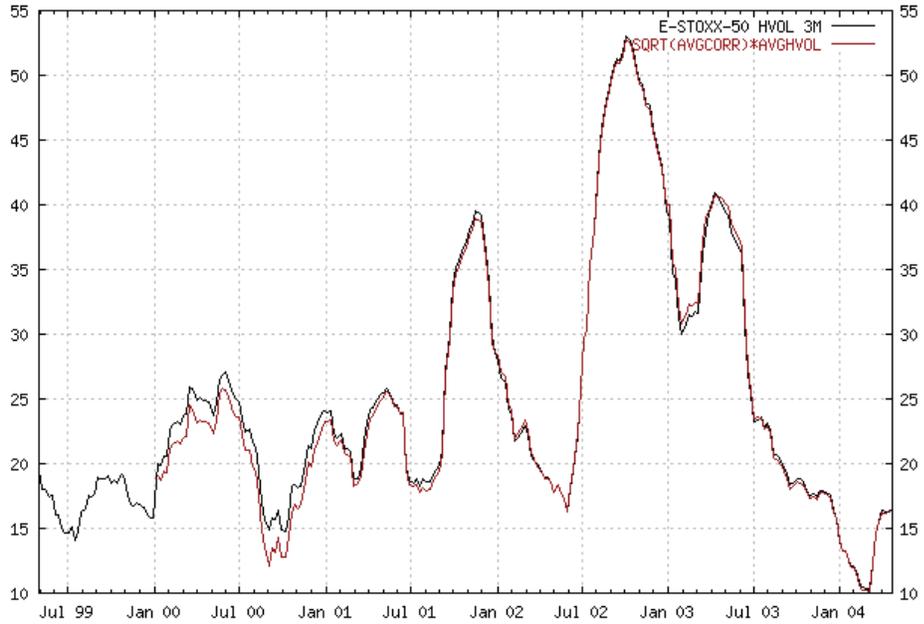
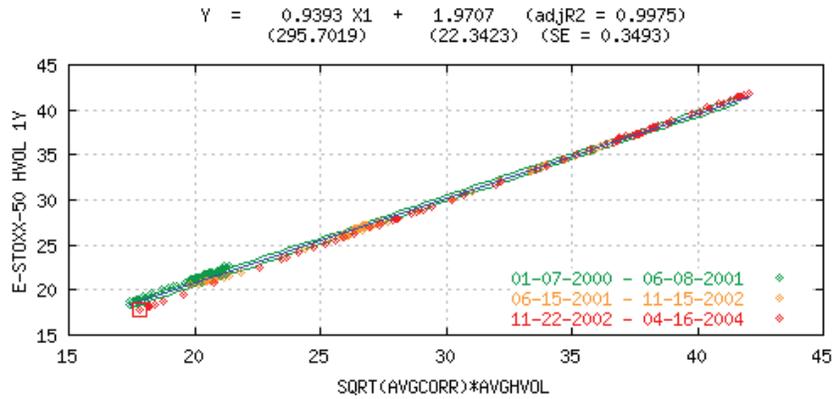


Exhibit 9

1Y realised volatility of EuroStoxx50 vs. fundamentals—Regression

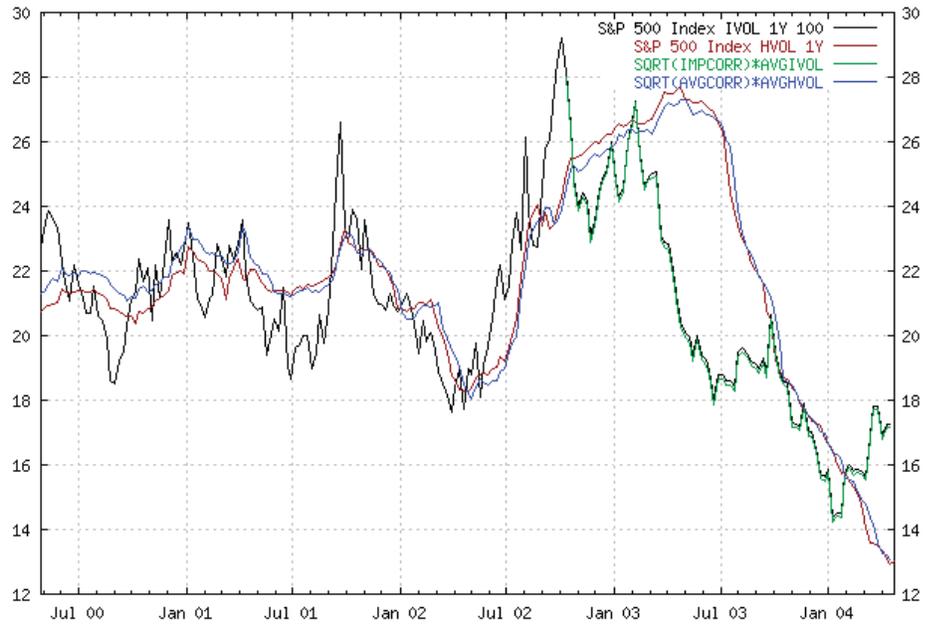


### Other Indices

The relationship was also tested on the S&P 500. Exhibit 10 displays the results for both implied and realised data. Again, strong accuracy was observed.

Exhibit 10

**1Y implied and realised volatility of S&P 500 vs. fundamentals**



FUNDAMENTAL RELATIONSHIP BETWEEN AN INDEX'S VOLATILITY AND THE CORRELATION AND AVERAGE VOLATILITY OF ITS COMPONENTS

## Appendix A—Quick Reference of Statistical Formulas

The following formulas hold for **complete** statistical series of length  $N$ , e.g.  $X = (x_i) = (x_1, \dots, x_N)$ . Please refer to the sub-section on Sample Estimation for formulas on incomplete series.

### Mean

The **mean** of a series  $X$  is:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N x_i$$

### Variance. Standard Deviation

The **variance** of a **complete** series  $X$  is:

$$\text{Var}(X) = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{X})^2$$

and the **standard deviation** is the square root of the variance (also called **volatility** in finance):

$$\sigma_X = \sqrt{\text{Var}(X)}$$

Note:  $\text{Var}(\alpha X + \beta) = \alpha^2 \text{Var}(X)$ .

### Covariance

The **covariance** between two **complete** series  $X$  and  $Y$  is:

$$\text{Cov}(X, Y) = N \sum_{i=1}^N (x_i - \bar{X})(y_i - \bar{Y})$$

Covariance appears in the variance of a sum of two series:

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

and more generally, for  $n$  series:

$$\text{Var}\left(\sum_{k=1}^n X_k\right) = \sum_{k=1}^n \text{Var}(X_k) + 2 \sum_{1 \leq j < k \leq n} \text{Cov}(X_j, X_k)$$

Note:  $\text{Cov}(\alpha X + Y, Z) = \alpha \text{Cov}(X, Z) + \text{Cov}(Y, Z)$  (bilinearity)

$$\text{Cov}(X, X) = \text{Var}(X)$$

### Correlation

The **coefficient of correlation** between two series  $X$  and  $Y$  is:

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Correlation appears in the variance of a sum of two series:

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y\rho_{X,Y}$$

whence: 
$$\rho_{X,Y} = \frac{\sigma_{X+Y}^2 - (\sigma_X^2 + \sigma_Y^2)}{2\sigma_X\sigma_Y}$$

and since  $(\sigma_X + \sigma_Y)^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y$ , we also have:

$$\rho_{X,Y} = \frac{\sigma_{X+Y}^2 - (\sigma_X^2 + \sigma_Y^2)}{(\sigma_X + \sigma_Y)^2 - (\sigma_X^2 + \sigma_Y^2)}$$

### Notes

Correlation is a statistical measure for the level of interdependence between two variables. In the case of the correlation of returns, it is a measure of the "joint directionality" of two assets: the higher the correlation, the more frequently the assets move upwards/downwards together.

A well-known property of correlation is that it is a number comprised between -1 and 1. The three perfect cases are:

Correlation	Interpretation
+1	The two variables are related with 100% certainty by a linear formula: $X = a + bY$ , where $b > 0$ .
0	The two variables are independent <sup>5</sup> from each other.
-1	The two variables are related with 100% certainty by a linear formula: $X = a - bY$ , where $b > 0$ .

The definition of correlation expands as follows:

$$\rho_{X,Y} = \frac{1}{N} \sum_{i=1}^N \left[ \left( \frac{x_i - \bar{X}}{\sigma_X} \right) \left( \frac{y_i - \bar{Y}}{\sigma_Y} \right) \right]$$

where  $N$  is the length of the series (number of observation dates),

$x_i, y_i$  are the  $i^{\text{th}}$  observations of  $X$  and  $Y$  respectively,

$\bar{x}, \bar{y}$  are the means (average returns), and

$\sigma_X, \sigma_Y$  are the standard deviations (volatilities).

A closer look at this formula yields the following observations:

<sup>5</sup> As understood in general language. In mathematics, two independent variables must have zero correlation, but the converse is not necessarily true.

- $\left(\frac{x_i - \bar{x}}{\sigma_x}\right)$  is the number of standard deviations from the mean for observation  $x_i$ : for instance, if the mean return is 5%, volatility is 10% and  $x_i = 20\%$ , this observation diverges by +1.5 standard deviations from its mean
- $\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\left(\frac{y_i - \bar{y}}{\sigma_y}\right)$  hence measures the joint deviation of  $x_i$  and  $y_i$ : it is a positive number if  $x_i$  and  $y_i$  'move' in the same direction, a negative one if they move in opposite directions, and close to zero otherwise.

	$y_i$ : upwards	$y_i$ : no deviation	$y_i$ : downwards
$x_i$ : upwards	Joint deviation >0	Nil	Joint deviation <0
$x_i$ : no deviation	Nil	Nil	Nil
$x_i$ : downwards	Joint deviation <0	Nil	Joint deviation >0

- Thus, correlation is nothing else than the average of joint deviations.

Because a single number cannot summarise the complexity of the dynamics between two statistical series, it does not replace a thorough pattern analysis, as illustrated below.

	$x_i$	$y_i$
Observation 1	1	1
Observation 2	-1	-1
Observation 3	1	-1
Observation 4	-1	1
Mean	0	0
Standard Deviation	1	1
Correlation	0	

In this example, the first two observations are perfectly correlated, while the last two are perfectly anti-correlated. This could reflect that some event between observations 2 and 3 had changed the nature of the relationship between X and Y. However, the overall correlation is 0.

### Sample Estimation

The formulas above are valid when the data series are 'complete'. When dealing with samples, however, we can only *estimate* the mean and variance. A Theory of Estimation was developed accordingly, and the main finding is that an **unbiased estimate** of the variance of a sample is:

$$\text{Var}(X) = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{X})^2$$

and similarly for covariance.

This subtlety has little impact for large  $N$ , and has no effect on the correlation coefficient as the averaging weights cancel out:

$$\begin{aligned}\rho_{x,y} &= \frac{\frac{1}{N-1} \sum (x_i - \bar{X})(y_i - \bar{Y})}{\sqrt{\frac{1}{N-1} \sum (x_i - \bar{X})^2} \sqrt{\frac{1}{N-1} \sum (y_i - \bar{Y})^2}} \\ &= \frac{\sum (x_i - \bar{X})(y_i - \bar{Y})}{\sqrt{\sum (x_i - \bar{X})^2} \sqrt{\sum (y_i - \bar{Y})^2}}\end{aligned}$$

#### Zero-mean assumption

If the true mean is assumed to be zero, an unbiased estimate of the variance is simply:

$$\text{Var}(X) = \frac{1}{N} \sum_{i=1}^N x_i^2$$

#### Notes

- Other estimation techniques, such as Maximum Likelihood, may yield different formulas.
- The vast majority of estimation formulas are based on the assumption that observations are independent.

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## 5.4 “Introduction to Variance Swaps”, Wilmott Magazine, 2006

# Introduction to Variance Swaps

**Sebastien Bossu**, Dresdner Kleinwort Wasserstein,  
Equity Derivatives Structuring

### Abstract

The purpose of this article is to introduce the properties of variance swaps, and give insights into the hedging and valuation of these instruments from the particular lens of an option trader.

- Section 1 gives general details about variance swaps and their applications.
- Section 2 explains in ‘intuitive’ financial mathematics terms how variance swaps are hedged and priced.

### Keywords

Variance swap, volatility, path-dependent, gamma risk, static hedge.

### Disclaimer

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### Payoff

A variance swap is a derivative contract which allows investors to trade future realized (or historical) volatility against current implied volatility. The reason why the contract is based on variance—the squared volatility—is that only the former can be replicated with a static hedge, as explained in the penultimate Section of this article.

Sample terms are given in Exhibit 1 in the next page.

These sample terms reflect current market practices. In particular:

1. Asset returns are computed on a logarithmic basis rather than arithmetic.
2. The mean return, which appears in the habitual statistics formula for variance, is ditched. This has the benefit of making the payoff perfectly additive (i.e. 1-year variance can be split into two 6-month segments.)
3. The 252 scaling factor corresponds to the standard number of trading days in a year. The  $10,000 = 100^2$  scaling factor corresponds to the conversion from decimal (0.01) to percentage point (1%).

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4. The notional is specified in volatility terms (here €50,000 per ‘vega’ or volatility point.) The true notional of the trade, called variance notional or variance units, is given as:

$$\text{Variance Notional} = \frac{\text{Vega Notional}}{2 \times \text{Strike}}$$

With this convention, if realized volatility is 1 point above the strike at maturity, the payoff will approximately be equal to the Vega Notional.

### Variance Swaps vs. Volatility Swaps

The fair strike of a variance swap is slightly higher than that of a volatility swap. This is to compensate for the fact that variance is convex in volatility, as illustrated in Exhibit 2 in the next page. Identical strikes for the two instruments would otherwise lead to an arbitrage.

Intuitively, the difference in fair strikes is related to the volatility of volatility: the higher the ‘vol of vol’, the more expensive the convexity effect of variance<sup>1</sup>. This phenomenon is clearly observed when the implied volatility skew is steep, as skew accounts for the empirical fact that volatility is

Exhibit 1—Variance Swap on Dow Jones Euro Stoxx 50 Index: sample terms and conditions

General Terms	
Swap Buyer (Party A)	TBD [e.g. Investor]
Swap Seller (Party B)	TBD [e.g. Dresdner Bank AG]
Trade date	TBD
Expiration Date	TBD [e.g. Trade date + 1 year]
Swap	Variance Swap on Equity Index
Index	.STOXX50E
Currency	EUR
Observation Frequency	Daily
Approximate Vega Notional	EUR 50,000
Variance Units	1157.41 (Approximate Vega Notional divided by 2 x Volatility Strike)
Volatility Strike	21.6
Variance Strike	466.56 (square of the Volatility Strike)
Cash Settlement Date	Two business days after the Expiration Date
Payment Amount	The Payment Amount is calculated as: $\text{Variance Units} \times (\sigma^2 - \text{Variance Strike})$ If the Payment Amount is positive, the Swap Seller (B) pays to the Swap Buyer (A); if the Payment Amount is negative, the Swap Buyer (A) pays to the Swap Seller (B) the absolute value.
Variance Calculation	$\sigma^2$ is the observed realized variance of the Index between the Trade Date and the Expiration Date, given as: $\sigma^2 = \frac{10,000 \times 252 \times \sum_{i=1}^{N_{Actual}} Return_i^2}{N_{Expected}}$ Where: $Return_i = \ln \left( \frac{E_i}{E_{i-1}} \right)$ $N_{Expected}$ is the expected number of trading days from, but not including, the Trade Date, up to and including the Expiration Date. $N_{Actual}$ is the actual number of trading days on which no market disruption event occurs from, but not including, the Trade Date, up to and including the Expiration Date. $E_0$ is the closing level of the index on Trade Date. $E_i$ is the closing level of the index on date $i$ or, at Expiration Date, the options final settlement level.

non constant. In fact, the fair strike of variance is often in line with the implied volatility of the 30% delta put.

**Rule of Thumb**

Demeterfi–Derman–Kamal–Zou (1999) derive the following rule of thumb when skew is linear in strike:

$$K_{var} \approx \sigma_{ATMF} \sqrt{1 + 3T \times skew^2}$$

where  $\sigma_{ATMF}$  is the at-the-money-forward volatility,  $T$  is the maturity, and  $skew$  is the slope of the skew curve. For example, with  $\sigma_{ATMF} = 20\%$ ,  $T = 2$  years, and a 90–100 skew of 2 vegas, we have  $K_{var} \approx 22.3\%$ . In comparison, a 30% delta put would have an implied volatility of 22.2% assuming a linear skew.

However, this rule of thumb becomes inaccurate when skew is steep.

**Applications**

**Bets on Future Realized Volatility**

Variance swaps are ideal instruments to bet on volatility:

- Unlike vanilla options, variance swaps do not require any delta-hedging
- Unlike the P&L of a delta-hedged vanilla option, the payoff at maturity of a long variance position will always be positive when realized volatility exceeds the strike<sup>2</sup>. (See the next Section on the path-dependency of vanilla options for more details.)
- The sensitivity of a variance swap to changes in (squared) implied volatility linearly collapses through time.

Furthermore, volatility sellers will find variance swaps more attractive than at-the-money options due to their higher variance strike. However this excess profit reflects the higher risk in case realized volatility jumps well above the strike.

**Bets on Forward Realized Volatility**

Forward-starting variance swaps can be synthesized with a calendar spread of two spot-starting variance swaps, with appropriate notionals. This is because the variance formula is designed to be perfectly additive. Taking annualization into account, we can indeed write:

$$3 \times Realized_{3Y} = Realized_{1Y} + 2 \times Forward Realized_{1Y \times 2Y}$$

where  $Realized_{1Y}$  is the future 1-year realized volatility,  $Realized_{3Y}$  is the future 3-year realized volatility, and  $Forward Realized_{1Y \times 2Y}$  is the future 2-year realized volatility starting in 1 year.

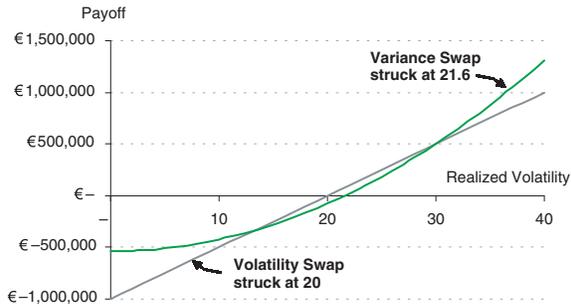
Thus, for a given forward variance notional, we must adjust the spot variance notionals as follows:

$$Variance\ Notional_{1Y} = \frac{1 \times Forward\ Variance\ Notional_{1Y \times 2Y}}{2}$$

$$Variance\ Notional_{3Y} = \frac{3 \times Forward\ Variance\ Notional_{1Y \times 2Y}}{2}$$



**Exhibit 2—Variance swaps are convex in volatility**



The resulting implicit fair strike for the forward variance swap is:

$$\sqrt{\frac{3 \times K_{3Y \text{ var}}^2 - 1 \times K_{1Y \text{ var}}^2}{2}}$$

For example, with  $K_{1Y \text{ var}} = 18.5$ ,  $K_{3Y \text{ var}} = 19.5$ , the fair strike of a 2-year variance swap starting in 1 year would be:

$$\sqrt{\frac{3 \times 19.5^2 - 1 \times 18.5^2}{2}} \approx 20.0$$

The corresponding replication strategy for a long €100,000 forward vega notional position (equivalent to 2,500 forward variance units) would be to buy  $3 \times 2,500/2 = 3,750$  variance units of the 3-year variance swap and sell  $2,500/2 = 1,250$  variance units of the 1-year.

### Correlation Trading

By simultaneously selling a variance swap on an index and buying variance swaps on the constituents, an investor effectively takes a short position on realized correlation. This type of trade is known as a variance dispersion. A proxy for the implied correlation level sold through a variance dispersion trade is given as the squared ratio of the index variance strike to the average of constituents' variance strikes.

Note that in order to offset the vega exposure between the two legs we must adjust the vega notionals of the constituents by a factor equal to the square root of implied correlation. It can be shown that by dynamically trading vega-neutral variance dispersions until maturity we would almost replicate the payoff of a correlation swap<sup>3</sup>.

## Trading P&L of a Vanilla Option

### Daily Trading P&L

Following the theory developed by Black–Scholes and Merton in 1973, the sensitivity of a vanilla option to changes in the stock price, or delta, can be completely offset by holding a reverse position in the stock in quantity equal to the delta. The iteration of this strategy until maturity is known as delta-hedging.

Once the delta is hedged, an option trader is primarily left with three risks:

- Gamma: sensitivity of the option delta to changes in the underlying stock price;
- Theta or time decay: sensitivity of the option price to the passage of time;
- Vega: sensitivity of the option price to changes in the market's expectation of future volatility (i.e. implied volatility.)

We can break down the daily P&L on a delta-neutral option position along these risks:

$$\text{Daily P\&L} = \text{Gamma P\&L} + \text{Theta P\&L} + \text{Vega P\&L} + \text{Other} \quad (\text{Eq. 1})$$

Here 'Other' includes the P&L from financing the reverse delta position on the underlying, as well as the P&L due to changes in interest rates, dividend expectations, and high-order sensitivities (e.g. sensitivity of Vega to changes in stock price, etc.)

Using Greek letters, we can rewrite Equation 1 as:

$$\text{Daily P\&L} = \frac{1}{2} \Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) + V \times (\Delta \sigma) + \dots$$

where  $\Delta S$  is the change in the underlying stock price,  $\Delta t$  is the fraction of time elapsed (typically 1/365), and  $\Delta \sigma$  is the change in implied volatility.

Assuming a zero interest rate, constant volatility and negligible high-order sensitivities, we can further reduce this equation to the first two terms:

$$\text{Daily P\&L} = \frac{1}{2} \Gamma \times (\Delta S)^2 + \Theta \times (\Delta t) \quad (\text{Eq. 2})$$

Equation 2 can be further expanded to be interpreted in terms of realized and implied volatility. This is because in our zero-interest rate world Theta can be re-expressed with Gamma<sup>4</sup>:

$$\Theta = -\frac{1}{2} \Gamma S^2 \sigma^2 \quad (\text{Eq. 3})$$

Plugging Equation 3 into Equation 2, we obtain a characterization of the daily P&L in terms of squared return and squared implied volatility:

$$\text{Daily P\&L} = \frac{1}{2} \Gamma S^2 \times \left[ \left( \frac{\Delta S}{S} \right)^2 - \sigma^2 \Delta t \right] \quad (\text{Eq. 4})$$

The first term in the bracket,  $\frac{\Delta S}{S}$ , is the percent change in the stock price—in other words, the one-day stock return. Squared, it can be interpreted as the *realized* one-day variance. The second term in the bracket,  $\sigma^2 \Delta t$ , is the squared daily *implied* volatility, which one could name the daily implied variance. Finally, the factor in front of the bracket,  $\frac{1}{2} \Gamma S^2$ , is known as *Dollar Gamma*: an adjusted measure for the second-order sensitivity of the option price to a squared percent change in the stock price.

In short, Equation 4 tells us that the daily P&L of a delta-hedged option position is driven by the difference between realized and implied variance, multiplied by the Dollar Gamma.

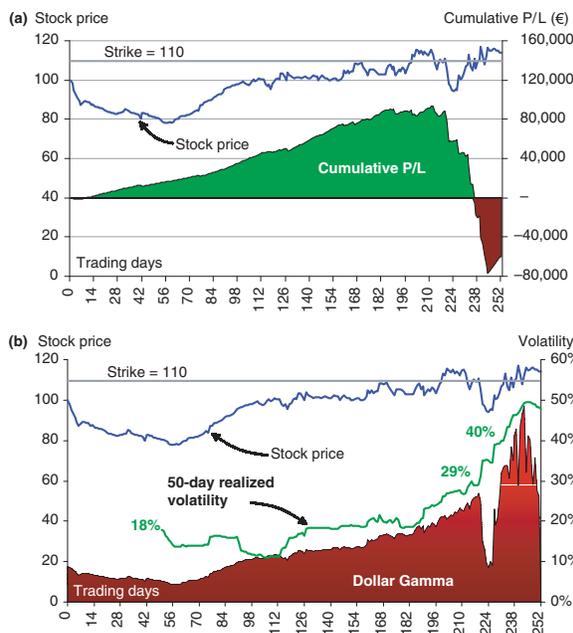
### Path Dependency

One can already see the connection between Equation 4 and variance swaps: if we sum all daily P&L's until maturity, we have an expression for the final trading P&L on a delta-neutral option position:

$$\text{Final P\&L} = \sum_{t=0}^n \gamma_t [r_t^2 - \sigma^2 \Delta t] \quad (\text{Eq. 5})$$

**Exhibit 3—Path-dependency of the cumulative P&L for a dynamically hedged option position**

In this simulation a trader issued 25,000 1-year calls struck at €110 for an implied volatility of 30%, and followed a daily delta-hedging strategy. The 1-year realized volatility at maturity was 27.6%, yet the cumulative trading P&L was down €60k. In figure (a) we can see that the strategy was up €100,000 two months before maturity and suddenly dropped. In figure (b) we can see that in the final two months the 50-day realized volatility rose well above 30% while the (short) dollar gamma peaked. Because the daily P&L of an option position is weighted by the dollar gamma, and because the volatility spread between implied and realized was negative, the final P&L plunged, even though the 1-year realized volatility was below 30%!



where the subscript  $t$  denotes time dependence,  $r_t$  is the stock daily return at time  $t$ ,  $\gamma_t$  is the dollar gamma, and  $n$  is the number of trading days until maturity.

Equation 5 is close to the payoff of a variance swap: it is a weighted sum of squared realized returns minus a constant that has the same role as a strike. But in a variance swap the weights are constant, while here the weights depend on the option gamma through time. This explains an option trading phenomenon known as path-dependency, illustrated in Exhibit 3.

**Static Replication of Variance Swaps**

In the previous section, we saw that a trader who follows a delta-hedging strategy is basically replicating the payoff of a weighted variance swap,

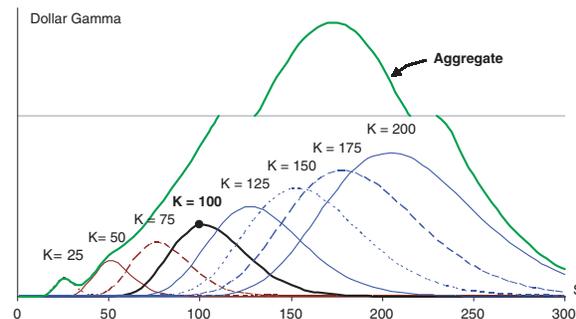
with weights equal to the dollar gamma. This result also holds for a portfolio of options. If we could find a combination of calls and puts such that their aggregate dollar gamma is always constant, we would have a semi-static hedge for variance swaps<sup>5</sup>.

Exhibit 4 shows the dollar gamma of options with various strikes in function of the underlying level. We can see that the contribution of low-strike options to the aggregate gamma is small compared to high-strike options. Hence, we need to increase the weights of low-strike options and decrease the weights of high-strike options.

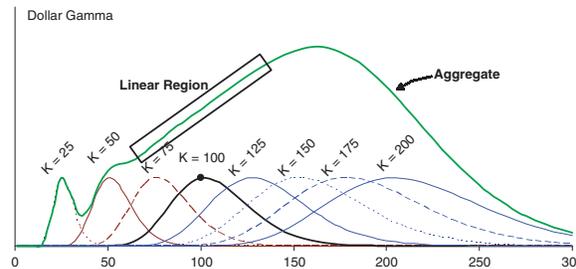
One 'naïve' idea is to use weights inversely proportional to the strike so as to scale all individual dollar gammas to the same peak level<sup>6</sup>, as illustrated in Exhibit 5. We can see that the aggregate dollar gamma is still non-constant, but we can notice the existence of a linear region.

This observation is crucial: if we can regionally obtain a linear aggregate dollar gamma with a certain weighting scheme  $w(K)$ , then the transformed weights  $w'(K) = w(K)/K$  will produce a constant dollar gamma in that region. Since the naïve weights are inversely proportional to the strike  $K$ , the correct weights should be chosen to be inversely proportional to the squared strike, i.e.:  $w'(K) = 1/K^2$ .

**Exhibit 4—Dollar Gamma of vanilla options with strikes 25 to 200 spaced 25 apart**



**Exhibit 5—Weighted Dollar Gamma of vanilla options: weights inversely proportional to the strike K**



**Exhibit 6—Weighted Dollar Gamma of vanillas: weights inversely proportional to the square of strike**

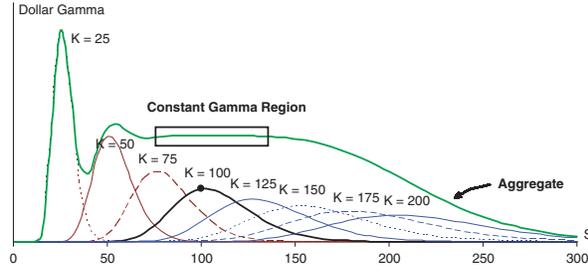


Exhibit 6 shows the results for the transformed weights: as expected, we now have a constant region for the aggregate dollar gamma.

To obtain a perfect constant aggregate gamma through all underlying levels would take infinitely many options struck along a continuum between 0 and infinity.

**Interpretation**

Having mentioned that a variance swap could be perfectly hedged with an infinite portfolio of puts and calls of constant Dollar Gamma, one might want to view this portfolio as a new kind of derivative and speculate on its nature. Denoting  $f$  the price of the derivative and  $S$  that of the underlying, the Dollar Gamma is given as:

$$\Gamma^S(S) = \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \times S^2$$

It follows that a constant Dollar Gamma derivative satisfies:

$$\frac{\partial^2 f}{\partial S^2} = \frac{a}{S^2}$$

for some constant  $a$ . The solution to this second-order differential equation is of the form:

$$f(S) = -a \ln(S) + bS + c$$

where  $a, b, c$  are constants and  $\ln(\cdot)$  is the natural logarithm.

This means that the perfect hedge for a variance swap would be a contract paying the log-price of the underlying stock at maturity, and a combination of the stock and cash. Unfortunately such log-contract does not trade—or, rather, it trades in the format of a variance swap.

**Valuation**

In the absence of arbitrage, the fair market value of a variance swap must equal the price of its replicating portfolio of puts and calls. As such, no model specification is required: the theoretical price can be calculated for any reasonable volatility smile. Note, however, that this approach does not take into account the impact of jumps or discrete dividend payments.

Given a set of  $N$  strikes  $(k_1, k_2, \dots, k_n, \dots, k_N)$  where  $k_n = 1$  denotes the split between out-of-the-money-forward puts and calls, a quick proxy for

**Exhibit 7—Fair value decomposition of a variance swap through a replicating portfolio of puts and calls**

In this example, we consider a variance swap on the S&P 500 index expiring on 15 December 2006. The time to maturity  $T$  is 1.1032 and the discount factor to maturity is  $DF = 0.94889$ . The total cost of the replicating portfolio (i.e. the weighted sum of put and call prices multiplied by  $2/T$ ) is 2.45%, which corresponds to a fair strike of 16.06%. A more accurate model gave 15.83%.

Weight = 5% Strike%	Call / Put	Strike	Strike (%Forward)	Implied Vol	Price (%Forward)
20.00%	Put	629.88	50.0%	24.53	0.02%
16.53%	Put	692.87	55.0%	24.39	0.06%
13.89%	Put	755.86	60.0%	23.91	0.14%
11.83%	Put	818.85	65.0%	23.06	0.28%
10.20%	Put	881.83	70.0%	21.99	0.48%
8.89%	Put	944.82	75.0%	20.81	0.78%
7.81%	Put	1,007.81	80.0%	19.56	1.22%
6.92%	Put	1,070.80	85.0%	18.26	1.84%
6.17%	Put	1,133.79	90.0%	16.92	2.72%
5.54%	Put	1,196.78	95.0%	15.55	3.94%
2.50%	Put	1,259.76	100.0%	14.20	5.64%
2.50%	Call	1,259.76	100.0%	14.20	5.64%
4.54%	Call	1,322.75	105.0%	13.12	3.30%
4.13%	Call	1,385.74	110.0%	12.34	1.73%
3.78%	Call	1,448.73	115.0%	11.83	0.82%
3.47%	Call	1,511.72	120.0%	11.55	0.37%
3.20%	Call	1,574.71	125.0%	11.45	0.16%
2.96%	Call	1,637.69	130.0%	11.40	0.06%
2.74%	Call	1,700.68	135.0%	11.38	0.03%
2.55%	Call	1,763.67	140.0%	11.37	0.01%
2.38%	Call	1,826.66	145.0%	11.38	0.00%
2.22%	Call	1,889.65	150.0%	11.39	0.00%

Source: DrKW.

the fair value of a variance swap with \$1 variance notional is:

$$\text{VarSwap}_{FV} \approx \frac{2}{T} \left[ \sum_{i=1}^n \frac{(k_i - k_{i-1})}{k_i^2} \text{put}_{\%F}(k_i) + \sum_{i=n+1}^N \frac{(k_i - k_{i-1})}{k_i^2} \text{call}_{\%F}(k_i) \right] - DF(0, T) \times K_{\text{var}}^2$$

where  $T$  is the maturity,  $K_{\text{var}}$  is the variance strike,  $DF(0, T)$  is the present value of \$1 collected at maturity,  $\text{put}_{\%F}(k)$  or  $\text{call}_{\%F}(k)$  is the price of a European put or call struck at  $k$ , and  $k_0 = 0$ . Note that the strikes and option prices are expressed in percentage of the forward price.

Exhibit 7 above gives a calculation example with strikes between 50% and 150% spaced 5% apart.

**Conclusion**

**Looking Forward**

Variance swaps have become an increasingly popular type of 'light exotic' derivative instrument. Market participants are the major derivatives houses, hedge funds, and institutional investors. An unofficial estimate of the typical inter-broker trading volume is between \$1,000,000 and \$7,000,000 total vega notional in the European and American markets every day.

With the commoditization of variance swaps, variance is becoming an asset class of its own. A number of volatility indices have been launched or

adjusted to follow the weighting methodology of the replicating portfolio, in particular the new Chicago Board Options Exchange SPX Volatility Index (VIX) and the Deutsche Börse VSTOXX Volatility Index. The current hot development is options on realized volatility, with recent research results by Dunamu (2004) and Carr-Lee (2005).

**ACKNOWLEDGEMENTS**

I thank my colleagues Alexandre Capez, Sophie Granchi, Assad Bouayoun for their helpful comments. Any remaining error is mine.

**FOOTNOTES & REFERENCES**

- 1. Readers with a mathematical background will also recall Jensen’s inequality:  $E(\sqrt{\text{variance}}) \leq \sqrt{E(\text{variance})}$ .
- 2. This is because the sign of  $(R^2 - K^2) = (R - K)(R + K)$  is determined by  $R - K$ , where  $R$  is the realized volatility and  $K$  is the strike.
- 3. A correlation swap is a derivative contract on several assets where counterparties exchange a fixed cash flow against a variable amount equal to the notional multiplied by the average of the pair-wise correlation coefficients between the assets.

- 4. In a zero interest-rate world the Black-Scholes partial differential equation becomes:  $\frac{1}{2}S^2\sigma^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} = 0$ .
- 5. The hedge is semi-static because the portfolio of puts and calls still needs to be delta-hedged. However, no dynamic trading of options is required.
- 6. This is because the dollar gamma peaks around the strike. Specifically, it can be shown that the peak is reached when the stock price is equal to  $S^* = Ke^{a\sqrt{T} - a^2T/2} \approx K$ , with a peak level proportional to  $S^*$ .

- Bossu, Strasser, Guichard (2005), *Just What You Need To Know About Variance Swaps*, JPMorgan Equity Derivatives report.
- Bossu (2005), *Arbitrage Pricing of Equity Correlation Swaps*, JPMorgan Equity Derivatives report.
- Carr, Lee (2005), *Robust Replication of Volatility Derivatives*, Bloomberg LP, Courant Institute and University of Chicago Working paper.
- Demeterfi, Derman, Kamal, Zou (1999), *More Than You Ever Wanted To Know About Volatility Swaps*, Goldman Sachs Quantitative Strategies Research Notes.
- Duanmu (2004), *Rational Pricing of Options on Realized Volatility*. Global Derivatives and Risk Management Conference, Madrid.
- Iberrakene (2005), *Hot Topics In Light Exotics*, The Euromoney Derivatives & Risk Management Handbook 2005/06.



**Titre:** La réplication statique des options européennes et la réplication dynamique des swaps de corrélation

**Mots clés:** options exotiques, formule de Breeden-Litzenberger, transformée de Radon, équations intégrales, intégrales régularisées

**Résumé:** Cette thèse de doctorat reproduit trois articles de recherche académique concernant la réplication statique des options européennes, et reprend des éléments de recherche antérieurs sur la réplication dynamique des swaps de corrélation, publiés sous forme de rapports professionnels ou de chapitres de livre. Une introduction générale proposée au chapitre 1 résume les principaux résultats théoriques obtenus et les relie à la théorie de valorisation des actifs financiers par arbitrage (asset and arbitrage pricing theory), ainsi qu'à plusieurs autres domaines des sciences économiques et de gestion. Le chapitre 2 étend la théorie de réplication des options européennes sur un seul actif sous-jacent, initiée par Breeden et Litzenberger en 1978 et complétée vingt ans plus tard par Carr et Madan, à un noyau intégral

arbitraire, tel que les stratégies d'options de straddle (straddle) et en papillon (butterfly). Les chapitres 3 et 4 généralisent la même théorie aux options sur plusieurs actifs sous-jacents à l'aide de plusieurs outils mathématiques avancés tels que les équations intégrales, l'analyse fractionnaire, les transformées de Radon et la régularisation des intégrales. En conséquence, les options européennes multi-actifs admettent un prix de non-arbitrage sans modèle qui est déterminé par les prix d'options d'achat sur panier. Enfin, le chapitre 5 montre comment les swaps de corrélation peuvent être répliqués de manière dynamique avec des contrats en dispersion de variance, et examine également les propriétés mathématiques des matrices de corrélation financière.

**Title:** Static replication of European options and dynamic replication of correlation swaps

**Keywords:** exotic option, Breeden-Litzenberger formula, Radon transforms, integral equations, integral regularization

**Abstract:** This doctoral thesis consists of three recent academic articles about the replication of European options, together with prior scholarly work on the dynamic replication of correlation swaps that was published in the form of industry reports or book chapters. A general introduction presented in Chapter 1 summarizes the main theoretical results and explains how they are connected with asset and arbitrage pricing theory as well as other theories from economics and management science. Chapter 2 extends the theory of static replication of European options on a single underlying asset, spearheaded by Breeden and Litzenberger in 1978 and completed by Carr and

Madan in 1998, to arbitrary integral kernels, including straddle and butterfly option strategies. Chapters 3 and 4 generalize the same theory to multiple underlying assets with the help of advanced mathematical tools such as integral equations, fractional calculus, Radon transforms and integral regularization techniques. Consequently, multi-asset European options admit a model-free arbitrage price consistent with vanilla basket calls. Finally, Chapter 5 shows how correlation swaps may be replicated by dynamic trading of zero-cost variance dispersion trades, and investigates the mathematical properties of financial correlation matrices.

