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Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

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Discipline : Mathématiques

présentée par

Benoît JOLY

About Barcodes and Calabi invariant for Hamiltonian homeomorphisms of surfaces

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dirigée par Patrice Le Calvez et Frédéric Le Roux

Soutenue le 15 Octobre 2021 devant le jury composé de :

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Résumé

Nous cherchons à expliciter certains liens entre la topologie symplectique et l'étude des systèmes dynamiques à travers la notion de code barres d'homéomorphismes hamiltoniens de surfaces et de l'invariant de Calabi de difféomorphismes hamiltoniens du disque unité. Ces deux objets représentent de puissants invariants en topologie symplectique. Plus précisément, nous visons à mettre en avant une interprétation dynamique de ces objets.

Cette thèse se divise en deux parties.

Dans une première partie nous étudierons les codes barres de Floer d'un point de vue dynamique. Notre motivation provient en particulier de l'utilisation récente des codes barres en topologie symplectique permettant d'obtenir des résultats purement dynamiques. Ainsi, nous expliciterons des constructions de codes barres pour certains homéomorphismes hamiltoniens de surfaces à l'aide de la théorie des feuilletages transverses de Le Calvez. Notre stratégie consistera à calquer la construction de l'homologie de Floer et de l'homologie de Morse à l'aide d'outils de systèmes dynamiques tels que des feuilletages. Nous prouverons en particulier que dans les cas les plus simples, nos constructions correspondent aux codes barres de Floer.

Dans une seconde partie nous nous intéresserons à l'invariant de Calabi pour les difféomorphismes hamiltoniens du disque unité. Usuellement, l'invariant de Calabi est bien défini sur l'ensemble des difféomorphismes hamiltoniens à support compact du disque unité. Inspirés par l'interprétation dynamique de cet object donné par Fathi dans sa thèse, nous étendrons la définition de ce dernier au groupe des C^1 difféomorphismes hamiltoniens du disque. En particulier, cela nous permettra de calculer l'invariant de Calabi de certaines pseudo-rotations irrationnelles du disque.

Abstract

The goal of this thesis is to give some links between sympletic topology and the study of dynamical systems through the notion of barcodes of Hamiltonian homeomorphisms of surfaces and the Calabi invariant of Hamiltonian diffeomorphisms of the unit disk. These two objects represent powerful invariants in symplectic topology. More precisely, we aim at giving a dynamical interpretation of these objects.

This thesis is divided into two parts.

In a first part we will study the Floer Homology barcodes from a dynamical point of view. Our motivation comes from recent results in symplectic topology using barcodes to obtain dynamical results. We will give some constructions of barcodes of some Hamiltonian homeomorphisms of surfaces using Le Calvez's transverse foliation theory. The strategy consists in copying the construction of the Floer and Morse Homologies using dynamical tools like Le Calvez's foliations. In particular, we will prove that for the simplest cases, our constructions coincide with the Floer Homology barcodes.

In a second part we will deal with the Calabi invariant of the Hamiltonian diffeomorphisms of the unit disk. Inspired by the dynamical interpretation of this object developed by Fathi in his thesis, we will extend it to the group of C^1 Hamiltonian diffeomorphisms of the disk. In particular, we will be able to compute the Calabi invariant of some irrational pseudo-rotations of the disk.

Contents

I Barcodes for Hamiltonian homeomorphisms of surfaces **13**

1 Introduction **15**
1.1 Goals and motivations . . . 15
1.2 Results . . . 18

2 Preliminaries **21**
2.1 Morse Homology . . . 21
2.2 Symplectic geometry . . . 24
2.2.1 Hamiltonian diffeomorphisms . . . 24
2.2.2 Hamiltonian Action . . . 25
2.2.3 Floer Homology . . . 25
2.2.4 Filtered Floer Homology . . . 28
2.2.5 Conley-Zehnder index and Maslov index . . . 29
2.3 Dynamical systems . . . 31
2.3.1 Isotopies and maximal Isotopies . . . 31
2.3.2 Lefschetz index . . . 32
2.3.3 Linking number . . . 32
2.3.4 Rotation vectors . . . 33
2.3.5 Local isotopies and local rotation set . . . 33
2.3.6 The blow-up at a fixed point . . . 35
2.3.7 Positively transverse foliations . . . 36
2.3.8 Generalized Isotopies . . . 38
2.3.9 Intersection number . . . 40
2.3.10 Action function of a Hamiltonian homeomorphism . . . 40

3 Introduction to barcodes and persistence modules **45**

4 The simplest case of barcode for Hamiltonian homeomorphisms **49**

5 First step into the non generic case, construction of the map $\mathcal{B}$ **57**

6 The barcode of a gradient-like foliation **61**
6.1 Geometric properties of a gradient-like foliation . . . 61
6.2 Some properties of the Barcode $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$ . . . 71

7 A barcode with an order on a maximal unlinked set of fixed points **83**

8 Equalities of the previous constructions and independence of the foliation 89
8.1 Equality between the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ and the barcode $\boldsymbol{\beta}_{>}$ 89
8.2 Equality between the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ and the barcode $B_{\text{gen}}(f, \mathcal{F})$ in the generic case 93

9 Perspectives 99

II Calabi invariant for Hamiltonian diffeomorphism of the unit disk 101

1 Introduction 103

2 Preliminaries 111

3 Three extensions 115
3.1 Action function 115
3.2 Angle function 117
3.3 Hamiltonian function 120

4 Proof of Theorem 1.0.7 123
4.1 Equality between $\widetilde{\text{Cal}}_2$ and $\widetilde{\text{Cal}}_3$. 123
4.2 Continuity of $\widetilde{\text{Cal}}$. 128

5 Computation of Cal_1 in some rigid cases 133
5.1 A simple case of C^1-rigidity 133
5.2 C^0-rigidity 134

6 Examples 139
6.1 An example of C^0-rigidity, the super Liouville type 139
6.2 An example of C^1-rigidity, the non Bruno type 141

Bibliography 143

Context

Let us begin with some basic definitions of symplectic geometry.

Let us consider (M^{2n}, ω) a *symplectic manifold*, meaning that M is an even dimensional manifold equipped with a closed non-degenerate differential 2-form ω called the *symplectic form*. In particular, if M is a symplectic surface, the symplectic form is an *area form*.

Let us consider a time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ defined by the equation

$$dH_t = \omega(X_t, \cdot),$$

where $H : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function 1-periodic in t, meaning that $H_{t+1} = H_t$ for every $t \in \mathbb{R}$. The function H is called a *Hamiltonian function*. If the vector field $(X_t)_{t\in\mathbb{R}}$ is complete, it induces a *Hamiltonian flow* which is a family $(f_t)_{t\in\mathbb{R}}$ of diffeomorphisms of M preserving ω and satisfying the equation

$$\frac{\partial}{\partial t} f_t(z) = X_t(f_t(z)).$$

The time one map f_1 of the isotopy $(f_t)_{t\in[0,1]}$ is called a *Hamiltonian diffeomorphism*. In particular, a Hamiltonian diffeomorphism on a surface preserves the area.

The case of *autonomous Hamiltonian diffeomorphisms* can be kept in mind. Considering a C^1 function H on a surface, the previous hamiltonian formalism provides a Hamiltonian flow which follows the level sets of H such that the *flux* passing through any loop is zero. The time-one map of such a Hamiltonian flow will be called an autonomous Hamiltonian diffeomorphism.

Birkhoff proved [10] a celebrated result, conjectured and proved in some cases by Poincaré [65], known as the Poincaré-Birkhoff theorem, that asserts that an area-preserving homeomorphism of a closed annulus that satisfies some "twist conditions" admits at least two fixed points. Further generalizations have been obtained by Franks [31], using Brouwer's lemma on translation arcs, and other authors.

In one hand, the Poincaré-Birkhoff theorem led to many questions of symplectic geometry such as the Arnold conjecture [2] and the developement of the Floer Homology theory [26, 27, 28, 29, 30]. Floer introduced the Floer Homology by combining the variational approach of Conley and Zehnder, the elliptic techniques of Gromov and the Morse-Smale-Witten complex in order to answer the Arnold conjecture stated as follows.

Conjecture 0.0.1. *A Hamiltonian diffeomorphism of a symplectic manifold M must have at least as many fixed points as the minimal number of critical points of a smooth function on M.*

On the other hand, the Poincaré-Birkhoff theorem led to the study of periodic points of homeomorphisms of surfaces and more generally to the study of the dynamics of such homeomorphisms.

The main goal of this thesis is to study some links between the symplectic geometry and the dynamical systems of surfaces. In a first part we will study *barcodes* for Hamiltonian homeomorphisms on surfaces. In a second part we will study the *Calabi invariant* for Hamiltonian diffeomorphisms of the unit disk.

Both parts of the thesis contain their own introduction and preliminaries chapters. They are independant.

Part I

Barcodes for Hamiltonian homeomorphisms of surfaces

Chapter 1

Introduction

1.1 Goals and motivations

Main question

In the first part of this thesis, we will think about the following question.

Question 1.1.1. *Can we construct barcodes for Hamiltonian homeomorphisms of surfaces, equal to the Floer homology barcodes, using dynamical objects as Le Calvez's transverse foliations?*

Barcodes

Given a Hamiltonian function $(H_t)_{t\in[0,1]}$ on a symplectic manifold (M,ω), we define the *action function* A_H on the space of contractible loops of M by

$$A_H(\gamma) = -\int_D u^*\omega + \int_0^1 H_t(\gamma(t))dt,$$

where u is an extension to the disk of the contractible loop $\gamma : \mathbb{S}^1 \to M$, that is, a map $u : D = \{z \in \mathbb{C} | |z| \leqslant 1\} \to M$ such that $u(\mathrm{e}^{2i\pi t}) = \gamma(t)$. If we suppose that $\pi_2(M) = 0$, the function A_H does not depend on the choice of u and it will always be the case in this thesis. We will see in the preliminaries that for a Hamiltonian diffeomorphism f, the action function A_H does not depend on the choice of the Hamiltonian function H which induces f, hence it defines an action function A_f asociated to f.

For example, on surfaces, the difference of action between two points x and y fixed by a Hamiltonian flow can be interpreted as the flux of this flow through any oriented path γ joining x and y.

An important fact is that the critical points of an action function A_f are the trajectories of the contractible fixed points of f. The study of the critical values of A_f will play a key role in this thesis.

The barcode of a Hamiltonian diffeomorphism f is a countable collection of intervals, called *bars* whose extremities are the critical values of its action function A_f. In the particular case of a generic hamiltonian diffeomorphism, each critical value of A_f is the end

of one and only one bar.

The construction of these barcodes, recalled in Chapter 3, is based on the *Floer Homology* theory.

Let us begin with a simple example. We consider a Hamiltonian flow $(f_t)_{t\in[0,1]}$ induced by an autonomous Hamiltonian function H. In this case, for t small enough, the barcode of the Hamiltonian diffeomorphism f_t is equal to the filtered Morse Homology $(\mathrm{HM}_*^t(H))_{t\in\mathbb{R}}$ of H.
To give more details about this example we explain how the filtered homology of H can be interpreted as a collection of bars. In general, the bars of a barcode are of the form $I_j = (a_j, b_j]$, $a_j \in \mathbb{R}, b_j \in \mathbb{R} \cup \{+\infty\}$ and satisfy certain finiteness assumptions. The ends of these bars are in correspondence with the critical points of H and can be classified as follows.

- There are the *death points* which are the critical points x of H ending some homology, meaning that the dimension of the vector spaces $(\mathrm{HM}_*^t(H))_{t\in\mathbb{R}}$ decreases at $H(x)$.
- There are the *birth points* which are the critical points x of H generating homology in $\mathrm{HM}_*H(x)(H)$, meaning that the dimension of the vector spaces $(\mathrm{HM}_*^t(H))_{t\in\mathbb{R}}$ increases at $H(x)$. The value $H(x)$ of a birth point x will be the begining of a bar.

The bars of a barcode can be described by the following classification of the birth points.

- A birth point can be "homological" and associated to the semi-infinite bar $(H(s), +\infty)$ in the barcode if the homology it generates in $\mathrm{HM}_*^{H(x)}$ persists in the vector spaces $(\mathrm{HM}_*^t)_{t\geqslant H(s)}$.
- A birth point can be "bound to die" and associated to a death point y and a finite bar $(H(x), H(y)]$ in the barcode if the homology it generates in $\mathrm{HM}_*^{H(x)}$ disapears in $\mathrm{HM}_*^{H(y)}$.

The previous filtered homology is an example of a *persistence module*. In fact, we will see that barcodes use to classify persistence modules up to isomorphisms. Roughly speaking, it is equivalent to consider a barcode as a set of bars or as a filtered homology.

Following this idea, we associate, canonically, a barcode $B(f)$ to every Hamiltonian diffeomorphism f by considering the filtered Floer Homology of f where the filtration is given by the action function A_f.

The barcode $B(f)$ gives information about the structure of the set of fixed points and the *spectral invariants* of f. The spectral invariants have been introduced by Viterbo [73]. They have been used in numerous deep applications and their theory has been developped in many contexts, we can cite for example the work of Schwarz [**?**] and Oh [**?**]. They are powerful tools which took an important place in the development of symplectic topology.

The notion of Barcode provides, in some topology, a continuous invariant of conjugacy on the set of smooth Hamiltonian diffeomorphisms of symplectic manifolds.

Hamiltonian homeomorphisms

In symplectic geometry, we can define the notion of *Hamiltonian homeomorphism* of a surface Σ by taking the closure of the Hamiltonian diffeomorphisms of Σ. This definition comes from the Gromov-Eliashberg theorem [19] which states that if a sequence of symplectomorphisms of a symplectic manifold (M, ω) converges in the C^0 topology to a diffeomorphism then this diffeomorphism is a symplectomorphism as well.

For a Hamiltonian homeomorphism of a surface, we are not able to consider directly its Floer Homology as the construction requires at least a C^2 setting. Howeover, on surfaces, the barcode $B(f)$ depends continuously, in the uniform topology, on f and moreover, extends to Hamiltonian homeomorphisms, see [61] for more details.

The barcode of a Hamiltonian homeomorphism f is defined by a limiting process and it is natural to wonder if it is possible to describe a direct construction.

Moreover, the notion of Hamiltonian homeomorphism of surfaces is well-known in dynamical systems and has a dynamical interpretation thanks to the notion of rotation vectors. On a symplectic surface (Σ, ω), ω is an area form which induces a Borel probability measure μ. We will say that a homeomorphism f of an oriented compact surface is *Hamiltonian* if it is isotopic to the identity and preserves a Borel probability measure μ whose support is the whole surface and whose *rotation vector* $\rho(\mu)$ is zero.

Le Calvez's transverse foliations

A key motivation for this thesis is to bring a dynamical interpretation of the barcodes for Hamiltonian homeomorphisms of surfaces. Taking this direction, we will give some constructions of barcodes, inspired by the Floer homology constructions, using Le Calvez's foliation theory.

Le Calvez's foliations theory has many applications in the study of dynamical systems of surfaces. For example in the study of prime ends by Koropecki, Le Calvez and Nassiri [48], the study of homoclinic orbits for area preserving diffeomorphisms by Sambarino and Le Calvez [52] or the results about the forcing theory of Le Calvez and Tal [53, 54].

Nowadays, Le Calvez's foliations theory [49] represents one of the most important dynamical tool in the study of the dynamics of homeomorphisms of surfaces. This theory already found applications to Barcodes of Hamiltonian homeomorphisms of surfaces. For example, for a homeomorphism f which preserves the area, Le Roux, Seyfaddini and Viterbo in [61] used Le Calvez's foliations theory to extract dynamical informations of the barcode of f without Kislev-Shelukhin's result [47].

Here are some details about transverse foliations. Let us consider a homeomorphism f on a surface. There are sets X of fixed points of f, called *maximal unlinked sets*, such

that there exists an isotopy, called *maximal isotopy*, from id to f fixing all points of X and which are maximal for the inclusion.

Le Calvez proved that given a maximal unlinked set X of fixed points of f and an isotopy I fixing all the fixed points of X, there exist oriented foliations $\mathcal{F}$ *positively transverse* to the isotopy I. Roughly speaking, this means that, given a point in the complement of X, its trajectory along the isotopy I is homotopic in $\Sigma \backslash X$ to a path transverse to $\mathcal{F}$.

Moreover, if we suppose that f is area-preserving, we will see in 2.3.7 that those foliations are *gradient-like*. To keep it simple, this means that we can see such a foliation as the gradient lines of a function defined on the surface. In particular, every leaf of a gradient-like foliation is an injective path, called a *connexion*, between two singularities of $\mathcal{F}$ and there is no cycle of connexions.
In the particular case where f has finitely many fixed points, by a result of Wang [74], the notion of action function can be extended. A key point is that, for every leaf ϕ of $\mathcal{F}$ the action function A_f of f satifsies $A_f(\alpha(\phi)) > A_f(\omega(\phi))$.

To give an example, we can consider again a Hamiltonian diffeomorphism f induced by an autonomous Hamiltonian function H on a surface. The induced Hamiltonian flow is a maximal isotopy I of f and the gradient flow of H is a gradient-like foliation positively transverse to I. In this case, the only maximal unlinked set of fixed points fixed by I is the set of critical points of H.

1.2 Results

We describe briefly the results of the first part of this thesis. We provide distinct constructions of barcodes for Hamiltonian homeomorphisms of surfaces.

First construction

We will describe a first construction in Chapter 4 under some generic hypothesis which is inspired from the Morse and Floer homology constructions. We will consider a Hamiltonian homeomorphism f of an oriented compact surface Σ with a finite number of fixed points which are, in a sense, non degenerate and such that the set of fixed points is *unlinked*, meaning that there exists an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f fixing all the fixed points of f. By Le Calvez's theorem we can consider a gradient-like foliation $\mathcal{F}$ transverse to I. We will suppose that $\mathcal{F}$ satisfies some "generic" hypothesis which allows us to construct a chain complex inducing a filtered homology and then a barcode denoted $B_{\text{gen}}(\mathcal{F})$.

We have the following theorem proved in Chapter 8.

Theorem 1.2.1. *The Barcode $B_{\text{gen}}(\mathcal{F})$ does not depend on the choice of the foliation $\mathcal{F} \in \mathcal{F}_{\text{gen}}(I)$.*

In the case of a Hamiltonian diffeomorphism close enough of the identity and generated by an autonomous hamiltonian function we will obtain the following result.

Theorem 1.2.2. *If we consider a Hamiltonian diffeomorphism f with a finite number of fixed points which is C^2-close to the identity and generated by an autonomous Hamiltonian function then the barcode $B_{\text{gen}}(\mathcal{F})$ is equal to the Floer homology barcode of f.*

Let us give the idea of the construction. Since f is area-preserving, we will see that there are three kinds of singularities for the foliation $\mathcal{F}$: sinks, sources and saddle points. We will suppose that $\mathcal{F}$ is in the set $\mathcal{F}_{\text{gen}}(I)$ of "generic" foliations positively transverse to I, meaning that there are finitely many leaves between sources and saddle points and between sinks and saddle points. In the Morse Homology theory, the chain complex is defined by counting modulo 2 the number of trajectories between the critical points of a Morse function f. Following the same ideas we will be able to define a chain complex associated to $\mathcal{F}$ by counting modulo 2 the number of leaves between singularities of $\mathcal{F}$ and more precisely the number of leaves between sinks and saddle points and between sources and saddle points.

A natural question appears.

Question 1.2.3. *Can we generalize the construction to barcodes for every Hamiltonian homeomorphisms of surfaces?*

Second construction

In general there is no natural way to construct a chain complex from a positively transverse foliation. The difficulties come from geometrical limitations of the foliations.

Nevertheless, given a Hamiltonian homeomorphism f, we will construct barcodes associated to maximal unlinked sets of fixed points of F.

Let us consider a maximal unlinked set X of fixed points of f, a maximal isotopy $I = (f_t)_{t\in[0,1]}$ fixing all the fixed points of X, and a gradient-like foliation $\mathcal{F}$, positively transverse to I. We will begin by associating a graph $G(\mathcal{F})$ to the foliation $\mathcal{F}$ whose set of vertices is equal to X and for every couple (x, y) of vertices there is an edge from x to y if there is a leaf ϕ of $\mathcal{F}$ starting at x and ending at y.

In Chapter 5 we will construct an application β which associates a barcode to triplets (G, A, i) where G is an oriented graph on the set of vertices X equipped with an action function A defined on X, meaning that for every edge e of G from x to y we have $A(x) > A(y)$, and an index function $i : X \to \mathbb{Z}$.

In Chapter 6, we will consider the barcode $\beta(G(\mathcal{F}), A_f, \text{ind}(\mathcal{F}, \cdot))$, denoted $\boldsymbol{\beta}_{\mathcal{F}}$, where A_f is the action function of f and $\text{ind}(\mathcal{F}, \cdot)$ the index function induced by $\mathcal{F}$ and prove some useful properties.

We will prove in Chapter 8 the following result.

Theorem 1.2.4. *The barcode $\boldsymbol{\beta}_{\mathcal{F}}$ does not depend on the choice of $\mathcal{F}$ and only depends on the maximal unlinked set of fixed points X.*

Third construction

To prove Theorem 1.2.4 we will construct another barcode associated to X as follows.

In Chapter 7, we will introduce an order on the fixed points of X. For two fixed points $x, y \in X$ we will say that $x > y$ if there exists an oriented path γ from x to y of which any lift $\widetilde{\gamma}$ on the universal cover $\mathbb{D}^2$ of $\Sigma \backslash X$ is a *Brouwer line* for the natural lift $\widetilde{f}$ of f, meaning that $\widetilde{\gamma}$ is the boundary of an attractor of $\widetilde{f}$. In the same ideas, we associate to this order a graph $G(>)$ whose set of vertices is equal to X and for every couple of vertices x and y there is an edge from x to y if $x > y$.

We will consider the barcode $\boldsymbol{\beta}_> = (G(>), A_f, \mathrm{ind}(I, \cdot))$ which depends only on X and we will prove the following result in Chapter 8.

Theorem 1.2.5. *For every foliation $\mathcal{F}$ positively transverse to the isotopy I we have*

$$\boldsymbol{\beta}_> = \boldsymbol{\beta}_{\mathcal{F}}.$$

In the same chapter, we will prove the following result which enlighten the link between the barcode associated to a maximal set of fixed points and the first construction in a more generic case.

Theorem 1.2.6. *Let us consider a Hamiltonian homeomorphism f on a compact surface Σ whose set of fixed points is finite, unlinked, and such that each fixed point $x \in \mathrm{Fix}(f)$ is not degenerate.*
We consider a maximal isotopy I such that $\mathrm{Sing}(I) = \mathrm{Fix}(f)$ then for a foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$ we have

$$B_{\mathrm{gen}}(\mathcal{F}) = \boldsymbol{\beta}_{\mathcal{F}}.$$

In fact, Theorems 1.0.3, **??** and 1.0.4 will be consequences of the two previous theorems.

Chapter 2

Preliminaries

2.1 Morse Homology

We give a quick presentation of Morse homology, largely inspired by the presentation of M. Audin and M. Damian in [3].

We fix a n-dimensional compact smooth manifold M. For a function $F : M \to \mathbb{R}$, a point x is said to be a critical point if $dF_x = 0$. The function F is said to be a *Morse function* if each critical point x of F is non degenerate, i.e. D^2F_x is non degenerate.

The local theory of critical points of Morse functions is well understood and we have the following lemma.

Lemma 2.1.1. *Let $x \in M$ a critical point of a Morse function $F : M \to \mathbb{R}$. There exists a neighborhood U of x and a diffeomorphism $\psi : (U, x) \to (\mathbb{R}^n, 0)$, called a Morse chart, such that*

$$F \circ \psi^{-1}(x_1, ..., x_n) = F(x) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

The integer i is called the *Morse index*, denoted $\mathrm{ind}(F, x)$, of the critical point x and does not depend on the choice of the diffeomorphism ψ. We denote by $\mathrm{Crit}_i(F)$ the set of critical points of F of index i.

Let us consider a Morse function $F : M \to \mathbb{R}$. A pseudo gradient vector field adapted to F is a vector field X on M such that for all $x \in M$ we have $dF_x(X_x) \leqslant 0$ with equality if and only if x is a critical point of F and for a Morse chart near a critical point of F, the vector field X is equal to the opposite of the gradient vector of F for the canonical metric on $\mathbb{R}^n$. That is to say that, in local coordinates, we have

$$X = \sum_{j=1}^{i} 2x_j \frac{\partial}{\partial x_j} - \sum_{j=i+1}^{n} 2x_j \frac{\partial}{\partial x_j}.$$

Notice that such a vector field always exists. If we denote by ϕ^s the flow of X, for x a critical point of F we define its stable manifold to be

$$W^s(x) = \left\{ y \in M | \lim_{s \to +\infty} \phi^s(y) = x \right\}$$

and its unstable manifold to be

$$W^u(x) = \left\{ y \in M | \lim_{s \to -\infty} \phi^s(y) = x \right\}.$$

Those manifolds satisfy $\dim(W^u(x)) = \operatorname{codim}(W^s(x)) = \operatorname{ind}(F, x)$.

Let us consider a pseudo gradient vector field X adapted to a Morse function $F : M \to \mathbb{R}$. We say that X satisfies the *Smale condition* if all stable and unstable manifolds of its critical points meet transversely.

Moreover Smale's Theorem assures that we can find a vector field Y on M C^1-close to X which satisfies the Smale condition.

Let us describe the Morse chain complex $C_*(F)$ of a Morse function $F : M \to \mathbb{R}$ and a pseudo gradient X of F on M which satisfy the Smale condition. The ith group of the chain complex $C_i(f)$ is given by

$$C_i(f) = \left\{ \sum_{y \in \mathrm{Crit}_i(f)} \lambda_y \cdot y, \lambda_y \in \mathbb{Z}/2\mathbb{Z} \right\}.$$

We define the differential map $\partial_X : C_i(f) \to C_{i-1}(f)$ as follows. For two critical points x_+ and x_- of F we define the set

$$\mathcal{M}(x_-, x_+; F, X) = \left\{ x \in M | \lim_{t \to \pm\infty} \phi^t(x) = x_\pm \right\}.$$

We have that $\mathcal{M}(x_-, x_+; F, X) \cong W^u(x_-) \cap W^s(x_+)$, so the transversality condition assures that $\dim(\mathcal{M}(x_-, x_+; F, X)) = \operatorname{ind}(F, x_-) - \operatorname{ind}(F, x_+)$.

For all $c \in \mathbb{R}$, if x is in $\mathcal{M}(x_-, x_+; F, X)$ then we have $\lim_{t \to \pm\infty} \phi^{t+c}(x) = x_\pm$. So it gives a free and proper action of $\mathbb{R}$ on $\mathcal{M}(x_-, x_+; F, X)$. Thus we can define the quotient $\widehat{\mathcal{M}}(x_-, x_+; F, X)$ of $\mathcal{M}(x_-, x_+; F, X)$ by this action. The dimension of the manifold $\widehat{\mathcal{M}}(x_-, x_+; F, X)$ is equal to $\operatorname{ind}(F, x_-) - \operatorname{ind}(F, x_+) - 1$.

For all critical points x_- of F of index i we define

$$\partial_X(x_-) = \sum_{x_+ \in \mathrm{Crit}_{i-1}} n(x_-, x_+; F, X) \cdot x_+,$$

where $n(x_-, x_+; F, X)$ denotes the cardinal of $\widehat{\mathcal{M}}(x_-, x_+; F, X)$ modulo 2.

Thus we have to verify that $\partial_X \circ \partial_X = 0$. First, for $x \in \mathrm{Crit}_{i+2}(F)$ we compute

$$\partial_X \circ \partial_X(x) = \sum_{z \in \mathrm{Crit}_i(F)} \sum_{y \in \mathrm{Crit}_{i+1}(F)} (n(x, y; F, X) \times n(y, z; F, X)) \cdot z$$

To prove that the previous sum is zero it suffices to prove that given two critical points, x of index $i+2$ and z of index i the sum

$$\sum_{y \in \mathrm{Crit}_{i+1}(F)} n(x, y; F, X) \times n(y, z; F, X) \tag{2.1}$$

is zero. This number equals the cardinal of the union

$$\bigcup_{y\in \mathrm{Crit}_{i+1}(F)} \widehat{\mathcal{M}}(x,y;F,X)) \times \widehat{\mathcal{M}}(y,z;F,X)).$$

This union is a set of points and the idea is to prove that it is the boundary of a manifold of dimension 1 which is an even number of points. We introduce the concept of broken gradient trajectories.

Definition 2.1.2. *A broken gradient trajectory between two critical points x_- and x_+ is a family $(x_1, ..., x_p)$ of points such that there exists a sequence $(y_1, ..., y_{p+1})$ of critical points of F satisfying*

1. *for all i, $x_i \in \widehat{\mathcal{M}}(y_i, y_{i+1}; F, X)$*
2. *$y_1 = x_-$ and $y_{p+1} = x_+$.*

For two critical points x and z we denote $\overline{\widehat{\mathcal{M}}}(x,z;F,X)$ the space of broken gradient trajectories from x to z. We have the following two theorems.

Theorem 2.1.3. *The space $\overline{\widehat{\mathcal{M}}}(x,z;F,X)$ is compact for all critical points x and z.*

The topology on $\overline{\widehat{\mathcal{M}}}(x,z;F,X)$ is induced by the topology on M. It admits a countable fundamental system of open neighborhoods and the compactness is proved using sequences. We refer to [3] for more details.

Theorem 2.1.4. *Let us consider $(x,y,z) \in \mathrm{Crit}_{i+1}(F) \times \mathrm{Crit}_i(F) \times \mathrm{Crit}_{i-1}(F)$, $x' \in \widehat{\mathcal{M}}(x,y;F,X)$ and $x'' \in \widehat{\mathcal{M}}(y,z;F,X)$. There is a continuous embedding ψ, differentiable on the interior of its definition domain, from an interval $[0,\delta)$, $\delta > 0$ to a neighborhood of (x',x'') in $\overline{\widehat{\mathcal{M}}}(x,z;F,X))$ such that*

1. *$\psi(0) = (x',x'') \in \overline{\widehat{\mathcal{M}}}(x,z;F,X)$,*
2. *$\psi(s) \in \widehat{\mathcal{M}}(x,z;F,X)$ for all $s \neq 0$.*

Moreover, for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\widehat{\mathcal{M}}(x,z;F,X)$ converging to (x',x'') and for n large enough, x_n lies in the image of ψ.

With some properties about the index, we obtain that

$$\bigcup_{y\in \mathrm{Crit}_i(F)} \widehat{\mathcal{M}}(x,y;F,X) \times \widehat{\mathcal{M}}(y,z;F,X) = \partial\overline{\widehat{\mathcal{M}}}(x,z;F,X)),$$

where (x,y,z) is defined as in the above theorem. Moreover, $\overline{\widehat{\mathcal{M}}}(x_-,x_+;F,X))$ is a one dimensional manifold with boundary. His boundary is an even number of points and hence we obtain that $\partial_X^2 = 0$.

The Morse homolgy of a Morse function F will be denoted $\mathrm{HM}_*(F)$ and $\mathrm{HM}_*^t(F)$ will refer to the naturally filtered Morse homology induced.

Remark 2.1.5. There is an important fact that we will use in the construction of barcodes and persistence modules: given a Morse function $F : M \to \mathbb{R}$ and a pseudo gradient vector field adapted to F, the value of F decreases along the flow of a point. Which means that, for every non critical point x we have

$$F(\lim_{s\to-\infty} \phi^s(x)) \geqslant F(\lim_{s\to+\infty} \phi^s(x)).$$

If we consider a Morse function F of a surface Σ then there are three categories of critical points of F, the critical points of index 0 called the *sinks*, corresponding to the local minimum, those of index 1 called the *saddle points* and those of index 2 called the *sources*, corresponding to the maximum local.

2.2 Symplectic geometry

For the remainder of this section, we consider a connected symplectic surface (Σ, ω) such that $\pi_2(M) = 0$ and where ω is a 2-form which is closed and nondegenerate. A *symplectic diffeomorphism* is a diffeomorphism $f : \Sigma \to \Sigma$ such that $f^*\omega = \omega$.

2.2.1 Hamiltonian diffeomorphisms

A Hamiltonian function on M is a time dependent function

$$H : \mathbb{S}^1 \times M \to \mathbb{R}.$$

The Hamiltonian function generates a Hamiltonian vector field X_H defined by the equation

$$dH_t = \omega(X_H, \cdot),$$

where we denote $H_t(x) = H(t, x)$. The flow $(f_t)_{t\in[0,1]}$ of this vector field is called the Hamiltonian isotopy generated by H. A *Hamiltonian diffeomorphism* is a symplectomorphism that can be written as the time 1 map of a Hamiltonian isotopy.

We consider two Hamiltonian diffeomorphisms f and g on a symplectic manifold (M, ω). We denote $H : \mathbb{S}^1 \times M \to \mathbb{R}$ and $G : \mathbb{S}^1 \times M \to \mathbb{R}$ two Hamiltonian functions such that f and g are the time-one map of the induced Hamiltonian flows $(f_t)_{t\in[0,1]}$ and $(g_t)_{t\in[0,1]}$. Then the Hamiltonian $K : \mathbb{S}^1 \times M \to \mathbb{R}$ given by

$$K_t(z) = H_t + G_t \circ f_t^{-1}(z),$$

induces a Hamiltonian flow such that $f \circ g$ is its time-one map. Moreover, the Hamiltonian $\bar{H} : \mathbb{S}^1 \times M \to \mathbb{R}$ defined by

$$\bar{H}_t(z) = -H_t(f^t(z)),$$

induces a Hamiltonian flow such that f^{-1} is its time-one map.

Hence the set of Hamiltonian diffeomorphisms of a symplectic manifold is a group that we denote $\mathrm{Ham}(M, \omega)$.

2.2.2 Hamiltonian Action

Let us consider (M, ω) a symplectic manifold and $H : \mathbb{R} \times M \to \mathbb{R}$ a Hamiltonian function which is periodic in t and satisfies $H_{t+1} = H_t$ for all $t \in \mathbb{R}$. We denote $(\phi^t)_{t\in\mathbb{R}}$ the Hamiltonian flow defined by H.

We consider a contractible loop $\gamma = (\gamma(t))_{t\in\mathbb{S}^1}$ in M and we denote by Ω the set of loops in M. We can consider the expression

$$A_H(\gamma) = -\int_D u^*\omega + \int_0^1 H_t(\gamma(t))dt,$$

where u is an extension of $\gamma : \mathbb{S}^1 \to M$ to the disk, that is, a map $u : D = \{z \in \mathbb{C} | |z| \leqslant 1\} \to M$ such that $u(\mathrm{e}^{2i\pi t}) = \gamma(t)$.

The integral does not depend on the choice of the extension u. Indeed if we consider another extension v then

$$\int_D u^*\omega - \int_D v^*\omega = \int_{\mathbb{S}^2} w^*\omega,$$

where w is defined by gluing the two disks along their common boundary. Since we assume that $\pi_2(M) = 0$ we have that the previous equation is equal to zero.

The function A_H will be called the *action function* and satisfies the following property.

Proposition 2.2.1. *A loop is a critical point of A_H if and only if $t \mapsto \gamma(t)$ is a 1 periodic solution of the Hamiltonian system $\dot{\gamma}(t) = X_t(\gamma(t))$.*

Let us sketch the proof. For a loop $\gamma \in \Omega$, the tangent space $T_\gamma\Omega$ at γ consists of the smooth vector fields $\xi \in C^\infty(\gamma^*TM)$ along γ satisfying $\xi(t+1) = \xi(t)$. Then the computation of the action function at γ in the direction of ξ gives

$$dA_H(\gamma)\xi = \int_0^1 \{\omega(\dot{\gamma}, \xi) + dH_t(\gamma(t))[\xi]\}dt,$$

which vanishes for every $\xi \in T_\gamma\Omega$ if and only if the loop γ is a solution of the Hamiltonian system

$$\dot{\gamma}(t) = X_t(\gamma(t)).$$

The periodic solutions of the flow induced by H will be denoted $\mathcal{P}_H$.

2.2.3 Floer Homology

We sketch the construction of the Floer homology in this section. There are many difficulties in making this construction and the purpurse of this section is only to give ideas of how Floer homology works. Thus, we may ignore some of these difficulties to set an understandable and short introduction to Floer homology. The section is inspired by Audin-Damian [3] and Hofer-Zehnder [40] presentations.

Let us consider a symplectic manifold (M, ω) and let us choose an almost complex structure J on M compatible with ω. The almost complex structure J is a smooth endomorphism of TM, such that for all $x \in M$, $J_x \in \mathcal{L}(T_xM, T_xM)$ satisfies $J_x^2 = -1$ and such that

$$g(\xi, \eta) = \omega(\xi, J_x\eta), \ \ \eta, \xi \in T_xM$$

defines a Riemannian metric g on M. We denote ∇H the gradient of H on M with respect to the x-variable in the metric g. Note that we have $\nabla H = -JX_H$.

The crucial objects are the solutions $u : \mathbb{R} \times \mathbb{S}^1 \to M$ of the gradient flow equation (also called the Floer equation)

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t, u) = 0. \tag{2.2}$$

We denote by $\mathcal{M}$ the set of "bounded solutions" of equation 2.2. This set is defined as the set of smooth solutions $u : \mathbb{R} \times \mathbb{S}^1 \to M$ of equation 2.2 which are contractible, and have finite energy, i.e. such that the number

$$E(u) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_0^1 \left\{ \left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_H(t, u)\right|^2 \right\} dsdt \tag{2.3}$$

is finite. Floer proved in [29] that the space $\mathcal{M}$ has a structure similar to the set of broken trajectories defined in section 2.1. The group $\mathbb{R}$ acts naturally on $\mathcal{M}$ by shifting $u(s, t)$ in the s direction which defines a continuous flow on $\mathcal{M}$. Moreover, for every bounded orbit $u \in \mathcal{M}$ there exists a pair $x, y \in \mathcal{P}_H$ such that u is a connecting orbit from y to x, i.e.,

$$\lim_{s \to -\infty} u(s, t) = y(t), \ \ \lim_{s \to +\infty} u(s, t) = x(t), \tag{2.4}$$

the convergence being uniform in t as $|s| \to \infty$ and $\frac{\partial u}{\partial s}$ converging to zero again uniformly in t. Given two periodic solutions $x, y \in \mathcal{P}_H$ we denote $\mathcal{M}(y, x)$ the set of solutions $u \in \mathcal{M}$ satisfying the asymptotic boundary conditions 2.4. Thus we have

$$\mathcal{M} = \bigcup_{y, x \in \mathcal{P}_H} \mathcal{M}(y, x).$$

For two critical points y, x, the set $\mathcal{M}(y, x)$ is an invariant subspace. The compactness can be formulated analogously to the finite dimensional Morse theory that we developed in section 2.1. We have the following proposition of Schwarz book [**?**].

Proposition 2.2.2. *Let us consider a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{M}(y, x)$. Up to a subsequence, the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the following property: there is a sequence $s_n^j \in \mathbb{R}$ of times, $j = 1, 2, ..., m$, such that $u_n(s + s_n^j)$ converges together with all derivatives uniformly on compact sets to solutions $u^j \in \mathcal{M}(x^j, x^{j-1})$ where $x^j \in \mathcal{P}_H$ for $j = 0, ..., m$, with $x^0 = x$ and $x^m = y$.*

We describe the previous proposition by the following Figure 2.1.

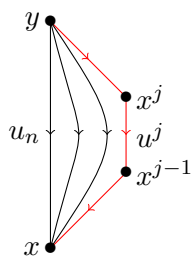


Figure 2.1: illustration of proposition 2.2.2

One may prove using Fredholm theory in the appropriate functional analytic setting that for a "generic" choice of the pair (H, J) the sets $\mathcal{M}(y, x)$ are smooth and finite dimensional manifolds such that the dimension of a set $\mathcal{M}(y, x)$ is equal to the difference $\mathrm{ind}_{CZ}(y) - \mathrm{ind}_{CZ}(x)$, where $\mathrm{ind}_{CZ}(x)$ is the Conley-Zehnder index of x whose definition will be recalled in section 2.2.5.

Then we can define the homology groups associated to a pair (H, J) on (M, ω). The grading of the chain complex $(C_k)_{k\in\mathbb{Z}}$ is given by the Conley-Zenhder index that we define later in section 2.2.5 and for all $k \in \mathbb{Z}$ we have

$$C_k = \bigoplus\{\mathbb{Z}/2\mathbb{Z} \cdot x | x \in P_H \ \& \ \mathrm{ind}_{CZ}(x) = k\},$$

where $\mathcal{P}_H$ is the set of non degenerate contractible periodic orbits of H.

If we consider a pair $y, x \in P_H$ such that $\mathrm{ind}_{CZ}(y) - \mathrm{ind}_{CZ}(x) = 1$ then $\mathcal{M}(y, x)$ is a one-dimensional manifold and more precisely has finitely many components, each component consists of a connecting orbit together with all its translates by the time s shift. We can now define the differential map $\partial_k : C_k \to C_{k-1}$ for $y \in P_H$ of index k as follows.

$$\partial_k y = \sum_{x\in\mathcal{P}_H | \mathrm{ind}_{CZ}(x)=k-1} n(y, x)x,$$

where $n(y, x)$ is the number of connected components of $\mathcal{M}(y, x)$ counted modulo 2.

Floer proved that $\partial_k \circ \partial_{k+1} = 0$, for each $k \in \mathbb{Z}$. We explain the idea of the proof. Assume that for three elements $x \in C_{k+1}$, $y \in C_k$ and $z \in C_{k-1}$ there is a connexion u in $\mathcal{M}(x, y)$ and a connexion v in $\mathcal{M}(y, z)$. We may view the pair (u, v) as a "broken trajectory" connecting x with z. In this case the set $\mathcal{M}(x, z)$ has dimension two. Floer proved that by a pertubation argument called the gluing method [29] that there exists a unique one parameter family of connexions in $\mathcal{M}(x, z)$. By taking the quotient by the $\mathbb{R}$-action of the time s-shift one finds a connected 1-dimensional manifold without boundaries of unparametrized orbits which represents one component of $\mathcal{M}(x, z)$. Such a manifold is either a circle or an interval with two ends. By Proposition 2.2.2 each end converges in a suitable sense to a well-defined *broken trajectory* $(u', v') \in \mathcal{M}(x, y') \times \mathcal{M}(y', z)$ for some $y' \in C_k$. One may prove by the same gluing arguments that there is a correspondence between the "broken trajectories" and the ends of connected component of $\mathcal{M}(x, z)$. Thus the 1-dimensional manifold $\mathcal{M}(x, z)/\mathbb{R}$ has an even number of ends and the broken trajectories between x and z occur in pairs. We obtain that

$$\partial_k \circ \partial_{k+1}(x) = \sum_{\mathrm{ind}_{CZ}(z)=k-1} \left(\sum_{\mathrm{ind}_{CZ}(y)=k} n(x,y)n(y,z) \right) z$$

is equal to 0 modulo 2.

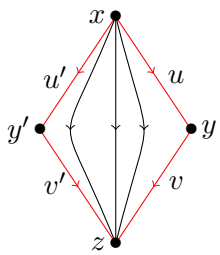


Figure 2.2

We can define the Floer homology groups $(\mathrm{HF}_k(M,H,J))_{k\in\mathbb{Z}}$ by

$$\mathrm{HF}_k(M,H,J) = \mathrm{Ker}(\partial_k)/\mathrm{Im}\partial_{k+1}.$$

Remark 2.2.3. Notice that the energy of a solution $u \in \mathcal{M}(y,x)$ is equal to $E(u) = A_H(y) - A_H(x)$ and is positive. We deduce that the action function A_H decreases along the solution u. We can compare this result to Remark 2.1.5 where a Morse funtion F on M is decreasing along the solutions of a pseudo gradient vector field.

2.2.4 Filtered Floer Homology

Let us consider a non-degenerate Hamiltonian function H on a symplectic manifold (M,ω) which satisfies the hypothesis of the previous section 2.2.3 and let us fix J an almost complex structure on M. We use the same notation as in Section 2.2.3 to define the filtered Floer homology of H from the Floer homology of H.

We consider the natural filtered chain complex $(C_k^t)_{k\in\mathbb{Z},t\in\mathbb{R}}$ where $C_k^t = \bigoplus\{\mathbb{Z}/2\mathbb{Z}\cdot x | x \in \mathcal{P}_H,\ \mathrm{ind}_{CZ}(H,x) = k,\ A_H(x) < t\}$ and the natural filtered differential application $\partial_k^t : C_k^t \to C_{k-1}^t$ defined as the restriction of ∂_k on C_k^t.

The filtered chain complex $(C_k^t, \partial_k^t)_{k\in\mathbb{Z},t\in\mathbb{R}}$ induces an homology denoted $(\mathrm{HF}_*^t)_{t\in R}$. This homology is referred to as the *filtered Floer homology* of the Hamiltonian H. One may prove that the filtered Floer homology of H does not depend on the choice of the almost complex structure J on M, see [3] for example.

We have the following property.

Proposition 2.2.4. *We consider two Hamiltonian flows $(\phi_{H_0}^t)_{t\in[0,1]}$ and $(\phi_{H_1}^t)_{t\in[0,1]}$ of two Hamiltonian functions H_0 and H_1 on $\mathbb{S}^1 \times M$. Let us suppose that $(\phi_{H_0}^t)_{t\in[0,1]}$ and $(\phi_{H_1}^t)_{t\in[0,1]}$ are homotopic relative to the endpoints in $\mathrm{Ham}(M,\omega)$. Then there exists a constant $c \in \mathbb{R}$ such that*

$$\mathrm{HF}_*^t(H_0) = \mathrm{HF}_*^{t+c}(H_1,1), \forall t \in \mathbb{R}.$$

2.2.5 Conley-Zehnder index and Maslov index

The Conley-Zehnder index is an important tool in the definition of Barcodes and we will discuss some properties of this index in our constructions. We give a short version of the definition although we will not use it directly.

Given a Hamiltonian function H of a symplectic manifold (M^{2n}, ω) we want to define the Conley-Zehnder index of any contractible 1-periodic solution $x(t)$ of $\dot{x}(t) = X_t(x)$.

We consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$ written in the coordinates $z = (x_1, ..., x_n, y_1, ..., y_n) \in \mathbb{R}^{2n}$ as follows

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

We denote J_0 the $2n \times 2n$ matrix

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix},$$

which represents a rotation by $\pi/2$ and satisfies $J_0^2 = -\mathbb{1}_{2n}$. We denote the group of symplectic matrices by

$$\mathrm{Sp}(n) = \{M \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} | M^T J_0 M = J_0\},$$

where M^T is the transpose matrix of M. We also denote $\mathrm{SP}(n)$ the set of paths γ in $\mathrm{Sp}(n)$ from id to a matrix A which do not have eigenvalue 1.

Let us consider a non degenerate orbit x. There are two steps to compute the index of the critical point x. We associate to the orbit a path $\psi : t \mapsto A(t)$ of matrices in $\mathrm{Sp}(2n)$. Then to a path ψ we associate an integer which is the Conley-Zenhder index of x.

First step

We fix the orbit $x(t) = \phi^t(x(0))$ then we can choose a family of symplectic bases, see [3] for example, $Z(t) = (Z_1(t), ..., Z_{2n}(t))$ of $T_{x(t)}M$ that depends smoothly on t. For every $t \in \mathbb{R}$, we can consider the matrix $A(t)$ of the linear map $T_{x(0)}\phi^t$ in the bases $Z(0)$ and $Z(t)$ and we obtain a path $\psi : t \to A(t)$ such that $A(0) = \mathrm{id}$ and such that $A(1)$ does not have eigenvalue 1 because the orbit is supposed to be nondegenerate.

Second step

Definition 2.2.5. *Let $\rho : \mathrm{Sp}(n) \to \mathbb{S}^1$ be the continuous map defined as follows. Given $A \in \mathrm{Sp}(n)$, we consider its positive eigenvalues $\{\lambda_i\}$. For an eigenvalue $\lambda = \mathrm{e}^{i\phi} \in \mathbb{S}^1 \backslash \{\pm 1\}$, let $m^+(\lambda)$ be the number of positive eigenvalues of the symmetric non degenerate 2-form Q defined on the generalized eigenspace E_λ by*

$$Q : E_\lambda \times E_\lambda \to \mathbb{R} : (z, z') \to \omega(z, \bar{z}').$$

Hence we have

$$\rho(A) = (-1)^{\frac{1}{2}m^-} \prod_{\lambda \in \mathbb{S}^1 \backslash \{\pm 1\}} \lambda^{\frac{1}{2}m^+(\lambda)}, \tag{2.5}$$

where m^- is the sum of the algebraic multiplicities $m_\lambda = \dim_{\mathbb{C}} E_\lambda$ of the real negative eigenvalues.

Theorem 2.2.6. *The map $\rho : \mathrm{Sp}(n) \to \mathbb{S}^1$ satisfies the following properties:*

1. *determinant: if $A \in U(n) = \mathrm{Sp}(n) \cap O(2n)$, then*

$$\rho(A) = \det_{\mathbb{C}}(X + iY), \text{ where } A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

2. *invariance: ρ is invariant under conjugation, i.e. for all $B \in \mathrm{Sp}(n)$ we have $\rho(BAB^{-1}) = \rho(A)$;*

3. *normalisation: $\rho(A) = \pm 1$ for matrices which have no eigenvalue on the unit circle;*

4. *multiplication: ρ behaves multiplicatively with respect to direct sums e.g if we consider $A \in \mathrm{Sp}(m)$ and $B \in \mathrm{Sp}(n)$ then we have*

$$\rho(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = \rho(A)\rho(B).$$

Moreover, the set $\mathrm{Sp}^*(n) = \{A \in \mathrm{Sp}(n) | \det(A - \mathrm{id}) \neq 0\}$ has two connected components. There are the connected component $\mathrm{Sp}^-(n) = \{A \in \mathrm{Sp}(n) | \det(A - \mathrm{id}) < 0\}$ which contains the matrix $-\mathrm{id}$, denoted W^-, and the connected component $\mathrm{Sp}^+(n) = \{A \in \mathrm{Sp}(n) | \det(A - \mathrm{id}) > 0\}$ which contains the matrix $\mathrm{diag}(2, 1/2, -1, ..., -1)$, denoted W^+. Notice that any loop in $\mathrm{Sp}^*(n)$ is contractible in $\mathrm{Sp}(n)$.

Then any path $\psi : [0, 1] \to \mathrm{Sp}(n)$ in $\mathrm{SP}(n)$ such that $\psi(1)$ is in $\mathrm{Sp}^*(n)$ can be extended to a path $\tilde{\psi} : [0, 2] \to \mathrm{Sp}(n)$ such that

1. $\widetilde{\psi}(t) = \psi(t)$ for $t \leqslant 1$;

2. $\widetilde{\psi}(t)$ is in $\mathrm{Sp}^*(n)$ for any $t \geqslant 1$;

3. $\widetilde{\psi}(2) \in \{W^\pm\}$.

Since $(\rho(\mathrm{id}))^2 = 1$ and $(\rho(W^\pm))^2 = 1$ we have that $\rho^2 \circ \widetilde{\psi} : [0, 2] \to \mathbb{S}^1$ is a loop in $\mathbb{S}^1$. Moreover one may prove that its degree does not depend on the extension $\widetilde{\psi}$ of ψ.

Definition 2.2.7. *The Maslov index of an element ψ of $\mathrm{Sp}(n)$ is defined by:*

$$\mu_M : \mathrm{Sp}(n) \to \mathbb{Z} \mid \psi \mapsto \deg(\rho^2 \circ \tilde{\psi}), \tag{2.6}$$

where $\tilde{\psi}$ is an extension of ψ as above.

Then we define the Conley-Zendher index of a critical point x of H as the Maslov index of the path of symplectic matrices associated to x in the first step.

Remark 2.2.8. If we suppose that the Hamiltonian diffeomorphism f is given by the 1-time map flow of an autonomous Hamiltonian function $H : M \to \mathbb{R}$ which is C^2-close to the identity then the Conley-Zehnder index of a fixed point x of f is equal to the Morse index of x where H is seen as a Morse function on M. One may refer to [68] for more details.

2.3 Dynamical systems

From now we consider a connected, compact and oriented surface Σ without boundary. Let $\mathrm{Homeo}(\Sigma)$ be the space of homeomorphisms of Σ equipped with the topology of uniform convergence on Σ. For $f \in \mathrm{Homeo}(\Sigma)$, $\mathrm{Fix}(f)$ represents the set of fixed points of f.

2.3.1 Isotopies and maximal Isotopies

An isotopy is a continuous path $t \mapsto f_t$ from $[0,1]$ to $\mathrm{Homeo}(\Sigma)$. We say that $f \in \mathrm{Homeo}(\Sigma)$ is isotopic to the identity if there exists an isotopy $I = (f_t)_{t\in[0,1]}$ such that $f_0 = \mathrm{id}$ and $f_1 = f$. We denote by $\mathrm{Homeo}_0(\Sigma)$ the set of those homeomorphisms.

Given an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f, we can extend it to an isotopy defined on $\mathbb{R}$ by the periodic relation $f_{t+1} = f_t \circ f_1$. We define the set of singularities $\mathrm{Sing}(I)$ of I as follows.

$$\mathrm{Sing}(I) = \{x \in \Sigma |\ \forall t \in [0,1],\ f_t(x) = x\}.$$

The complement of $\mathrm{Sing}(I)$ in Σ is called the domain of I and denoted $\mathrm{Dom}(I)$.

For a point $z \in \Sigma$, the arc $\gamma : [0,1] \to \Sigma$ where for each t$\in [0,1]$, $\gamma(t) = f_t(z)$ is called the *trajectory* of z along the isotopy I. For every $n \geqslant 0$, we denote by $\gamma_n(z)$ the concatenation of the trajectories of $z, f(z), ..., f^{n-1}(z)$.

We fix a homeomorphism $f \in \mathrm{Homeo}_0(\Sigma)$. A set $X \subset \mathrm{Fix}(f)$ is say to be *unlinked* if there exists an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f such that X is included in the set of singularities of I.
We denote by $\mathcal{I}(f)$ the set of couples (X, I) such that I is an isotopy from id to f and $X \subset \mathrm{Sing}(I)$. The set $\mathcal{I}(f)$ is naturally equipped with a pre-order $\leqslant$, where

$$(X, I) \leqslant (X', I'),$$

if $X \subset X'$ and for each $z \in \Sigma\backslash X$, its trajectory along I' and I are homotopic in $\Sigma\backslash X$. The couple (X', I) is called *an extension* of (X, I). An isotopy $I \in \mathcal{I}$ is called a *maximal isotopy* in $\mathcal{I}$ if the couple $(\mathrm{Sing}(I), I)$ is a maximal element of $(\mathcal{I}, \leqslant)$.

A recent result by F. Béguin, S. Crovisier and F. Le Roux [8] asserts that for a homeomorphism $f \in \mathrm{Homeo}_0(\Sigma)$ isotopic to the identity there always exists a maximal isotopy (a weaker result was previously proved by O. Jaulent [44]). We will often use Corollary 1.3 of [8] which we write as the following theorem:

Theorem 2.3.1. *Let us consider $f \in \mathrm{Homeo}_0(\Sigma)$. For each element $(X, I) \in \mathcal{I}(f)$ there is a maximal element $(X', I') \in \mathcal{I}(f)$ such that (X', I') is an extension of (X, I).*

In the case of the 2-sphere $\mathbb{S}^2$ we have the following result, which can be found in [50], about the homotopy classes of isotopies of an orientation preserving homeomorphism f of the sphere $\mathbb{S}^2$. We consider the isotopy $R_\infty = (r_t)_{t\in[0,1]}$ where r_t is the rotation of angle $2\pi t$ i.e $r_t(r, \theta) = (r, \theta + 2\pi t)$ in radial coordinates. The isotopy extends into an isotopy $\mathbb{R}^2 \sqcup \{\infty\}$ on the sphere. For $z \in \mathbb{S}^2$, we choose an orientation preserving homeomorphism $h_z : \mathbb{R}^2 \to \mathbb{S}^2\backslash\{z\}$ and we define the isotopy $R_z = h_z \circ R_\infty \circ h_z^{-1}$. If we consider

two points z and z' of the sphere we choose an orientation preserving homeomorphism $h_{z,z'} : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{z\}$ such that $h_{z,z'}(0) = z'$ and we define the isotopy $R_{z,z'} = h_{z,z'} \circ R_\infty \circ h_{z,z'}^{-1}$ which fixes the points z and z'.

Proposition 2.3.2. *Let us consider an orientation preserving homeomorphism f of the sphere $\mathbb{S}^2$.*

1. *For each fixed point $z \in \mathrm{Fix}(f)$, the set of isotopies from* id *to f which fix z is not empty. For two such isotopies I and I', there exists a unique integer $k \in \mathbb{Z}$ such that I' is homotopic to $R_z^k I$ relatively to $\{z\}$.*

2. *If f has at least two fixed points, then for each couple (z, z') of distinct fixed points the set of isotopies from* id *to f which fix z and z' is not empty. For two such isotopies, there exists a unique integer $k \in \mathbb{Z}$ such that I' is homotopic to $R_{z,z'}^k I$ relatively to $\{z, z'\}$.*

3. *If f has at least three fixed points, then for each triplet (z, z', z'') of distinct fixed points the set of isotopies from* id *to f which fix z, z' and z'' is not empty. All those isotopies are homotopic relatively to $\{z, z', z''\}$.*

2.3.2 Lefschetz index

For a homeomorphism $f \in \mathrm{Homeo}(\Sigma)$ and an isolated fixed point x of f, we define the Lefschetz index $\mathrm{ind}(f, x)$ of x as follows. let U be a chart centered at x and we denote by Γ a small oriented circle in U around x. For Γ sufficiently small, the map

$$z \mapsto \frac{f(z) - z}{||f(z) - z||},$$

is well defined on Γ and we denote by $\mathrm{ind}(f, x)$ the degree of this map.

2.3.3 Linking number

Let us consider an orientation preserving homeomorphism f of the plane isotopic to the identity and $I = (f_t)_{t \in [0,1]}$ an isotopy from id to f. Let us suppose that there exists a periodic point z^* of f of period $q \geqslant 1$. If z is a fixed point of f, the quotient of the map

$$t \mapsto \frac{f_t(z^*) - f_t(z)}{||f_t(z^*) - f_t(z)||},$$

defines a continuous function of the circle $\mathbb{R}/q\mathbb{Z}$ to $\mathbb{S}^1$. The degree of this application is called the *real linking number* of z^* and is denoted by $l_{I,z^*}(z)$. It depends only on the homotopy class of the isotopy I. For another isotopy I' of f there exists $k \in \mathbb{Z}$ such that I' is homotopic to $R_z^k I$, where R_z^k was defined in section 2.3.1. We verify that $l_{I',z^*}(z) = l_{I,z^*}(z) - kq$. Then the *linking number* $L_{f,z^*}(z) = l_{I,z^*}(z) + q\mathbb{Z} \in \mathbb{Z}/q\mathbb{Z}$ is independent on the choice of the isotopy.

2.3.4 Rotation vectors

Let $f \in \mathrm{Homeo}_0(\Sigma)$ be the time one map of an isotopy $I = (f_t)_{t\in[0,1]}$ from the identity to f. Among the many ways to define the *rotation vector*, we restrict ourselves to *positively recurrent points*. A point $z \in \Sigma$ is a *positively recurrent point* of f if for each neighborhood $U \subset \Sigma$ of z there exists an integer $n \in \mathbb{N}$ such that $f^n(z) \in U$. The integer $n > 0$ which is minimal for the previous property is called the *first return time* and is denoted by $\tau(z)$. The set of positively recurrent points is denoted by $\mathrm{Rec}^+(f)$.

Let $z \in \Sigma$ be a positively recurrent point. Fix a 2-ball $U \subset \Sigma$ containing z and let $(f^{n_k}(z))_{k\geqslant 0}$ be a subsequence of the positive orbit of z obtained by keeping the iterates of z by f that are in U. For any $k \geqslant 0$, choose an arc γ_k in U from $f^{n_k}(z)$ to z. The homology class $[\Gamma_k] \in H_1(\Sigma, \mathbb{Z})$ where Γ_k is the concatenation of $\gamma_{n_k-1}(z)$ and γ_k do not depend on the choice of γ_k. We say that z has a rotation vector $\rho(z) \in \mathrm{H}_1(\Sigma, \mathbb{R})$ if

$$\lim_{l\to+\infty} \frac{1}{n_{k_l}}[\Gamma_{k_l}] = \rho(z),$$

for any subsequence $(f^{n_{k_l}}(z))_{l\geqslant 0}$ which converges to z. Notice that the linking number of a periodic point z^* of an orientation preserving homeomorphism of the plane is equal to the rotation number of z^* in $\mathbb{R}^2\backslash\{z\}$.

In the case where f preserves a Borel probability measure μ, one applies Birkhoff's ergodic theorem to the first return map in U and proves that μ-a.e. point z is positively recurrent and has a rotation vector $\rho(z)$. Moreover, the measurable map ρ is bounded, and one may define the rotation vector of the measure

$$\rho(\mu) = \int_\Sigma \rho \; d\mu \in H_1(\Sigma, \mathbb{R}).$$

We say that $f \in \mathrm{Homeo}_0(\Sigma)$ is a *Hamiltonian homeomorphism* if it preserves a Borel probability measure whose support is the whole surface and rotation vector is zero. We denote by $\mathrm{Ham}(\Sigma)$ the set of Hamiltonians on Σ.

2.3.5 Local isotopies and local rotation set

Let Σ be a connected oriented surface. We write $f : (W, z_0) \to (W', z_0)$ for an orientation preserving homeomorphism between two neighborhoods W and W' of $z_0 \in \Sigma$ such that $f(z_0) = z_0$. Such a local homeomorphism f is called an *orientation preserving local homeomorphism at* z_0. We recall the definition of local isotopies of Le Calvez [50]: a *local isotopy* $I = (f_t)_{t\in[0,1]}$ from id to f is a continuous family of local homeomorphisms $(f_t)_{t\in[0,1]}$ fixing z_0 such that

- each f_t is a homeomorphism of a neigborhood $V_t \subset W$ of z into a neighborhood $V'_t \subset W'$ of z ;
- the sets $\{(z,t) \in \Sigma \times [0,1] | z \in V_t\}$ and $\{(z,t) \in \Sigma \times [0,1] \mid z \in V'_t\}$ are open in $\Sigma \times [0,1]$;
- the map $(z,t) \mapsto f_t(z)$ is continuous on $\{(z,t) \in \Sigma \times [0,1] \mid z \in V_t\}$;

- the map $(z,t) \mapsto f_t^{-1}(z)$ is continuous on $\{(z,t) \in \Sigma \times [0,1] \mid z \in V_t'\}$;
- we have $f_0 = \mathrm{id}_{V_0}$ and $f_1 = f|_{V_1}$;
- for all $t \in [0,1]$, we have $f_t(z_0) = z_0$.

Let us consider a local orientation preserving homeomorphism $f : (W, z_0) \to (W', z_0)$ and $I = (f_t)_{t\in[0,1]}$ a local isotopy from id to f. We want to define the local rotation set of the isotopy I at z_0. Given two neighborhoods $V \subset U$ of z_0 included in W and an integer $n \geqslant 1$ we define

$$E(U,V,n) = \{z \in U \mid z \notin V, f^n(z) \notin V, f^i(z) \in U \text{ for all } 1 \leqslant i \leqslant n\}.$$

We define the rotation set of I relative to U and V by

$$\rho_{U,V}(I) = \bigcap_{m\geqslant 1} \overline{\bigcup_{n\geqslant m} \{\rho_n(z) \mid z \in E(U,V,n)\}} \subset [-\infty, \infty],$$

where $\rho_n(z)$ is the average change of angular coordinate along the trajectory of z during n iterates. We define the local rotation set of I to be

$$\rho_s(I, z_0) = \bigcap_U \overline{\bigcup_V \rho_{U,V}(I)} \subset [-\infty, \infty],$$

where $V \subset U \subset W$ are neighborhoods of z_0.

The local rotation set is an invariant of local conjugacy in the following sense: let us say that an isotopy $I' = (f_t')_{t\in[0,1]}$ is locally conjugated to I if there exists a homeomorphism $\phi : W \to W''$ between two neighborhood of z_0 which preserves the orientation and fixes z_0 such that for each $t \in [0,1]$ we have $f_t' = \phi \circ f_t \circ \phi^{-1}$. For each neighborhoods V and U of z_0 such that $V \subset U \subset W$ we have

$$\rho_{U,V}(I) = \rho_{\phi(U),\phi(V)}(\phi I \phi^{-1}).$$

In particular we deduce that

$$\rho_s(I) = \rho_s(\phi I \phi^{-1}).$$

Let us consider a homeomorphism of the plane f isotopic to the identity which preserves the orientation and fixes the origin and an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f which fixes the origin. Recall that $R = (R_t)_{t\in[0,1]}$ is the isotopy of the rotation of angle 2π such that $R_t(z) = z\mathrm{e}^{2i\pi t}$ for each $z \in \mathbb{R}^2$ and $t \in [0,1]$. We have the following result about the local rotation set.

Lemma 2.3.3. *For each $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ we have*

$$\rho_s(R^p I^q) = q\rho_s(I) + p.$$

We say that f satisfies the *local intersection property* at z_0 if we have:

$$\text{For each non contractible loop } \gamma \text{ of } W\backslash\{z_0\} \text{ we have } f(\gamma) \cap \gamma \neq \varnothing. \tag{P2.6}$$

Example 2.3.4. Let us consider a fiber rotation $h_\alpha : (r,\theta) \to (r, \theta+\alpha(r))$ on the plane where $\alpha : (0,\infty) \to \mathbb{R}$ is continuous and an isotopy $I = (h_t)_{t\in[0,1]}$ such that $h_t(r,\theta) = (r, \theta+t\alpha(r))$ for $t \in [0,1]$. The local rotation set $\rho_s(I)$ of I at the origin is equal to the set of accumulation points of α at 0.

F. Le Roux proved [56, 59] that a homeomorphism of the plane which preserves the orientation and which fixes the origin has an empty local rotation set at 0 if and only if it is locally conjugated to the following maps:

- the contraction $z \mapsto \frac{z}{2}$,
- the expansion $z \mapsto 2z$,
- a holomorphic function $z \mapsto \mathrm{e}^{2i\pi\frac{p}{q}} z(1+z^{qr})$ where $q, r \in \mathbb{N}^+$ and $p \in \mathbb{Z}$.

Remark 2.3.5. In particular, in the case where f is area-preserving, Gambaudo and Pécout [34] proved that none of those above cases occurs, then the local rotation set is not empty. Moreover, if we suppose that $\mathrm{Fix}(f)$ is finite, 0 is not accumulated by fixed points then if the local rotation set is not empty it does not contain an integer in its interior. Notice that the result holds if we suppose that f satisfies the local intersection property, meaning that for each non contractible loop γ of $W\backslash\{z_0\}$ we have $f(\gamma) \cap \gamma \neq \varnothing$.

The *rotation number* classify the homotopy classes of the isotopies at z_0. Let us consider a local orientation preserving homeomorphism $f : (W, z_0) \to (W', z_0)$ of a surface Σ such that $f(z_0) = z_0$ and $I = (f_t)_{t\in[0,1]}$ a local isotopy from id to f which fix z_0. Let us consider a closed disk $D \subset \Sigma$ containing z_0 in its interior. For every point $z \in D\backslash\{z_0\}$ close enough to z_0, the trajectory of z_0 along I is a loop included in $D\backslash\{z_0\}$. There exists an integer $k \in \mathbb{Z}$ such that this trajectory is freely homotopic in $D\backslash\{z_0\}$ to $(\partial D)^k$. The integer k depends only on the choice of the isotopy I, it is the *rotation number* $k = \rho(I, z_0)$ of I at z_0. We consider the isotopy R_∞ defined in the previous section 2.3.1. The isotopy R_∞ extends into an isotopy on the sphere $\mathbb{R}^2 \sqcup \{\infty\}$ and we have $\rho(R_\infty, \infty) = 1$ while $\rho(R_\infty, 0) = -1$. We refer to [50] for more details.

2.3.6 The blow-up at a fixed point

Let us consider $f : (W, z_0) \to (W', z_0)$ an orientation preserving homeomorphism. We say that f can be blown-up at z_0, if we can "replace" z_0 by a unit circle $\mathbb{S}^1$ and extend $f|_{W\backslash\{z_0\}}$ continuously to a homeomorphism between $W\backslash\{z_0\} \sqcup \mathbb{S}^1$ and $W\backslash\{z_0\} \sqcup \mathbb{S}^1$, see [59] for more details. In particular, when f is a diffeomorphism, the extension can be induced by the map

$$v \mapsto \frac{Df(z_0)v}{||Df(z_0)v||}$$

on the space of unit tangent vectors.

Let us suppose that f can be blown-up at z_0, is isotopic to the identity and is not conjugate to a contraction or an expansion. We denote by h the extension of f on $\mathbb{S}^1$ and by $I = (f_t)_{t\in[0,1]}$ a local isotopy of f. We choose a small disk D which contains z_0 and we consider the universal cover $\pi : \widetilde{D} \to D\backslash\{z_0\}$. We consider the isotopy $(\widetilde{f}_t)_{t\in[0,1]}$ from id to $\widetilde{f}$ obtain by lifting I and we consider $\widetilde{h}$ the lift of h to $\mathbb{R}$ which is a continuous extension

of $\widetilde{f}_1$. We define the blown-up rotation number $\rho(I, z_0) \in \mathbb{R}$ to be the rotation number of $\widetilde{h}$. J.-M. Gambaudo,P. Le-Calvez and E. Pécou [33] proved that the blown-up rotation numbers does not depend on the choice of h.

Naturally we have the following property [59].

Proposition 2.3.6. *Let us consider an isotopy $I = (f_t)_{t\in[0,1]}$ from* id *to a homeomorphism of the plane f which preserves the orientation, fixes the origin and can be blown-up at* 0. *If the local rotation set $\rho_s(I)$ is not empty then it is equal to the singleton $\{\rho(I, z_0)\}$.*

2.3.7 Positively transverse foliations

Let us consider an oriented topological foliation $\mathcal{F}$ on the complement of a compact set X of a surface Σ. The set X will be called the set of *singularities* of $\mathcal{F}$. An open flow box of $\mathcal{F}$ is a couple (V, h), where V is an open set of Σ and $h : V \to (-1, 1)^2$ is an orientation-preserving homeomorphism that sends the foliation $\mathcal{F}|_V$ on the vertical foliation oriented with y decreasing. Writing $p_1 : \mathbb{R}^2 \to \mathbb{R}$ for the first projection, we say that an arc $\gamma : I \to \Sigma$ is *positively transverse* to the foliation $\mathcal{F}$ if for every $t_0 \in I$, there exists an open flow-box (V, h) such that $\gamma(t_0) \in V$ and the map $t \mapsto p_1(h(\gamma(t)))$ defined in a neighborhood of t_0 is strictly increasing.

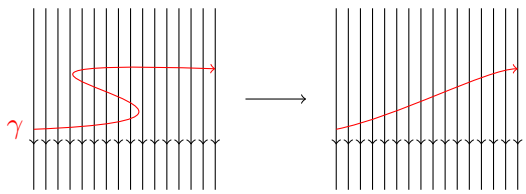


Figure 2.3: An example of a flow box

For $z \in \Sigma$, we write ϕ_z the leaf passing through z and ϕ_z^+ for the positive half-leaf from z. We consider an isolated singularity x of the foliation $\mathcal{F}$, we can define the index $\mathrm{ind}(\mathcal{F}, x)$ of x for the foliation $\mathcal{F}$ as follows. We consider a sufficiently small open chart U containing x and an orientation preserving homeomorphism $h : U \to D\backslash\{0\}$ which sends x to 0. We denote $\mathcal{F}_h$ the image of the foliation $\mathcal{F}|_U$ by h and we consider a simple loop $\Gamma : \mathbb{S}^1 \to D\backslash\{x\}$, one may cover Γ by a finite family $(V_i)_{i\in J}$ of flow-boxes of the foliation $\mathcal{F}_h$ included in $D\backslash\{0\}$. We denote by $\phi^+_{V_i,z}$ the positive half-leaf from z of the restricted foliation $\mathcal{F}_h|_{V_i}$. We can find a continuous map ψ defined from the loop Γ to $D_1\backslash\{x\}$ such that $\psi(z) \in \phi^+_{V_i,z}$ for every $i \in J$ and any $z \in V_i$. The map

$$\theta \to \frac{\psi(\Gamma(\theta)) - \Gamma(\theta)}{||\psi(\Gamma(\theta)) - \Gamma(\theta)||},$$

is well defined on Γ and $\mathrm{ind}(\mathcal{F}, x)$ is the degree of this map.

We say that a singularity x of an oriented foliation $\mathcal{F}$ is a sink (resp. source) if there is a neighborhood V of x such that the omega-limit point (resp. the alpha-limit point) of each leaf ϕ of $\mathcal{F}$ which is passing through V is equal to x. The sinks and sources of a

foliation $\mathcal{F}$ have an index equal to 1. Let us draw an example of the neighborhood of a sink on the left and a neighborhood of a source on the right of the following figure.

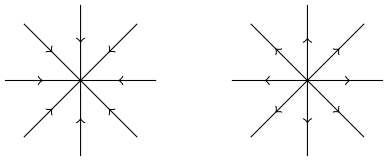


Figure 2.4: An example of a sink and a source of a foliation

We say that a singularity x of an oriented foliation $\mathcal{F}$ is a saddle point of the foliation $\mathcal{F}$ if the foliation is locally homeomorphic to a foliation as the one on Figure 2.5, we refer to [59] for more details on saddle points. A maximal connected union of leaves such that their alpha (resp. omega) limit is equal to x is called an unstable (resp. stable) cone of x. A saddle point has $1-\mathrm{ind}(\mathcal{F}, x)$ unstable cones and three stable cones which are alternated in the cyclic order. In Figure 2.5 we draw an example of a foliation near a saddle point of index -2.

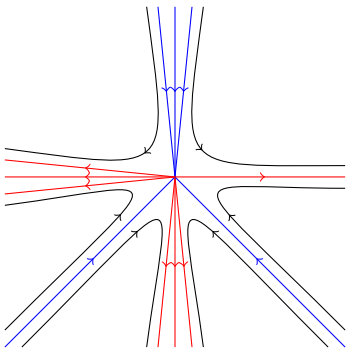


Figure 2.5

A leaf of an oriented foliation $\mathcal{F}$ whose alpha-limit point and omega-limit point are distinct singularities of $\mathcal{F}$ will be called a *connexion*.

Let us consider $f \in \mathrm{Homeo}_0(\Sigma)$ and a maximal isotopy $I = (f_t)_{t\in[0,1]}$ from the identity to f. A foliation $\mathcal{F}$ is said to be *positively transverse* to the isotopy I if $\mathrm{Sing}(I) = \mathrm{Sing}(\mathcal{F})$ and for every $z \in \mathrm{Dom}(I)$, the trajectory $\gamma(z)$ of z is homotopic in $\mathrm{Dom}(I)$, relatively to its endpoints, to a path γ which is positively transverse to the foliation $\mathcal{F}$. The following fondamental result of Le Calvez [49] asserts that for each maximal isotopy I there exists a dynamically transverse foliation to the isotopy I.

Theorem 2.3.7. *Let us consider a homeomorphism $f \in \mathrm{Homeo}_0(\Sigma)$ and an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f, such that $\mathrm{Sing}(I)$ is a maximal unlinked set of fixed points of f. There exists a foliation $\mathcal{F}$ which is dynamically transverse to the isotopy I.*

We denote by $\mathcal{F}(I)$ the set of foliations positively transverse to I. We will use the following definition of gradient-like foliations.

Definition 2.3.8. *A foliation $\mathcal{F}$ is said to be* gradient-like *if*

- *The number of singularities is finite.*
- *Every leaf defines a connexion.*
- *There is no closed leaf.*
- *There is no family $(\phi_i)_{i\in\mathbb{Z}/k\mathbb{Z}}$, $k \geqslant 1$ of leaves such that $\omega(\phi_i) = \alpha(\phi_{i+1})$, $i \in \mathbb{Z}/k\mathbb{Z}$.*

For the remaining of the thesis, most of the transverse foliations we will meet will be gradient-like foliations. The notion of connexion will be generalized in chapter 7, but until this chapter, a connexion will always refer to a leaf of a gradient-like foliation.

We refer to [49] for the proof of the following important properties.

Proposition 2.3.9. *Consider a Hamiltonian homeomorphism of a surface Σ with a finite number of fixed points then for each maximal isotopy I from* id *to f, a foliation $\mathcal{F}$ positively transverse to I is gradient-like. Moreover we have*

- $\mathrm{ind}(\mathcal{F}, x) \leqslant 1$ *for every point $x \in \mathrm{Sing}(I)$.*
- $\mathrm{ind}(\mathcal{F}, x) = 1$ *for every sink or source $x \in \mathrm{Sing}(I)$*
- $\mathrm{ind}(\mathcal{F}, x) = \mathrm{ind}(f, x)$ *for every saddle point $x \in \mathrm{Sing}(I)$, where* $\mathrm{ind}(f, \cdot)$ *is the Lefschetz index.*
- *For every leaf $\phi \in \mathcal{F}$, the action function A_f, defined later, of f satisfies $A_f(\alpha(\phi)) > A_f(\omega(\phi))$.*

For the remaining, we can keep in mind that, for a gradient-like foliation, there are three kinds of singularities: sinks, sources and saddle points.

Remark 2.3.10. For a maximal isotopy I of a Hamiltonian homeomorphism f of a surface Σ and a foliation $\mathcal{F} \in \mathcal{F}(I)$, if Σ is not the sphere, then the index function $\mathrm{ind}(\mathcal{F}, \cdot)$ defined on $\mathrm{Sing}(I)$ does not depend on the choice of $\mathcal{F} \in \mathcal{F}(I)$ and can be denoted $\mathrm{ind}(I, \cdot)$.

Let us consider a gradient-like foliation $\mathcal{F}$ of a surface Σ and a leaf ϕ of $\mathcal{F}$. By definition, the omega-limit set (resp. the alpha-limit set) of ϕ exists and is equal to a singleton $\{x\}$. To simplify the notations, x will be called *the omega-limit point* also denoted $\omega(\phi)$ (resp. *the alpha-limit point* also denoted $\alpha(\phi)$) of ϕ.

2.3.8 Generalized Isotopies

In this section, we consider a Hamiltonian homeomorphism f of a compact surface Σ such that $\mathrm{Fix}(f)$ is finite.

We consider the compactification $\hat{\Sigma} = \widetilde{\Sigma} \cup \{\infty\}$ of $\widetilde{\Sigma}$ into a 2-sphere.

Let us consider a maximal isotopy I of f on Σ and its natural lift $\widetilde{I}$ on $\widetilde{\Sigma}$. The isotopy $\widetilde{I}$ has an infinite number of singularities but for a non zero integer $k \in \mathbb{Z}^*$ and a fixed point $\widetilde{x}$ of $\widetilde{f}$, $R^k_{\widetilde{x}}\widetilde{I}$ has a finite number of singularities and can be extended to an isotopy

$\hat{I}$ of a homeomorphism $\hat{f}$ on $\hat{\Sigma}$ which has a finite number of fixed points. The point at infinity in $\hat{\Sigma}$ becomes a fixed point of such a homeomorphism $\hat{f}$ and its rotation number for $\hat{I}$ satisfies:

$$\rho_{\hat{I}}(\infty) = -k.$$

An isotopy $\hat{I}$ from id to $\hat{f}$ which is homotopic to $R^k_{\widetilde{x},\infty}\widetilde{I}$ relatively to $\hat{\Sigma}\backslash\{\widetilde{x},\infty\}$ such that the rotation number of ∞ is equal to $-k$ is called a *generalized isotopy* of f. We denote by $\hat{\mathcal{I}}_k$ the set of couples $(X,\hat{I})$ where $\hat{I}$ is a generalized isotopy of f such that $\rho_{\hat{I}}(\infty) = -k$ and $X \subset \mathrm{Sing}(\hat{I})$. To simplify notations, we can consider $\hat{I} \in \hat{\mathcal{I}}_k$ which refers to the couple $(\mathrm{Sing}(\hat{I}),\hat{I})$.

The set $\hat{\mathcal{I}}_k$ is naturally equipped with a pre-order $\leqslant$, where $(X,\hat{I}) \leqslant (X',\hat{I}')$ if $\infty \in X \subset X'$ are unlinked sets of fixed points and for each $z \in \hat{\Sigma}\backslash X$, its trajectory along $\hat{I}'$ and $\hat{I}$ are homotopic in $\hat{\Sigma}\backslash X$. The couple $(X',\hat{I})$ is called *an extension* of $(X,\hat{I})$. An isotopy $\hat{I} \in \hat{\mathcal{I}}_k$ is called a *maximal generalized isotopy* in $\hat{\mathcal{I}}_k$ if the couple $(\mathrm{Sing}(\hat{I}),\hat{I})$ is a maximal element of $(\hat{\mathcal{I}}_k,\leqslant)$.

Lemma 2.3.11. *Let us consider a generalized isotopy $\hat{I} \in \hat{\mathcal{I}}_k$ of f with $k \in \mathbb{Z}^*$,*

$$\#\mathrm{Sing}(\hat{I}) \leqslant \#\mathrm{Fix}(f) + 1,$$

and for each $z \in \mathrm{Fix}(f)$ we have

$$\#(\mathrm{Sing}(\hat{I}) \cap \pi^{-1}(z)) \leqslant 1.$$

Proof. By contradiction we prove the second inequality, the first one will follow.

Let us consider a generalized isotopy $\hat{I} \in \hat{\mathcal{I}}$ such that there exists $x \in \mathrm{Fix}(f)$ satisfying $\#(\mathrm{Sing}(\hat{I}) \cap \pi^{-1}(x)) > 1$ or $\#\mathrm{Sing}(\hat{I}) \geqslant \mathrm{Fix}(f) + 2$. We consider $\widetilde{I}$ the isotopy from id to f whose compactification is the isotopy $\hat{I}$. There exists two singularities $\widetilde{x}$ and $\widetilde{x}'$ of $\hat{I}$ which are in $\pi^{-1}(\{x\})$. The linking number between $\widetilde{x}$ and $\widetilde{x}'$ for the isotopy $\widetilde{I}$ is equal to zero.

We consider I' a maximal isotopy from id to f which fixes x and we denote $\widetilde{I}'$ the isotopy obtained by lifting I' on $\widetilde{\Sigma}$. We have that $\widetilde{I}'$ is homotopic to $R^{-k}_{\widetilde{x}}\widetilde{I}$ relatively to $\hat{\Sigma}\backslash\{x,\infty\}$, see Proposition 2.3.2 for more details, and we have $\pi^{-1}(\{x\}) \subset \mathrm{Sing}(\widetilde{I}')$. So, the linking number between $\widetilde{x}$ and $\widetilde{x}'$ for the isotopy $\widetilde{I}'$ is equal to zero but the linking number between $\widetilde{x}$ and $\widetilde{x}'$ for the isotopy $R^{-k}_{\widetilde{x}}\widetilde{I}$ is equal to $-k$. Hence we obtain our contradiction.

We deduce the inequality $\#\mathrm{Sing}(\hat{I}) \leqslant \#\mathrm{Fix}(f) + 1$. □

Lemma 2.3.12. *Let us consider a maximal generalized isotopy $\hat{I} \in \hat{\mathcal{I}}_k$ of f where $k \in \mathbb{Z}^*$. There exists a foliation $\hat{\mathcal{F}}$ on the 2-sphere such that $\hat{\mathcal{F}}$ is positively transverse to $\hat{I}$ and every foliation which is positively transverse to $\hat{I}$ is gradient-like.*

Remark 2.3.13. i. For such a foliation $\hat{\mathcal{F}}$ of a maximal generalized isotopy $\hat{I}$, the action function is decreasing along the leaves of $\hat{\mathcal{F}}$.

ii. The fixed point ∞ is a source of $\hat{\mathcal{F}}$ if $k < 0$ and a sink of $\hat{\mathcal{F}}$ if $k > 0$.

2.3.9 Intersection number

Let Γ and Γ' be two oriented, transverse and simple closed curves on an oriented surface Σ. The algebraic intersection number $\Gamma \wedge \Gamma'$ is defined as the sum of the indices of the intersection points of Γ and Γ', where an index of an intersection point is $+1$ if the orientation of the intersection agrees with the orientation of Σ and -1 otherwise.

We keep the same notation $\Gamma \wedge \gamma$ for the algebraic intersection number between a loop Γ and a path γ when it is defined, for example, when γ is proper or when γ is a compact path whose extremities are not in Γ. Similarly, we write $\gamma \wedge \gamma'$ for the algebraic intersection number of two path γ and γ' when it is defined, for example, when γ and γ' are compact paths and the ends of γ (resp. γ') are not on γ' (resp. γ).

2.3.10 Action function of a Hamiltonian homeomorphism

In this section we define dynamically the action function of a Hamiltonian homeomorphism f of a compact surface Σ with a finite number of fixed points. Notice that this definition extends the notion of action function defined in section 2.2.2 for Hamiltonian diffeomprhisms.

Let us consider two unlinked fixed points $x, y \in \mathrm{Fix}(f)$ of f and an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f such that $x, y \in \mathrm{Sing}(I)$. Let γ be a simple path from x to y and define the map $\rho_{f,\gamma}$ on Σ by $\rho_{f,\gamma}(z) = \gamma \wedge \gamma(z)$ where $\gamma(z)$ is the trajectory of z under the isotopy I and $\gamma \wedge \gamma(z)$ is the intersection number between γ and $\gamma(z)$. We define the difference of action between y and x by

$$A_f(x) - A_f(y) = \int_\Sigma \rho_{f,\gamma}(z)dz, \tag{2.7}$$

which does not depend on the choice of γ. Notice that in general for a homeomorphism f, the map $\rho_{f,\gamma}$ is not integrable. In our case, f admits a finite number of fixed points and one may prove that the previous integral exists, see [49].

Unfortunately, if we consider two fixed points $x, y \in \mathrm{Fix}(f)$ they may not be unlinked. The previous arguments fail and to define the action difference between y and x we have to consider the universal cover of Σ. We denote by $\widetilde{\Sigma}$ the universal cover of Σ, $\pi : \widetilde{\Sigma} \to \Sigma$ the covering map and for a homeomorphism f we set G^* the group of automorphisms of Σ which commute with the lift $\widetilde{f}$ of f on Σ.

The following definition comes from a more general work of Wang [74]. The construction is more difficult, first we have to extend the linking number used in equation 2.7 then thanks to the work of Wang if the number of fixed points of f is finite then this linking number exists and we can define the action function by integrating this linking number.

Extension of the linking number for a positively recurrent point

Let us consider f the time one map of an isotopy $I = (f_t)_{t\in[0,1]}$ on Σ and $\widetilde{f}$ the time 1-map of the lifted identity isotopy $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ to the universal cover $\widetilde{\Sigma}$ of Σ. For every distinct fixed points $\widetilde{x}$ and $\widetilde{y}$ of $\widetilde{f}$ there exists a non-equivariant isotopy $\widetilde{I}_1$ from id to $\widetilde{f}$

that fixes $\widetilde{x}$ and $\widetilde{y}$.

Recall that the set of positively recurrent points is denoted by $\mathrm{Rec}^+(f)$.

We consider $z \in \mathrm{Rec}^+(f)\backslash\pi(\{\widetilde{x},\widetilde{y}\})$ and an open disk $U \subset \Sigma\backslash\pi(\{\widetilde{x},\widetilde{y}\})$ which contains z. For each couple $(z', z'') \in U^2$, we choose an oriented simple path $\gamma_{z',z''}$ in U from z' to z''. We define the function $\widetilde{\Phi}$ by:

$$\begin{aligned}\widetilde{\Phi} : \pi^{-1}(\mathrm{Rec}^+(f)) \cap \pi^{-1}(U) &\to \pi^{-1}(\mathrm{Rec}^+(f)) \cap \pi^{-1}(U)\\ \widetilde{z} &\mapsto \widetilde{f}^{\tau(z)}(\widetilde{z}),\end{aligned}$$

where $z = \pi(\widetilde{z})$ and $\tau(z)$ is the first return map in U. $\widetilde{\Phi}$ is the lifted function of the first return map Φ define on the recurrent points of U by $\Phi(z) = f^{\tau(z)}(z)$ where $z \in \mathrm{Rec}^+(f)\cap U$.

For any $\widetilde{z} \in \pi^{-1}(U)$, write $\widetilde{U}_{\widetilde{z}}$ the connected component of $\pi^{-1}(U)$ that contains $\widetilde{z}$. For every $j \geqslant 1$, recall that $\tau_j(z) = \sum_{i=0}^{j-1} \tau(\Phi^i(z))$. For every $n \geqslant 1$, consider the following curve in $\widetilde{\Sigma}$:

$$\widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}} = \widetilde{I}_1^{\tau_n(z)}(\widetilde{z})\widetilde{\gamma}_{\widetilde{\Phi}^n(\widetilde{z}),\widetilde{z}_n},$$

where $\widetilde{z}_n \in \pi^{-1}(\{z\}) \cap \widetilde{U}_{\widetilde{\Phi}^n(\widetilde{z})}$, and $\widetilde{\gamma}_{\widetilde{\Phi}^n(\widetilde{z}),\widetilde{z}_n}$ is the lift of $\gamma_{\Phi^n(z),z}$ which is contained in $\widetilde{U}_{\widetilde{\Phi}^n(\widetilde{z})}$. We can define the infinite product

$$\widetilde{\Gamma}^n_{\widetilde{I}_1,z} = \prod_{\pi(\widetilde{z})=z} \widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}}.$$

In the annulus $A_{\widetilde{x},\widetilde{y}}$ we can see $\widetilde{\Gamma}^n_{\widetilde{I}_1,z}$ as a multi-loop with finite homology. So, if we consider $\widetilde{\gamma}$ an oriented path in $\widetilde{\Sigma}$ from $\widetilde{x}$ to $\widetilde{y}$, the intersection number $\widetilde{\gamma} \wedge \widetilde{\Gamma}^n_{\widetilde{I}_1,z}$ is well defined and does not depend on $\widetilde{\gamma}$ nor the isotopy $\widetilde{I}_1$ but depends on the open set U. We have:

$$\widetilde{\gamma} \wedge \widetilde{\Gamma}^n_{\widetilde{I}_1,z} = \widetilde{\gamma} \wedge \prod_{\pi(\widetilde{z})=z} \widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}} = \sum_{j=0}^{n-1} \widetilde{\gamma} \wedge \widetilde{\Gamma}^j_{\widetilde{I}_1,\widetilde{z}}.$$

For $n \geqslant 1$ we define the functions

$$L_n : ((\mathrm{Fix}(\widetilde{f}) \times \mathrm{Fix}(\widetilde{f}))\backslash\widetilde{\Delta}) \times (\mathrm{Rec}^+(f) \cap U) \to \mathbb{Z},$$

by

$$L_n(\widetilde{f};\widetilde{x},\widetilde{y},z) = \widetilde{\gamma} \wedge \widetilde{\Gamma}^n_{\widetilde{I}_1,z} = \sum_{j=0}^{n-1} L_1(\widetilde{f};\widetilde{x},\widetilde{y},\Phi^j(z)),$$

where $U \subset \Sigma\backslash\pi(\{\widetilde{x},\widetilde{y}\})$. Again, the function L_n depends on U but not on the choice of $\gamma_{\Phi^n(z),z}$.

Action function

Definition 2.3.14. *Let us consider $z \in \mathrm{Rec}^+(f)\backslash\pi(\{\widetilde{x},\widetilde{y}\}$. We say that the linking number $i(\widetilde{f};\widetilde{x},\widetilde{y},z) \in \mathbb{R}$ is defined if*

$$\lim_{k\to\infty} \frac{L_{n_k}(\widetilde{f};\widetilde{x},\widetilde{y},z)}{\tau_{n_k}(z)} = i(\widetilde{f};\widetilde{x},\widetilde{y},z)$$

for any subsequence $\{\Phi^{n_k}(z)\}_{k\geqslant 1}$ *of* $\{\Phi^n(z)\}_{n\geqslant 1}$ *which converges to* z.

J. Wang proved that the linking number $i(\widetilde{f};\widetilde{x},\widetilde{y},z)$ does not depend on U.

Let us consider $\mu \in \mathcal{M}(f)$ such that μ is ergodic or the support of μ is the whole surface Σ. In our particular case where I has a finite number of fixed points, the function $i(\widetilde{f};\widetilde{x},\widetilde{y},z)$ is μ-integrable and we define a function $i_\mu(\widetilde{f};\widetilde{x},\widetilde{y})$ as follows.

$$i_\mu(\widetilde{f};\widetilde{x},\widetilde{y}) = \int_{\Sigma\setminus\pi(\{\widetilde{x},\widetilde{y}\}} i(\widetilde{f};\widetilde{x},\widetilde{y},z)d\mu.$$

Proposition 2.3.15. *For any distinct fixed points* $\widetilde{x},\widetilde{y}$ *and* $\widetilde{z}$ *of* $\widetilde{f}$*, we have*

$$i_\mu(\widetilde{f};\widetilde{x},\widetilde{y}) + i_\mu(\widetilde{f};\widetilde{y},\widetilde{z}) + i_\mu(\widetilde{f};\widetilde{z},\widetilde{x}) = 0.$$

So there exists a function $l_\mu : \mathrm{Fix}(\widetilde{f}) \to \mathbb{R}$ *defined up to an additive constant, such that*

$$i_\mu(\widetilde{f};\widetilde{x},\widetilde{y}) = l_\mu(\widetilde{f},\widetilde{y}) - l_\mu(\widetilde{f},\widetilde{x}).$$

If f is a Hamiltonian homeomorphism then $\rho_{\Sigma,I}(\mathrm{Leb}) = 0$ for each isotopy I from id to f on Σ and we have the following result.

Proposition 2.3.16. *If we suppose that* $\rho_{\Sigma,I}(\mu) = 0$*,* $i_\mu(\widetilde{f};\widetilde{x},T(\widetilde{x})) = 0$ *for every* $\widetilde{x} \in \mathrm{Fix}(\widetilde{f})$ *and every automorphism* $T \in G^*$ *then there exists a function* $L_\mu(\widetilde{f},\cdot)$ *defined on* $\mathrm{Fix}(\widetilde{f})$ *such that for every two distinct fixed points* $\widetilde{x}$ *and* $\widetilde{y}$ *of* $\widetilde{f}$ *we have*

$$i_\mu(\widetilde{f};\widetilde{x},\widetilde{y}) = L_\mu(\widetilde{f},\pi(\widetilde{y})) - L_\mu(\widetilde{f},\pi(\widetilde{x})).$$

The function $L_\mu(\widetilde{f},\cdot)$ is called *the action function* defined on $\mathrm{Fix}(f)$ for the measure μ. If $\mu = \mathrm{Leb}$ then the function $L_\mu(\widetilde{f},\cdot)$ will be denoted A_f.

Proposition 2.3.17. *Let us consider a smooth diffeomorphism* f *of a surface* Σ*. The action function* A_f *is equal to the action function of section 2.2.2.*

Action function along a leaf of a transverse foliation

In particular, if we consider a maximal isotopy $I = (f_t)_{t\in[0,1]}$ from id to f and a foliation $\mathcal{F}$ positively transverse to I we can give a short proof of the last point of Proposition 2.3.9 stated as the following lemma.

Lemma 2.3.18. *For every leaf* $\phi \in \mathcal{F}$ *we have* $A_f(\alpha(\phi)) > A_f(\omega(\phi))$.

Proof. We set $x = \alpha(\phi)$ and $y = \omega(\phi)$. We give the ideas of the proof using Wang's work.

Let us consider a small open disk $U \subset \Sigma\setminus X$. For almost every point $z \in U$ we can define $\tau(z)$ the first return map. Meaning that $\tau(z)$ is the first integer $n > 0$ such that $f^n(z) \in U$. We consider the loop $\Gamma(z) = I^{\tau(z)-1}(z)\gamma(z)$ where $\gamma(x) \subset U$ is a path which joins $f^{\tau(z)}(z)$ to z. The algebraic intersection of $\Gamma(z)$ and ϕ does not depend on the choice of γ and is well defined. We will denote $\delta(z) = \Gamma(z)\wedge\phi$ the algebraic intersection number defined on U.

There is a finite number of fixed points, so Wang proved that δ/τ is bounded and then integrable. We can define the limit of Birkhoff's average functions δ^* and τ^* of δ and τ. One may prove that the function $\eta = \delta^*/\tau^*$ is defined almost everywhere on U and does not depend on the choice of U. So we obtain a function defined almost everywhere on $\Sigma\backslash X$ such that, by the construction of Wang, its integral is equal to the action difference between x and y. Thus, it is enough to prove that $\delta(z)$ is positive and not zero to obtain the result.

We consider the universal cover $\widetilde{\Sigma\backslash X}$ of $\Sigma\backslash X$, which can be identify as the open disk $\mathbb{D}$.

We fix a lift $\widetilde{\phi}$ of ϕ on $\mathbb{D}$, hence $\delta(z)$ is equal to the finite sum of the algebraic intersection numbers of $\widetilde{\phi}$ and the lifts of $\Gamma(z)$. Let us consider a lift $\widetilde{\Gamma}(z)$ of $\Gamma(z)$ whose algebraic intersection number with $\widetilde{\phi}$ is not zero. Roughly speaking, $\widetilde{\Gamma}(z)$ is an oriented path going from one side of $\mathbb{D}\backslash\widetilde{\phi}$ to another. Moreover, $\widetilde{\phi}$ is a Brouwer line for $\widetilde{f}$, so $\widetilde{\Gamma}$ is going from the right hand side of $\widetilde{\phi}$ to the left hand side of $\widetilde{\phi}$. Hence the intersection number $\widetilde{\Gamma}(z) \wedge \widetilde{\phi} > 0$ and then we have $\delta(z) > 0$.

$\square$

Chapter 3

Introduction to barcodes and persistence modules

The notion of barcodes and persistence modules was used in topological data analysis, see for example G. Carlsson in [15] or R. Ghrist in [35]. Barannikov already noticed the existence of a filtration of the Morse homology in [6] and we can find the notion of persistence modules in Usher's work [70, 71] but the barcodes have been introduced in symplectic topology by Polterovich and Shelukhin [67]. The same year, without the terminology of the barcodes Usher and Zang published some results about the persistent homology in [72]. Recently, the notion of barcodes appears as a great tool to study C^0 symplectic geometry, let us cite for example the work of Buhovski-Humilière-Seyfaddini [14], Jannaud [**?**] and Le Roux-Viterbo-Seyfaddini [61].

Most of the following definitions and results are coming from [61]. One can also refer to Chazal, De Silva, Glisse and Outdot's book [16] or to [18].

Barcodes

Let us consider a special family of intervals $\mathbf{B}$ of the form $((a_j, b_j])_{j\in\{1,\dots,n\}}$, with $-\infty \leqslant a_j \leqslant b_j \leqslant +\infty$, where we allow trivial intervals of the form $(a, a]$. We say that two families are equivalent if removing all intervals of the form $(a, a]$ from them yields the same family.

Definition 3.0.1. *A* barcode *B is an equivalence class of family of intervals $\boldsymbol{B}$.*

By convenience, we will often identify a list of intervals with the corresponding barcode.

Let $a \leqslant b, c \leqslant d$ be four elements of $\mathbb{R} \cup \pm\{\infty\}$. We set $d((a, b], (c, d]) = \max\{|c-a|, |d-b|\}$, with convention that $d(\infty, \infty) = 0$. Note that if $c = d = \frac{a+b}{2}$, then $d((a, b], (c, d]) = \frac{b-a}{2}$.

Definition 3.0.2. *Let B_1, B_2 be barcodes and take representatives $\boldsymbol{B}_1 = (I^1_j)_{j\in J}$, $\boldsymbol{B}_2 = (I^2_k)_{k\in K}$. The bottleneck distance between B_1, B_2, denoted by $\mathrm{d_{bot}}(B_1, B_2)$, is the infimum of the set of ϵ such that there is a bijection σ between two subsets J', K' of J, K with the property that for every $j \in J'$, $d(I^1_j, I^2_{\sigma(j)}) \leqslant \epsilon$ and all the remaining intervals I^1_j, I^2_k for $j \in J\backslash J'$, $k \in K\backslash K'$ have length less than 2ϵ.*

We will denote Barcode the set of barcodes in the next sections.

Persistence module

Definition 3.0.3. *A* persistence module V *is a family* $(V_t)_{t\in\mathbb{R}}$ *of vector spaces equipped with morphisms* $i_{s,t} : V_s \to V_t$*, for* $s \leqslant t$*, satisfying:*

1. *For all* $t \in \mathbb{R}$ *we have* $i_{t,t} = \mathrm{id}$ *and for every* $s \leqslant t \leqslant u$ *we have* $i_{t,u} \circ i_{s,t} = i_{s,u}$*,*
2. *There exists a finite subset* $F \subset \mathbb{R}$*, often referred to the spectrum of* V*, such that* $i_{s,t}$ *is an isomorphism whenever* s, t *belong to the same connected component of* $\mathbb{R}\backslash F$*,*
3. *For all* $t \in \mathbb{R}$*,* $\lim\limits_{s\to t, s<t} V_s = V_t$ *; equivalently, for fixed* t*,* $i_{s,t}$ *is an isomorphism for* $s < t$ *sufficiently close to* t*.*

Let us consider a persitence module $(V_t)_{t\in\mathbb{R}}$ equipped with the morphisms $(i_{s,t})_{s\leqslant t}$. For any $t \in \mathbb{R}$, there exists ϵ such that $i_{s,u} : V_s \to V_u$ is an isomorphism if $s, u \in (t-\epsilon, t]$ or if $s, u \in (t, t+\epsilon)$. Choose $t^- \in (t-\epsilon, t]$ and $t^+ \in (t, t+\epsilon)$ and let $j(t) = \dim(\mathrm{Ker}(i_{t^-,t^+})) + \mathrm{codim}(\mathrm{Im}(i_{t^-,t^+}))$. Notice that $j(t)$ is zero except for t in the spectrum of $\mathbf{V}$. We say that $\mathbf{V}$ is *generic* if $j(t) \leqslant 1$ for all $t \in \mathbb{R}$.

Functorial relations between the barcodes and the persistence modules

To establish the link between the previous objects we consider two functors as follows.

(i) Consider an interval I of the form $(a, b]$ and define $Q_t(I) = \mathbb{Z}/2\mathbb{Z}$, if $t \in I$, and $Q_s(I) = \{0\}$, if $s \notin I$. $Q_s(I)$ is a persistence module, with $i_{s,t}$ equal to id if $s, t \in I$ and 0 otherwise. For a set of intervals $\mathcal{I}$ for each $t \in \mathbb{R}$ we define

$$Q_t(\mathcal{I}) = \bigoplus_{I\in\mathcal{I}} Q_t(I).$$

(ii) We define a functor β from the set of generic persistence modules into the set of barcodes which associate to a generic persistence module $\mathbf{V} = (V_s)_{s\in\mathbb{R}}$ a barcode. We denote $(i_{s,t})_{s\leqslant t}$ the family of morphisms equipped with $\mathbf{V}$. Let us consider the set of t in the spectrum of V such that $\dim(\mathrm{Ker}(i_{t^-,t^+})) = 1$ and label its elements $b_1, ..., b_n$. For each b_j, there exists a unique $a_j \in \mathbb{R}$ with the following property: Let $x \in V_{b_j^-}$ represents a non-zero element in $\mathrm{Ker}(i_{t^-,t^+})$, the element x is in the image of $i_{a_j^+,b_j^-}$ but x is not in the image of $i_{a_j^-,b_j^-}$. We label the remaining elements of the spectrum of $\mathbf{V}$ by $\{c_1, ..., c_m\}$. The barcode $\beta(\mathbf{V})$ consists of the list of intervals: $((a_j, b_j], (c_k, +\infty))$, where $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$.
One may prove that the functor β extends to the set of persistence modules, we refer to [61] for more details.

The following theorem holds.

Theorem 3.0.4. *The functors defined above satisfy the following properties.*

1. $\beta \circ Q = \mathrm{id}_{\mathrm{Barcode}}$.

2. *β and Q are isometries for the interleaving distance (see next definition).*

We define the interleaving distance.

Definition 3.0.5. *Let $\boldsymbol{V} = (V_s)_{s\in\mathbb{R}}$ and $\boldsymbol{W} = (W_s)_{s\in\mathbb{R}}$ be two persistence modules, the pseudo-distance $d_{int}(\mathbf{V},\mathbf{W})$, called the interleaving distance, is defined as the infimum of the set of ϵ such that there are morphisms $\phi_s : V_s \to W_{s+\epsilon}$ and $\psi_s : W_s \to V_{s+\epsilon}$ "compatible" with the $i_{s,t}, j_{s,t}$ in the following sense:*

$$\begin{tikzcd}
V_{s-\epsilon} \arrow[r, "\phi_{s-\epsilon}"] \arrow[d, "i_{s-\epsilon,t-\epsilon}"'] & W_s \arrow[r, "\psi_s"] \arrow[d, "j_{s,t}"] & V_{s+\epsilon} \arrow[r, "\phi_{s+\epsilon}"] \arrow[d, "i_{s+\epsilon,t+\epsilon}"] & W_{s+2\epsilon} \arrow[d, "i_{s+2\epsilon,t+2\epsilon}"] \\
V_{t-\epsilon} \arrow[r, "\phi_{t-\epsilon}"] & W_t \arrow[r, "\psi_t"] & V_{t+\epsilon} \arrow[r, "\phi_{t+\epsilon}"] & W_{t+2\epsilon}
\end{tikzcd}$$

where $\psi_s \circ \phi_{s-\epsilon} = i_{s-\epsilon,t+\epsilon}$ and $\phi_{s+\epsilon} \circ \psi_s = j_{s,s+2\epsilon}$ s.t. the diagrams commute for all $s \leqslant t$.

The Morse example. To give a good idea of what a barcode is, we describe the case of a Morse function. Let Σ be a compact surface and $H : \Sigma \to \mathbb{R}$ a Morse function. The filtered Morse homology $(H_*(\{H < t\}))_{t\in\mathbb{R}}$ is a persistence module where the set $(i_{s,t})_{s\leqslant t}$ is given by the inclusions $i_{s,t} : H_*(\{H < s\}) \to H_*(\{H < t\})$. The Figure 3.1 give an example of the barcode of such a Morse function on the sphere.
In Figure 3.1 we consider the height function H on the 2-sphere which is a Morse function. In this case, H admits six critical points: two sinks p_1, p_2, two saddle points x_1, x_2, and two sources s_1 and s_2. If we compute the filtered homology of F we obtain the bars which are described on the right of the vertical axis. Notice that there are two semi-infinite bars, one which starts at $H(p_1)$ and the other one which start at $H(s_1)$ and there are two finite bars $(H(p_2), H(x_1)]$ and $(H(x_2), H(s_2)]$ as follows.

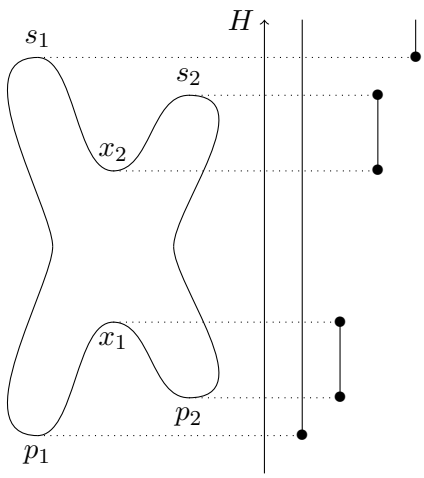


Figure 3.1

Chapter 4

The simplest case of barcode for Hamiltonian homeomorphisms

In this chapter, we consider a Hamiltonian homeomorphism f on a closed surface Σ which satisfies the following assumptions.

1. The set of fixed points is finite and is unlinked, in particular every fixed point is contractible.
2. The fixed points have distinct action values.
3. For every $x \in \mathrm{Fix}(f)$, $\mathrm{ind}(f, x)$ is either 1 or -1.

Let $I = (f_t)_{t\in[0,1]}$ be a maximal isotopy from identity to f. Let $\mathcal{F} \in \mathcal{F}(I)$ be a positively transverse foliation associated to I which satisfies the following "generic" assumptions.

1. There is no leaf joining two saddles points.
2. For every saddle point $x \in \mathrm{Fix}(f)$, there are exactly two unstable cones composed of one leaf whose alpha-limit point is x and two stable cones composed of one leaf whose omega-limit point is x.

The set of those foliations will be denoted $\mathcal{F}_{\mathrm{gen}}(I)$. An important fact is that, for every saddle point x of a foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$, the dynamic of $\mathcal{F}$ in a neighborhood of x is locally homeomorphic to a foliation as in the following figure.

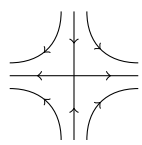


Figure 4.1

We denote A_f the action functional of f defined on $\mathrm{Fix}(f)$.

We have the following result from [49].

Lemma 4.0.1. *For every $x \in \mathrm{Fix}(f)$ and every $\mathcal{F} \in \mathcal{F}(I)$ we have $\mathrm{ind}(f,x) = \mathrm{ind}(\mathcal{F},x)$. So if $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$ then we have*

- $\mathrm{ind}(f,x) = 1$ *if x is a source or a sink of $\mathcal{F}$.*
- $\mathrm{ind}(f,x) = -1$ *if x is a saddle pont of $\mathcal{F}$.*

For the remainder of the section, we consider a foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(f)$. Recall that the foliation $\mathcal{F}$ is gradient-like and we will use the analogy between Morse Theory and gradient-like foliations to construct a filtered homology from the foliation $\mathcal{F}$. We define a graph associated to the foliation $\mathcal{F}$ and we associate to this graph a chain complex in order to compute its filtered homology and obtain a persistence module.

Remember that for a fixed point x of f, being a sink or a source of the foliation $\mathcal{F}$ does not depend on the choice of $\mathcal{F} \in \mathcal{F}(I)$. We define the index $\mathrm{ind}_{CZ}(f,\cdot)$ on the set of fixed points of f as follows. For $x \in \mathrm{Fix}(f)$ we set

- $\mathrm{ind}_{CZ}(f,x) = 0$ if x is a sink of $\mathcal{F}$,
- $\mathrm{ind}_{CZ}(f,x) = 1$ if x is a saddle point of $\mathcal{F}$,
- $\mathrm{ind}_{CZ}(f,x) = 2$ if x is a source of $\mathcal{F}$.

The notation ind_{CZ} of the index function refers to the Conley-Zehnder index function as they are equals under these assumptions.

Definition 4.0.2. *Let $G(\mathcal{F})$ be the graph whose set of vertices is the set $\mathrm{Fix}(f)$ and whose set of edges corresponds to the set of leaves ϕ of $\mathcal{F}$ such that $\mathrm{ind}_{CZ}(f,\alpha(\phi)) = \mathrm{ind}_{CZ}(f,\omega(\phi)) - 1$.*

For $i \in \mathbb{N}$ we consider the set $\mathrm{Fix}_i(f)$ of fixed points $x \in \mathrm{Fix}(f)$ which satisfy $\mathrm{ind}_{CZ}(f,x) = i$. Note that $\mathrm{Fix}_i(f) = \varnothing$ if $i \geqslant 3$. We define a chain complex associated to the graph $G(\mathcal{F})$ following the ideas from Morse homology .

Definition 4.0.3. *For $t \in \mathbb{R}$ and $i \in \mathbb{N}$, we define the chain complex*

$$C_i^t = \bigoplus_{\substack{z \in \mathrm{Fix}_i(f) \\ A_f(z) < t}} \mathbb{Z}/2\mathbb{Z} \cdot z,$$

and the maps $\partial_i^t : C_i^t \to C_{i-1}^t$ such that for every $z \in C_i^t$

$$\partial_i^t(z) = \sum_{b \in \mathrm{Fix}_{i-1}(f)} n(z,b)b,$$

where $n(z,b)$ is the number modulo 2 of edges from z to b in $G(\mathcal{F})$. If i is distinct from 1 or 2 then ∂_i^t is equal to 0 for every $t \in \mathbb{R}$.

Remark 4.0.4. For a fixed point $z \in \mathrm{Fix}(f)$, if there exists an edge from z to b in $G(\mathcal{F})$ then by Proposition 2.3.9 we have $A_f(z) > A_f(b)$ and for every $t > A_f(z)$, the element $\partial_i^t(z)$ belongs to C_{i-1}^t. So the map ∂_i^t is well defined.

We obtain that, for every $t \in \mathbb{R}$, (C_i^t, ∂_i^t) is a chain complex thanks to the following property.

Proposition 4.0.5. *For each $t \in \mathbb{R}$ and every $i \in \mathbb{N}$ the maps ∂_i^t satisfy $\partial_i^t \circ \partial_{i+1}^t = 0$.*

We prove Proposition 4.0.5 after the following definition and lemma.

Definition 4.0.6. *Let $x \in \mathrm{Fix}(f)$ be a source of the foliation $\mathcal{F}$, the subset $\bigcup\{\phi \in \mathcal{F} \mid \alpha(\phi) = x\} \cup \{x\}$ of Σ will be called the repulsive basin of x and denoted $W^u(x)$.*

For a source x of $\mathcal{F}$, we want to describe $W^u(x)$. Let $P_n \subset \mathbb{C}$ be the filled regular polygon of vertices $\mathrm{e}^{\frac{i\pi k}{n}}$, with $k \in \{0, ..., 2n-1\}$. We have the following lemma.

Lemma 4.0.7. *Let $x \in \mathrm{Fix}(f)$ be a source of the foliation $\mathcal{F}$. There exist $n \geqslant 1$ and a continuous map $d : P_n \to \Sigma$ such that*

- *$d(\mathrm{int}(P_n))$ is the repulsive basin of x.*
- *$d(\mathrm{e}^{\frac{i\pi k}{2n}})$ is a sink of $\mathcal{F}$ if k is even and a saddle point of $\mathcal{F}$ if k is odd.*
- *The image of a side of P_n is the closure of a leaf of $\mathcal{F}$.*

Proof of Lemma 4.0.7. Let us consider a source $x \in \mathrm{Fix}(f)$ of $\mathcal{F}$. There exists a homeomorphism $h : \mathbb{D} \to W^u(x)$ such that $h(0) = x$ and such that the leaves from x are the images by h of the segments $t\mathrm{e}^{i\theta}$, $t \in [0,1]$. For $\theta \in [0, 2\pi)$ we will denote ϕ_θ the image by h of the segment $t\mathrm{e}^{i\theta}$, $t \in [0,1]$.

There are a finite number of angles $(\theta_k)_{k \in \mathbb{Z}/n\mathbb{Z}}$ such that the omega-limit point of ϕ_{θ_k} is a saddle point x_k of $\mathcal{F}$.

Moreover the *attractive basin* of a sink x of a foliation is the union of x and the leaves whose omega-limit point is equal to x. The attractive basin of a sink is an open set. So, by connectedness, for every $k \in \mathbb{Z}/n\mathbb{Z}$ there exists a sink of $\mathcal{F}$, denoted s_k, such that for every leaf ϕ_θ of angle $\theta \in (\theta_k, \theta_{k+1})$, the omega-limit point of ϕ_θ is equal to s_k. We denote U_k the union of the leaves ϕ_θ, with $\theta \in (\theta_k, \theta_{k+1})$. We draw an example of such a set in Figure 4.2. The set U_k is a topological open disk on Σ whose boundary is the closure of four distinct leaves of $\mathcal{F}$: the leaves ϕ_{θ_k} and $\phi_{\theta_{k+1}}$, a leaf ψ_k from x_k to s_k and a leaf ϕ_k from x_{k+1} to s_k. The existence of the leaves ϕ_k and ψ_k is deduced from the dynamic of the foliation near the saddle points x_k and x_{k+1} described in Figure 4.1.

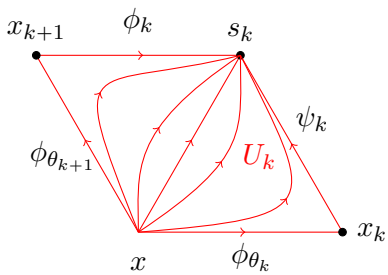


Figure 4.2: Example of a set U_k

We obtain that the repulsive basin of x is equal to the union $[\phi_{\theta_0} \cup U_0 \cup \phi_{\theta_1} \cup U_1 \cup ... \cup U_{n-1}] \cup \{x\}$. We define the map $d : P_n \to \Sigma$ given by Lemma 4.2 as follows. For every $k \in \{0, ..., n-1\}$ we set

- $d(\mathrm{e}^{\frac{i2k\pi}{2n}}) = x_k$,
- $d(\mathrm{e}^{\frac{i(2k+1)\pi}{2n}}) = s_k$.

The map d naturally extends to ∂P_n by sending the edges of the boundary of ∂P_n alternatively, in cyclic order, to the leaves ϕ_k, $k \in \mathbb{Z}/n\mathbb{Z}$ and the leaves ψ_k, $k \in \mathbb{Z}/n\mathbb{Z}$. Finally, the map d extends naturally on the interior of P_n as follows.

For $k \in \mathbb{Z}/n\mathbb{Z}$, we consider the slice S_k of the polygon P_n defined as the set of points of P_n whose angle θ in polar coordinates satisfies $\theta \in [\frac{ik\pi}{n}, \frac{i(k+1)\pi}{n}]$. We extend d by sending the slice S_k, $k \in \mathbb{Z}/n\mathbb{Z}$, of P_n on the closure of the set U_k defined previously. The map d is well-defined and continuous. Notice that d may not be injective (we give an example at the end of the proof).

Let us draw a repulsive basin of a source x of the foliation $\mathcal{F}$ in Figure 4.3. We represent the leaves of U_0 and its boundary in red in Figure 4.3.

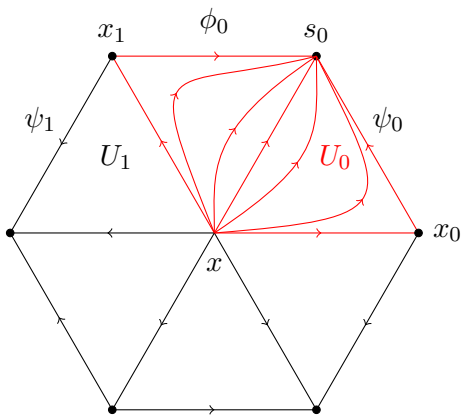


Figure 4.3: The repulsive basin of a source x

□

Example. We give an example of a source x of a foliation such that the map d provided by Lemma 4.0.7 is not injective. We consider the foliation $\mathcal{F}$ on the 2-sphere as in Figure 4.4, the repulsive basin of the source x of $\mathcal{F}$ is composed of all leaves of $\mathcal{F}$ except the two leaves ϕ_1 and ϕ_2 whose alpha-limit points are equals to the saddle point y of $\mathcal{F}$. The boundary of the repulsive basin of x is equal to the union $\phi_1 \cup \{y\} \cup \phi_2$ and is represented in blue on the Figure.

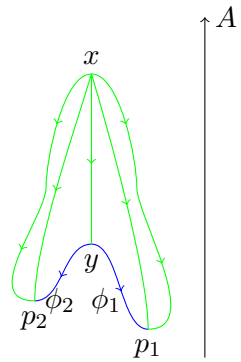


Figure 4.4

The polygon P provided by Lemma 4.0.7 has four edges which are sent alternatively on ϕ_1 and ϕ_2. We can represent P as in Figure 4.5.

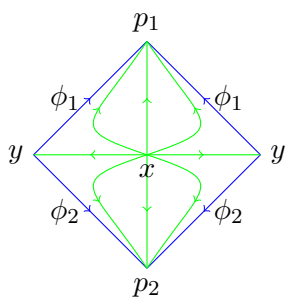


Figure 4.5

Now, we can give the proof of proposition 4.0.5.

Proof of proposition 4.0.5. We consider $t \in \mathbb{R}$ since ∂_i^t is 0 for i distinct from 1 or 2, it is enough to prove that for every source $s \in \mathrm{Fix}(f)$ we have $\partial_1^t \circ \partial_2^t(s) = 0$.
Let $x \in \mathrm{Fix}(f)$ be a source of the foliation $\mathcal{F}$. Using the same notations of the proof of Lemma 4.0.7, there exists an integer $n > 0$ and n leaves which were denoted $(\phi_{\theta_k})_{k \in \mathbb{Z}/n\mathbb{Z}}$ whose omega-limit points, denoted $(x_k)_{k \in \mathbb{Z}/n\mathbb{Z}}$ are exactly the saddle points of the foliation $\mathcal{F}$ which are connected to x. So we have $\partial_2^t(s) = \sum_{k=0}^{n-1} \omega(\phi_{\theta_k})$.

Moreover, for every $k \in \mathbb{Z}/n\mathbb{Z}$ the leaves ϕ_{k-1} and ψ_k of the proof of Lemma 4.0.7 are exactly the leaves of the foliation $\mathcal{F}$ whose alpha-limit point is x_k. So we can compute

$$
\begin{aligned}
\partial_1^t \circ \partial_2^t(s) &= \partial_1^t(\sum_{k=0}^{n-1} \omega(\phi_{\theta_k})), \\
&= \sum_{k=0}^{n-1} \partial_1^t(\omega(\phi_{\theta_k})), \\
&= \sum_{k=0}^{n-1} \left(\omega(\phi_{k-1}) + \omega(\psi_k)\right), \\
&= \sum_{k=0}^{n-1} \omega(\phi_k) + \omega(\psi_k), \\
&= \sum_{k=0}^{n-1} 2s_k, \\
&= 0.
\end{aligned}
$$

Hence we obtain the result of Proposition 4.0.5. □

Definition 4.0.8. *The image of the persistence module $H_*((C_i^t, \partial_i^t)_{i,t})$ under the functor β is called the barcode of f for the foliation $\mathcal{F}$ and we will denote it $B_{\text{gen}}(f, \mathcal{F})$.*

Remark 4.0.9. For a foliation $\mathcal{F} \in \mathcal{F}_{\text{gen}}(I)$, each value b of the action function A_f is the end of a unique bar of the barcode $B_{\text{gen}}(f, \mathcal{F})$.

Remark 4.0.10 (Similarities with the Morse example). Let us consider a Morse function H on the 2-sphere as in Figure 4.6. We suppose that H induces a Hamiltonian diffeomorphism f whose set of fixed points is equal to the set of critical points of H. In particular, $\text{Fix}(f)$ is unlinked. If we consider a Riemannian structure on Σ, the gradient flow of H induced by the Riemannian metric defines a foliation $\mathcal{F}$ positively transverse to the natural isotopy induced by the Hamiltonian function H. Moreover the action function A_f is given by $A_f(x) = H(x)$ for every $x \in \text{Sing}(\mathcal{F})$.
In this example f has six fixed points, two sinks p_1, p_2 two saddle points x_1, x_1 and two sources s_1, s_2. We draw the graph $G(\mathcal{F})$ on the left side of the figure and the barcode $B_{\text{gen}}(f, \mathcal{F})$, as intervals of $\mathbb{R}$, on the right side.

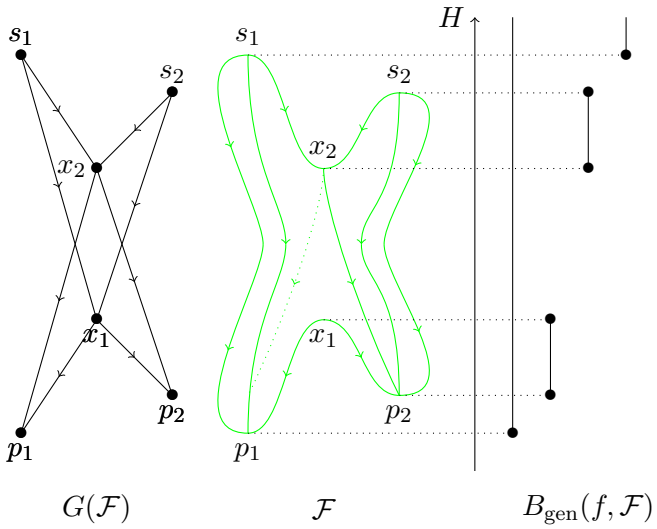


Figure 4.6

In this example the barcode $B_{\text{gen}}(f, \mathcal{F})$ is equal to the filtered Morse homology of the function H.

This is a general phenomenon. Indeed, we will prove in section 8 the following two results.

Proposition 4.0.11. *The barcode $B_{\text{gen}}(f, \mathcal{F})$ defined for a foliation $\mathcal{F}$ does not depend on the choice of $\mathcal{F} \in \mathcal{F}_{\text{gen}}(f)$.*

Hence we can denote $B_{\text{gen}}(f) = B_{\text{gen}}(f, \mathcal{F})$ for any choice of $\mathcal{F} \in \mathcal{F}_{\text{gen}}(f)$. With this notation, we have the following theorem.

Theorem 4.0.12. *If we consider a Hamiltonian diffeomorphism f with a finite number of fixed points which is C^2-close to the identity and generated by an autonomous Hamiltonian function then the barcode $B_{\text{gen}}(f)$ is equal to the Floer homology barcode of f.*

We would like to prove in a near future the more general result.

Question 4.0.13. *Is the result of Theorem 4.0.12 holds if only consider a Hamiltonian homeormorphism f whose set of fixed points is finite and unlinked?*

Chapter 5

First step into the non generic case, construction of the map $\mathcal{B}$

We give an algorithmic way to determine the barcode of certain type of finite graphs. We consider the set $\mathcal{G}$ of elements (G, A, ind) such that G is a finite oriented and connected graph equipped with a function, called *action function*, $A : V \to \mathbb{R}$ decreasing along the edges and a map $\mathrm{ind} : V \to \mathbb{Z}$ where V is the set of vertices of G. We construct a map

$$\mathcal{B} : \mathcal{G} \to \text{Barcode}.$$

For an element $(G, A, \mathrm{ind}) \in \mathcal{G}$, and a vertex x of G, we will say that x is a *sink* (resp. a *source*) of the graph G if there is no edge which begins with x (resp. if there is no edge which ends with x). For any other vertex x of G, we will say that x is a *saddle point* of the graph G.

For an element $(G, A, \mathrm{ind}) \in \mathcal{G}$, we could suppose that for a vertex x of G $\mathrm{ind}(x)$ is non positive if x is a saddle point of G and $\mathrm{ind}(x)$ is equal to 1 if x is a sink or a source of G as it will always be the case in our future applications. Howeover, we do not need to make these assumptions to construct the map $\mathcal{B}$.

Definition 5.0.1. *Let us consider an element $(G, A, \mathrm{ind}) \in \mathcal{G}$. For a subgraph G' of G we define*

$$L(G') = \min\{A(x) | x \in V \cap G'\},$$
$$D(G') = \max\{A(x) | x \in V \cap G'\}.$$

Let us consider an element $(G, A, \mathrm{ind}) \in \mathcal{G}$ and let us denote by V the set of vertices of G. For $t \in \mathbb{R}$, we define two subgraphs G_t^- and G_t^+ as follows.

Definition 5.0.2. *For $t \in \mathbb{R}$ we denote by G_t^- the maximal subgraph of G whose set of vertices is $V \cap A^{-1}((-\infty, t))$.*
Symmetrically, for $t \in \mathbb{R}$ we denote by G_t^+ the maximal subgraph of G whose set of vertices is $V \cap A^{-1}((t, +\infty))$.

Let us consider $t \in \mathbb{R}$ such that there exists $x \in V$ satisfying $A(x) = t$. Since V is finite, we can define the graphs $G_{t^+}^- = G_{t+\epsilon}^-$ and $G_{t^-}^+ = G_{t-\epsilon}^+$ where $\epsilon > 0$ satisfies $\Im(A) \cap ((t - \epsilon, t + \epsilon)) = \{t\}$.

Definition 5.0.3. *Let us considet $t \in \mathbb{R}$.*
We denote by $\mathcal{C}_t^-$ (resp. $\mathcal{C}_t^+$) the set of connected components of G_t^- (resp. of connected components of G_t^+).
We denote by $\mathcal{C}_{t^+}^-$ the set of connected components of $G_{t^+}^-$ and we denote by $\mathcal{C}_{t^-}^+$ the set of connected components of $G_{t^-}^+$.

Definition 5.0.4. *The inclusions $G_t^- \subset G_{t^+}^-$ and $G_t^+ \subset G_{t^-}^+$ induce natural maps $j_t : \mathcal{C}_t^- \to \mathcal{C}_{t^+}^-$ and $j'_t : \mathcal{C}_t^+ \to \mathcal{C}_{t^-}^+$ where for $C \in \mathcal{C}_t^-$, $j_t(C)$ is the connected component of $G_{t^+}^-$ which contains C and for $C' \in \mathcal{C}_t^+$, $j'_t(C')$ is the connected component of $G_{t^-}^+$ which contains C'.*

Now, we can give the definition of the map $\mathcal{B}$. Given an element (G, A, ind) of $\mathcal{G}$ we describe the bars of $\mathcal{B}(G, A, \mathrm{ind})$.

The map $\mathcal{B}$. The barcode $\mathcal{B}(G, A, \mathrm{ind})$ is composed of the bars of the following four categories.

Category 0. The bars $(L(G), +\infty)$ and $(D(G), +\infty)$ are bars of $\mathcal{B}(G, A, \mathrm{ind})$.

For every $t \in \Im(A)$ there are three categories of bars *Category 1.* For each element C of $\mathcal{C}_{t^+}^-$ such that $j_t^{-1}(C)$ is not empty, the barcode $\mathcal{B}(G, A, \mathrm{ind})$ contains $\#j_t^{-1}(C) - 1$ bars as follows.
We label $C_1, ..., C_n$ the elements of $j_t^{-1}(C)$ and we choose $i_0 \in [1, n]$ an integer such that $L(C_{i_0}) = \min_{i \in [1,n]} L(C_i)$.
The bars of category 1 associated to t are the bars $(L(C_i), t]$ for $i \neq i_0$.

Category 2. For each element C' of $\mathcal{C}_{t^-}^+$ such that $j_t'^{-1}(C')$ is not empty, the barcode $\mathcal{B}(G, A, \mathrm{ind})$ contains $\#j_t'^{-1}(C') - 1$ bars as follows.
We label $C'_1, ..., C'_n$ the elements of $j_t'^{-1}(C')$ and we choose $i_0 \in [1, n]$ an integer such that $D(C'_{i_0}) = \max_{i \in [1,n]} D(C'_i)$.
The bars of category 2 associated to t are the bars $(t, D(C'_i)]$ for $i \neq i_0$.

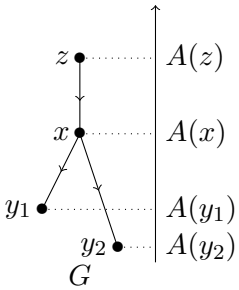
Category 3. We define $k = \sum\{|\mathrm{ind}(x)| \mid x \text{ saddle point}, \ A(x) = t\}$. Let us denote k' equal to k minus the number of bars of categories 1 and 2 associated to t. If $k' > 0$ then the bars of category 3 associated to t are k' bars $(t, +\infty)$ and if $k' \leqslant 0$ there is no bar of category 3 associated to t.

Remark 5.0.5. We refer to Proposition 6.2.4 to enlight the definition of the bars of category 3.

Remark 5.0.6. By construction for every bar $I = (a, b]$ or $J = (c, \infty)$ in the barcode $\mathcal{B}(G, A, \mathrm{ind})$ we have that a, b and c are values of the action function A.

Examples. We compute the barcode of two simple examples.

Example 1. We consider $(G, A, \mathrm{ind}) \in \mathcal{G}$ as follows.



The map ind satisfies $\mathrm{ind}(x) = -1$ and $\mathrm{ind}(y_1) = \mathrm{ind}(y_2) = \mathrm{ind}(z) = 1$. The values of the map A are represented on the vertical line on the right of the graph.

The bars of category 0 are $(A(y_2), +\infty)$ and $(A(z), +\infty)$.

The vertex x is the unique saddle point of the graph G. We describe the bars associated to $A(x)$ as follows.

The subgraph $G^-_{A(x)^+}$ has only one connected component $\mathcal{C}^-$ and $j^{-1}_{A(x)}(\{\mathcal{C}^-\}) = G^-_{A(x)^-}$ has two connected components $\mathcal{C} = \{y_1\}$ and $\mathcal{C}' = \{y_2\}$. In this example we have $L(\mathcal{C}) = A(y_1) > L(\mathcal{C}') = A(y_2)$ so by construction the bar $(A(y_1), A(x)]$ is the only bar of category 1 of the barcode $\mathcal{B}(G, A, \mathrm{ind})$.

The subgraph that $G^+_{A(x)^-}$ has only one connected component $\mathcal{C}^+$ and $j'^{-1}_{A(x)}(\{\mathcal{C}^+\}) = G^+_{A(x)^+}$ has one connected component $\mathcal{C} = \{z\}$. So by construction there is no bar of category 2 in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.

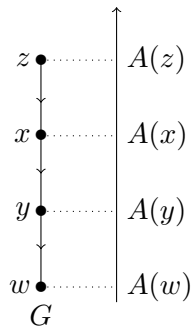
The index of x is equal to -1 and there is one bar of category 1 and zero bar of category 2 thus there is no bars of category 3 in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.

Finally we obtain the barcode

$$\mathcal{B}(G, A, \mathrm{ind}) = \{(A(y_2), +\infty),\ (A(y_1), A(x)],\ (A(z), +\infty)\}.$$

This example corresponds to the barcode of the example Figure 4.4 where the graph corresponds to the connexions of the gradient of the Morse function. See next Chapter 6 for a more precise definition.

Example 2. We consider $(G, A, \mathrm{ind}) \in \mathcal{G}$ as follows.



The map ind satisfies $\mathrm{ind}(x) = \mathrm{ind}(y) = -1$ and $\mathrm{ind}(z) = \mathrm{ind}(w) = 1$. The values of the map A are represented on the vertical line on the right of the graph.

The bars of category 0 are the bars $(A(w), +\infty)$ and $(A(z), +\infty)$.

The vertex x and y are the saddle points of the graph G. We describe the bars associated to $A(x)$ and $A(y)$ as follows.

First we compute the bars associated to $A(y)$. The subgraph $G^-_{A(y)^+}$ has only one connected component $\mathcal{C}^-_y$ and $j^{-1}_{A(y)}(\{\mathcal{C}^-_y\}) = G^-_{A(y)^-}$ has one connected component equal to $\{w\}$. By construction there is no bar of category 1 associated to the saddle point y in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.
In this example $G^+_{A(y)^-}$ has only one connected component ${\mathcal{C}'}^+_y$ and $j'^{-1}_{A(y)}(\{{\mathcal{C}'}^+_y\}) = G^+_{A(y)^+}$ has one connected component which contains the vertices z and x and one edge. Again, by construction there is no bar of category 2 associated to y in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.
The index of y is equal to -1 and there is no bar of category 1 and 2 thus there is a bar $(A(y), +\infty)$ of category 3 in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.

Secondly we compute the bars associated to $A(x)$. The subgraph $G^-_{A(x)^+}$ has only one connected component $\mathcal{C}^-_x$ and $j^{-1}_{A(x)}(\{\mathcal{C}^-_x\}) = G^-_{A(x)^-}$ has one connected component which contains y and w. By construction there is no bar of category 1 associated to the saddle point y in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.
The subgraph $G^+_{A(x)^-}$ has only one connected component ${\mathcal{C}'}^+_x$ and $j'^{-1}_{A(x)}(\{{\mathcal{C}'}^+_x\}) = G^+_{A(x)^+}$ has one connected component equal to $\{z\}$. So by construction there is no bar of category 2 associated to x in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.
The index of x is equal to -1 and there is no bar of category 1 and 2 thus there is a bar $(A(x), +\infty)$ of category 3 in the barcode $\mathcal{B}(G, A, \mathrm{ind})$.

Finally we obtain

$$\mathcal{B}(G, A, \mathrm{ind}) = \{(A(w), +\infty),\ (A(y), +\infty),\ (A(x), +\infty),\ (A(z), +\infty)\}.$$

This example corresponds to the barcode associated to a Morse function on the 2-torus where the graph is given by the connexions of the gradient lines.

Chapter 6

The barcode of a gradient-like foliation

Let us consider a gradient-like foliation $\mathcal{F}$, whose set of singularities X is finite, defined on the complement of X in a compact surface Σ. Recall that a gradient-like foliation is a foliation such that every leaf is a connexion and where there is no cycle of connexions, see section 2.3.7 of the preliminaries for more details. In particular, the singularities of $\mathcal{F}$ are isolated and are classified in three categories: the sinks, the sources and the saddle points. We suppose that the set of singularities X of $\mathcal{F}$ is equipped with an action function $A : X \to \mathbb{R}$ such that for each leaf ϕ we have $A(\alpha(\phi)) > A(\omega(\phi))$.

We will consider the oriented graph $G(\mathcal{F})$ of the foliation $\mathcal{F}$ whose set of vertices is X and for every couple of vertices x and y of $G(\mathcal{F})$ there exists an edge from x to y if and only if there exists a leaf ϕ in $\mathcal{F}$ such that $\alpha(\phi) = x$ and $\omega(\phi) = y$. We want to study the barcode $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$ associated to $\mathcal{F}$ defined in Chapter 5.

Notice that the graph $G(\mathcal{F})$ is not constructed as the graph of a generic foliation as in chapter 4 but it remains a finite oriented graph. The differences will be enlightened in section 8.

In a first section we give some geometrical properties of the foliation $\mathcal{F}$ and in a second section we prove some results about the barcode $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$.

6.1 Geometric properties of a gradient-like foliation

We introduce some useful definitions and notation.

Saturated set. A subset of $\Sigma \backslash X$ is said to be *saturated* if it is equal to a union of leaves of $\mathcal{F}$. We will use the fact that the closure in $\Sigma \backslash X$ of a saturated set is saturated.

Chain of connexions. A *chain of connexions* in Σ is a finite union of the closure of leaves $\psi_1, ..., \psi_k$ of $\mathcal{F}$ such that $\alpha(\psi_1) = x$, $\omega(\psi_i) = \alpha(\psi_{i+1})$ for every $i \in [1, k-1]$ and $\omega(\psi_k) = y$. We will say that a chain of connexions is associated to the leaves $\psi_1, ..., \psi_k$. If we consider two singularities x and y of $\mathcal{F}$, we say that there is a chain of connexions from x to y if there exists a chain of connexions, associated to leaves $\psi_1, ..., \psi_k$ such that $\alpha(\phi_1) = x$ and $\omega(\phi_k) = y$. In this case, x will be called the *starting point* of the chain and

y its *ending point.*

Trivialization. Let us consider a saddle point $x \in X$ of $\mathcal{F}$. We denote by Σ_x^- the union of leaves $\phi \in \mathcal{F}$ whose alpha-limit point is equal to x and by Σ_x^+ the union of leaves $\psi \in \mathcal{F}$ whose omega-limit point is equal to x. We will call a *trivialization* of $\mathcal{F}$ at x a couple (h, V) where V is a neighborhood of x such that $X \cap V = \{x\}$ and $h : V \to \mathbb{D}$ is a map that sends the foliation $\mathcal{F}|_V$ to the model foliation described in the appendix of [59] proposition B.5.4. which we now describe. To simplify the notations we set $n = 1 - \mathrm{ind}(\mathcal{F}, x)$. In this model foliation, for each leaf ϕ of $\mathcal{F}$, $\phi \cap V$ is connected and h sends Σ_x^- to n cones centered around the angles $\frac{2\pi(2k)}{2n}$ where $0 \leqslant k \leqslant n-1$ and sends Σ_x^+ to n cones centered around the angles $\frac{2\pi(2k+1)}{2n}$ where $0 \leqslant k \leqslant n-1$. We have

- The two sets Σ_x^- and Σ_x^+ are composed of n connected components. Using the map h we can label these connected components $(\sigma_k^-)_{0 \leqslant k \leqslant n-1}$ and $(\sigma_k^+)_{0 \leqslant k \leqslant n-1}$ in cyclic order around x.
- The connected components $(\sigma_k^-)_{0 \leqslant k \leqslant n-1}$ will be called the *unstable cones* of x and the connected components $(\sigma_k^+)_{0 \leqslant k \leqslant n-1}$ will be called the *stable cones* of x.
- Every stable cone and an unstable cone which are consecutive, in cyclic order, are separated by *hyperbolic sectors* of x. We denote U_k, $0 \leqslant k \leqslant 2n-1$ the hyperbolic sectors such that we have in cyclic order $V = \sigma_0^- \cup U_0 \cup \sigma_0^+ \cup U_1 \cup ... \cup U_{2n-1}$.

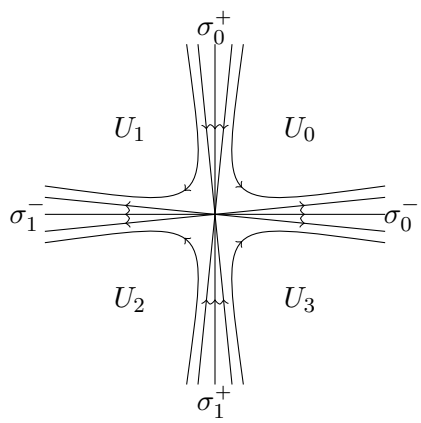


Figure 6.1: An example of a trivialization of a saddle point x of index -1

Notice that a stable or unstable cone of x can be composed of a unique leaf, see Figure 2.5 for example.

Local model. Let us consider a leaf ϕ of $\mathcal{F}$. We describe in Σ the leaves of $\mathcal{F}$ which are close to ϕ. It will be called the *local model* near ϕ. We parametrize the leaves of $\mathcal{F}$ near ϕ by a small arc $\gamma : (-1, 1) \to \Sigma \backslash X$ transversal to $\mathcal{F}$ such that $\gamma(0) \in \phi$. For every $t \in (-1, 1)$ we denote ϕ_t the leaf of $\mathcal{F}$ passing through $\gamma(t)$.

We prove the following property.

Proposition 6.1.1. *We have that the sets*

$$\Gamma_\phi^+ = \bigcap_{\epsilon>0} \overline{\bigcup_{t\in(0,\epsilon)} \phi_t}, \text{ and } \Gamma_\phi^- = \bigcap_{\epsilon>0} \overline{\bigcup_{t\in(-\epsilon,0)} \phi_t},$$

are chains of connexions containing the leaf ϕ and there exists $s \in (0,1)$ such that

- *Every leaf ϕ_t, $t \in (0,s)$ satisfies: $\alpha(\phi_t)$ is equal to the start of Γ_ϕ^+ and $\omega(\phi_t)$ is equal to its end.*
- *Every leaf ϕ_t, $t \in (-s,0)$ satisfies: $\alpha(\phi_t)$ is equal to the start of Γ_ϕ^- and $\omega(\phi_t)$ is equal to its end.*

Proof. We prove the result for the leaves ϕ_t, $t > 0$ as it is the same proof for the leaves ϕ_t, $t < 0$.

We start by studying the "future" of the connexions $(\phi_t)_{t\in(0,1)}$ and then, symmetrically, we study the "past" of these connexions.

We will prove that there exists a chain of connexions contained in Γ_ϕ^+ passing through ϕ from x to a singularity y^+ such that for every $t > 0$ small enough we have $\omega(\phi_t) = y$. The omega-limit point x_1 of ϕ is a sink of $\mathcal{F}$ or a saddle point. If x_1 is a saddle point then ϕ is in the interior of a stable cone of x_1 or in its boundary. We split the discussion into three cases.

Case 1. Suppose that x_1 is a sink of $\mathcal{F}$, since the set of leaves of which x_1 is the ending point is open, there exists $t_1 \in (0,1]$ such that for each $t \in [0,t_1)$, $\omega(\phi_t)$ is equal to x_1 and the chain of connexions we are looking for is associated to the leaf ϕ.

Case 2. We suppose that x_1 is a saddle point and the leaf ϕ is in the interior of a stable cone σ^+ of x_1. There exists $t_1 \in (0,1]$ such that for each $t \in [0,t_1)$, the leaf ϕ_t satisfies $\omega(\phi_t) = x_1$. The chain of connexions we are looking for is associated to the leaf ϕ.

Case 3. We suppose that x_1 is a saddle point and the leaf ϕ is in the boundary of a stable cone σ^+ of x_1. In this case, the leaf ϕ is in the boundary of the hyperbolic sector U of x_1 preceding σ^-. We can consider a trivialisation of $\mathcal{F}$ at x_1 and $t_1 \in (0,1)$ such that each leaf ϕ_t, $t \in (0,t_1]$, is a leaf of the sector U. The closure of the union of the leaves $(\phi_t)_{t\in(0,t_1]}$ contains a leaf ϕ_1 of the unstable cone of x which is adjacent to U. Notice that we have a chain of connexions associated to the leaves ϕ and ϕ_1.

We do the same discussion about the omega limit point of ϕ_1. If the omega limit point of ϕ_1 corresponds to the case 1 or the case 2, then we stop the process and if we are in case 3 then, we do the same discussion with the leaf ϕ_2 provided by case 3. If so, a chain of connexions is associated to the leaves ϕ, ϕ_1, ϕ_2. Since the number of singularities of $\mathcal{F}$ is finite, the process stops after a finite number of steps and we finally obtain a chain of connexions Γ_{future} associated to a finite number of leaves $\phi, \phi_1, ..., \phi_n$. We denote y the omega-limit point of the leave ϕ_n and there exists $t_n \in (0,1)$ such that each leaf ϕ_t with

$t \in (0, t_n]$ satisfies $\omega(\phi_t) = y^+$.

Let us draw an example of a chain of connexions provided by the previous process. In Figure 6.2 the horizontal line represents the chain of connexions from x to $\omega(\phi_n)$ and each line above represents a leaf ϕ_t where $t \in (0, 1]$. It is enough to re-parametrize the trivialization h to obtain the result we are looking for.

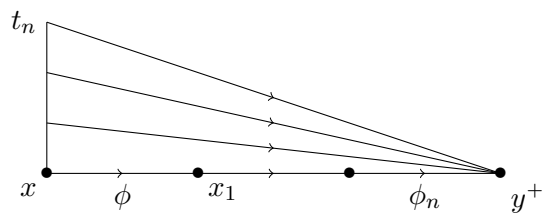


Figure 6.2

By symmetrical arguments there is another chain of connexions Γ_{past} from a singularity z^+ to $\omega(\phi)$, passing through ϕ, such that every leaf ϕ_t, $t > 0$, satisfies $\alpha(\phi_t) = z^+$.

Moreover, by the previous construction, we obtain that Γ_ϕ^+ is a chain of connexions and is equal to the union of Γ_{past} and Γ_{future}. So z^+ is its starting point and y^+ its ending point.

□

Remark 6.1.2. The space of leaves of a gradient-like foliation $\mathcal{F}$ is a non-Hausdorff manifold. The chains of connexions correspond to the set of non separated leaves.

Let us draw two examples of a local model of a leaf ϕ.

We consider a first example in Figure 6.3. The leaves above ϕ represent the leaves ϕ_t with $t \in (0, 1)$ and the leaves below ϕ represent the leaves ϕ_t with $t \in (-1, 0)$. The chain of connexions Γ_ϕ^+ is a chain from z^+ to y^+ passing through ϕ and the chain of connexions Γ_ϕ^- is a chain of connexions from z^- to y^- passing through ϕ.

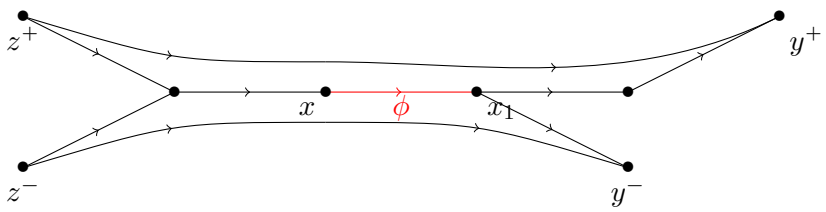


Figure 6.3: First example of the local model of a leaf ϕ

We draw a second example of a leaf ϕ in Figure 6.4. We consider γ in blue and the chains of connexions Γ_ϕ^- and Γ_ϕ^+ are both equal to $\overline{\phi}$.

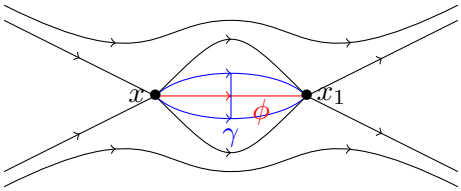


Figure 6.4: Second example of the local model of a leaf ϕ

The graph of the foliation $\mathcal{F}$. We consider the oriented graph $G(\mathcal{F})$ whose set of vertices is equal to $\mathrm{Sing}(\mathcal{F})$ and for every couple of vertices x and y there exists an edge from x to y if and only if there exists a leaf ϕ of $\mathcal{F}$ such that $\alpha(\phi) = x$ and $\omega(\phi) = y$. For $t \in \mathbb{R}$ we consider the subgraphs $G_t^-(\mathcal{F})$ (resp. $G_t^+\mathcal{F})$) which is the maximal subgraph of $G(\mathcal{F})$ whose set of vertices is $X \cap A^{-1}((-\infty, t))$ (resp. $X \cap A^{-1}((t, +\infty))$).

Attractive basin. Let us consider $t \in \mathbb{R}$ and a connected component $\mathcal{C}$ of $G_t^-(\mathcal{F})$. We define the *attractive basin* of $\mathcal{C}$, denoted $W^s(\mathcal{C})$, as the union of the leaves of $\mathcal{F}$ whose omega-limit point is a singularity of $\mathcal{C}$. Notice that it is a subset of Σ.
In particular we have

$$W^s(\mathcal{C}) = \bigcup_{x \in X \cap \mathcal{C}} W^s(x).$$

Lemma 6.1.3. *Let us consider a saddle point x in the frontier of $W^s(\mathcal{C})$. There exists a neighborhood V of x such that each hyperbolic sector U of x in V is either included in $W^s(\mathcal{C})$ or disjoint of it.*

Proof. Let us fix a neighborhood V of x. We consider an unstable cone σ^- of x and U a hyperbolic sector of x in V adjacent to σ^-. We denote ϕ the leaf of $\mathcal{F}$ such that $\phi = \sigma^- \cap \overline{U}$. Let $\gamma : [0, 1) \to U$ be a small arc transverse to the foliation $\mathcal{F}$ and such that $\gamma(0) \in \phi$. For every $t \in [0, 1)$ we denote ϕ_t the leaf of $\mathcal{F}$ passing through $\gamma(t)$.

We denote by y^+ the ending point of $\Gamma_\phi^+ = \bigcap_{\epsilon>0} \overline{\bigcup_{t\in(0,\epsilon)} \phi_t}$.

By the local model there exists $s \in (0, 1)$ such that every leaf ϕ_t, $t \in (0, s)$, satisfies $\omega(\phi_t) = y^+$.

So if y^+ is in $\mathcal{C}$ then every leaf ϕ_t, $t \in (0, s)$, is in $W^s(\mathcal{C})$ and if y^+ is not in $\mathcal{C}$ then no leaf ϕ_t, $t \in (0, s)$, is in $W^s(\mathcal{C})$. Hence, up to a smaller neighborhood V' of x we can suppose that every leaf of the hyperbolic sector U in V' is either in $W^s(\mathcal{C})$ or disjoint of it. Moreover, x has finitely many hyperbolic sectors so we can suppose that we have the same property for each one of them and we obtain the result. □

In the next chapter we will need a precise description of $W^s(\mathcal{C})$. We describe it with the following proposition. Let us consider a stable or unstable cone σ of a saddle point x in the frontier of $W^s(\mathcal{C})$, we say that σ is *adjacent* to $W^s(\mathcal{C})$ if one and only one of the two hyperbolic sectors adjacent to σ is in $W^s(\mathcal{C})$ and we say that σ is *surrounded* by $W^s(\mathcal{C})$ if the two hyperbolic sectors of x which are adjacent to σ is $W^s(\mathcal{C})$.

Proposition 6.1.4. *The set $W^s(\mathcal{C})$, is open, connected and its frontier is the closure of a finite union of leaves of $\mathcal{F}$ contained in stable or unstable cones of saddle points.*
More precisely, for each saddle point x in the frontier of $W^s(\mathcal{C})$, the stable and unstable cones satisfy the following properties.

1. *Let us consider a stable cone σ^+ of x. If σ^+ is surrounded by $W^s(\mathcal{C})$ then the leaves of $\partial\sigma^+$ are the only leaves of σ^+ in the frontier of $W^s(\mathcal{C})$. If σ^+ is adjacent to $W^s(\mathcal{C})$ then the leaf of $\partial\sigma^+ \cap \partial U$, where U is the adjacent hyperbolic sector of x in $W^s(\mathcal{C})$ adjacent to σ^+, is the only leaf of σ^+ in the frontier of $W^s(\mathcal{C})$. If none of the previous situation holds then σ^+ is disjoint from the frontier of $W^s(\mathcal{C})$.*

2. *Let us consider an unstable cone σ^- of x. There is a finite set of leaves of σ^-, possibly empty if σ^- is not adjacent to $W^s(\mathcal{C})$, in the frontier of $W^s(\mathcal{C})$.*

Every property of Proposition 6.1.4 is obvious except the finiteness property which is deduced from Lemma 6.1.11.

The first point of Proposition 6.1.4 is a straigthforward consequence of the definition of $W^s(\mathcal{C})$. Unfortunately, the second point of Proposition 6.1.4 can not be more precise and we draw three examples to illustrate this. After these examples, we prove that $W^s(\mathcal{C})$ is open in Σ and that there are finitely many leaves of $\mathcal{F}$ in its frontier.

First example. Let us consider a sink y of a gradient-like foliation $\mathcal{F}$ such that there exists a saddle point x in the frontier of $W^s(y)$ as in the Figure 6.5. In this example we draw in red the leaves of $W^s(y)$ and in blue the leaves in its frontier whose omega-limit point equals to x. There exists an unstable cone σ^- of x surrounded by $W^s(y)$. The sectors U and U' of x which are adjacent to σ^- are in $W^s(y)$ and by the first point of Proposition 6.1.4 there are two leaves ϕ and ϕ' in the stable cones of x which are adjacent to U and U' and in the frontier of $W^s(y)$. In our example there exists a leaf ψ of the cone σ^- which is in the frontier of $W^s(y)$. It is enough to suppose that $A(x) > A(z) > A(y)$ to obtain this example. In our example we choose to draw ψ in the boundary of the unstable cone σ^- but it could be any leaf of σ^-.

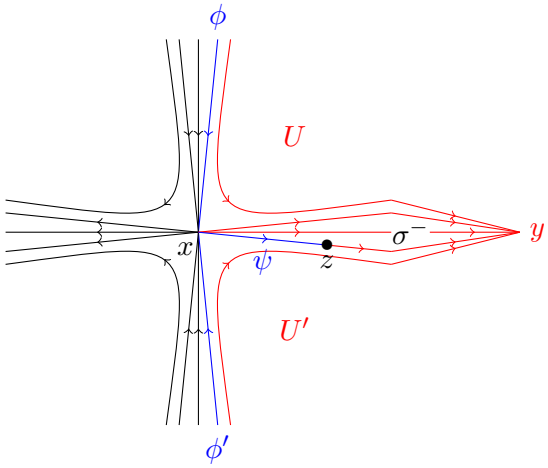


Figure 6.5

Second example. Let us consider a sink y of a gradient-like foliation $\mathcal{F}$ such that there exists a saddle point x in the frontier of $W^s(y)$ as in the Figure 6.6. In this example we draw in red the leaves of $W^s(y)$ and in blue the leaves of its frontier whose alpha-limit point is equal to x. The unstable cone σ^- of x is adjacent to $W^s(y)$. In this example there exist two leaves ψ and ψ' of the unstable cone σ^- from x to two singularities z and z' which are in the frontier of $W^s(y)$.

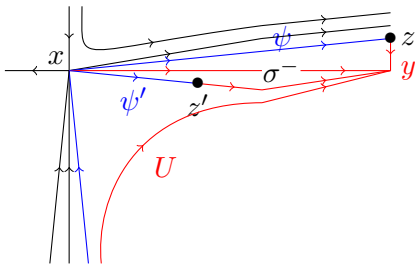


Figure 6.6

Third example. Let us consider a sink y of a gradient foliation $\mathcal{F}$ such that there exists a saddle point x in the frontier of $W^s(y)$ as in Figure 6.7. In this example we draw in red leaves of $W^s(y)$ and in blue the leaves on its frontier. The unstable cone σ^- is not adjacent to $W^s(y)$. In this example there exist leaves in the interior of σ^- whose omega-limit point is y and two leaves ψ of $\partial\sigma^-$ and ψ' in the interior of σ^- are in the frontier of $W^s(y)$ The omega limit-points of ψ and ψ' are two saddle points z and z' of $\mathcal{F}$.

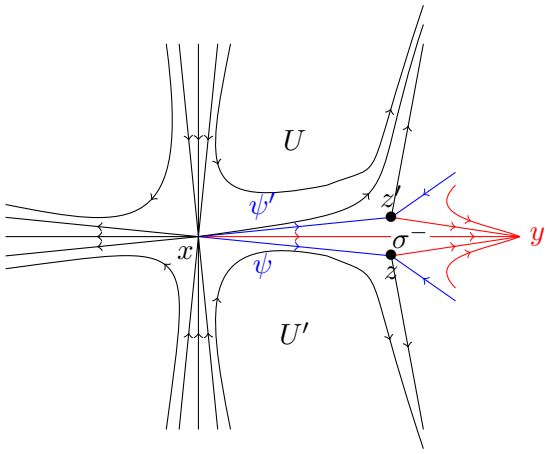


Figure 6.7

The remaining part of this chapter aims to give the proof of Proposition 6.1.4. The proof is divided into several lemmas. Notice that the following lemma will be useful in the next section.

Lemma 6.1.5. *Let us consider a saddle point x of $\mathcal{F}$. For each unstable cone σ^- of x there exists a unique connected component $\mathcal{C}$ of $G^-_{A(x)}(\mathcal{F})$ such that each leaf ϕ of the cone σ^- is in the attractive basin of $\mathcal{C}$.*

Proof. Let us consider an unstable cone σ^- of x and ϕ a leaf of σ^-. We denote by $\mathcal{C}$ the connected component of $G^-_{A(x)}(\mathcal{F})$ which contains the singularity $\omega(\phi)$. By the local model there exist two chains of connexions Γ^+_ϕ and Γ^-_ϕ, whose ending points are denoted y^+ and y^-, containing ϕ such that each leaf ψ of σ^- close enough to ϕ satisfies $\omega(\psi) = y^+$ or $\omega(\psi) = y^-$. We have that Γ^+_ϕ and Γ^-_ϕ contain ϕ, so by definition y^+ and y^- are also in $\mathcal{C}$ and we deduce that every leaf ψ of σ^- close enough to ϕ is in $W^s(\mathcal{C})$.

Moreover, if we consider a leaf $\phi' \in \sigma^-$ in the closure of a sequence $(\phi'_n)_{n\in\mathbb{N}}$ of leaves in $W^s(\mathcal{C})$ then by the local model, there is a chain of connexions Γ in the closure of the sequence $(\phi'_n)_{n\in\mathbb{N}}$ which contains ϕ'. Notice that $A(\omega(\phi')) < A(x)$ so $\omega(\phi')$ and the ending point of the chain Γ are in the same connected component of $G^-_{A(x)}(\mathcal{F})$. So $\omega(\phi')$ is in $\mathcal{C}$ and then ϕ' is in $W^s(\mathcal{C})$.

Thus we obtain that the intersection of σ^- and $W^s(\mathcal{C})$ is open and closed in σ^-. So, since σ^- is connected we have that every leaf of σ^- is in $W^s(\mathcal{C})$. □

Remark 6.1.6. Lemma 6.1.5 does not hold if $A(x)$ is replaced by some $t < A(x)$. Indeed if we only suppose that $\mathcal{C}$ is a connected component of $G^-_t(\mathcal{F})$ with $t < A(x)$ then the unstable cone σ^- is not necessarily included in the attractive basin of $\mathcal{C}$. See example 6.6 above where the leaf ψ' is not a leaf of the attractive basin of the connected component $\{y\}$ of $G^-_{A(y)^+}(\mathcal{F})$.

We prove that the attractive basin of a connected component of $G^-_t(\mathcal{F})$ with $t \in \mathbb{R}$ is open with the two following lemmas.

Lemma 6.1.7. *Let us consider $t \in \mathbb{R}$ and $\mathcal{C}$ a connected component of $G_t^-(\mathcal{F})$. Each singularity x of $\mathcal{C}$ is in the interior of $W^s(\mathcal{C})$.*

Proof. Let us consider a singularity x of $\mathcal{F}$ in $\mathcal{C}$. It is either a sink, a source or saddle. We separate those three cases.

1. If x is a sink, then each leaf in a neighborhood of x is in $W^s(\mathcal{C})$.

2. If x is a source, then each leaf ϕ in a neighborhood of x satisfies $A(\omega(\phi)) < A(x)$ and is in $W^s(\mathcal{C})$.

3. If x is a saddle point, we consider a trivialization (h, V) of $\mathcal{F}$ at x on a neighborhood V given by Lemma 6.1.3. By definition of $W^s(\mathcal{C})$, each leaf of Σ_x^- and Σ_x^+ is in $W^s(\mathcal{C})$. Moreover, let us consider a leaf ϕ in the boundary of an unstable cone σ^- of x and U the hyperbolic sector adjacent to ϕ, by the local model there exists a chain of connexions from x to a singularity y such that every leaf of the hyperbolic sector U admits y as its omega limit. By definition, the singularity y is in $\mathcal{C}$ hence the leaves of the hyperbolic sector U are in $W^s(\mathcal{C})$. Since it holds for each hyperbolic sector of x we obtain the result.

□

Let us consider $t \in \mathbb{R}$ and $\mathcal{C}$ a connected component of $G_t^-(\mathcal{F})$. We describe the attractive basin of $\mathcal{C}$ and its frontier.

Lemma 6.1.8. *The intersection of $W^s(\mathcal{C})$ and $\Sigma \backslash X$ is open.*

Proof. We consider a leaf ϕ in $W^s(\mathcal{C})$, by the local model there exists a small neighborhood V of ϕ and two chains of connexions $\phi_1, ..., \phi_k$ and $\phi'_1, ..., \phi'_{k'}$ of $\mathcal{F}$ which contain ϕ such that each leaf ψ passing through V satisfies $\omega(\psi) = \omega(\phi_k)$ or $\omega(\psi) = \omega(\phi'_{k'})$. By construction $\omega(\phi_k)$ and $\omega(\phi'_{k'})$ are in the same connected component of $G_t^-(\mathcal{F})$ and so every leaf passing through V is in $W^s(\mathcal{C})$. □

The following corollary is a consequence of Lemmas 6.1.7 and 6.1.8.

Corollary 6.1.9. *The attractive basin of $\mathcal{C}$ is an open surface in Σ.*

We deduce the following lemma.

Lemma 6.1.10. *The attractive basin of $\mathcal{C}$ is connected.*

Proof. For every $x \in X \cap \mathcal{C}$, $W^s(\S)$ is connected as it is arc-connected. Let us consider two singularities x and y of $\mathcal{C}$ such that there exists an edge from x to y. So there is a leaf ϕ such that $\alpha(\phi) = x$ and $\omega(\phi) = y$. The point x is in the frontier of $W^s(y)$ and is either a saddle point or a source. Since we proved previously that x is in the interior of $W^s(\mathcal{C})$ we deduce that $\overline{\phi} \subset W^s(x) \cup W^s(y)$.
We deduce easily that $W^s(\mathcal{C}) = \bigcup_{x \in X \cap \mathcal{C}} W^s(x)$ is connected. □

We prove a last result which concludes the proof of Proposition 6.1.4.

Lemma 6.1.11. *The number of leaves included in the frontier of the attractive basin of $\mathcal{C}$ is finite.*

Proof of Lemma 6.1.11. Let us consider a leaf ϕ of the frontier of $W^s(\mathcal{C})$. The singularity $\omega(\phi)$ can not be a sink nor a source so it is a saddle point. So ϕ is a leaf of a stable cone of $\omega(\phi)$. So by the first point of Proposition 6.1.4 and the fact that the number of singularities of $\mathcal{F}$ is finite, there exists a finite number of such leaves and we obtain the result. $\square$

Remark 6.1.12. Let us describe an example. We consider a foliation $\mathcal{F}$ on a 2-sphere such that $\mathcal{F}$ has two sources s_1, s_2, one saddle point x one sink p and the foliation is the gradient-lines of the Morse function $A : \mathbb{S}^2 \to \mathbb{R}$ represented in green in figure 6.8.

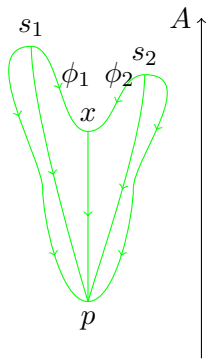


Figure 6.8

We consider the connected component $\mathcal{C} = \{p\}$ of $G^-_{A(x)}(\mathcal{F})$ and we want to describe the attractive basin of $\mathcal{C}$. In our example, we see that each leaf ϕ of $\mathcal{F}$ satisfies $\omega(\phi) = p$ except two leaves ϕ_1 and ϕ_2 such that ϕ_1 is a leaf between s_1 and x and ϕ_2 is a leaf between s_2 and x. Let us consider a trivialization (h, V) at x of the foliation $\mathcal{F}$ represented in figure 6.9.

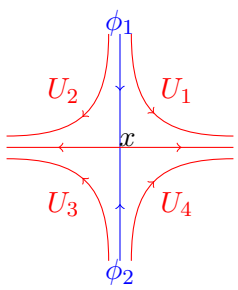


Figure 6.9

In the example the four hyperbolic sectors $U_1, ..., U_4$ of x are in $W^s(\mathcal{C})$ and in Figure 6.9 the leaves of $W^s(\mathcal{C})$ are represented in red. The frontier of $W^s(\mathcal{C})$ is the union of the three singularities x,s_1 and s_2 and the two leaves ϕ_1 and ϕ_2 in Σ.

Remark 6.1.13. Each result about a connected components of $G^-_t(\mathcal{F})$, where $t \in \mathbb{R}$, has its symmetrical result for a connected component of $G^+_t(\mathcal{F})$.

6.2 Some properties of the Barcode $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$

Let us consider a gradient-like foliation $\mathcal{F}$ defined on the complement of a finite set X of a compact surface Σ. We suppose that X is equipped with an action function $A : X \to \mathbb{R}$ such that for each leaf ϕ we have $A(\alpha(\phi)) > A(\omega(\phi))$. We prove some properties about the Barcode $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$, which will be denoted $\boldsymbol{\beta}_{\mathcal{F}}$.

We set A_m the minimum value of the action function A on X and A_M its maximum value.

Notice that, by construction of the map $\mathcal{B}$, for every singularity z of $\mathcal{F}$ of index zero there is no bar in $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$ one end of whose is equal to $A(z)$. Indeed, the point z does not connect distinct connected components of the graphs $G^-_{A_f(z)}(\mathcal{F})$ and $G^+_{A_f(z}(\mathcal{F})$ so if we follow the construction in Chapter 5, there is no bar of category 1 or 2 induced by z. So, since $\mathrm{ind}(\mathcal{F}, z) = 0$ there is no bar of category 3 induced by z neither.

We consider the graph $G(\mathcal{F})$ associated to $\mathcal{F}$ and the subgraphs $(G^-_t(\mathcal{F}))_{t\in\mathbb{R}}$ and $(G^+_t(\mathcal{F}))_{t\in\mathbb{R}}$. We will use some notations from Chapter 5. For $t \in \mathbb{R}$ we denote $\mathcal{C}^-_t$ and $\mathcal{C}^+_t$ the set of connected components of $G^-_t(\mathcal{F})$ and $G^+_t(\mathcal{F})$.
We set the maps $j_t : \mathcal{C}^-_t \to \mathcal{C}^-_{t^+}$ and $j'_t : \mathcal{C}^+_t \to \mathcal{C}^+_{t^-}$ where for $C \in \mathcal{C}^-_t$, $j_t(C)$ is the connected component of $G^-_{t^+}$ which contains C and for $C' \in \mathcal{C}^+_t$ $j'_t(C')$ is the connected component of $G^+_{t^-}$ which contains C'. For a connected component $\mathcal{C}$ of $G^-_t(\mathcal{F})$ we consider $L(\mathcal{C}) = \min\{A(y) \mid y \text{ vertex of } \mathcal{C}\}$.

Lemma 6.2.1. *For each sink y of $\mathcal{F}$ satisfying $A_m < A(y)$ there exists at least one saddle point x of $\mathcal{F}$ such that the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ contains a bar $b = (A(y), A(x)]$.*
For each source y of $\mathcal{F}$ satisfying $A(y) < A_M$ there exists at least one saddle point x of $\mathcal{F}$ such that the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ contains a bar $b = (A(x), A(y)]$.

In the situation of Lemma 6.2.1 we say that the pair (x, y) is associated to the finite bar b of $\boldsymbol{\beta}_{\mathcal{F}}$.

Proof. Let us fix a value $t > A_m$ of the action function A. We label $y_1, ..., y_n$ the sinks of X whose action value is t. We prove that for every $i \in \{1, ..., n\}$ there exists a saddle point x_i of $\mathcal{F}$ such that the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ contains a bar $(A(y_i), A(x_i)]$.

For $i \in [1, n]$ there is a map $c_i : s \mapsto \mathcal{C}^s_i$ defined for $s > t$ where $\mathcal{C}^s_i$ is the connected component of the graph $G^-_s(\mathcal{F})$ which contains y_i.

For a value $s > t$ such that $A^{-1}((t, s)) = \varnothing$, the connected components $\mathcal{C}^s_1, ..., \mathcal{C}^s_n$ are distinct.

For $s > t$ close to t, the elements $c_i(s)$, $i \in [1, n]$, are distincts and for $s > A_M$, we have that $c_i(s) = G(\mathcal{F})$ for every $i \in [1, n]$, hence we have $L(c_i(s)) = t$ for $s > 0$ close to t and $L(c_i(s)) = A_m$ for $t > A_M$.

Thus, for every $i \in [1, n]$, there exists an action value, denoted s_i, such that $A(y_i) = L(c_i(s_i))$ and $t > L(j_s(c_i(s_i)))$. In other words, there exists a saddle point x_i of $\mathcal{F}$ of action s_i such that

- $L(c_i(s_i)) = t = A(y_i)$
- x_i connects the connected component $c_i(s_i)$ and another connected component $\mathcal{C}$ of $G_s^-(\mathcal{F})$ such that $L(\mathcal{C}) < t$.

By construction of the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ there exists a bar $(A(y_i), s_i]$ of category 1 (see Chapter 5).

Notice that the points x_i, $i \in \{1, ..., n\}$ are not sources. Indeed, by contradiction we suppose that a source z of action s connects two distinct connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ of the graph $G_s^-(\mathcal{F})$. Thus, by hypothesis, there exist two singularities $x_1 \in \mathcal{C}_1$ and $x_2 \in \mathcal{C}_2$ in the frontier of $W^u(z)$. By Proposition 6.1.4, there exists a path of connexions Γ between x_1 and x_2 such that every singularity y in Γ satisfies $A(y) < s$, so x_1 and x_2 are in the same connected component of $G_s^-(\mathcal{F})$ and we have a contradiction.

We obtain the symmetrical results for sources of $\mathcal{F}$ by considering bars of category 2. □

Remark 6.2.2. For a singularity y of $\mathcal{F}$ of index 1, there may exist two saddle points x and x' with $A(x) = A(x')$ such that the couples (x, y) and (x', y) are associated to the same bar b.

Lemma 6.2.3. *We label $y_1, ..., y_n$ the sinks of $\mathcal{F}$ of action A_m. There exist $n-1$ finite bars $J_1, ..., J_{n-1}$ whose lower bound is equal to A_m and upper bound is the action value of a saddle point of $\mathcal{F}$ and one semi-infinite bar (A_m, ∞) in the barcode $\boldsymbol{\beta}_{\mathcal{F}}$.*

Proof. For $s > A_m$ such that $A^{-1}((A_m, s)) = \varnothing$, we denote $\mathcal{C}_i^s$ the connected component of $G_s^-(\mathcal{F})$ which contains the singularity y_i.

For $i \in [1, n]$ there is a map $c_i : s \mapsto \mathcal{C}_i^s$ defined for $s > t$ where $\mathcal{C}_i^s$ is the connected component of the graph $G_s^-(\mathcal{F})$ which contains y_i.

For $s > A_m$ close to A_m, we have $\mathcal{C}_i^s = \{y_i\}$, $i \in [1, n]$.

Let us consider $s > A_m$, such that there exists a subset K of $\{1, ..., n\}$ of cardinal at least two such that the connected components $c_i(s)$, $i \in K$ are distinct but the connected component $j_s(c_i(s))$, $i \in K$, are equal. We have $L(c_i(s)) = A_m$ for $i \in K$ so, by construction, it induces the existence of $\#K - 1$ bars $(A_m, s]$ in the barcode $\boldsymbol{\beta}_{\mathcal{F}}$.

Moreover, for each $i \neq j$ in $\{1, ..., n\}$ there exists $s > A_m$ such that $j_s(c_i(s)) = j_s(c_j(s))$ and $c_i(s) \neq c_j(s)$. Hence we obtain $n-1$ finite bars such that A_m is the lower bound of the bar.

The existence of a semi-infinite bar (A_m, ∞) in the barcode $\mathcal{B}(G(\mathcal{F}), A, \mathrm{ind}(\mathcal{F}, \cdot))$ is provided by the construction of $\mathcal{B}$: it is a bar of category 0. □

Let us consider a sink y_m of the foliation $\mathcal{F}$ such that $A(y_m) = A_m$ and a source y_M of the foliation $\mathcal{F}$ such that $A(y_M) = A_M$. We denote $X_{<0} \subset X$ the set of negative index singularities of $\mathcal{F}$ and $X_1 \subset X$ the set of singularities of index 1.

By Lemmas 6.2.1 and 6.2.3 there exists a map, which may be not unique,

$$\iota : X_1 \backslash \{y_m, y_M\} \to X_{<0}, \tag{6.1}$$

where $\iota(y) = x$ is a saddle point x of the foliation $\mathcal{F}$ given by Lemmas 6.2.1 and 6.2.3 such that the couple (x, y) is associated to a finite bar of $\boldsymbol{\beta}_{\mathcal{F}}$.

In particular, if A is injective and each singularity x of $X_{<0}$ has an index equal to -1 then, every bar $(a, b]$ of the barcode is naturally associated to a unique couple $x, y \in X$ by Lemma 6.2.1. In this case, ι is unique and an injection. Indeed, by construction, for every saddle point x of $X_{<0}$ there exists a unique bar one end of which is equal to $A(x)$. notice that this bar is either a finite interval whose infimum is $A(x)$, maximum is $A(x)$ or a semi-infinite interval whose infimum is $A(x)$.

We have the following result.

Proposition 6.2.4. *Let x be a saddle point of $\mathcal{F}$ of action value $t \in \mathbb{R}$ such that $A^{-1}(t) = \{x\}$. We have*

$$\#j_t^{-1}(\mathcal{C}_x) + \#j_t'^{-1}(\mathcal{C}'_x) \leqslant |\mathrm{ind}(\mathcal{F}, x)| + 2,$$

where $\mathcal{C}_x$ (resp. $\mathcal{C}'_x$) is the connected component of $G^-_{t^+}(\mathcal{F})$ (resp. $G^+_{t^-}(\mathcal{F})$) which contains x.

In general, Proposition 6.2.4 certifies that for a saddle point x of $\mathcal{F}$ such that $A^{-1}(t) = \{x\}$, there are $|\mathrm{ind}(f, x)|$ bars of which t is an end. We can link this result to Proposition 28 of [61] which asserts that for a Hamiltonian function H on a manifold the number of bars of which $t \in \mathbb{R}$ is an ending value is given by the dimension of the local Floer Homology at t.

Proof of Proposition 6.2.4. We set $t = A(x)$ and $i = \mathrm{ind}(\mathcal{F}, x)$. We label Σ_x^- the set of the unstable cones of x and Σ_x^+ the set of the stable cones of x. Both Σ_x^- and Σ_x^+ are equipped with a cyclic order and a natural *cyclic transformation* $\tau_x : \Sigma_x^- \cup \Sigma_x^+ \to \Sigma_x^- \cup \Sigma_x^+$ which sends Σ_x^- into Σ_x^+ and Σ_x^+ into Σ_x^- such that every element $\sigma^- \in \Sigma_x^-$ (resp. $\sigma^+ \in \Sigma_x^+$) is sent to the element of Σ_x^+ (resp. Σ_x^-) right after it in cyclic order.

By Lemma 6.1.5, for every $\sigma^- \in \Sigma_x^-$ there is a connected component $\mathcal{C}^-$of $G_t^-(\mathcal{F})$ such that $\omega(\phi)$ is a vertex of $\mathcal{C}^-$ for every leaf $\phi \in \sigma^-$. So we can define a map $\omega : \Sigma_x^- \to \mathcal{C}_t^-(\mathcal{F})$ given by $\omega(\sigma^-) = \mathcal{C}^-$.

Symmetrically, for every $\sigma^+ \in \Sigma_x^+$ there is a connected component $\mathcal{C}^+$of $G_t^+(\mathcal{F})$ such that $\alpha(\phi)$ is a vertex of $\mathcal{C}^+$ for every leaf $\phi \in \sigma^+$. So we can define a map $\alpha : \Sigma_x^+ \to \mathcal{C}_t^+(\mathcal{F})$ given by $\alpha(\sigma^+) = \mathcal{C}^+$.

We will denote by $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\omega)$ the image sets of α and ω and the result consists in establishing the inequality

$$\#\mathrm{Im}(\alpha) + \#\mathrm{Im}(\omega) \leqslant i + 2. \tag{6.2}$$

We introduce a combinatorial context to facilitate the proof.

Let us consider two sets E^+ and E^- and a couple of maps $\alpha : \Sigma^+ \to E^+$ and $\omega\Sigma^- \to E^-$ defined on sets $\Sigma^+ \subset \Sigma_x^+$, $\Sigma^- \subset \Sigma_x^-$ such that there is, in cyclic order, alternatively an element of Σ^+ and an element of Σ^- in $\Sigma_x^- \cup \Sigma_x^+$. In particular, Σ^+ and Σ^- have the same cardinal and are naturally equipped with a cyclic transformation τ. We define some useful notation.

Let c be in the image of ω. We set $J = \omega^{-1}(c)$ and we will say that

- An element $\sigma^+ \in \Sigma^+$ is *adjacent to J in Σ^+* if one and only one of the two elements $\tau(\sigma^+)$ and $\tau^{-1}(\sigma^+)$ is in J. More precisely we will say that an element $\sigma^+ \in \Sigma^+$ is *adjacent and before J in Σ^+* if $\tau(\sigma^+)$ is in J and $\tau^{-1}(\sigma^+)$ is not. Symmetrically we will say that an element $\sigma^+ \in \Sigma^+$ is *adjacent and after J in Σ^+* if $\tau^{-1}(\sigma^+)$ is in J and $\tau(\sigma^+)$ is not.

- An element $\sigma^+ \in \Sigma^+$ is *surrounded by J in Σ^+* if the two elements $\tau(\sigma^+)$ and $\tau^{-1}(\sigma^+)$ are in J.

A maximal set of consecutive elements of J in Σ^- will be called a maximal interval of J.

We prove the following lemma about the maps α and ω.

Lemma 6.2.5. *Suppose that for every $c \in \mathrm{Im}(\omega)$ and every $\sigma^+ \in \Sigma^+$ adjacent to $\omega^{-1}(c)$ in Σ^+, there exists $\sigma'^+ \in \Sigma^+ \backslash \{\sigma^+\}$ adjacent to $\omega^{-1}(c)$ in Σ^+ such that $\alpha(\sigma'^+) = \alpha(\sigma^+)$. Then we have*

$$\#\mathrm{Im}(\alpha) + \#\mathrm{Im}(\omega) \leqslant i + 2,$$

where $i + 1$ is the cardinal of the sets Σ^+ and Σ^-.

Proof. Let $i = 0$, for a couple (α, ω) defined on sets Σ^+ and Σ^- of cardinal equal to 1, the inequality is trivial.

Let $i \geqslant 1$, we suppose by induction that couples defined on sets of cardinal less than i satisfying the hypothesis of Lemma 6.2.5 satisfy the result.

We consider a couple (α, ω) defined on sets Σ^+ and Σ^- of cardinal equals to $i + 1$ satisfying the hypothesis of Lemma 6.2.5. We prove that $\#\mathrm{Im}(\alpha) + \#\mathrm{Im}(\omega) \leqslant i + 2$.

We divide the proof of the inequality into three cases.

Case 1. Suppose that ω is constant. In this case we have by definition $\#\mathrm{Im}(\omega) = 1$ and $\#\mathrm{Im}(\alpha) \leqslant i + 1$ so the result is trivial.

Case 2. Suppose that for each $c \in \mathrm{Im}(\omega)$ we have $\#\omega^{-1}(c) = 1$. By hypothesis, we obtain that α is constant on Σ^+. So we compute

$$\#\mathrm{Im}(\alpha) + \#\mathrm{Im}(\omega) = 1 + (i + 1),$$

and we obtain the result.

Case 3. Suppose that there exists $c \in \mathrm{Im}(\omega)$ such that $i + 1 > \#\omega^{-1}(c) > 1$. We will modify the couple (α, ω) into another couple (α_1, ω_1) defined on subsets of Σ^+ and Σ^- and

we prove that the couple (α_1, ω_1) satisfies the hypothesis of Lemma 6.2.5.

We denote $J = \omega^{-1}(c)$ and we define new maps on $\Sigma_1^- = \Sigma^- \backslash J$ and $\Sigma_1^+ = \Sigma^+ \backslash \tau(J)$.
We set $\omega_1 : \Sigma_1^- \to E^- \backslash \{c\}$, defined as the restriction of ω.
We define $\alpha_1 : \Sigma_1^+ \to E^+/\alpha(\tau(J)\Delta\tau^{-1}(J))$, where we have

- $\tau(J)\Delta\tau^{-1}(J)$ is the symmetric difference of $\tau(J)$ and $\tau^{-1}(J)$.
- $E^+/\alpha(\tau(J)\Delta\tau^{-1}(J))$ is the set obtained by identifiyng the elements of $\alpha(\tau(J)\Delta\tau^{-1}(J))$.
- α_1 is the natural map induced by α.

Notice that the set $\tau(J)\Delta\tau^{-1}(J)$ is composed of the elements of Σ^+ which are adjacent to J. Moreover, the sets Σ_1^- and Σ_1^+ are not empty.

We prove that (α_1, ω_1) satisfies the hypothesis of Lemma 6.2.5.

Let us consider $c_1 \in \mathrm{Im}(\omega_1)$. We set $J_1 = \omega_1^{-1}(c_1)$ and we consider $\sigma_1^+ \in \Sigma_1^+$ adjacent to J_1 in Σ_1^+. Our goal is to find an element of Σ_1^+ distinct from σ_1^+ and adjacent to J_1 in Σ_1^+ whose α_1 value is equal to the α_1 value of σ_1^+.
The element σ_1^+ may not be adjacent to J_1 in Σ^+. There are two possibilities.
1) σ_1^+ is adjacent to J_1 in Σ^+.
2) σ_1^+ is adjacent and before a maximal interval K of J in Σ^+ and there exists ${\sigma'}_1^+ \in J$ which is adjacent and after the interval K of J in Σ^+ and adjacent and before J_1 in Σ^+.

We separate these two cases.

In case 1), by hypothesis, there exists $\sigma_2^+ \in \Sigma^+ \backslash \{\sigma_1^+\}$ adjacent to J_1 in Σ^+ such that $\alpha(\sigma_1^+) = \alpha(\sigma_2^+)$. Again, there are two possibilities.

i) If $\sigma_2^+ \notin \tau(J)$ then σ_2^+ is adjacent to J_1 in Σ_1^+ and $\alpha_1(\sigma_1^+) = \alpha_1(\sigma_2^+)$ by definition. We obtain the result.

ii) If $\sigma_2^+ \in \tau(J)$ then σ_2^+ is adjacent and after a maximal interval K_2 of J in Σ^+. So σ_2^+ is before an interval K_1 of J_1 in Σ^+ and after the interval K_2 of J in Σ^+. We consider ${\sigma'}_2^+$ in Σ^+ just before the interval K_2 in Σ^+. We have that ${\sigma'}_2^+ \notin \tau(J)$ and ${\sigma'}_2^+ \neq \sigma_1^+$. Indeed, if we supposed that $\sigma_1^+ = {\sigma'}_2^+$ then since σ_1^+ is adjacent to J_1 in Σ^+ it is after an interval K_1' of J_1 in Σ^+ and just before the interval K_2 of J in Σ^+. Hence in the cyclic order we have in Σ^+

$$K_1', \sigma_1^+ = {\sigma'}_2^+, K_2, \sigma_2^+, K_1.$$

So in Σ_1^+ we obtain in the cyclic order

$$K_1', \sigma_1^+, K_1.$$

Meaning that σ_1^+ is surrounded by J_1 in Σ_1^+ and so it contradicts the fact that σ_1^+ is adjacent to J_1 in Σ_1^+.
Moreover, by hyothesis, $\alpha(\sigma_1^+) = \alpha(\sigma_2^+) \in \alpha(\tau(J)\Delta\tau^{-1}(J))$ so by construction $\alpha_1(\sigma_1^+) = \alpha_1({\sigma'}_2^+)$ and we obtain the result.

In case 2), ${\sigma'}_1^+$ is just before a maximal interval K_1' of J_1 in Σ^+. Moreover, by hypothesis, there exists $\sigma_2^+ \in \Sigma^+ \backslash \{{\sigma'}_1^+\}$ adjacent to J_1 in Σ^+ such that $\alpha({\sigma'}_1^+) = \alpha(\sigma_2^+)$. Again, there are two possibilities.

i) If $\sigma_2^+ \notin \tau(J)$ then σ_2^+ is adjacent to J_1 in Σ_1^+ and distinct from σ_1^+. Indeed, if we suppose that $\sigma_2^+ = \sigma_1^+$ then σ_1^+ is before the maximal interval K of J in Σ^+ and after a maximal interval K_1 of J_1 in Σ^+. Hence, in Σ^+, we have and in the cyclic order

$$K_1, \sigma_1^+ = \sigma_2^+, K, {\sigma'}_1^+, K_1'.$$

So in Σ_1^+ we obtain in the cyclic order

$$K_1, \sigma_1^+, K_1'.$$

Meaning that σ_1^+ is surrounded by J_1 in Σ_1^+ and it contradicts the fact that σ_1^+ is adjacent to J_1 in Σ_1^+.
Moreover, by hypothesis, $\alpha({\sigma'}_1^+) = \alpha(\sigma_2^+) \in \alpha(\tau(J)\Delta\tau^{-1}(J))$ and by definition $\alpha(\sigma_1^+) \in \alpha(\tau(J)\Delta\tau^{-1}(J))$. So, by construction, we have $\alpha_1(\sigma_1^+) = \alpha_1(\sigma_2^+)$ and we obtain the result.

ii) If $\sigma_2^+ \in \tau(J)$ then σ_2^+ is adjacent and after a maximal interval K_2 of J in Σ^+. We consider ${\sigma'}_2^+$ in Σ^+ just before the interval K_2 in Σ^+. We have $K \neq K_2$ then ${\sigma'}_2^+ \neq \sigma_1^+$ and $\alpha({\sigma'}_2^+) = \alpha(\sigma_1^+)$. Indeed, if we suppose that $K_2 = K$ then it contradicts the fact that $\sigma_2^+ \neq {\sigma'}_2^+$ and we obtain the result.

Now, we prove that $\#\mathrm{Im}(\alpha) + \#\mathrm{Im}(\omega) \leqslant i + 2$.

The image of ω is the union of the image of ω_1 and the singleton $\{c\}$. So we have $\#\mathrm{Im}(\omega) = \#\mathrm{Im}(\omega_1) + 1$.

Let us consider the natural projection $p : E^+ \to E^+/\alpha(\tau(J)\Delta\tau^{-1}(J))$, we have the following diagram

$$\begin{tikzcd} \Sigma_1^+ \arrow[r, "\alpha"] \arrow[dr, "\alpha_1"'] & E^+ \arrow[d, "p"] \\ & E^+/\alpha(\tau(J)\Delta\tau^{-1}(J)) \end{tikzcd}$$

We denote $e \in E^+/\alpha(\tau(J)\Delta\tau^{-1}(J))$ such that $p(\alpha(\sigma)) = e$, for every $p(c) \in \tau(J)\Delta\tau^{-1}(J)$.

The image of α satisfies

$$\mathrm{Im}(\alpha) = \alpha(\tau(J) \cup \tau^{-1}(J)) \cup \alpha(\Sigma^+ \backslash \{\tau(J) \cup \tau^{-1}(J)\}).$$

By definition, we have

$$\tau(J) \cup \tau^{-1}(J) = (\tau(J)\Delta\tau^{-1}(J)) \cup (\tau(J) \cap \tau^{-1}(J)).$$

So, we deduce

$$\alpha(\tau(J) \cup \tau^{-1}(J)) = \alpha(\tau(J)\Delta\tau^{-1}(J)) \cup \alpha(\tau(J) \cap \tau^{-1}(J)).$$

We denote by K the number of maximum intervals of J.Since $J \neq \Sigma^+$, we have $\#\tau(J)\Delta\tau^{-1}(J) = 2K$ and $\#(\tau(J) \cap \tau^{-1}(J)) = \#J - K$. Indeed, the set $\tau(J)\Delta\tau^{-1}(J)$ is the set of elements of Σ^+ which are adjacent to J and the set $\tau(J) \cap \tau^{-1}(J)$ is the set of elements of Σ^+ which are surrounded by J.

By hypothesis, for every element $\sigma^+ \in \tau(J)\Delta\tau^{-1}(J)$ there exists another element $\sigma'^+ \in \tau(J)\Delta\tau^{-1}(J)$ such that $\alpha(\sigma^+) = \alpha(\sigma'^+)$. So we obtain

$$\#\alpha(\tau(J)\Delta\tau^{-1}(J)) \leqslant \frac{2K}{2} \leqslant K.$$

Hence we compute

$$\begin{aligned}\#\alpha(\tau(J) \cup \tau^{-1}(J)) &\leqslant \#\alpha(\tau(J)\Delta\tau^{-1}(J)) + \#\alpha(\tau(J) \cap \tau^{-1}(J)) \\ &\leqslant K + \#J - K \\ &\leqslant \#J.\end{aligned}$$

It remains to estimate the cardinal of $C = \mathrm{Im}(\alpha)\backslash\{\alpha(\tau(J) \cup \tau^{-1}(J)\})\}$. For every $c \in C$ and $\sigma \in \alpha^{-1}(c)$, we have $\sigma \notin \alpha(\tau(J) \cup \tau^{-1}(J))$. So in particular, $\sigma \in \Sigma_1^+$ and $p(c) \neq e$. We deduce that

$$p(c) = \alpha_1(\sigma).$$

It implies that

$$p(C) \subset \mathrm{Im}(\alpha_1)\backslash\{e\}.$$

Moreover, by definition $C \cap p^{-1}(e) = \varnothing$, so $p|_C$ is a bijection and, since $\{e\} \in \mathrm{Im}(\alpha_1)$, we obtain

$$\#C \leqslant \#\mathrm{Im}(\alpha_1) - 1.$$

Thus we have

$$\mathrm{Im}(\alpha) \leqslant \#J + \#\mathrm{Im}(\alpha_1) - 1.$$

The couple (α_1, ω_1) is defined on a set of cardinal $i - \#J + 1$ and satisfies the hypothesis of Lemma 6.2.5, so we compute

$$\begin{aligned}\#\mathrm{Im}(\alpha) + \#\mathrm{Im}(\omega) &\leqslant (\#J + \#\mathrm{Im}(\alpha_1) - 1) + \#\mathrm{Im}(\omega_1) + 1 \\ &\leqslant \#\mathrm{Im}(\alpha_1) + \#\mathrm{Im}(\omega_1) + \#J \\ &\leqslant (i - \#J + 2) + \#J \\ &\leqslant i + 2,\end{aligned}$$

where the third inequality is given by the induction step.

□

Notice that if J is composed of at least 2 maximal intervals, the previous inequality is strict.

To complete the proof of Proposition 6.2.4 we will prove that the couple of maps $\alpha : \Sigma_x^+ \to \mathcal{C}_t^+(\mathcal{F})$ and $\omega : \Sigma_x^- \to \mathcal{C}_t^-(\mathcal{F})$ defined at the begining of the proof satisfies the

hypothesis of Lemma 6.2.5.

Let $\mathcal{C}$ be a connected component of $G_t^-(\mathcal{F})$ in the image of ω. We denote $B = W^s(\mathcal{C})$ and we want to desingularize its frontier $\mathrm{Fr}(B)$ as follows.

Desingularization. We cut the surface Σ along $\mathrm{Fr}(B)$ (see [11] for example) to obtain a manifold with boundary $\hat{B}$ and a natural projection $\pi : \hat{B} \to \overline{B}$ such that

- $\pi(\partial\hat{B}) = \mathrm{Fr}(B)$.
- $\pi(\hat{B}\backslash\partial\hat{B}) \simeq B$.

Let us draw simple examples to explain what we are doing.

First example. In Figure 6.10, we consider on the left a saddle point y in $\mathrm{Fr}(B)$ such that

- There are two hyperbolic sectors U' and U'' separated by a cone σ in B . We denote ϕ' and ϕ'' the leaves of U' and U'' in $\mathrm{Fr}(B)$.
- There exists a leaf ψ in $\sigma \cap \mathrm{Fr}(B)$.

We draw in red the leaves in B and in blue the leaves in $\mathrm{Fr}(B)$. We cut along leaves of $\mathrm{Fr}(B)$ in blue to obtain $\hat{B}$ on the right and we have

- $\pi^{-1}(y) = \{\hat{y}_1, \hat{y}_2\}$.
- $\pi^{-1}(\psi) = \{\hat{\psi}_1, \hat{\psi}_2\}$.
- $\pi^{-1}(\phi'') = \{\hat{\phi}'\}$.
- $\pi^{-1}(\phi') = \{\hat{\phi}''\}$.

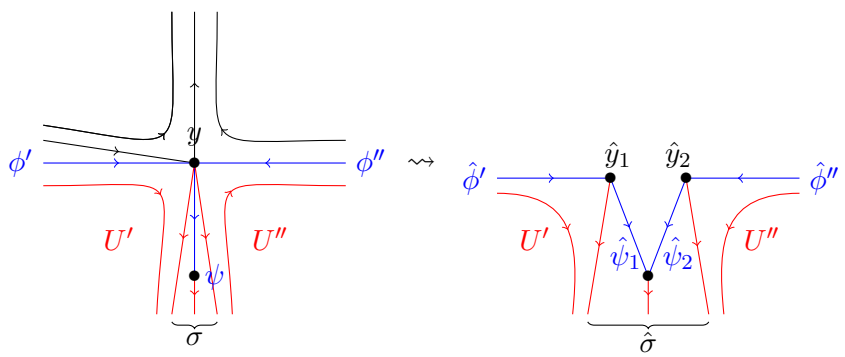


Figure 6.10

Second example. In Figure 6.11 we consider a saddle point y in $\mathrm{Fr}(B)$ such that

- There is a stable cone surrounded by B which contains two leaves ϕ_1 and ϕ_2 in $\mathrm{Fr}(B)$.
- There are two leaves ϕ' and ϕ'' of stable cones of y in $\mathrm{Fr}(B)$.

We draw in red the leaves in B, in blue the leaves in $\mathrm{Fr}(B)$. We cut along the leaves of $\mathrm{Fr}(B)$ to obtain $\hat{B}$ on the right and we have

- $\pi^{-1}(y) = \{\hat{y}_1, \hat{y}_2\}$.
- $\pi^{-1}(\psi_1) = \{\hat{\psi}_1\}$.
- $\pi^{-1}(\psi_2) = \{\hat{\psi}_2\}$
- $\pi^{-1}(\phi'') = \{\hat{\phi}'\}$.
- $\pi^{-1}(\phi') = \{\hat{\phi}''\}$.

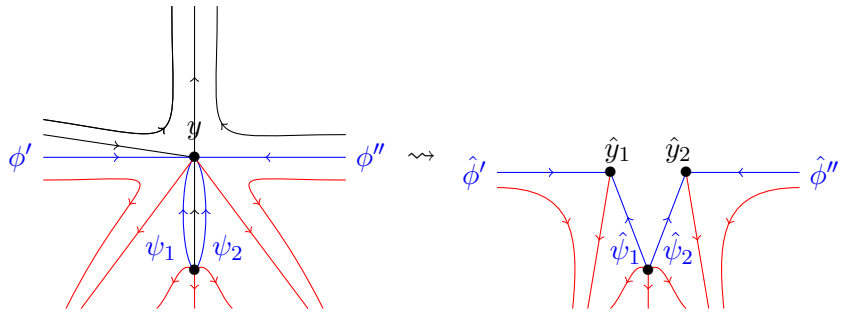


Figure 6.11

We have that $\hat{B}$ is a manifold with boundary whose boundary is a union of disjoint circles. By Proposition 6.1.4, each circle of $\partial\hat{B}$ is composed of chains of connexions of the foliation $\mathcal{F}$ such that for every leaf ϕ in $\mathrm{Fr}(B)$ we have

- $\#\pi^{-1}(\phi) = 1$ if ϕ is adjacent to B.
- $\#\pi^{-1}(\phi) = 2$ if ϕ is in the interior of $\overline{B}$.

Remark 6.2.6. As we saw in the second example with the leaves ψ_1 and ψ_2, a leaf ϕ can be adjacent to B and in a stable cone of a saddle point of $\mathrm{Fr}(B)$ which is surrounded by B

We set $J = \omega^{-1}(\mathcal{C})$ and recall that we set $t = A(x)$ at the begining. By definition, the saddle point x is in the boundary of B. We consider a stable cone $\sigma^+ \in \Sigma^+$ of x adjacent to J in Σ^+. To prove that the applications α and ω satisfy the hypothesis of Lemma 6.2.5 it is sufficient to prove that there exists another stable cone σ'^+ of x adjacent to J in Σ^+ such that $\alpha(\sigma'^+) = \alpha(\sigma^+)$.

The cone σ^+ is adjacent to B so, by Proposition 6.1.4, there is a unique leaf ϕ_0^+ in $\sigma^+ \cap \mathrm{Fr}(B)$. The cone σ^+ is adjacent to J in Σ^+, so we can denote $\pi^{-1}(\phi_0^+) = \{\hat{\phi}_0^+\} \subset \partial\hat{B}$.

Moreover, x is the only singularity of action t, so each singularity $\hat{y}$ of $\partial\hat{B}\backslash\pi^{-1}(x)$ satisfies $A_f(\pi(\hat{y})) > t$. So, if we consider a circle γ of $\partial\hat{B}$, the singularities in a same connected component of $\gamma\backslash\pi^{-1}(x)$ are the lift of singularities which are in the same connected component of $G_t^+(\mathcal{F})$. Notice that $\gamma\backslash\pi^{-1}(x)$ may be composed of more than one connected components.

We consider the circle γ_0 of $\partial\hat{B}$ containing $\hat{\phi}_0^+$. The connected component of $\gamma_0\backslash\pi^{-1}(x)$ containing $\hat{\phi}_0^+$ contains a leaf $\hat{\phi'}_0^+$ distinct from $\hat{\phi}_0^+$ such that $\omega(\pi(\hat{\phi'}_0)) = x$. There are two cases.

1) If $\pi(\hat{\phi'}_0)$ is in a stable cone of x adjacent to J in Σ^+ then we obtain the result.

2) Suppose that $\pi(\hat{\phi'}_0)$ is in a stable cone σ_1^+ of x surrounded by J in Σ^+. By construction, there exists another leaf $\hat{\phi}_1^+ \in \partial\hat{B}$ distinct from $\hat{\phi'}_0^+$ such that $\pi(\hat{\phi}_1^+)$, which may be equal to $\pi(\hat{\phi'}_0^+)$, is in σ_1^+.

By Lemma 6.1.5, we have that $\alpha(\pi(\hat{\phi}_1^+)) = \alpha(\pi(\hat{\phi'}_0^+))$. So we apply the same arguments to $\hat{\phi}_1^+$ and then there exists a leaf $\hat{\phi'}_1^+ \subset \partial\hat{B}$ such that $\hat{\phi'}_1^+$ and ϕ_1^+ are in the same connected component of $\gamma_1\backslash\{\pi^{-1}(x)\}$ and $\omega(\pi(\hat{\phi'}_1^+)) = x$. If ϕ'^+_1 is in a stable cone of x adjacent to J in Σ^+ we stop the process and if not we do the same discussion for ϕ'^+_1 as we did for ϕ'^+_0 in case 2).

Since x has finitely many stable cones, the process stops after a finite number of times and we obtain a leaf ϕ'^+ in the frontier of the attractive basin of $\mathcal{C}$ distinct from ϕ^+ which is in a stable cone σ'^+ of x adjacent to J in Σ^+ such that $\alpha(\sigma'^+) = \alpha(\sigma^+)$ and this ends the proof of Proposition 6.2.4. □

By construction of $\mathcal{B}$, we deduce the following corollary.

Corollary 6.2.7. *Let us suppose that the singularities of $\mathcal{F}$ have distinct action values and consider a saddle point x of $\mathcal{F}$. There exist exactly $-\mathrm{ind}(f,x)$ bars $J_1, ..., J_{-\mathrm{ind}(f,x)}$ of which $A(x)$ is an end point.*
Moreover, for each source or sink y of X there exists exactly one bar J of which $A(y)$ is an end point.

Proof. Let us consider a saddle point x of $\mathcal{F}$ of action t. We denote $\mathcal{C}_x$ (resp. $\mathcal{C}'_x$ the connected component of $G^-_{t^+}(\mathcal{F})$ (resp. $G^+_{t^-}(\mathcal{F})$) which contains x. By construction, there are $\#j_t^{-1}(\mathcal{C}_x)$ bars of category 1 such that $A(x)$ is the maximum and $\#j'^{-1}_t(\mathcal{C}'_x)$ bars of category 2 such that $A(x)$ is the infimum. Finally, by Proposition 6.2.4 we have $k = |\mathrm{ind}(\mathcal{F},x)| - \#j_t^{-1}(\mathcal{C}_x) - \#j_t'^{-1}(\mathcal{C}'_x) \geqslant 0$ so there exists k bars $(A_f(x), +\infty)$ of category 3. Thus there are exactly $-\mathrm{ind}(\mathcal{F},x)$ bars of which $A_f(x)$ is an end. □

Recall that $X_<$ is the set of singularities of X of negative index and X_1 the set of singularities of X of index 1. We set $S_{<0} = A(X_{<0})$ and we consider a sink y_m of the foliation $\mathcal{F}$ such that $A(y_m) = A_m$ and a source y_M of the foliation $\mathcal{F}$ such that $A(y_M) = A_M$ and an application $\iota : X\backslash\{y_m, y_M\} \to X_{<0}$ given by 6.1. We will prove the following result about the semi-infinite bars of $\boldsymbol{\beta}_{\mathcal{F}}$.

Lemma 6.2.8. *There exist exactly $2g+2$ semi-infinite bars in $\boldsymbol{\beta}_{\mathcal{F}}$, where g is the genus of Σ.*

Proof. By Lemma 6.2.1 for each value $t \in S_{<0}$ the number of finite bars of which one end

is the action value of t is equal to

$$\left(\sum_{C\in\mathcal{C}^-_{t^+}}(\#j_t^{-1}(\{C\})-1)+\sum_{C'\in\mathcal{C}^+_{t^-}}(\#{j'}_t^{-1}(\{C'\})-1)\right).$$

Moreover, the existence of the application ι asserts that the total number of finite bars is equal to the number of singularities of index 1 minus two. So we compute

$$\begin{aligned}\sum_{y\in X_1}1 &= 2+\sum_{y\in X_1\setminus\{y_m,y_M\}}\operatorname{ind}(\mathcal{F},y)\\ &= 2+\sum_{t\in S_{<0}}\left(\sum_{C\in\mathcal{C}^-_{t^+}}(\#j_t^{-1}(\{C\})-1)+\sum_{C\in\mathcal{C}^+_{t^-}}(\#{j'}_t^{-1}(\{C'\})-1)\right).\end{aligned}\tag{6.3}$$

Moreover, we have

$$\begin{aligned}2-2g &= \sum_{x\in X}\operatorname{ind}(\mathcal{F},x)\\ &= 2+\sum_{x\in X\setminus\{y_m,y_M\}}\operatorname{ind}(\mathcal{F},x).\\ &= 2+\sum_{x\in X_{<0}}\operatorname{ind}(\mathcal{F},x)+\sum_{y\in X_1\setminus\{y_m,y_M\}}\operatorname{ind}(\mathcal{F},y).\end{aligned}$$

Where the last equality is given by separating the fixed points of negative index and the fixed points of index 1.

For $t\in S_{<0}$ we define

$$k_t=-\sum_{\substack{x\in X_{<0}\\ A(x)=t}}\operatorname{ind}(\mathcal{F},x).$$

By equation 6.3, we get

$$\begin{aligned}2-2g &= 2+\sum_{t\in S_{<0}}\left(\sum_{\substack{x\in X_{<0}\\ A(x)=t}}\operatorname{ind}(\mathcal{F},x)\right)\\ &\quad+\sum_{t\in S_{<0}}\left(\sum_{C\in\mathcal{C}^-_{t^+}}(\#j_t^{-1}(\{C\})-1)+\sum_{C'\in\mathcal{C}^+_{t^-}}(\#{j'}_t^{-1}(\{C'\})-1)\right)\\ &= 2-\sum_{t\in S_{<0}}\left(k_t-\sum_{C\in\mathcal{C}^-_{t^+}}(\#j_t^{-1}(\{C\})-1)-\sum_{C'\in\mathcal{C}^+_{t^-}}(\#({j'}_t^{-1}(\{C'\})-1)\right)\\ &= 2-(\#\{\text{semi-infinite bars}\}-2).\end{aligned}$$

The last equation is given by the construction of the bars of category 3 in the construction of $\mathcal{B}$. Indeed, for action value t, the number of bars $(t,+\infty)$ in the barcode is equal, by

definition, to $k_t - \sum_{C \in \mathcal{C}^-_{t^+}} (\# j_t^{-1}(\{C\}) - 1) - \sum_{C' \in \mathcal{C}^+_{t^-}} (\# (j'^{-1}_t(\{C'\}) - 1)$. The -2 comes from the two semi-infinite bars $(A_m, +\infty)$ and $(A_M, +\infty)$. So the number of semi-infinite bars is equal to $2g + 2$. $\square$

Remark 6.2.9. Let us suppose that for each value t in the image of A, the set $A^{-1}(\{t\})$ is a singleton and each singularity x of $X_{<0}$ has index -1. In this case the proof is simpler to understand because ι is an injection.

The number of semi-infinite bars is equal to $\#(X_{<0} \backslash \mathrm{Im}(\iota)) + 2$ and we compute

$$\begin{aligned}
2 - 2g &= \sum_{x \in X} \mathrm{ind}(\mathcal{F}, x) \\
&= 2 + \sum_{y \in X_1 \backslash \{y_m, y_M\}} \mathrm{ind}(\mathcal{F}, y) + \sum_{x \in X_{<0}} \mathrm{ind}(\mathcal{F}, x) \\
&= 2 + \sum_{y \in X_1 \backslash \{y_m, y_M\}} 1 - \sum_{x \in \mathrm{Im}(\iota)} 1 - (\#\{\text{semi-infinite bars}\} - 2) \\
&= 4 - \#\{\text{semi-infinite bars}\}.
\end{aligned}$$

We obtain the result.

Chapter 7

A barcode with an order on a maximal unlinked set of fixed points

We consider a homeomorphism f of a compact surface Σ with a finite number of fixed points. We fix a maximal unlinked set of fixed points X of f. We denote by $\mathbb{D}$ the unit disk.

For a set $U \subset \mathbb{D}$ we will denote by $\mathrm{Adh}_{\mathbb{D}}(U)$ its closure in $\mathbb{D}$ and by $\mathrm{Adh}_{\mathbb{R}^2}U$ its closure in $\mathbb{R}^2$. A line γ will be a proper oriented topological embedding of the interval $(0,1)$. If an oriented line $\gamma : (0,1) \to \mathbb{D}$ separates $\mathbb{D}$ in two connected components, we will consider its left hand side, denoted $L(\widetilde{\gamma})$, and its right-hand side, denoted $R(\widetilde{\phi})$.

Let us consider a maximal isotopy I from id to f such that $\mathrm{Sing}(I) = X$. We equipped $\Sigma \backslash X$ with a hyperbolic metric m such that the universal cover $\widetilde{\Sigma \backslash X}$ of $\Sigma \backslash X$ with the pull back metric is isomorphic to $\mathbb{D}$.

We consider a singularity $x \in X$ and a path $\gamma : [0,1] \to \Sigma$ such that $\lim_{t \to 1} \gamma(t) = x$ and $\gamma((0,1]) \subset \Sigma \backslash X$. We fix a lift $\widetilde{\gamma} : (0,1) \to \mathbb{D}$ of $\gamma|_{(0,1)}$. We consider a horospherical neighborhood V of x in $(\Sigma \backslash X, m)$. Meaning that $\pi^{-1}(V)$ is a disjoint union of horodisks where a horodisk is a disk internally tangent to the unit circle. Notice that $V \cup \{x\}$ is a topological neighborhood of x in Σ.

Moreover, $\widetilde{\gamma}((0,\epsilon])$ is connected so there exists a unique horoball $\widetilde{V} \subset \pi^{-1}(V)$ which contains $\widetilde{\gamma}((0,\epsilon])$. By definition, the closure of a horoball intersects the boundary $\mathbb{S}^1$ of $\mathbb{D}$ in exactly one point and we set $\widetilde{x} = \mathrm{Adh}_{\mathbb{R}^2}(V) \cap \mathbb{S}^1$. Since the alpha-limit point of $\widetilde{\gamma}$ is a point of $\mathbb{S}^1$ we obtain that $\lim_{t \to 0} \widetilde{\gamma}(t) = \widetilde{x}$.

Thus, if we consider a path $\gamma : [0,1] \to \Sigma$ such that $\alpha(\gamma) = x \in X$, $\omega(\gamma) = y \in X$ and $\gamma((0,1)) \subset \Sigma \backslash X$, then for every lift $\widetilde{\gamma}$ of γ is line in $\mathbb{D}$ such that there are two points $\widetilde{x}$ and $\widetilde{y}$ of $\mathbb{S}^1$ which satisfy

$$\lim_{t \to 0} \widetilde{\gamma}(t) = \widetilde{x}, \text{ and } \lim_{t \to 1} \widetilde{\gamma}(t) = \widetilde{y}.$$

The point $\widetilde{x}$ (resp. $\widetilde{y}$) will be denoted $\alpha(\widetilde{\gamma})$ (resp. $\omega(\widetilde{\phi})$). We refer to Ratcliffe's book [**?**], chapter 9.8 for more details.

For two distinct points $\widetilde{x}$ and $\widetilde{y}$ of $\mathbb{S}^1$, we define $[\widetilde{x},\widetilde{y}] \subset \mathbb{S}^1$ the arc joining $\widetilde{x}$ to $\widetilde{y}$ for the usual orientation. We set $(\widetilde{x},\widetilde{y}) = [\widetilde{x},\widetilde{y}]\backslash\{\widetilde{x},\widetilde{y}\}$.

Let us consider two proper paths $\phi : [0,1] \to \Sigma\backslash X$ and $\phi' : [0,1] \to \Sigma\backslash X$ whose alpha and omega limit points are in X and satisfy $\phi((0,1)) \subset \Sigma\backslash X$ and $\phi'((0,1)) \subset \Sigma\backslash X$. We say that ϕ and ϕ' *strongly intersect* if there exist two lifts $\widetilde{\phi}$ and $\widetilde{\phi}'$ of ϕ and ϕ' as follows.

If we denote

- $\alpha(\widetilde{\phi}) = \widetilde{x}$,
- $\omega(\widetilde{\phi}) = \widetilde{y}$,
- $\alpha(\widetilde{\phi}') = \widetilde{x}'$,
- $\omega(\widetilde{\phi}') = \widetilde{y}'$,

then we have

- $\widetilde{x},\ \widetilde{y},\ \widetilde{x}',\ \widetilde{y}'$ are distinct,
- $\begin{cases}\widetilde{x}' \in (\widetilde{x},\widetilde{y})\\ \widetilde{y}' \in (\widetilde{y},\widetilde{x})\end{cases}$ or $\begin{cases}\widetilde{x}' \in (\widetilde{y},\widetilde{x})\\ \widetilde{y}' \in (\widetilde{x},\widetilde{y})\end{cases}$.

The homeomorphism $\widetilde{f}$ can be extended on the closed unit disk $\mathrm{Adh}_{\mathbb{R}^2}(\mathbb{D})$. A line $\gamma \subset \mathbb{D}$ is said to be a *Brouwer line* of $\widetilde{f}$ if it separates $\mathbb{D}$ in two connected components such that the one on the left-hand side, denoted $L(\gamma)$, contains $\widetilde{f}(\gamma)$ and the one on the right-hand side, denoted $R(\gamma)$, contains $\widetilde{f}^{-1}(\gamma)$. In particular, we have $\widetilde{f}(\mathrm{Adh}_{\mathbb{D}}(L(\gamma))) \subset L(\gamma)$. We have the following definition.

Definition 7.0.1. *For a couple $x, y \in X$, an oriented path $\gamma : [0,1] \to \Sigma$ such that $\alpha(\gamma) = x \in X$, $\omega(\gamma) = y \in X$ and $\gamma((0,1)) \subset \Sigma\backslash X$ is called a connexion from x to y if every lift $\widetilde{\gamma}$ of γ is an oriented Brouwer line of $\widetilde{f}$.*

Remark 7.0.2. If we consider a maximal isotopy $I = (f_t)_{t\in[0,1]}$ from id to f and a foliation $\mathcal{F}$ positively transverse to I then each leaf ϕ of $\mathcal{F}$ is a connexion in the previous sense. Indeed, if we consider a leaf ϕ of $\mathcal{F}$, then a lift $\widetilde{\phi}$ of ϕ is an oriented line which separates $\mathbb{D}$ in two connected components $L(\widetilde{\phi})$ and $R(\widetilde{\phi})$ such that $\widetilde{f}(\widetilde{\phi}) \subset L(\widetilde{\phi})$ and we deduce that $\widetilde{f}(\mathrm{Adh}_{\mathbb{D}}(L(\widetilde{\phi}))) \subset L(\widetilde{\phi})$.

We will prove the following lemma which will be useful in Chapter 8.

Lemma 7.0.3. *Let us consider $x, x', y, y' \in X$ fixed points of f such that there exist a connexion ϕ from x to y and a connexion ϕ' from x' to y'. If ϕ and ϕ' strongly intersect then there exists a connexion from x to y' and a connexion from x' to y.*

Remark 7.0.4. The result stands for non area-preserving homeomorphisms isotopic to the identity and no foliations are involved in the statement. Moreover, the fixed points x, y, x', y' do not have to be distincts to obtain the result. Nevertheless, if we suppose that f is a Hamiltonian homeomorphism, then for every connexion ϕ between two fixed points

x and y, we have $A_f(x) > A_f(y)$ where A_f is the action function of f, see 2.3.18 in the preliminaries. So, if we consider four fixed points satisfying the hypothesis of Lemma 7.0.3, then, by hypothesis, we have $x \neq y$ and $x' \neq y'$ and the result implies that $x \neq y'$ and $x' \neq y$.

We will need a result of Kerékjártó [46] which asserts that each connected component of the intersection of two Jordan domains is a Jordan domain. A *Jordan domain* is the relatively compact connected component of the complement of a simple closed curve of the plane, called a *Jordan curve*. We refer to [55] for the proof of the following result.

Theorem 7.0.5. *Let U and U' be two Jordan domains of the plane. Every connected component of $U \cap U'$ is a Jordan domain.*

We now prove Lemma 7.0.3.

Proof of Lemma 7.0.3. We consider the lifts $\widetilde{\phi}$ and $\widetilde{\phi}'$ of ϕ and ϕ' given by the hypothesis. We denote

- $\alpha(\widetilde{\phi}) = \widetilde{x}$,
- $\omega(\widetilde{\phi}) = \widetilde{y}$,
- $\alpha(\widetilde{\phi}') = \widetilde{x}'$,
- $\omega(\widetilde{\phi}') = \widetilde{y}'$.

By symmetry we can suppose that $\widetilde{x}' \in (\widetilde{y}, \widetilde{x})$ as in Figure 7.1.

We consider the oriented loops $\Gamma_{\widetilde{\phi}} = \widetilde{\phi} \cup [\widetilde{y}, \widetilde{x}]$ and $\Gamma_{\widetilde{\phi}'} = \phi' \cup [\widetilde{y}', \widetilde{x}']$ in $\mathbb{R}^2$. The loops $\Gamma_{\widetilde{\phi}}$ and $\Gamma_{\widetilde{\phi}'}$ are the frontier of the domains $\mathrm{Adh}_{\mathbb{R}^2}(L(\widetilde{\phi}))$ and $\mathrm{Adh}_{\mathbb{R}^2}(L(\widetilde{\phi}'))$ and are Jordan curves.

For every $\widetilde{z} = \mathrm{e}^{i\theta} \in (\widetilde{y}, \widetilde{x})$ there exist $\epsilon > 0$ and $\eta > 0$ such that

$$|\theta' - \theta| < \eta \text{ and } 1 - \epsilon < r < 1 \Rightarrow r\mathrm{e}^{i\theta'} \in L(\widetilde{\phi}).$$

Symmetrically, for every $\widetilde{z} = \mathrm{e}^{i\theta} \in (\widetilde{y}', \widetilde{x}')$ there exist $\epsilon > 0$ and $\eta > 0$ such that

$$|\theta' - \theta| < \eta \text{ and } 1 - \epsilon < r < 1 \Rightarrow r\mathrm{e}^{i\theta'} \in L(\widetilde{\phi}').$$

We denote $\widetilde{x}' = e^{i\alpha}$ and $\widetilde{y} = e^{i\beta}$ with $\beta < \alpha < \beta + 2\pi$. There exists a continuous map $\psi : (\beta, \alpha) \to [0, 1]$ such that for every $\theta \in (\beta, \alpha)$, we have

$$\psi(\theta) < r < 1 \Rightarrow r\mathrm{e}^{i\theta} \in L(\widetilde{\phi}) \cap L(\widetilde{\phi}').$$

We will consider the small croissant $K \subset \mathbb{D}$, as in figure 7.1, defined by

$$K = \{z = r\mathrm{e}^{i\theta} |\ \theta \in (\beta, \alpha),\ \psi(\theta) < r < 1\}.$$

By Theorem 7.0.5, the connected component U of $L(\widetilde{\phi}' \cap L(\widetilde{\phi}')$ which contains K is a Jordan domain in $\mathbb{R}^2$. Since $K \subset U$, the boundary of $\mathrm{Adh}_{\mathbb{R}^2}(U)$ is the union of an arc in

$\mathbb{S}^1$ and an oriented curve J in $\mathbb{D}$ from $\widetilde{x}'$ to $\widetilde{y}$.

Since $\widetilde{\phi}$ and $\widetilde{\phi}'$ are Brouwer lines of $\widetilde{f}$, we have that $\widetilde{f}(\mathrm{Adh}_{\mathbb{D}}(U))$ is connected and satisfies $\widetilde{f}(\mathrm{Adh}_{\mathbb{D}}(U)) \subset L(\widetilde{\phi}) \cap L(\widetilde{\phi}')$. Moreover, we have $\widetilde{f}(K) \subset U$, so we deduce

$$\widetilde{f}(\mathrm{Adh}_{\mathbb{D}}(U)) \subset U.$$

In other words, the line J is a Brouwer line for $\widetilde{f}$. Hence J induces a connexion in Σ from x' to y.

By considering the intersection $R(\widetilde{\phi}) \cap R(\widetilde{\phi}')$, with the same arguments we obtain a connexion in Σ from x to y'.

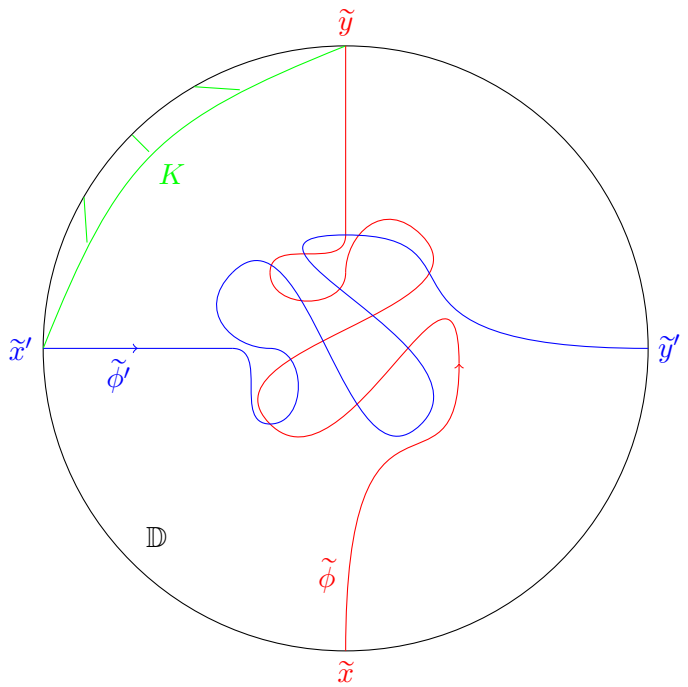


Figure 7.1

□

For the remainderof the section, we also suppose that f is a Hamiltonian homeomorphism, we will define a barcode associated to X.

The notion of connexion induces an order on X where, for $x, y \in X$, we say that $x > y$ if there exists a connexion from x to y.

We saw in section 2.3.7 that the index function $\mathrm{ind}(\mathcal{F}, \cdot)$ defined on X does not depend on the choice of the foliation $\mathcal{F}$ positively transverse to I and we denote this index function $\mathrm{ind}(I, \cdot)$. We will study the following barcode.

Definition 7.0.6. *We define the graph of connexion $G(>)$ whose set of vertices is equal to X and in which there is an edge between two vertices x and y if and only if $x > y$ in Σ. We denote by $\boldsymbol{\beta}_>$ the barcode $\mathcal{B}(G(>), A_f|_X, \mathrm{ind}(I, \cdot))$.*

Remark 7.0.7. If we consider a foliation $\mathcal{F} \in \mathcal{F}(I)$, the graph $G(\mathcal{F})$ is a subgraph of $G(>)$.

Remark 7.0.8 (The Morse example). Let us consider a Morse function H on the 2-sphere as in Figure 4.6. We suppose that H induces a Hamiltonian diffeomorphism f with a finite number of fixed points such that $\mathrm{Fix}(f)$ is unlinked. On Figure 7.2, we draw on the left the graph $G(>)$ associated to the isotopy $I = (f_t)_{t\in[0,1]}$ induced by H. Notice that this graph is distinct from the graph $G(\mathcal{F})$ of Figure 4.6 which were associated to a foliation $\mathcal{F} \in \mathcal{F}(I)$ positively transverse to I. To simplify the reading of the graph, the connexions between sources and sinks are represented by dotted edges. We draw on the right part the barcode $\boldsymbol{\beta}_>$ of this example. We can also notice that the barcode $\boldsymbol{\beta}_>$ of Figure 7.2 is equal to the barcode $B_{\mathrm{gen}}(\mathcal{F})$ of Figure 4.6.

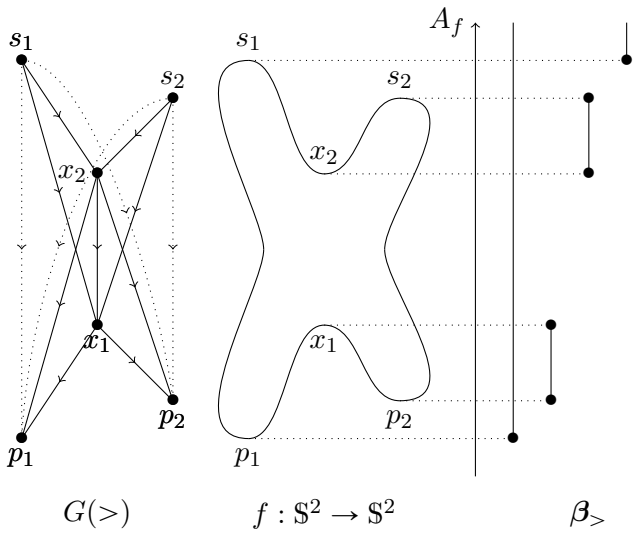


Figure 7.2

Chapter 8

Equalities of the previous constructions and independence of the foliation

In this chapter, we fix a Hamiltonian homeomorphism f of a closed and oriented surface Σ with a finite number of fixed points. Let $X \subset \mathrm{Fix}(f)$ be a maximal unlinked set of fixed points and $I = (f_t)_{t\in[0,1]}$ an isotopy from id to f such that $\mathrm{Sing}(I) = X$. The action functional of f will be denoted A_f.
For a foliation $\mathcal{F}$ positively transverse to the isotopy I, we denote by $G(\mathcal{F})$ the graph, defined in Chapter 6, whose set of vertices is the set X such that for every couple of vertices x and y of $G(\mathcal{F})$ there exists an edge from x to y if and only if there exists a leaf in $\mathcal{F}$ from x to y. We will also consider the subgraphs $(G_t^-(\mathcal{F}))_{t\in\mathbb{R}}$ and $(G_t^+(\mathcal{F}))_{t\in\mathbb{R}}$ given by the natural filtration of $G(\mathcal{F})$ by A_f.

We will use some notation of Chapter 5. For $t \in \mathbb{R}$ we denote $\mathcal{C}_t^-$ and $\mathcal{C}_t^+$ the sets of connected components of $G_t^-(\mathcal{F})$ and $G_t^+(\mathcal{F})$ and for a connected component $\mathcal{C}$ of $G_t^-(\mathcal{F})$ we define $L(\mathcal{C}) = \min\{A_f(y) \mid y \text{ vertex of } \mathcal{C}\}$.

We consider the graph of connexion $G(>)$, defined in Chapter 7, whose set of vertices is equal to X such that there is an edge between two vertices x and y if and only if there exists a connexion between x and y in Σ.

We will consider the barcode $\mathcal{B}(G(\mathcal{F}), A_f|_X, \mathrm{ind}(\mathcal{F}, \cdot))$, denoted $\boldsymbol{\beta}_{\mathcal{F}}$, constructed in Chapter 6 and associated to the foliation $\mathcal{F}$. Recall that the index function $\mathrm{ind}(\mathcal{F}, \cdot)$ defined on X does not depend on the choice of $\mathcal{F} \in \mathcal{F}(I)$ and we denote this index function $\mathrm{ind}(I, \cdot)$. We will consider the barcode $\mathcal{B}(G(>), A_f|_X, \mathrm{ind}(I, \cdot))$, denoted $\boldsymbol{\beta}_>$, defined in Chapter 7.

In the first section, we compare these two barcodes and in the second section we compare the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ with the barcode constructed in Chapter 4 in the generic case.

8.1 Equality between the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ and the barcode $\boldsymbol{\beta}_>$

In this section, we prove the following theorem.

Theorem 8.1.1. *The barcode $\boldsymbol{\beta}_{\mathcal{F}}$ does not depend on the choice of $\mathcal{F} \in \mathcal{F}(I)$ and satisfies $\boldsymbol{\beta}_{\mathcal{F}} = \boldsymbol{\beta}_{>}$.*

For the proof, we fix a foliation $\mathcal{F} \in \mathcal{F}(I)$ positively transverse to the isotopy I. We will need the following lemma.

Lemma 8.1.2 (Fundamental)**.** *For each $t \in \mathbb{R}$, the set of connected components of $G_t^-(\mathcal{F})$ defines the same partition of $X \cap A_f^{-1}((-\infty, t))$ than the set of connected components of $G_t^-(>)$.*

If we suppose Lemma 8.1.2 true, the proof of Theorem 8.1.1 is straigthforward since the action functions and the index functions are equals and the constructions of the barcodes $\boldsymbol{\beta}_{>}$ and $\boldsymbol{\beta}_{\mathcal{F}}$ in Chapter 5 depend only on the connected components of the graphs $G(\mathcal{F})$ and $G(>)$ and the action and index values of the singularities of X.

Proof of Lemma 8.1.2. For $t \in \mathbb{R}$ and for each fixed points $x, y \in X$ of action less than t, we want to prove the equivalence of the following two properties.

(i) The elements x, y are in the same connected component $\mathcal{C}_{>}$ of $G_t^-(>)$,

(ii) The elements x, y are in the same connected component $\mathcal{C}_{\mathcal{F}}$ of $G_t^-(\mathcal{F})$.

(ii) $\Rightarrow$ (i). Since x and y are in the same connected component $\mathcal{C}$ of $G_t^-(>)$, there exists a family $(x_i)_{0 \leqslant i \leqslant k}$ of singularities of X such that

- $x_0 = x$ and $x_k = y$.
- $A_f(x_i) < t$ for every $i \in \{0, ..., k\}$
- For every $i \in \{1, ..., k\}$ there exists a leaf ϕ_i of $\mathcal{F}$ either from x_i to x_{i-1} or from x_{i-1} to x_i

Moreover, each leaf of the foliation $\mathcal{F}$ is by definition a connexion, so $(\phi_i)_{0 \leqslant i \leqslant k}$ is a family of connexions and then the singularities x_i, $i \in \{0, ..., k\}$, are in the same connected component of $G_t^-(>)$ and we obtain the result.

(i) $\Rightarrow$ (ii). We will prove the following lemma.

Lemma 8.1.3. *Let us consider $\mathcal{F} \in \mathcal{F}(I)$ and $(x, y) \in X^2$. If $x > y$ then x and y are in the same connected component of $G_t^-(\mathcal{F})$ for every $t > A_f(x)$.*

Let us assume that Lemma 8.1.3 is true and consider $x, y \in X$ and $t > A_f(x)$ such that (i) is satisfied, we prove that (ii) holds.

Since x and y are in the same connected component $\mathcal{C}$ of $G_t^-(>)$, there exists a family $(x_i)_{0 \leqslant i \leqslant k}$ of singularities of X such that

- $x_0 = x$ and $x_k = y$.
- $A_f(x_i) < t$ for every $i \in \{0, ..., k\}$
- For every $i \in \{1, ..., k\}$ there exists a connexion ϕ_i either from x_i to x_{i-1} or from x_{i-1} to x_i

Moreover, by Lemma 8.1.3 x_{i-1} and x_i are in the same connected component of $G_t^-(\mathcal{F})$ for every $i \in \{1, ..., k\}$ so x and y are also in the same connected component of $G_t^-(\mathcal{F})$. $\square$

To complete the proof of Lemma 8.1.2, it remains to prove Lemma 8.1.3.

Proof of Lemma 8.1.3. We consider two fixed points $x, y \in X$ such that $x > y$. Every attractive or repulsive basin in this proof will be defined relatively to the foliation $\mathcal{F}$. We will divide the proof in three cases, in the first one x will be a sink of $\mathcal{F}$, in the second one x will be a saddle and in the last one x will be a source of $\mathcal{F}$.

First case. We suppose that x is a sink. We will prove that there is no connexion from x to another singularity of $\mathcal{F}$.

We say that the orbit of a q-periodic point z of f is *contractible* if the concatenation of the trajectories of the points $f^k(z)$, $k \in \{0, ..., q-1\}$, along the isotopy I is a contractible loop , denoted γ_z, in Σ. The loop γ_z is called the trajectory of the periodic orbit of z. We say that a contractible q-periodic orbit has *type* (p, q) associated to I at $x \in \mathrm{Fix}(f)$ if its trajectory along I is homotopic to $p\Gamma$ in $\Sigma \backslash \mathrm{Sing}(I)$, where Γ is the boundary of a sufficiently small Jordan domain containing x.

We will use the following version of a result of Yan and we refer to [75], Theorem 1.1, for a proof.

Theorem 8.1.4. *Let us consider a fixed point z of f of Lefschetz index equal to 1 fixed by the isotopy I, and such that the rotation set $\rho_s(I, z)$ is reduced to $\{0\}$. The point z is accumulated by periodic points. More precisely, the following property holds: there exists $\epsilon > 0$, such that, for every neighborhood of z, either for every irreducible $p/q \in (0, \epsilon)$, or for every irreducible $p/q \in (-\epsilon, 0)$, there exists a contractible periodic orbit $O_{p/q}$ of type (p, q).*

Let us prove that there is no connexion from x to another singularity of $\mathcal{F}$. We suppose that there exists a connexion ϕ from x to another singularity y of $\mathcal{F}$, we want to find a contradiction.

The singularity x is a sink of the foliation $\mathcal{F}$ which is positively transverse to the isotopy I, so the local rotation set $\rho_{s,I}(x)$ of x, introduced in section 2.3.5, is included in $(-\infty, 0]$. Moreover, it is not difficult to prove that the existence of the connexion ϕ implies that the rotation set $\rho_{s,I}(x)$ of x is included in $[0, +\infty)$. Indeed, locally, a connexion whose alpha-limit is x is a *positive arc*, which means that in polar coordinates where γ corresponds to the semi-line $\{\theta = 0\}$, for every point z close enough to x, the variation of θ along the trajectory is positive. We refer to Theorem 3.2.4 and section 2.4 of [59] for more details. So the local rotation set of x for the isotopy I is reduced to the integer $\{0\}$.

If the Lefschetz index of x is not equal to 1, by a result of Le Roux, see [59] Theorem 4.1.1, the foliation $\mathcal{F}$ and the homeomorphism f have the same index at x for the isotopy I. But, by hypothesis, x is a sink of the foliation $\mathcal{F}$ so $\mathrm{ind}(\mathcal{F}, x) = 1$ and we obtain a contradiction.

If the Lefschetz index of x is equal to 1 then we can apply Theorem 8.1.4. More precisely, the singularity x is a sink of $\mathcal{F}$ so there exists $\epsilon > 0$ such that x is accumumated by periodic orbits $O_{p/q}$ of type (p, q) where $p/q \in (0, \epsilon)$. So the rotation number $\rho_{s,I}(x)$ is not reduced to $\{0\}$ and we obtain a contradiction.

Second case. We suppose that x is a saddle point. We prove the result by contradiction.

We suppose that there exists a connexion $\gamma : [0, 1] \to \Sigma$ from x to some y and that x and y are not in the same connected component of $G^-_{A_f(x)^+}(\mathcal{F})$. We denote by $W^s(\mathcal{C}^-_x) \subset \Sigma$ the attractive basin of $\mathcal{C}^-_x$, where $\mathcal{C}^-_x$ is the connected component of $G^-_{A_f(x)^+}(\mathcal{F})$ which contains x. By Lemma 6.1.7, the fixed point x is in the interior of $W^s(\mathcal{C}^-_x)$ and every singularity z of $\mathcal{F}$ in the frontier of $W^s(\mathcal{C}^-_x)$ satisfies $A_f(z) > A_f(x)$. The existence of the connexion γ implies that $A_f(x) > A_f(y)$ so the fixed point y is not in the frontier of $W^s(\mathcal{C}^-_x)$ and is in $\Sigma \backslash \overline{W^s(\mathcal{C}^-_x)}$.
We consider the universal cover $\widetilde{\Sigma \backslash X}$ of $\Sigma \backslash X$ which is identified to the unit disk $\mathbb{D}$ and $\pi : \mathbb{D} \to \Sigma \backslash X$ the universal covering map. Let U be a connected component of $\pi^{-1}(\mathcal{C}^-_x \backslash X)$ and $\widetilde{\gamma}$ a lift of the connexion γ such that there exists $\epsilon > 0$ such that $\widetilde{\gamma}((0, \epsilon)) \subset U$. By hypothesis on y, there exists $\epsilon' > 0$ such that $\widetilde{\gamma}((1 - \epsilon', 1)) \notin U$.
Recall that π naturally extends to $\mathbb{S}^1$. Moreover, we saw that $\lim_{t \to 0} \widetilde{\gamma}(t)$ and $\lim_{t \to 1} \widetilde{\gamma}(t)$ are well-defined on $\mathbb{S}^1$ and will be denoted $\widetilde{x}$ and $\widetilde{y}$.

The set U is an open connected set of $\mathbb{D}$ whose frontier is a union of lifted leaves. Then there exists a lifted leaf $\widetilde{\psi}$ of $\mathrm{Fr}(U)$ which separates $\mathbb{D}$ such that U is on one side and $\widetilde{y}$ is in the other. By hypothesis, the points $\omega(\widetilde{\psi})$ and $\alpha(\widetilde{\psi})$ are distinct from $\widetilde{x}$ and $\widetilde{y}$, indeed, x is in the interior of $W^s(\mathcal{C}^-_x)$ and $y \notin \mathrm{Fr}(W^s(\mathcal{C}^-_x))$. Since $\widetilde{\psi}$ is the lift of a connexion, we obtain that ψ and γ are two connections which intersect strongly, hence by Lemma 7.0.3 there exists a connexion from x to $\omega(\psi)$ which is impossible because by definition $\omega(\psi) \in \mathrm{Fr}(W^s(\mathcal{C}^-_x))$ and then $A_f(\omega(\psi)) > A_f(x)$.

Third case. Suppose that x is a source of $\mathcal{F}$. The point y is either in the frontier of the repulsive basin $W^u(x)$ of x or in the complement of $\overline{W^u(x)}$. We separate these two cases.

1) We suppose that y is in the frontier of $W^u(x)$. There exists a chain of connexions from x to y and so we deduce the result. Indeed, by definition, the singularity y is accumulated by leaves $(\phi_j)_{j \in J}$ of $W^u(x)$ whose alpha-limit point is x. By the local model described in Chapter 6.1, the closure of these leaves contains a chain of connexions which starts at x and also contains y.

2) We suppose that y is in the complement of $\overline{W^u(x)}$. We consider U a connected component of $\pi^{-1}(W^u(x) \backslash \{x\})$ where $\pi : \mathbb{D} \to \Sigma \backslash X$ the covering map defined in the second case. We can consider a lift $\widetilde{\gamma}$ of the connexion γ such that there exists $\epsilon > 0$ such that $\widetilde{\gamma}((0, \epsilon)) \subset U$. By hypothesis on y, there exists $\epsilon' > 0$ such that $\widetilde{\gamma}((1 - \epsilon', 1)) \notin U$. The limits $\lim_{t \to 0} \widetilde{\gamma}(t)$ and $\lim_{t \to 1} \widetilde{\gamma}(t)$ are well-defined on $\mathbb{S}^1$ and will be denoted $\widetilde{x}$ and $\widetilde{y}$.

We apply similar arguments as in the second case. The set U is an open connected set of $\mathbb{D}$ whose boundary is a union of lifted leaves. Then there exists a lifted leaf $\widetilde{\psi}$ of

$\mathrm{Fr}(U)$ separating $\mathbb{D}$ such that U is on one side and $\widetilde{y}$ is on the other. By hypothesis, the points $\omega(\widetilde{\psi})$ and $\alpha(\widetilde{\psi})$ are distinct from $\widetilde{x}$ and $\widetilde{y}$, indeed, x is in the interior of $W^s(\mathcal{C}_x^-)$ and $y \notin \mathrm{Fr}(W^s(\mathcal{C}_x^-))$. Since $\widetilde{\psi}$ is the lift of a connexion, we obtain that ψ and γ are two connections which intersect strongly, hence by Lemma 7.0.3 there exists a connexion from $\alpha(\psi)$ to y. The singularity $\alpha(\psi)$ can not be a source nor a sink so it is a saddle point of $\mathcal{F}$, hence we apply the result of the second case which asserts that the existence of a connexion from $\alpha(\psi)$ to y implies that $\alpha(\psi)$ and y are in the same connected component of $G^-_{A_f(\alpha(\psi))^+}(\mathcal{F})$. Moreover, by hypothesis, $\alpha(\psi)$ and x are in the same connected component of $G^-_{A_f(x)^+}(\mathcal{F})$ thus we deduce that x and y are also in the same connected component of $G^-_{A_f(x)^+}(\mathcal{F})$.

□

8.2 Equality between the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ and the barcode $B_{\mathrm{gen}}(f, \mathcal{F})$ in the generic case

We consider a Hamiltonian homeomorphism f of a closed and oriented surface Σ with a finite number of fixed points. We suppose that $\mathrm{Fix}(f)$ is finite and unlinked, each fixed point $x \in \mathrm{Fix}(f)$ satisfies $\mathrm{ind}(f, x) \in \{-1, 1\}$ and that the action function $A_f : \mathrm{Fix}(f) \to \mathbb{R}$ is injective. Let $I = (f_t)_{t\in[0,1]}$ be a maximal isotopy from id to f fixing all fixed points of f. We denote A_f the action functional of f.

Recall that a foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$ does not have connexions between saddle points, and the stable and unstable cones of a saddle point x of $\mathcal{F}$ are both composed of a unique leaf which will be referred to as the stable and unstable leaves of x.

Let us consider the graph $G_{\mathrm{gen}}(\mathcal{F})$ given by Definition 4.0.2. Remember that $G_{\mathrm{gen}}(\mathcal{F})$ is the graph whose set of vertices is the set $\mathrm{Fix}(f)$ and whose edges correspond to leaves ϕ of $\mathcal{F}$ such that $\mathrm{ind}_{CZ}(f, \alpha(\phi)) = \mathrm{ind}_{CZ}(f, \omega(\phi)) - 1$, where $\mathrm{ind}_{CZ}(f, \cdot)$ is the Conley-Zehnder index, defined in Chapter 4, equals to 1 on sources and sinks and equals to -1 on saddle points. Notice that G_{gen} is distinct from the graph $G(\mathcal{F})$ given in the introduction of this chapter.

In this section we want to compare the barcode $B_{\mathrm{gen}}(\mathcal{F})$ to the barcode

$$\mathcal{B}(G(\mathcal{F}), A_f|_X, \mathrm{ind}(\mathcal{F}, \cdot)),$$

denoted $\boldsymbol{\beta}_{\mathcal{F}}$, constructed in Chapter 6. We will prove the following result.

Theorem 8.2.1. *Let us consider a Hamiltonian homeomorphism f on a compact surface Σ. We suppose that $\mathrm{Fix}(f)$ is finite and unlinked, each fixed point $x \in \mathrm{Fix}(f)$ satisfies $\mathrm{ind}(f, x) \in \{-1, 1\}$ and the action function is injective. We consider a maximal isotopy I such that $\mathrm{Sing}(I) = \mathrm{Fix}(f)$ then for a foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$ we have*

$$B_{\mathrm{gen}}(\mathcal{F}) = \boldsymbol{\beta}_{\mathcal{F}}.$$

We recall the definition of the functor β. Let $\mathbf{V} = (V_s)_{s\in\mathbb{R}}$ be a persistence module. Let us consider the set of $t \in \mathbb{R}$ in the spectrum of V such that $\dim(\mathrm{Ker}(i_{t^-,t^+}))$ is equal to 1 and label its elements $b_1, ..., b_n$. For each b_j, there exists a unique $a_j \in \mathbb{R}$ with the following

property: Let $x \in V_{b_j^-}$ represents a non-zero element in $\mathrm{Ker}(i_{t^-,t^+}) = 1$, the element x is in the image of $i_{a_j^+,b_j^-}$ but x is not in the image of $i_{a_j^-,b_j^-}$; if we label the remaining elements of the spectrum of $\mathbf{V}$ by $\{c_1, ..., c_m\}$ then the barcode $\beta(\mathbf{V})$ consists of the list of intervals $(a_j, b_j]$ and $(c_k, +\infty)$, where $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant m$.

We will consider $G^-_{\mathrm{gen},t}(\mathcal{F}), G^+_{\mathrm{gen},t}(\mathcal{F})$ the associated filtered graphs and we denote by $(H^t_*)_{t\in\mathbb{R}}$ the persistence module of the chain complex $(C^t_i)_{i\in\{0,1,2\},t\in\mathbb{R}}$ of Definition 4.0.3. Finally, we denote by $B_{\mathrm{gen}}(\mathcal{F})$ the barcode $\beta((H^t_*)_{t\in\mathbb{R}})$ where β is the functor defined in Chapter 3.

To avoid any confusion, we will always refer to the chain complex by C^t_i where $i \in \{0, 1, 2\}$ and $t \in \mathbb{R}$. We will refer to connected components of the graph $G(\mathcal{F})$ by $\mathcal{C}$ or $\mathcal{C}'$ and to a connected component of the graph $G_{\mathrm{gen}}(\mathcal{F})$ by $\mathcal{C}_{\mathrm{gen}}$. For $t \in \mathbb{R}$ and a connected component $\mathcal{C}$ of $G^-_t(\mathcal{F})$ we denote $L(C)$ the minimum of the action function on the sinks of $\mathcal{C}$ and for $t \in \mathbb{R}$ and a connected component $\mathcal{C}'$ of $G^+_t(\mathcal{F})$ we denote $D(C)$ the maximum of the action function on the sources of $\mathcal{C}'$. Moreover, to simplify the notation, we provide the filtered chain complex $(C^i_t)_{i\in\{0,1,2\},t\in\mathbb{R}}$ with a natural scalar product $\langle .|. \rangle$ associated to the canonical basis. Meaning that we consider the bilinear function $\langle .|. \rangle$ on the space C^i_t such that for every couple of fixed points x and y of f in C^i_t, we have $\langle x, y\rangle = 1$ if and only if $x = y$ and $\langle x, y\rangle = 0$ otherwise.

Theorem 8.2.1 allows us to prove Property 4.0.11 of Chapter 4 which states that the barcode $B_{\mathrm{gen}}(f, \mathcal{F})$ does not depend on the choice of $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$. We recall that assumptions of Theorem 8.2.1 are satisfies in this particular case.

Proof of Proposition 4.0.11. By Theorem 8.2.1, for each foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$ we have that $B_{\mathrm{gen}}(\mathcal{F}) = \boldsymbol{\beta}_{\mathcal{F}}$. Moreover, by Theorem 8.1.1, the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ does not depend on the choice of the foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$. So we obtain that $B_{\mathrm{gen}}(f, \mathcal{F})$ does not depend on the choice of $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}(I)$ which is the result. □

We fix a foliation $\mathcal{F} \in \mathcal{F}_{\mathrm{gen}}$ for the remaining of the section.

Proof of Theorem 8.2.1. By Remark 4.0.9, each action value of A_f is the end of a unique bar of the barcode $B_{\mathrm{gen}}(\mathcal{F})$ and by Corollary 6.2.7, we have the same result for the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ so it is enough to prove the inclusion $\boldsymbol{\beta}_{\mathcal{F}} \subset B_{\mathrm{gen}}(\mathcal{F})$ to prove that these barcodes are equal.

Moreover, Corollary 6.2.7 states that exactly one end point of every bar of the barcode $\boldsymbol{\beta}_{\mathcal{F}}$, except the bars $(\min A_f, +\infty)$ and $(\max A_f, +\infty)$, is the action value of a saddle point of $\mathcal{F}$. So it is enough to prove that finite bars of the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ are also bars of the barcode B_{gen} to prove the inclusion $\boldsymbol{\beta}_{\mathcal{F}} \subset B_{\mathrm{gen}}(\mathcal{F})$. Indeed, the remaining bars of the barcode would be the same semi-infinite bars as they would be associated to the same saddle points.

We will prove that for every saddle point x of f, if the bar J of $\boldsymbol{\beta}_{\mathcal{F}}$, of which $A_f(x)$ is an end, is a finite bar, then it is also a bar of the barcode $B_{\mathrm{gen}}(\mathcal{F})$. Notice that, by construc-

tion, the bars $(\min A_f, +\infty)$ and $(\max A_f, +\infty)$ of $\boldsymbol{\beta}_{\mathcal{F}}$ are also bars of the barcode $B_{\text{gen}}(\mathcal{F})$.

For the remainder of the proof, we consider a saddle point x of f, we denote by t its action value and by $\mathcal{C}_x$ the connected component of $G^-_{t^+}(\mathcal{F})$ which contains x. By Lemma 6.2.4 the set of connected components of $G^-_t(\mathcal{F})$ which are included in $\mathcal{C}_x$, which were labeled $j_t^{-1}(\mathcal{C}_x)$ in Chapter 5, has one or two elements. We separate those two cases.

Case 1. The set $j_t^{-1}(\mathcal{C}_x)$ consists of two connected components of $G^-_t(\mathcal{F})$ denoted $\mathcal{C}$ and $\mathcal{C}'$. By symmetry, we can suppose that $L(\mathcal{C}) > L(\mathcal{C}')$ and, by the construction described in Chapter 5, there is a bar $(L(\mathcal{C}), t]$ in the barcode $\boldsymbol{\beta}_{\mathcal{F}}$. Let us prove that this bar is also a bar of the barcode $B_{\text{gen}}(\mathcal{F})$. It means that there is an element of $\text{Ker}(i_{t^-,t^+})$ which is in the image of $i_{L(\mathcal{C})^+,t^-}$ but not in the image of $i_{L(\mathcal{C})^-,t^-}$.

By hypothesis, the omega-limit points of the unstable leaves of x are distinct sinks y and y' of $\mathcal{F}$ where $y \in \mathcal{C}$ and $y' \in \mathcal{C}'$. We have $\partial_1^{t^+}(x) = y + y'$ so the element $y + y' \in C_0^{t^+}$ satisfies $[y + y'] \in \text{Ker}(i_{t^-,t^+})$. It remains to prove that $[y + y']$ is in the image of $i_{L(\mathcal{C})^+,t^-}$ and not in the image of $i_{L(\mathcal{C})^-,t^-}$. For that, we will consider another cycle in $C_0^{t^+}$ representing $[y + y']$ in homology.

We will use some geometric lemmas.

We will call a *path of leaves* a path Γ in Σ which is the concatenation of leaves of $\mathcal{F}$. The singularities of a path of leaves will refer to the alpha-limit points and omega-limit points of those leaves.

Lemma 8.2.2. *Let us consider $s \in \mathbb{R}$, and two sinks y_1 and y_2 of $\mathcal{F}$ in the same connected component $\mathcal{C}_s$ of $G^-_s(\mathcal{F})$. There exists a path of leaves from y_1 to y_2 whose singularities are alternatively sinks and saddle points of $\mathcal{C}_s$.*

Proof of Lemma 8.2.2. By definition of the connected component $\mathcal{C}_s$ of G^-_s, there exists a path of leaves Γ from y_1 to y_2 in Σ. The path Γ may contain sources. For a source z in Γ we will modify Γ into a path which does not contain z.

If there is a source z in Γ, there exist two leaves $\phi_1 \subset \Gamma$ and $\phi_2 \subset \Gamma$ whose alpha-limit points are equal to z and omega-limit points are either saddle points or sinks of $\mathcal{F}$ that we denote x_1 and x_2. The singularities x_1 and x_2 are in the repulsive basin of z for $\mathcal{F}$ so, by Lemma 4.0.7, there exists a path γ of leaves of $G^-_s(\mathcal{F})$ from x_1 to x_2 whose singularities are alternatively saddle points and sinks of $\mathcal{F}$.
We cut the union $\phi_1 \cup \{z\} \cup \phi_2$ from the path Γ and replace this portion by the path γ given by Lemma 4.0.7. We obtain a new path Γ' from y to y' such that the source z is not in Γ'.

We do the same process for every source of Γ and we finally obtain a path from y to y' as wanted. □

We prove the following lemma.

Lemma 8.2.3. *For every $s \in \mathbb{R}$ and every couple of sinks y_1 and y_2 of $\mathcal{F}$ in the same connected component $\mathcal{C}_s$ of $G_s^-(\mathcal{F})$ we have $[y_1] = [y_2]$ in H_0^s.*

Proof of Lemma 8.2.3. Let us consider $s \in \mathbb{R}$, and two sinks y_1 and y_2 of $\mathcal{F}$ in the same connected component $\mathcal{C}_s$ of $G_s^-(\mathcal{F})$. By Lemma 8.2.2 there exists a path of leaves in Σ from y_1 to y_2 whose singularities are alternatively sinks and saddle points of $\mathcal{C}_s$. We denote by $(x_i)_{0\leqslant i\leqslant n]}$ the saddle points of the path Γ and by a simple computation we obtain

$$\partial_1^s\left(\sum_{i=0}^{n} x_i\right) = y_1 + y_2.$$

So, by definition, $[y_1] = [y_2]$ in H_0^s . □

Let us come back to the first case of the proof of Theorem 8.2.1. By Lemma 8.2.3 each sink $z \in \mathcal{C}$ of $\mathcal{F}$ (resp. each sink $z' \in \mathcal{C}'$ of $\mathcal{F}$) satisfies $[z] = [y]$ (resp. $[z'] = [y']$) in H_0^t. So for every couple of sinks $z \in \mathcal{C}$ and $z \in \mathcal{C}'$ of $\mathcal{F}$, the element $z + z' \in C_0^{t^+}$ satisfies $[z + z'] = [y + y'] \in \mathrm{Ker}(i_{t^-,t^+})$. We denote by $z_{\mathcal{C}}$ and $z_{\mathcal{C}'}$ the sinks of $\mathcal{C}$ and $\mathcal{C}'$ such that $A_f(z_{\mathcal{C}}) = L(\mathcal{C}),\ A_f(z_{\mathcal{C}'}) = L(\mathcal{C}')$.
We supposed that $A_f(z_{\mathcal{C}}) = L(\mathcal{C}) > A_f(z_{\mathcal{C}'}) = L(\mathcal{C}')$ so the sink $z_{\mathcal{C}}$ is not a cycle in $C_0^{L(\mathcal{C})^-}$ so the element $[z_{\mathcal{C}} + z_{\mathcal{C}'}]$ is not in the image of $i_{L(\mathcal{C})^-,t^-}$. Moreover, the sinks $z_{\mathcal{C}}$ and $z_{\mathcal{C}'}$ are not in the same connected component of $G_{t^-}(\mathcal{F})$ and so we deduce that $[z_{\mathcal{C}} + z_{\mathcal{C}'}]$ is in the image of $i_{L(\mathcal{C})^+,t^-}$.

So, by construction, there exists a bar $(L(C), t]$ in the barcode $B_{\mathrm{gen}}(\mathcal{F})$.

Case 2. The set $j_t^{-1}(\mathcal{C}_x)$ is a unique element. We will consider the connected components of the subgraphs $(G_t^+)_{t\in\mathbb{R}}$ instead of connected components of the subgraphs $(G_t^-)_{t\in\mathbb{R}}$. We consider the connected component $\mathcal{C}'_x$ of $G_{t^-}^+(\mathcal{F})$ which contains x. By Lemma 6.2.4 the set of connected components of $G_t^+(\mathcal{F})$ included in $\mathcal{C}'_x$, which were labeled $j_t'^{-1}(\mathcal{C}_x^-)$ in Chapter 5, is composed of 1 or 2 elements. We separate those two cases.

1) Suppose that $j_t'^{-1}(\mathcal{C}'_x)$ is composed of one connected component, then, by construction, there is no finite bar J in the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ of which t is an end point. We have nothing to prove in this case.

2) Now we suppose that $j_t'^{-1}(\mathcal{C}'_x)$ is composed of two connected components of the graph $G_{t^+}^+(\mathcal{F})$ denoted $\mathcal{C}$ and $\mathcal{C}'$. By symmetry we can suppose that $D(\mathcal{C}) < D(\mathcal{C}')$ and by construction there is a bar $(t, D(\mathcal{C})]$ in the barcode $\boldsymbol{\beta}_{\mathcal{F}}$. Let us prove that this bar is also a bar of the barcode $B_{\mathrm{gen}}(\mathcal{F})$. It means that there is an element in $\mathrm{Ker}(i_{D(\mathcal{C})^-,D(\mathcal{C})^+})$ which is in the image of $i_{t^+,D(\mathcal{C})^-}$ but not in the image of $i_{t^-,D(\mathcal{C})^-}$. We will need the following lemma about the repulsive basin $W^s(\mathcal{C})$ of $\mathcal{C}$.

Lemma 8.2.4. *We label $x_1, ..., x_n$ the saddle points in the frontier of $\overline{W^s(\mathcal{C})}$. Then, for every $T > D(\mathcal{C})$, the element $Y = \sum_{\substack{y\in\mathcal{C}\\ y \text{ source}}} y$ of C_2^T satisfies*

$$\partial_2^T(Y) = \sum_{i=1}^{n} x_i.$$

Proof. For each source y of $\mathcal{C}$, by definition, $\partial_2^T(y)$ is equal to the sum of the saddle points in the frontier of the repulsive basin of y. These saddle points have either one or both of their stable leaves in $W^s(\mathcal{C})$. We separate those cases.

Firstly, we label $x_1, ..., x_n$ the saddle points of $\mathcal{F}$ of which only one stable leaf have its alpha-limit point in $\mathcal{C}$. For every $i \in [0, n]$ we have $\langle \partial_2^T(Y)|x_i\rangle = 1$ for every $i \in [0, n]$. Moreover, the action values of these saddle points is less then or equal to t and it is simple to see that this belong to the frontier of $\overline{W^s(\mathcal{C})}$.

Secondly, we label $x'_1, ..., x'_m$ the saddle points such that both stable leaves have their alpha-limit points in $\mathcal{C}$. For every $i \in [0, m]$ we have $\langle \partial_2^T(Y)|x'_i\rangle = 2$. Those saddle points are nondegenerate saddle points of $\mathcal{F}$ so that they are in the interior $\overline{W^s(\mathcal{C})}$ and not in its frontier. Indeed, both stable cones of a saddle point x whose action satisfies $A_f(x) > t$ are leaves of $\mathcal{F}$ whose alpha-limit points are in the same connected component of $G_t^+(\mathcal{F})$.

Finally, we compute $\partial_2^T(Y)$ as follows.

$$\begin{aligned}\partial_2^T(Y) &= \sum_{\substack{y\in\mathcal{C},\\ y \text{ source}}} \partial_2^T(y)\\ &= \sum_{i=1}^n x_i + \sum_{i=1}^m 2x'_i\\ &= \sum_{i=1}^n x_i.\end{aligned}$$

And we obtain the result. □

Let us denote $c = D(\mathcal{C})$, we will consider the element $Y = \sum_{\substack{y\in\mathcal{C}\\ y \text{ source}}} y$ in $C_2^{c^+}$. By Lemma 8.2.4, Y satisfies

$$\partial_2^{c^+}(Y) = \sum_{i=1}^n x_i,$$

where $x_1, ..., x_n$ are the saddle points of the frontier of $\overline{W^s(\mathcal{C})}$. So we have $[\sum_{i=1}^n x_i] \in \mathrm{Ker}(i_{c^-,c^+})$.

By hypothesis, the saddle point x is one of the saddle points $(x_i)_{i\in[1,n]}$ and each x_i satisfies $A(x_i) \leqslant A(x)$ since $\mathcal{C}$ is a connected component of $G_t^+(\mathcal{F})$. So we have that $[\sum_{i=1}^n x_i]$ is in the image of i_{t^+,c^-}.

Moreover, the singularity x is not homologous in $C_1^{c^-}$ to a chain of singularities of $C_1^{t^-}$. Indeed, if it was the case then, by definition, it would exist $X' \in C_1^{t^-}$ and $Y' \in C_2^{c^-}$ such that $x = X' + \partial_2^{c^-}(Y')$.

We set $y_x \in \mathcal{C}$ and $y'_x \in \mathcal{C}'$ the only two sources of $\mathcal{F}$ such that x is in the frontier of the sets $W^u(y_x)$ and $W^u(y'_x)$. The equality $x = X' + \partial_2^{c^-}(Y')$ would imply that $\langle Y'|y_x\rangle = 1$

or $\langle Y'|y'_x\rangle = 1$, which is impossible because, by hypothesis, $A_f(y'_x) > A_f(y_x) = c$. Indeed, if we have $\partial_2^{c^-}(Y') = x - X'$ then, there exists a source y such that $\langle Y|y\rangle = 1$ and $\langle \partial_2^{c^-}(y)|x\rangle = 1$, which means that x is in the frontier of $W^u(y)$. So y is either equal to y_x or y'_x.

So, we have the same result for $\sum_{i=1}^n x_i$ and so $[\sum_{i=1}^n x_i] \notin \mathrm{Im}(i_{t^-,c^-})$.

Thus, by construction, there exists a bar $(A(x), D(\mathcal{C})]$ in the barcode $B_{\mathrm{gen}}(\mathcal{F})$.

□

Now we can prove Theorem 4.0.12 from Chapter 4 stated as follows.

Theorem 8.2.5. *If we consider a Hamiltonian diffeomorphism f with a finite number of fixed points which is C^2-close to the identity and generated by an autonomous Hamiltonian function then the barcode $B_{\mathrm{gen}}(\mathcal{F})$ is equal to the Floer homology barcode of f.*

Proof. If we suppose that the autonomous Hamiltonian function H is C^2 close to a constant then the Floer homology of H is equal to the Morse homology of H, we refer to [3] for a proof. We deduce that the Morse Homology barcode $\beta(\mathrm{HM}_*^t(H))_{t\in\mathbb{R}})$ of H is equal to the Floer Homology barcode $\beta(\mathrm{HF}_*^t(H))_{t\in\mathbb{R}})$ of H, where β is the functor defined in Chapter 3 which associate a persistence module to its barcode.

The time one map $f_1 = f$ of the Hamiltonian flow is C^1 close the the identity and its set of fixed points is unlinked.

Moreover, the gradient-lines of H provides a C^1 foliation $\mathcal{F}$ positively transverse to the natural Hamiltonian isotopy induces by H. This isotopy is maximal and so fixes every fixed points of f. The foliation $\mathcal{F}$ is gradient-like and there is no cone of leaves at the saddle points of $\mathcal{F}$.

Moreover, the construction of the map $\mathcal{B}$ in Chapter 5 follows the ideas of the Morse homology theory then we can assert that the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ is equal to the barcode $\beta(\mathrm{HM}_*^t(H))_{t\in\mathbb{R}}$,

Thus, by Theorem 8.2.1 we have

$$B_{\mathrm{gen}}(f) = \boldsymbol{\beta}_{\mathcal{F}} = \beta(\mathrm{HM}_*^t(H))_{t\in\mathbb{R}} = \beta(\mathrm{HF}_*^t(H))_{t\in\mathbb{R}}.$$

So we obtain the result. □

Chapter 9

Perspectives

Let us consider a Hamiltonian homeomorphism f of a surface with a finite number of fixed points. In the previous chapter, every construction of barcodes depends on the choice of a maximal unlinked set of fixed points X of f. A natural question remains.

Question 9.0.1. *Can we construct a barcode associated to f which is equal to the Floer Homology barcode in the case of a generic diffeomorphism?*

We have some ideas to study this question. In this short chapter we explain the difficulties to generalize our constructions and the objects we may use in a near future.

Conley-Zehnder index

The first problem we will have to deal with is the Conley-Zehnder index. The Floer Homology is indexed by the Conley-Zehnder index, denoted $\mathrm{ind}_{CZ}(\cdot)$, which is well-defined for C^1 diffeomorphisms.

Nevertheless, the Conley-Zehnder index can be extended for Hamiltonian homeomorphisms at isolated fixed points. We found an interesting way to describe this extension using generalized isotopies, defined in the preliminaries. Let us recall a quick definition. If we consider a Hamiltonian homeomorphism f on Σ, we denote $\widetilde{\Sigma}$ the universal cover of Σ and we can consider its compactification into a 2-sphere $\mathbb{S}^2$ by adding a point ∞, then a lift $\widetilde{f}$ of f can be compactified into a homeomorphism $\hat{f}$ of $\mathbb{S}^2$. An isotopy from id to $\hat{f}$ on $\mathbb{S}^1$ which fixes ∞ is called a generalized isotopy. In the next section we will see why considering generalized isotopies can be useful in our studies.

Moreover, let us consider the Floer homology barcode of a Hamiltonian diffeomorphism f. We denote A_f the action function of f. For every finite bar $J = (A_f(x), A_f(y)]$ where $x, y \in \mathrm{Fix}_c(f)$, we have $\mathrm{ind}_{CZ}(y) - \mathrm{ind}_{CZ}(x) = 1$. We refer to the construction of Floer Homology for more details [3].

Now, let us consider a Hamiltonian homeomorphism f with a finite number of fixed points, a maximal unlinked set of fixed points X of f and an isotopy I from id to f which fixes X and a foliation $\mathcal{F}$ positively transverse to I. We denote $\mathrm{ind}_{CZ}(\cdot)$ the extension of the Conley-Zehnder index defined on $\mathrm{Fix}_c(f)$. For every fixed point $x \in X$ one may prove that we have

- if $\mathrm{ind}_{CZ}(x) = 1$, then x is a saddle point for $\mathcal{F}$,

- if $\mathrm{ind}_{CZ}(x) \neq 1$, then x is either a sink or a source for $\mathcal{F}$.

So, the barcode $\boldsymbol{\beta}_{\mathcal{F}}$ defined in Chapter 6 may have finite bars $J = (A_f(x), A_f(y)]$ such that $\mathrm{ind}_{CZ}(y) - \mathrm{ind}_{CZ}(x) \neq 1$. Indeed, it can be the case if $\mathrm{Fix}_c(f)$ is not unlinked. Thus it is not enough to work only with maximal isotopies of f.

Torsion-low isotopies

Let us consider of a Hamiltonian homeomorphism f on a surface Σ. Yan [76] introduced the notion of torsion-low isotopies. Roughly speaking, an isotopy I from id to f is said to be *torsion-low* at $x \in \mathrm{Sing}(I)$ if $\rho_s(I, x) \subset [-1, 1]$, where $\rho_s(I, x)$ is the local rotation set of x for I, defined in the preliminaries.

Now, we consider a maximal generalized isotopy $\hat{I}$ of a Hamiltonian homeomorphism f on Σ. Every fixed point $\widetilde{x}$ of $\hat{I}$ is a lift of a fixed point x of f. The isotopy $\hat{I}$ is said to be a *maximal torsion-low generalized isotopy* if $\hat{I}$ is torsion-low at every fixed point of $\mathrm{Sing}(\hat{I})$ except at ∞ and $\hat{I}$ is a maximal isotopy.

A result of Yan [76] asserts the existence of maximal torsion-low generalized isotopies.

Moreover, for a maximal torsion-low generalized isotopy $\hat{I}$ of f, the Conley-Zehnder indices of the points of $\mathrm{Sing}(\hat{I})\backslash\{\infty\}$ is linked to the rotation number of the point ∞. Indeed, one may prove that if the rotation number of ∞ for the isotopy $\hat{I}$ is equal to $-k$, then it holds that:

- every saddle point $\widetilde{x}$ of $\hat{I}$ is the lift of a fixed point x of f such that $\mathrm{ind}_{CZ}(x) = 2k+1$,
- every sink $\widetilde{x}$ of $\hat{I}$ is the lift of a fixed point x of f such that $\mathrm{ind}_{CZ}(x) = 2k$,
- every source $\widetilde{x}$ of $\hat{I}$ is the lift of a fixed point x of f such that $\mathrm{ind}_{CZ}(x) = 2k + 2$.

We can consider a gradient-like foliation $\hat{\mathcal{F}}$ positively transverse to the maximal torsion-low generalized isotopy $\hat{I}$. Moreover, $\hat{\mathcal{F}}$ is equipped with the action function A_f of f and an index function $\mathrm{ind}(\hat{\mathcal{F}}, \cdot)$.

Thus, we can consider the barcode $\boldsymbol{\beta}_{\hat{\mathcal{F}}}$ associated to $\hat{\mathcal{F}}$ as defined in Chapter 6. We can prove that for every finite bar J of $\boldsymbol{\beta}_{\hat{\mathcal{F}}}$, there exist $x, y \in \mathrm{Fix}_c(f)$ and lifts $\widetilde{x}, \widetilde{y}$ of x and y fixed by $\hat{I}$ such that $J = (A_f(x), A_f(y)]$ and $\mathrm{ind}_{CZ}(y) - \mathrm{ind}_{CZ}(x) = 1$.

In a near future, we hope to define a Barcode using the barcodes associated to maximal torsion-low generalized isotopies but two important questions remain.

Question 9.0.2. *We consider a Hamiltonian diffeomorphism f and a bar J of its Floer homology barcode. Does there exist a torsion-low generalized isotopy of f such that for a positively transverse foliation $\hat{\mathcal{F}}$, J is a bar of $\boldsymbol{\beta}_{\hat{\mathcal{F}}}$?*

If we consider every torsion-low generalized isotopy of a Hamiltonian diffeomorphism f and every bar of their associated barcodes, then we may have too many bars to obtain a barcode equal to the Floer homology barcode of f.

Question 9.0.3. *Is there a natural way to select the "good" bars to keep?*

Part II

Calabi invariant for Hamiltonian diffeomorphism of the unit disk

Chapter 1

Introduction

In this second part of the thesis, we study the Calabi invariant on the unit disk usually defined on compactly supported Hamiltonian diffeomorphisms of the open disk. In particular we extend the Calabi invariant to the group of C^1 diffeomorphisms of the closed disk which preserves the standard symplectic form. We also compute the Calabi invariant for some diffeomorphisms of the disk which satisfy some rigidity hypothesis.

Let us begin with some basic definitions of symplectic geometry.

Let us consider (M^{2n}, ω) a *symplectic manifold*, meaning that M is an even dimensional manifold equipped with a closed non-degenerate differential 2-form ω called the *symplectic form*. We suppose that $\pi_2(M) = 0$ and that ω is exact, meaning that there exists a 1-form λ, called a *Liouville form*, which satisfies $d\lambda = \omega$.

Let us consider a time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ defined by the equation

$$dH_t = \omega(X_t, .), \tag{1.1}$$

where

$$\begin{aligned} H : \mathbb{R} \times M &\to \mathbb{R} \\ (t, x) &\mapsto H_t(x) \end{aligned}$$

is a smooth function 1-periodic on t, meaning that $H_{t+1} = H_t$ for every $t \in \mathbb{R}$. The function H is called a *Hamiltonian function*. If the vector field $(X_t)_{t\in\mathbb{R}}$ is complete, it induces a family $(f_t)_{t\in\mathbb{R}}$ of diffeomorphisms of M that preserve ω, also called *symplectomorphisms* or *symplectic diffeomorphisms*, satisfying the equation

$$f_0 = \mathrm{id} \text{ and } \frac{\partial}{\partial t} f_t(z) = X_t(f_t(z)).$$

In particular the family $I = (f_t)_{t\in[0,1]}$ defines an isotopy from id to f_1. The map f_1 is called a *Hamiltonian diffeomorphism*. It is well known that the set of Hamiltonian diffeomorphisms of a symplectic manifold M is a group which we denote $\mathrm{Ham}(M, \omega)$, we refer to [62] for more details.

Let us consider (M, ω) a symplectic manifold which is boundaryless, $\pi_2(M) = 0$ and such that ω is exact. We say that H is a *compactly supported Hamiltonian function* if

there exists a compact set $K \subset M$ such that H_t vanishes outside K for every $t \in \mathbb{R}$. A compactly supported Hamiltonian function induces a *compactly supported Hamiltonian diffeomorphism* f. Such a map is equal to the identity outside a compact subset of M. Let us consider a compactly supported Hamiltonian diffeomorphism f and λ a Liouville form on M. The form $f^*\lambda - \lambda$ is closed because f is symplectic but we have more, it is exact. More precisely there exists a unique compactly supported function $A_f : M \to \mathbb{R}$, also called *action function*, such that

$$dA_f = f^*\lambda - \lambda.$$

In the literature the *Calabi invariant* $\mathrm{Cal}(f)$ of f is defined as the mean of the function A_f and we have

$$\mathrm{Cal}(f) = \int_M A_f \omega^n, \tag{1.2}$$

where $\omega^n = \omega \wedge ... \wedge \omega$ is the volume form induced by ω, see [62] for more details. We will prove later that the number $\mathrm{Cal}(f)$ does not depend on the choice of λ.

Let us give another equivalent definition of the Calabi invariant for a compactly supported Hamiltonian diffeomorphism f. We note H a compactly supported Hamiltonian function defining f. The Calabi invariant of f can also be defined by the equation

$$\mathrm{Cal}(f) = (n+1) \int_0^1 \int_M H_t \omega^n dt. \tag{1.3}$$

To prove that $\int_M A_f \omega^n$ does not depend on the choice of the Liouville form λ, one may use the fact that the action function A_f satisfies

$$A_f(z) = \int_0^1 (\iota(X_s)\lambda + H_s) \circ f_s(z) ds, \tag{1.4}$$

where $(X_s)_{s\in\mathbb{R}}$ is the time dependent vector field induced by H by equation (1.1) and $(f_s)_{s\in\mathbb{R}}$ is the isotopy induced by the vector field $(X_s)_{s\in\mathbb{R}}$. Moreover, $\int_0^1 \int_M H_t \omega^n dt$ does not depend on the compactly supported Hamiltonian function H defining f.

The function Cal defines a real valued morphism on the group of compactly supported Hamiltonian diffeomorphisms of M and thus it is a conjugacy invariant. It is an important tool in the study of difficult problems such as the description of the algebraic structure of the groups $\mathrm{Ham}(M, \omega)$: A.Banyaga proved in [5] that the kernel of the Calabi invariant is always simple, which means that it does not contain nontrivial normal subgroups.

In this article, we study the case of the dimension two and more precisely the case of the closed unit disk which is a surface with boundary. We denote by $||.||$ the usual Euclidian norm on $\mathbb{R}^2$, by $\mathbb{D}$ the closed unit disk and by $\mathbb{S}^1$ its boundary. The group of C^1 orientation preserving diffeomorphisms of $\mathbb{D}$ will be denoted by $\mathrm{Diff}^1_+(\mathbb{D})$. We consider $\mathrm{Diff}^1_\omega(\mathbb{D})$ the group of C^1 symplectomorphisms of $\mathbb{D}$ which preserve the normalized standard symplectic form $\omega = \frac{1}{\pi} du \wedge dv$, written in cartesian coordinates (u, v). In the case of the disk, the group $\mathrm{Diff}^1_\omega(\mathbb{D})$ is contractible, see [39] for a proof, and coincides with the group of Hamiltonian diffeomorphisms of $\mathbb{D}$. Moreover, the 2-form ω induces the Lebesgue probability measure denoted by Leb and the symplectic diffeomorphisms are the C^1 diffeomorphisms

of $\mathbb{D}$ which preserve the Lebesgue measure and the orientation.

Let us begin by the case of the unit open disk $\mathring{\mathbb{D}}$. The open disk is boundaryless hence we already have two equivalent definitions of the Calabi invariant given by equations 1.2 and 1.3 on the set of compactly supported symplectic diffeomorphisms of $\mathring{\mathbb{D}}$. Let us give a third one. A. Fathi in his thesis [22] gave a dynamical definition which is also described by J.-M. Gambaudo and É. Ghys in [32]: if we consider an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f, there exists an angle function $\mathrm{Ang}_I : \mathring{\mathbb{D}} \times \mathring{\mathbb{D}}\backslash\Delta \to \mathbb{R}$ where Δ is the diagonal of $\mathring{\mathbb{D}} \times \mathring{\mathbb{D}}$ such that for each $(x, y) \in \mathring{\mathbb{D}} \times \mathring{\mathbb{D}}\backslash\Delta$, the quantity $2\pi\mathrm{Ang}_I(x, y)$ is the variation of angle of the vector $f_t(y) - f_t(x)$ between $t = 0$ and $t = 1$. If f is a compactly supported C^1 symplectic diffeomorphism then this angle function is integrable (see section 3) and it holds that

$$\mathrm{Cal}(f) = \int_{\mathring{\mathbb{D}}\times\mathring{\mathbb{D}}\backslash\Delta} \mathrm{Ang}_I(x, y)d\mathrm{Leb}(x)d\mathrm{Leb}(y), \tag{1.5}$$

where the integral does not depend on the choice of the isotopy.

In this article we will give an answer to the following question.

Question 1.0.1. *How to define an extension of the Calabi invariant to the group* $\mathrm{Diff}^1_\omega(\mathbb{D})$*?*

M. Hutchings [43] extended the definition given by equation 1.3 to the C^1 symplectic diffeomorphisms which are equal to a rotation near the boundary. In another point of view, V. Humilière [42] extended the definition given by equation 1.3 to certain group of compactly supported symplectic homeomorphisms of an exact symplectic manifold (M, ω) where a compactly supported symplectic homeomorphism f of M is a C^0 limit of a sequence of Hamiltonian diffeomorphisms of M supported on a common compact subset of M.

In the case of the open disk, for a compactly supported symplectomorphism f, the choice of the isotopy class of f is natural. But if f is a symplectic diffeomorphism of the closed disk such that its restriction to the open disk is not compactly supported then there is no such natural choice of an isotopy from id to f.

The rotation number is a well-known dynamical tool introduced by Poincaré in [66] on the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of homeomorphisms of $\mathbb{S}^1$ which preserve the orientation. Let us consider the set of homeomorphisms $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{g}(x + 1) = \widetilde{g}(x)$, denoted $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$. One may prove that there exists a unique $\widetilde{\rho} \in \mathbb{R}$ such that for each $z \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have $|\widetilde{g}^n(z) - z - n\widetilde{\rho}| < 1$. The number $\widetilde{\rho} = \widetilde{\rho}(\widetilde{g})$ is called the *rotation number* of $\widetilde{g}$. Let us consider $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ and two lifts $\widetilde{g}$ and $\widetilde{g}'$ of g in $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$, there exists $k \in \mathbb{Z}$ such that $\widetilde{g} = \widetilde{g}' + k$ and so $\widetilde{\rho}(\widetilde{g}) = \widetilde{\rho}(\widetilde{g}') + k$. Consequently we can define a map $\rho : \mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{T}^1$ such that $\rho(g) = \widetilde{\rho}(\widetilde{g}) + \mathbb{Z}$ where $\widetilde{g}$ is a lift of g. The number $\rho(g)$ is called the *rotation number* of g. We give further details about the rotation number in the next section.

We now state the results of this article. The following proposition allows us to consider a natural choice of an action function of a symplectomorphism of the closed disk.

Proposition 1.0.2. *Let us consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$, $A_f : \mathbb{D} \to \mathbb{R}$ a C^1 function such that $dA_f = f^*\lambda - \lambda$ and μ an f invariant Borel probability measure supported on $\mathbb{S}^1$. Then the number $\int_{\mathbb{S}^1} A_f d\mu$ does not depend on the choice of μ and λ.*

The first theorem follows.

Theorem 1.0.3. *For each $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ there exists a unique function $A_f : \mathbb{D} \to \mathbb{R}$ such that $dA_f = f^*\lambda - \lambda$ and $\int_{\mathbb{S}^1} A_f d\mu = 0$ where λ is a Liouville form and μ a f-invariant probability measure on $\mathbb{S}^1$. The map $\mathrm{Cal}_1 : \mathrm{Diff}^1_\omega(\mathbb{D}) \to \mathbb{R}$ defined by*

$$\mathrm{Cal}_1(f) = \int_{\mathbb{D}} A_f(z)\omega(z)$$

does not depend on the choice of λ and μ. Moreover the map Cal_1 is a homogeneous quasi-morphism that extends the Calabi invariant.

In another direction, the definition given by equation 1.3 and the definition given by equation 1.5 are based on isotopies. Then we consider the universal cover $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ of $\mathrm{Diff}^1_\omega(\mathbb{D})$ which is composed of couples $\widetilde{f} = (f, [I])$ where $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ and $[I]$ is an homotopy class of isotopies from id to f. We will prove that for $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ and I an isotopy from id to f, the angle function Ang_I does not depend on the choice of $I \in [I]$. Hence, for $\widetilde{f} = (f, [I]) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ we can denote $\mathrm{Ang}_{\widetilde{f}} = \mathrm{Ang}_I$ for $I \in [I]$.
Moreover, for a diffeomorphism $f \in \mathrm{Diff}^1(\mathbb{D})$ two isotopies $I = (f_t)_{t\in[0,1]}$ and $I' = (f'_t)_{t\in[0,1]}$ from id to f are homotopic if and only if their restriction $I|_{\mathbb{S}^1}$ and $I'|_{\mathbb{S}^1}$ to $\mathbb{S}^1$ are homotopic and so define the same lift $\widetilde{f|_{\mathbb{S}^1}}$ of $f|_{\mathbb{S}^1}$ on the universal cover of $\mathbb{S}^1$. Hence it is equivalent to consider $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ as the set of couples $\widetilde{f} = (f, \widetilde{\phi})$ where $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ and $\widetilde{\phi}$ a lift of $f|_{\mathbb{S}^1}$ to the universal cover of $\mathbb{S}^1$.

Theorem 1.0.4. *Let us consider an element $\widetilde{f}$ of $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. The number*

$$\widetilde{\mathrm{Cal}}_2(\widetilde{f}) = \int_{\mathbb{D}^2\setminus\Delta} \mathrm{Ang}_{\widetilde{f}}(x, y)\omega(x)\omega(y),$$

defines a morphism $\widetilde{\mathrm{Cal}}_2 : \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \to \mathbb{R}$ which induces a morphism $\mathrm{Cal}_2 : \mathrm{Diff}^1_\omega(\mathbb{D}) \to \mathbb{T}^1$ defined for every $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ by

$$\mathrm{Cal}_2(f) = \widetilde{\mathrm{Cal}}_2(\widetilde{f}) + \mathbb{Z},$$

where $\widetilde{f}$ is a lift of f to $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$.

Along the same lines, we have the following result.

Theorem 1.0.5. *Let us consider an element $(f, \widetilde{\phi})$ of $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. There exists a Hamiltonian function $H : \mathbb{T}^1 \times M \to \mathbb{R}$ such that H_t is equal to 0 on $\mathbb{S}^1$ for every $t \in \mathbb{R}$ which induces an isotopy $(\phi_t)_{t\in[0,1]}$ from id to f where the lifted isotopy $(\widetilde{\phi}_t)_{t\in[0,1]}$ satisfies $\widetilde{\phi}_1 = \widetilde{\phi}$. The number*

$$\widetilde{\mathrm{Cal}}_3(f, \widetilde{\phi}) = \int_0^1 \int_{\mathbb{D}} H_t(z)\omega(z)dt,$$

does not depend on the choice of such a Hamiltonian function H. Moreover the map $\widetilde{\mathrm{Cal}}_3 : \widetilde{\mathrm{Diff}}^1_{\omega(\mathbb{D})} \to \mathbb{R}$ is a morphism and induces a morphism $\mathrm{Cal}_3 : \mathrm{Diff}^1_{\omega(\mathbb{D})} \to \mathbb{T}^1$ defined by

$$\mathrm{Cal}_3(f) = \widetilde{\mathrm{Cal}}_3(f, \widetilde{\phi}) + \mathbb{Z}.$$

Remark 1.0.6. We have the following commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \arrow[r, "\tilde{\pi}"] \arrow[d, "\widetilde{\mathrm{Cal}}_i"'] & \mathrm{Diff}^1_\omega(\mathbb{D}) \arrow[d, "\mathrm{Cal}_i"] \\ \mathbb{R} \arrow[r, "\pi"] & \mathbb{T}^1 \end{tikzcd}$$

where $i \in \{2, 3\}$.

The link between these three extensions is given by the following result:

Theorem 1.0.7. *The morphisms* $\widetilde{\mathrm{Cal}}_2$ *and* $\widetilde{\mathrm{Cal}}_3$ *are equal and for* $\tilde{f} = (f, \tilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ *we have the following equality*

$$\widetilde{\mathrm{Cal}}_2(\tilde{f}) = \mathrm{Cal}_1(f) + \tilde{\rho}(\tilde{\phi}).$$

Moreover the maps Cal_1, $\widetilde{\mathrm{Cal}}_2$, Cal_2, $\widetilde{\mathrm{Cal}}_3$ *and* Cal_3 *are continuous in the* C^1 *topology.*

In the following, $\widetilde{\mathrm{Cal}}_2$ and $\widetilde{\mathrm{Cal}}_3$ will be denoted $\widetilde{\mathrm{Cal}}$. Since the morphism $\widetilde{\mathrm{Cal}}$ and the quasi-morphism Cal_1 are not trivial we obtain the following corollary about the perfectness of the groups $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and $\mathrm{Diff}^1_\omega(\mathbb{D})$. Recall that a group G is said to be perfect if it is equal to its commutator subgroup $[G, G]$ which is generated by the commutators $[f, g] = f^{-1}g^{-1}fg$ where f and g are elements of G.

Corollary 1.0.8. *The groups* $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ *and* $\mathrm{Diff}^1_\omega(\mathbb{D})$ *are not perfect.*

The non simplicity of those groups was already known since the group of compactly supported Hamiltonian diffeomorphisms is a non trivial normal subgroup of $\mathrm{Diff}^1_\omega(\mathbb{D})$. The questions of the simplicity and the perfectness of groups of diffeomorphisms and Hamiltonian diffeomorphism have a long story, especially the case of the group of area-preserving and compactly supported homeomorphisms of the disk $\mathbb{D}$. The question appears on McDuff and Salamon's list of open problems in [62] and we can refer for example to [5, 11, 20, 21, 58, 57, 63, 64]. Recently D. Cristofaro-Gardiner, V. Humilière, S. Seyfaddini in [17] proved that the connected component of id in the group of area-preserving homeomorphisms of the unit disk $\mathbb{D}$ is not simple. The proof requires the study of the Calabi invariant on the group of compactly supported Hamiltonian of $\mathbb{D}$ but also strong arguments of symplectic geometry as Embedded Contact Homology (also called ECH) developed by M. Hutchings and D. Cristofaro-Gardiner in [17].

To give an illustration of the extension we compute the Calabi invariant Cal_1 of non trivial symplectomorphisms in Sections 5 and 6. We study the Calabi invariant Cal_1 of some *irrational pseudo-rotations*. An irrational pseudo-rotation of the disk is an area-preserving homeomorphism f of $\mathbb{D}$ that fixes 0 and that does not possess any other periodic point. To such a homeomorphism is associated an irrational number $\overline{\alpha} \notin \mathbb{Q}/\mathbb{Z}$, called the *rotation number* of f that measures the rotation number of every orbit around 0 and consequently is equal to the rotation number of the restriction of f to $\mathbb{S}^1$. We refer to the next section for more details.

The following results of this paper are well-inspired by M. Hutchings's recent work. M. Hutching proved as a corollary in [43] that the Calabi invariant Cal_3 of every C^∞

irrational pseudo-rotation f of the closed unit disk $\mathbb{D}$ such that f is equal to a rotation near the boundary is equal to the rotation number of f. This means that for an irrational pseudo-rotation f which is equal to a rotation near the boundary, $\mathrm{Cal}_1(f)$ is equal to 0. The proof uses strong arguments of symplectic geometry such as the notion of open-books introduced by Giroux (see [37] for example) and the Embedded Contact Homology theory. We want to adopt a more dynamical point of view and we partially answer the following question.

Question 1.0.9. *Is the Calabi invariant* $\mathrm{Cal}_1(f)$ *equal to* 0 *for every* C^1 *irrational pseudo-rotation* f *of* $\mathbb{D}$*?*

With the continuity of $\widetilde{\mathrm{Cal}}$ in the C^1 topology, we can deduce the first result of C^1-rigidity as the following result.

Theorem 1.0.10. *Let* f *be a* C^1 *irrational pseudo-rotation of* $\mathbb{D}$*. If there exists a sequence* $(g_n)_{n\in\mathbb{N}}$ *in* $\mathrm{Diff}^1_\omega(\mathbb{D})$ *of* C^1 *diffeomorphisms of finite order which converges to* f *for the* C^1 *topology, then*

$$\mathrm{Cal}_1(f) = 0.$$

Corollary 1.0.11. *Let* f *be a* C^1 *irrational pseudo-rotation of* $\mathbb{D}$*. If there exists a sequence* $(n_k)_{k\in\mathbb{N}}$ *such that* f^{n_k} *converges to the identity in the* C^1 *topology, then we have*

$$\mathrm{Cal}_1(f) = 0.$$

The morphisms $\widetilde{\mathrm{Cal}}$ and Cal are not continuous in the C^0 topology, see proposition 4.2.5. Nevertheless, by a more precise study of the definition of Cal we obtain a C^0-rigidity result as follows.

Theorem 1.0.12. *Let* f *be a* C^1 *irrational pseudo-rotation of* $\mathbb{D}$*. If there exists a sequence* $(n_k)_{k\in\mathbb{N}}$ *of integers such that* $(f^{n_k})_{k\in\mathbb{N}}$ *converges to the identity in the* C^0 *topology, then we have*

$$\mathrm{Cal}_1(f) = 0.$$

There are already general results of C^0-rigidity of pseudo-rotations. Bramham proved [12] that every C^∞ irrational pseudo-rotation f is the limit, for the C^0 topology, of a sequence of periodic C^∞ diffeomorphisms. Bramham [13] also proved that if we consider an irrational pseudo-rotation f whose rotation number is super Liouville (we will recall what it means later) then f is C^0-rigid. That is, there exists a sequence of iterates f^{n_j} that converges to the identity in the C^0-topology as $n_j \to \infty$. Le Calvez [51] proved similar results for C^1 irrational pseudo-rotation f whose restriction to $\mathbb{S}^1$ is C^1 conjugate to a rotation.

Then for f a C^1 pseudo-rotation of the disk $\mathbb{D}$ the results of Bramham and Le Calvez provide a sequence of periodic diffeomorphisms $(g_n)_{n\in\mathbb{N}}$ which converges to f, the diffeomorphism g_n may not be area-preserving but let us hope to completely answer question 1.0.9.

In the last section we give examples where the rotation number of a pseudo-rotation satisfies some algebraic properties and where the hypothesis of Theorem 1.0.12 and Corollary 1.0.11 are satisfied.

Organization

We begin to give some additional preliminaries in chapter 1. In a second chapter we give the formal definitions of the Calabi invariant of equations 1.2, 1.3 and 1.5 and their natural extensions given by Theorems 1.0.3, 1.0.4 and 1.0.5. In chapter 3 we give the proof of the link between these extensions given by Theorem 1.0.7. The last chapter concerns the results about the computation of the Calabi invariant for pseudo-rotations.

Chapter 2

Preliminaries

This chapter aims to complete the preliminaries of part I. To simplify the reading, some notions and notation are re-introduced.

Invariant measures. Let us consider f a homeomorphism of a topological space X. A Borel probability measure μ is f-invariant if for each Borel set A we have

$$\mu(f^{-1}(A)) = \mu(A).$$

In other terms, the push forward measure $f_*\mu$ is equal to μ. We denote by $\mathcal{M}(f)$ the set of f-invariant probability measures on X. It is well-known that the set $\mathcal{M}(f)$ is not empty if X is compact.

For a probability measure μ on $\mathbb{D}$ we will note $\mathrm{Diff}^1_\mu(\mathbb{D})$ the subgroup of $\mathrm{Diff}^1_+(\mathbb{D})$ that is the set of orientation preserving C^1 diffeomorphisms which preserve μ.

Quasi-morphism. A function $F : G \to \mathbb{R}$ defined on a group G is a *homogeneous quasi-morphism* if

1. there exists a constant $C \geqslant 0$ such that for each couple f, g in G we have $|F(f \circ g) - F(f) - F(g)| < C$,
2. for each $n \in \mathbb{Z}$ we have $F(f^n) = nF(f)$.

Rotation numbers of homeomorphisms of the circle. The rotation number is defined on the group $\mathrm{Homeo}_+(\mathbb{S}^1)$ of homeomorphisms of $\mathbb{S}^1$ which preserve the orientation. We begin to give the definition of the rotation number on the lifted group $\widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ which is the set of homeomorphisms $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{g}(x+1) = \widetilde{g}(x) + 1$. There exists $\widetilde{\rho} \in \mathbb{R}$ such that for each $z \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have $|\widetilde{g}^n(z) - z - n\widetilde{\rho}| < 1$, see [45] for example. The number $\widetilde{\rho}$ is called the *rotation number* of $\widetilde{g}$ and denoted $\widetilde{\rho}(\widetilde{g})$. It defines a map $\widetilde{\rho} : \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1) \to \mathbb{R}$.

Moreover, $\widetilde{\rho}(\widetilde{g})$ naturally lifts a map $\rho : \mathrm{Homeo}_+(\mathbb{S}^1) \to \mathbb{T}^1$. Indeed, if we consider $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ and two lifts $\widetilde{g}$ and $\widetilde{g}'$ of g there exists $k \in \mathbb{Z}$ such that $\widetilde{g}' = \widetilde{g}$ hence we have $\widetilde{\rho}(\widetilde{g}') = \widetilde{\rho}(\widetilde{g}) + k$. By the Birkhoff ergodic theorem for every g-invariant measure μ we have

$$\widetilde{\rho}(\widetilde{g}) = \int_{\mathbb{S}^1} \delta d\mu.$$

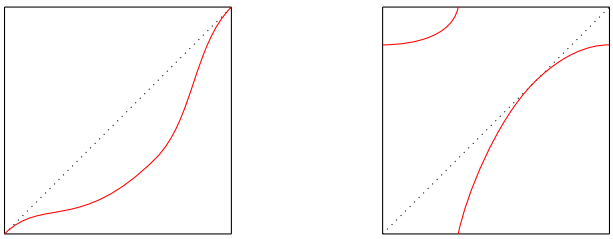
We denote by $\widetilde{\delta} : \mathbb{R} \to \mathbb{R}$ the displacement function of $\widetilde{g}$ where $\widetilde{\delta}(z) = \widetilde{g}(z) - z$ is one-periodic and lifts a function δ such that for every $\widetilde{g} \in \mathrm{Homeo}_+(\mathbb{S}^1)$ lifting a homeomorphism g and every g-invariant measure μ we have

$$\widetilde{\rho}(\widetilde{g}) = \int_{\mathbb{S}^1} \delta d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta(g^i(z)),$$

for every $z \in \mathbb{S}^1$.
The map $\widetilde{\rho}$ is the unique homogeneous quasi-morphism from $\widetilde{\mathrm{Diff}}^1_+(\mathbb{S}^1)$ to $\mathbb{R}$ which takes the value 1 on the translation by 1, see [36] for example. More precisely for each $\widetilde{f}, \widetilde{g} \in \widetilde{\mathrm{Homeo}}_+(\mathbb{S}^1)$ it holds that $|\widetilde{\rho}(\widetilde{f}) - \widetilde{\rho}(\widetilde{g})| < 1$ and for each $n \in \mathbb{Z}$ we have $\widetilde{\rho}(\widetilde{f}^n) = n\widetilde{\rho}(\widetilde{f})$.

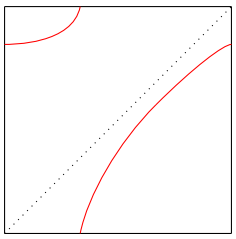
Let us describe why $\widetilde{\rho}$ is not a morphism and only a quasi-morphism. A homeomorphism of the circle has a fixed point if and only if its rotation number is zero, see [45] chapter 11 for more details. Below we give an example of two homeomorphisms ϕ and ψ of $\mathbb{S}^1$ of rotation number zero such that the composition $\phi \circ \psi$ gives us a homeomorphism as in Figure 2.2 without fixed point and so the rotation number of the composition is not equal to 0.

Let us consider the two homeomorphisms of rotation number 0 with one fixed point as in Figures 2.1 and 2.2.



ϕ ψ

Figure 2.1



$\phi \circ \psi$

Figure 2.2

For $g \in \mathrm{Homeo}_+(\mathbb{S}^1)$ there is a bijection between the lifts of g to $\mathbb{R}$ and the isotopies from id to g as follows. Let $I = (g_t)_{t\in[0,1]}$ be an isotopy from id to g, the lifted isotopy $\widetilde{I} = (\widetilde{g}_t)_{t\in[0,1]}$ of I defines a unique lift $\widetilde{g}_1$ of g. Then for an isotopy I from id to g, let us denote $\widetilde{g}$ the time-one map of the lifted isotopy $\widetilde{I}$ on $\mathbb{R}$, we can define the rotation number

$\widetilde{\rho}(I) \in \mathbb{R}$ of I to be the rotation number $\widetilde{\rho}(\widetilde{g})$ of $\widetilde{g}$. If we consider f a homeomorphism of the disk isotopic to the identity and $I = (f_t)_{t\in[0,1]}$ an isotopy from id to f then we will denote $\widetilde{\rho}(I|_{\mathbb{S}^1}) \in \mathbb{R}$ the rotation number of the restriction of the isotopy I to $\mathbb{S}^1$. If we consider another isotopy I' from id to g one may prove that there exists an integer $k \in \mathbb{Z}$ such that I' is homotopic to $R^k I$ where the isotopy $R = (R_t)_{t\in[0,1]}$ satisfies $R_t(z) = z\mathrm{e}^{2\pi i t}$ for every $z \in \mathbb{S}^1$ and every $t \in [0,1]$. We consider $\widetilde{I}$ the lifted isotopy of I' and we denote $\widetilde{g}'$ its time-one map. Hence $\widetilde{g}$ and $\widetilde{g}'$ are two lifts of g such that $\widetilde{g}' = \widetilde{g} + k$ and $\widetilde{\rho}(\widetilde{g}') = \widetilde{\rho}(\widetilde{g}) + k$ and so the number $\widetilde{\rho}(I)$ does not depend on the choice of the isotopy in the homotopy class of I.

Irrational pseudo-rotation. An irrational pseudo-rotation is an area-preserving homeomorphism f of $\mathbb{D}$ that fixes 0 and that does not possess any other periodic point. To such a homeomorphism is associated an irrational number $\alpha \in \mathbb{R}/\mathbb{Z}\backslash\mathbb{Q}/\mathbb{Z}$, called the *rotation number* of f, characterized by the following : every point admits α as a rotation number around the origin. To be more precise, choose a lift $\widetilde{f}$ of $f|_{\mathbb{D}\backslash\{0\}}$ to the universal covering space $\widetilde{D} = \mathbb{R} \times (0,1]$. There exists $\widetilde{\alpha} \in \mathbb{R}$ such that $\widetilde{\alpha} + \mathbb{Z} = \alpha$ and for every compact set $K \subset \mathbb{D}\backslash\{0\}$ and every $\epsilon > 0$, one can find $N \geqslant 1$ such that

$$\forall n \geqslant N,\ \widetilde{z} \in \pi^{-1}(K) \cap \widetilde{f}^{-n}(\pi^{-1}(K)) \Rightarrow |\frac{p_2(\widetilde{f}^n(\widetilde{z})) - p_2(\widetilde{z})}{n} - \widetilde{\alpha}| \leqslant \epsilon,$$

where $\pi : (r,\theta) \mapsto (r\cos(2\pi\theta), r\sin(2\pi\theta))$ is the covering projection and $p_2 : (r,\theta) \mapsto \theta$ the projection on the second coordinate. If moreover f is a C^k diffeomorphism $1 \leqslant k \leqslant +\infty$ we will call f a C^k irrational pseudo-rotation.

Notice that the rotation number α of an irrational pseudo-rotation f is equal to $\rho(f|_{\mathbb{S}^1})$.

One can construct irrational pseudo-rotations with the method of fast periodic approximations, presented by Anosov and Katok [1]. One may see [23, 24, 25, 38, 60] for further developments about this method and see [9, 7] for other results on irrational pseudo-rotations.

Chapter 3

Three extensions

In this section we will explain why the functions Cal_1, $\widetilde{\mathrm{Cal}}_2$ and $\widetilde{\mathrm{Cal}}_3$ are well-defined and we will establish the relations between them. The full statement like the continuity or the quasi-morphism property will be proved in the next section.

3.1 Action function

Let us consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ and λ a Liouville 1-form such that $d\lambda = \omega$. The fact that $H^1(\mathbb{D}, \mathbb{R}) = 0$ implies that the closed 1-form $f^*\lambda - \lambda$ is exact. More precisely its integral along each loop $\gamma \subset \mathbb{D}$ is zero. Consequently the map $z \mapsto \int_{\gamma_z} f^*\lambda - \lambda$ is a C^1 primitive of $f^*\lambda - \lambda$, equal to 0 at the origin, where for every $z \in \mathbb{D}$ the path $\gamma_z : [0,1] \to \mathbb{D}$ is such that $\gamma_z(t) = tz$.

If we suppose that f is compactly supported on $\mathring{\mathbb{D}}$ then it is natural to consider the unique C^1 function $A : \mathbb{D} \to \mathbb{R}$ that is zero near the boundary of $\mathbb{D}$ and that satisfies

$$dA = f^*\lambda - \lambda. \tag{3.1}$$

Without the compact support hypothesis we have the following proposition.

Proposition 3.1.1. *If we consider a C^1 function $A : \mathbb{D} \to \mathbb{R}$ such that $dA = f^*\lambda - \lambda$ then the number*

$$\int_{\mathbb{S}^1} A|_{\partial\mathbb{D}} d\mu$$

does not depend on the choice of μ in $\mathcal{M}(f|_{\mathbb{S}^1})$.

Proof. To prove the independence over μ there are two cases to consider.

• If there exists only one $f|_{\mathbb{S}^1}$-invariant probability measure on $\mathbb{S}^1$ the result is obvious. In this case $f|_{\mathbb{S}^1}$ is said to be uniquely ergodic.

• If $f|_{\mathbb{S}^1}$ is not uniquely ergodic then by Poincaré's theory $\rho(f|_{\mathbb{S}^1}) = \frac{p}{q} + \mathbb{Z}$ is rational with $p \wedge q = 1$. The ergodic decomposition theorem, see [45] for example, tells us that an $f|_{\mathbb{S}^1}$-invariant measure is the barycenter of ergodic $f|_{\mathbb{S}^1}$-invariant measures. Moreover, each

ergodic measure of $f|_{\mathbb{S}^1}$ is supported on a q-periodic orbit as follows. For z a q-periodic point of $f|_{\mathbb{S}^1}$, we define the probability measure μ_z supported on the orbit of z by

$$\mu_z = \frac{1}{q}\sum_{k=0}^{q-1} \delta_{f^k(z)},$$

where δ_z is the Dirac measure on the point $z \in \mathbb{S}^1$. Hence it is sufficient to prove that $\int_{\mathbb{D}} A(f,\lambda,\mu_z)\omega$ does not depend on the choice of a periodic point $z \in \mathbb{S}^1$.

Let us consider two periodic points z and w of $f|_{\mathbb{S}^1}$. We consider an oriented path $\gamma \subset \mathbb{S}^1$ from z to w. We compute

$$\begin{aligned}
\int_{\mathbb{S}^1} A d\mu_z - \int_{\mathbb{S}^1} A d\mu_w &= \frac{1}{q}\sum_{k=0}^{q-1} A(f^k(z)) - A(f^k(w)) \\
&= \frac{1}{q}\sum_{k=0}^{q-1} \int_{f^k(\gamma)} dA \\
&= \frac{1}{q}\sum_{k=0}^{q-1} \int_{f^k(\gamma)} f^*(\lambda) - \lambda \\
&= \frac{1}{q}\left(\int_{f^q(\gamma)} \lambda - \int_{\gamma} \lambda\right) \\
&= 0
\end{aligned}$$

where the last equality is due to the fact that $f^q(\gamma)$ is a reparametrization of the path γ. □

Proposition 3.1.1 allows us to make a natural choice of the action function to define an extension of the Calabi invariant as follows.

Theorem 3.1.2. *For each $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ we consider the unique C^1 function A_f of f such that $dA_f = f^*\lambda - \lambda$ and $\int_{\mathbb{S}^1} A_f d\mu = 0$ where λ is a Liouville form of ω and μ an f-invariant probability measure on $\mathbb{S}^1$. The number*

$$\mathrm{Cal}_1(f) = \int_{\mathbb{D}} A_f(z)\omega(z)$$

does not depend on the choice of λ or μ.

Proof. The independence on the measure μ comes from Proposition 3.1.1 and it remains to prove the independence on λ.
Let us consider another primitive λ' of ω. We denote A and A' the two functions such that $dA = f^*\lambda - \lambda$ and $dA' = f^*\lambda' - \lambda'$ and such that for each $\mu \in \mathcal{M}(f|_{\mathbb{S}^1})$ we have $\int_{\mathbb{S}^1} A d\mu = \int_{\mathbb{S}^1} A' d\mu = 0$.

The 1-form $\lambda - \lambda'$ is closed because $d\lambda - d\lambda' = \omega - \omega = 0$. So there exists a smooth function $u : \mathbb{D} \to \mathbb{R}$ such that $\lambda' = \lambda + du$. We compute

$$\begin{aligned}
dA' &= f^*(\lambda + du) - (\lambda + du) \\
&= f^*\lambda - \lambda + d(u \circ f - u) \\
&= dA + d(u \circ f - u).
\end{aligned}$$

Thus there exists a constant c such that

$$A' = A + u \circ f - u + c.$$

For a measure $\mu \in \mathcal{M}(f|_{\mathbb{S}^1})$ the condition $\int_{\mathbb{S}^1} A' d\mu = 0 = \int_{\mathbb{S}^1} A d\mu$ implies that

$$\int_{\mathbb{S}^1} A' d\mu = \int_{\mathbb{S}^1} A d\mu + \int_{\mathbb{S}^1} (u \circ f|_{\mathbb{S}^1} - u) d\mu + c = \int_{\mathbb{S}^1} A d\mu,$$

Howeover $\int_{\mathbb{S}^1} (u \circ f|_{\mathbb{S}^1} - u) d\mu = 0$ since $f|_{\mathbb{S}^1}$ preserves μ we have

$$c = 0.$$

Finally f preserves ω hence $\int_{\mathbb{D}} (u \circ f - u)\omega = 0$ and we can conclude that

$$\int_{\mathbb{D}} A'\omega = \int_{\mathbb{D}} A\omega.$$

□

We show that the extension Cal_1 vanishes on rotations of the disk.

Proposition 3.1.3. *For $\theta \in \mathbb{R}$ the rotation R_θ of angle θ satisfies*

$$\mathrm{Cal}_1(R_\theta) = 0.$$

Proof. For the Liouville form $\lambda = \frac{r^2}{2\pi} d\theta$ of ω we have $R_\theta^* \lambda - \lambda = 0$ thus the action function A is constant. So it is equal to 0 and we obtain the result. □

3.2 Angle function

The following interpretation is due to Fathi in his thesis [22] in the case of compactly supported symplectic diffeomorphisms of the unit disk. This interpretation is also developped by Ghys and Gambaudo in [32].

Let us consider $f \in \mathrm{Diff}^1_+(\mathbb{D})$ and $I = (f_t)_{t \in [0,1]}$ an isotopy from id to f. For $x, y \in \mathbb{D}$ distinct we can consider the vector v_t from $f_t(x)$ to $f_t(y)$ and we denote by $\mathrm{Ang}_I(x, y)$ the angle variation of the vector v_t for $t \in [0, 1]$ defined as follows.

We have the polar coordinates (r, θ) and a differential form

$$d\theta = \frac{udv - vdu}{u^2 + v^2},$$

where (u, v) are the cartesian coordinates. For every couple $(x, y) \in \mathbb{D}^2 \backslash \Delta$ we define

$$\mathrm{Ang}_I(x, y) = \frac{1}{2\pi} \int_\gamma d\theta, \tag{3.2}$$

where $\gamma : t \mapsto f_t(x) - f_t(y)$.

The function Ang_I is continuous on the complement of the diagonal of $\mathbb{D} \times \mathbb{D}$. Moreover, if f is at least C^1 then the function Ang_I can be extended on the diagonal into a bounded function on $\mathbb{D} \times \mathbb{D}$. Indeed, we consider K the compact set of triplets (x, y, d) where $(x, y) \in \mathbb{D} \times \mathbb{D}$ and d a half line in $\mathbb{R}^2$ containing x and y and oriented by the vector joining x to y if $x \neq y$. If x and y are distinct, the half line d is uniquely determined and $\mathbb{D} \times \mathbb{D} \backslash \Delta$ can be embedded in K as a dense and open set. We define $\mathrm{Ang}_I(x, x, d)$ as the variation of angle of the half lines $df_t(d)$ for $t \in [0, 1]$. This number is well-defined and extends Ang_I into a continuous function on K.

For $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and two Hamiltonian isotopies $I = (f_t)_{t \in [0,1]}$ and $I' = (f'_t)_{t \in [0,1]}$ from id to f associated to $\widetilde{\phi}$. The isotopies I' and I are homotopic so for every couple $(x, y) \in \mathbb{D}^2 \backslash \Delta$ we have

$$\int_\gamma d\theta = \int_{\gamma'} d\theta,$$

where $\gamma : t \mapsto f_t(x) - f_t(y)$ and $\gamma' : t \mapsto f'_t(x) - f'_t(y)$. Hence, we can define the angle function $\mathrm{Ang}_{\widetilde{f}}$ of $\widetilde{f}$ by

$$\mathrm{Ang}_{\widetilde{f}} = \mathrm{Ang}_I.$$

We have the following lemma.

Lemma 3.2.1. *Let us consider $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. For every $(x, y) \in \mathbb{D}^2 \backslash \Delta$ the number $\mathrm{Ang}_{\widetilde{f}}(x, y) - \widetilde{\rho}(\widetilde{\phi})$ only depends on f.*

Proof. Let us consider I' another isotopy from id to f.

There exists $k \in \mathbb{Z}$ such that I' is homotopic to $R^k_{2\pi} I$ and by definition of Ang_h given by equation 4.1 we have $\mathrm{Ang}_{R^k_{2\pi} I} = \mathrm{Ang}_I + k$. Moreover I' is in the same homotopy class of $R^k_{2\pi} I$ and we obtain $\mathrm{Ang}_{I'} = \mathrm{Ang}_I + k$. Since the rotation number also satisfies $\widetilde{\rho}(I'|_{\mathbb{S}^1}) = \widetilde{\rho}(I|_{\mathbb{S}^1}) + k$, the result follows. □

Lemma 3.2.1 allows us to extend the Calabi invariant on the lifted group $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ as follows.

Theorem 3.2.2. *Let us consider $\widetilde{f} = (\widetilde{f}, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. The number*

$$\widetilde{\mathrm{Cal}}_2(\widetilde{f}) = \int_{\mathbb{D}^2 \backslash \Delta} \mathrm{Ang}_{\widetilde{f}}(x, y) \omega(x) \omega(y),$$

defines a morphism $\widetilde{\mathrm{Cal}}_2 : \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \to \mathbb{R}$ and induces a morphism on $\mathrm{Diff}^1_\omega(\mathbb{D})$ defined by

$$\mathrm{Cal}_2(f) = \widetilde{\mathrm{Cal}}_2(\widetilde{f}) + \mathbb{Z},$$

where $\widetilde{f} \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ is a lift of f.

Proof. First, $\widetilde{\mathrm{Cal}}$ is well-defined since the angle function $\mathrm{Ang}_{\widetilde{f}}$ is integrable on $\mathbb{D}^2 \backslash \Delta$.

Let us consider $\widetilde{f} = (f, \widetilde{\phi})$ and $\widetilde{g} = (g, \widetilde{\phi}')$ two elements of $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and two isotopies $I = (f_t)_{t \in [0,1]} \in [I]$ from id to f associated to $\widetilde{\phi}$ and $I' = (g_t)_{t \in [0,1]}$ from id to g associated to $\widetilde{\phi}'$. We consider the concatenation $I \cdot I'$ of the isotopy I and I' which gives an isotopy

from id to $f\circ g$ associated to $\widetilde{\phi}\circ\widetilde{\phi}'$ and we define the element $\widetilde{f}\circ\widetilde{g} = (f\circ g, \widetilde{\phi}\circ\widetilde{\phi}') \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. For each $(x,y) \in \mathbb{D}^2\backslash\Delta$ we have

$$\mathrm{Ang}_{I\cdot I'}(x,y) = \mathrm{Ang}_{I'}(x,y) + \mathrm{Ang}_I(g(x), g(y)).$$

Hence we obtain

$$\mathrm{Ang}_{\widetilde{f}\circ\widetilde{g}}(x,y) = \mathrm{Ang}_{\widetilde{g}}(x,y) + \mathrm{Ang}_{\widetilde{f}}(g(x), g(y)).$$

We integrate the previous equality and since g preserves ω we deduce that $\widetilde{\mathrm{Cal}}_2$ is a morphism from $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ to $\mathbb{R}$.

Moreover, Lemma 3.2.1 assures that $\widetilde{\mathrm{Cal}}_2$ induces the morphism Cal_2 from $\mathrm{Diff}^1_\omega(\mathbb{D})$ to $\mathbb{T}^1$. □

Notice that the morphisms $\widetilde{\mathrm{Cal}}_2$ and Cal_2 satisfy the following commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \arrow[r] \arrow[d, "\widetilde{\mathrm{Cal}}_2"'] & \mathrm{Diff}^1_\omega(\mathbb{D}) \arrow[d, "\mathrm{Cal}_2"] \\
\mathbb{R} \arrow[r] & \mathbb{T}^1
\end{tikzcd}$$

where the horizontal arrows are the covering maps.

This interpretation allows us to generalize the definition to other invariant measures of the disk. Let us consider $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1(\mathbb{D})$ and an isotopy I from id to f associated to $\widetilde{\phi}$. We consider a probability measure μ on $\mathbb{D}$ without atom which is f-invariant. We define the number $\widetilde{\mathcal{C}}_\mu(I)$ by

$$\widetilde{\mathcal{C}}_\mu(\widetilde{f}) = \int\int_{\mathbb{D}^2\backslash\Delta} \mathrm{Ang}_{\widetilde{f}}(x,y)d\mu(x)d\mu(y).$$

By Lemma 3.2.1 we obtain the following corollary.

Corollary 3.2.3. *Let us consider $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. For every $(x,y) \in \mathbb{D}^2\backslash\Delta$ the number $\widetilde{\mathcal{C}}_\mu(\widetilde{f}) - \widetilde{\rho}(\widetilde{\phi})$ only depends on f.*

Birkhoff ergodic theorem gives another way to compute $\widetilde{\mathcal{C}}_\mu(\widetilde{f})$ for $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1(\mathbb{D})$. Let us consider an isotopy $I = (f_t)_{t\in[0,1]}$ from id to f associated to $\widetilde{\phi}$. For $(x,y) \in \mathbb{D}\times\mathbb{D}\backslash\Delta$ we have

$$\mathrm{Ang}_{I^n}(x,y) = \mathrm{Ang}_I(x,y) + \mathrm{Ang}_I(f(x), f(y)) + ... + \mathrm{Ang}_I(f^{n-1}(x), f^{n-1}(x)). \tag{3.3}$$

The function Ang_I is bounded so the function

$$\widehat{\mathrm{Ang}}_I(x,y) = \lim_{n\to\infty} \frac{1}{n}\mathrm{Ang}_{I^n}(x,y),$$

is defined $\mu\times\mu$ almost everywhere and depends only on the homotopy class of I. Hence we can define $\widehat{\mathrm{Ang}}_{\widetilde{f}} = \widehat{\mathrm{Ang}}_I$. Thus we obtain the following equality

$$\widetilde{\mathcal{C}}_\mu(\widetilde{f}) = \int\int_{\mathbb{D}\times\mathbb{D}} \widehat{\mathrm{Ang}}_{\widetilde{f}}(x,y)d\mu(x)d\mu(y). \tag{3.4}$$

We state the proposition of topological invariance, see [32].

Proposition 3.2.4. *Let us consider two probability measures μ_1 and μ_2 of $\mathbb{D}$ without atom and two compactly supported elements of $\mathrm{Diff}^1_{\mu_1}(\mathbb{D})$ and $\mathrm{Diff}^1_{\mu_2}(\mathbb{D})$ denoted ϕ_1 and ϕ_2 such that there exists a homeomorphism $h \in \mathrm{Diff}^0_+(\mathbb{D})$ satisfying $\phi_2 = h \circ \phi_1 \circ h^{-1}$ and $h_*(\mu_1) = \mu_2$. We have that*

$$\mathcal{C}_{\mu_1}(\phi_1) = \mathcal{C}_{\mu_2}(\phi_2).$$

For a probability measure μ of the disk, there is the equivalent result to extend the invariant $\mathcal{C}_\mu$.

Theorem 3.2.5. *Let us consider an element $\widetilde{f} \in \widetilde{\mathrm{Diff}}^1_\mu(\mathbb{D})$. The number*

$$\widetilde{\mathcal{C}}_\mu(\widetilde{f}) = \int_{\mathbb{D}^2\backslash\Delta} \mathrm{Ang}_{\widetilde{f}}(x,y)d\mu(x)d\mu(y),$$

defines a morphism $\widetilde{\mathcal{C}}_\mu : \widetilde{\mathrm{Diff}}^1_\mu(\mathbb{D}) \to \mathbb{R}$ which induces a morphism $\mathcal{C}_\mu : \mathrm{Diff}^1_\mu(\mathbb{D}) \to \mathbb{T}^1$ defined for every $f \in \mathrm{Diff}^1_\mu(\mathbb{D})$ by

$$\mathcal{C}_\mu(f) = \widetilde{\mathcal{C}}_\mu(\widetilde{f}) + \mathbb{Z},$$

where $\widetilde{f} \in \widetilde{\mathrm{Diff}}^1_\mu(\mathbb{D})$ is a lift of f.

The proof of the previous theorem is basically the same as Theorem 3.2.2 and if we consider the Lesbegue measure Leb then we have

$$\widetilde{\mathcal{C}}_{\mathrm{Leb}} = \widetilde{\mathrm{Cal}}_2.$$

We have the following computation in the case of the rotations.

Lemma 3.2.6. *For $\theta \in \mathbb{R}$ we consider $\widetilde{R}_\theta = (R_\theta, \widetilde{r}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ where R_θ is the rotation $\mathbb{D} \to \mathbb{D}$ of angle θ. We have*

$$\widetilde{\mathrm{Cal}}_2(\widetilde{R}) = \widetilde{\rho}(\widetilde{r}).$$

Proof. Let us consider $R = (R_t)_{t\in[0,1]}$ the isotopy from id to R_θ given in section 2. For a couple $(x,y) \in \mathbb{D} \times \mathbb{D}\backslash\Delta$ we consider the complex $z = x - y$ and we have for each $t \in [0,1]$ $R_t(z) = z\mathrm{e}^{it\theta}$ and we can compute $\mathrm{Ang}_R(x,y) = \theta$. By integration on $\mathbb{D} \times \mathbb{D}\backslash\Delta$ we obtain

$$\widetilde{\mathrm{Cal}}_2(\widetilde{R}_\theta) = \theta = \widetilde{\rho}(\widetilde{r}).$$

□

3.3 Hamiltonian function

In this section, the goal is to detail the construction of the Calabi invariant given by equation 1.3 in the case of compactly supported diffeomorphisms of the disk. This construction leads to Theorem 1.0.5 and we explain the definition of $\widetilde{\mathrm{Cal}}_3$ given by this theorem but we refer to the next section for the proofs of certain results.

Let us consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ and a Hamiltonian isotopy $I = (f_t)_{t\in[0,1]}$ from id to f. We consider the Hamiltonian function $(H_t)_{t\in\mathbb{R}}$ which induces the isotopy I. We denote $(X_t)_{t\in\mathbb{R}}$ the associated vector field. We have that for every $t \in \mathbb{R}$, X_t is tangent to $\mathbb{S}^1$. So each H_t is constant on $\mathbb{S}^1$ and we can consider $(H_t)_{t\in\mathbb{R}}$ the associated Hamiltonian function such that

$$H_t|_{\mathbb{S}^1} = 0.$$

We have the following lemma.

Lemma 3.3.1. *The integral*

$$2\int_{z\in\mathbb{D}}\int_0^1 H_t(z)\omega(z)dt - \widetilde{\rho}(I|_{\mathbb{S}^1}),$$

depends only on f.

Proof. The result will be a corollary of Theorem 1.0.7. □

Theorem 3.3.2. *Let us consider an element $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and a Hamiltonian function $H : \mathbb{S}^1 \times \mathbb{D} \to \mathbb{R}$ of f which induces the flow $(\phi_t)_{t\in[0,1]}$ such that the lift of $\phi_1|_{\mathbb{S}^1}$ is equal to $\widetilde{\phi}$ and such that H_t is equal to 0 on $\mathbb{S}^1$ for every $t \in \mathbb{R}$. The number*

$$\widetilde{\mathrm{Cal}}_3(\widetilde{f}) = 2\int_0^1\int_{\mathbb{D}} H_t(z)\omega(z)dt,$$

does not depend on the choice of H. Moreover the map $\widetilde{\mathrm{Cal}}_3 : \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \to \mathbb{R}$ is a morphism and $\widetilde{\mathrm{Cal}}(\widetilde{f}) + \mathbb{Z}$ depends only on f. It induces a morphism

$$\mathrm{Cal}_3(f) = 2\int_0^1\int_{\mathbb{D}} H_t(z)\omega(z)dt + \mathbb{Z},$$

defined on $\mathrm{Diff}^1_\omega(\mathbb{D})$.

The proof comes from the equality between $\widetilde{\mathrm{Cal}}_2$ and $\widetilde{\mathrm{Cal}}_3$ which will be proven in the next section. Moreover, the definition of Cal_3 comes from Lemma 3.3.1 and we obtain the following commutative diagram where the horizontal arrows are the universal covering maps.

$$\begin{tikzcd}
\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \arrow[r] \arrow[d, "\widetilde{\mathrm{Cal}}_3"'] & \mathrm{Diff}^1_\omega(\mathbb{D}) \arrow[d, "\mathrm{Cal}_3"] \\
\mathbb{R} \arrow[r] & \mathbb{T}^1
\end{tikzcd}$$

Chapter 4

Proof of Theorem 1.0.7

In this section, we prove Theorem 1.0.7.

Theorem 4.0.1. *The morphisms* $\widetilde{\mathrm{Cal}}_2$ *and* $\widetilde{\mathrm{Cal}}_3$ *are equal. For* $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ *we have the following equality*

$$\widetilde{\mathrm{Cal}}_2(\widetilde{f}) = \mathrm{Cal}_1(f) + \widetilde{\rho}(\widetilde{\phi}).$$

Moreover Cal_1*,* $\widetilde{\mathrm{Cal}}_2$ *and* $\widetilde{\mathrm{Cal}}_3$ *are continuous in the* C^1 *topology.*

We separate the proof into two subsections, in the first one we establish the links between the previous definitions then we prove the continuity of $\widetilde{\mathrm{Cal}}_2$ and $\widetilde{\mathrm{Cal}}_3$.

4.1 Equality between $\widetilde{\mathrm{Cal}}_2$ and $\widetilde{\mathrm{Cal}}_3$.

Proposition 4.1.1. *The morphisms* $\widetilde{\mathrm{Cal}}_2$ *and* $\widetilde{\mathrm{Cal}}_3$ *are equal.*

Proof. The proof is essentially the same as in [69], the only difference is that our symplectic form is normalized and the Hamiltonian diffeomorphisms that we consider is not compactly supported in the open unit disk. Nevertheless, we verify that the proof is still relevant in our case.

Let us consider $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and a Hamiltonian isotopy $I = (f_t)_{t\in[0,1]}$ from id to f associated to $\widetilde{\phi}$. For the proof we will give a definition of the angle function Ang_I in complex coordinates as follows. We define a 1-form α by

$$\alpha = \frac{1}{2\pi}\frac{d(z_1 - z_2)}{z_1 - z_2}.$$

The imaginary part satisfies

$$d\theta = 2\pi \mathrm{Im}(\alpha),$$

where θ is the angle coordinate in radial coordinates. For an element $Z = (z_1, z_2) \in \mathbb{D}^2 \backslash \Delta$ we consider the curve $I_Z \subset \mathbb{D} \times \mathbb{D} \backslash \Delta$ defined by

$$t \mapsto I_Z(t) = (f_t(z_1), f_t(z_2)),$$

for each $t \in [0, 1]$ and that for every element $Z = (z_1, z_2) \in \mathbb{D} \times \mathbb{D} \backslash \Delta$ we have

$$\mathrm{Ang}_I(z_1, z_2) = \frac{1}{2\pi}\int_{I_Z} d\theta. \tag{4.1}$$

Let us consider the Hamiltonian $(H_t)_{t\in[0,1]}$ which induces the flow of the isotopy I and which is equal to 0 on the boundary of $\mathbb{D}$. We consider the symplectic form $\omega = \frac{i}{2\pi} dz \wedge d\overline{z}$ written in complex coordinates on $\mathbb{D}$. We define $\xi_t = dz(X_t)$ and then it satisfies

$$i_{X_t}\left(\frac{i}{2\pi} dz \wedge d\overline{z}\right) = \frac{i}{2\pi}\xi_t d\overline{z} - \frac{i}{2\pi}\overline{\xi}_t dz.$$

By definition

$$dH_t = -\frac{\partial H_t}{\partial \overline{z}} d\overline{z} - \frac{\partial H_t}{\partial z} dz,$$

so we have

$$\xi_t = -2i\pi \frac{\partial H_t}{\partial \overline{z}}. \tag{4.2}$$

We compute the integral of the angle function

$$\begin{aligned}\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta} \mathrm{Ang}_I(z_1,z_2)\omega(z_1)\omega(z_2) &= \int_{\mathbb{D}\times\mathbb{D}\setminus\Delta}\int_{I_{(z_1,z_2)}} \frac{1}{2\pi} d\theta\ \omega(z_1)\omega(z_2)\\ &= \mathrm{Im}\left(\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta}\int_{I_{(z_1,z_2)}} \alpha\ \omega(z_1)\omega(z_2)\right).\end{aligned}$$

The following computation is well-inspired by the proof in [69].

$$\begin{aligned}\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta}\int_{I_{(z_1,z_2)}} \alpha\ \omega(z_1)\omega(z_2) &= \frac{1}{2\pi}\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta}\int_{I_{(z_1,z_2)}} \frac{d(z_1-z_2)}{z_1-z_2}\omega(z_1)\omega(z_2)\\ &= \frac{1}{2\pi}\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta}\int_{t=0}^1 \frac{\xi_t(f_t(z_1))-\xi_t(f_t(z_2))}{f_t(z_1)-f_t(z_2)} dt\omega(z_1)\omega(z_2),\\ &= \frac{1}{2\pi}\int_{t=0}^1\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta} \frac{\xi_t(f_t(z_1))-\xi_t(f_t(z_2))}{f_t(z_1)-f_t(z_2)}\omega(z_1)\omega(z_2)dt,\\ &= 2\times\frac{1}{2\pi}\int_{t=0}^1\int_{z_2\in\mathbb{D}}\int_{z_1\in\mathbb{D}\setminus\{z_2\}} \frac{\xi_t(z_1)}{z_1-z_2}\omega(z_1)\omega(z_2)dt\\ &= \frac{1}{\pi}\int_0^1\int_{\mathbb{D}}\int_{\mathbb{D}\setminus\{z_2\}} -2i\pi\frac{\partial H_t}{\partial\overline{z}}\frac{i}{2\pi}\frac{dz_1\wedge d\overline{z_1}}{z_1-z_2}\omega(z_2)dt\\ &= 2i\int_0^1\int_{\mathbb{D}}\int_{\mathbb{D}\setminus\{z_2\}} \frac{1}{2i\pi}\frac{\partial H_t}{\partial\overline{z}}\frac{dz_1\wedge d\overline{z_1}}{z_1-z_2}\omega(z_2)dt.\end{aligned}$$

The third equality is obtained by Fubini because the integral is absolutely integrable, see Lemma 4.1.2 below. The fourth equality is due to the absolute integrability of both terms. We established the penultimate with equation 4.2 and the definition of ω.

We use the Cauchy formula for smooth functions (see [41]). For any C^1-function $g : \mathbb{D} \to \mathbb{C}$, we have

$$g(w) = \frac{1}{2i\pi}\int_{\mathbb{S}^1}\frac{g(z)}{z-w}dz + \frac{1}{2i\pi}\int_{\mathbb{D}}\frac{\partial f}{\partial\overline{z}}\frac{dz\wedge d\overline{z}}{z-w}.$$

Moreover H_t is equal to zero on the boundary $\mathbb{S}^1$ and we have

$$\int_{\mathbb{D}\times\mathbb{D}\setminus\Delta}\int_{I_{(z_1,z_2)}} \alpha\ \omega(z_1)\omega(z_2) = 2i\int_0^1\int_{\mathbb{D}} H_t(z_2)\omega(z_2)dt.$$

It leads to

$$\int_{\mathbb{D}\times\mathbb{D}\backslash\Delta} \mathrm{Ang}_I(z_1,z_2)\omega(z_1)\omega(z_2) = 2\int_0^1\int_{\mathbb{D}} H_t(z)\omega(z)dt.$$

To obtain the result it remains to prove the absolute integrability we used in the computation.

Lemma 4.1.2. *We have the following inequality*

$$\int_{\mathbb{D}\times\mathbb{D}\backslash\Delta}\int_{t=0}^1 \left|\frac{\xi_t(f_t(z_1))-\xi_t(f_t(z_2))}{f_t(z_1)-f_t(z_2)}\right|\omega(z_1)\omega(z_2)dt < \infty.$$

Proof. The total measure of $\mathbb{D}\times\mathbb{D}\backslash\Delta$ for ω and $[0,1]$ for the Lebesgue measure is finite so by Tonnelli's theorem it is sufficient to have the following inequalities

$$\begin{aligned}\int_{t=0}^1\int_{\mathbb{D}\times\mathbb{D}\backslash\Delta}\left|\frac{\xi_t(f_t(z_1))-\xi_t(f_t(z_2))}{f_t(z_1)-f_t(z_2)}\right|\omega(z_1)\omega(z_2)dt &= \int_{t=0}^1\int_{\mathbb{D}\times\mathbb{D}\backslash\Delta}\left|\frac{\xi_t(z_1)-\xi_t(z_2)}{z_1-z_2}\right|\omega(z_1)\omega(z_2)dt\\ &\leqslant 2\int_{t=0}^1\int_{z_1\in\mathbb{D}}|\xi_t(z_1)|\int_{z_2\in\mathbb{D}\backslash\{z_1\}}\frac{1}{|z_1-z_2|}\omega(z_1)\omega(z_2)dt\\ &\leqslant 8\pi\int_{t=0}^1\int_{z_1\in\mathbb{D}}|\xi_t(z_1)|\omega(z_1)dt\\ &< \infty.\end{aligned}$$

To prove the second last inequality one may prove that

$$\int_{z_2\in\mathbb{D}\backslash\{z_1\}}\frac{1}{|z_1-z_2|}\omega(z_2)\leqslant 4\pi.$$

□

□

Remark 4.1.3. The number $\widetilde{\mathrm{Cal}}_2(f,\widetilde{\phi})$ does not depend on the choice of the isotopy in the homotopy class of I, we obtain the same result for the construction of $\widetilde{\mathrm{Cal}}_3(f,\widetilde{\phi})$ which completes the proof of Lemma 3.3.1.

Proposition 4.1.4. *For each element* $\widetilde{f}=(f,\widetilde{\phi})\in\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ *we have*

$$\widetilde{\mathrm{Cal}}_3(\widetilde{f}) = \mathrm{Cal}_1(f)+\widetilde{\rho}(\widetilde{\phi}).$$

Proof. Let us consider an element $\widetilde{f}=(f,\widetilde{\phi})\in\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and a Hamiltonian isotopy $I=(f_t)_{t\in[0,1]}$ from id to f associated to $\widetilde{\phi}$. There exists a unique Hamiltonian function $(H_t)_{t\in\mathbb{R}}$ which induces the isotopy I and such that H_t is zero on the boundary $\mathbb{S}^1$ of $\mathbb{D}$ for each $t\in\mathbb{R}$.

We know that Cal_1 does not depend on the choice of the primitive of ω. We consider the Liouville 1-form $\lambda=\frac{r^2}{2\pi}d\theta$ in radial coordinates. We consider a probability measure $\mu\in\mathcal{M}(f|_{\mathbb{S}^1})$.

We describe the link between the action function of the first definition and the Hamiltonian of the third definition. We consider a C^1 family of functions $(A_t)_{t\in[0,1]}$, where $A_t : \mathbb{D} \to \mathbb{R}$ satisfies for each $t \in [0,1]$

$$dA_t = f_t^*\lambda - \lambda,$$

and such that the map A_1 is equal to $A(f, \lambda, \mu)$. So the isotopy $(A_t)_{t\in[0,1]}$ satisfies

$$\begin{aligned} d\dot{A}_t &= \frac{d}{dt}(f_t^*\lambda) \\ &= f_t^*\mathcal{L}_{X_t} \\ &= f_t^*(i_{X_t}(d\lambda) + d(\lambda(X_t))) \\ &= d(H_t \circ f_t + \lambda(X_t) \circ f_t). \end{aligned}$$

Then, for every $t \in [0,1]$, there exists a constant c_t such that

$$\dot{A}_t = H_t \circ f_t + \lambda(X_t) \circ f_t + c_t,$$

and the map $A : \mathbb{D} \to \mathbb{R}$ satisfies for each $z \in \mathbb{D}$

$$A_1(z) = \int_0^1 (H_t + i_{X_t}\lambda)(f_t(z))dt + \int_0^1 c_t dt.$$

We denote by C the constant $\int_0^1 c_t dt$. Since the restriction of λ to $\mathbb{S}^1$ is equal to $\frac{1}{2\pi}d\theta$ then for every $z \in \mathbb{S}^1$ we have

$$\int_0^1 i_{X_t}\lambda(f_t(z))dt = \frac{1}{2\pi}\int_0^1 d\theta(\frac{\partial}{\partial t}f_t(z))dt.$$

Notice that the last integral is equal to the displacement function $\delta : \mathbb{R} \to \mathbb{R}$ of $\widetilde{\phi}$.

Moreover, the rotation number $\widetilde{\rho}(\widetilde{\phi})$ of the isotopy I satisfies for each $z \in \mathbb{S}^1$

$$\widetilde{\rho}(\widetilde{\phi}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\delta(\widetilde{\phi}^k(z)).$$

The map $z \mapsto \delta(z)$ is μ integrable and the Birkhoff ergodic theorem gives us

$$\int_{\mathbb{S}^1}\widetilde{\rho}(\widetilde{\phi})d\mu(z) = \int_{\mathbb{S}^1}\delta(z)d\mu(z).$$

We obtain

$$\int_{\mathbb{S}^1}\int_0^1 i_{X_t}\lambda(f_t(z))dtd\mu(z) = \int_{\mathbb{S}^1}\widetilde{\rho}(\widetilde{\phi})d\mu(z) = \widetilde{\rho}(I|_{\mathbb{S}^1}).$$

Moreover, the Hamiltonian H_t is equal to zero on $\mathbb{S}^1$. So if $z \in \mathbb{S}^1$ it holds that $A_1(z) = \delta(z) + C$ and consequently

$$\int_{\mathbb{S}^1} A_1(z)d\mu(z) = C + \widetilde{\rho}(\widetilde{\phi}).$$

So the condition on A implies that

$$C = -\widetilde{\rho}(\widetilde{\phi}).$$

Thus

$$\begin{aligned}\int_{\mathbb{D}} A(z)\omega(z) &= \int_{\mathbb{D}}\int_0^1 (H_t + i_{X_t}\lambda)(f_t(z))dt\omega(z) - \widetilde{\rho}(\widetilde{\phi}) \\ &= \int_{\mathbb{D}}\int_0^1 H_t(f_t(z))dt\omega(z) + \int_{\mathbb{D}}\int_0^1 i_{X_t}(\lambda)(f_t(z))dt\omega(z) - \widetilde{\rho}(\widetilde{\phi}).\end{aligned}$$

We compute $\int_{\mathbb{D}}\int_0^1 i_{X_t}(\lambda)(f_t(z))dt\omega(z)$. Each 3-form is zero on the disk so we have

$$\begin{aligned}0 &= i_{X_t}(\lambda \wedge \omega) \\ &= i_{X_t}(\lambda)\omega - \lambda \wedge i_{X_t}(\omega) \\ &= i_{X_t}(\lambda)\omega - \lambda \wedge dH_t \\ &= i_{X_t}(\lambda)\omega + dH_t \wedge \lambda \\ &= i_{X_t}(\lambda)\omega + d(H_t\lambda) - H_t\omega.\end{aligned}$$

We deduce that

$$\begin{aligned}\int_{\mathbb{D}}\int_0^1 i_{X_t}(\lambda)(f_t(z))dt\omega(z) &= \int_{\mathbb{D}}\int_0^1 (H_t\omega - d(H_t\lambda))dt \\ &= \int_{\mathbb{D}}\int_0^1 H_t\omega dt - \int_0^1\int_{\mathbb{S}^1} H_t\lambda dt \\ &= \int_{\mathbb{D}}\int_0^1 H_t\omega dt,\end{aligned}$$

where the first equality is due to the fact that f_t preserves ω. Moreover H_t is equal to zero on the boundary $\mathbb{S}^1$. We obtain

$$\int_{\mathbb{D}} A(z)\omega(z) = 2\int_{\mathbb{D}}\int_0^1 H_t(z)\omega(z)dt - \widetilde{\rho}(\widetilde{\phi}).$$

□

We know that $\widetilde{\rho}$ is a homogeneous quasi-morphism, it gives us the following corollary.

Corollary 4.1.5. *The map* $\mathrm{Cal}_1 : \mathrm{Diff}^1_\omega(\mathbb{D}) \to \mathbb{R}$ *is a homogeneous quasi-morphism.*

Proof. The result is straightforward because Cal_1 is equal to the sum of a morphism and a homogeneous quasi-morphism. □

Notice that Lemma 3.2.6 ensures that the morphisms $\widetilde{\mathrm{Cal}}$ (resp. Cal) is not zero, then its kernel is a normal non trivial subgroup of $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ (resp. $\mathrm{Diff}^1_\omega(\mathbb{D})$) and we obtain the following corollary.

Corollary 4.1.6. *The groups* $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ *and* $\mathrm{Diff}^1_\omega(\mathbb{D})$ *are not perfect.*

4.2 Continuity of $\widetilde{\mathrm{Cal}}$.

For every continuous map f from $\mathbb{D}$ to $\mathbb{C}$ we set $||f||_\infty = \max_{x\in\mathbb{D}} |f(x)|$.
We denote d_0 the distance between two maps f and g of $\mathrm{Diff}^0(\mathbb{D})$ defined by

$$d_0(f,g) = \max(||f-g||_\infty, ||f^{-1}-g^{-1}||_\infty).$$

We denote d_1 the distance between two maps f and g of $\mathrm{Diff}^1(\mathbb{D})$ defined by

$$d_1(f,g) = \max(d_0(f,g), ||Df-Dg||_\infty, ||Df^{-1}-Dg^{-1}||_\infty),$$

where for every C^1 diffeomorphism f of $\mathbb{D}$, $||Df||_\infty = \max_{x\in\mathbb{D}} ||D_x f||$.

The distances d_0 and d_1 define naturally two distances, denoted $\widetilde{d}_0$ and $\widetilde{d}_1$, on $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ defined as follows. Let us consider $\widetilde{f} = (f, \widetilde{\phi})$ and $\widetilde{g} = (g, \widetilde{\psi})$ in $\widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$, we have

$$\widetilde{d}_0(\widetilde{f},\widetilde{g}) = \max(d_0(f,g), ||\widetilde{\phi}-\widetilde{\psi}||_\infty, ||\widetilde{\phi}^{-1}-\widetilde{\psi}^{-1}||_\infty),$$
$$\widetilde{d}_1(\widetilde{f},\widetilde{g}) = \max(d_1(f,g), ||\widetilde{\phi}-\widetilde{\psi}||_\infty, ||\widetilde{\phi}^{-1}-\widetilde{\psi}^{-1}||_\infty).$$

We denote $\widetilde{\mathrm{id}} = (\mathrm{id}_{\mathbb{D}}, \mathrm{id}_{\mathbb{R}}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$. In this section we prove the following result.

Theorem 4.2.1. *The map $\widetilde{\mathrm{Cal}} : \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D}) \to \mathbb{R}$ is continuous in C^1 topology.*

We need some results about the angle function.

Lemma 4.2.2. *Let us consider $\widetilde{f} = (f, \widetilde{\phi}) \in \mathrm{Diff}^1_+(\mathbb{D})$ such that $\widetilde{d}_1(\widetilde{f}, \widetilde{\mathrm{id}}) \leqslant \epsilon \leqslant 1/2$, then for every $(x,y) \in \mathbb{D}^2\backslash\Delta$, it holds that*

$$|\cos(2\pi \mathrm{Ang}_{\widetilde{f}}(x,y)) - 1| \leqslant 2\epsilon.$$

Proof of Lemma 4.2.2. The proof is a simple computation. Let us consider $x, y \in \mathbb{D}$ such that $x \neq y$. One can write $f = \mathrm{id} + h$ where $||h||_\infty \leqslant \epsilon$ and $||Dh||_\infty \leqslant \epsilon$. By the mean theorem we have

$$\left|\frac{h(y)-h(x)}{y-x}\right| \leqslant \epsilon. \tag{4.3}$$

We have

$$\cos(2\pi \mathrm{Ang}_{\widetilde{f}}(x,y)) = \left\langle \frac{f(y)-f(x)}{|f(y)-f(x)|} \Big| \frac{y-x}{|y-x|} \right\rangle,$$

where $\langle .|. \rangle$ is the canonical scalar product on $\mathbb{R}^2$. We compute

$$\begin{aligned}
|\cos(2\pi \mathrm{Ang}_{\widetilde{f}}(x,y)) - 1| &= \left|\left\langle \frac{f(y)-f(x)}{|f(y)-f(x)|} - \frac{y-x}{|y-x|} \Big| \frac{y-x}{|y-x|} \right\rangle\right| \\
&\leqslant \left|\frac{f(y)-f(x)}{|f(y)-f(x)|} - \frac{y-x}{|y-x|}\right|.
\end{aligned}$$

We compute

$$\begin{aligned}
\left|\frac{f(y)-f(x)}{|f(y)-f(x)|} - \frac{y-x}{|y-x|}\right| &\leqslant \left|\frac{f(y)-f(x)-(y-x)}{|y-x|}\right| + |f(y)-f(x)| \left|\frac{1}{|f(y)-f(x)|} - \frac{1}{|y-x|}\right| \\
&\leqslant \left|\frac{h(y)-h(x)}{|y-x|}\right| + \left|\frac{|y-x|-|f(y)-f(x)|}{|y-x|}\right| \\
&\leqslant 2\left|\frac{h(y)-h(x)}{y-x}\right| \\
&\leqslant 2\epsilon.
\end{aligned}$$

□

From Lemma 4.2.2, we deduce the following result.

Corollary 4.2.3. *Let us consider* $\widetilde{f} \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ *such that* $d_1(\widetilde{f}, \widetilde{\mathrm{id}}) \leqslant \epsilon \leqslant 1/2$. *The angle function satisfies*

$$||\mathrm{Ang}_{\widetilde{f}}||_\infty \leqslant \sqrt{\epsilon}/\pi.$$

Proof. For every couple $(x, y) \in \mathbb{D}^2 \backslash \Delta$ there exists a unique $k \in \mathbb{Z}$ such that $\mathrm{Ang}_{\widetilde{f}}(x, y) - k \in [-1/2, 1/2)$. So by Lemma 4.2.2 we have

$$1 \geqslant \cos(|2\pi \mathrm{Ang}_{\widetilde{f}}(x, y) - k|) \geqslant 1 - 2\epsilon \geqslant 0.$$

The function arccos is decreasing so we obtain

$$0 \leqslant \arccos(\cos(|2\pi \mathrm{Ang}_{\widetilde{f}}(x, y) - k|)) \leqslant \arccos(1 - 2\epsilon).$$

Moreover the function arccos is defined on $[0, 1]$ and of class C^1 on $[0, 1)$ such that for every $x \in (0, 1]$ we have

$$(\arccos(1 - x))' = \frac{1}{\sqrt{2x - x^2}} \leqslant \frac{1}{\sqrt{x}}.$$

We obtain that for every $x \in [0, 1]$ we have

$$\arccos(1 - x) \leqslant 2\sqrt{x}.$$

Hence we have

$$2\pi |\mathrm{Ang}_{\widetilde{f}}(x, y) - k| \leqslant 2\sqrt{2\epsilon}.$$

And so

$$|\mathrm{Ang}_{\widetilde{f}}(x, y) - k| \leqslant \frac{\sqrt{2\epsilon}}{\pi} < 1/2.$$

Moreover $\mathbb{D}^2 \backslash \Delta$ is path connected. Indeed, let us prove that every couple $(x, y) \in \mathbb{D}^2 \backslash \Delta$ is connected to $((0, 0), (1, 0))$ by a path as follows. We set d the line of $\mathbb{D}^2$ passing through x and y. The line d intersects $\mathbb{S}^1$ in two points which we denote $\hat{x}$ and $\hat{y}$ such that $\hat{x}$ is closer to x than y and $\hat{y}$ is closer to y than x as in figure 4.1

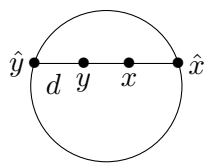


Figure 4.1

Let us consider the path $\gamma_y : [0, 1] \to \mathbb{D}$ defined by $\gamma_y(t) = t(\hat{y} - y) + y$ from y to $\hat{y}$. The path $\Gamma_y : t \to (x, \gamma_y(t))$ defined on $[0, 1]$ sends the couple (x, y) to $(x, \hat{y})$.

Let us consider the path $\gamma_x : [0, 1] \to \mathbb{D}$ defined by $\gamma_x(t) = (1 - t)x$ from x to $(0, 0)$. The path $\Gamma_x : t \to (\gamma_x(t), \hat{y})$ defined on $[0, 1]$ sends the couple $(x, \hat{y})$ to $(0, \hat{y})$.

Now we consider R_α the rotation of $\mathbb{D}$ of angle $\alpha = arg(\hat{y})$. The rotation $\mathbb{R}_\alpha^{-1}$ sends $\hat{y}$ to $(1,0)$. We denote $(R_t)_{t\in[0,1]}$ the isotopy from id to R_α such that for every $t \in [0,1]$ R_t is the rotation of angle $t\alpha$.

Hence the composition of the path Γ_y, Γ_x and $t \to ((0,0), R_t^{-1}(\hat{y}))$ sends (x,y) to $((0,0),(0,1))$.

Moreover $\mathrm{Ang}_{\tilde{f}}$ is continuous on $\mathbb{D}^2\backslash\Delta$ we deduce from the last inequality that k does not depend on the choice of (x,y). The fact that $\widetilde{d}_1(\widetilde{f},\widetilde{\mathrm{id}}) \leqslant \epsilon \leqslant 1/2$ implies that $k=0$ and we obtain that for every $(x,y) \in \mathbb{D}^2\backslash\Delta$

$$|\mathrm{Ang}_{\tilde{f}}(x,y)| \leqslant \sqrt{\epsilon}/\pi.$$

□

We now prove the continuity of Cal_1 in C^1 topology.

Proof of theorem 4.2.1. By Theorem 3.2.2 we know that $\widetilde{\mathrm{Cal}}$ is a group morphism. So it is sufficient to prove the continuity at the identity. Let us consider $\widetilde{f} = (f,\widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ such that $\widetilde{d}_1(\widetilde{f},\widetilde{\mathrm{id}}) \leqslant \epsilon \leqslant 1/2$. By Corollary 4.2.3 we have for every couple $(x,y) \in \mathbb{D}^2\backslash\Delta$

$$|\mathrm{Ang}_{\tilde{f}}(x,y)| \leqslant \sqrt{\epsilon}/\pi.$$

By integration on $\mathbb{D}^2\backslash\Delta$ we obtain that

$$|\widetilde{\mathrm{Cal}}(\widetilde{f})| \leqslant \frac{\sqrt{2\epsilon}}{\pi}.$$

Hence $\widetilde{\mathrm{Cal}}$ is continuous at the identity. □

Moreover, it is well-known that the rotation number $\widetilde{\rho} : \widetilde{\mathrm{Homeo}}^+(\mathbb{S}^1) \to \mathbb{R}$ is continuous and we deduce from Theorem 4.0.1 the following corollary.

Corollary 4.2.4. *The map* $\mathrm{Cal}_1 : \mathrm{Diff}^1_\omega(\mathbb{D}) \to \mathbb{R}$ *is continuous in* C^1 *topology.*

Let us prove that the Calabi invariant is not continuous in C^0 topology.

Proposition 4.2.5. *The morphism* $\widetilde{\mathrm{Cal}}$ *is not continuous in* C^0 *topology.*

We give a counterexample which also prove that the Calabi invariant defined in the introduction is also not continuous in the C^0 topology, this counterexample can be found in [32]

Proof. Let us consider a sequence $(h_n)_{n\geqslant 1}$ of smooth functions $h_n : [0,1] \to \mathbb{R}$ such that

1. h_n is constant near the origin,
2. $h_n(r)$ is zero for $r > 1/n$,
3. $\int_0^1 h_n(r) 2\pi r dr = 1$.

We consider the Hamiltonian functions $H_n : \mathbb{D} \to \mathbb{R}$ by $H_n(z) = h_n(|z|)$. Each function H_n defines a time independent vector field X_n, whose induced flow is denoted ϕ_n^t. We have the following property [32] about the computation of the Calabi invariant for compactly supported and autonomous Hamiltonian functions

Proposition 4.2.6. *Let us consider $H : \mathbb{D} \to \mathbb{R}$ a Hamiltonian function with compact support. We denote ϕ^t the induced Hamiltonian flow and we have*

$$\mathrm{Cal}(\phi^t) = -2\pi t \int_{\mathbb{D}} H(z)\omega(z),$$

where Cal *is the Calabi invariant defined by equation 1.3.*

This result allows us to compute the Calabi invariant for ϕ_n^1 and we obtain for each $n \geqslant 1$

$$\mathrm{Cal}(\phi_n^1) = -2\pi.$$

For each $n \geqslant 1$ we consider $(\phi_n^1, \mathrm{id}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ and we have

$$\widetilde{\mathrm{Cal}}_2((\phi_n^1, \mathrm{id})) = -2\pi.$$

Moreover, ϕ_n^1 converges to the identity in C^0 topology and we obtain the result. $\square$

Chapter 5

Computation of Cal_1 in some rigid cases

In this section, we prove several results about the Calabi invariant of irrational pseudo-rotations.

5.1 A simple case of C^1-rigidity

Let us begin by the simple computation of the Calabi invariant for periodic symplectic maps.

Lemma 5.1.1. *If $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ has a finite order, then we have*

$$\mathrm{Cal}_1(f) = 0.$$

Proof. By assumption there exists $p \geqslant 1$ such that $f^p = \mathrm{id}$ and so $\mathrm{Cal}_1(f^p) = p\,\mathrm{Cal}_1(\mathrm{id}) = 0$. □

We deduce the following properties

Proposition 5.1.2. *Let us consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$. If there exists a sequence of periodic diffeomorphisms $(g_k)_{k\in\mathbb{N}}$ in $\mathrm{Diff}^1_\omega(\mathbb{D})$ which converges to f fin C^1 topology, then we have*

$$\mathrm{Cal}_1(f) = 0.$$

Proof. By Lemma 5.1.1 for each $n \in \mathbb{N}$ we have $\mathrm{Cal}_1(g_n) = 0$ and we obtain the result by the continuity of the map Cal_1 in C^1 topology. □

Proposition 5.1.3. *Let us consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$. If there exists a sequence $(q_k)_{k\in\mathbb{N}}$ such that f^{q_k} converges to the identity in C^1 topology then we have*

$$\mathrm{Cal}_1(f) = 0.$$

Proof. We have $\mathrm{Cal}_1(f^{q_k}) = q_k\mathrm{Cal}_1(f)$ and $\mathrm{Cal}_1(f^{q_k})$ converges to $\mathrm{Cal}_1(\mathrm{id}) = 0$ so $\mathrm{Cal}_1(f) = 0$. □

5.2 C^0-rigidity

The following theorem is a stronger version of Corollary 5.1.3.

Theorem 5.2.1. *Let us consider $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$. If there exists a sequence $(q_k)_{k\in\mathbb{N}}$ of integers such that $(f^{q_k})_{k\in\mathbb{N}}$ converges to the identity in C^0 topology then we have*

$$\mathrm{Cal}_1(f) = 0.$$

To prove the previous statement we will give an estimation of the angle function of f^{q_n} for a given isotopy I from id to f. For that we will consider two cases, the first one if x is *close* to y and the other if x is not *close* to y. The following lemma gives us an evaluation of what *close* means.

Lemma 5.2.2. *Let us consider f a C^1 diffeomorphism of the unit disc $\mathbb{D}$, I an isotopy from* id *to f. If $d_0(f, \mathrm{id}) \leqslant \epsilon \leqslant 1/4$ then for every couple $(x, y) \in \mathbb{D} \times \mathbb{D}$ which satisfies $|x - y| \geqslant \sqrt{\epsilon}$, we have*

$$|\cos(2\pi \mathrm{Ang}_I(x, y)) - 1| \leqslant 4\sqrt{\epsilon}.$$

Proof. Let $(x, y) \in \mathbb{D} \times \mathbb{D}$ be a couple such that $|x - y| \geqslant \sqrt{\epsilon}$. One can write $f = \mathrm{id} + h$ where $h : \mathbb{D} \to \mathbb{R}^2$ satisfies $||h||_\infty \leqslant \epsilon$ and we have

$$\left|\frac{h(y) - h(x)}{y - x}\right| \leqslant 2\frac{\epsilon}{\sqrt{\epsilon}} = 2\sqrt{\epsilon}. \tag{5.1}$$

We use the equation

$$\cos(2\pi \mathrm{Ang}_I(x, y)) = \frac{\langle f(y) - f(x), y - x\rangle}{|f(y) - f(x)|\ |y - x|} \tag{5.2}$$

Moreover, if we write $1 = \langle \frac{y-x}{|y-x|}, \frac{y-x}{|y-x|}\rangle$ we obtain

$$\cos(\mathrm{Ang}_I(x, y)) - 1 = \langle \frac{f(y) - f(x)}{|f(y) - f(x)|} - \frac{y - x}{|y - x|}, \frac{y - x}{|y - x|}\rangle \tag{5.3}$$

Equation 5.3 becomes

$$\begin{aligned}
\left|\frac{f(y) - f(x)}{|f(y) - f(x)|} - \frac{y - x}{|y - x|}\right| &\leqslant |f(y) - f(x)| \left|\frac{1}{|f(y) - f(x)|} - \frac{1}{|y - x|}\right| + \left|\frac{f(y) - f(x) - (y - x)}{|y - x|}\right| \\
&= \left|\frac{|y - x| - |f(y) - f(x)|}{|y - x|}\right| + \left|\frac{h(y) - h(x)}{|y - x|}\right| \\
&\leqslant 2\left|\frac{h(y) - h(x)}{y - x}\right| \\
&\leqslant 4\sqrt{\epsilon}.
\end{aligned}$$

□

We obtain the following lemma.

Lemma 5.2.3. *Under the same hypothesis, there exists an integer $k \in \mathbb{Z}$, uniquely defined, such that for every couple $(x, y) \in \mathbb{D} \times \mathbb{D}$ such that $|x - y| \geqslant \sqrt{\epsilon}$, we have*

$$|\mathrm{Ang}_I(x, y) - k| \leqslant 2\sqrt[4]{\epsilon}/\pi < 1/2. \tag{5.4}$$

Proof. We consider $\epsilon \in (0, 1/16)$ and a couple $(x, y) \in \mathbb{D}$ such that $|y - x| \geqslant \sqrt{\epsilon}$. By definition of the floor function there exists a unique $k \in \mathbb{Z}$ such that $2\pi \mathrm{Ang}_I(x, y) - 2\pi k \in [-\pi, \pi)$ and we have

$$1 \geqslant \cos(|2\pi \mathrm{Ang}_I(x, y) - 2\pi k|) \geqslant 1 - 4\sqrt{\epsilon} \geqslant 0.$$

The function arccos is decreasing so we obtain

$$0 \leqslant \arccos(\cos(|2\pi \mathrm{Ang}_I(x, y) - 2\pi k|)) \leqslant \arccos(1 - 4\sqrt{\epsilon}).$$

The function arccos is defined on $[0, 1]$ and of class C^1 on $[0, 1)$. Moreover we have for every $x \in [0, 1)$

$$(\arccos(1 - x))' = \frac{1}{\sqrt{2x - x^2}} \leqslant \frac{1}{\sqrt{x}}.$$

We obtain that for every $x \in [0, 1]$

$$\arccos(1 - x) \leqslant 2\sqrt{x}.$$

Hence we have

$$\begin{aligned} |2\pi \mathrm{Ang}_I(x, y) - 2\pi k| &\leqslant \arccos(1 - 4\sqrt{\epsilon}) \\ &\leqslant 4\sqrt[4]{\epsilon}. \end{aligned}$$

Thus we have

$$|\mathrm{Ang}_I(x, y) - k| \leqslant 2\sqrt[4]{\epsilon}/\pi < 1/2.$$

Now we prove that k does not depend of (x, y). Indeed the set of couples $(x, y) \in \mathbb{D}^2$ such that $|x - y| \geqslant \sqrt{\epsilon}$ is connected in $\mathbb{D}^2$. Indeed for a couple $(x, y) \in \mathbb{D}^2$ such that $|x - y| \geqslant \sqrt{\epsilon}$, let us construct a path from (x, y) to $((-1, 0), (1, 0))$.
We set d the line of $\mathbb{D}^2$ passing through x and y. The line d intersects $\mathbb{S}^1$ in two points which we denote $\hat{x}$ and $\hat{y}$ such that $\hat{x}$ is closer to x than y and $\hat{y}$ is closer to y than x as in the previous figure 4.1.

Let us consider the path $\gamma_x : [0, 1] \to \mathbb{D}$ defined by $\gamma_x(t) = t(\hat{x} - x) + x$ from x to $\hat{x}$ and the path $\gamma_y : [0, 1]$ defines by $\gamma_y(t) = t(\hat{y} - y) + y$ from y to $\hat{y}$. So the path $\Gamma : t \mapsto (\gamma_x(t), \gamma_y(t))$ defined on $[0, 1]$ sends the couple (x, y) to $(\hat{x}, \hat{y})$.

We consider a path $\Gamma' : t \to (\hat{x}, \gamma(t))$ which fixes $\hat{x}$ and send $\hat{y}$ on $\hat{x}$ along the arc of $\mathbb{S}^1$ such that $\gamma(t) \in \mathbb{S}^1 \backslash \{\hat{x}\}$.

Now we consider R_α the rotation of $\mathbb{D}$ of angle $\alpha = \arg(\hat{x})$. Notice that the rotation R_α^{-1} sends $\hat{x}$ to $(1, 0)$. We denote $(R_t)_{t \in [0,1]}$ the isotopy from id to R_α such that for every $t \in [0, 1]$ R_t is the rotation of angle $t\alpha$.

Hence the composition of the path Γ, $Gamma'$ and the path $t \mapsto (R_t^{-1}(\hat{x}), R_t^{-1}(\hat{y}))$ sends (x, y) to $((1, 0), (-1, 0))$.

Moreover, $2\sqrt[4]{\epsilon}/\pi < 1/2$ so k does not depend on the choice of $(x, y) \in \mathbb{D}$ such that $|x - y| > \sqrt{\epsilon}$.

□

With these two lemmas we can give a proof of Theorem 5.2.1.

Proof of Theorem 5.2.1. We can consider $I = (f_t)_{t\in[0,1]}$ an isotopy from id to f which fixes a point of $\mathring{\mathbb{D}}$. Up to conjugacy we can suppose that I fixes the origin and we denote $I|_{\mathbb{S}^1}$ the restriction of I on $\mathbb{S}^1$. We lift $I|_{\mathbb{S}^1}$ to an isotopy $(\widetilde{\phi}_t)_{t\in[0,1]}$ on the universal covering space $\mathbb{R}$ of $\mathbb{S}^1$ such that $\widetilde{\phi}_0 = \mathrm{id}$ and set $\widetilde{\phi} = \widetilde{\phi}_1$. We will prove that $\widetilde{\mathrm{Cal}}_2(f, \widetilde{\phi}) = \widetilde{\rho}(\widetilde{\phi})$ and from Theorem 4.0.1 we will obtain

$$\widetilde{\mathrm{Cal}}_2(f, \widetilde{\phi}) - \widetilde{\rho}(\widetilde{\phi}) = \mathrm{Cal}_1(f) = 0.$$

For $q \in \mathbb{N}$ we define the isotopy I^q from id to f^q as follows. We write $I^q = (f^q_t)_{t\in[0,1]}$ and for every $z \in \mathbb{D}$ and $t \in [\frac{k-1}{q}, \frac{k}{q}]$ we set

$$f^q_t(z) = f_{qt-k+1} \circ \underbrace{(f \circ \ldots \circ f)}_{k-1 \text{ times}}.$$

We will denote $\epsilon_n = d_0(f^{q_n}, \mathrm{id})$. For every $k \in \mathbb{Z}$ we can separate the difference between the integral of the angle function of f^{q_n} and k into two parts as follows

$$\begin{aligned}\int\int_{\mathbb{D}\times\mathbb{D}} \mathrm{Ang}_{I^{q_n}}(x,y)\omega(y)\omega(x) - k = \int_{\mathbb{D}} \left(\int_{B_{\sqrt{\epsilon_n}}(x)} \mathrm{Ang}_{I^{q_n}}(x,y)\omega(y) - k\right)\omega(x) \\ + \int_{\mathbb{D}} \left(\int_{B^c_{\sqrt{\epsilon_n}}(x)} \mathrm{Ang}_{I^{q_n}}(x,y)\omega(y) - k\right)\omega(x),\end{aligned} \tag{5.5}$$

where $B^c_{\sqrt{\epsilon_n}}(x)$ is the complementary of $B_{\sqrt{\epsilon_n}}(x)$ in $\mathbb{D}$.

We can suppose that $\epsilon_n < 1/16$ and by Lemma 5.2.3, there exists a unique $k_n \in \mathbb{Z}$ such that for each couple $(x, y) \in \mathbb{D} \times \mathbb{D}$ such that $|y - x| \geqslant \sqrt{\epsilon}_n$ we have

$$|\mathrm{Ang}_{I^{q_n}}(x,y) - k_n| \leqslant 2\sqrt[4]{\epsilon_n}/\pi. \tag{5.6}$$

Moreover, by definition of the rotation number there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of 1-periodic functions $\xi_n : \mathbb{R} \to \mathbb{R}$ such that $||\xi_n||_\infty \leqslant 1$ for every $n \in \mathbb{N}$ and such that for every $y \in \mathbb{S}^1$ and every lift $\widetilde{y} \in \mathbb{R}$ of y we have

$$\mathrm{Ang}_{I^{q_n}}(0,y) = \widetilde{\phi}^{q_n}(\widetilde{y}) - \widetilde{y} = q_n\widetilde{\rho}(\widetilde{\phi}) + \xi_n(\widetilde{y}).$$

So, for every $y \in \mathbb{S}^1$ we have

$$|\mathrm{Ang}_{I^{q_n}}(0,y) - k_n| = |q_n\widetilde{\rho}(\widetilde{\phi}) + \xi_n(\widetilde{y}) - k_n| \leqslant 2\sqrt[4]{\epsilon_n}/\pi,$$

where $\widetilde{y}$ is a lift of y. Hence we obtain

$$|q_n\widetilde{\rho}(\widetilde{\phi}) - k_n| \leqslant 2\sqrt[4]{\epsilon_n}/\pi + 1.$$

Thus we have

$$\widetilde{\rho}(\widetilde{\phi}) = \lim_{n\to\infty} \frac{k_n}{q_n}.$$

By equation 5.6 we obtain

$$\left|\int_{\mathbb{D}} \left(\int_{B^c_{\sqrt{\epsilon_n}}(x)} (\mathrm{Ang}_{I^{q_n}}(x,y) - k_n)\omega(y)\right)\omega(x)\right| \leqslant 2\sqrt[4]{\epsilon_n}/\pi. \tag{5.7}$$

We know that for every couple $(x,y) \in \mathbb{D}^2 \backslash \Delta$ and for every $n \in \mathbb{N}$ we have

$$\mathrm{Ang}_{I^{q_n}}(x,y) = \mathrm{Ang}_I(x,y) + \mathrm{Ang}_I(f(x),f(y)) + ... + \mathrm{Ang}_I(f^{q_n-1}(x), f^{q_n-1}(y)). \quad (5.8)$$

Hence for every $n \in \mathbb{N}$ the angle function satisfies

$$||\mathrm{Ang}_{I^{q_n}}||_\infty \leqslant q_n ||\mathrm{Ang}_I||_\infty. \quad (5.9)$$

We can estimate the first integral of equation 5.5 as follows

$$\left| \int_{\mathbb{D}} \left(\int_{B_{\sqrt{\epsilon_n}}(x)} (\mathrm{Ang}_{I^{q_n}}(x,y) - k_n)\omega(y) \right) \omega(x) \right| \leqslant \epsilon_n (q_n ||\mathrm{Ang}_I||_\infty + |k_n|). \quad (5.10)$$

So we can deduce from the previous equations a new estimation of the Calabi invariant

$$\left| \int \int_{\mathbb{D}\times\mathbb{D}} \mathrm{Ang}_{I^{q_n}}(x,y)\omega(y)\omega(x) - k_n \right| \leqslant 2\sqrt[4]{\epsilon_n}/\pi + \epsilon_n (q_n ||\mathrm{Ang}_f||_\infty + |k_n|). \quad (5.11)$$

By definition we obtain

$$\left| \widetilde{\mathrm{Cal}}_2(f,\widetilde{\phi}) - \frac{k_n}{q_n} \right| \leqslant \frac{2\sqrt[4]{\epsilon_n}}{q_n \pi} + \epsilon_n ||\mathrm{Ang}_f||_\infty + \epsilon_n \frac{|k_n|}{q_n}. \quad (5.12)$$

Hence we have

$$\begin{aligned} \left| \widetilde{\mathrm{Cal}}_2(f,\widetilde{\phi}) - \widetilde{\rho}(\widetilde{\phi}) \right| &\leqslant \left| \widetilde{\mathrm{Cal}}(f,\widetilde{\phi}) - \frac{k_n}{q_n} \right| + \left| \widetilde{\rho}(\widetilde{\phi}) - \frac{k_n}{q_n} \right| \\ &\leqslant \frac{4\sqrt[4]{\epsilon_n}}{q_n \pi} + \epsilon_n ||\mathrm{Ang}_f||_\infty + \frac{1}{q_n} + \epsilon_n \frac{|k_n|}{q_n}. \end{aligned}$$

By taking the limit on $n \in \mathbb{N}$, we conclude that

$$\widetilde{\mathrm{Cal}}_2(f,\widetilde{\phi}) = \widetilde{\rho}(\widetilde{\phi}).$$

□

Remark 5.2.4. If we consider a sequence $(\widetilde{g}_n = (g_n, \widetilde{\phi}_n))_{n\in\mathbb{N}} \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ which converges to $\widetilde{f} = (f, \widetilde{\phi}) \in \widetilde{\mathrm{Diff}}^1_\omega(\mathbb{D})$ in C^0 topology where for each $n \in \mathbb{N}$, g_n is a periodic diffeomorphism of the disk and f is an irrational pseudo-rotation, then the previous method fails to prove that $\widetilde{\mathrm{Cal}}_2(g_n, \widetilde{\phi}_n)$ converges to $\widetilde{\mathrm{Cal}}_2(f, \widetilde{\phi})$. It is easy to see that $\mathrm{Ang}_{\widetilde{f}}$ is close to $\mathrm{Ang}_{\widetilde{g}_n}$ but if we compute the difference $\widetilde{\mathrm{Cal}}(g_n, \widetilde{\phi}_n) - \widetilde{\mathrm{Cal}}(f, \widetilde{\phi})$, as we did in equation 5.5, we do not have a control of $||\mathrm{Ang}_{\widetilde{g}_n}||_\infty$ so we cannot estimate properly the integral

$$\int_x \int_{y \in B_{\sqrt{\epsilon_n(x)}}} \mathrm{Ang}_{\widetilde{g}_n}(x,y)\omega(x)\omega(y),$$

where $\epsilon_n = ||g_n - f||_\infty$.

Chapter 6

Examples

In this section, we will be interested in irrational pseudo-rotations with specific rotation numbers.

Best approximation: Any irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ can be written as a continued fraction where $(a_i)_{i\geqslant 1}$ is a sequence of integers $\geqslant 1$ and $a_0 = \lfloor\alpha\rfloor$. Conversely, any sequence $(a_i)_{i\in\mathbb{N}}$ corresponds to a unique number α. We define two sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n_2} \text{ for } n \geqslant 2, & p_0 &= a_0,\, p_1 = a_0 a_1 + 1 \\ q_n &= a_n q_{n-1} + q_{n-2} \text{ for } n \geqslant 2, & q_0 &= 1,\, q_1 = a_1. \end{aligned}$$

The sequence $(p_n/q_n)_{n\in\mathbb{N}}$ is called the *best approximation* of α and for every $n \geqslant 1$ we have

$$\{q_{n-1}\alpha\} \leqslant \{k\alpha\},\ \forall k < q_n$$

where $\{x\}$ is the fractional part of $x \in \mathbb{R}$. And for every $n \in \mathbb{N}$ we have

$$\frac{1}{q_n(q_n + q_{n+1})} \leqslant (-1)^n(\alpha - p_n/q_n,) \leqslant \frac{1}{q_n q_{n+1}}. \tag{6.1}$$

The numbers q_n are called the *approximation denominators* of α.

6.1 An example of C^0-rigidity, the super Liouville type

In this section, we show that a C^1 irrational pseudo-rotation with a super Liouville rotation number satisfies the assumptions of Theorem 5.2.1.

Super Liouville. A real number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is called *super Liouville* if the sequence $(q_n)_{n\in\mathbb{N}}$ of the approximation denominators of α satisfies

$$\limsup_n q_n^{-1} \log(q_{n+1}) = +\infty. \tag{6.2}$$

If we consider a real $\alpha \in \mathbb{R}$ which has super Liouville type then for each $k \in \mathbb{Z}$ the real $\alpha + k$ is also super Liouville and to simplify the notations we will say that an element

$\widetilde{\alpha} \in \mathbb{T}^1$ is super Liouville.

Bramham already showed in [13] that any C^∞ irrational pseudo-rotation f of the disk with super Liouville rotation number is C^0 rigid, meaning that f is the C^0-limit of a sequence of periodic diffeomorphisms. More recently Le Calvez [51] proved that any C^1 irrational pseudo-rotation which is C^1 conjugated to a rotation on the boundary is C^0 rigid. These results go as follows.

Theorem 6.1.1. *Let us consider either a C^∞ irrational pseudo-rotation or a C^1 irrational pseudo-rotation f which is C^1 conjugated to a rotation on the boundary. We consider $\alpha \in \mathbb{R}$ such that $\alpha + \mathbb{Z}$ is equal to the rotation number of f. For a sequence of rationals $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ which converges to α there exists a sequence $(g_n)_{n\in\mathbb{N}} : \mathbb{D} \to \mathbb{D}$ of q_n-periodic diffeomorphims of the unit disk which converges to f for the C^0 topology.*
Moreover there exists a constant C depending on f such that for every $n \in \mathbb{N}$ we have

$$d_0(f, g_n) < C(q_n\alpha - p_n)^{\frac{1}{2}}.$$

We deduce the following corollary.

Corollary 6.1.2. *Let us consider either a C^∞ irrational pseudo-rotation or a C^1 irrational pseudo-rotation f which is C^1 conjugated to a rotation on the boundary. If the rotation number of f is super Liouville then we have*

$$\mathrm{Cal}_1(f) = 0.$$

Proof of Corollary 6.1.2. Let us consider f which is either a C^∞ irrational pseudo-rotation or a C^1 irrational pseudo-rotation. We consider $\alpha \in \mathbb{R}$ such that $\alpha + \mathbb{Z}$ is equal to the rotation number of f. We will prove that f satisfies the hypothesis of Theorem 5.2.1. We consider $\alpha \in \mathbb{R}$ such that $\alpha + \mathbb{Z}$ is equal to the rotation number of f and we consider the sequence of rationals $(p_n/q_n)_{n\in\mathbb{N}}$, defined above, which converges to α such that q_n satisfies equation 6.1. Let $(g_n)_{n\in\mathbb{N}}$ be the sequence of q_n periodic diffeomorphisms given by Theorem 6.1.1 associated to f and the sequence $(p_n/q_n)_{n\in\mathbb{N}}$. We denote by K the C^1 norm of f and we set $\epsilon_n = C(q_n\alpha - p_n)^{1/2}$ where C is the constant given by Theorem 6.1.1.

For all $k \in \mathbb{N}$ and each $n \in \mathbb{N}$ the following inequality holds

$$d_0(f^k, g_n^k) < K^k \epsilon_n. \tag{6.3}$$

By equation 6.1 we can majorate ϵ_n by $\frac{C}{(q_{n+1})^{\frac{1}{2}}}$ to obtain for $k = q_n$ the inequality

$$d(f^{q_n}, id) < K^{q_n} \frac{C}{(q_{n+1})^{\frac{1}{2}}}. \tag{6.4}$$

Equation 6.2 assures that

$$\limsup_n \frac{K^{q_n}}{(q_{n+1})^{1/2}} = 0.$$

Thus we obtain that

$$\limsup_n d_0(f^{q_n}, \mathrm{id}) = 0.$$

Hence up to a subsequence we can suppose that

$$d_0(f^{q_n}, id) \to 0.$$

So f satisfies the hypothesis of Theorem 5.2.1 and we conclude

$$\mathrm{Cal}_1(f) = 0.$$

□

6.2 An example of C^1-rigidity, the non Bruno type

Bruno type. A number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ will be said to be Bruno type if the sequence $(q_n)_{n\in\mathbb{N}}$ of the approximation denominators of α satisfies

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1})}{q_n} < +\infty.$$

If we consider $\alpha \in \mathbb{R}$ which is not Bruno type then for each $k \in \mathbb{Z}$ the real $\alpha + k$ is also not Bruno type and to simplify the notation we will say that an element $\tilde{\alpha} \in \mathbb{T}^1$ is non Bruno type.

Avila, Fayad, Le Calvez, Xu and Zhang proved in [4] that if we consider a number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ which is not Bruno type, for $H > 1$ there exists a subsequence q_{n_k} of the sequence of the approximation denominators of α such that for every $n \in \mathbb{N}$ $q_{n_{j+1}} \geqslant H^{q_{n_j}}$ and there exists an infinite set $\mathrm{J} \subset \mathbb{N}$ such that for every $j \in \mathrm{J}$ we have

$$\{q_{n_j}\alpha\} < \mathrm{e}^{-\frac{q_{n_j}}{j^2}}. \tag{6.5}$$

We can also find the following result in the same paper.

Proposition 6.2.1. *Let us consider a C^2 irrational pseudo-rotation $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$. Suppose that $\rho(f|_{\mathbb{S}^1})$ is not Bruno type, then the sequence q_{n_j} satisfies*

$$d_1(f^{q_{n_j}}, Id) \to 0.$$

Hence a C^2 irrational pseudo-rotation $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$ satisfies the hypothesis of Corollary 5.1.3 and we obtain the following corollary.

Corollary 6.2.2. *Let us consider a C^2 irrational pseudo-rotation $f \in \mathrm{Diff}^1_\omega(\mathbb{D})$. Suppose that $\rho(f)$ is not Bruno type, then we have*

$$\mathrm{Cal}_1(f) = 0.$$

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