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Quentin Vigneron

Influence of topology in inhomogeneous cosmology through non-Euclidean extensions of Newton's theory and relativistic numerical schemes

Influence de la topologie en cosmologie inhomogène à travers des extensions non-euclidiennes de la théorie de Newton et des schémas numériques relativistes

Devant le jury composé de :

ROUKEMA Boudewijn, Professeur (Université de Toruń)

TRIAIY Roland, Professeur (Université Aix-Marseille)

HUNEAU Cécile, Chargée de recherche (École Polytechnique)

MAHMOUDI Nazila, Professeure (UCBL)

SALATI Pierre, Professeur (USMB)

BUCHERT Thomas, Professeur (UCBL)

Rapporteur

Rapporteur

Examinatrice

Examinatrice

Examineur

Directeur de thèse

Résumé

Le modèle standard de la cosmologie suppose que les inhomogénéités à petite échelle n'affectent pas l'expansion des plus grandes échelles. Cependant, un tel phénomène, appelé rétroaction, pourrait exister et être suffisamment important pour expliquer l'énergie noire. Alors que la plupart des études sur la rétroaction se sont concentrées sur sa relation avec la formation des structures, peu de choses ont été faites pour comprendre sa dépendance à la topologie de notre Univers. Ma thèse de doctorat vise à combler cette lacune en suivant deux stratégies.

Dans un premier temps, j'essaie de définir une extension non-euclidienne de la théorie de Newton à partir de la relativité générale et de généraliser le théorème de Buchert-Ehlers, qui stipule que la rétroaction est nulle en gravité newtonienne. Comme première étape vers cette définition, je montre que la cosmologie newtonienne peut être obtenue à partir de la théorie de Newton-Cartan. Dans ce cas, l'expansion apparaît comme un champ fondamental de la théorie. Je propose ensuite deux "théories newtoniennes non-euclidiennes" basées sur le formalisme de Newton-Cartan. La première théorie comporte une rétroaction, tandis que l'autre n'en a aucune. Enfin j'essaie de justifier l'une d'entre elles en utilisant la limite galiléenne de la relativité générale. Je montre que pour permettre des géométries non-euclidiennes à la limite, un terme supplémentaire lié à la courbure spatiale doit être ajouté dans le tenseur énergie-impulsion d'un fluide géodésique. L'une des conséquences de cette modification est que le système d'équations à l'ordre dominant n'est pas fermé, laissant ouverte la question de la "bonne" théorie newtonienne non-euclidienne compatible avec la relativité générale.

Dans un deuxième temps, j'étudie la possibilité de faire des simulations cosmologiques relativistes dans des géométries non-euclidiennes. Les simulations relativistes commencent à être utilisées en tant que nouvelles méthodes indépendantes pour quantifier la rétroaction. Cependant, jusqu'à maintenant elles ont toutes été réalisées dans une géométrie euclidienne et reposent toutes sur le formalisme BSSN pour résoudre l'équation d'Einstein. Je montre que ce schéma numérique pourrait ne pas être adapté aux géométries non-euclidiennes et je suggère d'utiliser sa version covariante.

Abstract

The Standard Model of Cosmology assumes that the small scale inhomogeneities do not affect the expansion of the largest scales. However, such a phenomenon, named *backreaction*, could exist and be significant enough to explain the dark energy. While most of the studies about the backreaction focused on its relation with structure formation, little has been done to understand its dependence on the topology of our Universe. My PhD thesis aims at filling this gap, following two strategies.

Firstly, I try to define a non-Euclidean extension of Newton's theory from general relativity and to generalise the Buchert-Ehlers theorem, which states that the backreaction is zero in Newton's theory. As a first step towards this definition I show that the Newtonian cosmology can be derived from the Newton-Cartan theory. In this case, the expansion arises as a fundamental field of the theory. I then propose two 'non-Euclidean Newtonian theories' based on the Newton-Cartan formalism. The first theory features a backreaction, while the other features none. Finally I try to justify one of them using the Galilean limit of general relativity. I show that to allow for non-Euclidean geometries in the limit, an additional term related to the spatial curvature needs to be added to the energy-momentum tensor of a geodesic fluid. One of the consequence of this modification is that the system of equations at leading order is not closed, leaving open the question of the 'right' non-Euclidean Newtonian theory compatible with general relativity.

Secondly, I study the possibility of making relativistic cosmological simulations in non-Euclidean geometries. Relativistic simulations are starting to be used as new independent methods to quantify the backreaction. However, until now they have all been done in an Euclidean geometry, and have all relied on the BSSN formalism to solve the Einstein equation. I show that this numerical scheme might not be appropriate for non-Euclidean geometries and I suggest to use its covariant version.

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Albert Einstein montant le Mont Ventoux pendant le Tour de France 1933, Archive du Californian Institut of Technology ([url](#))

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Notations

References

When available the bibliography style features three different links associated with three different colors: links to the journal/editor website or to a numerical version of the paper are in **red**, links to the ADS website are in **blue** and links to the arXiv website are in **green**.

Acronyms

CDM	Cold-Dark-Matter
CPAE	Collected Papers of Albert Einstein (url)
CMB	Cosmic Microwave Background
<i>cst</i>	constant
Λ CDM	Λ -Cold-Dark-Matter
lhs	left hand-side
LTB	Lemaître-Tolman-Bondi
NEN	Non-Euclidean Newtonian
rhs	right hand-side
SMC	Standard Model of Cosmology

Results

The important passages of the thesis are highlighted in grey boxes.

Mathematical conventions

We denote a tensor of any type, except scalars, in bold (example: \mathbf{g}). In the case where the type is of importance, a tensor of type (n, m) will feature n over-bars and m under-bars [example: $\overline{\underline{\mathbf{g}}}$ for a type $(0, 2)$ -tensor].

We define the symmetric part $T_{(ab)}$, the antisymmetric part $T_{[ab]}$ and the sym-

metric traceless part $T_{\langle ab \rangle}$ of a rank-2 tensor \mathbf{T} as

$$\begin{aligned} T_{(ab)} &:= \frac{1}{2}(T_{ab} + T_{ba}) \quad ; \quad T_{[ab]} := \frac{1}{2}(T_{ab} - T_{ba}) ; \\ T_{\langle ab \rangle} &:= T_{(ab)} - \frac{T}{D}g_{ab}, \end{aligned}$$

where \mathbf{g} is the metric of the manifold on which \mathbf{T} is defined and D the dimension of this manifold.

An antisymmetrisation over three indices is defined as

$$T_{[abc]} := \frac{1}{3} \left(T_{a[bc]} + T_{c[ab]} + T_{b[ca]} \right). \quad (1)$$

An index which should not be included in a antisymmetrisation (over two or three indices) or a symmetrisation is denoted between vertical bars. For instance, in the case of two indices antisymmetrisation:

$$T_{[a|b|c]} := \frac{1}{2} (T_{abc} - T_{cba}). \quad (2)$$

The Lie derivative on a manifold \mathcal{M} of a tensor \mathbf{T} along a vector field $\bar{\mathbf{A}}$ is denoted $\mathcal{L}_{\bar{\mathbf{A}}}\mathbf{T}$. The Lie derivative does not commute with the metric, so for instance, for a rank-1 tensor \mathbf{B} , $\mathbf{g}(\mathcal{L}_{\bar{\mathbf{A}}}\bar{\mathbf{B}}, \cdot) \neq \mathcal{L}_{\bar{\mathbf{A}}}\mathbf{B}$. We will then use $\mathcal{L}_{\bar{\mathbf{A}}}B^a$, respectively $\mathcal{L}_{\bar{\mathbf{A}}}B_a$, to denote the coordinate components of $\mathcal{L}_{\bar{\mathbf{A}}}\bar{\mathbf{B}}$, respectively $\mathcal{L}_{\bar{\mathbf{A}}}\underline{\mathbf{B}}$.

Then for a vector \mathbf{A} and a tensor \mathbf{T} on a manifold \mathcal{M} , we have

$$\begin{aligned} \mathcal{L}_{\bar{\mathbf{A}}}T^{a_1 \dots b_1 \dots} &:= A^c \nabla_c T^{a_1 \dots b_1 \dots} \\ &+ \sum_i T^{a_1 \dots \underset{\uparrow}{c} \dots} \nabla_{b_i} A^c - \sum_j T^{\dots \overset{\downarrow}{c} \dots}_{b_1 \dots} \nabla_c A^{a_j}, \end{aligned} \quad (3)$$

where ∇ is any connection on \mathcal{M} . The $\overset{\circ}{\uparrow}_i$ notation means that c is the i^{th} index.

Finally, we denote indices running from 0 to 3 by Greek letters ($\alpha, \beta, \gamma, \dots$) and indices running from 1 to 3 by Roman letters (a, b, c, \dots).

Variables

g	Lorentzian metric (-+++)
∇	Spacetime connection (Lorentzian or Galilean)
$\mathbf{Riem} (R^\sigma_{\alpha\beta\gamma})$	Spacetime Riemann curvature tensor (Lorentzian or Galilean)
R	Spacetime Ricci curvature tensor (Lorentzian or Galilean)
R	Spacetime scalar curvature (Lorentzian or Galilean)
τ	Time metric
h	Spatial metric
D	Spatial connection related to h
$\mathcal{Riem} (\mathcal{R}^\sigma_{\alpha\beta\gamma})$	Spatial Riemann curvature tensor
\mathcal{R}	Spatial Ricci tensor
\mathcal{R}	Spatial scalar curvature
K	Extrinsic curvature
Θ	Expansion tensor
Ω	Vorticity tensor

Foreword

IN 1915, when Einstein finalised his theory of general relativity (Einstein 1915a,b,d and summarised by Einstein, 1916), he published in the same time his paper on the “explanation of the perihelion motion of Mercury from general relativity” (Einstein, 1915c), thus providing a first evidence of the coherence of the theory on solar system scales. However, it was crucial for him to determine whether or not the theory leads to contradictions if applied on higher scales. For this purpose he developed the first relativistic cosmological model (Einstein, 1917): modern cosmology was born.

At that time, for him, the Universe was necessarily static, i.e. not in expansion, and due to his interpretation of Mach’s principle it was also closed. His model then represented a static, closed Universe with an uniform distribution of matter. For the solution to be non-null, he had to introduce an additional fundamental constant to the field equations of general relativity: the cosmological constant. While he was never convinced about this modification of the theory (see O’Raifeartaigh & Mitton, 2018, for a historical analysis of Einstein’s “biggest blunder”), it is only years later he accepted to remove the constant. This followed from the theoretical work of Friedmann (1922) and Lemaître (1927) and the observations of Hubble (1929) who proved that the spiral nebulae (shown to be recessing by Slipher, 1917; Wirtz, 1918) were in fact extragalactic objects moving away from the solar system, thus providing a clear evidence for the expansion of the Universe. Neither the cosmological constant, nor the closed nature of the Universe (needed for the static model of Einstein) was anymore needed in the cosmological models.

The only energy sources of these models were radiation, ordinary matter and *dark matter* which was later introduced to account for the observations of Zwicky (1933) and Rubin et al. (1980). This additional type of matter is only subject to the gravitational force, is decoupled from the other interactions and dominates in proportions the baryonic matter. Until the 80s, the model which prevailed was the Cold-Dark-Matter model (CDM model) describing an homogeneous expanding Universe with these sources of energy and without cosmological constant. The model was definitely adopted with the discovery of the Cosmic Microwave Background by McKellar (1941); Penzias & Wilson (1965) which confirmed that the Universe expanded from a very dense state.

With the growing amount of observations, it appeared clear in the late 80s and 90s that the measured matter density implied an age of the Universe, predicted under the hypothesis of the CDM model, which was too young compared to certain astronomical objects (e.g. [Efstathiou et al., 1990](#)). In 1998, the observation of distant supernovae with the newcomer Hubble telescope ([Perlmutter et al., 1999](#); [Riess et al., 1998](#)) showed that the local scale factor of our Universe was accelerating, which, under the hypothesis of a homogeneous model, corresponds to an acceleration of the expansion of the Universe: something impossible with the CDM model. Among the proposed solutions to solve this problem, were the introduction of a new energy source named *dark energy*, acting against gravitational collapse, and which appeared compatible with the reintroduction of a positive cosmological constant (e.g. [Efstathiou et al., 1990](#); [Riess et al., 1998](#)): this is the Λ -Cold-Dark-Matter model (Λ CDM model).

One remarkable thing is that the energy budget associated to this constant is of the same order as the one of the dark matter. Because the cosmological constant is unconstrained by general relativity and that its energy density is constant over time in the Λ CDM model, this requires fined-tuned initial conditions of the early Universe to fit today's expansion, which also seems to indicate that we live in a special epoch of cosmic evolution. This is the so-called *coincidence problem* formulated by [Steinhardt \(1997\)](#). To tackle this problem other models of dark energy have been proposed, eventually addressing in the same time the problem of dark matter, and removing the cosmological constant. This includes models with a fundamental *quintessence* scalar field whose energy density 'tracks' the one of dark matter (e.g. [Steinhardt et al., 1999](#)) so that they are comparable today; or alternative theories of gravity modifying the Einstein equation (e.g. [Popławski, 2006](#)).

Another approach is to consider a more general class of cosmological solutions of Einstein's equation, especially solutions removing the hypothesis of homogeneity generally kept in the previous models. General relativity is left untouched and no new source of 'dark' energy is added. This approach accounts for the large inhomogeneities in the distribution of matter in the Universe. It was originally proposed by [Einstein \(1931\)](#) to tackle a problem on the determination of the age of the Universe. But at that time, this was solved with a better measurement of the local expansion rate, while still keeping the homogeneous Lemaître's model. It is only in the 80s, with [Ellis \(1984\)](#), that a renewed interest arrived for inhomogeneous relativistic solutions for cosmology. This approach also has the advantage of solving another formulation of the coincidence problem (e.g. [Buchert, 2008](#); [Buchert & Räsänen, 2012](#); [Roukema et al., 2013](#)): the fact that the apparent acceleration of the expansion coincides with the formation of the large-scale structures of the Universe, known as the *cosmic web*.

In the standard model of cosmology, expansion and structure formation are decoupled and the latter is studied as a perturbation around a homogeneous expand-

ing background given by the Λ CDM model. This view is idealised and in reality a ‘backreaction’ of the structures on the expansion is *a priori* expected. The field of inhomogeneous cosmology tries to quantify this coupling and to understand whether or not it is sufficient to explain the apparent acceleration found by (Perlmutter et al., 1999; Riess et al., 1998): this is the *problem of backreaction in cosmology* (see e.g. Buchert, 2008). Inhomogeneous cosmology also aims at explaining current ‘tensions’ between the standard model and observations (see Buchert et al., 2016, for a review on this topic), such as the tension on the local measurement of the expansion rate (Riess, 2019; Heinesen & Buchert, 2020).

A popular approach to the backreaction problem is the Buchert formalism, based on an averaging over ‘space’ of scalar quantities. Under this formalism the average properties of the inhomogeneities become an effective energy source for the apparent acceleration of the expansion. The Buchert formalism has the advantage of being able to be used in the framework of general relativity (Buchert, 2000, 2001), or Newtonian gravitation (Buchert & Ehlers, 1997). But while in general relativity, there is *a priori* no constraint on the backreaction, in Newtonian cosmology, Buchert & Ehlers (1997) showed that it is exactly zero if expanding periodic boundary conditions are chosen (that can be interpreted as a closed Universe in terms of its topology): this is the *Buchert-Ehlers theorem*. Because such conditions are necessary for analytical or numerical models, a first consequence of this theorem is to imply that the study of backreaction needs to be done in general relativity (if this theory is considered to be the genuine theory of gravitation). But it also shows that the backreaction might be highly dependent on the topology of the Universe. While the question of the dependence of the backreaction on the kinematical properties of the matter fluid has been widely studied (e.g. Wiegand & Buchert, 2010; Buchert et al., 2013; Vigneron & Buchert, 2019), little has been made to study its dependence on topology (see Brunswic & Buchert, 2020, for a recent study).

During my PhD thesis under the direction of Thomas Buchert, I have been working within the field of inhomogeneous cosmology. My main focus has been the study of *the effect of the topology of the Universe on the backreaction*. The main strategy I have been following is to define a non-Euclidean extension of Newton’s theory from general relativity. This follows from Kaiser’s interpretation (Kaiser, 2017; Buchert, 2018) of the Buchert-Ehlers theorem: an implication of the theorem might be that a universe with a locally Newtonian dynamics should have negligible backreaction. As the local dynamics of our Universe appears to be compatible with Newtonian dynamics, this would imply that the backreaction of structures on the expansion cannot explain dark energy. However this statement can be true only if the Universe is locally *and* globally Newtonian, which corresponds to the hypothesis needed to apply the Buchert-Ehlers theorem. By *globally Newtonian*, we mean having a topology which is the one on which Newton’s theory is defined, i.e. an Euclidean topology. But while we can consider the fluid dynamics to be approximately Newtonian on domains small compared with the size of the Universe,

i.e. locally, we cannot consider the topology to be approximately Euclidean. As general relativity allows for non-Euclidean topologies, this global condition is then a strict condition that has to be respected by the relativistic description of the Universe, and not in a certain limit. From this, there are no reasons that we could not consider an universe which is locally Newtonian, but with a non-Euclidean topology. For such an universe, Newton's theory could not be used to study the global dynamics, i.e. the expansion; the Buchert-Ehlers theorem would not apply; and backreaction might not be negligible anymore. This is the main idea I want to develop in this thesis, and the first stage is the definition of a non-Euclidean Newtonian theory. This approach lead to two papers (Vigneron, 2020, 2021).

Studying the backreaction can also be made with relativistic cosmological simulations. Such simulations are starting to be used as new, independent cosmological tests (e.g. Giblin et al., 2016; Macpherson et al., 2019). However, as for now, they all rely on the assumption of an Euclidean topology: the 3-Torus. In the view of understanding the behaviour of the backreaction under different topologies, it is essential to be able to perform relativistic simulations on non-Euclidean topologies. This is the study of my second project.

Outline of the thesis

This manuscript is divided into 6 chapters. Chapter 1 details the theoretical background of the thesis, and notably the $1+3/3+1$ formalisms of general relativity along with the averaging procedure for the backreaction. Chapters 2 to 4 constitute the different stages in the development of a non-Euclidean Newtonian theory, aiming at generalising the Buchert-Ehlers theorem. Chapter 2 is a proof that Newtonian cosmology, on which the theorem is based, arise from the Newton-Cartan theory; in chapter 3 I review the only existing non-Euclidean Newtonian theory, and proposes two new ones based on the preceding chapter. Chapter 4 aims at developing a non-Euclidean Newtonian limit from Einstein's theory, using an existing Lorentzian to Galilean structure limit. As a complementary approach to the issues of topology and backreaction, chapter 5 discusses the recent relativistic cosmological simulations and proposes a numerical scheme allowing for future non-Euclidean relativistic numerical simulations. Finally, chapter 6 concludes this thesis and details the perspectives for future investigations.

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Chapter 1

Introduction

THE mathematical formalisms used for the Buchert approach to the backreaction problem are the 3+1 and 1+3 formalisms of general relativity. They will also be used for the construction of a non-Euclidean Newtonian theory. We will therefore firstly present them in section 1.1. Then the standard model of cosmology will be introduced in section 1.2 and the Buchert's approach in section 1.3. We will also detail the Newtonian approach to cosmology in section 1.4. This will be especially important to understand how and why we must generalise the Buchert-Ehlers theorem.

1.1 The 3+1 and 1+3 formalisms of general relativity

1.1.1 The ingredients of general relativity

1.1.1.1 The Einstein equation

One of the main principle that drove Einstein to the construction of general relativity from special relativity, is *the equivalence principle*.¹ It demands that test observers only subject to gravitation cannot perform any experiment that would allow them to determine whether or not they are subject to gravitation. This principle implies, under the framework of differential geometry used in special relativity, that any free-falling particle follows the geodesics of the spacetime (see [Weinberg, 1972](#), for a derivation of the geodesic motion from the equivalence principle). In this view the (flat) Minkowski spacetime of special relativity was not suited anymore as its geodesics are straight lines, not influenced by the gravitational content. Then the general theory of relativity that Einstein wanted required a curved spacetime the curvature of which depends on the gravitational field, and therefore on the mass content. With the knowledge of the relation between mass and energy obtained in special relativity, Einstein went even further by demanding that the curvature

¹See the [Apollo 15 experiment](#) for a beautiful illustration of this principle.

should depend on the whole energy content of the spacetime. The theory of general relativity he finally defined in 1915 describes the local coupling of the energy content of the Universe with its spacetime geometrical (curvature) properties.

A physical system studied under this theory is associated to a 4-dimensional (4D) differentiable manifold \mathcal{M} , which is equipped with a Lorentzian metric \mathbf{g} , i.e. a symmetric (0,2)-tensor of rank 4 and signature $(-, +, +, +)$, and a torsion-free compatible connection ∇ (with coefficients denoted $\Gamma_{\alpha\beta}^{\gamma}$), called the Levi-Civita connection. The geometrical properties of $(\mathcal{M}, \mathbf{g}, \nabla)$ are encoded in the Riemann curvature tensor **Riem** of the connection ∇ , defined as

$$R^{\mu}{}_{\alpha\beta\gamma}A^{\alpha} := (\nabla_{\beta}\nabla_{\gamma} - \nabla_{\gamma}\nabla_{\beta})A^{\mu}, \quad (1.1)$$

where $R^{\mu}{}_{\alpha\beta\gamma}$ are the coordinate components of **Riem** and \mathbf{A} is any vector on \mathcal{M} . From the Riemann tensor we also define the (symmetric) Ricci curvature tensor \mathbf{R} , with components $R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta}$ and the scalar curvature $R := g^{\mu\nu}R_{\mu\nu}$.

By analogy with the Maxwell energy-momentum tensor in classical electrodynamics (see section 1.1.1.2), the energy-momentum tensor in general relativity is a symmetric (2,0)-tensor, denoted \mathbf{T} . As a second fundamental principle of general relativity, the energy needs to be conserved, which implies that \mathbf{T} is divergence free:

$$\nabla_{\mu}T^{\mu\alpha} = 0. \quad (1.2)$$

Then, under the framework defined above, coupling the energy content with the geometrical properties of spacetime means giving an equation linking the tensors \mathbf{T} and **Riem**, while ensuring the property (1.2). The ‘simplest’ equation, i.e. involving only first order terms of the curvature tensors, is

$$\mathbf{R} - \frac{1}{2}R\mathbf{g} + \Lambda\mathbf{g} = 8\pi G\mathbf{T}, \quad (1.3)$$

where G is the gravitational constant, and Λ is a constant called *the cosmological constant*. We assumed the speed of light c to be $c := 1$. This equation was first proposed by Einstein (1915); Hilbert (1915), and was later shown by Lovelock (1972) to be the only possible equation relating a tensor featuring first order curvature terms and the energy-momentum tensor, while conserving the latter.

1.1.1.2 Energy sources in general relativity

When solving the Einstein equation, one has to specify the sources that compose the energy-momentum tensor. Most of the sources considered in cosmology are either matter fluids, radiative fluids or scalar field fluids.

1.1.1.2.a Matter fluids. Such fluids are composed of massive particles, which can eventually interact via non-gravitational forces. The 4-velocity of a fluid element is a time-like vector \mathbf{u} and the energy-momentum tensor takes the form

$$T_{\alpha\beta} = \epsilon u_\alpha u_\beta + p b_{\alpha\beta} + 2q_{(\alpha} u_{\beta)} + \pi_{\alpha\beta}, \quad (1.4)$$

where ϵ is the energy density, p the pressure, \mathbf{q} the heat flux and $\boldsymbol{\pi}$ the anisotropic stress of the fluid as measured in its rest frames. \mathbf{b} is the projector on the rest frames of the fluid, with $\mathbf{b} := \mathbf{g} + \mathbf{u} \otimes \mathbf{u}$. By definition, $q_\mu u^\mu = 0$ and $\pi_{\alpha\beta} = \pi_{(\alpha\beta)}$ with $\pi_{\alpha\mu} u^\mu = 0$ and $\pi_\mu{}^\mu = 0$.

The variables p , \mathbf{q} and $\boldsymbol{\pi}$ take into account eventual collisions and the non-gravitational interactions. However, due to velocity dispersion of the fluid particles inside a fluid element, a matter fluid composed of particles only subject to gravity will not necessarily have $p = 0$, $\mathbf{q} = 0$ and $\boldsymbol{\pi} = 0$. The fluid 4-velocity can even be non-geodesic, while the particles are. Therefore we distinguish two types of purely gravitational matter fluid:

- (i) the non-relativistic fluids, or *dust fluids*, where velocity dispersion inside a fluid element is negligible. Their energy-momentum tensor is $T_{\alpha\beta}^{(\text{dust})} = \epsilon u_\alpha u_\beta$, and the 4-velocity \mathbf{u} is geodesic.
- (ii) the ultra-relativistic fluids, where velocity dispersion inside a fluid element approximates the speed of light. Their energy-momentum tensor is $T_{\alpha\beta}^{(\text{ultra})} = \epsilon u_\alpha u_\beta + \frac{\epsilon}{3} b_{\alpha\beta}$, and the 4-velocity \mathbf{u} is not necessarily geodesic.

The process of averaging the 4-velocities of the particles inside a small space-time volume, or fluid element, may be ill-defined (see the relativistic kinetic theory in section 5.4 of [Ellis et al., 2012](#), for a proposed framework). Numerical schemes which mix a fluid description of matter with a particle description have been developed ([Daverio et al., 2019](#); [Barrera-Hinojosa & Li, 2020](#)), but they are not based on a well-defined mathematical framework. Furthermore, the velocity dispersion depends on the precise scale of the fluid element, or mesoscopic scale, which, in a cosmological context, is difficult to assess. That is why the relevance of the dust energy-momentum tensor for a Universe only filled with gravitationally interacting non-relativistic particles is still debated (see [Wiltshire, 2011](#); [Coley & Wiltshire, 2017](#), and references therein). A solution would be to solve the Einstein equation in vacuum, assuming the matter is described by a distribution of infinitely small and dense particles. This requires either black hole singularities for each particle [see the method of geometrostatics in [Lindquist & Wheeler \(1957\)](#) and its application to cosmology (e.g. [Clifton, 2015](#))], or surgery on the manifold \mathcal{M} to prevent these singularities. However such methods are hardly tractable for a cosmological model representative of our Universe. In this thesis, we will not be concerned with such issues and we will only take either $\mathbf{T}^{(\text{dust})}$ or $\mathbf{T}^{(\text{ultra})}$ when considering a purely gravitational matter fluid.

1.1.1.2.b Radiative fluid. An electro-magnetic fluid, or *radiative fluid*, is a fluid described by the electro-magnetic energy-momentum tensor:

$$T_{\alpha\beta}^{(\text{rad})} = \frac{1}{\mu_0} \left(F_{\alpha\mu} F^\mu_{\beta} - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right), \quad (1.5)$$

where \mathbf{F} is the Maxwell electro-magnetic tensor.

In general no preferred time-like vector can be attributed to such a radiative fluid. However it is still possible to decompose (1.5) with respect to a family of observers described by a time-like vector field \mathbf{o} as follows

$$T_{\alpha\beta}^{(\text{rad})} = {}^{(\text{rad})}\epsilon_{\alpha} o_{\beta} + 2S_{(\alpha} o_{\beta)} + \sigma_{\alpha\beta}, \quad (1.6)$$

where $S_{\mu} o^{\mu} := 0$, $o^{\mu} \sigma_{\mu\alpha} := 0$ and $\sigma^{\mu}_{\mu} = -\epsilon^{(\text{rad})}$. The variables $\epsilon^{(\text{rad})}$, \mathbf{S} and $\boldsymbol{\sigma}$ are respectively the energy density, the Poynting vector and the Maxwell stress tensor of the electro-magnetic fluid as observed by the family of observers \mathbf{o} .

1.1.1.2.c Scalar fields. In the standard model of cosmology, the description of inflation, dark energy, or even dark matter is often made with scalar fields. They are generally chosen to be minimally coupled with a potential $V(\Phi)$, the scalar field being denoted Φ , and therefore are described by an energy-momentum tensor in the form

$$T_{\alpha\beta}^{(\Phi)} = \nabla_{\alpha} \Phi \nabla_{\beta} \Phi - \left[\frac{1}{2} \nabla_{\mu} \Phi \nabla^{\mu} \Phi + V(\Phi) \right] g_{\alpha\beta}. \quad (1.7)$$

Its conservation leads to the Klein-Gordon equation

$$\nabla_{\mu} \nabla^{\mu} \Phi - V'(\Phi) = 0. \quad (1.8)$$

If $\nabla\Phi$ is time-like, it can be considered as the (non-normalised) 4-velocity of the scalar field.

1.1.2 The 3+1 formalism of general relativity

Under the theory of general relativity, a cosmological model is generally built using a split of the spacetime manifold \mathcal{M} into a family $\{\Sigma_t\}_{t \in \mathbb{R}}$ of spatial 3-dimensional (3D) hypersurfaces Σ_t which properties are parametrised by the time t . This is the 3+1 picture of general relativity. It is especially suited for a Newtonian interpretation of the Einstein equation, where spatial properties are evolved with time. But while this construction follows from the Newtonian idea of an absolute time and an absolute space, it is still compatible with Einstein's theory. The split is however not unique and other such families can be defined in the same manifold \mathcal{M} .

The 3+1 split can be performed on the Einstein equation itself, giving 3D-equations parametrised by time, called *the 3+1-Einstein equations*. In the following sections, we will present these equations, but also their construction as this will be adapted for the Newton-Cartan theory in chapter 2.

1.1.2.1 Foliation variables

The family $\{\Sigma_t\}_{t \in \mathbb{R}}$ of hypersurfaces in \mathcal{M} is called a *foliation* and can be uniquely defined by the level surfaces of a smooth scalar field \hat{t} in \mathcal{M} . In order to be able to interpret $\{\Sigma_t\}_{t \in \mathbb{R}}$ as a space-like hypersurface evolving with time, each Σ_t needs to be a Cauchy surface: a space-like hypersurface which is intersected by all the time-like or light-like curves in \mathcal{M} only once. Having a Cauchy foliation in \mathcal{M} is allowed only if \mathcal{M} is globally hyperbolic, which we will suppose from now.

The 3+1-Einstein equations are the projections of the Einstein equation onto and normal to the foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$. To be able to perform these projections one has to define a normal unit vector field to the family of hypersurfaces. The gradient $\nabla \hat{t}$ of the scalar field \hat{t} naturally defines a normal timelike vector field to the hypersurfaces. In general this vector is not a unit vector. We then define the timelike unit vector field to the hypersurfaces Σ_t as

$$\mathbf{n} := -N \nabla \hat{t}, \quad (1.9)$$

where $N := \left(-\nabla_\mu \hat{t} \nabla^\mu \hat{t}\right)^{-1/2}$ is called the *lapse*, is positive by convention, and is a property of the foliation. The global minus sign in the definition of \mathbf{n} is a convention imposing this vector to be future oriented with respect to the time scalar field \hat{t} . The 3+1-Einstein equations we will get do not depend on this convention.

The projection operator onto the hypersurfaces is the tensor

$$\mathbf{h} := \mathbf{g} + \mathbf{n} \otimes \mathbf{n}. \quad (1.10)$$

This tensor also corresponds to metric on Σ_t induced by \mathbf{g} . A *spatial tensor* is defined as having no normal part with respect to the hypersurfaces Σ_t . The spatial covariant derivative \mathbf{D} applied on a spatial tensor \mathbf{T} is defined as

$$D_\mu T^{\alpha_1 \dots \beta_1 \dots} := h^\sigma{}_\mu (h^{\alpha_1}{}_{\mu_1 \dots}) (h^{\nu_1}{}_{\beta_1 \dots}) \nabla_\sigma T^{\mu_1 \dots \nu_1 \dots}. \quad (1.11)$$

We define two more spatial rank-2 tensors: the intrinsic Ricci curvature \mathbf{R} of the hypersurfaces Σ_t , defined like \mathbf{R} but with respect to \mathbf{D} instead of ∇ ; and the extrinsic curvature \mathbf{K} of the hypersurfaces Σ_t embedded in \mathcal{M} . The extrinsic curvature makes the link between the geometrical properties of the hypersurfaces Σ_t and the ones of \mathcal{M} . We can write the components of \mathbf{K} as

$$K_{\alpha\beta} = -h^\mu{}_\alpha h^\nu{}_\beta \nabla_\nu n_\mu, \quad (1.12)$$

or

$$K_{\alpha\beta} = -\frac{1}{2N} \mathcal{L}_{N\mathbf{n}} h_{\alpha\beta}. \quad (1.13)$$

The negative sign is a convention. Because \mathbf{n} is proportional to a gradient, \mathbf{K} is a symmetric tensor. Then the gradient of the normal vector can be decomposed as

$$\nabla_\alpha n_\beta = -K_{\beta\alpha} - n_\alpha {}^n a_\beta, \quad (1.14)$$

where ${}^n \mathbf{a}$ is the 4-acceleration of the normal vector with ${}^n a_\alpha := n^\mu \nabla_\mu n_\alpha = D_\alpha \ln N$.

1.1.2.2 3+1 decomposition of the spacetime Ricci tensor

The Einstein equation (1.3) features the Ricci curvature tensor \mathbf{R} . We then give in the present section its decomposition onto and orthogonal to the foliation. \mathbf{R} being a symmetric tensor, we will have 10 projection equations. Details for the derivation of these equations can be found in Gourgoulhon (2012).

The two times projection onto Σ_t gives the 3+1-Ricci² equation

$$h^\mu{}_\alpha h^\nu{}_\beta R_{\mu\nu} = -\frac{1}{N} \mathcal{L}_{N\mathbf{n}} K_{\alpha\beta} - \frac{1}{N} D_\alpha D_\beta N + \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - 2K_{\alpha\mu} K^\mu{}_\beta, \quad (1.15)$$

where K is the trace of \mathbf{K} . Note that this equation features only spatial tensors as the Lie derivatives of a spatial tensor along $N\mathbf{n}$ (or along \mathbf{n}) is a spatial tensor. The spatial and orthogonal projection gives the 3+1-Codazzi equation

$$h^\mu{}_\alpha n^\nu R_{\mu\nu} = D_\alpha K - D_\mu K^\mu{}_\alpha. \quad (1.16)$$

The twice orthogonal projection gives the 3+1-Raychaudhuri equation

$$n^\mu n^\nu R_{\mu\nu} = \frac{1}{N} \mathcal{L}_{N\mathbf{n}} K - K_{\mu\nu} K^{\mu\nu} + \frac{1}{N} D_\mu D^\mu N. \quad (1.17)$$

Combining the trace of the 3+1-Ricci equation (1.15) with the 3+1-Raychaudhuri equation (1.17) we obtain the 3+1-Gauss equation

$$R + 2 R_{\mu\nu} n^\mu n^\nu = \mathcal{R} + K^2 - K_{\mu\nu} K^{\mu\nu}. \quad (1.18)$$

Note that 3+1-Gauss is redundant with the 3+1-Ricci and 3+1-Raychaudhuri equations together, it is however essential when solving the Cauchy problem in general relativity.

1.1.2.3 The 3+1-Einstein equations

We consider now that $(\mathcal{M}, \mathbf{g}, \nabla)$ is solution of the Einstein equation (1.3). The energy-momentum tensor \mathbf{T} can be decomposed with respect to the foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$, i.e. with respect to the observer \mathbf{n} , as

$$T_{\alpha\beta} = E n_\alpha n_\beta + P h_{\alpha\beta} + 2Q_{(\alpha} n_{\beta)} + \Pi_{\alpha\beta}. \quad (1.19)$$

The variables E , P , \mathbf{Q} , $\mathbf{\Pi}$ are respectively the energy density, the pressure, the heat flux, and the anisotropic stress of the energy content as measured by an observer whose 4-velocity is \mathbf{n} . We call such an observer an *Eulerian observer*. By definition $Q_\mu n^\mu = 0$ and $\Pi_{\alpha\beta} = \Pi_{(\alpha\beta)}$ with $\Pi_{\alpha\mu} n^\mu = 0$ and $\Pi_\mu{}^\mu = 0$.

²We name the equations with the suffix ‘3+1’ to distinguish them from their equivalent in the 1+3 formalism of general relativity presented in section 1.1.3.

The 3+1-Einstein system of equations is obtained from equations (1.15)–(1.18) when introducing the previous matter variables. Written as a Cauchy system it is composed of 6 evolution equations, obtained from the 3+1-Ricci equation,

$$\begin{aligned} \frac{1}{N}\mathcal{L}_{Nn}K_{\alpha\beta} &= 4\pi G[-(E-P)h_{\alpha\beta} - 2\Pi_{\alpha\beta}] - \Lambda h_{\alpha\beta} \\ &\quad - \frac{1}{N}D_{\alpha}D_{\beta}N + \mathcal{R}_{\alpha\beta} + K K_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}_{\beta}, \end{aligned} \quad (1.20)$$

and two constraint equations, the momentum constraint (1.21) (or 3+1-Codazzi equation) and the Hamilton constraint (1.22) (or 3+1-Gauss equation):

$$-8\pi GQ_{\alpha} = D_{\alpha}K - D_{\mu}K^{\mu}_{\alpha}, \quad (1.21)$$

$$16\pi GE + 2\Lambda = \mathcal{R} + K^2 - K_{\mu\nu}K^{\mu\nu}. \quad (1.22)$$

Like in the previous section, combining the 3+1-Ricci equation (1.20) and the Hamilton constraint (1.22), we get the 3+1-Raychaudhuri equation

$$\frac{1}{N}\mathcal{L}_{Nn}K = -4\pi G(E+3P) + \Lambda + K_{\mu\nu}K^{\mu\nu} - \frac{1}{N}D_{\mu}D^{\mu}N. \quad (1.23)$$

While this last equation is not part of the Cauchy problem of the 3+1-Einstein equations, we keep it as it will be useful later in this thesis.

1.1.2.4 The 3+1-conservation equations

Solving the set of equations (1.20)–(1.22) is sufficient to solve the Einstein equation. It is however of physical relevance to give two additional equations, that is the *3+1-energy conservation* and the *3+1-momentum conservation*, both coming from the projections of the conservation equation $\nabla_{\mu}T^{\mu\alpha} = 0$ with respect to $\{\Sigma_t\}_{t \in \mathbb{R}}$. The 3+1-energy conservation is

$$\frac{1}{N}\mathcal{L}_{Nn}E = K(E+P) - D_{\mu}Q^{\mu} - 2Q^{\mu}D_{\mu}\ln N + K_{\mu\nu}\Pi^{\mu\nu}, \quad (1.24)$$

and the 3+1-momentum conservation is

$$\begin{aligned} \frac{1}{N}\mathcal{L}_{Nn}Q^{\alpha} &= -(E+P)D^{\alpha}\ln N - D^{\alpha}P + KQ^{\alpha} + 2Q^{\mu}K_{\mu}^{\alpha} \\ &\quad - D_{\mu}\Pi^{\mu\alpha} - \Pi^{\mu\alpha}D_{\mu}\ln N. \end{aligned} \quad (1.25)$$

1.1.2.5 Matter fluid in 3+1

We assume the energy content is a matter fluid as described by the energy-momentum tensor (1.4). For a general foliation, $\mathbf{n} \neq \mathbf{u}$ and the fluid variables measured in its rest frames ($\epsilon, p, \mathbf{q}, \boldsymbol{\pi}$) are different from the ones measured by the Eulerian observer ($E, P, \mathbf{Q}, \boldsymbol{\Pi}$). The nature of the matter fluid is however given by the variables measured in its rest frames. Therefore it is often useful to express the variables measured

by \mathbf{n} [defined in (1.19)] as function of the ones measured by \mathbf{u} [defined in (1.4)]. For this we introduce the *tilt velocity* \mathbf{w} of the fluid 4-velocity with respect to the foliation as

$$\mathbf{w} := \frac{1}{\gamma} \mathbf{u} - \mathbf{n}, \quad (1.26)$$

where $\gamma := (1 - w_\mu w^\mu)^{-1/2}$ is the Lorentz factor. \mathbf{w} is spatial by definition. We then have the following relations:

$$E = \gamma^2 \epsilon + (\gamma^2 - 1) p + 2\gamma w^\mu q_\mu + w^\mu w^\nu \pi_{\mu\nu}, \quad (1.27)$$

$$P = (\gamma^2 - 1) \epsilon + (\gamma^2 + 2) p + 2\gamma w^\mu q_\mu + w^\mu w^\nu \pi_{\mu\nu}, \quad (1.28)$$

$$Q_\alpha = \gamma^2 (\epsilon + p) w_\alpha + \gamma w^\mu q_\mu w_\alpha + \gamma h^\mu{}_\alpha q_\mu - h^\mu{}_\alpha w^\nu \pi_{\mu\nu}, \quad (1.29)$$

$$\Pi_{\alpha\beta} = \gamma^2 \epsilon w_\alpha w_\beta + p (h_{\alpha\beta} + \gamma^2 w_\alpha w_\beta) + 2\gamma w_{(\alpha} h_{\beta)}^\mu q_\mu + h_{\alpha}{}^\mu h_{\beta}{}^\nu \pi_{\mu\nu} - P h_{\alpha\beta}, \quad (1.30)$$

which, in the case of a dust fluid, become

$$E^{(\text{dust})} = \gamma^2 \epsilon, \quad (1.31)$$

$$P^{(\text{dust})} = \frac{\gamma^2 - 1}{3} \epsilon, \quad (1.32)$$

$$Q_\mu^{(\text{dust})} = \gamma^2 \epsilon w_\mu, \quad (1.33)$$

$$\Pi_{\mu\nu}^{(\text{dust})} = \epsilon \left(\gamma^2 w_\mu w_\nu - \frac{\gamma^2 - 1}{3} h_{\mu\nu} \right). \quad (1.34)$$

1.1.2.6 Foliation adapted coordinates

While the 3+1-Einstein equations (1.20)–(1.23) are spatial equations, they are still defined on the 4-manifold \mathcal{M} . Using specific coordinates, it is possible to write them as equations defined on a 3-manifold Σ , thus completing the 3+1-picture. We detail this final construction in the present section.

1.1.2.6.a Shift vector and classes of adapted coordinates. Until now we wrote the 3+1-Einstein equations for any coordinate system. We however often want to introduce one, and especially one which is adapted to the foliation. In such a coordinate system, the coordinate vector basis $\{\partial_\alpha\}_{\alpha=0,1,2,3}$ features three spatial vectors: ∂_1 , ∂_2 and ∂_3 . The 0-coordinate is chosen to correspond to the scalar field \hat{t} . We then write $\partial_t := \partial_0$ and call it the *time vector*. By definition, ∂_t is not spatial.³

In general, $\partial_t \neq \mathbf{n}$, and we have $\partial_t^\mu n_\mu = -N$. We then define the *shift vector* β as

$$\beta := \partial_t - N \mathbf{n}. \quad (1.35)$$

³This does not imply that ∂_t is timelike (see section 5.2 in Gourgoulhon, 2012).

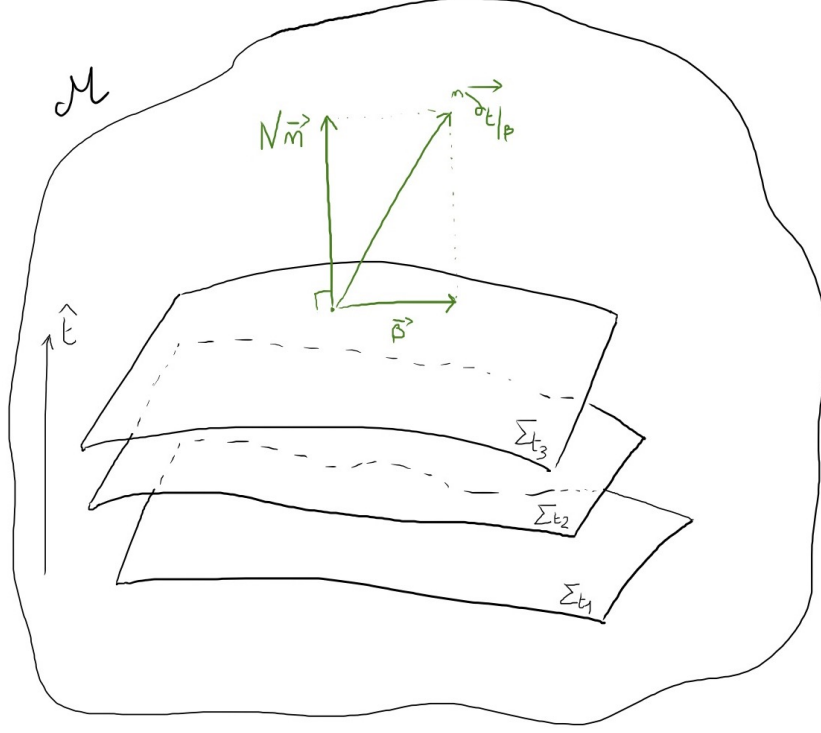


Figure 1.1: Scheme of the foliation related to \mathbf{n} in \mathcal{M} . The different slices are curved in \mathcal{M} due to the extrinsic curvature \mathbf{K} . The time vector ${}^n\partial_{t|\beta}$ and shift vector β of an adapted coordinate system are also represented.

By definition β is spatial. Reversing the definition (1.35): to a spatial vector β corresponds a time vector

$${}^n\partial_{t|\beta} := N\mathbf{n} + \beta. \quad (1.36)$$

But instead of defining a single adapted coordinate system in \mathcal{M} , the shift vector defines a class of coordinate systems adapted to the foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ in \mathcal{M} . We write this class \mathcal{X}_β^n . By definition, any coordinate system in \mathcal{X}_β^n can be obtained from a coordinate system having β as shift vector with a time independent spatial change of coordinates, i.e. $(t \rightarrow t, x^a \rightarrow y^b(x^a))_{a,b=1,2,3}$. This relation between the shift vector and the choice of coordinates is especially important for the comparison between Newton's theory and general relativity which will be done in chapter 2.

The class with a zero shift, denoted \mathcal{X}_0^n and its time vector ${}^n\partial_{t|0}$, is said to be comoving with respect to the Eulerian observer. We call these coordinates *Eulerian comoving coordinates*. Then any shift β corresponds to the coordinate velocity vector of the class \mathcal{X}_β^n with respect to these coordinates.

1.1.2.6.b Pull-back. Once we chose an adapted coordinate system $\{t, x^a\}_{a=1,2,3}$, characterised by its shift β , it is possible to write the 3+1-Einstein equations (1.20)–

(1.23) with indices running from 1 to 3. This comes from the fact that the 4D-components $T^{\alpha_1 \dots \beta_1 \dots}$ of any spatial tensor \mathbf{T} are totally determined by the spatial components $T^{a_1 \dots b_1 \dots}$ and by the spatial components β^a of the shift. Note that the shift is only needed for covariant components.

For instance, in a class \mathcal{X}_β^n , the contravariant components of a vector \mathbf{A} are $A^\alpha = (0, V^a)$, and its covariant components are $V_\alpha = (\beta^c V_c, V_a)$. The spatial covariant components can be obtained from the contravariant ones by lowering with the spatial components h_{ab} of the spatial metric, i.e. $V_a = V^c h_{ac}$. The same can be done with an order 2 tensor \mathbf{T} :

$$T^{\alpha\beta} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & T^{ab} \end{array} \right) ; \quad T_{\alpha\beta} = \left(\begin{array}{c|c} \beta^c \beta^d T_{cd} & \beta^c T_{ca} \\ \hline T_{ac} \beta^c & T_{ab} \end{array} \right). \quad (1.37)$$

The operation $T_{\alpha\beta} \rightarrow T_{ab}$ is called a *pull-back*. It links spatial tensors on \mathcal{M} to tensors on a single 3-manifold Σ . The components of the pulled-back tensor \mathbf{T} on Σ are T_{ab} . As for each hypersurface Σ_t corresponds a pull-back to Σ , the global pull-back from \mathcal{M} to Σ is said to be parametrised by time. This implies that the properties of the 3-manifold Σ and the tensors defined on it are parametrised by time. This situation is similar to the classical formulation of Newton's theory (presented in section 1.4).

Applying the pull-back operation on the 3+1-Einstein equations (1.20)–(1.23) allows us to have equations living on the 3-manifold Σ and parametrised by time.

1.1.2.6.c 3+1-Einstein equations on Σ . To write the 3+1-equations on Σ we need to make to pull-back operation on each term of these equations. It is only not trivial for the Lie derivative $\mathcal{L}_{N\mathbf{n}}$ present in the 3+1-Ricci (1.20) and 3+1-Raychaudhuri (1.23) equations, as it still explicitly features a non-spatial tensor, i.e. \mathbf{n} . To remove this dependence, we use the definition of the shift (1.35) and the fact that $\mathcal{L}_{n\partial_{t|\beta}} T^{\alpha_1 \dots \beta_1 \dots} = {}^n\partial_{t|\beta} T^{\alpha_1 \dots \beta_1 \dots}$. Then for a spatial (n, m) -tensor \mathbf{T} , the spatial components of $\mathcal{L}_{N\mathbf{n}}\mathbf{T}$ are

$$(\mathcal{L}_{N\mathbf{n}}\mathbf{T})^{a_1 \dots b_1 \dots} = {}^n\partial_{t|\beta} T^{a_1 \dots b_1 \dots} - \mathcal{L}_\beta T^{a_1 \dots b_1 \dots}. \quad (1.38)$$

Remark. $\mathcal{L}_\beta T^{a_1 \dots b_1 \dots}$ corresponds to the spatial components of the spatial projection of $\mathcal{L}_\beta T^{\alpha_1 \dots \beta_1 \dots}$. The latter is however not necessarily spatial (see appendix A.1).

Equations (1.37) and (1.38) allow us to write the 3+1-Einstein equations as equations living on the 3-manifold Σ parametrised by the time t . Then the 3+1-Einstein equations become:

(i) *the 3+1-Raychaudhuri evolution equation:*

$$\frac{1}{N} \left({}^n\partial_{t|\beta} - {}^\Sigma\mathcal{L}_\beta \right) K = -4\pi G(E + 3P) + \Lambda + K_{cd}K^{cd} - \frac{1}{N} D_c D^c N, \quad (1.39)$$

(ii) *the 3+1-Ricci evolution equation:*

$$\begin{aligned} \frac{1}{N} \left({}^n\partial_{t|\beta} - {}^\Sigma\mathcal{L}_\beta \right) K_{ab} &= 4\pi G [-(E - P) h_{ab} - 2\Pi_{ab}] - \Lambda h_{ab} - \frac{1}{N} D_a D_b N \\ &\quad + \mathcal{R}_{ab} + K K_{ab} - 2K_{ac} K^c_b, \end{aligned} \quad (1.40)$$

(iii) *the 3+1-Codazzi equation* (or 3+1-momentum constraint):

$$-8\pi G Q_a = D_a K - D_c K^c_a. \quad (1.41)$$

Equations (1.39) and (1.40) give the 3+1-Gauss equation (or 3+1-Hamilton constraint)

$$16\pi G E + 2\Lambda = \mathcal{R} + K^2 - K_{cd} K^{cd}. \quad (1.42)$$

This system needs to be completed by the relation between the extrinsic curvature and the spatial metric:

$$K_{ab} = -\frac{1}{2N} \left({}^n\partial_{t|\beta} - {}^\Sigma\mathcal{L}_\beta \right) h_{ab}. \quad (1.43)$$

This concludes the construction of the 3+1-Einstein equations on a time-parametrised 3-manifold Σ from the Einstein equation.

1.1.2.7 Discussion

The Choquet-Bruhat & Geroch theorem (Choquet-Bruhat & Geroch, 1969) states that from an initial condition, given as a Cauchy hypersurface, and solution of the two constraint equations (1.41) and (1.42), the solution of the 3+1-Einstein equations exists and is unique. This shows that the 3+1-formalism is a powerful tool to solve the Einstein equation.

It allows for the description of a Lorentzian spacetime solution of the Einstein equation into spatial properties evolving with time via the introduction of a spatial foliation. But due to our incapacity of observing a spatial hypersurface, because of the speed limit of information, the properties of such a foliation will never completely correspond to our observations. This does not mean that the 3+1-formalism prevents the prediction of observations as one can still reconstruct the light-cone of any event from a solution of the 3+1-Einstein equations.

As the formalism allows for an interpretation of the Einstein equation, and its solutions, close to the Newtonian picture of spacetime, it is still interesting to look at the spatial properties given by the foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$, even if these are not totally representative of observations. The main problem arising when one follows this line of interpretation is related to the arbitrariness of the choice of foliation: one

could choose another Cauchy family in \mathcal{M} , and have different spatial properties from the same solution of the Einstein equation. The physical interpretation of the solution is therefore dependent on the choice of foliation. From this point, one could seek for a preferred foliation which would be the most representative of physical observations (see Buchert et al., 2018, 2020, for a detailed study of the foliation choice and especially its consequence for the backreaction in cosmology). A natural choice would be to take the foliation related to the fluid 4-velocity itself: $\mathbf{n} = \mathbf{u}$, as it would directly probe the physics of the fluid. This is however not possible in general as a vector defines a foliation if and only if its vorticity (definition given below in section 1.1.3.1) is zero: this is the Frobenius theorem.

Despite these issues, it is still possible to perform a split of the Einstein equations with respect to the fluid 4-velocity. This split, called *the 1+3-split of general relativity*, is however different from the 3+1-split. It allows to obtain similar equations, called the 1+3-Einstein equations, which cannot be ‘spatialised’ with a pull-back as in section 1.1.2.6. These equations does not suffer from an arbitrary choice of foliation and are only functions of the fluid physical properties. In this sense they are particularly suited for the study of a matter fluid in a cosmological context. We present the 1+3-formalism in the next section.

1.1.3 The 1+3 formalism of general relativity

In this section we present the 1+3-formalism of general relativity. However, we will not give its full derivation as in the previous section for 3+1. A detailed description of this formalism can be found in (Ellis, 2009; Roy, 2014).

The 1+3-Einstein equations are obtained similarly as the 3+1-Einstein equations by projecting the Einstein equation along and orthogonal to \mathbf{u} . But instead of the extrinsic curvature, the 1+3-Einstein equations involve the expansion and the vorticity tensors of the fluid, known as the kinematical variables, and presented below.

For this section only, we reintroduce the speed of light c , as it will be useful for chapter 4.

1.1.3.1 Kinematical variables

The expansion tensor Θ and the vorticity tensor Ω of the fluid correspond respectively to the symmetric and antisymmetric parts of the 4-velocity gradient $\nabla\mathbf{u}$ projected on the rest frames of the fluid:

$$\Theta_{\alpha\beta} := b^\mu{}_{(\alpha} b^\nu{}_{\beta)} \nabla_\mu u_\nu, \quad (1.44)$$

$$\Omega_{\alpha\beta} := b^\mu{}_{[\alpha} b^\nu{}_{\beta]} \nabla_\mu u_\nu. \quad (1.45)$$

The 4-velocity gradient can be decomposed as

$$\nabla_{\alpha} u_{\beta} = \Theta_{\alpha\beta} + \Omega_{\alpha\beta} - \frac{u_{\alpha} u_{\beta}}{c^2}. \quad (1.46)$$

where ${}^u a^{\alpha} := u^{\mu} \nabla_{\mu} u^{\alpha}$ is the 4-acceleration of the fluid. We also define the trace $\theta := \Theta_{\mu}{}^{\mu}$ and the *shear tensor* σ : $\sigma_{\alpha\beta} := \Theta_{\alpha\beta} - \frac{\theta}{3} b_{\alpha\beta}$.

In the case of a flow orthogonal foliation, i.e. $\mathbf{n} = \mathbf{u}$, we have the following relation between the extrinsic curvature and the expansion tensor, $\mathbf{K} = -\Theta$, and the fluid is necessarily irrotational, with $\Omega = 0$.

1.1.3.1.a Interpretation of Θ and Ω . To understand what is the physical interpretation of these tensors, let us assume that the world lines related by \mathbf{u} correspond to galaxies. We consider one such galaxy denoted O , and its neighbour galaxies which form a small volume δV around O on the rest frames of \mathbf{u} . Then θ corresponds to the rate of change (as function of the proper time of O) of the volume δV ; the shear σ characterises the distortion (without change of volume or rotation) of δV ; the vorticity Ω represents a pure rotation of the volume δV , without distortion or expansion. This is schematised in figure 1.2. But while the shear preserves the volume, it does not preserve the distances, and one direction might be expanding, while another might be contracting. This is not the case for the vorticity tensor which both preserve distances and volume. That is why the whole tensor Θ , and not just its trace, is called the expansion tensor. See sections 4.5 and 4.6 of [Ellis et al. \(2012\)](#) for more details on the interpretation of the kinematical quantities.

1.1.3.2 Rest frame spatial derivative and rest frame curvature

A tensor \mathbf{T} is said to be a *rest frame tensor* if its projection along \mathbf{u} is null. We note the rest frame covariant derivative ${}^u \mathbf{D}$ applied on a rest frame vector \mathbf{T} as

$${}^u D_{\mu} T^{\alpha_1 \dots \beta_1 \dots} := b^{\sigma}_{\mu} (b^{\alpha_1}_{\mu_1 \dots}) (b^{\beta_1}_{\nu_1 \dots}) \nabla_{\sigma} T^{\mu_1 \dots \nu_1 \dots}. \quad (1.47)$$

While \mathbf{D} in 3+1 is related to a metric on a spatial hypersurface, this is not the case for ${}^u \mathbf{D}$, as the rest frames do not correspond in general to spatial hypersurfaces. The main consequence is that ${}^u \mathbf{D}$ features torsion:

$${}^u D_{[\alpha} {}^u D_{\beta]} \phi = \frac{1}{c^2} \Omega_{\alpha\beta} u^{\mu} \nabla_{\mu} \phi \neq 0, \quad (1.48)$$

for a scalar field ϕ . The torsion tensor ${}^u \mathbf{T}$ is given by

$${}^u T^{\gamma}_{\alpha\beta} = -\frac{2}{c^2} \Omega_{\alpha\beta} u^{\gamma}. \quad (1.49)$$

The definition of the Riemann tensor ${}^u \mathbf{Riem}$ associated to ${}^u \mathbf{D}$, called *the rest frames Riemann tensor*, with components denoted ${}^u R^{\gamma}_{\mu\alpha\beta}$, is

$${}^u \mathcal{R}^{\gamma}_{\mu\alpha\beta} v^{\mu} := ({}^u D_{\alpha} {}^u D_{\beta} - {}^u D_{\beta} {}^u D_{\alpha}) v^{\gamma} + b^{\gamma}_{\nu} {}^u T^{\mu}_{\alpha\beta} \nabla_{\mu} v^{\nu}, \quad (1.50)$$

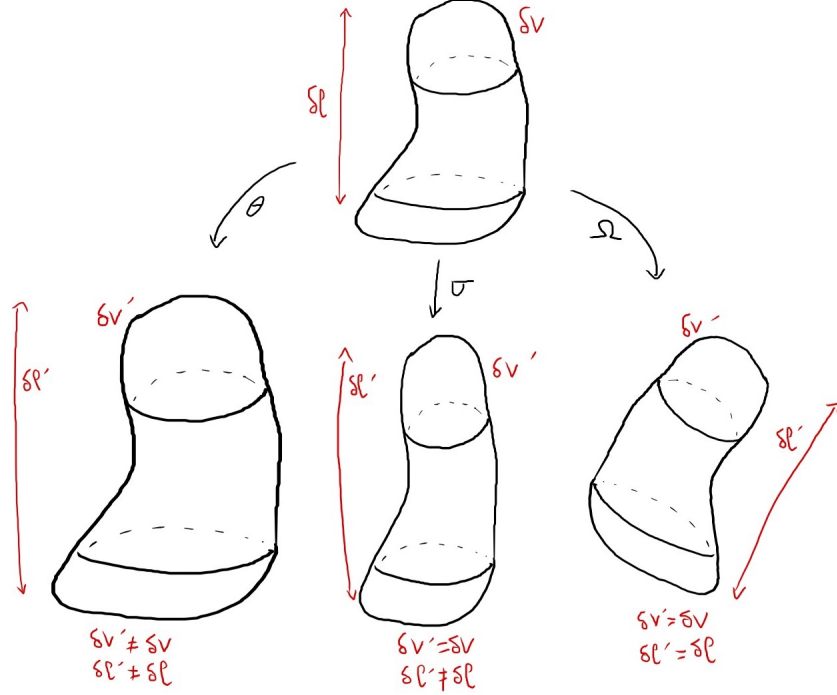


Figure 1.2: Scheme of the deformation underwent by the small volume δV due to θ , σ and Ω .

and the rest frames Ricci curvature tensor ${}^u\mathcal{R}$ is

$${}^u\mathcal{R}_{\alpha\beta} := {}^u\mathcal{R}^{\mu}_{\alpha\mu\beta}. \quad (1.51)$$

The rest frames scalar curvature is denoted ${}^u\mathcal{R} := {}^uR_{\mu\nu}g^{\mu\nu}$. Because of the torsion, the first and second Bianchi identities are only recovered if the fluid is irrotational. Furthermore, the rest frames Ricci tensor is not symmetric anymore. These additional properties can be found in Roy (2014).

Remark. In chapter 4, we will see that in the Galilean limit the rest frames are hypersurfaces; the torsion tensor vanishes; and the rest frames Riemann curvature tensor retrieves all the symmetries of an usual Riemann tensor and can be related to a spatial metric.

1.1.3.3 The 1+3-Einstein equations

Projecting the Ricci tensor \mathbf{R} along and orthogonal to \mathbf{u} allows us to obtain the 1+3-Einstein equations:

The 1+3-Einstein equations:

(i) *The 1+3-Raychaudhuri equation:*

$$\mathcal{L}_u\theta = -4\pi G \left(\epsilon + \frac{3p}{c^2} \right) - \Theta_{\mu\nu}\Theta^{\mu\nu} + \Omega_{\mu\nu}\Omega^{\mu\nu} + {}^u D_\mu u^\mu + \frac{{}^u a_\mu u^\mu}{c^2}, \quad (1.52)$$

(ii) *The 1+3-Ricci equation:*

$$\begin{aligned} \mathcal{L}_u \Theta_{\alpha\beta} = 4\pi G \left[\left(\epsilon - \frac{p}{c^2} \right) + 2 \frac{\pi_{\alpha\beta}}{c^2} \right] + \Lambda b_{\alpha\beta} + {}^u D_{(\alpha} u_{\beta)} + \frac{{}^u a_{(\alpha} u_{\beta)}}{c^2} \\ - c^2 {}^u \mathcal{R}_{(\alpha\beta)} - \theta \Theta_{\alpha\beta} - 2\Theta^\mu{}_{(\alpha} \Theta_{\beta)\mu} - 2\Theta^\mu{}_{(\alpha} \Omega_{\beta)\mu}, \end{aligned} \quad (1.53)$$

(iii) *The 1+3-Codazzi equation (or 1+3-momentum constraint):*

$${}^u D^\alpha \theta - {}^u D_\mu (\Theta^{\alpha\mu} + \Omega^{\alpha\mu}) + 2 \frac{{}^u a_\mu \Omega^{\mu\alpha}}{c^2} = \frac{8\pi G}{c^2} q^\alpha. \quad (1.54)$$

We can also project the first Bianchi identity for **Riem**, giving an evolution equation and a constraint equation for the vorticity:

(iv) *the 1+3-vorticity evolution equation:*

$$\mathcal{L}_u \Omega_{\alpha\beta} = {}^u D_{[\alpha} u_{\beta]}, \quad (1.55)$$

(v) *the 1+3-vorticity constraint equation:*

$${}^u D_{[\gamma} \Omega_{\alpha\beta]} = \frac{{}^u a_{[\gamma} \Omega_{\alpha\beta]}}{c^2}. \quad (1.56)$$

These last equations are geometrical constraints and does not require the Einstein equation to be valid. Finally the conservation equation $\nabla_\mu T^\mu{}_\alpha = 0$ for the fluid energy-momentum tensor gives:

(vi) *1+3-energy conservation:*

$$\mathcal{L}_u \epsilon = -\theta \left(\epsilon + \frac{p}{c^2} \right) - \nabla_\mu q^\mu - \frac{{}^u a_\mu q^\mu}{c^2} - \frac{\Theta_{\mu\nu} \pi^{\mu\nu}}{c^2}, \quad (1.57)$$

(vii) *the 1+3-momentum conservation:*

$$b^\alpha{}_\mu \mathcal{L}_u q^\mu = - \left(\epsilon + \frac{p}{c^2} \right) u^\alpha - {}^u D^\alpha (p) - \theta q^\alpha - 2q^\mu (\Theta_\mu{}^\alpha + \Omega_\mu{}^\alpha) - b^\alpha{}_\mu \nabla_\nu \pi^{\nu\mu}. \quad (1.58)$$

This system can be supplemented by *the 1+3-Gauss equation* (or 1+3-Hamilton constraint), which is redundant with equations (1.52) and (1.53):

$$c^2 {}^u \mathcal{R} + \theta^2 - \Theta_{\mu\nu} \Theta^{\mu\nu} + \Omega_{\mu\nu} \Omega^{\mu\nu} = 16\pi G \epsilon + 2\Lambda. \quad (1.59)$$

1.2 The standard model of cosmology

1.2.1 The cosmological principle

The standard model of cosmology (SMC) is based on the *cosmological principle* which states that there exists a family of free-falling observers for which the Universe looks the same in all directions. The main observation from which this principle results is the Cosmic Microwave Background radiation (CMB). This is an isotropic constant flux of radiation (presented in more details in section 1.2.3) the properties of which, especially its spectrum, are isotropic. Assuming that we do not occupy a privileged place in the Universe (this is the Copernican principle), we can therefore suppose that we are part of a family of observers seeing the same isotropic radiation. Consequently, any observation will be equivalent for all the members of this family, implying that the Universe is homogeneous for these observers.

The apparent homogeneity is also an observed feature of our Universe. Using catalogues of galaxies, it appears that from a scale between 70 Mpc/h and 150 Mpc/h⁴ (e.g. Gonçalves et al., 2018; Avila et al., 2018, and references therein) the distribution of matter is statistically homogeneous. But this result may vary as function of the set of data (galaxy catalogues, quasar catalogues, ...), and the method used to derive it (see Heinesen, 2020, and references therein). Therefore, the value of the homogeneity scale and even its existence are still under debate.

1.2.2 Homogeneous and isotropic solutions

1.2.2.1 The Friedmann equations

The cosmological principle is equivalent as saying that there exists a time-like vector field \mathbf{n} in the Universe manifold \mathcal{M} , such that it is geodesic and vorticity-free. Furthermore, the hypersurfaces of the space-like foliation orthogonal to \mathbf{n} , along with the energy-momentum variables defined on this foliation [see equation (1.19)] have isotropic properties (see section 6.3 in Chruściel, 2019, for a mathematical approach to the cosmological principle). We derive in this section solutions to the Einstein equation that satisfy this cosmological principle.

The presence of the time-like vector \mathbf{n} in \mathcal{M} as described above implies the following relations for the 3+1-variables related to it:

$$Q = 0 \quad ; \quad \Pi = 0, \quad (1.60)$$

$$\mathcal{R} = \frac{\mathcal{R}}{3}\mathbf{h} \quad ; \quad \mathbf{K} = \frac{K}{3}\mathbf{h} \quad ; \quad N = 1, \quad (1.61)$$

and the remaining scalar quantities K , \mathcal{R} , E and P all depend only on the time \hat{t} . This time is also the proper time of \mathbf{n} and is called *the cosmic time*.

⁴The parameter $h \sim 70$ is the dimensionless factor $(H_0/100)$ km/s/Mpc, where H_0 is the local measured expansion rate (presented in section 1.2.2).

Following the interpretation made in section 1.1.3.1, the tensor $-\mathbf{K}$ corresponds to the expansion tensor related to the observer \mathbf{n} . Because we have $\mathbf{K} = \frac{K}{3}\mathbf{h}$, the expansion between two world lines of \mathbf{n} is the same everywhere on the hypersurfaces Σ_t and does not depend on the direction: expansion is homogeneous and isotropic. Then we introduce the scale factor $a(t)$, such that $K = -3\frac{\dot{a}}{a}$. It corresponds to the spatial distance between two world lines of \mathbf{n} as function of time, and from K is defined up to a constant factor.

Using the relation (1.43) in the class of coordinates \mathcal{X}_0^n , i.e. comoving with \mathbf{n} , we can show that the space-time line element takes the form

$$ds^2 = -dt^2 + a^2(t)\tilde{h}_{ab}(x^i)dx^a dx^b, \quad (1.62)$$

where the Ricci curvature tensor $\tilde{\mathcal{R}}$ related to the metric $\tilde{\mathbf{h}}$ is $\tilde{\mathcal{R}} = \frac{\tilde{\mathcal{R}}}{3}\tilde{\mathbf{h}}$, with the scalar curvature $\tilde{\mathcal{R}}$ being a space-time constant. The line element (1.62) is called the Friedmann-Lemaître-Roberston-Walker (FLRW) line element, or *Friedmann-Lemaître-Roberston-Walker metric* for the related metric components. The scalar curvature $\mathcal{R}(t)$ can therefore be written as:

$$\mathcal{R}(t) = \frac{\mathcal{R}_i}{a^2(t)}, \quad (1.63)$$

where \mathcal{R}_i is the value of $\mathcal{R}(t)$ at an initial time t_i , and where we choose $a(t_i) = 1$.

Finally, the 3+1-Raychaudhuri equation (1.39) and the Hamilton constraint (1.42) become respectively:

$$3\frac{\ddot{a}}{a} = -4\pi G(E + 3P) + \Lambda, \quad (1.64)$$

$$3\left(\frac{\dot{a}}{a}\right) = 8\pi GE - \frac{\mathcal{R}_i}{2a^2} + \Lambda. \quad (1.65)$$

These equations are called the *Friedmann equations*. We can complement them by the conservation of the energy-momentum tensor with

$$\dot{E} + 3\frac{\dot{a}}{a}(E + P) = 0. \quad (1.66)$$

The Friedmann equations are evolution equations for the scale factor $a(t)$. They relate the expansion, which can be totally determined by the expansion rate $H := \frac{\dot{a}}{a}$, as function of the energy content of the Universe (with E and P) and the geometrical content (with \mathcal{R}).

1.2.2.2 Sources in the standard model

The only unknowns remaining in the Friedmann equations are the energy density E and pressure P , especially their dependance on time. To determine them, we

need to specify the sources in the energy-momentum tensor. In the SMC, several sources are considered, which do not interact. The total energy-momentum tensor can therefore be decomposed as

$$T_{\alpha\beta} = \sum_{n \in \{\text{sources}\}} T_{\alpha\beta}^{(n)}, \quad \text{with} \quad \nabla_{\mu} T^{(n),\mu\alpha} = 0 \quad \forall n \in \{\text{sources}\}. \quad (1.67)$$

This implies that each source comes with its energy density $E^{(n)}$ and pressure $P^{(n)}$, which are conserved with $\dot{E}^{(n)} + 3\frac{\dot{a}}{a}(E^{(n)} + P^{(n)}) = 0$. Most of the sources considered in the SMC can be described by a linear equation of state, $P^{(n)} = \omega^{(n)}E^{(n)}$, with $\omega^{(n)} = cst$. This implies that

$$E^{(n)} = E_{\mathbf{i}}^{(n)} a^{-3(1+\omega^{(n)})}, \quad (1.68)$$

where $E_{\mathbf{i}}^{(n)}$ is the value of $E^{(n)}$ at the initial time $t_{\mathbf{i}}$.

In the SMC, the following sources are considered

- $\omega^{(n)} = 0$: this corresponds to a dust fluid. In this case, the energy $E^{(\text{dust})} \propto a^{-3}$ corresponds to the rest mass density of the fluid, and the 4-velocity \mathbf{u} of the dust fluid coincides with the one of the fundamental observer, $\mathbf{u} = \mathbf{n}$. This equation of state can be used to describe either the cold dark matter (non-relativistic dark matter), or non-relativistic baryonic matter when non-gravitational interactions can be neglected.
- $\omega^{(n)} = \frac{1}{3}$: this corresponds to a radiative fluid as described in section 1.1.1.2. In this case $E^{(\text{rad})} \propto a^{-4}$ corresponds to the electro-magnetic energy density as observed by the fundamental observer \mathbf{n} . Such an equation of state can also describe a fluid of ultra-relativistic particles (ultra-relativistic with respect to the fundamental observer), such as neutrinos.

The curvature and cosmological constant terms in the Friedmann equations can be seen as coming from an effective fluid with energy density and pressure:

$$E^{\mathcal{R}} := -\frac{\mathcal{R}}{16\pi G} \quad ; \quad P^{\mathcal{R}} := \frac{\mathcal{R}/3}{16\pi G} \quad (\omega^{\mathcal{R}} = -1/3), \quad (1.69)$$

$$E^{\Lambda} := \frac{\Lambda}{8\pi G} \quad ; \quad P^{\Lambda} := -\frac{\Lambda}{8\pi G} \quad (\omega^{\Lambda} = -1). \quad (1.70)$$

1.2.2.3 Dynamics of the expansion

We introduce the (dimensionless) cosmological parameters, or Ω -parameters, defined as

$$\Omega_{(n)} := \frac{8\pi G E^{(n)}}{6H^2}, \quad (1.71)$$

and similarly for the curvature and the cosmological constant. The Friedmann equations (1.64) and (1.65) may be rewritten as

$$1 = \Omega_{(\text{dust})} + \Omega_{(\text{rad})} + \Omega_{\mathcal{R}} + \Omega_{\Lambda}, \quad (1.72)$$

$$q := -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2} \left(\Omega_{(\text{dust})} + \Omega_{(\text{rad})} - \Omega_{\Lambda} \right), \quad (1.73)$$

where the parameter q quantifies the deceleration of the expansion. From the knowledge of the cosmological parameters and the expansion rate H at a time t_0 , the expansion rate at any time t can be known. The measured made by the Planck collaboration (Planck Collaboration et al., 2020) lead the following values of the cosmological parameters at the current cosmological epoch:

$$H_0 = (67.4 \pm 0.5) \text{ km/s/Mpc}, \quad (1.74)$$

$$\Omega_{(\text{dust})}^0 = 0.3111 \pm 0.0056, \quad (1.75)$$

$$\Omega_{(\text{rad})}^0 \sim 10^{-5}, \quad (1.76)$$

$$\Omega_{\mathcal{R}}^0 = 0.001 \pm 0.002, \quad (1.77)$$

$$\Omega_{\Lambda}^0 = 0.6889 \pm 0.005. \quad (1.78)$$

Then under the SMC, the Universe energy content is currently dominated by the cosmological constant and the dust matter, which is decomposed into baryonic matter with $\Omega_{\text{b}}^0 \simeq 0.0490$ and dark matter with $\Omega_{\text{dm}}^0 \simeq 0.261$. The value of the curvature term is compatible with a spatially flat Universe, which is generally assumed. As the radiation, matter and Λ terms evolve respectively as a^{-4} , a^{-3} and a constant, and because today's expansion rate is positive, we can infer from the Friedmann equations that $a(t) \rightarrow 0$ at a finite time in the past, implying a 'Big Bang'. Three different areas also appear: a radiation dominated area at early times, when the scale factor can be approximated by $a(t) \propto \sqrt{t}$; a matter dominated area; and an area in accelerated expansion dominated only by the cosmological constant. Assuming the SMC model, our Universe is currently in transition between the matter and the Λ areas.

1.2.2.4 Inflation area

In the model described above, the deceleration parameter is always positive, except at late times. It can be shown that with a never negative q , for any observer there will always be a part of the Universe which will never be causally related to this observer, implying a 'causal horizon'. At the time of emission of the CMB, the size of this horizon was much smaller than the distance between the two ends of the last scattering surface. And yet the observation of the CMB suggests that these two ends should have been correlated at early times. This is known as *the horizon problem*. To solve it, Guth (1981); Starobinsky (1982); Linde (1983) proposed that the radiative area was preceded by an accelerating area, called *inflation*, where $q < 0$ such that the observed Universe was entirely in causal contact at early times. However, there

is no consensus on the mechanism behind this phase of acceleration, and several models are competing. The need for inflation to solve the horizon problem can also be questioned (e.g. [Triay, 2003](#)).

1.2.2.5 What means ‘dark energy’?

The reintroduction of the cosmological constant results from the measure of a negative q_0 by [Perlmutter et al. \(1999\)](#); [Riess et al. \(1998\)](#), corresponding to an acceleration of the scale factor. However, other proposals have been suggested to explain this observation. Most of them consist of a ‘quintessence’ scalar field with energy-momentum tensor as described in section 1.1.1.2. The origin of this scalar field can be either phenomenological, or it can arise from quantum gravity theories. Some studies also try to explain dark matter in the same time ([Sahni & Wang, 2000](#); [Matos & Arturo Ureña-López, 2001](#)). In any case, the SMC requires an additional unknown energy source, fundamental with quintessence fields, or effective with Λ , to explain $q_0 < 0$, which is called *the dark energy*.

Remark. [Perlmutter et al. \(1999\)](#); [Riess et al. \(1998\)](#) observed an acceleration of the local scale factor, i.e. along the light-cone. However, talking about an acceleration of the spatial expansion, and even about dark energy, is model dependant. Therefore, in the remaining of the thesis, when talking about an observed acceleration or about the dark energy, we will refer to the acceleration of the local scale factor.

1.2.3 The Cosmic Microwave Background

The fact that $a(t) \rightarrow 0$ at a finite time suggests that the Universe was initially very dense and hot, with non-negligible interactions between matter and radiation. As the Universe expanded, the density of photons and baryons became low enough so that the formers did not scatter anymore with the latters and became free. After this decoupling between mater and radiation, the Universe has remained transparent. The remanent of the last scattered emission of photons, called the *last scattering surface*, can be seen in microwave length. This is the Cosmic Microwave Background.

As shown in figure 1.3, this radiation is very homogeneous with a temperature contrast of the order 10^{-5} . Its analysis allows us to determine the value of the cosmological parameters presented above. As the matter-dominated area begins after the emission of the CMB, it can be used to construct initial conditions for the formation of the matter structures. However such structures cannot be studied within the class of FLRW solutions. This is usually done either using perturbative approach around the FLRW geometry within the framework of general relativity, or using Newton’s theory of gravitation (see section 1.2.5).

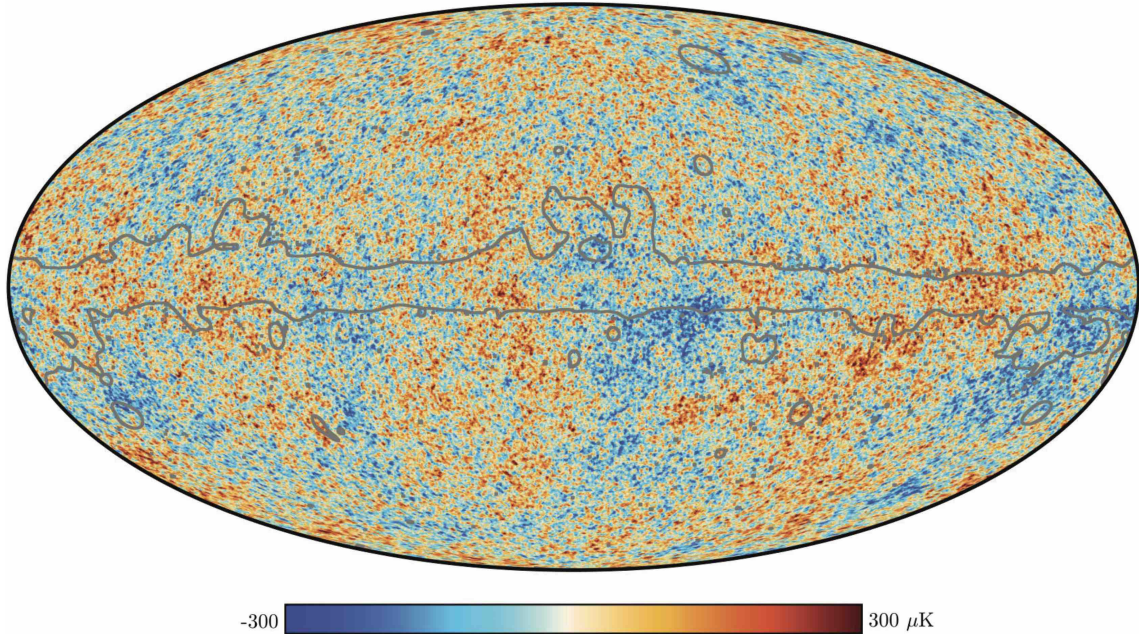


Figure 1.3: Map of the temperature fluctuations around the mean temperature (~ 2.7 K) of the CMB and without the dipole moment. The areas encircled by grey lines correspond to the removed contribution of the Milky way and the nearby (dwarf) galaxies. *Credits: ESA, Planck collaboration.*

1.2.4 Cosmic topology

1.2.4.1 The Thurston geometries

The Thurston geometrisation conjecture (Thurston, 1982), proven by Perelman (2002, 2003a,b), is a classification of the closed (i.e. compact with no boundary) 3-manifolds. It states that any closed 3-manifold can be decomposed into pieces that each has one of the eight Thurston geometric structures: the spherical geometry, the Euclidean geometry, the hyperbolic geometry, the geometry of $\mathcal{S}^2 \times \mathbb{R}$, the geometry of $\mathcal{H}^2 \times \mathbb{R}$, the geometry of $\tilde{S}L(2, \mathbb{R})$, the Nil geometry, and the Sol geometry. The precise mathematical definition of these structures can be found in Lachieze-Rey & Luminet (1995).

Remark. The notion of geometry used in this classification is a bite more restrictive than the notion of topology. We will however use both terms, as it is often the case in cosmology.

1.2.4.2 Representations of a topology

A 3-manifold with a single Thurston geometry can be represented in three different ways, which are summarised in figure 1.4 for the 2-torus:

- The 3-manifold can be embedded in \mathbb{R}^n , with $3 + 1 < n \leq 3 * 2 + 1$, but at the

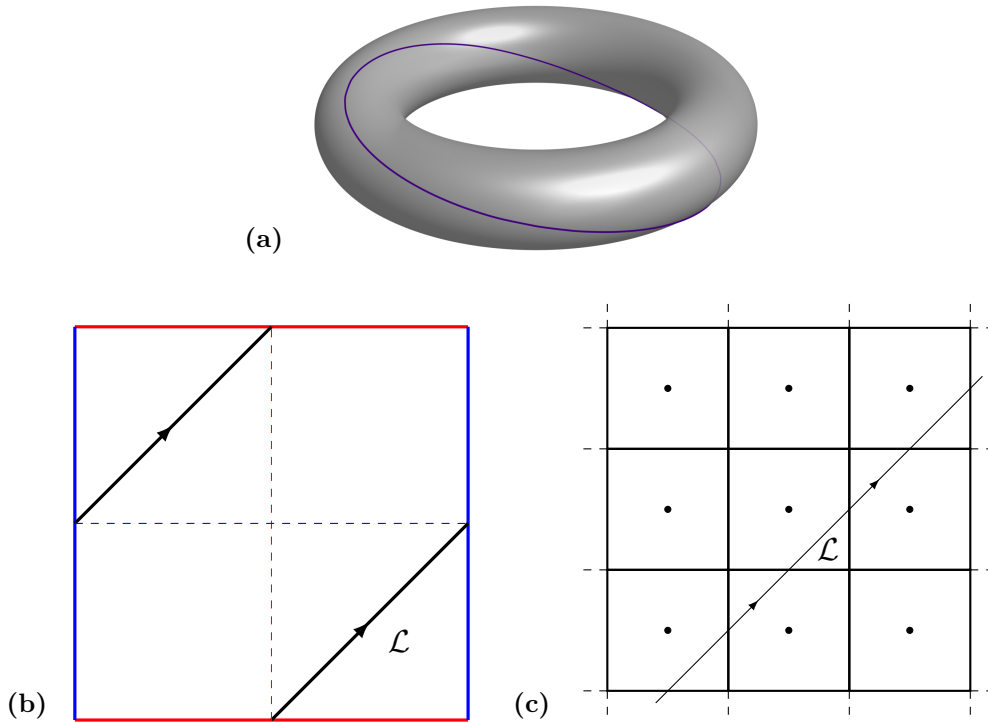


Figure 1.4: Scheme of the three ways of representing the flat 2-dimensional torus T^2 . **(a)** T^2 is embedded into \mathcal{E}^3 and possesses an extrinsic curvature. **(b)** Fundamental domain of T^2 . This is a square in which we identify the opposite faces. T^2 is a multi-connected space since the closed loop \mathcal{L} cannot be contracted continuously into a point. **(c)** Tiling of the universal covering space \mathcal{E}^2 by the fundamental domain (the square) using the holonomy group. We clearly see in this representation that the loop \mathcal{L} is an infinite line, but closed in the fundamental domain. It cannot be contracted into a point.

cost of distortions, via an extrinsic curvature.

- The space can be defined by its *fundamental domain*⁵ which is a polyhedron for which we identify properly the faces together. For \mathbb{T}^2 , the fundamental domain is a square with opposite sides identified.
- The space can be represented as a tiling by the fundamental domain of a simply connected space, which is then called the *universal covering space* $\tilde{\mathcal{M}}$. The tiling is realised with the *holonomy group* Γ . Thus a point at the center of a fundamental domain will be at the center of all the fundamental domains tiling the covering space. Taking the example of the Poincaré space (spherical geometry): in this case we tile the 3-sphere \mathcal{S}^3 with 120 dodecahedrons for which the opposite faces are identified with a rotation of $\pi/10$. Mathematically, we define the topological 3-space \mathcal{M} as $\mathcal{M} := \tilde{\mathcal{M}}/\Gamma$.

⁵The fundamental domain is not necessarily unique and can be position dependent.

1.2.4.3 The spatial geometry of the SMC

Once we suppose global hyperbolicity, the question of the topology of \mathcal{M} narrows to the topology, or Thurston geometry, of the hypersurfaces of any spacelike Cauchy foliation. The Einstein equation being a local equation, it does not give constraints on this geometry. Therefore, such constraints can only come from the initial conditions. In certain cases, there can also be a simple relation between the Ricci tensor \mathcal{R} of the hypersurfaces Σ_t and their geometry, which is the same for any of these hypersurfaces. This is the case for the FLRW solutions, where the Ricci tensor $\mathcal{R} = \frac{\mathcal{R}}{3}\mathbf{h}$ allows only for three types of 3-spaces:

- $\mathcal{R} > 0$: Σ has a spherical geometry. The spatial manifold is necessarily closed.
- $\mathcal{R} = 0$: Σ is the Euclidean space E^3 , and if it is closed has an Euclidean geometry,
- $\mathcal{R} < 0$: Σ is the hyperbolic space \mathcal{H}^3 , and if it is closed has an hyperbolic geometry.

However, the FLRW metric does not constraint the precise topology within a type, and it can be either simply connected or multi-connected, open or closed (except for the spherical case). The expansion law is also independent of this precise topology, which therefore, in a strictly homogeneous model, is degenerate.

1.2.4.4 Method of observation of the spatial geometry

Once we consider the inhomogeneities in the matter distribution of the Universe, it is theoretically possible to determine the precise topology within a type. The following methods exist:

- The cosmic crystallography ([Lehoucq et al., 1996](#)): this method is based on the correlation in the distribution of galaxies. The topology is determined by the spikes in the separation histogram of objects in catalogues of galaxies. Using the third representation [panel (c) in figure 1.4], the spikes appear because of the presence of image galaxies in the covering space.
- Circles in the CMB (e.g. [Luminet & Roukema, 1999](#); [Luminet et al., 2003](#)): this method is based on the correlation of temperature fluctuations in the CMB. For a closed topology smaller than the size of the CMB, identical circles could be found in the CMB map lying in different directions. The size and place of these circles determine the precise topology. However, if the Universe is closed but bigger than the size of the CMB, no circle is visible.

As for now, these methods only give a lower bound for the volume of the Universe, of around 10 to 20 $(\text{Gpc}/h)^3$ ([Aurich et al., 2004](#); [Aurich, 2008](#); [Roukema et al., 2014](#); [Planck Collaboration et al., 2020](#)).

These two methods are all based on the correlation of the matter in a single slice Σ_t . In this sense they can be called ‘static methods’. However, while in the FLRW solution the precise topology does not play a role on the dynamics of the Universe, it might be possible that once the inhomogeneities are considered, a lift a degeneracy occurs. In this case, the dynamics of the structures or even the expansion could be probes for the measurement of the topology. The non-Euclidean Newtonian theory we want to develop in this thesis will be a tool to understand the effects of specific topologies on the dynamics of the Universe.

1.2.5 Inhomogeneities under the standard model

The study of inhomogeneities in the Universe (CMB anisotropies and structure formation) cannot be done under the FLRW solution of general relativity. In the SMC, the CMB anisotropies are studied using perturbations of the metric around a FLRW background (Bardeen, 1980). In this approach, the spatial metric is decomposed into a scalar, a vector and a tensor part which can be measured using the CMB map.

Regarding the formation of the structures after the surface of last scattering, it is often assumed that they can be described with Newtonian gravitation, due to their low velocity dispersion. That is why most of the simulations of structure formation are done using Newtonian cosmology (e.g. Alimi et al., 2012; Nelson et al., 2015), which is presented in section 1.4. A downside of this approach is that it might miss relativistic effects not related to spatial (dispersion) velocities, but to the spatial curvature (Buchert & Carfora, 2008; Vigneron & Buchert, 2019; Roukema & Ostrowski, 2019). Furthermore, as we shall see in section 1.4 and in chapter 2, the (isotropic) expansion law in Newtonian cosmology is necessarily given by the Friedmann equations (this is the Buchert-Ehlers theorem, Buchert & Ehlers, 1997), and this for any inhomogeneous solutions. This contrasts with the purely relativistic case (presented in section 1.3), which features an additional effective source term in the Friedmann equations, coming from the inhomogeneities: the global backreaction term \mathcal{Q}_Σ . This global effect is necessarily missing in the Newtonian simulations. Finally, the Newtonian approach imposes the simulated Universe to have an Euclidean geometry, which is a restriction with respect to general relativity.

In section 1.3, we introduce a formalism which allows for the description of the relativistic effects of inhomogeneities on the expansion. This formalism is based on a scalar averaging of the Einstein equation and was first introduced by Buchert (2000). In section 1.4 we present how cosmology is studied under Newton’s theory, and we outline the strategy of the thesis in section 1.5.

1.3 The backreaction of inhomogeneities

1.3.1 The backreaction problem in cosmology

The cosmological principle asserts that there exists a scale above which the Universe can be considered as homogeneous and isotropic. The SMC then assumes that the inhomogeneities under this scale do not affect the dynamics of domains of size bigger than the homogeneity scale. This implies that the expansion of such domains is given by the (homogeneous) FLRW solution of general relativity, and the Friedmann expansion laws. That is why the dynamics of structure, under the SMC, is generally solved as a deviation around a FLRW background expansion.

This assumption is not a consequence of the cosmological principle, and is an additional hypothesis. In reality the inhomogeneities (with scale necessarily smaller than the homogeneity scale) might affect the expansion at large scales. This effect is called *the backreaction*. The main question surrounding this phenomenon is to know whether or not it could be important enough to explain the recent acceleration of the local scale factor, and if the dark energy could be entirely explained by the inhomogeneities. This is *the problem of backreaction in cosmology* (see [Buchert, 2008](#)).

A way to tackle this problem is by averaging the 3+1-Einstein equations on a spatial domain supposedly bigger than the homogeneity scale, and derive the expansion law of this domain. This approach was followed by [Buchert \(2000\)](#) for irrotational dust fluids, [Buchert \(2001\)](#) for irrotational perfect fluids, and by [Buchert et al. \(2020\)](#) for general fluids. In the following sections we present the case of an irrotational dust fluid.

1.3.2 Averaging procedure

As we seek for information on the average properties of the fluid, this requires the use of equations featuring the kinematical variables related to this fluid. These equations correspond to the 1+3-Einstein equations. We also want to perform an average over a spatial domain, which requires the introduction of a foliation and the use of the 3+1-Einstein equations. A way to deal with both of these systems is to assume that the fluid is a Cauchy fluid, i.e. irrotational and such that the foliation it defines is a Cauchy foliation. We will then perform the spatial average on this foliation. We also take a dust fluid, which is generally assumed for the description of the Universe after the surface of last scattering. Therefore, we have $N = 1$, $\mathbf{n} = \mathbf{u}$, $\mathbf{K} = -\Theta$, $\Omega = 0$, and $\epsilon = \rho$ the rest mass density.

Let us consider a compact domain \mathcal{D} on the fluid orthogonal spatial hypersurfaces, which is propagated between each slice along the fluid flow. It is defined as a time independent domain in the comoving class of coordinates \mathcal{X}_0^u , i.e. a Lagrangian domain. This ensures that the total fluid rest mass in \mathcal{D} is conserved through time.

We define the spatial average on the domain \mathcal{D} of a scalar field ψ as:

$$\langle \psi \rangle_{\mathcal{D}}(t) := \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} \psi \sqrt{\det(h_{ab})} d^3x, \quad (1.79)$$

where $V_{\mathcal{D}}(t) := \int_{\mathcal{D}} \sqrt{\det(h_{ab})} d^3x$ is the volume of \mathcal{D} . This definition ensures that for a spatially constant scalar $\psi(t)$, we have $\langle \psi \rangle_{\mathcal{D}} = \psi$.

The averaging procedure (1.79) is only well defined for scalar fields. For tensor fields, it would require the comparison of the components of the tensor at different points in Σ_t , which would depend on the coordinates, and therefore would not be covariant. The problem of averaging procedures on tensors, see [Zalaletdinov \(1992\)](#) for a proposed formalism, will not be addressed in this thesis.

1.3.3 Averaged expansion laws and the backreaction

1.3.3.1 Commutation rule

A key point in the emergence of backreaction is the non-commutativity of averaging and dynamics ([Ellis, 1984](#); [Ellis & Buchert, 2005](#)): the averaging operator $\langle \cdot \rangle_{\mathcal{D}}$ do not commute with the Lagrangian evolution operator ${}^u\partial_{t|_0}$ (this notation is defined in section 1.1.2.6). The commutation rule is ([Buchert & Ehlers, 1997](#); [Buchert, 2000](#))

$$\langle \dot{\psi} \rangle_{\mathcal{D}} - \langle \dot{\psi} \rangle_{\mathcal{D}} = \langle \theta \psi \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}}, \quad (1.80)$$

where we denote $\dot{\psi} := {}^u\partial_{t|_0} \psi$. Applied for $\psi = 1$, the commutation rule leads to

$$\langle \theta \rangle_{\mathcal{D}} = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}}. \quad (1.81)$$

Therefore $\langle \theta \rangle_{\mathcal{D}}$ corresponds to the expansion rate of the volume of the domain \mathcal{D} .

1.3.3.2 Averaged expansion law

We introduce the dimensionless ‘effective’ scale factor

$$a_{\mathcal{D}} := \left(\frac{V_{\mathcal{D}}}{V_{\mathcal{D},i}} \right)^{1/3}, \quad (1.82)$$

with $V_{\mathcal{D},i}$ the (initial) reference volume.

The expansion laws for the effective scale factor are obtained by taking the average of the 1+3-Einstein equations. We are only able to perform this average on the scalar equations of the 1+3-system (1.52)–(1.59), i.e. on the Raychaudhuri

equation, the Gauss equation and the energy conservation. We obtain

$$3 \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -4\pi G \rho + \Lambda + \mathcal{Q}_{\mathcal{D}}, \quad (1.83)$$

$$3 \left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = 8\pi G \langle \rho \rangle_{\mathcal{D}} + \Lambda - \frac{\langle \mathcal{R} \rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}}{2}, \quad (1.84)$$

$$\langle \rho \rangle_{\mathcal{D}} \dot{} = -3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \rho \rangle_{\mathcal{D}}, \quad (1.85)$$

where we defined

$$\mathcal{Q}_{\mathcal{D}} := \frac{2}{3} \left(\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2 \right) - \langle \sigma_{cd} \sigma^{cd} \rangle_{\mathcal{D}}. \quad (1.86)$$

The averaged conservation equation (1.85) leads to $(V_{\mathcal{D}} \langle \rho \rangle_{\mathcal{D}})' = 0$ which corresponds to the conservation of the total rest mass $M_{\mathcal{D}} := V_{\mathcal{D}} \langle \rho \rangle_{\mathcal{D}}$ in \mathcal{D} .

From equations (1.83)–(1.84), one can derive the *integrability condition*:

$$\dot{\mathcal{Q}}_{\mathcal{D}} + 6 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \mathcal{Q}_{\mathcal{D}} + \langle \mathcal{R} \rangle_{\mathcal{D}} \dot{\phantom{\mathcal{R}}} + 2 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \mathcal{R} \rangle_{\mathcal{D}} = 0. \quad (1.87)$$

1.3.4 Averaged equations VS Friedmann equations

We can see that the equations (1.83) and (1.84) are similar to the Friedmann equations (1.64) and (1.65). But important differences also appear:

- All the terms (except Λ) depend on the averaging region,
- The curvature term $\frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{2}$ does not necessarily behave as $\propto a_{\mathcal{D}}^{-2}$ like in the FLRW case. We introduce *the curvature backreaction* $\mathcal{W}_{\mathcal{D}} := \langle \mathcal{R} \rangle_{\mathcal{D}} - \frac{\langle \mathcal{R} \rangle_{\mathcal{D}_1}}{a_{\mathcal{D}}^2(t)}$ which quantifies the deviation of the dynamics of the average curvature with respect to the one of a FLRW solution.
- There is an additional term $\mathcal{Q}_{\mathcal{D}}$. It compares two positive contributions of the inhomogeneities: the variance of the volume expansion $(\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2)$ and the shear expansion $\langle \sigma_{ab} \sigma^{ab} \rangle_{\mathcal{D}}$. Furthermore it influences the dynamics of the expansion. Therefore $\mathcal{Q}_{\mathcal{D}}$ quantifies the backreaction of the inhomogeneities on the volume expansion of the domain \mathcal{D} . We call it, *the kinematical backreaction term*.⁶

⁶The term ‘kinematical’ distinguishes with the ‘dynamical’ backreaction term, which is an additional term present in the case of non-dust perfect fluids (Buchert, 2001).

Following section 1.2.2.2, we introduce the effective backreaction energy density $\epsilon_{\text{eff}}^{\mathcal{D}}$ and pressure $p_{\text{eff}}^{\mathcal{D}}$ as:

$$\epsilon_{\text{eff}}^{\mathcal{D}} := -\frac{\mathcal{Q}_{\mathcal{D}} + \mathcal{W}_{\mathcal{D}}}{16\pi G} \quad ; \quad p_{\text{eff}}^{\mathcal{D}} := -\frac{\mathcal{Q}_{\mathcal{D}} - \frac{1}{3}\mathcal{W}_{\mathcal{D}}}{16\pi G}. \quad (1.88)$$

Remark. Using the integrability condition (1.87), a vanishing $\mathcal{Q}_{\mathcal{D}}$ is necessary and sufficient to have a Friedmannian effective scale factor.

1.3.5 Backreaction, dark energy and dark matter

From the expansion law (1.83) we can see that the kinematical backreaction term is an additional source for the acceleration. Therefore if $\mathcal{Q}_{\mathcal{D}} > 0$, the effect of the backreaction on the expansion of \mathcal{D} is an acceleration. In this sense it can mimic the behaviour of the dark energy.

While the motivation for the introduction of the backreaction was the quantification of the effects of the small scale inhomogeneities on the homogeneity scale, the formalism described above and the related averaged expansion laws do not depend on the existence of such a homogeneity scale. Therefore we are not obliged to assume that $(V_{\mathcal{D}})^{1/3}$ is bigger than this homogeneity scale, and we can study the backreaction of small scale inhomogeneities on intermediate scale inhomogeneities. In such cases, if $\mathcal{Q}_{\mathcal{D}} < 0$ the kinematical backreaction can mimic dark matter as it slows down the expansion of the domain (e.g. Buchert & Carfora, 2008; Wiegand & Buchert, 2010; Vigneron & Buchert, 2019).

1.3.6 Caveats of this approach to the backreaction problem

One of the caveats of the above approach to the backreaction problem at large scales, is the arbitrariness of the domain \mathcal{D} . A natural way to deal with this problem is to consider a spatially closed universe Σ and to take the domain \mathcal{D} to be Σ . In this case, the backreaction variables \mathcal{Q}_{Σ} and \mathcal{W}_{Σ} quantify the effect of all the inhomogeneities of this universe on its expansion. In this thesis, we will assume that our Universe is closed, and will only study the backreaction on its whole volume. Note that this might be a restriction to the study of the backreaction as the current acceleration of the scale factor could be a local effect, only needing a backreaction on medium scales.

Because the average procedure can only be done on the scalar equations, part of the dynamics is lost in the average process. As a consequence, the equations (1.83)–(1.85) are not closed: there are four variables $(a_{\mathcal{D}}, \langle \rho \rangle_{\mathcal{D}}, \mathcal{Q}_{\mathcal{D}}, \langle \mathcal{R} \rangle_{\mathcal{D}})$ for three equations. Therefore closure conditions are generally assumed using cosmological models. Another way to deal with this problem would be to assume that our Universe is well described by Newtonian gravitation, and study the backreaction under the Newtonian cosmology. This is presented in the next section.

1.4 Newtonian cosmology

Cosmology within Newton's theory, and especially the description of expansion was developed after general relativity by Heckmann & Schücking (1955, 1956). The general expansion law was finally derived by Buchert & Ehlers (1997).

1.4.1 Kinematical variables in Newton

The classical Newton system of equations describes the time evolution of a fluid characterised by a scalar field ρ , the fluid density, and a vector field \mathbf{v} , the fluid velocity. These two tensors are defined in a 3-dimensional manifold, denoted Σ^N , equipped with a flat Riemannian metric \mathbf{h} ⁷ and are parametrised by the time t . The system of equations is composed of two evolution equations, one for the scalar ρ and one for the vector \mathbf{v} , and two constraint equations. Given a fixed coordinate basis vector $\{\mathbf{e}_a\}_{a=1,2,3}$ on Σ^N , i.e. the vectors \mathbf{e}_a are not parametrised by time, the evolution equations in the corresponding coordinate system $\{x^a\}_{a=1,2,3}$ are

(i) the *mass conservation equation*

$$(\partial_t + v^c D_c) \rho = -\rho D_c v^c, \quad (1.89)$$

(ii) the *Euler equation*

$$(\partial_t + v^c D_c) v^a = g^a + a_{\neq \text{grav}}^a, \quad (1.90)$$

where D_a are the components of the Levi-Civita connection of \mathbf{h} , and \mathbf{g} is the gravitational vector field constraint by the following equations:

(i) the *Newton-Gauss equation*

$$D_c g^c = -4\pi G \rho + \Lambda, \quad (1.91)$$

(ii) the *Newton-Faraday equation*

$$D_{[a} g_{b]} = 0. \quad (1.92)$$

The vector $\mathbf{a}_{\neq \text{grav}}$ corresponds to the non-gravitational acceleration.

Due to the equivalence principle, the Euler equation (1.90) can be seen as a definition of the gravitational vector field. Then apart from this equation, the Newton system can be written independently of \mathbf{g} . To do so, we introduce the expansion

⁷The only requirement on Σ^N is to be Euclidean, but its topology is not necessarily \mathbb{R}^3 .

tensor Θ and the vorticity tensor Ω of the vector field \mathbf{v} being respectively the symmetric and the antisymmetric part of the velocity gradient $D\mathbf{v}$, with⁸

$$\Theta_{ab} := D_{(a}v_{b)} \quad ; \quad \Omega_{ab} := D_{[a}v_{b]}, \quad (1.93)$$

and we note the trace $\Theta_c^c =: \theta$. The interpretation of these two tensors is the same as for the ones introduced in general relativity (see section 1.1.3.1).

Rewriting equation (1.89), (1.91) and (1.92), we obtain *the kinematical Newton system*:

(i) the mass conservation:

$$(\partial_t + \mathcal{L}_v)\rho = -\rho\theta, \quad (1.94)$$

(ii) the Raychaudhuri equation:

$$(\partial_t + \mathcal{L}_v)\theta = -4\pi G\rho + \Lambda - \Theta_{cd}\Theta^{cd} + \Omega_{cd}\Omega^{cd} + D_c a_{\neq\text{grav}}^c, \quad (1.95)$$

(iii) the vorticity conservation:

$$(\partial_t + \mathcal{L}_v)\Omega_{ab} = D_{[a}(a_{\neq\text{grav}})_{b]}. \quad (1.96)$$

These equations need to be completed by the definition (1.93) of Θ and Ω . Then, the system (1.93)–(1.96) is closed and equivalent to the system (1.89)–(1.92). The gravitational vector field \mathbf{g} is defined as

$$g^a := (\partial_t + \mathcal{L}_v)v^a + v^c(\Theta_c^a + \Omega_c^a) - (a_{\neq\text{grav}})^a. \quad (1.97)$$

Introducing the Lie derivative in this last equation allows us to have the same differential operator acting on ρ , Θ_{ab} and Ω_{ab} . Furthermore, this formulation of \mathbf{g} is closer to the general one we will give in chapter 2.

Remark. While the expansion and vorticity tensors are explicitly covariant under any change of coordinates, parametrised by time or not, the differential operator ∂_t is not. In appendix B we derive the form of the kinematical Newton system in any time-parametrised frame, i.e. for any time-parametrised coordinate system. This generalised system features a shift freedom similar to the 3+1-Einstein system.

1.4.2 Homogeneous deformation

For the above system to be well defined, closing conditions need to be added. There are two possibilities:

⁸Here we adopt the sign convention $\Omega_{ab} := +D_{[a}v_{b]}$. This implies the relation: $\text{curl } v_a = \epsilon_{acd}\Omega^{cd}$ where ϵ_{acd} is the Levi-Civita tensor. The inverse relation is $\Omega_{ab} = \frac{1}{2}\epsilon_{abc}\text{curl } v^c$.

- (i) the system is isolated, and fall-off conditions are taken at infinity, i.e. $\rho \xrightarrow{r \rightarrow \infty} 0$ and $\mathbf{v} \xrightarrow{r \rightarrow \infty} \mathbf{0}$. This situation is used for astrophysical flows.
- (ii) the velocity \mathbf{v} is decomposed into a *homogeneous deformation* velocity \mathbf{v}_H with $v_H^a := H_c^a(t) x^c$ (in Cartesian coordinates) where H_c^a are only functions of t , and a peculiar-velocity \mathbf{P} : we have $\mathbf{v} = \mathbf{v}_H + \mathbf{P}$. The peculiar-velocity is periodically defined on the absolute Euclidean space \mathcal{E}^3 . This situation is used in cosmology.

In the second choice, the periodicity of the peculiar-velocity corresponds to a cubic domain, denoted \mathbb{T}^3 , in the comoving coordinates $\{\tilde{x}^a\}_{a=1,2,3}$, which are defined as

$$\tilde{x}^a := A_c^a x^c, \quad (1.98)$$

with $A_c^a(t) := \int_0^t H_c^a dt$. These coordinates follow the flow of the velocity \mathbf{v}_H . The domain \mathbb{T}^3 is distorted by the homogeneous deformation H_a^b in the absolute Euclidean space \mathcal{E}^3 :

- $H := \frac{1}{3} H_c^c$ corresponds to the volume expansion of the domain \mathbb{T}^3 ,
- $H_{\langle ab \rangle}$ corresponds to an anisotropic expansion of \mathbb{T}^3 in \mathcal{E}^3 ,
- $H_{[ab]}$ corresponds to the rotation of \mathbb{T}^3 in the Euclidean space \mathcal{E}^3 .

Because the absolute Euclidean space is periodically tiled with the domain \mathbb{T}^3 , a natural interpretation of the homogeneous deformation is to represent an effective flat 3-torus which is expanding (covering space representation, see section 1.2.4.2 and figure 1.4), i.e. a closed Universe. That is why this approach is well suited for cosmology. In this interpretation, once the homogeneous deformation is introduced, the spatial velocity which describes the fluid is considered to be the peculiar-velocity \mathbf{P} . The evolution equations for this velocity are called the cosmological Newton equations. We derive them in what follows.

1.4.3 The cosmological Newton equations

We can show the following relation between the time partial derives ∂_t and $\tilde{\partial}_t := \partial_{t_{\tilde{x}}}$ (see section B.1.2 in appendix B for the derivation):

$$\tilde{\partial}_t = \partial_t + \mathcal{L}_{\mathbf{v}_H}. \quad (1.99)$$

This relation is valid when applied to tensors of any types.

The expansion and vorticity tensor can be rewritten as:

$$\Theta_{ab} = H_{\langle ab \rangle} + {}^P\Theta_{ab} \quad ; \quad \Omega_{ab} = H_{[ab]} + {}^P\Omega_{ab}, \quad (1.100)$$

where we introduced the expansion ${}^P\Theta_{ab} := D_{\langle a} P_{b \rangle}$ and vorticity ${}^P\Omega_{ab} := D_{[a} P_{b]}$ of the peculiar-velocity.

The homogeneous deformation $H_a{}^b$ being an additional unconstrained field in the Newton system, we need evolution equations for its different parts (trace, symmetric traceless part, antisymmetric part). As $H_a{}^b$ is only a function of time (in Cartesian coordinates), we can obtain these evolution equations by spatially averaging the kinematical Newton system on the periodic domain. We use the same average operator as for backreaction in general relativity [defined in equation (1.79)].

As the periodic boundary conditions defining \mathbb{T}^3 are expanding, the volume $V_{\mathbb{T}^3}$ depends on time with the relation

$$\frac{1}{3} \frac{\tilde{\partial}_t V_{\mathbb{T}^3}}{V_{\mathbb{T}^3}} = H(t). \quad (1.101)$$

The average of the Newton-Raychaudhuri equation (1.95) gives an evolution equation for H , i.e. an expansion law, while the average of the vorticity conservation (1.96) gives an evolution equation for $H_{[ab]}$.⁹ However, it is not possible to obtain an evolution equation for the anisotropic expansion $H_{\langle ab \rangle}$, which remains free.

The cosmological Newton equations are:

(i) the mass conservation:

$$\left(\tilde{\partial}_t + \mathcal{L}_P\right) \rho = -3H\rho - {}^P\theta\rho \quad ; \quad \langle \rho \rangle_{\mathbb{T}^3} = \frac{M_{\text{tot}}}{V_{\mathbb{T}^3}(t)}, \quad (1.102)$$

where M_{tot} is the total mass in \mathbb{T}^3 .

(ii) the cosmological Newton-Gauss equation:

$$D_c {}^P g^c = -4\pi G \hat{\rho} - \widehat{H_{\langle cd \rangle} H^{\langle cd \rangle}} + \widehat{H_{[cd]} H^{[cd]}}, \quad (1.103)$$

where $\hat{\psi} := \psi - \langle \psi \rangle_{\mathbb{T}^3}$

(iii) the cosmological Newton-Faraday equation:

$$D_{[a} {}^P g_{b]} = -\tilde{\partial}_t H_{[ab]}, \quad (1.104)$$

where we introduced the peculiar (periodic) gravitational field as

$${}^P g^a := \left(\tilde{\partial}_t + P^c D_c\right) P^a + 2P^c H_c{}^a - (a_{\neq \text{grav}})^a, \quad (1.105)$$

⁹The average of the vorticity equation is performed in Cartesian coordinates.

(iv) the expansion law:

$$3 \left[\tilde{\partial}_t H + H^2 \right] = -4\pi G \langle \rho \rangle_{\mathbb{T}^3} + \Lambda - \langle H_{\langle cd \rangle} H^{\langle cd \rangle} \rangle_{\mathbb{T}^3} + \langle H_{[cd]} H^{[cd]} \rangle_{\mathbb{T}^3}. \quad (1.106)$$

To get this last equation we used the fact that divergences of functions of \mathbf{P} averaged over a periodic domain \mathbb{T}^3 are zero due to Stokes's theorem.

(v) the evolution equation of $H_{[ab]}$:

$$\partial_t H_{[ab]} + H_{[cb]} H_a^c + H_{[ac]} H_b^c = 0. \quad (1.107)$$

1.4.3.1 Backreaction in Newtonian cosmology

The formalism developed in section 1.3 can also be used to describe the backreaction in Newton's theory. We end up with a similar kinematical backreaction, which features an additional term related to the vorticity:

$$\mathcal{Q}_{\mathcal{D}} := \frac{2}{3} \left(\langle \theta^2 \rangle_{\mathcal{D}} - \langle \theta \rangle_{\mathcal{D}}^2 \right) - \langle \sigma_{cd} \sigma^{cd} \rangle_{\mathcal{D}} + \langle \Omega_{cd} \Omega^{cd} \rangle_{\mathcal{D}}. \quad (1.108)$$

In the case of the Newtonian cosmology, and assuming we average over the periodic domain \mathbb{T}^3 , this expression becomes:

$$\mathcal{Q}_{\mathbb{T}^3} := - \langle H_{\langle cd \rangle} H^{\langle cd \rangle} \rangle_{\mathbb{T}^3} + \langle H_{[cd]} H^{[cd]} \rangle_{\mathbb{T}^3}, \quad (1.109)$$

and the expansion law (1.106) can be rewritten as

$$3 \left[\tilde{\partial}_t H + H^2 \right] = -4\pi G \langle \rho \rangle_{\mathbb{T}^3} + \Lambda + \mathcal{Q}_{\mathbb{T}^3}. \quad (1.110)$$

1.4.3.2 Interpretation

Equation (1.106) is similar to the Friedmann equation (1.64). However, it features two additional terms: $\langle H_{\langle cd \rangle} H^{\langle cd \rangle} \rangle_{\mathbb{T}^3}$ and $\langle H_{[cd]} H^{[cd]} \rangle_{\mathbb{T}^3}$. This means that the anisotropic expansion and the global rotation corresponds to additional effective source terms in the Friedmann equation. As both terms are positive, the anisotropic expansion source creates a deceleration of the volume expansion, whereas the global rotation source creates an acceleration. Furthermore, equation (1.104) shows that the global rotation $H_{[ab]}$ also corresponds to a *gravitomagnetic field* for the peculiar gravitational field ${}^{\text{P}}\mathbf{g}$. Gravitomagnetism is generally considered to be a post-Newtonian effect, and in this sense it is generally assumed to be zero in the cosmological Newton system.

Remark. In chapter 4 we will show that when the cosmological Newton system is derived from general relativity, the global rotation is necessarily zero.

The system (1.102)–(1.107) is used in the Newtonian cosmological simulations. Because the anisotropic expansion is unconstrained, and because there is no evidence

for such feature in the expansion of our Universe, $H_{\langle cd \rangle}$ is assumed to be zero. In this case the expansion law (1.106) is equivalent to the Friedmann equation. This means that the expansion of the periodic domain \mathbb{T}^3 in Newton is equivalent to the expansion of a homogeneous Universe in general relativity, whatever the distribution of matter and peculiar-velocities in the domain \mathbb{T}^3 . In other words, there is no backreaction of the inhomogeneities on the (isotropic) expansion of the domain \mathbb{T}^3 ($\mathcal{Q}_{\mathbb{T}^3} = 0$). This result is called the Buchert-Ehlers theorem, and was first derived by Buchert & Ehlers (1997).

1.4.4 Caveats of the Newtonian cosmology

The main consequence of the Buchert-Ehlers theorem is to imply that the study of the backreaction of inhomogeneities on the expansion of a closed universe needs to be made in general relativity: the backreaction is a relativistic effect.¹⁰

However, the Buchert-Ehlers theorem relies on the interpretation of the homogeneous deformation made in section 1.4.2, in particular on the fact that we considered the universe represented by the cosmological Newton system to be an expanding 3-torus. In reality the topology of the spatial manifold is still \mathbb{R}^3 , and the torus is only an interpretation of the periodic boundary conditions imposed on the peculiar-velocity. This contrasts with general relativity, where we can assume the spatial manifold to be a (closed) 3-torus which is expanding. In this sense, the physics described by the cosmological Newton system might not necessarily be compatible with general relativity.

Another caveat of the Buchert-Ehlers theorem and its interpretation is that it imposes the geometry of the model universe to be Euclidean. As general relativity allows for all the Thurston geometries, this is a clear restriction with respect to this theory. This raises the following question: **can we generalise the Buchert-Ehlers theorem to non-Euclidean geometries ?** Answering this question requires a generalisation of Newton's theory to non-Euclidean geometries. If this is possible and that the backreaction is still zero in these geometries, this would definitely show that this effect is a purely relativistic effect. Otherwise, if there is a non-zero backreaction in non-Euclidean geometries, this would show that the topology plays a major role on the expansion of our Universe.

¹⁰Strictly speaking, only the backreaction on an entire closed universe is relativistic. As shown in section 1.4.3.1 it is also possible to have a backreaction on smaller domains in Newton (Buchert et al., 2000).

1.5 Strategy of the thesis

Generalising the Buchert-Ehlers theorem to non-Euclidean geometries is the main goal of this thesis. To do it I will pursue the following strategy:

- (i) I will first show that the Newtonian cosmology and the interpretation in terms of a spatially closed manifold is compatible with general relativity. This is done in chapter 2 using the Newton-Cartan theory, as it can be derived via a well defined geometrical limit of general relativity.
- (ii) Generalising the Buchert-Ehlers theorem requires the definition of a non-Euclidean Newtonian (NEN) theory. In chapter 3, I review the existing approaches to such a theory, and I propose two new ones based on the results of chapter 2. I also derive the generalised Buchert-Ehlers theorem for each of these approaches.
- (iii) The final stage aims at justifying one of my proposed NEN theories from a limit of general relativity. This is done in chapter 4.

As a complementary approach to study of the relation between topology and global backreaction, I will investigate in chapter 5 the possibility of doing relativistic numerical simulations in non-Euclidean geometries.

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Chapter 2

1+3-Newton-Cartan system and Newton-Cartan cosmology

2.1 Why Newton-Cartan?

THE Newton-Cartan theory is a formulation of Newton’s theory of gravitation in a Galilean 4-manifold. The structure on this manifold, called a Galilean structure, follows from Newton’s ideas of absolute time and absolute space. Similarly to general relativity, the goal behind the Newton-Cartan formulation is to describe the gravitational force with a spacetime connection. The physical equations constraining this connection, called the Newton-Cartan equations, are equations relating the Riemann tensor associated to this connection and the energy content.

This theory, originally introduced as a spacetime geometrisation of Newton’s theory by [Cartan \(1923, 1924\)](#), has then been developed to study the Newtonian limit (e.g. [Künzle, 1976](#)) and to define post-Newtonian approximations to general relativity (e.g. [Dautcourt, 1997](#); [Tichy & Flanagan, 2011](#)). [Ehlers \(1981\)](#); [Ehlers \(2019\)](#) also proposed a unification of Newton-Cartan theory and general relativity, within his frame theory.

The limit allowing to derive the Newton-Cartan theory from general relativity is a well defined limit of a 1-parameter family of solutions to the Einstein equation. The limit is also covariant, i.e. it is not performed in a specific coordinate system. Therefore we can consider that the Newton-Cartan theory, firstly, is compatible with general relativity, and secondly corresponds to the ‘true’ formulation of Newton’s theory. This second statement will be strengthened by the results we will obtain at the end of this chapter. In this sense Newton-Cartan provides the right framework to study the compatibility of the Newtonian cosmology with general relativity. To do it, we first perform a covariant 1+3-split of the Newton-Cartan equations with respect to the fluid 4-velocity, similarly to the 1+3 and 3+1 split in general relativity presented in sections [1.1.2](#) and [1.1.3](#). The resulting system of equations will be called *the 1+3-Newton-Cartan system*. We will show that this system is equivalent to the

cosmological Newton system of section 1.4.3. This will complete the work of [Ruede & Straumann \(1997\)](#) who also studied the Newton-Cartan cosmology, but only for homogeneous solutions.

In section 2.2 we recall the definition and properties of Galilean spacetimes. Section 2.3 presents the construction of the 1+3-Newton-Cartan equations. We solve the system of equations in section 2.4 and show that space expansion arises as a fundamental field of the theory. In section 2.5, we define the gravitational field and rewrite the 1+3-NC system so that it features the Cosmological Newton-Gauss equation. Then we compare with the cosmological Newton equations (1.102)–(1.107). Finally we define different observers in section 2.6. Note that the limit of general relativity to Newton-Cartan will only be detailed (and needed) in chapter 4.

This chapter is based on my paper, [Vigneron \(2021\)](#).

2.2 Galilean spacetimes

2.2.1 Galilean structure

This section is largely inspired by the nice presentation of Galilean spacetimes by [Künzle \(1972\)](#), which, to my opinion, should be considered as the reference paper for the Newton-Cartan theory.

2.2.1.1 Definition

A *Galilean spacetime* is a 4-dimensional differentiable manifold \mathcal{M} equipped with a *Galilean structure* $(\underline{\tau}, \overline{\mathbf{h}}, \underline{\nabla})$, where $\underline{\tau}$ is an exact 1-form, $\overline{\mathbf{h}}$ is a symmetric (2,0)-tensor of rank 3, with $h^{\alpha\mu}\tau_\mu = 0$, and $\underline{\nabla}$ is a connection compatible with $\underline{\tau}$ and $\overline{\mathbf{h}}$, called a *Galilean connection*:

$$\nabla_\alpha \tau_\beta = 0 \quad ; \quad \nabla_\gamma h^{\alpha\beta} = 0. \quad (2.1)$$

A vector $\bar{\mathbf{u}}$ is called a *unit timelike vector* if $u^\mu \tau_\mu = 1$, and an (n,0)-tensor \mathbf{T} is called *spatial* if $\tau_\mu T^{\dots\overset{\alpha}{\downarrow}\dots} = 0$ for all $\alpha \in \llbracket 1, n \rrbracket$. The exact 1-form $\underline{\tau}$ defines a foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ in \mathcal{M} , where Σ_t are spatial hypersurfaces in \mathcal{M} defined as the level surfaces of the scalar field t , with $\underline{\tau} = \mathbf{d}t$. In this chapter we will not use a Lorentzian connection. So $\underline{\nabla}$ and $\Gamma_{\alpha\beta}^\gamma$ will always refer to the Galilean connection and its coefficients.

No spacetime metric, i.e. a symmetric (0,2)-tensor of rank 4, is part of the Galilean structure. Furthermore it is not possible to define a spacetime metric compatible with the connection (2.1) (see chapter 12 of [Misner et al., 1973](#)). Thus raising and lowering indices is not possible *a priori*. Then, to avoid confusion when defining new tensors for the first time, we will use the over and under bars notation

introduced in the beginning of this thesis. Once they have been defined, we will however return to the simpler bold notation.

The time metric $\boldsymbol{\tau}$ and the space metric \boldsymbol{h} define what we can call an absolute (or preferred) time and an absolute (or preferred) space. This is the Newtonian picture of spacetime.

Remark. The structure defined above is called ‘Galilean’ because the components of the tensors $\boldsymbol{\tau}$ and \boldsymbol{h} are invariant under a local Galilean transformation. We can also show that the set of local Galilean transformations is totally determined by a Galilean structure (see [Künzle, 1972](#)). Therefore such structures are the most general structure which are invariant under Galilean transformations.

2.2.1.2 Properties

From the knowledge of $\boldsymbol{\tau}$ and \boldsymbol{h} , the connection $\boldsymbol{\nabla}$ is not unique. Its coefficients $\Gamma_{\alpha\beta}^\gamma$ are defined up to a unit timelike vector $\underline{\boldsymbol{B}}$ and a two form $\underline{\boldsymbol{\kappa}}$ as follows:

$$\Gamma_{\alpha\beta}^\gamma = {}^B\Gamma_{\alpha\beta}^\gamma + 2\tau_{(\alpha}\kappa_{\beta)\mu}h^{\mu\gamma}, \quad (2.2)$$

where

$${}^B\Gamma_{\alpha\beta}^\gamma := h^{\gamma\mu} \left(\partial_{(\alpha} {}^B b_{\beta)\mu} - \frac{1}{2} \partial_\mu {}^B b_{\alpha\beta} \right) + B^\gamma \partial_{(\alpha} \tau_{\beta)}, \quad (2.3)$$

and where $\underline{{}^B\boldsymbol{b}}$ is the projector orthonormal to the vector \boldsymbol{B} with

$${}^B b_{\alpha\mu} B^\mu := 0 \quad ; \quad {}^B b_{\alpha\mu} h^{\mu\beta} := \delta_\alpha^\beta - \tau_\alpha B^\beta. \quad (2.4)$$

We have the following properties

$$B^\mu {}^B \nabla_\mu B^\alpha = 0 \quad ; \quad h^{\mu[\alpha} {}^B \nabla_\mu B^{\beta]} = 0, \quad (2.5)$$

where ${}^B \boldsymbol{\nabla}$ is the connection associated with the coefficients ${}^B \Gamma_{\alpha\beta}^\gamma$.

The connection coefficients (2.2) naturally define a Riemann tensor $R^\sigma{}_{\alpha\beta\gamma}$ with the usual formula

$$R^\sigma{}_{\alpha\beta\gamma} := 2 \partial_{[\beta} \Gamma_{\gamma]\alpha}^\sigma + 2 \Gamma_{\mu[\beta}^\sigma \Gamma_{\gamma]\alpha}^\mu. \quad (2.6)$$

The Ricci and Bianchi identities are still satisfied for this Riemann tensor:

$$R^\sigma{}_{[\alpha\beta\gamma]} = 0, \quad (2.7)$$

$$\nabla_{[\mu} R^\sigma{}_{|\alpha|\beta\gamma]} = 0. \quad (2.8)$$

The definition (2.1) leads to the following additional relations:

$$\tau_\mu R^\mu{}_{\alpha\beta\gamma} = 0, \quad (2.9)$$

$$h^{\mu(\alpha} R^{\beta)}{}_{\mu\gamma\sigma} = 0. \quad (2.10)$$

The Ricci tensor $R_{\alpha\beta}$ is defined as $R_{\alpha\beta} := R^\mu{}_{\alpha\mu\beta}$.

2.2.2 Coordinates in a Galilean spacetime

In this section, we define objects which will be used in the construction of the 1+3-Newton-Cartan equations.

2.2.2.1 Adapted coordinate systems

A coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$, which is associated with the coordinate basis vectors $\{\bar{\partial}_\alpha\}_{\alpha=0,1,2,3}$, is said to be adapted to the foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ if the three vectors $\{\bar{\partial}_a\}_{a=1,2,3}$ are spatial and $\bar{\partial}_0$ is unit timelike.¹ Such coordinates are not unique and are determined up to the spatial vector freedom in the definition of the unit timelike vector $\bar{\partial}_0$. In any adapted coordinate system

$$\tau_\alpha = \delta_\alpha^0 \quad ; \quad T^{\alpha_1 \dots \alpha_n} = T^{a_1 \dots a_n} \delta_{a_1}^{\alpha_1} \dots \delta_{a_n}^{\alpha_n}, \quad (2.11)$$

where \mathbf{T} is a spatial tensor.

2.2.2.2 Pull-back

The relation (2.11) shows that any spatial tensor \mathbf{T} is totally determined by its components $T^{a_1 \dots a_n}$ in an adapted coordinate system. We can then consider that $T^{a_1 \dots a_n}$ are the components of a tensor living in a Riemannian 3-manifold Σ whose metric contravariant components are h^{ab} , thus defining a pull-back $T^{\alpha_1 \dots \alpha_n} \rightarrow T^{a_1 \dots a_n}$.

As Σ is a Riemannian manifold, indices of tensor components on this manifold can be raised and lowered with the metric \mathbf{h} on Σ . Then, we define $T_{a_1 \dots a_n}$ as

$$T_{a_1 \dots a_n} := T^{c_1 \dots c_n} h_{c_1 a_1} \dots h_{c_n a_n}, \quad (2.12)$$

where h_{ab} is the inverse matrix of h^{ab} and corresponds to the covariant components of the Riemannian metric on Σ .

2.2.2.3 Classes of adapted coordinates

Given a unit timelike vector \mathbf{u} , one can characterise with respect to this vector any adapted system $\{\bar{\partial}_\alpha\}_{\alpha=0,1,2,3}$ by introducing a vector $\bar{\beta}$ as

$$\bar{\beta} := \bar{\partial}_0 - \mathbf{u}. \quad (2.13)$$

The vector $\bar{\beta}$ is spatial and is called the *shift vector* of the system $\{\bar{\partial}_\alpha\}_{\alpha=0,1,2,3}$ with respect to \mathbf{u} . The shift vector defines a class of adapted coordinate systems, denoted \mathcal{X}_β^u . This class is the set of all adapted coordinate systems whose shift vector with respect to \mathbf{u} is $\bar{\beta}$. The systems inside a class are related by time-independent spatial changes of coordinates.

¹Actually $\bar{\partial}_0$ only needs *not* to be spatial, i.e. timelike in Newton-Cartan. But by convention we take it to be unit timelike.

2.2.2.4 Spatial covariant derivative

In an adapted coordinate system, the spatial projection Γ_{ab}^γ of the connection coefficients (2.2) are

$$\Gamma_{ab}^\gamma = \delta_c^\gamma h^{cd} \left(\partial_{(a} h_{b)d} - \frac{1}{2} \partial_d h_{ab} \right), \quad (2.14)$$

for any unit timelike vector \mathbf{B} chosen in the relation (2.2). This comes from the fact that ${}^{\mathbf{B}}b_{ab} = h_{ab}$ for any unit timelike vector \mathbf{B} . The coefficients (2.14) correspond to the coefficients of the Levi-Civita connection \mathbf{D} of the metric \mathbf{h} on Σ . Then the pull-back of $h^{\beta\gamma} \nabla_\gamma T^{\alpha_1 \dots \alpha_n}$ on Σ with \mathbf{T} a spatial tensor gives

$$h^{\beta\gamma} \nabla_\gamma T^{\alpha_1 \dots \alpha_n} \rightarrow D^b T^{a_1 \dots a_n}, \quad (2.15)$$

and the pull-back of the divergence $\nabla_\gamma T^{\alpha_1 \dots \gamma \dots \alpha_n}$ gives

$$\nabla_\gamma T^{\alpha_1 \dots \gamma \dots \alpha_n} \rightarrow D_c T^{a_1 \dots c \dots a_n}. \quad (2.16)$$

2.3 The 1+3-Newton-Cartan equations

2.3.1 The Newton-Cartan equations

The Newton-Cartan (NC) equations are:

$$\nabla_\mu T^{\mu\alpha} = 0, \quad (2.17)$$

$$R_{\alpha\beta} = \tau_\alpha \tau_\beta (4\pi G \tau_\mu \tau_\nu T^{\mu\nu} - \Lambda), \quad (2.18)$$

$$h^{\mu[\alpha} R^{\beta]}_{(\gamma\sigma)\mu} = 0, \quad (2.19)$$

where Λ is the cosmological constant and $\bar{\mathbf{T}}$ is symmetric and corresponds to the energy-momentum tensor of the matter.

Equation (2.17) is the energy and momentum conservation; equation (2.18) is the equivalent to the Einstein equation and links the geometry of \mathcal{M} to its energy content; equation (2.19) is the Trautman-Künzle condition.

Remark. In all the literature, the condition (2.19) is called the Trautman condition, citing Trautman (1963).² However Trautman originally gave the condition $h^{\mu[\alpha} R^{\beta]}_{\gamma\sigma\mu} = 0$, i.e. without the symmetrisation. This original condition is stronger than (2.19). In particular it already implies the proportionality $R_{\alpha\beta} \propto \tau_\alpha \tau_\beta$, i.e. equation (2.18). Based on a count of the remaining degrees of freedom in the Riemann tensor, Künzle (1972) proposed instead the condition (2.19). It has the advantage of still implying the irrotationality of the gravitational field and the closeness of κ , i.e. the original reasons for the introduction of the condition by Trautman in 1963, but without the stronger proportionality implication. That is why I propose to call this condition the *Trautman-Künzle condition*.

²Paper written in French and available on the website of the Bibliothèque Nationale de France at the following web page: <https://gallica.bnf.fr/ark:/12148/bpt6k4007z/f639.image>.

2.3.2 Matter fluid

For the remainder of this paper, we will only consider that the energy-momentum tensor describes a matter fluid,³ the 4-velocity of which is a unit timelike vector $\bar{\mathbf{u}}$ such that:

$$T^{\alpha\beta} := \rho u^\alpha u^\beta + p h^{\alpha\beta} + 2q^{(\alpha} u^{\beta)} + \pi^{\alpha\beta}, \quad (2.20)$$

where ρ is the mass density, p the pressure, $\bar{\mathbf{q}}$ the heat flux and $\bar{\pi}$ the anisotropic stress of the fluid. By definition $q^\mu \tau_\mu = 0$, $\pi^{\mu\alpha} \tau_\mu = 0$ and ${}^u b_{\mu\nu} \pi^{\mu\nu} = 0$.

2.3.3 1+3 split in Newton-Cartan

The basics behind the 1+3 split of the NC equations is to decompose the Ricci tensor along and normal to the fluid velocity \mathbf{u} . This is similar to the 1+3-split in general relativity (section 1.1.3), but also to the 3+1 split (section 1.1.2) as we will be able to write the final equations on the 3-manifold Σ . To do so, we first introduce the kinematical variables of the fluid in section 2.3.3.1 and then realise the split in section 2.3.3.2.

2.3.3.1 The kinematical variables

Similar to general relativity, we introduce the expansion tensor $\bar{\Theta}$ and the vorticity tensor $\bar{\Omega}$ of the fluid as the projection orthogonal to the fluid of the 4-velocity gradient $\nabla \bar{\mathbf{u}}$:

$$\Theta^{\alpha\beta} := {}^u P_\nu^{(\beta} h^{\alpha)\mu} \nabla_\mu u^\nu \quad ; \quad \Omega^{\alpha\beta} := {}^u P_\nu^{[\beta} h^{\alpha]\mu} \nabla_\mu u^\nu, \quad (2.21)$$

with ${}^u P_\alpha^\beta := {}^u b_{\mu\alpha} h^{\beta\mu} = \delta_\alpha^\beta - \tau_\alpha u^\beta$. We denote $\theta := {}^u b_{\mu\nu} \Theta^{\mu\nu}$. We also introduce the acceleration ${}^u \bar{\mathbf{a}}$ of the fluid 4-velocity \mathbf{u} as

$${}^u a^\alpha := u^\mu \nabla_\mu u^\alpha. \quad (2.22)$$

The tensors ${}^u \mathbf{a}$, Θ and Ω are all spatial. We have the following additional relation

$$\Theta^{\alpha\beta} = -\frac{1}{2} \mathcal{L}_u h^{\alpha\beta}. \quad (2.23)$$

Remark. Relation (2.23) was originally introduced by Toupin (1957) as the definition for Θ . Note that the relation “ ${}^u b_{\alpha\gamma} {}^u b_{\beta\sigma} \Theta^{\gamma\sigma} = \frac{1}{2} \mathcal{L}_u {}^u b_{\alpha\beta}$ ” given by Künzle (1976) is incorrect.

³For an electromagnetic fluid in the NC theory, see Künzle (1976).

2.3.3.2 1+3-Newton-Cartan equations

In this section we project the NC equations (2.17)–(2.19) with respect to $\boldsymbol{\tau}$, \boldsymbol{u} , ${}^u\boldsymbol{b}$ and \boldsymbol{h} . The conservation equation (2.17) projected along ${}^u\boldsymbol{b}$ and $\boldsymbol{\tau}$ gives

$$\mathcal{L}_u \rho = -\rho\theta - \nabla_\mu q^\mu, \quad (2.24)$$

$$\rho^u a^\alpha = -h^{\mu\alpha} \nabla_\mu P - \nabla_\mu \pi^{\mu\alpha} - [\mathcal{L}_u q^\alpha + q^\alpha \theta + 2{}^u b_{\mu\nu} q^\mu (\Theta^{\nu\alpha} + \Omega^{\nu\alpha})]. \quad (2.25)$$

The NC equation (2.18) projected respectively twice along \boldsymbol{u} , along \boldsymbol{u} and \boldsymbol{h} , and twice along \boldsymbol{h} gives:

$$\mathcal{L}_u \theta = -4\pi G \rho + \Lambda + \nabla_\mu {}^u a^\mu - {}^u b_{\alpha\mu} {}^u b_{\beta\nu} \Theta^{\alpha\beta} \Theta^{\mu\nu} + {}^u b_{\alpha\mu} {}^u b_{\beta\nu} \Omega^{\alpha\beta} \Omega^{\mu\nu}, \quad (2.26)$$

$$h^{\mu\alpha} \nabla_\mu \theta - \nabla_\mu (\Theta^{\alpha\mu} + \Omega^{\alpha\mu}) = 0, \quad (2.27)$$

$$h^{\mu\alpha} h^{\nu\beta} R_{\mu\nu} = 0. \quad (2.28)$$

In the Trautman-Künzle condition (2.19), the indices α and β are purely spatial. Then only the indices γ and σ need to be projected along \boldsymbol{u} and \boldsymbol{h} . The condition projected respectively twice along \boldsymbol{u} , along \boldsymbol{u} and \boldsymbol{h} , and twice along \boldsymbol{h} gives:

$$\mathcal{L}_u \Omega^{\alpha\beta} = 4{}^u b_{\mu\nu} \Theta^{\mu[\alpha} \Omega^{\beta]\nu} + h^{\mu[\alpha} \nabla_\mu {}^u a^{\beta]}, \quad (2.29)$$

$$h^{\mu[\alpha} \nabla_\mu \Omega^{\beta\gamma]} = 0, \quad (2.30)$$

$$h^{\mu\gamma} h^{\nu\sigma} h^{\zeta[\alpha} R^{\beta]}_{(\mu\nu)\zeta} = 0. \quad (2.31)$$

For the system to be closed, the relation (2.23) needs to be added.

2.3.4 The equations

The 1+3-NC equations (2.23)–(2.31) are all scalar or spatial equations on \mathcal{M} . By pulling them back they become 3D-equations living on a Riemannian 3-manifold Σ . A pull-back is defined for each Σ_t , i.e. each time t . This implies that the geometrical properties (Riemann tensor and metric) of the manifold Σ , as long as all the other tensors defined on it, are parametrised by the time.

In section 2.2.2.4, we detailed the pull-back of the spatial derivative, but to fully write the 1+3-NC equations as 3D-equations, there remains to pull-back the operator \mathcal{L}_u present in the evolution equations. This is done by introducing a class \mathcal{X}_β^u of coordinates. Then \mathcal{L}_u applied on a spatial tensor \boldsymbol{T} becomes, under the pull-back,

$$\mathcal{L}_u T^{\alpha_1 \dots \alpha_n} \rightarrow (\partial_t - \mathcal{L}_\beta) T^{a_1 \dots a_n}, \quad (2.32)$$

where the Lie derivative $\mathcal{L}_\beta T^{a_1 \dots a_n}$ applied on the spatial components of a spatial tensor corresponds to the Lie derivative on Σ . Then, the pull-back of the system (2.23)–(2.31) on Σ gives:

The 1+3-Newton-Cartan equations:

- Evolution equations:

$$(\partial_t - \mathcal{L}_\beta) \rho = -\rho\theta - D_c q^c, \quad (2.33)$$

$$(\partial_t - \mathcal{L}_\beta) h_{ab} = 2 \Theta_{ab}, \quad (2.34)$$

$$(\partial_t - \mathcal{L}_\beta) \theta = -4\pi G\rho + \Lambda + D_c {}^u a^c - \Theta^{cd} \Theta_{cd} + \Omega^{cd} \Omega_{cd}, \quad (2.35)$$

$$(\partial_t - \mathcal{L}_\beta) \Omega_{ab} = D_{[a} {}^u a_{b]}, \quad (2.36)$$

- Constraint equations:

$$D_c (\Theta^{ac} + \Omega^{ac}) - D^a \theta = 0, \quad (2.37)$$

$$D_{[a} \Omega_{bc]} = 0, \quad (2.38)$$

$$\mathcal{R}^{ab} = 0, \quad (2.39)$$

$$\mathcal{R}^{d[abc]} = 0, \quad (2.40)$$

where \mathbf{Riem} and \mathcal{R} are the Riemann and Ricci curvature tensors of the spatial metric \mathbf{h} , and with

$$\rho {}^u a^a = -D^a P - D_c \pi^{ca} - [(\partial_t - \mathcal{L}_\beta) q^a + q^a \theta + 2q_c (\Theta^{ca} + \Omega^{ca})]. \quad (2.41)$$

Equation (2.40) is the Bianchi identity for \mathbf{Riem} , and thus is not a constraint. The shift vector β is not physical and corresponds to a choice of coordinates (as in 3+1-Einstein). In the section 2.6 we will see what choice of β leads to a Galilean coordinate system.

Remark. We show in Vignerón (2020), see appendix B, that even in the classical formulation of Newton's theory, the choice of time-parametrised coordinate systems is characterised by a vector, similar to a shift vector.

2.3.5 Discussion

In the 1+3-NC system, the expansion and vorticity tensors are not defined as the symmetric and antisymmetric gradient of a spatial vector. Instead, they are defined via the constraints (2.37) and (2.38). This is a major difference with the classical definition of Newton's equations (see section 1.4.1). In section 2.4.1, we will see what are the consequences for those two tensors.

The 1+3-NC system is nearly formally equivalent to the 1+3-Einstein system of equations (section 1.1.3.3). This was also spotted by the seminal paper on the comparison between Newton's theory and general relativity (Ellis, 1967). However in that paper, Θ and Ω were not defined via the constraints (2.37) and (2.38), but directly as the gradients of a spatial vector.

The main difference between the 1+3-NC and 1+3-Einstein systems is the missing of the 1+3-Ricci equation in the former. It is replaced by the flatness of space, i.e. equation (2.39). Vigneron (2020) showed that this results from the fact that this equation becomes a relation for the second order, in $1/c^2$, of the spatial curvature in the Newtonian limit (appendix C).

The 1+3-NC system does not suffer from the lack of the Hamilton constraint of general relativity. This is because this constraint is needed when only the 1+3-Ricci equation is considered, without the Raychaudhuri equation. As the latter equation is present, only the 1+3-Ricci equation is missing.

The term $D_c q^c$ in equation (2.33) implies that the mass of a fluid element is not conserved, but the total mass inside Σ is still conserved [see equation (2.60)]. Therefore this term should be zero if we want to recover the classical mass conservation equation. In this chapter we will keep it non-zero to keep the study of the NC-equations the most general possible. However, we will drop it in the next chapter. Also, in chapter 4, we will show that it still possible to consider a non-zero heat-flux \mathbf{q} while locally conserving the mass.

The 1+3-NC system does not feature a dependence on the choice of Galilean connection. This is discussed in section 2.6.4.

2.4 Space expansion in Newton-Cartan

In order to derive the expansion law (in section 2.4.2), we first need solve the constraint equations (2.37) and (2.38) (in section 2.4.1).

2.4.1 Solving the constraint equations

Equation (2.38) implies that the 2-form Ω is closed, which translates into

$$\Omega_{ab} = D_{[a} w_{b]} + \omega_{ab}, \quad (2.42)$$

where ω is a harmonic 2-form on Σ , i.e. $D_c D^c \omega_{ab} = 0$.

Because the metric \mathbf{h} on the 3-manifold Σ is flat [with equation (2.39)], we can use the decomposition theorem showed by Straumann (2008) to uniquely decompose the expansion tensor into scalar, vector and tensor parts (hereafter SVT decomposition) as

$$\Theta_{ab} = \chi h_{ab} + D_{(a} v_{b)} + \Xi_{ab}, \quad (2.43)$$

with χ a scalar field and Ξ is a transverse-traceless (TT) tensor, i.e. $\Xi_c^c := 0$ and $D_c \Xi^{ca} := 0$. The theorem is valid for a Riemannian metric of constant scalar curvature, with zero traceless Ricci curvature. Fall-off conditions at infinity or compactness of Σ also have to be added.

The link between the vectors in the expansion and vorticity tensors is made by the momentum constraint (2.37), which becomes, with the decomposition (2.43),

$$2D_a\chi = D^c \left(D_{[c}v_{a]} - D_{[c}w_{a]} \right), \quad (2.44)$$

using the fact that $D^c\omega_{ca} = 0$ for a harmonic 2-form. The right-hand side (hereafter rhs) is divergence free, whereas the left-hand side (hereafter lhs) is vorticity free. Then the Hodge decomposition implies that

$$D_a\chi = 0 \quad ; \quad D^c \left(D_{[c}v_{a]} - D_{[c}w_{a]} \right) = 0. \quad (2.45)$$

The first equation implies that χ is only a function of time. The second equation implies that the 2-form $(D_{[c}v_{a]} - D_{[c}w_{a]})$ is co-closed, but as it is also exact, it is zero. We finally have

$$\Theta_{ab} = \chi(t)h_{ab} + D_{(a}v_{b)} + \Xi_{ab} \quad ; \quad \Omega_{ab} = D_{[a}v_{b]} + \omega_{ab}. \quad (2.46)$$

We see that in general, the tensor Θ is not the gradient of a vector, but also features non-zero scalar and tensor parts. The same applies for the vorticity tensor which features a non-zero harmonic part ω . The physical role of these terms will be discussed in the next section.

Remark. From the NC equations (2.17)–(2.19), there are no more constraints on the harmonic 2-form ω . However, if one derives these equations from a limit of general relativity, an additional constraint appears on the Galilean connection, which eventually implies $\omega = 0$. Then, the only physical choice on ω that is compatible with general relativity is $\omega = 0$. This is shown in chapter 4. As we did not consider the NC theory as a limit of general relativity in the present chapter, we will keep $\omega \neq 0$.

2.4.2 Space expansion in the Newton-Cartan theory

The expansion rate $H(t)$ of Σ is defined as

$$H(t) := \frac{1}{3} \frac{\partial_t \mathcal{V}_\Sigma}{\mathcal{V}_\Sigma} = \frac{1}{3} \langle \theta \rangle_\Sigma(t), \quad (2.47)$$

where $\langle \bullet \rangle_\Sigma := \frac{1}{V_\Sigma} \int_\Sigma \bullet \sqrt{\det(h_{ab})} d^3x$ is the spatial average over the whole manifold Σ and $V_\Sigma := \int_\Sigma \sqrt{\det(h_{ab})} d^3x$ is the volume of Σ . Equation (2.46) implies

$$H(t) = \chi(t), \quad (2.48)$$

which means that χ corresponds to the expansion rate of Σ .

As we explained in the previous section, the scalar χ enters in the SVT decomposition of the expansion tensor Θ . This tensor, along with the vorticity tensor,

characterises the fluid. Then any part of the decomposition of Θ corresponds to a physical fundamental field characterising the fluid. In particular, equation (2.48) shows that *the expansion*, through χ , is a *fundamental physical field* of the 1+3-NC system (2.33)–(2.41).

This system does not explicitly feature an evolution equation for the scalar χ . Such an equation can be obtained by taking the spatial average of the Raychaudhuri equation (2.35), which gives

$$3[\dot{H} + H^2] = -4\pi G \langle \rho \rangle_\Sigma + \Lambda - \langle \Xi_{cd} \Xi^{cd} \rangle_\Sigma + \langle \omega_{cd} \omega^{cd} \rangle_\Sigma.$$

To get this equation we used the fact that divergences averaged over a closed domain are zero due to Stokes' theorem. This expansion law is equivalent to the one in Newtonian cosmology [equation (1.106)]: we retrieved the Buchert-Ehlers theorem.

While we have an evolution equation for χ , this is not the case for the TT tensor Ξ and the harmonic 2-form ω which are totally free, in their space and time dependence. We call Ξ the *transverse shear*, and ω the *Coriolis field*. The reason for this name will be given in section 2.6.3.

2.4.3 Spatial metric separability

The time dependence of the spatial metric is given by equation (2.34). Using the class of coordinates \mathcal{X}_{-v}^u , the equation becomes

$$\partial_t h_{ab} - 2H(t)h_{ab} = 2\Xi_{ab}, \quad (2.49)$$

which, in the case $\Xi = 0$, leads to

$$h_{ab}(t, x^i) = a^2(t) \tilde{h}_{ab}(x^i), \quad (2.50)$$

where \tilde{h} is a flat metric.

Equation (2.50) shows that the metric components can be separated into space and time dependence. This is a major property as it ensures that in the coordinates \mathcal{X}_{-v}^u , we can solve the 1+3-NC system assuming a time-independent background flat metric \tilde{h} , or comoving background metric.

In the case $\Xi \neq 0$, the equation (2.49) does not lead to the separation of the metric components. Because Ξ is a symmetric traceless-transverse tensor, it cannot be written as the symmetric gradient of a vector. This means that there is no coordinate system in which equation (2.34) becomes $\partial_{t|\beta} h_{ab} + 2H(t)h_{ab} = 0$. Therefore, the space and time separation of h is only possible if $\Xi = 0$.

2.5 Gravitational field and cosmological equations in Newton-Cartan

2.5.1 The gravitational field

In the literature concerning the Newton-Cartan theory, the gravitational field is often defined using the coefficients $\Gamma_{\alpha\beta}^\gamma$ of the Galilean connection as (e.g. Ehlers, 1981; Ehlers, 2019; Dautcourt, 1990a)

$${}^g g^a := \Gamma_{00}^a. \quad (2.51)$$

This definition is however only valid in a specific adapted coordinate system (defined by a time vector ${}^g \partial_t$) such that

$${}^g \partial_t := \mathbf{B},$$

where \mathbf{B} is the vector freedom in the definition of the Galilean connection of equation (2.2). The definition (2.51) implies that the gravitational field is the opposite of the 4-acceleration of \mathbf{B} , with $B^\mu \nabla_\mu B^\alpha = -g^\alpha$, and that it is solution of the cosmological Poisson equation (equation (16) in Dautcourt, 1990a).

In this chapter, we are interested in giving a purely coordinate independent definition of g . We propose the following definition:

$$g^a := (\partial_t - \mathcal{L}_\beta) v^a + 2v^c (\Theta_c^a + \Omega_c^a) - v^c D_c v^a - u^a. \quad (2.52)$$

In the following and in section 2.6.2 we will see that it is coherent with the standard definition (2.51).

This definition could be used as a covariant definition of the gravitational field in GR, as it uses the same kinematical variables Θ and Ω . However it requires the knowledge of the vector \mathbf{v} defined from Θ using the SVT decomposition. This decomposition is *a priori* not possible in general in GR as the curvature orthogonal to the fluid is not necessarily zero. Thus it seems non-trivial to adapt the definition (2.52) in this theory.

2.5.2 The cosmological equations

We can introduce the gravitational field (2.52) in the 1+3-NC system. The vorticity equation (2.36) becomes the Newton-Faraday equation, and the Raychaudhuri equation (2.35) becomes the Newton-Gauss equation when subtracting the expansion law. Then the 1+3-NC system can be rewritten in a form equivalent to the cosmological Newton system in section 1.4.3:

The cosmological system in Newton-Cartan: The gravitational field is given by

$$g^a = (\partial_t - \mathcal{L}_\beta) v^a + v^c D_c v^a + 2v^c (H\delta_c^a + \Xi_c^a + \omega_c^a) - (a_{\neq \text{grav}})^a, \quad (2.53)$$

$$D_c g^c = -4\pi G \widehat{\rho} - \widehat{\Xi_{cd}\Xi^{cd}} + \widehat{\omega_{cd}\omega^{cd}}, \quad (2.54)$$

$$D_{[a} g_{b]} = -(\partial_t - \mathcal{L}_{\beta+v}) \omega_{ab}, \quad (2.55)$$

where $\widehat{\psi} := \psi - \langle \psi \rangle_\Sigma$. These equations are completed by

$$\mathcal{R}_{ab} = 0, \quad (2.56)$$

$$(\partial_t - \mathcal{L}_\beta) h_{ab} = 2 \left(H(t) h_{ab} + D_{(a} v_{b)} + \Xi_{ab} \right), \quad (2.57)$$

$$(\partial_t - \mathcal{L}_\beta) \rho = -\rho (3H + D_c v^c), \quad (2.58)$$

and the averaged equations

$$3 \left(\dot{H} + H^2 \right) = -4\pi G \langle \rho \rangle_\Sigma + \Lambda + \mathcal{Q}_\Sigma, \quad (2.59)$$

$$\langle \rho \rangle_\Sigma = \frac{M_{\text{tot}}}{V_\Sigma(t)}, \quad (2.60)$$

with

$$\mathcal{Q}_\Sigma = - \left\langle \Xi_{cd} \Xi^{cd} \right\rangle_\Sigma + \left\langle \omega_{cd} \omega^{cd} \right\rangle_\Sigma. \quad (2.61)$$

We recall that Ξ is a traceless-transverse tensor and ω is a harmonic 2-form.

Remark. Equations (2.54) and (2.55) were also obtained from NC by Ehlers (1981); Ehlers (2019); Dautcourt (1990b). But these studies neither assume expansion, nor transverse shear and therefore do not have the term $\widehat{\Xi_{cd}\Xi^{cd}}$ and the averages given by the operator $\widehat{\bullet}$ in the gravitational field source equation (2.54).

2.5.3 Newton-Cartan cosmology VS Classical Newtonian cosmology

The expansion and vorticity tensors in the classical theory with homogeneous deformation can be written as

$$\Theta_{ab} = H(t) h_{ab} + D_{(a} P_{b)} + H_{(ab)}; \quad \Omega_{ab} = D_{[c} P_{a]} + H_{[ab]}.$$

These expressions are equivalent to those in NC [equation (2.46)], if we make the following associations:

$$\mathbf{P} \rightarrow \mathbf{v}; \quad H \rightarrow \chi; \quad H_{(ab)} \rightarrow \Xi_{ab}; \quad H_{[ab]} \rightarrow \omega_{ab}.$$

As H_{ab} is only a function of time in Cartesian coordinates, this implies: firstly, that $H_{(ab)}$ is divergence-free and falls into the class of TT-tensors, and secondly, that

$H_{[ab]}$ is harmonic. This justifies the last two associations. Furthermore, because the Newton-Gauss and Newton-Faraday equations are formally the same as in the classical Newtonian cosmology [equations (2.54) and (2.55) in NC; equations (1.103) and (1.104) in classical Newton], we can make the following association for the gravitational field:

$${}^P\mathbf{g} \rightarrow \mathbf{g}.$$

Then, with these associations, the equations governing \mathbf{P} , H_{ab} , and ${}^P\mathbf{g}$ are exactly the cosmological Newton-Cartan equations of section 2.5.2 in a closed space. This shows that the solutions to the classical cosmological Newton system are equivalent to the solutions of Newton-Cartan, *and* that the interpretation of the expanding periodic boundary conditions in terms of an expanding closed space is correct.

2.5.4 Interpretation of Ξ

2.5.4.1 Anisotropic expansion

From the association $H_{\langle ab \rangle} \rightarrow \Xi_{ab}$ we can interpret the transverse shear Ξ as an anisotropic expansion if its components are spatially constant in Cartesian coordinates. This requires the following condition: $D_c \Xi_{ab} = 0$. This is more restrictive than the TT condition of Ξ (see the study of the TT tensors in flat spaces by Tafel, 2018), which allows for non-constant components in Cartesian coordinates. For instance, we can assume that in the Galilean class of coordinates \mathcal{X}^u_v (see section 2.6.2) the metric can be written as

$$h_{ab} = \text{diag} \left[A^2(t); B^2(t); C^2(t) \right].$$

This metric represents an anisotropic expansion of Σ : the edges of the cubic domain (of the 3-torus) have different lengths and expands differently. From this metric we obtain

$$3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C},$$

$$\Xi_{ab} = \text{diag} \left[\frac{\dot{A}}{A} - H; \frac{\dot{B}}{B} - H; \frac{\dot{C}}{C} - H \right].$$

But while $H(t)$ has an evolution equation, the two degrees of freedom of Ξ are unconstrained.

Remark. The condition $D_c \Xi_{ab} = 0$ is equivalent to the one imposed on H_{ab} in classical Newtonian cosmology. However, we could also have defined the homogeneous deformation to be only divergence free, and not gradient free. In this case the Newtonian cosmology is equivalent to the NC cosmology.

2.5.4.2 Gravitational waves from Ξ

As Ξ is a TT tensor corresponding to the time derivative of the spatial metric, we could consider that it also encompasses gravitational waves phenomenon, which are described by the same kind of tensors. This is not incompatible with Newton's theory as shown by Ehlers (1997). However, while general relativity provides an evolution equation, in the form of a wave equation, for $\int_t \Xi$, this is not the case *a priori* in Newton-Cartan. Therefore we need to impose Ξ to represent a gravitational wave, for instance by taking (in the class \mathcal{X}_{-v}^u)

$$\Xi_{ab} \propto \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin[\omega(t - z/c)] \quad (2.62)$$

where a and b are constant amplitudes and ω is the pulsation frequency of the wave. This ansatz represents a plane wave propagating in the z -direction.

From equation (2.54), we see that even in vacuum, i.e. $\rho = 0$, a gravitational field is created by the gravitational wave due to the term $\widehat{\Xi_{cd}\Xi^{cd}}$:

$$D_c g^c = -\widehat{\Xi_{cd}\Xi^{cd}}. \quad (2.63)$$

This term corresponds to an effective energy density of the wave, similar to the one defined in general relativity.

Looking at the expansion law (2.59), the average energy density of the gravitational waves has a deceleration effect on the spatial expansion. It might be interesting to know how $\langle \Xi_{cd}\Xi^{cd} \rangle_\Sigma$ scales as function of the scale factor $a(t)$, to see if additional expansion scenarios, with respect to FLRW solutions, are possible.

2.5.4.3 Dark matter?

Following what we just said concerning the effect of Ξ on the gravitational field and the expansion, the transverse shear might play the role of dark matter:

- $\langle \Xi_{cd}\Xi^{cd} \rangle_\Sigma$ acts as an additional effective mass in the expansion law,
- $\widehat{\Xi_{cd}\Xi^{cd}}$ affects the gravitational field in the Newton-Gauss equation,
- a test particle undergoes an additional force due to the transverse shear (see section 2.6.3) of the form $'-2V^c + \Xi_c^a'$.

It remains to find if there exists a Ξ which reproduces the dark matter observations, and if it can be justified other than empirically.

2.6 Observers in the Newton-Cartan theory

2.6.1 General observers

An *observer*, or family of observers, in the NC theory is described by a unit timelike vector \mathbf{o} , as in general relativity. A choice of coordinates can be associated to a choice of observer whose unit timelike vector is the time basis vector, i.e. $\mathbf{o} = \partial_t$. In this sense, solving the NC equations (2.33)–(2.40) in such coordinates corresponds to solving the dynamics of the fluid \mathbf{u} with respect to the observer \mathbf{o} .

For a general observer $\mathbf{o} \neq \mathbf{u}$, and we define the spatial vector \mathbf{V} as

$$\mathbf{V} := \mathbf{o} - \mathbf{u} + \mathbf{v}. \quad (2.64)$$

We have $\mathbf{V} = \boldsymbol{\beta} + \mathbf{v}$. The acceleration ${}^o a$ of this observer can then be written as

$${}^o a = -g^a + (\partial_t - \mathcal{L}_{\boldsymbol{\beta} + \mathbf{v}}) V^a + V^c D_c V^a + 2V^c (\Theta_c^a + \Omega_c^a - D_c v^a), \quad (2.65)$$

which simplifies into

$${}^o a = -g^a + \partial_t V^a + V^c D_c V^a + 2V^c (\chi \delta_c^a + \Xi_c^a + \omega_c^a). \quad (2.66)$$

2.6.2 Galilean observers

When the observer is chosen such that $\mathbf{V} = 0$, then

$${}^o a = -g^a. \quad (2.67)$$

This corresponds to an observer whose acceleration is the opposite of the gravitational field created by the fluid \mathbf{u} . Such an observer is called a *Galilean observer*. A class of coordinates associated to a Galilean observer is called a *Galilean class*.

With the above definition, there seems to be a unique Galilean observer, i.e. the observer $\mathbf{o} = \mathbf{u} - \mathbf{v}$. However, this is true only if \mathbf{v} is unique from the knowledge of $\boldsymbol{\Theta}$. This is not the case as \mathbf{v} is defined up to a spatial vector \mathbf{A} whose spatial gradient $D_a A_b$ is zero. For a flat space, the solution to the equation $D_a A_b = 0$ is a ‘constant’ spatial vector, i.e. corresponding to a global translation. Thus \mathbf{v} is defined up to a global translation, and therefore a Galilean observer is also defined up to a global translation. This is an expected freedom of Galilean coordinates.

However once we choose the vector \mathbf{v} this fixes the Galilean observer. We denote this observer with the vector \mathbf{G} , where $\mathbf{G} := \mathbf{u} - \mathbf{v}$. Then the vector $\mathbf{V} = \mathbf{o} - \mathbf{G}$ of a general observer \mathbf{o} corresponds to its spatial velocity with respect to the Galilean observer \mathbf{G} . Therefore, the vector $\mathbf{v} = \mathbf{u} - \mathbf{G}$ is *the spatial velocity of the fluid with respect to the Galilean observer*. General and Galilean observers are represented with respect to the fluid in figure 2.1.

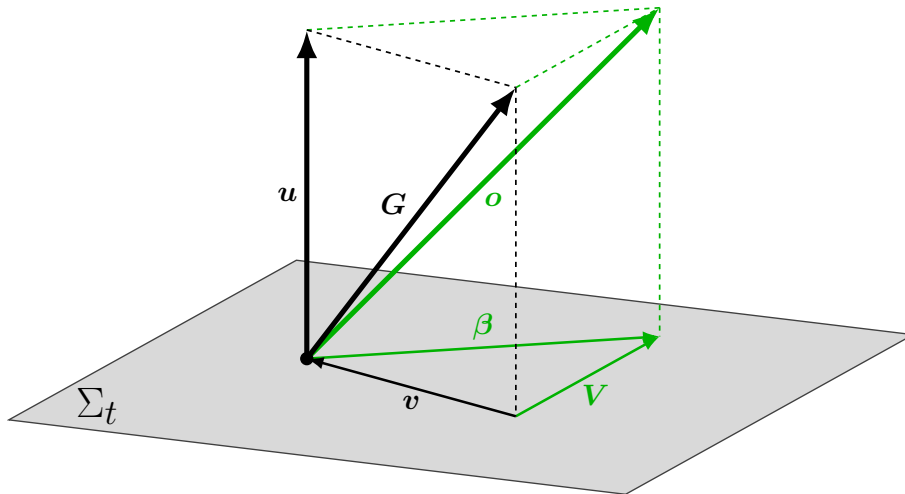


Figure 2.1: Representation of the different vectors involved in the definition of the fluid \mathbf{u} , the Galilean observer \mathbf{G} and a general observer \mathbf{o} . These vectors are represented with respect to a slice Σ_t of the foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$. In black are the vectors defining the fluid; in green are the vectors defining a general observer. Note that because there is no global metric in the structure of the embedding Galilean manifold \mathcal{M} , the orthogonality on the figure between the unit timelike vector \mathbf{u} and the slice Σ_t has no signification and is just a representation convention.

If the Galilean connection is chosen such that $\mathbf{B} := \mathbf{G}$, then in the coordinate class \mathcal{X}_{-v}^u , i.e. $\partial_t := \mathbf{G}$, we retrieve the usual definition (2.51) present in Ehlers (1981); Ehlers (2019); Dautcourt (1990a) for the gravitational field. Therefore, we can consider that equation (2.67) is also a definition of the gravitational field in Newton-Cartan. This definition only requires the SVT decomposition of Θ to be valid, i.e. so that we can uniquely define $D\mathbf{v}$ from Θ , and does not depend on the NC equations. This will be especially important in the next chapter, when defining non-Euclidean Newtonian theories.

2.6.3 Test observers

We define a *test observer* with unit timelike vector \mathbf{T} as a geodesic observer, i.e. ${}^T\mathbf{a} = 0$. The equation of motion of these observers with respect to the Galilean observer, i.e. the evolution equation for \mathbf{V} , is

$$\partial_t V^a + V^c D_c V^a = g^a - 2V^c (\chi \delta_c^a + \Xi_c^a + \omega_c^a). \quad (2.68)$$

This corresponds to the second law of Newton with the velocity acceleration on the lhs and three non-inertial terms on the rhs: $-2\chi V^a$, $-2V^c \Xi_c^a$ and $-2V^c \omega_c^a$. The first one corresponds to an expansion force; the second one to an anisotropic force resulting from the transverse shear; the third one is a Coriolis force created by the field ω , hence the name ‘Coriolis field’.

The term $-2V^c\omega_c^a$ corresponds to the Coriolis force created by a global rotation only if the Coriolis field components ω_{ab} are constants in Cartesian coordinates. As ω is harmonic along with compactness or fall-off conditions at infinity, this is the case. This confirms the result of [Dautcourt \(1990b\)](#). Note that [Ehlers \(1981\)](#); [Ehlers \(2019\)](#) does not suppose closing conditions, but instead adds an additional constraint to the NC system in order for ω to be a global rotation: the “law of existence of absolute rotation” (equation (23) in [Ehlers, 1981](#); [Ehlers, 2019](#)). However, this equation does not have a relativistic equivalent, and therefore cannot be obtained from a Newtonian limit.

We emphasise that while ω is called a Coriolis field, it differs from the Coriolis field introduced in the classical formulation of Newton’s theory, as the force $-2V^c\omega_c^a$ cannot be removed by a change of coordinates. This is because ω is harmonic and cannot be written as the anti-symmetric gradient of a spatial vector.

2.6.4 Choice of Galilean connection

Throughout this chapter we made no assumption on the choice of Galilean connection, i.e. choice of \mathbf{B} and $\boldsymbol{\kappa}$ in equation (2.2). However once we wrote the 1+3-NC equations and pulled them back on Σ , the Galilean connection disappears. Then the intrinsic freedom on the definition of this connection disappears too. As in most of the literature on NC, reasoning is often made with \mathbf{B} and $\boldsymbol{\kappa}$, we detail in the present section the relation between these two tensors and the kinematical variables.

We have

$$\Theta^{\alpha\beta} = h^{\mu(\alpha\mathbf{B}\nabla_{\mu}u^{\beta)}, \quad (2.69)$$

$$\Omega^{\alpha\beta} = h^{\mu[\alpha\mathbf{B}\nabla_{\mu}u^{\beta]} + h^{\mu\alpha}\kappa_{\mu\nu}h^{\beta\nu}, \quad (2.70)$$

$${}^u a^{\alpha} = u^{\mu\mathbf{B}\nabla_{\mu}u^{\alpha} + 2u^{\mu}\kappa_{\mu\nu}h^{\alpha\nu}. \quad (2.71)$$

If we choose $\mathbf{B} = \mathbf{u}$, then using equation (2.5) we have

$$\Theta^{\alpha\beta} = h^{\mu(\alpha\mathbf{u}\nabla_{\mu}u^{\beta)}, \quad (2.72)$$

$$\Omega^{\alpha\beta} = h^{\mu\alpha}\kappa_{\mu\nu}h^{\beta\nu}, \quad (2.73)$$

$${}^u a^{\alpha} = 2u^{\mu}\kappa_{\mu\nu}h^{\alpha\nu}. \quad (2.74)$$

In this choice of connection, we see that the spatial projection of $\boldsymbol{\kappa}$ is the vorticity of the fluid. Then this projection cannot be taken to zero as this would be a physical restriction to the fluid.

Only if $\mathbf{B} \neq \mathbf{u}$, one is allowed to take $h^{\mu\alpha}\kappa_{\mu\nu}h^{\beta\nu} = 0$ without loss of generality. But in any case, the tensors \mathbf{B} and $\boldsymbol{\kappa}$ do not appear in the 1+3-NC equations on Σ , and thus their choice, in addition to having no physical implications, is not relevant to the solving of these equations. Only the choice of adapted coordinates via $\boldsymbol{\beta}$, defining the partial time derivative, i.e. the choice of observer, plays a role in equations (2.33)–(2.41), or equivalently in the cosmological system (2.53)–(2.61).

2.7 Conclusion

This chapter aimed at presenting the equations resulting from a covariant 1+3-split of the Newton-Cartan equations, called the *1+3-Newton-Cartan equations* [equations (2.33)–(2.41)], along with the cosmological system (2.53)–(2.61), featuring the gravitational field. The main results are:

- (i) The 1+3-Newton-Cartan equations have the same algebraic structure as the 1+3-Einstein evolution *and* constraint equations (apart for the 1+3-Ricci equation). In particular, as in the relativistic theory, a choice of adapted coordinates in Newton-Cartan corresponds to a choice of shift vector.
- (ii) We give a covariant definition of the gravitational field [equation (2.52) in section 2.5.1],
- (iii) We exactly retrieve the cosmological Newton system [equations (2.53)–(2.61)].
- (iv) When solving the constraint equations, the space expansion arises as a fundamental physical field in the theory. It corresponds to the expansion of a real closed manifold. This contrasts with the Newtonian cosmology where the expansion is only a construction corresponding to the expansion of periodic boundary conditions, and the manifold is still \mathbb{R}^3 .
- (v) We retrieve the Buchert-Ehlers theorem in the Newton-Cartan theory, with the expansion law (2.59).
- (vi) We saw that Ξ can describe gravitational waves with an effective energy density $\widehat{\Xi_{cd}\Xi^{cd}}$.

These results show that the Newtonian cosmology and especially its interpretation in terms of a spatially closed manifold is compatible and equivalent to the Newton-Cartan theory, itself compatible with general relativity. Thus, we can consider that the 1+3-NC equations (or equivalently the cosmological equations) are the ‘true’ 3D-formulation of Newton’s theory. Then from now, we will refer to the Newton-Cartan cosmological system (2.53)–(2.61) when talking about Newtonian cosmology.

An interesting feature of the Newton-Cartan cosmology, which requires to be extensively studied, is the transverse shear Ξ , especially whether or not it can explain the dark matter problem.

In the next chapter, we will tackle the second point of our strategy presented in section 1.5: defining non-Euclidean-Newtonian theories.

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Chapter 3

On non-Euclidean Newtonian theories and their global backreaction

THE non-Euclidean Newtonian theory required to generalise the Buchert-Ehlers theorem needs to be derived from general relativity with a well defined limit: a non-Euclidean Newtonian limit. Finding such a limit is not an easy task as we will see in chapter 4. A preliminary approach to the definition of a NEN theory is to stay under the framework of the Newton-Cartan theory and assume we change the spatial curvature such that non-Euclidean geometries are allowed. While this approach is not justified from general relativity *a priori*, it gives a first insight in to what could be the right NEN theory.

In this chapter, we will first review the only approach of the literature aiming at defining a NEN theory (Roukema & Rózański, 2009; Barrow, 2020). Then we will propose new ones based on Künzle (1976) and the 1+3-Newton-Cartan equations developed in the previous chapter. The expansion law, and therefore, the generalised Buchert-Ehlers theorem, will be derived in each case. The results are summarised in table 3.1.

3.1 Introduction of a spatial Ricci curvature

As seen in chapter 2, the spatial geometry in Newton's theory is Euclidean because the spatial Ricci curvature \mathcal{R} is zero. Such a curvature is not the only possible one in Euclidean geometries, i.e. in an Euclidean 3-manifold it is possible to define a metric whose Ricci curvature is non-zero. However, the null curvature is the one which requires the less parameters to be determined, i.e. zero. As we want to introduce a non-zero Ricci curvature for our NEN theories, to face the problem of the choice of this curvature, we can assume that we take the one with the less parameters to determine.

In this thesis we will mostly be interested in either the spherical or hyperbolic geometries. Therefore, following the restriction we made above for the curvature choice, we assume that

$$\mathcal{R} := \frac{\mathcal{R}(t)}{3}\mathbf{h}. \quad (3.1)$$

We can see with the above formula that there is only one unknown parameter: the scalar curvature. It depends only on t as a consequence of the contracted Bianchi identity.

The main reasons to consider either spherical or hyperbolic geometries are the following:

- These are the ‘simplest’ geometries after the Euclidean geometry.
- They are the only geometries where the SVT decomposition of symmetric 2-tensors has been proven to be unique (the derivation of the theorem requires an isotropic Ricci curvature: [Straumann, 2008](#)). As seen in the previous chapter, this decomposition is required to define a spatial velocity \mathbf{v} from the expansion tensor Θ . This will also be the case in this chapter.

Remark. The ‘most simple’ Ricci tensor for the other five Thurston geometries can be found in section 5.1 of [Lachieze-Rey & Luminet \(1995\)](#). For these Ricci tensors, the SVT decomposition cannot be used *a priori*. However, it might be possible that a generalisation exists, especially for the Ricci curvatures proposed in [Lachieze-Rey & Luminet \(1995\)](#).

3.2 Poisson equation approach

In the NEN theory defined in [Roukema & Rózański \(2009\)](#) and [Barrow \(2020\)](#), the Newton-Gauss equation, or equivalently the Poisson equation, is considered as a fundamental feature of Newton’s theory. For this reason their approach is to keep the algebraic structure of this equation while changing the metric such that it has a non-zero Ricci curvature of the form (3.1). Therefore, they have:

$$D_c g^c = -4\pi G\rho \quad ; \quad \mathcal{R}_{ab} = \frac{\mathcal{R}(t)}{3}h_{ab}, \quad (3.2)$$

where the spatial connection \mathbf{D} is related to the metric with the above Ricci curvature.

Remark. This approach is similar to the way quantum fields are treated in curved spaces: the field equations are kept in their algebraic form, and the Minkowski metric is changed for a ‘curved metric’, not influenced by the fields.

3.2.1 The mass point gravitational field

The gravitational field created by one mass point M is given by the spherical solution of equation (3.2) with $\rho = \delta_\Sigma$, where δ_Σ is the Dirac distribution of the manifold Σ . The solution can be found when choosing hyperspherical coordinates, in which the line element takes the form:

$$dl^2 = R_c^2 \left[d\xi^2 + \text{sinn}^2(\xi) \left(d\theta^2 + \sin^2 \theta d\varphi \right) \right], \quad (3.3)$$

where

$$\text{sinn}(\xi) := \begin{cases} \sinh(\xi), & \text{if } \mathcal{R} < 0 \text{ (hyperbolic)} \\ \sin(\xi), & \text{if } \mathcal{R} > 0 \text{ (spherical)} \end{cases} \quad (3.4)$$

where $\xi \in [0, \pi]$ if $\mathcal{R} > 0$ and $\xi \in \mathbb{R}^+$ if $\mathcal{R} < 0$, and $R_c := \sqrt{6/|\mathcal{R}|}$ is the curvature radius of the spherical or hyperbolic space. In these coordinates, $r := R_c \xi$ corresponds to the distance to the origin $(0, 0, 0)$.

We suppose that the point mass is located at the origin. Then the gravitational field $g^a = (g^\xi, 0, 0)$ it creates is

$$g^\xi = \begin{cases} \frac{A}{R_c^2 \sinh^2(\xi)}, & \text{if } \mathcal{R} < 0 \text{ (hyperbolic)} \\ \frac{A}{R_c^2 \sin^2(\xi)}, & \text{if } \mathcal{R} > 0 \text{ (spherical)} \end{cases} \quad (3.5)$$

with A a constant. To be compatible with the (Euclidean) Newton theory close to the point mass, one must retrieve $g^\xi(\xi \sim 0) \simeq -GM/r^2$. Thus we need to take $A = -GM$.

We see that in the case of a spherical geometry, the gravitational field diverges at the point mass position ($\xi = 0$) as expected, but also at the pole opposed to this position ($\xi = \pi$). The field around this pole is given by:

$$g^\xi(\xi \sim \pi) = -\frac{GM}{R_c^2 (\pi - \xi)^2} + \mathcal{O}(1). \quad (3.6)$$

This field points in direction of the origin, i.e. outwards from the pole $\xi = \pi$, which means that this pole corresponds to a white hole, i.e. a repulsive gravitational field.

Remark. Because there are two divergences at $\xi = 0$ and $\xi = \pi$, the equation we solved is not the one with “ $\rho = \delta(\xi)$ ”, but with $\rho = \delta(\xi) - \delta(\xi - \pi)$.

3.2.2 Caveats of the approach

The NEN theory defined above has two major caveats:

- The Poisson equation (3.2) requires that the total mass must be zero in a closed space. This can be derived when averaging the equation over the whole space, and using Stoke's theorem. This explains why the point mass solution in spherical geometry (necessarily closed) has two diverging points: one corresponding to an attractive (positive) mass and one to a repulsive (negative) mass, such that the sum if these two masses gives zero. This solution is not physical as a spherical Universe described by this NEN theory would be full of white holes.
- The proposed NEN theory cannot account for the spatial expansion, which, as we saw in the previous chapter, should be a fundamental field present in the equations.

As we saw in the previous chapter, the cosmological and 1+3-NC systems both arise from a more fundamental formulation of the theory, where all the physics is put into a spacetime connection. Then it might seem more natural to start from this formulation to construct the NEN theory by a change of spatial curvature. This approach also ensures that the Galilean invariance of spacetime, which could be considered as a fundamental principle of Newton's theory, is kept. This was proposed by [Künzle \(1976\)](#). However, the expansion law, along with the full 3D-system of equations, were not derived. This is done in the next sections.

3.3 The Newton-Cartan approach

3.3.1 A curvature in Newton-Cartan

As we saw in chapter 2, the NC equation (2.18) implies that the spatial Ricci curvature is necessarily zero. To allow for $\mathcal{R} \neq 0$, [Künzle \(1976\)](#) proposed¹ to modify the NC equation (2.18) by adding a 'curvature term':

$$R_{\alpha\beta} - \frac{\mathcal{R}}{3} B_{\alpha\beta} = \tau_\alpha \tau_\beta (4\pi G \tau_\mu \tau_\nu T^{\mu\nu} - \Lambda), \quad (3.7)$$

where \mathbf{B} is a unit timelike vector. When projected twice along h^{ab} , this equation indeed leads to $\mathcal{R}^{ab} = \frac{\mathcal{R}}{3} h^{ab}$. This is the simplest modification of the NC equation allowing for a non-zero spatial Ricci curvature. In chapter 4, we will try to justify it from general relativity.

In the NC theory, any orthonormal projector $B_{\alpha\beta}$ can play the role of a twice covariant spatial metric. This is because in any adapted coordinate system we have $B_{ab} = h_{ab}$, with h_{ab} the inverse matrix of h^{ab} . That is why the additional term in equation (3.7) features $B_{\alpha\beta}$. As a consequence there is a freedom on the choice of the vector \mathbf{B} , i.e. on the choice of observer to which the curvature term is related.

¹Künzle claims to give references for the modification (3.7), but these are unrelated to this equation. We can therefore consider [Künzle \(1976\)](#) to be the first occurrence of this modification.

We can choose \mathbf{B} to be a ‘preferred’ unit timelike vector. Following the previous chapter, two such vectors can be chosen:

- $\mathbf{B} = \mathbf{u}$: the curvature term is related to the fluid observer.
- $\mathbf{B} = \mathbf{G}$: the curvature term is related to the Galilean observer. Its definition ($\mathbf{G} := \mathbf{u} - \mathbf{v}$, see section 2.6.2) only requires that Θ has a SVT decomposition (so that the spatial velocity \mathbf{v} can be uniquely defined from Θ), which is the case as the spatial Ricci curvature is of the form $\mathcal{R} = \frac{\mathcal{R}}{3}\mathbf{h}$.

In the following sections we will develop the NEN theory in both cases: we will derive the 1+3-system (similarly as in chapter 2), and the cosmological system, i.e. featuring the gravitational field and the expansion law. The backreaction will be given, thus generalising the Buchert-Ehlers theorem in both cases.

3.3.2 NEN theory I: Fluid observer curvature term

In this section, we consider $\mathbf{B} := \mathbf{u}$.

3.3.2.1 The system of equations

The derivation of the 3D-system of equations is done by projecting the modified NC equation with respect to \mathbf{u} and \mathbf{h} . We will not detail its full derivation, which is similar to the one made in section 2.3.3. The main difference comes from the projections of the additional curvature term in the NC equation. We have

$$\frac{\mathcal{R}}{3} u_{\mu\nu} h^{a\mu} h^{b\nu} = \frac{\mathcal{R}}{3} h^{ab} \quad ; \quad \frac{\mathcal{R}}{3} u_{\mu\nu} u^{\mu} h^{a\nu} = 0 \quad ; \quad \frac{\mathcal{R}}{3} u_{\mu\nu} u^{\mu} u^{\nu} = 0. \quad (3.8)$$

Then the 3D-system of equations corresponds to: the following evolution equations

$$(\partial_t - \mathcal{L}_{\beta}) \rho = -\rho\theta - D_c q^c, \quad (3.9)$$

$$(\partial_t - \mathcal{L}_{\beta}) h_{ab} = 2\Theta_{ab}, \quad (3.10)$$

$$(\partial_t - \mathcal{L}_{\beta}) \theta = -4\pi G\rho + \Lambda + D_c u^a{}^c - \Theta^{cd}\Theta_{cd} + \Omega^{cd}\Omega_{cd}, \quad (3.11)$$

$$(\partial_t - \mathcal{L}_{\beta}) \Omega_{ab} = D_{[a} u_{b]}, \quad (3.12)$$

and constraint equations

$$D_c (\Theta^{ac} + \Omega^{ac}) - D^a \theta = 0, \quad (3.13)$$

$$D_{[a} \Omega_{bc]} = 0, \quad (3.14)$$

$$\mathcal{R}^{ab} = \frac{\mathcal{R}}{3} h^{ab}, \quad (3.15)$$

$$\mathcal{R}^{d[abc]} = 0, \quad (3.16)$$

with

$$\rho {}^u a^a = -D^a P - D_c \pi^{ca} - [(\partial_t - \mathcal{L}_\beta) q^a + q^a \theta + 2q_c (\Theta^{ca} + \Omega^{ca})]. \quad (3.17)$$

We recall that ${}^u \mathbf{a}$ plays the role of $\mathbf{a}_{\neq \text{grav}}$.

The only difference with the (Euclidean) 1+3-NC equations (2.33)–(2.41) is the curvature equation (3.15).

3.3.2.2 Solving the constraint equations

While the constraint equations are algebraically the same as in the Euclidean case, because the spatial curvature is not zero anymore, their solution is different. The constraint (3.14) still implies that Ω can be written as

$$\Omega_{ab} = D_{[a} w_{b]} + \omega_{ab}, \quad (3.18)$$

with ω a harmonic 2-form. In the case of a spherical geometry, ω is necessarily zero.

The Ricci curvature (3.15) still allows us to use the Straumann theorem (Straumann, 2008), implying that the SVT decomposition (2.43) for Θ is still valid. Then the momentum constraint implies

$$D^c \Delta \Omega_{ca} - 2D_a \chi - v_a \frac{\mathcal{R}}{3} = 0, \quad (3.19)$$

where $\Delta \Omega_{ab} := D_{[a} w_{b]} - D_{[a} v_{b]}$. We introduce the Hodge decomposition on \mathbf{v} : $\mathbf{v} = \mathbf{D}\Phi_v + \text{curl}\mathbf{A} + \boldsymbol{\lambda}$, where Φ_v and \mathbf{A} are respectively the scalar potential and the vector potential of \mathbf{v} , and $\boldsymbol{\lambda}$ is a harmonic 1-form. As the term $D_a \chi$ is vorticity-free and the term $D^c \Delta \Omega_{ca}$ is divergence-free, we have

$$D_a \chi = D_a \left(\frac{\mathcal{R}}{6} \Phi_v \right), \quad (3.20)$$

$$D^c \Delta \Omega_{ca} = \frac{\mathcal{R}}{3} \left(\epsilon_{acd} D^c A^d + \lambda_a \right). \quad (3.21)$$

We see that, contrary to the Euclidean case, the first equation (3.20) does not imply anymore that χ is only a function of time. Instead we have

$$\chi(t, x^i) = cst(t) + \frac{\mathcal{R}(t)}{6} \Phi_v(t, x^i). \quad (3.22)$$

This can be interpreted as *an inhomogeneous volume expansion*. The second equation (3.21) shows that $\mathbf{v} \neq \mathbf{w}$ in general. This is a radical difference with the Euclidean case: the vectors inside the expansion and vorticity tensors are different. We finally have

$$\Theta_{ab} = \chi(t, x^i) h_{ab} + D_{(a} v_{b)} + \Xi_{ab} \quad ; \quad \Omega_{ab} = D_{[c} v_{a]} + \Delta \Omega_{ab} + \omega_{ab}. \quad (3.23)$$

3.3.2.3 The cosmological system

The system of equations (3.9)–(3.17) of the present NEN theory can be rewritten to feature the gravitational field and the analogue to the Newton-Gauss equation. The gravitational field is still defined as the opposite to the 4-acceleration of the Galilean observer and is given by the formula (2.52). Then, accounting for the expression of Θ and Ω in equation (3.23), the cosmological system takes the form (defining $\mathcal{T} := {}^\Delta\Omega + \omega$):

Non-Euclidean-Newtonian theory I: The gravitational field is given by

$$g^a = (\partial_t - \mathcal{L}_\beta) v^a + v^c D_c v^a + 2v^c (\chi \delta_c^a + \Xi_c^a + \mathcal{T}_c^a) - (a_{\neq \text{grav}})^a, \quad (3.24)$$

$$D_c g^c = -4\pi G \rho - \Xi_{cd} \Xi^{cd} + \mathcal{T}_{cd} \mathcal{T}^{cd} - \frac{\mathcal{R}}{3} v_c v^c - 3 \left[(\partial_t - \mathcal{L}_\beta) \chi + \chi^2 - v^c D_c \chi \right], \quad (3.25)$$

$$D_{[a} g_{b]} = -(\partial_t - \mathcal{L}_{\beta+v}) \mathcal{T}_{ab}, \quad (3.26)$$

with

$$\chi = cst(t) + \frac{\mathcal{R}(t)}{6} \Phi_v, \quad (3.27)$$

$$D^c \mathcal{T}_{ca} = \frac{\mathcal{R}}{3} (V_a - D_a \Phi_v). \quad (3.28)$$

where Φ_v is the scalar potential of \mathbf{v} , and \mathcal{T} a closed 2-form. These equations are completed by

$$\mathcal{R}_{ab} = \frac{\mathcal{R}(t)}{3} h_{ab}, \quad (3.29)$$

$$(\partial_t - \mathcal{L}_\beta) h_{ab} = 2 (\chi h_{ab} + D_{(a} v_{b)} + \Xi_{ab}), \quad (3.30)$$

$$(\partial_t + \mathcal{L}_\beta) \rho = -\rho (3H + D_c v^c), \quad (3.31)$$

and the averaged equations

$$3 (\dot{H} + H^2) = -4\pi G \langle \rho \rangle_\Sigma + \Lambda + \mathcal{Q}_\Sigma, \quad (3.32)$$

$$\langle \rho \rangle_\Sigma = \frac{M_{\text{tot}}}{V_\Sigma(t)}, \quad (3.33)$$

with $H(t) := \langle \chi \rangle_\Sigma$ and where

$$\mathcal{Q}_\Sigma = - \langle \Xi_{cd} \Xi^{cd} \rangle_\Sigma + \langle \mathcal{T}_{cd} \mathcal{T}^{cd} \rangle_\Sigma - \frac{\mathcal{R}}{3} \langle v_c v^c \rangle_\Sigma + \frac{\mathcal{R}}{6} \left(\langle \Phi_v^2 \rangle_\Sigma - \langle \Phi_v \rangle_\Sigma^2 \right). \quad (3.34)$$

3.3.2.4 Gravitomagnetism

In the above system we introduced the closed 2-form $\mathcal{T} := {}^\Delta\Omega + \omega$. This field corresponds to a gravitomagnetic field which Newton-Faraday equation is equation (3.26) and which Newton-Ampère equation is equation (3.28). The later does not feature any displacement current. The analogue to the current density is $J_a := \frac{\mathcal{R}}{3}(V_a - D_a\Phi_v)$.

3.3.2.5 Expansion law or generalised Buchert-Ehlers theorem

The backreaction term (3.34) in the expansion law (3.32) features additional terms with respect to the Euclidean case. These terms depend on the scalar curvature, the velocity \mathbf{v} , and the gravitomagnetic field \mathcal{T} . This shows that the backreaction depends on the type of geometry, via \mathcal{R} , but also on the dynamical properties of the fluid, via \mathbf{v} and \mathcal{T} . Especially the term $-\frac{\mathcal{R}(t)}{3}\langle v^c v_c \rangle_\Sigma$ shows that the total bulk velocity of the fluid affects the backreaction.

We expect this bulk velocity to be more important in the late Universe when structure formation increases. Then, the formula (3.34) suggests that the structure formation might play a major role on the global expansion of our Universe if the geometry is non-Euclidean. It also shows that the Universe might have a local Newtonian dynamics, but a global dynamics which differs from the one of an homogeneous Universe. However, quantifying the bulk velocity remains difficult because it depends on the mesoscopic scale, which is not well defined in cosmology.

3.3.2.6 Space–time separation of the spatial metric?

Because χ depends on time and space, there does not exist a coordinate system in which the spatial metric components take the form (2.50), even if $\Xi = 0$. Instead we can only have the conformal form (which is valid in the Galilean coordinates: $\beta = -\mathbf{v}$):

$$h_{ab}(t, x^i) = \psi^2(t, x^i)\tilde{h}_{ab}(x^i), \quad (3.35)$$

where $\partial_t\psi/\psi := \chi$. We can still define a background metric $\tilde{\mathbf{h}}$ independent of time, but its Ricci curvature tensor $\tilde{\mathcal{R}}$ has a traceless part, and its scalar curvature $\tilde{\mathcal{R}}$ is not constant anymore (see the conformal relation (7.42) in Gourgoulhon, 2012). Such a feature makes the equation hardly tractable, which is something we don't expect for a (non-Euclidean) Newtonian theory.

The only way to have a complete separation is to suppose $D\Phi_v = 0$. But in that case the vector \mathbf{v} , which is interpreted as the spatial velocity of the fluid, is purely rotational. Therefore it is difficult to consider that the proposed NEN theory can be called 'Newtonian'.

3.3.2.7 Advantages and caveats of the approach

The advantages are:

- The approach is based on the NC formulation and a minimal modification of the NC equation.
- The additional curvature term in the NC equation is defined with respect to the fluid 4-velocity.

The caveats are:

- There is a gravitomagnetic field which cannot be taken to be zero without loss of generality on \mathbf{v} .
- The vectors inside Θ and Ω are different.
- The spatial metric cannot be separated in space and time without loss of generality on \mathbf{v} . No ‘simple’ background metric can be defined, and the system of equations is hardly tractable.
- There is no N-body description, because the Newton-Gauss equation (3.25) is not linear and does not only feature the gravitational field, but also the velocity.

3.3.3 NEN theory II: Galilean observer curvature term

In this section, we consider $\mathbf{B} := \mathbf{G}$.

3.3.3.1 The system of equations

The system of 3D-equations is obtained by projecting the modified NC equation with respect to \mathbf{u} and \mathbf{h} . The main difference comes from the projections of the additional curvature term in the NC equation. We have

$$\frac{\mathcal{R}}{3} G_{b\mu\nu} h^{a\mu} h^{b\nu} = \frac{\mathcal{R}}{3} h^{ab} \quad ; \quad \frac{\mathcal{R}}{3} G_{b\mu\nu} u^\mu h^{a\nu} = \frac{\mathcal{R}}{3} v^a \quad ; \quad \frac{\mathcal{R}}{3} G_{b\mu\nu} u^\mu u^\nu = \frac{\mathcal{R}}{3} v^\mu v^\nu G_{b\mu\nu}. \quad (3.36)$$

Then the 3D system of equations corresponds to: the following evolution equations

$$(\partial_t - \mathcal{L}_\beta) \rho = -\rho\theta - D_c q^c, \quad (3.37)$$

$$(\partial_t - \mathcal{L}_\beta) h_{ab} = 2\Theta_{ab}, \quad (3.38)$$

$$(\partial_t - \mathcal{L}_\beta) \theta = -4\pi G\rho + \Lambda + D_c {}^u a^c - \Theta^{cd}\Theta_{cd} + \Omega^{cd}\Omega_{cd} - \frac{\mathcal{R}}{3} v^c v_c, \quad (3.39)$$

$$(\partial_t - \mathcal{L}_\beta) \Omega_{ab} = D_{[a} {}^u a_{b]}, \quad (3.40)$$

and constraint equations

$$D_c (\Theta^{ac} + \Omega^{ac}) - D^a \theta = \frac{\mathcal{R}}{3} v_a, \quad (3.41)$$

$$D_{[a} \Omega_{bc]} = 0, \quad (3.42)$$

$$\mathcal{R}^{ab} = \frac{\mathcal{R}}{3} h^{ab}, \quad (3.43)$$

$$\mathcal{R}^{d[abc]} = 0, \quad (3.44)$$

with

$$\rho^u a^a = -D^a P - D_c \pi^{ca} - [(\partial_t - \mathcal{L}_\beta) q^a + q^a \theta + 2q_c (\Theta^{ca} + \Omega^{ca})]. \quad (3.45)$$

With respect to the first proposed NEN theory, this system of equations features additional term sin the momentum constraint (3.41) and in the Raychaudhuri equation (3.39). These terms will radically change the solutions to the constraint equations and the expansion law, as presented in the following section.

3.3.3.2 Solving the constraint equations

Following the same logic than in section 3.3.2.2 for the first NEN theory, the momentum constraint implies

$$D_a \chi = 0, \quad (3.46)$$

$$D^c \Delta \Omega_{ca} = 0. \quad (3.47)$$

This is equivalent to the Euclidean case in chapter 2. Therefore χ depends only on t , and $\Delta \Omega = 0$. We finally have

$$\Theta_{ab} = H(t) h_{ab} + D_{(a} v_{b)} + \Xi_{ab} \quad ; \quad \Omega_{ab} = D_{[c} v_{a]} + \omega_{ab}, \quad (3.48)$$

with $H := \langle \chi \rangle_\Sigma = \chi$.

3.3.3.3 The cosmological system

The cosmological system is obtained when rewriting the system (3.9)–(3.17) as function of the gravitational field, and accounting for the expression of Θ and Ω in equation (3.48):

Non-Euclidean-Newtonian theory II: The gravitational field is given by

$$g^a = (\partial_t - \mathcal{L}_\beta) v^a + v^c D_c v^a + 2v^c (H \delta_c^a + \Xi_c^a + \omega_c^a) - (a_{\neq \text{grav}})^a, \quad (3.49)$$

$$D_c g^c = -4\pi G \widehat{\rho} - \widehat{\Xi_{cd} \Xi^{cd}} + \widehat{\omega_{cd} \omega^{cd}}, \quad (3.50)$$

$$D_{[a} g_{b]} = -(\partial_t - \mathcal{L}_{\beta+v}) \omega_{ab}. \quad (3.51)$$

These equations are completed by

$$\mathcal{R}_{ab} = \frac{\mathcal{R}(t)}{3} h_{ab}, \quad (3.52)$$

$$(\partial_t - \mathcal{L}_\beta) h_{ab} = 2 \left(H(t) h_{ab} + D_{(a} v_{b)} + \Xi_{ab} \right), \quad (3.53)$$

$$(\partial_t + \mathcal{L}_\beta) \rho = -\rho (3H + D_c v^c), \quad (3.54)$$

and the averaged equations

$$3 \left(\dot{H} + H^2 \right) = -4\pi G \langle \rho \rangle_\Sigma + \Lambda - \langle \Xi_{cd} \Xi^{cd} \rangle_\Sigma + \langle \omega_{cd} \omega^{cd} \rangle_\Sigma, \quad (3.55)$$

$$\langle \rho \rangle_\Sigma = \frac{M_{\text{tot}}}{V_\Sigma(t)}, \quad (3.56)$$

where we recall that β corresponds to the choice of coordinates (see section 2.2.2.3). Ξ is a traceless-transverse tensor and ω is a harmonic 2-form.

3.3.3.4 Expansion law or generalised Buchert-Ehlers theorem

The expansion law is the same as for the (Euclidean) Newton theory [equation (2.59)]. Therefore, the same conclusions apply: the expansion law for $\Xi = 0 = \omega$ is given by the acceleration Friedmann equation (1.64), and this for any inhomogeneous solutions. This means that there is no global backreaction of the inhomogeneities on the isotropic expansion. If $\Xi \neq 0$ and $\omega \neq 0$, the backreaction is

$$\mathcal{Q}_\Sigma = - \langle \Xi_{cd} \Xi^{cd} \rangle_\Sigma + \langle \omega_{cd} \omega^{cd} \rangle_\Sigma. \quad (3.57)$$

The fields Ξ and ω are totally free, and especially they are decoupled from the fluid spatial velocity \mathbf{v} , so unless there is a hidden (from general relativity for instance) condition relating these tensors to \mathbf{v} , it is not clear to me if we can consider that equation (3.57) corresponds to a physical backreaction. Still a discussion on the role(s) of Ξ is made in section 2.5.4.

If this NEN theory is the right one, then an Euclidean, spherical or hyperbolic universe, the dynamics of which is locally Newtonian, has no global backreaction. For such a universe dark energy cannot be described by the global backreaction.

Remark. In equation (3.57), the backreaction \mathcal{Q}_Σ is defined as the deviation with respect to the Friedmann equation: i.e. $\mathcal{Q}_\Sigma := 3 \left(\dot{H} + H^2 \right) + 4\pi G \langle \rho \rangle_\Sigma - \Lambda$. Because the Raychaudhuri equation (3.39) features an additional term, it is not algebraically equivalent to the Raychaudhuri equation (2.35) in (Euclidean) Newton's theory, and thus the formula (1.108) is not valid anymore. Instead the definition $\mathcal{Q}_\Sigma := 3 \left(\dot{H} + H^2 \right) + 4\pi G \langle \rho \rangle_\Sigma - \Lambda$ leads to

$$\mathcal{Q}_\Sigma := \frac{2}{3} \left(\langle \theta^2 \rangle_\Sigma - \langle \theta \rangle_\Sigma^2 \right) - \langle \sigma_{cd} \sigma^{cd} \rangle_\Sigma + \langle \Omega_{cd} \Omega^{cd} \rangle_\Sigma - \frac{\mathcal{R}}{3} \langle v^c v_c \rangle_\Sigma.$$

3.3.3.5 Space–time separation of the spatial metric

Taking equation (3.53) in the class of coordinates where $\beta = -\mathbf{v}$ (Galilean coordinates), the equation becomes

$$\partial_{t|_{-\mathbf{v}}} h_{ab} - 2H(t)h_{ab} = 2\Xi_{ab}, \quad (3.58)$$

which, in the case $\Xi = 0$, leads to

$$h_{ab}(t, x^i) = a^2(t)\tilde{h}_{ab}(x^i), \quad (3.59)$$

where the Ricci curvature tensor $\tilde{\mathcal{R}}$ related to the metric $\tilde{\mathbf{h}}$ is $\tilde{\mathcal{R}} = \frac{\mathcal{R}_i}{3}\tilde{\mathbf{h}}$.

Equation (3.59) shows that the metric components can be separated into space and time dependence, which was also the case in (Euclidean) Newton's theory. This is a major property as it ensures that in the coordinates $\beta = -\mathbf{v}$, we can solve the system (3.52)–(3.56) assuming a time-independent background metric $\tilde{\mathbf{h}}$.

In the case $\Xi \neq 0$, the equation (3.58) does not lead to the separation of the metric components. Because Ξ is a symmetric traceless-transverse tensor, it cannot be written as the symmetric gradient of a vector. This means that there is no coordinate system in which equation (3.53) becomes $\partial_{t|\beta} h_{ab} + 2H(t)h_{ab} = 0$. Therefore, the space and time separation of \mathbf{h} is only possible if $\Xi = 0$.

3.3.3.6 N-body description

A major strength of Newton's theory is the fact that its system of equations can be written either as a partial differential system (fluid description), or as an ordinary differential system (particle description). This results from the linearity of the Newton-Gauss equation. In the present proposed NEN theory, the same applies: the particle description is obtained when solving equation (3.50) for $\rho = \delta_\Sigma$. Then the second law of Newton is retrieved when introducing a particle position $y^a(t)$ and writing $v^a = \dot{y}^a$: equation (3.49) becomes

$$\ddot{y}^a + 2\dot{y}^a (H\delta_c^a + \Xi_c^a + \omega_c^a) = g_{\text{tot}}^a + (a_{\neq\text{grav}})^a, \quad (3.60)$$

where \mathbf{g}_{tot} is the gravitational field created by the set of particles.

3.3.3.7 The mass point gravitational field

The N-body description requires the mass point solution of the cosmological Newton-Gauss equation which corresponds to the solution of

$$D_c g^c = -4\pi G \left[\delta_\Sigma - \frac{M}{V_\Sigma} \right], \quad (3.61)$$

where V_Σ is the volume of the spatial manifold and M the mass of the Dirac field. We assumed $\Xi = 0 = \omega$. Choosing the hyperspherical coordinates (3.3), and the particle placed at the origin, we obtain

$$g^\xi = \begin{cases} -\frac{GM}{R_c^2 \sin^2(\xi)}, & \text{if } \mathcal{R} < 0 \text{ (hyperbolic)} \\ -\frac{GM}{\pi R_c^2} \left[\cot \xi + \frac{\pi - \xi}{\sin^2 \xi} \right], & \text{if } \mathcal{R} > 0 \text{ (spherical)} \end{cases} \quad (3.62)$$

with $R_c(t) = a(t)R_{c,i}$.

In the case of a spherical geometry, the behaviour at the poles is

$$g^\xi(\xi \sim 0) = -\frac{GM}{R_c^2 \xi^2} - \frac{GM}{3R_c^2} + \mathcal{O}(\xi), \quad (3.63)$$

$$g^\xi(\xi \sim \pi) = -\frac{2GM}{3\pi R_c^2} (\pi - \xi) + \mathcal{O}[(\pi - \xi)^3]. \quad (3.64)$$

We see that the first caveat of the proposition of Roukema & Rózański (2009) and Barrow (2020) is solved, i.e. the field at the opposite pole $\xi = \pi$ does not diverge anymore. It is still a repulsive field, but which tends to zero when $\xi \rightarrow \pi$. This is physically coherent.

The second term in the rhs of equation (3.63) is an additional term with respect to Newton's gravitational law. It is called *topological acceleration*. Such a zeroth order term is also present in the (Euclidean) Newton theory (but in a different form) in the case Σ is a 3-torus (Roukema et al., 2007). Therefore it is not necessarily linked to non-Euclidean geometries, but also to non-trivial topologies.

Remark. It is also possible to consider a mass point at the center of a spherical multi-connected topology. This was done in Roukema & Rózański (2009) for the topologies which fundamental domains are platonic solids. The method is to consider that the 3-sphere is tilted with the fundamental domain of the topology, i.e. the covering space representation (see section 1.2.4.2 and figure 1.4). Then, as the Newton-Gauss equation is linear, the gravitational field is given by the sum of the spherical mass point solution (3.62) over all the tilling mass points. However, if one wants to use this method for closed hyperbolic geometries, then the solution is not a sum of the first equation in (3.64), because $M = 0$ for this solution.

3.3.3.8 Advantages and caveats of the approach

The advantages are:

- The approach is based on the NC formulation and a minimal modification of the NC equation.

- As for the Euclidean case in section 2.4.3, equation (3.53) in Galilean coordinates, along with $\Xi = 0$, leads to the space-time separation of the spatial metric.
- N-body description is possible.
- There is no gravitomagnetic field if $\omega = 0$, which is expected from general relativity.

The caveats are:

- The additional curvature in the NC equation is not defined with respect to the fluid 4-velocity: the starting point is less ‘natural’ (but the final form of the cosmological system is more ‘natural’).

3.4 Lorentzian manifold from a solution of the Newton’s equations

This section presents the results of the second part of my paper [Vigneron \(2020\)](#). This study is slightly on the margin of the main research line of this thesis. It was a first stage in the construction of the NEN theory. However, it was a dead-end and failed at giving this theory. Furthermore, the study was done before the development of the 1+3-NC equations, and therefore only considers the classical Newton equations. While the results presented in this study are still correct, the approach became obsolete compared to the one presented in chapters 2 and 3 which uses the Newton-Cartan theory and Galilean spacetimes. For these reasons, in this section, I will only summarise the methods and results of this study. The full paper is put in appendices B and C.

3.4.1 The main results

In chapter 2 and in appendix B, we showed that Newton’s equations are (nearly) algebraically equivalent with the 1+3-Einstein equations. Furthermore, the degrees of freedom on the choice of spatial coordinates is also equivalent: it depends on a vector, called the shift vector. Using these similarities between Newton’s theory and general relativity, we show in section C.1 that we can construct a Lorentzian manifold \mathcal{M}^N equipped with a Lorentzian structure (\mathbf{g}, ∇) from a solution of the classical Newton equations. This is performed with a push-forward of the 1+3-Newton equations, which is the reverse of the construction of the 3+1-Einstein equations presented in section 1.1.2. The Lorentzian metric \mathbf{g} , obtained from a spatial velocity \mathbf{v} solution of the kinematical Newton system, is

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -1 + v_c v^c & v_b \\ \hline v_a & h_{ab}(x^i) \end{array} \right), \quad (3.65)$$

in Eulerian coordinates with h_{ab} a flat metric. Because this construction was done before the 1+3-NC equations, it lacks the fundamental expansion field χ .

The goal of this construction was to be able to compare directly a Newtonian solution to a GR solution. For both solutions to be said ‘equivalent’, the relativistic solution needs to be given by the metric (3.65).² In section C.1, we show that this is the case only if the fluid is irrotational and has a vanishing rest frames Ricci curvature.

As an example, we then show in section C.2.5 that the Schwarzschild geometry corresponds to such a solution: the Schwarzschild metric written in Gullstrand-Painlevé coordinates (MacLaurin, 2019), i.e. related a parabolic free-falling test fluid/observer, is exactly of the form (3.65), with $v^a = (\pm\sqrt{\frac{2GM}{r}}, 0, 0)$.

As a side result of the construction, a Newton-to-GR dictionary is also defined in section C.2.4.

3.4.2 What was expected next

The next step was to use these results to find the NEN theory: I expected that, as for the (Euclidean) Newton theory, there should exist solutions of GR which are exact NEN solutions in the sense given above. The guess was to say that these solutions would have a metric of the form (3.65), but with a positive/negative spatial scalar curvature (for spherical or hyperbolic geometries).

We can however show that a metric of this form, solution of the Einstein equation, and in spherical symmetry (LTB solutions), is necessarily homogeneous (Lasky et al., 2007), i.e. $\mathbf{v} = 0$. This means that with this guess, the spherical solutions of the NEN theory would necessarily be homogeneous, i.e. with a zero spatial velocity, which is obviously not expected.

Then I stopped searching the NEN theory in this direction, and focused on the Newton-Cartan approach, which ultimately lead to the NEN limit presented in the next chapter. This limit will be a powerful tool to study the link between Newton’s theory and general relativity, and in fact my entire paper (Vigneron, 2020) could be redone in an easier way and even be generalised using this tool.

3.5 Conclusion

In this chapter we presented two novel approaches to define a non-Euclidean Newtonian theory, based on the simplest modification of the Newton-Cartan equation, allowing for a non-zero Ricci curvature. We derived the expansion law in each case,

²The Newton-Cartan formalism provides a better definition of this equivalence. This will be presented in chapter 4.

thus generalising the Buchert-Ehlers theorem. In the first approach, a backreaction appears which depends, in particular, on the averaged bulk velocity of the fluid (an expected feature of backreaction), and also on the type of geometry. The second approach has the same expansion law as in (Euclidean) Newton's theory. We also studied the possibility of doing N-body calculations in each case, along with the possibility of separating the spatial metric in space and time, i.e. having a background description. These results are summarised in table 3.1.

None of the proposed theories is directly derived from general relativity. This has to be done using a limit, which is presented in chapter 4.

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NEN theory	$3(\dot{H} + H^2) + 4\pi G \langle \rho \rangle_\Sigma - \Lambda =$	N-body	Separable spatial metric	Gravitomagnetism
Based on the NC equation: $R_{ab} + {}^u b_{ab} \frac{\mathcal{R}}{3} = -4\pi G \rho \tau_a \tau_b$ (Fluid observer term)	$-\langle \Xi_{cd} \Xi^{cd} \rangle_\Sigma + \langle \omega_{cd} \omega^{cd} \rangle_\Sigma$ $-\frac{\mathcal{R}}{3} \langle v_c v^c \rangle_\Sigma + \frac{\mathcal{R}}{6} (\langle \Phi_v^2 \rangle_\Sigma - \langle \Phi_v \rangle_\Sigma^2)$ $+ \langle \Delta \Omega_{cd} \Delta \Omega^{cd} \rangle_\Sigma$ <p>with $v_a = D_a \Phi_v + \epsilon_{acd} D^c A^d$</p>	No	No, unless $\Xi = 0$ and $\Phi_v = 0$	Yes: $\mathcal{T}_{ab} := \Delta \Omega_{ab} + \omega_{ab}$ $D_{[a} g_{b]} = -\tilde{\partial}_t \mathcal{T}_{ab}$ $D^c \mathcal{T}_{ca} = J_a$ with $J_a := \frac{\mathcal{R}}{3} \epsilon_{acd} D^c A^d$ and no displacement current
Based on the NC equation: $R_{ab} + {}^G b_{ab} \frac{\mathcal{R}}{3} = -4\pi G \rho \tau_a \tau_b$ (Galilean observer term)	$-\langle \Xi_{cd} \Xi^{cd} \rangle_\Sigma + \langle \omega_{cd} \omega^{cd} \rangle_\Sigma$	Yes	Yes, if $\Xi = 0$	No, if $\omega = 0$

Table 3.1: Summary of the proposed NEN theories. The second column quantifies the difference with a FLRW expansion. In each case, the field Ξ and ω can be set to zero without loss of generality on \mathbf{v} . This is done in (Euclidean) Newtonian simulations. A discussion on the role(s) of Ξ is made in section 2.5.4.

Chapter 4

Galilean limit of general relativity

THE final step in the generalisation of the Buchert-Ehlers theorem is to develop a non-Euclidean Newtonian limit of general relativity, aiming at recovering one (or another) of the NEN theories proposed in chapter 3. The strategy I followed was to adapt an existing limit to allow for non-Euclidean geometries: *the Galilean limit of general relativity* (Künzle, 1976). I will first give the motivations for this choice (section 4.1), then I will present the limit itself and its properties in section 4.2. Section 4.3 presents the Euclidean case, i.e. the Newtonian limit, and section 4.4 presents the non-Euclidean case.

4.1 Motivations

4.1.1 The classical Newtonian limit and its limitations

The Newtonian limit is generally defined as a weak field limit of general relativity, where the Lorentzian metric g is a perturbation of the Minkowski metric η as $g_{\alpha\beta} = \eta_{\alpha\beta} + f_{\alpha\beta}$ where $|f_{\alpha\beta}| \ll 1$ and $|\partial_\gamma f_{\alpha\beta}| \ll 1$. The Einstein equation is then linearised at first order in $f_{\alpha\beta}$ and its derivatives. Details of how the Poisson equation is recovered from this limit can be found, e.g., in chapter 4 of Chruściel (2019).

This approach to the Newtonian limit has many drawbacks. First it is not a well-defined limit of a 1-parameter family of solutions of the Einstein equations. There is only an implicit small parameter ϵ with respect to which the metric is expanded. So it is difficult to assess which properties the relativistic solutions should have so that they admit such a limit. The limit is also built such that the Poisson equation is recovered. However, we saw in chapter 2 that the complete Newton theory should feature the cosmological Poisson equation (2.54), and that the classical Poisson equation (1.91) is only valid for isolated systems in \mathbb{R}^3 . So the limit does not allow us to recover the full Newton theory.

Because the perturbation is defined with respect to the Minkowski metric, the limit is also necessarily Euclidean for the spatial sections. One way to allow for

non-Euclidean sections would be to replace $\boldsymbol{\eta}$ by an FLRW metric. This is the approach of the standard perturbation theory in cosmology (Bardeen, 1980). This would imply by definition that the spatial expansion is necessarily given by the Friedmann equations. This is obviously not appropriate as we want to derive the expansion law rather than assuming it as a background. This means that we need to perform a perturbation around a metric that we do not know *a priori*, which is difficult to define.

4.1.2 The ‘right’ definition of a NEN theory

Another way to tackle the problem of a non-Euclidean Newtonian limit is to redefine what we called a non-Euclidean Newtonian theory. In the previous chapters, we defined such a theory as a theory whose equations should reduce to Newton’s equations on scales small with respect to the spatial curvature (curvature introduced to allow for non-Euclidean geometries). This was indeed the case for the NEN theories proposed in chapter 3. However this is not a precise mathematical definition because it does not give any information on the theory before this small-curvature limit is taken. Furthermore we saw in chapter 2 that Newton’s theory is best formulated, and argued that it must be, in its geometrised form, i.e. the Newton-Cartan theory. In particular, we saw that from this formulation, the expansion arises as an emerging field of the theory. Another fundamental aspect of this formulation is that the Galilean invariance of spacetime is taken as a fundamental principle on which the theory is based. We can therefore redefine a NEN theory as follows:

Definition. A **non-Euclidean Newtonian theory** is a Galilean theory whose spatial sections have a non-Euclidean geometry.

where we define

Definition. A **Galilean theory** is a theory defined on a Galilean manifold. It is given by an equation relating the Riemann tensor of the Galilean structure to the energy content.

Remark. The Galilean invariance at the origin of the definition of Galilean structures is defined as a *local* invariance between frames (see Künzle, 1972), whereas a global invariance would necessarily require a flat Euclidean space. That is why Galilean manifolds can have non-Euclidean spatial sections.

It comes from this definition that the NEN limit of general relativity can be defined as a limit of structures: from a Lorentzian structure $(\mathcal{M}, \boldsymbol{g}, \overset{g}{\nabla})$ solution of the Einstein equation, to a Galilean structure $(\mathcal{M}, \boldsymbol{h}, \boldsymbol{\tau}, \nabla)$ with non-Euclidean sections. More generally, we call *Galilean limit of general relativity*, such a limit of structures, but without assumptions on the geometry of the spatial sections. Thus

the Newtonian and the NEN limits are special cases of Galilean limits of general relativity.

Remark. *A priori*, the Newtonian and NEN limits do not cover all the possible Galilean limits, because one could conceive a Galilean limit which is Euclidean, but with non-zero spatial Ricci curvature, hence being different from the Newtonian limit.

There remains to find a 1-parameter family of structures $(\mathbf{g}, \overset{g}{\nabla})$ solution of the Einstein equation, which corresponds to a Galilean structure at leading order. Such a family was constructed by [Künzle \(1976\)](#) and [Dautcourt \(1964\)](#)¹ to derive the (Euclidean) Newton-Cartan theory from general relativity. In this chapter we will see to what extent it can be used for non-Euclidean geometries.

4.2 Galilean limit of Lorentzian structures

We present in this section the limit, and its properties, of a Lorentzian structure to a Galilean structure. For now we will not assume that the Lorentzian structure is solution of the Einstein equation. This will be done in sections [4.3](#) and [4.4](#).

4.2.1 The limit

Main hypothesis of the limit: We consider a 4-manifold \mathcal{M} and a 1-parameter family of smooth Lorentzian metrics $\{\overset{\lambda}{\mathbf{g}}\}_{\lambda>0}$ on \mathcal{M} that depends smoothly on λ , such that

$$\overset{\lambda}{g}^{\alpha\beta} = h^{\alpha\beta} + \lambda \overset{1}{g}^{\alpha\beta} + \mathcal{O}(\lambda^2), \quad (4.1)$$

where $\overset{=}{\mathbf{h}}$ is a positive semi-definite tensor field of rank 3 on \mathcal{M} , i.e. its signature is $(0, +, +, +)$.

From this hypothesis, it can be shown that the covariant components $\overset{\lambda}{g}_{\alpha\beta}$ of the Lorentzian metric are decomposed with respect to λ as follows

$$\overset{\lambda}{g}_{\alpha\beta} = -\frac{1}{\lambda} \tau_\alpha \tau_\beta + \overset{0}{g}_{\alpha\beta} + \mathcal{O}(\lambda), \quad (4.2)$$

where the 1-form τ lies in the kernel of $\overset{=}{\mathbf{h}}$, i.e. $\tau_\mu h^{\mu\alpha} = 0$, and such that $\tau_\mu \tau_\nu \overset{1}{g}^{\mu\nu} = -1$. The proof is given in section [D.1](#) of appendix [D](#).

By definition we have $\overset{\lambda}{g}^{\alpha\mu} \overset{\lambda}{g}_{\mu\beta} = \delta_\beta^\alpha$, which implies that $h^{\alpha\mu} \overset{0}{g}_{\mu\beta} = \delta_\beta^\alpha - \tau_\beta B^\alpha$, where we denote $B^\alpha := -\tau_\mu \overset{1}{g}^{\mu\alpha}$. Then we can write $\overset{0}{g}_{\alpha\beta}$ as

$$\overset{0}{g}_{\alpha\beta} = \overset{B}{b}_{\alpha\beta} - 2\phi \tau_\alpha \tau_\beta, \quad (4.3)$$

¹[Künzle \(1976\)](#) claimed that his limit was an improvement of the original limit proposed by ([Dautcourt, 1964](#)). However, I was not able to get access to the latter paper, and therefore I cannot present the original version of the limit.

where ϕ is an arbitrary scalar and ${}^B b_{\alpha\beta}$ is defined in equation (2.4) and corresponds to the projector orthogonal to $\bar{\mathbf{B}}$ with respect to the metrics $\underline{\boldsymbol{\tau}}$ and $\bar{\mathbf{h}}$. Then, computing the Levi-Civita connection of $\hat{\mathbf{g}}$, we find

$$\begin{aligned} \hat{\Gamma}_{\alpha\beta}^{\gamma} &= \frac{1}{\lambda} h^{\gamma\mu} \left(\tau_{\alpha} \partial_{[\mu} \tau_{\beta]} + \tau_{\beta} \partial_{[\mu} \tau_{\alpha]} \right) + \left(\phi h^{\gamma\mu} + \hat{g}^{\gamma\mu} \right) \left(\tau_{\alpha} \partial_{[\mu} \tau_{\beta]} + \tau_{\beta} \partial_{[\mu} \tau_{\alpha]} \right) \\ &\quad + h^{\gamma\mu} \left(\partial_{(\alpha} {}^B b_{\beta)\mu} - \frac{1}{2} \partial_{\mu} {}^B b_{\alpha\beta} \right) + B^{\gamma} \partial_{(\alpha} \tau_{\beta)} + \tau_{\alpha} \tau_{\beta} h^{\gamma\mu} \partial_{\mu} \phi + \mathcal{O}(\lambda). \end{aligned} \quad (4.4)$$

If the 1-form $\underline{\boldsymbol{\tau}}$ is closed, which we will suppose from now, the connection $\hat{\Gamma}_{\alpha\beta}^{\gamma}$ has a regular limit for $\lambda \rightarrow 0$, which correspond to a Galilean connection compatible with $\underline{\boldsymbol{\tau}}$ and $\bar{\mathbf{h}}$:

$$\begin{aligned} \hat{\Gamma}_{\alpha\beta}^{\gamma} &= \overset{0}{\Gamma}_{\alpha\beta}^{\gamma} + \mathcal{O}(\lambda) \\ &= {}^B \Gamma_{\alpha\beta}^{\gamma} + \tau_{\alpha} \tau_{\beta} h^{\gamma\mu} \partial_{\mu} \phi + \mathcal{O}(\lambda). \end{aligned} \quad (4.5)$$

This Galilean connection is not general as the 2-form $\kappa_{\alpha\beta}$ [see equation (2.2)] necessarily takes the form $\kappa_{\alpha\beta} = \tau_{[\alpha} \partial_{\beta]} \phi$.

This shows that the Lorentzian structure $(\hat{\mathbf{g}}, \hat{\Gamma}_{\alpha\beta}^{\gamma})$ becomes at leading order the Galilean structure $(\underline{\boldsymbol{\tau}}, \bar{\mathbf{h}}, \overset{0}{\Gamma}_{\alpha\beta}^{\gamma})$.

4.2.2 Some remarks

We defined a 1-parameter family of Lorentzian structures on a 4-manifold \mathcal{M} which becomes at leading order a Galilean structure on \mathcal{M} . The only additional hypothesis made after the definition of the family $\{(\hat{\mathbf{g}}, \hat{\Gamma}_{\alpha\beta}^{\gamma})\}_{\lambda>0}$ is the closedness of the 1-form $\underline{\boldsymbol{\tau}}$.

This is a covariant limit, as it does not require to be done in a specific coordinate system.

While the limit of the covariant components of the Lorentzian metric is not regular, i.e. the leading order is a ‘-1’ order, the Lorentzian connection is regular for $\lambda \rightarrow 0$, which implies that the Riemann tensor $\hat{R}^{\sigma}_{\alpha\beta\gamma}$ is also regular.

Under the framework of this limit, the manifold \mathcal{M} is the same, only the structure changes with the parameter λ . This contrasts with perturbation approaches where a 1-parameter family of 4-manifolds $\{\hat{\mathcal{M}}_{\lambda>0}\}$ is also defined (e.g. [Bardeen, 1980](#); [Ellis & Bruni, 1989](#); [Green & Wald, 2012](#)).

As the computation of spacetime distances is made using the metric(s) of the structure, this means that, if the Lorentzian structure is well approximated by its leading order, physical measures of distances in \mathcal{M} can be made using a Galilean structure, i.e. by assuming a preferred space and time.

A condition for the existence of this limit given a 4-manifold \mathcal{M} is that there must exist a time and space orientable Lorentzian structure in this manifold, or in other words \mathcal{M} must be orientable and non-compact [see section 3 of [Künzle \(1972\)](#) and [Künzle \(1976\)](#)].

At leading order the contravariant and covariant components of the Lorentzian metric are not each others inverse anymore.

4.2.3 Interpretation of λ

The Minkowski metric, in coordinates where x^0 has the dimension of a time, can be written as $\eta_{\alpha\beta} = \text{diag}(-c^2, 1, 1, 1)$, where c is the speed of light. Its inverse is $\eta^{\alpha\beta} = \text{diag}(-c^{-2}, 1, 1, 1)$. Then assuming $c \rightarrow \infty$ these two matrices become

$$\begin{aligned}\eta_{\alpha\beta} &\stackrel{c \rightarrow \infty}{\sim} \text{diag}(-c^2, 0, 0, 0), \\ \eta^{\alpha\beta} &\stackrel{c \rightarrow \infty}{\sim} \text{diag}(0, 1, 1, 1).\end{aligned}$$

These leading orders for $c \rightarrow \infty$ have the same form as the leading order of $\overset{\lambda}{\mathbf{g}}$ for $\lambda \rightarrow 0$: in coordinates adapted to the foliation given by $\boldsymbol{\tau}$, i.e. $\tau_\alpha = \delta_\alpha^0$ and $h^{\alpha\beta} = h^{ab}$, we have $\overset{0}{\mathbf{g}}^{\alpha\beta} = h^{ab}$ and $\overset{-1}{\mathbf{g}} = \text{diag}(-1/\lambda, 0, 0, 0)$. This shows that the limit $\lambda \rightarrow 0$ can be seen as a limit $c \rightarrow \infty$ where $\lambda = c^{-2}$.

Remark. The leading order of $\eta^{\alpha\beta}$ can be seen as a qualitative justification of the initial hypothesis we made for the Galilean limit.

Another way to interpret λ is to say that the speed of light related to a Lorentzian metric $\overset{\lambda}{\mathbf{g}}$ of the family $\{\overset{\lambda}{\mathbf{g}}\}_{\lambda>0}$ depends on λ and is given by

$$c_\lambda = \lambda^{-1/2}. \tag{4.6}$$

This means that the family $\{\overset{\lambda}{\mathbf{g}}\}_{\lambda>0}$ of Lorentzian metrics defines a family of light-cones at each events of \mathcal{M} . The light-cone related to a metric $\overset{\lambda_1}{\mathbf{g}}$ will be more open than the one related to a metric $\overset{\lambda_2}{\mathbf{g}}$ if $\lambda_1 < \lambda_2$. This is represented in figure 4.1.

If one wants to set the speed of light to be 1, which corresponds to choosing a coordinate system such that $x^0 = ct$, this is only possible for one Lorentzian metric $\overset{\lambda}{\mathbf{g}}$. For all the other metrics, the speed of light in this coordinate system will differ from 1. This property is really important as it tells us that when we will consider equations which should feature the speed of light, we cannot take $c = 1$, and are obliged to take $c = \lambda^{-1/2}$. This will be the case for the norm of timelike vectors and for the Einstein equation.

Remark. As the light-cones represent the causality of the spacetime structure, the parameter λ was also called *causality constant* by [Ehlers \(1981\)](#); [Ehlers \(2019\)](#).

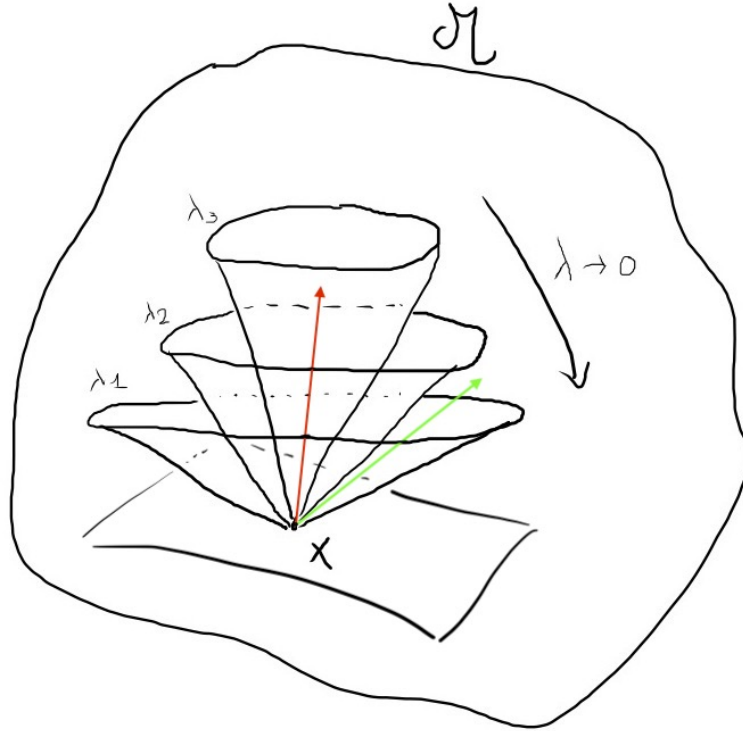


Figure 4.1: Scheme of three light-cones of the family $\{\hat{\mathbf{g}}^\lambda\}_{\lambda>0}$ of Lorentzian metrics at an event X in \mathcal{M} . The smaller λ is, the more open the light-cone is: $\lambda_1 < \lambda_2 < \lambda_3$. The slice corresponds to the only hypersurface member of the foliation defined by τ and passing through X . We see that the limit of the family of light-cones is this hypersurface. The red vector represents the 4-velocity of the event X and is proportional to τ . The green vector is an example of a τ -timelike vector which is g-timelike for the Lorentzian metric $\hat{\mathbf{g}}^{\lambda_1}$, but g-spacelike for $\hat{\mathbf{g}}^{\lambda_2}$ and $\hat{\mathbf{g}}^{\lambda_3}$.

4.2.4 Limit of timelike and lightlike vectors

Because the notion of timelike vectors is defined in both structures, but is not equivalent, to distinguish between the two, we will call g-timelike vectors the ones related to the Lorentzian structure, and τ -timelike vectors the ones related to the Galilean structure. This notation will also be applied for the other common notions.

Let us consider a g-timelike vector $\bar{\mathbf{u}}$ which is ‘unit’ for every member of the family $\{\hat{\mathbf{g}}^\lambda\}_{\lambda>0}$, i.e. $u^\mu u^\nu \hat{g}_{\mu\nu} = -1/\lambda$. If we assume that this vector does not depend on λ , then for a sufficiently small λ we have $(u^\mu \tau_\mu)^2 = 1$. As $\bar{\mathbf{u}}$ is any unit g-timelike vector, this cannot be possible. So, $\bar{\mathbf{u}}$ needs to depend on λ . We write it $\hat{\mathbf{u}}$ (more precisely we define a family $\{\hat{\mathbf{u}}^\lambda\}_{\lambda>0}$ of vectors). Because it is unit for all $\hat{\mathbf{g}}^\lambda$ then we have

$$\hat{u}^\alpha = \overset{\circ}{u}^\alpha + \lambda \overset{\dagger}{u}^\alpha + \mathcal{O}(\lambda),$$

with $\dot{u}^\mu \tau_\mu = 1$, and $\dot{u}^\mu \tau_\mu = \frac{1}{2} \dot{u}^\mu \dot{u}^\nu \dot{g}_{\mu\nu}$. Concerning the covariant components of $\dot{\mathbf{u}}$, we have:

$$\dot{u}_\alpha := \dot{u}^\mu \dot{g}_{\mu\alpha} = -\frac{1}{\lambda} \tau_\alpha + \dot{u}_\alpha + \mathcal{O}(\lambda^2),$$

where $\dot{u}_\alpha = \dot{u}^\mu \dot{g}_{\mu\alpha} - \frac{1}{2} \dot{u}^\mu \dot{u}^\nu \dot{g}_{\mu\nu} \tau_\alpha$. Note that $\dot{u}_\alpha \neq \dot{u}^\mu \dot{g}_{\mu\alpha}$.

This result means that a unit g-timelike vector for all members of $\{\dot{\mathbf{g}}\}_{\lambda>0}$ corresponds to a unit τ -timelike vector at the limit $\lambda \rightarrow 0$. However, the reverse is not possible: a τ -timelike vector can never correspond to a g-timelike vector for all λ . This can easily be seen using figure 4.1: for a given λ -independent vector, there always exists a λ_0 such that, for all $\lambda > \lambda_0$, the vector lies outside of the lightcone given by $\dot{\mathbf{g}}$.

Let us consider the case of lightlike vectors. We define a family vector $\dot{\mathbf{u}}$ such that $\dot{u}^\mu \dot{u}^\nu \dot{g}_{\mu\nu} = 0$ for all λ . Then if we assume that $\dot{\mathbf{u}}$ can be decomposed into Laurent series of λ , then each term in this series would be zero. This means that we cannot define the limit of a lightlike vector. In other words, a Galilean structure has no equivalent of the lightlike vectors of Lorentzian structures.

Remark. A possibility would be to consider non-integer powers of λ .

4.2.5 Limit of the Riemann tensor

The family $\{\dot{\mathbf{g}}\}_{\lambda>0}$ defines a family of Riemann tensors $\{\mathbf{Riem}^\lambda\}_{\lambda>0}$ on \mathcal{M} . Using the expression of the Riemann tensor in terms of the coefficients of the connection,

$$\dot{R}^\sigma{}_{\gamma\alpha\beta} = 2\partial_{[\alpha} \dot{\Gamma}^\sigma{}_{\beta]\gamma} + 2\dot{\Gamma}^\sigma{}_{\mu[\alpha} \dot{\Gamma}^\mu{}_{\beta]\gamma}, \quad (4.7)$$

we easily see that the development of \mathbf{Riem}^λ takes the form

$$\dot{R}^\sigma{}_{\gamma\alpha\beta} = \overset{0}{R}^\sigma{}_{\gamma\alpha\beta} + \mathcal{O}(\lambda),$$

where $\overset{0}{R}^\sigma{}_{\gamma\alpha\beta}$ is the Riemann tensor associated to the Galilean connection coefficients $\overset{0}{\Gamma}^\sigma{}_{\alpha\beta}$. It comes from this that the leading order of the Ricci tensor coefficients $\dot{R}_{\alpha\beta}$ correspond to the Ricci tensor of the Galilean connection:

$$\dot{R}_{\alpha\beta} = \overset{0}{R}_{\alpha\beta} + \mathcal{O}(\lambda).$$

We recall that the Riemann and Ricci tensors of Galilean connections are only defined by the connection commutation (1.1) as, respectively, a (1,3)-tensor and a (0,2)-tensor. This is because Galilean structures do not feature non-degenerate metrics to raise and lower indices. For instance we have

$$\dot{R}^{\sigma\gamma}{}_{\alpha\beta} := \dot{R}^\sigma{}_{\mu\alpha\beta} \dot{g}^{\mu\gamma} = \overset{0}{R}^\sigma{}_{\mu\alpha\beta} h^{\mu\gamma} + \mathcal{O}(\lambda).$$

The limit of the spacetime scalar curvature is

$$\overset{\lambda}{R} := \overset{\lambda}{R}_{\mu\nu} \overset{\lambda}{g}^{\mu\nu} = {}^h\mathcal{R} + \lambda \left(\overset{1}{g}^{\mu\nu} \overset{0}{R}_{\mu\nu} + h^{\mu\nu} \overset{1}{R}_{\mu\nu} \right) + \mathcal{O}(\lambda^2),$$

where ${}^h\mathcal{R}$ and ${}^h\mathcal{R}^{\alpha\beta}$ are respectively the spatial scalar curvature and the spatial Ricci tensor of the τ -foliation induced by the Galilean structure, i.e. related to the spatial metric $\overset{\bar{1}}{h}$. The limit of the Einstein tensor is

$$\overset{\lambda}{G}_{\alpha\beta} = \frac{1}{\lambda} \left[\frac{{}^h\mathcal{R}}{2} \tau_\alpha \tau_\beta \right] + \overset{0}{R}_{\alpha\beta} - \frac{1}{2} \left[{}^h\mathcal{R} \overset{1}{g}_{\alpha\beta} - \left(\overset{1}{g}^{\mu\nu} \overset{0}{R}_{\mu\nu} + h^{\mu\nu} \overset{1}{R}_{\mu\nu} \right) \tau_\alpha \tau_\beta \right] + \mathcal{O}(\lambda^2), \quad (4.8)$$

$$\overset{\lambda}{G}^{\alpha}_{\beta} = h^{\alpha\mu} \overset{0}{R}_{\mu\beta} - \frac{{}^h\mathcal{R}}{2} \delta_{\beta}^{\alpha} + \mathcal{O}(\lambda), \quad (4.9)$$

$$\overset{\lambda}{G}^{\alpha\beta} = {}^h\mathcal{R}^{\alpha\beta} - \frac{{}^h\mathcal{R}}{2} h^{\alpha\beta} + \mathcal{O}(\lambda), \quad (4.10)$$

where $\overset{1}{R}_{\alpha\beta}$ depends on the first order of the Lorentzian connection (not specified here).

Because the contracted Bianchi identity features raised Riemann tensor indices, it is not defined for Galilean structures. Its limit is

$$\overset{\lambda}{\nabla}_{\mu} \overset{\lambda}{G}^{\mu}_{\beta} = h^{\mu\nu} \overset{0}{\nabla}_{\mu} \overset{0}{R}_{\nu\alpha} - \overset{0}{\nabla}_{\alpha} \frac{{}^h\mathcal{R}}{2} + \mathcal{O}(\lambda) = 0. \quad (4.11)$$

The same applies for the interchange symmetry relation, which follows from the first Bianchi identity, $\overset{\lambda}{R}^{\alpha}_{\gamma\sigma} = \overset{\lambda}{R}^{\beta}_{\sigma\gamma}$:

$$h^{\mu\alpha} \overset{0}{R}^{\alpha}_{\gamma\mu\sigma} - h^{\mu\beta} \overset{0}{R}^{\beta}_{\sigma\mu\gamma} + \mathcal{O}(\lambda) = 0. \quad (4.12)$$

The leading order of this equation is equivalent to the Trautman-Künzle condition (2.19). Therefore this condition, which was initially added to Newton-Cartan to obtain the conservation of vorticity and close the system of equations (see the remark in section 2.3.1), naturally follows from the Galilean limit of Lorentzian structures. As in the relativistic case, it is a purely geometrical condition as it does not require the Einstein equation.

Remark. The condition under this form is also used by Ehlers in his frame theory. But it is miswritten in the original paper Ehlers (1981); Ehlers (2019) [see equation (7)], while it is under the right form in Ehlers (1997).

Apart for the leading orders of the Bianchi identities, no other constraint on the Galilean structure appears. In particular, from this limit alone, the spatial Ricci curvature ${}^h\mathcal{R}^{\alpha\beta}$ is totally free. Constraints on this tensor can only appear once we assume the Lorentzian structures to be solutions of the Einstein equation.

4.3 The (Euclidean) Newtonian limit

4.3.1 Limit of the energy-momentum tensor

The (Euclidean) Newtonian limit is obtained when we consider the Lorentzian structures to be solutions of the Einstein equation *and* that the energy-momentum (2,0)-tensor is regular for $\lambda \rightarrow 0$. In the present chapter, we will assume this tensor to be related to a matter fluid as described by equation (1.4) in section 1.1.1.2. We then have

$$\hat{T}^{\alpha\beta} = \hat{\epsilon} \hat{u}^\alpha \hat{u}^\beta + 2\hat{q}^{\lambda(\alpha} \hat{u}^{\lambda\beta)} + \hat{p} \hat{u}^\lambda \hat{b}^{\alpha\beta} + \hat{\pi}^{\alpha\beta}, \quad (4.13)$$

where the expansion of $\hat{u}^\lambda \hat{b}^{\alpha\beta}$ in terms of λ can be found in section D.2 of appendix D. Because the g-timelike vector $\hat{\mathbf{u}}$ ‘moves’ with λ , its rest frames also depend on λ , i.e. its orthogonal projector depends on λ . Therefore, any tensor which is defined as being orthogonal to $\hat{\mathbf{u}}$ will also ‘move’ with λ . That is why the heat flux and the anisotropic pressure depend on λ .

As \hat{q}^α , $\hat{\pi}^{\alpha\beta}$ and $\hat{u}^\lambda \hat{b}_{\alpha\beta}$ are regular, then $\hat{q}_\alpha := \hat{q}^\mu \hat{u}_\mu^\lambda$ and $\hat{\pi}_{\alpha\beta} := \hat{\pi}^{\mu\nu} \hat{u}_\mu^\lambda \hat{u}_\nu^\lambda$ are also regular for $\lambda \rightarrow 0$. We have

$$\hat{T}^{\alpha\beta} = \hat{\epsilon}^0 \hat{u}^0 \hat{u}^0 \alpha\beta + 2\hat{q}^0(\alpha \hat{u}^0 \beta) + \hat{p}^0 \hat{u}^0 \hat{b}^0 \alpha\beta + \hat{\pi}^0 \alpha\beta + \mathcal{O}(\lambda), \quad (4.14)$$

$$\hat{T}^\alpha_\beta = \frac{1}{\lambda} \left[-\hat{\epsilon}^0 \hat{u}^0 \tau_\beta - \hat{q}^0 \tau_\beta \right] + \mathcal{O}(\lambda), \quad (4.15)$$

$$\hat{T}_{\alpha\beta} = \frac{1}{\lambda^2} \left[\hat{\epsilon}^0 \tau_\alpha \tau_\beta \right] + \frac{1}{\lambda} \left[\hat{\epsilon}^1 \tau_\alpha \tau_\beta - \hat{\epsilon}^0 \tau_{(\alpha} \hat{u}_{\beta)} - 2\hat{q}^0_{(\alpha} \tau_{\beta)} \right] + \mathcal{O}(1), \quad (4.16)$$

$$\hat{T} := \hat{T}^{\mu\nu} \hat{g}_{\mu\nu} = \frac{1}{\lambda} \left[\hat{\epsilon}^0 \right] + \left[\hat{\epsilon}^1 + 3\hat{p}^0 \right] + \mathcal{O}(\lambda), \quad (4.17)$$

with $\hat{q}^\mu \tau_\mu = 0$ and $\hat{\pi}^{\mu\alpha} \tau_\mu = 0$. The variable $\hat{\epsilon}^0$ corresponds to the rest mass density ρ of the fluid, and $\hat{\epsilon}^1$ to an internal energy density. Therefore if we consider only a dust fluid, the energy density $\hat{\epsilon}$ is only a zero order term. The other orders depend on the type of fluid. For a perfect fluid, we expect $\hat{\epsilon}^{n+1} = 0$ and $\hat{p}^n = 0$ for all $n \geq 1$, so that the additional source of energy is only internal.

4.3.2 Limit of the Einstein equation

The Einstein equation (1.3) features two constants: the cosmological constant Λ and the gravitational constant G . From the Einstein equation alone these constants do not have, *a priori*, a preferred dimension. For instance, if we consider Λ to have the dimension of a curvature, then we will have the term $\Lambda \hat{g}_{\alpha\beta}$ in the Einstein equation. Instead, if we consider that it has the dimension of a time⁻², then we will have $\lambda \Lambda \hat{g}_{\alpha\beta}$. However, only the second choice leads to the Newton-Cartan equations at the limit. Concerning the gravitational constant in front of the energy-momentum tensor, it must appear as $8\pi G \lambda^2$ to lead to the right limit. Therefore we consider

that the Lorentzian metrics $\{\hat{g}\}_{\lambda>0}$ are solutions of the following Einstein equations:

$$\hat{G}_{\alpha\beta} + \lambda\Lambda\hat{g}_{\alpha\beta} = 8\pi G\lambda^2\hat{T}_{\alpha\beta}, \quad (4.18)$$

or equivalently in the form

$$\hat{R}_{\alpha\beta} = 8\pi G\lambda^2 \left(\hat{T}_{\alpha\beta} - \frac{1}{2}\hat{T}\hat{g}_{\alpha\beta} \right) + \lambda\Lambda\hat{g}_{\alpha\beta}. \quad (4.19)$$

Using the contravariant form of equation (4.18) along with equation (4.9) we immediately see that the regular hypothesis of the energy-momentum tensor (4.13) implies that the spatial Ricci curvature ${}^h\mathcal{R}^{\alpha\beta}$ of the Galilean structure is necessarily zero. While this is expected for a (Euclidean) Newtonian limit, this is a problem if we want a non-Euclidean spatial geometry. In section 4.4 we will see what are the solutions to tackle this problem.

Using the Einstein equation in the form (4.19) we obtain

$$\overset{0}{R}_{\alpha\beta} = (4\pi G\rho - \Lambda)\tau_\alpha\tau_\beta, \quad (4.20)$$

which corresponds to the Newton-Cartan equation (2.18). The other two equations (2.17) and (2.19) correspond respectively to the contracted Bianchi identity (4.11) and the first Bianchi identity (4.12). In particular, the former written in terms of fluid variables and projected along τ_α leads to the conservation of the mass density (2.24):

$$\overset{0}{\nabla}_\mu \left(\overset{0}{\epsilon} \overset{0}{u}^\mu \right) = -\overset{0}{\nabla}_\mu \overset{0}{q}^\mu. \quad (4.21)$$

Then as discussed in section 2.3.5, we must take $\overset{0}{q}^\alpha = 0$ so that the mass of a fluid element is conserved.

The conservation of the internal energy $\overset{1}{\epsilon}$ is obtained by taking the next order of the conservation of the energy-momentum tensor. It features the first order of the connection $\overset{1}{\Gamma}^\gamma_{\alpha\beta}$ which, in this equation, can be expressed as function of $\overset{0}{u}^\alpha$ and $\overset{0}{g}_{\alpha\beta}$ which are known. We do not present the calculation in the present study.

This concludes the derivation of the Newton-Cartan equations from the Einstein equation using the Galilean limit of Lorentzian structures.

4.3.3 Constraint on the Coriolis field

Contrary to the general Galilean structure considered in chapter 2, the one obtained from general relativity is constrained: equation (4.5) implies that the vorticity tensor $\Omega^{\alpha\beta}$ of the fluid 4-velocity $\overset{0}{u}^\alpha$, defined as

$$\Omega^{\alpha\beta} = h^{\mu[\alpha} \overset{0}{\nabla}_\mu \overset{0}{u}^{\beta]}, \quad (4.22)$$

reduces to

$$\Omega^{\alpha\beta} = h^{\mu[\alpha} B^{\beta]} \nabla_{\mu} \left(\overset{0}{u} - B \right). \quad (4.23)$$

As $\overset{0}{u} - B^{\alpha}$ is spatial, then $\Omega^{ab} = D^{[a} \left(\overset{0}{u} - B \right)^{b]}$, which means that Ω^{ab} is exact. This shows that the harmonic 2-form ω , called Coriolis field in chapter 2, is necessarily zero when the Newton-Cartan equations are derived as a limit of general relativity.

4.3.4 Preferred foliation?

One criticism we can make about the Galilean limit is that it seems to imply a preferred foliation in the manifold \mathcal{M} , the τ -foliation, on which the limit must be taken. This would mean that only a certain class of solutions to the Einstein equation would be suited for a Newtonian limit. However, this interpretation of the limit is *a priori* wrong. To understand why, one can take the example of the Schwarzschild metric in two coordinate systems adapted to two different foliations:

- (i) the Schwarzschild coordinates (corresponding to an accelerated observer at fixed distance from the singularity):

$$g^{\alpha\beta} = \text{diag} \left[- \left(\frac{1}{\lambda} - \frac{2GM}{r} \right)^{-1}; \left(1 - \lambda \frac{2GM}{r} \right); \frac{1}{r^2}; \frac{1}{r^2 \sin^2 \theta} \right], \quad (4.24)$$

$$g_{\alpha\beta} = \text{diag} \left[- \left(\frac{1}{\lambda} - \frac{2GM}{r} \right); \left(1 - \lambda \frac{2GM}{r} \right)^{-1}; r^2; r^2 \sin^2 \theta \right]. \quad (4.25)$$

- (ii) the Gullstrand-Painlevé coordinates (corresponding to a geodesic observer in parabolic radial free-fall):

$$\overset{\lambda}{g}^{\alpha\beta} = \begin{pmatrix} -\lambda & \pm \lambda \sqrt{\frac{2GM}{r}} & 0 & 0 \\ \pm \lambda \sqrt{\frac{2GM}{r}} & 1 - \lambda \frac{2GM}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix}, \quad (4.26)$$

$$\overset{\lambda}{g}_{\alpha\beta} = \begin{pmatrix} \left(\frac{1}{\lambda} - \frac{2GM}{r} \right) & \pm \sqrt{\frac{2GM}{r}} & 0 & 0 \\ \pm \sqrt{\frac{2GM}{r}} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (4.27)$$

In both cases the coefficients $\overset{\lambda}{g}^{\alpha\beta}$ reduce to $\text{diag} \left[0; 1; \frac{1}{r^2}; \frac{1}{r^2 \sin^2 \theta} \right]$ at leading order, and $\overset{\lambda}{g}_{\alpha\beta}$ to $\text{diag} \left[-\frac{1}{\lambda}; 0; 0; 0 \right]$, which both correspond to the leading orders of the Galilean limit. The time and space metric are however related to a different foliation.

This shows that the Galilean limit of the Schwarzschild metric can be performed in at least two different foliations, i.e. there is no preferred foliation in which we

need to perform this limit. So in the end, the only requirement to be able to perform the Newtonian limit from a solution of the Einstein equation is on the manifold \mathcal{M} : its topology needs to be $\Sigma \times \mathbb{R}$ where Σ has an Euclidean geometry.

4.4 The non-Euclidean Newtonian limit

It is always assumed that Einstein's theory must contain Newton's theory in some limit, and this because the later works quite well in several situations. That is how, using the Newtonian limit, we can derive the constant $\frac{8\pi G}{c^4}$ in front of the energy-momentum tensor in the Einstein equation, and the form $\mathbf{T} = \rho \mathbf{u} \otimes \mathbf{u}$ for a dust fluid. However the situations in which Newton's theory was tested to be valid does not rule out Galilean theories (for instance, the second NEN theory of chapter 2) with spatial curvature small in front of the size of the domain of test. In this sense, general relativity should also contain any Galilean theory in some limit. But as shown above, once we suppose that $\hat{T}^{\alpha\beta}$ is regular for $\lambda \rightarrow 0$, the spatiale curvature of the Galilean limit is necessarily zero. In particular this is also true if we assume a dust fluid, where the energy-momentum tensor reduces to $\hat{T}^{\alpha\beta} = \rho \hat{u}^\alpha \hat{u}^\beta + \mathcal{O}(\lambda)$. So it seems that the only way to obtain a non-flat and non-Euclidean Galilean theory at the limit is to add non-regular terms in the energy-momentum tensor. We follow this possibility in the following section.

4.4.1 Non-regular energy-momentum tensor

We assume that the fluid variables $\hat{\epsilon}$, \hat{q}^α , \hat{p} , $\hat{\pi}^{\alpha\beta}$ are not regular for $\lambda \rightarrow 0$. Using the Einstein equation in the form (4.18) and relations (4.8) and (4.9) we obtain that

$$\hat{\epsilon} = \frac{1}{\lambda} \left[\frac{{}^h\mathcal{R}}{2 * 8\pi G} \right] + \overset{0}{\epsilon} + \mathcal{O}(\lambda), \quad (4.28)$$

$$\hat{p} = \frac{1}{\lambda^2} \left[-\frac{{}^h\mathcal{R}}{6 * 8\pi G} \right] + \frac{1}{\lambda} \overset{-1}{p} + \overset{0}{p} + \mathcal{O}(\lambda), \quad (4.29)$$

$$\hat{\pi}^{\alpha\beta} = \frac{1}{\lambda^2} \left[\frac{{}^h\mathcal{R}^{(\alpha\beta)}}{8\pi G} \right] + \frac{1}{\lambda} \overset{-1}{\pi}^{\alpha\beta} + \overset{0}{\pi}^{\alpha\beta} + \mathcal{O}(\lambda). \quad (4.30)$$

The Einstein equation does not allow us to specify $\overset{-1}{p}$ and $\overset{-1}{\pi}^{\alpha\beta}$ in terms of the two leading orders of the Lorentzian metric.

Using the conservation of the energy-momentum tensor $\hat{\nabla}_\mu \hat{T}^{\mu\alpha} = 0$ projected along \hat{u}_α , we obtain that the leading order of the heat flux needs to be a -1 order:

$$\hat{q}^\alpha = \frac{1}{\lambda} \overset{-1}{q}^\alpha + \overset{0}{q}^\alpha + \mathcal{O}(\lambda), \quad (4.31)$$

with $\overset{-1}{q}^\mu \tau_\mu = 0$ and that

$$\overset{0}{u}^\mu \hat{\nabla}_\mu \overset{-1}{\epsilon} + \hat{\nabla}_\mu \overset{0}{u}^\mu \left(\overset{-1}{\epsilon} + \overset{-1}{p} \right) + \hat{\nabla}_\mu \overset{-1}{q}^\mu + \hat{\nabla}_\mu \overset{0}{u}_\nu \overset{-2}{\pi}^{\mu\nu} = 0, \quad (4.32)$$

which can also be written as

$$\dot{u}^\mu \overset{0}{\nabla}_\mu {}^h\mathcal{R} + \frac{{}^h\mathcal{R}}{3} \overset{0}{\nabla}_\mu \dot{u}^\mu + 16\pi G \overset{0}{\nabla}_\mu \bar{q}^\mu + 2 \overset{0}{\nabla}_\mu \dot{u}_\nu {}^h\mathcal{R}^{(\mu\nu)} = 0. \quad (4.33)$$

As for \bar{p}^{-1} and $\bar{\pi}^{-1\alpha\beta}$, the leading order $\bar{q}^{\flat\alpha}$ cannot be specified in terms of the two leading orders of the Lorentzian metric.

4.4.1.1 Limit of the 1+3-Einstein equations

It is also possible to retrieve the above Laurent expansion series of the fluid variables from the 1+3-Einstein equations (section 1.1.3.3). For this we need the leading order of the expansion tensor $\overset{\lambda}{\Theta}$ and vorticity tensors of $\overset{\lambda}{\Omega}$ of the fluid 4-velocity $\overset{\lambda}{u}$. Using appendix D.2, we obtain

$$\overset{\lambda}{\Theta}^{\alpha\beta} = \Theta^{\alpha\beta} + \mathcal{O}(\lambda), \quad (4.34)$$

$$\overset{\lambda}{\Omega}^{\alpha\beta} = \Omega^{\alpha\beta} + \mathcal{O}(\lambda), \quad (4.35)$$

where $\Theta^{\alpha\beta}$ and $\Omega^{\alpha\beta}$ correspond to the expansion and vorticity tensors of the vector $\overset{0}{u}^\alpha$ as defined by the Galilean structure [see equation (2.21)].

Concerning the rest frame connection $\overset{\lambda}{D}_\alpha$ defined in equation (1.47), it reduces to the spatial connection D_α related to $h^{\alpha\beta}$:

$$\overset{\lambda}{D}_\alpha = D_\alpha + \mathcal{O}(\lambda). \quad (4.36)$$

In particular the torsion of the rest frame connection, defined in equation (1.49), is a first order term, and thus vanishes at leading order.

The rest frame Ricci curvature, defined in equation (1.50), reduces to the spatial Ricci curvature related to $h^{\alpha\beta}$:

$$\overset{\lambda}{\mathcal{R}}^{\alpha\beta} = {}^h\mathcal{R}^{\alpha\beta} + \mathcal{O}(\lambda). \quad (4.37)$$

In particular, $\overset{\lambda}{\mathcal{R}}^{\alpha\beta}$ is symmetric at leading order.

Then the lhs of the 1+3-Gauss equation (1.59), features a -1 order (the curvature term) implying that the rhs, which corresponds to the energy density $\hat{\epsilon}$, needs to feature a -1 order of the form (4.28). The developments (4.29) of $\overset{\lambda}{p}$ and (4.30) of $\overset{\lambda}{\pi}^{\alpha\beta}$ are obtained similarly using the 1+3-Ricci equation (1.53).

For $\overset{\lambda}{q}^\alpha$, the development (4.31) is obtained when considering the 1+3-momentum constraint (1.54) which is at leading order:

$$D^\alpha \theta - D_\mu (\Theta^{\alpha\mu} + \Omega^{\alpha\mu}) = 8\pi G \bar{q}^{\flat\alpha}.$$

From this equation, and the knowledge we gained in chapter 2 and 3 on the role of the momentum constraint on the space-time separation of the spatial metric, we see that we necessarily need $\bar{q}^{\flat\alpha} \neq 0$ for this separation to be possible. The momentum constraint needs to be of the form (3.41).

4.4.1.2 The 1+3-Galilean system

From the Laurent expansion series of the fluid variables, the Einstein equation (4.19) becomes at leading order

$$\overset{\circ}{R}_{\alpha\beta} - {}^h\mathcal{R}^{\mu\nu}\overset{\circ}{b}_{\mu\alpha}\overset{\circ}{b}_{\nu\beta} = 8\pi G \left[\frac{1}{2} \left(\overset{\circ}{\epsilon} + 3\overset{\circ}{p} \right) \tau_\alpha \tau_\beta - 2\bar{q}^{\mu\alpha}\overset{\circ}{b}_{\mu(\alpha}\tau_{\beta)} \right] - \Lambda \tau_\alpha \tau_\beta. \quad (4.38)$$

The projection along and orthogonal to $\overset{\circ}{u}$ if this equation, along with the Bianchi identities, gives the 1+3-Galilean system

$$(\partial_t - \mathcal{L}_\beta) {}^h\mathcal{R} = -\frac{{}^h\mathcal{R}}{3}\theta - D_c \bar{q}^b - \Theta_{cd} {}^h\mathcal{R}^{(cd)}, \quad (4.39)$$

$$(\partial_t - \mathcal{L}_\beta) h^{ab} = -2\Theta^{ab}, \quad (4.40)$$

$$(\partial_t - \mathcal{L}_\beta) \theta = -4\pi G \left(\overset{\circ}{\epsilon} + 3\overset{\circ}{p} \right) + \Lambda + D_c {}^u a^c - \Theta^{cd} \Theta_{cd} + \Omega^{cd} \Omega_{cd}, \quad (4.41)$$

$$(\partial_t - \mathcal{L}_\beta) \Omega_{ab} = D_{[a} {}^u a_{b]}, \quad (4.42)$$

with the constraint equations

$$D_c (\Theta^{ac} + \Omega^{ac}) - D^a \theta = 8\pi G \bar{q}^b, \quad (4.43)$$

$$D_{[a} \Omega_{bc]} = 0, \quad (4.44)$$

$${}^h\mathcal{R}^{ab} = {}^h\mathcal{R}^{ab}, \quad (4.45)$$

$${}^h\mathcal{R}^{d[abc]} = 0. \quad (4.46)$$

We see that the NEN theory depends on the unknown fluid variables \bar{p}^1 , \bar{q}^b and $\bar{\pi}^{1ab}$. This means that without additional hypothesis on these variables, the system is not closed.

As a natural hypothesis we can assume that the spatial Ricci curvature is purely scalar, implying the τ -sections to have either spherical or hyperbolic geometries. In this case $\bar{\pi}^{2\alpha\beta} = 0$, then $\bar{\pi}^{1\alpha\beta}$ is spatial and we have an additional equation linking \bar{q}^b , \bar{p}^1 and $\bar{\pi}^{1\alpha\beta}$:

$$(\partial_t - \mathcal{L}_\beta) \bar{q}^b + \theta \bar{q}^b + 2\bar{q}^b \Theta_c^a + h^{ca} D_c \bar{p}^1 + D_c \bar{\pi}^{1ca} = 0. \quad (4.47)$$

It comes from the conservation of the energy-momentum tensor. To obtain it, we used that fact that \bar{p}^2 depends only on t implying the term ${}^u b^{\alpha\mu} \partial_\mu \bar{p}^2$, which should be present in the above equation, to vanish.

4.4.1.3 Interpretation of the non-regular energy-momentum tensor

We saw that all the fluid variables need to have negative leading orders, related to the spatial curvature, so that a non-Euclidian limit is possible. This means, that

even for a dust fluid, or geodesic fluid, the energy momentum tensor needs to be of the form

$$\hat{T}^{\lambda\alpha\beta} = \rho \hat{u}^\alpha \hat{u}^\beta + \mathcal{R}\hat{T}^{\lambda\alpha\beta}, \quad (4.48)$$

where the leading orders of the decomposition of $\mathcal{R}\hat{T}^{\lambda\alpha\beta}$ with respect to \hat{u}^α are equations (4.28)-(4.30).

Interpretation of the additional curvature term: We can interpret the additional term $\mathcal{R}\hat{T}^{\lambda\alpha\beta}$ as a term we missed when extrapolating the energy-momentum tensor of a dust fluid from Newton's theory, because we only had the knowledge of an Euclidean Newtonian theory.

For this interpretation to be valid, the fluid described by (4.48) needs to be geodesic, i.e. a dust fluid. This geodesic nature of \hat{u}^α needs to be implied by the conservation of $\hat{T}^{\lambda\alpha\beta}$:

$$\hat{\nabla}_\mu \hat{T}^{\lambda\mu\alpha} = 0 \longrightarrow \hat{u}^\mu \hat{\nabla}_\mu \hat{u}^\alpha = 0. \quad (4.49)$$

In this sense we can see (4.48) as a generalisation of the energy-momentum tensor for a geodesic fluid: generalisation in the sense that $\hat{T}^{\lambda\alpha\beta} = \rho \hat{u}^\alpha \hat{u}^\beta$ is not the unique energy-momentum tensor implying that the fluid is geodesic.

Remark. As an additional requirement, we can also suppose that $\mathcal{R}\hat{T}^{\lambda\alpha\beta}$ is decoupled from $\rho \hat{u}^\alpha \hat{u}^\beta$ so that both terms are subject to the contracted Bianchi identity independently. This is less general, but would imply, in addition to the geodesic equation for \mathbf{u} , the conservation of the density ρ .

4.4.1.4 The main issue

While the above interpretation justifies the additional term, it does not provide a way to determine it. So we need to guess what is the full order form of $\mathcal{R}\hat{T}^{\lambda\alpha\beta}$. This is where the main issue in obtaining the NEN theory arises: there is *a priori* not a unique possibility for $\mathcal{R}\hat{T}^{\lambda\alpha\beta}$ which both leads to the right leading orders and preserves the geodesic nature of the fluid. Then there is not a unique choice for \bar{p}^{-1} , \bar{q}^b and $\bar{\pi}^{ab}$, which means that closing the 1+3-system of section (4.4.1.2), i.e. obtaining the NEN theory, is not unique.

As for now, I have tried a lot of guesses for the term $\mathcal{R}\hat{T}^{\lambda\alpha\beta}$ to see if there is a 'more natural one', but without success...

4.4.2 Specific choices of the non-regular terms

Both NEN theories proposed in chapter 3 can be retrieved by specific choices of the non-regular terms:

- the NEN theory I is retrieved with

$$\bar{\pi}^{2\alpha\beta} = 0 \quad ; \quad \bar{p}^2 = -\frac{{}^h\mathcal{R}(t)}{6 * 8\pi G} \quad ; \quad \bar{q}^{bx} = 0 \quad ; \quad \bar{\epsilon}^0 + 3\bar{p}^1 = \rho, \quad (4.50)$$

- the NEN theory II is retrieved with

$$\bar{\pi}^{2\alpha\beta} = 0 \quad ; \quad \bar{p}^2 = -\frac{{}^h\mathcal{R}(t)}{6 * 8\pi G} \quad ; \quad \bar{q}^{bx} = -\frac{{}^h\mathcal{R}}{3 * 8\pi G} v^a \quad ; \quad \bar{\epsilon}^0 + 3\bar{p}^1 = \rho + \frac{2}{3} \frac{{}^h\mathcal{R}}{8\pi G} v^c v_c, \quad (4.51)$$

with v^a the vector in the SVT decomposition of Θ^{ab} .

In both cases, it is not clear to me how the conservation of ρ can be retrieved. It corresponds to the second leading order of the 1+3-conservation equation (1.57) and requires the knowledge of $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ and \bar{q}^{α} .

While we still do not know what is the right choice for these non-regular terms, requiring the space–time separability of the spatial metric seems to be physically relevant. This is done by assuming $\bar{q}^{bx} = -\frac{{}^h\mathcal{R}}{3 * 8\pi G} v^a$. Without further hypothesis on the other non-regular terms, the expansion law of Σ is in this case

$$3(\dot{H} + H^2) = -4\pi G \left(\langle \bar{\epsilon}^0 \rangle_{\Sigma} + 3 \langle \bar{p}^1 \rangle_{\Sigma} \right) + \Lambda + \frac{\mathcal{R}}{3} \langle v_c v^c \rangle_{\Sigma}. \quad (4.52)$$

But as we do not know *a priori* what is the expression of $\bar{\epsilon}^0$ and \bar{p}^1 as function of ρ , v^a and ${}^h\mathcal{R}$, we cannot know the deviation of this expansion law with the FLRW one, and we cannot answer the main question of this thesis.

4.5 Backreaction from post-Newtonian terms

We focused until now on the leading order of the Einstein equation. Deriving the next order might be interesting for the study of the global backreaction, and this even in the Newtonian case with $\hat{T}_{\alpha\beta}^{\lambda} = \rho \hat{u}_{\alpha}^{\lambda} \hat{u}_{\beta}^{\lambda}$. Indeed, we can consider that the global backreaction in Newton is the zeroth order of the full relativistic backreaction. Then accounting for $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ and \bar{u}^{α} , we can derive $\bar{\Theta}^{\alpha\beta}$ and $\bar{\Omega}^{\alpha\beta}$. The first order backreaction $\bar{\mathcal{Q}}_{\Sigma}$ is

$$\bar{\mathcal{Q}}_{\Sigma} = \frac{4}{3} \left(\langle \bar{\theta}^0 \bar{\theta}^1 \rangle_{\Sigma} - \langle \bar{\theta}^0 \rangle_{\Sigma} \langle \bar{\theta}^1 \rangle_{\Sigma} \right) - \langle \bar{\sigma}_{cd}^0 \bar{\sigma}^{1cd} + \bar{\sigma}_{cd}^1 \bar{\sigma}^0 cd \rangle_{\Sigma} + \langle \bar{\Omega}_{cd}^0 \bar{\Omega}^{1cd} + \bar{\Omega}_{cd}^1 \bar{\Omega}^0 cd \rangle_{\Sigma}. \quad (4.53)$$

This backreaction depends on $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ which requires the knowledge of $\bar{g}^{2\alpha\beta}$ and $\bar{g}_{\alpha\beta}^1$. These are post-Newtonian corrections to the Lorentzian metric and come with their own equations (see [Dautcourt, 1997](#); [Tichy & Flanagan, 2011](#)).

4.6 Conclusion

This chapter aimed at studying the possibility of obtaining Galilean theories with non-zero spatial Ricci curvature from the Einstein equation. The main results are the following:

- (i) The Galilean limit allows, as expected, to recover the Newton-Cartan equations.
- (ii) In order to have non-zero spatial Ricci curvature at the limit, the energy-momentum (2,0)-tensor must be non-regular for $\lambda \rightarrow 0$.
- (iii) We interpreted the non-regular orders as coming from an additional term ${}^{\mathcal{R}}T^{\alpha\beta}$ in the energy-momentum tensor of a dust fluid. This interpretation is valid if the conservation of $T^{\alpha\beta} = \rho u^\alpha u^\beta + {}^{\mathcal{R}}T^{\alpha\beta}$ implies the geodesic equation for u^α .
- (iv) The NEN theory depends on the second leading order of this term, which is not known.

What is the full order form of ${}^{\mathcal{R}}T^{\alpha\beta}$ remains an open question, as well as what is the ‘right’ NEN theory compatible with general relativity. A lot of questions remain to be solved to arrive at that goal:

- Did we use the most general way to make a Galilean limit of a Lorentzian structure?
- Is there a way to justify the non-regular terms other than with the Galilean limit?
- *A priori* the limit allows for an Euclidean Galilean limit with non-zero spatial curvature. Is this case physical?

While the goal of this chapter was to derive the global backreaction at leading order in non-Euclidean geometries, the tools we developed can be used to compute the first order global backreaction in the Euclidean case (section 4.5). This is left for a future study.

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Chapter 5

Towards non-Euclidean relativistic simulations

IN numerical cosmology, two kinds of simulations are usually conducted, using either Newtonian gravity or general relativity. The former is the most common kind of simulations. They are usually used to create mock catalogues for different observables, such as weak lensing signals, X-ray luminosity or statistics on galaxy clusters (see e.g. [Springel et al., 2006](#); [Angulo et al., 2012](#)). As all these simulations use the Newtonian cosmology with an isotropic expansion, the Buchert-Ehlers theorem prevents them to study the global backreaction of structures on the expansion. That is why, in the past decade, the field of numerical cosmology has seen the growth of non-perturbative general relativistic simulations. Various methods have been used, from approximate or non general schemes to exact schemes using an exact resolution of the Einstein equation. But while most of them aim at probing the backreaction effect, the issue of topology is generally not studied, and the flat 3-torus topology is generally assumed.

The only non-Euclidean cosmological simulations known to me lie in the field of lattice cosmological models. These are solutions of the vacuum Einstein equation which can be interpreted as representing the evolution of black holes in \mathcal{S}^3 . As we will see, they are hardly representative of our Universe. Still this method is conceptually interesting and will be presented in section 5.1. We will then focus on relativistic numerical cosmology with fluid models. Section 5.2 presents the current status of such numerical studies. We will see that they allow for a first quantification of the backreaction in a relatively realistic model universe. However, the numerical formalism used in these simulations, the BSSN formalism, is not adapted for non-Euclidian geometries. In section 5.3, after introducing this formalism, we will see how it can be adapted to allow for non-Euclidian geometries.

This chapter does not aim at providing a full numerical simulation; it is rather a feasibility study.

5.1 Lattice cosmology

The exact relativistic approach for describing a lattice of black holes in \mathcal{S}^3 has been extensively studied the past decade (e.g. Clifton et al., 2012; Bentivegna & Korzyński, 2013; Clifton et al., 2013; Clifton, 2014; Korzyński, 2014; Bentivegna, 2014; Clifton, 2015; Durk & Clifton, 2017; Bentivegna et al., 2018). It is based on the initial work of Misner (1963) who studied exact solutions of the 3+1-constraint equations in vacuum, i.e. in terms of initial conditions of the Cauchy problem, representing a lattice of black holes with asymptotically flat space at infinity. The formalism used is called *geometrostatics*. If applied to cosmology, it allows for the definition of exact inhomogeneous initial conditions, which can be evolved subsequently. I introduce the *geometrostatics* in section 5.1.1 and describe how it can be applied to cosmology in section 5.1.2. Finally, in section 5.1.3, I present an overview of the literature on lattice cosmological models.

5.1.1 Principle of geometrostatics

The *geometrostatics* is a method to find more easily initial conditions fulfilling the 3+1-constraint equations [equations (1.41) and (1.42)]. It is generally used in vacuum and is based on the Lichnerowicz-York equations, which are obtained from the Gauss-Codazzi constraints with a *conformal transformation* of the spatial metric:

$$h_{ab} =: \psi^4 \tilde{h}_{ab}, \quad (5.1)$$

where \tilde{h} is called the *conformal metric* and ψ is a never vanishing scalar field called *the conformal factor*. We also define the conformal traceless extrinsic curvature

$$\tilde{A}_{ab} := \psi^2 K_{\langle ab \rangle},$$

and its contravariant components, which are raised with the conformal metric

$$\tilde{A}^{ab} := \tilde{A}_{cd} \tilde{h}^{ca} \tilde{h}^{db} = \psi^{-10} K^{(ab)}.$$

Then introducing the conformal traceless extrinsic curvature and the conformal metric in the 3+1-constraints, we obtain:

The Lichnerowicz-York equations (in vacuum):

$$\tilde{\Delta}\psi - \frac{\tilde{\mathcal{R}}}{8}\psi - \frac{K^2}{12}\psi^5 + \frac{1}{8}\tilde{A}_{cd}\tilde{A}^{cd}\psi^{-7} = 0, \quad (5.2)$$

$$\tilde{D}_c \tilde{A}^{ca} - \frac{2}{3}\psi^6 \tilde{\gamma}^{ca} \tilde{\nabla}_c K = 0, \quad (5.3)$$

where $\tilde{\Delta}$, $\tilde{\mathcal{R}}$ and \tilde{D}_i are respectively the Laplace-Beltrami operator, the scalar curvature and the covariant derivative associated with \tilde{h}_{ab} .

The goal of making the conformal transformation (5.1) is to transform the

Hamilton constraint into a constraint on a single scalar which is the conformal factor. Hence, it is possible to set the conformal metric and then solve equation (5.2) for ψ . In this way, if the initial condition we want is a conformal equivalent of a given metric $\tilde{\gamma}_{ij}$, the Lichnerowicz-York equations give a ‘simple’ way to find the full physical metric \mathbf{h} . To simplify the system even more, it is also generally assumed that the initial condition is time-symmetric, i.e. $K_{ab} = 0$, which leads to the linear equation

$$\tilde{\Delta}\psi - \frac{\tilde{\mathcal{R}}}{8}\psi = 0. \quad (5.4)$$

Misner (1963) made a first application of equation (5.4) in the case of a conformally flat manifold, i.e. $\tilde{h}_{ab} = \text{diag}[1, 1, 1]$, which implies $\tilde{\mathcal{R}}_{ab} = 0$. He wanted to find an initial condition with a lattice of black holes in an asymptotically flat manifold. Since this equation is linear in ψ , the solution for a lattice of black holes is the sum of the solution for one black hole, i.e. a sum of spherically symmetric solutions. It takes the form

$$\psi(x) = \sum_{n=1}^N \frac{M_n}{d[x, x_n]},$$

where N is the number of black holes, M_n are their mass, x_n is the position of the n^{th} black hole and $d[x, x_n]$ is the distance (computed with the conformal metric) between the n^{th} black hole and the point of coordinates x .

This is the strength of the lattice method: we just have to solve a linear equation, and since it is exact, the non-linearity of the Einstein equation is conserved.

Remark. While the resolution of equation (5.4) was done with the conformal metric, the physical metric is still $\mathbf{h} := \psi^4 \tilde{\mathbf{h}}$. Then the physical distances need to be computed using the following line element:

$$dl^2 = \left(\sum_{n=1}^N \frac{M_n}{d[x, x_n]} \right)^4 \{dx^2 + dy^2 + dz^2\}.$$

5.1.2 Application to cosmology

5.1.2.1 Lattice for conformally spherical spaces

Initially the lattice method was developed for asymptotically flat spaces with a compact distribution of matter, i.e. a finite number of black holes in \mathbb{R}^3 . If one wants to apply this method to cosmology, a first solution is to consider an infinite regular distribution corresponding to the images in \mathbb{R}^3 of one black hole in \mathbb{T}^3 (the

covering space representation, see section 1.2.4.2):

$$\psi(x) = \sum_{x_n \in \text{cell}(\mathbb{T}^3)} \frac{M}{d[x, x_n]},$$

where the sum is over the images of the black hole in \mathbb{R}^3 . However, the spatial average, with respect to $\tilde{\mathbf{h}}$, over \mathbb{T}^3 of the Laplacian in equation (5.4) implies that M must be zero. Or in other words, the above sum diverges, unless $M = 0$. This results from the fact that there exists non time-symmetric hypersurfaces, i.e. $K_{ab} = 0$, for spacetimes with closed Euclidean spatial sections (see [Bentivegna et al., 2018](#)). A similar problem arises if we consider a conformally hyperbolic geometry and impose $\tilde{\mathcal{R}}_{ab} = \frac{\tilde{\mathcal{R}}}{3} \tilde{h}_{ab}$, with $\tilde{\mathcal{R}} < 0$. A solution would be to drop the time-symmetry hypothesis, but the Lichnerowicz-York equations could not be solved analytically anymore.

As shown in [Bentivegna et al. \(2018\)](#), the application of the lattice method to cosmology can only be done in conformally spherical spaces, i.e. where $\tilde{\mathcal{R}}_{ab} = \frac{\tilde{\mathcal{R}}}{3} \tilde{h}_{ab}$ with $\tilde{\mathcal{R}} > 0$. In this case, the conformal factor can be written as

$$\psi(x) = \sum_{n=1}^N \frac{m_n}{R_c \sin \frac{d[x, x_n]}{2}}, \quad (5.5)$$

where $R_c := \sqrt{6/\mathcal{R}}$ is the curvature radius and the physical spatial line element is

$$d^2l = \frac{1}{R_c^2} \left(\sum_{n=1}^N \frac{m_n}{2 \sin \frac{d[x, x_n]}{2}} \right)^4 \left\{ d^2\chi + \sin^2 \chi \left(d^2\theta + \sin^2 \theta d^2\varphi \right) \right\}. \quad (5.6)$$

A representation of such an initial condition is given in figure 5.1 in the case of 8 black holes regularly placed on the 3-sphere.

5.1.2.2 Interpretation

While we assumed the conformal metric to be spherical, with $\tilde{\mathcal{R}}_{ab} = \frac{\tilde{\mathcal{R}}}{3} \tilde{h}_{ab}$, this does not mean that the topology of the initial condition is \mathcal{S}^3 . Actually, the physical spatial metric \mathbf{h} , related to the line element (5.6), implies that the volume of each singularity is infinite and that the spatial Ricci curvature \mathcal{R}_{ab} tends to zero. Therefore, each singularity corresponds to an asymptotically infinite flat space. This is represented in figure 5.1 for the top singularity.

To be able to interpret this initial condition as a lattice of black holes in \mathcal{S}^3 , and hence as a cosmological model, each singularity must have horizons which causally separate the spatial slice so that the domain delimited by the set of horizons is compact and non-null. The location of these horizons corresponds to the *marginally outer trapped surfaces* which technical details can be found in [Gibbons \(1972\)](#). We

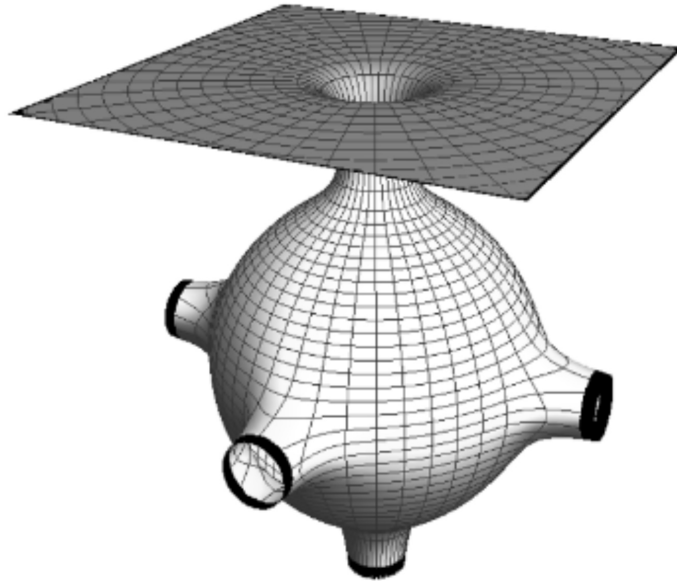


Figure 5.1: Representation of a lattice of 8 black holes in \mathcal{S}^3 (from Clifton, 2015). It shows a 2D slice of the initial condition, passing by 6 black holes and embedded in \mathbb{R}^3 . The infinite asymptotic flat space of the top singularity is represented. The black areas correspond to the horizons of each singularity.

can show that the non-null and compactness condition is fulfilled once we have more than 3 singularities.¹ In figure 5.1, the horizons are represented by the black thick circles. They delimit a compact space.

Remark. To my knowledge, the marginally outer trapped surface gives the exact position of the horizon for static spacetimes. Therefore, it remains well suited for an estimation of this position on a time-symmetric initial slice, i.e. $K_{ab} = 0$. However, it is not clear to me at which point this estimation remains correct during the evolution of this slice. As the interpretation of the model, in terms of black holes in \mathcal{S}^3 , is based on the domain delimited by the set of horizons, this questions the validity of this interpretation during the evolution.

5.1.2.3 Multi-connected spherical topologies

Once we make the above interpretation, the effective topology the initial condition represents can be any multi-connected spherical topology, and not necessarily \mathcal{S}^3 . For this, the black holes need to be regularly placed in \mathcal{S}^3 so that their position correspond to the multiple images of the desired topology in the covering space \mathcal{S}^3 .

¹For two singularities, the two horizons coincide (see figure 1 in Clifton, 2014), and the topology corresponds to a wormhole between two asymptotically flat spaces. Actually this solution corresponds to the spatial sections of the Schwarzschild spacetime in isotropic coordinates.

For instance, if we consider one black hole in the center of the spherical torus, i.e. the spherical space which fundamental domain is a cube, then the images in \mathcal{S}^3 of this black hole are at the positions $\left\{ (0, 0, 0), (\pi, 0, 0), \left(\frac{\pi}{2}, 0, 0\right), \left(\frac{\pi}{2}, \pi, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right), \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\right) \right\}$, in hyperspherical coordinates (see section 3.2.1). The cases in which fundamental domain is one of the other platonic solids are given by Clifton et al. (2012).

5.1.2.4 Evolution of the initial condition

Once the initial condition is set, it remains to evolve it through time with the 3+1-Ricci equation (1.40). This can only be done numerically. As will be presented in section 5.2 and 5.3 of this chapter, there exists no implemented algorithm that allows to simulate non-Euclidean spaces. However, the geometry of the above initial conditions is not spherical because the other side of each singularity is an infinite flat space. Therefore it is possible, by a change of coordinates, to ‘open-up’ one singularity so that the spatial metric becomes conformally flat and the slice represents $N - 1$ blackholes in \mathbb{R}^3 . From this new coordinate system it is then possible to perform the evolution of the initial condition using existing codes (the details of this method can be found in Korzyński, 2014).

To conclude on the method of geometrostatics, we saw that it allows to build cosmological models which can be evolved with existing codes, but which can represent (effective) spherical geometries. In this sense it might be a powerful tool for the study of the influence of topology on the global backreaction. In the next section, we review the literature on this topic.

Remark. For a while I tried to find the NEN theory using the lattice cosmology method. The main reason for which it failed is that this method does not provide the g_{00} components of the spacetime metric, which should lead to the gravitational potential in the classical Newtonian limit.

5.1.3 Overview of the literature on lattice cosmologies

The following studies solved the 3+1-Einstein equations with the initial conditions presented above. The evolution was done with the BSSN formalism, which will be presented in section 5.3.2. Though there is no fluid in the lattice cosmology approach because the Einstein equation is solved in vacuum, it is still possible to define an effective density arising from the black holes masses and compare the expansion of the 3-sphere with the one predicted for a homogeneous universe. Bentivegna & Korzyński (2013) performed the simulation for a lattice of 8 black holes regularly placed on the sphere. Clifton et al. (2013) tested all other possible regular lattices of black holes on \mathcal{S}^3 , i.e. 5, 8, 16, 24, 120 and 600. These setups are equivalent to solving the dynamics of one black hole in the respective topology, i.e. in the spherical spaces which fundamental domains are a tetrahedron, a cube, an octahedron or a

dodecahedron (see [Cavicchioli et al., 2009, 2010](#), for the classification of all compact topologies from Platonic solids).

Both [Bentivegna & Korzyński \(2013\)](#) and [Clifton et al. \(2013\)](#) found little discrepancies with respect to the spherical FLRW evolution for any number of black holes they tested, implying a negligible backreaction. However their very symmetric and simple setup makes difficult the interpretation of these results as representative of the Universe. But still, these studies paved the way for the first exact numerical quantification of the backreaction in non-Euclidean geometries. A more realistic test would be to conduct the same simulations but with a randomly placed increased number of black holes. This would allow to probe the influence of virialisation on the expansion. To my knowledge, it is not possible to perform the full evolution with such number of black holes in a reasonable time. However, it is still possible to analyse the properties of the initial slice with this increased number of black holes, as done by [Durk & Clifton \(2017\)](#). It is also possible to study the continuum limit when this number tends to infinity ([Korzyński, 2014](#)).

This limitation in the use of the lattice cosmology to model the Universe implies that reproducing the formation of large scale structures and their influence on the global expansion requires a fluid approach.

5.2 Relativistic numerical cosmology with fluid models

In this section we present an overview of the literature on the relativistic numerical cosmology using exact fluid schemes. We will focus on the studies which aimed at probing the global backreaction.

5.2.1 Exact fluid schemes

A growing number of papers using an exact relativistic fluid scheme for cosmological simulations have been released the past decade (e.g. [Bentivegna, 2014](#); [Torres et al., 2014](#); [Rekier et al., 2015](#); [Bentivegna & Bruni, 2016](#); [Giblin et al., 2016a](#); [Mertens et al., 2016](#); [Giblin et al., 2016b](#); [Macpherson et al., 2017, 2018, 2019](#)). Most of them do not assume any specific local symmetry during the evolution.² However, they all conduct their simulation in a cubic box with periodic boundary conditions, thus implicitly assuming the topology of a 3-torus for their spatial sections. The main reason for this choice of topology is because it is possible to define a spatial reference metric of the form $h_{ab} = \delta_{ab}$, which is needed for the numerical scheme

²Only [Torres et al. \(2014\)](#) and [Rekier et al. \(2015\)](#) assume spherical symmetry, which reproduces the Lemaître-Tolman-Bondi class of solutions ([Lemaître, 1933](#); [Tolman, 1934](#); [Bondi, 1947](#)). But these do not seek to probe the backreaction, especially [Torres et al. \(2014\)](#) who focuses on the influence of exotic matter on structure formation.

they use (see section 5.3.3). The main difference between these simulations resides in the power spectrum they take for their initial conditions.

[Bentivegna \(2014\)](#) focuses on solving the Einstein constraints, i.e. the Hamilton and the momentum constraints, in compact periodic spaces allowing to properly set the initial conditions. This study is then used in [Bentivegna & Bruni \(2016\)](#) to evolve a dust fluid using the BSSN formalism. The initial density fluctuation is a 1st order harmonic, with initial density contrast of $\delta\rho_1 = 10^{-2}$. The time span of the simulation corresponds to a relative increase in the scale factor of 100, compared to the usually estimated ~ 1000 relative increase between the CMB time and the current epoch. They found negligible backreaction with an evolution of $\mathcal{Q}_{\mathcal{D}} \propto \frac{1}{a}$, consistent with the expected first order behaviour (e.g. [Buchert et al., 2000](#); [Kolb et al., 2005](#)). While there is no specific symmetry in their setup, the simple harmonic they took for the initial condition is not realistic.

[Giblin et al. \(2016a\)](#); [Mertens et al. \(2016\)](#) are the first exact relativistic simulations reproducing structure formation.³ They have a richer initial power spectrum than [Bentivegna & Bruni \(2016\)](#) but it is still not fully realistic since they cut it at high frequencies ($k_{\max} \sim 40 \text{ Mpc}^{-1}$) preventing the smaller structures to form before current epoch. Moreover the density contrast they take is smaller than the expected one using the Planck data and the simulation is only run between redshifts $z = 2$ and $z = 0$. While the expansion of their spatial sections are close to the FLRW solution, they cannot probe the backreaction of the small scale structures because their power spectrum is cut at high frequencies, and their resolution is limited ($\Delta x \sim 16 \text{ Mpc}$).

Because all observations are made in the past light-cone of the Earth, there could be an acceleration of the redshift growth (as observed by [Riess et al., 1998](#); [Perlmutter et al., 1999](#)) without an acceleration of the spatial expansion. Therefore it is also important to analyse the deviation properties of the light-cone around the expectation of the Standard Model of Cosmology. A first study is made in [Giblin et al. \(2016b\)](#), who found little deviations. They use the simulations of [Giblin et al. \(2016a\)](#) and [Mertens et al. \(2016\)](#), and thus have the same limitations in terms of accuracy and realism.

[Giblin et al. \(2017\)](#) developed a scheme aiming at improving this accuracy and the efficiency of simulations using the BSSN formalism for cosmological setups. They use a reference analytical solution to define the numerical variables as deviations (not necessarily small) from this solution. However the introduction of a homogeneous reference solution *could* imply a gauge (or foliation) dependence (see [Bardeen, 1980](#); [Ellis & Bruni, 1989](#)).

In a series of papers, [Macpherson et al. \(2017\)](#), [Macpherson et al. \(2018\)](#) and [Macpherson et al. \(2019\)](#) performed the most realistic cosmological relativistic sim-

³The code used in this series of papers is not currently publicly available.

ulations to date. As with the previous studies they use the BSSN formalism for the evolution of the spatial variables. Their time span is from the CMB time to the current epoch, and they have a resolution of ~ 4 Mpc in a 3-torus of 1 Gpc edges (compared to the 2 Gpc edges and ~ 16 Mpc resolution of [Giblin et al., 2016a](#); [Mertens et al., 2016](#)). The power spectrum is the full power spectrum of the CMB up to the spatial precision of their grid code. They found little global backreaction, with $\Omega_{\mathcal{Q}}^{\mathcal{D}} \sim \Omega_{\mathcal{R}}^{\mathcal{D}} \sim -10^{-8}$. While this is the first realistic numerical evaluation of the backreaction on large scales, we cannot assert that they closed the problem. Indeed, there is place for improvement in many aspects of their simulations and analysis:

- (i) Their time step is at constant conformal time, meaning that the late universe (when backreaction is expected to be the highest) is not well resolved in time.
- (ii) They have a violation of the Hamilton constraint several orders bigger than the measured backreaction.
- (iii) As their fluid is rotational, they cannot use the formalism of [Buchert \(2000, 2001\)](#) to calculate the backreaction. They used the formalism of [Larena \(2009\)](#) which is based on an average of the extrinsic curvature of the chosen foliation (necessarily different from the fluid orthogonal rest frames). This formalism is highly dependent on the foliation, and a better choice would have been the one proposed by [Buchert et al. \(2020\)](#), which is based on an average of the fluid variables, as the original papers ([Buchert, 2000, 2001](#)).

5.2.2 Relativistic fluid/N-body schemes

The previous simulations were all dust: the pressure related to the velocity dispersion inside a fluid element is neglected. But while the energy density associated with this pressure is small in front of the mass density, the velocity dispersion, via shell-crossing, still plays a major role on the stabilisation of structures in the late Universe. In this sense, the pure dust simulation might not capture all the physics of structure formation, and especially the backreaction. That is why schemes mixing a fluid and an N-body approach in general relativity have been developed ([Adamek et al., 2016](#); [Daverio et al., 2019](#); [Barrera-Hinojosa & Li, 2020](#)). However, the numerical schemes used in these studies are not exact, and generally assume small perturbations of the spatial metric.

An exact way to deal with N-bodies in general relativity is to consider black holes and the lattice cosmological models of section 5.1. However, it is only possible to simulate a small number of black holes which is far less than the $10^6 - 10^9$ needed number to fully account for structure formation ([Barrera-Hinojosa & Li, 2020](#)).

5.2.3 The issue of topology

In general relativity, any type of geometry can be considered. For now only the 3-torus has been investigated, mainly for simplicity reasons and because the goal of these studies was not to probe the effect of topology. However, a similar theorem as the Buchert-Ehlers theorem could exist in general relativity. Though it would not imply a strictly zero global backreaction in closed spaces, it could prevent it to grow significantly. Since I was not able to define a non-Euclidean Newtonian theory, it is not possible to have an insight on a possible extension of this theorem in non-Euclidean spaces. This justifies the need for probing the effect of cosmic topology on the global backreaction using relativistic cosmological simulations with fluid matter models.

The studies presented in section 5.2.1 all use the BSSN formalism. However, in its original form introduced by [Shibata & Nakamura \(1995\)](#); [Baumgarte & Shapiro \(1999\)](#), it is best suited for Euclidean geometries. We will see in section 5.3.4 how it can be adapted to allow for other types of geometries.

5.3 Probing the effect of topology with the BSSN formalism

The field of numerical relativity was born in the 60s with the first proposed numerical scheme to solve the Einstein equations ([Hahn & Lindquist, 1964](#)). From then various schemes have been proposed, solving either approximately or exactly these equations. We will only focus on exact schemes.

5.3.1 Numerical schemes in general relativity

Most of the numerical scheme in general relativity are based on the 3+1 formalism in which the 4D space-time is decomposed into space-like hypersurfaces which are evolved with time. However, using the 3+1-equations (or ADM equations) in the form presented in section 1.1.2.6.c [equations (1.40)-(1.43)] as a free evolution scheme, i.e. by solving the constraints only initially, has proven to be numerically unstable, leading the constraint violating modes. That is why other formalisms have been developed, generally based on the ADM equations, but which introduce additional evolution variables in order to stabilise the numerical resolution. We can quote the constrained scheme of [Bonazzola et al. \(2004\)](#). Among the free-schemes, the most popular are the Bona-Masso formalism ([Bona et al., 1995](#)), the NOR formalism ([Nagy et al., 2004](#)), and the BSSN formalism ([Nakamura et al., 1987](#); [Shibata & Nakamura, 1995](#); [Baumgarte & Shapiro, 1999](#)). The latter has proven to be really stable for a wide range of physical problems. It is used by the recent relativistic cosmological simulations of [Bentivegna & Bruni \(2016\)](#); [Giblin et al. \(2016a\)](#); [Macpherson et al. \(2019\)](#), and is the one we will be concerned with in this chapter.

Apart for the recent cosmological applications, these numerical schemes have until now only been tested for isolated systems, i.e. with asymptotic flatness on the spatial hypersurfaces. Thus most of the stability studies concerning either the specific scheme used or the foliation choice (also called slicing condition) concern those kinds of systems. Removing the asymptotic flatness condition and changing the topology could affect the reasons why the code is stable. Thus it is important to study whether or not the usual numerical schemes and the standard foliation choices are suited for any topology. We discuss this issue for the BSSN formalism in section 5.3.4.

5.3.2 The BSSN formalism

5.3.2.1 The BSSN equations

The BSSN formalism is based on the 3+1-Einstein equations but uses a different set of variables. As for the lattice cosmology, it introduces a conformal decomposition of the spatial metric:

$$h_{ab} = \psi^4 \tilde{h}_{ab}, \quad (5.7)$$

along with the conformal traceless extrinsic curvature \tilde{A}_{ij} :

$$\tilde{A}_{ij} := \psi^{-4} K_{(ab)}. \quad (5.8)$$

However, a condition is imposed on the conformal factor

$$\psi := \det(h_{ab})^{1/12}, \quad (5.9)$$

which leads to the following constraint on the conformal metric: $\det(\tilde{h}_{ij}) = 1$. As we will see in section 5.3.3, the condition (5.9) implies that the new variables \tilde{h}_{ij} and ψ are not tensors but tensor densities.

The heart of the BSSN formalism is the introduction of *the conformal connection functions*

$$\tilde{\Gamma}^a := \tilde{h}^{cd} \tilde{\Gamma}_{cd}^a = -\partial_c \tilde{h}^{ca}, \quad (5.10)$$

with

$$\tilde{\Gamma}_{cd}^a := \frac{1}{2} \tilde{h}^{ak} \left(\partial_c \tilde{h}_{dk} + \partial_d \tilde{h}_{ck} - \partial_k \tilde{h}_{cd} \right).$$

Standard slicing conditions are usually based on this variable.

Then the set of independent variables used in the BSSN formalism is

$$\psi \ ; \ \tilde{h}_{ij} \ ; \ K \ ; \ \tilde{A}_{ij} \ ; \ \tilde{\Gamma}^i \ ; \ N \ ; \ \beta^i.$$

The set evolution and constraint equations of these variables is obtained when inserting them in the 3+1-Einstein equations.

The BSSN equations:

Evolution equations:

$$\partial_0 \psi = -\frac{\psi}{6} \left(NK - \partial_k \beta^k \right), \quad (5.11)$$

$$\partial_0 \tilde{h}_{ij} - 2\tilde{h}_{k(i} \partial_j) \beta^k = -2N \tilde{A}_{ij} - \frac{2}{3} \tilde{h}_{ij} \partial_k \beta^k, \quad (5.12)$$

$$\begin{aligned} \partial_0 K &= -\psi^{-4} \tilde{h}^{kl} \left(\tilde{D}_k \partial_l N + 2 \partial_k \ln \psi \partial_l N \right) \\ &\quad + N \left[4\pi (E + 3P) + \tilde{A}_{kl} \tilde{A}^{kl} + \frac{1}{3} K^2 \right], \end{aligned} \quad (5.13)$$

$$\begin{aligned} \partial_0 \tilde{A}_{ij} - 2\tilde{A}_{k(i} \partial_j) \beta^k &= \psi^{-4} \left[-\tilde{D}_i \partial_j N + 4 \partial_{(i} \ln \psi \partial_{j)} N + N \left(\tilde{R}_{ij} + \tilde{R}_{ij}^\psi - 8\pi \Pi_{ij} \right) \right] \\ &\quad + NK \tilde{A}_{ij} - 2N \tilde{h}^{kl} \tilde{A}_{ik} \tilde{A}_{lj} - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k \end{aligned} \quad (5.14)$$

$$\begin{aligned} &\quad - \frac{\tilde{h}_{ij}}{3} \psi^{-4} \left[-\tilde{h}^{kl} \left(\tilde{D}_k \partial_l N + 4 \partial_k \ln \psi \partial_l N \right) \right. \\ &\quad \left. + N \left(\tilde{R} + \tilde{R}^\psi - 24\pi \psi^4 P \right) \right], \end{aligned}$$

$$\begin{aligned} \partial_0 \tilde{\Gamma}^i + \tilde{\Gamma}^k \partial_k \beta^i &= 2N \left(6\tilde{A}^{ik} \partial_k \ln \psi + \tilde{A}^{kl} \tilde{\Gamma}^i{}_{kl} - \frac{2}{3} \tilde{h}^{ik} \partial_k K - 8\pi \psi^4 p^i \right) \\ &\quad + \tilde{h}^{kl} \partial_k \partial_l \beta^i + \frac{1}{3} \tilde{h}^{il} \partial_l \partial_k \beta^k - 2\tilde{A}^{ik} \partial_k N + \frac{2}{3} \tilde{\Gamma}^i \partial_k \beta^k, \end{aligned} \quad (5.15)$$

with the constraint equations

$$\psi^{-4} \left(\tilde{R} + \tilde{R}^\psi \right) + \frac{2}{3} K^2 - \tilde{A}_{ij} \tilde{A}^{ij} = 16\pi E, \quad (5.16)$$

$$\partial_k \tilde{A}^{ik} + \tilde{A}^{kl} \tilde{\Gamma}^i{}_{kl} + 6\tilde{A}^{ik} \partial_k \ln \psi - \frac{2}{3} \tilde{h}^{ik} \partial_k K = 8\pi \psi^4 Q^i, \quad (5.17)$$

where $\partial_0 := \partial_t - \beta^k \partial_k$. \tilde{D}_i , \tilde{R}_{ij} and \tilde{R} are respectively the covariant derivative and the Ricci tensor and scalar of the conformal metric. We also have

$$\begin{aligned} \tilde{R}_{ij}^\psi &:= -2 \left(\partial_i \partial_j \ln \psi - \tilde{\Gamma}^k{}_{ij} \partial_k \ln \psi \right) - 2\tilde{h}_{ij} \tilde{h}^{kl} \left(\partial_k \partial_l \ln \psi - \tilde{\Gamma}^n{}_{kl} \partial_n \ln \psi \right) \\ &\quad + 4 \partial_i \ln \psi \partial_j \ln \psi - 4\tilde{h}_{ij} \tilde{h}^{kl} \partial_k \ln \psi \partial_l \ln \psi, \end{aligned} \quad (5.18)$$

$$\tilde{R}^\psi := \tilde{h}^{kl} \tilde{R}_{kl}^\psi = -8 \tilde{h}^{kl} \left(\partial_k \partial_l \ln \psi - \tilde{\Gamma}^n{}_{kl} \partial_n \ln \psi + \partial_k \ln \psi \partial_l \ln \psi \right). \quad (5.19)$$

The conformal Ricci tensor and scalar are

$$\tilde{R}_{ij} = -\frac{1}{2} \tilde{h}^{kl} \partial_k \partial_l \tilde{h}_{ij} + \tilde{h}_{k(i} \partial_{j)} \tilde{\Gamma}^k + \mathcal{Q}_{ij}(\tilde{\mathbf{h}}, \partial \tilde{\mathbf{h}}), \quad (5.20)$$

$$\tilde{R} = \partial_k \tilde{\Gamma}^k + \frac{1}{2} \tilde{h}^{kl} \partial_k \tilde{h}^{ij} \partial_l \tilde{h}_{ij} + \tilde{h}^{ij} \mathcal{Q}_{ij}(\tilde{\mathbf{h}}, \partial \tilde{\mathbf{h}}), \quad (5.21)$$

with

$$\mathcal{Q}_{ij}(\tilde{\mathbf{h}}, \partial \tilde{\mathbf{h}}) = -\partial_k \tilde{h}_{l(i} \partial_{j)} \tilde{h}^{kl} + \frac{1}{2} \tilde{\Gamma}^k \partial_k \tilde{h}_{ij} - \tilde{\Gamma}^k{}_{il} \tilde{\Gamma}^l{}_{jk}. \quad (5.22)$$

5.3.2.2 Stability of the BSSN formalism

When deriving the evolution equation for $\tilde{\Gamma}^i$, the ‘divergence’ of $\partial_k \tilde{A}^{ik}$ appears, i.e. $\tilde{D}_i \partial_k \tilde{A}^{ik}$. The standard BSSN equations use the momentum constraint to remove this divergence. This is a key point for the stability of this numerical scheme. [Alcubierre et al. \(2000\)](#) showed that this is likely due to the constraint violating modes which are damped if we perform this analytical trick. Removing the Ricci scalar in the evolution of K is also essential for the system of equations to be strongly hyperbolic. In other words, the scheme is more stable if, instead of the trace of the 3+1-Ricci equation, the 3+1-Raychaudhuri equation is used to evolve K .

The introduction of the $\tilde{\Gamma}^i$ functions allows to write a wave equation for the conformal spatial metric (see section 11.4.3 in [Gourgoulhon, 2012](#)). The time derivative of equation (5.12) along with equation (5.14) lead to

$$\left[(\partial_t - \mathcal{L}_\beta)^2 - \frac{N^2}{\psi^4} \tilde{h}^{kl} \partial_k \partial_l \right] \tilde{h}_{ij} = \mathcal{F}(\tilde{h}_{ij}, \partial \tilde{h}_{ij}, \psi, K, \tilde{A}_{ij}) + 2 \frac{N^2}{\psi^4} \tilde{h}_{k(i} \partial_{j)} \tilde{\Gamma}^k, \quad (5.23)$$

where \mathcal{F} is a function which does not depend on second derivatives of \tilde{h}_{ij} . The left hand-side of this equation is a wave operator in the limit where \tilde{h}_{ij} is a perturbation around the Euclidean metric in Cartesian coordinates. The right hand-side can be interpreted as a source term of the wave equation as it features no second spatial derivatives of \tilde{h}_{ij} .

Obtaining this wave equation for the conformal metric is a key point which participates to the numerical stability of the BSSN formalism ([Gourgoulhon, 2012](#)). This has been made possible with the introduction of $\tilde{\Gamma}^i$ as a new independent variable by [Shibata & Nakamura \(1995\)](#); [Baumgarte & Shapiro \(1999\)](#).

Remark. It is also possible to define $\tilde{\Gamma}^i$ without introducing the conformal transformation. This is the NOR formalism of [Nagy et al. \(2004\)](#).

5.3.3 Coordinate system in the BSSN formalism

As quoted previously, ψ , \tilde{h}_{ij} , \tilde{A}_{ij} and $\tilde{\Gamma}^i$ are tensor densities because of the constraint (5.9). A tensor density of weight w , denoted ${}^w T^{\mu_1 \dots \nu_1 \dots}$, is an object which, under a coordinate transformation, transforms as

$$({}^w T')^{\mu_1 \dots \nu_1 \dots} = ({}^0 T')^{\mu_1 \dots \nu_1 \dots} J^w, \quad (5.24)$$

where J is the determinant of the Jacobian of the coordinate transformation and ${}^0 T'$ is the tensor in the new coordinate system if it had a weight of zero.

Remark. Due to this difference of transformation with standard tensors, the covariant and Lie derivatives have extra terms depending on the weight w of the tensor density they are applied on (see appendix A.2). Thus care must be taken when writing the BSSN equations

with the covariant and Lie derivative notations as quoted in Baumgarte & Shapiro (1999) and Alcubierre et al. (2003). In section 5.3.2.1, all the equations are written in a way that avoids this confusion, i.e. the only times covariant derivatives appear, they are applied on true tensors.

So because most of the variables are tensor densities, the BSSN equations are not covariant. This in itself is not necessarily a major problem as these equations are still general and can be used in any coordinate system. But in practice, some limitations appear:

- Dealing with tensor densities makes it difficult to compare evolutions done with the same slicing conditions but different spatial coordinate systems.
- In cosmology, the initial condition for the spatial metric is usually set as a conformal perturbation around a flat metric (e.g. Giblin et al., 2016a; Macpherson et al., 2019), i.e. with $\tilde{h}_{ab} = \delta_{ab}$ for Cartesian coordinates. This is compatible with the condition $\det(\tilde{h}_{ij}) = 1$ imposed in section 5.3.2.1. However, if one wants to use non-Cartesian coordinates, or even a non flat conformal metric, it appears difficult to keep this condition (see Alcubierre & Mendez, 2011).
- A third drawback linked to the non-covariance of the BSSN equations concerns the numerical stability of the scheme. The stability analysis of Alcubierre et al. (2000) relies on a perturbation of the spatial metric around a flat metric in Cartesian coordinates. Such a perturbation requires the spatial sections to have an Euclidean geometry, so that it is possible to define a flat metric. But in the case we want to impose a non-Euclidean spatial geometry, it is not sure at which point the stability of the BSSN equations is kept. Furthermore, we saw in section 5.3.2.2 that part of the stability is also due to the presence of equation (5.23) which becomes a wave equation for the conformal metric if \tilde{h}_{ab} is a perturbation around a flat metric in Cartesian coordinates (otherwise there is not anymore a Laplace-Beltrami operator in the lhs). Then, again, assuming a non-Euclidean spatial geometry might lead to a loss of stability.

To get rid of these problems, Brown (2009) proposed a *covariant formulation of the BSSN formalism*. It does not deal anymore with tensor densities, the condition $\det(\tilde{h}_{ij}) = 1$ is not present and the wave equation can be obtained in any considered geometry. As a side effect, Alcubierre & Mendez (2011) showed that using this covariant formulation also allows them to get rid of most of the coordinate singularities and poles of this coordinate system. These artificial singularities could indeed be difficult to treat numerically. In the next section we detail the equations of the covariant BSSN formalism.

5.3.4 The covariant BSSN formalism

5.3.4.1 The reference metric

The key difference with the standard BSSN formalism is the introduction of a *reference metric*, denoted \hat{h} . At first, the only requirement on this metric is to be compatible with the spatial geometry of the numerical setup.

Remark. In the literature on the covariant BSSN formalism, \hat{h} is called the background metric (e.g. [Brown, 2009](#); [Alcubierre & Mendez, 2011](#); [Gourgoulhon, 2012](#)). But because this name already has a definition in cosmology, which generally implies perturbations around \hat{h} (which is not the case here), I prefer the name ‘reference metric’.

Then the condition (5.9) is replaced by

$$\psi := \left(\frac{h}{\hat{h}} \right)^{1/12}, \quad (5.25)$$

where we denote $h := \det h_{ab}$ and $\hat{h} := \det \hat{h}_{ab}$. This relation implies $\tilde{h} := \det \tilde{h}_{ab} = \hat{h}$. The gamma functions are also redefined to be

$$\tilde{\Gamma}^a := \hat{D}_k \tilde{h}^{ki}, \quad (5.26)$$

where \hat{D} is the connection associated with the reference metric \hat{h} . With these new definitions, the variables ψ , \tilde{h}_{ab} , \tilde{A}_{ab} , and $\tilde{\Gamma}^a$ are now true tensors, and not tensor densities anymore.

Then the set of independent variables used in the covariant BSSN formalism is

$$\psi \ ; \ \tilde{h}_{ij} \ ; \ K \ ; \ \tilde{A}_{ij} \ ; \ \tilde{\Gamma}^i \ ; \ N \ ; \ \beta^i \ ; \ \hat{h}_{ij}.$$

There is one additional variable which is the reference metric. The standard formalism correspond to $\hat{h}_{ab} = \delta_{ab}$.

Without additional hypothesis, the variables \hat{h}_a , β^a and N have no evolution equations. As already stated, such an equation for the last two variables is called a slicing condition and corresponds to a choice of foliation. However, with the introduction of the reference metric we need an additional evolution equation compared with the standard BSSN formalism.

5.3.4.2 Equations without condition choice

We write in this section the 3+1-Einstein equations with the previous variables, without specifying an evolution equation for \hat{h}_{ab} :

Evolution equations:

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta) \psi &= -\frac{\psi}{6} \left[NK + \frac{1}{2} (\partial_t - \mathcal{L}_\beta) \ln \hat{h} \right], \\
(\partial_t - \mathcal{L}_\beta) \tilde{h}_{ij} &= -2N \tilde{A}_{ij} + \frac{1}{3} \tilde{h}_{ij} (\partial_t - \mathcal{L}_\beta) \ln \hat{h}, \\
(\partial_t - \mathcal{L}_\beta) K &= -\psi^{-4} \tilde{h}^{kl} \left(\tilde{D}_k \partial_l N + 2 \partial_k \ln \psi \partial_l N \right) + N \left[4\pi (S + \rho) + \tilde{A}_{ij} \tilde{A}^{ij} + \frac{K^2}{3} \right], \\
(\partial_t - \mathcal{L}_\beta) \tilde{A}_{ij} &= \psi^{-4} \left[-\tilde{D}_i \partial_j N + 4 \partial_{(i} \ln \psi \partial_{j)} N + N \left(\tilde{R}_{ij} + \tilde{R}_{ij}^\psi - 8\pi S_{ij} \right) \right] \\
&\quad - \frac{\tilde{h}_{ij}}{3} \psi^{-4} \left[\tilde{h}^{kl} \left(-\tilde{D}_k \partial_l N + 4 \partial_k \ln \psi \partial_l N \right) + N \left(\tilde{R} + \tilde{R}^\psi - 8\pi \psi^4 S \right) \right] \\
&\quad + NK \tilde{A}_{ij} - 2N \tilde{h}^{kl} \tilde{A}_{ik} \tilde{A}_{lj} + \frac{1}{3} \tilde{A}_{ij} (\partial_t - \mathcal{L}_\beta) \ln \hat{h}, \\
(\partial_t - \mathcal{L}_\beta) \tilde{\Gamma}^i &= 2N \left(6 \tilde{A}^{ik} \partial_k \ln \psi + \tilde{A}^{kl} \Delta^i{}_{kl} - \frac{2}{3} \tilde{h}^{ki} \partial_k K - 8\pi \psi^4 p^i \right) + \tilde{h}^{kl} \hat{D}_k \hat{D}_l \beta^i \\
&\quad + \tilde{h}^{il} \hat{D}_l \hat{D}_k \beta^k - 2 \tilde{A}^{ik} \partial_k N + \beta^n \tilde{h}^{kl} \hat{R}^i{}_{knl} - \tilde{h}^{kl} \partial_t \hat{\Gamma}^i{}_{kl} \\
&\quad + \frac{1}{3} \tilde{h}^{ik} \hat{D}_k \left[(\partial_t - \mathcal{L}_\beta) \ln \hat{h} \right] - \frac{1}{3} \tilde{\Gamma}^i (\partial_t - \mathcal{L}_\beta) \ln \hat{h}.
\end{aligned}$$

Constraint equations:

$$\begin{aligned}
\psi^{-4} \left(\tilde{R} + \tilde{R}^\psi \right) + \frac{2}{3} K^2 - \tilde{A}_{ij} \tilde{A}^{ij} &= 16\pi \rho, \\
\hat{D}_k \tilde{A}^{ik} + \tilde{A}^{kl} \Delta^i{}_{kl} + 6 \tilde{A}^{ik} \partial_k \ln \psi - \frac{2}{3} \tilde{h}^{ik} \partial_k K &= 8\pi \psi^4 p^i,
\end{aligned}$$

with

$$\begin{aligned}
\Delta^i{}_{kl} &:= \tilde{\Gamma}^i{}_{kl} - \hat{\Gamma}^i{}_{kl}, \\
\tilde{R}_{ij}^\psi &:= -2 \tilde{D}_i \partial_j \ln \psi - 2 \tilde{h}_{ij} \tilde{h}^{kl} \tilde{D}_k \partial_l \ln \psi + 4 \partial_i \ln \psi \partial_j \ln \psi - 4 \tilde{h}_{ij} \tilde{h}^{kl} \partial_k \ln \psi \partial_l \ln \psi, \\
\tilde{R}^\psi &:= \tilde{h}^{kl} \tilde{R}_{kl}^\psi = -8 \tilde{h}^{kl} \left(\partial_k \partial_l \ln \psi - \tilde{\Gamma}^n{}_{kl} \partial_n \ln \psi + \partial_k \ln \psi \partial_l \ln \psi \right).
\end{aligned}$$

Conformal Ricci and scalar curvature:

$$\begin{aligned}
\tilde{R}_{ij} &= -\frac{1}{2} \tilde{h}^{kl} \hat{D}_k \hat{D}_l \tilde{h}_{ij} + \tilde{h}_{k(i} \hat{D}_{j)} \tilde{\Gamma}^k + \mathcal{Q}_{ij}(\tilde{\mathbf{h}}, \hat{\mathbf{D}}\tilde{\mathbf{h}}) + \tilde{h}_{n(i} \tilde{h}^{kl} \hat{R}^n{}_{kj)l}, \\
\tilde{R} &= \hat{D}_k \tilde{\Gamma}^k + \frac{1}{2} \tilde{h}^{kl} \hat{D}_k \tilde{h}^{ij} \hat{D}_l \tilde{h}_{ij} + \tilde{h}^{ij} \mathcal{Q}_{ij}(\tilde{\mathbf{h}}, \hat{\mathbf{D}}\tilde{\mathbf{h}}) + \tilde{h}^{nk} \hat{R}_{nk},
\end{aligned}$$

with

$$\mathcal{Q}_{ij}(\tilde{\mathbf{h}}, \hat{\mathbf{D}}\tilde{\mathbf{h}}) = -\hat{D}_k \tilde{h}_{l(i} \hat{D}_{j)} \tilde{h}^{kl} + \frac{1}{2} \tilde{\Gamma}^k \hat{D}_k \tilde{h}_{ij} - \Delta^k{}_{il} \Delta^l{}_{kj}.$$

The coloured terms are additional terms compared with the standard BSSN formalism: the blue terms are related to the reference determinant, the green terms to the time derivative of the reference connection coefficients and the red terms to the

Riemann curvature of the reference metric. In addition to these additional terms, the partial derivatives of the standard formulation are replaced by the reference covariant derivative.

Remark. [Brown \(2009\)](#) defines a reference connection $\hat{\Gamma}^c_{ab}$ without the introduction of a reference metric. This is possible as \hat{h}_{ab} only appears in the above system with its determinant \hat{h} , which can be replaced by \tilde{h} (this slightly differs from our approach which corresponds to the one proposed by [Gourgoulhon, 2012](#)). He imposes that this reference connection is independent of time. Hence he only has the additional blue and red terms.

5.3.4.3 Equations with condition choice

For the sake of generality we wrote the covariant BSSN equations without giving an evolution equation for the reference metric. However, in practice the reference metric is always assumed to be independent of time (e.g. [Bonazzola et al., 2004](#); [Alcubierre & Mendez, 2011](#); [Gourgoulhon, 2012](#)), i.e. $\partial_t \hat{h}_{ij} = 0$, essentially because there is no numerical reasons to take a more complex evolution equation. With this choice the reference metric coefficients are constant along the flow lines of the time basis vector, and we have $\partial_t \hat{\Gamma}^i_{kl} = 0$ and $\partial_t \ln \hat{h} = 0$. Then the 3+1-Einstein equations written in terms of ψ , \tilde{h}_{ij} , K , \tilde{A}_{ij} , $\tilde{\Gamma}^i$, N , β^i and \hat{h}_{ij} become:

Covariant BSSN formalism with “ $\partial_t \hat{h}_{ij} = 0$ ”: Evolution equations:

$$(\partial_t - \mathcal{L}_\beta) \psi = -\frac{\psi}{6} \left(NK - \tilde{D}_k \beta^k - \frac{1}{2} \mathcal{L}_\beta \ln \hat{h} \right), \quad (5.27)$$

$$(\partial_t - \mathcal{L}_\beta) \tilde{h}_{ij} = -2N \tilde{A}_{ij} - \frac{2}{3} \tilde{h}_{ij} \tilde{D}_k \beta^k - \frac{1}{3} \tilde{h}_{ij} \mathcal{L}_\beta \ln \hat{h}, \quad (5.28)$$

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta) K &= -\psi^{-4} \tilde{h}^{kl} \left(\tilde{D}_k \partial_l N + 2 \partial_k \ln \psi \partial_l N \right) \\ &\quad + N \left[4\pi (E + 3P) + \tilde{A}_{ij} \tilde{A}^{ij} + \frac{K^2}{3} \right], \end{aligned} \quad (5.29)$$

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta) \tilde{A}_{ij} &= \psi^{-4} \left[-\tilde{D}_i \partial_j N + 4 \partial_{(i} \ln \psi \partial_{j)} N + N \left(\tilde{R}_{ij} + \tilde{R}_{ij}^\psi - 8\pi \Pi_{ij} \right) \right] \\ &\quad - \frac{\tilde{h}_{ij}}{3\psi^4} \left[\tilde{h}^{kl} \left(-\tilde{D}_k \partial_l N + 4 \partial_k \ln \psi \partial_l N \right) + N \left(\tilde{R} + \tilde{R}^\psi - 24\pi \psi^4 P \right) \right] \\ &\quad + NK \tilde{A}_{ij} - 2N \tilde{h}^{kl} \tilde{A}_{ik} \tilde{A}_{lj} - \frac{2}{3} \tilde{A}_{ij} \tilde{D}_k \beta^k - \frac{1}{3} \tilde{A}_{ij} \mathcal{L}_\beta \ln \hat{h}, \end{aligned} \quad (5.30)$$

$$\begin{aligned} (\partial_t - \mathcal{L}_\beta) \tilde{\Gamma}^i &= 2N \left(6 \tilde{A}^{ik} \partial_k \ln \psi + \tilde{A}^{kl} \Delta^i{}_{kl} - \frac{2}{3} \tilde{h}^{ki} \partial_k K - 8\pi \psi^4 p^i \right) + \tilde{h}^{kl} \hat{D}_k \hat{D}_l \beta^i \\ &\quad + \frac{1}{3} \tilde{h}^{il} \hat{D}_l \hat{D}_k \beta^k - 2 \tilde{A}^{ik} \partial_k N + \frac{2}{3} \tilde{\Gamma}^i \hat{D}_k \beta^k + \beta^n \tilde{h}^{kl} \hat{R}^i{}_{knl} \\ &\quad + \frac{1}{3} \left(\tilde{\Gamma}^i - \tilde{h}^{ik} \partial_k \right) \mathcal{L}_\beta \ln \hat{h}. \end{aligned} \quad (5.31)$$

Constraint equations:

$$\psi^{-4} \left(\tilde{R} + \tilde{R}^\psi \right) + \frac{2}{3} K^2 - \tilde{A}_{ij} \tilde{A}^{ij} = 16\pi E, \quad (5.32)$$

$$\hat{D}_k \tilde{A}^{ik} + \tilde{A}^{kl} \Delta^i{}_{kl} + 6 \tilde{A}^{ik} \partial_k \ln \psi - \frac{2}{3} \tilde{h}^{ik} \partial_k K = 8\pi \psi^4 Q^i, \quad (5.33)$$

with

$$\Delta^i{}_{kl} := \tilde{\Gamma}^i{}_{kl} - \hat{\Gamma}^i{}_{kl}, \quad (5.34)$$

$$\tilde{R}_{ij}^\psi := -2 \tilde{D}_i \partial_j \ln \psi - 2 \tilde{h}_{ij} \tilde{h}^{kl} \tilde{D}_k \partial_l \ln \psi + 4 \partial_i \ln \psi \partial_j \ln \psi - 4 \tilde{h}_{ij} \tilde{h}^{kl} \partial_k \ln \psi \partial_l \ln \psi, \quad (5.35)$$

$$\tilde{R}^\psi := \tilde{h}^{kl} \tilde{R}_{kl}^\psi = -8 \tilde{h}^{kl} \left(\partial_k \partial_l \ln \psi - \tilde{\Gamma}^n{}_{kl} \partial_n \ln \psi + \partial_k \ln \psi \partial_l \ln \psi \right). \quad (5.36)$$

Conformal Ricci and scalar curvature:

$$\tilde{R}_{ij} = -\frac{1}{2} \tilde{h}^{kl} \hat{D}_k \hat{D}_l \tilde{h}_{ij} + \tilde{h}_{k(i} \hat{D}_{j)} \tilde{\Gamma}^k + \mathcal{Q}_{ij} \left(\tilde{\mathbf{h}}, \hat{\mathbf{D}} \tilde{\mathbf{h}} \right) + \tilde{h}_{n(i} \tilde{h}^{kl} \hat{R}^n{}_{kj} \tilde{h}_{l)}, \quad (5.37)$$

$$\tilde{R} = \hat{D}_k \tilde{\Gamma}^k + \frac{1}{2} \tilde{h}^{kl} \hat{D}_k \tilde{h}^{ij} \hat{D}_l \tilde{h}_{ij} + \tilde{h}^{ij} \mathcal{Q}_{ij} \left(\tilde{\mathbf{h}}, \hat{\mathbf{D}} \tilde{\mathbf{h}} \right) + \tilde{h}^{nk} \hat{R}_{nk}, \quad (5.38)$$

with

$$\mathcal{Q}_{ij} \left(\tilde{\mathbf{h}}, \hat{\mathbf{D}} \tilde{\mathbf{h}} \right) = -\hat{D}_k \tilde{h}_{l(i} \hat{D}_{j)} \tilde{h}^{kl} + \frac{1}{2} \tilde{\Gamma}^k \hat{D}_k \tilde{h}_{ij} - \Delta^k{}_{il} \Delta^l{}_{kj}. \quad (5.39)$$

This system of equations ensures to keep the numerical stability of the BSSN scheme, along with allowing for non-Euclidean spatial geometries. In particular, one reason for which we introduced the reference metric was to recover the wave equation (5.23) in non-Euclidean geometries. In the case of the covariant BSSN formalism, this equation becomes

$$\left[(\partial_t - \mathcal{L}_\beta)^2 - \frac{N^2}{\psi^4} \tilde{h}^{kl} \hat{D}_k \hat{D}_l \right] \tilde{h}_{ij} = \mathcal{F}(\tilde{h}_{ij}, \hat{D}\tilde{h}_{ij}, \psi, K, \tilde{A}_{ij}) + 2 \frac{N^2}{\psi^4} \tilde{h}_{k(i} \hat{D}_{j)} \tilde{\Gamma}^k, \quad (5.40)$$

where the lhs indeed corresponds to a wave operator if we assume that \tilde{h}_{ab} is a perturbation around \hat{h}_{ab} .

5.3.5 Slicing conditions

In the 3+1-Einstein equations, the lapse and the shift are non-dynamical variables as they do not have an evolution equation. Imposing such an equation for N and β is called a *slicing condition* and corresponds to choosing a foliation and an adapted coordinate system in the spacetime manifold. Many different conditions can be chosen depending on what we want to simulate and the degree of accuracy we need to keep during the numerical evolution (see chapter 10 of [Gourgoulhon, 2012](#)). The situations in which the different slicing conditions have been studied are generally black holes or Euclidean cosmological simulations. But none of them are in non-Euclidean spaces. Therefore it is important to assess whether or not we need to adapt the usual slicing conditions as function of the topology. Such a study is beyond the scope of this thesis.

5.4 Which topology to choose?

Two factors drive the choice of topology for the first non-Euclidean cosmological simulation:

- (i) The practical factor: the topology should be implementable on existing softwares (which aim at numerically solving the Einstein equation) without major modifications of the code. Ideally, the main modification should only concern the evolution equations, without changing the type of coordinate system and the shape of the domain of simulation, which is generally cubic.
- (ii) The physical factor: as a first requirement, the space needs to be orientable. A reason for this is that in a non-orientable space in which cosmic neutrinos could have travelled at least once the whole Universe, we could observe both helicities for these particles, which is not the case for now.⁴ Furthermore, for simplicity, and because there are no observational reasons for another choice, we focus on

⁴This argument only work if the size of the Universe is smaller than the CMB.

globally isotropic topologies, i.e. topologies for which each axis of symmetry is equivalent. This concerns all the topologies in which the fundamental domain is a Platonic solid, which is only possible for either spherical, Euclidean or hyperbolic geometries (Cavicchioli et al., 2009, 2010). Ideally, the volume of the simulated universe should be compatible with actual constraints giving a lower bound of around 10 to 20 (Gpc/h)³ (see section 1.2.4.4). But in practice, we will only be able to simulate up to a 1 Gpc³ size universe in a reasonable time (Giblin et al., 2016a; Macpherson et al., 2018).

Remark. Because the size of the simulation can only be around 1 Gpc³, the scalar curvature is expected to be much higher than the one of our Universe. If the backreaction effect is amplified by this curvature, as is the case for the first NEN theory in chapter 3, I expect this phenomenon to be overestimated by this simulation. This should not be a major problem: firstly it would allow for a more robust detection of the curvature dependence of the backreaction, and secondly, by applying a scaling law, we could estimate the value of the backreaction in a non-Euclidean universe compatible with observations, i.e. with a smaller scalar curvature.

We could think about the 3-sphere to be the first non-Euclidean geometry that should be tested, essentially because it is the only simply-connected closed topology, and hence could be called the ‘simplest’ closed topology. However the main drawback of this choice is that there exists no non-singular coordinate system on this manifold. Therefore the necessary coordinate singularities require regularisations which need to be studied and developed specifically for the chosen coordinate system (as done for spherical coordinates by Alcubierre & Mendez, 2011). Instead I propose *the spherical torus*,⁵ i.e. the well known quaternionic space $\mathcal{S}^3/_{(222)}$ (the manifold M3 in Cavicchioli et al., 2009, Freudet, 2020). The reasons for this choice are the following:

- For the same volume (or size), the scalar curvature is smaller smaller than that of \mathcal{S}^3 . In this sense for a 1 Gpc³ size spherical torus, the corresponding scalar curvature is closer to the restrictions given by the Planck data (Planck Collaboration et al., 2020) than the curvature of a 1 Gpc³ 3-sphere.
- There exists a non-degenerate coordinate system defined everywhere on the manifold.
- The fundamental domain is a cube. This implies that there exists a coordinate system (as for now I have not found it) for which the interval of definition of the coordinates is a cube (in \mathbb{R}^3), i.e. the interval of definition of one coordinate is independent of the other coordinates. This makes the adaptation of existing codes (like *Einstein ToolKit*) simpler, as the already implemented domain is generally a cube.
- All the axes of symmetry of the spherical torus are equivalent. This is not the case of the lens space $L(8, 3)$ (the manifold M2 in Cavicchioli et al., 2009),

⁵I am not sure at which point ‘spherical torus’ is a used name for this 3-manifold in mathematics.

which is also a spherical 3-manifold for which one of the choices of fundamental domain is a cube, but whose face associations are different and not equivalent.

A hyperbolic geometry could not fulfil the third point because there exists no hyperbolic 3-manifold for which the fundamental domain is a cube (see [Cavicchioli et al., 2009, 2010](#), for a classification of hyperbolic 3-manifolds for which the fundamental domain is a platonic solid).

5.5 Conclusion

The goal of this chapter was to present an overview of the literature concerning relativistic cosmological simulations, along with an analysis on how to perform such simulations in non-Euclidean geometries. We saw that while there is an increasing precision of exact relativistic cosmological simulations ([Mertens et al., 2016](#); [Macpherson et al., 2018](#), being the most realistic simulations to date) allowing them to give a first numerical value of the backreaction, they are still limited for two main reasons:

- The matter model is a dust fluid. Hence, they cannot account for the shell-crossing and the virialisation, i.e. the stabilisation of the structures.
- The geometry is always Euclidean, with the 3-torus.

The first point is important as an expected behaviour of the backreaction is to be dominant during virialisation of the structures. A solution to account for shell-crossing is to perform simulations mixing a fluid and an N-body approach as proposed by [Adamek et al. \(2016\)](#); [Daverio et al. \(2019\)](#); [Barrera-Hinojosa & Li \(2020\)](#). However, the numerical schemes used in these studies are not exact, and generally assume small perturbations of the spatial metric.

The second point is also a major issue because the global backreaction might be highly dependent on the topology, as explained throughout this thesis. We first saw how it is possible to perform numerical simulations of spherical topologies using geometrostatics: this is the method of lattice cosmology. However, it is limited as it cannot describe small structures, and the topology is only ‘effectively’ spherical. Secondly we analysed why the most widely used (in cosmology) relativistic fluid scheme, called BSSN formalism, can only simulate Euclidean geometries. To remedy this problem, we showed that the covariant BSSN formalism (introduced by [Brown, 2009](#)) is the solution, and we provided the system of equations [equations (5.27)–(5.39)].

Finally we argued that the spherical torus, i.e. the topological space $\mathcal{S}^3/_{(222)}$, should be the simplest spherical geometry for a first exact non-Euclidean relativistic cosmological simulation.

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Chapter 6

Conclusion and perspectives

IN this PhD thesis I developed tools aiming at studying the effects of topology in inhomogeneous cosmology. More particularly, I have put a strong emphasis on the construction of a non-Euclidean Newtonian theory in order to generalise the Buchert-Ehlers theorem. In what follows I present a summary of the main results and the perspectives for future works.

In chapter 1, I presented the context of this thesis. We saw that the current Standard Model of Cosmology suffers from a major hypothesis: the global expansion of our Universe is supposed to be not affected by its small scale inhomogeneities. This neglected phenomenon, called *backreaction*, could actually play a major role in cosmology (Buchert, 2008), and possibly explain the dark energy problem and the discrepancy in the measure of the Hubble constant (Riess, 2019). While most of the studies concerning the backreaction focused on its relation with structure formation, little has been made to understand its dependence on the topology of our Universe. To tackle this problem I followed two different strategies:

- (i) developing a non-Euclidean Newtonian theory from general relativity (chapters 2 to 4). While such a theory would still allow for the encoding of the non-linearities and all the global conditions (i.e. topology) of general relativity, it would also be simpler to use.
- (ii) proposing a numerical relativistic scheme allowing for the simulation of non-Euclidean universes (chapter 5).

At the end of chapter 1 we saw that the Newtonian cosmology lacks a clear justification from general relativity. Especially, the way it deals with the spatial expansion relies on the interpretation of expanding periodic boundary conditions as a representation of a closed Euclidean space. Then justifying the Newtonian cosmology from general relativity was an important step in the definition of a non-Euclidean Newtonian theory. This was the work presented in chapter 2. I used the Newton-Cartan theory which can be derived with a well-defined covariant limit of general relativity. I performed a 1+3 split of the Newton-Cartan equations, which resulted in a system

of equations similar to the 1+3-Einstein system: *the 1+3-Newton-Cartan system* [equation (2.33)–(2.41)]. When introducing the gravitational field to this system, I retrieved the Newtonian cosmological equations [equation (2.53)–(2.61)]. However, a major difference of the Newton-Cartan approach is the expansion which arises as a fundamental emerging field of the theory. The equations also feature an additional unconstrained *transverse shear*, which can model an anisotropic expansion. We retrieve the Buchert-Ehlers theorem, stating in particular that in Newton’s theory the isotropic expansion of a closed universe is given by the Friedmann equations for any inhomogeneous solution.

In chapter 3, I used the tools developed for what precedes to define two non-Euclidean Newtonian theories. They are based on a suggestion of Künzle (1976) to allow for a non-zero spatial curvature with a minimal modification of the Newton-Cartan equations. If this modification is made with respect to the fluid observer, this leads to the non-Euclidean Newtonian theory of section 3.3.2. It features a non-zero global backreaction which depends on the spatial geometry, through the spatial curvature, and on the average bulk velocity of the fluid, an expected behaviour of the backreaction. The main drawback of this approach is the fact that the spatial vectors in the vorticity and expansion tensors are different, making difficult the interpretation of the theory. To my opinion, this disqualifies this proposal to be physically relevant. The second non-Euclidean Newtonian theory does not suffer from this problem. In this case the modification of the Newton-Cartan equation is made with respect to the Galilean observer (section 3.3.3). The cosmological system of this theory [equations (3.49)–(3.56)] is algebraically equivalent to the cosmological Newton equations, but with a non-zero spatial scalar curvature. Hence the Buchert-Ehlers theorem also applies and no backreaction appears if the expansion is isotropic. This approach also improves the one of Roukema & Róžański (2009); Barrow (2020) as the gravitational field at the opposite pole of a mass point in \mathcal{S}^3 is not divergent anymore [equation (3.64)].

Chapter 4 is the final stage in the construction of a non-Euclidean Newtonian theory. I try to derive it using the Galilean limit of general relativity (Künzle, 1976) which is a limit of structures, from a Lorentzian to a Galilean structure. We saw that to obtain a non-zero spatial Ricci curvature at the limit, allowing for non-Euclidean spatial geometries, an additional source term needs to be added in the energy-momentum tensor. I interpreted it as a generalisation of this tensor for a geodesic, or dust, fluid. While the Galilean limit provides the leading orders of the additional term, its full order form is unknown and needs to be extrapolated. We then saw that the non-Euclidean Newtonian theory depends on this extrapolation, and therefore is unknown as for now.

In chapter 5, I studied the possibility of performing numerical relativistic simulations of non-Euclidean geometries in cosmology. We saw that the method of lattice cosmology allows for such simulations (Clifton, 2015), but at the cost of a

low resolution of the structures as they are modelled with black holes which can be evolved only in few number. Then I presented the most popular relativistic scheme, the BSSN formalism, which is used in the recent cosmological simulations of [Giblin et al. \(2016\)](#); [Macpherson et al. \(2018\)](#). In its original form, this formalism is not suited for non-Euclidean geometries. I showed that the covariant BSSN formalism of [Brown \(2009\)](#), the equations of which are presented in section 5.3.4.3, is the right choice to simulate such geometries. Finally I argued that the spherical torus should be the best choice for a first exact non-Euclidean relativistic cosmological simulation.

While we tackled successfully most of the issues we raised in the beginning of this thesis (justifying the Newtonian cosmology, proposing non-Euclidean Newtonian theories, finding a non-Euclidean Newtonian limit, finding a non-Euclidean relativistic numerical scheme), other questions appeared which I intend to study:

- What can model the transverse shear? In addition to gravitational waves, I suggested that the dark matter might be included in Ξ . To understand if this is possible, a first investigation is to try to reproduce existing models of dark matter.
- Is the additional energy-momentum curvature term imposed by the Galilean limit physical? We could search for a justification of this term through a variational principle. In this view, maximally coupled scalar fields (see [Madsen, 1988](#)) appear as an interesting approach.
- What is the first order, i.e. first post-Newtonian correction, of the backreaction? Computing it would provide an estimation of the relativistic contribution to the backreaction using Newtonian calculations.

The question of the ‘right’ non-Euclidean Newtonian theory remains also open, as well as the generalisation of the Buchert-Ehlers theorem. To my opinion, the second proposed theory in chapter 3 should be this ‘right’ one due to its simplicity in terms of modifications of the Newton-Cartan equations and of its cosmological equations. In this sense the extrapolation of the additional term in the energy-momentum tensor should be made so that we retrieve this theory. Then, performing N-body simulations with this theory would provide a first insight into the effects of topology on structure formation in cosmology.

Concerning my study on non-Euclidean relativistic schemes, the logical next step is to implement the covariant BSSN formalism. This could be done using softwares such as *Einstein Toolkit*. The final step would be to compare non-Euclidean simulations made using this scheme with the ones of [Giblin et al. \(2016\)](#); [Macpherson et al. \(2018\)](#).

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Appendix A

Formulas

A.1 Lie derivative in 3+1

We consider a foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ in a Lorentzian manifold $(\mathcal{M}, \mathbf{g}, \nabla)$. Let \mathbf{A} and \mathbf{T} be a spatial vector and a spatial tensor with respect to this foliation. Then the Lie derivative of \mathbf{T} along \mathbf{A} is not necessarily spatial. It is spatial if \mathbf{T} has only contravariant indices. In general we have the relation

$$\mathcal{L}_{\mathbf{A}} T^{\alpha_1 \dots \beta_1 \dots} = (h^{\alpha_1}_{\mu_1} \dots) (h^{\beta_1}_{\nu_1} \dots) \mathcal{L}_{\mathbf{A}} T^{\mu_1 \dots \nu_1 \dots} - 2 \sum_i n_{\beta_i} T^{\alpha_1 \dots \nu_1 \dots \mu_1 \dots} n_{\mu_1} \nabla^{(\mu_1} A^{\nu)}. \quad (\text{A.1})$$

Under the formalism of the 3+1-Einstein equations in section 1.1.2, given a class of adapted coordinates \mathcal{X}_{β}^n , this means that $\mathcal{L}_{\beta} K_{\alpha\beta}$ is not a spatial tensor in general. This implies that ${}^{\beta} \partial_t K_{\alpha\beta}$ are not the spacetime components of a spatial tensor (contrary to what is stated in section 5.3.1 of [Gourgoulhon, 2012](#)). However because the Lie derivative $\mathcal{L}_{Nn} = {}^{\beta} \partial_t - \mathcal{L}_{\beta}$ applied on a spatial tensor is spatial, then ${}^{\beta} \partial_t K_{\alpha\beta} - \mathcal{L}_{\beta} K_{\alpha\beta}$ remains spatial, and the pull-back ${}^{\beta} \partial_t K_{\alpha\beta} - \mathcal{L}_{\beta} K_{\alpha\beta} \rightarrow {}^{\beta} \partial_t K_{ab} - \mathcal{L}_{\beta} K_{ab}$ is correct.

The normal part of the partial time derivative ${}^{\beta} \partial_t K_{\gamma\alpha}$ is

$$n^{\gamma} {}^{\beta} \partial_t K_{\gamma\alpha} = 4K_{\alpha}^{\mu} n^{\nu} \nabla_{(\mu} \beta_{\nu)}. \quad (\text{A.2})$$

A.2 Tensor densities

We consider a tensor density ${}^w T^{\mu_1 \dots \nu_1 \dots}$ of weight w . The covariant derivative on this tensor is defined as

$$\nabla_{\alpha} {}^w T^{\mu_1 \dots \nu_1 \dots} := {}^0 \nabla_{\alpha} {}^w T^{\mu_1 \dots \nu_1 \dots} - w {}^w T^{\mu_1 \dots \nu_1 \dots} \Gamma_{\mu\alpha}^{\mu}, \quad (\text{A.3})$$

where ${}^0 \nabla_{\alpha}$ refers to the usual covariant derivative on a tensor of weight 0. The Lie derivative of ${}^w T^{\mu_1 \dots \nu_1 \dots}$ along a vector \mathbf{u} is defined as

$$\mathcal{L}_{\mathbf{u}} {}^w T^{\mu_1 \dots \nu_1 \dots} := {}^0 \mathcal{L}_{\mathbf{u}} {}^w T^{\mu_1 \dots \nu_1 \dots} + w {}^w T^{\mu_1 \dots \nu_1 \dots} \nabla_{\mu} u^{\mu}, \quad (\text{A.4})$$

where ${}^0 \mathcal{L}_{\mathbf{u}}$ refers to the usual Lie derivative on a tensor of weight 0.

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Appendix B

Shift freedom in the classical formulation of Newton's theory

As shown in chapter 2, the degrees of freedom associated with the choice of coordinates in the Newton-Cartan theory corresponds to a vector, called the shift vector. The work presented in this appendix is a first derivation of this shift freedom but in case of the classical formulation of Newton's theory. It was done before obtaining this result in Newton-Cartan and appears in the first part of my paper [Vigneron \(2020\)](#).

B.1 General form of the Newton system

B.1.1 In fixed coordinates

We only consider dust fluids, implying the pressure and the non-ideal fluid terms to be zero.

The Newton system of equations describes the time evolution of a fluid characterised by a scalar field ρ , the fluid density, and a vector field \mathbf{v} , the fluid velocity. These two tensors are defined in a 3-dimensional flat manifold denoted Σ^N and are parametrised by the time t . They are thus function of t and the position on Σ^N . The metric on Σ^N is denoted \mathbf{h} . The system of equations is composed of two evolution equations, one for the scalar ρ and one for the vector \mathbf{v} , and two constraint equations. Given a fixed coordinate basis vector $\{\mathbf{e}_i\}_{i=1,2,3}$ on Σ^N , i.e. the vectors \mathbf{e}_i are not parametrised by time, the evolution equations in the corresponding coordinate system $\{x^i\}_{i=1,2,3}$ are

- the *mass conservation equation*

$$\left(\partial_{t|_x} + v^k D_k\right) \rho = -\rho D_k v^k, \quad (\text{B.1})$$

- the *Euler equation*

$$\left(\partial_{t|x} + v^k D_k\right) v^i = g^i, \quad (\text{B.2})$$

where D_i are the components of the Levi-Civita connection on Σ^N in the coordinates x^i and \mathbf{g} is the gravitational vector field constraint by the following equations:

- the *Newton-Gauss equation*

$$D_k g^k = -4\pi G\rho + \Lambda, \quad (\text{B.3})$$

- the *Newton-Faraday equation*

$$D_{[i} g_{j]} = 0. \quad (\text{B.4})$$

with Λ the cosmological constant.

Due to the equivalence principle, the Euler equation (B.2) can be seen as a definition of the gravitational vector field. Then apart from this equation, the Newton system can be written independently of \mathbf{g} . To do so, we introduce the expansion tensor Θ and the vorticity tensor Ω of the vector field \mathbf{v} being respectively the symmetric and the antisymmetric part of the velocity gradient $D\mathbf{v}$, with

$$\Theta_{ij} := D_{(i} v_{j)} \quad ; \quad \Omega_{ij} := D_{[i} v_{j]}, \quad (\text{B.5})$$

and we note the trace $\Theta_k^k =: \theta$. The indices are lowered and raised by the metric \mathbf{h} . We can then rewrite equations (B.1), (B.3) and (B.4) respectively as

$$\left(\partial_{t|x} + \mathcal{L}_v\right) \rho = -\rho\theta, \quad (\text{B.6})$$

$$\left(\partial_{t|x} + \mathcal{L}_v\right) \theta = -4\pi G\rho + \Lambda - \Theta_{ij}\Theta^{ij} + \Omega_{ij}\Omega^{ij}, \quad (\text{B.7})$$

$$\left(\partial_{t|x} + \mathcal{L}_v\right) \Omega_{ij} = 0. \quad (\text{B.8})$$

The gravitational vector field \mathbf{g} is defined as

$$g^i := \left(\partial_{t|x} + \mathcal{L}_v\right) v^i + v^k \left(\Theta_k^i + \Omega_k^i\right). \quad (\text{B.9})$$

Introducing the Lie derivative in this last equation allows us to have the same differential operator acting on ρ , Θ_{ij} and Ω_{ij} .

The system (B.5)-(B.8) is closed and equivalent to the system (B.1)-(B.4).

While the expansion and vorticity tensors are explicitly covariant under any change of coordinates, parametrised by time or not, the differential operator $\partial_{t|x}$ is not. In the next section we will see how it changes as function of the time parametrisation of the coordinate transformation. This will allow us to write the Newton system for any time parametrised coordinate system.

B.1.2 In general parametrised coordinates – v description

We consider a coordinate vector basis $\{\tilde{\mathbf{e}}_a\}_{a=1,2,3}$ on Σ^N . If the vectors $\tilde{\mathbf{e}}_a$ are parametrised by time, the coordinate system they define is called a *parametrised coordinate system*. We consider such a coordinate system on Σ^N and note it $\{\tilde{x}^a\}_{a=1,2,3}$. For this section, any component of a tensor in the fixed coordinates x^i will use the Roman letters i, j, k, l , etc (example: T_{ij}) and the same applies for the partial derivatives with $\partial_t := \partial_{t|x}$ and $\partial_i := \partial_{x^i}$; any component of a tensor in the parametrised coordinates \tilde{x}^a will be denoted with a tilde and will use the Roman letters a, b, c, d , etc (example: \tilde{T}_{ab}) and the same applies for the partial derivatives with $\tilde{\partial}_t := \partial_{t|\tilde{x}}$ and $\tilde{\partial}_a := \partial_{\tilde{x}^a}$.

To be able to write the Newton equations in the \tilde{x}^a coordinates from the equations in the x^i coordinates, we need to consider the coordinate transformation between x^i and \tilde{x}^a . This allows us to write \tilde{x}^a as functions of x^i and t , and inversely x^i as functions of \tilde{x}^a and t . The Jacobian matrix J^i_a of this transformation, and its inverse J_i^a , are then

$$J^i_a := \tilde{\partial}_a x^i \quad ; \quad J_i^a := \partial_i \tilde{x}^a.$$

Because the change of coordinates $x^i \rightarrow \tilde{x}^a$ depends on time, in general the Jacobian will also depend explicitly on time.

The components $\tilde{T}^{ab\dots}_{cd\dots}$ of any tensor \mathbf{T} in Σ^N are related to the components $T^{ij\dots}_{kl\dots}$ of that same tensor by

$$\tilde{T}^{ab\dots}_{cd\dots} := \left(J_i^a J_j^b \dots \right) T^{ij\dots}_{kl\dots} \left(J^k_c J^l_d \dots \right).$$

We consider now a tensor \mathbf{W} whose components in the $\{\mathbf{e}_i\}_{i=1,2,3}$ basis are $W^{ij\dots}_{kl\dots} := \partial_t T^{ij\dots}_{kl\dots}$ with \mathbf{T} a parametrised tensor. As mentioned in the previous section, because the derivative ∂_t is not explicitly covariant under the change of coordinates $x^i \rightarrow \tilde{x}^a(t, x^i)$, the relation $\tilde{W}^{ab\dots}_{cd\dots} = \tilde{\partial}_t \tilde{T}^{ab\dots}_{cd\dots}$ does not hold in general. It only holds if the \tilde{x}^a coordinates do not depend on time. Instead we have the relation

$$\begin{aligned} \left(J_i^a J_j^b \dots \right) \partial_t \left(T^{ij\dots}_{kl\dots} \right) \left(J^k_c J^l_d \dots \right) = \\ \tilde{\partial}_t \tilde{T}^{ab\dots}_{cd\dots} - \mathcal{L}_U \tilde{T}^{ab\dots}_{cd\dots}, \end{aligned} \tag{B.10}$$

where \mathbf{U} is the coordinate velocity vector of the \tilde{x}^a coordinates with respect to the x^i coordinates and is defined such as

$$U^i := \tilde{\partial}_t x^i, \tag{B.11}$$

which implies $\tilde{U}^a = -\partial_t \tilde{x}^a$ using (B.10).

Proof. For simplicity we show the proof for a rank-1 tensor; it can easily be

generalised for any tensor. Making use of $\partial_t = \tilde{\partial}_t - \tilde{\partial}_t x^k \partial_k$, we have

$$\begin{aligned}
 J^i_a \partial_t T_i &= J^i_a \left(\tilde{\partial}_t T_i - \tilde{\partial}_t x^k \partial_k T_i \right) \\
 &= \tilde{\partial}_t \tilde{T}_a - T_i \tilde{\partial}_t J^i_a - J^i_a U^k \partial_k T_i \\
 &= \tilde{\partial}_t \tilde{T}_a - J^i_a \left(T_k \partial_i U^k + U^k \partial_k T_i \right) \\
 &= \tilde{\partial}_t \tilde{T}_a - J^i_a (\mathcal{L}_U \mathbf{T})_i \\
 &= \tilde{\partial}_t \tilde{T}_a - \mathcal{L}_U \tilde{T}_a. \quad \square
 \end{aligned}$$

We can then write the system (B.6)-(B.9) in the coordinates \tilde{x}^a . This gives *the generalised Newton equations for the fluid velocity vector \mathbf{v}* in a time parametrised coordinate system:

$$(\tilde{\partial}_t + \mathcal{L}_{\mathbf{v}-U}) \rho = -\rho \theta, \quad (\text{B.12})$$

$$(\tilde{\partial}_t + \mathcal{L}_{\mathbf{v}-U}) \theta = -4\pi G \rho + \Lambda - \tilde{\Theta}_{cd} \tilde{\Theta}^{cd} + \tilde{\Omega}_{cd} \tilde{\Omega}^{cd}, \quad (\text{B.13})$$

$$(\tilde{\partial}_t + \mathcal{L}_{\mathbf{v}-U}) \tilde{\Omega}_{cd} = 0, \quad (\text{B.14})$$

and the definition of the gravitational field

$$\tilde{g}^a := \left(\tilde{\partial}_t + \mathcal{L}_{\mathbf{v}-U} \right) \tilde{v}^a + \tilde{v}^c \left(\tilde{\Theta}_c^a + \tilde{\Omega}_c^a \right). \quad (\text{B.15})$$

These equations, while written for any parametrised coordinate system, still require reference coordinates, i.e. the fixed coordinates x^i , to be able to define the tensor \mathbf{U} . This is discussed in section B.1.4.

The system (B.12)-(B.15) depends on the vectors \mathbf{U} and \mathbf{v} . The latter can be called the velocity of the fluid with respect to the fixed coordinates. However when taking a non-zero coordinate velocity \mathbf{U} , it might be useful to work with the velocity vector \mathbf{V} of the fluid with respect to the parametrised coordinates defined as

$$\mathbf{V} := \mathbf{v} - \mathbf{U}. \quad (\text{B.16})$$

In the next section we develop the Newton equations as functions of \mathbf{U} and \mathbf{V} .

B.1.3 In general parametrised coordinates – \mathbf{V} description

We introduce the expansion tensors ${}^V \Theta$ and ${}^U \Theta$, and the vorticity tensors ${}^V \Omega$ and ${}^U \Omega$ of the vectors \mathbf{V} and \mathbf{U} as

$$\begin{aligned}
 {}^V \tilde{\Theta}_{ab} &:= \tilde{D}_{(a} \tilde{V}_{b)} & ; & \quad {}^U \tilde{\Theta}_{ab} := \tilde{D}_{(a} \tilde{U}_{b)} & ; \\
 {}^V \tilde{\Omega}_{ab} &:= \tilde{D}_{[a} \tilde{V}_{b]} & ; & \quad {}^U \tilde{\Omega}_{ab} := \tilde{D}_{[a} \tilde{U}_{b]},
 \end{aligned}$$

and their trace ${}^V \theta := {}^V \tilde{\Theta}_c^c$ and ${}^U \theta := {}^U \tilde{\Theta}_c^c$.

We can then write the system (B.12)-(B.15) as function of \mathbf{V} and \mathbf{U} . This gives the generalised Newton equations for the fluid coordinates velocity vector \mathbf{V} in a time parametrised coordinate system:

$$(\tilde{\partial}_t + \mathcal{L}_{\mathbf{V}}) \rho = -\rho ({}^V\theta + {}^U\theta), \quad (\text{B.17})$$

$$(\tilde{\partial}_t + \mathcal{L}_{\mathbf{V}}) ({}^V\theta + {}^U\theta) = -4\pi G\rho + \Lambda \quad (\text{B.18})$$

$$\begin{aligned} & - ({}^V\tilde{\Theta}_{cd} + {}^U\tilde{\Theta}_{cd}) ({}^V\tilde{\Theta}^{cd} + {}^U\tilde{\Theta}^{cd}) \\ & + ({}^V\tilde{\Omega}_{cd} + {}^U\tilde{\Omega}_{cd}) ({}^V\tilde{\Omega}^{cd} + {}^U\tilde{\Omega}^{cd}), \\ (\tilde{\partial}_t + \mathcal{L}_{\mathbf{V}}) ({}^V\tilde{\Omega}_{cd} + {}^U\tilde{\Omega}_{cd}) & = 0, \end{aligned} \quad (\text{B.19})$$

and the definition of the gravitational field

$$\begin{aligned} \tilde{g}^a & := (\tilde{\partial}_t + \mathcal{L}_{\mathbf{V}}) (\tilde{V}^a + \tilde{U}^a) \\ & + (\tilde{V}^c + \tilde{U}^c) ({}^V\tilde{\Theta}_c^a + {}^U\tilde{\Theta}_c^a + {}^V\tilde{\Omega}_c^a + {}^U\tilde{\Omega}_c^a). \end{aligned} \quad (\text{B.20})$$

B.1.4 Class of coordinates

We define the following mathematical object:

Definition. Given a coordinate system $\{y^a\}_{a=1,2,3}$, parametrised or not, we define **the class of y coordinates**, denoted \mathcal{X}_y , as the ensemble of coordinate systems which can be obtained from the system $\{y^a\}_{a=1,2,3}$ with a time-independent coordinate transformation.

The equations developed in sections B.1.1, B.1.2 and B.1.3 were defined, directly or indirectly, with respect to a chosen fixed coordinate system $\{x^i\}_{i=1,2,3}$ on Σ^N . They however do not depend on this system as all these equations are invariant under a time-independent change of coordinates. Instead, any system of coordinates in the class of fixed coordinates can be chosen. The same applies for the definition of the vector \mathbf{U} , and so of the vector \mathbf{V} .

Proof. We consider two fixed coordinate systems $\{x^i\}_{i=1,2,3}$ and $\{y^I\}_{I=1,2,3}$ and a parametrised coordinate system $\{\tilde{x}^a\}_{a=1,2,3}$. The components of a tensor in the y^I coordinates will be denoted with capital Roman letters I, J, \dots . Let \mathbf{U} be the coordinate velocity vector of the coordinates \tilde{x}^a with respect to the coordinates x^i . Then

$$\begin{aligned} U^i & := \tilde{\partial}_t x^i \\ & = \tilde{\partial}_t x^i (y^I) \\ & = \partial_t y^K \partial_{y^K} x^i. \end{aligned}$$

$\partial_{y^K} x^i$ is the Jacobian of the coordinate transformation between x^i and y^K . Then $U^I := \tilde{\partial}_t y^I$. This means that the definition of \mathbf{U} is unchanged if the fixed system of reference is $\{y^I\}_{I=1,2,3}$. \square

The choice of parametrised coordinates \tilde{x}^a then defines uniquely the vector \mathbf{U} . The opposite is wrong: defining a vector field \mathbf{U} on Σ^N does not determine uniquely a parametrised system $\{\tilde{x}^a\}_{a=1,2,3}$. Instead \mathbf{U} uniquely defines a **class of parametrised coordinate systems** which we can write $\mathcal{X}_{\mathbf{U}}$. Then \mathbf{U} is the coordinate velocity of any system in $\mathcal{X}_{\mathbf{U}}$ with respect to any system in \mathcal{X}_0 , where \mathcal{X}_0 is the class of fixed coordinates.

The Newton system (B.12)-(B.15), or equivalently the system (B.17)-(B.20), then corresponds to the original Newton system (B.6)-(B.9) written in any class of coordinates. It is the most general writing of the original equations (B.6)-(B.9), assuming the time parameter is unchanged.

But while the original set of equations required the definition of only one vector field, the fluid velocity vector \mathbf{v} , the general equations of section B.1.2 require the definition of a second vector field, the coordinate velocity vector \mathbf{U} of the chosen class of coordinates to work in. If one chooses the point of view of section B.1.3, the pair of vectors (\mathbf{v}, \mathbf{U}) is replaced by the pair (\mathbf{V}, \mathbf{U}) . However only \mathbf{v} is physical as it is the fluid velocity vector and does not depend on a chosen class of coordinates: taking $\mathbf{v} = 0$ changes the generality of the equations as it implies $\partial_t \rho = 0$, while \mathbf{U} or \mathbf{V} can be taken to 0 without loss of generality.

A non-trivial choice of \mathbf{U} can however be of physical interest depending on the physical system studied. In the next section we present specific examples of parametrised coordinates.

B.2 Specific choices of coordinates

In this section we will always use the Newton equations in the same class of coordinates as the vector \mathbf{U} we will choose. We can then omit the tilde notation. The partial time derivative will also always be partial time derivative at fixed $\mathcal{X}_{\mathbf{U}}$ coordinates, we will note it $\partial_{t|_{\mathbf{U}}}$.¹

B.2.1 Galilean coordinates

Galilean coordinates are the classes of coordinates for which $\partial_{t|_{\mathbf{U}}} U^a = 0$ and $\nabla \mathbf{U} = 0$, i.e. the coordinates $\mathcal{X}_{\mathbf{U}}$ are uniformly moving with respect to the class of fixed coordinates. If one chooses the fluid description in terms of (\mathbf{v}, \mathbf{U}) , then the corresponding Newton system (B.12)-(B.15) is not equivalent for all Galilean coordinates due to the terms $U^c D_c$. The Galilean invariance only appears in the (\mathbf{V}, \mathbf{U}) description of the fluid as the corresponding equations (B.17)-(B.20) are formally equivalent for all Galilean coordinates.

¹To avoid possible confusions, we precise that this notation does not imply $\partial_{t|_{\mathbf{U}}} U^a = 0$.

This shows that the description in terms of the fluid coordinate velocity is more appropriate when \mathbf{U} is non-zero as it will encode the non-inertial effect due to \mathbf{U} . Indeed, we can rewrite the Euler equation (B.20) to feature the non-inertial terms acting on \mathbf{V}

$$\begin{aligned} \left(\partial_{t|\mathbf{U}} + V^c D_c\right) V^a &= g^a - \left(\partial_{t|\mathbf{U}} + U^c D_c\right) U^a \\ &\quad - 2V^c \left({}^U\theta_c^a + {}^U\tilde{\Omega}_c^a\right). \end{aligned} \quad (\text{B.21})$$

We see that the acceleration of \mathbf{V} , on the left-hand side of the equation, is affected by the gravitational field and the non-inertial terms, depending on \mathbf{U} . We however recall that these effects are only gauge effects as the true dynamics of the fluid is given by \mathbf{v} .

B.2.2 Globally translating and rotating coordinates

In classical mechanics, the most general coordinates are usually globally rotating and translating with respect to \mathcal{X}_0 . In this case they are called *frames*. They correspond to all the classes of coordinates where \mathbf{U} can be decomposed as

$$\mathbf{U} = \mathbf{U}_{\text{tr}} + \mathbf{U}_{\text{rot}}, \quad (\text{B.22})$$

where $D\mathbf{U}_{\text{tr}} = 0$, ${}^U\Theta_{\text{rot}} = 0$ and $\epsilon_{acd} D_b D^c U_{\text{rot}}^d = 0$ where ϵ is the Levi-Civita tensor.

The condition $D\mathbf{U}_{\text{tr}} = 0$ ensures that \mathbf{U}_{tr} is a global translation of the $\mathcal{X}_{\mathbf{U}}$ coordinates with respect to the Galilean classes of coordinates; ${}^U\Theta_{\text{rot}} = 0$ ensures that \mathbf{U}_{rot} is only rotational. The rotation vector of the frame is $\omega^a := \epsilon^a{}_{cd} D^c U_{\text{rot}}^d$. Then $\epsilon_{bcd} D_a D^c U_{\text{rot}}^d = D_a \omega_b = 0$ ensures that this rotation is also global.

In these conditions, the Euler equation (B.21) becomes

$$\begin{aligned} \left(\partial_{t|\mathbf{U}} + V^c D_c\right) V^a &= g^a - \partial_{t|\mathbf{U}} U_{\text{tr}}^a - \partial_{t|\mathbf{U}} U_{\text{rot}}^a \\ &\quad - (U_{\text{tr}}^c + U_{\text{rot}}^c) \Omega_c^a - 2V^c {}^U\Omega_c^a. \end{aligned} \quad (\text{B.23})$$

The term $\partial_{t|\mathbf{U}} U_{\text{tr}}^a + \partial_{t|\mathbf{U}} U_{\text{rot}}^a + (U_{\text{tr}}^c + U_{\text{rot}}^c) \Omega_c^a$ is the *centrifugal acceleration*, and $2V^c {}^U\Omega_c^a$ is the *Coriolis acceleration*. We retrieve the usual Euler equation in a non-inertial frame where the vorticity of \mathbf{U} corresponds to the global rotation of that frame with respect to a Galilean frame. There is however no contribution of the expansion tensor of \mathbf{U} in that case, as it is zero. In the next section we will show to what corresponds a non-zero ${}^U\Theta$.

B.2.3 Homogeneous deformation

The expansion tensor of the coordinate velocity vector \mathbf{U} can be linked to the time variations of the metric in the $\mathcal{X}_{\mathbf{U}}$ coordinates. We have the following relation:

$$\frac{1}{2} \partial_{t|\mathbf{U}} h_{ab} = {}^U\Theta_{ab}. \quad (\text{B.24})$$

Proof. For this proof only we reintroduce the tilde and untilde notations of section B.1.2 concerning parametrised and fixed coordinates. Using property (B.10), we have

$$\begin{aligned} J^i{}_a J^j{}_b \partial_{t|_0} h_{ij} &= (\partial_{t|_U} - \mathcal{L}_U) \tilde{h}_{ab}, \\ &= \partial_{t|_U} \tilde{h}_{ab} - 2\tilde{D}_{(a} \tilde{U}_{b)}. \end{aligned}$$

$\partial_{t|_0}$ is the time derivative with respect to the fixed coordinate class. Because x^i are fixed coordinates, $\partial_{t|_0} h_{ij} = 0$. Then $\partial_{t|_U} \tilde{h}_{ab} - 2\tilde{D}_{(a} \tilde{U}_{b)} = 0$. \square

Remark. In a frame coordinate system, i.e. globally translating and rotating, the metric is static as ${}^U\Theta = 0$.

Relation (B.24) implies that with a change of coordinates from fixed coordinates, we can simulate space expansion. This expansion is always a *gradient expansion*, i.e. $\frac{1}{2}\partial_{t|_U} h_{ab}$ is the gradient of a vector. Taking ${}^U\Theta$ such that $\mathbf{D}{}^U\Theta = 0$ implies that the expansion is global: this is called a *homogeneous deformation*. Furthermore, when it is isotropic, the coordinate velocity vector corresponds to the *position vector*, i.e. in Cartesian coordinates $U^a \propto x^a$. In this case this is called a *Hubble flow*. However one has to remember that the physical vector is \mathbf{v} . Therefore the expansion due to ${}^U\Theta$ is strictly speaking a fluid expansion and not a space expansion.

The main consequence of the gradient expansion is that no global expansion is possible if the 3D-manifold Σ^N has a compact topology.² Indeed in such a topology x^a cannot be the components of a tensor as they do not respect the global symmetry of a compact space. So strictly speaking, Newtonian cosmological simulations, said to be realised in a 3-torus with global isotropic expansion, are actually simulating an infinite 3D-manifold. The 3-torus symmetry is only set on \mathbf{V} and not \mathbf{U} , thus the physical vector \mathbf{v} lies in an infinite 3D-manifold.

B.2.4 Lagrangian coordinates

We saw that the physical dynamical properties of the fluid are encoded in \mathbf{v} . By working in parametrised coordinates, we split these properties into \mathbf{U} and \mathbf{V} . Then, taking $\mathbf{V} = 0$ implies that the coordinate velocity \mathbf{U} is the velocity of the fluid \mathbf{v} . Coordinates such as $\mathbf{V} = 0$ are called *Lagrangian coordinates* as they follow the fluid flows given by \mathbf{v} . In Lagrangian coordinates, part of the fluid dynamics, the pure expansion θ and the shear $\sigma_{ab} := \Theta_{\langle ab \rangle}$ of \mathbf{v} , is put into the time variation of the metric. The other part, the vorticity of \mathbf{v} , does not affect the metric.

²The only possible compact oriented topology is the flat 3-torus \mathbb{T}^3 up to a finite covering.

B.3 Similarities with the 3+1 and 1+3 formalisms of general relativity

In section B.1, we derived the Newton system of equations in an arbitrary class of coordinates. We saw that the freedom associated with a choice of class is a vector \mathbf{U} . Furthermore the difference between two partial time derivatives is a Lie derivative. These two properties resemble the properties of the shift freedom in the 3+1 formalism of general relativity (see section 1.1.2). We could add that Newton's equations live on a time-parametrised 3D-manifold³ which is the same situation as for the 3+1-Einstein equations.

This shows that apart from the known formal equivalence between the Newtonian system (B.6)-(B.8) and the 1+3-Einstein equations explained in Ellis (2009), Newton's theory also features similarities with the 3+1 construction of the Einstein equation.

References

- Ellis G., 2009, *Relativistic cosmology*, *Gen. Rel. Grav.*, 581–660
 Vigneron Q., 2020, *1 +3 formulation of Newton's equations*, *Phys. Rev. D*, 102, 124005

³Actually, among the tensors defining Σ^N , only the metric can depend on time, the Riemann tensor being zero in any class of coordinates.

Appendix C

Newton's equations on a Lorentzian manifold

This appendix is the second part of my paper [Vigeneron \(2020\)](#), the first part being in appendix [B](#). For this reason some notations are only common to those two appendices, and not introduced in the main body of the thesis. This is especially the case for the denomination ‘1+3-Newton equations’, which refer to another set of equations than the 1+3-NC equations of chapter [2](#).

C.1 Construction of the 4D-Newton equations

In section [1.1.2](#) we detailed the construction of the 3+1-Einstein equations on a time parametrised 3D-manifold Σ . In the present section we will reverse this construction for the case of the Newton equations: from the parametrised manifold Σ^N , we will define a spacetime manifold \mathcal{M}^N and write the Newton equations on this manifold.

C.1.1 Push-forward of the Newton equations

In order to write the Newton equations as equations living in a 4D-manifold \mathcal{M}^N , we reverse the pull-back operation of section [1.1.2.6](#). The operation $\Sigma^N \rightarrow \mathcal{M}^N$ is called a *push-forward* of Σ^N in \mathcal{M}^N . While the pull-back in general relativity defined the parametrised manifold Σ , the push-forward here will define the spacetime manifold \mathcal{M}^N .

The push-forward is parametrised by t . It hence defines a set $\{\Sigma_t^N\}_{t \in \mathbb{R}}$ of hypersurfaces embedded in \mathcal{M}^N . At this stage, for a general push-forward, these hypersurfaces can intersect. We however impose that the family $\{\Sigma_t^N\}_{t \in \mathbb{R}}$ defines a foliation in \mathcal{M}^N . We note \mathbf{n} and \mathcal{N} the normal vector and the lapse of this foliation.

In the 3+1-Einstein equations [\(1.40\)](#) and [\(1.39\)](#) on Σ , the partial time derivative ${}^n\partial_{t|_\beta}$ carries the information on the shift of the adapted coordinates in which the pull-back was made. This is not the case for the classical 3D-Newton equations [\(1.93\)](#)-[\(1.96\)](#) as no pull-back is at their origin. This means that the derivative ∂_{t_0} in the

Newton equations does not necessarily correspond to the derivative ${}^n\partial_{t|_0}$ in \mathcal{M}^N . Instead in a general push-forward of the Newton equations, $\partial_{t|_0}$ becomes ${}^n\partial_{t|\mathcal{B}}$, where \mathcal{B} is a spatial vector relative to the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$.

So a class \mathcal{X}_U in Σ_t^N corresponds to an adapted class $\mathcal{X}_{\mathcal{B}+U}^n$ in \mathcal{M}^N . This is schematised in Fig. C.1 where we represent a slice Σ_t^N and the vectors \mathbf{n} , \mathcal{B} and \mathbf{v} . We also represent in blue the shift β and the time vector ${}^\beta\partial_t$ of a general adapted class \mathcal{X}_β^n as well as the vectors \mathbf{U} and \mathbf{V} defined in section B.1.2 with respect to this class.

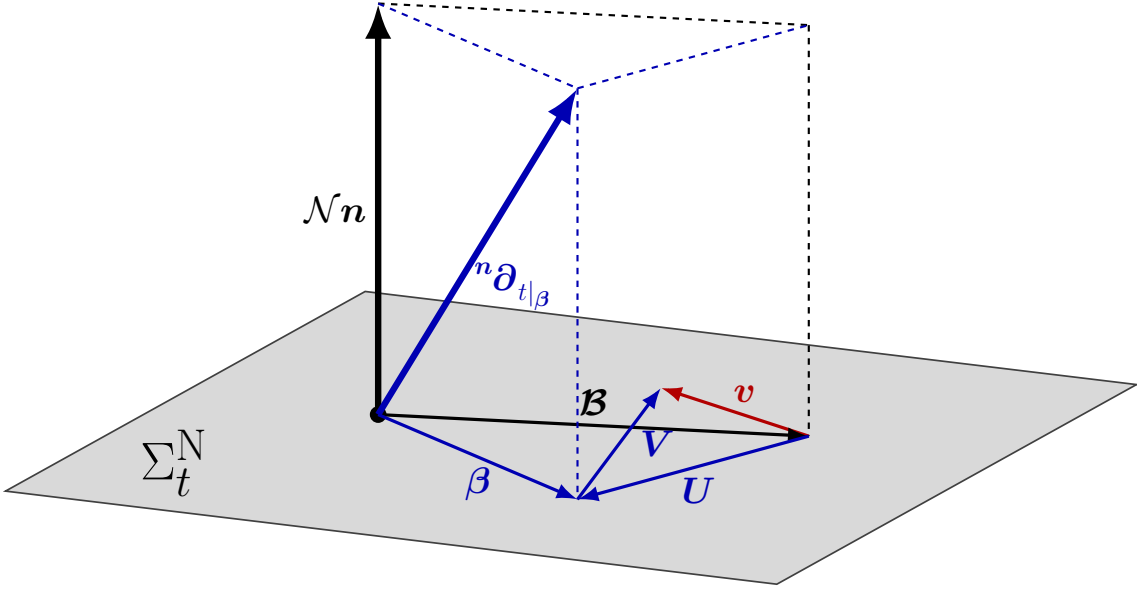


Figure C.1: Representation of a slice Σ_t^N of the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$. We show the vectors defining the 4D-manifold \mathcal{M}^N (in black); the Newtonian fluid velocity (in red); the vectors relative to a general adapted class \mathcal{X}_β^n (in blue).

The only constraint on the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$, and so on \mathcal{M}^N , is to be spatially flat and to have an adapted coordinate system in which the spatial components of the spatial metric do not depend on time. This coordinate system is \mathcal{X}_β^n . There are however no constraints on \mathcal{N} or \mathcal{B} from the Newton equations.

In the coordinates \mathcal{X}_β^n , the spacetime metric is

$$g_{\alpha\beta} = \left(\begin{array}{c|c} \pm\mathcal{N}^2 + \mathcal{B}_c\mathcal{B}^c & \mathcal{B}_b \\ \hline \mathcal{B}_a & h_{ab}(x^c) \end{array} \right), \quad (\text{C.1})$$

where $h_{ab}(x^c)$ are the spatial components of the flat spatial metric in the chosen adapted coordinates. The \pm sign depends on the choice of signature for the metric: $+$ for $(++++)$ signature and $-$ for $(-+++)$ signature. This is discussed in section C.1.3. As \mathcal{M}^N is determined by the metric (C.1), then the choice of \mathcal{N} and \mathcal{B} determines this manifold.

Remark. The push-forward $\partial_{t|_0} \rightarrow \mathcal{L}^n_{\partial_{t|\mathcal{B}}}$ is only possible if the derivative is applied on a contravariant tensor as $\mathcal{L}^n_{\partial_{t|\mathcal{B}}}$ applied on a covariant tensor is not spatial (see appendix A.1). This is also true for $\Sigma^N \mathcal{L}_v \rightarrow \mathcal{M}^N \mathcal{L}_v$, where $\Sigma^N \mathcal{L}_v$ and $\mathcal{M}^N \mathcal{L}_v$ are respectively the Lie derivative in Σ^N and in \mathcal{M}^N . Therefore, the push-forward of the Newton-vorticity equation (1.96) has to be done when written in the contravariant form.

C.1.2 4D-Newton equations

The push-forward on \mathcal{M}^N of the 3D-Newton equations (1.93)-(1.97) gives the 4D-Newton equations

$$\mathcal{L}_{\mathcal{N}n+\mathcal{B}+v} \rho = -\rho\theta, \quad (\text{C.2})$$

$$\mathcal{L}_{\mathcal{N}n+\mathcal{B}+v} \theta = -4\pi G\rho + \Lambda - \Theta_{\mu\nu}\Theta^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu}, \quad (\text{C.3})$$

$$\mathcal{L}_{\mathcal{N}n+\mathcal{B}+v} \omega^{\alpha\beta} = -4\omega^{\mu[\alpha}\Theta^{\beta]}_{\mu}, \quad (\text{C.4})$$

with the definition of the gravitational field

$$g^\alpha := \mathcal{L}_{\mathcal{N}n+\mathcal{B}+v} v^\alpha + v^\mu (\Theta_\mu^\alpha + \omega_\mu^\alpha), \quad (\text{C.5})$$

where $\Theta_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are defined as

$$\Theta_{\alpha\beta} := D_{(\alpha} v_{\beta)} \quad ; \quad \omega_{\alpha\beta} := D_{[\alpha} v_{\beta]}. \quad (\text{C.6})$$

The constraints on the foliation are that the Ricci tensor of the hypersurfaces Σ_t^N is zero for all t and that their extrinsic curvature is

$$K_{\alpha\beta} := \mathcal{N}D_{(\alpha} \mathcal{B}_{\beta)}. \quad (\text{C.7})$$

This amounts to saying that the spatial components of the spatial metric in the coordinates $\mathcal{X}_{\mathcal{B}}^n$ do not depend on time. We call such a foliation, a *Newtonian foliation*.

The system (C.2)-(C.6) is equivalent to the original system (1.93)-(1.97), i.e. both systems can be derived from the other. The solutions for \mathbf{v} in the 4D-system are then the same as for the original system. Furthermore, it is still possible to write the 4D Newton-Raychaudhuri equation (C.3) like the Newton-Gauss equation (1.91). This means that we have the relation $D_\mu g^\mu = -4\pi G\rho + \Lambda$ and this for any choice of \mathcal{N} and \mathcal{B} . The same applies for the 4D Newton-vorticity equation (C.4) which can be written as $D_{[\alpha} g_{\beta]} = 0$.

As said before, the only constraint at that point on \mathcal{M}^N is to have a Newtonian foliation. So in the general case where \mathcal{N} and \mathcal{B} are not chosen, \mathcal{M}^N is not influenced by the dynamics of \mathbf{v} . However, choices on \mathcal{N} and \mathcal{B} can be made such that the properties of this manifold will depend on \mathbf{v} . Such a choice is the subject of section C.2. Also in section C.1.5 we discuss a choice where \mathcal{M}^N is a homogeneous expanding background manifold.

Remark. Making the push-forward from the Newton equations in $\mathcal{X}_{\mathcal{U}}$ [equations (B.12)-(B.15)] is equivalent as from the same equations in \mathcal{X}_0 , which is done in this section. The equations from $\mathcal{X}_{\mathcal{U}}$ are obtained from (C.2)-(C.5) by changing \mathcal{B} into $\mathcal{B} + \mathcal{U}$.

C.1.3 Signature of \mathcal{M}^N

While constructing \mathcal{M}^N with its metric given by (C.1), we made no assumptions on its signature. The push-forward manoeuvre made in section C.1.1 is independent of this signature. So the metric of \mathcal{M}^N can be either of Lorentzian $(-+++)$, or Euclidean $(++++)$ signature. It is an additional freedom to \mathcal{N} and \mathcal{B} of the 4D-Newton equations. We will however take only Lorentzian manifolds. The only argument to take such manifolds is to enable us to directly compare \mathcal{M}^N with solutions of the Einstein equations.

The Lorentzian choice might seem in contradiction with the Galilean invariance of Newton's theory. This is only the case if we ask the connection related to the metric (C.1), defined on the spacetime manifold \mathcal{M}^N , to have this invariance. This property is however not imposed by the axioms of the classical formulation of Newton's theory on a 3D-manifold, i.e. the one presented in section ???. In this formulation, no spacetime manifold is defined. That is why, when constructing \mathcal{M}^N from the classical formulation, some freedom appears on the properties of this manifold. This view is different from the Newton-Cartan theory, where the structure on the manifold, defined by two degenerate metrics and a compatible connection, is imposed to be invariant under Galilean transformations. This structure is called a Galilei structure, and the related manifold, a Galilei manifold (see [Künzle, 1976](#)).

The Lorentzian choice might also seem in contradiction with the fact that there is no speed limit in Newton's theory, something linked to the notion of causality. We clarify why there is no such contradiction hereafter.

The causality is the relationship between causes and effects of an event, or observer. Thus this notion depends on the definition of observers. In general relativity, the manifold of work is a 4D-manifold, on which an observer is defined by a 4-vector such that the spatial velocity of an event he measures in his rest frames cannot be greater than c . This implies that the spacetime manifold is a Lorentzian manifold and that the 4-vector of this observer is a unit, timelike vector.

In the classical formulation of Newton's theory, an observer is described by a velocity vector field \mathbf{p} in the Euclidean 3-space not limited by the speed of light. If we push-forward this observer in \mathcal{M}^N , it is still defined by \mathbf{p} which is spatial. Then an observer in the 4D-formulation of Newton's theory is not described by a unit 4-velocity vector, but by a spatial vector field, not limited by c . On the one hand, contrary to general relativity, this definition of an observer does not require \mathcal{M}^N to be Lorentzian; reversely, choosing \mathcal{M}^N to be Lorentzian does not impose constraints on the definition of an observer in 4D-Newton. On the other hand, this Newtonian definition of an observer, and therefore of causality, naturally allows for the measure of superluminal velocities on the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$ by any observers as their spatial velocities can themselves be arbitrarily large.

In the next subsection we will see that it is possible to physically define a 4-velocity vector ${}^N\mathbf{u}$ for the Newtonian fluid. We will however necessarily have an

additional constraint if we want this vector to be a unit vector [see equation (C.10)].

Remark. The push-forward used to construct \mathcal{M}^N is taken from the 3+1-formalism of GR; it thus automatically defines a spacetime metric, implying \mathcal{M}^N to be (pseudo)-Riemannian. It might however be allowed to use a push-forward which does not necessarily lead to such a manifold. Recovering the Newton-Cartan theory using the method of section C.1 should in this case be possible.

C.1.4 Newtonian 4-velocity

The equations of section C.1.2 describe the evolution of a Newtonian fluid in \mathcal{M}^N . This fluid is defined by the spatial vector \mathbf{v} . We would like to define a vector ${}^N\mathbf{u}$ which we can call the 4-velocity of the Newtonian fluid. The definition of this vector is not constrained by the 4D-equations, so it remains a choice.

The choice we make is physically motivated by the definition of the Lagrangian coordinates (see section B.2.4). In general relativity, Lagrangian coordinates are defined to be comoving with the fluid 4-velocity, i.e. $\partial_t \propto \mathbf{u}$. For a foliation defined by the normal vector \mathbf{n} and the lapse N , and a tilt velocity vector \mathbf{w} of the fluid with respect to that foliation, the Lagrangian coordinates correspond to the adapted class $\mathcal{X}_{N\mathbf{w}}^n$.

In the classical Newton theory, these coordinates correspond to the class $\mathcal{X}_{\mathbf{v}}$ on Σ^N . Its equivalent on the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$ is the adapted class $\mathcal{X}_{\mathcal{B}+\mathbf{v}}^n$ (see section C.1.1). Then we demand that the tilt velocity vector of ${}^N\mathbf{u}$ with respect to the foliation defined by \mathbf{n} and \mathcal{N} be $\frac{1}{N}(\mathcal{B} + \mathbf{v})$.

However there remains a freedom on the choice of the normal part of ${}^N\mathbf{u}$ with respect to $\{\Sigma_t^N\}_{t \in \mathbb{R}}$. Two natural choices are possible:

- *The Newton-Cartan choice:* this 4D theory features a 1-form ψ^1 which defines an absolute time and a foliation. An observer in this theory, described by a vector \mathbf{u} , is defined with respect to this absolute time. The vector \mathbf{u} has then the following property $\psi_\mu u^\mu = 1$. The analogue to this definition in our case would be to impose $n_\mu {}^N u^\mu = -1$. This leads to a first definition of ${}^N\mathbf{u}$:

$${}^N\mathbf{u} := \frac{1}{N} (\mathcal{N}\mathbf{n} + \mathcal{B} + \mathbf{v}), \quad (\text{C.8})$$

- *The relativistic choice:* in GR, an observer is described by a unit vector \mathbf{u} , with $u_\mu u^\mu = -1$. In our case, this translates into ${}^N u_\mu {}^N u^\mu = -1$. This leads to a second natural definition of ${}^N\mathbf{u}$:

$${}^N\mathbf{u} := \frac{\gamma}{N} (\mathcal{N}\mathbf{n} + \mathcal{B} + \mathbf{v}), \quad (\text{C.9})$$

¹Using the notation of Künzle [Künzle \(1976\)](#).

with $\gamma := \left[1 - \frac{1}{\mathcal{N}^2} (\mathcal{B}_\mu + v_\mu) (\mathcal{B}^\mu + v^\mu)\right]^{-1/2}$. The downside of this definition is that it adds the following constraint to the 4D-Newton equations (C.2)-(C.6):

$$(\mathcal{B}_\mu + v_\mu) (\mathcal{B}^\mu + v^\mu) < \mathcal{N}^2. \quad (\text{C.10})$$

This is indeed a constraint, as if we take $\mathcal{N} = 1$ and $\mathcal{B} = 0$, equation (C.10) imposes $v_\mu v^\mu < 1$. Such a constraint is not implied by the first definition (C.8).

We take the first definition (C.8), as it remains general with respect to the 4D-Newton equations. This 4-velocity is illustrated in Fig. C.2, along with \mathbf{n} , \mathcal{B} and \mathbf{v} .

With this choice, we can interpret the 4-velocity ${}^N\mathbf{u}$ as follows: ${}^N\mathbf{u}$ corresponds to the covered distance Δx^μ in spacetime per unit of proper time ${}^n\tau$, where ${}^n\tau$ refers to the proper time of \mathbf{n} . This vector \mathbf{n} and its induced foliation then define a fundamental time (as in the NC theory) with respect to which Newtonian 4-velocities are defined. The situation is different in general relativity, where the 4-velocity of a fluid element is defined as the covered distance Δx^μ in spacetime per unit of fluid element proper time ${}^u\tau$.

Remark. With what precedes, we can complete the definition of a *Newtonian observer* in the 4D-Newton theory, as being described by a vector \mathbf{m} such that $m^\mu n_\mu = -1$. This is the equivalent definition of an observer in the NC theory. The observer given by ${}^N\mathbf{u}$ is then the fluid itself.

Remark. For both definitions, \mathbf{v} corresponds to the coordinate velocity of the fluid 4-velocity ${}^N\mathbf{u}$ with respect to the coordinates $\mathcal{X}_\mathcal{B}^n$ (see Fig. C.2). The tilt velocity is however still not the physical vector. The latter remains \mathbf{v} as taking $\mathbf{v} = 0$ still implies a constraint on ρ with the 4D equation (C.2). This is not the case if we take $\frac{1}{\mathcal{N}} (\mathcal{B} + \mathbf{v}) = 0$.

C.1.5 Background homogeneous expanding spacetime

In this section, we present a first choice for the manifold \mathcal{M}^N .

Taking $\mathcal{N} := 1$ and $K_{\alpha\beta} = D_{(\alpha}\mathcal{B}_{\beta)} := -H_{\alpha\beta}$, with $D_\mu H_{\alpha\beta} := 0$, implies that \mathcal{M}^N is a homogeneous globally expanding spacetime. This expansion is anisotropic, unless $H_{\alpha\beta} \propto h_{\alpha\beta}$ which corresponds to the Einstein-de Sitter spacetime.

The tilt velocity of ${}^N\mathbf{u}$ is then $\mathbf{w} = \mathcal{B} + \mathbf{v}$. The expansion tensor can be rewritten $\Theta_{\alpha\beta} := H_{\alpha\beta} + D_{(\alpha}w_{\beta)}$. Then, in Eulerian comoving coordinates, the 4D-Newton equations for the vector \mathbf{w} become the usual Newton equations with a homogeneous deformation (equations for \mathbf{V} introduced in section B.1.3 with the homogeneous deformation of section B.2.3).

We still have the same results concerning expansion in a compact topology. If we impose the hypersurfaces Σ_t^N to have a compact topology, then $H_{\alpha\beta}$, being a constant gradient, has to be zero. It is still not possible to have an expanding compact topology in Newton, even when using the 4D-Newton formalism. This was expected as the two formulations are equivalent. In order to do it, the trick is to

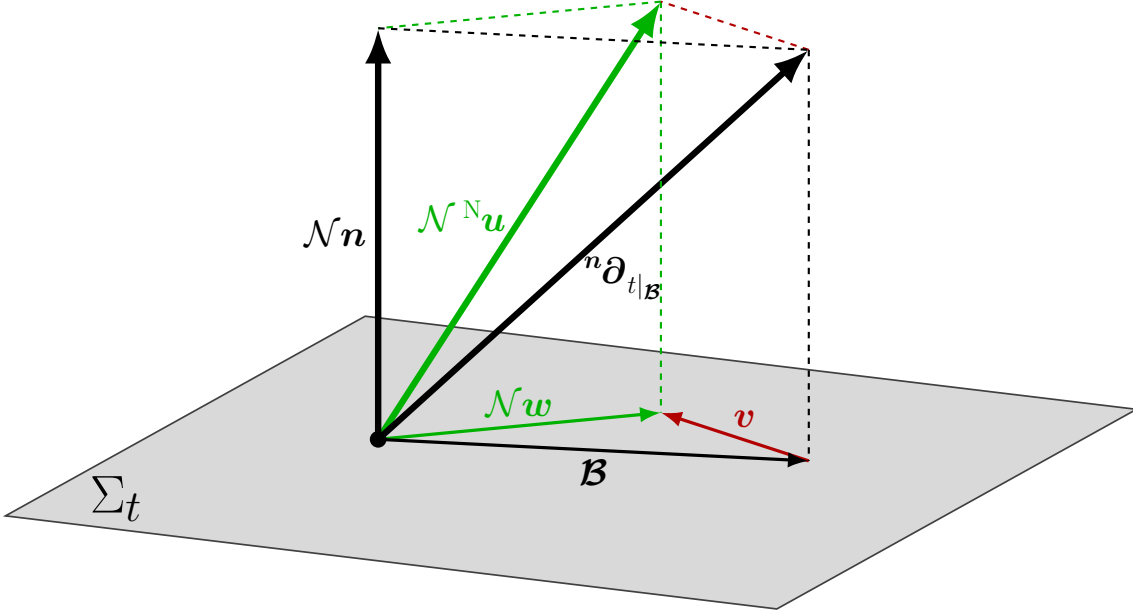


Figure C.2: Representation of the chosen Newtonian 4-velocity vector ${}^N\mathbf{u}$ with respect to the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$. Using the analogue between Lagrangian coordinates in Newton and in GR, we imposed the tilt velocity of ${}^N\mathbf{u}$ to be $\boldsymbol{\omega} := \frac{1}{N}(\mathcal{B} + \mathbf{v})$. The normal part is chosen to be \mathbf{n} , which is the equivalent to what is taken in the Newton-Cartan theory for the 4-velocity of an observer.

consider periodic boundaries only on the vector \mathbf{w} as explained in section B.2.3. In this case the topology of the hypersurfaces Σ_t^N is still \mathbb{R}^3 as \mathcal{B} is not periodically defined.

The choice of 4D-manifold \mathcal{M}^N of this section is independent of the fluid kinematical quantities. It is then only a background manifold. We therefore cannot draw any dictionary between the Newtonian fluid quantities and the relativistic fluid quantities defined via the Einstein equation for the 4D-manifold. In the next section, \mathcal{M}^N will depend on the Newtonian fluid enabling, the definition of a dictionary in section C.2.4.

Remark. The choice we make here cannot strictly be called a foliation choice as this would imply that another choice would describe the same equations but in another foliation, the 4D-manifold being unchanged. This is not true as, in general, another choice for \mathcal{N} and \mathcal{B} changes \mathcal{M}^{New} .

C.2 1+3-Newton equations

C.2.1 The choice

A natural choice coming from the definition of the Newtonian 4-velocity (C.8) is to take \mathcal{N} and \mathcal{B} such that the foliation is orthogonal to ${}^N\mathbf{u}$, implying ${}^N\mathbf{u} = \mathbf{n}$. This is done by taking $\mathcal{B} = -\mathbf{v}$. The lapse \mathcal{N} remains unknown. In analogy with GR, as we deal with a dust fluid, we choose the 4-acceleration of ${}^N\mathbf{u}$ to be zero, which is imposed by $\mathcal{N} = 1$. We expect this choice to be different in the case of non-dust fluids (this is discussed in section C.3.3).

Remark. Interestingly, with the above choice, the two definitions (C.8) and (C.9) are equivalent.

Under the present choice the 4D-Newton equations become

$$\mathcal{L}_{N\mathbf{u}} \rho = -\rho\theta, \quad (\text{C.11})$$

$$\mathcal{L}_{N\mathbf{u}} \theta = -4\pi G\rho + \Lambda - \Theta_{\mu\nu}\Theta^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu}, \quad (\text{C.12})$$

$$\mathcal{L}_{N\mathbf{u}} \omega_{\alpha\beta} = 0, \quad (\text{C.13})$$

with the definition of the gravitational field

$$g^\alpha := \mathcal{L}_{N\mathbf{u}} v^\alpha + v^\mu (\Theta_\mu^\alpha + \omega_\mu^\alpha), \quad (\text{C.14})$$

where

$$\Theta_{\alpha\beta} := D_{(\alpha} v_{\beta)} \quad ; \quad \omega_{\alpha\beta} := D_{[\alpha} v_{\beta]}, \quad (\text{C.15})$$

$$= \nabla_{(\alpha} {}^N u_{\beta)}, \quad (\text{C.16})$$

and with ${}^N u^\mu \nabla_\mu {}^N u^\alpha = 0$ and $\nabla_{[\alpha} {}^N u_{\beta]} = 0$, so that ${}^N\mathbf{u}$ defines a foliation. Note that the covariant form of the vorticity equation is now possible as only the normal vector remains in the Lie derivative.

The gravitational field definition (C.5) can be rewritten as

$$g^\alpha := {}^N u^\mu \nabla_\mu v^\alpha. \quad (\text{C.17})$$

The RHS is spatial as ${}^N\mathbf{u}$ has no 4-acceleration. We see that the gravitational field corresponds to the 4-acceleration, with respect to the observer ${}^N\mathbf{u}$, of the Newtonian fluid velocity \mathbf{v} .

The properties of the foliation $\{\Sigma_t^N\}_{t \in \mathbb{R}}$ are now linked to those of the fluid with the relation $K_{\alpha\beta} = -\Theta_{\alpha\beta}$. Then the 4D-Newton equations (C.11)-(C.13) closely resemble the 1+3-Einstein equations (1.52)-(1.57) for a dust fluid: on a formal aspect and on the fact that they are expressed in the rest frames of the fluid. We call them *the 1+3-Newton equations*.

The main difference between the 1+3-Newton and the 1+3-Einstein equations remains in the definition of the vorticity. In the Einstein equations, it is defined as the antisymmetric rest frame projection of the gradient $\nabla \mathbf{u}$ [definition (1.45)]. However in Newton, the antisymmetric part of $\nabla^N \mathbf{u}$ is zero as ${}^N \mathbf{u}$ defines a foliation. Instead the vorticity is defined as the antisymmetric part of a spatial vector gradient [second equation in (C.15)], the symmetric part of that gradient being the expansion tensor [first equation in (C.15)]. This is the reason why we will define the Newtonian limit (see section C.2.3) and the Newton-GR dictionary (see section C.2.4) for irrotational flows.

With the choice made in the present section, the spacetime metric of the manifold \mathcal{M}^N in the adapted coordinates \mathcal{X}_{-v}^n is

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -1 + v_c v^c & v_b \\ \hline v_a & h_{ab}(x^c) \end{array} \right). \quad (\text{C.18})$$

where $h_{ab}(x^c)$ are the spatial components of the flat spatial metric in the fixed coordinates used to derive the solution for \mathbf{v} .

Remark. As said previously, the norm of the Newtonian spatial velocity \mathbf{v} is not bounded by c . Where \mathbf{v} is superluminal, the time vector ${}^n \partial_{t|-\mathbf{v}}$ is spacelike and the points in \mathcal{M}^N where $v_\mu v^\mu = c^2$ correspond to coordinate singularities of the class $\mathcal{X}_{\mathcal{B}}^n$. As we will see in section C.2.5, this is not necessarily unphysical.

C.2.2 3+1-Newton equations

Once we have chosen \mathcal{N} and \mathcal{B} , the manifold \mathcal{M}^N is set. The choice made in section C.2.1, leading to equations (C.11)-(C.14), is such that these equations are written with respect to the foliation orthogonal to the Newtonian fluid 4-velocity ${}^N \mathbf{u}$ we defined. This is why they are called 1+3-Newton equations. We can however change this foliation.

We define a timelike unit vector field \mathbf{m} on \mathcal{M}^N , defining a foliation $\{\Sigma_t^{N,m}\}_{t \in \mathbb{R}}$ of lapse M in \mathcal{M}^N . We can then decompose ${}^N \mathbf{u}$, Θ and ω with respect to $\{\Sigma_t^{N,m}\}_{t \in \mathbb{R}}$. The same can be done for the 1+3-Newton equations. Equations (C.11) and (C.12) are scalar equations and do not need to be projected, contrary to equations (C.13) and (C.14). As for the Lie derivative $\mathcal{L}_{N\mathbf{u}}$, it becomes $\mathcal{L}_{\gamma\mathbf{m}+\gamma\mathbf{w}}$ with the usual decomposition of ${}^N \mathbf{u}$ with respect to \mathbf{m} defined in (1.26).

Then writing the 1+3-Newton equations in terms of the variables ${}^N \mathbf{u}$, Θ and ω projected with respect to $\{\Sigma_t^{N,m}\}_{t \in \mathbb{R}}$ gives the 3+1-Newton equations. We do not give these equations here but discuss in section C.3.2.1 a possible use of them in relation with dictionary defined in section C.2.4.

C.2.3 1+3-Newton from GR

The choice of \mathcal{N} and \mathcal{B} leading to the 1+3-Newton equations implies that the properties of the Lorentzian manifold \mathcal{M}^N depend on the dynamics of the Newtonian velocity \mathbf{v} . We however do not know at which point \mathcal{M}^N with the metric (C.18) is a solution of the Einstein equations for the same fluid as the one in 1+3-Newton, i.e. a dust fluid. In this section we will recover the 1+3-Newton equations from GR, enabling us to answer this question in section C.2.4.

C.2.3.1 Expansion tensor decomposition

We want to recover the 1+3-Newton equations from general relativity, thus defining a Newtonian limit. Our approach will need the definition of a flow orthogonal foliation. However, as explained in section C.2.1, the difference in the definition of vorticity between Newton and GR implies that such a foliation cannot be built in the latter theory as opposed to the former. So we expect that recovering the 1+3-Newton equations from the 1+3-Einstein equations will be more complicated for vortical flows.

We then only take irrotational fluids in both theories. Note that a solution to allow for vorticity, but still dealing with foliations, is to make the limit between the 3+1-Newton equations (presented in C.2.2) and the 3+1-Einstein equations. This will not be studied in this appendix but it is discussed in section C.3.2.1.

We consider the 3+1-Einstein equations (1.20), (1.21) and (1.23) in the orthogonal foliation of an irrotational dust fluid of 4-velocity \mathbf{u} (they are equivalent to the 1+3-Einstein equations for irrotational flows). The Hamilton constraint (1.22) is redundant with the other equations and not needed for the discussion. For this section only we use spatial indices and reintroduce the light speed c .

In a cosmological setup, we suppose that we can decompose the expansion tensor into scalar, vector and tensor parts as in standard perturbation theory²:

$$\Theta_{ab} = \chi h_{ab} + D_{(a} v_{b)} + \Xi_{ab} \quad (\text{C.19})$$

with $\Xi_c^c = 0$ and $D_c \Xi_a^c = 0$. h_{ab} is the spatial metric of the orthogonal foliation. The irreducibility of this decomposition is discussed in section C.5.1.

We take \mathbf{v} to be irrotational, i.e. $D_{[a} v_{b]} = 0$. This is a choice motivated by the 1+3-Newton equations in which $D_{[a} v_{b]}$ plays the role of the vorticity. In section C.3.2.2 we discuss what $D_{[a} v_{b]}$ should be in the case of rotational fluids.

Remark. The scalar-vector-tensor decomposition we made is fully covariant (it does not depend on an adapted class of coordinates). It is also independent of a choice of foliation as the spatial projection used is defined with respect to the fluid. This is not the case for the decomposition of the spatial metric in standard perturbation theory.

²In standard perturbation theory this is done for the spatial metric.

The parameter χ is interpreted as the *scalar expansion*, $D_{(a}v_{b)}$ as the *gradient expansion* and Ξ as the *gravitational wave term*. Ξ is only a shear term as its trace is zero. While the trace of the gradient expansion is on average zero for a compact space, the scalar expansion is not. Then global expansion in a compact space is driven by χ . Both χ and Ξ are not present in Newton, where only $D_{(a}v_{b)}$ is. This is coherent with the fact that there are no gravitational waves nor global expansion in a compact space for this theory. In this view, we can then interpret $D\mathbf{v}$ to be the *Newtonian fluid expansion* and χ to be the *space expansion*.

C.2.3.2 The limit

The first approximation we make is to neglect the space expansion and the gravitational wave term compared to the Newtonian fluid expansion (this is discussed in section C.5):

$$\Theta_{ab} \simeq D_a v_b. \quad (\text{C.20})$$

This implies that in the adapted class $\mathcal{X}_{-\mathbf{v}}^u$

$${}^u\partial_{t|-\mathbf{v}} h_{ab} \ll D_a v_b,$$

and thus

$${}^u\partial_{t|-\mathbf{v}} (D_a v_b) \simeq D_a ({}^u\partial_{t|-\mathbf{v}} v_b). \quad (\text{C.21})$$

In this commutation relation we neglected the time variation of the spatial metric.

We define the beta-factor $\beta_v := |\mathbf{v}|/c$ and the following length scales:

- the typical length scale $L_{v,l}$ of the spatial variation of the vector \mathbf{v} , i.e. $\frac{1}{c}D_a v_b = \frac{1}{c}\Theta_{ab} \sim \beta_v/L_{v,l}$,
- the typical length scale $L_{v,t}$ of the time variation of the vector \mathbf{v} , i.e. $\frac{1}{c^2}{}^u\partial_{t|-\mathbf{v}} v^a \sim \beta_v/L_{v,t}$,
- the Schwarzschild density length scale $L_\epsilon := \left(\frac{G\epsilon}{c^4}\right)^{(-1/2)}$,
- the typical local curvature radius L_R of the spatial Ricci tensor, i.e. $R_{ab} \sim 1/L_R^2$.

By defining the Newtonian gravitational field as $g^a := ({}^u\partial_{t|-\mathbf{v}} + v^c D_c)v^a$ [justified by the 1+3-Newton equation (C.17)], we can say that in a Newtonian regime, ${}^u\partial_{t|-\mathbf{v}} v^a$ will be of the same order as $v^c D_c v^a$ which implies

$$L_{v,l}/L_{v,t} \sim \beta_v.$$

Assuming that $\frac{1}{c}D_a D_b v_c \sim \beta_v/L_{v,l}^2$ and using the 3+1-Raychaudhuri equation (1.39) with the commutation relation (C.21) we have

$$\left(\frac{L_{v,l}}{L_\epsilon}\right)^2 \sim \beta_v^2.$$

This relation along with the 3+1-Ricci equation (1.40) leads to

$$\left(\frac{L_{v,l}}{L_R}\right)^2 \sim \beta_v^2. \quad (\text{C.22})$$

For the usual condition $\beta_v \ll 1$ on the Newtonian velocity \mathbf{v} , relation (C.22) shows that the spatial variations of the Newtonian velocity are small in front of the typical length scale given by the spatial curvature. This shows that the curvature is of second order in β_v .

We can then consider that, at leading order, the spatial curvature does not affect the dynamics of \mathbf{v} , i.e. $\mathbf{D}\mathbf{v} \sim \hat{\mathbf{D}}\mathbf{v}$, where $\hat{\mathbf{D}}$ is the connection of a flat metric $\hat{\mathbf{h}}$ (a more quantitative justification of this approximation is given in section C.5). Then, at leading order, the 3+1-Ricci equation is not an evolution equation anymore but becomes a relation giving the spatial curvature orthogonal to the fluid as function of the kinematical quantities of that fluid. In this view, we then have $R_{ab} = R_{ab}^{(2)}$, where $R_{ab}^{(2)}$ is of second order in β_v with

$$R_{ab}^{(2)} = \frac{-1}{c^2} \left[\left({}^u\partial_{t|_v} + {}^\Sigma\mathcal{L}_v \right) \Theta_{ab} + \left(\frac{4\pi G\epsilon}{c^2} + \Lambda \right) \hat{h}_{ab} + \theta\Theta_{ab} - 2\Theta_{ac}\Theta^c_b \right], \quad (\text{C.23})$$

with $\Theta = \hat{\mathbf{D}}\mathbf{v}$.

As for the momentum constraint (1.41), it becomes $\hat{D}_{[a}\hat{D}_{b]}v_c = 0$ at leading order in β_v . This is consistent with a zero curvature at first order. Thus equation (1.41) is not a constraint anymore.

We give an additional relation for the Weyl tensor in this limit (see Maartens & Bassett, 1998, for the expression of the Weyl tensor in terms of the kinematical quantities Θ and ω). Its electric part E_{ab} is

$$E_{ab} = -D_{\langle a}g_{b\rangle}. \quad (\text{C.24})$$

This relation is true to any order in β_v once assumption (C.20) is made. The magnetic part H_{ab} is zero. Note that if $D_{[a}v_{b]} \neq 0$, this is not true anymore. This is discussed in section C.3.2.2.

Remark. Using the decomposition of the expansion tensor (C.19), we can see that the solutions of the Einstein equations which do not feature the Newtonian fluid expansion term will not have a Newtonian limit. In particular, this is the case for purely gravitational waves solutions.

C.2.3.3 Recovering the 1+3-Newton equations

The limit introduced in the previous section implies that at leading order in β_v the Ricci equation is a relation for the spatial curvature and not an evolution equation anymore. This spatial curvature orthogonal to the fluid 4-velocity is of second order in β_v . The expansion tensor is a gradient, $\Theta_{ab} = D_{(a}v_{b)}$. The momentum constraint is then trivial at leading order. The only 3+1-Einstein equations remaining to determine the evolution of Θ_{ab} are the 3+1-Raychaudhuri equation (1.23) and the 3+1-energy conservation (1.24), which are respectively equivalent to equations (C.12) and (C.11) of the 1+3-Newton system. The Newton-vorticity equation (C.13) is trivially recovered as the limit is done for irrotational flows.

With the Newtonian limit defined in the previous subsection, we recovered the 1+3-Newton equations in the irrotational case. This formulation of Newton's equations is then supported. In the next section we will use the Newtonian limit of the present section to define a Newton-GR dictionary.

C.2.4 Newton-GR dictionary

In the previous section we showed that we can recover the 1+3-Newton equations for irrotational fluids from GR with a limit at leading order in β_v . The limit also defines a spacetime manifold \mathcal{M}_{lim} solution of the Einstein equations at leading order. This manifold however differs from the manifold \mathcal{M}^{N} given by the metric (C.18). Indeed, \mathcal{M}^{N} has strictly flat spatial sections orthogonal to the fluid, whereas the curvature of the same sections in \mathcal{M}_{lim} is non-zero and of second order. This implies that \mathcal{M}^{N} with the metric (C.18) is not solution of the Einstein equations at leading order. The dictionary will therefore be given by \mathcal{M}_{lim} and not \mathcal{M}^{N} . For the remainder of this appendix \mathcal{M}_{lim} will then be denoted $\mathcal{M}^{(\text{dic})}$.

We define the following *Newton-GR dictionary for irrotational dust fluids*: given a solution of the Newton equations for \mathbf{v} , the relativistic quantities, denoted with the upper-script $^{(\text{dic})}$, are determined by the following relations:

$$\epsilon^{(\text{dic})} = \rho, \quad (\text{C.25})$$

$$\Theta_{ab}^{(\text{dic})} = \hat{D}_a v_b, \quad (\text{C.26})$$

$$R_{ab}^{(\text{dic})} = -\hat{D}_a g_b - \hat{D}_c v^c \hat{D}_a v_b + \hat{D}_a v_c \hat{D}^c v_b + (4\pi G\rho + \Lambda) \hat{h}_{ab}, \quad (\text{C.27})$$

where the RHS are the Newtonian quantities. $\Theta^{(\text{dic})}$ is the expansion tensor of the relativistic fluid, $\mathbf{R}^{(\text{dic})}$ is the spatial curvature orthogonal to the relativistic fluid, $\hat{\mathbf{D}}$ is a flat connection, \mathbf{g} is the Newtonian gravitational field constrained by the Newton-Gauss equation (1.91) and the Newton-Faraday equation (1.92).

Remark. As we only give the Ricci tensor $R_{ab}^{(\text{dic})}$, the spatial metric orthogonal to the fluid 4-velocity cannot be explicitly constructed. However, raising or lowering the dictionary quantities at leading order only requires the flat spatial metric \hat{h}_{ab} .

Studying the light ray trajectories with a Newtonian solution using our dictionary requires the 3+1 light-geodesic equation of the manifold $\mathcal{M}^{(\text{dic})}$. This equation can be found in [Vincent et al. \(2012\)](#).

In section [C.2.5](#) we will take the example of an exact solution of the Newton equations to test the dictionary.

As the 1+3-Newton equations were recovered from GR only for irrotational fluids, we were only able to draw a dictionary for these kinds of fluids. In section [C.3.2](#) we discuss the possibility of a dictionary with vorticity.

C.2.5 Schwarzschild geometry

C.2.5.1 Point mass Newtonian solution

In this section we study an exact vorticity-free solution of the 1+3-Newton equations.

We begin with a 3D-Newtonian calculation. We consider a point mass of mass M creating a gravitational field $g^a = (-GM/r^2, 0, 0)$ in spherical coordinates (r, θ, φ) . We then solve the Euler equation [\(1.90\)](#) for a *stationary*, irrotational fluid of test observers of velocity v . We have

$$\begin{aligned} D_a(v_c v^c) &= 2g_a, \\ v_a &= D_a \Psi, \end{aligned}$$

with Ψ a scalar field depending only on the radial coordinates. The general solution is

$$v^a = \left(\pm \sqrt{2E + \frac{2GM}{r}}, 0, 0 \right), \quad (\text{C.28})$$

where E is a constant corresponding to the energy of the fluid particles. This solution corresponds to a radially ingoing or outgoing free-falling fluid of test observers. If $E < 0$, the solution is valid in the region $r < -\frac{2GM}{E}$ and corresponds to fluid particles with bounded orbits, i.e. elliptic orbits. If $E = 0$, the orbits are parabolic and for $E > 0$ they are hyperbolic. Note that all the particles have the same type of orbit as E is a constant of space.

Remark. In the case of the 1+3-Newton equations, this solution implies the following spacetime line element for \mathcal{M}^N in the adapted class \mathcal{X}_{-v}^{Nu} (we recall that \mathcal{X}_{-v}^{Nu} corresponds to the class \mathcal{X}_0):

$$\begin{aligned} ds^2 &= \left(-1 + 2E + \frac{2GM}{r} \right) dt^2 \mp 2\sqrt{2E + \frac{2GM}{r}} dt dr \\ &\quad + dr^2 + r^2 d\Omega^2, \end{aligned} \quad (\text{C.29})$$

with $d\Omega^2 := d\theta^2 + \sin^2 \theta d\varphi^2$.

C.2.5.2 Relativistic quantities from the Newtonian solution

Using the dictionary (C.25)-(C.27), we can derive the relativistic quantities corresponding to the Newtonian solution (C.28). We obtain

$$\epsilon^{(\text{dic})} = 0, \quad (\text{C.30})$$

$$\Theta_{ab}^{(\text{dic})} = \text{diag} \left(-\frac{GM}{r^2 v^r}, r v^r, r v^r \sin^2 \theta \right), \quad (\text{C.31})$$

$$R_{ab}^{(\text{dic})} = -2E \text{diag} (0, 1, \sin^2 \theta), \quad (\text{C.32})$$

with $v^r := \pm \sqrt{2E + \frac{2GM}{r}}$.

C.2.5.3 Radially free-falling test fluids in GR

We want to know if the manifold defined by equations (C.30)-(C.32) is solution of the Einstein equations at leading order in β_v , and if it describes the same physical system as the Newtonian solution, i.e. a radially free-falling test fluid on a mass point.

The solution for this physical system in GR is given by the Schwarzschild manifold and the adapted coordinates corresponding to a free-falling observer are the generalised Gullstrand-Painlevé coordinates (see MacLaurin, 2019³). The Schwarzschild line element in these coordinates is

$$\begin{aligned} ds^2 = & \frac{-1 + \frac{2GM}{r}}{2E + 1} dt^2 \mp 2 \frac{\sqrt{2E + \frac{2GM}{r}}}{2E + 1} dt dr \\ & + \frac{1}{2E + 1} dr^2 + r^2 d\Omega^2. \end{aligned} \quad (\text{C.33})$$

where E can be interpreted as the Newtonian energy of the fluid particles. For $E = 0$, the observer associated with the generalised Gullstrand-Painlevé coordinates is a parabolic radially free-falling test fluid. For $E < 0$ and $E > 0$ the free-fall is respectively elliptic and hyperbolic.

We note with the upper-script ^(GP) the relativistic quantities corresponding to an observer associated with the generalised Gullstrand-Painlevé coordinates. These quantities are

$$\epsilon^{(\text{GP})} = 0, \quad (\text{C.34})$$

$$\Theta_{ab}^{(\text{GP})} = \text{diag} \left(-\frac{GM}{r^2 v^r (2E + 1)}, r v^r, r v^r \sin^2 \theta \right), \quad (\text{C.35})$$

$$R_{ab}^{(\text{GP})} = -2E \text{diag} (0, 1, \sin^2 \theta), \quad (\text{C.36})$$

with $v^r := \pm \sqrt{2E + \frac{2GM}{r}}$.

³MacLaurin (2019) uses the Killing energy $e := -n_{\text{ff}}^\mu \xi_\mu$ where \mathbf{n}_{ff} is the 4-velocity of the free falling observer and $\boldsymbol{\xi}$ is a static Killing vector. The energy definition we use is linked to e with the relation $E = e^2 - 1$.

C.2.5.4 Comparison

In the present section, we compare the relativistic quantities (C.30)-(C.32) obtained from the Newton-GR dictionary with the ones of the Schwarzschild metric (C.34)-(C.36). For simplicity, we will not consider the case $E < 0$. Then E corresponds to the Newtonian energy of the test particles at infinity.

The energy densities $\epsilon^{(\text{dic})}$ and $\epsilon^{(\text{GP})}$ are the same. This was expected as the Newtonian and the GR solutions are both vacuum solutions. The covariant components of the spatial curvatures are also the same, with $R_{ab}^{(\text{dic})} = R_{ab}^{(\text{GP})}$. The covariant components of the expansion tensors differ only for the component $_{rr}$, with $\Theta_{rr}^{(\text{dic})} = (2E + 1)\Theta_{rr}^{(\text{GP})}$.

The limit under which the dictionary is defined implies $|\mathbf{v}| \ll c$ for all r . Taking $r \rightarrow \infty$, this implies $E \ll 1$ which in turn implies that the comparison should be done in the region $r \gg GM$. Then at leading order in E and $\frac{GM}{r}$, the dictionary quantities are the same as those of Gullstrand-Painlevé. This supports our dictionary.

C.2.5.5 The parabolic free-fall: $E = 0$

In the case $E = 0$, the dictionary quantities are exactly equal to the general Gullstrand-Painlevé ones. Furthermore the metric (C.29) of the manifold \mathcal{M}^N , constructed from the Newtonian solution (C.28), is exactly the Schwarzschild metric in generalised Gullstrand-Painlevé coordinates, which implies that \mathcal{M}^N is the Schwarzschild manifold. This result is true without any approximation. Then the 4D construction of Newton's equations we introduced in this appendix, and in particular the case of the 1+3-Newton equations, allows us to recover exactly a physical solution of the Einstein equations. This further supports the choice ${}^N\mathbf{u} = \mathbf{n}$ made in section C.2.1.

Note that it was already known that the velocity as a function of the point mass distance of a parabolic radially free-falling observer was the same in Newton and in GR. What we showed is that this solution allows us to recover from Newton the full spacetime metric of Schwarzschild. This was possible because the foliation of the generalised Gullstrand-Painlevé coordinates with $E = 0$ has flat spatial sections, which is required by the 1+3-Newton equations.

We see from this solution that even if the Newtonian velocity \mathbf{v} can, at certain points of \mathcal{M}^N , be comparable to the speed of light, and even exceed it, it is still physical. We know that because it is the Schwarzschild spacetime. This means that solutions of the Newton equations are not necessarily unphysical for $v^\mu v_\mu \sim c^2$. We however expect this statement to be true in few cases only.

Remark. Strangely, this exact correspondence between a Newtonian solution and a GR one arises for a Newtonian fluid whose energy is zero for any fluid particles. This leads to the following question: is there a link, in general, between the energy of a Newtonian fluid and the validity of the related Newtonian solution with respect to GR? If this is the case,

this would be true only for one gravitational potential energy convention. For an isolated system like a mass point, we saw that it works if the gravitational potential is taken to be zero at infinity. What convention should be taken in the case of a compact spacelike \mathbb{T}^3 remains to be determined.

C.3 Discussions

C.3.1 Remarks on the Newtonian limit

The Newtonian limit of general relativity and the corresponding dictionaries (e.g. [Green & Wald, 2012](#)) are usually done with respect to an accelerated observer. We note its 4-velocity \mathbf{n} . The foliation corresponding to this observer has then a lapse N , the spatial gradient of which is the acceleration of the observer. Hence in the adapted coordinates \mathcal{X}_0^n , the component g_{00} of the spacetime metric is N^2 . The acceleration of the observer is considered in these limits to be the gravitational field of Newton theory. Using ${}^n a_\alpha := D_\alpha \ln N$, this is why, at leading order, the lapse, and therefore the component g_{00} , gives the Newtonian gravitational potential. In this leading order approximation, the accelerated observer is considered to be only slightly tilted with respect to the fluid.

The Newtonian limit we defined in section [C.2.3](#) is however made in the rest frames of the fluid, which is not accelerated, being a dust fluid. The Newtonian gravitational field then cannot be the lapse, which is fixed to 1. In place, this field is defined as the acceleration of the spatial vector present in the decomposition of the expansion tensor [see equation [\(C.17\)](#)].

However, the interpretation of g_{00} as the gravitational potential, in coordinates adapted to the fluid rest frames, still holds in the case of stationary irrotational fluids. In these cases,

$$g_\alpha = D_\alpha \left(\frac{1}{2} v_\mu v^\mu \right).$$

In the coordinates \mathcal{X}_{-v}^{Nu} , we have $g_{00} = -1 + v_\mu v^\mu$ which implies

$$g_\alpha = \frac{1}{2} D_\alpha (g_{00} + 1), \tag{C.37}$$

and g_{00} can be interpreted as the gravitational potential.

Note that this interpretation is not valid in the case of the Lagrangian coordinates \mathcal{X}_0^{Nu} where $g_{00} = 1$ as all the dynamics of the fluid is put in the time variations of the spatial metric. Furthermore, in the case of non-stationary fluids, the term g_{00} is not the gravitational potential anymore, as [\(C.37\)](#) does not hold.

C.3.2 Dictionary with vorticity

In section C.2.4 we drew our Newton-GR dictionary in the case of irrotational fluids. The reason for this was that, in general relativity, no orthogonal foliation can be defined for a rotational fluid. But as we made the dictionary in the rest frames of the fluid, we needed such a foliation.

We will not detail the construction of a dictionary with vorticity in this article. We however present two possibilities that should allow for it.

C.3.2.1 Tilted dictionary

The first, and most promising possibility, is to make the dictionary in a tilted foliation with respect to the fluid. In general relativity, the 3+1-Einstein equations provide the tilted description of a fluid and allow for vortical flows. In our formulation of Newton's theory, the equations where the fluid is tilted are the 3+1-Newton equations presented in section C.2.2. They are derived from the 1+3-Newton equations by making a change of foliation ${}^N\mathbf{u} \rightarrow \mathbf{m}$.

One strength of a tilted dictionary would be to show that Newton's theory can be obtained from any foliation.⁴ But as the choice of this foliation is not necessarily physically motivated (see also Buchert et al., 2020, for a discussion of this topic), we would prefer making the dictionary with respect to the fluid rest frames. We discuss this in the next subsection.

C.3.2.2 Orthogonal dictionary

Constructing a dictionary in the rest frames of the fluid might be more complicated as no foliation can be defined in general relativity, contrary to the Newtonian case.

It is however possible to define a rest frame Riemann tensor ${}^u\mathbf{Riem}$ and a rest frame covariant derivative ${}^u\mathbf{D}$ (see Roy, 2014). They do not have the same properties as the ones defined on hypersurfaces. The first Bianchi identity for ${}^u\mathbf{Riem}$ will feature the vorticity of the fluid and ${}^u\mathbf{D}$ will have torsion. The latter is however of second order in β_v . We then hope that at leading order the rest frames can be approximated to be a family of hypersurfaces.

It remains to be shown that the projection along \mathbf{u} of the Lie brackets of rest frame vectors is also of second order.⁵ This would indicate that we could maybe define a coordinate basis on the rest frames at leading order.

Remark. In this dictionary, the gradient in the decomposition (C.19) of the expansion tensor would feature a non-zero antisymmetric part which would be the vorticity tensor. Subsequently, the magnetic part of the Weyl tensor would not be zero anymore.

⁴ In standard perturbation theory, this means that any gauge choice would be suited for a Newton-GR dictionary.

⁵As there is no foliation, there is no coordinate basis in the rest frames of the fluid. This implies that the Lie brackets of any vectors in these rest frames feature a non-zero part along \mathbf{u} .

C.3.2.3 Is it really possible?

It is known that Newton's theory features gravitational phenomena which are not described by GR. Assuming that the latter is the genuine theory of gravitation, these phenomena are not physical. We mentioned in the introduction the case of a shear-free dust fluid which can both rotate and expand in Newton, but cannot in GR. This implies that no limit exists which allows us to recover the full Newtonian theory from GR.

To our knowledge, there exists no example of a phenomenon like the one just mentioned, i.e. present in Newton but not in GR, for a vorticity-free fluid. If this is indeed the case, this might imply that the impossibility at fully recovering Newton from GR, is due to the vorticity. Then constructing a dictionary with vorticity, as we presented in the previous subsections, would need require additional approximation than just $|\mathbf{v}| \ll c$.

C.3.3 1+3-Newton for non-dust fluids

We assumed until now the Newtonian fluid to be a dust fluid. This was done to simplify the interpretations made while constructing the 1+3-Newton equations and the related dictionary. We briefly study the case of a non-dust fluid in this section.

Such a fluid is influenced by additional forces, other than the gravitational force, described by a vector field \mathbf{F} . These forces can be either internal, linked to the fluid properties (density, pressure, viscosity, ...), or external. The changes in the Newton system for a non-dust fluid is given by the second law of Newton. This is translated by the addition of \mathbf{F} in the Euler equation (1.90):

$$\left(\partial_{t|x} + v^k D_k\right) v^i = g^i + F^i/\rho, \quad (\text{C.38})$$

with \mathbf{g} still solution of the Newton-Gauss (1.91) and Newton-Faraday (1.92) constraints.

In the Newton system (1.93)-(1.96) written in terms of kinematical quantities of the fluid, the change is made by adding the divergence of \mathbf{F} in the Newton-Raychaudhuri equation (1.95) and the vorticity of \mathbf{F} in the Newton-vorticity equation (1.96). These additional terms are then present in the 4D-Newton equations.

The 1+3-Newton equations should not however be obtained with the choice $\mathcal{N} = 1$ and $\mathcal{B} = -\mathbf{v}$ but rather with

$$\mathcal{B} = -\mathbf{v} \quad ; \quad D \ln \mathcal{N} = \mathbf{F}. \quad (\text{C.39})$$

This choice would not change the Newton-GR dictionary much. Essentially the interpretation of the Ricci equation to be a relation for the spatial curvature tensor would remain valid.

C.3.4 Cosmological models from 1+3-Newton

In the introduction of [Vigneron \(2020\)](#) we motivated the construction of the 1+3-Newton system as a way to better understand why Newton's theory, compared to GR, is lacking the phenomenon of backreaction (in a compact space). Ultimately this would be used to define simple models suited for the study of backreaction and global topology in cosmology. This section aims at presenting how we could define such models from the 1+3-Newton formulation and GR. However, we leave the precise construction of these models for another paper.

In section [C.3.4.1](#), we present an extension of our dictionary to allow for global expansion of a compact space, but still without backreaction. The next two subsections focus on possible strategies enabling the construction of the cosmological models.

C.3.4.1 1+3-Newton equations and dictionary for a globally expanding compact space

In sections [B.2.3](#) and [C.1.5](#) we showed that no global expansion is possible in a compact space in Newton's theory. A solution to allow for expansion was to decompose the fluid velocity \mathbf{v} into a homogeneous deformation vector \mathbf{U} and a peculiar velocity \mathbf{V} , the latter having periodic boundary conditions. As explained in section [B.2.3](#), this is an effective picture of the expansion in a compact space, as Σ^N (or equivalently the hypersurfaces Σ_t^N for the 4D formulation) is still \mathbb{R}^3 .

Having Σ^N compact with a global expansion is possible with a modification of the Newton equations based on the effective picture of section [B.2.3](#) and the decomposition introduced in section [C.2.3.1](#). We will focus on a modification allowing for isotropic global expansion.

The modification is to replace the definition ([C.15](#)) for the expansion tensor by

$$\Theta_{\alpha\beta} := Hh_{\alpha\beta} + D_{(\alpha}v_{\beta)}. \quad (\text{C.40})$$

where H is a homogeneous Hubble expansion rate (i.e. $D_\alpha H = 0$), while still using the 1+3-Newton equations ([C.11](#))-([C.14](#)). H an additional fundamental variable in the theory. These equations, along with the definition ([C.40](#)), are equivalent to the Hubble flow equations of section [B.2.3](#) but allow Σ_t^N to be compact. Note that the evolution equation for H , being a spatial constant, is given by the spatial average of the Raychaudhuri equation over the whole manifold Σ^N .⁶ This average equation then depends on the boundary conditions at infinity if $\Sigma^N = \mathbb{R}^3$ or on the topology if Σ^N is compact.

We can then redefine the dictionary of section [C.2.4](#) to feature the global expansion. We then have a *Newton-GR dictionary for irrotational dust fluids and globally*

⁶In the case $\Sigma^N = \mathbb{R}^3$, the spatial average requires boundary conditions at infinity to be defined.

expanding compact spaces:

$$\epsilon^{(\text{dic})} = \rho, \quad (\text{C.41})$$

$$\Theta_{ab}^{(\text{dic})} = \Theta_{ab}, \quad (\text{C.42})$$

$$R_{ab}^{(\text{dic})} = -\hat{D}_a g_b - \theta \Theta_{ab} + \Theta_{ac} \Theta_b^c + (4\pi G \rho + \Lambda) \hat{h}_{ab}, \quad (\text{C.43})$$

The RHS side are the Newtonian quantities with $\Theta_{ab} := H h_{ab} + \hat{D}_a v_b$ and $\hat{D}_{[a} v_{b]} = 0$.

The modification (C.40) can be justified by the limit introduced in section C.2.3.2 and section C.5. When neglecting the space expansion term in section C.5.2.1, a freedom remained on χ from equation (C.46) as a spatial constant freedom. This constant is H . We took it to zero in section C.5.2.1 in order to recover the 1+3-Newton equation as defined in section C.2.1.

As we included global expansion, the dictionary (C.40)-(C.43) can be used to compare Newtonian and relativistic cosmological simulations. It is however still a bit limited as it requires irrotational fluids.

The modified 1+3-Newton equations of this section, and the related dictionary still do not feature backreaction in a compact space (the theorem of Buchert & Ehlers still holds). Furthermore, the spatial sections being flat, we are not able to study structure formation in spherical or hyperbolic spaces, and the only oriented compact topology available is \mathbb{T}^3 , up to a finite covering. In the next two subsections we will discuss possible GR based modifications of the 1+3-Newton equations which would allow these studies.

C.3.4.2 Models for the study of the backreaction

The Newton theory for fluid dynamics is a scalar-vector theory, i.e. the dynamical variables are a scalar and a vector. The scalar is the rotational free part of \mathbf{v} and the vector is the divergence free part of \mathbf{v} . The scalar part is evolved with the Raychaudhuri equation (C.12) and the vector part with the vorticity equation (C.13).

General relativity is a scalar-vector-tensor theory, i.e. there are dynamical variables, called tensorial variables, which cannot be written as function of a scalar or a vector. This is the case of the gravitational wave term in the decomposition (C.19).

What we mean by *defining a GR based model from Newton's equations* is to keep the scalar-vector theory of Newton but with additional non-tensorial variables, terms and/or equations motivated by GR. Keeping a scalar-vector theory ensures a relative simplicity compared to tensor theories like GR. Such a model would enable the study of relativistic effects not present in Newton's theory using the simple tools of this theory.

In particular, we would like to focus on models implementing the backreaction which is a missing phenomenon of Newton's theory (for compact spaces). A possible model to study backreaction while allowing for non-linear structure formation would

be to consider $\chi \neq 0$ in the decomposition (C.19) along with $D_\alpha \chi \neq 0$. As in section C.3.4.1, the expansion tensor features an additional term. But we consider here that χ is not a constant of space. The space expansion is thus local and global. For this model to be well defined one has to derive an evolution/constraint equation for the fundamental field χ from GR. This equation will feature \mathbf{v} . Thus the Newtonian fluid dynamics will affect the space expansion. In this sense this model could be useful to probe the backreaction effect.

C.3.4.3 For non-flat topologies

In section C.5, we assumed the spatial metric to be conformally flat to justify our Newton-GR dictionary. Relaxing this hypothesis and supposing h_{ab} to be conformal to a constant curvature metric might be a way to define a Newtonian-like theory on a non-flat space.

Such a theory was heuristically defined in Roukema & Rózański (2009) to probe the topological acceleration in different spherical topologies. There were however different possibilities in this heuristic definition which were not relativistically motivated.

Adapting the limit leading to the 1+3-Newton formalism from GR (by changing the flat conformal hypothesis) could provide a non-flat Newtonian like theory coherent with general relativity. Along with the additional term χ in the expansion tensor, this theory if well defined, will be a tool to probe the effect of topology on the backreaction.

As an example, we give a possible model, but we do not try to justify it from GR. We consider, similarly to Roukema & Rózański (2009), that the Newton equations (1.93)-(1.97) are also valid if Σ^N is a constant curvature space,⁷ i.e. its Ricci tensor is $R_{ab} = \frac{R}{3}h_{ab}$, where R is the scalar curvature. Then if we calculate the backreaction \mathcal{Q}_{Σ^N} , on the whole manifold Σ^N , defined by Buchert & Ehlers (1997) as

$$\mathcal{Q}_{\Sigma^N} := \left\langle \theta^2 - \Theta_{cd}\theta^{cd} + \omega_{cd}\omega^{cd} \right\rangle_{\Sigma^N} - \frac{2}{3}\langle \theta \rangle_{\Sigma^N}^2, \quad (\text{C.44})$$

it is not zero for a compact space (as for the flat case) anymore. Instead, we have the relation

$$\mathcal{Q}_{\Sigma^N} = \frac{R}{3}\langle v_c v^c \rangle_{\Sigma^N} \quad (\text{C.45})$$

where the brackets $\langle \cdot \rangle_{\Sigma^N}$ correspond to the spatial average over the compact space Σ^N . This relation implies a dependence of the backreaction on the type of the global topology (spherical, flat or hyperbolic) via $\frac{R}{3}$, as well as on the Newtonian dynamics of the fluid via $\langle v_c v^c \rangle_{\Sigma^N}$.

⁷In Roukema & Rózański (2009), the heuristic assumption regarding the curvature of Σ^N is made on the system (1.89)-(1.92) which is not equivalent to doing it from the system (1.93)-(1.97).

This model is however only heuristically defined, and thus we cannot be sure that the result (C.45) is physically relevant. The 1+3-Newton formulation and the Newtonian limit we introduced might help justify, or disprove, this calculation.

C.3.5 Comparison with the Newton-Cartan theory

In the present section we explain the differences between our 4D formulation of Newton's equations and the Newton-Cartan theory.

The main difference is that we were able to define a non-degenerate metric on the spacetime manifold \mathcal{M}^N of our formulation, implying this manifold to be (pseudo)-Riemannian, whereas this is not case in NC.

In addition to this point, the 1+3-Newton formulation does not feature an absolute time or foliation. We are free to change the foliation in which we are writing the equations. This leads to the 3+1-Newton equations (see section C.2.2). At most we can say that the formulation implies, like in GR, a preferred foliation: the one defined with respect to the fluid 4-velocity. The situation is different in NC where an absolute time is defined, linked to an absolute foliation.

Finally the 1+3-Newton system of equations (C.11)-(C.15) is equivalent to the classical Newton system (1.93)-(1.97). The ensemble of solutions is then the same for both formulations. This is not the case in NC, where the theory is slightly more general than the classical formulation of Newton's theory Ehlers (see 1997).

C.4 Conclusion

The aim of this appendix was to introduce a new formulation of Newton's equations on a 4-dimensional Lorentzian manifold.

To get to this formulation, we started from the classical Newton equations (1.89)-(1.92) written in a Galilean frame on a 3-dimensional manifold Σ^N . We generalised these equations by writing them for any time-parametrised coordinate system [equations (B.12)-(B.15)]. We showed that the freedom on the choice of this coordinate system corresponds to a vector \mathbf{U} , defining what we called a class of coordinates $\mathcal{X}_{\mathbf{U}}$. This vector in general is not uniform, implying that its gradient is not zero. The symmetric part of the latter corresponds to the time variation of components of the metric [equation (B.24)], the antisymmetric part, if chosen to be a constant of space, corresponds to a global rotation of the coordinates $\mathcal{X}_{\mathbf{U}}$ with respect to a Galilean frame.

The freedom on \mathbf{U} and the role it plays in the Newton equations (B.12)-(B.15) makes it very similar to the shift vector in the 3+1-formalism of general relativity. This allowed us to write the Newton equations as living in a 4-dimensional manifold \mathcal{M}^N . This was done using a push-forward on \mathcal{M}^N of the classical Newton equations (1.89)-(1.92) (see section C.1.1). The way the push-forward is done was inspired by the 3+1-Einstein equations. It was however not unique, which therefore

implies that some freedom remains on the properties of \mathcal{M}^N . Regarding the signature freedom, we chose this manifold to be Lorentzian and argued that this was not in contradiction with Newton's theory (section C.1.3). The remaining freedom (present as a lapse and a shift freedom) was chosen so that the foliation in which the equations are written corresponds to the rest frames of the Newtonian 4-velocity ${}^N\mathbf{u}$ we defined in equation (C.8). This led to the *1+3-Newton equations* (C.11)-(C.15). This set of equations is equivalent to the classical Newton equations, i.e. both can be derived from the other. This implies that the solutions described by the 1+3-Newton equations for the spatial fluid velocity \mathbf{v} are the same as the solutions of the classical Newton equations.

We then showed in section C.2.2 that these equations, in the case of irrotational flows, can be recovered from general relativity in a limit $|\mathbf{v}| \ll c$. This limit was performed with respect to a non-accelerating observer, the fluid itself. The Newtonian gravitational field \mathbf{g} then does not correspond to the 4-acceleration of a relativistic observer. Instead, it is defined as the acceleration, with respect to the fluid, of the spatial velocity \mathbf{v} [see equation (C.17)]. The limit also showed what happens to the 3+1-Ricci equation of general relativity. This equation, not needed for a vector theory like Newton, is shown to be a relation for the second order rest frames curvature in the limit we introduced.

A first consequence of this limit is that the classical interpretation of the component g_{00} of the spacetime metric as the gravitational potential still holds for coordinates adapted to a non-accelerating observer (section C.3.1). This is however true only for stationary and irrotational fluids.

Another consequence of the limit is to define a dictionary (for irrotational flows) between the Newtonian fluid variables and general relativity (section C.2.4). The spacetime manifold, denoted $\mathcal{M}^{(\text{dic})}$, given by this dictionary as function of the Newtonian variables is solution of the Einstein equations at leading order in $|\mathbf{v}|/c$. In general $\mathcal{M}^{(\text{dic})} \neq \mathcal{M}^N$, implying that \mathcal{M}^N is not solution of the Einstein equations at leading order. The difference between these two manifolds, $\mathcal{M}^{(\text{dic})}$ defined with the dictionary, and \mathcal{M}^N defined with the 1+3-Newton equations, resides in the curvature of the fluid rest frames. It is exactly zero for \mathcal{M}^N , which is not the case for $\mathcal{M}^{(\text{dic})}$ where the curvature is of second order in $|\mathbf{v}|/c$.

The dictionary was then tested for spherically symmetric vacuum solutions of Newton's and Einstein's theories of gravitation. For Newton this corresponded to a radially free-falling test fluid, and for general relativity to an observer associated with the generalised Gullstrand-Painlevé coordinates of the Schwarzschild spacetime. We showed that the dictionary allows us to recover the relativistic solution in the Newtonian limit. But in the specific case of a parabolic free-falling Newtonian fluid, the translation to general relativity is exact. This means that the Schwarzschild spacetime manifold is an exact solution, in terms of the manifold \mathcal{M}^N , of the 1+3-Newton equations. This supports our formulation.

The 1+3 formulation of Newton's equations might be seen as a new approach to

evaluate the link between Newton's theory and general relativity. What is essentially new compared to other approaches (e.g. [Green & Wald, 2012](#)) is that the comparison is done in the rest frames of the fluid, thus a non-accelerating observer. Furthermore we were able to construct a Lorentzian manifold on which the Newton equations are defined, contrasting with the Galilei manifold of the Newton-Cartan theory.

When developing this formulation we had in mind a future use for the study of the backreaction problem in cosmology and the effect of global topology. We think that the formulation might enable us to identify what is missing in Newton's theory for this study (due to the Buchert-Ehlers theorem, the backreaction is exactly zero for compact spaces in this theory). The final objective is then to use the 1+3-Newton equations and the scalar-vector-tensor decomposition of the expansion tensor in general relativity to define relatively simple models aimed at probing the backreaction and the effect of global topology. We give an example of what could be such a model in section [C.3.4.3](#).

Two things remain to be done before reaching this objective. The first one is to further test the dictionary for non-vacuum, non-stationary and non-isolated solutions. This can be done by comparing spherically symmetric solutions of Newton's equations with the Lemaitre-Tolman-Bondi class of solutions in general relativity. The second one is to upgrade the dictionary for vortical flows. We discussed this possibility in section [C.3.2](#).

C.5 Details on the approximations for the Newton-GR dictionary

We detail in this section arguments for the approximations made in section [C.2.3.1](#) regarding the decomposition of the expansion tensor and the covariant spatial derivative.

C.5.1 Decomposition theorem?

Equation [\(C.19\)](#) is a decomposition only if each term in the RHS of this equation can be uniquely defined from Θ_{ab} . [Straumann \(2008\)](#) showed that the decomposition [\(C.19\)](#) for rank-2 tensors is always possible in compact⁸ constant curvature spaces (constant scalar curvature and zero trace-less curvature). To our knowledge, no similar theorem exists for any curvature, which implies that the decomposition might be ill-defined in a general space. We however expect it to be reasonably valid for generally curved spaces in the context of cosmology. However the case of vortical flows remains problematic as no hypersurface orthogonal to the fluid 4-velocity can be defined.

⁸For non-compact spaces, fall-off conditions have to be used.

C.5.2 Approximation on the decomposition

The only approximation which did not rely on $\beta_v \ll 1$ regards equation (C.20) where we neglected the space expansion and the gravitational wave term. We will see in this section at which conditions it is consistent with a leading order approximation in β_v .

C.5.2.1 Neglecting the space expansion

Let us consider the momentum constraint (1.41) with the decomposition (C.19). It becomes

$$D_a \chi = v^c R_{ac}, \quad (\text{C.46})$$

using $D_{[a} v_{b]} = 0$, and the traceless and divergence free properties of the gravitational wave term.

We introduce the typical length scale L_χ of the space expansion. The RHS of equation (C.46) is of order $\beta_v^3/L_{v,l}^2$. So unless $L_{v,l}/L_\chi \gg 1$, which we assume is not the case for a cosmological setup, χ/c is at least of order $\beta_v^2/L_{v,l}$. Then neglecting the space expansion is coherent with a leading order approximation in β_v .

Remark. Actually this only shows that the space expansion term is a spatial constant at leading order. To recover the 1+3-Newton equation we take this constant to zero. However, letting it unspecified might be more interesting as it allows for expansion in a compact space (see section C.3.4.1).

C.5.2.2 Neglecting the gravitational wave term

Let us consider that $\Xi = 0$ and $\chi \neq 0$, so that $\Theta = \chi \mathbf{h} + \mathbf{D}v$. Then in the adapted coordinates \mathcal{X}_{-v}^u

$${}^u \partial_{t|_{-v}} h_{ab} = 2\chi h_{ab}.$$

The solution to this differential equation can be written as

$$h_{ab} = \psi^2 \hat{h}_{ab}, \quad (\text{C.47})$$

where \hat{h}_{ab} is called the *background metric* with ${}^u \partial_{t|_{-v}} \hat{h}_{ab} = 0$ and $\chi = {}^u \partial_{t|_{-v}} \ln \psi$. Note that this solution, while derived from a specific coordinate system, is however covariantly defined.

We introduce the covariant derivative \hat{D} of \hat{h} . The conformal relation (C.47) implies (see chapter 7 of Gourgoulhon, 2012, for details on this calculation⁹)

$$\begin{aligned} R_{ab} &= \hat{R}_{ab} - \hat{D}_a \hat{D}_b \ln \psi - \hat{h}_{ab} \hat{D}_c \hat{D}^c \ln \psi \\ &\quad + \hat{D}_a \ln \psi \hat{D}_b \ln \psi - \hat{h}_{ab} \hat{D}_c \ln \psi \hat{D}^c \ln \psi, \end{aligned} \quad (\text{C.48})$$

⁹ In Gourgoulhon (2012), the conformal relation is $h_{ab} = \psi^4 \hat{h}_{ab}$ instead of our $h_{ab} = \psi^2 \hat{h}_{ab}$.

where $\hat{\mathbf{R}}$ is the Ricci tensor relative to the metric \hat{h} , with ${}^u\partial_{t|v}\hat{R}_{ab} = 0$.

In general there does not exist a scalar ψ and a time independent spatial metric \hat{h}_{ab} such that any Ricci tensor R_{ab} can be written as in equation (C.48). This equation is then a restriction for the form of R_{ab} due to the initial assumption $\Xi = 0$. Consequently we expect the assumption $\Theta_{ab} \simeq D_a v_b$ made in section C.2.3.1 to *not* be valid if, at leading order in β_v , the spatial Ricci tensor \mathbf{R} is not of the form of equation (C.48). This can be seen as a test for the dictionary.

C.5.3 Flat covariant derivative approximation

We assume in this section that the spatial metric can be written as in equation (C.47), thus dropping the gravitational wave term. The Ricci tensor have the form (C.48). This form suggests two typical curvature radius associated with R_{ab} : one for \hat{R}_{ab} and one for the spatial variation of $\ln \psi$. We suppose the conformal metric \hat{h}_{ab} to be flat, so that $\hat{\mathbf{R}} = 0$ and \mathbf{R} has only one typical radius as assumed in section C.2.3.1.¹⁰ The relation $R_{ab} \sim 1/L_R^2$ implies that $\mathbf{D} \ln \psi \sim 1/L_R$.

The covariant derivative \mathbf{D} can be written as function of ψ and the covariant derivative $\hat{\mathbf{D}}$ of \hat{h}_{ab} (see chapter 7 of Gourgoulhon, 2012, for details on this calculation⁹):

$$D_a v_b = \hat{D}_a v_b + h_{ab} v^c \hat{D}_c \ln \psi - 2 v_{(a} \hat{D}_{b)} \ln \psi. \quad (\text{C.49})$$

Using equation (C.22), the second and third terms of the RHS of the decomposition (C.49) are of the order β_v^2 . So to first and leading order in β_v , we have $\mathbf{D}v = \hat{\mathbf{D}}v$ which is what we assumed in the Newtonian limit of section C.2.3.2.

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¹⁰Relaxing this hypothesis might lead to non-flat Newtonian-like theories as discussed in section C.3.4.3.

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Appendix D

Proofs and formulas for the Galilean limit

D.1 Covariant components of the Lorentzian metric

The covariant components of the Lorentzian metric can be decomposed into Laurent series of λ as

$$\hat{g}_{\alpha\beta} = \lambda^{n_0} g_{\alpha\beta}^{n_0} + \lambda^{n_0+1} g_{\alpha\beta}^{n_0+1} + \mathcal{O}(\lambda^{n_0+2}),$$

with $n_0 \in \mathbb{Z}$. By definition $\hat{g}^{\alpha\mu} \hat{g}_{\mu\beta} = \delta_{\beta}^{\alpha}$ which implies that

$$\delta_{\beta}^{\alpha} = \lambda^{n_0} g_{\beta\mu}^{n_0} h^{\mu\alpha} + \lambda^{n_0+1} [g_{\beta\mu}^{n_0} g^{\alpha\mu} + g_{\beta\mu}^{n_0+1} h^{\alpha\mu}] + \mathcal{O}(\lambda^{n_0+2}). \quad (\text{D.1})$$

As δ_{β}^{α} is a zero order, we have $n_0 \leq 0$.

If $n_0 = 0$, then $g_{\beta\mu}^{n_0} h^{\mu\alpha} = \delta_{\beta}^{\alpha}$, which is not possible since $h^{\alpha\beta}$ is of rank 3. Then $n_0 < 0$, and $g_{\beta\mu}^{n_0} h^{\mu\alpha} = 0$. This implies that there exists a 1-form $\underline{\tau} \in \text{Ker}(h^{\alpha\beta})$ such that

$$g_{\alpha\beta}^{n_0} = A \tau_{\alpha} \tau_{\beta}, \quad (\text{D.2})$$

where $A \neq 0$ is an arbitrary scalar.

If $n_0 \leq -2$, equations (D.1) and (D.2) imply that $\hat{g}^{\mu\nu} \tau_{\mu} \tau_{\nu} = 0$. This is not compatible with the signature of \hat{g} , which requires $\hat{g}^{\mu\nu} \tau_{\mu} \tau_{\nu} < 0$ for any $\tau_{\alpha} \in \text{Ker}(h^{\alpha\beta})$.

Proof. The signature of $h^{\alpha\beta}$ is (0+++), and the one of $\hat{g}^{\alpha\beta}$ is (-++++). This implies that for sufficiently small positive λ s, $\hat{g}^{\mu\nu} \tau_{\mu} \tau_{\nu} < 0$ for all $\tau_{\alpha} \in \text{Ker}(h^{\alpha\beta})$, which traduces into $\hat{g}^{\mu\nu} \tau_{\mu} \tau_{\nu} < 0$. \square

Remark. It might also be possible to assume $\hat{g}^{\mu\nu} \tau_{\mu} \tau_{\nu} = 0$ and $\hat{g}^{\mu\nu} \tau_{\mu} \tau_{\nu} < 0$, and still be compatible with the signature of the Lorentzian metric. This was proposed by [Dautcourt](#)

(1997), who argued that this choice leads to ‘non-Newtonian shadow fields’ when the limit of the Einstein equation is taken. However the argument behind this statement was unclear. So this point is a weakness in the proof of equation (4.2) from the initial hypothesis (4.1).

Finally we necessarily have $n_0 = -1$, and $A\tau_\beta\tau_\beta\dot{g}^{\alpha\mu} + \dot{g}_{\beta\mu}h^{\alpha\mu} = \delta_\beta^\alpha$. This last equation multiplied by τ_α leads to $A\tau_\mu\tau_\nu\dot{g}^{\mu\nu} = 1$. As $\tau_\mu\tau_\nu\dot{g}^{\mu\nu} < 0$, then $A < 0$. Therefore we can rescale τ_α such that

$$\dot{g}_{\alpha\beta}^{-1} = -\tau_\alpha\tau_\beta, \quad (\text{D.3})$$

with $\tau_\mu\tau_\nu\dot{g}^{\mu\nu} = -1$. This concludes the proof of equation (4.2).

D.2 Formulas

The projector orthogonal to a g-timelike vector \dot{u} is defined as

$${}^u\dot{b}_{\alpha\beta}^\lambda := \dot{g}_{\alpha\beta} + \lambda\dot{u}_\alpha\dot{u}_\beta. \quad (\text{D.4})$$

Its limit is

$${}^u\dot{b}_{\alpha\beta}^\lambda = {}^B b_{\alpha\beta} + \tau_\alpha\tau_\beta\dot{u}^\mu\dot{u}^\nu{}^B b_{\mu\nu} - \tau_{(\alpha}\dot{u}^\mu{}^B b_{\beta)\mu} + \mathcal{O}(\lambda), \quad (\text{D.5})$$

$${}^u\dot{b}^\alpha_\beta = \delta^\alpha_\beta - \dot{u}^\alpha\tau_\beta + \lambda\left[\dot{u}^\alpha\dot{u}_\beta - \dot{u}^\alpha\tau_\beta\right] + \mathcal{O}(\lambda^2), \quad (\text{D.6})$$

$${}^u\dot{b}^{\alpha\beta} = h^{\alpha\beta} + \lambda\left[\dot{u}^\alpha\dot{u}^\beta + \dot{g}^{\alpha\beta}\right] + \mathcal{O}(\lambda^2). \quad (\text{D.7})$$

Equation (D.5) can also be written as

$${}^u\dot{b}_{\alpha\beta}^\lambda = {}^0\dot{b}_{\alpha\beta} + \mathcal{O}(\lambda), \quad (\text{D.8})$$

where ${}^0\dot{b}_{\alpha\beta}$ is the Galilean projector orthogonal to \dot{u}^α [defined by equation (2.4)].

References

Dautcourt G., 1997, *Post-Newtonian extension of the Newton-Cartan theory*, *Classical and Quantum Gravity*, **14**, A109-A117