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Regularization phenomena for stochastic (partial) differential equations via Itô- and pathwise stochastic calculi

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Laboratoire de Probabilités, Statistique et Modélisation

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présentée par

Florian BECHTOLD

Regularization phenomena for stochastic (partial) differential equations via Itô- and pathwise stochastic calculi

dirigée par **Lorenzo ZAMBOTTI**

Rapporteurs :

M. **Paul GASSIAT** Université Paris-Dauphine
M. **Benjamin GESS** Universität Bielefeld

Soutenue le 27 Septembre 2021 devant le jury composé de :

M.	**Paul GASSIAT**	Université Paris-Dauphine	Rapporteur
Mme.	**Martina HOFMANOVA**	Universität Bielefeld	Examinatrice
M.	**Cyril LABBÉ**	Université de Paris	Examinateur
M.	**Thierry LÉVY**	Sorbonne Université	Examinateur
M.	**Nicolas PERKOWSKI**	Freie Universität Berlin	Examinateur
M.	**Lorenzo ZAMBOTTI**	Sorbonne Université	Directeur

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Und in zittrigen Fingern die Bachelorarbeit,
Ein Fazit zu zieh'n, jetzt wäre Zeit

Das Lumpenpack, *Die Zukunft wird groß*

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Résumé

Dans cette thèse, nous étudions trois cas de phénomènes de régularisation pour des équations aux dérivées partielles stochastiques (EDPS). Dans un premier temps, nous nous intéressons à des EDPS semi-linéaires à diffusion non bornée. En établissant une généralisation de l'inégalité maximale pour des convolutions stochastiques grâce à l'effet régularisant du semi-groupe sous-jacent, nous prouvons l'existence de solutions fortes dans un régime sous-critique. Par ailleurs, en exploitant une approximation par des solutions sous-critiques, nous démontrons dans le régime critique l'existence de solutions martingales à l'aide de la méthode de compacité de Flandoli-Gątarek.

Ensuite, nous établissons une loi des grands nombres pour des systèmes de particules en interaction sans hypothèse d'indépendance ou de moment sur les conditions initiales. Dans ce but, nous déterminons une équation non fermée satisfaite au sens faible par la mesure empirique associée, qui ne diffère de l'EDP de McKean-Vlasov attendue à la limite que par un terme de bruit. Dans le traitement de ce dernier terme, nous utilisons de façon complémentaire des bornes trajectorielles issues des chemins rugueux et des arguments fondés sur le calcul d'Itô, qui permettent d'établir la loi des grands nombres désirée.

Enfin, nous décrivons des phénomènes de régularisation qui apparaissent en moyennant le long des trajectoires. En se fondant sur des estimations récentes de régularité spatio-temporelle pour les temps locaux du mouvement brownien fractionnaire en une dimension, nous étudions les équations de transport moyennées grâce à leurs caractéristiques régularisées associées. Un argument de point fixe sur les équations de transport nous permet ensuite de nous intéresser à une équation de type Burgers moyennée le long des chemins du mouvement brownien fractionnaire. A chaque étape, les arguments sont conditionnels à ce que le paramètre de Hurst satisfasse certaines conditions explicites.

Abstract

In this thesis, we study three instances of regularization phenomena for stochastic (partial) differential equations (SPDEs). We first study semilinear SPDEs with unbounded diffusion terms: By deriving a generalization to the maximal inequality for stochastic convolutions harnessing the regularizing effect of the appearing semigroup, we are able to establish existence of strong solutions in the subcritical regime. We moreover use the associated sequence of subcritical solutions to establish existence of a martingale solution in the critical case via the Flandoli-Gątarek compactness method.

Secondly, we establish a law of large numbers for interacting particle systems without imposing independence or finite moment assumptions on the initial conditions: Towards this end, we establish a non-closed equation satisfied by the associated empirical measure in a mild sense that differs from the expect limiting McKean-Vlasov PDE only by a certain noise term. In treating said noise term, we employ pathwise rough path bounds and arguments based on Itô-calculus in a complementary fashion that allow to establish the desired law of large numbers.

Finally we investigate regularization phenomena through averaging along curves. Based on recent space-time regularity estimates for local times of fractional Brownian motion in one dimension, we study averaged transport equations in passing by their associated regularized characteristics. By employing a fixed point argument on the level of transport equations, we are able to subsequently pass to a Burgers' type equation averaged along paths of fractional Brownian motion. The arguments at each step are conditional on the Hurst parameter satisfying explicitly established conditions.

Keywords Chapter 2: semilinear Stochastic partial differential equations, stochastic integration in Banach spaces, maximal inequality for stochastic convolutions, factorization method, stochastic compactness method.

Keywords Chapter 3: Interacting particle system, McKean-Vlasov equation, Semigroup approach, Rough paths, Self-normalized processes

Keywords Chapter 4: Averaging along irregular curves, non-linear Young integration, local times for fractional Brownian motion

Notation

In the following, we list some notation frequently employed throughout this manuscript.

H, U	Generic separable Hilbert spaces
X, Y	Generic separable Banach spaces
$L_2(U, H)$	The space of Hilbert Schmidt operators from U to H
$\gamma(U, X)$	The space of γ-radonifying operators from U to X
$X \hookrightarrow Y$	X embeds continuously into Y
$X \hookrightarrow\hookrightarrow Y$	X embeds compactly into Y
$\mathbb{T}^N$	The N-dimensional torus
$W^{\alpha,p}$	Sobolev–Slobodeckij space for $\alpha \in \mathbb{R}_+$ and $p \in [1, \infty]$
$H^{\alpha,p}$	Bessel potential space for $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$
$B^{\alpha}_{p,q}$	inhomogeneous Besov space for $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$
$u^n \rightharpoonup u$	weak convergence of $(u^n)_n$ to u in X
$u^n \overset{*}{\rightharpoonup} u$	weak-*-convergence of $(u^n)_n$ to u in X

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Chapter 1

Introduction

The present thesis studies three particular instances of regularization phenomena for stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs). In the following introduction, we discuss the different approaches exploited in this thesis to study such equations and what we understand by a regularization phenomenom in each of the instances.

According to Zambotti, a stochastic partial differential equation is "a PDE which contains some stochastic process (or field) and cannot be defined with standard analytic techniques" [Zam21, p.2]. As is classically known, already the study of stochastic differential equations driven by Brownian motion W in $\mathbb{R}^d$ of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

are confronted with this challenge. This is due to the fact that almost every sample path of Brownian motion does not enjoy sufficient regularity for the above problem to be analytically well posed. Typically, for stochastic partial differential equations with space-time noise, this problem becomes even more challenging as even less analytical regularity of the noise is available. However, going beyond a purely analytical framework, stochastic processes as Brownian motion or other forms of noises typically enjoy some probabilistic structure that can be harnessed to study otherwise ill-posed problems. For SDEs driven by semi-martingale noises for example, this is achieved by the classical Itô stochastic calculus [RY99], [Gal16] which also admits an infinite dimensional generalization [DPZ14] [LR15] (and therefore also an approach to study SPDEs). Such infinite dimensional generalizations typically come already at the price of some imposed "damping" or regularization[1] by either imposing sufficient spatial coloring on the noise or demanding integrands of stochastic integrals to be Hilbert-Schmidt- (and not only bounded-) operator valued[2]. In the case of semilinear problems and their mild formulation, further regularization due to the presence of a semigroup can be harnessed to establish higher spatial regularity of solutions and balance out other appearing unbounded operators that would otherwise break up the Hilbert-Schmidt property of the

[1] Refer however to recent approaches to singular SPDEs briefly discussed in the end of Section 1.2

[2] For a more precise formulation of this instance we refer to Section 1.1

diffusion term. This idea is further explained and contextualized in Sections 1.1 and 1.2 of this Introduction and developed in Chapter 2.

Another approach that has enjoyed considerable attention in studying S(P)DEs for a very general class of noises is given by the theory of rough paths (refer to [Lyo98], [Gub04], [FH14] and Section 1.3 for a discussion of some main ideas). A main difference between this approach and the Itô stochastic calculus is that the former achieves a very clear "factorization" into a probabilistic and an analytical step: In fixing the local behaviour of stochastic integrals by the prescription of certain iterated integrals in a first probabilistic step, a subsequent completely analytical theory for the study of associated differential equations is provided by the theory of rough paths. While this point of view comes with many advantages such as a pathwise construction of stochastic integrals and associated almost sure bounds as well as several stability results typically unavailable to Itô calculus, the fact that the second purely analytical step does not take into account any further probabilistic structure renders certain estimates inaccessible to this approach that are readily available via the classical Itô calculus. In Chapter 3, we show how both perspectives and their respective merits can be employed in synergy in order to study a law of large numbers for interacting particle systems under less restrictive assumptions on the initial condition than usually demanded.

Finally we study regularization by noise phenomena for ODEs, transport equations and Burgers' equation along paths of fractional Brownian motion following recent approaches to these problems by [CG16], [Cat16], [GG20a] and in particular [HP21]. Similar in spirit to the theory of rough paths, a main idea to the latter approach consists in a certain probabilistic-analytic "factorization": While in a first step one re-examines the local behavior of certain (Lebesgue) integrals via probabilistic tools (in particular relying on novel regularity results for the local time associated with fractional Brownian motion in [HP21]), a subsequent purely analytical theory for ODEs perturbed by additive fractional Brownian motion can be developed based essentially on the Sewing Lemma. In Chapter 4, we first extend this approach from [HP21] to non-autonomous ODEs. Subsequently, we exploit the so obtained regularization by noise phenomenon on the level of characteristics in order to study regularized transport equations with non-differentiable initial conditions. Finally, in performing a fixed point argument on the level of transport equations, we are able to study averaging effects along paths of fractional Brownian motion for Burgers' equation.

In the following sections, let us elaborate more closely the context of the results presented in later chapters.

1.1 Semilinear non-singular SPDEs

As is true for the theory of PDEs, SPDEs come with a zoo of different possible approaches respectively adapted to different kinds of equations. One commonly studied type of equations are semilinear parabolic SPDEs that are of the form

$$\partial_t u = \Delta u + f(u) + g(u)\xi \qquad u(0) = u_0 \tag{1.1.1}$$

where ξ is some random field or process. A standard approach to studying problems of the above form consists in the so called mild formulation [DPZ14]. This approach views (1.1.1) as a stochastic differential equation in an infinite dimensional Hilbert space[3] H, while extensively relying on semigroup theory: One calls an adapted process $u = (u_t)_t$ a mild solution to the problem (1.1.1) if

$$u_t = S_t u_0 + \int_0^t S_{t-s} f(u_s) ds + \int_0^t S_{t-s} g(u_s)\xi_s ds \tag{1.1.2}$$

holds as an equality in H, where $(S_t)_t$ denotes the semigroup generated by the Laplacian. The main challenge then becomes to make sense of the integral on the right hand side: Since from a purely analytic perspective, the random field ξ only enjoys some distributional regularity the integral is often ill defined due to the product term $g(u_s)\xi_s$.

In the particular case of ξ being the time derivative of a cylindrical Brownian motion W taking values in a Hilbert space U, an abstract analogue of the classical stochastic calculus in infinite dimensions is available requiring that g takes values in the space of Hilbert-Schmidt operators $L_2(U, H)$ [DPZ14], [LR15]. A key tool in establishing the fixed point argument for (1.1.2) is then the maximal inequality for stochastic convolutions

$$\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t S_{t-s} g(u_s) dW_s \right\|_H^2\right] \leqslant c\mathbb{E}\left[\int_0^T \|g(u_s)\|_{L_2(U,H)}^2 ds\right]. \tag{1.1.3}$$

From the above and the triangle inequality for Bochner integrals, it can be easily seen that if we can interpret the functions f, g as abstract Nemytskii operators $f : H \to H$ and $g : H \to L_2(U, H)$ that satisfy standard growth and Lipschitz assumptions, then the above problem admits a unique solution thanks to a Banach fixed point argument in an appropriate space.

Note that in the argument sketched above, one tacitly exploits that the semigroup is a family of uniformly bounded operators on $[0, T]$ in order to treat the drift term. However, in the case of an analytic semigroup generated by A, we can moreover harness the regularizing effect of $(S_t)_t$ on H expressed quantitatively by

$$\left\|A^\delta S_t\right\|_{L(H)} \leqslant ct^{-\delta} \tag{1.1.4}$$

for $\delta > 0$ [Paz83]. This allows to study problems with unbouded drift such as

$$du = \Delta u dt + (-\Delta)^\delta f(u) dt + g(u) dW_t, \qquad u(0) = u_0 \tag{1.1.5}$$

[3] A common example is $H = L^2(\mathbb{T}^d)$

for $\delta \in [0,1)$. Moreover, other types of unboundedness in the drift term such as divergences can be treated by formulating the problem as

$$du = \Delta u dt + (-\Delta)^{1/2}(-\Delta)^{-1/2}\mathrm{div}(F(u))dt + g(u)dW_t, \qquad u(0) = u_0, \tag{1.1.6}$$

as was done in [Hof13]. There Hofmanová also shows that in the case of finite dimensional noise (i.e. U finite dimensional) the obtained mild solutions can in fact be identified as a strong one: The solution $u = (u_t)_t$ satisfies

$$u_t = u_0 + \int_0^t \Delta u_s ds + \int_0^t \mathrm{div}F(u_s)ds + \int_0^t g(u_s)dW_s,$$

meaning in particular that u is two times continuously differentiable in space. This is accomplished by establishing uniform bounds on the sequences of Picard iterations $(u^n)_n$ in Sobolev spaces of sufficiently high enough order and exploiting the Sobolev embedding theorem. Towards this end, she crucially exploits the Nemytskii operator result

$$\|f(h)\|_{W^{m,p}} \leqslant c(1 + \|h\|_{W^{m,p}} + \|h\|^m_{W^{1,mp}}) \tag{1.1.7}$$

for $f \in C^m$ and $h \in W^{1,mp} \cap W^{m,p}$ suggesting that for $m > 1$ (as required in order to embed into C^2), one has to leave the setting of stochastic integration in Hilbert spaces and consider stochastic integration in 2-smooth Banach spaces [vNVW15].

1.2 Semilinear SPDEs with unbounded diffusion

Having in mind inequality (1.1.4) and the problem (1.1.5), it is natural to study the generalization

$$du = (\Delta - 1)udt + (-\Delta + 1)^{\delta_0} f(u)dt + (-\Delta + 1)^{\delta_1/2} g(u)dW_t, \qquad u(0) = u_0 \tag{1.2.1}$$

in its mild formulation

$$u_t = S_t u_0 + \int_0^t (-\Delta + 1)^{\delta_0} S_{t-s} f(u_s)ds + \int_0^t (-\Delta + 1)^{\delta_1/2} S_{t-s} g(u_s)dW_s \tag{1.2.2}$$

for $\delta_0, \delta_1 \in [0,1)$. This problem is studied in chapter 2 based on [Bec20]. By considering the linear case, it can be easily seen that indeed $\delta_1 = 1$ is critical by Itô's isometry and (1.1.4). The crucial step to study (1.2.2) then becomes a generalization to the maximal inequality (1.1.3) for stochastic convolutions. Once this have been established in a sufficiently general setting, similar arguments as in [Hof13] may be employed to establish existence strong solutions for finite dimensional driving noise W. Moreover, the critical case $\delta_1 \nearrow 1$ is studied using the Flandoli-Gątarek compactness method, whose main idea we sketch briefly.

For better readability, consider the problem

$$du^\delta = (\Delta - 1)u^\delta dt + \mu(-\Delta + 1)^{\delta/2} g(u^\delta)dW_t, \qquad u(0) = u_0 \tag{1.2.3}$$

for some $\mu > 0$. Having established strong solutions in the case of finite dimensional noise W, one can conclude by Itô's formula an a priori energy estimate for the solution u^δ in $L^2(\Omega \times [0,T], W^{1,2})$ uniformly in $\delta \in [0,1)$. However such an energy estimate is not sufficient in order to pass to the limit, as the embedding

$$L^2(\Omega \times [0,T], W^{1,2}) \hookrightarrow L^2(\Omega \times [0,T], L^2)$$

is only continuous and not compact, thus impeaching the implementation of a tightness argument for the laws of the familiy $(u^\delta)_{\delta\in[0,1)}$. To address this problem in a different context, Flandoli and Gątarek remarked in [FG95] that for u^δ being a strong solution satisfying

$$u_t^\delta = u_0 + \int_0^t \Delta u_s^\delta ds + \mu \int_0^t (-\Delta)^\delta g(u_s^\delta) dW_s$$

one can establish improved time regularity on a fractional Sobolev-Slobodeckij scale at the cost of some spatial regularity due to the regularizing effect of stochastic integration in time. This permits to obtain a second bound of u^δ in $L^2(\Omega, W^{1/2-\epsilon}([0,T], H^{-1}))$ for any $\epsilon > 0$ uniformly in $\delta \in [0,1)$. As the embedding

$$L^2([0,T], W^{1,2}) \cap W^{1/2-\epsilon}([0,T], H^{-1}) \hookrightarrow\hookrightarrow L^2(\Omega \times [0,T], L^2)$$

now is indeed compact, one can proceed with a classical tightness argument using the Skorokhod representation theorem to deduce the existence of a martingale solution to the critical equation

$$du^\delta = \Delta u^\delta dt + \mu(-\Delta)^{1/2} g(u^\delta) dW_t, \qquad u(0) = u_0. \tag{1.2.4}$$

We stress that the above considerations and the mild solution approach to SPDEs approach heavily rely on Nemytskii operator results that are available only in the finite dimensional noise setting. In particular, arguments of the above kind often break down in dealing with space-time white noise ξ: Even for $f = 0$ and $g(x) = \mathrm{Id}$, it fails for $H = L^2(\mathbb{T}^d)$ in any dimension d, since g does not take values in $L_2(H,H)$. To bypass this difficulty in dimension $d = 1$, one can again harnesses the regularizing effect of the semigroup, ensuring for example that the stochastic convolution

$$\int_0^t S_{t-s}\xi_s ds$$

does indeed take values in $L^2(\mathbb{T}^1)$ and thus allowing to study more general nonlinearities f, g. Already in dimension $d = 2$ though, this approach breaks down due to the singularity of the noise. While in the particular case of polynomial nonlinearities and additive noise, the Da Prato Debussche trick based on renormalization techniques can be employed [DPD03] in dimension $d = 2$, dimension $d = 3$ was only recently addressed in the break through theories of regularity structures [Hai14] and paracontrolled distributions [GIP15]. A common central element in all these theories remains the exploitation of the regularizing effect of convolving with the semigroup[4].

[4] We remark however that extensions to problems that don't fall into the class of semilinear SPDEs have been addressed with the latter two approaches, see [GH19], [FG19] for quasilinear SPDEs for example

1.3 Elements of the theory of rough paths and comparison to the theory of stochastic calculus

As already mentioned in the previous section, a major component in treating stochastic (partial) differential equations consists in the appearance of products that are a priori ill defined from a purely pathwise point of view. Indeed, already for standard d-dimensional Brownian motion W the expression

$$\int_0^t W_s(\omega) \otimes dW_s(\omega) \tag{1.3.1}$$

does not admit a canonical pathwise definition. As already mentioned in the previous section, this obstacle is typically circumnavigated by replacing the pathwise perspective by Itô-calculus based on martingale theory [RY99] (or refer to [DPZ14], [LR15] for the infinite dimensional setting). In particular, controlled ordinary differential equations of the form

$$dY_t = f(Y)dW_t \qquad Y_0 = y_0 \in \mathbb{R}^d \tag{1.3.2}$$

can be treated with this theory along with numerous extensions. Itô-calculus comes however with certain theoretical and practical drawbacks, a major one being from an analytical point of view that it is inherently an $L^2(\Omega)$ theory and therefore pathwise arguments are usually not available.

An alternative viewpoint on Itô and more general stochastic calculi was proposed by Lyons in his theory of rough paths [Lyo98] and further developed by Gubinelli in [Gub04], obtaining a clear separation between pathwise considerations and probabilistic arguments. As a guiding question, one can ask under which condition on a generic continuous paths $X : [0, T] \to \mathbb{R}^d$ the expression

$$\int_s^t f(X_u)dX_u \tag{1.3.3}$$

is well defined for any smooth $f : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$. By using approximating Riemann-Stiltjes sums

$$\sum_{[u,v]\in\mathcal{P}^n} f(X_u)(X_v - X_u) \tag{1.3.4}$$

for a sequence of partitions $(\mathcal{P}^n)_n$ of $[s, t]$, it was already observed in [You36] and [Kon37] that for α-Hölder continuous paths X and $\alpha > 1/2$, (1.3.3) can be defined canonically by considering the limit as $|\mathcal{P}^n| \to 0$ in the above Riemann-Stiltjes sums. A heuristic intuition why in this case the above Riemann-Stiltjes summands accurately mimic the small scale behavior of (1.3.3) can be given by considering an order zero Taylor expansion of the integrand: Since for $u \in [s, t]$, we have

$$f(X_u) = f(X_s) + O(|t - s|^\alpha)$$

one obtains heuristically

$$\int_s^t f(X_u)dX_u = f(X_s)X_{s,t} + O(|t-s|^{2\alpha}),$$

and therefore

$$\frac{1}{|t-s|}\left(\int_s^t f(X_u)dX_u - f(X_s)X_{s,t}\right) = O(|t-s|^{2\alpha-1}),$$

where we used the shorthand notation $X_{s,t} = X_t - X_s$. This suggests that indeed for $\alpha > 1/2$, the above Riemann-Stiltjes summands do accurately portray the small scale behavior of (1.3.3). In particular, since sample paths of Brownian motion are $\mathbb{P}$-almost surely α-Hölder continuous for $\alpha < 1/2$, the above reasoning fails and indeed the sequence of Riemann-Stiltjes sums defined in (1.3.4) doesn't converge almost surely for Brownian motion[5]. Notice however that provided one was given an expression as in (1.3.3) (thanks to Itô calculus for example in the case of sample paths of Brownian motion), then the above heuristics can also be applied to an order one Taylor expansion, i.e. for $u \in [s,t]$, one would have $f(X_u) = f(X_s) + D_x f(X_s)X_{s,u} + O(|t-s|^{2\alpha})$ and therefore

$$\frac{1}{|t-s|}\left(\int_s^t f(X_u)dX_u - f(X_s)X_{s,t} - D_x f(X_s)\mathbb{X}_{s,t}\right) = O(|t-s|^{3\alpha-1}),$$

where $\mathbb{X}_{s,t} = \int_s^t X_{s,u} \otimes dX_u$ is the iterated integral of the path X against itself. In the case of X being a realization of d-dimensional Brownian motion W and its Itô iterated integral $\mathbb{W}^{Ito}_{s,t}$ the above expansion suggest that the integral (1.3.3) understood in the Itô sense should locally behave like

$$f(W_s)W_{s,t} + D_x f(W_s)\mathbb{W}^{Ito}_{s,t} \tag{1.3.5}$$

and that therefore it should be possible to define (1.3.3) as the almost sure limit of the modified Riemann-Stiltjes sums

$$\sum_{[u,v]\in\mathcal{P}^n} f(W_u)W_{u,v} + D_x f(W_u)\mathbb{W}^{Ito}_{u,v}. \tag{1.3.6}$$

The fact that indeed the local approximations (1.3.5) can be "sown together" to Riemann sums (1.3.6) giving rise in the limit $|\mathcal{P}^n| \to 0$ to the integral (1.3.3) is ensured by Gubinelli's by now classical Sewing Lemma [Gub04].

By prescribing only the iterated integral (1.3.1) (by probabilistic tools for example), it is thus possible to obtain a pathwise theory capable of defining expressions of the form (1.3.3) in the Itô sense for Brownian motion. From this point of view, the construction of the Itô integral (1.3.3) is therefore factorized in two steps: Associating to the path W

[5]They do however converge in $L^2(\Omega)$ to the corresponding Itô integral.

the couple $\mathbf{W}^{Ito} = (W, \mathbb{W}^{Ito})$ also called Itô lift over W and then establishing the convergence of the Riemann sums (1.3.6) with purely analytical i.e. pathwise arguments. A particular advantage of this approach yielding the same object (1.3.3) consists in the *pathwise* a priori bound

$$\left| \left(\int_0^t f(W_u) dW_u \right)(\omega) \right| \leqslant C(\omega) \, \|f\|_{C_b^2} \, T^{1/2}, \tag{1.3.7}$$

available for any $t \in [0, T]$, $T > 1$ and almost every $\omega \in \Omega$, where $C(\omega)$ is a random constant admitting arbitrarily high moments. In particular, this a priori bound allows to interpret the mapping

$$K_t(\omega) : f \to \left(\int_0^t f(W_u) dW_u \right)(\omega)$$

as a bounded linear functional on C_b^2 and therefore $K = (K_t)_t$ as a stochastic process taking values in the dual $(C_b^2)^*$ such that

$$\sup_{t \in [0,T]} \|K_t(\omega)\|_{(C_b^2)^*} = \sup_{t \in [0,T]} \sup_{\|f\|_{C_b^2} \leqslant 1} \left| \left(\int_0^t f(W_u) dW_u \right)(\omega) \right| \leqslant C(\omega) T^{1/2}. \tag{1.3.8}$$

As $C(\omega)$ is also of finite second moment, this implies moreover the bound

$$\mathbb{E} \left[\sup_{t \in [0,T]} \sup_{\|f\|_{C_b^2} \leqslant 1} \left| \int_0^t f(W_u) dW_u \right|^2 \right] \leqslant CT. \tag{1.3.9}$$

Let us stress that the supremum over functions of norm one in C_b^2 is taken first in the above inequality, i.e. inside the expectation and before taking the supremum over time, making of (1.3.9) an inequality typically unavailable by standard stochastic calculus arguments à la Burkholder-Davis-Gundy. Even more so, bounds as in (1.3.8) are completely inaccessible by Itô calculus by the very fact that it is not a pathwise theory.

1.4 Application of rough path theory and stochastic calculus to laws of large number for interacting particle systems

This alternative rough path perspective on Itô calculus and the available bounds of the from of (1.3.8) that come with it are exploited in chapter 3 based on the work [BC20] to study a law of large numbers for interacting particle systems. For $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ bounded Lipschitz, let $X = (X^j)_{j \leqslant n}$ be the unique solution to the interacting particle system

$$\begin{aligned} dX_t^j &= \frac{1}{n} \sum_{i=1}^n \Gamma(X_t^i, X_t^j) dt + dB_t^j \\ X_0^j &= x_0^j. \end{aligned} \tag{1.4.1}$$

By considering the associated empirical measure

$$\nu_t^n = \frac{1}{n}\sum_{j=1}^n \delta_{X_t^j}$$

one can show classically that the sequence $(\nu^n)_n$ converges to ν, solution to the associated McKean-Vlasov PDE

$$\begin{aligned} \partial_t \nu &= \frac{1}{2}\Delta\nu + \operatorname{div}(\nu(\Gamma * \nu)) \\ \nu(0) &= \nu_0 \end{aligned} \tag{1.4.2}$$

provided ν_0^n converges to ν_0. This result is typically called a law of large numbers for the sequence of empirical measures $(\nu^n)_n$. Notice however that such results typically require either a bounded moment assumption [CG19], [Gä88] or independence on the initial conditions $(x_0^j)_{j \leqslant N}$ [Szn91] [Tan84]. This contrasts the deterministic setting, i.e. (1.4.1) without additive noise, where such assumptions aren't required to establish similar results [Dob79], [Neu84].

The main contribution established in chapter 3 consists in exploiting the above rough path perspective in combination with fine estimates for self-normalized processes in order to establish a law of large numbers for the interacting particle system (1.4.1) without imposing bounded moment or independence assumptions on the initial condition.

Towards this end, we pass by an equation satisfied by ν^n in the mild sense, namely we establish that for any testfunction h chosen in a space of sufficiently high regularity, we have

$$\langle \nu_t^n, h\rangle = \langle \nu_0^n, S_t h\rangle + \int_0^t \langle \nu_s^n, (\nabla S_{t-s} h)(\Gamma * \nu_s^n)\rangle \mathrm{d}s + w_t^n(h), \tag{1.4.3}$$

where

$$w_t^n(h) = \frac{1}{n}\sum_{j=1}^n \int_0^t [\nabla S_{t-s} h]\,(X_s^j)\cdot \mathrm{d}B_s^j. \tag{1.4.4}$$

Comparing this equation with the one satisfied by the solution ν to the McKean-Vlasov equation (1.4.2), that is

$$\langle \nu_t, h\rangle = \langle \nu_0, S_t h\rangle + \int_0^t \langle \nu_s, (\nabla S_{t-s} h)(\Gamma * \nu_s)\rangle \mathrm{d}s, \tag{1.4.5}$$

one observes that heuristically, a law of large numbers can be established by showing that the noise term w^n vanishes in the limit $n \to \infty$. Towards this end, we establish two distinct bounds on w^n, allowing to render this heuristic rigorous: In a first step, relying on rough path theory and estimates similar to (1.3.8), we establish the almost sure uniform bound

$$\sup_{\|h\|_{H^m}=1} |w_t^n(h)(\omega)| \leqslant C_T(\omega)$$

allowing to treat w^n as a process taking values in $H^{-m}(\mathbb{R}^d)$ for $m > 3 + d/2$. Notice that as we exploit pathwise arguments at this stage, the information of independence of the additive noises $(B^j)_{j \leqslant n}$ is lost and therefore sharper bounds in n are unavailable. In passing by a Gronwall type argument in (1.4.3), this bound permits to establish a uniform bound on the sequence of empirical measures in $L^\infty([0,T], H^{-m}(\mathbb{R}^d))$ almost surely and therefore to extract a weak-*-convergent subsequence. Provided with this convergent subsequence, we establish a second bound

$$\mathbb{E}\left[\sup_{t\in[0,T]} |w_t^n(h)|^2\right] \leqslant \frac{C}{n} \|h\|_{H^m}^2$$

via maximal inequalities for self-normalized processes. This second bound on w^n evaluated on a fixed test function $h \in H^m$ then allows to identify the limit to the convergent subsequence as the unique solution to (1.4.2). In particular, note that while we exploit a moment bound on the noise term w^n, the a priori bound on the sequence of empirical measures $(\nu^n)_n$ is established uniformly on a set of full probability thanks to the rough path perspective discussed above, which thus allows to avoid additional moment assumptions on the initial condition ν_0.

1.5 Local times and Nonlinear Young integration vs. Lebesgue integration

As discussed in the previous section, the theory of rough paths allows for an alternative point of view on Itô integration by first coming up with adequate local approximations via prescribed iterated intergrals and second sewing together such local approximations to modified Riemann sums whose convergence is assured by the Sewing Lemma. Note however that the scope of the Sewing Lemma is much broader and not restricted to a particular local approximation chosen - indeed, it formulates very general conditions under which given local approximations can be sown together to a global object. Another instance where this Lemma is fruitfully employed in passing by different local approximations is given by a particular approach to non-linear Young integration as propoed in [HP21], which we discuss in the present section.

As a motivating example, consider the following problem

$$x_t = x_0 + \int_0^t b(x_s)ds - w_t^H, \tag{1.5.1}$$

where $(w_t^H)_t$ is some H-Hölder continuous path for $H \in (0,1)$. By the substitution $y = x + w$, the above can be reformulated as

$$y_t = y_0 + \int_0^t b(y_s - w_s)ds. \tag{1.5.2}$$

Observe that for smooth functions $b : \mathbb{R}^d \to \mathbb{R}^d$, y will be continuously differentiable and therefore smoother than the perturbative path w^H. Heuristically, the oscillations

of w^H therefore dominate the oscillations of y, meaning that one might expect an averaging effect to be produced in the Lebesgue integral in (1.5.2) of b along w^H. In order to render this intuition rigorous, let us further assume that w admits a local time $L = (L_t(x))_{t\in[0,T],x\in\mathbb{R}^d}$, i.e. a measurable function such that the occupation times formula

$$\int_0^t f(w_s)ds = \int_{\mathbb{R}^d} L_t(x)f(x)dx \tag{1.5.3}$$

holds for any positive measurable function f. We stress that the assumption of the existence of a local time is far from trivial and has so far only been demonstrated for realisations of certain stochastic processes. In particular, the problem of finding a concrete deterministic path that admits a local time remains an open problem at the moment of writing this manuscript (see however recent progress made in [IMPdRR21] and [GG20a], [GG20b]).

Assuming the occupation times formula (1.5.3) holds, notice in particular that we have for any $x \in \mathbb{R}^d$

$$\int_0^t b(x - w_s)ds = \int_{\mathbb{R}^d} L_t(z)b(x - z)dz = b * L_t(x). \tag{1.5.4}$$

Note that since heuristically[6] the regularity of the convolution $b * L_t$ is given by the sum of the spatial regularities of b and L_t, the above convolution might still be a well defined function in space even though b only enjoys distributional regularity, provided the local time L is sufficiently regular.

This observation motivates a quantitative study of the space-time regularity of local times associated with a given path. Since already the existence of local times associated with a given deterministic path remains a challenging problem as mentioned above, this study of space time regularity of local times is mainly done for certain classes of stochastic processes, in particular locally non-deterministic processes. A particular case of a result by Harang and Perkowski establish for example the following regularity estimate in [HP21]:

Theorem 1.5.1 (Harang, Perkowski 21)**.** *Let $H \in [0, 1/d)$ and $(w_t^H)_{t\in[0,T]}$ be fractional Brownian motion of Hurst parameter H in d dimensions. Then almost every sample path admits a local time $L = (L_t(x))_{t\geqslant 0,x\in\mathbb{R}^d}$ and moreover for any $\lambda < \frac{1}{2H} - \frac{d}{2}$ and $\gamma \in [0, 1 - (\lambda + \frac{d}{2})H)$ we have*

$$\|L_t(\cdot) - L_s(\cdot)\|_{H^\lambda} \leqslant C_T|t - s|^\gamma.$$

for any $t, s \in [0, T]$.

Remark that in combination with Young's inequality in Besov spaces, this regularity estimate allows for a quantifiable regularization effect in (1.5.4): For example in

[6] A rigorous formulation to this statement is given by Young's inequality in Besov spaces, refer to Lemma 4.B.1

dimension $d = 1$ and for $b \in B^{-\beta}_{\infty,\infty}$, the convolution $b * L_t$ will be in $B^{\alpha}_{\infty,\infty}$ provided

$$\alpha < \frac{1}{2H} - \left(\beta + \frac{1}{2}\right).$$

Notice in particular that the convolution $b * L_t$ might therefore be a continuous function even if b only enjoys distributional regularity as long as H is chosen sufficiently small.

With this observation in mind, it is suggestive to approximate the Lebesgue integral in (1.5.2) locally by

$$\int_s^t b(y_u - w_u)du \simeq \int_s^t b(y_s - w_u)du = (b * L_{s,t})(y_s) =: A^y_{s,t}, \tag{1.5.5}$$

where we used the shorthand notation $L_{s,t}(x) := L_t(x) - L_s(x)$. As with the local approximation (1.3.5) in the context of rough path theory, one can now use the Sewing Lemma in order to verify under which conditions on the nonlinearity b and the local time L the above germ (1.5.5) can be sown together. Notice that in the case of averaging along fractional Brownian motion, the above result by Harang and Perkowski also provides us with regularity estimates in time for the the local time that are crucial in the application of said Sewing Lemma.

Once the conditions of the Sewing Lemma are met for the germ $A^y_{s,t}$, one can show in the case of smooth b that

$$\int_0^t b(y_s - w_s)ds := (\Xi A^y)_{s,t} = \lim_{|\mathcal{P}^n| \searrow 0} \sum_{[u,v] \in \mathcal{P}^n} A^y_{u,v} = \lim_{|\mathcal{P}^n| \searrow 0} \sum_{[u,v] \in \mathcal{P}^n} (b * L_{u,v})(y_u),$$

where Ξ denotes the Sewing operator[7] and $\mathcal{P} = (\mathcal{P}^n)_n$ any sequence of partitions of $[0, t]$ whose mesh size tends to zero. Comparing this definition of the above integral to its classical in the setting of Riemann integration where

$$\int_0^t b(y_s - w_s)ds := \lim_{|\mathcal{P}^n| \searrow 0} \sum_{[u,v] \in \mathcal{P}^n} B^y_{u,v}$$

for $B^y_{s,t} = b(y_s - w_s)(t - s)$, note that while the Riemann summands $B^y_{s,t}$ are linear in the integrator in the sense that for any $s \in [0, T]$ fixed the mapping $t \in [s, T] \to B^y_{s,t}$ is affine linear, the modified Riemann summands $A^y_{s,t}$ lack this property. For this reason the above approach to define Riemann integrals averaged along a path w can be seen as a particular instance of the theory of non-linear integration or more precisely non-linear Young integration, as no further correction terms need to be introduced in order to assure an accurate local description of the integral by Riemann summands (as was the case for stochastic integration in the rough path setting above).

[7] Refer to Lemma 4.A.1

As on the level of local approximations we can exploit the regularizing effect due to the local time as described above, it then becomes possible to study the generalization

$$y_t = x_0 + (\Xi A^y)_t$$

to the problem (1.5.2) even for non-linearities b that enjoy less regularity than typically required in the study of (1.5.2). Moreover, even in cases where b is sufficiently regular for (1.5.2) to be well posed this approach can be fruitful as it can be exploited to establish higher spatial regularity of the associated flow.

In chapter 4 of this thesis based on a work in progress with Nicolas Perkowski, we exploit the above approach due to [HP21] in order to study associated phenomena of averaging along fractional Brownian motion w^H for the transport equation

$$\begin{aligned} \partial_t u + \widetilde{b}\partial_x u &= 0 \\ u(0) &= u_0 \end{aligned} \tag{1.5.6}$$

and Burgers' equation

$$\begin{aligned} \partial_t u + \widetilde{u}\partial_x u &= 0 \\ u(0) &= u_0, \end{aligned} \tag{1.5.7}$$

where for a function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ we denoted $\widetilde{f}(t,x) = f(t, x - w_t^H)$. Recall that solutions to (1.5.6) are closely linked to the associated characteristic equation given by

$$X_t = x + \int \widetilde{b}(s, X_s)ds = x + \int b(s, X_s - w_s^H)ds. \tag{1.5.8}$$

By interpreting the above as the non-linear Young integral problem

$$X_t = x + (\Xi A^X)_t \tag{1.5.9}$$

where $A^X_{s,t} = b_s * L_{s,t}(X_s)$ as discussed above, we are able to leverage the regularization effect due to the local time of fractional Brownian motion in such a way as to obtain existence of weak solutions to (1.5.6) for $b \in C_t^{0,\gamma} C_x^b \cap C_t^b C_x^\gamma$ and L^1 initial data provided the Hurst parameter H is chosen sufficiently small. We moreover show that provided $b \in C_t^{0,\gamma} C_x^b \cap C_t^b C_x^\gamma \cap L_t^\infty L_x^1$, then so is the constructed solution. Furthermore, we provide general a priori bounds to solutions to (1.5.6) and discuss uniqueness in the case of differentiable initial conditions.

By exploiting slightly more regularisation, i.e. imposing H to be slightly smaller and assuming the initial condition to be sufficiently small, we are able to perform a fixed point argument on the level of the transport equation (1.5.6) in an appropriately chosen space, establishing existence of solutions to (1.5.7).

Chapter 2

Strong solutions to semilinear SPDEs with unbounded diffusion

We prove a modification to the classical maximal inequality for stochastic convolutions in 2-smooth Banach spaces using the factorization method. This permits to study semilinear stochastic partial differential equations with unbounded diffusion operators driven by cylindrical Brownian motion via the mild solution approach. In the case of finite dimensional driving noise, provided sufficient regularity on the coefficients, we establish existence and uniqueness of strong solutions. In the case of "critical unboundedness", we show how to use the stochastic compactness method to obtain a martingale solution.

This chapter is based on the work [Bec20].

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2.1 Introduction

We consider the problem

$$\begin{aligned} du_t &= (\Delta - 1)u_t dt + (-\Delta + 1)^{\delta_0} F(u_t)dt + (-\Delta + 1)^{\delta_1/2} B(u_t)dW_t \\ u(0) &= u_0 \end{aligned} \tag{2.1.1}$$

for $t \in [0, T]$ where $\delta_0, \delta_1 \in [0, 1)$, Δ is the Laplacian with periodic boundary conditions on the N-dimensional torus $\mathbb{T}^N$and W is cylindrical Brownian motion over a separable Hilbert space U.

A canonical setting for the investigation of well-posedness to the above problem would be assuming $F : L^2(\mathbb{T}^N) \to L^2(\mathbb{T}^N)$ and $B : L^2(\mathbb{T}^N) \to L_2(U, L^2(\mathbb{T}^N))$ to be Lipschitz. Yet, due to the unbounded fractional Laplacian in the diffusion term $(-\Delta+1)^{\delta_1/2}B(u_t)$, one loses the Hilbert-Schmidt property and is thus unable to define a stochastic integral with values in $L^2(\mathbb{T}^N)$, as typically required in the variational approach to stochastic partial differential equations [KR81], [LR15], [Gyö98].

In keeping the above assumptions, an alternative approach consists in the mild formulation to the problem given by

$$u_t = S_t u_0 + \int_0^t S_{t-s}(-\Delta + 1)^{\delta_0} F(u_s)ds + \int_0^t S_{t-s}(-\Delta + 1)^{\delta_1/2} B(u_s)dW_s,$$

where S denotes the semigroup generated by $\Delta - 1$. Exploiting this formulation in the particular case of a bounded diffusion, i.e. $\delta_1 = 0$, Hofmanová is able to apply a fixed point theorem in $L^2(\Omega \times [0, T], L^2(\mathbb{T}^N)))$ [Hof13]. To this end, she crucially exploits the fact that the fractional power of a generator A composed with its corresponding semigroup $(S_t)_{t \geqslant 0}$ yields a bounded operator, or more precisely

$$\left\|(-A)^\delta S_t\right\| \leqslant C_\delta t^{-\delta},$$

provided the semigroup is analytic [Paz83]. Applying this bound in combination with the triangle inequality on the drift term, she is able to close the argument in a classical way. In particular, this encompasses the use of the maximal inequality for stochastic convolutions, which in this context reads

$$\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t S_{t-s} B(u_s)dW_s \right\|^2_{L^2(\mathbb{T}^N)}\right] \leqslant C\mathbb{E}\left[\int_0^T \|B(u_s)\|^2_{L_2(U,L^2(\mathbb{T}^N))}\, ds\right].$$

Notice that in the present context of an unbounded diffusion coefficient, i.e. $\delta_1 \neq 0$, the above bound is not available since the diffusion term is no longer Hilbert-Schmidt, therefore preventing an immediate generalization of [Hof13].

In this chapter, we show how to generalise the above maximal inequality in a weaker

form to the present context of an unbounded diffusion coefficient, notably[1]

$$\mathbb{E}\left[\sup_{t\leqslant T}\left\|\int_0^t(-\Delta+1)^{\delta_1/2}S_{t-s}B(u_s)dW_s\right\|^p_{L^p(\mathbb{T}^N)}\right]$$
$$\leqslant CT^{p/2(1-\delta_1)}\mathbb{E}\left[\sup_{t\leqslant T}\|B(u_t)\|^p_{\gamma(U,L^p(\mathbb{T}^N))}\right]$$

for $p>2/(1-\delta_1)$. The reasoning we follow exploits a combination of the above trade-off between fractional powers of generators and their analytic semigroups as well as the factorization method [DPKZ88], [DPZ14]. In particular, the new maximal inequality derived permits to set up a fixed point argument in $L^q(\Omega, C([0,T], L^p(\mathbb{T}^N)))$ for $p\geqslant 2$ and q sufficiently large, which establishes existence and uniqueness of mild solutions to the equation in question.

In the particular case of finite dimensional noise, i.e. U being of finite dimension, we show that the corresponding sequence of Picard iterations is uniformly bounded in Sobolev spaces, provided sufficient regularity of F and B. By using the Sobolev embedding theorem, this allows to identify the constructed mild solutions as strong ones. In order to use the embedding theorem, we have be able to pass to Sobolev spaces of high enough order. This necessitates to leave the Hilbert space framework and consider the more general theory of stochastic integration in 2-smooth Banach spaces, which is why our maximal inequality is stated in this more general setting. For a concise introduction to this theory, we refer the reader to [Ond04] and [Brz95]. Let us mention that in the case of finite dimensional noise an alternative approach using rough path theory to define and control the stochastic convolution and subsequently solve the associated semilinear problem was recently presented in [GHN19] building upon earlier results in [GT10]. Note that the constraint $\delta_0,\delta_1\in[0,1)$ ensures that the problem (2.1.1) is subcritical in the terminology of [GHN19][2].

Finally, we discuss the limit case $\delta_1\nearrow 1$, to which the present maximal inequality can not be applied. We show that provided B is sufficiently small, the corresponding sequence of strong solutions can be used to establish existence of a martingale solution of the corresponding limiting equation via the stochastic compactness method due to [FG95].

2.2 Setting and main results

Throughout this chapter we fix a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},\mathbb{P})$ with the filtration satisfying the usual conditions. Let U be a separable Hilbert space and W a cylindrical Brownian motion on U. In case U is of finite dimension, this means that for any

[1]Notice that the above inequality is expressed in the more general context of stochastic integration in the Banach space $L^p(\mathbb{T}^N)$, necessitating to replace the usual Hilbert-Schmidt norm in $L_2(U,L^2(\mathbb{T}^N))$ by the corresponding norm in the space of γ-radonifying operators $\gamma(U,L^p(\mathbb{T}^N))$ on the right hand side of the inequality.

[2]Refer to the discussion following the "Metatheorem" therein.

orthonormal basis $(e_i)_{i \leqslant d}$ of U, W admits the expansion

$$W_t = \sum_{i=1}^{d} e_i \beta_t^i,$$

where $(\beta^i)_{i \leqslant d}$ are independent standard Brownian motions. Let X be a 2-smooth Banach space and denote by $\gamma(U, X)$ the space of γ-radonifying operators from U to X. Let $\mathbb{T}^N$ denote the N-dimensional torus and $L^p(\mathbb{T}^N)$, $W^{m,p}(\mathbb{T}^N)$, $H^{\alpha,p}(\mathbb{T}^{\mathbb{N}})$ the associated Lebesgue, respectively Sobolev, respectively Bessel potential spaces, which we recall all fall into the class of 2-smooth Banach spaces for $p \geqslant 2$ [Ond04]. We fix moreover a finite time horizon $T \in \mathbb{R}^+$.

For convenience we introduce for $q \geqslant 2$ and X a separable Banach space the space

$$Z_{q,X} := L^q(\Omega, C([0,T], X)),$$

which endowed with its naturally inherited norm

$$\|u\|_{Z_{q,X}}^q := \mathbb{E}\left[\sup_{t \leqslant T} \|u_t\|_X^q\right]$$

is itself a separable Banach space. Throughout the chapter C shall denote an unessential constant that may change from one line to the next. Dependencies of the constant C on parameters are indicated by corresponding subscripts.

Theorem 2.2.1 (A maximal inequality)**.** *Let $\delta \in [0,1)$ and $T > 0$. Let X be a 2-smooth Banach space, $A : D(A) \subset X \to X$ be generator of an analytic contraction semigroup of operators $(S_t)_{t \geqslant 0}$. Let W be a cylindrical Wiener process on a separable Hilbert space U. Suppose a measurable $B : X \to \gamma(U, X)$ satisfies for $q > \frac{2}{1-\delta}$*

$$\|B(u)\|_{\gamma(U,X)}^q \leqslant C(1 + \|u\|_X^q),$$

then for every progressively measurable $u \in Z_{q,X}$ the process

$$t \to \int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s$$

admits a $\mathbb{P}$-almost surely continuous modification and we have

$$\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s \right\|_X^q\right] \leqslant C T^{q/2(1-\delta)} \mathbb{E}\left[\sup_{t \leqslant T} \|B(u_t)\|_{\gamma(U,X)}^q\right].$$

Remark 2.2.2 (Comparison to classical maximal inequality)**.** *Recall that the classical maximal inequality for stochastic convolutions in 2-smooth Banach spaces reads for $q > 0$*

$$\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t S_{t-s} B(u_s) dW_s \right\|_X^q\right] \leqslant C \mathbb{E}\left[\left(\int_0^T \|B(u_s)\|_{\gamma(U,X)}^2 ds\right)^{q/2}\right],$$

(see [Brz97], [vNZ11]) which is considerably sharper than the above inequality for the case $\delta = 0$. Yet, we remark that for the fixed point argument in $Z_{q,X}$ below, the coarser inequality of Theorem 2.2.1 is sufficient. For sharper maximal inequalities in the infinite time horizon case, we refer to [VW11] and [vNVW12]. We also refer to the recent survey on maximal inequalities for stochastic convolutions in 2-smooth Banach spaces [vNV20].

Remark 2.2.3 (The stochastic integral is well defined)**.** *Note first that due to the assumption of A being the generator of an analytic semigroup on X, we have*

$$||(-A)^{\delta} S_t||_{L(X)} \leqslant C_\delta t^{-\delta}.$$

Moreover, by the the ideal property of γ-radonifying operators in the space of boudned operators and Itô's isomorphism in 2-smooth Banach spaces we have for any $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s\right\|_X^q\right] \\
&\leqslant C\mathbb{E}\left[\left(\int_0^t \left\|(-A)^{\delta/2} S_{t-s} B(u_s)\right\|_{\gamma(U,X)}^2 ds\right)^{q/2}\right] \\
&\leqslant C\mathbb{E}\left[\left(\int_0^t \left\|(-A)^{\delta/2} S_{t-s}\right\|_{L(X)}^2 \|B(u_s)\|_{\gamma(U,X)}^2 ds\right)^{q/2}\right] \\
&\leqslant C\mathbb{E}\left[\left(\int_0^t \frac{1}{(t-s)^\delta} \|B(u_s)\|_{\gamma(U,X)}^2 ds\right)^{q/2}\right] \\
&\leqslant C\mathbb{E}\left[\sup_{t \leqslant T} \|B(u_t)\|_{\gamma(U,X)}^q\right] \left(\int_0^t \frac{1}{s^\delta} ds\right)^{q/2} \\
&\leqslant Ct^{q/2(1-\delta)}\mathbb{E}\left[1 + \sup_{t \leqslant T} \|u_t\|_X^q\right] \\
&\leqslant CT^{q/2(1-\delta)}(1 + \|u\|_{Z_{q,X}}^q)
\end{aligned}$$

hence, the stochastic integral in question is well defined provided $u \in Z_{q,X}$.

Corollary 2.2.4 (Distributional regularity for $\delta \geqslant 1$)**.** *Consider the case $X = L^p(\mathbb{T}^N)$ for $p \geqslant 2$ and $A = \Delta - 1$. Note that due to*

$$\left(D((-\Delta+1)^{\alpha/2}), ||(-\Delta+1)^{\alpha/2} \cdot ||_{L^p(\mathbb{T}^N)}\right) \simeq \left(H^{\alpha,p}(\mathbb{T}^N), ||\cdot||_{H^{\alpha,p}(\mathbb{T}^N)}\right)$$

one obtains for $\alpha > (\delta-1)^+$ and $q > 2/(1-(\delta-\alpha))$, provided all other conditions of Theorem 2.2.1 are met,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \leqslant T}\left\|\int_0^t (-\Delta+1)^{\delta/2} S_{t-s} B(u_s) dW_s\right\|_{H^{-\alpha,p}(\mathbb{T}^N)}^q\right] \\
&\leqslant C\mathbb{E}\left[\sup_{t \leqslant T}\left\|\int_0^t (-\Delta+1)^{(\delta-\alpha)/2} S_{t-s} B(u_s) dW_s\right\|_{L^p(\mathbb{T}^N)}^q\right] \\
&\leqslant CT^{q/2(1-(\delta-\alpha))}\mathbb{E}\left[\sup_{t \leqslant T} \|B(u_t)\|_{\gamma(U,L^p(\mathbb{T}))}^q\right]
\end{aligned}$$

where we exploited that for $\alpha \geqslant 0$, one can pass the bounded operator $(-\Delta+1)^{-\alpha/2}$ into the stochastic integral. The above permits thus to conclude that for $\delta \geqslant 1$ the convolution process

$$t \to \int_0^t (-\Delta+1)^{\delta/2} S_{t-s} B(u_s) dW_s$$

takes values in $H^{-\alpha,p}(\mathbb{T}^N)$ for all $\alpha > \delta - 1$, $\mathbb{P}$-almost surely.

We now turn to the main theorem to be demonstrated in the following section.

Theorem 2.2.5 (Mild solutions in $Z_{q,W^{m,p}(\mathbb{T}^N)}$)**.** *Let $B_1, \dots, B_d \in C^m(\mathbb{T}^N \times \mathbb{R})$ and $F \in C^m(\mathbb{R})$ have bounded derivatives up to order m satisfying*

$$\sum_{i=1}^d |B_i(x,\xi)|^2 \leqslant C(1+|\xi|^2).$$

Then for $\delta_0, \delta_1 \in [0,1)$, $q > 2/(1-\delta_1)$, $p \geqslant 2$ and $u_0 \in L^q(\Omega, W^{m,p}(\mathbb{T}^N)) \cap L^{mq}(\Omega, W^{1,mp}(\mathbb{T}^N))$ such that u_0 is $\mathcal{F}_0$-measurable, the equation

$$\begin{aligned} du_t &= (\Delta-1)u_t dt + (-\Delta+1)^{\delta_0} F(u_t) dt + (-\Delta+1)^{\delta_1/2} \sum_{k=1}^d B_i(u_t) d\beta_t^i \\ u(0) &= u_0 \end{aligned} \tag{2.2.1}$$

admits a unique mild solution $u \in Z_{q,W^{m,p}(\mathbb{T}^N)} \cap Z_{mq,W^{1,mp}(\mathbb{T}^N)}$ satisfying

$$\begin{aligned} &||u||^q_{Z_{q,W^{m,p}(\mathbb{T}^N)}} + ||u||^{mq}_{Z_{mq,W^{1,mp}(\mathbb{T}^N)}} \\ &\leqslant C\left(1 + \mathbb{E}||u_0||^q_{W^{m,p}(\mathbb{T}^N)} + \mathbb{E}||u_0||^{mq}_{W^{1,mp}(\mathbb{T}^N)}\right). \end{aligned}$$

Remark 2.2.6 (Strong solutions)**.** *Note that by the Sobolev embedding theorem, the above Theorem 2.2.5 implies that the constructed mild solutions lie in $Z_{q,C^{m-1,\lambda}(\mathbb{T}^N)}$ for $\lambda \in (0, 1-N/p)$. In particular, for $m \geqslant 3$, this permits to identify the mild solution as a strong one.*

Remark 2.2.7 (Other types of unboundedness)**.** *The statement of Theorem 2.2.5 can be extended to equations of the form*

$$du_t = (\Delta-1)u_t dt + div(F(u_t))dt + (-\Delta+1)^{\delta_1/2} \sum_{i=1}^d B_i(u_t) d\beta_t^i$$

for $F \in C^m(\mathbb{R}, \mathbb{R}^N)$ by interpreting them as

$$\begin{aligned} du_t &= (\Delta-1)u_t dt + (-\Delta+1)^{1/2}\left((-\Delta+1)^{-1/2} div(F(u_t))\right) dt \\ &\quad + (-\Delta+1)^{\delta_1/2} \sum_{i=1}^d B_i(u_t) d\beta_t^i \end{aligned}$$

and verifying that the associated Nemytskii operator

$$\begin{aligned} f : X &\to \gamma(U, X) \\ u &\to (-\Delta + 1)^{-1/2} div(F(u)) \end{aligned}$$

satisfies the necessary conditions stated in Lemmas 2.3.6, 2.3.9 and 2.3.12. Heuristically, this is possible as the inverse root of the shifted Laplacian compensates for the unboundedness of the divergence. For the precise statement refer to [Hof13, Theorem 2.1].

2.3 Proof of the main results

2.3.1 Proof of Theorem 2.2.1

For the proof of the maximal inequality, we use the factorization method due to [DPKZ88]. Exploiting the identity

$$\int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha} ds = \frac{\pi}{\sin \pi\alpha}$$

which holds for any $\sigma \leqslant t$ in $[0, T]$ and $\alpha \in (0, 1)$, we have due to Fubini's theorem (which we may apply due to Remark 2.2.3)

$$\begin{aligned} &\int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s \\ =&\frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} (-A)^{\delta/2} S_{t-s} \underbrace{\left(\int_0^s (s-\sigma)^{-\alpha} S_{s-\sigma} B(u_\sigma) dW_\sigma \right)}_{=:(Yu)_s} ds. \end{aligned}$$

We proceed with the proof in two steps, made up of the following two Lemmata. Let us recall that throughout this subsection, X denotes a generic 2-smooth Banach space and $A : D(A) \subset X \to X$ is generator of an analytic semigroup of operators $(S_t)_{t \geqslant 0}$.

Lemma 2.3.1. *Let $\alpha \in (0, 1)$, $\delta \in [0, 1)$ and $p > 1$ such that*

$$\lambda := \frac{p}{p-1}(1 + \delta/2 - \alpha) < 1.$$

Then the family of operators $(G_t)_t$ defined via

$$\begin{aligned} G_t : L^p([0, T], X) &\to X \\ f &\to \textstyle\int_0^t (t-s)^{\alpha-1} (-A)^{\delta/2} S_{t-s} f(s) ds \end{aligned}$$

is uniformly continuous in the sense that

$$\sup_{t \leqslant T} ||G_t f||_X \leqslant C \left(\frac{1}{1-\lambda} \right)^{\frac{p-1}{p}} T^{\frac{p-1}{p}(1-\lambda)} ||f||_{L^p([0,T],X)}.$$

Moreover, for every fixed $f \in L^p([0, T], X)$ the mapping $t \to G_t f$ is continuous as a mapping from $[0, T]$ to X.

Proof. Note again that one has

$$||(-A)^{\delta/2}S_{t-s}||_{L(X)} \leqslant C(t-s)^{-\delta/2}$$

and therefore by the triangle inequality for Bochner integrals and Hölder's inequality

$$\begin{aligned}
||G_t f||_X &\leqslant C\int_0^t (t-s)^{\alpha-1-\delta/2}||f(s)||_X ds \\
&\leqslant C\left(\int_0^t (t-s)^{-\lambda}ds\right)^{\frac{p-1}{p}} ||f||_{L^p([0,T],X)} \\
&\leqslant C\left(\frac{1}{1-\lambda}\right)^{\frac{p-1}{p}} T^{(-\lambda+1)\frac{p-1}{p}}||f||_{L^p([0,T],X)}.
\end{aligned}$$

Towards continuity, suppose $f \in C([0,T],X)$, then

$$\begin{aligned}
\|G_t f - G_s f\|_X \leqslant& \left\|\int_0^s u^{\alpha-1}(-A)^{\delta/2}S_u\left(f(t-u)-f(s-u)\right)du\right\|_X \\
&+\left\|\int_s^t u^{1-\alpha}(-A)^{\delta/2}S_u f(t-u)du\right\|_X \\
&\leqslant C\left(\int_0^s u^{-\lambda}du\right)^{\frac{p-1}{p}}\left(\int_0^s \|f(t-u)-f(s-u)\|_X^p\,du\right)^{1/p} \\
&+C\left(\int_s^t u^{-\lambda}du\right)^{\frac{p-1}{p}}\|f\|_{L^p([0,T],X)}.
\end{aligned}$$

Due to the assumed continuity of f, the first expression in the above estimate vanishes as t goes to s, which together with the continuity of the Lebesgue integral in the second expression yields continuity of $t \to G_t f$ for $f \in C([0,T],X)$. Together with the already established uniform bound on the operator family, this permits to employ a density argument, establishing continuity for $f \in L^p([0,T],X)$. □

Lemma 2.3.2. *Suppose that $\alpha \in (0,1/2)$. Then the mapping*

$$\begin{aligned}
Y : L^p(\Omega, C([0,T],X)) &\to L^p(\Omega, L^p([0,T],X)) \\
u &\to Yu
\end{aligned}$$

defined via

$$(Yu)_s := \int_0^s (s-\sigma)^{-\alpha}S_{s-\sigma}B(u_\sigma)dW_\sigma$$

satisfies

$$||Yu||^p_{L^p(\Omega,L^p([0,T],X))} \leqslant C_{\alpha,p}T^{p/2(1-2\alpha)+1}\mathbb{E}\left[\sup_{s\leqslant T}||B(u_s)||^p_{\gamma(U,X)}\right]$$

where

$$C_{\alpha,p} = C\left(\frac{1}{1-2\alpha}\right)^{p/2}\frac{1}{p/2(1-2\alpha)+1}.$$

Proof. One has

$$
\begin{aligned}
&||Yu||^p_{L^p(\Omega,L^p([0,T],X))} \\
&= \mathbb{E}\left[\int_0^T \left\|\int_0^s (s-\sigma)^{-\alpha} S_{s-\sigma} B(u_\sigma) dW_\sigma\right\|_X^p ds\right] \\
&= \int_0^T \mathbb{E}\left[\left\|\int_0^s (s-\sigma)^{-\alpha} S_{s-\sigma} B(u_\sigma) dW_\sigma\right\|_X^p\right] ds \\
&\leqslant C\int_0^T \mathbb{E}\left[\left(\int_0^s (s-\sigma)^{-2\alpha} ||B(u_\sigma)||^2_{\gamma(U,X)} d\sigma\right)^{p/2}\right] ds \\
&\leqslant C\mathbb{E}\left[\sup_{\sigma\leqslant T} ||B(u_\sigma)||^p_{\gamma(U,X)}\right] \int_0^T (\int_0^s (s-\sigma)^{-2\alpha} d\sigma)^{p/2} ds \\
&\leqslant C_{\alpha,p} T^{p/2(1-2\alpha)+1} \mathbb{E}\left[\sup_{\sigma\leqslant T} ||B(u_\sigma)||^p_{\gamma(U,X)}\right]
\end{aligned}
$$

where we crucially exploited Itô's isomorphism for X a 2-smooth Banach space. This inequality is also what one obtains from the classical maximal inequality for stochastic convolutions in [vNZ11] by considering the trivial semigroup $S_t = \mathrm{Id}$. □

Remark 2.3.3. *We wish to next combine the two previous Lemmata 2.3.1 and 2.3.2. Note that to this end, we need to demand for $\delta \in [0,1)$ that*

$$\lambda := \frac{p}{p-1}(1+\delta/2-\alpha) < 1$$

and $\alpha \in (0,1/2)$. Remark that the least restrictive condition on λ is obtained by choosing α as large as possible, i.e. setting for some $\epsilon > 0$ small $\alpha := 1/2 - \epsilon/2$, we obtain

$$p > \frac{2}{1-\epsilon-\delta}$$

which, since $\epsilon > 0$ can be chosen arbitrarily small, resorts to demanding

$$p > \frac{2}{1-\delta}$$

as required in the statement of Theorem 2.2.1.

Hence, assuming $p > 2/(1-\delta)$, we may put together the two previous Lemmata 2.3.1 and 2.3.2 and we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\leqslant T}\left\|\int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s\right\|_X^p\right] \\
&\leqslant CT^{(-\lambda+1)(p-1)} \mathbb{E}\left[||Y||^p_{L^p([0,T],X)}\right] \\
&= CT^{p-1-p(1+\delta/2-\alpha)} ||Y||^p_{L^p(\Omega,L^p([0,T],X)} \\
&\leqslant C_{\alpha,p,\delta} T^{p/2(1-\delta)} \mathbb{E}\left[\sup_{t\leqslant T} ||B(u_t)||^p_{\gamma(U,X)}\right]
\end{aligned}
$$

where

$$C_{\alpha,p,\delta} = C\left(\frac{1}{1-2\alpha}\right)^{p/2} \frac{1}{p/2(1-2\alpha)+1}\left(\frac{1}{1-\lambda}\right)^{\frac{p-1}{p}}$$

completing the proof to Theorem 2.2.1.

Remark 2.3.4. *We stress the fact that this proof using the factorization method can not be used to recover the sharper classical maximal inequality in the case of a bounded diffusion term, i.e. $\delta = 0$. Indeed, the factorization method was originally used in [DPKZ88] to prove existence of a continuous version of the stochastic convolution (among other regularity results) and not to prove a maximal inequality. As will be seen in the proof of Theorem 2.2.5 though, this coarser (yet in our context more general) inequality is sufficient to implement a fixed point argument nonetheless.*

2.3.2 Proof of Theorem 2.2.5

Having established the maximal inequality of Theorem 2.2.1, we can now generalize the strategy employed by Hofmanová in [Hof13] in order to prove Theorem 2.2.5. This strategy consists in showing that first there exists a unique mild solution to (2.1.1) in $Z_{q,L^p(\mathbb{T}^N)}$ via a Banach fixed point argument. Considering the corresponding sequence of Picard iterations $(u^n)_n \subset Z_{q,L^p(\mathbb{T}^N)}$, one is able to show - thanks to the maximal inequality of Theorem 2.2.1 - that the sequence $(u^n)_n$ is also uniformly bounded in $Z_{q,W^{m,p}(\mathbb{T}^N)}$ (provided the initial condition lies in this space), allowing to conclude that the corresponding limit (in the topology of $Z_{q,L^p(\mathbb{T}^N)}$), actually already lies in $Z_{q,W^{m,p}(\mathbb{T}^N)}$. By the Sobolev embedding theorem, this permits to conclude that the unique mild solution is differentiable in space and hence a strong solution to (2.1.1).

For the sake of readability, we choose to split up the steps mentioned above into two parts: In a first part (Lemmata denoted as abstract statements), we consider the generic setting in which F and B are seen as operators with suitable properties. In a second part, we recall for the convenience of the reader results of Hofmanová in [Hof13] to justify why the given functions F and $(B_i)_{i \leqslant d}$ give rise to associated Nemytskii operators with such properties (Lemmata denoted Nemytskii operator type results).

Mild solutions in $Z_{q,L^p(\mathbb{T}^N)}$

Lemma 2.3.5 (Abstract statement)**.** *For $\delta_0, \delta_1 \in [0,1)$, let $q > \frac{2}{1-\delta_1}$. Let X be a 2-smooth Banach space, let W be a cylindrical Brownian motion on a separable Hilbert space U. Suppose that $B : X \to \gamma(U,X)$ and $F : X \to X$ are Lipschitz continuous. Let $A : D(A) \subset X \to X$ be generator of an analytic contraction semigroup $(S_t)_{t \leqslant T} \subset L(X)$. Then for any $T < \infty$ and any $\mathcal{F}_0$-measurable $u_0 \in L^q(\Omega, X)$, the stochastic partial differential equation*

$$\begin{aligned} du_t &= Au_t dt + (-A)^{\delta_0} F(u)dt + (-A)^{\delta_1/2} B(u_t)dW_t \\ u(0) &= u_0 \end{aligned} \tag{2.3.1}$$

admits a unique mild solution, meaning there exists a unique progressively measurable process $u \in Z_{q,X}$ such that

$$u_t = S_t u_0 + \int_0^t (-A)^{\delta_0} S_{t-s} F(u_s) ds + \int_0^t (-A)^{\delta_1/2} S_{t-s} B(u_s) dW_s.$$

Moreover u satisfies

$$||u||^q_{Z_{q,X}} \leqslant \mathbb{E}||u_0||^q_X + C_T.$$

Proof. One uses the classical Banach fixed point theorem, i.e. a contraction argument in $Z_{q,X}$ for a sufficiently small time horizon. Consider the operator $\mathcal{K}: \ Z_{q,X} \to Z_{q,X}$, defined by

$$(\mathcal{K}u)_t := S_t u_0 + \int_0^t (-A)^{\delta_0} S_{t-s} F(u_s) ds + \int_0^t (-A)^{\delta_1/2} S_{t-s} B(u_s) dW_s.$$

Notice that the assumed Lipschitz continuity of B and F imply growth conditions of the form

$$\|B(u)\|^q_{\gamma(U,X)} \leqslant C(1 + \|u\|^q_X),$$

and

$$\|F(u)\|^q_X \leqslant C(1 + \|u\|^q_X).$$

Concerning the term associated with the drift, one has therefore

$$\begin{aligned}
\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t (-A)^{\delta_0} S_{t-s} F(u_s) ds \right\|^q\right] &\leqslant C\mathbb{E}\left[\left(\int_0^t \frac{1}{(t-s)^{\delta_0}} ||F(u_s)||_X ds\right)^q\right] \\
&\leqslant CT^{q(1-\delta_0)}\mathbb{E}\left[\sup_{t \leqslant T} ||F(u_t)||^q_X\right] \\
&\leqslant CT^{q(1-\delta_0)}(1 + ||u||^q_{Z_{q,X}}),
\end{aligned}$$

as well as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t (-A)^{\delta_0} S_{t-s} (F(u_s) - F(v_s)) ds \right\|^q_X\right] \\
&\leqslant CT^{q(1-\delta_0)}\mathbb{E}\left[\sup_{t \leqslant T} ||F(u_t) - F(v_t)||^q_X\right] \\
&\leqslant CT^{q(1-\delta_0)}||u - v||^q_{Z_{q,X}}.
\end{aligned}$$

Concerning the term associated with the diffusion, note that because of the maximal inequality of Theorem 2.2.1 one has

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t \leqslant T} \left\| \int_0^t (-A)^{\delta/2} S_{t-s} B(u_s) dW_s \right\|^q_X\right] \\
&\leqslant CT^{q/2(1-\delta)}\mathbb{E}\left[\sup_{t \leqslant T} \|B(u_t)\|^q_X\right] \\
&\leqslant CT^{q/2(1-\delta)}(1 + ||u||^q_{Z_{q,X}}),
\end{aligned}$$

as well as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leqslant T}||\int_0^t(-\Delta)^{\delta/2}S_{t-s}(B(u_s)-B(v_s))dW_s||_X^q\right]\\
\leqslant& CT^{q/2(1-\delta)}\mathbb{E}\left[\sup_{t\leqslant t}||B(u_t)-B(v_t)||^q_{\gamma(U,X)}\right]\\
\leqslant& CT^{q/2(1-\delta)}\mathbb{E}\left[\sup_{t\leqslant T}||u_t-v_t||_X^q\right]\\
=&CT^{q/2(1-\delta)}||u-v||^q_{Z_{q,X}},
\end{aligned}$$

due to the assumed Lipschitz continuity of B. Overall, one concludes that

$$||\mathcal{K}(u)||^q_{Z_{q,X}}\leqslant C\left(||u_0||^q_{Z_q}+(T^{q(1-\delta_0)}+T^{q/2(1-\delta_1)})(1+||u||^q_{Z_{q,X}})\right) \tag{2.3.2}$$

meaning that $\mathcal{K}$ maps $Z_{q,X}$ into itself. Moreover, we have

$$||\mathcal{K}(u)-\mathcal{K}(v)||^q_{Z_{q,X}}\leqslant C\left(T^{q(1-\delta_0)}+T^{q/2(1-\delta_1)}\right)||u-v||^q_{Z_{q,X}}.$$

By choosing T sufficiently small, $\mathcal{K}$ is a contraction on $Z_{q,X}$ and hence admits a unique fixed point that by definition coincides with a mild solution.

Continuity in time is a consequence of the existence of a continuous modification of the stochastic convolution in Theorem 2.2.1 as well as the continuity of the Bochner integral and of the semigroup. This permits to compute the unique solution to the equation in question on $[T,2T]$ with initial condition u_T calculated previously, etc., thus recovering existence and uniqueness of mild solutions on arbitrary finite time horizons.

Finally, the bound on solutions is derived from weak-*-lower semicontinuity of the norm $||\cdot||_{Z_{q,X}}$, the strong convergence of Picard iterations and the above estimate (2.3.2). □

Lemma 2.3.6 (A Nemytskii operator type result for $L^p(\mathbb{T}^N)$, [Hof13, Proposition 4.1])**.** *Let U be a d-dimensional Hilbert space with orthonormal basis $(e_i)_{i\leqslant d}$. Let $B_1,\dots,B_d\in C^1(\mathbb{T}^N\times\mathbb{R};\mathbb{R})$ have bounded derivative and satisfy the growth condition*

$$\sum_{i=1}^d|B_i(x,\xi)|^2\leqslant C(1+|\xi|^2).$$

Then for $p\geqslant 2$ the associated Nemytskii operator

$$\begin{aligned}
B:L^p(\mathbb{T}^N)&\to\gamma(U,L^p(\mathbb{T}^N))\\
z&\to\left(u\to\sum_{i=1}^dB_i(\cdot,z(\cdot))\langle u,e_i\rangle\right)
\end{aligned}$$

is well defined and Lipschitz continuous. Suppose $F\in C^1(\mathbb{R})$ is of bounded derivative. Then the associated Nemytskii operator

$$\begin{aligned}
F:L^p(\mathbb{T}^N)&\to L^p(\mathbb{T}^N)\\
u&\to F(u)
\end{aligned}$$

is well defined and Lipschitz continuous.

Corollary 2.3.7. *Suppose the conditions of Theorem 2.2.5 are satisfied and $p \geqslant 2$. Then there exists a unique mild solution $u \in Z_{q,L^p(\mathbb{T}^N)}$ to the problem* (2.2.1).

Mild solutions in $Z_{q,W^{1,p}(\mathbb{T}^N)}$

Lemma 2.3.8 (Abstract statement)**.** *Suppose all conditions of Lemma 2.3.5 are met. Let $\widetilde{X}$ be another 2-smooth Banach space, continuously embedded into X, the embedding operator being the identity and suppose $u_0 \in L^q(\Omega, \widetilde{X})$. Suppose that B seen as an operator $B : \widetilde{X} \to \gamma(U, \widetilde{X})$ is well defined and satisfies the growth condition*

$$||B(u)||^q_{\gamma(U,\widetilde{X})} \leqslant C(1 + ||u||^q_{\widetilde{X}}).$$

Suppose that F seen as an operator $F : \widetilde{X} \to \widetilde{X}$ is well defined satisfying the growth condition

$$||F(u)||^q_{\widetilde{X}} \leqslant C(1 + ||u||^q_{\widetilde{X}}).$$

Suppose moreover that $A|_{\widetilde{X}}$ generates a strongly continuous contraction semigroup $(\widetilde{S}_t)_t \subset L(\widetilde{X})$ such that $S_t|_{\widetilde{X}} = \widetilde{S}_t$. Then the unique mild solution $u \in Z_{q,X}$ to (2.3.1) *is also the unique mild solution $u \in Z_{q,\widetilde{X}}$ to* (2.3.1) *satisfying*

$$||u||^q_{Z_{q,\widetilde{X}}} \leqslant C(1 + \mathbb{E}||u_0||^q_{\widetilde{X}}).$$

Proof. Since the conditions of the previous theorem are met, there exists a unique $u \in Z_{q,X}$ which is the strong limit of Picard iterations, given via the recursive formula $u^0 = u_0$ and

$$u^n_t = \underbrace{S_t u_0 + \int_0^t (-A)^{\delta_0} S_{t-s} F(u^{n-1}_s) ds + \int_0^t (-A)^{-\delta_1/2} S_{t-s} B(u^{n-1}_s) dW_s}_{=:\mathcal{K}(u^{n-1})_t}$$

for $n \geqslant 1$. Note that due to the maximal inequality of Theorem 2.2.1 for the stochastic integral and the triangle inequality for the Bochner integral, one obtains an estimate similar to (2.3.2) but in the space $Z_{q,\widetilde{X}}$ namely

$$||\mathcal{K}(u^n)||^q_{Z_{q,\widetilde{X}}} \leqslant \Big(||u_0||^q_{Z_{q,\widetilde{X}}} + (T^{q(1-\delta_0)} + T^{q/2(1-\delta_1)})(1 + ||u^{n-1}_t||^q_{Z_{q\widetilde{X}}}) \Big)$$

We conclude recursively that

$$||u^n||^p_{Z_X} \leqslant C(1 + ||u_0||^q_{Z_{q,\widetilde{X}}}) \sum_{k=0}^{n-1} (CT^{q/2(1-\delta_1)})^k + (CT^{q(1-\delta_0)})^k.$$

For T sufficiently small the above geometric series converges and one obtains a uniform bound on the sequence of Picard iterations $(u^n)_n$ in the the space $Z_{q,\widetilde{X}}$. By Alaoglu's theorem, one can extract a weak-*-convergent subsequence with limit $v \in Z_{\widetilde{X}}$. Since $Z_{\widetilde{X}} \hookrightarrow Z_X$, one also has

$$u^n \overset{*}{\rightharpoonup} v$$

in Z_X and by uniqueness of limits, $v = u$, meaning the limit of Picard iterations already lies in $Z_{\widetilde{X}}$. By Lemma 2.A.2, one can identify the stochastic integral in X with the stochastic integral in $\widetilde{X}$, meaning in the notation introduced in Lemma 2.A.2 below

$$\begin{aligned} u_t &= S_t u_0 + \int_0^t (-A)^{\delta_0} S_{t-s} F(u_s) ds + \left(X \int_0^t \right) (-A)^{-\delta_1/2} S_{t-s} B(u_s) dW_s \\ &= \widetilde{S}_t u_0 + \int_0^t \widetilde{S}_{t-s} (-A)^{\delta_0} F(u_s) ds + \left(\widetilde{X} \int_0^t \right) (-A)^{-\delta_1/2} \widetilde{S}_{t-s} B(u_s) dW_s \end{aligned}$$

which is the definition of u being a mild solution in $Z_{q,\widetilde{X}}$. Uniqueness follows from the embedding $Z_{\widetilde{X}} \hookrightarrow Z_X$. Continuity follows from the continuous version of the modified stochastic convolution thanks to Theorem 2.2.1. The estimate of the solutions follows from the weak-*-lower semicontinuity of the norm and the uniform bound on Picard iterations established. □

Lemma 2.3.9. *(A Nemytskii operator type result for $W^{1,p}(\mathbb{T}^N)$, [Hof13, Proposition 4.2]) Let U be a d-dimensional Hilbert space with orthonormal basis $(e_i)_{i \leqslant d}$. Let $B_1, \dots, B_d \in C^1(\mathbb{T}^N \times \mathbb{R})$ be differentiable and of bounded derivative. Let $p \geqslant 2$. Then the associated Nemytskii operator*

$$\begin{aligned} B : W^{1,p}(\mathbb{T}^N) &\to \gamma(U, W^{1,p}(\mathbb{T}^N)) \\ z &\to \left(u \to \sum_{i=1}^d B_i(\cdot, z(\cdot)) \langle u, e_i \rangle \right) \end{aligned}$$

is well defined and satisfies the linear growth condition

$$||B(z)||_{\gamma(U, W^{1,p}(\mathbb{T}^N))} \leqslant C(1 + ||z||_{W^{1,p}(\mathbb{T}^N)}).$$

Suppose $F \in C^1(\mathbb{R})$ has bounded derivative, then the operator

$$\begin{aligned} W^{1,p}(\mathbb{T}^N) &\to W^{1,p}(\mathbb{T}^N) \\ u &\to F(u) \end{aligned}$$

is well defined and satisfies

$$||F(u)||_{W^{1,p}(\mathbb{T}^N)} \leqslant C(1 + ||u||_{W^{1,p}(\mathbb{T}^N)})$$

Corollary 2.3.10. *Suppose the conditions of Theorem 2.2.5 and $p \geqslant 2$. Then there exists a unique mild solution $u \in Z_{q,W^{1,p}(\mathbb{T}^N)}$ to the problem* (2.2.1).

Mild solutions in $Z_{q,W^{m,p}(\mathbb{T}^N)}$

Constructing mild solutions in $Z_{q,W^{1,p}(\mathbb{T}^N)}$ essentially relied on the fact that Nemytskii operators associated to the $C^1(\mathbb{T}^N \times \mathbb{R})$ functions $B_1, \dots, B_d$ and the $C^1(\mathbb{R})$ function F are of linear growth thanks to Lemma 2.A.1. This reasoning breaks down when considering Sobolev spaces of higher order (as required in order to be able to implement the Sobolev embedding theorem). Nontheless, one is able to recover (polynomial) growth conditions in this setting, which will turn out to be sufficient to again control Picard iterations in $Z_{q,W^{m,p}(\mathbb{T}^N)}$.

Lemma 2.3.11. *(Abstract statement) Suppose all conditions of Lemma 2.3.5 are satisfied. Let X_1, X_2 be 2-smooth Banach spaces such that $X_1 \hookrightarrow X$ the embedding operator being the identity and suppose $u_0 \in L^q(\Omega, X_1) \cap L^{mq}(\Omega, X_2)$ for some $m \geqslant 1$. Suppose that B seen as an operator $B : X_1 \cap X_2 \to \gamma(U, X_i)$ is well defined for both $i = 1, 2$ and satisfies the growth conditions*

$$||B(u)||^q_{\gamma(U,X_1)} \leqslant C(1 + ||u||^q_{X_1} + ||u||^{mq}_{X_2})$$

and

$$||B(u)||^q_{\gamma(U,X_2)} \leqslant C(1 + ||u||^q_{X_2})$$

for $u \in X_1 \cap X_2$. Suppose that F seen as an operator $F : X_1 \cap X_2 \to X_i$ is well defined for both $i = 1, 2$ and satisfies the growth conditions

$$||F(u)||^q_{X_1} \leqslant C(1 + ||u||^q_{X_1} + ||u||^{mq}_{X_2})$$

and

$$||F(u)||^q_{X_2} \leqslant C(1 + ||u||^q_{X_2})$$

for $u \in X_1 \cap X_2$. Suppose moreover that $A|_{X_i}$ generates a strongly continuous contraction semigroup $(S^i_t)_t \subset L(X_i)$ such that $S_t|_{X_i} = S^i_t$ for both $i = 1, 2$. Then the unique mild solution $u \in Z_{q,X}$ of Lemma 2.3.5 lies in $Z_{q,X_1} \cap Z_{mq,X_2}$, and is a continuous in time mild solution in Z_{q,X_1}.

Proof. Due to Lemma 2.3.5, there exists a unique continuous solution in $Z_{q,X}$. Thanks to the growth condition imposed, one can check that $\mathcal{K}$ maps the space $Z_{q,X_1} \cap Z_{mq,X_2}$ onto itself, meaning the sequence of Picard iterations lies in $Z_{q,X_2} \cap Z_{mq,X_2}$. Due to the growth condition imposed on the Nemytskii operators, we obtain again a uniform bound on this sequence of Picard iterations in both Z_{q,X_1} and Z_{mq,X_2} for T sufficiently small. This means there exist a weak* convergent subsequence of Picard iterations in Z_{q,X_1} and a weak* convergent subsequence in Z_{mq,X_2}. Due to the same reasoning as in Lemma 2.3.8, both weak* limits need to coincide with the fixed point $u \in Z_{q,X}$ obtained from the application of Lemma 2.3.5, i.e. the solution lies in $Z_{q,X_1} \cap Z_{mq,X_2}$. Exploiting Lemma 2.A.2 we deduce that u is a mild solution in Z_{q,X_1} thanks to the continuous embedding $X_1 \hookrightarrow X$. Continuity follows again from Theorem 2.2.1. □

Lemma 2.3.12. *(A Nemytskii operator type result for $W^{m,p}(\mathbb{T}^N)$, [Hof13, Proposition 4.3]) Let U be a d-dimensional Hilbert space with orthonormal basis $(e_i)_{i \leqslant d}$. Let $B_1, \dots, B_d \in C^m(\mathbb{T}^N \times \mathbb{R})$ with bounded m-th derivative. Then the associated Nemytskii operator*

$$\begin{aligned} B : W^{1,mp}(\mathbb{T}^N) \cap W^{m,p}(\mathbb{T}^N) &\to \gamma(U, W^{m,p}(\mathbb{T}^N)) \\ z &\to \left(u \to \sum_{i=1}^{d} B_i(\cdot, z(\cdot))\langle u, e_i \rangle \right) \end{aligned}$$

is well defined and satisfies the growth condition

$$||B(z)||_{\gamma(U,W^{m,p}(\mathbb{T}^N))} \leqslant C(1 + ||z||^m_{W^{1,mp}(\mathbb{T}^N)} + ||z||_{W^{m,p}(\mathbb{T}^N)})$$

for all $p \geqslant 2$. Suppose $F \in C^m(\mathbb{R})$ is of bounded m-th derivative. Then the associated Nemytskii operator

$$\begin{aligned} F : W^{1,mp}(\mathbb{T}^N) \cap W^{m,p}(\mathbb{T}^N) &\to W^{m,p}(\mathbb{T}^N) \\ u &\to F(u) \end{aligned}$$

is well defined and satisfies the growth condition

$$||F(u)||_{W^{m,p}(\mathbb{T}^N)} \leqslant C \left(1 + ||u||_{W^{m,p}(\mathbb{T}^N)} + ||u||^m_{W^{1,mp}(\mathbb{T}^N)}\right)$$

Remark 2.3.13. *The preceding Lemma 2.3.11 whose proof is essentially based on Lemma 2.A.1 conveys the reason why one has to leave the Hilbert space framework of stochastic integration and rather work in the setting of stochastic integration with values in 2-smooth Banach spaces: Since one needs to control norms in $W^{1,mp}(\mathbb{T}^N)$ and $W^{m,p}(\mathbb{T}^N)$ one leaves the Hilbert space setting $p = 2$ as soon as one intends to consider orders higher than $m = 1$. Note moreover that the preceding Lemma implies Lemma 2.3.9.*

Combing the results of this subsection, we conclude that under the conditions of Theorem 2.2.5 there exists a unique mild solution $u \in Z_{q,W^{m,p}(\mathbb{T}^N)} \cap Z_{mq,W^{1,mp}(\mathbb{T}^N)}$ to the problem (2.2.1). The bound stated in Theorem 2.2.5 follows again from weak-*-lower semicontinuity of the norms. In particular, by the Sobolev embedding theorem, this implies also that u is a strong solution for $m \geqslant 3$.

Remark 2.3.14. *Note that while the abstract existence and uniqueness statements of this section are formulated for general U-cylindrical Brownian motion, i.e. hold in the infinite dimensional setting, Lemmas 2.3.6, 2.3.9, 2.3.12 concerning Nemytskii operator results crucially rely on U being finite dimensional.*

2.4 The critical equation $\delta_1 = 1$

We wish to study the critical problem[3]

$$\begin{cases} du_t &= (\Delta - 1)u_t dt + \mu(-\Delta + 1)^{1/2} \sum_{i=1}^d B_i(u_t) d\beta_t^i \\ u(0) &= u_0 \end{cases} \tag{2.4.1}$$

which we will also occasionally write more compactly as

$$\begin{cases} du_t &= (\Delta - 1)u_t dt + \mu(-\Delta + 1)^{1/2} B(u_t) dW_t \\ u(0) &= u_0. \end{cases}$$

Towards this end, we pass by the corresponding sequence of subcritical problems in the following sense: For $\delta \in [0, 1)$, $q > 2/(1 - \delta)$ and $p \geqslant 2$, let u^δ be the unique mild solution to

$$\begin{cases} du_t^\delta &= (\Delta - 1)u_t^\delta dt + \mu(-\Delta + 1)^{\delta/2} \sum_{i=1}^d B_i(u_t^\delta) d\beta_t^i \\ u^\delta(0) &= u_0 \end{cases}$$

[3]For easier reading, we suppress the additional nonlinear drift.

in $Z_{q,W^{1,p}(\mathbb{T}^N)}$ where $\mu \in \mathbb{R}$ and $B_1, \dots, B_d \in C^1(\mathbb{T}^N)$ have bounded derivative satisfying the growth condition stated in Theorem 2.2.5. In particular $(u^\delta)_{\delta\in[0,1)} \subset L^2(\Omega \times [0,T], W^{1,2}(\mathbb{T}^N))$. We show that for μ sufficiently small, $(u^\delta)_{\delta\in[0,1)}$ is uniformly bounded in this space. Together with a second uniform bound in a space of higher time and lower space regularity, this allows us to apply the stochastic compactness method due to Flandoli and Gątarek (refer to [FG95] for the original article and the lecture notes [Hof16] for a pedagogical introduction). Upon the establishment of uniform bounds in the aforementioned spaces, one is able to establish tightness of the laws of $(u^\delta)_{\delta\in[0,1)}$, which in combination with Prokhorov's theorem and the Skorokhod representation theorem permits to conclude the existence of a martingale solution in the following sense.

Definition 2.4.1. *We say the problem* (2.4.1) *admits a martingale solution if there exists a filtered probability space* $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_t, \mathbb{P}')$*, a d-dimensional* $(\mathcal{F}'_t)_t$ *Brownian motion* W' *and a progressively measurable process* $u : \Omega' \times [0,T] \to L^2(\mathbb{T}^N)$ *such that for some* $\alpha > 0$ *we have* $\mathbb{P}'$*-almost surely*

$$u \in C([0,T], H^{-\alpha}(\mathbb{T}^N)) \cap L^2([0,T], W^{1,2}(\mathbb{T}^N))$$

as well as for any $\varphi \in H^\alpha(\mathbb{T}^N)$

$$\langle u_t, \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle(\Delta - 1)u_s, \varphi\rangle ds + \mu \int_0^t \langle(-\Delta + 1)^{1/2} B(u_s), \varphi\rangle dW'_s.$$

2.4.1 A first a priori bound

Lemma 2.4.2. *There exists* $\mu_0 \in \mathbb{R}$ *such that for all* $\mu^2 < \mu_0^2$*, the family* $(u^\delta)_{\delta\in[0,1)}$ *is uniformly bounded in* $L^2(\Omega \times [0,T], W^{1,2}(\mathbb{T}^N))$ *and in* $L^p(\Omega \times [0,T], L^2(\mathbb{T}^N))$ *for* $p \geqslant 2$*, meaning*

$$\mathbb{E}\left[\int_0^T \left\|u_s^\delta\right\|^2_{W^{1,2}(\mathbb{T}^N)} ds\right] < \infty$$

and

$$\mathbb{E}\left[\int_0^T \left\|u_s^\delta\right\|^p_{L^2(\mathbb{T}^N)} ds\right] < \infty$$

uniformly in $\delta \in [0,1)$*.*

Proof. By Theorem 2.2.5, solutions u^δ take values in $Z_{q,W^{1,2}(\mathbb{T}^N)}$ for $q > 2/(1-\delta)$. We apply Itô's formula in the Hilbert space $L^2(\mathbb{T}^N)$ to the functional $F(u) := \|u\|^p_{L^2(\mathbb{T}^n)}$ for $p \geqslant 2$. Note that

$$F'(u) = p\,\|u\|^{p-2}_{L^2(\mathbb{T}^N)}, \qquad F''(u) = p(p-1)\,\|u\|^{p-2}_{L^2(\mathbb{T}^N)}\,\mathrm{Id}.$$

We therefore obtain

$$\begin{aligned} ||u_t^\delta||^p_{L^2(\mathbb{T}^N)} &= ||u_0||^p_{L^2(\mathbb{T}^N)} - p\int_0^t \left\|u_s^\delta\right\|^{p-2}_{L^2(\mathbb{T}^N)} \int_{\mathbb{T}^N} (|u_s^\delta|^2 + |\nabla u_s^\delta|^2) dx ds + M_t \\ &\quad + \mu^2 \frac{p(p-1)}{2} \sum_{i=1}^d \int_0^t \left\|u_s^\delta\right\|^{p-2}_{L^2(\mathbb{T}^N)} \int_{\mathbb{T}^N} |(-\Delta+1)^{\delta/2} B_i(u_s^\delta)|^2 dx ds, \end{aligned} \tag{2.4.2}$$

where

$$M_t = \mu p \sum_{i=1}^{d} \int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \int_{\mathbb{T}^N} u_s^\delta \cdot (-\Delta+1)^{\delta/2} B_i(u_s^\delta) dx d\beta_s^i.$$

Note also that by

$$\begin{aligned}\left|\int_{\mathbb{T}^N} u_s^\delta \cdot (-\Delta+1)^{\delta/2} B_i(u_s^\delta) dx\right| &\leqslant \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)} \left\|B(u^\delta)\right\|_{H^1(\mathbb{T}^N)} \\ &\leqslant C \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)} (1 + \left\|u_s^\delta\right\|_{H^1(\mathbb{T}^N)})\end{aligned}$$

and the fact that $u^\delta \in L^q(\Omega \times [0,T], W^{1,2}(\mathbb{T}^N))$, we conclude that indeed M is a martingale. Hence, taking expectations in (2.4.2), we obtain

$$\begin{aligned}\mathbb{E}[||u_t^\delta||_{L^2(\mathbb{T}^N)}^p] &= \mathbb{E}[||u_0||_{L^2(\mathbb{T}^N)}^p] - p\mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \int_{\mathbb{T}^N} (|u_s^\delta|^2 + |\nabla u_s^\delta|^2) dx ds\right] \\ &\quad + \mu^2 \frac{p(p-1)}{2} \sum_{i=1}^{d} \mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \int_{\mathbb{T}^N} |(-\Delta+1)^{\delta/2} B_i(u_s^\delta)|^2 dx ds\right].\end{aligned}$$

Observe that by a similar reasoning as above, we have

$$\left|\int_{\mathbb{T}^N} |(-\Delta+1)^{\delta/2} B_i(u_s^\delta)|^2 dx\right| \leqslant C_i(1 + \left\|u_s^\delta\right\|_{H^1(\mathbb{T}^N)}^2)$$

and therefore

$$\begin{aligned}\mathbb{E}[||u_t^\delta||_{L^2(\mathbb{T}^N)}^p] &\leqslant \mathbb{E}[||u_0||_{L^2(\mathbb{T}^N)}^p] - p\mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \int_{\mathbb{T}^N} (|u_s^\delta|^2 + |\nabla u_s^\delta|^2) dx ds\right] \\ &\quad + C\mu^2 \frac{p(p-1)}{2} \mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \left(1 + \int_{\mathbb{T}^N} |(|u_s^\delta|^2 + |\nabla u_s^\delta|^2) dx\right) ds\right] \\ &\leqslant \mathbb{E}[||u_0||_{L^2(\mathbb{T}^N)}^p] - p\mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \int_{\mathbb{T}^N} (|u_s^\delta|^2 + |\nabla u_s^\delta|^2) dx ds\right] \\ &\quad + C\mu^2 \frac{p(p-1)}{2} \mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{L^2(\mathbb{T}^N)}^{p-2} \int_{\mathbb{T}^N} |(|u_s^\delta|^2 + |\nabla u_s^\delta|^2) dx ds\right]\end{aligned}$$

where we exploited the embedding $L^{p+\epsilon}(\Omega \times [0,T]) \hookrightarrow L^{p+\epsilon}(\Omega \times [0,T])$ for any $\epsilon > 0$. Hence, we may conclude that indeed, for μ sufficiently small, we have

$$\mathbb{E}[\int_0^T \left\|u_t^\delta\right\|_{L^2(\mathbb{T}^N)}^p dt] = \int_0^T \mathbb{E}[||u_t^\delta||_{L^2(\mathbb{T}^N)}^p] dt \leqslant T\mathbb{E}[||u_0||_{L^2(\mathbb{T}^N)}^p]$$

Moreover, in the case $p = 2$, we have

$$\begin{aligned}\mathbb{E}[||u_t^\delta||_{L^2(\mathbb{T}^N)}^2] &\leqslant \mathbb{E}[||u_0||_{L^2(\mathbb{T}^N)}^2] - 2\mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{W^{1,2}(\mathbb{T}^N)}^2 ds\right] \\ &\quad + C\mu^2 \mathbb{E}\left[\int_0^t \left\|u_s^\delta\right\|_{W^{1,2}(\mathbb{T}^N)}^2 ds\right]\end{aligned}$$

meaning that we also have for μ sufficiently small

$$\mathbb{E}\left[\int_0^T \|u_s^\delta\|^2_{W^{1,2}(\mathbb{T}^N)}\,ds\right] \leqslant \frac{1}{2-C\mu^2}\mathbb{E}[||u_0||^2_{L^2(\mathbb{T}^N)}].$$

□

2.4.2 A second a priori bound

We begin by recalling the following Lemma due to [FG95].

Lemma 2.4.3 ([FG95, Lemma 2.1])**.** *Let $p \geqslant 2$, $\alpha < 1/2$. Let U, H be separable Hilbert spaces and W a U-cylindrical Brownian motion. Then for any progressively measurable process*

$$f \in L^p(\Omega \times [0,T], L_2(U,H))$$

it holds that

$$I(f) := \int_0^{(\cdot)} f_s dW_s \in L^p(\Omega, W^{\alpha,p}([0,T],H))$$

and there exists a constant $C = C(\alpha,p) > 0$ independent of f such that

$$\mathbb{E}\left[\|I(f)\|^p_{W^{\alpha,p}([0,T],H)}\right] \leqslant C\mathbb{E}\left[\int_0^T \|f_s\|^p_{L_2(U,H)}\,ds\right].$$

While we can not apply this Lemma in the above context for $H = L^2(\mathbb{T}^N)$ due to the unbounded operator in front of the diffusion term, we can still obtain a bound in a suitable Bessel potential space of distributions, which will turn out to be sufficient for the stochastic compactness method.

Lemma 2.4.4. *Let $u \in L^p(\Omega \times [0,T], L^2(\mathbb{T}^N))$ be a progressively measurable process. Let $\alpha \in (0,1/2)$ and $p \geqslant 2$. Then for $B_1, \dots, B_d \in C^1(\mathbb{T}^N)$ having bounded derivatives, we have that*

$$I(u) := \int_0^{(\cdot)} (-\Delta+1)^{\delta/2} \sum_{i=1}^d B_i(u_s)d\beta_s^i \in L^p(\Omega, W^{\alpha,p}([0,T], H^{-1}(\mathbb{T}^N)))$$

and moreover

$$\mathbb{E}\left[\|I(u)\|^2_{W^{\alpha,p}([0,T],H^{-1}(\mathbb{T}^N))}\right] \leqslant C\left(1 + \mathbb{E}\left[\int_0^T \|u_s\|^p_{L^2(\mathbb{T}^N)}\,ds\right]\right)$$

Proof. We consider as above for $U = \mathbb{R}^d$ the Nemytskii operator

$$\begin{aligned} B : L^2(\mathbb{T}^N) &\to L_2(U, L^2(\mathbb{T}^N)) \\ z &\to \left(u \to \sum_{i=1}^d B_i(z)\langle u, e_i\rangle\right) \end{aligned}$$

that as we know from Lemma 2.3.6 is of linear growth. We then have by the above Lemma 2.4.3 that

$$\begin{aligned}\mathbb{E}\left[\|I(u)\|^p_{W^{\alpha,p}([0,T],H^{-1}(\mathbb{T}^N))}\right] &\leqslant C\mathbb{E}[\int_0^T \left\|(-\Delta+1)^{\delta/2}B(u_s)\right\|^p_{L_2(U,H^{-1}(\mathbb{T}^N))}\,ds]\\ &\leqslant C\mathbb{E}[\int_0^T \|B(u_s)\|^p_{L_2(U,L^2(\mathbb{T}^N))}\,ds]\\ &\leqslant C\left(1+\mathbb{E}[\int_0^T \|u_s\|^p_{L^2(\mathbb{T}^N)}\,ds]\right).\end{aligned}$$

□

In combination with the first a priori bound of Lemma 2.4.2, this permits the derivation of a second one.

Lemma 2.4.5. *For $p \geqslant 2$ and $\alpha \in (0,1/2)$ the family $(u^\delta)_{\delta\in[0,1)}$ is uniformly bounded in $L^p(\Omega, W^{\alpha,p}([0,T],H^{-1}(\mathbb{T}^N)))$.*

Proof. As u^δ is strong solution to

$$u^\delta_t = u_0 + \int_0^t (-\Delta+1)u^\delta_s ds + \mu\int_0^t (-\Delta+1)^{\delta/2}\sum_{i=1}^d B_i(u^\delta_s)d\beta^i_s$$

we need to bound the two time dependent terms on the right hand side. It is immediate that

$$\mathbb{E}\left[\left\|\int_0^{(\cdot)}(-\Delta+1)u^\delta_s ds\right\|^p_{W^{1,2}([0,T],H^{-1}(\mathbb{T}^N))}\right] \leqslant C\mathbb{E}\left[\left\|u^\delta\right\|^2_{L^2([0,T],H^1(\mathbb{T}^N))}\right] < \infty.$$

Moreover, by Lemma 2.4.4

$$\begin{aligned}&\mathbb{E}\left[\left\|\int_0^{(\cdot)}(-\Delta+1)^{\delta/2}\sum_{i=1}^d B_i(u^\delta_s)d\beta^i_s\right\|^p_{W^{\alpha,p}([0,T],H^{-1}(\mathbb{T}^N)}\right]\\ &\leqslant C\left(1+\mathbb{E}[\int_0^T \left\|u^\delta\right\|^p_{L^2(\mathbb{T}^N)}]\right) < \infty.\end{aligned}$$

By the embedding for Sobolev-Slobodeckij spaces $W^{1,2} \hookrightarrow W^{\alpha,p}$ which holds for any $\alpha \in (0,1/2)$ and $p \in [1,\infty)$, this shows overall that indeed,

$$\mathbb{E}\left[\left\|u^\delta\right\|^p_{W^{\alpha,p}([0,T],H^{-1}(\mathbb{T}^N))}\right] < \infty.$$

□

2.4.3 Tightness

We next show that the laws of $(u^\delta)_\delta$ are tight in $L^2([0,T]\times\mathbb{T}^N)\cap C([0,T],H^{-(1+\epsilon)}(\mathbb{T}^N))$ for any $\epsilon > 0$. In order to do so, we use the following result originally due to [Sim86], which is central to the argument in [FG95].

Theorem 2.4.6. *Let B_0, B, B_1 be Banach spaces satisfying the following embedding property*

$$B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1,$$

where by $B_0 \hookrightarrow\hookrightarrow B$ we understand that the embedding $B_0 \hookrightarrow B$ is compact. Then for $p \in (1,\infty)$ and $\alpha \in (0,1)$, one has the following compact embedding

$$L^p([0,T],B_0) \cap W^{\alpha,p}([0,T],B_1) \hookrightarrow\hookrightarrow L^p([0,T],B).$$

Suppose moreover that

$$\alpha p > 1,$$

then we also have the compact embedding

$$W^{\alpha,p}([0,T],B_0) \hookrightarrow\hookrightarrow C([0,T],B).$$

Lemma 2.4.7. *The laws of $(u^\delta)_{\delta\in[0,1)}$ are tight in $L^2([0,T]\times\mathbb{T}^N)\cap C([0,T],H^{-(1+\epsilon)}(\mathbb{T}^N))$ for any $\epsilon > 0$.*

Proof. Note that with $B_0 = W^{1,p}(\mathbb{T}^N)$, $B = L^2(\mathbb{T}^N)$ and $B_1 = H^{-1}(\mathbb{T}^N)$, we can apply the previous Theorem 2.4.6 to conclude that we have the compact embedding

$$Y := L^2([0,T],W^{1,2}(\mathbb{T}^N)) \cap W^{\alpha,2}([0,T],H^{-1}(\mathbb{T}^N)) \hookrightarrow\hookrightarrow L^2([0,T]\times\mathbb{T}^N).$$

We conclude that for any $R > 0$, we set

$$B_R := \{u \in Y \mid \|u\|_{L^2([0,T],W^{1,2}(\mathbb{T}^N))} + \|u\|_{W^{\alpha,2}([0,T],H^{-1}(\mathbb{T}^N))} \leqslant R\}$$

is compact in $L^2([0,T]\times\mathbb{T}^N)$. Hence by Markov's inequality and the a priori bounds established in Lemmas 2.4.2 and 2.4.5 we obtain

$$\begin{aligned}
\mathbb{P}(u^\delta \in B_R^c) &\leqslant \mathbb{P}(\left\|u^\delta\right\|_{L^2([0,T],W^{1,2}(\mathbb{T}^N))} > R/2) \\
&\quad + \mathbb{P}(\left\|u^\delta\right\|_{W^{\alpha,2}([0,T],H^{-1}(\mathbb{T}^N))} > R/2) \\
&\leqslant \frac{4}{R^2}\mathbb{E}\left[\left\|u^\delta\right\|^2_{L^2([0,T],W^{1,2}(\mathbb{T}^N))} + \left\|u^\delta\right\|^2_{W^{\alpha,2}([0,T],H^{-1}(\mathbb{T}^N))}\right] \\
&\leqslant \frac{4C}{R^2}
\end{aligned}$$

from which we conclude that $(u^\delta)_{\delta\in[0,1)}$ is tight in $L^2([0,T]\times\mathbb{T}^N)$. Moreover, notice that due to the second compact embedding stated in the previous Theorem 2.4.6, we also obtain similarly tightness in $C([0,T],H^{-(1+\epsilon)}(\mathbb{T}^N))$ for any $\epsilon > 0$ due to the established a priori bound of Lemma 2.4.5. □

2.4.4 Prokhorov, Skorokhod and Martingale representation theorem

Consider the sequence $(u^\delta)_{\delta\in[0,1)\cap\mathbb{Q}}$. By Prokhorov's theorem, we can find a subsequence we shall also note $(u^\delta)_{\delta\in[0,1)\cap\mathbb{Q}}$ whose laws converge weakly in $L^2([0,T]\times\mathbb{T}^N)\cap C([0,T],H^{-(1+\epsilon)}(\mathbb{T}^N))$. By Skorokhod's representation theorem, we then find a stochastic basis $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_t,\widetilde{\mathbb{P}})$ as well as random variables $\widetilde{u},(\widetilde{u}^\delta)_\delta$ on this stochastic basis taking values in $L^2([0,T]\times\mathbb{T}^N)\cap C([0,T],H^{-(1+\epsilon)}(\mathbb{T}^N))$ such that $\widetilde{\mathbb{P}}$-almost surely

$$\widetilde{u}_\delta\to\widetilde{u}\qquad\text{in } L^2([0,T]\times\mathbb{T}^N)\cap C([0,T],H^{-(1+\epsilon)}(\mathbb{T}^N)),$$

as $\delta\to 1$. Note moreover that

$$\widetilde{\mathbb{E}}\left[\int_0^T\left\|\widetilde{u}_s^{\,\delta}\right\|^2_{W^{1,2}(\mathbb{T}^N)}ds\right]=\mathbb{E}\left[\int_0^T\left\|u_s^\delta\right\|^2_{W^{1,2}(\mathbb{T}^N)}ds\right]<\infty,$$

and thus in particular $\widetilde{u}\in L^2(\Omega\times[0,T],W^{1,2}(\mathbb{T}^N))$ and $\widetilde{u}^\delta\rightharpoonup\widetilde{u}$ in $L^2([0,T],W^{1,2}(\mathbb{T}^N))$, $\widetilde{\mathbb{P}}$-almost surely and also in $L^2(\Omega)$ due to Vitali's convergence theorem. In particular, by definition for any $\varphi\in H^1(\mathbb{T}^N)$ and $t\leqslant T$ we have

$$\int_0^t\langle(-\Delta+1)\widetilde{u}_s^\delta,\varphi\rangle_{L^2(\mathbb{T}^N)}ds\to\int_0^t\langle(-\Delta+1)\widetilde{u}_s,\varphi\rangle_{L^2(\mathbb{T}^N)}ds\tag{2.4.3}$$

in $L^2(\Omega)$. Let us now define the sequence of $L^2(\mathbb{T}^N)$ valued $(\mathcal{F}_t)_t$-martingales $(M^\delta)_{\delta\in[0,1)\cap\mathbb{Q}}$ via

$$\begin{aligned}M_t^\delta&:=(-\Delta+1)^{-(1+\epsilon)/2}\left(u_t^\delta-u_0-\int_0^t(\Delta-1)u_s^\delta ds\right)\\&=\int_0^t\mu(-\Delta+1)^{(\delta-1-\epsilon)/2}B(u_s^\delta)dW_s.\end{aligned}$$

Since $\mathcal{L}(u^\delta)=\mathcal{L}(\widetilde{u}^\delta)$, it is easy to see that

$$\widetilde{M}_t^\delta:=(-\Delta+1)^{-(1+\epsilon)/2}\left(\widetilde{u}_t^\delta-u_0-\int_0^t(\Delta-1)\widetilde{u}_s^\delta ds\right)$$

is a martingale with respect to the filtration $(\mathcal{G}_t)_t$, where $\mathcal{G}_t:=\sigma(\{\widetilde{u}_s^\delta,s\leqslant t\})$, whose quadratic variation is given by

$$\langle\widetilde{M}^\delta\rangle_t=\mu^2\int_0^t\left((-\Delta+1)^{(\delta-1-\epsilon)/2}B(\widetilde{u}_s^\delta)\right)\left((-\Delta+1)^{(\delta-1-\epsilon)/2}B(\widetilde{u}_s^\delta)\right)^*ds.$$

Defining finally the process

$$\widetilde{M}_t:=(-\Delta+1)^{-(1+\epsilon)/2}\left(\widetilde{u}_t-u_0-\int_0^t(\Delta-1)\widetilde{u}_s ds\right),$$

we need to show that it is a martingale with quadratic variation given by

$$\langle \widetilde{M} \rangle_t = \mu^2 \int_0^t \left((-\Delta+1)^{-(1+\epsilon)/2} B(\widetilde{u}_s)\right) \left((-\Delta+1)^{-(1+\epsilon)/2} B(\widetilde{u}_s)\right)^* ds.$$

Indeed, we know that for any bounded continuous functional ϕ on $L^2([0,T]\times\mathbb{T}^N)\cap C([0,T],H^{-(1+\epsilon)})$ and $v,z\in H^{1+\epsilon}(\mathbb{T}^N)$ we have

$$\mathbb{E}\left[\langle \widetilde{M}^\delta_t - \widetilde{M}^\delta_s, v\rangle \phi(\widetilde{u}^\delta|_{[0,s]})\right] = 0$$

and due to (2.4.3), we can pass to the limit concluding that

$$\mathbb{E}\left[\langle \widetilde{M}_t - \widetilde{M}_s, v\rangle \phi(\widetilde{u}|_{[0,s]})\right] = 0,$$

i.e. $(\widetilde{M}_t)_t$ is a martingale with respect to the filtration generated by the process $(\widetilde{u}_t)_t$. Moreover, concerning its quadratic variation, note that

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle \widetilde{M}^\delta_t, v\rangle\langle \widetilde{M}^\delta_t, z\rangle - \langle \widetilde{M}^\delta_s, v\rangle\langle \widetilde{M}^\delta_s, z\rangle\right)\phi(\widetilde{u}^\delta|_{[0,s]})\right]\\
=&\mathbb{E}\Bigg[\left(\int_s^t \sum_{i=1}^d \langle (-\Delta+1)^{(\delta-1-\epsilon)/2} B_i(\widetilde{u}^\delta_r), v\rangle \langle (-\Delta+1)^{(\delta-1-\epsilon)/2} B_i(\widetilde{u}^\delta_r), z\rangle dr\right)\\
&\phi(\widetilde{u}^\delta|_{[0,s]})\Bigg]\mu^2.
\end{aligned}$$

Since the family of operators

$$u \to (-\Delta+1)^{(\delta-1-\epsilon)/2} B_i(u)$$

for $\delta\in[0,1)$ is uniformly continuous on $L^2(\mathbb{T}^N)$ and $\widetilde{u}^\delta\to\widetilde{u}$ in $L^2([0,T]\times\mathbb{T}^N)$ $\widetilde{\mathbb{P}}$-almost surely and in $L^2(\Omega)$ due to Vitali's convergence theorem, we can again pass to the limit $\delta\nearrow 1$ obtaining

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle \widetilde{M}_t, v\rangle\langle \widetilde{M}_t, z\rangle - \langle \widetilde{M}_s, v\rangle\langle \widetilde{M}_s, z\rangle\right)\phi(\widetilde{u}|_{[0,s]})\right]\\
&= \mathbb{E}[\mu^2\left(\int_s^t \sum_{i=1}^d \langle (-\Delta+1)^{-\epsilon/2} B_i(\widetilde{u}_r), v\rangle \langle (-\Delta+1)^{-\epsilon/2} B_i(\widetilde{u}_r), z\rangle dr\right)\phi(\widetilde{u}|_{[0,s]})],
\end{aligned}$$

meaning that indeed

$$\langle \widetilde{M} \rangle_t = \int_0^t \left((-\Delta+1)^{-(1+\epsilon)/2} B(\widetilde{u}_s)\right) \left((-\Delta+1)^{-(1+\epsilon)/2} B(\widetilde{u}_s)\right)^* ds.$$

Similar to section 8.4 in [DPZ14], we can then use the martingale representation theorem 8.2 in [DPZ14] to conclude the existence of a filtered probability space

$(\Omega', \mathcal{F}', (\mathcal{F}'_t)_t, \mathbb{P}')$, a d-dimensional $(\mathcal{F}'_t)_t$-Brownian motion W' and a predictable process $(u'_t)_t$ taking values in $L^2([0,T]\times\mathbb{T}^N)\cap C([0,T], H^{-(1+\epsilon)}(\mathbb{T}^N))$ such that

$$\begin{aligned}(-\Delta+1)^{-(1+\epsilon)/2}u'_t &= (-\Delta+1)^{-(1+\epsilon)/2}u_0\\ &+ (-\Delta+1)^{-(1+\epsilon)/2}\int_0^t(\Delta-1)u'_s ds\\ &+ \mu(-\Delta+1)^{-(1+\epsilon)/2}\int_0^t(-\Delta+1)^{1/2}B(u'_s)dW'_s,\end{aligned}$$

or in other words for any $\varphi\in H^{1+\epsilon}(\mathbb{T}^N)$, we have

$$\langle u'_t,\varphi\rangle = \langle u_0,\varphi\rangle + \left\langle \int_0^t(\Delta-1)u'_s ds,\varphi\right\rangle + \left\langle \int_0^t \mu(-\Delta+1)^{1/2}B(u'_s)dW'_s,\varphi\right\rangle,$$

meaning we have indeed a martingale solution.

Example 2.4.8. *As an example one can consider the problem*

$$\begin{cases} du_t &= (\Delta-1)u_t dt + \mu\sum_{i=1}^d div(B_i(u_t))d\beta^i_t\\ u(0) &= u_0\end{cases} \tag{2.4.4}$$

for $B_1,\dots,B_d\in C^1(\mathbb{R},\mathbb{R}^N)$ with bounded derivative which can be seen as

$$\begin{cases} du_t &= (\Delta-1)u_t dt + \mu(-\Delta+1)^{1/2}\sum_{i=1}^d(-\Delta+1)^{-1/2}div(B_i(u_t))d\beta^i_t\\ u(0) &= u_0.\end{cases}$$

Having in mind the abstract results in the proof of Theorem 2.2.5, one has to consider the Nemytskii operators

$$\begin{aligned}B^0 : L^p(\mathbb{T}^N) &\to \gamma(U, L^p(\mathbb{T}^N))\\ z &\to \left(u\to\sum_{i=1}^d(-\Delta+1)^{-1/2}div(B_i(z))\langle u,e_i\rangle\right)\end{aligned}$$

and

$$\begin{aligned}B^1 : W^{1,p}(\mathbb{T}^N) &\to \gamma(U, W^{1,p}(\mathbb{T}^N))\\ z &\to \left(u\to\sum_{i=1}^d(-\Delta+1)^{-1/2}div(B_i(z))\langle u,e_i\rangle\right).\end{aligned}$$

After showing that B^0 is Lipschitz and B^1 admits a growth condition in the above setting, one can for each $\delta<1$ construct a unique mild solution $u^\delta\in Z_{q,W^{1,p}(\mathbb{T}^N)}$ to the regularized problem

$$\begin{cases} du^\delta_t &= (\Delta-1)u^\delta_t dt + \mu(-\Delta+1)^{\delta/2}\sum_{i=1}^d(-\Delta+1)^{-1/2}divB_i(u^\delta_t)dW^i_t\\ u^\delta(0) &= u_0.\end{cases}$$

Provided μ is sufficiently small, one can exploit the reasoning of this section to deduce the existence of a martingale solution to the problem (2.4.4).

Remark 2.4.9. *Existence of weak solutions in the linear setting of this example i.e. $(B_i)_{i \leqslant d}$ linear is classical, see Example 7.22 in [DPZ14] or section three in [KR81]. In this light, the present subsection provides an alternative approximation scheme to Yosida approximations used in [DPZ14] and Galerkin approximations employed in [KR81], capable of treating non-linear diffusion coefficients in this setting.*

2.A Appendix

Lemma 2.A.1. *Let $G \in C^m(\mathbb{T}^N \times \mathbb{R})$ and $F \in C^m(\mathbb{R})$ have bounded derivative and such that $F(0) = 0$. Then for all $p \geqslant 1$ and $h \in W^{m,p}(\mathbb{T}^N) \cap W^{1,mp}(\mathbb{T}^N)$ one has*

$$||G(\cdot, h(\cdot))||_{W^{m,p}(\mathbb{T}^N)} \leqslant C(1 + ||h||^m_{W^{1,mp}(\mathbb{T}^N)} + ||h||_{W^{m,p}(\mathbb{T}^N)})$$

and

$$||F(h)||_{W^{m,p}(\mathbb{T}^N)} \leqslant C(||h||^m_{W^{1,mp}(\mathbb{T}^N)} + ||h||_{W^{m,p}(\mathbb{T}^N)}).$$

(refer to [Hof13] and the references therein).

On stochastic integrals in different Banach spaces

Note that the stochastic integral in a Banach space X is defined as a certain limit in the Bochner space $L^2(\Omega; X)$, i.e. it depends in particular on the topology induced by the norm on X. To underline this fact, introduce for $\psi \in L^2(\Omega \times [0,T]; \gamma(U,X))$ the notation

$$(X \int_0^T) \psi(t) dW_t \in L^2(\Omega; X)$$

Heuristically speaking, one should expect the stochastic integral not to change if one looks at it in a "larger" Banach space. A bit more formally, for $X \hookrightarrow Y$, one would expect

$$(X \int_0^T) \psi(t) dW_t = (Y \int_0^T) \psi(t) dW_t.$$

The following Lemma formalizes this consideration.

Lemma 2.A.2. *(Banach space consistency of stochastic integration) Let X, Y be 2-smooth Banach spaces such that $X \hookrightarrow Y$, where the embedding operator is the identity operator. Suppose that $\psi \in L^2(\Omega \times [0,T]; \gamma(U,X))$. Then one has $\psi \in L^2(\Omega \times [0,T]; \gamma(U,Y))$ and thus the stochastic integral*

$$(Y \int_0^T) \psi(t) dW_t \in L^2(\Omega; Y)$$

is well defined. Moreover this Y-stochastic integral also lies in $L^2(\Omega; X)$ and one has

$$(X \int_0^T) \psi(t) dW_t = (Y \int_0^T) \psi(t) dW_t \qquad \text{in } L^2(\Omega; X)$$

Proof. Note that by the continuous embedding $X \hookrightarrow Y$we have $\gamma(U,X) \hookrightarrow \gamma(U,Y)$ and therefore also $L^2(\Omega \times [0,T];\gamma(U,X)) \hookrightarrow L^2(\Omega \times [0,T];\gamma(U,Y))$, i.e. in particular $\psi \in L^2(\Omega \times [0,T];\gamma(U,Y))$.

For the second part of the Lemma, note that stochastic integrals are defined as L^2 limits of stochastic integrals over approximating elementary processes. On the level of elementary processes, the norms of X and Y do not come into play and the canonical definitions of stochastic integrals with respect to elementary processes coincide therefore in $L^2(\Omega, X)$ and $L^2(\Omega, Y)$ as the spaces of reference in which the stochastic integrals live.

Let $(\psi^n)_n$ be a sequence of elementary processes approximating $\psi \in L^2(\Omega \times [0,T];\gamma(U;X))$, i.e. one has

$$\mathbb{E}\left[\int_0^T ||\psi(t) - \psi^n(t)||^2_{\gamma(U,X)} ds\right] \to 0$$

Due to the demonstrated embedding result one also has $(\psi^n)_n \subset L^2(\Omega \times [0,T];\gamma(U;Y))$ and moreover, for the same reason

$$\begin{aligned}\mathbb{E}\left[\int_0^T ||\psi(t) - \psi^n(t)||^2_{\gamma(U,Y)} dt\right] &\leqslant C\mathbb{E}\left[\int_0^T ||\psi(t) - \psi^n(t)||^2_{\gamma(U,X)} dt\right] \\ &\to 0.\end{aligned}$$

Hence, any sequence of elementary processes $(\psi^n)_n$ approximating ψ in the space $L^2(\Omega \times [0,T];\gamma(U;X))$ also approximates ψ in $L^2(\Omega \times [0,T];\gamma(U;Y))$. Moreover, by Itô's inequality applied in the Banach space Y one has

$$\begin{aligned}\mathbb{E}\left[|| \int_0^T \psi^n(t) dW_s - \int_0^T \psi^m(t) dW_s||^2_Y\right] &\leqslant \mathbb{E}\left[\int_0^T ||\psi^n(t) - \psi^m(t)||^2_{\gamma(U,Y)} ds\right] \\ &\to 0\end{aligned}$$

The sequence

$$\left(\int_0^t \psi^n(t) dW_s\right)_{n \geqslant 1}$$

is therefore Cauchy in $L^2(\Omega, Y)$. By definition, the stochastic integral $I_Y := (Y \int_0^T)\psi(t)dW_t$ is the $L^2(\Omega; Y)$ limit of the above sequence. Note however that

$$\begin{aligned}&\mathbb{E}\left[||(X \int_0^T)\psi(t)dW_s - \int_0^T \psi^n(t)dW_s||^2_Y\right] \\ \leqslant\ &\mathbb{E}\left[||(X \int_0^T)\psi(t)dW_s - \int_0^T \psi^n(t)dW_s||^2_X\right] \to 0\end{aligned}$$

since $(\psi^n)_n$ is a sequence of elementary processes used to define the stochastic integral in the Banach space X. By the uniqueness of limits, one concludes therefore

$$(X \int_0^T)\psi(t)dW_s = (Y \int_0^T)\psi(t)dW_s.$$

□

Chapter 3

A Law of Large Numbers for interacting diffusions via a mild formulation

We consider a system of n weakly interacting particles driven by independent Brownian motions. In many instances, it is well known that the empirical measure converges to the solution of a partial differential equation, usually called McKean-Vlasov or Fokker-Planck equation, as n tends to infinity. We propose a relatively new approach to show this convergence by directly studying an equation that the empirical measure satisfies for each fixed n. Under a suitable control on the noise term, which appears in said equation due to the finiteness of the system, we are able to prove that the stochastic perturbation goes to zero, showing that the limiting measure is a solution to the classical McKean-Vlasov equation. In contrast with known results, we do not require any independence or finite moment assumption on the initial condition, but only weak convergence. The evolution of the empirical measure is studied in a suitable class of Hilbert spaces where the noise term is controlled using two distinct but complementary techniques: rough paths theory and maximal inequalities for self-normalized processes.

This chapter is based on the joint work [BC20] with Fabio Coppini.

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3.1 Introduction

The theory of weakly interacting particle systems has received great attention in the last fifty years. On the one hand, its mathematical tractability has allowed to obtain a deep understanding of the behavior of the empirical measure for such systems: law of large numbers [Oel84, CDFM20], fluctuations and central limit theorems [Tan84, FM97], large deviations [FK06, Gä88] and propagation of chaos properties [Szn91] are by now established. On the other hand, the theory of weakly interacting particles enters in several areas of applied mathematics such as mean-field games or finance models [CDLL19], making it an area of active research.

Depending on the context of application, several results are available. The class of mean-field systems under the name of weakly interacting particles is rather large and models may substantially vary from one another depending on the regularity of the coefficients or the noise. This richness in models is reflected in a variety of different techniques implemented in their study (see e.g. [CDFM20, Oel84, Szn91] for three very different approaches).

If one focuses on models where the interaction function is regular enough, e.g. bounded and globally Lipschitz, one of the aspects that has not been completely investigated so far, concerns the initial condition. To the authors' knowledge, most of known results require a finite moment condition in order to prove tightness properties of the general sequence (e.g. [Gä88]) or to apply a fixed-point argument in a suitable topological space (e.g. [CDFM20]). The only exceptions are given by [Szn91, Tan84], although they require independent and identically distributed (IID) initial conditions. We want to point out that existence of a solution to the limiting system, a non-linear partial differential equation (PDE) known as Fokker-Planck or McKean-Vlasov equation, does not require any finite moment condition on the initial measure, see e.g. [Szn91, Theorem 1.1]. Furthermore, whenever the particle system is deterministic, there is no need to assume independence (or any finite moment) for this same convergence, see, e.g., [Dob79, Neu84].

We present a result in the spirit of a law of large numbers, without requiring any assumption on the initial conditions but the convergence of the associated empirical measure. Our main idea consists in exploiting a mild formulation associated to a stochastic equation satisfied by the empirical measure for a fixed (finite!) population. The main difficulty is giving a meaning to the noise term appearing in such formulation: exploiting the regularizing properties of the semigroup generated by the Laplacian in two different ways, using rough paths theory and maximal inequalities for self normalized processes respectively, we are able to adequately control it. By taking the limit

for the size of the population which tends to infinity, the stochastic term vanishes and the limiting measure satisfies the well-known McKean-Vlasov equation.

Organization

The chapter is organized as follows. In the rest of this section we present the model, known results and introduce the set-up in which the evolution of the empirical measure is studied along with notation used.

In Section 2 we give the definition of our notion of solution as well as a corresponding uniqueness statement. The law of large numbers, Theorem 3.2.3, is presented right after; the section ends with a discussion, a comparison with the existing literature and the strategy of the proof.

The noise perturbation mentioned in the introduction is tackled in Section 3 where rough paths techniques and maximal inequalities for self-normalized processes are exploited. The proof of Theorem 3.2.3 is given at the end of this section.

Appendix A recalls general properties of analytic semigroups; Appendix B provides an extension of Gubinelli's theory of rough integration to our setting.

3.1.1 The model and known results

Consider $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P}\big)$ a filtered probability space, the filtration satisfying the usual conditions. Fix $d \in \mathbb{N}$, let $(B^i)_{i\in\mathbb{N}}$ be a sequence of IID $\mathbb{R}^d$-valued Brownian motions adapted to the filtration $(\mathcal{F}_t)_{t \geqslant 0}$.

Fix $n \in \mathbb{N}$ and $T > 0$ a finite time horizon. Let $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded function that is Lipschitz in each argument uniformly in the second. Let $(x^{i,n})_{1 \leqslant i \leqslant n}$ be the unique strong solution[1] to

$$\begin{cases} \mathrm{d}x_t^{i,n} = \frac{1}{n}\sum_{j=1}^n \Gamma(x_t^{i,n}, x_t^{j,n})\mathrm{d}t + \mathrm{d}B_t^i, \\ x_0^{i,n} = x_0^i, \end{cases} \tag{3.1.1}$$

for $t \in [0,T]$ and $i = 1, \dots, n$. The initial conditions are denoted by the sequence $(x_0^i)_{i\in\mathbb{N}} \subset \mathbb{R}^d$, whenever they are random they are taken independent of the Brownian motions.

The main quantity of interest in system (3.1.1) is the empirical measure $\nu^n = (\nu_t^n)_{t\in[0,T]}$, a random variable with values on the space of probability measures. It is defined for $t \in [0,T]$ by

$$\nu_t^n := \frac{1}{n}\sum_{j=1}^n \delta_{x_t^{j,n}}. \tag{3.1.2}$$

Observe that ν^n is apriori a probability measure on the continuous trajectories with values in $\mathbb{R}^d$, i.e. $\nu^n \in \mathcal{P}(C([0,T],\mathbb{R}^d))$, however in many instances we rather consider its projection $(\nu_t^n)_{t\in[0,T]} \in C([0,T],\mathcal{P}(\mathbb{R}^d))$ as continuous function over the probability

[1] For the classical existence and uniqueness result, refer to [Szn91, Chapter I, Theorem 1.1] for example.

measures on $\mathbb{R}^d$. This last object does not carry the information of the time dependencies between time marginals, but is in our case more suitable when studying (3.1.1) in the limit for n which tends to infinity.

Known results

Fix an initial probability measure $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$. Whenever $(x_0^i)_{i\in\mathbb{N}}$ are taken to be IID random variables sampled from ν_0, or ν_0 has a finite moment and ν_0^n weakly converges to it, it is well known (e.g. [Szn91, Theorem 1.4] and [CDFM20, Theorem 3.1]) that ν^n converges (in a precise sense depending on the setting) to the solution of the following PDE

$$\begin{cases} \partial_t \nu_t = \frac{1}{2}\Delta\nu_t - \operatorname{div}[\nu_t(\Gamma * \nu_t)], \\ \nu_{|t=0} = \nu_0, \end{cases} \tag{3.1.3}$$

for $t \in [0,T]$ and where $*$ denotes the integration with respect to the second argument, i.e. for $\mu \in \mathcal{P}(\mathbb{R}^d)$

$$(\Gamma * \mu)(x) = \int_{\mathbb{R}^d} \Gamma(x,y)\,\mu(\mathrm{d}y), \quad x \in \mathbb{R}^d.$$

Equation (3.1.3) is usually called McKean-Vlasov or non-linear Fokker-Planck equation.

Remark 3.1.1. *Observe that requiring IID initial conditions is not an innocent assumption as they are, in particular, exchangeable, see [Szn91, §I.2] for more on this perspective. From an applied viewpoint, independence is often a hypothesis that we do not want to assume, see e.g. [DGL16, Example II].*

A solution to (3.1.3) is linked to the following non-linear process:

$$\begin{cases} x_t = x_0 + \int_0^t \int_{\mathbb{R}^d} \Gamma(x_s,y)\,\nu_s(\mathrm{d}y)\mathrm{d}s + B_t, \\ \nu_t = Law\,(x_t), \end{cases} \tag{3.1.4}$$

where B is a Brownian motion independent of $(B^i)_{i\in\mathbb{N}}$ and x_0. It is well-known that $\nu = (\nu_t)_{t\in[0,T]}$ is a solution to (3.1.3) if and only if the non-linear process $(x_t)_{t\in[0,T]}$ in (3.1.4) exists and is such that $Law\,(x_t) = \nu_t$ for every $t \in [0,T]$.

We have to following theorem.

Theorem 3.1.2 ([Szn91, Theorem 1.1])**.** *Suppose Γ is bounded and Lipschitz and x_0 is a random variable with law $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$. Then, system* (3.1.4) *has a unique solution $(x_t)_{t\in[0,T]}$.*

Moreover, if $\nu = (\nu_t)_{t\in[0,T]}$ is the law of $(x_t)_{t\in[0,T]}$, then $\nu \in C([0,T],\mathcal{P}(\mathbb{R}^d))$ and it solves the McKean-Vlasov equation (3.1.3) *in the weak sense.*

3.1.2 Set-up and notations

Let $W^{m,p} = W^{m,p}(\mathbb{R}^d)$ be the standard Sobolev space with $m \in \mathbb{N}$ and $p \in [1, \infty)$. Classical results as [AF03, Theorem 4.12] assure that

$$W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d) \quad \text{whenever } mp > d, \tag{3.1.5}$$

where $C_b(\mathbb{R}^d)$ is the space of continuous bounded functions on $\mathbb{R}^d$. The space $W_0^{m,p}(\mathbb{R}^d)$ is the closure of $C_0^\infty(\mathbb{R}^d)$, i.e., the space of smooth functions with compact support, with respect to the norm

$$\|\varphi\|_{W^{m,p}} := \left(\sum_{0 \leqslant |\alpha| \leqslant m} \int_{\mathbb{R}^d} |\partial^\alpha \varphi(x)|^p \, \mathrm{d}x \right)^p, \quad \varphi \in C_0^\infty(\mathbb{R}^d),$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\partial^\alpha = (\partial_{x_1})^{\alpha_1} (\partial_{x_2})^{\alpha_2} ... (\partial_{x_d})^{\alpha_d}$.

Fix $p = 2$ and $m > d/2$, we consider the Hilbert space $H^m := W^{m,2}(\mathbb{R}^d)$, with norm denoted by $\|\cdot\|_m$ and its dual space $H^{-m} := (H^m)^*$ with the standard dual norm defined by $\|\mu\|_{-m} := \sup_{\|h\|_m \leqslant 1} \langle \mu, h \rangle_{-m,m}$. The action of H^{-m} on H^m is denoted by $\langle \cdot, \cdot \rangle_{-m,m}$. By duality, if follows from (3.1.5) that

$$\mathcal{P}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)^* \subset H^{-m}.$$

We denote by $(\cdot, \cdot)_m$ the scalar product in H^m and by $\langle \cdot, \cdot \rangle$ the natural action of a probability measure on test functions, i.e., for $\nu \in \mathcal{P}(\mathbb{R}^d)$ and a smooth function h, we write $\langle \nu, h \rangle = \int_{\mathbb{R}^d} h(x) \nu(\mathrm{d}x)$. We often abuse of notation denoting the density of a probability measure by the probability measure itself.

Let $\nu \in \mathcal{P}(\mathbb{R}^d)$, and thus $\nu \in H^{-m}$, and let $\widetilde{\nu} \in H^m$ be its Riesz representative, then we have for any $h \in H^m$

$$\langle \nu, h \rangle = \nu(h) = (\widetilde{\nu}, h)_m = \langle \nu, h \rangle_{-m,m}$$

and therefore

$$|\langle \nu, h \rangle| \leqslant \|\nu\|_{-m} \|h\|_m .$$

In particular

$$\sup_{\|h\|_m \leqslant 1} \langle \nu, h \rangle = \|\nu\|_{-m} .$$

If $(\mu^n)_{n \in \mathbb{N}}$ is a sequence of probability measures which weakly converges to some $\mu \in \mathcal{P}(\mathbb{R}^d)$, we use the notation $\mu^n \rightharpoonup \mu$. For weak convergence and weak-*-convergence of a sequence $(x_n)_n \subset X$ to some $x \in X$, X being a Banach space, we use the standard notations $x_n \rightharpoonup x$ and $x_n \overset{*}{\rightharpoonup} x$ respectively.

As introduced in [Mź82], we will use $\|\cdot\|_{-m}$ as distance between probability measures and our results will be expressed with respect to this topology.

The various constants in the chapter will always be denoted by C or C_α to emphasize the dependence on some parameter α. Their value may change from line to line.

3.2 Main result

Before stating the main result, we give the definition of weak-mild solutions to (3.1.3) in the Hilbert space H^m. We denote by $S = (S_t)_{t\in[0,T]}$ the analytic semigroup generated by the Laplacian operator $\frac{\Delta}{2}$ on H^m. We refer to Appendix A for general properties of S.

Definition 3.2.1 (m-weak-mild solutions to McKean-Vlasov PDEs)**.** *Let ν_0 be an element in H^{-m}. We call $\nu \in L^\infty([0,T], H^{-m})$ an m-weak-mild solution to the problem* (3.1.3)*, if for every $h \in H^m$ and $t \in [0,T]$, it holds*

$$\langle \nu_t, h\rangle_{-m,m} = \langle \nu_0, S_t h\rangle_{-m,m} + \int_0^t \langle \nu_s, (\nabla S_{t-s}h)(\Gamma * \nu_s)\rangle_{-m,m}\mathrm{d}s. \tag{3.2.1}$$

If Γ is sufficiently regular, uniqueness can be readily established by using a classical argument. This is illustrated in the next proposition.

Proposition 3.2.2 (Uniqueness)**.** *Suppose that $\Gamma(\cdot_x, \cdot_y) \in H^m_y W^{m,\infty}_x$, i.e.,*

$$\|\Gamma(\cdot_x, \cdot_y)\|_{H^m_y W^{m,\infty}_x} = \max_{|\beta| \leqslant m} \left\| \sum_{|\alpha| \leqslant m} \int_{\mathbb{R}^d} \left(\partial^\beta_x \partial^\alpha_y \Gamma(x,y)\right)^2 \mathrm{d}y \right\|_{L^\infty_x} < \infty. \tag{3.2.2}$$

Then, m-weak mild solutions to (3.1.3) *are unique.*

Proof. Suppose $\nu, \rho \in L^\infty([0,T], H^{-m})$ are two m-weak mild solutions. Then, taking the difference between the two equations (3.2.1), one obtains that for every $h \in H^m$

$$\begin{aligned}\langle \nu_t - \rho_t, h\rangle_{-m,m} = &\int_0^t \langle \nu_s - \rho_s, (\nabla S_{t-s}h)(\Gamma * \nu_s)\rangle_{-m,m}\mathrm{d}s \\ &+ \int_0^t \langle \rho_s, (\nabla S_{t-s}h)(\Gamma * (\nu_s - \rho_s))\rangle_{-m,m}\mathrm{d}s.\end{aligned}$$

In particular,

$$\begin{aligned}\|\nu_t - \rho_t\|_{-m} \leqslant &\int_0^t \|\nu_s - \rho_s\|_{-m} \sup_{\|h\|_m \leqslant 1} \|(\nabla S_{t-s}h)(\Gamma * \nu_s)\|_m \,\mathrm{d}s \\ &+ \int_0^t \|\rho_s\|_{-m} \sup_{\|h\|_m \leqslant 1} \|(\nabla S_{t-s}h)(\Gamma * (\nu_s - \rho_s))\|_m \,\mathrm{d}s.\end{aligned}$$

Observe that, for $\mu \in H^{-m}$ it holds that

$$\begin{aligned}\|(\nabla S_{t-s}h)(\Gamma * \mu)\|_m &\leqslant \|\nabla S_{t-s}h\|_m \|\Gamma * \mu\|_{W^{m,\infty}} \\ &\leqslant \frac{C}{\sqrt{t-s}} \|h\|_m \|\mu\|_{-m} \|\Gamma(\cdot_x, \cdot_y)\|_{H^m_y W^{m,\infty}_x},\end{aligned} \tag{3.2.3}$$

where we have used the properties of the semigroup (see Lemma 3.A.1). Using the continuous embedding of $\mathcal{P}(\mathbb{R}^d)$ into H^{-m}, we conclude that there exists a (new) constant $C > 0$:

$$\|\nu_t - \rho_t\|_{-m} \leqslant C \, \|\Gamma(\cdot_x, \cdot_y)\|_{H_y^m W_x^{m,\infty}} \int_0^t \frac{1}{\sqrt{t-s}} \, \|\nu_s - \rho_s\|_{-m} \, \mathrm{d}s.$$

A Gronwall-like lemma yields the proof. □

We are ready to state the main result.

Theorem 3.2.3. *Assume $m > d/2 + 3$ and $\Gamma(\cdot_x, \cdot_y) \in H_y^m W_x^{m,\infty}$. If $\nu_0 \in H^{-m}$, then there exists $\nu \in L^\infty([0,T], H^{-m})$, unique m-weak-mild solution to* (3.2.1)*. Suppose that the initial empirical measure associated to the particle system* (3.1.1) *is such that*

$$\nu_0^n \rightharpoonup \nu_0 \qquad \text{in } H^{-m}$$

in probability. Then, the empirical measure ν^n of (3.1.1) *satisfies*

$$\nu^n \overset{*}{\rightharpoonup} \nu \qquad \text{in } L^\infty([0,T], H^{-m})$$

in probability.

Moreover, if $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$, then ν is the unique weak solution of the McKean-Vlasov equation (3.1.3) *and, in particular, $\nu \in C([0,T], \mathcal{P}(\mathbb{R}^d))$.*

3.2.1 Discussion

Theorem 3.2.3 shows a law of large numbers in $L^\infty([0,T], H^{-m})$ by directly studying the evolution of the empirical measure. Contrary to most of the existing proofs in the literature, it does not establish any trajectorial estimates on system (3.1.1) and does not invoke propagation of chaos techniques, as, e.g., in [MM13, Szn91]. This allows to deal with very general initial data: the weak convergence of $(\nu_0^n)_{n\in\mathbb{N}}$ in H^{-m} – which is implied by the weak convergence in $\mathcal{P}(\mathbb{R}^d)$ – suffices.

Working in H^{-m} for $m > d/2$ assures a bound on $\|\nu\|_{-m}$ which is uniform in $\nu \in \mathcal{P}(\mathbb{R}^d)$ thanks to the continuous embedding of $\mathcal{P}(\mathbb{R}^d)$ in H^{-m} and the duality properties of probability measures, see Lemma 3.A.3. By exploiting the equation satisfied by ν^n, we are able to establish a compactness property for $(\nu^n)_{n\in\mathbb{N}}$, usually hard to obtain in $\mathcal{P}(\mathbb{R}^d)$, and which represents our main tool for obtaining the existence both of the limit solution and of a convergent subsequence.

Weak-mild solutions make sense for any $m > d/2$, yet we have to require the stronger condition $m > d/2 + 3$ in order to give a pathwise meaning to the stochastic term present in the dynamics. This implies that Γ is C^3. In this last case, it is already known that a weak solution to the McKean-Vlasov equation (3.1.3) exists for any initial probability measure ν_0. Since weak solutions are weak-mild solutions, as we will show in the sequel, a byproduct of our main result is the uniqueness of (weak) solutions to equation (3.1.3).

The particle system (3.1.1) represents an interaction setting where no transport is present in the dynamics. We have decided not to include other terms so as to keep the underlying ideas and techniques as clear as possible. However, all our arguments readily extend to the more general case of interacting particles given by

$$\mathrm{d}x_t^{i,n} = F(x_t^{i,n})\mathrm{d}t + \frac{1}{n}\sum_{j=1}^{n}\Gamma(x_t^{i,n}, x_t^{j,n})\mathrm{d}t + \mathrm{d}B_t^i, \tag{3.2.4}$$

provided that $F \in H^m$.

Finally, we point out that the need of rather high regularity in Γ (and F) is an intrinsic requirement of rough paths theory and not of the particular class of models we are working with. In particular, proving Theorem 3.2.3 independently of rough paths arguments would likely yield less restrictive regularity constraints on Γ. On the other hand, rough paths theory allows to give a pathwise definition of the stochastic partial differential equation satisfied by the empirical measure. Such viewpoint appears to be new in the literature. Finally, we observe that the proposed strategy represents an application of the algebraic integration with respect to semigroups, presented in [GT10], that can be interesting on its own.

3.2.2 Comparison with the existing literature

Proving a law of large numbers by directly studying the empirical measure and not the single trajectories is the classical approach in the deterministic setting [Neu84, Dob79], i.e., when no Brownian motions are acting on system (3.1.1). In the case of interacting diffusions, the idea of studying the equation satisfied by the empirical measure for a fixed n, comes from the two articles [BGP14, LP17] and the recent [Cop19], where a weak-mild formulation is derived and carefully studied. Contrary to our case, in [BGP14, Cop19, LP17] the particles live in the one dimensional torus which considerably simplifies the analysis; we refer to Remark 3.3.5.

A Hilbertian approach for particle systems has already been discussed in [FM97], where it is used to study the fluctuations of the empirical measure around the McKean-Vlasov limit. However, [FM97] does not make use of the theory of semigroups but instead requires strong hypothesis on the initial conditions which have to be IID and with finite $(4d+1)$-moment (see [FM97, §3]). The evolution of the empirical measure (3.1.2) is then studied in weighted Hilbert spaces (or, more precisely, in spaces of Bessel potentials) so as to fully exploit the properties of mass concentration given by the condition on the moments. Observe that we are not able to present a fluctuation result, given the lack of a suitable uniform estimate on the noise term.

Studying the action of an analytic semigroup in the evolution of an interacting particle system has been recently proposed in similar settings; we refer to [FLO19, FOS20] and references therein. This method is referred to as the semigroup approach. We want to stress that the cited works deal with smooth mollified empirical measures and work in a weaker topology (with respect to the time variable) than the one expressed in Theorem 3.2.3.

The strategies developed in [MMW15, MM13, Kol11], and further applied in the case of mean-field games in [CDLL19], study the evolution of the joint law of system (3.1.1) and take a more abstract viewpoint. In particular, they study the system dynamics at the level of the flows and not directly addressing the empirical measure.

Finally, observe that under a suitable change of the time-scale, the n-dependent SPDE satisfied by the empirical measure (3.1.2) is the mild formulation of the Dean-Kawasaki equation [KLvR20, Theorem 1] and [KLvR19].

3.2.3 Strategy of the proof

Using Itô's formula, we derive an equation satisfied by ν^n for every fixed $n \in \mathbb{N}$, which turns out to be the McKean-Vlasov PDE perturbed by some noise w^n, see Lemma 3.3.1. This equation makes sense in $L^\infty([0,T], H^{-m})$ and in this space we study the convergence of $(\nu^n)_{n\in\mathbb{N}}$.

The main challenge towards the proof of Theorem 3.2.3 is giving a meaning to w^n and suitably controlling it. In Lemma 3.3.2, we first give a pathwise definition of this term through rough paths theory, referring to Appendix 3.B for a suitable theory of rough integration in our setting. This in turn will allow to show that $(\nu^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T], H^{-m})$ and to extract a weak-* converging subsequence, see Lemma 3.3.8.

To show that a converging subsequence satisfies the weak-mild formulation (3.2.1) in the limit, as shown in Lemma 3.3.9, we need a further step: the pointwise estimate of $w^n(h)$, for a fixed $h \in H^m$. Using a suitable decomposition of the semigroup and a maximal inequality for self-normalized processes, we are able to prove that $w^n(h)$ converges to zero in probability as n diverges, see Lemma 3.3.6. If on the one hand the rough paths bound cannot take advantage of the statistical independence of the Brownian motions and thus, cannot be improved in n; on the other hand the probability estimate does not suffice to define w^n as an element of $L^\infty([0,T], H^{-m})$. We refer to Subsection 3.3.2 and Remark 3.3.7 for more on this aspect.

The uniqueness of weak-mild solution, Proposition 3.2.2, is the last ingredient to obtain that any convergent subsequence of $(\nu^n)_{n\in\mathbb{N}}$ admits a further subsequence that converges $\mathbb{P}$-a.s. to the same ν satisfying equation (3.2.1). This is equivalent to the weak-* convergence in probability to the weak-mild solution ν.

3.3 Proofs

We start by giving the n-dependent stochastic equation satisfied by the empirical measure for each $n \in \mathbb{N}$. We then move to the control on the noise term and, finally, the proof of Theorem 3.2.3.

3.3.1 A weak-mild formulation satisfied by the empirical measure

Recall that $(S_t)_{t\in[0,T]}$ denotes the semigroup generated by $\frac{\Delta}{2}$ on H^m.

Lemma 3.3.1. *Assume $m > d/2+2$. The empirical measure* (3.1.2) *associated to the particle systems* (3.1.1) *satisfies for every $h \in H^m$ and $t \in [0,T]$*

$$\langle \nu_t^n, h\rangle_{-m,m} = \langle \nu_0^n, S_t h\rangle_{-m,m} + \int_0^t \langle \nu_s^n, (\nabla S_{t-s}h)(\Gamma * \nu_s^n)\rangle_{-m,m} \mathrm{d}s + w_t^n(h), \tag{3.3.1}$$

where

$$w_t^n(h) = \frac{1}{n}\sum_{j=1}^n \int_0^t \left[\nabla S_{t-s}h\right](x_s^{j,n}) \cdot \mathrm{d}B_s^j. \tag{3.3.2}$$

Proof. Fix $t \in [0,T]$ and $h \in H^m$, by (3.1.5) h is $C^2(\mathbb{R}^d)$. For $s < t$, applying Itô's formula onto the test function $\varphi(x,s) = (S_{t-s}h)(x)$, we obtain

$$\begin{aligned} h(x_t^{i,n}) =& (S_t h)(x_0^{i,n}) + \frac{1}{n}\sum_{j=1}^n \int_0^t (\nabla S_{t-s}h)(x_s^{i,n})\Gamma(x_s^{i,n}, x_s^{j,n})\mathrm{d}s \\ &+ \int_0^t (\nabla S_{t-s}h)(x_s^{j,n}) \cdot \mathrm{d}B_s^j. \end{aligned}$$

Summing over all particles and dividing by $1/n$, the claim is proved modulo well-posedness of the noise term w^n which is presented in the following subsection. □

3.3.2 Controlling the noise term

The aim of this subsection is to control the noise term w^n appearing in the weak-mild formulation (3.3.1) for the empirical measure. We start by giving a pathwise definition of the integral (3.3.2), i.e. for any $\omega \in A \subset \Omega$ where $\mathbb{P}(A) = 1$ and any $h \in H^m$ we define

$$w_t^n(h)(\omega) = \left(\frac{1}{n}\sum_{j=1}^n \int_0^t \left[\nabla S_{t-s}h\right](x_s^{j,n}) \cdot \mathrm{d}B_s^j\right)(\omega),$$

which in turn allows to define w^n as an element of $L^\infty([0,T], H^{-m})$, via an inequality of the form

$$\sup_{\|h\|_m=1} |w_t^n(h)(\omega)| \leqslant C_T(\omega)$$

for $\omega \in A$, see Lemma 3.3.2. For this purpose, we extend Gubinelli's theory for rough integration (see [Gub04] and [GT10, §3 and 4]) to our setting, see Appendix 3.B for notations and precise results on this extension.

A probabilistic estimate is then given, exploiting the independence of the Brownian motions; Lemma 3.3.6 shows that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |w_t^n(h)|^2\right] \leqslant \frac{C}{n}\|h\|_m^2, \quad h \in H^m.$$

This estimate will allow us to prove the convergence of (3.3.1) to (3.2.1) for every fixed $h \in H^m$, see Lemma 3.3.9.

Pathwise definition via rough paths theory for semigroup functionals

We start by observing that the noise term $w_t^n(h)$ in (3.3.2) is neither a stochastic convolution that could be treated using a maximal inequality in Hilbert spaces (e.g. [DPZ14, §6.4] and [Bec20] in the context of an unbounded diffusion operator), nor a classical controlled rough path integral (e.g. [FH14]) as the integrand depends on the upper integration limit.

We combine the strategies in [GT10, Gub04] so to define $w_t^n(h)$ in a pathwise sense. Note that our setting is different from [GT10], where an infinite dimensional theory à la Da Prato-Zabczyk is constructed, while we are interested in finite dimensional stochastic integrals over functionals of such objects. Our construction is nonetheless similar to [GT10]: we fix the Itô-rough path lift associated to Brownian motion and extend the algebraic integration in [GT10] to our setting of semigroup functionals. This extension is presented in detail in Appendix B, where the main ingredient, the Sewing lemma, is proven. Before stating Lemma 3.3.2, we present in a heuristic fashion the main ideas towards a rough path construction of (3.3.2). Let us remark in this context that a more general theory of nonlinear Volterra equations with singular kernels as recently put forward in [HT21] [HTW21] might alternatively be employed.

Note that it suffices to define integrals of the form

$$\int_s^t \nabla S_{t-u} f(x_u) \cdot \mathrm{d}B_u \tag{3.3.3}$$

in a pathwise sense for a class of sufficiently regular functions f and where $(x_u)_u$ is an $\mathbb{R}^d$-valued process controlled by the Brownian motion $(B_u)_u$, such that

$$x_t - x_s = B_t - B_s + O(|t-s|), \quad \text{for } s,t \in [0,T],\ \mathbb{P}\text{-a.s.}. \tag{3.3.4}$$

Recall that in the classical setting of rough paths theory, one has for $s \leqslant t$

$$\int_s^t f(x_u)\mathrm{d}B_u = f(x_s)B_{ts} + (D_x f)(x_u)\mathbb{B}_{ts} + R_{ts}$$

where we have used the notation $B_{ts} := B_t - B_s$ as well as

$$\mathbb{B}_{ts} := \int_s^t B_{us} \otimes \mathrm{d}B_u, \quad t \geqslant s \in [0,T].$$

In particular, $A_{ts} := f(x_s)B_{ts} + (D_x f)(x_u)\mathbb{B}_{ts}$ is a germ and, thanks to (3.3.4), $R_{ts} = o(|t-s|)$ is a remainder in the terminology of [Gub04]. In the same spirit of [Gub04], we rewrite the left hand side of this expression as

$$\int_s^t f(x_u)\mathrm{d}B_u = [\delta I]_{ts} = I_t - I_s$$

where

$$I_t = \int_0^t f(x_u)\mathrm{d}B_u.$$

We are thus left with

$$[\delta I]_{ts} = A_{ts} + R_{ts}. \tag{3.3.5}$$

Recall that Gubinelli's Sewing Lemma formulates precise conditions under which a given germ A gives rise to a unique remainder term $R_{ts} = o(|t-s|)$ and such that I can be obtained as

$$I_t := \lim_{|\mathcal{P}[0,t]|\downarrow 0} \sum_{[u,v]\in\mathcal{P}[0,t]} A_{vu}.$$

If one tries to follow a similar approach for the quantity of interest (3.3.3), a canonical candidate for local approximations to (3.3.3) would be

$$\int_s^t (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u = (\nabla S_{t-s} f)(x_s)B_{ts} + (D\nabla S_{t-s} f)(x_s)\mathbb{B}_{ts} + R_{ts}.$$

However, notice that if we were to set

$$I_t(f) := \int_0^t (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u$$

then, we would obtain

$$\begin{aligned}
&[\delta I(f)]_{ts} = I_t(f) - I_s(f) \\
&= \int_0^t (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u + \int_0^s (\nabla S_{s-u}(S_{t-s} - \mathrm{Id})f)(x_u)\mathrm{d}B_u - \int_0^s (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u \\
&= \int_s^t (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u + I_s((S_{t-s} - \mathrm{Id})f) \\
&\neq \int_s^t (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u,
\end{aligned}$$

in contrast to the above setting, meaning the standard approach of [Gub04] fails. If one defines, following Gubinelli and Tindel [GT10, p.16], the operator ϕ via

$$[\phi I(f)]_{ts} = I_s((S_{t-s} - \mathrm{Id})f)$$

as well as the operator $\hat{\delta}$ via

$$[\hat{\delta} I(f)]_{ts} = [\delta I(f)]_{ts} - [\phi I(f)]_{ts},$$

the desired relationship is recovered, indeed

$$\begin{aligned}
[\hat{\delta} I(f)]_{ts} &= \int_s^t (\nabla S_{t-u} f)(x_u)\mathrm{d}B_u \\
&= (\nabla S_{t-s} f)(x_s)B_{ts} + (D\nabla S_{t-s} f)(x_s)\mathbb{B}_{ts} + R_{ts}.
\end{aligned}$$

The idea is hence to change the cochain complex in [Gub04] and to consider a perturbed version of it associated to the operator $\hat{\delta}$, this is done in Lemma 3.B.1. Lemma 3.B.2

proves a Sewing Lemma in this modified setting, which in turn allows to construct the above remainder R_{ts}. The germ will therefore be

$$[Af]_{ts} = (\nabla S_{t-s} f)(x_s) B_{ts} + (D\nabla S_{t-s} f)(x_s) \mathbb{B}_{ts}.$$

For $0 = t_0 < \cdots < t_{n+1} = t$, note that due to

$$\begin{aligned} I_t(f) &= \int_0^t (\nabla S_{t-u} f)(x_u) \mathrm{d}B_u \\ &= \sum_{k=0}^n \int_{t_k}^{t_{k+1}} (\nabla S_{t-u} f)(x_u) \mathrm{d}B_u \\ &= \sum_{k=0}^n \int_{t_k}^{t_{k+1}} (\nabla S_{t_{k+1}-u}(S_{t-t_{k+1}} f))(x_u) \mathrm{d}B_u \\ &= \sum_{k=0}^n [A(S_{t-t_{k+1}} f)]_{t_{k+1}t_k} + \sum_{k=0}^n R_{t_{k+1}t_k}, \end{aligned}$$

the correct way of sewing together the germs is given by

$$I_t(f) = \lim_{n\to\infty} \sum_{k=0}^n [A(S_{t-t_{k+1}} f)]_{t_{k+1}t_k},$$

which is reflected in equation (3.B.3) in Corollary 3.B.3. In particular, note that this Corollary comes with the stability estimate (3.B.2) which allows to eventually deduce the first crucial estimate (3.3.6) on the noise term, as shown in the next Lemma.

Lemma 3.3.2. *Suppose $m > d/2 + 3$. For every $\alpha \in (1/3, 1/2)$, there exists a positive random constant $C = C_\alpha$ that is finite $\mathbb{P}$-a.s.(and of finite moments for all orders) such that $\mathbb{P}$-a.s.*

$$|w_t^n(h)| \leqslant C_\alpha (1+t)^{3\alpha} \|h\|_m \tag{3.3.6}$$

for any $t \geqslant 0$ and $h \in H^m$.

Proof. We follow the notations of Appendix 3.B. Fix $\alpha \in (1/3, 1/2)$ and recall that $(B, \mathbb{B})$ is the Itô rough path lift, with

$$\mathbb{B}_{ts} := \int_s^t B_{us} \otimes \mathrm{d}B_u, \quad s \leqslant t \in [0, T],$$

where $B_{us} := B_u - B_s$. Note that the above stochastic integral is understood in the Itô sense.

We use Lemma 3.B.3 to define the Itô integral (3.3.3). This in turn will imply the well-posedness of $w_t^n(h)$ with the choice $f = h$, $x = x^{i,n}$ and $B = B^i$ for $i = 1, \ldots, n$. Indeed, $x^{i,n}$ is controlled by B^i, as

$$(x_t^{i,n} - x_s^{i,n}) - (B_t^i - B_s^i) = \frac{1}{n} \sum_{j=1}^n \int_s^t \Gamma(x_u^{i,n}, x_u^{j,n}) du = O(|t-s|) = o(|t-s|^{2\alpha})$$

since Γ is bounded. We therefore have

$$|w_t^n(h)| \leqslant \frac{1}{n}\sum_{i=1}^n \left|\int_0^t (\nabla S_{t-s}h)(x_s^{i,n})\mathrm{d}B_s^i\right| =: \frac{1}{n}\sum_{i=1}^n p_t^{i,n}.$$

Hence bounding each $p^{i,n}$ individually as done below will allow us to conclude the claim of the Lemma by the strong law of large numbers.

Define the operator A acting on $f \in H^m$ into $C(\Delta_2, \mathbb{R})$ via

$$[Af]_{ts} := (\nabla S_{t-s}f)(x_s) \cdot B_{ts} + (D_x \nabla S_{t-s}f)(x_s) \cdot \mathbb{B}_{ts},$$

where D_x denotes the Jacobian in $\mathbb{R}^d$ and $\cdot$ the scalar product between tensors of the same dimension. In the sequel, we adopt the following shorter notation

$$[Af]_{ts} := \nabla S_{ts} f_s \, B_{ts} + D_x \nabla S_{ts} f_s \, \mathbb{B}_{ts}.$$

As in classical rough paths theory $[Af]_{ts}$ is not a 1-increment (i.e. a difference as B_{ts}) but a continuous function of the two variables s and t. In particular $A \in D_2$, i.e. A is a linear operator from the Banach space H^m to $C(\Delta_2, \mathbb{R})$.

One can actually prove that $A \in D_2^\alpha$: for $0 \leqslant s \leqslant t \leqslant T$ and $f \in H^m$

$$|[Af]_{ts}| \leqslant \|\nabla S_{ts} f\|_\infty |B_{ts}| + \|D_x \nabla S_{ts} f\|_\infty |\mathbb{B}_{ts}| \leqslant C_\alpha \|f\|_m |t-s|^\alpha,$$

where $C_\alpha = C_\alpha(\omega)$ depends on the α-Hölder norm of $B(\omega)$ and $\mathbb{B}(\omega)$ and we have used the properties of S, see Lemma 3.A.2. Note in particular that $C_\alpha < \infty$, $\mathbb{P}$-a.s. and that C_α has finite moments of all orders.

Recall the definition of $\hat{\delta}$ (Lemma 3.B.1), in order to apply Lemma 3.B.2 and Corollary 3.B.3 we need to show that $\hat{\delta}A \in D_3^{1+}$. Let $f \in H^m$ and $s < u < t$, one has

$$\begin{aligned}[\hat{\delta}Af]_{tus} &= [\delta Af]_{tus} - [\phi Af]_{tus} = [Af]_{ts} - [Af]_{tu} - [Af]_{us} - [A(S_{t\cdot} - \mathrm{Id})f]_{us} \\ &= [Af]_{ts} - [Af]_{tu} - [AS_{t\cdot}f]_{us}.\end{aligned}$$

Observe that thanks to the properties of the semigroup

$$[AS_{t\cdot}f]_{us} = \nabla S_{us} S_{tu} f_s \, B_{us} + D_x \nabla S_{us} S_{tu} f_s \, \mathbb{B}_{us} = \nabla S_{ts} f_s \, B_{us} + D_x \nabla S_{ts} f_s \, \mathbb{B}_{us}.$$

In particular, using Chen's relation

$$\mathbb{B}_{ts} = \mathbb{B}_{us} + \mathbb{B}_{tu} + B_{tu} \otimes B_{us}$$

we obtain

$$\begin{aligned}[\hat{\delta}Af]_{tus} &= \nabla S_{ts} f_s \, B_{tu} - \nabla S_{tu} f_u \, B_{tu} + D_x \nabla S_{ts} f_s \, (\mathbb{B}_{ts} - \mathbb{B}_{us}) - D_x \nabla S_{tu} f_u \, \mathbb{B}_{tu} \\ &= (\nabla S_{ts} f_s - \nabla S_{tu} f_u) \, B_{tu} + D_x (\nabla S_{ts} f_s - \nabla S_{tu} f_u) \, \mathbb{B}_{tu} + D_x S_{ts} f_s \, B_{tu} \otimes B_{us}.\end{aligned}$$

We rewrite everything as the sum of four terms

$$\begin{aligned}[\hat{\delta}Af]_{tus} &= \nabla (S_{ts} - S_{tu}) f_u \, B_{tu} + D_x \nabla (S_{ts} - S_{tu}) f_u \, \mathbb{B}_{tu} + \\ &\quad + (D_x \nabla S_{ts} f_s - D_x \nabla S_{ts} f_u) \, \mathbb{B}_{tu} + (\nabla S_{ts} f_s - \nabla S_{ts} f_u + D_x \nabla S_{ts} f_s B_{us}) \, B_{tu} \\ &=: A^1 + A^2 + A^3 + A^4.\end{aligned}$$

For A^1 we obtain

$$|\nabla(S_{ts} - S_{tu}) f_u \, B_{tu}| \leqslant \|\nabla(S_{ts} - S_{tu}) f_u\|_\infty \, |B_{tu}| \leqslant C_\alpha \, \|f\|_m \, |t-u|^\alpha \, |u-s| \, ,$$

where $C_\alpha = C_\alpha(\omega)$ depends on the α-Hölder norm of $B(\omega)$ and we have used the properties of S, see Lemma 3.A.2. Note in particular that $C_\alpha < \infty$, $\mathbb{P}$-a.s. and that C_α has finite moments of all orders. Similarly, for A^2 (with a different C_α)

$$|D_x \nabla(S_{ts} - S_{tu}) f_u \, \mathbb{B}_{tu}| \leqslant C_\alpha \, \|f\|_m \, |t-u|^{2\alpha} \, |u-s|^{1/2} \, .$$

Observe now that, since $f \in C_b^3$, the function $D_x \nabla S_{ts} f$ is Lipschitz uniformly in s and t, from which we extract that

$$\begin{aligned} |(D_x \nabla S_{ts} f_s - D_x \nabla S_{ts} f_u) \, \mathbb{B}_{tu}| &\leqslant C_\alpha \, \|f\|_m \, |x_s - x_u| \, |t-u|^{2\alpha} \\ &\leqslant C_\alpha \, \|f\|_m \, |t-u|^{2\alpha} \, |u-s|^\alpha \, . \end{aligned}$$

Using (3.3.4), we recognize in A^4 the Taylor expansion of $\nabla S_{ts} f$ around x_s, i.e.

$$\begin{aligned} &|\nabla S_{ts} f_u - \nabla S_{ts} f_u - D_x \nabla S_{ts} f_s \, B_{us}| \\ &\leqslant |\nabla S_{ts} f_u - \nabla S_{ts} f_u - D_x \nabla S_{ts} f_s \, x_{us}| + c \, |D_x \nabla S_{tu} f_s| \, |u-s| \\ &\leqslant c \, \|f\|_m \, |x_{us}|^2 + c \, \|f\|_m \, |u-s| \leqslant c \, \|f\|_m \, |u-s|^{2\alpha} \, . \end{aligned}$$

We conclude that

$$\left|A^4\right| \leqslant C_\alpha \, \|f\|_m \, |t-u|^\alpha \, |u-s|^{2\alpha} \, .$$

Putting the four estimates together, we have just shown $\hat{\delta} A \in D_3^{1+}$ and, in particular, that

$$\left\|\hat{\delta} A\right\|_{D_3^{3\alpha}} \leqslant C_\alpha$$

for some C_α which is finite $\mathbb{P}$-a.s. and admits moments of all orders. As $\alpha > 1/3$, by Corollary 3.B.3, we know that there exists $I \in D_1$ such that

$$[\hat{\delta} I f]_{ts} = \lim_{|\mathcal{P}^n[s,t]| \to 0} \sum_{[u,v] \in \mathcal{P}^n[s,t]} [A S_{t\cdot} f]_{vu}$$

is well defined. For $0 \leqslant s \leqslant t \leqslant T$, se set

$$\int_s^t \nabla S_{t-u} f(x_u) \cdot \mathrm{d}B_u := [\hat{\delta} I f]_{ts}.$$

Again Corollary 3.B.3 assures that there exists a (new) constant C_α, depending on the norm of A in D_2^α and the norm of $\hat{\delta} A$ in $D_3^{3\alpha}$, such that

$$\left| \int_0^t \nabla S_{t-u} f(x_u) \cdot \mathrm{d}B_u \right| \leqslant C_\alpha \, \|f\|_m \, (1+t)^{3\alpha}.$$

The proof is concluded. □

Controlling $w_t^n(h)$ via a maximal inequality for self-normalized processes

The aim of this subsection is to give a probabilistic bound on

$$w_t^n(h) = \frac{1}{n}\sum_{j=1}^{n}\int_0^t [\nabla S_{t-s}h]\,(x_s^{j,n})\mathrm{d}B_s^j$$

by exploiting the independence of the Brownian motions (we have removed the product symbol $\cdot$ for the sake of notation).

Observe that if $w_t^n(h)$ didn't involve a convolution with the semigroup S, $w_t^n(h)$ would be a standard martingale and classical estimates like the Burkholder-Davis-Gundy inequality could be used to establish the desired bound. While the convolution with the semigroup S destroys the martingale property, $w_t^n(h)$ is still closely related to maximal inequalities for self-normalized martingales for which the following fine estimate due to Graversen and Peskir [GP00] is available.

Lemma 3.3.3 ([GP00, Corollary 2.8] and [JZ20, Corollary 2.4])**.** *Let $(M_t)_{t\in[0,T]}$ be a continuous local martingale. There exists a universal constant C such that*

$$\mathbb{E}\left[\sup_{t\in[0,\tau]}\frac{|M_t|^2}{1+\langle M\rangle_t}\right] \leqslant C\,\mathbb{E}\left[\log(1+\log(1+\langle M\rangle_\tau))\right]$$

for every stopping time $\tau \leqslant T$.

Observe that this result is a consequence of more general bounds on self-normalized processes of the form $X_t = A_t/B_t$ (e.g. [dlPKL04]), where in this case $A_t = M_t$ is a martingale and $B_t^2 - 1 = \langle M\rangle_t$ its quadratic variation.

Let us illustrate in the following example how this interpretation can be used to directly obtain a bound on

$$v_t = \frac{1}{n}\sum_{j=1}^{n}\int_0^t e^{-a(t-s)}\mathrm{d}B_s^j, \quad a>0,$$

which could be seen as a most simple toy model for $w_t^n(h)$.

Example 3.3.4. *Let $(B^j)_{j\leqslant n}$ be independent Brownian motions on a common filtered probability space $(\Omega,\mathcal{F},(\mathcal{F})_t)_t,\mathbb{P})$. For $a>0$, let $(X^j)_{j\leqslant n}$ be the following associated familiy of Ornstein Uhlenbeck processes:*

$$X_t^j := \int_0^t e^{-a(t-s)}\mathrm{d}B_s^j, \quad t\in[0,T]$$

and consider the quantity

$$v_t := \frac{1}{n}\sum_{j=1}^{n} X_t^j.$$

We remark that we may rewrite

$$\sum_{j=1}^{n} X_t^j = \sqrt{\frac{n}{2a}} e^{-at} \left(\sum_{j=1}^{n} \sqrt{\frac{2a}{n}} \int_0^t e^{as} \mathrm{d}B_s^j \right) =: \sqrt{\frac{n}{2a}} e^{-at} M_t.$$

Notice that M is a martingale of quadratic variation

$$\langle M \rangle_t = (e^{2at} - 1)$$

and therefore, by Lemma 3.3.3, we conclude that

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} |v_t|^2 \right] &= \frac{1}{2na} \mathbb{E}\left[\sup_{t\in[0,T]} |e^{-at} M_t|^2 \right] \\ &= \frac{1}{2na} \mathbb{E}\left[\sup_{t\in[0,T]} \frac{|M_t|^2}{1 + \langle M \rangle_t} \right] \\ &\leqslant C \frac{1}{2na} \log(1 + 2aT). \end{aligned}$$

Note that we crucially exploited the splitting $e^{-a(t-s)} = e^{-at} e^{as}$, which is not available in the semigroup setting we are concerned with[2]. Intending to employ such a step suggests to pass by a functional calculus for the semigroup, which we briefly discuss next.

Recall that an analytic semigroup is a bounded linear operator that can be expressed by means of a Dunford integral (e.g. [Hen81, Lun95] and Appendix 3.A). The integral representation of S is given for every $t \in [0, T]$ by

$$S_t = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, \tfrac{\Delta}{2}) \mathrm{d}\lambda, \tag{3.3.7}$$

where $R(\lambda, \frac{\Delta}{2}) = (\lambda \mathrm{Id} - \frac{\Delta}{2})^{-1}$ denotes the resolvent of $\frac{\Delta}{2}$ and where, for $r > 0$ and $\eta \in (\pi/2, \pi)$, $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg \lambda| = \eta, |\lambda| \geqslant r\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| \leqslant \eta, |\lambda| = r\}$, oriented counterclockwise.

Plugging (3.3.7) into the expression of $w_t^n(h)$ yields

$$w_t^n(h) = \frac{1}{2\pi i n} \sum_{j=1}^{n} \int_0^t \int_{\gamma_{r,\eta}} e^{(t-s)\lambda} \left[\nabla R(\lambda, \tfrac{\Delta}{2}) h \right] (x_s^{j,n}) \mathrm{d}\lambda \mathrm{d}B_s^j,$$

splitting the complex integral into three real integrals parametrizing $\gamma_{r,\eta}$, and then using stochastic Fubini, one is left with expressions similar to

$$\frac{1}{2\pi i n} \sum_{j=1}^{n} \int_0^t e^{(t-s)\rho e^{i\eta}} \left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2}) h \right] (x_s^{j,n}) e^{i\eta} \mathrm{d}B_s^j, \quad \rho > r,$$

[2]Provided $-A$ generates a semigroup S and admits some functional calculus, one has $S_t = e^{-tA}$. Yet, in this case "e^{sA}" will be ill defined. Indeed, note that the semigroup property is not a group property, as would be required in the above splitting.

which remind us of 1-dimensional self-normalized martingale for every ρ, similar to the process $(v_t)_t$ considered in Example 3.3.4.

It remains to establish a suitable bound on the expression

$$\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})$$

and to ensure that this bound is integrable for $\rho \in (r, \infty)$, see Lemma 3.A.4.

Putting all the above considerations together with care, one obtains a maximal inequality for $w_t^n(h)$ that we present in Lemma 3.3.6.

Remark 3.3.5. *A similar control has already been used in [Cop19, Lemma 3.3], see also [BGP14, §3.1] and [LP17, §4] for an estimate using the Rodemich-Garsia-Rumsey lemma. However, in all these cases the particles are living in the one dimensional torus, making the (still highly technical) noise analysis considerably simpler due to the decomposition in Fourier series.*

Lemma 3.3.6. *Assume $m > d/2$. There exists a constant $C \geqslant 1$, independent of n and $h \in H^m$, such that for every $h \in H^m$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |w_t^n(h)|^2\right] \leqslant \frac{C}{n}\|h\|_m^2 . \tag{3.3.8}$$

Proof. Let $h \in H^m$ and $\gamma_{r,\eta}$ be the curve in (3.3.7) with $\eta \in (\pi/2, \pi)$ and $r > 0$. Since the real values of η and r are not crucial for the proof, we may suppose $r > 1$. Using the decomposition of S we obtain:

$$\begin{aligned}
w_t^n(h) &= \frac{1}{n}\sum_{j=1}^n \int_0^t [\nabla S_{t-s}h](x_s^{j,n})\mathrm{d}B_s^j \\
&= \frac{1}{2\pi i n}\sum_{j=1}^n \int_0^t \left[\nabla \int_{\gamma_{r,\eta}} e^{(t-s)\lambda} R(\lambda, \tfrac{\Delta}{2})h\mathrm{d}\lambda\right](x_s^{j,n})\mathrm{d}B_s^j \\
&= \frac{1}{2\pi i n}\sum_{j=1}^n \int_0^t \int_{\gamma_{r,\eta}} e^{(t-s)\lambda}\left[\nabla R(\lambda, \tfrac{\Delta}{2})h\right](x_s^{j,n})\mathrm{d}\lambda\mathrm{d}B_s^j \\
&= Z_t^1(h) + Z_t^2(h) + Z_t^3(h),
\end{aligned}$$

for

$$\begin{aligned}
Z_t^1(h) &:= \frac{1}{2\pi i n}\sum_{j=1}^n \int_0^t \int_r^\infty e^{(t-s)\rho e^{i\eta}}\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})e^{i\eta}\mathrm{d}\rho\mathrm{d}B_s^j, \\
Z_t^2(h) &:= \frac{1}{2\pi i n}\sum_{j=1}^n \int_0^t \int_{-\eta}^\eta e^{(t-s)re^{i\alpha}}\left[\nabla R(re^{i\alpha}, \tfrac{\Delta}{2})h\right](x_s^{j,n})ire^{i\alpha}\mathrm{d}\alpha\mathrm{d}B_s^j, \\
Z_t^3(h) &:= -\frac{1}{2\pi i n}\sum_{j=1}^n \int_0^t \int_r^\infty e^{(t-s)\rho e^{-i\eta}}\left[\nabla R(\rho e^{-i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})e^{-i\eta}\mathrm{d}\rho\mathrm{d}B_s^j.
\end{aligned} \tag{3.3.9}$$

where in the third step we have used that ∇ is a closed linear operator on $D(\frac{\Delta}{2})$ and may therefore be drawn into the above Dunford integral. Using the classical estimate $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, it follows that

$$|w_t^n(h)|^2 \leqslant 3\left[\left|Z_t^1(h)\right|^2 + \left|Z_t^2(h)\right|^2 + \left|Z_t^3(h)\right|^2\right].$$

We focus on $Z_t^1(h)$, but similar estimates for $Z_t^2(h)$ and $Z_t^2(h)$ follow in exactly the same way.

Fix $\epsilon > 0$ small, the stochastic Fubini theorem (e.g. [DPZ14, §4.5]) and Cauchy-Schwartz inequality imply that

$$\begin{aligned}
\left|Z_t^1(h)\right|^2 &= \left|\int_r^\infty \left[\int_0^t \frac{\rho^{\frac{1+\epsilon}{2}}}{2\pi i n}\sum_{j=1}^n e^{(t-s)\rho e^{i\eta}}\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})e^{i\eta}\mathrm{d}B_s^j\right]\frac{\mathrm{d}\rho}{\rho^{\frac{1+\epsilon}{2}}}\right|^2 \\
&\leqslant C\int_r^\infty \left|\int_0^t \frac{1}{n}\sum_{j=1}^n e^{(t-s)\rho e^{i\eta}}\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})e^{i\eta}\mathrm{d}B_s^j\right|^2 \rho^{1+\epsilon}\mathrm{d}\rho \\
&= \frac{C}{n^2}\int_r^\infty e^{-2t\rho(-\cos\eta)}\left|\underbrace{\int_0^t \sum_{j=1}^n e^{-s\rho e^{i\eta}}\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})\mathrm{d}B_s^j}_{=:M_t}\right|^2 \rho^{1+\epsilon}\mathrm{d}\rho
\end{aligned}$$

where $C = \frac{1}{4\pi^2}\int_r^\infty \frac{\mathrm{d}\rho}{\rho^{1+\epsilon}}$.

We introduce the continuous martingale $X_\cdot^{\epsilon,\rho}(h)$ defined for $t \geqslant 0$ by

$$X_t^{\epsilon,\rho}(h) := \rho^{1/2+\epsilon}\sqrt{\frac{-2\rho\cos\eta}{\|h\|_m^2\, n}}M_t,$$

so to obtain

$$\begin{aligned}
\left|Z_t^1(h)\right|^2 &\leqslant \frac{C}{-2n\cos\eta}\|h\|_m^2\int_r^\infty e^{2t\rho\cos\eta}\left|X_t^{\epsilon,\rho}(h)\right|^2\frac{\mathrm{d}\rho}{\rho^{1+\epsilon}} \\
&\leqslant \frac{C}{n}\|h\|_m^2\int_r^\infty e^{2t\rho\cos\eta}\left|X_t^{\epsilon,\rho}(h)\right|^2\frac{\mathrm{d}\rho}{\rho^{1+\epsilon}}.
\end{aligned}$$

where we absorbed the factor $(-2\cos\eta)^{-1}$ in the unessential constant C.

We compute the quadratic variation of $X_t^{\epsilon,\rho}(h)$:

$$\langle X^{\epsilon,\rho}(h)\rangle_t = \rho^{1+2\epsilon}\frac{(-2\rho\cos\eta)}{\|h\|_m^2\, n}\sum_{j=1}^n\int_0^t e^{-2s\rho\cos\eta}\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right]^2(x_s^{j,n})\mathrm{d}s.$$

Lemma 3.A.3 assures that for every ϵ such that $0 < 2\epsilon < (m-d/2)\wedge 1$, $\mathcal{P}(\mathbb{R}^d)$ is

continuously embedded in $H^{-m+2\epsilon}$, in particular

$$\begin{aligned}\left|\left[\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right](x_s^{j,n})\right| &= \left|\langle \delta_{x_s^{j,n}}\,,\, \nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\rangle_{-m,m}\right| \\ &= \left|\langle \delta_{x_s^{j,n}}, \nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\rangle_{-m+2\epsilon,m-2\epsilon}\right| \\ &\leqslant \left\|\delta_{x_s^{j,n}}\right\|_{-m+2\epsilon} \left\|\nabla R(\rho e^{i\eta}, \tfrac{\Delta}{2})h\right\|_{m-2\epsilon} \leqslant C\frac{\|h\|_m}{\rho^{1/2+\epsilon}},\end{aligned} \tag{3.3.10}$$

where we have exploited the properties of the resolvent operator R, see Lemma 3.A.4.

Thus, the quadratic variation of $X_t^{\epsilon,\rho}(h)$ is bounded $\mathbb{P}$-a.s. by

$$\langle X^{\epsilon,\rho}(h)\rangle_t \leqslant C(-2\rho\cos\eta)\int_0^t e^{-2s\rho\cos\eta}\mathrm{d}s = C\left(e^{-2t\rho\cos\eta}-1\right). \tag{3.3.11}$$

Observe then

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|Z_t^1(h)\right|^2\right] &\leqslant \frac{C}{n}\|h\|_m^2\int_r^\infty \mathbb{E}\left[\sup_{t\in[0,T]} e^{2t\rho\cos\eta}\,|X_t^{\epsilon,\rho}(h)|^2\right]\frac{\mathrm{d}\rho}{\rho^{1+\epsilon}} \\ \leqslant \frac{C}{n}\|h\|_m^2\int_r^\infty \mathbb{E}&\left[\sup_{t\in[0,T]}\frac{|X_t^{\epsilon,\rho}(h)|^2}{1+\langle X^{\epsilon,\rho}(h)\rangle_t}\left(\sup_{t\in[0,T]}\frac{1+\langle X^{\epsilon,\rho}(h)\rangle_t}{e^{-2t\rho\cos\eta}}\right)\right]\frac{\mathrm{d}\rho}{\rho^{1+\epsilon}}.\end{aligned}$$

The term $\sup_{t\in[0,T]}\frac{1+\langle X^{\epsilon,\rho}(h)\rangle_t}{e^{-2t\rho\cos\eta}}$ is bounded using (3.3.11) by a constant, wherefore we are left with

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|Z_t^1(h)\right|^2\right] \leqslant \frac{C}{n}\|h\|_m^2\int_r^\infty \mathbb{E}\left[\sup_{t\in[0,T]}\frac{|X_t^{\epsilon,\rho}(h)|^2}{1+\langle X^{\epsilon,\rho}(h)\rangle_t}\right]\frac{\mathrm{d}\rho}{\rho^{1+\epsilon}}.$$

We now invoke Lemma 3.3.3, which in conjunction with (3.3.11) allows to deduce that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\frac{|X_t^{\epsilon,\rho}(h)|^2}{1+\langle X^{\epsilon,\rho}(h)\rangle_t}\right] &\leqslant C\,\mathbb{E}\left[\log\left(1+\log\left(1+\langle X^{\epsilon,\rho}(h)\rangle_T\right)\right)\right] \\ &\leqslant C\mathbb{E}\left[\log\left(1-2T\rho\cos\eta+\log(C)\right)\right]\end{aligned}$$

where in the last inequality, we have bounded the constant C appearing in (3.3.11) by $\max\{1, C\}$. Further modifying C accordingly, we are thus left with

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|Z_t^1(h)\right|^2\right] \leqslant \frac{C}{n}\|h\|_m^2\int_r^\infty \log(1-2T\rho\cos\eta+\log(C))\frac{\mathrm{d}\rho}{\rho^{1+\epsilon}} \leqslant \frac{C}{n}\|h\|_m^2.$$

Concerning $Z_t^3(h)$, computations are the same if one replaces η by $-\eta$. Concerning $Z_t^2(h)$, computations are easier since there is no a priori diverging integral to deal with and we omit the proof. The overall bound on $w_t^n(h)$ is thus obtained by summing the three estimates and choosing the constant C accordingly. □

Remark 3.3.7. *Note that Lemma 3.3.6 implies by Jensen's inequality the following bound*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |w_t^n(h)|\right] \leqslant \frac{C}{\sqrt{n}} \|h\|_m ,$$

which is sharper in n with respect to (3.3.6)*, but in a weaker topology. One could ask if it is possible to establish a similar $O(1/\sqrt{n})$ bound for*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|w_t^n\|_{-m}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} \sup_{\|h\|_m \leqslant 1} |w_t^n(h)|\right].$$

Such a bound cannot be obtained by rough paths theory and a full probabilistic proof, which takes the independence between the Brownian motions into account, is desirable. To the authors' knowledge, this has been established only in the case of interacting oscillators; we refer to the noise term analysis in [BGP14, Cop19, LP17].

3.3.3 Proving Theorem 3.2.3

The proof of Theorem 3.2.3 consists in two steps: using the pathwise bound on w^n, Lemma 3.3.8 shows that we can extract from $(\nu^n)_{n\in\mathbb{N}}$ a weak-*-convergence subsequence; then, by exploiting the probabilistic bound on $w_t^n(h)$ for a fixed $h \in H^m$, we identify through Lemma 3.3.9 the limit with a solution to (3.2.1).

Extraction of a weak-*-convergent subsequence

The main result of this subsection is given by the next lemma.

Lemma 3.3.8. *The sequence $(\nu^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T], H^{-m})$ and thus admits a subsequence that converges weak-* to some $\nu \in H^{-m}$, $\mathbb{P}$-a.s..*

Proof. It suffices to show that $(\nu^n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,T], H^{-m})$ $\mathbb{P}$-a.s., an application of Banach-Alaoglu yields the existence of a convergent subsequence.

Exploiting the mild formulation in Lemma 3.3.1 and the bound on $w_t^n(h)$ in Lemma 3.3.2 for some $\alpha \in (1/3, 1/2)$, one obtains that

$$\begin{aligned}\|\nu_t^n\|_{-m} &\leqslant \|\nu_0^n\|_{-m} + \int_0^t \|\nu_s^n\|_{-m} \sup_{\|h\|_m \leqslant 1} \|(\nabla S_{t-s}h)(\Gamma * \nu_s^n)\|_m \,\mathrm{d}s + \|w_t^n\|_{-m} \\ &\leqslant \|\nu_0^n\|_{-m} + \int_0^t \frac{C}{\sqrt{t-s}} \|\nu_s^n\|_{-m} \,\mathrm{d}s + C_\alpha (1+t)^{3\alpha},\end{aligned}$$

where we have exploited the properties of the semigroup and the bound already used in (3.2.3). A Gronwall-like argument implies the existence of a constant a independent of n and T such that

$$\sup_{t\in[0,T]} \|\nu_t^n\|_{-m} \leqslant 2\left(\|\nu_0^n\|_{-m} + C_\alpha (1+T)^{3\alpha}\right)\sqrt{T} e^{a\sqrt{T}}.$$

In particular, using Lemma 3.A.3, we conclude

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\nu_t^n\|_{-m} \leqslant C_{\alpha,T}.$$

□

We move to the identification of the limit $\nu \in L^\infty([0,T], H^{-m})$.

The limit coincides with an m-weak-mild solution

We prove that any possible limit of $(\nu^n)_{n\in\mathbb{N}}$ is a weak-mild solution (3.2.1). Given the uniqueness of (3.2.1), this implies the weak-* convergence in $L^\infty([0,T], H^{-m})$ of $(\nu^n)_{n\in\mathbb{N}}$ to the element ν given in Lemma 3.3.8.

Lemma 3.3.9. *Let $(\nu^n)_{n\in\mathbb{N}}$ be converging weak-* to some $\bar\nu \in L^\infty([0,T], H^{-m})$ $\mathbb{P}$-a.s. along a subsequence that we denote by $(\nu^{n_k})_{k\in\mathbb{N}}$. Then $\bar\nu$ satisfies* (3.2.1)*, i.e.*

$$\langle \bar\nu_t, h\rangle_{-m,m} = \langle \bar\nu_0, S_t h\rangle_{-m,m} + \int_0^t \langle \bar\nu_s, (\nabla S_{t-s}h)(\Gamma * \bar\nu_s)\rangle_{-m,m} \mathrm{d}s,$$

meaning $\bar\nu$ is an m-weak-mild solution to (3.1.3)*.*

Proof. Recall that for every n, ν^n solves the mild formulation (3.3.1), i.e. for $t \in [0,T]$

$$\langle \nu_t^n, h\rangle_{-m,m} = \langle \nu_0^n, S_t h\rangle_{-m,m} + \int_0^t \langle \nu_s^n, (\nabla S_{t-s}h)(\Gamma * \nu_s^n)\rangle_{-m,m} \mathrm{d}s + w_t^n(h).$$

By hypothesis we have that for every $t \in [0,T]$ and $h \in H^m$

$$\lim_{k\to\infty} \langle \nu_t^{n_k}, h\rangle_{-m,m} = \langle \bar\nu_t, h\rangle_{-m,m}, \quad \mathbb{P}\text{-a.s..}$$

In particular, this is true for $(\nu_0^{n_k})_k$ since $S_t h \in H^m$. Furthermore, Lemma 3.3.6 implies that

$$\lim_{k\to\infty} w_t^{n_k}(h) = 0, \quad \text{in } \mathbb{P}\text{-probability}$$

and thus in particular the convergence holds $\mathbb{P}$-a.s. along a sub-subsequence $(n_{k_j})_j$. Thus, it remains to show that $\mathbb{P}$-a.s.

$$\lim_{j\to\infty} \int_0^t \langle \nu_s^{n_{k_j}}, (\nabla S_{t-s}h)(\Gamma * \nu_s^{n_{k_j}})\rangle_{-m,m} \mathrm{d}s = \int_0^t \langle \bar\nu_s, (\nabla S_{t-s}h)(\Gamma * \bar\nu_s)\rangle_{-m,m} \mathrm{d}s. \quad (3.3.12)$$

For better readability and lighter notation, we will not distinguish between n and n_{k_j} in the following, understanding that we continue to work on the sub-subsequence. Consider then

$$\begin{aligned}
&\langle \bar\nu_s^n, (\nabla S_{t-s}h)(\Gamma * \bar\nu_s^n)\rangle_{-m,m} - \langle \bar\nu_s, (\nabla S_{t-s}h)(\Gamma * \bar\nu_s)\rangle_{-m,m} \\
&= \langle \bar\nu_s^n - \bar\nu_s, (\nabla S_{t-s}h)(\Gamma * \bar\nu_s)\rangle_{-m,m} + \langle \bar\nu_s^n, (\nabla S_{t-s}h)(\Gamma * (\bar\nu_s^n - \bar\nu_s))\rangle_{-m,m}.
\end{aligned}$$

Using again (3.2.3), it is easy to see that $\mathbb{P}$-a.s.

$$\mathbf{1}_{[0,t]}(s)(\nabla S_{t-s}h)(\Gamma * \nu_s) \in L^1([0,T], H^m)$$

wherefore it is indeed an element of the predual to $L^\infty([0,T], H^{-m})$ and thus by weak-* convergence in this space, we have

$$\int_0^T \langle \nu_s^n - \nu_s, \mathbf{1}_{[0,t]}(s)(\nabla S_{t-s}h)(\Gamma * \nu_s)\rangle \mathrm{d}s = \int_0^t \langle \nu_s^n - \nu_s, (\nabla S_{t-s}h)(\Gamma * \nu_s)\rangle \mathrm{d}s \to 0.$$

For the second term, note that

$$\begin{aligned}&\langle \bar{\nu}_s^n, (\nabla S_{t-s}h)(\Gamma * (\bar{\nu}_s^n - \bar{\nu}_s))\rangle_{-m,m}\\ &= \langle \bar{\nu}_s^n - \bar{\nu}_s, \langle \bar{\nu}_s^n(\mathrm{d}x), (\nabla S_{t-s}h)(x)(\Gamma(x,\cdot))\rangle_{-m,m}\rangle_{-m,m}\end{aligned}$$

and that the function

$$\begin{aligned}y \mapsto \langle \bar{\nu}_s^n(\mathrm{d}x), (\nabla S_{t-s}h)(x)(\Gamma(x,y))\rangle_{-m,m} &=\\ = \frac{1}{n}\sum_{j=1}^n (\nabla S_{t-s}h(x_s^{j,n})) \cdot (\Gamma(x_s^{j,n}, y))&\end{aligned}$$

is in H^m for every $s \in [0,t]$, since $\Gamma(x,\cdot) \in H^m$ for a.e. $x \in \mathbb{R}^d$, recall (3.2.2). Namely,

$$\begin{aligned}\left\| \frac{1}{n}\sum_{j=1}^n (\nabla S_{t-s}h(x_s^{j,n})) \cdot (\Gamma(x_s^{j,n}, \cdot)) \right\|_m &\leqslant \|\nabla S_{t-s}h\|_\infty \|\Gamma\|_{H_y^m L_x^\infty}\\ &\leqslant C\frac{\|h\|_m}{\sqrt{t-s}} \|\Gamma\|_{H_y^m W_x^{m,\infty}}.\end{aligned}$$

We conclude that

$$\mathbf{1}_{[0,t]}\langle \bar{\nu}_s^n(\mathrm{d}x), (\nabla S_{t-s}h)(x)(\Gamma(x,\cdot))\rangle_{-m,m} \in L^1([0,T], H^m)$$

and in particular

$$\int_0^T \langle \bar{\nu}_s^n - \bar{\nu}_s, \langle \bar{\nu}_s^n(\mathrm{d}x), \mathbf{1}_{[0,t]}(s)\,(\nabla S_{t-s}h)(x) \cdot (\Gamma(x,\cdot))\rangle_{-m,m}\rangle_{-m,m} \mathrm{d}s \to 0.$$

This establishes (3.3.12).

Overall, we have thus shown that any subsequence of $(\nu^n)_n$ converges along some further subsequence $\mathbb{P}$-a.s. weak-* in $L^\infty([0,T], H^{-m})$, the limit $\bar{\nu}$ satisfying for every $h \in H^m$ the equation

$$\langle \bar{\nu}_t, h\rangle_{-m,m} = \langle \bar{\nu}_0, S_t h\rangle_{-m,m} + \int_0^t \langle \bar{\nu}_s, (\nabla S_{t-s}h)(\Gamma * \bar{\nu}_s)\rangle_{-m,m} \mathrm{d}s,$$

meaning that $\bar{\nu}$ is indeed an m-weak-mild solution. □

Proof of Theorem 3.2.3

Proof. In order to show that $\nu^n \overset{*}{\rightharpoonup} \nu$ in $L^\infty([0,T], H^{-m})$ in probability, we show that any subsequence $(\nu^{n_k})_k$ admits a further subsequence that converges $\mathbb{P}$-a.s. in weak-* topology of $L^\infty([0,T], H^{-m})$ to ν.

Let $(\nu^{n_k})_k$ be hence a subsequence. By assumption of the Theorem, Lemmas 3.3.6 and 3.3.8, we find a further subsequence $(\nu^{n_{k_j}})_j$, along which

$$\begin{aligned} w_t^{n_{k_j}}(h) &\to 0 \quad \forall h \in H^m \\ \langle \nu_0^{n_{k_j}}, h\rangle &\to \langle \nu_0, h\rangle \quad \forall h \in H^m \\ \nu^{n_{k_j}} &\overset{*}{\rightharpoonup} \bar{\nu} \quad \text{in } L^\infty([0,T], H^{-m}) \end{aligned} \tag{3.3.13}$$

$\mathbb{P}$-a.s., where the limit $\bar{\nu} \in L^\infty([0,T], H^{-m})$ may apriori depend on the subsequence chosen. Notice however that due to Lemma 3.3.9, any such limit is a m-weak-mild solution to (3.1.3). By the uniqueness result of Proposition 3.2.2, we conclude that the limit $\bar{\nu} = \nu$ must be the same for any subsequence chosen.

The first part of the Theorem is proved. Note that apriori, our limit ν is only a distribution in H^{-m} at each fixed timepoint.

Suppose $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$. In order to show that ν_t is actually a probability measure for each $t \in [0,T]$, we observe that a weak solution $\mu \in C([0,T], \mathcal{P}(\mathbb{R}^d))$ to (3.1.3) (which exists due to Theorem 3.1.2) is a weak-mild solution (3.2.1).

Indeed, let $\mu = (\mu_t)_{t\in[0,T]} \in C([0,T], \mathcal{P}(\mathbb{R}^d))$ be a weak solution to (3.1.3). As done in Lemma 3.3.1, one can show that for every $f \in C_0^\infty$ and $t \in [0,T]$

$$\langle \mu_t, f\rangle_{-m,m} = \langle \mu_0, S_t f\rangle_{-m,m} + \int_0^t \langle \mu_s, (\nabla S_{t-s} f) \cdot (\Gamma * \mu_s)\rangle_{-m,m} \mathrm{d}s \tag{3.3.14}$$

holds. Note that by standard approximation, (3.3.14) holds also for $f \in H^m \subset C_b^3$, meaning that μ is indeed a weak-mild solution. By the uniqueness statement of Proposition 3.2.2 we conclude $\mu = \nu$ and thus in particular $\nu \in C([0,T], \mathcal{P}(\mathbb{R}^d))$. This concludes the second part and thus the entire proof of the Theorem. □

3.A Hilbert spaces and Semigroups

The Laplacian semigroup

The following definitions are taken from [Hen81, Lun95]. For the sake of notation, we focus on Δ, the standard Laplacian on $L^2(\mathbb{R}^d)$, instead of $\frac{\Delta}{2}$. We can consider the part of Δ on (the complexification e.g. [Lun95, Appendix A] of) H^m:

$$\Delta : \mathcal{D}(\Delta) \subset H^m \longrightarrow H^m.$$

It is not difficult to see that $H^{m+2} \subset \mathcal{D}(\Delta)$, where the inclusion is dense, and that Δ is a sectorial operator with spectrum given by $(-\infty, 0]$. In particular, it generates an

analytic strongly continuous semigroup denoted for all $t \geqslant 0$ by S_t; recall that $S_0 := \mathrm{Id}$ is the identity operator.

We represent S for $t \in [0, T]$ as the following Dunford integral

$$S_t = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, \Delta) \mathrm{d}\lambda,$$

where $R(\lambda, \Delta) = (\lambda \mathrm{Id} - \Delta)^{-1}$ denotes the resolvent of Δ and where, for $r > 0$ and $\eta \in (\pi/2, \pi)$, $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg \lambda| = \eta, |\lambda| \geqslant r\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| \leqslant \eta, |\lambda| = r\}$, oriented counterclockwise.

Observe that $\gamma_{r,\eta}$ is contained in the resolvent set of Δ, i.e. $\gamma_{r,\eta} \subset \rho(\Delta)$, and that, for all regular values $\lambda \in \rho(\Delta)$, $R(\lambda, \Delta)$ is a bounded linear operator on H^m.

When computing the semigroup against a function h through (3.3.7), we use the following decomposition into three real integrals:

$$\begin{aligned} S_t h \;=\; & \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, \Delta) h \mathrm{d}\lambda = \frac{1}{2\pi i} \Bigg[\int_r^\infty e^{t\rho e^{i\eta}} R(\rho e^{i\eta}, \Delta) e^{i\eta} \mathrm{d}\rho \\ & + \int_{-\eta}^{\eta} e^{tr(\cos\alpha + i \sin\alpha)} R(re^{i\alpha}, \Delta) ire^{i\alpha} \mathrm{d}\alpha - \int_r^\infty e^{t\rho e^{-i\eta}} R(\rho e^{-i\eta}, \Delta) e^{-i\eta} \mathrm{d}\rho \Bigg]. \end{aligned} \tag{3.A.1}$$

The section ends with some estimates basic concerning the regularity of S.

Lemma 3.A.1. *Let $(S_t)_{t \leqslant T}$ be the heat semigroup acting on H^m. Then for $\alpha \geqslant 0$ and $t > 0$ one has*

$$\|(-\Delta)^\alpha S_t h\|_m \leqslant C \frac{1}{t^\alpha} \|h\|_m .$$

In particular, we also have

$$\|\nabla S_t h\|_m \leqslant C \frac{1}{\sqrt{t}} \|h\|_m$$

Proof. Refer to [Paz83, Chapter II, Theorem 6.13] and the proof of [Hof13, Proposition 4.3] for example. □

Lemma 3.A.2. *Assume $m > d/2 + 3$. Let $(S_t)_{t \leqslant T}$ be the heat semigroup acting on H^m. For $f \in H^m$, it holds that*

$$\begin{aligned} \|\nabla (S_t - Id) f\|_\infty &\leqslant \sqrt{t} \left\|D^2 f\right\|_\infty \leqslant C \sqrt{t} \|f\|_m , \\ \|\nabla (S_t - Id) f\|_\infty &\leqslant \frac{1}{2} t \left\|D^3 f\right\|_\infty \leqslant \frac{C}{2} t \|f\|_m , \end{aligned}$$

where D^2 is the Hessian and D^3 the tensor with third-order derivatives. In particular

$$\begin{aligned} \|\nabla (S_t - S_s) f\|_\infty &\leqslant \sqrt{|t-s|} \left\|D^2 f\right\|_\infty \leqslant C \sqrt{|t-s|} \|f\|_m , \\ \|\nabla (S_t - S_s) f\|_\infty &\leqslant \frac{1}{2} |t-s| \left\|D^3 f\right\|_\infty \leqslant \frac{C}{2} |t-s| \|f\|_m . \end{aligned}$$

Proof. We calculate explicitly

$$\begin{aligned}
|\nabla(S_t - \mathrm{Id})f(x)| &= \left| \int_{\mathbb{R}^d} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} (\nabla f(x-y) - \nabla f(x)) \mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}^d} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} \left(\nabla f(x-y) - \nabla f(x) + (-y)^T (D\nabla f)(x) \right) \mathrm{d}y \right| \\
&\leqslant \frac{1}{2} \left\| D^3 f \right\|_\infty \int_{\mathbb{R}^d} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|y|^2}{2t}} |y|^2 \mathrm{d}y \\
&= \frac{1}{2} \left\| D^3 f \right\|_\infty t
\end{aligned}$$

where we exploited the asymmetry of the first Taylor component. The first statement follows from a similar consideration, considering an order one Taylor expansion of $\nabla f(x-y)$ around x instead of an order two Taylor expansion. The proof follows by Sobolev's embeddings. □

The Hilbert space H^s

It is useful to give an explicit definition of H^m through the Fourier transform (e.g. [AF03, 7.62]). Let $s > 0$, define $(H^s, \|\cdot\|_s)$ by

$$\begin{aligned}
H^s &= \left\{ u \in L^2(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| \mathcal{F}(u)(\xi) \right|^2 \mathrm{d}\xi < \infty \right\}, \\
\|u\|_s^2 &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \left| \mathcal{F}(u)(\xi) \right|^2 \mathrm{d}\xi.
\end{aligned} \tag{3.A.2}$$

Whenever s is an integer, it is well known that this definition coincides with the standard definition of the Sobolev space $W^{s,2}(\mathbb{R}^d)$.

The next lemma extends the embedding (3.1.5) to H^s and its relationship with the space of probability measures.

Lemma 3.A.3. *For all $s > d/2$, one has the following continuous embedding*

$$H^s \subset C_b(\mathbb{R}^d). \tag{3.A.3}$$

Moreover, there exists $C > 0$ (depending on s only) such that

$$\sup_{\mu \in \mathcal{P}(\mathbb{R}^d)} \|\mu\|_{-s} \leqslant C. \tag{3.A.4}$$

Proof. The continuous embedding (3.A.3) is a consequence of the embedding of Besov spaces into the space of continuous bounded functions (e.g. [AF03, Theorem 7.34]) and the fact that they coincide with H_s for a particular choice of the indices.

Turning to (3.A.4), let $\mu \in \mathcal{P}(\mathbb{R}^d)$, then

$$\|\mu\|_{-s} = \sup_{h \in H^s} \frac{\langle \mu, h \rangle_{-s,s}}{\|h\|_s} = \sup_{h \in H^s} \frac{\langle \mu, h \rangle}{\|h\|_s} \leqslant \sup_{h \in H^s} \frac{\|h\|_\infty}{\|h\|_s} \leqslant C,$$

where C is the norm of the identity operator between H^s and $C_b(\mathbb{R}^d)$. □

Fractional operators on H^s

We have the following lemma.

Lemma 3.A.4. *Let $\lambda = \rho e^{i\eta} \in \rho(\Delta)$ and suppose $\rho > 1$. There exists a positive constant $C = C_\eta$ such that for every $\epsilon \in (0, 1/2)$*

$$\left\|\nabla R(\rho e^{i\eta}, \Delta)h\right\|_{m-2\epsilon}^2 \leqslant C_\eta \frac{\|h\|_m^2}{\rho^{1+2\epsilon}}, \quad h \in H^m. \tag{3.A.5}$$

Proof. Exploiting the Fourier multipliers associated to ∇ and R, one obtains

$$\begin{aligned}
&\left\|\nabla R(\rho e^{i\eta}, \Delta)h\right\|_{m-2\epsilon}^2 = \int_{\mathbb{R}^d} (1+|\xi|^2)^{m-2\epsilon} \left|\mathcal{F}(\nabla R(\rho e^{i\eta}, \Delta)h)(\xi)\right|^2 \mathrm{d}\xi \\
&\leqslant \int_{\mathbb{R}^d} (1+|\xi|^2)^{m-2\epsilon} \left|\mathcal{F}(h)(\xi)\right|^2 \left|\frac{\xi}{\rho e^{i\eta} + |\xi|^2}\right|^2 \mathrm{d}\xi \\
&\leqslant \int_{\mathbb{R}^d} (1+|\xi|^2)^{m} \left|\mathcal{F}(h)(\xi)\right|^2 \frac{|\xi|^2}{|\rho e^{i\eta} + |\xi|^2|^2} \frac{1}{(1+|\xi|^2)^{2\epsilon}} \mathrm{d}\xi.
\end{aligned}$$

Since we are assuming $\rho > 1$, we have that

$$\frac{|\xi|^2}{|\rho e^{i\eta} + |\xi|^2|^2} \frac{1}{(1+|\xi|^2)^{2\epsilon}} \leqslant \frac{(\rho + |\xi|^2)^{1-2\epsilon}}{|\rho e^{i\eta} + |\xi|^2|^2} \leqslant C_\eta \frac{1}{(\rho + |\xi|^2)^{1+2\epsilon}} \leqslant C_\eta \frac{1}{\rho^{1+2\epsilon}},$$

with $C_\eta = (\sup_{x \geqslant 0}(1+x)/|e^{i\eta} + x|)^2 < \infty$ since $\eta \neq \pi$. □

3.B Rough integration associated to semigroup functionals

We mostly follow [GT10] and use very similar notations. Let $k \geqslant 1$ and Δ_k be the k-dimensional simplex given by

$$\Delta_k = \{t_1, \ldots, t_k \in [0,T] \,:\, T \geqslant t_1 \geqslant t_2 \geqslant \ldots \geqslant t_k \geqslant 0\}.$$

Let $C_k = C(\Delta_k; \mathbb{R})$ and W a Banach space with a strongly continuous semigroup $(S_t)_{t\in[0,T]}$ acting on it. Define D_k as the space of linear operators from W to C_k. Furthermore, let $D_* = \bigcup_{k \geqslant 1} D_k$ and define the following operators on D_*:

$$\delta : D_k \to D_{k+1}, \quad \phi : D_k \to D_{k+1}, \qquad k \geqslant 1.$$

For $A \in D_k$ and $f \in W$, they are defined as

$$\begin{aligned}
[\delta A f]_{t_1 \ldots t_{k+1}} &= \sum_{i=1}^{k+1} (-1)^{i+1} [Af]_{t_1 \ldots \cancel{t_i} \ldots t_{k+1}}, \\
[\phi A f]_{t_1 \ldots t_{k+1}} &= [A(S_{t_1 t_2} - \mathrm{Id})f]_{t_2 \ldots t_{k+1}},
\end{aligned}$$

where $\cancel{t_i}$ means that the argument t_i is omitted and $S_{t_1 t_2}$ stands for $S_{t_1 - t_2}$.

We are ready for the first lemma.

Lemma 3.B.1. *Let $\hat{\delta} := \delta - \phi$. Then $(D_*, \hat{\delta})$ is an acyclic cochain complex. In particular*

$$\operatorname{Ker} \hat{\delta}|_{D_{k+1}} = \operatorname{Im} \hat{\delta}|_{D_k}, \quad \textit{for any } k \geqslant 1.$$

Proof. The proof mimics [GT10, Proposition 3.1]. We only mention that for proving $\operatorname{Ker} \hat{\delta}|_{D_{k+1}} \subset \operatorname{Im} \hat{\delta}|_{D_k}$, a possible choice for $A \in \operatorname{Ker} \hat{\delta}|_{D_{k+1}}$ is given by $B \in D_k$ defined as

$$[Bf]_{t_1 \dots t_k} = (-1)^{k+1} [Af]_{t_1 \dots t_k 0}, \quad f \in W.$$

□

We now introduce some analytical assumptions on the previous function spaces. We start with a Hölder-like norm on C_k for $k = 2, 3$. For $\mu > 0$ and $g \in C_2$, define

$$\|g\|_\mu := \sup_{t,s \in \Delta_2} \frac{|g_{ts}|}{|t-s|^\mu},$$

and consequently

$$C_2^\mu := \left\{ g \in C_2 \, ; \, \|g\|_\mu < \infty \right\}.$$

For $\gamma, \rho > 0$ and $g \in C_3$, define

$$\|g\|_{\gamma,\rho} := \sup_{t,u,s \in \Delta_3} \frac{|g_{tus}|}{|t-u|^\mu \, |u-s|^\rho}$$

and

$$\|g\|_\mu := \inf \left\{ \sum_i \|g_i\|_{\rho_i, \mu - \rho_i} \, ; \, g = \sum_i g_i \, , \, 0 < \rho_i < \mu \right\},$$

where the infimum is taken on all sequences $(g_i) \subset C_3$ such that $g = \sum_i g_i$ and $\rho_i \in (0, \mu)$. Again, $\|\cdot\|_\mu$ defines a norm on C_3 and we denote the induced subspace by

$$C_3^\mu := \left\{ g \in C_3 \, ; \, \|g\|_\mu < \infty \right\}.$$

With these definitions in mind, let

$$D_k^\mu := L(W, C_k^\mu), \quad D_k^{1+} := \bigcup_{\mu > 1} D_k^\mu, \qquad k = 2, 3.$$

The space $L(W, C_k^\mu)$ is the space of linear bounded operators from W to C_k^μ equipped with its corresponding operator norm, i.e.

$$\|A\|_{D_k^\mu} := \sup_{\|f\|_W \leqslant 1} \|Af\|_\mu, \quad f \in D_k^\mu.$$

The main tool for constructing the pathwise integral associated to semigroup functionals is given by the next lemma and the following corollary. We use the notation $\hat{\delta}(D_k) := \operatorname{Im} \hat{\delta}|_{D_k}$ for $k \geqslant 1$.

Lemma 3.B.2 (Sewing)**.** *There exists a unique linear operator*

$$\Lambda : D_3^{1+} \cap \hat{\delta}(D_2) \to D_2^{1+}$$

such that

$$\hat{\delta}\Lambda = \mathrm{Id}|_{D_3^{1+} \cap \hat{\delta}(D_2)}.$$

Moreover, if $\eta > 1$ then Λ is a continuous operator from $D_3^\eta \cap \hat{\delta}(D_2)$ to D_2^η, i.e. there exists a constant $C = C_\eta > 0$ such that

$$\|\Lambda A\|_\eta \leqslant C_\eta \|A\|_\eta, \quad A \in D_3^\eta \cap \hat{\delta}(D_2). \tag{3.B.1}$$

Proof. Concerning uniqueness, let $\widetilde{\Lambda}$ be another map satisfying the conditions stated in the Lemma. Then for $A \in D_3^{1+} \cap \hat{\delta} D_2$ we have

$$\hat{\delta}(\widetilde{\Lambda} A - \Lambda A) = A - A = 0$$

hence $Q := \widetilde{\Lambda} A - \Lambda A \in \mathrm{Ker}(\hat{\delta}) \cap D_2$. By Lemma 3.B.1 there exists $q \in D_1$ such that $Q = \hat{\delta} q$. Note that for any partition $\mathcal{P}^n([s,t]) = (t_i)_{0 \leqslant i \leqslant n+1}$ of the interval $[s,t] \subset [0,T]$ such that $t_0 = s$ and $t_{n+1} = t$, we have the following telescopic sum expansion

$$\begin{aligned}
\sum_{i=0}^n [\hat{\delta} q S_{tt_{i+1}} f]_{t_{i+1}t_i} &= \sum_{i=0}^n [q S_{tt_{i+1}} f]_{t_{i+1}} - [q S_{tt_{i+1}}]_{t_i} - [q S_{tt_{i+1}} (S_{t_{i+1}t_i} - \mathrm{Id}) f]_{t_i} \\
&= \sum_{i=0}^n [q S_{tt_{i+1}} f]_{t_{i+1}} - [q S_{tt_i} f]_{t_i} \\
&= [qf]_t - [q S_{ts} f]_s \\
&= [\hat{\delta} q f]_{ts}
\end{aligned}$$

for any $f \in W$. We conclude

$$[Qf]_{ts} = [\hat{\delta} q f]_{ts} = \sum_{i=0}^n [\hat{\delta} q S_{tt_{i+1}} f]_{t_{i+1}t_i} = \sum_{i=0}^n [Q S_{tt_{i+1}} f]_{t_{i+1}t_i}.$$

Letting $\mathcal{P}^n([s,t])$ be for example be the dyadic partition, one obtains for $Q \in D_2^\gamma$, with $\gamma > 1$, the estimate

$$|[Q S_{tt_{i+1}} f]_{t_{i+1}t_i}| \leqslant 2^{-n\gamma} \left\| Q S_{tt_{i+1}} f \right\|_\gamma |t-s|^\gamma \leqslant 2^{-n\gamma} \|Q\|_{D_2^\gamma} \|f\|_W |t-s|^\gamma$$

where we exploited that S is a contraction semigroup. Returning to the telescope sum, we obtain

$$|[Qf]_{ts}| \leqslant 2^{n(1-\gamma)} \|Q\|_{D_2^\gamma} \|f\|_W |t-s|^\gamma.$$

By passing to the limit for n which tends to infinity, we conclude that for any $f \in W$ and any $[s,t] \subset [0,T]$

$$[Qf]_{ts} = 0$$

yielding $Q = 0$, i.e. $\Lambda A = \widetilde{\Lambda} A$ for any $A \in D_3^{1+} \cap \hat{\delta}(D_2)$, concluding uniqueness.

Towards existence, let $A \in D_3^{1+} \cap \hat{\delta}(D_2)$, i.e. there exist a $B \in D_2$ and $\eta > 1$ such that $\hat{\delta} B = A \in D_3^{\eta}$. Let $(r_k^n)_{0 \leqslant k \leqslant 2^n}$ be the dyadic partition of $[s,t]$. We set, following [GT10]

$$\begin{aligned} M^n : D_3^{1+} \cap \hat{\delta}(D_2) &\to D_2^{1+} \\ \hat{\delta} B &\mapsto M^n \hat{\delta} B \end{aligned}$$

where

$$[(M^n \hat{\delta} B) f]_{ts} := [B(f)]_{ts} - \sum_{k=0}^{2^n - 1} [B(S_{tr_{k+1}^n} f)]_{r_{k+1}^n r_k^n}.$$

Note in particular that $[M^n \hat{\delta} f]_{ts} = 0$. We show that $(M^n \hat{\delta} B)_n$ is Cauchy in D_2^{η}. Note that

$$\begin{aligned} &[(M^n \hat{\delta} B) f]_{ts} - [(M^{n+1} \hat{\delta} B) f]_{ts} = \\ &= \sum_{k=0}^{2^n - 1} [B(S_{tr_{2k+2}^n} f)]_{r_{2k+2}^n r_{2k}^n} - [B(S_{tr_{2k+2}^n} f)]_{r_{2k+2}^n r_{2k+1}^n} - [B(S_{tr_{2k+1}^n} f)]_{r_{2k+1}^n r_{2k}^n} \\ &= \sum_{k=0}^{2^n - 1} [\delta B(S_{tr_{2k+2}^n} f)]_{r_{2k+2}^n r_{2k+1}^n r_{2k}^n} - \sum_{k=0}^{2^n - 1} [\phi B(S_{tr_{2k+2}^n} f)]_{r_{2k+2}^n r_{2k+1}^n r_{2k}^n} \\ &= \sum_{k=0}^{2^n - 1} [\hat{\delta} B(S_{tr_{2k+2}^n} f)]_{r_{2k+2}^n r_{2k+1}^n r_{2k}^n} \leqslant (t-s)^{\eta} \sum_{k=0}^{2^n - 1} \left\| \hat{\delta} B(S_{tr_{2k+2}^n} f) \right\|_{\eta} 2^{-n\eta} \\ &\leqslant (t-s)^{\eta} \left\| \hat{\delta} B \right\|_{D_3^{\eta}} \|f\|_W \, 2^{-n(\eta - 1)} \end{aligned}$$

From which we deduce that $(M^n \hat{\delta} B)_{n \in \mathbb{N}}$ is a Cauchy sequence in D_2^{η}, indeed

$$\left\| M^n \hat{\delta} B - M^{n+1} \hat{\delta} B \right\|_{D_2^{\eta}} \leqslant \left\| \hat{\delta} B \right\|_{D_3^{\eta}} 2^{-n(\eta - 1)}.$$

Let $\Lambda \hat{\delta} B \in D_2^{\eta}$ be its limit. By a telescope argument

$$\left\| M^n \hat{\delta} B \right\|_{D_2^{\eta}} = \left\| \sum_{k=0}^{n-1} M^k \hat{\delta} B - M^{k+1} \hat{\delta} B \right\|_{D_2^{\eta}} \leqslant \sum_{k=0}^{n-1} 2^{-k(\eta - 1)} \left\| \hat{\delta} B \right\|_{D_3^{\eta}} \leqslant C_{\eta} \left\| \hat{\delta} B \right\|_{D_3^{\eta}}$$

from which we obtain (3.B.1) using to weak-*-lower semicontinuity of the norm. One can prove that the limit does not depend on the particular sequence, see [GT10, Proposition 2.3].

Finally, let $u = 2^m$ for some $m \in 0, \dots, n$ and note that

$$\begin{aligned}
[(\hat{\delta} M^n \hat{\delta} B) f]_{tus} &= [(M^n \hat{\delta} B) f]_{ts} - [(M^n \hat{\delta} B) f]_{tu} - [(M^n \hat{\delta} B) f]_{us} - [(M^n \hat{\delta} B (S_{tu} - \mathrm{Id})) f]_{us} \\
&= [Bf]_{ts} - [Bf]_{tu} - [BS_{tu} f]_{us} + \sum_{k=0}^{2^n - 1} [B(S_{tr^n_{k+1}} f)]_{r^n_{k+1} r^n_k} \\
&- \sum_{k=2^m}^{2^n - 1} [B(S_{tr^n_{k+1}} f)]_{r^n_{k+1} r^n_k} - \sum_{k=0}^{2^m - 1} [B(S_{ur^n_{k+1}} S_{tu} f)]_{r^n_{k+1} r^n_k} \\
&= [Bf]_{ts} - [Bf]_{tu} - [BS_{tu} f]_{us} \\
&= [\hat{\delta} B f]_{sut}
\end{aligned}$$

from which we recover, in the limit $n \to \infty$, that $\hat{\delta}\Lambda = \mathrm{Id}|_{D_3^{1+} \cap \hat{\delta}(D_2)}$. □

Corollary 3.B.3. *Suppose that $A \in D_2$ is such that $\hat{\delta} A \in D_3^{1+}$. Then there exists $I \in D_1$ such that*

$$\hat{\delta} I = \left(\mathrm{Id} - \Lambda \hat{\delta}\right) A,$$

i.e. for every $f \in W$ and $(t, s) \in \Delta_2$, $\left[\hat{\delta} I f\right]_{ts} = [Af]_{ts} - \left[\Lambda \hat{\delta} A f\right]_{ts}$. In particular, if $A \in D_2^{\mu}$ with $\mu > 0$ and $\hat{\delta} A \in D_3^{\eta}$ with $\eta > 1$, then for every $f \in W$

$$\left|[\hat{\delta} I f]_{ts}\right| \leqslant \left(\|A\|_{D_2^{\mu}} (t - s)^{\mu} + \left\|\hat{\delta} A\right\|_{D_3^{\eta}} (t - s)^{\eta}\right) \|f\|_W . \tag{3.B.2}$$

Finally

$$\left[\hat{\delta} I f\right]_{ts} = \lim_{|\mathcal{P}[s,t]| \downarrow 0} \sum_{[v,u] \in \mathcal{P}[s,t]} [AS_{tu} f]_{uv} , \tag{3.B.3}$$

where the limit is over any partition of $[s, t]$ whose mesh tends to zero.

Proof. The proof is an easy application of the Sewing Lemma and the properties of $(D_*, \hat{\delta})$. Indeed, observe that $\hat{\delta}(Af - \Lambda \hat{\delta} A f) = 0$ for any $f \in W$, which means that $A - \Lambda \hat{\delta} A \in \operatorname{Ker} \hat{\delta}|_{D_2}$, and thus there exists $I \in D_1$ such that $\hat{\delta} I = A - \Lambda \hat{\delta} A$.

The estimate (3.B.2) follows from (3.B.1). Concerning (3.B.3), observe that for a partition $|\mathcal{P}[s, t]|$, using the properties of $\hat{\delta}$, one obtains

$$\begin{aligned}
\sum_{[v,u] \in \mathcal{P}[s,t]} [AS_{tu} f]_{uv} &= \sum_{[v,u] \in \mathcal{P}[s,t]} \left[\hat{\delta} I S_{tu} f\right]_{uv} + \left[\Lambda \hat{\delta} A S_{tu} f\right]_{uv} = \\
&= \left[\hat{\delta} I f\right]_{ts} + \sum_{[v,u] \in \mathcal{P}[s,t]} \left[\Lambda \hat{\delta} A S_{tu} f\right]_{uv} .
\end{aligned}$$

By taking the limit for the mesh which tends to zero and using the fact that $\Lambda \hat{\delta} A \in D_2^{1+}$, the last sum converges to zero. □

Chapter 4

Averaging along irregular curves for Burgers' equation

We study a regularization by noise phenomena for a modified Burgers' equation through fractional Brownian motion of Hurst parameter H. Our analysis builds upon a recent approach discussed in [HP21] to study regularization by noise for ODEs that relies on quantifying the space-time regularities of local times associated with locally non-deterministic processes. We establish existence of weak solutions to such equations, provided the initial condition is chosen sufficiently small depending on the time horizon.

This chapter is based on a joint work in progress with Nicolas Perkowski.

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4.1 Introduction

In this chapter we consider the problem

$$\begin{aligned} \partial_t f + \widetilde{f}\partial_x f &= 0 \\ f(0) &= f_0, \end{aligned} \tag{4.1.1}$$

where for a function g, we employed the notation $\widetilde{g}(t,x) = g(t, x - w_t^H)$ and where $w^H : [0,T] \to \mathbb{R}$ is a fractional Brownian motion of Hurst parameter H. We show that one can establish existence of solutions to the above equation provided H is chosen sufficiently small and $f_0 \in C^{0,\gamma} \cap W^{1,1}$ is chosen sufficiently small depending on the time horizon.

We will do so by employing a fixed point argument on the level of corresponding transport equations

$$\begin{aligned} \partial_t f + \widetilde{b}\partial_x f &= 0 \\ f(0) &= f_0. \end{aligned} \tag{4.1.2}$$

Recall that (4.1.2) is intimately related to its associated characteristic equation

$$X_t = x + \int_0^t \widetilde{b}(s, X_s)ds = x + \int_0^t b(s, X_s - w_s^H)ds. \tag{4.1.3}$$

Heuristically, as for bounded continuous functions $b : [0,T] \times \mathbb{R} \to \mathbb{R}$, X can easily be seen to be Lipschitz, the oscillations of sample paths of fractional Brownian motion dominate the oscillations of X. This suggests to expect an averaging effect in the Lebesgue integral of the above expression (4.1.3). In the case of the related problem

$$Y_t = x + \int_0^t b(s, Y_s)ds - w_t^H, \tag{4.1.4}$$

obtained through the substitution $Y = X - w^H$ and $H = 1/2$ such studies of regularization by noise go back to the pioneering works of [Zvo74] and [Ver81] (see also the more recent work [KR04]). Regularisation by noise for the transport equation was famously studied in the case of Brownian motion by [FGP09]. For a comprehensive study of regularization by noise phenomena for ODEs and PDEs in the case of Brownian motion, we refer the to St. Flour lecture notes [Fla11]. In the case of $H \neq 1/2$ (4.1.4) was studied in [NO02], [BNP19] and very recently in [NS21] for the case of two fractional Brownian motions.

All of the above can be seen as mainly probabilistic approaches relying extensively on Girsanov transforms or Malliavin calculus. An alternative approach to regularization by noise for fractional Brownian motion (and even for more generic random perturbations) that has enjoyed considerable interest since the past year goes back to the works [CG16] and [Cat16] for ODEs and the transport equation respectively. Similar in spirit to rough path theory this approach "factorizes" the problem into an

initial probabilistic step and subsequent purely analytical considerations. The central object of study linking these steps is the averaging operator $T^{-w^H}b$ associated with the averaging of the non-linearity b along the path $-w^H$, defined by

$$(T_t^{-w^H}b)(x) := \int_0^t b(s, x - w_s^H)ds. \tag{4.1.5}$$

In a first probabilistic step to this approach, one establishes space-time regularity estimates for $T_t^{-w^H}b$ as a function of the regularity of b. Exploiting the so obtained regularity estimates, one proceeds with a so called non-linear Young construction of Lebesgue integrals. This approach to regularisation by noise for ODEs is then capable of consistently solving (4.1.4) even in the case of nonlinearities that only enjoy distributional regularity.

Recently, the above approach to ODEs and transport equations has been revisited and extended in [GG20a], [GG20b] and moreover has been also applied to interacting particle systems [HM20], distribution dependent SDEs [GHM21b], [GHM21a] and the multiplicative stochastic heat equation [CH21].

A variation to this approach was recently proposed in [HP21] and will be the main subject of investigation in the following. Let us sketch some main ideas of this approach before proceeding to the outline of the chapter.

4.1.1 Non-linear Young theory

In the following, let us sketch the main idea to non-linear Young integration in the spirit of [HP21]. Suppose that a path $w : [0, T] \to \mathbb{R}^d$ admits a local time $L : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, then by the occupation times formula we have for any bounded and measurable $b : \mathbb{R}^d \to \mathbb{R}^d$ and $x \in \mathbb{R}^d$

$$\int_0^t b(x - w_u)du = \int_{\mathbb{R}} b(x - a)L_t(a)da = (b * L_t)(x).$$

Notice in particular that due to Young's convolutional inequality in Lemma 4.B.1, the right hand side of this equality will be a well defined function in $x \in \mathbb{R}^d$ for certain distributions b, provided the local time L enjoys sufficient regularity. In the case of w being a fractional Brownian motion, we have the following quantitative result in [HP21] for almost every realization of the local time.

Lemma 4.1.1 ([HP21, Theorem 17]). *Let w be a d-dimensional fractional Brownian motion of Hurst parameter $H < 1/d$ on $[0, T]$. Then there exists a null set $\mathcal{N}$ such that for all $\omega \in \mathcal{N}^c$, the path $w(\omega)$ has a local time $L(\omega)$ and for $\lambda < \frac{1}{2H} - \frac{d}{2}$ and $\gamma \in [0, 1 - (\lambda + \frac{d}{2})H)$ we have*

$$\|L_{s,t}(\omega)\|_{H^\lambda} \leqslant C_T(\omega)|t - s|^\gamma. \tag{4.1.6}$$

for any $s, t \in [0, T]$, where $L_{s,t} = L_t - L_s$.

Remark 4.1.2 (Dependency on the time horizon)**.** *Remark that in the above Lemma, one fixes a fractional Brownian motion on a given time horizon, wherefore the random constant $C_T(\omega)$ appearing in* (4.1.6) *also depends on T. However, note that by monotonicity, we have for any fixed T^* and any $T < T^*$ that*

$$\sup_{s<t\in[0,T]} \frac{\|L_{s,t}\|}{|t-s|^\gamma} \leqslant \sup_{s<t\in[0,T^*]} \frac{\|L_{s,t}\|}{|t-s|^\gamma} \leqslant C_{T^*}(\omega)$$

meaning we may choose the random constant $C_T(\omega)$ in (4.1.6) *uniformly for any $T \leqslant T^*$.*

Remark 4.1.3 (Altering integrability)**.** *Taking the above result as starting point, observe that we can obtain similar estimates on $\|L_{s,t}\|_{B^\alpha_{p,q}}$ for $p \geqslant 2$ via Besov space embeddings as in 4.B.4. Moreover, note that by monotonicity, the support of $L_t(\omega)$ is contained in the compact support of $L_T(\omega)$ for any $t \in [0,T]$. Denoting $K_{T^*} = supp(L_{T^*}(\omega))$, we therefore have by the embedding in Besov spaces of compactly supported functions $B^\alpha_{p+r,q}(K) \hookrightarrow B^\alpha_{p,q}(K)$ for $\alpha \in \mathbb{R}$, $p,q \in [1,\infty]$ and $r \geqslant 0$ that for any $s,t \in [0,T]$ and $T < T^*$*

$$\|L_{s,t}\|_{B^\alpha_{p,q}} = \|L_{s,t}\|_{B^\alpha_{p,q}(K)} \leqslant c_{K_{T^*}} \|L_{s,t}\|_{B^\alpha_{p+r,q}(K)} = c_{K_{T^*}} \|L_{s,t}\|_{B^\alpha_{p+r,q}} .$$

In particular, we have for example

$$\|L_{s,t}\|_{W^{1,1}} \leqslant C_{T^*} \|L_{s,t}\|_{H^1} .$$

Once existence and quantitative estimates on the local time associated with w are established, problems of the form

$$X_t = x + \int_s^t b(X_u - w_u)du$$

allow for the following observation: Note that by the occupation times formula, we have in the case of continuous functions b and for $|t-s| \ll 1$ the local approximation

$$\int_s^t b(X_u - w_u)du \simeq \int_s^t b(X_s - w_u)du = \int_{\mathbb{R}^d} b(X_s - y)L_{s,t}(y)dy = (b * L_{s,t})(X_s) =: A^X_{s,t}.$$

Seen from the point of view of Young integration, it then becomes natural to propose the following definition replacing the above Lebesgue integral:

$$\int_s^t b(X_u - w_u)du := (\Xi A^X)_{s,t},$$

where Ξ denotes Gubinelli's sewing operator (see [Gub04] or [FH14]). While it can be easily checked that in the case of continuous and bounded functions b the proposed definition coincides with the classical Lebesgue integral, the definition as a Sewing provides a canonical extension to functions potentially enjoying only distributional regularity by harnessing the averaging effects of the perturbing path w. With this definition at hand, Harang and Perkowski then proceed to study problems of the form

$$X_{s,t} = (\Xi A^X)_{s,t}, \qquad X_0 = x.$$

and their associated flow.

4.1.2 Organization

In the following section, we extend the results of [HP21] to non-autonomous problems and add results derived in [GG20a] in the slightly different setting of averaging operators. In particular this allows to establish invertibility and differentiability of the associated flow as well as comparison principles and a priori bounds even in the case of non-differentiable functions b. This is possible as the regularizing effect of the local time L carries over from local approximations to the sewing. In Section 4.3 this differentiability of the flow and its inverse allows us to study associated transport equations and establish well-posedness in the classical sense provided differentiable initial conditions. For the case of less regular initial data, we moreover establish existence and provide a priori bounds for solutions to the transport equation that might be of independent interest. Finally, in Section 4.4, we show how by employing a fixed point argument on the level of transport equations, we are able to address (4.1.1).

4.1.3 Notation

Let us fix some notation to be employed throughout the remaining chapter. Throughout the remainder of this chapter, we shall fix a $T^* > 0$ and consider all appearing problems on time intervals $[0,T]$ for $T < T^*$. For a Banach space X and a given time interval $[0,T]$, we shall denote

$$\|f\|_{C_T^\gamma X} := \sup_{s<t\in[0,T]} \frac{\|f_t - f_s\|_X}{|t-s|^\gamma}, \qquad \|f\|_{C_T^{0,\gamma} X} := \sup_{t\in[0,T]} \|f_t\|_X + \|f\|_{C_T^\gamma X}\,.$$

We denote by $C_x^0 = C^0(\mathbb{R})$ the space of bounded uniformly continuous functions on $\mathbb{R}$, C_x^b the space ob bounded continuous functions on $\mathbb{R}$ and by $C_x^\alpha = C^\alpha(\mathbb{R})$ the space of globally Hölder-continuous functions on $\mathbb{R}$. $H_x^{s,p}$ and $B_{p,q}^s$ denote Bessel potential spaces respectively non-homogeneous Besov space on $\mathbb{R}$. We use the convention $H_x^s = H_x^{s,2}$. Moreover, for $\gamma \in (1/2, 1)$ we define

$$Z_T^\gamma := C_T^{0,\gamma} C_x^b \cap C_T^b C_x^\gamma \cap L_T^\infty L_x^1.$$

4.2 ODEs and a priori bounds

Let us begin by considering a smooth and bounded function $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and the associated problem

$$X_t = x + \int_0^t b(s, X_s - w_s)ds$$

As b is assumed to be smooth, we know that there exists a unique Lipschitz continuous solution X. Notice that for the integral on the right hand side, we then have

heuristically

$$\begin{aligned}\int_0^t b(s, X_s - w_s)ds &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} b(u, X_u - w_u)(u-v) \\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \int_u^v b(u, X_u - w_s)ds \\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} b_u * L_{u,v}(X_u).\end{aligned}$$

We next show that the right hand side can indeed be interpreted as the sewing of the germ $A^X_{s,t} := b_s * L_{s,t}(X_s)$, provided some further conditions are imposed on b that however do not require Lipschitz continuity.

Lemma 4.2.1 (Identification of Riemann integral with Sewings)**.** *Let $\gamma \in (1/2, 1)$ and $H \in (0, 1/3)$. Suppose that $b \in C^\gamma_T C^b_x$. Let L be the local time of fractional Brownian w^H motion with Hurst parameter H in one dimension and $X : [0, T] \to \mathbb{R}$ such that $\|X\|_{C^\gamma_T} < \infty$. Then the germ*

$$(A^X)_{s,t} := (b_s * L_{s,t})(X_s)$$

admits a Sewing. Assume moreover that for some $\epsilon > 0$ we have $b \in C^b_t C^\epsilon_x$. Then the germ

$$(T^X)_{s,t} = T^{-w^H}_{s,t}(X_s) := \int_s^t b(r, X_s - w^H_r)dr$$

admits a sewing and moreover, we have

$$\int_0^t b(s, X_s - w^H_s)ds = (\Xi A^X)_{0,t} = (\Xi T^X)_{0,t}.$$

Proof. Remark first that by Lemma 4.1.1 and $H < 1/3$ we have that $L \in C^{1/2}_t H^1_x$. By Remark 4.1.3 we therefore have

$$\begin{aligned}|(\delta A^X)_{s,u,t}| &= |b_s * L_{u,t}(X_s) - b_u * L_{u,t}(X_u)| \\ &= |b_{s,u} * L_{u,t}(X_s) + b_u * L_{u,t}(X_s) - b_u * L_{u,t}(X_u)| \\ &\leqslant \|b_{s,u} * L_{u,t}\|_\infty + \|b_u * L_{u,t}\|_{C^1_b} |X_u - X_s| \\ &\leqslant C_{T^*} \|b\|_{C^\gamma_t L^\infty_x} \|L\|_{C^{1/2}_t L^2} |u-s|^\gamma |t-u|^{1/2} \\ &+ C_{T^*} \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2}_t H^1_x} |t-u|^{1/2} \|X\|_{C^\gamma} |u-s|^\gamma \\ &= O(|t-s|^{1/2+\gamma}),\end{aligned}$$

where we used that due to the continuity of the translation map in L^1 and Young's inequality, $x \to (b_u * \partial_x L_{u,t})(x)$ is indeed continuous. Hence A^X does indeed admit a Sewing and we have

$$|A^X_{s,t} - (\Xi A^X)_{s,t}| = O(|t-s|^{1/2+\gamma}).$$

Moreover for $b \in C_T^b C_x^\epsilon$ and for any $s \in [0,T]$, it can be easily seen that the germ $B^s_{u,v} = b(s, X_s - w_u^H)(v-u)$ admits a Sewing for which we have by definition

$$(\Xi B^s)_{s,t} = \int_s^t b(s, X_s - w_r^H)dr = (b_s * L_{s,t})(X_s) = A^X_{s,t}.$$

Similarly, it can be easliy checked that also the germ $C_{u,v} = b(u, X_u - w_u^H)(v-u)$ admits a Sewing for which we have by definition

$$(\Xi C)_{u,v} = \int_u^v b(r, X_r - w_r^H)dr.$$

Remark in particular that $B^s_{s,t} = C_{s,t}$. We therefore conclude that

$$\begin{aligned}
&\left|\int_s^t b(r, X_r - w_r^H)dr - (\Xi A^X)_{s,t}\right| \\
&\leqslant |(\Xi C)_{s,t} - C_{s,t}| + |B^s_{s,t} - (\Xi B^s)_{s,t}| + |A^X_{s,t} - (\Xi A^X)_{s,t}| \\
&= O(|t-s|^{1+\epsilon H}).
\end{aligned}$$

Hence the function

$$t \to \int_0^t b(r, X_r - w_r^H)dr - (\Xi A^X)_t$$

is constant and since it starts in zero. We have therefore established

$$\int_0^t b(r, X_r - w_r^H)dr = (\Xi A^X)_t.$$

Concerning the second germ, note that

$$\begin{aligned}
|(\delta T^X)_{s,u,t}| &= \left|\int_u^t b(r, X_u - w_r^H) - b(r, X_s - w_r^H)dr\right| \\
&\leqslant \|b\|_{C_t^b C_x^\epsilon} |t-u||X_u - X_s|^\epsilon \\
&\leqslant \|b\|_{C_t^b C_x^\epsilon} \|X\|_{C^\gamma} |t-s|^{1+\epsilon\gamma}.
\end{aligned}$$

Hence it admits a Sewing and we have

$$|(T^X)_{s,t} - (\Xi T^X)_{s,t}| = O(|t-s|^{1+\epsilon\gamma}).$$

Moreover, it can be easily seen that the germ $\widetilde{B}^s_{u,v} := b(u, X_s - w_u^H)(v-u)$ admits a Sewing for which we have by definition

$$(\Xi \widetilde{B}^s)_{s,t} = \int_s^t b(r, X_s - w_r^H)dr = (T^X)_{s,t}.$$

Similarly to above, we therefore conclude that since $C_{s,t} = \widetilde{B}^s_{s,t}$, we have

$$\begin{aligned}
&\left|\int_s^t b(r, X_r - w_r^H)dr - (\Xi T^X)_{s,t}\right| \\
&\leqslant |(\Xi C)_{s,t} - C_{s,t}| + |\widetilde{B}^s_{s,t} - (\Xi \widetilde{B}^s)_{s,t}| + |T^X_{s,t} - (\Xi T^X)_{s,t}| \\
&= O(|t-s|^{1+\epsilon H}),
\end{aligned}$$

which completes the proof. □

Remark 4.2.2. *Notice that in locally approximating the object*

$$\int_s^t b(r, X_r - w_r^H)dr$$

the averaging operator approach only replaces X_r by X_s, while the approach due to [HP21] in the non-autonomous setting requires to replace both X_r by X_s and $b(r,\cdot)$ by $b(s,\cdot)$. On the level of Sewings, this is the reason why results in [GG20a] don't require additional Hölder regularity in time, while in our setting, we will always require $b \in C_t^{0,\gamma}C_x^b$. Notice also, that in the autonomous setting, this issue doesn't arise and the approaches can thus be seen as equivalent in spirit.

Lemma 4.2.3 (ODEs and priori bounds I)**.** *Let $T^* > T > 0$. Assume $b \in C_t^{0,\gamma}C_x^b$ for $\gamma \in (1/2, 1)$ and $H < 1/5 \wedge 2(1-\gamma)$. Then for $A_{s,t}^X := b_s * L_{s,t}(X_s)$ the problem*

$$X_{s,t} = (\Xi A^X)_{s,t}, \qquad X_0 = x \in \mathbb{R}, \tag{4.2.1}$$

where $\Xi : C_2^{\gamma,2\gamma} \to C^\gamma$ denotes the sewing operator (see Lemma 4.A.1) admits a unique solution $X \in C^\gamma$ on $[0,T]$. Moreover if in addition $\|b\|_{C_t^{0,\gamma}C_x^b} \leqslant 1$, we have the a priori bounds

$$\|X\|_{C^\gamma} \leqslant C_{T^*},$$

and

$$\sup_{t \leqslant T} |X_t| \leqslant |x| + T^\gamma C_{T^*},$$

for some constant C_{T^*}. Finally, assuming in addition that $b \in C_t^b C_x^\gamma$, then X is the unique solution in C^γ to

$$X_t = x + \int_0^t b(s, X_s - w_s^H)ds.$$

Proof. The proof is very similar to the one establishing existence and uniqueness to Young differential equations. Being able to leverage the regularity of the local time, we can in fact set up a fixed point argument in C^γ (i.e. not in $C^{\gamma-\epsilon}$ for some small $\epsilon > 0$ as usually done, refer to [FH14, Chapter 8.3] for example).

Let us define $\mathcal{B} = \{Y \in C^\gamma : Y_0 = x, \|Y\|_{C^\gamma} \leqslant 1\}$. Equipped with the metric $(X,Y) \to \|X - Y\|_{C^\gamma}$, $\mathcal{B}$ is a complete metric space. We show that the mapping $\mathcal{M} : \mathcal{B} \to \mathcal{B}$ defined by

$$\mathcal{M} : X \to x + (\Xi A^X)_{0,(\cdot)}$$

is well defined and a contraction provided T is chosen sufficiently small. Remark first that by Lemma 4.1.1 and $H < 1/3$ we have that $L \in C_t^{1/2+\epsilon}H_x^1$. Similarly, by $H < 2(1-\gamma)$, we have and $L \in C_t^{\gamma+\epsilon}L^2$ for some $\epsilon > 0$. By Remark 4.1.3 we therefore have

$$\begin{aligned}
|A_{s,t}^X| &\leqslant \|b_s * L_{s,t}\|_\infty \\
&\leqslant \|b\|_{L_{t,x}^\infty} \|L\|_{C_t^{\gamma+\epsilon}L_x^1} T^\epsilon |t-s|^\gamma \\
&\leqslant C_{T^*}T^\epsilon \|b\|_{L_{t,x}^\infty} \|L\|_{C_t^{\gamma+\epsilon}L_x^2} |t-s|^\gamma \\
&\leqslant C_{T^*}T^\epsilon \|b\|_{L_{t,x}^\infty} |t-s|^\gamma,
\end{aligned}$$

as well as

$$
\begin{aligned}
|(\delta A^X)_{s,u,t}| &= |b_s * L_{u,t}(X_s) - b_u * L_{u,t}(X_u)| \\
&\leqslant |b_{s,u} * L_{u,t}(X_s)| + |b_u * L_{u,t}(X_s) - b_u * L_{u,t}(X_u)| \\
&\leqslant \|b_{s,u} * L_{u,t}\|_\infty + \|b_u * L_{u,t}\|_{C_b^1} |X_u - X_s| \\
&\leqslant C_{T^*} T^\epsilon \|b\|_{C_t^\gamma L_x^\infty} \|L\|_{C_t^{1/2+\epsilon} L^2} |u-s|^\gamma |t-u|^{1/2} \\
&+ C_{T^*} T^\epsilon \|b\|_{L^\infty_{t,x}} \|L\|_{C_t^{1/2+\epsilon} H_x^1} |t-u|^{1/2} \|X\|_{C^\gamma} |u-s|^\gamma \\
&\leqslant C_{T^*} T^\epsilon (\|b\|_{L^\infty_{t,x}} (1 + \|X\|_{C^\gamma}) + \|b\|_{C_t^\gamma L_x^\infty}) |t-s|^{1/2+\gamma},
\end{aligned}
$$

where we used that due to the continuity of the translation map in L^1 and Young's inequality, $x \to (b_u * \partial_x L_{u,t})(x)$ is indeed continuous. We therefore conclude that

$$
\begin{aligned}
\left\|\Xi A^X\right\|_\gamma &\leqslant \left\|A^X\right\|_\gamma + \left\|A - \Xi A^X\right\|_\gamma \\
&\leqslant C_{T^*} T^\epsilon (\|b\|_{L^\infty_{t,x}} (1 + \|X\|_{C^\gamma}) + \|b\|_{C_t^\gamma L_x^\infty})
\end{aligned}
\tag{4.2.2}
$$

and thus for T sufficiently small depending on C_{T^*} and b and $\|X\|_{C^\gamma} \leqslant 1$ we have

$$
\|\mathcal{M}(X)\|_{C^\gamma} = \left\|\Xi A^X\right\|_{C^\gamma} \leqslant 1
$$

This shows that the mapping $\mathcal{M}$ leaves $\mathcal{B}$ invariant. Moreover, let $X^1, X^2 \in \mathcal{B}$, then by the linearity of the Sewing operator we have

$$
(\mathcal{M}(X^1) - \mathcal{M}(X^2))_{s,t} = (\Xi(A^{X^1} - A^{X^2}))_{s,t}.
$$

Note that we have

$$
\begin{aligned}
(A^{X^1} - A^{X^2})_{s,t} &= b_s * L_{s,t}(X_s^1) - b_s * L_{s,t}(X_s^2) \\
&\leqslant C_{T^*} \|b\|_{L^\infty_{t,x}} \|L\|_{C_t^\gamma H_x^1} T^\gamma \left\|X^1 - X^2\right\|_{C^\gamma} |t-s|^\gamma \\
&\leqslant C_{T^*} T^\gamma \|b\|_{L^\infty_{t,x}} \left\|X^1 - X^2\right\|_{C^\gamma} |t-s|^\gamma.
\end{aligned}
$$

Since we also have $H < 1/5$, by Lemma 4.1.1 we have $L \in C_t^{1/2+\epsilon} H_x^2$. Moreover, by the division property (see Lemma 4.A.2) we also have

$$
\begin{aligned}
|(\delta A^{X^1} - A^{X^2})_{s,u,t}| &\leqslant (|b_{s,u}) * L_{u,t}(X_s^1) - (b_{s,u}) * L_{u,t}(X_s^2)| \\
&+ |b_u * L_{u,t}(X_s^1) - b_u * L_{u,t}(X_u^1) - b_u * L_{u,t}(X_s^2) + b_u * L_{u,t}(X_u^2)| \\
&\leqslant C_{T^*} \|b\|_{C_t^\gamma L_x^\infty} |u-s|^\gamma \|L\|_{C_t^{1/2} H_x^1} |t-u|^{1/2} \left\|X^1 - X^2\right\|_{C^\gamma} T^\gamma \\
&+ C_{T^*} \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2+\epsilon} H_x^2} T^\epsilon |t-u|^{1/2} \left\|X^1 - X^2\right\|_{C^\gamma} |u-s|^\gamma \\
&\leqslant C_{T^*} T^\gamma \|b\|_{C_t^\gamma L_x^\infty} \left\|X^1 - X^2\right\|_{C^\gamma} |t-s|^{1/2+\gamma} \\
&+ C_{T^*} T^\epsilon \|b\|_{L^\infty_{t,x}} \left\|X^1 - X^2\right\|_{C^\gamma} |t-s|^{1/2+\gamma}.
\end{aligned}
$$

We therefore conclude again that

$$\begin{aligned}
&\left\|\mathcal{M}(X^1)-\mathcal{M}(X^2)\right\|_{C^\gamma}\\
&\leqslant \left\|A^{X^1}-A^{X^2}\right\|_{C^\gamma}+\left\|A^{X^1}-A^{X^2}-\Xi(A^{X^1}-A^{X^2})\right\|_{C^\gamma}\\
&\leqslant C_{T^*}T^\epsilon\left(T^{\gamma-\epsilon}\left\|b\right\|_{L^\infty_{t,x}}+T^{\gamma-\epsilon}\left\|b\right\|_{C^\gamma_t L^\infty_x}\left\|b\right\|_{L^\infty_{t,x}}\right)\left\|X^1-X^2\right\|_{C^\gamma},
\end{aligned}$$

yielding that $\mathcal{M}$ is indeed a contraction, provided T is chosen sufficiently small. As this choice can be done independent of the initial condition, by iteration we obtain a unique solution X to (4.2.1) on any time interval $[0,T]$ for $T<T^*$. Moreover, provided $\|b\|_{C^{0,\gamma}_t C^b_x}\leqslant 1$, we can now go back to (4.2.2) and observe that for this unique solution X to (4.2.1) we moreover have in the notation of Lemma 4.A.3 and $h<T$

$$\|X\|_{\gamma,h}=\left\|\Xi A^X\right\|_{\gamma,h}\leqslant C_{T^*}h^\epsilon(1+\|X\|_{\gamma,h})$$

Hence, choosing $h=(2C_{T^*})^{-1/\epsilon}$, we infer that

$$\|X\|_{\gamma,h}\leqslant 1$$

and thus by Lemma 4.A.3, we obtain for a new constant depending only on T^* and still denoted C_{T^*} that

$$\|X\|_{C^\gamma}\leqslant C_{T^*}$$

Finally, the last claim of the statement is a direct consequence of Lemma 4.2.1. □

Remark 4.2.4. *Let us point out that at first sight, the condition $H<1/5\wedge 2(1-\gamma)$ might seem counter-intuitive, as it becomes more restrictive for γ close to one, i.e. very regular b. However, this restriction appears as the Lemma establishes not only existence of a unique solution X to* (4.2.1) *but also states that said solution lies in C^γ. Indeed, since the germ*

$$A^X_{s,t}=b_s*L_{s,t}(X_s)$$

inherits the Hölder regularity of the local time, demanding the solution to lie in C^γ requires more and more restrictive conditions on H. For $b\in C^b_tC^\gamma_x$ however we can by Lemma 4.2.1 identify the Sewings in question again as Lebesgue integrals, allowing to recover Lipschitz continuity in time and therefore to drop the condition $H<2(1-\gamma)$ (refer also to Lemma 4.2.8). This possibility will however be neither available in considering the derivative of the associated flow later on, nor in the following Lemma.

We next establish the following comparison principle

Lemma 4.2.5 (Comparison)**.** *Let $T^*>T>0$. Let $\gamma\in(1/2,1)$, $\alpha\geqslant 1$ and*

$$H<\frac{1}{2\alpha+3}\wedge\frac{2(1-\gamma)}{2\alpha+1}.$$

For $i\in\{1,2\}$ and $b^i\in C^{0,\gamma}_tC^b_x\cap C^b_tC^\gamma_x$ such that $\|b^i\|_{C^{0,\gamma}_tC^b_x}\leqslant 1$, let $X^i=X^i(x)$ be the unique solution to the problem

$$X^i_t=x+\int_0^t b^i(s,X^i_s-w^H_s)ds$$

on $[0,T]$ *or by Lemma 4.2.1 equivalently*

$$X^i_{s,t} = (\Xi A^i)_{s,t}, \qquad X_0 = x,$$

where

$$A^i_{s,t} = b^i_s * L_{s,t}(X^i_s).$$

We then have

$$\left\|X^1 - X^2\right\|_{C^\gamma_t L^\infty_x} \leqslant C_{T^*}\left(\left\|b^1_0 - b^2_0\right\|_{B^{-\alpha}_{\infty,\infty}} + \left\|b^1 - b^2\right\|_{C^\gamma_t B^{-\alpha}_{\infty,\infty}}\right).$$

Proof. Remark first that since b^i is Hölder continuous in space uniformly in time, it is in particular uniformly continuous in space, uniformly in time. Hence $b^i \in C^\gamma_t C^0$ and therefore by the Besov space embedding $C^0 \hookrightarrow B^0_{\infty,\infty} \hookrightarrow B^{-\alpha}_{\infty,\infty}$ we have indeed $b^i \in C^\gamma_t B^{-\alpha}_{\infty,\infty}$.

Note that by $H < 2(1-\gamma)/(2\alpha+1)$, we have $L \in C^\gamma_t H^{\alpha+\epsilon}$ and by $H < 1/(2\alpha+3)$ we have $L \in C^{1/2+\epsilon} H^{\alpha+1+\epsilon}$ for some $\epsilon > 0$. We consider the sewing of the germ

$$\begin{aligned}
A_{s,t} :&= |b^1_s * L_{s,t}(X^1_s) - b^2_s * L_{s,t}(X^2_s)| \\
&\leqslant |(b^1_s - b^2_s) * L_{s,t}(X^1_s)| + |b^2_s * L_{s,t}(X^1_s) - b^2_s * L_{s,t}(X^2_s)| \\
&\leqslant \left\|(b^1_s - b^2_s) * L_{s,t}\right\|_\infty + \left\|b^2_s * \partial_x L_{s,t}\right\|_\infty |X^1_s - X^2_s| \\
&\leqslant \left\|(b^1_s - b^2_s) * L_{s,t}\right\|_{B^0_{\infty,1}} + \left\|b^2_s * \partial_x L_{s,t}\right\|_\infty |X^1_s - X^2_s| \\
&\leqslant C_{T^*}\left\|b^1_s - b^2_s\right\|_{B^{-\alpha}_{\infty,\infty}} \|L_{s,t}\|_{B^\alpha_{1,1}} + \left\|b^2_s\right\|_{L^\infty} \|L_{s,t}\|_{W^{1,1}} |X^1_s - X^2_s| \\
&\leqslant C_{T^*}\left(\left\|b^1_0 - b^2_0\right\|_{B^{-\alpha}_{\infty,\infty}} + \left\|b^1 - b^2\right\|_{C^\gamma_t B^{-\alpha}_{\infty,\infty}} T^\gamma\right) \|L\|_{C^\gamma H^{\alpha+\epsilon}} |t-s|^\gamma \\
&+ C_{T^*} T^\gamma \|L\|_{C^\gamma H^1} |t-s|^\gamma \left\|X^1 - X^2\right\|_{C^\gamma_t L^\infty_x} \\
&\leqslant C_{T^*}\left(\left\|b^1_0 - b^2_0\right\|_{B^{-\alpha}_{\infty,\infty}} + \left\|b^1 - b^2\right\|_{C^\gamma_t B^{-\alpha}_{\infty,\infty}} T^\gamma\right) |t-s|^\gamma \\
&+ C_{T^*} T^\gamma \left\|X^1 - X^2\right\|_{C^\gamma_t L^\infty_x} |t-s|^\gamma,
\end{aligned}$$

for which we also have by division property (see Lemma 4.A.2)

$$\begin{aligned}
&(\delta A)_{s,u,t} \\
=&|(b^1_s - b^1_u) * L_{u,t}(X^1_s) - (b^1_s - b^1_u) * L_{u,t}(X^2_s)| \\
&+ |(b^1_{s,u} - b^2_{s,u}) * L_{u,t}(X^2_s)| \\
&+ |(b^1_u - b^2_u) * L_{u,t}(X^1_s) - (b^1_u - b^2_u) * L_{u,t}(X^1_u)| \\
&+ |b^2_u * L_{u,t}(X^1_s) - b^2_u * L_{u,t}(X^1_u) - b^2_u * L_{u,t}(X^2_s) + b^2_u * L_{u,t}(X^2_u)| \\
&\leqslant C_{T^*} T^\epsilon \left\|b^1\right\|_{C^\gamma L^\infty} |u-s|^\gamma \|L\|_{C^{1/2+\epsilon}_t H^1} |t-u|^{1/2} \left\|X^1 - X^2\right\|_{C^\gamma} T^\gamma \\
&+ C_{T^*} T^\epsilon \left\|b^1 - b^2\right\|_{C^\gamma B^{-\alpha}_{\infty,\infty}} |u-s|^\gamma \|L\|_{C^{1/2+\epsilon} H^{\alpha+\epsilon}} |t-u|^{1/2} \\
&+ C_{T^*} T^\epsilon (\left\|b^1_0 - b^2_0\right\|_{B^{-\alpha}_{\infty,\infty}} + T^\gamma \left\|b^1 - b^2\right\|_{C^\gamma_t B^{-\alpha}_{\infty,\infty}}) \|L\|_{C^{1/2+\epsilon} H^{\alpha+1+\epsilon}} |t-u|^{1/2} \left\|X^1\right\|_{C^\gamma} |u-s|^\gamma \\
&+ C_{T^*} T^\epsilon \left\|b^2\right\|_{L^\infty_{t,x}} \|L\|_{C^{1/2+\epsilon} H^2_x} |t-u|^{1/2} \left\|X^1 - X^2\right\|_{C^\gamma} |u-s|^\gamma.
\end{aligned}$$

The conclusion now follows as in the proof of Lemma 4.2.3.

□

After establishing well posedness above via the Sewing Lemma, let us reference [GG20a] adapted to our setting to establish existence of a differentiable flow for $b \in Z_T^\gamma$. This adaption is possible in the setting we are concerned with thanks to Lemma 4.2.1.

Lemma 4.2.6 (Differentiable flow, [GG20a, Theorems 16, 17])**.** *Let $b \in L^\infty_{t,x}$ and $\gamma > 1/2$. Assume that for the averaging operator*

$$T_{s,t}(\cdot) = \int_s^t b(r, \cdot - w_r^H)dr$$

we have

$$\sup_{t \neq s} \frac{\left\| T_{s,t}^{(\cdot)} \right\|_{C_x^{3/2}}}{|t-s|^\gamma} < \infty$$

Then for $(T^X)_{s,t} = T_{s,t}(X_s)$, the problem

$$X_{s,t} = (\Xi T^X)_{s,t}, \qquad X_0 = x$$

admits a flow of diffeomorphisms $\Phi : \Delta_T^2 \times \mathbb{R} \to \mathbb{R}$ belonging to $C_t^\gamma C^1_{loc}$. Denoting $\Phi_t(x) = \Phi(0,t,x)$, it satisfies for a constant $C = C(\gamma, T, \|b\|_{L^\infty_{t,x}} \vee \left\|A^{(\cdot)}\right\|_{C_t^\gamma C_x^{3/2}})$

$$|\Phi_t(x) - \Phi_t(y)| \leqslant C(1+T^\gamma)|x-y|$$

and the same inequality holds for it's spatial inverse $\psi_t(\cdot) = (\Phi_t(\cdot))^{-1}$. Moreover, we have

$$D_x\Phi_{s,t}(x) = (\Xi \widetilde{B}^{(D_x\Phi,\Phi)})_{s,t}, \qquad D_x\Phi_0(x) = 1$$

where

$$\widetilde{B}_{s,t}^{(D_x\Phi,\Phi)} = ((\partial_x T)^{\Phi(x)})_{s,t}(D_x\Phi_s)(x).$$

Moreover, we have for another constant $c = c(\gamma, T, \|b\|_{L^\infty_{t,x}} \vee \left\|A^{(\cdot)}\right\|_{C_t^\gamma C_x^{3/2}})$ the bound

$$|D_x\Phi_t(x)| \leqslant c,$$

and similarly for the derivative of the inverse flow.

In the following, let us adapt the above to our context working with convolutions with the local time rather than averaging operators. For $\|b\|_{Z_T^\gamma} \leqslant 1$ we moreover establish a priori bounds uniform in space that depend only on T^*.

Lemma 4.2.7 (A priori bounds II)**.** *Let $T^* > T > 0$. Suppose $\gamma > 1/2$ and $H < 2(1-\gamma)/3$ and $b \in Z_T^\gamma$ such that $\|b\|_{Z_T^\gamma} \leqslant 1$. Then Lemma 4.2.6 applies. Assume in addition $H < 1/5$. Then we have in the notation of Lemma 4.2.6:*

$$\Phi_t(x) = x + (\Xi T^\Phi)_{0,t} = x + (\Xi A^\Phi)_{0,t} \tag{4.2.3}$$

for $t \in [0,T]$ *where* $(A^X)_{s,t} = b_s * L_{s,t}(X_s)$. *Moreover, we have*

$$|\Phi_t(x) - \Phi_t(y)| \leqslant C_{T*} t^\gamma |x-y|$$

and

$$|\Phi_t(x) - \Phi_s(x) - b_s * L_{s,t}(\Phi(_s(x))| \leqslant C_{T^*}|t-s|^{\gamma+1/2} \tag{4.2.4}$$

uniformy in $x \in \mathbb{R}$ *for* $(s,t) \in \Delta_2([0,T])$ *and some* $\epsilon > 0$. *Moreover, for* $D_x\Phi_t := D_x\Phi(0,t,x)$ *we have for*

$$B_{s,t} = (b_s * \partial_x L_{s,t})(\Phi_s(x)) D_x \Phi_s(x)$$

that

$$D_x\Phi_t(x) = 1 + (\Xi B)_{0,t} = 1 + (\Xi \widetilde{B})_{s,t}, \tag{4.2.5}$$

and

$$|(D_x\Phi(x))_{s,t} - B(x)_{s,t}| \leqslant C_{T^*}|t-s|^{\gamma+1/2} \tag{4.2.6}$$

uniformly in $x \in \mathbb{R}$ *for* $(s,t) \in \Delta_2([0,T])$ *and some* $\epsilon > 0$. *Moreover, we have the a priori bounds*

$$\|D_x\Phi\|_{C_t^\gamma L_x^\infty} \leqslant C_{T^*}$$

and

$$|D_x\Phi_t(x)| \leqslant 1 + T^\gamma C_{T^*}.$$

for $t \in [0,T], x \in \mathbb{R}$. *Moreover similar bounds hold for the inverse flow and its derivative.*

Proof. Note that we may apply Lemma 4.2.6 under the stated condition on H thanks to Lemma 4.A.4 (choosing $s = 0$, $\rho = 3/2+\delta$ for some small $\delta > 0$, p, q arbitrarily large and exploiting the corresponding Besov space embeddings in Lemma 4.B.4). Note that equations (4.2.3) and (4.2.5) are an immediate consequence of the previous Lemma 4.2.6 and Lemma 4.2.1 (respectively a slight variation of the latter one). Note that $L \in C_t^\gamma H^1 \cap C_t^{1/2+\epsilon} H^2$ for some $\epsilon > 0$. Let us note $\Phi_t(x) := \Phi(0,t,x)$, then we have

$$\Phi_t(x) - \Phi_t(y) = x - y + (\Xi A)_{0,t}$$

where $A_{s,t} = (b_s * L_s(\Phi_s(x)) - b_s * L_s(\Phi_s(y))$ and therefore

$$\begin{aligned} A_{s,t} &= (b_s * L_s(\Phi_s(x)) - b_s * L_s(\Phi_s(y)) \\ &\leqslant \|b_s * L_{s,t}\|_{C^1} |\Phi_s(x) - \Phi_s(y)| \\ &\leqslant C_{T^*} \|L\|_{C_t^\gamma H_x^1} |t-s|^\gamma (|x-y| + T^\gamma \|\Phi(x) - \Phi(y)\|_{C^\gamma}) \end{aligned}$$

Moreover, we have by the division property Lemma 4.A.2

$$\begin{aligned} (\delta A)_{s,u,t} &\leqslant |b_{s,u} * L_{u,t}(\Phi_s(x)) - b_{s,u} * L_{u,t}(\Phi_s(y))| \\ &+ |b_u * L_{u,t}(\Phi_s(x)) - b_u * L_{u,t}(\Phi_u(x)) - b_u * L_{u,t}(\Phi_s(y)) + b_u * L_{u,t}(\Phi_u(y))| \\ &\leqslant C_{T^*} T^\epsilon |u-s|^\gamma \|L\|_{C_t^{1/2+\epsilon} H_x^1} |t-u|^{1/2} (|x-y| + T^\gamma \|\Phi(x) - \Phi(y)\|_{C^\gamma}) \\ &+ C_{T^*} \|L_{u,t}\|_{H^2} (|x-y| + \|\Phi(x) - \Phi(y)\|_{C^\gamma}) |u-s|^\gamma \\ &\leqslant C_{T^*} T^\epsilon (|x-y| + T^\gamma \|\Phi(x) - \Phi(y)\|_{C^\gamma}) |t-s|^{\gamma+1/2} \\ &+ C_{T^*} T^\epsilon (|x-y| + \|\Phi(x) - \Phi(y)\|_{C^\gamma}) |t-s|^{\gamma+1/2} \end{aligned}$$

and therefore for T sufficiently small, we have indeed

$$\|\Phi(x) - \Phi(y)\|_{C^\gamma} \leqslant C_{T^*}|x-y|$$

where as in Lemma 4.2.3, we can pass to global bounds for any $T < T^*$ via Lemma 4.A.3. Concerning the bound on the derivative of the flow, we proceed similarly. Note that $D\Phi_0(x) = 1$ and thus

$$\begin{aligned}
|B_{s,t}| &= |b_s * \partial_x L_{s,t}(\Phi_s(x)) D_x\Phi_s(x)| \\
&\leqslant \|b_s * \partial_x L_{s,t}\|_\infty (1 + T^\gamma \|D_x\Phi\|_{C^\gamma}) \\
&\leqslant C_{T^*} T^\epsilon \|L\|_{C_t^{\gamma+\epsilon} H_x^1} |t-s|^\gamma (1 + T^\gamma \|D_x\Phi\|_{C^\gamma}) \\
&\leqslant C_{T^*} T^\epsilon (1 + T^\gamma \|D_x\Phi\|_{C^\gamma}) |t-s|^\gamma
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
|(\delta B)_{s,u,t}| &\leqslant |b_{s,u} * \partial_x L_{u,t}(\Phi_s) D_x\Phi_s| \\
&+ |b_u * \partial_x L_{u,t}(\Phi_s) D_x\Phi_s - b_u * \partial_x L_{u,t}(\Phi_u) D\Phi_s| \\
&+ |b_u * \partial_x L_{u,t}(\Phi_u)(D_x\Phi_u - D_x\Phi_s)| \\
&\leqslant C_{T^*} T^\epsilon \|L\|_{C_t^{1/2+\epsilon} H_x^1} (1 + T^\gamma \|D_x\Phi\|_{C^\gamma}) |t-s|^{\gamma+1/2} \\
&+ C_{T^*} T^\epsilon \|L\|_{C_t^{1/2+\epsilon} H_x^2} \|\Phi\|_{C^\gamma} (1 + T^\gamma \|D_x\Phi\|_{C^\gamma}) |t-s|^{\gamma+1/2} \\
&+ C_{T^*} T^\epsilon \|L\|_{C_t^{1/2+\epsilon} H_x^1} T^\gamma \|D_x\Phi\|_{C^\gamma} |t-s|^{\gamma+1/2}
\end{aligned}$$

and thus, as before for T sufficiently small, we do indeed recover the desired a priori bound on $\|D\Phi\|_{C^\gamma}$. Since moreover,

$$|D\Phi_t(x)| \leqslant 1 + T^\gamma \|D\Phi\|_{C^\gamma}$$

this also establishes the uniform bound on the derivative of the flow Φ. Finally, the expansion (4.2.4) is a direct consequence of Lemma 4.2.1 and the expansion (4.2.6) follows from the uniform bound on δB above and the Sewing Lemma. □

Remark 4.2.8 (Higher time regularity)**.** *Note that in our setting where we assume b to be bounded continuous, we also have from classical considerations and Lemma 4.2.1 that*

$$|\Phi_t(x) - \Phi_s(x)| = \left| \int_s^t b(u, \Phi_u(x) - w_u) du \right| \leqslant \|b\|_\infty |t-s|$$

Moreover, by differentiating the relation $\psi_t(\Phi_t(x)) = x$ in time, we have

$$|\psi_t(x) - \psi_s(x)| = \left| \int_s^t D_x\psi_u(x) b(u, x - w_u) du \right| \leqslant \|D\psi\|_\infty \|b\|_\infty |t-s|$$

4.3 The transport equation

Definition 4.3.1. *Let $b \in C_{t,x}$, $H < 1/3$. A function $f \in L^\infty_t L^1_x$ is called solution to the problem*

$$\begin{aligned} \partial_t f + \widetilde{b}\partial_x f &= 0 \\ f(0) &= f_0 \in L^1, \end{aligned}$$

where $\widetilde{b}(t,x) = b(t, x - w^H_t)$, if any $s \leqslant t \in [0,T]$ and any $\varphi \in C^2_b$

$$\langle f_t - f_s, \varphi\rangle - \langle f_s b_s * \partial_x L_{s,t}, \varphi\rangle - \langle f_s b_s * L_{s,t}, \partial_x \varphi\rangle = O(|t-s|^{1+\epsilon}),$$

for some $\epsilon > 0$. Note in particular that $H < 1/3$ implies that $b_s * \partial_x L_{s,t}$ is bounded continuous in space, uniformly in time and therefore the left hand side is meaningful.

Remark 4.3.2. *Note that for $f, b \in Z^\gamma_T$ with $\gamma > 1/2$, the germ $A_{s,t} = \langle f_s b_s * \partial_x L_{s,t}, \varphi\rangle$ admits a sewing. Moreover, note that for $s \in [0,T]$ the product $f_s \partial_x \widetilde{b}_s$ is well defined as an element in $B^{\gamma-1}_{\infty,\infty}$ (refer to Lemmata 4.B.2 and 4.B.3) and the function $s \to \langle f_s \partial_x \widetilde{b}_s, \varphi\rangle$ is continuous, meaning that similar to Lemma 4.2.1 one has*

$$(\Xi A)_{s,t} = \int_s^t \langle f_r \partial_x \widetilde{b}_r, \varphi\rangle dr.$$

Similarly, we can identify the sewing of the term $B_{s,t} = \langle f_s b_s * L_{s,t}, \partial_x \varphi\rangle$ as

$$(\Xi B)_{s,t} = \int_s^t \langle f_r \widetilde{b}_r, \partial_x \varphi\rangle dr.$$

Hence, by applying the sewing operator Ξ to the equality

$$\langle f_t - f_s, \varphi\rangle - \langle f_s b_s * \partial_x L_{s,t}, \varphi\rangle - \langle f_s b_s * L_{s,t}, \partial_x \varphi\rangle = O(|t-s|^{1+\epsilon})$$

satisfied by our solution in terms of Definition 4.3.1, we obtain

$$\langle f_t - f_0, \varphi\rangle = \int_0^t \langle f_s \partial_x \widetilde{b}_s, \varphi\rangle + \langle f_s \widetilde{b}_s, \partial_x \varphi\rangle ds.$$

Therefore, the solution to be constructed in Lemma 4.3.3 will actually be a weak solution in the classical PDE sense. Similarly for $\gamma > 1/2$ and $b \in Z^\gamma_T$ any weak solution f in the above PDE sense such that $f \in Z^\gamma_T$ is also a solution in the sense of Definition 4.3.1 thanks to an adaption of Lemma 4.2.1.

4.3.1 Existence

With the above at hand, we are ready to establish existence of solutions to the above transport equation under less restrictive conditions on the initial condition.

Lemma 4.3.3. *Let $f_0 \in L^1_x$, $\gamma > 1/2$ and $H < 1/5 \wedge 2(1-\gamma)/3$. let $b \in Z^\gamma_T$. Let $\psi : \Delta_T \times \mathbb{R} \to \mathbb{R}$ be the inverse flow associated to the problem*

$$X_{s,t} = (\Xi A^X)_{s,t} \qquad X_0 = x$$

where $(A^X)_{s,t} = b_s * L_{s,t}(X_s)$. Then $f(t,x) := f_0(\psi(0,t,x))$ is a solution to

$$\partial_t f + \widetilde{b}\partial_x f = 0$$
$$f(0) = f_0$$

in the sense of Definition 4.3.1.

Proof. Note that by change of variables and a Taylor expansion in φ we have for some point $p(s,t,x)$ between $\Phi_t(x)$ and $\Phi_s(x)$

$$\begin{aligned}
\langle f_t, \varphi\rangle &= \int_{\mathbb{R}} f_0(\psi_t(x))\varphi(x)dx \\
&= \int_{\mathbb{R}} f_0(x)\varphi(\Phi_t(x))D_x\Phi_t(x)dx \\
&= \int_{\mathbb{R}} f_0(x)\varphi(\Phi_s(x))D_x\Phi_t(x)dx + \int_{\mathbb{R}} f_0(x)(\partial_x\varphi)(\Phi_s(x))(\Phi_t(x)-\Phi_s(x))D_x\Phi_t(x)dx \\
&+ \int_{\mathbb{R}} f_0(x)\frac{1}{2}(\partial_x^2\varphi)(p(s,t,x))|\Phi_t(x)-\Phi_s(x)|^2 D_x\Phi_t(x)dx.
\end{aligned}$$

Note that by the a priori bounds in Lemmata 4.2.3 and 4.2.7, we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} f_0(x)\frac{1}{2}(\partial_x^2\varphi)(p(s,t,x))|\Phi_t(x)-\Phi_s(x)|^2 D_x\Phi_t(x)dx\right| \\
&\leqslant \|f_0\|_{L^1}\|\varphi\|_{C^2_b}\|\Phi\|^2_{C^\gamma_t L^\infty_x}|t-s|^{2\gamma}\|D_x\Phi\|_{L^\infty{}_{t,x}} = O(|t-s|^{2\gamma})
\end{aligned}$$

Note moreover that by Lemma 4.2.7

$$|D_x\Phi_t(x) - D_x\Phi_s(x) - (b_s * \partial_x L_{s,t})(\Phi_s(x))D_x\Phi_s(x)| = O(|t-s|^{\gamma+1/2})$$

and

$$|\Phi_t(x) - \Phi_s(x) - (b_s * L_{s,t})(\Phi_s(x))| = O(|t-s|^{\gamma+1/2})$$

uniformly in $x \in \mathbb{R}$. Hence, we obtain

$$\begin{aligned}
&\langle f_t - f_s, \varphi\rangle - \langle (b_s * \partial_x L_{s,t})f_s, \varphi\rangle - \langle (b_s * L_{s,t})f_s, \partial_x\varphi\rangle \\
&= \int_{\mathbb{R}} f_0(x)\varphi(\Phi_s(x))\left(D_x\Phi_t(x) - D_x\Phi_s(x)\right)dx \\
&+ \int_{\mathbb{R}} f_0(x)(\partial_x\varphi)(\Phi_s(x))(\Phi_t(x)-\Phi_s(x))D_x\Phi_t(x)dx \\
&- \langle (b_s * \partial_x L_{s,t})f_s, \varphi\rangle - \langle (b_s * L_{s,t})f_s, \partial_x\varphi\rangle \\
&= \int_{\mathbb{R}} f_0(x)\varphi(\Phi_s(x))\left(D_x\Phi_t(x) - D_x\Phi_s(x) - (\partial_x b * L_{s,t})(\Phi_s(x))D_x\Phi_s(x)\right)dx \\
&+ \int_{\mathbb{R}} f_0(x)(\partial_x\varphi)(\Phi_s(x))(\Phi_t(x)-\Phi_s(x)-(b * L_{s,t})(\Phi_s(x)))D_x\Phi_t(x)dx \\
&= O(|t-s|^{\gamma+1/2}),
\end{aligned}$$

where again we exploited $f_0 \in L^1_x$, $\varphi \in C^2_b$ and $D_x\Phi \in L^\infty_{t,x}$. □

4.3.2 Conservation of regularity

We next show that in our framework, the solution f described above preserves the regularity of b, i.e. if $b \in Z_T^\gamma$, then $f \in Z_T^\gamma$.

Lemma 4.3.4 (Conservation of regularity)**.** *Let $T^* > T > 0$ Let $\gamma > 1/2$ and $H < 1/5 \wedge 2(1-\gamma)/3$. Assume $b \in Z_T^\gamma$ with $\|b\|_{Z_T^\gamma} \leqslant 1$. Let $\psi : \Delta_2([0,T]) \times \mathbb{R} \to \mathbb{R}$ be the inverse flow associated with the problem*

$$X_{s,t} = (\Xi A^X)_{s,t}, \qquad X_0 = x,$$

where $A^X_{s,t} = b_s * L_{s,t}(X_s)$. Let $f_0 \in C^\gamma \cap L^1 \cap L^\infty$. Then for $f(t,x) := f_0(\psi(0,t,x))$, we have

$$\|f\|_{Z_T^\gamma} \leqslant (1 + C_{T^*}T^\gamma)(\|f_0\|_{C^\gamma} + \|f_0\|_{L^1} + \|f_0\|_{L^\infty}).$$

Proof. Note by Lemma 4.2.7, we have

$$|\psi_t(x) - \psi_t(y)| \leqslant C_{T*}T^\gamma |x-y|$$

and therefore in particular

$$|f(t,x) - f(t,y)| \leqslant \|f_0\|_{C^\gamma} |\psi_t(x) - \psi_t(y)|^\gamma \leqslant \|f_0\|_{C^\gamma} C_{T^*}T^\gamma |x-y|^\gamma$$

which shows that $f \in C_t^0 C_x^\gamma$. Moreover, we have by Remark 4.2.8

$$|\psi_t(x) - \psi_s(x)| \leqslant |t-s| \, \|D_x\psi\|_\infty \|b\|_\infty \leqslant (1 + C_{T^*}T^\gamma)|t-s|$$

Combined with

$$|f(t,x) - f(s,x)| \leqslant \|f_0\|_{C^\gamma} |\psi_t(x) - \psi_s(x)|^\gamma$$

we see that $f \in C_t^\gamma L_x^\infty$. Next, we have

$$\int_{\mathbb{R}} |f(t,x)| dx = \int_{\mathbb{R}} |f_0(\psi_t(x))| dx = \int_{\mathbb{R}} |f_0(x)| D_x\Phi_t(x) dx \leqslant (1 + C_{T^*}T^\gamma) \|f_0\|_{L_x^1}$$

and therefore $f \in L_t^\infty L_x^1$. Finally, it is immediate that $\|f\|_{L^\infty_{t,x}} \leqslant \|f_0\|_{L^\infty}$.

□

4.3.3 Uniqueness

In the following subsection, we establish uniqueness of solutions to the transport equation for $b \in Z_T^\gamma$ under the additional assumption of $f_0 \in C^1$. Let us first remark that in this setting, the solution f constructed in Lemma 4.3.3 is actually continuously differentiable in both time and space and therefore a strong solution. Indeed, we have

$$\partial_t f(t,x) = (\partial_x f_0)(\psi(0,t,x)) \frac{d}{dt}\psi(0,t,x) = (\partial_x f_0)(\psi(0,t,x)) D_x\psi(0,t,x) b(t, x - w_t)$$

and

$$\partial_x f(t,x) = (\partial_x f_0)(\psi(0,t,x)) \frac{d}{dt}\psi(0,t,x) = D_x\psi(0,t,x).$$

Lemma 4.3.5. *Let $b \in Z_T^\gamma$ and $f_0 \in C^1$. Then the function $f = f_0(\psi(0,t,x))$ given in Lemma 4.3.3 is the unique classical solutions to the transport equation*

$$\partial_t f + \widetilde{b}\partial_x f = 0$$
$$f(0) = f_0 \in C^1 \cap L^1 \cap L^\infty.$$

Proof. The proof we give for the sake of completeness is classical, refer to Proposition 1 of Section 5.1 in [GG20a] for example. Let u be a classical solution to the transport equation and Φ the flow associated to the problem

$$X_{s,t} = (\Xi A)_{s,t}, \qquad X_0 = x \tag{4.3.1}$$

where $A_{s,t} = b_s * L_{s,t}(X_s)$. Notice that under our regularity assumption on b (4.3.1) is equivalent to

$$X_t = x + \int_0^t b(s, X_s - w_s^H)ds$$

We therefore have that

$$\begin{aligned}\frac{d}{dt}u(t,\Phi(0,t,x)) &= \partial_t u(t,\Phi(0,t,x)) + (\partial_x u)(t,\Phi(0,t,x))b(t,\Phi(0,t,x) - w_t^H)\\ &= \left(\partial_t u + \widetilde{b}\partial_x u\right)(t,\Phi(0,t,x)) = 0\end{aligned}$$

meaning that u is constant along the characteristics implying uniqueness. □

4.3.4 Conservation of regularity in the regular case

Let us finally show that for $b \in Z^1 := C_t^1 C_x^0 \cap C_t^0 C_x^1 \cap L_t^\infty L_x^1$ and $f_0 \in C^1 \cap L^1$ the solution f to the corresponding transport equation constructed in Lemma 4.3.3 also lies in Z^1.

Lemma 4.3.6 (Conservation of regularity II)**.** *Let $b \in Z^1$ with $\|b\|_{Z^1} \leqslant 1$. Let $\psi : \Delta_2 \times \mathbb{R} \to \mathbb{R}$ be the inverse flow associated with the problem*

$$X_{s,t} = (\Xi A^X)_{s,t} \qquad X_0 = x$$

where $A_{s,t}^X = b_s * L_{s,t}(X_s)$. Let $f_0 \in C^1 \cap L^1 \cap$. Then for $f(t,x) := f_0(\psi(0,t,x))$, we have

$$\|f\|_{Z^1} \leqslant (1 + C_{T^*}T^\gamma)(\|f_0\|_{C^1} + \|f_0\|_{L^1} + \|f_0\|_{L^\infty}).$$

Proof. Remark that

$$\frac{d}{dx}f(t,x) = (\frac{d}{dx}f_0)(\psi(0,t,x))D_x\psi(0,t,x) \leqslant \|f_0\|_{C^1}(1 + C_{T^*}T^\gamma))$$

as well as

$$\frac{d}{dt}f(t,x) = (\frac{d}{dx}f_0)(\psi(0,t,x))D_x\psi(0,t,x)b(t,x - w_t^H) \leqslant \|f_0\|_{C^1}(1 + C_{T^*}T^\gamma)).$$

Finally, the last bound in $L_t^\infty L_x^1$ is the same as in Lemma 4.3.4. □

4.3.5 A priori bounds for solutions to the transport equation

In the following section, we establish some a priori bounds for solutions to the transport equation without assuming differentiable initial conditions. While these estimates do not allow to deduce uniqueness they do appear interesting in their own right. Towards this end, let us establish the following Lemma based on a DiPerna-Lions commutator argument. Note that in the following, we will not as usual require $\partial_x b \in L^\infty$, as we may harness again the regularity of the appearing local time.

Lemma 4.3.7. *For $\gamma > 1/2$, $H < 1/5 \wedge (1-\gamma)/3$ and $b \in Z_T^\gamma$ such that $\|b\|_{Z_T^\gamma} \leqslant 1$, suppose there exists a solution $f \in Z_T^\gamma$ with $\|f\|_{Z_T^\gamma} \leqslant 1$ to the problem*

$$\begin{aligned} \partial_t f + \widetilde{b}\partial_x f &= 0 \\ f(0) &= f_0 \in C^\gamma \cap L^1 \cap L^\infty \end{aligned}$$

where $\widetilde{b}(t,x) = b(t, x - w_t^H)$. Then we have

$$\|f_t\|_{L^2}^2 - \|f_s\|_{L^2}^2 = (\Xi A)_{s,t} \tag{4.3.2}$$

where

$$A_{s,t} := \int_{\mathbb{R}} (b_s * \partial_x L_{s,t})(x)(f_s^2)(x)dx$$

Proof. The proof follows the classical strategy of [DL89] based on a commutator argument. Crucially, we replace Lebesgue integration in time by corresponding sewings capable of leveraging the smoothness of the local time L associated with the regularizing path w^H. Once this regularization is exploited, we show that we can pass to the limit in the mollification, letting the intermediately appearing commutator disappear again.

Let $(\rho^\epsilon)_\epsilon$ be a sequence of positive mollifiers with ρ supported in B_1 such that $\|\rho\|_{L^1} = 1$. For a distribution ϕ we will use the abbreviation $\phi^\epsilon := \rho^\epsilon * \phi$. Suppose $f \in Z_T^\gamma$ is a solution to the transport equation

$$\partial_t f + \widetilde{b}\partial_x f = 0 \tag{4.3.3}$$

with $\widetilde{b}(x,t) = b(x - w_t, t)$ such that $b \in Z_T^\gamma$. Note that under this assumption, in particular the product

$$\widetilde{b}\partial_x f$$

is well defined for any $t \in [0,T]$ (refer to Lemmata 4.B.2 and 4.B.3). As in DiPerna-Lions we then mollify equation (4.3.3), obtaining

$$\partial_t f^\epsilon + \widetilde{b}\partial f^\epsilon = -R^\epsilon(f, \widetilde{b}) \tag{4.3.4}$$

where

$$\begin{aligned}
R^\epsilon(f,\widetilde{b})(x) &= (\widetilde{b}\partial_x f)^\epsilon - \widetilde{b}\partial_x(f^\epsilon) \\
&= \int_{\mathbb{R}} \widetilde{b}(y)(\partial_y f)(y)\rho^\epsilon(x-y)dy - \widetilde{b}(x)\int_{\mathbb{R}} f(y)\partial_x\rho^\epsilon(x-y)dy \\
&= -\int_{\mathbb{R}} \partial_y(\widetilde{b}(y)\rho^\epsilon(x-y))f(y)dy - \widetilde{b}(x)\int_{\mathbb{R}} f(y)\partial_x\rho^\epsilon(x-y)dy \\
&= -\int_{\mathbb{R}} (\partial_y\widetilde{b})(y)\rho^\epsilon(x-y)f(y)dy \\
&+ \int_{\mathbb{R}} \widetilde{b}(y)(\partial_x\rho^\epsilon)(x-y)f(y) - \widetilde{b}(x)\int_{\mathbb{R}} f(y)\partial_x\rho^\epsilon(x-y)dy \\
&= \int_{\mathbb{R}} f(y)\left(\widetilde{b}(x)-\widetilde{b}(y)\right)\partial_x\rho^\epsilon(x-y)dy - (f\partial_x\widetilde{b})^\epsilon(x) \\
&= \int_{B_1} f(x-\epsilon z)\frac{\widetilde{b}(x-\epsilon z)-\widetilde{b}(x)}{\epsilon}(\partial_x\rho)(z)dz - (f\partial_x\widetilde{b})^\epsilon(x).
\end{aligned}$$

Notice we used integration by parts above, which poses no problem though as ρ^ϵ is of compact support. Multiplying both sides of (4.3.4) by f^ϵ, we obtain

$$\partial_t(f^\epsilon)^2 + \widetilde{b}\partial(f^\epsilon)^2 = -2R^\epsilon(f,\widetilde{b})f^\epsilon$$

integrating in space and time we obtain

$$\|f_t^\epsilon\|_{L^2}^2 - \|f_s^\epsilon\|_{L^2}^2 + \int_s^t\int_{\mathbb{R}} \widetilde{b}\partial_x(f^\epsilon)^2 dudx = -2\int_s^t\int_{\mathbb{R}} R^\epsilon(f,\widetilde{b})f^\epsilon dudx \tag{4.3.5}$$

Now let us define the germ

$$A_{s,t}^\epsilon := \int_{\mathbb{R}} (b_s * \partial_x L_{s,t})(x)(f_s^\epsilon)^2(x)dx.$$

Provided the Sewing Lemma can be applied, we have by integration by parts, a Fubini for the Sewing map [Gal21, Lemma A.1] and a classical Fubini

$$\begin{aligned}
(\Xi A^\epsilon)_{s,t} &= \left(\Xi\int_{\mathbb{R}} (b_s * \partial_x L_{s,t})(x)(f_s^\epsilon)^2(x)dx\right)_{s,t} \\
&= -\left(\Xi\int_{\mathbb{R}} (b_s * L_{s,t})(x)\partial_x(f_s^\epsilon)^2(x)dx\right)_{s,t} \\
&= -\int_{\mathbb{R}} \left(\Xi(b_s * L_{s,t})(x)\partial_x(f_s^\epsilon)^2(x)\right)_{s,t} dx \\
&= -\int_{\mathbb{R}}\int_s^t \widetilde{b}_u(x)\partial_x(f_u^\epsilon)^2(x)dudx \\
&= -\int_s^t\int_{\mathbb{R}} \widetilde{b}_u(x)\partial_x(f_u^\epsilon)^2(x)dxdu.
\end{aligned}$$

Note that the integration by parts is justified, since for almost any $s \leqslant t \in [0,T]$ $\lim_{|x|\to\infty} b_s * L_{s,t}(x) = \lim_{|x|\to\infty}(f_s^\epsilon)^2(x) = 0$, which is due to the fact that $b, f \in L^\infty L^1_x \cap C^0_t C^\gamma_x$ and hence in particular for almost any $t \in [0,T]$ the functions $b_s * L_{s,t}(\cdot), (f^\epsilon)^2_s(\cdot)$ are uniformly continuous and integrable. We are therefore left with

$$\|f_t^\epsilon\|^2_{L^2} - \|f_s^\epsilon\|^2_{L^2} - (\Xi A^\epsilon)_{s,t} = -2\int_{\mathbb{R}}\int_s^t R^\epsilon(f,\widetilde{b})f^\epsilon du dx$$

By a similar reasoning, provided the Sewing Lemma can be applied, one can show that the right hand side can be given by the sewing of the germ

$$\begin{aligned}
B^\epsilon_{s,t} &= \int_{\mathbb{R}} f_s^\epsilon(x)\int_{B_1} f_s(x-\epsilon z)\frac{(b_s * L_{s,t})(x-\epsilon z) - (b_s * L_{s,t})(x)}{\epsilon}(\partial_x\rho)(z)dzdx \\
&- \int_{\mathbb{R}}(f_s b_s * \partial_x L_{s,t})^\epsilon(x) f_s^\epsilon(x)dx \\
&=: \widetilde{B^{\epsilon,1}}_{s,t} - \widetilde{B^{\epsilon,2}}_{s,t}
\end{aligned}$$

Meaning we are left with the equation

$$\|f_t^\epsilon\|^2_{L^2} - \|f_s^\epsilon\|^2_{L^2} = (\Xi A^\epsilon)_{s,t} - 2(\Xi B^\epsilon)_{s,t}. \tag{4.3.6}$$

Let us verify that indeed, A^ϵ and B^ϵ admit a sewing to establish the above equality, for which we need to check the condition of the Sewing Lemma 4.A.1. Note that

$$(\delta A^\epsilon)_{s,u,t} = \int_{\mathbb{R}}\left((f_s^\epsilon)^2(x)(b_{s,u} * \partial_x L_{u,t})(x) - ((f_s^\epsilon)^2 - (f_u^\epsilon)^2)(x)(b_u * \partial_x L_{u,t})(x)\right)dx.$$

Since $H < (1-\gamma)/3$, we have $L \in C_T^{\gamma/2+1/2}H^1_x$ and hence by L^p space interpolation

$$\begin{aligned}
&|(\delta A^\epsilon)_{s,u,t}| \\
&\leqslant \|f_s^\epsilon\|^2_{L^2}\|b_{s,u} * \partial_x L_{u,t}\|_\infty + \left\|(f_s^\epsilon)^2 - (f_u^\epsilon)^2\right\|_{L^1}\|b_s * \partial_x L_{u,t}\|_\infty \\
&\leqslant C_{T^*}\|f_s^\epsilon\|^2_{L^2}\|b\|_{C^\gamma_t L^\infty_x}|u-s|^\gamma\|L\|_{C^{1/2}_t H^1}|t-u|^{1/2} \\
&+ 2\sup_{r\in[s,t]}\|f_r^\epsilon\|_{L^2}\|f_s^\epsilon - f_u^\epsilon\|_{L^2}\|b_s * \partial_x L_{u,t}\|_\infty \\
&\leqslant C_{T^*}\|f_s^\epsilon\|^2_{L^2}|t-s|^{\gamma+1/2} + C_{T^*}\sup_{r\in[s,t]}\|f_r^\epsilon\|_{L^2}\|f_s^\epsilon - f_u^\epsilon\|^{1/2}_{L^1}\|f_s^\epsilon - f_u^\epsilon\|^{1/2}_{L^\infty}\|L_{u,t}\|_{H^1} \\
&\leqslant C_{T^*}\|f_s^\epsilon\|^2_{L^2}|t-s|^{\gamma+1/2} \\
&+ C_{T^*}\sup_{r\in[s,t]}\|f_r^\epsilon\|_{L^2}\|f\|^{1/2}_{L^\infty_t L^1_x}\|f\|^{1/2}_{C^\gamma_t L^\infty_x}|u-s|^{\gamma/2}\|L\|_{C^{\gamma/2+1/2}_t H^1_x}|t-u|^{\gamma/2+1/2} \\
&\leqslant C_{T^*}(\sup_{r\in[s,t]}\|f_r^\epsilon\|^2_{L^2} + \sup_{r\in[s,t]}\|f_s^\epsilon\|_{L^2})|t-s|^{\gamma+1/2},
\end{aligned}$$

which yields the desired error bound. Now we need to bound $(\delta B^\epsilon)_{s,u,t}$ for which we

consider the two relevant expressions $\widetilde{B^{\epsilon,1}}_{s,t}$ and $\widetilde{B^{\epsilon,2}}_{s,t}$ separately. First we have

$$
\begin{aligned}
&|\delta(\widetilde{B^{\epsilon,2}})_{s,u,t}| \\
&\leqslant \int_{\mathbb{R}} |((f_s \partial_x b_s * L_{s,t})^\epsilon(x) f_s^\epsilon) dx| \\
&\leqslant \int_{\mathbb{R}} |(\rho^\epsilon * ((f_{s,u}) b_s * \partial_x L_{u,t}))(x) f_s^\epsilon(x)| dx \\
&+ \int_{\mathbb{R}} |(\rho^\epsilon * (f_u b_{s,u} * \partial_x L_{u,t}))(x) f_s^\epsilon(x)| dx \\
&+ \int_{\mathbb{R}} |(\rho^\epsilon * (f_u b_u * \partial_x L_{u,t}))(x)(f_s^\epsilon(x) - f_u^\epsilon(x))| dx \\
&\leqslant \|(f_{s,u}) b_s * \partial_x L_{u,t}\|_{L^2} \|f_s^\epsilon\|_{L^2} \\
&+ \|(f_u) b_{s,u} * \partial_x L_{u,t}\|_{L^2} \|f_s^\epsilon\|_{L^2} + \|(f_u) b_u * \partial_x L_{u,t}\|_{L^2} \|f_s^\epsilon - f_u^\epsilon\|_{L^2} \\
&\leqslant \|f_s - f_u\|_{L^1}^{1/2} \|f_s - f_u\|_{L^\infty}^{1/2} \|b_s * \partial_x L_{u,t}\|_\infty \|f_s^\epsilon\|_{L^2} \\
&+ \|f_u\|_{L^2} \|b_{s,u} * \partial_x L_{u,t}\|_\infty \|f_s^\epsilon\|_{L^2} + \|f_u\|_{L^2} \|b_u * \partial_x L_{u,t}\|_\infty \|f_s^\epsilon - f_u^\epsilon\|_{L^1}^{1/2} \|f_s^\epsilon - f_u^\epsilon\|_{L^\infty}^{1/2} \\
&\leqslant C_{T^*} |u-s|^{\gamma/2} \|L\|_{C_t^{\gamma/2+1/2} H_x^1} |t-u|^{\gamma/2+1/2} \|f_s^\epsilon\|_{L^2} \\
&+ C_{T^*} |u-s|^{\gamma} \|L\|_{C_t^{1/2} H_x^1} |t-u|^{\gamma+1/2} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2}^2 \\
&+ C_{T^*} |u-s|^{\gamma/2} \|L\|_{C_t^{\gamma/2+1/2} H_x^1} |t-u|^{\gamma/2+1/2} \\
&\leqslant C_{T^*} \left(\sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} (1 + \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2}) \right) |t-s|^{\gamma+1/2},
\end{aligned}
$$

where we used again Fubini and the Fatou property. Secondly we have similarly

$$
\begin{aligned}
&\left|(\delta\widetilde{B^{\epsilon,1}})_{s,u,t}\right| \\
&\leqslant \left| \int_{\mathbb{R}} (f_s^\epsilon(x) - f_u^\epsilon(x)) \int_{B_1} f_s(x-\epsilon z) \frac{(b_s * L_{u,t})(x-\epsilon z) - (b_s * L_{u,t})(x)}{\epsilon z} z(\partial_x \rho)(z) dz dx \right| \\
&+ \left| \int_{\mathbb{R}} f_u^\epsilon(x) \int_{B_1} (f_s(x-\epsilon z) - f_u(x-\epsilon z)) \frac{(b_s * L_{u,t})(x-\epsilon z) - (b_s * L_{u,t})(x)}{\epsilon z} z(\partial_x \rho)(z) dz dx \right| \\
&+ \left| \int_{\mathbb{R}} f_u^\epsilon(x) \int_{B_1} f_u(x-\epsilon z) \frac{(b_{s,u} * L_{u,t})(x-\epsilon z) - (b_{s,u} * L_{u,t})(x)}{\epsilon z} z(\partial_x \rho)(z) dz dx \right| \\
&\leqslant \|f_s^\epsilon - f_u^\epsilon\|_{L^1}^{1/2} \|f_s^\epsilon - f_u^\epsilon\|_{L^\infty}^{1/2} \|f_s\|_{L^2} \|b_s * \partial_x L_{u,t}\|_\infty \\
&+ \|f_u^\epsilon\|_{L^2} \|f_u - f_s\|_{L^2} \|b_s * \partial_x L_{u,t}\|_\infty \\
&+ \|f_u^\epsilon\|_{L^2} \|f_u\|_{L^2} \|b_{s,u} * \partial_x L_{u,t}\|_\infty \\
&\leqslant C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} |u-s|^{\gamma/2} \|L\|_{C_t^{1+\epsilon-\gamma/2} H^1} |t-u|^{1+\epsilon-\gamma/2} \\
&+ C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} |u-s|^{\gamma/2} \|L\|_{C_t^{\gamma/2+1/2} H^1} |t-u|^{\gamma/2+1/2} \\
&+ C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2}^2 |u-s|^{\gamma} \|L\|_{C_t^{\gamma/2+1/2} H^1} |t-u|^{\gamma/2+1/2}.
\end{aligned}
$$

This shows that indeed, the germ $B^\epsilon_{s,t}$ admits a sewing as well, meaning we did establish that (4.3.5) is equivalent to (4.3.6) thanks to a version of Lemma 4.2.1. We next show that in the limit $\epsilon \to 0$, we obtain the claimed equation

$$\|f_t\|^2_{L^2} - \|f_s\|^2 = (\Xi A)_{s,t} \tag{4.3.7}$$

where

$$A_{s,t} := \int_{\mathbb{R}} (b_s * \partial_x L_{s,t})(x)(f_s)^2(x)dx.$$

Towards this end, we show that

$$\lim_{\epsilon\to 0} \sup_{s<t} \frac{(\Xi(A^\epsilon - A))_{s,t}}{|t-s|^\gamma} = 0, \qquad \lim_{\epsilon\to 0} \sup_{s<t} \frac{(\Xi B^\epsilon - 0)_{s,t}}{|t-s|^\gamma} = 0.$$

Note that towards this end that by Sewing Lemma, this can be shown by establishing

$$\lim_{\epsilon\to 0} \sup_{s<t} \frac{(A^\epsilon - A)_{s,t}}{|t-s|^\gamma} = 0, \qquad \lim_{\epsilon\to 0} \sup_{s<u<t} \frac{(\delta(A^\epsilon - A))_{s,u,t}}{|t-s|^\eta} = 0$$

as well as

$$\lim_{\epsilon\to 0} \sup_{s<t} \frac{(B^\epsilon)_{s,t}}{|t-s|^\gamma} = 0, \qquad \lim_{\epsilon\to 0} \sup_{s<u<t} \frac{(\delta B^\epsilon)_{s,u,t}}{|t-s|^\eta} = 0$$

for some $\eta > 1$. Towards this end, we exploit that in our setting,

$$\lim_{\epsilon\to 0} \sup_{s\in[0,T]} \|f^\epsilon_s - f_s\|_{L^p} = 0$$

for $p \in [1,\infty)$. This follows essentially from the fact that $f \in C^0_t C^\gamma_x$ and a proof is given in the Appendix in Lemma 4.A.5.

Concerning the first germ A^ϵ, we have

$$\begin{aligned}
A^\epsilon_{s,t} - A_{s,t} &= \int_{\mathbb{R}} (b_s * \partial_x L_{s,t})(x)((f^\epsilon_s)^2 - (f_s)^2)(x)dx \\
&\leqslant \|b_s * \partial_x L_{s,t}\|_\infty \left\|(f^\epsilon_s)^2 - (f_s)^2\right\|_1 \\
&\leqslant C_{T^*} \|b\|_{L^\infty_{t,x}} \|L\|_{C^\gamma_t H^1_x} \|f^\epsilon_s + f_s\|_{L^2} \|f^\epsilon_s - f_s\|_{L^2} |t-s|^\gamma \\
&\leqslant C_{T^*} \sup_{r\in[s,t]} \|f^\epsilon_r\|_{L^2} \sup_{s\in[0,T]} \|f^\epsilon_s - f_s\|_{L^2} |t-s|^\gamma,
\end{aligned}$$

from which we conclude that indeed $\|A^\epsilon - A\|_{C^\gamma} \to 0$. Moreover, we have

$$
\begin{aligned}
&(\delta A^\epsilon)_{s,u,t} - (\delta A)_{s,u,t} \\
&= -\int_{\mathbb{R}} \big((f_s^\epsilon)^2(x)(b_{s,u} * \partial_x L_{u,t})(x) - ((f_s^\epsilon)^2 - (f_u^\epsilon)^2)(x)(b_u * \partial_x L_{u,t})(x)\big)\, dx \\
&+ \int_{\mathbb{R}} \big((f_s)^2(x)(b_{s,u} * \partial_x L_{u,t})(x) - ((f_s)^2 - (f_u)^2)(x)(b_u * \partial_x L_{u,t})(x)\big)\, dx \\
&= \int_{\mathbb{R}} (b_{s,u} * \partial_x L_{u,t})(x)((f_s^\epsilon)^2 - (f_s)^2)(x) dx \\
&+ \int_{\mathbb{R}} ((f_s^\epsilon)^2 - (f_s)^2 - (f_u^\epsilon)^2 + (f_u)^2)(x)(b_u * \partial_x L_{u,t})(x) dx \\
&\leqslant C_{T^*} \|L\|_{C_t^{1/2} H_x^1} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} \sup_{s\in[0,T]} \|f_s^\epsilon - f_s\|_{L^2} |u-s|^\gamma |t-u|^{1/2} \\
&+ \int_{\mathbb{R}} [(f_s^\epsilon + f_u^\epsilon)(f_{u,s}^\epsilon - f_{u,s})](x)(b_u * \partial_x L_{u,t})(x) dx \\
&+ \int_{\mathbb{R}} [(f_s - f_u)(f_u^\epsilon + f_s^\epsilon - f_s - f_u)](x)(b_u * \partial_x L_{u,t})(x) dx \\
&\leqslant C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} \sup_{s\in[0,T]} \|f_s^\epsilon - f_s\|_{L^2} |t-s|^{\gamma+1/2} \\
&+ C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} \left\|f_{u,s}^\epsilon - f_{u,s}\right\|_{L^1}^{1/2} \left\|f_{u,s}^\epsilon - f_{u,s}\right\|_{L^\infty}^{1/2} \|L\|_{C_t^{\gamma/2+1/2} H_x^1} |t-u|^{\gamma/2+1/2} \\
&+ C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon - f_r\|_{L^2} \|f_s - f_u\|_{L^1}^{1/2} \|f_s - f_u\|_{L^\infty}^{1/2} \|L\|_{C_t^{\gamma/2+1/2} H_x^1} |t-u|^{\gamma/2+1/2} \\
&\leqslant C_{T^*} \sup_{s\in[0,T]} \|f_s^\epsilon - f_s\|_{L^2} |t-s|^{\gamma+1/2} \\
&+ C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon\|_{L^2} \sup_{r\in[s,t]} \|f_r^\epsilon - f_r\|_{L^1}^{1/2} \|f\|_{C_t^\gamma L_x^\infty}^{1/2} |u-s|^{\gamma/2} |t-u|^{\gamma/2+1/2} \\
&+ C_{T^*} \sup_{r\in[s,t]} \|f_r^\epsilon - f_r\|_{L^2} \|f\|_{L_t^\infty L_x^1}^{1/2} \|f\|_{C_t^\gamma L_x^\infty}^{1/2} |u-s|^{\gamma/2} |t-u|^{\gamma/2+1/2},
\end{aligned}
$$

from which we conclude that indeed

$$
\sup_{s<t} \frac{|(\delta A^\epsilon)_{s,u,t} - (\delta A)_{s,u,t}|}{|t-s|^{\gamma+1/2}} \to 0.
$$

We proceed with the second germ B^ϵ. Let us first remark that we may rewrite this germ as

$$
\begin{aligned}
B_{s,t}^\epsilon = &\int_{\mathbb{R}} f_s^\epsilon(x) \int_{B_1} f_s(x-\epsilon z) \frac{(b_s * L_{s,t})(x-\epsilon z) - (b_s * L_{s,t})(x)}{\epsilon} (\partial_x \rho)(z) dz dx \\
&- \int_{\mathbb{R}} f_s^\epsilon(x) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{s,t})(x-\epsilon z)\rho(z) dz dx.
\end{aligned}
$$

Employing the Taylor expansion

$$
(b_s * L_{s,t})(x) = (b_s * L_{s,t})(x-\epsilon z) + (\epsilon z)(b_s * \partial_x L_{s,t})(x-\epsilon z) + \frac{1}{2}(\epsilon z)^2 (b_s * \partial_x^2 L_{s,t})(\zeta_{x,z,\epsilon})
$$

with Lagrange remainder for some $\zeta_{x,z,\epsilon}$ we may further rewrite B^ϵ as

$$\begin{aligned}
B^\epsilon_{s,t} &= -\int_{\mathbb{R}} f^\epsilon_s(x) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{s,t})(x-\epsilon z) z (\partial_x \rho)(z) dz dx \\
&\quad - \int_{\mathbb{R}} f^\epsilon_s(x) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{s,t})(x-\epsilon z) \rho(z) dz dx - R^\epsilon_{s,t} \\
&= -(B^{1,\epsilon}_{s,t} + B^{2,\epsilon}_{s,t} + R^\epsilon_{s,t})
\end{aligned}$$

where

$$R^\epsilon_{s,t} = \epsilon \int_{\mathbb{R}} f^\epsilon_s(x) \int_{B_1} f(x-\epsilon z)(b_s * \partial_x^2 L_{s,t})(\zeta_{x,z}) z^2 (\partial_x \rho)(z) dz dx$$

We now show first that B^ϵ converges to

$$\begin{aligned}
&- \int_{\mathbb{R}} f_s(x) \int_{B_1} f_s(x)(b_s * \partial_x L_{s,t})(x) z (\partial_x \rho)(z) dz dx \\
&- \int_{\mathbb{R}} f_s(x) \int_{B_1} f_s(x)(b_s * \partial_x L_{s,t})(x) \rho(z) dz dx \\
&= -(B^1_{s,t} + B^2_{s,t}) \\
&= 0.
\end{aligned}$$

Notice that

$$\begin{aligned}
&B^{2,\epsilon}_{s,t} - B^2_{s,t} \\
&= \int_{\mathbb{R}} f^\epsilon_s(x) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{s,t})(x-\epsilon z) \rho(z) dz dx \\
&- \int_{\mathbb{R}} f_s(x) \int_{B_1} f_s(x)(b_s * \partial_x L_{s,t})(x) \rho(z) dz dx \\
&= \int_{\mathbb{R}} (f^\epsilon_s - f_s)(x) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{s,t})(x-\epsilon z) \rho(z) dz dx \\
&+ \int_{\mathbb{R}} f_s(x) \int_{B_1} (f_s(x-\epsilon z) - f_s(x))(b_s * \partial_x L_{s,t})(x-\epsilon z) \rho(z) dz dx \\
&+ \int_{\mathbb{R}} f_s(x) \int_{B_1} f_s(x)((b_s * \partial_x L_{s,t})(x-\epsilon z) - (b_s * \partial_x L_{s,t})(x)) \rho(z) dz dx \\
&\leqslant C_{T^*} \sup_{r \in [s,t]} \|f^\epsilon_r - f_r\|_{L^1} \|f\|_{L^\infty_{t,x}} \|b\|_{L^\infty_{t,x}} \|L\|_{C^\gamma_t H^1_x} |t-s|^\gamma \\
&+ \epsilon^\gamma C_{T^*} \|f\|_{L^\infty_t L^1_x} \|f\|_{C^0_t C^\gamma_x} \left(\int_{B_1} |z|^\gamma \rho(z) dz \right) \|b\|_{L^\infty_{t,x}} \|L\|_{C^\gamma_t H^1_x} |t-s|^\gamma \\
&+ \epsilon C_{T^*} \|f\|_{L^\infty_t L^1_x} \|f\|_{L^\infty_{t,x}} \left(\int_{B_1} |z| \rho(z) dz \right) \|b\|_{L^\infty_{t,x}} \|L\|_{C^\gamma_t H^2_x} |t-s|^\gamma.
\end{aligned}$$

Similar estimates hold for $B^{1,\epsilon} - B^1_{s,t}$. Moreover, it is easy to see that $\|R^\epsilon\|_{C^\gamma} \to 0$. This allows to conclude that indeed

$$\lim_{\epsilon \to 0} \sup_{s<t} \frac{|B^\epsilon_{s,t}|}{|t-s|^\gamma} = 0$$

Finally, we also need to check $(\delta B^\epsilon)_{s,u,t}$. We have $\delta B^\epsilon = -\delta B^{1,\epsilon} - \delta B^{2,1} + \delta R^\epsilon$. We show again that for

$$B^2_{s,t} = \int_{\mathbb{R}} f_s(x) \int_{B_1} f_s(x)(b_s * \partial_x L_{s,t})(x)\rho(z)dzdx$$

we have

$$\sup_{s<u<t} \frac{(\delta B^{2,\epsilon} - B^2)_{s,u,t}}{|t-s|^\eta} \to 0$$

Towards this end, note that

$$\begin{aligned}
(\delta B^{2,\epsilon})_{s,u,t} &= \int_{\mathbb{R}} (f^\epsilon_s(x) - f^\epsilon_u(x)) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f^\epsilon_u(x) \int_{B_1} (f_s(x-\epsilon z) - f_u(x-\epsilon z))(b_s * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f^\epsilon_u(x) \int_{B_1} f_u(x-\epsilon z)(b_{s,u} * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&= I^\epsilon_{s,u,t} + II^\epsilon_{s,u,t} + III^\epsilon_{s,u,t}
\end{aligned}$$

and

$$\begin{aligned}
(\delta B^2)_{s,u,t} &= \int_{\mathbb{R}} (f_s(x) - f_u(x)) \int_{B_1} f_s(x)(b_s * \partial_x L_{u,t})(xz)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f_u(x) \int_{B_1} (f_s(x) - f_u(x))(b_s * \partial_x L_{u,t})(x)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f_u(x) \int_{B_1} f_u(x)(b_{s,u} * \partial_x L_{u,t})(x)\rho(z)dzdx \\
&= I_{s,u,t} + II_{s,u,t} + III_{s,u,t}
\end{aligned}$$

We shall consider the differences $I^\epsilon - I$, $II^\epsilon - II$ and $III^\epsilon - III$ separately. We have that

$$\begin{aligned}
&III^\epsilon_{s,u,t} - III_{s,u,t} \\
&= \int_{\mathbb{R}} (f^\epsilon_u - f_u)(x) \int_{B_1} f_u(x-\epsilon z)(b_{s,u} * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f_u(x) \int_{B_1} (f_u(x-\epsilon z) - f_u(x))(b_{s,u} * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f_u(x) \int_{B_1} f_u(x) \left((b_{s,u} * \partial_x L_{u,t})(x-\epsilon z) - (b_{s,u} * \partial_x L_{u,t})(x)\right)\rho(z)dzdx \\
&\leqslant C_{T^*} \sup_{s\in[0,T]} \|f^\epsilon_s - f_s\|_{L^1} \|f\|_{L^\infty_{t,x}} \|b\|_{C^\gamma_t L^\infty_x} \|L\|_{C^{1/2}_t H^1_x} |t-s|^{\gamma+1/2} \\
&+ \epsilon^\gamma C_{T^*} \|f\|_{L^\infty_t L^1_x} \|f\|_{C^0_t C^\gamma_x} \left(\int_{B_1} |z|^\gamma \rho(z)dz\right) \|b\|_{C^\gamma_t L^\infty_x} \|L\|_{C^{1/2}_t H^1_x} |t-s|^{\gamma+1/2} \\
&+ \epsilon C_{T^*} \|f\|_{L^\infty_t L^1_x} \|f\|_{L^\infty_{t,x}} \left(\int_{B_1} |z|\rho(z)dz\right) \|b\|_{C^\gamma_t L^\infty_x} \|L\|_{C^{1/2}_t H^2_x} |t-s|^{\gamma+1/2}
\end{aligned}$$

and therefore $III^\epsilon - III \to 0$. Moreover, we have

$$
\begin{aligned}
&II^\epsilon_{s,u,t} - II_{s,u,t} \\
&= \int_{\mathbb{R}} (f^\epsilon_u - f_u)(x) \int_{B_1} (f_s(x-\epsilon z) - f_u(x-\epsilon z))(b_s * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f_u(x) \int_{B_1} (f_{u,s}(x-\epsilon z) - f_{u,s}(x))(b_s * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dzdx \\
&+ \int_{\mathbb{R}} f_u(x) \int_{B_1} f_{u,s}(x) \left((b_s * \partial_x L_{u,t})(x-\epsilon z) - (b_s * \partial_x L_{u,t})(x)\right)\rho(z)dzdx \\
&\leqslant C_{T^*} \sup_{r\in[0,T]} \|f^\epsilon_r - f_r\|_{L^1} \|f\|_{C^\gamma_t L^\infty_x} |u-s|^\gamma \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2}_t H^1_x} |t-u|^{1/2} \\
&+ \sup_{r\in[0,T]} \|f_r\|_{L^2} \left(\int_{B_1} \|f_{u,s}(\cdot-\epsilon z) - f_{u,s}(\cdot)\|_{L^2} \rho(z)dz\right) \|b_s * \partial_x L_{u,t}\|_{L^\infty} \\
&+ \epsilon C_{T^*} \|f\|_{L^\infty_t L^1_x} \|f\|_{C^\gamma_t L^\infty_x} |u-s|^\gamma \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2}_t H^2_x} |t-u|^{1/2} \left(\int_{B_1} |z|\rho(z)dz\right) \\
&\leqslant C_{T^*} \sup_{r\in[0,T]} \|f^\epsilon_r - f_r\|_{L^1} |t-s|^{\gamma+1/2} \\
&+ \sup_{r\in[0,T]} \|f_r\|_{L^2} \sqrt{2} \left(\sup_{z\in B_1} \sup_{r\in[0,T]} \|f_r(\cdot-\epsilon z) - f_r(\cdot)\|^{1/2}_{L^1}\right) \sqrt{2} \|f_{u,s}\|^{1/2}_{L^\infty} \|b_s * \partial_x L_{u,t}\|_{L^\infty} \\
&+ \epsilon C_{T^*} |t-s|^{\gamma+1/2} \\
&\leqslant C_{T^*} \sup_{r\in[0,T]} \|f^\epsilon_r - f_r\|_{L^1} |t-s|^{\gamma+1/2} \\
&+ C_{T^*} \left(\sup_{z\in B_1} \sup_{r\in[0,T]} \|f_r(\cdot-\epsilon z) - f_r(\cdot)\|^{1/2}_{L^1}\right) \|f\|_{C^\gamma_t L^\infty_x} \|L\|_{C^{\gamma/2+1/2}_t H^1_x} |t-s|^{\gamma+1/2} \\
&+ \epsilon C_{T^*} |t-s|^{\gamma+1/2},
\end{aligned}
$$

which vanishes in the limit due to the continuity of the translation operator in L^1. .

Finally, we have

$$
\begin{aligned}
&I^{\epsilon}_{s,u,t} - I_{s,u,t} \\
&= \int_{\mathbb{R}} (f^{\epsilon}_{u,s} - f_{u,s})(x) \int_{B_1} f_s(x-\epsilon z)(b_s * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dz \\
&+ \int_{\mathbb{R}} f_{u,s}(x) \int_{B_1} (f_s(x-\epsilon z) - f_s(x))(b_s * \partial_x L_{u,t})(x-\epsilon z)\rho(z)dz \\
&+ \int_{\mathbb{R}} f_{u,s}(x) \int_{B_1} f_s(x) \left((b_s * \partial_x L_{u,t})(x-\epsilon z) - (b_s * \partial_x L_{u,t})(x)\right)\rho(z)dz \\
&\leqslant \left\| f^{\epsilon}_{u,s} - f_{u,s} \right\|_{L^2} \|f\|_{L^\infty_t L^2_x} \|b_s * \partial_x L_{u,t}\|_{L^\infty} \\
&+ \|f_{s,u}\|_{L^\infty} \sup_{z\in B_1} \sup_{r\in[0,T]} \|f_r(\cdot - \epsilon z) - f_r(\cdot)\|_{L^1} \|b_s * \partial_x L_{u,t}\|_{L^\infty} \\
&+ \epsilon \|f_{u,s}\|_{L^\infty} \|f\|_{L^\infty_t L^1_x} \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2}_t H^2_x} \left(\int_{B_1} |z|\rho(z)\right) |t-u|^{1/2} \\
&\leqslant C_{T^*} \|f^{\epsilon} - f\|^{1/2}_{L^\infty_t L^1} \|f\|^{1/2}_{C^\gamma_t L^\infty} |u-s|^{\gamma/2} \|f\|_{L^\infty_t L^2_x} \|b\|_{L^\infty_{t,x}} \|L\|_{C^{\gamma/2+1/2}_t H^1_x} |t-u|^{\gamma/2+1/2} \\
&+ C_{T^*} \|f\|_{C^\gamma_t L^\infty} |u-s|^{\gamma} \sup_{z\in B_1} \sup_{r\in[0,T]} \|f_r(\cdot - \epsilon z) - f_r(\cdot)\|_{L^1} \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2}_t H^1_x} |t-u|^{1/2} \\
&+ \epsilon C_{T^*} \|f\|_{C^\gamma_t L^\infty} |u-s|^{\gamma} \|f\|_{L^\infty_t L^1_x} \|b\|_{L^\infty_{t,x}} \|L\|_{C^{1/2}_t H^2_x} \left(\int_{B_1} |z|\rho(z)\right) |t-u|^{1/2},
\end{aligned}
$$

which again vanishes due to the continuity of the translation operator in L^1. In total, we have shown that

$$\sup_{s<u<t} \frac{(\delta(B^{2,\epsilon} - B^2))_{s,u,t}}{|t-s|^{\gamma+1/2}} \to 0.$$

By simply replacing ρ by $z(\partial_x \rho)(z)$ in the above calculations, one obtains

$$\sup_{s<u<t} \frac{(\delta(B^{1,\epsilon} - B^1))_{s,u,t}}{|t-s|^{\gamma+1/2}} \to 0$$

Moreover, it is again easily seen that

$$\sup_{s<u<t} \frac{(\delta R^{\epsilon})_{s,u,t}}{|t-s|^{\gamma+1/2}} \to 0$$

As $B^1_{s,t} + B^2_{s,t} = (\delta(B^1 + B^2))_{s,u,t} = 0$, this mean that overall we can conclude

$$\lim_{\epsilon\to 0} \sup_{s<u<t} \frac{|\delta B^{\epsilon}_{s,u,t}|}{|t-s|^{\gamma+1/2}} = 0$$

Overall, recalling $\sup_{s\in[0,T]} \|f^{\epsilon}_s\|^2_{L^2} \to \sup_{s\in[0,T]} \|f^{\epsilon}_s\|^2_{L^2}$ this completes the proof. □

Remark 4.3.8. *Let us remark that equalities similar to* (4.3.2) *in the L^p setting for $p \in (1,\infty)$ can be readily obtained by similar arguments as presented above.*

Lemma 4.3.9. *Assume the conditions of Lemma 4.3.7. Assume in addition that for a given solution f to the transport equation, we have*

$$\int_{\mathbb{R}} |f_t^2 - f_s^2|(x)dx \leqslant c \sup_{r\in[0,t]} \|f_r\|_{L^2}^2 |t-s|^\gamma \tag{4.3.8}$$

Then we also have the a priori bound

$$\sup_{t\in[0,T]} \|f_t\|_{L^2}^2 \leqslant C_{T,L} \|f_0\|_{L^2}^2 .$$

Proof. Notice that thanks to our additional assumption we may establish a more refined a priori bound on the sewing $(\Xi A)_{s,t}$, notably we for

$$A_{s,t} = \int_{\mathbb{R}} (b_s * \partial_x L_{s,t})(x)(f_s)^2(x)dx$$

that

$$|A_{s,t}| \leqslant C_{T^*} \sup_{r\in[s,t]} \|f_r\|_{L^2}^2 \|b\|_{L^\infty_{t,x}} \|L\|_{C_t^\gamma H_x^1} |t-s|^\gamma$$

and in particular

$$\begin{aligned}(\delta A)_{s,u,t} &= \int_{\mathbb{R}} \big((f_s)^2(x)(b_{s,u} * \partial_x L_{u,t})(x) - ((f_s)^2 - (f_u)^2)(x)(b_u * \partial_x L_{u,t})(x)\big)\, dx \\ &\leqslant C_{T^*} \sup_{r\in[s,t]} \|f_r\|_{L^2}^2 \|b\|_{C_t^\gamma L_x^\infty} \|L\|_{C_t^{1/2} H_x^1} |t-s|^{\gamma+1/2} \\ &+ C_{T^*} \sup_{r\in[0,t]} \|f_r\|_{L^2}^2 |t-s|^\gamma \|b\|_{L^\infty_{t,x}} \|L\|_{C_t^{1/2} H_x^1} |t-s|^{1/2}\end{aligned}$$

These bounds then yield the following bound for the Sewing

$$|(\Xi A)_{0,t}| \leqslant C_{T^*} T^\gamma \sup_{r\in[0,t]} \|f_r\|_{L^2}^2 .$$

Going back to (4.3.2), this finally yields

$$\|f_t\|_{L^2}^2 \leqslant \|f_0\|_{L^2}^2 + C_{T^*} T^\gamma \sup_{r\in[0,t]} \|f_r\|_{L^2}^2$$

and therefore for T chosen sufficiently small (independent of the initial condition), we obtain the claim

$$\sup_{t\in[0,T]} \|f_t\|_{L^2}^2 \leqslant C_{T^*} \|f_0\|_{L^2}^2 .$$

□

Remark 4.3.10. *Notice that the additional nonlinear condition* (4.3.8) *we had to impose in the previous Lemma doesn't allow us to establish uniqueness to solutions of the transport equation, as it is not clear why the difference of two solutions should*

still satisfy (4.3.8). *Alternatively, by only exploiting an interpolation argument and the assumption that* $f \in Z_T^\gamma$, *one can observe that*

$$\begin{aligned}\left\|f_s^2 - f_u^2\right\|_{L^1} &\leqslant 2\left\|f\right\|_{L_t^\infty L_x^2}\left\|f_s - f_u\right\|_{L^2}\\ &\leqslant 2\sqrt{2}\sup_{r\in[0,T]}\left\|f_r\right\|_{L_x^2}\left\|f\right\|_{L_t^1 L_x^\infty}^{1/2}\left\|f\right\|_{C_t^\gamma L^\infty}|u-s|^\gamma.\end{aligned}$$

In this case however, note however that the bound is no longer quadratic in $\sup_{r\in[0,T]}\left\|f_r\right\|_{L^2}$, *meaning that we only have*

$$|(\Xi A)_{0,t}| \leqslant c_{T,L}(\sup_{r\in[0,T]}\left\|f_r\right\|_{L^2}^2 + \sup_{r\in[0,T]}\left\|f_r\right\|_{L^2})$$

which does not allow us to obtain a bound as in the previous Lemma. Essentially, exploiting some Hölder regularity in time of the expression $f_s^2 - f_u^2$ *in this setting (which is necessary in order to establish the Sewing), one automatically looses the quadratic bound in* $\sup_{r\in[0,T]}\left\|f_r\right\|_{L^2}$ *unless imposing the additional nonlinear constraint* (4.3.8).

4.4 Averaged Burgers' equation

From previous sections, we see that under the conditions of Lemma 4.3.3 and for $f_0 \in C^\gamma \cap L^1 \cap L^\infty$ we can define the mapping

$$\begin{aligned}K : Z_T^\gamma &\to Z_T^\gamma\\ b &\to f\end{aligned}$$

associating a given function b to the solution solution of the regularized transport equation

$$\begin{aligned}\partial_t f + \widetilde{b}\partial_x f &= 0\\ f(0) &= f_0,\end{aligned}$$

given in Lemma 4.3.3. Moreover, it is easy to see that for $f_0 \in C^1 \cap L^1 \cap L^\infty$ sufficiently small, the sequence of iterations in the map K, i.e. the sequence $(f^n)_n := (K^{(n)}(f_0))_n$ stays uniformly bounded in Z_T^γ. This is a direct consequence of the conservation of regularity in Lemma 4.3.4. We next show convergence of the sequence $(f^n)_n$ in a space of distributions, whose limit we can identify as an element Z_T^γ thanks to the a priori bounds established.

Lemma 4.4.1 (Contraction)**.** *Let* $\gamma > 1/2$ *and* $H < 1/7 \wedge 2(1-\gamma)/5$. *Let* $f_0 \in C^{0,\gamma} \cap W^{1,1}$ *and* $b^1, b^2 \in Z_T^\gamma$ *be some functions such that* $\left\|b^1\right\|_{Z_T^\gamma} \wedge \left\|b^2\right\|_{Z_T^\gamma} \leqslant 1$ *and* $b^1(0,\cdot) = b^2(0,\cdot)$. *For* $\widetilde{b}^i(t,x) = b^i(t, x - w_t^H)$, *let* f^i *be the solutions to the problem problem*

$$\begin{aligned}\partial_t f^i + \widetilde{b}^i\partial_x f^i &= 0\\ f^i(0) &= f_0\end{aligned} \tag{4.4.1}$$

given in Lemma 4.3.3. Then we have for some $\epsilon > 0$ *and* C_{T^*}

$$\left\|f^1 - f^2\right\|_{C_t^\gamma B_{\infty,\infty}^{-(2+\epsilon)}} \leqslant C_{T^*}\left\|f_0\right\|_{W^{1,1}}\left\|b^1 - b^2\right\|_{C_t^\gamma B_{\infty,\infty}^{-(2+\epsilon)}}.$$

Proof. Let $f^i = f_0(\psi^i_t(x))$ be the solution to 4.4.1 where ψ^i is the inverse flow generated by solutions associated with the problem

$$X^i_{s,t} = (\Xi A^{i,X^i})_{s,t}, \qquad X^i_0 = x$$

where $(A^{i,X^i})_{s,t} = b^i_s * L_{s,t}(X^i_s)$. For $\varphi \in \mathcal{S}$ and let $\bar{\varphi}(x) := \int_{-\infty}^x \varphi(y)dy$ be a primitive. Then for any $\epsilon > 0$ we have the continuous embeddings $B^{2+\epsilon}_{1,1} \hookrightarrow C^1_b$ and $B^{2+\epsilon}_{1,1} \hookrightarrow L^1$ and hence

$$\begin{aligned}
&|\langle f^1_{s,t} - f^2_{s,t}, \varphi\rangle| \\
&= \left| \int_{\mathbb{R}} \left(\varphi(X^1_t)D_xX^1_t - \varphi(X^1_s)D_xX^1_s - \varphi(X^2_t)D_xX^2_t + \varphi(X^2_s)D_xX^2_s\right) f_0(x)dx \right| \\
&= | \int_{\mathbb{R}} f_0 D_x \left(\bar{\varphi}(X^1_t) - \bar{\varphi}(X^1_s) - \bar{\varphi}(X^2_t) + \bar{\varphi}(X^2_s)\right) dx| \\
&= | - \int_{\mathbb{R}} D_x f_0 \left(\bar{\varphi}(X^1_t) - \bar{\varphi}(X^1_s) - \bar{\varphi}(X^2_t) + \bar{\varphi}(X^2_s)\right) dx| \\
&\leqslant \|f_0\|_{W^{1,1}} \left(\|\bar{\varphi}\|_{C^2_b}\right) \left\|X^1_{s,t} - X^2_{s,t}\right\|_{C^0} \\
&\leqslant \|f_0\|_{W^{1,1}} \left(\|\varphi\|_{C^1_b}) + \|\varphi\|_{L^1}\right) \left\|X^1_{s,t} - X^2_{s,t}\right\|_{C^0} \\
&\leqslant C \|f_0\|_{W^{1,1}} \|\varphi\|_{B^{2+\epsilon}_{1,1}} \left\|X^1_{s,t} - X^2_{s,t}\right\|_{C^0}
\end{aligned}$$

where we used again the division property (Lemma 4.A.2), uniformly in space. Choosing $\epsilon > 0$ sufficiently small, we can now use the comparison principle of Lemma 4.2.5 for $\alpha = 2 + \epsilon$ thanks to our assumption on H. We thus obtain

$$\left\|X^1_{s,t} - X^2_{s,t}\right\|_{C^0} \leqslant C_{T^*} \left\|b^1 - b^2\right\|_{C^\gamma_t B^{-(2+\epsilon)}_{\infty,\infty}} |t-s|^\gamma,$$

where we used that by assumption $(b^1 - b^2)(0,\cdot) = 0$. We therefore obtain indeed

$$\left\|f^1 - f^2\right\|_{C^\gamma_t B^{-(2+\epsilon)}_{\infty,\infty}} \leqslant C_{T^*} \|f_0\|_{W^{1,1}} \left\|b^1 - b^2\right\|_{C^\gamma_t B^{-(2+\epsilon)}_{\infty,\infty}}.$$

□

The above permits us to conclude:

Corollary 4.4.2. *For $\gamma > 1/2$, let $H < 1/7 \wedge 2(1-\gamma)/5$. Let $[0,T]$ be an arbitrary fixed time horizon. Let $f_0 \in C^{0,\gamma} \cap W^{1,1}$ such that $\|f_0\|_{C^{0,\gamma}} + \|f_0\|_{W^{1,1}} \leqslant 1$. Then there exists a constant C_{T^*} such that for any $\bar{f}_0 = \frac{1}{(C_{T^*}\vee 1)(1+T^\gamma)} f_0$, the problem*

$$\begin{aligned}
\partial_t f + \widetilde{f}\partial_x f &= 0 \\
f(0) &= \bar{f}_0
\end{aligned}$$

where $\widetilde{f}(t,x) = f(t, x - w^H_t)$ admits solution $f \in Z^\gamma_T$ in the sense of Definition 4.3.1 such that $\|f\|_{Z^\gamma_T} \leqslant 1$. By Remark 4.3.2, f is also a weak solution.

Proof. Let $\mathcal{B} = \{g \in Z_T^\gamma \mid g_0 = f_0, \|g\|_{Z_T^\gamma} \leqslant 1\}$. Equipped with the metric $(f,g) \to \|f-g\|_{C_t^\gamma B_{\infty,\infty}^{-(2+\epsilon)}}$ this makes $\mathcal{B}$ a complete metric space. We show that the mapping $K : Z_T^\gamma \to Z_T^\gamma$ associating to a function b the unique solution to

$$\begin{aligned} \partial_t f + \widetilde{b}\partial_x f &= 0 \\ f(0) &= f_0 \end{aligned} \tag{4.4.2}$$

admits a unique fixed point in $\mathcal{B}$. Note that by Lemma 4.3.4 we have

$$\|Kf\|_{Z_T^\gamma} \leqslant (1 + C_{T^*}T^\gamma)(\|f_0\|_{C^\gamma} + \|f_0\|_{L^1} + \|f_0\|_{L^\infty})$$

Hence we choosing as initial fondition $\bar{f}_0$, we see that $\|Kb\|_{Z_T^\gamma} \leqslant 1$. Together with the contraction property of Lemma 4.4.1, this permits to conclude that K admits a unique fixed point f in $\mathcal{B}$. The fact that the fixed point lies in Z_T^γ then allows to conclude that we have found a solution to (4.4.2) in the sense of Definition 4.3.1, which by Remark 4.3.2 is equivalent to a weak solution. □

Remark 4.4.3. *Let us compare the above result to the formation of shocks in the classical setting. Recall that shocks for Burgers' equation*

$$\partial_t f + f\partial_x f = 0, \qquad f(0) = f_0$$

appear the first time that two characteristics cross. Let x^1 be the characteristic starting in $x_1 \in \mathbb{R}$ and x^2 be the characteristic starting in $x_2 \in \mathbb{R}$. Note that since the characteristics are explicitly given by

$$x^i(t) = x_i + f_0(x_i)t$$

we obtain for the shock time t_s of the two characteristics x^1, x^2 that

$$t_s = -\frac{x_1 - x_2}{f_0(x_1) - f_0(x_2)},$$

provided this quantity is positive. The time T_s up to which no shocks appear is hence given by

$$T_s = \inf_{x_1 \neq x_2}\left(-\frac{x_1 - x_2}{f_0(x_1) - f_0(x_2)}\right) = \frac{1}{\sup_{x_1 \neq x_2} \frac{-(f_0(x_1) - f_0(x_2))}{x_1 - x_2}}.$$

Note that for $f(x) = -|x|^\gamma$ with $\gamma \in (1/2, 1)$, we have $T_s = 0$, i.e. shocks are produced instantly. This is not the case in our regularized setting. (note that in order to apply our result, we also have to cut off f smoothly at some point to ensure the other conditions besides Hölder regularity are met).

In the case of more regular initial conditions, i.e. $f_0 \in C^1 \cap W^{1,1}$, we may analogously conclude thanks to Lemma 4.3.6

Corollary 4.4.4. *For $\gamma > 1/2$, let $H < 1/7 \wedge 2(1-\gamma)/5$. Let $[0,T]$ be an arbitrary fixed time horizon. Let $f_0 \in C^1 \cap W^{1,1}$ such that $\|f_0\|_{C^1} + \|f_0\|_{W^{1,1}} \leqslant 1$. Then there exists a constant C_{T^*} such that for any $\bar{f}_0 = \frac{1}{1+C_{T^*}T^\gamma} f_0$, the problem*

$$\begin{aligned} \partial_t f + \widetilde{f}\partial_x f &= 0 \\ f(0) &= \bar{f}_0 \end{aligned}$$

where $\widetilde{f}(t,x) = f(t, x - w_t^H)$ admits a unique strong solution on $[0,T] \times \mathbb{R}$.

4.5 Perspectives

A first immediate question that raises itself in the light of the above Corollaries 4.4.2 and 4.4.4 would be: Can the appearing constant C_{T^*} be replaced by some function of $k(T)$? More precisely, is it true that we have the results stated above for initial conditions $\bar{f}_0$ of the form

$$\bar{f}_0 = C(\omega)\frac{1}{k(T)}f_0. \tag{4.5.1}$$

Let us recall that in our considerations, there were essentially two instances that necessitated the employment of the auxiliary time horizon and thus the constant C_{T^*}:

On the one hand, the passing from L^1 to L^2 integrability by Remark 4.1.3 - often done in the above proofs in order exploit Theorem 4.1.1 - comes at the price of a constant depending on the support of the local time L. Oberve that or fractional Brownian motion w^H of Hurst parameter H by [KMRS15, Proposition 3.1], there exists a non negative random variable ξ such that for any $\epsilon > 0$ and $T \geqslant 1$

$$\sup_{t\in[0,T]} |w_t^H(\omega)| < \xi(\omega)T^{H+\epsilon}$$

Hence, for the associated local time L we also have for all $t \in [0,T]$

$$\mathrm{supp}(L_t) \subset [-\xi(\omega)T^{H+\epsilon}, \xi(\omega)T^{H+\epsilon}].$$

which could therefore be used to replace the appearance of C_{T^*} in the $L^2 \hookrightarrow L^1$ embedding by

$$\|L_{s,t}(\omega)\|_{L^1} \leqslant \left(2\xi(\omega)T^{H+\epsilon}\right)^{1/2} \|L_{s,t}\|_{L^2}.$$

Alternatively, for $b \in Z_T^\gamma$, one could also employ interpolation again through

$$\begin{aligned}\|b_s * L_{s,t}\|_\infty &\leqslant \|b_s\|_{L^2} \|L_{s,t}\|_{L^2} \\ &\leqslant \|b\|^{1/2}_{L^\infty_t L^1_x} \|b\|^{1/2}_{L^\infty_{t,x}} \|L_{s,t}\|_{L^2},\end{aligned}$$

by which one does no longer require the embedding $L^2 \hookrightarrow L^1$ that was a first source for the appearance of C_{T^*}.

The second instance where a constant C_{T^*} needs to be employed can be found in equation (4.1.6) of Lemma 4.1.1: As the proof of equation (4.1.6) is obtained by employing Kolmogorov's continuity criterion yielding only *local* Hölder continuity, the constant $C_T(\omega)$ appearing on the righthand side of (4.1.6) does indeed depend on T. Not exploiting the auxilliary time T^* would therefore require to establish bounds on this constant of the form

$$C_T(\omega) \leqslant h(T)\zeta$$

for some non-negative random variable ζ. Provided this could be achieved (one approach could be revisiting the Kolmogorov continuity theorem) one could in principle re-asses the above calculations and eventually derive Corollaries 4.4.2 and 4.4.4 for

initial conditions normalized as in (4.5.1).

This would also shed some more light on the scaling relation between the initial condition and the time interval on which solutions exist. Let us explicit this point: In the setting of the classical Burgers' equation (refer to Remark 4.4.3) and in the case of a continuously differentiable initial condition, there will be no shocks produced up to time

$$T_s = \frac{-1}{\inf_{x\in\mathbb{R}} f_0'(x)}.$$

In particular, scaling $\widetilde{f_0} = f_0/\lambda$, we have $\widetilde{T_s} = \lambda T_s$. Hence, if f_0 has gives rise to a solution without shocks on $[0, T_s]$, then $f - 0 \cdot T_s/T$ gives rise to a solution without shocks on $[0, T]$. Hence, in the notation of these perspectives section, we hwould ave $k(T) = T$. In our regularized setting, we expect $k(T)$ to be some power of T depending on H and γ. Unfortunately, we can not yet explicit $k(T)$, as there is information hidden in the auxiliary C_{T^*} that appears due to the due reasons discussed in the perspectives section.

Finally, it would be of interest if one can establish uniqueness of weak solutions to the transport equation provided the initial condition f_0 is only in $C^{0,\gamma}$ and not differentiable.

4.A Some useful Lemmata

We recall the Sewing Lemma due to [Gub04] (see also [FH14, Lemma 4.2]). Let E be a Banach space, $[0, T]$ a given interval. Let Δ_n denote the n-th simplex of $[0, T]$, i.e. $\Delta_n : \{(t_1, \dots, t_n)|0 \leqslant t_1 \dots \leqslant t_n \leqslant T\}$. For a function $A : \Delta_2 \to E$ define the mapping $\delta A : \Delta_3 \to E$ via

$$(\delta A)_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$$

Provided $A_{t,t} = 0$ we say that for $\alpha, \beta > 0$ we have $A \in C_2^{\alpha,\beta}(E)$ if $\|A\|_{\alpha,\beta} < \infty$ where

$$\|A\|_\alpha := \sup_{(s,t)\in\Delta_2} \frac{\|A_{s,t}\|_E}{|t-s|^\alpha}, \qquad \|\delta A\|_\beta := \sup_{(s,u,t)\in\Delta_3} \frac{\|(\delta A)_{s,u,t}\|_E}{|t-s|^\beta} \qquad \|A\|_{\alpha,\beta} := \|A\|_\alpha + \|\delta A\|_\beta$$

For a function $f : [0, T] \to E$, we note $f_{s,t} := f_t - f_2$

Moreover, if for any sequence $(\mathcal{P}^n([s, t]))_n$ whose mesh size goes to zero, the quantity

$$\lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}^n([s,t])} A_{u,v}$$

converges to the same limit, we note

$$(\Xi A)_{s,t} := \lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}^n([s,t])} A_{u,v}.$$

Lemma 4.A.1 (Sewing)**.** *Let $0 < \alpha \leqslant 1 < \beta$. Then for any $A \in C_2^{\alpha,\beta}(E)$, (ΞA) is well defined. Moreover, denoting $(\Xi A)_t := (\Xi A)_{0,t}$, we have $(\Xi A) \in C^\alpha([0,T],E)$ and $(\Xi A)_0 = 0$ and for some constant $c > 0$ depending only on β we have*

$$\|(\Xi A)_t - (\Xi A)_s - A_{s,t}\|_E \leqslant c\,\|\delta A\|_\beta\,|t-s|^\beta.$$

Let us also recall the following result from for the convenience of the reader

Lemma 4.A.2 (Division property, [FH14, Lemma 8.2])**.** *Let $f \in C_b^2(\mathbb{R}^d,\mathbb{R}^d)$, $\alpha \in (0,1)$, $K \in [1,\infty)$ and $T \leqslant 1$. Then there exists a constant $C_{\alpha,K}$ such that for any $X, Y \in C^\alpha([0,T])$ with $\|X\|_{C^\gamma} \wedge \|Y\|_{C^\gamma} \leqslant K$ we have*

$$\|f(X) - f(Y)\|_{C^\gamma} \leqslant C_{\alpha,K}\,\|f\|_{C_b^2}\left(|X_0 - Y_0| + \|X - Y\|_{C^\gamma}\right).$$

The following Lemma allows to recover Hölder bounds on the full intervall $[0,T]$ from Hölder bounds on intervals of size h:

Lemma 4.A.3 ([FH14, Exercise 4.24])**.** *Fix $\alpha \in (0,1]$, $h > 0$ and $M > 0$. Let V be a Banach space and $Z : [0,T] \to V$ a path such that*

$$\|Z\|_{\gamma,h} := \sup_{\substack{0 \leqslant s<t \leqslant T \\ t-s<h}} \frac{\|Z_{s,t}\|_V}{|t-s|^\gamma} \leqslant M.$$

Then we have

$$\|Z\|_{C_t^\gamma V} \leqslant M(1 \vee 2h^{-(1-\alpha)}).$$

The following is a marginal straightforward generalization to Theorem 4 in [GG20a].

Lemma 4.A.4 (Regularity of averaging operators, [GG20a, Theorem 4])**.** *Let w^H be fractional Brownian motion of Hurst parameter H. For $p, q \in [2,\infty)$ let $b \in L_t^q H^{s,p}$ and $s \in \mathbb{R}$. Then for any ρ and $\gamma > 1/2$ satisfying*

$$\rho H + \frac{1}{q} < 1 - \gamma$$

the averaging operator

$$T_t(x) = \int_0^t b(r, x - w_r^H)dr$$

satisfies $T \in C_t^\gamma H_x^{s+\rho,p}$.

Lemma 4.A.5. *Let $f \in Z_T^\gamma$ with $\|f\|_{Z_T^\gamma} \leqslant 1$ and ρ be a positive mollifier supported in $B_1(0)$ such that $\|\rho\|_{L^1} = 1$. Then we have for $p \in [1,\infty)$*

$$\lim_{\epsilon\to 0} \sup_{s\in[0,T]} \|f_s^\epsilon - f_s\|_{L^p} = 0$$

Proof. Let q be the Hölder conjugate to p. Note that

$$\begin{aligned}\int_{\mathbb{R}} |f_s^\epsilon(x) - f_s(x)|^p dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f_s(x-y) - f_s(x))\rho^\epsilon(y) dy \right|^p dx \\ &\leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} |f_s(x-y) - f_s(x)|^p |\rho^\epsilon(y)| dy \left(\int_{\mathbb{R}} |\phi(z)| dz \right)^{p/q} dx \\ &\leqslant \int_{\mathbb{R}} |\rho^\epsilon(y)| \left(\int_{\mathbb{R}} |f_s(x-y) - f_s(x)|^p dx \right) dy\end{aligned}$$

Let $\eta > 0$. As $f \in L_t^\infty L_x^1 \cap L_t^\infty L_x^\infty$, we have by interpolation $f \in L_t^\infty L_x^p$. In particular, we find an N_η such that

$$2^p \int_{\mathbb{R}} \mathbf{1}_{[-N_\eta, N_\eta]^c}(x) \sup_{s\in[0,T]} |f_s(x)|^p dx < \eta$$

We therefore have for any $r > 0$

$$\begin{aligned}&\int_{\mathbb{R}} |\rho^\epsilon(y)| \left(\int_{\mathbb{R}} |f_s(x-y) - f_s(x)|^p dx \right) dy \\ &= \int_{[-r,r]^c} |\rho^\epsilon(y)| \left(\int_{\mathbb{R}} |f_s(x-y) - f_s(x)|^p dx \right) dy \\ &+ \int_{[-r,r]} |\rho^\epsilon(y)| \left(\int_{\mathbb{R}} \mathbf{1}_{[-N_\eta, N_\eta]}(x) |f_s(x-y) - f_s(x)|^p dx \right) dy \\ &+ \int_{[-r,r]} |\rho^\epsilon(y)| \left(\int_{\mathbb{R}} \mathbf{1}_{[-N_\eta, N_\eta]^c}(x) |f_s(x-y) - f_s(x)|^p dx \right) dy.\end{aligned}$$

Note that for any $r > 0$, there exists an $\epsilon' > 0$ such that for any $\epsilon \in (0, \epsilon')$ we have

$$\int_{[-r,r]^c} |\rho^\epsilon(y)| dy < \eta 2^{-p}$$

Choosing $r = (\eta/(2N_\eta))^{1/(p\gamma)}$, we have therefore due to $f \in C_t^0 C_x^\gamma$:

$$\begin{aligned}&\int_{\mathbb{R}} |\rho^\epsilon(y)| \left(\int_{\mathbb{R}} |f_s(x-y) - f_s(x)|^p dx \right) dy \\ &\leqslant 2^p \sup_{s\in[0,T]} \|f_s\|_{L^p}^p \, \eta 2^{-p} \\ &+ \int_{[-r,r]} |\rho^\epsilon(y)| 2N_\eta |y^{p\gamma}| dy \\ &+ \int_{[-r,r]} |\rho^\epsilon(y)| dy 2^p \int_{\mathbb{R}} \mathbf{1}_{[-N_\eta, N_\eta]^c}(x) \sup_{s\in[0,T]} |f_s(x)|^p dx \\ &\leqslant 3\eta\end{aligned}$$

□

4.B Some basic facts on Besov spaces

For the reader's convenience, let us state some basic results from the theory of (non-homegeneous) Besov spaces. A good overview to this topic can be found in [vZ20].

Lemma 4.B.1 (Young's inequality, [KS21, Theorem 2.2]). *Let $s, u \in \mathbb{R}$, $p, p_1, p_2 \in [1, \infty]$ and $q, q_1, q_2 \in (0, \infty]$ such that*

$$\frac{1}{q} \leqslant \frac{1}{q_1} + \frac{1}{q_2} \qquad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

Suppose that $f \in B^s_{p_1,q_1}$ and $g \in B^u_{p_2,q_2}$, then $f * g \in B^{s+u}_{p,q}$ and there exists a constant C independent of f, g such that

$$\|f * g\|_{B^{s+u}_{p,q}} \leqslant C \|f\|_{B^s_{p_1,q_1}} \|g\|_{B^u_{p_2,q_2}}$$

Lemma 4.B.2 (Derivative consistency [vZ20, Lemma 16.1]). *Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$. Then there exist a constant such that for all $u \in \mathcal{S}'$ we have*

$$\|\partial_x u\|_{B^{s-1}_{p,q}} \leqslant C \|u\|_{B^s_{p,q}}.$$

Lemma 4.B.3 (Multiplication, [vZ20, Theorem 19.7], [Mar18, Corollary 2.1.35]). *Let $\alpha, \beta \in R\backslash\{0\}$ such that $\alpha + \beta > 0$ and $\alpha \leqslant \beta$. Let $\delta > 0$. Then there exists a $C > 0$ such that for any $p, p_1, p_2 \in [1, \infty]$ and $q, q_1, q_2 \in [1, \infty]$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leqslant 1, \qquad \frac{1}{q} = 1 \wedge \frac{1}{q_1} + \frac{1}{q_2}$$

we have for all $u, v \in \mathcal{S}'$

$$\|u \cdot v\|_{B^{\alpha-\delta}_{p,q}} \leqslant C \|u\|_{B^{\alpha}_{p,q}} \|v\|_{B^{\beta}_{p,q}}$$

Lemma 4.B.4 (Besov space embeddings [BCD11, Proposition 2.71],[Tri83, p.89]). *Let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$. Then for Besov spaces over $\mathbb{R}^d$, we have the continuous embedding*

$$B^s_{p_1,q_1} \hookrightarrow B^{s-d(1/p_1-1/p_2)}_{p_2,q_2}$$

Moreover, if C^0 denotes the space of bounded uniformly continuous functions, we have the continuous embeddings

$$B^0_{\infty,1} \hookrightarrow C^0 \hookrightarrow B^0_{\infty,\infty}.$$

Moreover, we have by [Tri83, p.131] that for $s \in \mathbb{R}$ and $1 < p_0 < p_1 < \infty$ such that $s_0 - d/p_0 = s_1 - d/p_1$, there holds

$$H^{s_0,p_0} \hookrightarrow B^{s_1}_{p_1,p_0}.$$

Moreover, we have for $s \in \mathbb{R}$, $p, q \in [1, \infty]$ and any $\eta > 0$ that

$$B^s_{p,q} \hookrightarrow B^s_{p,q+\eta} \qquad B^{s+\eta}_{p,\infty} \hookrightarrow B^s_{p,1}.$$

and finally $B^s_{2,2} = H^s$ for any $s \in \mathbb{R}$.

Lemma 4.B.5 (Duality for Besov space, [BCD11, Proposition 2.76])**.** *Let $\phi \in \mathcal{S}$ be a Schwartz function. Then for any $p, q \in [1, \infty]$, $s \in \mathbb{R}$ and $u \in B^s_{p,q}$ the mapping*

$$\begin{aligned} \mathcal{S} &\to \mathbb{R} \\ \phi &\to \langle u, \phi \rangle \end{aligned}$$

extends to a linear functional on $B^{-s}_{p',q'}$, where p', q' are the Hölder conjugates of p, q respectively. Moreover, if $u \in \mathcal{S}'$, we have for $Q^{-s}_{p',q'}$ being the space of $\phi \in \mathcal{S}$ such that $\|\phi\|_{B^{s'}_{p',q'}} \leqslant 1$ that

$$\|u\|_{B^s_{p,q}} \leqslant C \sup_{\phi \in Q^{-s}_{p',q'}} \langle u, \phi \rangle.$$

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