Ensembles basiques pour les groupes symétriques et alternés, involution de Mullineux

Ana Bernal

Université de Reims Champagne-Ardenne
École Doctorale Sciences Fondamentales - Santé N° 619
Laboratoire de Mathématiques de Reims UMR 9008

Thèse

Pour obtenir le grade de

Docteur de l'Université de Reims Champagne-Ardenne

Discipline : Mathématiques

Présentée et soutenue publiquement par

Ana Bernal Arango

le 8 décembre 2021

Ensembles basiques pour les groupes symétriques et alternés, involution de Mullineux

---

Thèse dirigée par Nicolas Jacon et
co-encadrée par Loïc Poulain d'Andecy

Jury

Président du jury	Cédric Lecouvey	Université de Tours
Directeur de thèse	Nicolas Jacon	Université de Reims Champagne-Ardenne
Co-encadrant	Loïc Poulain d'Andecy	Université de Reims Champagne-Ardenne
Rapporteur	Matthew Fayers	Queen Mary University of London
Rapporteur	Jean-Baptiste Gramain	University of Aberdeen
Examinateur	Olivier Brunat	Université de Paris
Examinatrice	Maria Chlouveraki	UVSQ - Université Paris-Saclay
Examinateur	Rupert W. T. Yu	Université de Reims Champagne-Ardenne

Remerciements

Cette partie est consacrée à remercier à ceux que d'une manière ou une autre ont contribué dans le projet de doctorat qui se matérialise en partie dans cette thèse.

Je remercie d'abord mes directeurs de thèse Nicolas Jacon et Loïc Poulain d'Andecy. Je me sens très privilégiée d'avoir été encadrée par vous. Nicolas, merci d'avoir toujours eu confiance dans mes initiatives et de m'avoir incitée à programmer. Merci aussi de m'avoir laissé beaucoup de liberté de travailler à ma façon. Loïc, merci de m'avoir montré, dans nos discussions, la beauté dans les théories où j'ai navigué pendant ces trois ans (et d'autres théories) et d'avoir souvent suggéré une façon simple et transparente de mettre en forme les idées sur lesquelles nous avons travaillé. Merci à tous les deux d'avoir toujours été disponibles pour moi et mes nombreuses questions et inquiétudes, de votre patience et encouragement constant.

Je remercie spécialement les rapporteurs de cette thèse, Matthew Fayers et Jean-Baptiste Gramain. Je leur suis très reconnaissante d'avoir pris le temps de la lire en détail et s'y intéresser, c'est un honneur pour moi. Je souhaite remercier également Cédric Lecouvey, Olivier Brunat, Maria Chlouveraki et Rupert Yu d'avoir accepté de faire partie du jury et de se déplacer à Reims.

Un grand merci également à mes professeurs de Strasbourg, tout particulièrement à Frédéric Chapoton, qui a encadré mes mémoires de Master et à Vladimir Fock et Hubert Rubenthaler pour leurs cours d'Algèbre et Combinatoire et de Théorie des Représentations, qui ont été ma principale motivation pour travailler dans ces domaines.

Je remercie les personnes que j'ai rencontrées aux conférences de façon récurrente et avec qui on a eu des moments et discussions agréables, je pense en particulier à Olivier Brunat, Maria Chlouveraki, Jérémie Guilhot, Thomas Gerber et Cédric Lecouvey. Cela m'a toujours fait plaisir de voir vos noms à l'avance dans la liste de participants.

Je remercie également tous les membres du Laboratoire de Mathématiques de Reims. Pendant ces trois ans j'ai passé la plupart d'heures de mon existence ici, et partager ce temps avec vous a été très agréable. Merci à Christelle, Guillaume, Rupert, et les doctorants : Benjamin, David, Farah, Laurie, Océane, Pierre et Serge. Merci en particulier à Quentin pour son amitié et son soutien.

Je remercie chaleureusement mes amis à Strasbourg : Christian, Claudine et Luis. Leur amitié est un soutien constant pour moi, étant loin de ma famille.

Finalmente agradezco a mi familia Pa, Ma, Manyi y Tebi. Ustedes son mi hogar en este universo al que llegamos desamparados con tantas preguntas y ninguna respuesta, son las raíces fuertes que me dan estabilidad, seguridad, calma y mucha felicidad.

Résumé

Dans cette thèse nous nous intéressons aux propriétés combinatoires de la théorie des représentations modulaires des groupes symétriques et alternés.

Nous nous concentrons sur le problème de l'étiquetage des représentations irréductibles modulaires des groupes symétriques et alternés. Une façon naturelle d'aborder ce problème est de trouver des ensembles basiques unitriangulaires pour les matrices de décomposition. Une de nos principales motivations est liée à l'application de Mullineux, application qui contrôle la restriction du groupe symétrique au sous-groupe alterné. Dans le but d'avoir un étiquetage particulièrement bien adapté à cette restriction, on cherche des ensembles basiques unitriangulaires stables. De tels ensembles sont stables pour la conjugaison.

Il apparaît lors de notre étude le phénomène remarquable suivant : le nombre de partitions auto-conjuguées avec des longueurs des crochets diagonaux non divisibles par p, appelées BG-partitions, est égal au nombre de points fixes de l'application de Mullineux, ou partitions auto-Mullineux. Nous montrons une correspondance combinatoire explicite entre les deux ensembles de partitions.

Récemment il a été montré qu'il n'existe pas toujours un ensemble basique unitriangulaire pour le groupe alterné. Cependant, ces notions peuvent être définies au niveau des blocs. Nous étudions l'application de Mullineux dans les blocs auto-conjugués de p-poids 2 du groupe symétrique et nous exhibons un ensemble basique unitriangulaire stable pour ces blocs, ce qui implique l'existence d'ensembles basiques unitriangulaires pour certains blocs des groupes alternés.

Mots clés: Combinatoire algébrique, Théorie des représentations, Groupe symétrique, Groupe alterné, involution de Mullineux.

Abstract

This thesis concerns combinatorial properties of the modular representation theory of the symmetric and alternating groups.

We focus on the problem of labelling the modular irreducible representations of the symmetric and alternating groups. A natural way to approach this is through unitriangular basic sets for the decomposition matrices. One of our main motivations is related to the Mullineux map, which controls the restriction from the symmetric group to the alternating subgroup. In order to have a labelling which is particularly well adapted to this restriction, we look for stable unitriangular basic sets. Such sets are stable for conjugation.

In our study, the following remarkable phenomenon is observed: the number of self-conjugate partitions with diagonal hook-lengths not divisible by p, called BG-partitions, is equal to the number of fixed points of the Mullineux map, or self-Mullineux partitions. We give a combinatorial and explicit correspondence between the two sets of partitions.

Recently it has been shown that there is not always a unitriangular basic set for the alternating group. However, these notions can be defined at the level of blocks. In this thesis, we study the Mullineux map in self-conjugate blocks of p-weight 2 of the symmetric group and we construct a stable unitriangular basic set for these blocks, which implies the existence of unitriangular basic sets for some blocks of the alternating groups.

Keywords: Algebraic combinatorics, Representation theory, Symmetric group, Alternating group, Mullineux involution.

Contents

Table de matières

Index of notations

The following table is an index of notations used throughout this thesis.

Notation	Meaning	First on p.
$\mathfrak{A}_n$	the alternating group on n symbols	31
a_λ	the number of nodes in the p-rim of λ	52
a^*_λ	the number of nodes in the p-rim* of λ	61
$\mathcal{B}^\lambda_r$	a β-set associated to the partition λ	24
BG^n_p	the set of p-BG partitions of n	36
BG_p	the set of p-BG partitions	36
$\mathrm{bg}_p(\lambda)$	the (p-) BG-symbol of partition λ	63
$\mathrm{BG}^n_p(\gamma)$	the set of p-BG-partitions in the block $\mathfrak{B}_\gamma$	92
$({}_i\gamma_j)_{ij}$	the pyramid associated to a block with p-core γ	99
${}_i0_j$ or ${}_i1_j$	${}_i\gamma_j = 0$ or ${}_i\gamma_j = 1$ resp.	99
$\mathfrak{B}_\gamma$	the block of ordinary irreducibles of $\mathfrak{S}_n$ corresponding to the p-core γ	48
$\mathcal{C}^\lambda$	the conjugacy class of $\mathfrak{S}_n$ associated to the partition $\lambda \vdash n$	32
D^λ	the irreducible $\mathbb{K}\mathfrak{S}_n$-module associated to the p-regular partition λ	40
$\mathbf{D}_{n,p}$	the (p-modular) decomposition matrix of $\mathfrak{S}_n$	46
$\mathbf{D}^U_{n,p}$	the matrix formed by the rows in $\mathbf{D}_{n,p}$ labelled by elements in U	47
$d_{\lambda\mu}$	the decomposition number which is the composition multiplicity of D^μ in S^λ	46
$\partial\lambda$	Richards' ∂ function on weight 2 partition λ	104
∂_ℓ	set of weight 2 partitions in a block with $\partial = \ell$	104

$\partial_\ell^{\mathrm{reg}}$	set of weight 2 p-regular partitions in a block with $\partial = \ell$	104
∂_0^+	or ∂_0^-: $\partial_0 = \partial_0^+ \cup \partial_0^-$	105
$\varepsilon(\pi)$ or $\mathrm{sgn}(\pi)$	the sign of the permutation π	31
$G_p(\lambda)$	the (p-)Mullineux symbol of partition λ	54
$\mathcal{G}_p(n)$	the Grothendieck group of $\mathbb{F}_p\mathfrak{S}_n$	87
ε^*_λ	the parity of a^*_λ	61
$L^{(g)}$	the conjugate $\mathbb{K}H$-module of the $\mathbb{K}H$-module L, where $g \in G \triangleright H$	40
H^λ_{ij}	the (i,j)-th hook of the partition λ	22
h^λ_{ij}	(or h_{ij}) the length of the (i,j)-th hook of the partition λ	23
$\mathrm{Irr}_p(\mathfrak{B}_\gamma)$	or $\mathrm{Irr}(\mathfrak{B}_\gamma)$, the block of p-modular irreducibles corresponding to the p-core γ	48
$\mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n)$	the set of p-modular irreducibles of $\mathfrak{S}_n$ up to equivalence	87
$l(\lambda)$	the length of the partition λ	21
λ'	the conjugate partition of λ	21
$[\lambda]$	the Young diagram of the partition λ	21
$\overline{\lambda}_p$	the p-quotient $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p-1)})$ of partition λ	29
$\lambda^{(i)}$	the i-th partition in the process of removing p-rims recursively	52
$\lambda^{(i)*}$	the i-th partition in the process of removing p-rims* recursively	62
$M\downarrow_H$	the restriction of the $\mathbb{K}G$-module M to $H \leq M$	37
$\boldsymbol{m}_p(\lambda)$	the Mullineux map applied to partition λ	43
$\boldsymbol{m}_p^M(\lambda)$	Mullineux algorithm applied to a partition λ	54
M^ε	$M \otimes \mathrm{sgn}$, where sgn is the sign representation of $\mathfrak{S}_n$	41
M_p^n	the set of (p-)self-Mullineux partitions of $n \subseteq \mathrm{Reg}_p(n)$	43
M_p	the set of (p-)self-Mullineux partitions	43
$\mathcal{M}_p$	the set of Mullineux symbols of all the (p-)self-Mullineux partitions	69
$M_p^n(\gamma)$	the set of self-Mullineux partitions in the block $\mathfrak{B}_\gamma$	92
$\mathbb{N}$	$\{0,1,2,\ldots\}$	21
$N\uparrow^G$	the induction of the $\mathbb{K}H$-module N to $G \geq H$	37

$\mathbb{N}^*$	$\{1,2,3,\ldots\}$	24
$\mathrm{Par}(n)$	the set of partitions of n	21
$\mathrm{Par}^{\mathfrak{A}}(n)$	the set of partitions of n with an even number of even parts	33
$\mathrm{Par}^{\mathfrak{A}}_{+-}(n)$	the set of partitions of n formed by different odd parts	33
$\mathrm{Par}^{\mathfrak{A}}_{\circ}(n)$	$\mathrm{Par}^{\mathfrak{A}}(n) \setminus \mathrm{Par}^{\mathfrak{A}}_{+-}(n)$	33
$\mathrm{Rk}(\lambda)$	the rank of the partition λ	21
R^{λ}_{ij}	the (i,j)-th rim-hook of the partition λ	23
r^*_{λ}	the number of nodes in $(Rim)^*_p(\lambda)$ above or on the diagonal of λ	62
$\mathrm{Reg}_p(n)$	the set of p-regular partitions of n	21
$\mathrm{Reg}_p(\mathfrak{B}_\gamma)$	the set of Specht modules indexed by p-regular partitions of n in $\mathfrak{B}_\gamma$.	90
$\mathrm{Rim}_p(\lambda)$	the p-rim of partition λ	52
$\mathrm{Rim}^*_p(\lambda)$	the p-rim* of partition λ	61
$\mathrm{Rim}^X_p(\lambda)$	the truncated p-rim of partition λ	56
S^{λ}	the Specht module corresponding to partition λ	39
$\mathfrak{S}_n$	the symmetric group on n symbols	30
U_{λ}	the set of nodes in $\mathrm{Rim}_p(\lambda)$ above or on the diagonal	61
$\{\mathbf{t}\}$	the tabloid corresponding to the tableau t	38
$\{s,t\}$	notation by for partitions of weight 2 in a block	119
$\trianglelefteq$	the dominance order on partitions	22
$\leq$	the lexicographic order on partitions	22
$\langle\cdot\rangle$	$\langle i,j\rangle$, $\langle i\rangle$ or $\langle i^2\rangle$. Partitions defined with respect to certain indexes in the abacus.	100
$\lceil\cdot\rfloor$	$\lceil i,j\rfloor$, $\lceil i\rfloor$ or $\lceil i^2\rfloor$. p-regular partitions defined with respect to certain indexes in the abacus.	103

Introduction

L'origine de la théorie des représentations des groupes finis remonte à la fin du 19ème siècle, lorsque Frobenius a étudié la factorisation d'un certain polynôme associé à un groupe fini, appelé le "déterminant du groupe". Généralisant le cas connu d'un groupe abélien, il a prouvé que ce polynôme se factorise comme un produit de polynômes irréductibles, où chaque polynôme apparaît avec une multiplicité égale à son degré. Cette question, remarquée par Dedekind, est ce qui a conduit Frobenius à inventer la théorie des caractères pour un groupe fini arbitraire et à construire les bases de ce que nous appelons aujourd'hui la théorie des représentations. Pour des références historiques sur l'origine du sujet, dont les pionniers sont Frobenius, Burnside, Schur et Brauer, voir [Cur99,Haw71]. Le domaine s'est largement développé à ce jour et il a des liens avec la physique, la chimie et d'autres domaines. Il est intéressant de noter que la plupart des résultats sur la classification des représentations ordinaires du groupe symétrique étaient connus ou découverts au début de la théorie, et, en revanche, il y a encore beaucoup de questions sur les représentations modulaires qui ne sont pas résolues à ce jour.

En langage moderne, la théorie des représentations des groupes (finis) étudie les façons dont un groupe donné agit sur des espaces vectoriels en tant qu'opérateurs linéaires. Concrètement, une représentation de dimension finie d'un groupe G est un homomorphisme de groupes $G \to \mathrm{GL}_d(\mathbb{K})$, où $\mathbb{K}$ est un corps quelconque et $\mathrm{GL}_d(\mathbb{K})$ est le groupe des matrices inversibles $d \times d$. Une représentation peut alors être considérée comme une manière de "représenter" les éléments d'un groupe sous forme de matrices ; et ce faisant, de voir le groupe, qui est en principe un objet abstrait, comme un objet dans l'univers de l'algèbre linéaire.

Étudier la théorie des représentations du groupe G c'est étudier tous ces morphismes (ou, plus précisément, à une certaine équivalence près). Une représentation peut être construite à partir de représentations "plus petites". Cela se produit lorsque chaque matrice qui "représente" un élément du groupe, peut être mise simultanément dans une structure triangulaire par blocs, par un changement de base. Chacun des blocs diagonaux est alors une représentation "plus petite" (la taille de la matrice est plus petite). Si un tel changement de base n'existe pas, alors la représentation est dite irréductible. De plus, pour un groupe fini G et un corps $\mathbb{K}$ de caractéristique 0 (ou ne divisant pas l'ordre de G), la structure triangulaire par blocs peut toujours être réduite à une structure diagonale par blocs ; donc, dans ce cas, la tâche d'étudier tous les morphismes $G \to \mathrm{GL}_d(\mathbb{K})$ se réduit plus ou moins à étudier toutes les représentations irréductibles, à équivalence près. Pour un corps général $\mathbb{K}$, la situation n'est pas exactement la même, mais les représentations irréductibles restent des blocs fondamentaux de la théorie des représentations et on commence habituellement l'étude de la théorie des représentations d'un groupe G sur $\mathbb{K}$ par l'étude des représentations

irréductibles.

De nombreuses questions différentes se posent dans une telle entreprise ; si nous fixons un groupe, combien de représentations irréductibles existent (à équivalence près) ? à quel point sont-elles explicites ? quelles dimensions possibles d apparaissent ? existe-t-il une façon "naturelle" d'indexer les représentations irréductibles ? dans quelle mesure les réponses à ces questions changent-elles si nous changeons le corps $\mathbb{K}$? quelles propriétés du groupe se reflètent dans les représentations et vice versa ? etcetera. Des réponses, ou des réponses partielles, à certaines de ces questions ont été trouvées pour certains groupes ou familles de groupes et ont conduit à des applications intéressantes, par exemple la classification des groupes simples finis.

Dans cette thèse, notre attention se porte sur les représentations du groupe symétrique $\mathfrak{S}_n$ et du groupe alterné $\mathfrak{A}_n$. Le groupe symétrique $\mathfrak{S}_n$ est le groupe des permutations d'une liste ordonnée de n symboles, où l'opération est la composition. C'est un groupe d'ordre $n!$. Comme les permutations sont des objets assez naturels, les groupes symétriques et, plus généralement les groupes de permutation (sous-groupes d'un groupe symétrique), ont été largement étudiés et utilisés, bien avant la formalisation des groupes abstraits. Le groupe alterné $\mathfrak{A}_n$ est l'ensemble des permutations paires (permutations de signature 1) d'une liste ordonnée de n symboles. C'est donc un sous-groupe de $\mathfrak{S}_n$.

La théorie des représentations du groupe symétrique a été étudiée et développée pour la première fois par Frobenius, Schur et Young. Ils ont trouvé des réponses à plusieurs des questions ci-dessus lorsque $\mathbb{K} = \mathbb{C}$[1]. Dans ce cas, chaque représentation de $\mathfrak{S}_n$ se décompose en une somme de représentations irréductibles et il existe une classification de ces représentations irréductibles ; elles peuvent être indexées par l'ensemble des partitions de n :

$$\mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_n) = \{S^\lambda \mid \lambda \text{ est une partition de } n\}.$$

Les partitions de n sont toutes les différentes manières non ordonnées d'écrire n comme une somme d'entiers positifs. Depuis les travaux de Young, la combinatoire des partitions et d'autres objets qui leur sont associés sont fondamentaux dans le développement de la théorie des représentations du groupe symétrique : les représentations S^λ, appelées *modules de Specht*, peuvent être décrites complètement en termes de λ. Par exemple, le module de Specht $S^{(n)}$ correspondant à la partition $\lambda = (n)$ est la représentation *triviale*, et le module de Specht $S^{(1^n)}$ correspondant à la partition $\lambda = (1^n)$ est la représentation *signe*. Un autre exemple est la formule appelée formule des équerres qui donne facilement la dimension de S^λ en termes de λ. Pour citer un dernier exemple, la décomposition en représentations irréductibles de l'induction de S^λ de $\mathfrak{S}_n$ à $\mathfrak{S}_{n+1}$, peut être facilement décrite en termes de λ.

En caractéristique positive, disons $p = \mathrm{car}(\mathbb{K})$, si p ne divise pas $n!$ (de façon équivalente $p > n$), les représentations de $\mathfrak{S}_n$ ont la même classification qu'en caractéristique 0, c'est la théorie des représentations *ordinaires*. Par contre, lorsque p divise l'ordre du groupe, dans notre cas si p divise $n!$ (de façon équivalente $p \leq n$), on parle de théorie des représentations *modulaires*.

L'étude des représentations p-modulaires de $\mathfrak{S}_n$ a commencé avec les travaux de Nakayama dans les années 1940, après les travaux de Brauer sur les représentations

[1] En fait, toute représentation de $\mathfrak{S}_n$ sur $\mathbb{C}$ peut être réalisée sur $\mathbb{Q}$, ce qui fait que tout est similaire si on prend $\mathbb{K} = \mathbb{Q}$ ou n'importe quelle extension de $\mathbb{Q}$.

modulaires des groupes en général. Il y a différentes façons de de construire les représentations modulaires irréductibles $\mathrm{Irr}_p(\mathfrak{S}_n)$ de $\mathfrak{S}_n$. Pour une des ces constructions, les représentations peuvent être indexées par l'ensemble des partitions dites *p-régulières*, c'est-à-dire des partitions dans lesquelles aucune partie n'apparaît p fois,

$$\mathrm{Irr}_p(\mathfrak{S}_n) = \{D^\lambda \mid \lambda \text{ est une partition } p\text{-régulière de } n\}.$$

Cette construction n'est pas aussi explicite qu'en caractéristique zéro : D^λ est l'unique quotient simple de S^λ. En conséquence la façon d'associer à λ le module D^λ n'est pas aussi naturelle ; la partition (n) correspond à $D^{(n)}$ qui est la représentation triviale, comme précédemment. Mais (1^n) ne correspond pas à la représentation signe. Cependant l'étude de cette classification implique des objets combinatoires, et il y a un certain nombre de questions sans réponse sur les représentations modulaires, par exemple, il n'y a pas d'équivalent de la formule des équerres pour calculer la dimension de D^λ. Dans le Chapitre 1, nous rappelons plusieurs faits concernant la théorie des représentations de $\mathfrak{S}_n$. Pour un traitement très complet, voir par exemple [JK81, Sag01].

Les considérations ci-dessus nous mènent au sujet d'étude de cette thèse. Il y a deux résultats principaux dont chacun fait l'objet d'un chapitre. Dans ce qui suit, nous introduisons le contexte et les principales questions que nous traitons.

Partitions auto-Mullineux et partitions auto-conjuguées

Un des principaux résultats de cette thèse concerne une correspondance entre deux familles de partitions, l'une d'entre elles étant l'ensemble des partitions *auto-Mullineux*. Nous motivons l'intérêt pour de telles familles de partitions.

Soit $n \geq 2$. Si M est une représentation irréductible de $\mathfrak{S}_n$ sur un corps $\mathbb{K}$, la restriction $M\downarrow_{\mathfrak{A}_n}$ de M à $\mathfrak{A}_n$ est soit irréductible, soit elle se décompose en la somme directe de deux représentations irréductibles de $\mathfrak{A}_n$. De plus, toute représentation irréductible de $\mathfrak{A}_n$ sur $\mathbb{K}$ peut être obtenue de cette façon. Plus précisément, on sait que $M\downarrow_{\mathfrak{A}_n}$ se décompose en somme directe de deux modules simples si et seulement si $M \simeq (M \otimes \varepsilon)$, où ε est la représentation signe de $\mathfrak{S}_n$. Sinon, $M\downarrow_{\mathfrak{A}_n}$ est irréductible. Or, puisque ε est de dimension 1, $(M \otimes \varepsilon)$ est à nouveau une représentation irréductible. Détaillons cette discussion dans les cas ordinaire et modulaire :

Dans le cas ordinaire, on sait que toute représentation irréductible peut être indexée par une partition λ de n. Par conséquent, la discussion ci-dessus dit que $S^\lambda\downarrow_{\mathfrak{A}_n}$ se décompose si et seulement si $S^\lambda \simeq (S^\lambda \otimes \varepsilon)$. Il existe une partition μ telle que $(S^\lambda \otimes \varepsilon) \simeq S^\mu$. Une question naturelle est alors : comment calculer μ à partir de λ ? Dans le cas ordinaire, il est facile de répondre à cette question ; nous avons $\mu = \lambda'$ la partition conjuguée (ou transposée) de λ. Elle est obtenue simplement en transposant le diagramme de Young de λ. Par exemple, si $\lambda = (n)$, on sait que $S^{(n)}$ est la représentation triviale, alors $S^{(n)} \otimes \varepsilon \simeq \varepsilon$, et comme mentionné ci-dessus, la représentation signe $\varepsilon = S^\mu$ est indexée par $\mu = (1^n)$, qui est en effet la conjuguée de $\lambda = (n)$. Ainsi, la condition pour que la restriction de S^λ se décompose en somme de deux modules simples devient : $\lambda = \lambda'$. Sinon, si $\lambda \neq \lambda'$, la restriction de S^λ à $\mathfrak{A}_n$ est irréductible. Les détails seront donnés dans §1.3.3.

Dans le cas modulaire, par exemple si $\mathbb{K} = \mathbb{F}_p$ avec $2 < p < n$, on a un ensemble complet (à equivalence près) de représentations irréductibles indexées par les parti-

tions p-régulières, et pour λ p-régulière, la restriction $D^\lambda \downarrow_{\mathfrak{A}_n}$ de la représentation p-modulaire irréductible D^λ de $\mathfrak{S}_n$ se décompose en somme directe de deux modules simples si et seulement si $D^\lambda \simeq (D^\lambda \otimes \varepsilon)$. Il existe une partition p-régulière μ telle que $(D^\lambda \otimes \varepsilon) \simeq D^\mu$. Dans ce cas, comment calculer la partition p-régulière μ à partir de λ ? Soit $\mu = \boldsymbol{m}_p(\lambda)$. L'involution qui assigne à λ la partition $\boldsymbol{m}_p(\lambda)$ est l'*application de Mullineux* : $D^\lambda \otimes \varepsilon = D^{\boldsymbol{m}_p(\lambda)}$. Le problème qui consiste à trouver une description combinatoire de cette application est appelé le *problème de Mullineux*.

La réponse au problème de Mullineux n'est pas aussi simple que dans le cas ordinaire. Conjuguer une partition p-régulière ne résulte pas toujours en une partition p-régulière, alors μ n'est pas simplement la partition conjuguée de λ. Dans [Mul79a] Mullineux a défini une involution sur l'ensemble des partitions p-régulières de n, et a conjecturé que cette involution était $\boldsymbol{m}_p$. Plus tard, dans [Kle96], Kleshchev a décrit un algorithme différent pour calculer $\boldsymbol{m}_p(\lambda)$ et dans [FK97], Ford et Kleshchev ont montré que la conjecture de Mullineux était vraie. La conjecture de Mullineux a également été prouvée dans [BO98] par Bessenrodt et Olsson en utilisant un autre algorithme pour calculer $\boldsymbol{m}_p(\lambda)$[2].

Revenant à la restriction de D^λ à $\mathfrak{A}_n$ on a que, pour λ une partition p-régulière de $n \geq 2$,

- Si $\lambda \neq \boldsymbol{m}_p(\lambda)$ alors $D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\boldsymbol{m}_p(\lambda)} \downarrow_{\mathfrak{A}_n}$ est irréductible.
- Si $\lambda = \boldsymbol{m}_p(\lambda)$ alors $D^\lambda \downarrow_{\mathfrak{A}_n}$ se décompose en deux représentations irreductibles p-modulaires non équivalentes $D^{\lambda+}$ et $D^{\lambda-}$, de $\mathfrak{A}_n$:

$$D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\lambda+} \oplus D^{\lambda-},$$

et

$$\begin{aligned} \{D^\lambda \downarrow_{\mathfrak{A}_n} \mid \lambda \vdash n,\ \lambda\ p\text{-régulière et } \lambda \neq \boldsymbol{m}_p(\lambda)\} \sqcup \\ \{D^{\lambda+}, D^{\lambda-} \mid \lambda \vdash n,\ \lambda\ p\text{-régulière et } \lambda = \boldsymbol{m}_p(\lambda)\}, \end{aligned} \tag{1}$$

où seulement l'une des λ ou $\boldsymbol{m}_p(\lambda)$ est considérée dans le premier ensemble, est un ensemble complet de représentations irréductibles p-modulaires non équivalentes de $\mathfrak{A}_n$.

Étant donnée cette façon d'indexer les représentations p-modulaires irréductibles de $\mathfrak{A}_n$, nous nous intéressons en particulier à l'ensemble des points fixes de $\boldsymbol{m}_p$: les partitions λ de n telles que $\lambda = \boldsymbol{m}_p(\lambda)$ que nous appelons les partitions *auto-Mullineux*. Déterminer l'ensemble des partitions (p-)auto-Mullineux n'est pas une tâche facile, dans le sens où avec les algorithmes disponibles pour calculer l'involution de Mullineux, nous ne pouvons pas déterminer rapidement si une partition p-régulière est un point fixe. On se demande alors s'il n'existe pas un autre ensemble de partitions, plus facile à déterminer, qui soit en correspondance avec l'ensemble des partitions auto-Mullineux. En fait, le nombre de partitions (p-)self-Mullineux est égal au nombre de partitions auto-conjuguées dont la longueur des crochets diagonaux n'est pas divisible par p (voir §1.3.4) que nous appelons *(p-)BG-partitions*[3]. Ainsi, il

[2] D'autres algorithmes pour calculer $\boldsymbol{m}_p$ se trouvent dans [Kle96, Xu97, BK03, Fay21a, Jac21].

[3] "BG" est pour Brunat–Gramain, puisque ces partitions apparaissent dans [BG10] dans l'étude des ensembles basiques pour $\mathfrak{S}_n$ et $\mathfrak{A}_n$.

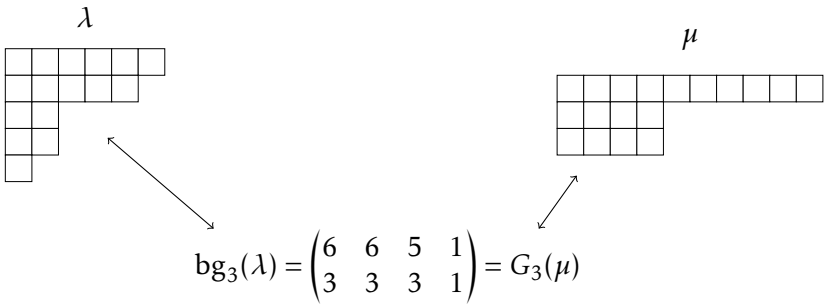
est naturel de demander une bijection explicite et naturelle entre les partitions auto-Mullineux et les BG-partitions. Dans le Chapitre 3 de cette thèse nous donnons une réponse à cette question[4].

Une autre motivation pour avoir une bijection explicite entre les deux ensembles de partitions se trouve dans le contexte des *ensembles p-basiques* : Dans [BG10], Brunat et Gramain ont montré l'existence d'un ensemble p-basique pour le groupe symétrique, qui, par restriction, donne un ensemble p-basique pour le groupe alterné. Cependant, cet ensemble, qui fournit de façon naturelle un ensemble d'indices pour les représentations irréductibles p-modulaires, n'est pas explicite et il serait idéal d'en donner une description complète. Une chose que nous savons à propos d'un tel ensemble est que les seules partitions auto-conjuguées qu'il contient sont les BG-partitions. Celles-ci indexent les points fixes pour l'application de Mullineux. Ceci donne une motivation pour obtenir une bijection explicite entre les BG-partitions et les partitions auto-Mullineux.

Aperçu du résultat principal du Chapitre 3. La construction qui mène à notre correspondance explicite peut être décrite en quelques mots comme suit. Tout d'abord, nous rappelons qu'à toute partition p-régulière λ, correspond de façon unique un *symbole de Mullineux* $G_p(\lambda)$. Ce symbole est un tableau formé de certains entiers positifs associés à λ. Maintenant, en nous inspirant de ce symbole de Mullineux, nous définissons un autre symbole associé, cette fois, aux partitions auto-conjuguées. Ce symbole est noté bg_p et nous l'appelons *BG-symbole*.

Nous sommes alors en mesure de prouver le fait remarquable, qui est au cœur de notre construction, que lorsque le BG-symbole est calculé sur une BG-partition λ, le tableau $\mathrm{bg}_p(\lambda)$ obtenu est exactement le symbole Mullineux $G_p(\mu)$ d'une certaine partition auto-Mullineux μ. En associant μ à λ, on obtient la correspondance souhaitée (puisqu'on montre en outre que le symbole BG est injectif).

Si nous prenons par example $p = 3$, la partition auto-Mullineux $\mu = (10, 4^2)$ correspond à la BG-partition $\lambda = (6, 5, 2^3, 1)$:



Nous soulignons que la correspondance que nous avons obtenue est explicite dans les deux sens. Les deux étapes de la bijection fonctionnent comme suit (comme illustré dans le diagramme ci-dessus). D'abord, on prend une BG-partition et on calcule son BG-symbole (un calcul direct). Ensuite, on considère le symbole obtenu

[4]Une bijection entre l'ensemble des partitions auto-Mullineux de n et les partitions de n à parties impaires différentes, aucune d'entre elles divisible par p, peut être dérivée d'une bijection entre deux ensembles plus généraux définis par C. Bessenrodt dans [Bes91]. Cependant, les deux approches sont très différentes car notre bijection est définie directement entre les ensembles qui nous intéressent. De plus, nous obtenons une bijection différente (voir Remark 3.2.4).

comme un symbole de Mullineux. Puisque toute partition p-régulière peut être facilement construite à partir de son symbole de Mullineux, nous pouvons reconstruire la partition p-régulière correspondante. La partition obtenue est une partition auto-Mullineux. Cette procédure est réversible, puisque reconstruire une partition auto-conjuguée à partir d'un BG-symbole est un procédé simple. En conclusion, nous avons une procédure explicite pour associer des BG-partitions à des partitions auto-Mullineux et vice versa.

En utilisant cette description, nous sommes en mesure de montrer la propriété intéressante suivante : le p-cœur d'une partition est préservé par la correspondance. Ceci a pour conséquence que notre bijection se restreint à une bijection entre les partitions auto-Mullineux et les BG-partitions dans chaque bloc du groupe symétrique.

Ensembles basiques unitriangulaires pour des p-blocs

Le deuxième résultat principal de cette thèse concerne la façon d'indexer les représentations irréductibles dans certains p-blocs du groupe symétrique. Introduisons le contexte.

Les modules de Specht S^λ sont en fait définis sur $\mathbb{Z}$ et il est donc possible de "réduire mod p" et de voir S^λ comme une représentation de $\mathfrak{S}_n$ sur $\mathbb{F}_p$, en d'autres termes, comme un $\mathbb{F}_p\mathfrak{S}_n$-module. En général, les modules de Specht sur $\mathbb{F}_p$ ne sont pas simples (irréductibles), ou même pas complètement réductibles, mais on peut regarder les multiplicités de composition des modules simples $\mathbb{F}_p\mathfrak{S}_n$-modules D^μ pour μ une partition p-régulière, dans S^λ. Ces multiplicités donnent des informations intéressantes pour étudier la théorie des représentations de $\mathfrak{S}_n$ sur $\mathbb{F}_p$. Pour un $\mathbb{F}_p\mathfrak{S}_n$-module simple D^μ, soit $d_{\lambda\mu} = [S^\lambda : D^\mu]$ la multiplicité de composition de D^μ dans le module de Specht S^λ. Les nombres $d_{\lambda\mu}$ forment alors une matrice $\mathbf{D}_{n,p}$ à coefficients entiers non négatifs, lignes indexées par les partitions de n et colonnes indexées par les $\mathbb{F}_p\mathfrak{S}_n$-modules simples. C'est la *matrice de décomposition* de $\mathbb{F}_p\mathfrak{S}_n$.

Lorsque les lignes de $\mathbf{D}_{n,p}$ sont ordonnées de manière décroissante selon un ordre total quelconque sur les partitions qui raffine l'ordre de dominance, il se produit quelque chose de particulièrement intéressant : Pour chaque colonne, la première entrée non nulle est 1 et la ligne associée correspond à une unique *partition p-régulière* de n. Cela donne une façon d'indexer les $\mathbb{F}_p\mathfrak{S}_n$-modules simples par l'ensemble des partitions p-régulières de n.

Par exemple, en prenant l'ordre lexicographique sur les partitions, $\mathbf{D}_{4,3}$ est

$$\begin{array}{l|cccc} 4 & 1 & & & \\ 3,1 & \cdot & 1 & & \\ 2^2 & 1 & \cdot & 1 & \\ 2,1^2 & \cdot & \cdot & \cdot & 1 \\ 1^4 & \cdot & \cdot & 1 & \cdot \end{array}$$

où les points et les entrées omises sont égales à 0. On voit en effet que, à ordre des lignes et colonnes près, la sous-matrice carrée formée par les lignes correspondant aux partitions p-régulières, a une forme unitriangulaire inférieure. Le fait que les $\mathbb{F}_p\mathfrak{S}_n$-modules simples puissent être indexés par l'ensemble des partitions p-régulières est connu avant d'observer la forme de la matrice $\mathbf{D}_{n,p}$. En effet, il existe autant de classes

de $\mathbb{F}_p\mathfrak{S}_n$-modules simples non équivalents que de classes de conjugaison de $\mathfrak{S}_n$ dont l'ordre est premier à p. De telles classes de conjugaison sont, à leur tour, en bijection avec les partitions p-régulières de de n. L'information supplémentaire qui provient de la matrice $\mathbf{D}_{n,p}$ ayant une telle forme, est que les classes des modules de Specht

$$\{S^\lambda \mid \lambda \text{ est une partition } p\text{-régulière de } n\}, \tag{2}$$

forment une base sur $\mathbb{Z}$ de ce qu'on appelle le *groupe de Grothendieck* de $\mathbb{F}_p\mathfrak{S}_n$ (en quelques mots, le groupe de Grothendieck d'une algèbre de dimension finie A est le $\mathbb{Z}$-module engendré par les classes d'isomorphisme des A-modules finiment engendrés, avec la relation qui dit que deux modules sont équivalents si et seulement si ils ont les mêmes facteurs de composition). Et la sous-matrice carrée de $\mathbf{D}_{n,p}$ dont les lignes sont indexées par les partitions p-régulières est précisément la matrice de transition entre cette base, formée par les classes des modules de Specht dans (2), et la base naturelle du groupe de Grothendieck, à savoir celle formée par les classes des $\mathbb{F}_p\mathfrak{S}_n$-modules simples. L'ensemble des partitions p-régulières est alors appelé un *ensemble basique unitriangulaire* (ou *UBS* en abrégé, pour ses sigles en anglais) pour $\mathbb{F}_p\mathfrak{S}_n$. En quelques mots, un UBS est un sous-ensemble des modules de Specht, avec un ordre total des lignes, pour lequel la matrice $\mathbf{D}_{n,p}$ prend une telle forme, c'est-à-dire qu'un tel ensemble indexe des modules simples de $\mathbb{F}_p\mathfrak{S}_n$, et à un tel ensemble on associe une $\mathbb{Z}$-base du groupe de Grothendieck de $\mathbb{F}_p\mathfrak{S}_n$. L'idée générale est que les modules de Specht sont beaucoup mieux connus que les modules simples (dimensions, caractères) et d'une certaine manière la matrice de décomposition permet de passer des uns aux autres de manière simple grâce à la unitriangularité.

Les ensembles basiques unitriangulaires pour $\mathbb{F}_p\mathfrak{S}_n$ ne sont pas uniques. Lors de l'étude de la théorie des représentations du groupe alterné $\mathfrak{A}_n$, la recherche d'un UBS plus "approprié" se pose. Une façon dont un UBS pour $\mathbb{F}_p\mathfrak{S}_n$ pourrait être plus appropriée pour cette étude est un UBS pour lequel faire le produit tensoriel avec la représentation signe, corresponde à la conjugaison des partitions, même pour $0 < p < n$. Un telle UBS est, en particulier, stable par la conjugaison de partitions. De plus, nous ajoutons la condition que les seules partitions auto-conjuguées dans une telle UBS sont des BG-partitions. Un *ensemble basique unitriangulaire stable* (SUBS pour ses sigles en anglais) est alors un UBS U, vu comme un sous-ensemble de l'ensemble des partitions, tel que :

(A) si $\mu \in U$, alors $\mu' \in U$, et

(B) si $\mu = \mu' \in U$, alors μ est une (p-)BG-partition.

La condition (B) est une condition technique qui permet aux SUBS U de se "restreindre" à un UBS pour $\mathbb{F}_p\mathfrak{A}_n$, voir [BGJ20, Theorem 12]. Cette condition implique également que le nombre de points fixes pour la conjugaison dans l'ensemble est égal au nombre de points fixes de l'application de Mullineux. Remarquons que l'UBS de $\mathbb{F}_p\mathfrak{S}_n$ formée par les partitions p-régulières n'est pas un SUBS, puisque par exemple, la conjuguée $\lambda' = (1^n)$ de la partition $\lambda = (n)$ est p-singulière lorsque $p \leq n$. La question est alors : existe-t-il toujours un SUBS pour le groupe symétrique ? et la réponse est non. Dans [BGJ20, §3], Brunat, Gramain et Jacon montrent que $\mathbb{F}_p\mathfrak{A}_n$ n'a pas toujours un UBS (par exemple lorsque $p = 3$ et $n = 18$), ce qui implique que le groupe symétrique ne peut pas toujours avoir un SUBS.

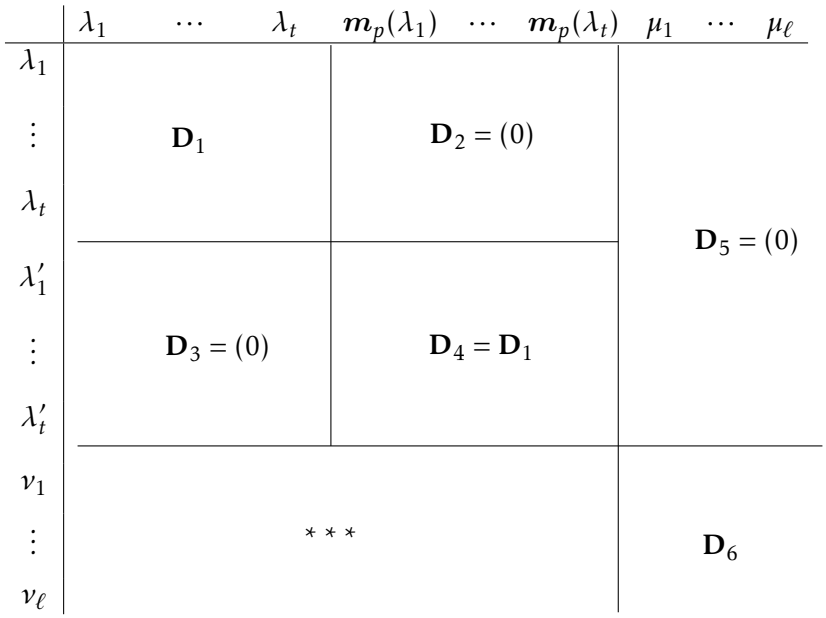
Malgré cela, la même question peut être abordée "bloc par bloc". Grâce à la théorie des représentations modulaires des groupes finis, on sait que l'ensemble de modules de Specht et celui des modules simples de $\mathbb{F}_p\mathfrak{S}_n$ se répartissent dans des *p-blocs*, que nous décrivons précisément dans §1.4. Cela signifie en particulier que la matrice de décomposition est une matrice diagonale par blocs :

$$\mathbf{D}_{n,p} = \begin{pmatrix} * & & 0 \\ & * & \\ & & \ddots \\ 0 & & * \end{pmatrix}$$

Les concepts d'UBS et de SUBS peuvent être adaptés pour avoir une signification pour chacun de ces p-blocs, nous les définissons précisément dans §4.2. Un entier non négatif est associé à chaque p-bloc ; son *p-poids*. Le poids d'un bloc peut être considéré comme une mesure de sa complexité. Par exemple, les blocs de poids 0 sont les plus faciles à décrire : ils sont formés d'un seul module. L'intérêt de définir des UBS et SUBS pour les blocs est de demander les mêmes propriétés dans le sens des blocs : la sous-matrice correspondant au bloc aura une forme unitriangulaire inférieure, cela donnera une façon naturelle d'indexer les $\mathbb{F}_p\mathfrak{S}_n$-modules simples, dans un bloc, pour lesquels l'application de Mullineux est la conjugaison. Un autre intérêt est d'obtenir un UBS pour les blocs de $\mathbb{F}_p\mathfrak{A}_n$: si un bloc de $\mathbb{F}_p\mathfrak{S}_n$ a un SUBS, alors il fournit par restriction, un UBS pour les blocs de $\mathbb{F}_p\mathfrak{A}_n$. Ainsi, la question est de déterminer si un bloc a une SUBS et si oui, d'en décrire un. Nous donnons une réponse à cette question dans le Chapitre 4 de cette thèse, pour tout p-bloc de $\mathbb{F}_p\mathfrak{S}_n$ de poids 2.

Aperçu du résultat principal du chapitre 4. Le résultat principal du Chapitre 4 est la construction d'un ensemble basique unitriangulaire stable pour les blocs auto-conjugués de poids 2. Notons d'abord que les blocs de poids 0 forment automatiquement un SUBS eux-mêmes et que les blocs de poids 1 ont des SUBS explicitement connus, voir [BGJ20, §5.2] et §4.4. Par conséquent, le poids 2 est naturellement la situation suivante à considérer. Néanmoins, nous étudions comment la compréhension de la combinatoire et de l'application de Mullineux pour les partitions de poids 1 peut aider à définir un ordre naturel sur les partitions pour ce SUBS. Dans un certain sens, notre étude des blocs de poids 2 montre qu'un tel bloc est formé de plusieurs parties disjointes qui se comportent indépendamment comme des blocs de poids 1.

Plus précisément, pour tout bloc auto-conjugué de poids 2, nous exhibons un sous-ensemble explicite de partitions tel que la matrice de décomposition prenne la forme :



où $\mathbf{D}_1$ et $\mathbf{D}_6$ sont unitriangulaires. Les partitions $\{\lambda_1,\dots,\lambda_t\}$ sont certaines partitions p-régulières dans le bloc qui ont été choisies de façon appropriée. Les partitions $\{\nu_1,\dots,\nu_\ell\}$ sont les BG-partitions dans le bloc et les partitions $\{\mu_1,\dots,\mu_\ell\}$ sont les partitions auto-Mullineux dans le bloc.

On remarque que l'ensemble des partitions qui indexe les lignes de la matrice est, par construction, stable par conjugaison, de sorte que l'UBS est en fait un SUBS.

Pour montrer l'unitriangularité de la matrice, nous nous appuyons sur des propriétés des nombres de décomposition qui s'expriment en termes d'ordre de dominance "$\trianglelefteq$" sur les partitions et l'application de Mullineux. Nous utilisons un étiquetage des modules de Specht dans de tels blocs qui provient de l'étude de Fayers et Richards sur les blocs de poids 2 [Fay, Ric96]. Dans [Ric96], Richards introduit un objet appelé *pyramide*, associé à un bloc. Nous étudions de façon détaillée l'ordre de dominance dans le bloc, qui est liée à la configuration de la pyramide et à d'autres notions combinatoires, qui nous permettent de montrer le résultat souhaité.

En pratique, l'unitriangularité de la matrice ci-dessus est montrée en quatre étapes, qui constituent la partie principale de notre travail :

D'abord, l'ensemble de partitions $\{\lambda_1,\dots,\lambda_t\}$ a la propriété que la matrice $\mathbf{D}_1 = (d_{\lambda_i\lambda_j})_{ij}$ est unitriangulaire. De plus, d'après les propriétés de la matrice de décomposition et de l'application de Mullineux, on a $\mathbf{D}_1 = \mathbf{D}_4$.

Notre étude détaillée de la pyramide nous permet de montrer la propriété combinatoire cruciale : $\lambda_i \trianglelefteq m_p(\lambda_j)$, pour tout i,j, ce qui implique que $\mathbf{D}_3 = (0)$.

Ensuite, on montre que $\lambda_i \trianglelefteq \mu_k$, pour tout i,k, ce qui implique que $\mathbf{D}_5 = (0)$.

La dernière étape consiste à montrer que la matrice $\mathbf{D}_6$ est unitriangulaire ; les entrées de cette matrice sont les nombres de décomposition qui relient les partitions auto-conjuguées et les partitions auto-Mullineux. Pour montrer l'unitriangularité, nous étudions la forme précise de la matrice $\mathbf{D}_6$ en utilisant les tables des nombres de décomposition dans [Ric96, Fay].

On signale que la bijection entre les BG-partitions et les partitions auto-Mullineux dans ce bloc, utilisée pour définir les SUBS, est spécifique au poids 2. Cependant, des

tests semblent montrer que cette bijection coïncide avec la bijection générale entre les deux familles de partitions définie au Chapitre 3.

Pour conclure, nous rappelons que l'existence d'un SUBS n'est pas assurée en général pour tout bloc. Notre résultat montre qu'il un existe toujours un pour les blocs de poids 2, et de plus, nous en avons donné un explicitement.

Organisation de cette thèse. Dans le Chapitre 1 nous rappelons les concepts fondamentaux impliqués dans les représentations du groupe symétrique, y compris la combinatoire des partitions et des p-crochets. Dans le Chapitre 2, nous donnons un aperçu historique du problème de Mullineux et nous rappelons quelques algorithmes pour calculer l'involution de Mullineux. Dans le chapitre 3, nous introduisons le BG-symbole qui permet ensuite de définir une bijection explicite entre les partitions auto-Mullineux et les BG-partitions. Ce chapitre est principalement basé sur [Ber21a]. Le Chapitre 4 concerne les ensembles basiques unitriangulaires stables et notre construction de tels ensembles pour les blocs de poids 2 de $\mathfrak{S}_n$. Ce chapitre est principalement basé sur [Ber21c]. Nous donnons enfin une annexe qui contient une liste de fonctions écrites en GAP, qui ont été utiles pour étudier les exemples concernant la bijection dans le Chapitre 3.

Introduction

The origin of representation theory of finite groups goes back to the end of the 19th century, when Frobenius studied the factorization of certain polynomial associated to a finite group, called the "group determinant". Generalizing the known case of an Abelian group, he proved that this polynomial factors as a product of irreducible polynomials, where each polynomial appears with a multiplicity equal to its degree. This question, noticed by Dedekind, is what led Frobenius to invent character theory for an arbitrary finite group and to build the foundations of what we now call representation theory. For a historical references on the origin of the subject, for which the pioneers are Frobenius, Burnside, Schur and Brauer, see [Cur99,Haw71]. The area has widely developed and has connections with physics, chemistry and other number of domains. It is interesting to note that most of the results on the classification of the ordinary representations of the symmetric group were known or discovered in the beginning of the theory, and, in contrast, there are still many questions on the modular representations that are unsolved to this day.

In modern language, representation theory of (finite) groups studies the ways in which a given group acts on vector spaces as linear operators. Concretely, a finite-dimensional representation of a group G is a group homomorphism $G \to \mathrm{GL}_d(\mathbb{K})$, where $\mathbb{K}$ is some field and $\mathrm{GL}_d(\mathbb{K})$ is the group of invertible $d \times d$ matrices. A representation can then be thought of as a way of "representing" elements of a group as matrices; and by doing so, bringing the group, which is an abstract object in principle, to the linear algebra universe.

Studying the representation theory of the group G means to study all such morphisms (or more precisely, up to certain equivalence). A representation may be built out of "smaller" representations. This happens when every matrix which represents an element of the group, can be put simultaneously in a triangular-by-blocks structure, by a change of basis. Each of the diagonal blocks is then a "smaller" representation (the size of the matrix is smaller). If no such a change of basis exists, then the representation is called irreducible. Moreover, for a finite group G and a field $\mathbb{K}$ of characteristic 0 (or not dividing the order of G), the triangular-by-blocks structure can always be reduced to a diagonal-by-blocks structure; hence, in this case, the task of studying all morphisms $G \to \mathrm{GL}_d(\mathbb{K})$ reduces more or less to studying all irreducible representations, up to equivalence. For a general field $\mathbb{K}$, the irreducible representations are anyway the fundamental building blocks of the representation theory and one usually starts the study of the representation theory of a group G over $\mathbb{K}$ by the study of the irreducible representations. Many different questions can be asked in such an enterprise; if we fix a group, how many nonequivalent irreducible representations exist? how explicit are they? what possible dimensions d come up? is there a "natural" labelling for the irreducible representations? how much do the answers to

these questions change if we change the field $\mathbb{K}$? what properties of the group reflect in the representations and vice versa? etcetera. Answers, or partial answers, to some of these questions have been found for certain groups or families of groups and have led to interesting applications, for example the classification of finite simple groups.

In this thesis our attention is focused on representations of the symmetric group $\mathfrak{S}_n$ and the alternating group $\mathfrak{A}_n$. The symmetric group $\mathfrak{S}_n$ is the group of permutations of an ordered list of n symbols, where the operation is composition. It is a group of order $n!$. As permutations are natural objects to consider, symmetric groups and more generally permutation groups (subgroups of a symmetric group), have been extensively studied and used, even long before the formalization of abstract groups. The alternating group $\mathfrak{A}_n$ is the set of even permutations (permutations with signature 1) of an ordered list of n symbols. It is then a subgroup of $\mathfrak{S}_n$.

The representation theory of the symmetric group was first studied and developed by Frobenius, Schur and Young. They found answers for several of the questions above when $\mathbb{K} = \mathbb{C}$[5]. In this case, every representation of $\mathfrak{S}_n$ splits as a sum of irreducible representations and there is a classification of these irreducible representations; they can be indexed by the set of partitions of n:

$$\mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_n) = \{S^\lambda \mid \lambda \text{ is a partition of } n\}.$$

Partitions of n are all the non-ordered different ways of writing n as a sum of positive integers. Since the work of Young, the combinatorics of partitions and a number of different objects associated to them are fundamental in the development of the representation theory of the symmetric group: the representations S^λ, called Specht modules, can be described completely in terms of λ. For example, the Specht module $S^{(n)}$ corresponding to the partition $\lambda = (n)$ is the *trivial* representation and the Specht module $S^{(1^n)}$ corresponding to the partition $\lambda = (1^n)$ is the *sign* representation. Another example is a formula called hook formula which easily gives the dimension of S^λ in terms of λ. To name one last example, the decomposition in irreducible representations of the induction of S^λ to $\mathfrak{S}_{n+1}$, can be easily described from λ.

In positive characteristic, say $p = \mathrm{char}(\mathbb{K})$, if p does not divide $n!$ (equivalently $p > n$), the representations of $\mathfrak{S}_n$ have the same classification as in characteristic 0, this is the *ordinary* representation theory. On the other hand, when p divides the order of the group, in our case if p divides $n!$ (equivalently $p \leq n$), we talk about *modular* representation theory.

The study of p-modular representation theory of $\mathfrak{S}_n$ started with the work of Nakayama in the 1940s, after the work of Brauer on modular representations of groups in general. There is a classification and a construction of the modular irreducible representations $\mathrm{Irr}_p(\mathfrak{S}_n)$ of $\mathfrak{S}_n$; they can be indexed by the set of p-regular partitions, that is, partitions in which no part occurs p times,

$$\mathrm{Irr}_p(\mathfrak{S}_n) = \{D^\lambda \mid \lambda \text{ is a } p\text{-regular partition of } n\}.$$

This construction is not as explicit as in characteristic zero: D^λ is the unique simple quotient of S^λ. Then the way of associating to λ the module D^λ is not as natural; the partition (n) corresponds to $D^{(n)}$ which is the trivial representation, as before.

[5] In fact, every representation of $\mathfrak{S}_n$ over $\mathbb{C}$ can be realized over $\mathbb{Q}$, then everything is similar if we take $\mathbb{K} = \mathbb{Q}$ or any extension of $\mathbb{Q}$.

But (1^n) does not correspond to the sign representation. However the study of this classification involves combinatorial objects, and there is a number of unanswered questions about the p-modular representations, for example, there is no equivalent of the hook formula for computing the dimension of D^λ. In Chapter 1 we recall several facts about the representation theory of $\mathfrak{S}_n$. For a very complete treatment see for example [JK81, Sag01].

The above considerations lead us to the subject of study in this thesis. There are two main results each of which makes the subject of a chapter. In what follows we introduce the context and the main questions that we treat.

Self-Mullineux partitions and self-conjugate partitions

One of the main results in this thesis concerns a correspondence between two families of partitions, one of them being the set of *self-Mullineux partitions*. We motivate the interest for such families.

Let $n \geq 2$. If M is an irreducible representation of $\mathfrak{S}_n$ over a field $\mathbb{K}$, the restriction $M\downarrow_{\mathfrak{A}_n}$ of M to $\mathfrak{A}_n$ is either irreducible, or it splits in the direct sum of two irreducible representations of $\mathfrak{A}_n$. Moreover every irreducible representation of $\mathfrak{A}_n$ over $\mathbb{K}$ arises in this way. More precisely, we know that $M\downarrow_{\mathfrak{A}_n}$ splits as a direct sum of two simple modules if and only if $M \simeq (M\otimes\varepsilon)$, where ε is the sign representation of $\mathfrak{S}_n$. Otherwise $M\downarrow_{\mathfrak{A}_n}$ is irreducible. Now, since ε is one-dimensional, $(M\otimes\varepsilon)$ is again an irreducible representation. Let us detail this discussion in both the ordinary and modular cases:

In the ordinary case, we know that every irreducible representation can be indexed by a partition λ of n. Hence, the above discussion says that $S^\lambda\downarrow_{\mathfrak{A}_n}$ splits if and only if $S^\lambda \simeq (S^\lambda\otimes\varepsilon)$. There exists a partition μ such that $(S^\lambda\otimes\varepsilon) \simeq S^\mu$. A natural question is then how to compute μ from λ. In the ordinary case, this question is easy to answer; we have $\mu = \lambda'$ the conjugate (or transposed) partition of λ. It is obtained just by transposing the Young diagram of λ. For example, if $\lambda = (n)$, we know that $S^{(n)}$ is the trivial representation, then $S^{(n)}\otimes\varepsilon \simeq \varepsilon$, and as mentioned above, the sign representation $\varepsilon = S^\mu$ is indexed by $\mu = (1^n)$, which is indeed the conjugate of $\lambda = (n)$. Hence the condition for the splitting of the restriction of S^λ as a direct sum of two simple modules becomes: $\lambda = \lambda'$. Otherwise, if $\lambda \neq \lambda'$, the restriction of S^λ to $\mathfrak{A}_n$ is irreducible. Details will be given in §1.3.3.

In the modular case, for exemple if $\mathbb{K} = \mathbb{F}_p$ with $2 < p < n$, we have a complete set of non-equivalent irreducible representations indexed by the p-regular partitions, and for λ p-regular, the restriction $D^\lambda\downarrow_{\mathfrak{A}_n}$ of the p-modular irreducible module D^λ of $\mathfrak{S}_n$ splits as a direct sum of two simple modules if and only if $D^\lambda \simeq (D^\lambda\otimes\varepsilon)$. Now, there is a p-regular partition μ such that $(D^\lambda\otimes\varepsilon) \simeq D^\mu$. In this case, how to compute the p-regular partition μ from λ ? Let $\mu = \boldsymbol{m}_p(\lambda)$. The involution which assigns to λ the partition $\boldsymbol{m}_p(\lambda)$ is the *Mullineux map*: $D^\lambda\otimes\varepsilon = D^{\boldsymbol{m}_p(\lambda)}$. The problem of finding a combinatorial description of this map is referred to as the *Mullineux problem*.

The answer to the Mullineux problem is not as straightforward as in the ordinary case. Conjugating a p-regular partition does not always result in a p-regular partition, then μ is not simply the conjugate partition of λ. In [Mul79a] Mullineux defined an involution on the set of p-regular partitions of n, and conjectured it to be $\boldsymbol{m}_p$. Later, in [Kle96], Kleshchev described a different algorithm to compute $\boldsymbol{m}_p(\lambda)$ and in [FK97],

Ford and Kleshchev proved Mullineux conjecture to be true. Mullineux conjecture was also proven to be true in [BO98] by Bessenrodt and Olsson by using yet another algorithm for computing $\boldsymbol{m}_p(\lambda)$[6].

Back to the restriction of D^λ to $\mathfrak{A}_n$ we have that, for λ a p-regular partition of $n \geq 2$,

- If $\lambda \neq \boldsymbol{m}_p(\lambda)$ then $D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\boldsymbol{m}_p(\lambda)} \downarrow_{\mathfrak{A}_n}$ is irreducible.
- If $\lambda = \boldsymbol{m}_p(\lambda)$ then $D^\lambda \downarrow_{\mathfrak{A}_n}$ splits into two non-equivalent irreducible p-modular representations $D^{\lambda+}$ and $D^{\lambda-}$ of $\mathfrak{A}_n$:

$$D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\lambda+} \oplus D^{\lambda-},$$

and

$$\begin{aligned}\{D^\lambda \downarrow_{\mathfrak{A}_n} | \ \lambda \vdash n,\ \lambda \text{ } p\text{-regular and } \lambda \neq \boldsymbol{m}_p(\lambda)\} \sqcup \\ \{D^{\lambda+}, D^{\lambda-} \mid \lambda \vdash n,\ \lambda \text{ } p\text{-regular and } \lambda = \boldsymbol{m}_p(\lambda)\},\end{aligned} \tag{3}$$

where only one of λ or $\boldsymbol{m}_p(\lambda)$ is considered in the first set, is a complete set of non-equivalent p-modular irreducible representations of $\mathfrak{A}_n$.

Following such an indexing of the irreducible p-modular representations of $\mathfrak{A}_n$, we are interested in particular in the set of fixed points of $\boldsymbol{m}_p$: the partitions λ of n such that $\lambda = \boldsymbol{m}_p(\lambda)$ which we call the self-Mullineux partitions. Determining the set of (p-)self-Mullineux partitions is not an easy task, in the sense that with the available algorithms for computing the Mullineux involution, we can not quickly determine if a p-regular partition is a fixed point. We then wonder if there is a different set of partitions, easier to determine, which is in correspondence with the set of self-Mullineux partitions. In fact, the number of (p-)self-Mullineux partitions is equal to the number of self-conjugate partitions with diagonal hook-lengths not divisible by p (see §1.3.4) which we refer to as *(p-)BG-partitions*[7]. Thus, it is natural to ask for an explicit and natural bijection between the self-Mullineux partitions and the BG-partitions. In Chapter 3 of this thesis we give an answer to this question[8].

A further motivation for having an explicit bijection between the two sets of partitions is found in the context of *p-basic sets*: In [BG10], Brunat and Gramain have shown the existence of a p-basic set for the symmetric group, which, by restriction, gives a p-basic set for the alternating group. However, this set, which provides a natural indexing set for the modular irreducible representations is not explicit and it would be ideal to give a complete description of it. One thing we know about such a set is that the only self-conjugate partitions that it contains are the BG-partitions, which index the fixed points of the Mullineux map. This gives a motivation for obtaining an explicit bijection between BG-partitions and self-Mullineux partitions.

[6] Other algorithms for computing $\boldsymbol{m}_p$ are found in [Kle96, Xu97, BK03, Fay21a, Jac21].

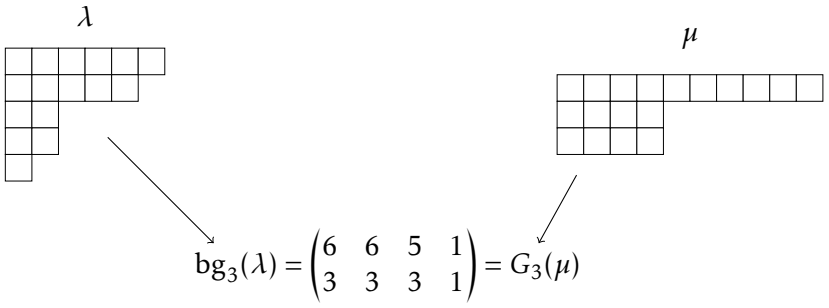
[7] "BG" is for Brunat–Gramain, since these partitions appear in [BG10] in the study of p-basic sets for $\mathfrak{S}_n$ and $\mathfrak{A}_n$.

[8] A bijection between the set of self-Mullineux partitions of n and partitions of n with different odd parts, none of them divisible by p can alternatively be derived from a bijection between two more general sets defined by C. Bessenrodt in [Bes91]. However, the two approaches are quite different because our bijection is defined directly between the sets of our interest. Moreover, we obtain a different bijection (see Remark 3.2.4).

Outline of the main result in Chapter 3. Roughly speaking, the construction leading to our explicit correspondence can be described as follows. First, we recall that to any p-regular partition λ, corresponds uniquely a *Mullineux symbol* $G_p(\lambda)$. This symbol is an array of certain positive integers associated to λ. Now, inspired by this Mullineux symbol, we define another symbol associated, this time, to self-conjugate partitions. We denote it bg_p and call it *BG-symbol*.

Then we are able to prove the remarkable fact, which lies at the heart of our construction, that when the BG-symbol is computed on a BG-partition λ, the array $\mathrm{bg}_p(\lambda)$ obtained is exactly the Mullineux symbol $G_p(\mu)$ of some self-Mullineux partition μ. Associating μ to λ, this provides the desired correspondence (since moreover the BG-symbol is shown to be injective).

If we take for example $p = 3$, the 3-self-Mullineux partition $\mu = (10, 4^2)$ corresponds to the 3-BG-partition $\lambda = (6, 5, 2^3, 1)$:



We emphasize that the correspondence we have obtained is really explicit, in both directions. The two steps work as follows (as illustrated in the diagram above). First take a BG-partition and calculate its BG-symbol (a straightforward calculation). Then see the obtained symbol as a Mullineux symbol. Since any p-regular partition can be easily constructed from its Mullineux symbol, we can reconstruct the corresponding p-regular partition. The partition obtained is a self-Mullineux partition. This procedure is reversible, since it is also a straightforward procedure to reconstruct a self-conjugate partition from its BG-symbol. In conclusion, we have an explicit procedure for associating BG-partitions to self-Mullineux partitions and vice versa.

Using this description we are able to show the following interesting property of this combinatorial procedure: the p-core of a partition is preserved under the correspondence. This has the consequence that our bijection restricts to a bijection between self-Mullineux partitions and BG-partitions within each block of the symmetric group.

Unitriangular basic sets for p-blocks

The second main result in this thesis concerns the labelling of irreducible representations in certain p-blocks of the symmetric group. Let us introduce the context.

Specht modules S^λ are actually defined over $\mathbb{Z}$ and thus it is possible to "reduce mod p" and see S^λ as a representation of $\mathfrak{S}_n$ over $\mathbb{F}_p$, in other words, as an $\mathbb{F}_p\mathfrak{S}_n$-module. In general, Specht modules over $\mathbb{F}_p$ are not simple (irreducible), or not even completely reducible, but one can look at the composition multiplicities of the simple $\mathbb{F}_p\mathfrak{S}_n$-modules D^μ for μ a p-regular partition, in S^λ. These multiplicities give interesting information to study the representation theory of $\mathfrak{S}_n$ over $\mathbb{F}_p$. For a simple

$\mathbb{F}_p\mathfrak{S}_n$-module D^μ, let $d_{\lambda\mu} = [S^\lambda : D^\mu]$ be the composition multiplicity of D^μ in the Specht module S^λ. The numbers $d_{\lambda\mu}$ form then a matrix $\mathbf{D}_{n,p}$ with non-negative integer coefficients, rows indexed by the partitions of n and columns indexed by the simple $\mathbb{F}_p\mathfrak{S}_n$-modules. This is the *decomposition matrix* of $\mathbb{F}_p\mathfrak{S}_n$.

When the rows of $\mathbf{D}_{n,p}$ are organised decreasingly according to any total order on partitions which refines the dominance order, something of special interest happens: For each column, the first non-zero entry is 1 and the associated row corresponds to a unique *p-regular partition* of n. This gives the mentioned indexing of the simple $\mathbb{F}_p\mathfrak{S}_n$-modules by the set of p-regular partitions of n.

For example, taking the lexicographic order on partitions, $\mathbf{D}_{4,3}$ is

$$\begin{array}{l|cccc} 4 & 1 & & & \\ 3,1 & \cdot & 1 & & \\ 2^2 & 1 & \cdot & 1 & \\ 2,1^2 & \cdot & \cdot & \cdot & 1 \\ 1^4 & \cdot & \cdot & 1 & \cdot \end{array}$$

where the dots and ommited entries are equal to 0. Indeed we can see that up to rearranging the rows and columns, the square submatrix formed by rows corresponding to p-regular partitions, has a lower unitriangular shape. The fact that the simple $\mathbb{F}_p\mathfrak{S}_n$-modules can be indexed by the set of p-regular partitions is known before observing the form of the matrix $\mathbf{D}_{n,p}$. Indeed, there are as many classes of nonequivalent simple $\mathbb{F}_p\mathfrak{S}_n$-modules as conjugacy classes of $\mathfrak{S}_n$ with order which is prime to p. Such conjugacy classes are, in turn, in bijection with the p-regular partitions of n. The additional information which comes from the matrix $\mathbf{D}_{n,p}$ having such a form, is that the classes of the Specht modules

$$\{S^\lambda \mid \lambda \text{ is a } p\text{-regular partition of } n\}, \tag{4}$$

form a $\mathbb{Z}$-basis of the so-called *Grothendieck group* of $\mathbb{F}_p\mathfrak{S}_n$ (roughly speaking, the Grothendieck group of a finite dimensional algebra A is the $\mathbb{Z}$-module generated by the isomorphism classes of the finitely generated A-modules, with the relation that says that two modules are equivalent if and only if they have the same composition factors). And the square sub-matrix of $\mathbf{D}_{n,p}$ with rows indexed by the p-regular partitions is precisely the transition matrix between this basis, formed by the classes of the Specht modules in (4), and the natural basis of the Grothendieck group; namely the one formed by the classes of simple $\mathbb{F}_p\mathfrak{S}_n$-modules. The set of p-regular partitions is then said to be a *unitriangular basic set*, or *UBS* for short, for $\mathbb{F}_p\mathfrak{S}_n$. Roughly speaking, a UBS is a subset of the Specht modules, together with a total order of the rows, for which the matrix $\mathbf{D}_{n,p}$ takes such a form i.e. such a set indexes simple $\mathbb{F}_p\mathfrak{S}_n$-modules, and to such a set we associate a $\mathbb{Z}$-basis of the Grothendieck group of $\mathbb{F}_p\mathfrak{S}_n$. The general idea is that the Specht modules are much better known than the simple modules (dimensions, characters) and in some way the decomposition matrix allows to simply pass from one the another by using the unitriangularity.

Unitriangular basic sets for $\mathbb{F}_p\mathfrak{S}_n$ are not unique. When studying the representation theory of the alternating group $\mathfrak{A}_n$, the quest of finding a more "suitable" UBS arises. Hence, one way in which a UBS for $\mathbb{F}_p\mathfrak{S}_n$ could be more suitable for this study is one for which tensoring with the sign representation corresponds to conjugation of partitions, even for $0 < p < n$. Such a UBS is, in particular, stable by conjugation of

partitions. Moreover, we add the condition that the only self-conjugate partitions in such a UBS are p-BG-partitions. A *stable unitriangular basic set* (SUBS) is then a UBS U, seen as a subset of the set of partitions, such that:

(A) if $\mu \in U$, then $\mu' \in U$, and

(B) if $\mu = \mu' \in U$, then μ is a (p-)BG-partition.

The condition (B) is a technical condition which allows the SUBS U to "restrict" to a UBS for $\mathbb{F}_p\mathfrak{A}_n$, see [BGJ20, Theorem 12]. This condition also implies that the number of fixed points of conjugation in the set is equal to the number of fixed points of the Mullineux map. Notice that the UBS of $\mathbb{F}_p\mathfrak{S}_n$ formed by p-regular partitions is not a SUBS, since for example, the conjugate $\lambda' = (1^n)$ of partition $\lambda = (n)$ is p-singular when $p \leq n$. The question is then: does a SUBS always exist for the symmetric group? and the answer is no. In [BGJ20, §3], Brunat, Gramain and Jacon show that $\mathbb{F}_p\mathfrak{A}_n$ does not always have a UBS (for example when $p = 3$ and $n = 18$), implying that the symmetric group cannot always have a SUBS.

Despite this, the same question can be addressed "blockwise". From the modular representation theory of finite groups we know that the Specht modules and the simple modules of $\mathbb{F}_p\mathfrak{S}_n$ fall into *p-blocks*, which we describe precisely in §1.4. This means in particular that the decomposition matrix is a block diagonal matrix:

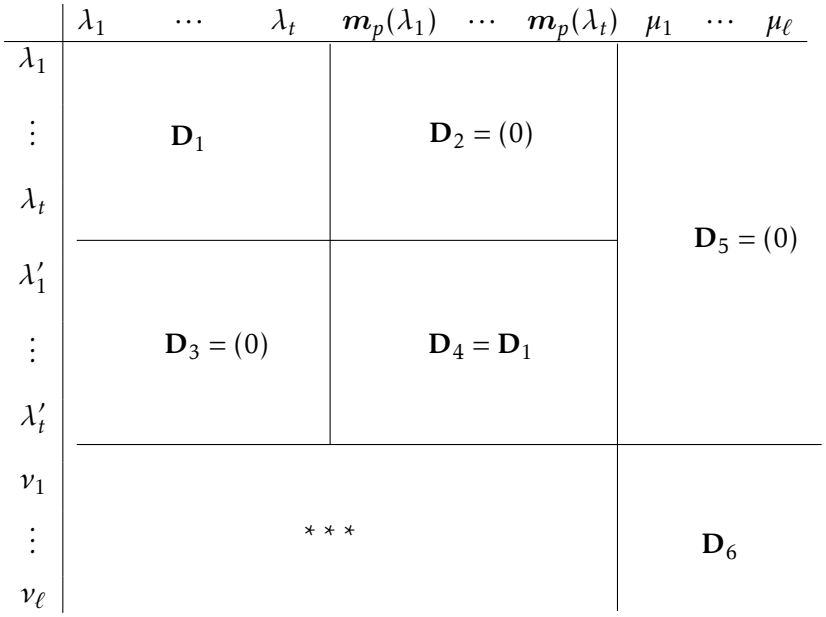
$$\mathbf{D}_{n,p} = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

The concepts of UBS and SUBS can be adapted to have a meaning for each of these p-blocks, we define them precisely in §4.2. There is a non-negative integer associated to each p-block; its *p-weight*. The weight of a block can be thought of a measure of its complexity, for example, blocks of weight 0 are the easiest to describe: they are formed by just one module. The interest of defining UBS and SUBS for blocks is to ask for the same properties blockwise: the submatrix corresponding to the block will have a lower unitriangular shape, this will give a natural labelling of the simple $\mathbb{F}_p\mathfrak{S}_n$-modules in a block, for which the Mullineux map is conjugation. Another interest is to obtain a UBS for blocks of $\mathbb{F}_p\mathfrak{A}_n$: if a block of $\mathbb{F}_p\mathfrak{S}_n$ has a SUBS, then it provides by restriction, a UBS for blocks of $\mathbb{F}_p\mathfrak{A}_n$. So that the question is to determine whether a block has a SUBS and if yes, to describe one. We give an answer to this question in Chapter 4 of this thesis, for any p-block of $\mathbb{F}_p\mathfrak{S}_n$ of weight 2.

Outline of the main result in Chapter 4. The main result in Chapter 4 is to provide a stable unitriangular basic set for self-conjugate blocks of p-weight 2. First we note that blocks of weight 0 form automatically a SUBS themselves and blocks of weight 1 have explicitely known SUBS, see [BGJ20, §5.2] and §4.4. Therefore the weight 2 is naturally the following situation to consider. Nevertheless, we study how understanding the combinatorics and Mullineux map for partitions of weight 1 can help to define a natural order on partitions for this SUBS. In a sense, our study of blocks

of weight 2 shows that such a block is formed of several disjoint parts which behave independently like blocks of weight 1.

More precisely, for any self-conjugate block of weight 2, we exhibit an explicit subset of partitions such that the decomposition matrix takes the form:



where $\mathbf{D}_1$ and $\mathbf{D}_6$ are unitriangular. The partitions $\{\lambda_1,\dots,\lambda_t\}$ are certain p-regular partitions in the block which have been appropriately chosen. The partitions $\{\nu_1,\dots,\nu_\ell\}$ are the BG-partitions in the block and the partitions $\{\mu_1,\dots,\mu_\ell\}$ are the self-Mullineux partitions in the block.

Note that the set of partitions forming the lines of the matrix is by construction stable by conjugation, so that the resulting UBS is indeed a SUBS.

To show the unitriangularity of the matrix, we base on properties of the decomposition numbers which are expressed in terms of the dominance order "$\trianglelefteq$" on partitions and the Mullineux map. We use a labelling of the Specht modules in such blocks which comes from Fayers' and Richards' study of blocks of weight 2 [Fay, Ric96]. In [Ric96], Richards introduces an object called a *pyramid*, associated to a block. We make a very detailed study of the dominance order, related to the pyramid and other combinatorial notions, which allow us to show the desired result.

In practice, the unitriangularity of the above matrix is then shown in four steps, which form the main part of our work:

First, the set $\{\lambda_1,\dots,\lambda_t\}$ has the property that the matrix $\mathbf{D}_1 = (d_{\lambda_i\lambda_j})_{ij}$ is unitriangular. Moreover, from properties of the decomposition matrix and the Mullineux map $\mathbf{D}_1 = \mathbf{D}_4$.

Our detailed study of the pyramid allows us to show the crucial combinatorial property: $\lambda_i \not\trianglelefteq m_p(\lambda_j)$, for any i,j, which implies that $\mathbf{D}_3 = (0)$.

Then, we show that $\lambda_i \not\trianglelefteq \mu_k$, for any i,k, which implies that $\mathbf{D}_5 = (0)$.

The last step is to show that the matrix $\mathbf{D}_6$ is unitriangular; the entries in this matrix are the decomposition numbers which relate self-conjugate partitions and self-Mullineux partitions. To show the unitriangularity we study the precise form of the

matrix using the tables of decomposition numbers in [Ric96, Fay]. We note that the bijection between the BG-partitions and the self-Mullineux partitions in this block, used to define the SUBS, is specific to weight 2. However, tests seem to show that this bijection coincides with the general bijection between the two families of partitions defined in Chapter 3.

To conclude, we recall that the existence of SUBS is not assured in general for any block. Our result shows that it always exists for blocks of weight 2, and moreover, we have provided one explicitly.

Organisation of this thesis. The organisation of this thesis is as follows. In Chapter 1 we recall the fundamentals concepts involved in the representations of the symmetric group, including the combinatorics of partitions and p-hooks. In Chapter 2 we give a historial background on the Mullineux problem and we recall some algorithms for computing the Mullineux involution. In Chapter 3 we introduce the BG-symbol which then allows to define the explicit bijection between self-Mullineux partitions and BG-partitions. This chapter is mainly based on [Ber21a]. Chapter 4 concerns stable unitriangular basic sets and our construction of these sets for blocks of weight 2 of $\mathfrak{S}_n$. This chapter is mainly based on [Ber21c]. Finally, the appendix contains a list of functions written in GAP, which were useful to study examples concerning the bijection in Chapter 3.

1 Preliminaries

In this chapter we recall the fundamentals of the representation theory of the symmetric group which are necessary for the following chapters. The organization of the present chapter is the following: The first section is about partitions and some combinatorial objects associated to them. In §1.2 we recall the description of conjugacy classes of the symmetric group and the alternating group. Then, in §1.3 we recall the classification of the modular and ordinary irreducible representations of the symmetric and alternating groups.

1.1 Combinatorics of partitions

Integer partitions are combinatorial objects that arise when studying the symmetric group. This section contains definitions related to integer partitions.

1.1.1 Partitions

Let $n \in \mathbb{N}$. A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers containing only finitely many non-zero terms such that $\lambda_1 + \lambda_2 + \cdots = n$. We write $\lambda \vdash n$. The number n is called the *rank* of λ, denoted $\mathrm{Rk}(\lambda)$.

We denote by $\mathrm{Par}(n)$ the set of partitions of n. The integers λ_i are called the *parts* of the partition λ. If there is a part that repeats k times, say $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+k-1}$, we abbreviate the writing of the part λ_i as $(\ldots, \lambda_i^k, \ldots)$. The number of non-zero parts is the *length* of λ and is denoted $l(\lambda)$. The *empty partition* is the only partition of 0, it is denoted $\emptyset$.

The *conjugate* (or *transpose*) partition of $\lambda = (\lambda_1, \ldots, \lambda_l)$, where $l = l(\lambda)$, is the partition of n denoted λ' and defined as $\lambda'_i = \#\{j \mid 1 \leq j \leq l \text{ and } \lambda_j \geq i\}$. The operation of conjugation of a partition is an involution. A partition which is equal to its conjugate is called a *self-conjugate* partition.

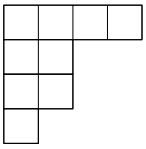
For a positive integer p, the partition λ is said to be *p-regular* if it does not contain p parts $\lambda_i \neq 0$ which are equal. A partition which is not p-regular is called p-singular. The partition $\lambda = (4, 2^2, 1)$ is not 2-regular but it is 3-regular. We denote by $\mathrm{Reg}_p(n)$ the set of p-regular partitions of n.

1.1.2 Young diagrams

A partition can be represented by its *Young diagram*. The Young diagram $[\lambda]$ of a partition λ is the set

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i \geq 1 \text{ and } 1 \leq j \leq \lambda_i\},$$

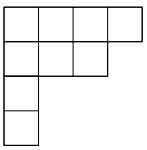
whose elements are called *nodes*. We represent $[\lambda]$ as an array of boxes in the plane with the convention that i increases downwards and j increases from left to right. For example, the Young diagram of $\lambda = (4, 2^2, 1)$ is

$[\lambda] =$ 

We will often identify a partition with its Young diagram.

Remark 1.1.1. A partition can also be graphically represented by what is known as its *Ferrers graph*. For which, instead of boxes, every node in the Young diagram is represented by a dot. △

The *diagonal* of $[\lambda]$ (or *diagonal* nodes) is the set of nodes of the form (i, i) for $1 \leq i \leq \lambda_i$ for every $1 \leq i \leq l(\lambda)$. Conjugation of partitions is easily seen in the Young diagram; the Young diagram of λ' is the reflection of the Young diagram of λ against the main diagonal. Then the i-th part of λ' is the number of nodes in the i-th column of λ. If $\lambda = (4, 2^2, 1)$, then $\lambda' = (4, 3, 1^2)$, its Young diagram is

$[\lambda'] =$ 

1.1.3 Orders on partitions

The *dominance order* "$\trianglelefteq$" is a partial order defined on the set of all partitions. It is defined as follows: let λ and μ be two partitions, we say that $\lambda \trianglelefteq \mu$ if and only if

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \quad \text{for every } k \geq 1.$$

We write $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ but $\lambda \neq \mu$. The *lexicographic order* "$\leq$" is a total order defined on the set of all partitions. It is defined as follows: let λ and μ be two partitions, we say that $\lambda \leq \mu$ if and only if the first non-vanishing difference $\mu_i - \lambda_i$ is positive. We write $\lambda < \mu$ if $\lambda \leq \mu$ but $\lambda \neq \mu$. The lexicographic order is a refinement of the dominance order in the set $\mathrm{Par}(n)$, and they are different if $n \geq 6$. Indeed, we have $(3^2) < (4, 1^2)$ but (3^2) and $(4, 1^2)$ are not comparable for the dominance order.

1.1.4 Rim, hooks and cores

Hooks are certain parts of the Young diagram of a partition, which are important when studying the modular representations of symmetric groups. They were first considered by Nakayama [Nak41a, Nak41b]. We recall some definitions.

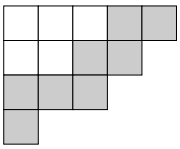
To any node (i, j) of the diagram of λ we can associate its (i, j)-*th hook*, denoted H_{ij}^{λ}, which consists of: the node (i, j), the nodes to the right of it in the same row (the *arm*),

and the nodes lower down in the same column (the *leg*). The cardinal of H_{ij}^{λ} is the *length* (or *hook-length*) of the (i,j)-th hook, it is denoted h_{ij}^{λ} and it is equal to

$$h_{ij}^{\lambda} = \lambda_i - j + \lambda'_j - i + 1.$$

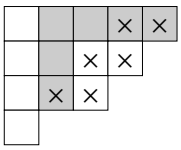
The number of nodes on the arm (resp. leg) of the hook is called the *arm-length* (resp. *leg-length*).

The *rim* of λ is the set of nodes $\{(i,j) \in [\lambda] \mid (i+1, j+1) \notin [\lambda]\}$, in words, it is the south-east border of $[\lambda]$. For example, the rim of $\lambda = (5,4,3,1)$ is formed by the shaded nodes in the following diagram



To H_{ij}^{λ} (or to the node (i,j)) we can associate a set R_{ij}^{λ} of the same size, consisting of the adjacent nodes forming a segment in the rim of λ between nodes (i,λ_i) and (λ'_j, j). The set R_{ij}^{λ} is called the (i,j)-th *rim-hook*. We call *p-rim-hook* any rim-hook of length p, for a positive integer p. We call (p)-rim-hook of (p)-hook any rim-hook of length **divisible** by p.

Example 1.1.2. Let $\lambda = (5,4,3,1)$. The hook $H_{1,2}^{\lambda}$ is illustrated by shaded nodes and the corresponding rim-hook $R_{1,2}^{\lambda}$ illustrated by nodes marked with "×" in the following diagram



The hook (and the rim-hook) in this diagram has length 6, the arm-length is 3 and the leg-length is 2. △

Given a partition λ and any node $(i,j) \in [\lambda]$, a new partition μ can be defined by *removing* the rim-hook R_{ij}^{λ} from λ. That is, μ is the partition with Young diagram $[\mu] = [\lambda] \setminus R_{ij}^{\lambda}$. The partition μ is well defined since a rim-hook is precisely a subset of adjacent nodes that can be removed from a Young diagram such that the remaining nodes form a Young diagram.

The operation of removing a rim-hook will be repeatedly used in this thesis; it is involved in some recursive algorithms. Let us start using the recursive removal of rim-hooks by defining the p-core of a partition.

Let p be a positive integer, and $\lambda \vdash n$ a partition. Consider the following procedure : If λ has a p-rim-hook, remove it from λ. We obtain a partition of $n-p$. If the partition obtained has a p-rim-hook, remove it, and we get a partition of $n-2p$. Continue removing p-rim-hooks recursively until obtaining a partition γ_λ, possibly the empty partition, for which the diagram does not have p-rim-hooks. Suppose that in total we removed w p-rim-hooks in this sequence of steps. The partition obtained $\gamma_\lambda \vdash (n-wp)$ does not depend on the way that the p-rim-hooks were removed (see [Nak41a, §4] or [JK81, Theorem 2.7.16]). This partition is called the *p-core* of λ. The number w is

also independent of the way they were removed and it is called the *p-weight* of λ. The p-weight of λ is also equal to the number of rim-hooks of λ of length divisible by p [JK81, 2.7.40]. A partition without p-rim-hooks is called a *p-core*, in other words, a partition of p-weight 0. The p-core of a p-core is equal to itself.

Example 1.1.3. Let $p = 3$ and $\lambda = (5,4,1)$. There are two possible sequences of rim-hook removals to obtain the p-core of λ. They are shown in the following diagrams: first the rim-hook with "×" nodes and then the rim-hook with "∘" nodes.

The p-core of λ is then $(2,1^2)$, which does not have 3-rim-hooks. For this example we show the possible sequences of rim-hooks removals, but it is sufficient to find one of these sequences to compute the p-core of a partition. Since a sequence in this example contains exactly two 3-rim-hooks, then λ is a partition of 3-weight 2. △

1.1.5 The abacus

The abacus display for a partition is a graphic representation which is convenient when considering the p-hooks of a partition and its p-core. It was introduced by James in [Jam78].

In order to define the abacus, we first recall a family of sets of integers associated to a partition. The following definition is based on [Fay14].

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition and r an integer. The *β-set* $\mathcal{B}_r^\lambda$ is the set defined as:

$$\mathcal{B}_r^\lambda := \{\lambda_i - i + r \mid i \in \mathbb{N}^*\}.$$

For every $r \in \mathbb{Z}$, the set $\mathcal{B}_r^\lambda$ is then an infinite set of integers bounded above. Since from a certain rank λ_i is zero, if we denote $\beta_i = \lambda_i - i + r$, then the sequence $\beta_1 > \beta_2 > \cdots$ is formed of consecutive negative integers, starting from a certain rank. Hence, the complement of $\mathcal{B}_r^\lambda$ in $\mathbb{Z}$ is bounded below. Conversely, let $\mathcal{B}$ be a set of integers formed by a sequence bounded above and whose complement in $\mathbb{Z}$ is bounded below. Denote the numbers in $\mathcal{B}$ as $\beta_1 > \beta_2 > \cdots$. There is a unique partition λ and an integer r such that $\mathcal{B} = \mathcal{B}_r^\lambda$. Indeed, let r be the number of positive integers in $\mathcal{B}$ minus the number of negative integers in $\mathbb{Z} \setminus \mathcal{B}$. Let $\lambda_i := \beta_i + i - r$. Then $\mathcal{B} = \mathcal{B}_r^\lambda$.

Example 1.1.4. Let $\lambda = (4,3^3,2,1^2)$. The β-set corresponding to $r = 0$ is

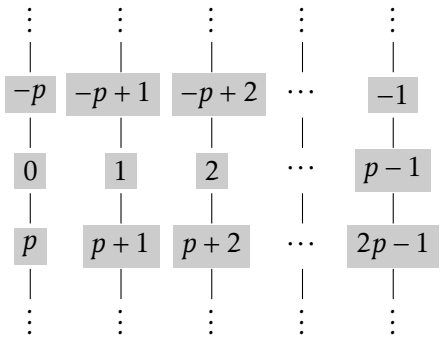
$$\mathcal{B}_0^\lambda = \{3,\ 1,\ 0,\ -1,\ -3,\ -5,\ -6,\ -8, -9,\ -10,\ \ldots\}.$$

The β-set corresponding to $r = 7$ is

$$\mathcal{B}_7^\lambda = \{10,\ 8,\ 7,\ 6,\ 4,\ 2,\ 1,\ -1,\ -2,\ -3,\ \ldots\}.$$

△

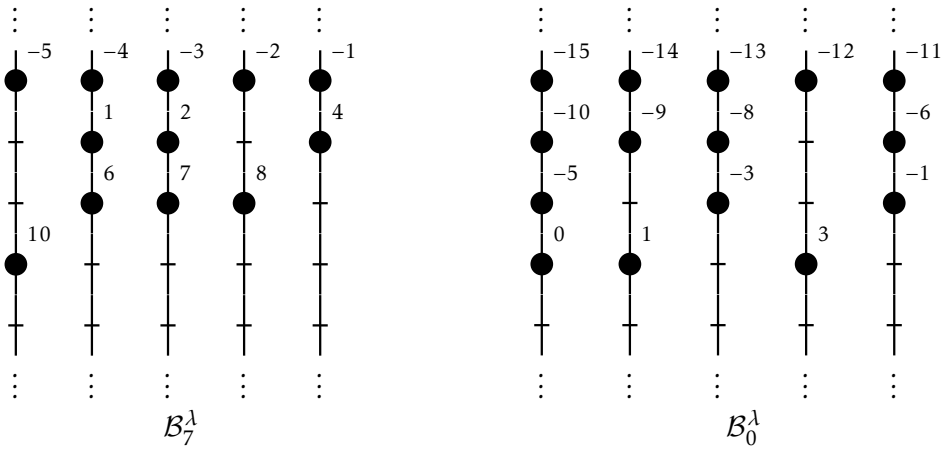
The *abacus* is a very convenient way of visualizing β-sets. For a positive integer p, consider p vertical lines called *runners* on which there are positions labelled by $\mathbb{Z}$ as follows.



(1.1)

Each position can be either empty or occupied by a bead. A *p-abacus* (or abacus) for a partition λ is the abacus with beads placed in positions corresponding to some β-set $\mathcal{B}_r^\lambda$ and empty spaces in positions corresponding to $\mathbb{Z} \setminus \mathcal{B}_r^\lambda$.

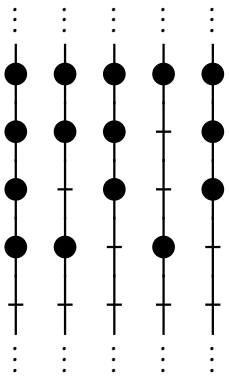
Example 1.1.5. Let $p = 5$ and $\lambda = (4, 3^3, 2, 1^2)$ as in Example 1.1.4. The p-abacus displays for λ corresponding to β-sets $\mathcal{B}_7^\lambda$ and $\mathcal{B}_0^\lambda$ are:



where only positions with beads are labelled. △

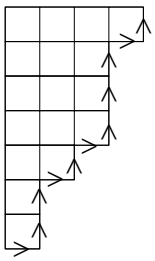
Since each p-abacus corresponds to a β-set for λ, there is a whole family of p-abacuses which represent graphically a same partition λ. But two β-sets differ only by shifting, that is, adding the same integer to each of the numbers in one β-set. Hence we can drop the labelling of beads in the abacus display without losing information. Unless otherwise specified, the preferred abacus for a partition λ is the one corresponding to the β-set $\mathcal{B}_0^\lambda$. For this abacus can be said to *have a number of beads* which is a multiple of p. By *having a number of beads* we mean all the beads in the set of rows of the abacus which contain at least one empty space. It is clear that this set of beads is finite since empty spaces correspond to the complement of $\mathcal{B}_0^\lambda$ which is bounded below. With these conventions, the 5-abacus of the partition $\lambda = (4, 3^3, 2, 1^2)$

from Example 1.1.5 is:



(1.2)

A practical way to obtain the p-abacus for a partition $\lambda = (\lambda_1, \ldots)$ from its Young diagram and vice versa is the following. Start with the Young diagram of λ. The south-east border of the diagram is made of segments. We can think of this segments as being oriented right "$\rightarrow$" and "$\uparrow$" segments which define then a lattice path starting from the bottom of the diagram to the top and following the south-east border, see the diagram below. We define an infinite word on the symbols "$\bullet$" and "$\circ$" as follows: form a finite word from the lattice path by replacing every "$\rightarrow$" for "$\circ$" and every "$\uparrow$" for "$\bullet$"; then concatenate an infinite number of "$\bullet$" to the left of the finite word and an infinite number of "$\circ$" to the right, we obtain an infinite $\bullet\circ$-*word*. For example, for the partition $\lambda = (4, 3^3, 2, 1^2)$, with Young diagram



the infinite word is

$$\cdots\bullet\bullet\bullet\circ\bullet\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\circ\circ\cdots$$

Now, if we divide the $\bullet\circ$-word in groups of consecutive p symbols (we can represent such a division by drawing vertical bars), then such a division gives a p-abacus for λ by associating each group of p symbols, from right to left, to a row in the abacus from top to bottom. Or, in other words, by *wrapping* the $\bullet\circ$-word in an a empty abacus with p runners in the direction of the labelling in Figure 1.1. Then the symbol "$\bullet$" corresponds to a bead and the symbol "$\circ$" to an empty position. For obtaining the 5-abacus (1.2) we draw vertical bars every 5 symbols as follows:

$$\underbrace{\cdots\bullet}_{a}\,|\underbrace{\bullet\bullet\bullet\circ\bullet}_{b}|\underbrace{\bullet\circ\bullet\circ\bullet}_{c}|\underbrace{\bullet\bullet\circ\bullet\circ}_{d}|\,\underbrace{\circ\cdots}_{e}$$

Finally, we have given the names a, b, c, d for the groups of symbols from left to right in this example. We now put them from top to bottom as follows:

$$\begin{array}{ll} a & \{\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ b & \{\bullet \quad \bullet \quad \bullet \quad \circ \quad \bullet \\ c & \{\bullet \quad \circ \quad \bullet \quad \circ \quad \bullet \\ d & \{\bullet \quad \bullet \quad \circ \quad \bullet \quad \circ \\ e & \{\circ \quad \circ \quad \circ \quad \circ \quad \circ \end{array}$$

We can see that we obtain the 5-abacus of $\lambda = (4, 3^3, 2, 1^2)$ shown in (1.2).

The correspondence Young diagram - $\bullet\circ$-word - abacus is well defined: to a Young diagram corresponds a unique family of β-sets, each one indexed by $r \in \mathbb{Z}$. For every r, a β-set $\mathcal{B}_r^\lambda \subseteq \mathbb{Z}$ determines an indicator function $\mathbb{1}_{\mathcal{B}_r^\lambda} : \mathbb{Z} \longrightarrow \{0,1\}$, which is 0 starting from a certain rank and 1 for all values less than certain rank, that is, there exist $m, M \in \mathbb{Z}$ such that $\mathbb{1}_{\mathcal{B}_r^\lambda}(l) = 0$ for all $l \geq M$ or $\mathbb{1}_{\mathcal{B}_r^\lambda}(l) = 1$ for all $l \leq m$. A $\bullet\circ$-word with divisions is completely determined by choosing the position of one division, hence, making correspond 0 and 1 to $\circ$ and $\bullet$, respectively, such a word is equivalent to a function $\mathbb{Z} \longrightarrow \{0,1\}$, which is 1 at $-\infty$ and 0 at $+\infty$.

The last fact for verifying that the correspondence is well defined is that the south-east lattice path encodes a β-set for λ. Consider the β-set $\mathcal{B}_0^\lambda$, where $\beta_i = \lambda_i - i$ for all $i \in \mathbb{N}^*$. And observe that for this β-set (hence for any β-set), the number of integers in the gap between two consecutive β-numbers, β_i and β_{i+1} is equal to $\beta_{i+1} - \beta_i - 1 = \lambda_{i+1} - \lambda_i$. That is, the difference between two parts of λ. In terms of the abacus, the gap between two consecutive beads has a number of empty spaces equal to a difference between consecutive parts of λ.

1.1.6 Core and quotient in the abacus

Let us see why the abacus display is particularly nice for computing the p-core and p-quotient in practice. Consider the β-set $\mathcal{B}_h^\lambda$, where $h = l(\lambda)$. The set $\mathcal{B}_h^\lambda$ is formed by the numbers

$$\lambda_1 + h - 1, \quad \lambda_2 + h - 2, \quad \ldots \quad , \quad \lambda_{h-1} + 1, \quad -1, \quad -2, \quad -3, \quad \ldots$$

The positive integers in $\mathcal{B}_h^\lambda$ are exactly the hook-lengths of the nodes in the first column of $[\lambda]$, that is, hooks $H_{1,1}, H_{2,1}, \ldots H_{h,1}$. Hence, if a p-rim-hook is removed from λ, for some positive integer p, then a β-set for the resulting partition is obtained from $\mathcal{B}_h^\lambda$ by replacing certain β_i for $\beta_i - p$, which in the p-abacus amounts to sliding a bead one position up. This is true in general for any β-set associated to λ, and it is the motivation for introducing β-sets:

Lemma 1.1.6 (Lemma 2.7.13, JK81)**.** *Removing a p-rim-hook $R_{i,j}^\lambda$ from $[\lambda]$ means for every β-set $\mathcal{B}_r^\lambda$ that a suitable $\beta_k^r \in \mathcal{B}_r^\lambda$ is changed into $\beta_k^r - p$, and the resulting set is a β-set for $[\lambda] \setminus R_{i,j}^\lambda$. Conversely, changing a number β_k in a β-set for λ into $\beta_k - p$ (if $\beta_k - p$ is not already in the β-set) results in a β-set for a partition μ which arises from λ by removing a p-rim-hook.*

β-sets are a way of formalizing the fact of removing p-rim-hooks from a partition. Hence they are useful for proving results involving p-rim-hooks, such as the uniqueness of the p-core for a partition.

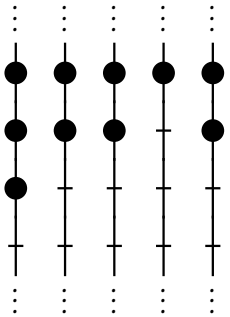
Remark 1.1.7. The leg-length of a p-hook is encoded in the p-abacus: let λ be any partition and consider its p-abacus display. Take any bead that can be moved one position down, say the bead is in position $t \in \mathbb{Z}$ and move it down one position. The new position is then $t+p$. The new abacus corresponds to a partition obtained from λ by adding some p-rim hook $H_{i,j}$. From the correspondence of Young diagrams and $\bullet\circ$-words, we have that the leg-length of $H_{i,j}$ is equal to the number of beads occuring in positions $t+1, t+2, \ldots, t+p-1$. That is the number of beads in positions between the initial and final position of the moved bead. Similarly, the arm-length of $H_{i,j}$ is equal to the number of empty spaces occuring in the same list of positions. △

A direct consequence of Lemma 1.1.6 is that a β-set for the p-core γ of the partition λ is obtained from a β-set $\mathcal{B}_r^\lambda$ of λ by replacing $\beta \in \mathcal{B}_r^\lambda$ with $\beta - p$ as far as possible. Which is then seen in the p-abacus as sliding the beads up as far as possible, that is, such that there are no empty positions between beads in a same runner. Also, the p-weight of λ is the number of such replacements, or bead slide movements. That confirms the fact that every partition has a uniquely determined p-core.

Example 1.1.8. The 5-core of the partition $\lambda = (4, 3^3, 2, 1^2)$ from Example 1.1.5 can be calculated by removing for example the 5-rim-hooks $H^\lambda_{1,3}, H^\lambda_{4,1}$, and $H^\lambda_{1,1}$ in that order. In the following diagram they are marked respectively as $0, 1$ and 2.

	2	0	0
	2	0	
2	2	0	
2	1	0	
1	1		
1			
1			

The resulting diagram (1^2) does not have any 5-rim-hooks. The partition (1^2) is then the 5-core of λ. In the abacus, the partition (1^2) is obtained from diagram (1.2) by sliding up the only two beads that can be moved, as far as possible. In total, 3 sliding movements have to be made, which agrees with the 3 hooks removed from λ. We obtain the following abacus:



△

We now recall the definition of the p-quotient of a partition λ. Consider the abacus corresponding to the β-set $\mathcal{B}_0^\lambda$, that is, the p-abacus for λ in which the number of beads is a multiple of p. This convention ensures that the abacus display for a partition is unique. Observe that each runner can be seen as a 1-abacus, which corresponds then

to some partition. Denote such a partition $\lambda^{(i)}$ if it corresponds to the i-th runner from left to right for $1 \leq i \leq p-1$. The p-tuple of these partitions

$$\overline{\lambda}_p := (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p-1)}),$$

is called the *p-quotient* of λ. In general, a k-tuple $\overline{\mu}$ of partitions is called a *k-partition*. For an integer m we call *rank* of $\overline{\mu}$ or we say that $\overline{\mu}$ is k-partition of m if the sum of the ranks of the partition which form $\overline{\mu}$ is equal to m. The p-quotient of λ is a p-partition of w, where w is the p-weight of λ.

Example 1.1.9. Let $p = 5$ and $\lambda = (4, 3^3, 2, 1^2)$ with abacus (1.2) in page 26. The p-quotient of λ is $\overline{\lambda}_p = (\emptyset, (1), \emptyset, (2), \emptyset)$. This is a p-partition of 3, the p-weight of λ. △

The uniqueness of the abacus, under the chosen convention and Lemma 1.1.6 gives

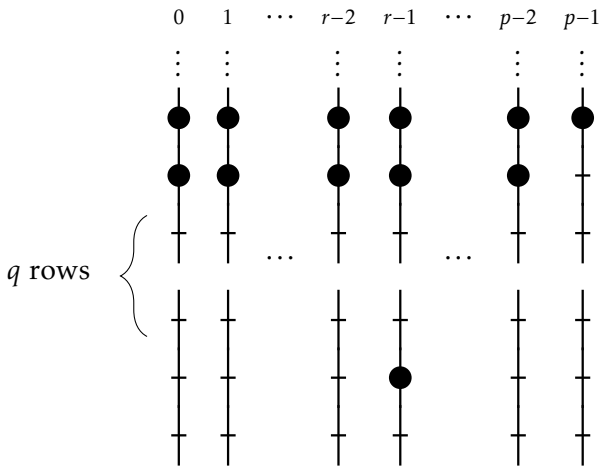
Theorem 1.1.10 ([JK81, Theorem 2.7.30]). *For every $p \in \mathbb{N}^*$, a partition is uniquely determined by its p-core and its p-quotient.*

Theorem 1.1.10 can be seen as a version of the Euclidean division for partitions. In fact, for a one-row partition, the p-core and the p-quotient are exactly the residue and quotient from the division by p, let us see this in detail.

Example 1.1.11. Let $n \in \mathbb{N}^*$ and λ be the one-row partition $\lambda = (n)$. Let $1 \leq p < n$. Euclidean division says that there exist unique integers q and r such that

$$n = pq + r, \qquad \text{and} \quad 0 \leq r < p.$$

The p-abacus of $\lambda = (n)$ has the following form:

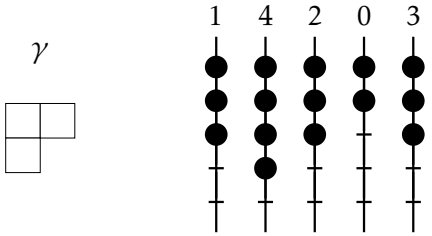


Then, the p-quotient of λ is then the p-partition $\overline{\lambda}_p = (\emptyset, \ldots, \emptyset, (q), \emptyset, \ldots, \emptyset)$, with partition (q) as the r-th component. The p-core of λ is obtained by sliding up the bead that can be moved in the $(r-1)$-th runner from left to right, all the way up, which gives partition (r). The p-core of λ is then (r). △

1.1.7 Labelling of runners in the abacus

We recall a useful labelling for the runners of the abacus of a p-core. Let γ be a p-core and consider its abacus. Take the positions of the lowest bead in each runner. This gives a list of p integers $\rho_0 < \rho_1 < \cdots < \rho_{p-1}$. Since each of these integers corresponds to (a position on) a runner, such a list yields a labelling $0, 1, \ldots, p-1$ of the runners, possibly different from the natural left-to-right labelling: now the 0-th runner is the leftmost with "*minimal number*" of beads, the $(p-1)$-th runner is the rightmost with the "*maximal number*" of beads. We can rephrase this by defining a total order on runners: Let "$\preccurlyeq$" be the following order on runners. For R and S two runners on the abacus of γ we say that $R \preccurlyeq S$ iff R has strictly less beads than S, or R and S have the same number of beads and R is to the left of S. Where, of course, we talk about "*number of beads*" in a figurative sense. Now, with this order, our new labelling $0, 1, \ldots, p-1$ coincides with increasingly ordering the runners with respect to "$\preccurlyeq$". From now on, if not otherwise specified, we use this labelling for runners on an abacus.

Example 1.1.12. Let $p = 5$ and $\gamma = (2, 1)$. The Young diagram and 5-abacus of γ, with the corresponding labelling of runners is



△

1.2 The symmetric and alternating groups

One of our principal objects of study is the symmetric group. In this section we recall some definitions about the symmetric group, its conjugacy classes and the alternating group.

Let $n \in \mathbb{N}$. The *symmetric group* $\mathfrak{S}_n$, on n symbols, is the set of bijections from $\{1, 2, \ldots, n\}$ to itself. The multiplication is composition. The degree of $\mathfrak{S}_n$ is then $|\mathfrak{S}_n| = n!$. An element $\pi \in \mathfrak{S}_n$ is called a *permutation* of $\{1, \ldots, n\}$ or simply a permutation. There are different ways to write a permutation. One way is two-line notation, that is:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

For example, the permutation defined by $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$, $4 \mapsto 5$ and $5 \mapsto 4$ is written

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}.$$

Our preferred way of denoting permutations is cycle notation: for a $i \in \{1, 2, \ldots, n\}$ consider $i, \pi(i), \pi^2(i), \ldots$. There exists $r \geq 1$ such that $\pi^r(i) = i$. The *cycle* containing i is

$$(\, i \quad \pi(i) \quad \pi^2(i) \quad \cdots \quad \pi^{(r-1)}(i) \,).$$

Conversely, a cycle $(i\ j\ k\ \cdots\ l)$ means that π sends i to j, j to k, ..., and finally l to i, cyclically. Now, if there is any, take $j \in \{1,2,\ldots,n\}$ which is not in the cycle containing i and iterate the process. The concatenation of the obtained cycles gives the cycle notation for π. For example, for the permutation π, above the cycle notation is (1 2 3)(4 5). Cyclically permuting elements within a cycle does not change the permutation. For example (1 2 3) = (2 3 1) = (3 1 2) in $\mathfrak{S}_3$. A cycle with k elements is called a *k-cycle*. Sometimes 1-cycles are omitted; by convention, omitted numbers are considered fixed points of the permutation.

Example 1.2.1. $\mathfrak{S}_3$ = {Id, (1 2), (1 3), (2 3), (1 2 3), (1 3 2)}. △

The 2-cycles in $\mathfrak{S}_n$ are an important kind of permutations, they are also referred to as *transpositions*. The group $\mathfrak{S}_n$ is generated by transpositions: Any cycle $(i\ j\ k\ \cdots\ l\ m)$ is equal to the product $(i\ j)(j\ k)\cdots(l\ m)$. Moreover, it can be shown that $\mathfrak{S}_n$ is generated by *adjacent* transpositions, that is, the transpositions $(1\ 2),(2\ 3),\ldots(n-1\ n)$. Given a permutation $\pi \in \mathfrak{S}_n$, there is not a unique way of writing π as a product of transpositions, for example $(1\ 2)^2 = (2\ 3)^2$ in $\mathfrak{S}_3$. However, it can be proved that the parity of the number of transpositions in any writing of π is always the same. In view of this fact, let $\pi = \tau_1\tau_2\cdots\tau_k$ with τ_i transpositions, the *sign* of π is defined as

$$\varepsilon(\pi) := (-1)^k.$$

Another notation for the sign of π is $\operatorname{sgn}(\pi)$. The map $\varepsilon : \mathfrak{S}_n \longrightarrow \{1,-1\}$ defines an homomorphism into the multiplicative group $\{1,-1\}$. The kernel of ε is the subgroup denoted $\mathfrak{A}_n$ and called the *alternating group*. According with the parity of transposition writing, the elements of $\mathfrak{A}_n$ are called *even* permutations and those in $\mathfrak{S}_n \setminus \mathfrak{A}_n$ are called odd.

Example 1.2.2. $\mathfrak{A}_3$ = {Id, (1 2 3), (1 3 2)}. △

The alternating group is a normal subgroup of $\mathfrak{S}_n$. From the isomorphism theorems for groups $|\mathfrak{A}_0| = |\mathfrak{A}_1| = 1$ and $|\mathfrak{A}_n| = \frac{n!}{2}$ for $n \geq 2$. A useful remark is that a k-cycle $(i_1\ i_2\ \cdots\ i_{k-1}\ i_k)$ is in $\mathfrak{A}_n$ (is even) if and only if k is odd. Indeed $(i_1\ i_2\ \cdots\ i_{k-1}\ i_k) = (i_1\ i_2)(i_2\ i_3)\cdots(i_{k-1}\ i_k)$ is written as the product of $k-1$ transpositions.

1.2.1 Conjugacy classes of $\mathfrak{S}_n$

The cycle notation for a permutation is unique up to permutation of the cycles, if the cycles are disjoint. Hence the set of lengths of the cycles is well defined for a permutation. Let $\pi \in \mathfrak{S}_n$. Write π in cycle notation and organize the cycles decreasingly by length

$$\pi = (i_1\ \cdots\ i_{\lambda_1})(j_1\ \cdots\ j_{\lambda_2})\cdots(k_1\ \cdots\ k_{\lambda_r}).$$

The *cycle type* of π is the partition λ of n whose parts are the lengths of the cycles of π

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r).$$

Example 1.2.3. The cycles types of the permutations Id, (1 2) and (1 2 3) in $\mathfrak{S}_3$ are, respectively the partitions (1^3), $(2,1^2)$ and (3) of 3. △

Note that every partition of n occurs as the cycle type of some permutation. This correspondence is not a bijection, though, since two different permutations can have the same cycle type. As a matter of fact, all permutations with a same cycle type form a conjugacy class of $\mathfrak{S}_n$. Let us observe this in detail.

Recall that two elements g, h in a group G are *conjugates* if there exists $k \in G$ such that $g = khk^{-1}$. This relation between pairs of elements in G defines an equivalence relation. The equivalence classes in this case are called *conjugacy classes*. In the symmetric group $\mathfrak{S}_n$, the action of conjugating a cycle has an interesting behavior: consider the cycle $(i_1\ i_2\ \cdots\ i_k) \in \mathfrak{S}_n$ and let $\pi \in \mathfrak{S}_n$. Then

$$\pi\ (i_1\ \ i_2\ \ \cdots\ \ i_k)\ \pi^{-1} = (\pi(i_1)\ \ \pi(i_2)\ \ \cdots\ \ \pi(i_k)).$$

Thus, in general, conjugating a permutation $\sigma \in \mathfrak{S}_n$ written in cycle notation by a permutation π is equal to applying π to every number $1, 2, \ldots, n$ keeping the same cycle structure of σ. Hence, conjugating does not change the cycle type of a permutation. Conversely, two permutations having the same cycle type are conjugated. For example, in $\mathfrak{S}_5$, the permutations $\pi = (2\ 3)(1\ 4\ 5)$ and $\sigma = (4\ 5)(1\ 2\ 3)$ are conjugated by any permutation mapping the 2-cycle to the 2-cycle and the 3-cycle to the 3-cycle; take

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}.$$

In consequence, the cycle type of permutation defines a bijection:

$$\begin{array}{ccc} \mathrm{Par}(n) & \longleftrightarrow & \{\, \mathcal{C} \mid \mathcal{C} \text{ is a conjugacy class of } \mathfrak{S}_n\} \\ \lambda & \longmapsto & \mathcal{C}^\lambda := \{\pi \in \mathfrak{S}_n \mid \text{the cycle type of } \pi \text{ is } \lambda\}. \end{array}$$

1.2.2 Conjugacy classes of $\mathfrak{A}_n$

We recall the structure of conjugacy classes of $\mathfrak{A}_n$. We saw that conjugacy classes of $\mathfrak{S}_n$ are indexed by the partitions of n by means of the cycle type of partitions. Then, all permutations in a conjugacy class of $\mathfrak{S}_n$ are either even or odd. That is, if $\mathcal{C}^\lambda$ is a conjugacy class of $\mathfrak{S}_n$ then $\mathcal{C}^\lambda \subseteq \mathfrak{A}_n$ or $\mathcal{C}^\lambda \subseteq \mathfrak{S}_n \setminus \mathfrak{A}_n$.

Let $\mathcal{C}^\lambda$ be a conjugacy class of $\mathfrak{S}_n$ of even elements, that is, $\mathcal{C}^\lambda \subseteq \mathfrak{A}_n$. That means that λ has an **even number of even parts**. Let $\pi \in \mathcal{C}^\lambda$. Consider the following two situations.

- Suppose that there is $\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$ such that π and σ commute. That is, $\sigma\pi\sigma^{-1} = \pi$. Let π' be such that $\pi' = x\pi x^{-1}$, for some $x \in \mathfrak{S}_n$.
  If $x \in \mathfrak{A}_n$, then π and π' are conjugates in $\mathfrak{A}_n$; if $x \in \mathfrak{S}_n \setminus \mathfrak{A}_n$ then $x\sigma \in \mathfrak{A}_n$ as it is the product of two odd permutations. Now, $\pi' = x\pi\ x^{-1} = x(\sigma\pi\sigma^{-1})x^{-1} = (x\sigma)\pi(x\sigma)^{-1}$, then π and π' are conjugates in $\mathfrak{A}_n$.
  Hence, in this case, the conjugacy class of π in $\mathfrak{A}_n$ coincides with the conjugacy class of π in $\mathfrak{S}_n$.

- If π does not commute with any element in $\mathfrak{S}_n \setminus \mathfrak{A}_n$, then the $\mathfrak{S}_n$-conjugacy class $\mathcal{C}^\lambda$ of π splits in two $\mathfrak{A}_n$-conjugacy classes of the same size, which are respectively formed by elements of type $x\pi x^{-1}$ with $x \in \mathfrak{A}_n$, and elements of type $x\pi x^{-1}$ with $x \in \mathfrak{S}_n \setminus \mathfrak{A}_n$. The two $\mathfrak{A}_n$-classes are of equal size since one is the image of the other under conjugation with any permutation in $\mathfrak{S}_n \setminus \mathfrak{A}_n$.

In conclusion, given a $\mathfrak{S}_n$-conjugacy class $\mathcal{C}^\lambda$, included in $\mathfrak{A}_n$ (λ has an even number of even parts), either all of the elements in $\mathcal{C}^\lambda$ form a conjugacy class of $\mathfrak{A}_n$, or $\mathcal{C}^\lambda$ splits in two $\mathfrak{A}_n$-conjugacy classes of the same size. Moreover, $\mathcal{C}^\lambda$ splits if and only if for every $\pi \in \mathcal{C}^\lambda$, π does not commute with any element $\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$.

Now, the only elements which commute with a given permutation π are generated by the product of powers of cycles in π, as well as elements permuting cycles in π of a same length. Hence, if there is a cycle of even length in π, then the permutation formed only by this cycle is in $\mathfrak{S}_n \setminus \mathfrak{A}_n$, and commutes with π. Also, if there are two cycles of the same odd length in π, the permutations which exchanges one by one elements in these two cycles, commutes with π and it is odd, as it is an odd product of transpositions. We then have the following known criterion for when an even $\mathfrak{S}_n$-conjugacy class splits as two $\mathfrak{A}_n$-conjugacy classes:

Proposition 1.2.4. *Let $\mathcal{C}^\lambda$ be a conjugacy class of $\mathfrak{S}_n$ such that $\mathcal{C}^\lambda \subseteq \mathfrak{A}_n$.*

- *$\mathcal{C}^\lambda$ is the disjoint union of two conjugacy classes $\mathcal{C}^{\lambda+} \sqcup \mathcal{C}^{\lambda-}$ of equal size of $\mathfrak{A}_n$ if and only if λ is formed by **different odd** parts.*
- *$\mathcal{C}^\lambda$ is a conjugacy class of $\mathfrak{A}_n$ otherwise.*

Furthermore, in the first case, we have that for any $\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$

$$\sigma \mathcal{C}^{\lambda+} \sigma^{-1} = \mathcal{C}^{\lambda-}.$$

That is, the conjugacy classes $\mathcal{C}^{\lambda+}$ and $\mathcal{C}^{\lambda-}$ are conjugate in $\mathfrak{S}_n$.

Example 1.2.5. Let $n = 4$ and consider the partitions of 4: $\lambda = (3,1)$ and $\mu = (2^2)$. The associated conjugacy classes of $\mathfrak{S}_4$ are, for λ the 3-cycles:

$$\mathcal{C}^\lambda = \{(1\ 2\ 3),\ (1\ 3\ 2),\ (2\ 3\ 4),\ (2\ 4\ 3),\ (1\ 3\ 4),\ (1\ 4\ 3),\ (1\ 2\ 4),\ (1\ 4\ 2)\},$$

and for μ the double-transpositions:

$$\mathcal{C}^\mu = \{(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\}.$$

Now, as sets, both conjugacy classes are included in $\mathfrak{A}_4$, but the partition λ is formed by different odd parts, then from Proposition 1.2.4 we know that $\mathcal{C}^\lambda = \mathcal{C}^{\lambda+} \sqcup \mathcal{C}^{\lambda-}$, where $\mathcal{C}^{\lambda+}$ and $\mathcal{C}^{\lambda-}$ are $\mathfrak{A}_4$-conjugacy classes. We have:

$$\mathcal{C}^{\lambda+} = \{(1\ 2\ 3),\ (1\ 3\ 4),\ (2\ 4\ 3),\ (1\ 4\ 2)\},$$

and

$$\mathcal{C}^{\lambda-} = \{(1\ 3\ 2),\ (1\ 2\ 4),\ (2\ 3\ 4),\ (1\ 4\ 3)\},$$

where the notations + and − where randomly assigned. On the other hand, μ is not a partition formed by different odd parts, then $\mathcal{C}^\mu$ is itself a $\mathfrak{A}_4$-conjugacy class. △

Proposition 1.2.4 gives an indexing of the conjugacy classes of $\mathfrak{A}_n$: Denote by $\mathrm{Par}^{\mathfrak{A}}(n)$ the set of partitions of n with an even number of even parts (the cycle types of permutations in $\mathfrak{A}_n$). Denote by $\mathrm{Par}^{\mathfrak{A}}_{+-}(n)$ the set of partitions of n formed by different

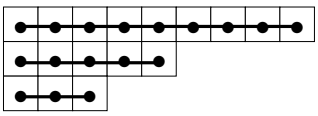
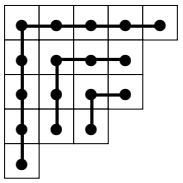
odd parts. We have $\mathrm{Par}^{\mathfrak{A}}_{+-}(n) \subseteq \mathrm{Par}^{\mathfrak{A}}(n)$. Finally let $\mathrm{Par}^{\mathfrak{A}}_{\circ}(n) = \mathrm{Par}^{\mathfrak{A}}(n) \setminus \mathrm{Par}^{\mathfrak{A}}_{+-}(n)$. There is a bijection:

$$\begin{array}{ccc} \mathrm{Par}^{\mathfrak{A}}_{\circ}(n) \sqcup \left(\mathrm{Par}^{\mathfrak{A}}_{+-}(n) \times \{+,-\}\right) & \longleftrightarrow & \{\, \mathcal{C} \mid \mathcal{C} \text{ is a conjugacy class of } \mathfrak{A}_n\} \\ \begin{cases} \lambda & \longmapsto & \mathcal{C}^{\lambda} \\ (\mu,+) \text{ or } (\mu,-) & \longmapsto & \mathcal{C}^{\mu+} \text{ or } \mathcal{C}^{\mu-}. \end{cases} & & \end{array} \tag{1.3}$$

A partition bijection The set $\mathrm{Par}^{\mathfrak{A}}_{+-}(n)$ is in bijection with the set of self-conjugate partitions of n. Indeed, let $\lambda \in \mathrm{Par}^{\mathfrak{A}}_{+-}(n)$. We define a partition μ as the self-conjugate partition with diagonal hook-lengths the parts of lambda:

$$h^{\mu}_{1,1} = \lambda_1, \quad h^{\mu}_{2,2} = \lambda_2, \quad \dots$$

Since the parts of λ are odd and different, μ is well-defined. Conversely, a self-conjugate partition μ gives a partition λ by defining the parts of λ as the diagonal hook-lengths of μ, as above. That way, the parts of λ are different and odd since μ is self-conjugate. This correspondence is easily seen in the Young diagrams: parts become diagonal hooks and vice versa. For example, if $\lambda = (9,5,3) \in \mathrm{Par}^{\mathfrak{A}}_{+-}(17)$ then $\mu = (5,4^4,3,1)$:

$[\lambda] =$  $\longleftrightarrow$ $[\mu] =$ 

(1.4)

Then, there is a bijection:

$$\mathrm{Par}^{\mathfrak{A}}_{+-}(n) = \{\, \lambda \mid \lambda \vdash n,\ \lambda_i \text{ are all odd and different} \,\} \longleftrightarrow \{\, \lambda \mid \lambda \vdash n,\ \lambda = \lambda' \,\}.$$

Remark 1.2.6. Later, when reviewing the representation theory of $\mathfrak{A}_n$ we will see that there exists a bijection between the set of conjugacy classes of $\mathfrak{A}_n$ and the set of partitions:

$$\{\, \lambda \mid \lambda \vdash n,\ \lambda > \lambda' \,\} \sqcup (\{\, \lambda \mid \lambda \vdash n,\ \lambda = \lambda' \,\} \times \{+,-\}),$$

(see Theorem 1.3.12). Implying that the set $\mathrm{Par}^{\mathfrak{A}}_{\circ}(n) \sqcup \left(\mathrm{Par}^{\mathfrak{A}}_{+-}(n) \times \{+,-\}\right)$ is also in bijection with the preceding set of partitions of n. Now, considering the fact that we have an explicit bijection $\mathrm{Par}^{\mathfrak{A}}_{+-}(n) \leftrightarrow \{\, \lambda \mid \lambda \vdash n,\ \lambda = \lambda' \,\}$, the natural question for an explicit bijection between the sets

$$\{\, \lambda \mid \lambda \vdash n,\ \lambda > \lambda' \,\} \leftrightarrow \mathrm{Par}^{\mathfrak{A}}_{\circ}(n),$$

arises. We ask for an explicit bijection between the set of partitions strictly greater than their conjugate (or a set of partitions on which we chose one of λ or λ', for every $\lambda \vdash n$) and the set of partitions with an even number of even parts such that if all the parts are odd, then there exist two equal parts. We do not have an answer to this combinatorial problem. △

1.2.3 p-prime conjugacy classes and BG-partitions

There is a family of conjugacy classes of $\mathfrak{S}_n$ which is involved in the p-modular representation theory of $\mathfrak{S}_n$. Before discussing this relation in detail, let us see a useful combinatorial correspondence.

Recall that the order of an element in a group depends only on the conjugacy class. We can then talk of the order of a conjugacy class. For a prime number p, we say that a conjugacy class is *p-prime* if the order is relatively prime to p.

In the symmetric group $\mathfrak{S}_n$, the order of a conjugacy class is easy to calculate: it is the least common multiple of the lengths of the cycles in the cycle structure. Hence, a conjugacy class $\mathcal{C}^\lambda$ is p-prime if and only if

$$\gcd(\mathrm{lcm}(\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}), p) = 1.$$

Which is equivalent to $p \nmid \lambda_i$ for all $i \leq \lambda \leq l(\lambda)$. Then the partitions indexing p-prime conjugacy classes of $\mathfrak{S}_n$ are those for which none of the parts are divisible by p. We call such partitions *p-prime*[1] partitions.

There are as many p-prime partitions of n as p-regular partitions. This is true in general for any $p \in \mathbb{N}$. Recall that a partition is p-regular if it does not contain p non-zero parts which are equal. One bijection between the two sets of partitions is due to Glaisher (1883, see [Bre99, Exercise 2.2.7]). Such bijection is a generalization of his correspondence between partitions formed of odd parts and partitions formed of different parts. A proof with generating functions, that these two sets of partitions have the same number of elements can be found in [JK81, proof of Lemma 6.1.2].

Glaisher's correspondence Glaisher's bijection from p-prime partitions to p-regular partitions is as follows. Take a p-prime partition. If it is already p-regular there is nothing to do. If it is not p-regular, there is a part λ_i repeated p times. Replace these p parts for one part of length $p\lambda_i$. Iterate this process until none of the parts repeats p times. The obtained partition is p-regular. This process is uniquely reversible.

Example 1.2.7. Let $p = 3$ and take $\lambda = (7, 2^3) \vdash 13$. The partition λ is not 3-regular, but it is 3-prime. Under Glaisher's correspondence, the three repeated parts (of length 2) become one part (of length 6). The obtained partition is $(7, 6)$ which is 3-regular. △

We have then a one-to-one correspondence:

$$\{\, p\text{-prime conjugacy classes of } \mathfrak{S}_n \,\} \longleftrightarrow \{\, p\text{-regular partitions of } n \,\}. \tag{1.5}$$

The importance of the fact that these two sets are in bijection will be more clear when studying the representation theory, where instead of indexing p-modular irreducible representations of $\mathfrak{S}_n$ by p-prime partitions, they will be indexed by p-regular partitions.

Let us now consider some p-prime conjugacy classes of $\mathfrak{A}_n$. We saw in Proposition 1.2.4 that there are two kinds of $\mathfrak{S}_n$-conjugacy classes which are included in $\mathfrak{A}_n$: those

[1] In the literature these partitions are sometimes called p-regular partitions, and our p-prime conjugacy classes are called p-regular conjugacy classes. Since we use the term p-regular for a different kind of partitions, we chose to leave the term p-prime for the corresponding conjugacy classes and partitions.

which split as two $\mathfrak{A}_n$-conjugacy classes and those which do not split. Those which split are indexed by the set of partitions of n formed with different odd parts. Now, from these conjugacy classes, choose the ones which are p-prime. From the correspondence (1.4), we obtain a bijection

$$\left\{ \mathcal{C}^\lambda \;\middle|\; \begin{array}{c} \mathcal{C}^\lambda \subseteq \mathfrak{A}_n \text{ is a } p\text{-prime} \\ \text{conj. class with } \lambda \in \mathrm{Par}^{\mathfrak{A}}_{+-}(n) \end{array} \right\} \longleftrightarrow \left\{ \lambda \;\middle|\; \begin{array}{c} \lambda \vdash n,\ \lambda \text{ is self-conjugate and} \\ p \nmid h^\lambda_{ii}, \text{ for all } \lambda_i \neq 0 \end{array} \right\} \tag{1.6}$$

The two sets are important in this work. Hence we give a name to the set of partitions in the right.

Definition 1.2.8. A partition λ of n is called a *p-BG-partition* (or a *BG-partition*) if λ is self-conjugate and none of the diagonal hook-lengths of λ is divisible by p. The set of p-BG-partitions of n is denoted BG^n_p. The set of all partitions which are p-BG partitions is denoted BG_p.

In terms of (p)-hooks, as defined in §1.1.4, λ is a p-BG-partition if and only if λ is self-conjugate and it does not have diagonal (p)-hooks.

Example 1.2.9. The set of 3-BG-partitions of 18 is

$$\mathrm{BG}^{18}_3 = \{(6,5,2^3,1),\ (7,4,2^2,1^3),\ (9,2,1^7)\}.$$

△

1.3 Representations of the symmetric and alternating groups

In this section we state well known facts about the representation theory of the symmetric and alternating groups. Statements and definitions are principally borrowed from [JK81, Sag01, CR62].

1.3.1 Fundamentals

Some vocabulary

Before discussing the representation theory of the symmetric group, we set some general vocabulary.

Let $\mathbb{K}$ be any field and G a finite group. A *representation* (π, V) of G over $\mathbb{K}$ (or a $\mathbb{K}$-representation of G) is a group homomorphism $\pi : G \longrightarrow \mathrm{GL}_{\mathbb{K}}(V)$, where V is a finite-dimensional vector space over $\mathbb{K}$ and $\mathrm{GL}_{\mathbb{K}}(V)$ is the group of $\mathbb{K}$-automorphisms of V. The *dimension* of the representation is the dimension of V. We refer to a representation by either the morphism π or the vector space V.

Consider the group algebra $\mathbb{K}G$. There is a one-to-one correspondence between the set of $\mathbb{K}$-representations of G and the set of finitely generated $\mathbb{K}G$-modules: Let $(M, *)$ be a $\mathbb{K}G$-module. In particular, M is a $\mathbb{K}$-vector space. For a fixed $g \in G$, the $\mathbb{K}$-linear map

$$\begin{array}{rccc} \pi_g : & M & \longrightarrow & M \\ & m & \longmapsto & g * m \end{array}$$

is a $\mathbb{K}$-automorphism of M, of inverse $\pi_{g^{-1}}$. Hence

$$\begin{array}{rccc} \pi: & G & \longrightarrow & \mathrm{GL}_{\mathbb{K}}(M) \\ & g & \longmapsto & \pi_g \end{array}$$

defines a representation of the group G. Conversely, any $\mathbb{K}$-representation (π, V) of G uniquely defines a $\mathbb{K}G$-module, with action $*$ defined linearly as

$$g * v := \pi(g)(v),$$

for $g \in G$ and $v \in V$. We use this correspondence throughout this text, by referring to either a representation of a group or a module.

Two representations (π, M) and (ρ, L) of G are said to be *equivalent* (or *isomorphic*) if there is an isomorphism of $\mathbb{K}$-vector spaces $\phi : M \longrightarrow L$ such that $\phi \circ \pi(g) = \rho(g) \circ \phi$. In this case, the isomorphism ϕ is called a G-isomorphism. We write $M \simeq L$. Equivalence of representations is, of course, an equivalence relation.

Let M be a $\mathbb{K}G$-module and let $H \leq G$ be a subgroup. The *restriction* of M to H is the $\mathbb{K}H$-module denoted $M \downarrow_H$ where the underlying module is M and the action is the action of G restricted to H. If N is a $\mathbb{K}H$-module, the *induced* $\mathbb{K}G$-module is $N \uparrow^G = \mathbb{K}G \otimes_{\mathbb{K}H} N$.

Reducibility and decomposition

A *submodule* of a $\mathbb{K}G$-module M is a subspace $N \subseteq M$ that is *stable*, that is, closed under the action of G. A $\mathbb{K}G$-module M is said to be *simple* (or *irreducible* for the corresponding representation), if the only submodules are 0 and M.

In representation theory of finite groups, we aim to classify the irreducible representations of a group, up to equivalence. The reason for this is that when the characteristic $\mathrm{char}(\mathbb{K})$ of $\mathbb{K}$ does not divide $|G|$, Maschke's theorem states that any representation V of G over $\mathbb{K}$ decomposes as a direct sum of irreducible representations

$$V = V_1 \oplus \cdots \oplus V_k.$$

In this case, we also say that V is *semisimple*. If $\mathrm{char}(\mathbb{K})$ divides $|G|$, there is not always such a decomposition. However, since $\mathbb{K}G$ is a finite-dimensional associative $\mathbb{K}$-algebra, the Jordan-Hölder theorem ensures a different kind of decomposition.

A *composition series* of a $\mathbb{K}G$-module M is a sequence of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M,$$

such that M_i/M_{i-1} is simple for all i. If M is, in particular, a finite-dimensional vector space over $\mathbb{K}$, as in our case, then M satisfies the so-called ascending chain condition and descending chain condition for submodules. This ensures the existence of a composition series ([CR62, 13.10]). The Jordan-Hölder theorem states that whenever composition series exist, the isomorphism classes of the simple quotients and their (composition) multiplicities are uniquely determined (they only depend on M). More precisely, if $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M$ and $0 = M_0' \subsetneq M_1' \subsetneq \cdots \subsetneq M_l' = M$ are two composition series for the $\mathbb{K}G$-module M, then $k = l$ and there exists a permutation σ of $1, \ldots, n$ such that M_i/M_{i-1} is isomorphic to $M'_{\sigma(i)}/M'_{\sigma(i)-1}$. The simple modules

M_i/M_{i-1} are called the *composition factors* of M, and the set of composition factors of a $\mathbb{K}G$-module is thus well-defined.

In both cases, roughly speaking, irreducible representations of G are the layers forming any representation of G. When $\mathrm{char}(\mathbb{K})$ does not divide $|G|$ we talk about *ordinary* representation theory, for example if $\mathrm{char}(\mathbb{K}) = 0$. Otherwise, we talk about *modular* representation theory.

1.3.2 Representations of the symmetric groups: Specht modules

For a finite group, it is well known that the number of nonequivalent irreducible representations is the same as the number of conjugacy classes, in the ordinary case, or p-prime conjugacy classes, in the modular case [CR62, 30.12 and 83.5]. Now, recall from Section 1.2.1 that conjugacy classes of $\mathfrak{S}_n$ are indexed by partitions of n, and p-prime conjugacy classes are, in turn, indexed by p-regular partitions of n. For every partition λ we are going to define a module S^λ. From such modules we are going to obtain the classification of ordinary and modular irreducible representations of $\mathfrak{S}_n$. From now on, let $\mathbb{K}$ be any field with $\mathrm{char}(\mathbb{K}) \neq 2$.

Let $\lambda \vdash n$. A *Young tableau of shape* λ (or a λ-*tableau*), is an array t obtained by replacing boxes in the Young diagram of λ with the numbers $1, 2, \ldots, n$, bijectively. There are $n!$ tableaux of shape λ. The symmetric group $\mathfrak{S}_n$ acts on the set of λ-tableaux. For example, let $\lambda = (3,2) \vdash 5$. A tableau of shape λ is:

$$t = \begin{array}{ccc} 3 & 2 & 4 \\ 5 & 1 \end{array} \tag{1.7}$$

The permutation $\pi = (23)(45) \in \mathfrak{S}_5$ acting on the tableau t gives:

$$\pi t = \begin{array}{ccc} 2 & 3 & 5 \\ 4 & 1 \end{array}$$

We say that two λ-tableaux t and t' are *row equivalent* if t' is obtained from t by a permutation which fixes the row. In other words, t and t' are equivalent if they are the same up to reordering each row. The equivalence class of t is denoted $\{\mathbf{t}\}$ and is called a *tabloid of shape* λ (or a λ-*tabloid*). We represent a tabloid like a tableau with lines between rows, indicating that the order in each row can be ignored. For example:

$$\{\mathbf{t}\} = \begin{array}{|ccc|} \hline 3 & 2 & 4 \\ \hline 5 & 1 \\ \cline{1-2} \end{array} = \begin{array}{|ccc|} \hline 4 & 3 & 2 \\ \hline 1 & 5 \\ \cline{1-2} \end{array}$$

There are $\frac{n!}{\lambda_1! \cdots \lambda_l!}$ tabloids of shape λ, where $l = l(\lambda)$.

The group $\mathfrak{S}_n$ acts on tabloids by $\pi\{\mathbf{t}\} = \{\pi\mathbf{t}\}$. This action gives rise to a $\mathbb{K}\mathfrak{S}_n$-module: Let $\lambda \vdash n$, and suppose that $\{\mathbf{t_1}\}, \ldots, \{\mathbf{t_k}\}$ is a complete list of λ-tabloids. We define the $\mathbb{K}\mathfrak{S}_n$-module M^λ as the $\mathbb{K}$-vector space spanned by the λ-tabloids:

$$M^\lambda := \mathbb{K}\{\{\mathbf{t_1}\}, \ldots, \{\mathbf{t_k}\}\}.$$

Now, for a λ-tableau t define a subgroup of $\mathfrak{S}_n$ as follows. Suppose that the columns of t are the sets $C_1, \ldots, C_m \subseteq \{1, \ldots, n\}$. The *column-stabilizer* of t is the subgroup

$$C_t := S_{C_1} \times \cdots \times S_{C_m},$$

of $\mathfrak{S}_n$, where S_T denotes the group of bijections of the set T. Define an element κ_t, in the group algebra $\mathbb{K}\mathfrak{S}_n$, associated to the tableau t as

$$\kappa_t := \sum_{\pi \in C_t} \varepsilon(\pi)\pi.$$

Associated to t, as well, we define the *polytabloid* as the following element in the module M^λ:

$$\mathbf{e_t} := \kappa_t\{\mathbf{t}\}.$$

If t is of shape λ we say that $\mathbf{e_t}$ is a polytabloid of shape λ or a λ-polytabloid.

Example 1.3.1. if t is the tableau in (1.7), the column-stabilizer of t is

$$\begin{aligned} C_t &= S_{\{3,5\}} \times S_{\{2,1\}} \times S_{\{4\}} \\ &= \{\mathrm{Id}, (3\ 5)\} \times \{\mathrm{Id}, (2\ 1)\} \times \{\mathrm{Id}\}, \end{aligned}$$

then, κ_t factors as

$$\kappa_t = (\mathrm{Id} - (3\ 5))\,(\mathrm{Id} - (2\ 1)).$$

Hence,

$$\mathbf{e_t} := \overline{\begin{array}{|c|c|c|}\hline 3 & 2 & 4 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 5 & 2 & 4 \\ \hline 3 & 1 \\ \cline{1-2}\end{array}} - \overline{\begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array}} + \overline{\begin{array}{|c|c|c|}\hline 5 & 1 & 4 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}}.$$

△

Definition 1.3.2. The *Specht module* S^λ is defined as the submodule of M^λ spanned by the polytabloids $\mathbf{e_t}$, where t is of shape λ.

Example 1.3.3. If $\lambda = (n)$, the Specht module S^λ is the *trivial representation*: there is only one polytabloid $\mathbf{e_t} = \overline{\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline \end{array}}$, and $\pi\mathbf{e_t} = \mathbf{e_t}$ for every $\pi \in \mathfrak{S}_n$.

If $\lambda = (1^n)$, it can be shown that S^λ is one-dimensional as well, and $\pi\mathbf{e_t} = \varepsilon(\pi)\mathbf{e_t}$ for every $\pi \in \mathfrak{S}_n$. The Specht module S^λ is the sign representation. △

Remark 1.3.4. There exists a basis for the Specht module S^λ indexed by the set of *standard tableaux*. A standard tableau of shape λ is a tableau in which the entries in each row and column are increasing. The dimension of S^λ is given by the *Hook length formula* in terms of the hook-lengths in the Young diagram $[\lambda]$, see [FRT54]. The dimension is

$$n! \prod_{(i,j)\in[\lambda]} \frac{1}{h^\lambda_{ij}}.$$

△

There is a bilinear form $\Phi(\cdot,\cdot)$ that can be defined in M^λ, which is symmetric, $\mathfrak{S}_n$-invariant and bilinear. For this bilinear form, the $\mathbb{K}\mathfrak{S}_n$-module $S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is either irreducible or zero (see [JK81, §7.1]). In the ordinary case, that is, when $\mathrm{char}(\mathbb{K})$ does not divide $|\mathfrak{S}_n| = n!$, we know that $(S^\lambda \cap S^{\lambda\perp}) = 0$, and since there are as many irreducible representations as conjugacy classes of $\mathfrak{S}_n$, we have:

Theorem 1.3.5 ([JK81, Theorem 7.1.9])**.** *The Specht modules S^λ for $\lambda \vdash n$ give a complete list of nonequivalent ordinary irreducible representations of $\mathfrak{S}_n$.*

In the modular case, that is when $p = \mathrm{char}(\mathbb{K})$ divides $|\mathfrak{S}_n| = n!$, we know that $S^\lambda/(S^\lambda \cap S^{\lambda\perp})$ is non-zero if and only if λ is a p-regular partition. We have:

Theorem 1.3.6 ([JK81, Theorem 7.1.14])**.** *Let $p = \mathrm{char}(\mathbb{K})$. As λ varies over p-regular partitions of n,*

$$D^\lambda := S^\lambda/(S^\lambda \cap S^{\lambda\perp})$$

varies over a complete set of nonequivalent modular irreducible representations of $\mathfrak{S}_n$.

The result in Theorem 1.3.6 explains, in particular, the interest of having a correspondence (1.5) between p-prime conjugacy classes and p-regular partitions.

Remark 1.3.7. Some authors (for example Mathas in [Mat99]) use a different way to define the Specht modules and the modular irreducible representations of $\mathfrak{S}_n$. We use James' definition. As James defined it, the Specht module S^λ is isomorphic to the *dual* module which Mathas indexed by λ', the conjugate of λ. Now, for us, the modular irreducible modules are indexed by p-regular partitions. For Mathas, they are indexed then by all the conjugates of p-regular partitions, which are called p-*restricted* partitions. A p-restricted partition λ is then a partition such that $\lambda_{i+1} - \lambda_i < p$ for every i. In other words, a partition is p-restricted if and only if its conjugate is p-regular. △

Remark 1.3.8. The construction of Specht modules shown here is due to James, and it is valid more generally for the representation theory of *Hecke algebras*, see [DJ86]. △

Remark 1.3.9 (About the field)**.** If $\overline{\mathbb{K}}$ is the algebraic closure of the field $\mathbb{K}$, a *splitting field* $\mathbb{E}$ for a finite group G, is a finite extension of $\mathbb{K}$ such that all irreducible $\overline{\mathbb{K}}G$-modules are realizable over $\mathbb{E}$. The symmetric group $\mathfrak{S}_n$ has the property that every field is a splitting field for $\mathfrak{S}_n$ ([JK81, Theorem 2.1.12]). Hence, in this thesis when we talk about ordinary representations of $\mathfrak{S}_n$, the field $\mathbb{K}$ can be taken as being for example $\mathbb{Q}$, or $\mathbb{F}_p$ with $p \nmid n!$, that is, $p > n$. When we talk about modular representations, for the field $\mathbb{K}$ we take $\mathbb{F}_p$ with $p \mid n!$, that is, $p < n$. △

1.3.3 Representations of the alternating groups

Let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$ and let $n \geq 2$. The irreducible representations of the alternating group $\mathfrak{A}_n$ can be derived from those of $\mathfrak{S}_n$ applying Clifford's theory of representations of normal subgroups. Our case is an easy application, since $[\mathfrak{S}_n : \mathfrak{A}_n] = 2$. Let us see this in detail.

For stating Clifford's theorem, we introduce the notion of *conjugate* modules. Let G be a finite group, and let $H \triangleleft G$ be a normal subgroup. Let (π, L) be a $\mathbb{K}H$-representation. For an element $g \in G$, the conjugate (by g)representation $(\pi^{(g)}, L^{(g)})$ is the $\mathbb{K}H$-representation with underlying vector space L and action defined as

$$\pi^{(g)}(h) \cdot l = \pi(ghg^{-1}) \cdot l.$$

Two $\mathbb{K}H$-modules N and N' are said to be conjugates if $N' = N^{(g)}$ for some $g \in G$. As stated in [CR62, 49.7], we have:

Theorem 1.3.10 (Frobenius, Clifford [Cli37]). *Let M be an irreducible $\mathbb{K}G$-module, and let $H \triangleleft G$. Then $M\downarrow_H$ is a semisimple $\mathbb{K}H$-module, and the irreducible $\mathbb{K}H$-submodules of $M\downarrow_H$ are all conjugates of each other. Moreover, if L is an irreducible $\mathbb{K}H$-submodule of $M\downarrow_H$, then*

$$M\downarrow_H = k(L^{(g_1)} \oplus \cdots \oplus L^{(g_r)})$$

where $\{L^{(g_1)}, \ldots, L^{(g_r)}\}$ is a full set of nonequivalent conjugates of L, and k is some positive integer.

Before applying Theorem 1.3.10, let us do some preparation by observing the effect of conjugating $\mathbb{K}\mathfrak{A}_n$-modules.

Lemma 1.3.11. *Let $n \geq 2$ and let L be a $\mathbb{K}\mathfrak{A}_n$-module. Then L and $L^{(1\ 2)}$ form a maximal set of nonequivalent conjugates of L. In other words, for any $\sigma \in \mathfrak{S}_n$, any conjugate $L^{(\sigma)}$ of L is equivalent to L or to $L^{(1\ 2)}$, or possibly both.*

Proof. Denote ρ the representation associated to the module L. For $\sigma \in \mathfrak{S}_n$, denote $(\rho^\sigma, L^{(\sigma)})$ the conjugate module of L by σ.

If $\sigma \in \mathfrak{A}_n$ we claim that $L^{(\sigma)} \simeq L$. Indeed, let $\tau \in \mathfrak{A}_n$. We have that $\rho^{(\sigma)}(\tau) = \rho(\sigma\tau\sigma^{-1})$. Then $\rho^{(\sigma)}(\tau) = \rho(\sigma)\rho(\tau)\rho(\sigma)^{-1}$. Thus $\rho(\sigma)^{-1}\rho^{(\sigma)}(\tau) = \rho(\tau)\rho(\sigma)^{-1}$. Then $\rho(\sigma)^{-1}$ is a $\mathfrak{A}_n$-isomorphism of L and $L^{(\sigma)}$.

If $\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$, we claim that $L^{(\sigma)} \simeq L^{(1\ 2)}$. For this, we first rewrite the element σ: we have $\sigma = \nu(1\ 2)$, where $\nu = \sigma(1\ 2) \in \mathfrak{A}_n$. Now, let $\tau \in \mathfrak{A}_n$. We have

$$\begin{aligned}\rho^{(\sigma)}(\tau) &= \rho(\sigma\tau\sigma^{-1}) \\ &= \rho(\nu(1\ 2)\tau(1\ 2)\nu^{-1}) \\ &= \rho(\nu)\rho((1\ 2)\tau(1\ 2))\rho(\nu^{-1}) \\ &= \rho(\nu)\rho^{(1\ 2)}(\tau)\rho(\nu^{-1}).\end{aligned}$$

Then $\rho(\nu^{-1})\rho^{(\sigma)}(\tau) = \rho^{(1\ 2)}(\tau)\rho(\nu^{-1})$, so that $\rho(\nu^{-1})$ is a $\mathfrak{A}_n$-isomorphism of $L^{(\sigma)}$ and $L^{(1\ 2)}$.

□

We now apply Theorem 1.3.10 to $G = \mathfrak{S}_n$ and $H = \mathfrak{A}_n$. The following explanation is based in [For97]. Let M be an irreducible $\mathbb{K}\mathfrak{S}_n$-module.

A preliminary remark is the following. Consider the $\mathbb{K}\mathfrak{S}_n$-module M^ε defined as

$$M^\varepsilon := M \otimes \varepsilon,$$

where ε is the sign representation of $\mathfrak{S}_n$. Since ε is a one-dimensional representation, M^ε is an irreducible $\mathbb{K}\mathfrak{S}_n$-module. On the other hand, notice that

$$M\downarrow_{\mathfrak{A}_n} \simeq M^\varepsilon\downarrow_{\mathfrak{A}_n},$$

since $\mathfrak{A}_n$ acts as the identity on ε. We keep this remark for later.

Now, Theorem 1.3.10 says that $M\downarrow_{\mathfrak{A}_n}$ is semisimple. Let D be an irreducible constituent of $M\downarrow_{\mathfrak{A}_n}$. From Lemma 1.3.11 we have either

$$M\downarrow_{\mathfrak{A}_n} = k(D) \qquad \text{or} \qquad M\downarrow_{\mathfrak{A}_n} = k\left(D \oplus D^{(1\ 2)}\right),$$

where k is as in Theorem 1.3.10.

- We are in the first case if and only if $D^{(1\,2)} \simeq D$. In this situation, from [Fei82, III.2.14], we have that
$$M\downarrow_{\mathfrak{A}_n} = D,$$
that is, $k = 1$. In this case M and M^ε are *not* isomorphic $\mathbb{K}\mathfrak{S}_n$-modules. Indeed, suppose that there is a $\mathfrak{S}_n$-isomorphism φ. Then, φ is also a $\mathfrak{A}_n$-isomorphism; but D is an irreducible $\mathbb{K}\mathfrak{A}_n$-module. Then φ must be a scalar multiple of the identity map; but this contradicts the assumption that this is an isomorphism between M and $M\otimes\varepsilon$.
- We are in the second case if and only if $D^{(1\,2)} \not\simeq D$. We have
$$M\downarrow_{\mathfrak{A}_n} = D \oplus D^{(1\,2)} = M^\varepsilon\downarrow_{\mathfrak{A}_n}.$$
In this case, the integer k from Theorem 1.3.10 is equal to 1 since $D\oplus D^{(1\,2)}$ is stable for the action of $\mathfrak{S}_n$, then it is a stable submodule of the (irreducible) $\mathbb{K}\mathfrak{S}_n$-module M. From Mackey's subgroup theorem ([CR62, §44]) we have that
$$M \simeq D\uparrow^{\mathfrak{S}_n} \simeq M^\varepsilon,$$
then in this case $M \simeq M\otimes\varepsilon$.

Summarizing, let M be an irreducible $\mathbb{K}\mathfrak{S}_n$-representation.

(i) If $M \not\simeq M^\varepsilon$, then $M\downarrow_{\mathfrak{A}_n} \simeq M^\varepsilon\downarrow_{\mathfrak{A}_n}$ is an irreducible $\mathbb{K}\mathfrak{A}_n$-representation, while

(ii) if $M \simeq M^\varepsilon$, then $M\downarrow_{\mathfrak{A}_n} \simeq M^\varepsilon\downarrow_{\mathfrak{A}_n}$ splits into two nonequivalent irreducible $\mathbb{K}\mathfrak{A}_n$-representations (conjugated by $\mathfrak{S}_n$).

With this characteristic free reasoning (provided that $\mathrm{char}(\mathbb{K}) \neq 2$), we are now ready to give the classification of ordinary and modular irreducible representations of $\mathfrak{A}_n$.

Ordinary case

The ordinary irreducible $\mathbb{K}\mathfrak{S}_n$-modules are the Specht modules S^λ, for every $\lambda \vdash n$. Since $S^\lambda\otimes\varepsilon$ is again irreducible, then there exists $\lambda^\varepsilon \vdash n$ such that $S^\lambda\otimes\varepsilon \simeq S^{\lambda^\varepsilon}$. From the construction of S^λ, it can be shown that λ^ε is the conjugate λ' of lambda:
$$S^\lambda\otimes\varepsilon \simeq S^{\lambda'}.$$

Hence, from the above discussion we obtain:

Theorem 1.3.12 (Ordinary irreducible $\mathfrak{A}_n$-representations)**.** *Let λ be a partition of $n \geq 2$.*

(i) If $\lambda \neq \lambda'$, then $S^\lambda\downarrow_{\mathfrak{A}_n} \simeq S^{\lambda'}\downarrow_{\mathfrak{A}_n}$ is an irreducible $\mathbb{K}\mathfrak{A}_n$-representation, while

(ii) if $\lambda = \lambda'$, then $S^\lambda\downarrow_{\mathfrak{A}_n} \simeq S^{\lambda'}\downarrow_{\mathfrak{A}_n}$ splits into two nonequivalent irreducible and conjugate representations $S^{\lambda+}$, $S^{\lambda-}$ of $\mathfrak{A}_n$ (conjugated by $\mathfrak{S}_n$).

A complete system of nonequivalent ordinary irreducible $\mathbb{K}\mathfrak{A}_n$-representations is
$$\{S^\lambda\downarrow_{\mathfrak{A}_n} \mid \lambda \vdash n,\ \lambda > \lambda'\} \sqcup \{S^{\lambda+}, S^{\lambda-} \mid \lambda \vdash n,\ \lambda = \lambda'\}.$$

Remark 1.3.13. In the first set of irreducible $\mathbb{K}\mathfrak{A}_n$-representations, since $S^\lambda\downarrow_{\mathfrak{A}_n} \simeq S^{\lambda'}\downarrow_{\mathfrak{A}_n}$ and we want to list a complete set of **nonequivalent** $\mathbb{K}\mathfrak{A}_n$-representations, we choose one of $S^\lambda\downarrow_{\mathfrak{A}_n}$ or $S^{\lambda'}\downarrow_{\mathfrak{A}_n}$ by taking only $\lambda > \lambda'$ △

Modular case

Let $p = \mathrm{char}(\mathbb{K})$. The modular irreducible $\mathbb{K}\mathfrak{S}_n$-modules are the modules D^λ, for every p-regular partition $\lambda \vdash n$. Then, being irreducible, the $\mathbb{K}\mathfrak{S}_n$-module $D^\lambda \otimes \varepsilon$ is equivalent to some D^{λ^ε}. In this case, in the construction of the modules D^λ, the connection between the partition λ and the module D^λ is not as transparent as in the ordinary case (construction of S^λ). Then, it is not straightforward to find the p-regular partition λ^ε corresponding to the irreducible module $D^\lambda \otimes \varepsilon$. Moreover, it is a difficult problem, for which Mullineux proposed a solution ([Mul79a]). Mullineux conjecture was later proven to be true. Let $\boldsymbol{m}_p(\lambda)$ denote the partition λ^ε (which depends on p). In §2 we explain Mullineux algorithm for finding $\boldsymbol{m}_p(\lambda)$ explicitly .

With such notations we have:

$$D^\lambda \otimes \varepsilon \simeq D^{\boldsymbol{m}_p(\lambda)}.$$

From the discussion above we obtain:

Theorem 1.3.14 (Modular irreducible $\mathfrak{A}_n$-representations)**.** *Let λ be a p-regular partition of $n \geq 2$.*

(i) If $\lambda \neq \boldsymbol{m}_p(\lambda)$, then $D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\boldsymbol{m}_p(\lambda)} \downarrow_{\mathfrak{A}_n}$ is an irreducible $\mathbb{K}\mathfrak{A}_n$-representation, while

(ii) if $\lambda = \boldsymbol{m}_p(\lambda)$, then $D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\boldsymbol{m}_p(\lambda)} \downarrow_{\mathfrak{A}_n}$ splits into two nonequivalent irreducible and conjugate representations $D^{\lambda+}$, $D^{\lambda-}$ of $\mathfrak{A}_n$ (conjugated by $\mathfrak{S}_n$).

A complete system of modular nonequivalent irreducible $\mathbb{K}\mathfrak{A}_n$-representations is

$$\{D^\lambda \downarrow_{\mathfrak{A}_n} | \, \lambda \in \mathrm{Reg}_p(n),\ \lambda > \boldsymbol{m}_p(\lambda)\} \sqcup \{D^{\lambda+}, D^{\lambda-} \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda)\}.$$

Remark 1.3.15. As in the ordinary case, in the first set of irreducible $\mathbb{K}\mathfrak{A}_n$-representations, since $D^\lambda \downarrow_{\mathfrak{A}_n} \simeq D^{\boldsymbol{m}_p(\lambda)} \downarrow_{\mathfrak{A}_n}$ and we want to list a complete set of **nonequivalent** $\mathbb{K}\mathfrak{A}_n$-representations, we choose one of $D^\lambda \downarrow_{\mathfrak{A}_n}$ or $D^{\boldsymbol{m}_p(\lambda)} \downarrow_{\mathfrak{A}_n}$ by taking only one of λ or $\boldsymbol{m}_p(\lambda)$. For making this choice we take $\lambda > \boldsymbol{m}_p(\lambda)$. △

1.3.4 On the number of self-Mullineux partitions

In the preceding section we saw representations of $\mathfrak{S}_n$ which split upon restriction to $\mathfrak{A}_n$. Recall, from Proposition 1.2.4, when discussing conjugacy classes of $\mathfrak{S}_n$, contained in $\mathfrak{A}_n$, that there was a similar splitting situation. Moreover, we know that ordinary (or modular) representations are indexed by conjugacy classes (or p-prime conjugacy classes) of $\mathfrak{S}_n$. These observations give interesting bijections, that allow to say more about the latter correspondence. Namely, we have the following proposition which follows from [AO91, Proposition 2]. We call *p-self-Mullineux* partition (or *self-Mullineux*), any partition λ such that $\lambda = \boldsymbol{m}_p(\lambda)$. The set of p-self-Mullineux partitions of n is denoted M_p^n, and the set of p-self-Mullineux partitions is denoted M_p.

Proposition 1.3.16. *Let $n \geq 2$ and let p be an odd positive integer. The number of p-BG partitions of n is equal to the number of p-self-Mullineux partitions of n.*

Remark 1.3.17. As mentioned, Proposition 1.3.16 follows from [AO91, Proposition 2]. Here, we give an alternative proof using characters and the study of $\mathfrak{A}_n$-conjugacy classes in §1.2.2. △

Before proving Proposition 1.3.16 we make some preparation. We know that the number of $\mathfrak{A}_n$-conjugacy classes is equal to the number of ordinary irreducible $\mathfrak{A}_n$-representations. The two sets can be indexed by sets of partitions (or partitions marked with a sign) of n. The first indexing comes from (1.3) after Example 1.2.5, and the second comes from Theorem 1.3.12. Hence, the following two sets of partitions have the same cardinality:

$$\left| \mathrm{Par}_{\circ}^{\mathfrak{A}}(n) \sqcup \left\{ \lambda+, \lambda- \mid \lambda \in \mathrm{Par}_{+-}^{\mathfrak{A}}(n) \right\} \right| = \left| \{ \lambda \vdash n \mid \lambda > \lambda' \} \sqcup \{ \lambda+, \lambda- \mid \lambda \vdash n,\ \lambda = \lambda' \} \right|$$

As tempting as it might be to affirm that the couples of corresponding subsets in the two sides of this relation have the same number of elements as well, it is not automatically true. It would be ideal, though, since the right-hand subsets on each side correspond to splitting conjugacy classes and splitting representations. Fortunately, we can make such affirmation since we know that there is an explicit bijection ((1.4) on page 34) between the two sets:

$$\mathrm{Par}_{+-}^{\mathfrak{A}}(n) \quad \longleftrightarrow \quad \{ \lambda \vdash n \mid \lambda = \lambda' \}.$$

We note that we do not know an explicit bijection between the sets $\mathrm{Par}_{\circ}^{\mathfrak{A}}(n)$ and $\{\lambda \vdash n \mid \lambda > \lambda'\}$, which, as a consequence of the latter, have the same number of elements.

Similarly, in the modular setting, if $p = \mathrm{char}(\mathbb{K})$, we know that the number of $\mathfrak{A}_n$-p-prime conjugacy classes is equal to the number of modular irreducible $\mathfrak{A}_n$-representations. The two sets can be indexed by sets of partitions (or partitions marked with a sign) of n. The first indexing comes from the restriction to p-regular partitions of (1.3) after Example 1.2.5, and the second comes from Theorem 1.3.14. Hence, the following two sets of partitions have the same cardinality:

$$\left| \{ \lambda \in \mathrm{Par}_{\circ}^{\mathfrak{A}}(n) \mid \lambda \text{ is } p\text{-prime } \} \sqcup \{ \lambda+, \lambda- \mid \lambda \in \mathrm{Par}_{+-}^{\mathfrak{A}}(n) \text{ and } \lambda \text{ is } p\text{-prime } \} \right| = \left| \{ \lambda \in \mathrm{Reg}_p(n) \mid \lambda > \boldsymbol{m}_p(\lambda) \} \sqcup \{ \lambda+, \lambda- \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda) \} \right|$$

Now, as before, we would like to affirm, in particular, that the set of partitions $\{\lambda \in \mathrm{Par}_{+-}^{\mathfrak{A}}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}$ has the same cardinality as the set of self-Mullineux partitions: $\{\lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda)\}$. Recall, from (1.6) in §1.2.3, that we have a bijection between $\{\lambda \in \mathrm{Par}_{+-}^{\mathfrak{A}}(n) \mid \lambda \text{ is } p\text{-prime}\}$ and BG_p^n. For now, we do not have an explicit bijection, as for the ordinary case above, however the equality of the cardinalities makes the content of Proposition 1.3.16.

Proof of Proposition 1.3.16. In this proof we use *Brauer characters*: To an irreducible $\mathbb{K}\mathfrak{A}_n$-module D we can associate a function χ_D which is called the *(irreducible) Brauer character* of $\mathfrak{A}_n$ afforded by D. This function χ_D is a complexed-valued function defined on the set of p-prime elements of $\mathfrak{A}_n$ (permutations with order which is relatively prime to p) and it is constant on conjugacy classes. Isomorphic $\mathbb{K}\mathfrak{A}_n$-modules are associated to equal Brauer characters. See [Isa06, §15] for the precise definition of Brauer character and for further information.

In particular [Isa06, Theorem 15.10] says that the set of irreducible Brauer characters of $\mathfrak{A}_n$ form a basis of the space of $\mathbb{C}$-valued functions defined on p-prime elements of $\mathfrak{A}_n$ and constant on conjugacy classes. This implies that there are as many irreducible Brauer characters of $\mathfrak{A}_n$ as p-prime conjugacy classes of $\mathfrak{A}_n$, and by [Isa06,

Corollary 15.11], this is also the number of isomorphism classes of $\mathbb{K}\mathfrak{A}_n$-modules. Therefore to each element λ of the set

$$\{\lambda \in \mathrm{Reg}_p(n) \mid \lambda > \boldsymbol{m}_p(\lambda)\} \sqcup \{\lambda+, \lambda- \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda)\},$$

we can associate an irreducible Brauer character $\chi_{[\mu]}$. That way, a basis of the space of $\mathbb{C}$-valued functions defined on p-prime elements of $\mathfrak{A}_n$ and constant on conjugacy classes is

$$\{\chi_{[\lambda]} \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda > m(\lambda)\} \sqcup \{\chi_{[\lambda^+]}, \chi_{[\lambda^-]} \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda)\}.$$

To prove that

$$|\{\lambda \mid \lambda \in \mathrm{Par}^{\mathfrak{A}}_{+-}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}| \ = \ |\{\lambda \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda)\}|,$$

we will give two bases of a same space of functions, and the equality of the cardinality of these bases will give the result.

Denote by E the space of $\mathbb{C}$-valued functions defined on p-regular elements of $\mathfrak{A}_n$ and constant on conjugacy classes

$$E = \{\ f : \{p\text{-regular elements of } \mathfrak{A}_n\} \longrightarrow \mathbb{C} \mid\ f \text{ is a class function of } \mathfrak{A}_n\ \}.$$

Define an action of $\mathfrak{S}_n$ on E by conjugation as follows: for $\sigma \in \mathfrak{S}_n$ and $f \in E$, f^σ is the class function

$$f^\sigma(\tau) := f(\sigma\tau\sigma^{-1}).$$

For $\sigma \in \mathfrak{S}_n \setminus \mathfrak{A}_n$, let E^σ be the set of class functions fixed by conjugation by σ:

$$E^\sigma = \{f \in E \mid f^\sigma = f\}.$$

This is a subspace of E. From Proposition 1.2.4 and the comment after such proposition, a basis for E^σ is

$$\{\mathbb{1}_{C^\lambda} \mid \lambda \in \mathrm{Par}^{\mathfrak{A}}_{\circ}(n),\ \lambda \text{ is } p\text{-prime}\} \sqcup \{\mathbb{1}_{C^{\lambda+}} + \mathbb{1}_{C^{\lambda-}} \mid \lambda \in \mathrm{Par}^{\mathfrak{A}}_{+-}(n)\},$$

where $\mathbb{1}_T$ denotes the indicator function for $T \subset \mathfrak{A}_n$. We claim that a basis for E^σ is

$$\{\chi_{[\lambda]} \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda > m(\lambda)\} \sqcup \{\chi_{[\lambda^+]} + \chi_{[\lambda^-]} \mid \lambda \in \mathrm{Reg}_p(n)\ \lambda = m(\lambda)\}.$$

Indeed, this comes from the fact that, as with usual characters of representations in characteristic zero, conjugation of the character of a representation is the character of the conjugate representation, here with Brauer characters. And also from the fact that conjugation by σ permutes the modules associated to λ^+ and λ^- above.

Now, we have two bases for E and two bases for E^σ. On one hand, from the characteristic function basis, the dimension of E is

$$|\{\lambda \mid \lambda \in \mathrm{Par}^{\mathfrak{A}}_{\circ}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}| \ + \ 2|\{\lambda \mid \lambda \in \mathrm{Par}^{\mathfrak{A}}_{+-}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}|,$$

or

$$|\{\lambda \mid \lambda \in \mathrm{Par}^{\mathfrak{A}}_{\circ}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}| + 2\,|\mathrm{BG}^n_p|,$$

from (1.4). From the Brauer character basis, the dimension of E is

$$|\{\lambda \in \mathrm{Reg}_p(n) \mid \lambda > \boldsymbol{m}_p(\lambda)\}| \ + \ 2|\{\lambda \mid \lambda \in \mathrm{Reg}_p(n),\ \lambda = \boldsymbol{m}_p(\lambda)\}|,$$

or

$$|\{\lambda \in \mathrm{Reg}_p(n) \mid \lambda > \boldsymbol{m}_p(\lambda)\}| \; + \; 2|\mathrm{M}_p^n|.$$

Hence

$$|\{\lambda \mid \lambda \in \mathrm{Par}_\circ^{\mathfrak{A}}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}| + 2|\mathrm{BG}_p^n| \; = \; |\{\lambda \in \mathrm{Reg}_p(n) \mid \lambda > \boldsymbol{m}_p(\lambda)\}| + 2|\mathrm{M}_p^n|.$$

Now, counting the elements on the two bases for E^σ, we obtain that the dimension of E^σ is

$$|\{\lambda \mid \lambda \in \mathrm{Par}_\circ^{\mathfrak{A}}(n) \text{ and } \lambda \text{ is } p\text{-prime }\}| + |\mathrm{BG}_p^n| \; = \; |\{\lambda \in \mathrm{Reg}_p(n) \mid \lambda > \boldsymbol{m}_p(\lambda)\}| + |\mathrm{M}_p^n|.$$

These two identities imply that $|\mathrm{BG}_p^n| = |\mathrm{M}_p^n|$. □

1.4 Decomposition matrix and blocks of the symmetric group

Decomposition matrices are important objects which allow to study the modular representation theory of $\mathfrak{S}_n$ in particular. They contain the information of the relationship between the ordinary and the modular irreducible representations. In this section we recall the definition of the decomposition matrix of the symmetric group $\mathfrak{S}_n$. Here, let $p < n$.

1.4.1 Decomposition matrix of the symmetric group

As mentioned in §1.3.1, $\mathbb{F}_p\mathfrak{S}_n$-modules are not always semisimple. However the Jordan-Hölder theorem allows to assert the uniqueness of the multiplicity of an irreducible $\mathbb{F}_p\mathfrak{S}_n$-module D^μ, as a composition factor of any $\mathbb{F}_p\mathfrak{S}_n$-module, where $\mu \vdash n$ is p-regular. In particular we can consider the Specht module S^λ, for any $\lambda \vdash n$, which is not in general a p-modular irreducible. Consider the composition multiplicity of D^μ for $\mu \vdash n$ p-regular. Denote this multiplicity

$$d_{\lambda\mu} = [S^\lambda : D^\mu]$$

.

If we denote by $\sim$ the equivalence relation on the set of $\mathbb{F}_p\mathfrak{S}_n$-modules given by: $M \sim N$ if and only if M and N have the same composition factors (with multiplicities), and $[M]$ denotes the equivalence class of M, we have:

$$[S^\lambda] \sim \left[\bigoplus_{\mu \in \mathrm{Reg}_p(n)} d_{\lambda\mu} D^\mu \right].$$

The multiplicities $d_{\lambda\mu}$ are called the (p-modular) *decomposition numbers* of S^λ. They can be arranged in a matrix

$$\mathbf{D}_{n,p} = (d_{\lambda\mu})_{\substack{\lambda \in \mathrm{Par}(n) \\ \mu \in \mathrm{Reg}_p(n)}},$$

called the (p-modular) *decomposition matrix*. It is a rectangular matrix with as many rows as partitions of n and as many columns as p-regular partitions of n.

A crucial fact about the modular irreducible representations of $\mathfrak{S}_n$, which reflects on the decomposition matrix is the following.

Theorem 1.4.1 ([JK81, Theorem 7.1.14]). *If D^μ is a composition factor of S^λ, where $\lambda, \mu \vdash n$ and μ p-regular, then $\mu \trianglerighteq \lambda$. The composition multiplicity of D^μ in S^μ is 1.*

That is, if we organize the rows of the matrix $\mathbf{D}_{n,p}$ downwards, placing the p-regular partitions before the p-singular partitions, in any total order which respects the dominance order (for example the lexicographic order), it takes the following form:

$$\begin{array}{r} \\ S^\lambda,\ \lambda \in \mathrm{Reg}_p(n) \\ S^\lambda,\ \lambda \in \mathrm{Par}(n) \setminus \mathrm{Reg}_p(n) \end{array} \overbrace{\begin{array}{l} \\ \left\{ \\ \left\{ \end{array}}^{D^\lambda,\ \lambda \in \mathrm{Reg}_p(n)} \begin{pmatrix} 1 & & & \\ & 1 & 0 & \\ & & \ddots & \\ & * & & 1 \\ & & * & \end{pmatrix}$$

In this form, we say that the matrix is *lower unitriangular*: it is lower triangular and the diagonal entries are equal to 1. Let us introduce some terminology:

Definition 1.4.2. Let $U \subseteq \{S^\lambda \mid \lambda \vdash n\}$. We denote by $\mathbf{D}^U_{n,p}$ the matrix formed by the rows in $\mathbf{D}_{n,p}$ labelled by elements in U, and all the columns in $\mathbf{D}_{n,p}$. We call $\mathbf{D}^U_{n,p}$ the *restriction of* $\mathbf{D}_{n,p}$ *to* U. In terms of [JK81, §6.3], we say that $\mathbf{D}^U_{n,p}$ has *wedge shape* if the rows and columns can be reordered in such a way that the resulting matrix is lower unitriangular. △

In such terms $\mathbf{D}^{\mathrm{Reg}_p(n)}_{n,p}$ has wedge shape (or $\mathbf{D}^U_{n,p}$ for any $\mathrm{Reg}_p(n) \subseteq U$).

Let us rephrase Theorem 1.4.1 in terms of the decomposition numbers.

Theorem 1.4.3 ([Mat99, Corollary 4.17]). *Let $\lambda \in \mathrm{Par}(n)$ and $\mu \in \mathrm{Reg}_p(n)$. Then,*

(i) $d_{\mu\mu} = 1$, *and*

(ii) if $d_{\lambda\mu} \neq 0$ *then* $\lambda \trianglelefteq \mu$.

Example 1.4.4. The decomposition matrix $\mathbf{D}_{4,3}$ is

	$D^{(4)}$	$D^{(3,1)}$	$D^{(2^2)}$	$D^{(2,1^2)}$
$S^{(4)}$	1			
$S^{(3,1)}$	·	1		
$S^{(2^2)}$	1	·	1	
$S^{(2,1^2)}$	·	·	·	1
$S^{(1^4)}$	·	·	1	·

where both dots · and blank spaces are 0. △

1.4.2 p-block structure of $\mathfrak{S}_n$

From the modular representation theory of finite groups (see for example [Isa06, Alp86]), the sets of ordinary irreducible modules and the p-modular irreducible modules of $\mathfrak{S}_n$ can be partitioned into *p-blocks*. It means that if rows and columns are arranged according to these p-blocks, then the decomposition matrix is a block matrix:

$$\mathbf{D}_{n,p} = \begin{pmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

In other words, we can define an equivalence relation $\sim_p$ on Specht modules as follows: for two partitions of n, $S^\lambda \sim_p S^\mu$ if and only if S^λ and S^μ have at least one composition factor in common; we let $\sim_p$ be the transitive closure of such relation. We also define $\sim_p$ on the set of p-modular irreducibles as $D^\lambda \sim_p D^\mu$ if and only if $S^\lambda \sim_p S^\mu$. An equivalence class of $\sim_p$ for the Specht modules is called a *p-block* of $\mathfrak{S}_n$.

An important result on the representation theory of $\mathfrak{S}_n$ says that p-blocks are characterized by the p-cores of the Young diagrams which index the modules. The following result was conjectured by Nakayama [Nak41b] and proved first by Brauer and Robinson [Bra47, Rob47].

Theorem 1.4.5 (Nakayama's Conjecture)**.** *Two ordinary irreducible representations S^λ and S^μ of $\mathfrak{S}_n$ lie in the same p-block if and only if the p-cores of λ and μ are equal.*

Hence, a p-block $\mathfrak{B}$ of $\mathfrak{S}_n$ can be indexed by a p-core, γ which is the p-core of every $\lambda \vdash n$ such that $S^\lambda \in \mathfrak{B}$.

Definition 1.4.6. Let $w \geq 0$ be an integer such that $n - wp \geq 0$. Let $\gamma \vdash n - wp$ be a p-core. The *p-block* (or *block*) $\mathfrak{B}_\gamma$ of $\mathbb{F}_p\mathfrak{S}_n$ is the following set of ordinary irreducible representations of $\mathfrak{S}_n$ up to equivalence:

$$\mathfrak{B}_\gamma := \{S^\lambda \mid \lambda \vdash n \text{ and the } p\text{-core of } \lambda \text{ is } \gamma\}.$$

The p-block $\mathfrak{B}_\gamma$ in the set of modular irreducible representations is the subset:

$$\mathrm{Irr}_p(\mathfrak{B}_\gamma) := \{D^\lambda \mid \lambda \in \mathrm{Reg}_p(n) \text{ and } p\text{-core of } \lambda \text{ is } \gamma\}.$$

The *p-weight* w of the block $\mathfrak{B}_\gamma$ (or $\mathrm{Irr}_p(\mathfrak{B}_\gamma)$) is defined as the p-weight of any partition in the block, which is equal to $w = \frac{n-\mathrm{Rk}(\gamma)}{p}$. The submatrix of $\mathbf{D}_{n,p}$ given by rows indexed by $\mathfrak{B}_\gamma$ and columns indexed by $\mathrm{Irr}_p(\mathfrak{B}_\gamma)$ is denoted $\mathbf{D}_\gamma$. △

By abuse of notation, we consider $\mathfrak{B}_\gamma$ and $\mathrm{Irr}_p(\mathfrak{B}_\gamma)$ as subsets of partitions of n; that is if we write $\lambda \in \mathfrak{B}_\gamma$, it really means $S^\lambda \in \mathfrak{B}_\gamma$.

Example 1.4.7. Let $p = 3$. There are 4 p-blocks of $\mathbb{F}_3\mathfrak{S}_8$. From Nakayama conjecture, the partitions of 8 with 3-core equal to (1^2) form a block, of weight 2:

$$\mathfrak{B}_{(1^2)} = \{(1^8), (2^3,1^2), (3,2,1^3), (3,2^2,1), (4,1^4), (4,2^2), (4^2), (6,2), (7,1)\}.$$

The other p-blocks of $\mathbb{F}_3\mathfrak{S}_8$ are indexed by p-cores (2), $(3,1^2)$ and $(4,2,1^2)$, they are:

$$\mathfrak{B}_{(2)} = \{(2,1^6),\ (2^2,1^4),\ (2^4),\ (3^2,1^2),\ (4,3,1),\ (5,1^3),\ (5,2,1),\ (5,3),\ (8)\},$$

$$\mathfrak{B}_{(3,1^2)} = \{(3,1^5),\ (3^2,2),\ (6,1^2)\},$$

$$\mathfrak{B}_{(4,2,1^2)} = \{(4,2,1^2)\}.$$

The respective p-weights are $2, 1$ and 0. △

1.4.3 About the number of partitions in a p-block

From Nakayama's conjecture, the number of partitions λ in a block labelled by the p-core γ is equal to the number of $\lambda \vdash n$ with p-core γ. We saw in §1.1.6, Theorem 1.1.10 that a partition is uniquely determined by its p-core and its p-quotient. Recall that the p-quotient of λ is a p-partition of the p-weight of λ. Then, if w is the weight of $\mathfrak{B}_\gamma$, the number of partitions in $\mathfrak{B}_\gamma$ is equal to the number of p-quotients, that is, of p-partitions of the weight w. Let $p(k)$ denote the number of partitions of k for $k \in \mathbb{N}$. We have

Theorem 1.4.8 ([JK81, 6.2.1]). *The number of ordinary irreducible representations in a p-block of weight w of $\mathfrak{S}_n$ **depends only on** p **and** w. It is equal to*

$$\sum_{\substack{(w_0,\dots,w_{p-1})\\ w_0+\cdots+w_{p-1}=w\\ w_i\in\mathbb{N}}} p(w_0)\cdots p(w_{p-1}) = \prod_{k\geq 1}\frac{1}{(1-t^k)^p}\Bigg|_{t^w}$$

which is equal to the number of p-partitions of w. The expression in the right means the coefficient of the monomial t^w in the corresponding formal series.

There is a corresponding result for the number of p-modular irreducible representations. It turns out that the number of p-modular irreducibles in a p-block is equal to the number of $(p-1)$-partitions of the weight:

Theorem 1.4.9 ([JK81, 6.2.2]). *The number of p-modular irreducible representations in a p-block of weight w of $\mathfrak{S}_n$ **depends only on** p **and** w. It is equal to*

$$\sum_{\substack{(w_1,\dots,w_{p-1})\\ w_1+\cdots+w_{p-1}=w\\ w_i\in\mathbb{N}}} p(w_1)\cdots p(w_{p-1}) = \prod_{k\geq 1}\frac{1}{(1-t^k)^{p-1}}\Bigg|_{t^w}$$

which is equal to the number of $(p-1)$-partitions of w. The expression in the right means the coefficient of the monomial t^w in the corresponding formal series.

In conclusion, if we fix a weight w, blocks of $\mathfrak{S}_n$ with that given weight have the same size. In fact, there are certain equivalences that can be defined in the set of p-blocks of the symmetric group for which two p-blocks with the same weight are equivalent, see for Example [Sco95, CR08]. Hence, roughly speaking, when looking at certain properties, blocks of the same weight have similar structures.

2 The Mullineux map

In this chapter we give a historical background on the Mullineux problem and we recall different algorithms to compute the Mullineux map. For these descriptions we follow closely [FK97, Fay21b].

2.1 On the Mullineux problem

Recall, from the introduction, that the *Mullineux map* $\boldsymbol{m}_p$ is the involution on $\mathrm{Reg}_p(n)$ such that, for $\lambda \in \mathrm{Reg}_p(n)$

$$D^\lambda \otimes \varepsilon \simeq D^{\boldsymbol{m}_p(\lambda)}.$$

When $p = 0$ or $p > n$, the Mullineux map is conjugation of partitions: $\boldsymbol{m}_p(\lambda) = \lambda'$ for every p-regular partition λ (which in this case is any partition of n, and $D^\lambda \simeq S^\lambda$) [JK81, 2.1.8]. When $p = 2$, the sign representation ε is equivalent to the trivial representation, then the Mullineux map is the identity map.

The Mullineux problem concerns then the case $p < n$, where, as explained in the introduction, we look for a combinatorial description of the Mullineux map.

In [Mul79a], Mullineux defined a bijection $\boldsymbol{m}_p^M$ in $\mathrm{Reg}_p(n)$, we refer to the computation as Mullineux's algorithm. This algorithm uses *Mullineux symbols* which are arrays of integers that record some information in the Young diagram of the partition. Mullineux conjectured that $\boldsymbol{m}_p^M = \boldsymbol{m}_p$. Later, in [Kle96], Kleshchev defined an algorithm for computing $\boldsymbol{m}_p$, different from Mullineux's algorithm, which provides a bijection $\boldsymbol{m}_p^K = \boldsymbol{m}_p$ in $\mathrm{Reg}_p(n)$. Kleshchev's algorithm uses the notion of *good node* of a p-regular partition, introduced in [Kle95] in the context of the study of the branching rules for modular representations of symmetric groups. That is, the study of the composition factors of $D^\lambda \downarrow_{\mathfrak{S}_{n-1}}$, where D^λ is an irreducible representation of $\mathfrak{S}_n$. In [Kle96], Kleshchev reduced the Mullineux conjecture to a combinatorial question about the compatibility of the Mullineux's algorithm with the removal of good nodes. This problem was then solved in [FK97] where Ford and Kleshchev showed that $\boldsymbol{m}_p^K = \boldsymbol{m}_p^M$, proving the Mullineux conjecture. In [BO94], Bessenrodt and Olsson gave a shorter and easier proof of the Mullineux conjecture, by introducing a *residue symbol* associated to p-regular partitions. Since these different algorithms compute the Mullineux map, we keep only the notation $\boldsymbol{m}_p$.

Other algorithms have been found to compute the Mullineux map. Xu's algorithm [Xu97] is in a similar spirit from the one of Mullineux, but it does not use Mullineux symbols. Brundan and Kujawa gave a different proof of the Mullineux conjecture by showing that Xu's algorithm computes the Mullineux map, see [BK03]. Their proof uses an analogue of the Schur–Weyl duality in the context of superalgebras. In [Fay14],

Fayers gives a new algorithm for computing the Mullineux map which uses crystal isomorphisms of $\widehat{\mathfrak{sl}}_p$. Then, in [Jac21], in the context of affine Hecke algebras, Jacon gives an algorithm which is equivalent to Xu's algorithm.

The history of this problem lets see the importance and interest of the question.

2.2 Mullineux's algorithm

Mullineux's algorithm is defined recursively : starting with a p-regular partition λ, and recursively removing certain subsets of the rim of λ, called *p-rims*. In this process, certain numbers are recorded in an array called *Mullineux symbol*. The array is then transformed and the output of this process is a partition recursively defined from the new array.

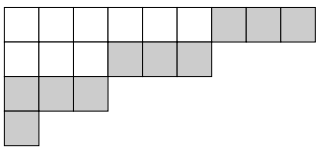
Let λ be a p-regular partition λ. Recall that the rim of λ is the rim-hook R^{λ}_{11}. We label the nodes of the rim with positive integers following a path from the top right to the bottom left. For example, if $\lambda = (8,5,3,1)$, the labelling is:

				4	3	2	1
		7	6	5			
10	9	8					
11							

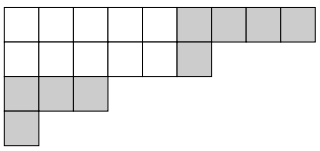
The first *p-segment* of the rim consists of the nodes corresponding to integers less or equal than p. If the last node (i,j) of the first p-segment is in the last row of $[\lambda]$, then $[\lambda]$ only has one p-segment. If not, let l be the smallest label on row $i+1$. The second p-segment of the rim consists of the nodes labelled by $l \leq m \leq l+p-1$. Repeating this procedure we will eventually reach the bottom row of the diagram and it is clear that all p-segments have p nodes, except possibly the last one. The *p-rim* of λ is defined as the union of all the p-segments.

Definition 2.2.1. The *p-rim* of λ is a subset of the rim defined as the union of all the p-segments. This set of nodes is denoted as $\mathrm{Rim}_p(\lambda)$. The cardinal of $\mathrm{Rim}_p(\lambda)$ is denoted a_λ. △

Example 2.2.2. The following two diagrams illustrate the p-rim of $\lambda = (9,6,3,1)$ for $p = 3$ and $p = 5$.



$\mathrm{Rim}_3(\lambda)$



$\mathrm{Rim}_5(\lambda)$

△

Define diagrams $\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(l)}$ as follows. Put $\lambda^{(0)} = \lambda$ and for $i \geq 1$ put

$$\lambda^{(i)} = \lambda^{(i-1)} \setminus \mathrm{Rim}_p(\lambda^{(i-1)}),$$

where we choose l maximal with respect to $\lambda^{(l)} \neq \emptyset$; so $\lambda^{(l+1)} = \emptyset$.

Definition 2.2.3. Fix a partition λ, and p an odd prime. For $\lambda^{(i)}$ in the sequence of partitions, depending on λ (and p), defined above, we associate the following information:

- The p-rim $\mathrm{Rim}_p(\lambda^{(i)})$ of $\lambda^{(i)}$ is called the *i-th p-rim of λ*,
- the cardinal $a_{\lambda^{(i)}}$ of $\mathrm{Rim}_p(\lambda^{(i)})$ is abbreviated as a_i,
- r_i denotes the number of rows of $\lambda^{(i)}$, that is, $r_i = l(\lambda^{(i)})$, and finally
- a number $\varepsilon_i \in \{0,1\}$ defined by

$$\varepsilon_i := \begin{cases} 0 & \text{if } p \mid a_i, \\ 1 & \text{if } p \nmid a_i. \end{cases}$$

△

The following definition was introduced in [BO94].

Definition 2.2.4. The *Mullineux symbol* of λ is

$$G_p(\lambda) := \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}. \tag{2.1}$$

△

Example 2.2.5. Let $p = 5$ and $\lambda = (9,6,3,1)$. In the following diagram we represent the i-th p-rim of λ with label i on its nodes.

2	2	2	2	1	0	0	0	0
2	1	1	1	1	0			
0	0	0						
0								

$$G_5(\lambda) = \begin{pmatrix} 9 & 5 & 5 \\ 4 & 2 & 2 \end{pmatrix}.$$

We have: $\lambda^{(1)} = (5^2)$ and $\lambda^{(2)} = (4,1)$. △

The following proposition is a reformulation ([AO91, §5]) of a result proved in [Mul79a, 3.6].

Proposition 2.2.6. *Let p be an odd prime and λ a p-regular partition of a non-negative integer n. The entries of $G_p(\lambda)$ satisfy*

1. $\varepsilon_i \le r_i - r_{i+1} < p + \varepsilon_i$ *for* $0 \le i < l$,
2. $1 \le r_l < p + \varepsilon_l$,
3. $r_i - r_{i+1} + \varepsilon_{i+1} \le a_i - a_{i+1} < p + r_i - r_{i+1} + \varepsilon_{i+1}$ *for* $0 \le i < l$,
4. $r_l \le a_l < p + r_l$,
5. $\sum_{i=0}^{l} a_i = n$.

Moreover, if $a_0,\dots,a_l,r_0,\dots,r_l$ are positive integers such that these inequalities are satisfied then there exists exactly one p-regular partition λ of n such that

$$G_p(\lambda)=\begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}.$$

Remark 2.2.7. It is easy to recover the p-regular partition λ from its Mullineux symbol $G_p(\lambda)$; start with the hook $\lambda^{(l)}$ of size a_l and length r_l, and for $i=l-1,l-2,\dots,0$, add the i-th p-rim (consisting of a_i nodes) to $\lambda^{(i+1)}$ from the bottom to the top, starting by placing a node on the first free placement in row r_i. Then, adding nodes either on top (whenever it is possible) or to the right of the last added node until having added the last node of the p-segment and add the following p-segment starting on the first free placement of the row on top of the last added node. This procedure finishes at the first row. This algorithm is precisely described in [FK97, §1]. △

Let λ be a p-regular partition of n, with Mullineux symbol

$$G_p(\lambda)=\begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}. \tag{2.2}$$

and let ε_i be as in Proposition 2.2.6. For $0\le i\le l$ define

$$s_i := a_i+\varepsilon_i-r_i.$$

In [Mul79a, 4.1] it is shown that the array

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ s_0 & s_1 & \cdots & s_l \end{pmatrix},$$

which is the Mullineux symbol of λ where r_i has been replaced by s_i for all $0\le i\le l$, corresponds to the Mullineux symbol of a unique p-regular partition. In view of this, we have the following definition.

Definition 2.2.8. With the above notations, let $\boldsymbol{m}_p(\lambda)$ be defined as the unique p-regular partition such that

$$G_p(\boldsymbol{m}_p(\lambda))=\begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ s_0 & s_1 & \cdots & s_l \end{pmatrix}.$$

△

Because of Proposition 2.2.6, $\boldsymbol{m}_p(\lambda)$ is well defined, and from the definition we can see that $\boldsymbol{m}_p$ is an involution.

Example 2.2.9. Let $p=5$ and $\lambda=(9,6,3,1)$ as in Example 2.2.5. Since

$$G_5(\lambda)=\begin{pmatrix} 9 & 5 & 5 \\ 4 & 2 & 2 \end{pmatrix},$$

We have that

$$G_5(\boldsymbol{m}_p(\lambda))=\begin{pmatrix} 9 & 5 & 5 \\ 6 & 3 & 3 \end{pmatrix}.$$

Now, from this Mullineux symbol we construct $\boldsymbol{m}_p(\lambda)$. Since the symbol has 3 columns, we know that there are 3 partitions $\boldsymbol{m}_p(\lambda)^{(2)}$, $\boldsymbol{m}_p(\lambda)^{(1)}$, and $\boldsymbol{m}_p(\lambda)^{(0)}$, that we construct in that order: the partition $\boldsymbol{m}_p(\lambda)^{(2)}$ is the only 5-regular partition with length 3; the partition $\boldsymbol{m}_p(\lambda)^{(1)}$ is the only 5-regular partition obtained by adding exactly 1 5-rim to $\boldsymbol{m}_p(\lambda)^{(2)}$, and keeping the length 3; finally the partition $\boldsymbol{m}_p(\lambda)^{(0)} = \boldsymbol{m}_p(\lambda)$ results from adding one 4-rim (in a way that the length is 6), and then one 5-rim starting in the row above, to $\boldsymbol{m}_p(\lambda)^{(1)}$:

$$\boldsymbol{m}_p(\lambda)^{(2)} = \begin{array}{|c|c|c|}\hline 2&2&2\\\hline 2\\\cline{1-1} 2\\\cline{1-1}\end{array} \qquad \boldsymbol{m}_p(\lambda)^{(1)} = \begin{array}{|c|c|c|c|}\hline 2&2&2&1\\\hline 2&1&1&1\\\hline 2&1\\\cline{1-2}\end{array} \qquad \boldsymbol{m}_p(\lambda)^{(0)} = \boldsymbol{m}_p(\lambda) = \begin{array}{|c|c|c|c|c|}\hline 2&2&2&1&0\\\hline 2&1&1&1&0\\\hline 2&1&0&0&0\\\hline 0&0\\\cline{1-2} 0\\\cline{1-1} 0\\\cline{1-1}\end{array}$$

Then, $\boldsymbol{m}_p(\lambda) = (5^3, 2, 1^2)$. △

Example 2.2.10. Let $2 < p < n$, and let λ be the one-row partition $\lambda = (n)$. We compute $\boldsymbol{m}_p(\lambda)$. The Euclidean division of n by p says that there are unique integers q and r such that

$$n = pq + r, \quad \text{and} \quad 0 \leq r < p.$$

We compute the Mullineux symbol of $\lambda = (n)$:

$$\lambda = \underbrace{\boxed{q}\cdots\boxed{q}}_{r \text{ times}}\underbrace{\boxed{1}\cdots\boxed{1}}_{p \text{ times}}\underbrace{\boxed{0}\cdots\boxed{0}}_{p \text{ times}}$$

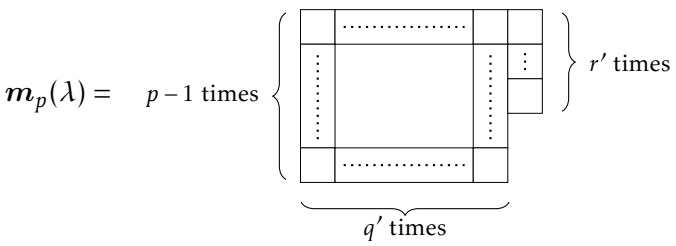
then

$$G_p(\lambda) = \begin{pmatrix} p & \cdots & p & r \\ \underbrace{1 & \cdots & 1}_{q \text{ times}} & 1 \end{pmatrix}.$$

Then, the Mullineux symbol of $G_p(\boldsymbol{m}_p(\lambda))$ is:

$$G_p(\boldsymbol{m}_p(\lambda)) = \begin{pmatrix} p & \cdots & p & r \\ \underbrace{p-1 & \cdots & p-1}_{q \text{ times}} & r \end{pmatrix}.$$

Hence if we construct $\boldsymbol{m}_p(\lambda)$ as described in Remark 2.2.7, we know that the Young diagram of $\boldsymbol{m}_p(\lambda)$ has the following shape



where q' and r' are some positive integers. Hence, to write the partition $\boldsymbol{m}_p(\lambda)$, we do the Euclidean division of n by $(p-1)$: the integers q' and r' are such that

$$n = (p-1)q' + r', \quad \text{and} \quad 0 \leq r' < p-1,$$

so that $n = r'(q'+1) + (p-1-r')q'$, and

$$\boldsymbol{m}_p(\lambda) = ((q'+1)^{r'}, q'^{(p-1-r')}).$$

△

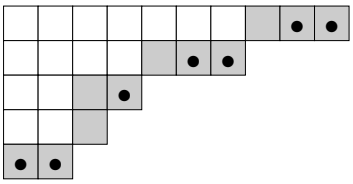
Remark 2.2.11. If $p > n$, then $\mathrm{Reg}_p(n) = \mathrm{Par}(n)$ and irreducible $\mathbb{K}\mathfrak{S}_n$-modules are therefore indexed by all partitions of n. In this case, the Mullineux's algorithm coincides with conjugation: $\boldsymbol{m}_p(\lambda) = \lambda'$. △

2.3 Xu's algorithm

Xu's algorithm is recursive as well in the sense that nodes are removed from the initial partition to obtain the empty partition. The number of nodes removed at each staged is recorded, and unlike Mullineux's algorithm, the output is not constructed recursively but these numbers give the column lengths of the final partition. Let us explain Xu's algorithm. The following description is based in [Fay08].

Let λ be a partition. The *truncated p-rim* of λ is the set of nodes (i,j) in the p-rim $\mathrm{Rim}_p(\lambda)$, of λ such that $(i,j-1)$ also lies in the p-rim, together with the node $(l(\lambda),1)$ if $p \nmid |\mathrm{Rim}_p(\lambda)|$. In other words to obtain the set of nodes in the truncated p-rim we take all the nodes on the p-rim, from these we remove the first appearing in each row except for the last row if $p \nmid |\mathrm{Rim}_p(\lambda)|$. Denote $\mathrm{Rim}_p^X(\lambda)$ the set of nodes in the truncated p-rim of λ.

Example 2.3.1. Let $p = 3$ and $\lambda = (10,7,4,3,2)$. In the following diagram we represent the p-rim $\mathrm{Rim}_p(\lambda)$ of λ with highlighted nodes (□) and the truncated p-rim is the set of nodes marked with a "•" (⊡).



△

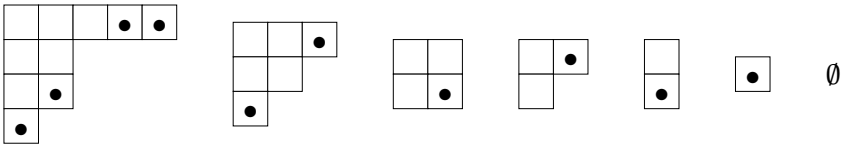
We now describe Xu's algorithm. Start with a p-regular partition λ.

1. Let μ_0 be the number of nodes in the truncated p-rim of λ: $\mu_0 := |\mathrm{Rim}_p^X(\lambda)|$. Let λ^1 be the partition obtained from λ by removing its truncated p-rim.
2. Let $\mu_1 := |\mathrm{Rim}_p^X(\lambda^1)|$ and let λ^2 be the partition obtained from λ^1 by removing its truncated p-rim.
3. Continue until obtaining the empty partition: $\lambda^{z+1} = \emptyset$.

4. The list of positive integers $\mu_0, \mu_1, \ldots, \mu_z$ is decreasing and defines a partition $\mu := (\mu_0, \ldots, \mu_z)$. These numbers are the column lengths of $\boldsymbol{m}_p(\lambda)$:

$$\boldsymbol{m}_p(\lambda) = \mu'.$$

Example 2.3.2. Let $p = 3$ and $\lambda = (5, 2^2, 1)$. We illustrate the sequence of removal of the truncated p-rim. The nodes in the truncated p-rim are marked with a "•".



The partition which records the size of the truncated p-rim at each stage is $\mu = (4, 2, 1^4)$. Then

$$\boldsymbol{m}_p(\lambda) = \mu' = (6, 2, 1^2).$$

△

2.4 Kleshchev's algorithm

Kleshchev's algorithm for computing the Mullineux map uses his results on modular branching rules for the symmetric group. Here we describe this algorithm in a completely combinatorial way. The notions and operations used in the algorithm have an algebraic sense that we do not explain here, for the details see [Kle95].

Let $p > 2$ be a prime. To each node $(r, c) \in \mathbb{N}^* \times \mathbb{N}^*$ we can associate its *p-residue* (or simply residue): $i = (c - r) \mod p$. In this case (r, c) is called an i-node. We can consider, for example, the p-residues of the nodes in the Young diagram of a partition λ. If $p = 3$ and $\lambda = (5, 2^2, 1)$, the p-residues of the nodes in the Young diagram of λ are:

0	1	2	0	1
2	0			
1	2			
0				

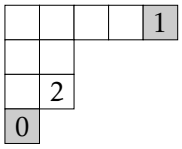
Let λ be a partition. For $i \in \mathbb{Z}/p\mathbb{Z}$, an i-node x in $[\lambda]$, the Young diagram of λ, is called *removable* if $[\lambda] \setminus \{x\}$ is the Young diagram of some partition. An *addable* i-node for λ is an i-node $x \in \mathbb{N}^* \times \mathbb{N}^*$ such that $x \notin [\lambda]$ and $[\lambda] \cup \{x\}$ is the Young diagram of some partition. For $p = 3$ and $\lambda = (5, 2^2, 1)$, the following diagram shows all the addable and removable i-nodes for λ that we indicate with the corresponding residues:

				1	2
		1			
	2				
0	1				
2					

In this example, λ has exactly one removable i-node for each $i \in \mathbb{Z}/p\mathbb{Z}$, it does not have addable 0-nodes, it has two addable 1-nodes and two addable 2-nodes.

For $i \in \mathbb{Z}/p\mathbb{Z}$, the *i-signature* of λ is a word on the symbols "+" and "−" constructed by following the south-east border of the Young diagram from the top to the bottom and writing "+" for each addable i-node and "−" for each removable i-node. In our example, the 0-signature is just −, the 1-signature is −++, and the 2-signature is +−+. The *reduced* i-signature is the i-signature after having erased every "+−" occurrence. In our example, the reduced i-signatures are then: for $i = 0$ we have −, for $i = 1$ we have −++ and finally for $i = 2$ we have +.

A *good* i-node, if there is one, is the removable i-node corresponding to the **last** "−" in the reduced i-signature. In our example there is a good 0-node and a good 1-node. The only removable 2-node is not a good node since it does not correspond to a "−" in the reduced 2-signature. Every non-empty p-regular partition has at least one good i-node for some $i \in \mathbb{Z}/p\mathbb{Z}$. The following diagram shows the good i-nodes, highlighted:



For $i \in \mathbb{Z}/p\mathbb{Z}$, denote by $\tilde{e}_i$ the operation that takes a partition having a good i-node and returns the partition obtained by removing this good i-node. The partition obtained is p-regular. In our example, we have $\tilde{e}_1(\lambda) = (4, 2^2, 1)$. The inverse operation can also be defined: a *cogood* i-node, if there is one, is the addable i-node corresponding to the **first** "+" in the reduced i-signature. For $i \in \mathbb{Z}/p\mathbb{Z}$, denote by $\tilde{f}_i$ the operation that takes a partition having a cogood i-node and returns the partition obtained by adding this good i-node. The partition obtained is p-regular. The operations $\tilde{e}_i$ and $\tilde{f}_i$ are inverses of each other ([MM90]). Indeed, in our example $\tilde{f}_1((4, 2^2, 1)) = \lambda$.

With these definitions we can now describe Kleshchev's algorithm for computing $\boldsymbol{m}_p$. Start with a p-regular partition λ.

1. Choose any good i-node of λ. Denote this p-residue as i_1. Remove this good i_1-node to obtain the partition $\tilde{e}_{i_1}(\lambda)$.

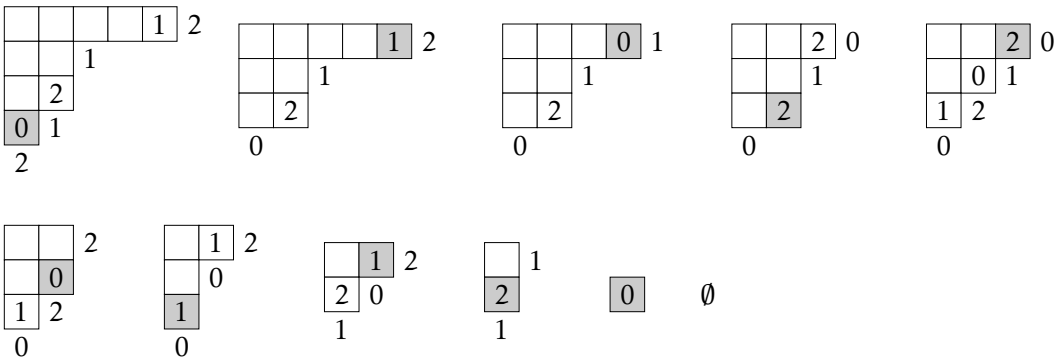
2. Choose any good i-node of $\tilde{e}_{i_1}(\lambda)$. Denote this p-residue as i_2. Remove this good i_2-node to obtain the partition $\tilde{e}_{i_2}\tilde{e}_{i_1}(\lambda)$.

3. Continue until obtaining the empty partition $\emptyset = \tilde{e}_{i_z} \cdots \tilde{e}_{i_2}\tilde{e}_{i_1}(\lambda)$. We get a list of residues $i_z, \ldots, i_2, i_1$.

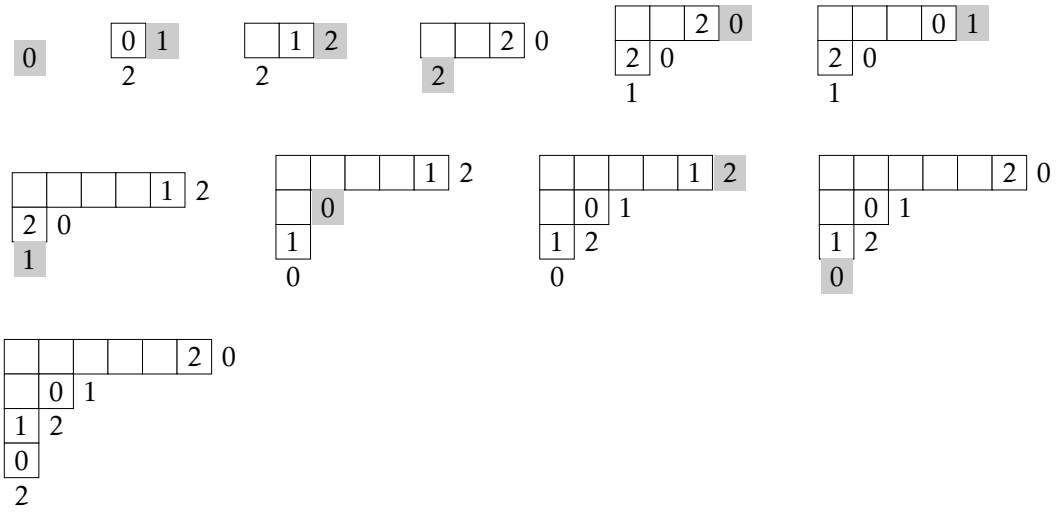
4. The partition $\boldsymbol{m}_p^K(\lambda)$ is constructed recursively by starting with the empty partition and adding cogood nodes with the residues $-i_1, -i_2, \ldots, -i_z$, where $-i_j$ is $-i_j$ mod p. That is
$$\boldsymbol{m}_p^K(\lambda) = \tilde{f}_{-i_z} \cdots \tilde{f}_{-i_2}\tilde{f}_{-i_1}(\emptyset).$$

The result is independent of the chosen sequence of good i-nodes. As announced $\boldsymbol{m}_p^K = \boldsymbol{m}_p$ ([Kle96]). Let us see an example.

Example 2.4.1. We continue the ongoing example: $p = 3$ and $\lambda = (5, 2^2, 1)$. Let us do the first three steps of the algorithm. We show the diagram at each stage, and we highlight the chosen good i-node to be removed.



We obtain the sequence of p-residues: 0 1 0 2 2 0 1 1 2 0. Hence, for the fourth step, we recursively add the cogood nodes with residues 0 1 2 2 0 1 1 0 2 0 in that order starting with the empty partition, which has a unique addable (and cogood) 0-node. We highlight the cogood node to be added:



Then $\boldsymbol{m}_p(\lambda) = (6, 2, 1^2)$.

△

Remark 2.4.2. The combinatorial algorithms for computing the Mullineux map do not depend on p being prime. Moreover there is an algebraic interpretation of the Mullineux map when $p > 1$ is any integer: let $q \in \mathbb{C}^*$ be a primitive root of unity of order $e > 1$. The Iwahori–Hecke algebra $H_n(q)$ of $\mathfrak{S}_n$ is an associative $\mathbb{C}$-algebra whose simple modules D^λ are also indexed by the e-regular partitions (see [Mat99]). When $q = 1$ and e is a prime number we encounter our particular case $H_n(q) \simeq \mathbb{F}_e\mathfrak{S}_n$, for this reason $H_n(q)$ is sometimes called a "deformation" of the group algebra of $\mathfrak{S}_n$. In the general case, there is an involution # on $H_n(q)$ with which the action on a $H_n(q)$-module V can be twisted to obtain another $H_n(q)$-module $V^\#$ (when $q = 1$, $V^\#$ is exactly $\varepsilon \otimes V$). When we apply this to a simple $H_n(q)$-module D^λ, where λ is a e-regular partition, we obtain a simple $H_n(q)$-module $D^{\boldsymbol{m}_e}(\lambda)$. The involution $\boldsymbol{m}_e$ on e-regular partitions is the Mullineux map in this more general setting. In [Bru98], Brundan proved the analogues of the modular branching rules for the symmetric group, by Kleshchev, for $H_n(q)$, showing in particular that Kleschev's description of the Mullineux map generalized for the Iwahori–Hecke algebra. Since all the other

algorithms do not depend on p being prime, they are also valid for $H_n(q)$, and any $e > 1$. △

3 A combinatorial bijection

This chapter is based on [Ber21a]. The organization of this chapter is as follows. In §3.1 we introduce a symbol, the BG-symbol defined on self-conjugate partitions and we show that it defines an injective mapping. Section 3.2 contains the main result: the BG-symbol computed on the set of BG-partitions produces Mullineux symbols associated to fixed points of the Mullineux map, defining a one-to-one correspondence, by an additional argument of cardinalities. This section also contains a proposition which is not included in [Ber21a], that says that the one-to-one correspondence is compatible with (fixes) the p-core. Finally, in §3.4 we show that the cardinality argument is not necessary: we show that every Mullineux symbol of a self-Mullineux partition is the BG-symbol of a BG-partition.

3.1 BG-partitions and the BG-symbol

We introduce a symbol, defined in general for self-conjugate partitions. This symbol is inspired by the Mullineux symbol. In a similar way as the Mullineux symbol, which is defined by counting nodes on the p-rims of a sequence of partitions, the BG-symbol is defined by counting elements in a set of nodes called the p-rim* which is a *symmetric* analogue of the p-rim.

Let λ be a self-conjugate partition. Set

$$U_\lambda = \{(i,j) \in \mathrm{Rim}_p(\lambda) \mid i \leq j\},$$

that is, U_λ consists of the nodes of the p-rim which are **above (or on) the diagonal** of $[\lambda]$. We denote $r^*_\lambda := |U_\lambda|$. Set

$$L_\lambda = \{(j,i) \mid (i,j) \in U_\lambda\}.$$

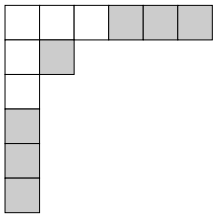
The set L_λ consists of the nodes in U_λ reflected across the diagonal of λ. Notice that $L_\lambda \subseteq [\lambda]$, since $\lambda = \lambda'$, so that $(i,j) \in [\lambda]$ if and only if $(j,i) \in [\lambda]$.

Definition 3.1.1. Let λ be a **self-conjugate** partition. The *p-rim** of λ is the set

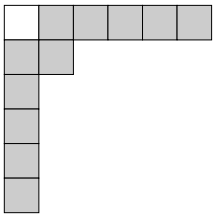
$$\mathrm{Rim}^*_p(\lambda) = U_\lambda \cup L_\lambda.$$

The cardinal of $\mathrm{Rim}^*_p(\lambda)$ is denoted a^*_λ. Define ε^*_λ as $\varepsilon^*_\lambda := a^*_\lambda \bmod 2$. △

Example 3.1.2. The following two diagrams illustrate the p-rim* of $\lambda = (6, 2, 1^4)$ in shaded boxes, for $p = 3$ and $p = 5$.



$\mathrm{Rim}_3^*(\lambda)$



$\mathrm{Rim}_5^*(\lambda)$

△

Remark 3.1.3. Notice that for a self-conjugate partition λ, from the definition of p-rim* we have

$$\begin{aligned} \varepsilon_\lambda^* = 0 &\Leftrightarrow a_\lambda^* \text{ is even.} \\ &\Leftrightarrow \mathrm{Rim}_p^*(\lambda) \text{ has no diagonal nodes.} \end{aligned}$$

△

Following this remark, we associate a number r_λ^* to λ as follows.

Definition 3.1.4. For λ, self-conjugate, the cardinal of U_λ, that is, the number of nodes of the p-rim* of λ which are **above (or on) the diagonal** is

$$r_\lambda^* = \begin{cases} \frac{a_\lambda^*}{2} & \text{if } a_\lambda^* \text{ is even,} \\ \frac{a_\lambda^*+1}{2} & \text{otherwise,} \end{cases}$$

thus

$$r_\lambda^* := \frac{a_\lambda^* + \varepsilon_\lambda^*}{2}.$$

△

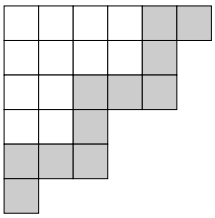
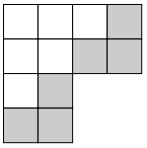
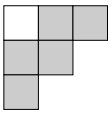
Now, let λ be a self-conjugate partition. We define diagrams $\lambda^{(0)*}, \lambda^{(1)*}, \ldots, \lambda^{(l)*}$ in an analogue way as for the Mullineux symbol, by considering the p-rim* instead of the p-rim. Put $\lambda^{(0)*} = \lambda$ and for $i \geq 1$ put

$$\lambda^{(i)*} = \lambda^{(i-1)*} \setminus \mathrm{Rim}_p^*(\lambda^{(i-1)}),$$

where we chose l maximal with respect to $\lambda^{(l)*} \neq \emptyset$; so $\lambda^{(l+1)*} = \emptyset$. We call the p-rim* of $\lambda^{(i)*}$ the *i-th p-rim** of λ.

Remark 3.1.5. Notice that the p-rim* is only defined for self-conjugate partitions, but we claim that the diagrams $\lambda^{(i)*}$ are well defined, given the fact that $\mathrm{Rim}_p^*(\lambda)$ is *symmetric* in the sense that $(u, v) \in \mathrm{Rim}_p^*(\lambda)$ if and only if $(v, u) \in \mathrm{Rim}_p^*(\lambda)$. Therefore, removing these nodes from $[\lambda]$ to obtain $\lambda^{(1)*}$ results again in a self-conjugate partition and then so it is for every $\lambda^{(i)*}$. In other words, if $\lambda^{(i)*}$ is self-conjugate, then $\lambda^{(i+1)*}$ is self-conjugate. △

Example 3.1.6. Let $p = 3$ and $\lambda = (6, 5^2, 3^2, 1)$. Then

$[\lambda] = \lambda^{(0)*} =$  $\lambda^{(1)*} =$  $\lambda^{(2)*} =$  $\lambda^{(3)*} =$ 

where shaded boxes represent the i-th p-rim* of λ. △

In a similar way as for p-regular partitions, for a fixed self-conjugate partition we associate a set of information:

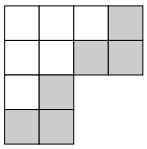
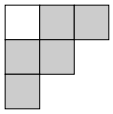
Definition 3.1.7. Fix a self-conjugate partition λ, and p an odd prime. For $\lambda^{(i)*}$ in the sequence of partitions depending on λ (and p) defined above, we associate the following information:

- The cardinal $a^*_{\lambda^{(i)*}}$ of $\mathrm{Rim}^*_p(\lambda^{(i)*})$ is abbreviated as a^*_i,
- the number $r^*_{\lambda^{i*}}$ of nodes on $\mathrm{Rim}^*_p(\lambda^{(i)*})$ which are above or on the diagonal of $\lambda^{(i)*}$ is abbreviated as r^*_i, and finally
- the parity $\varepsilon^*_{\lambda^{(i)*}} \in \{0, 1\}$ of $a^*_{\lambda^{(i)*}}$ is abbreviated as ε^*_i.

△

All these values associated to self-conjugate partitions may seem technical, and they are better understood by means of an example.

Example 3.1.8. Let $p = 3$, $\lambda = (4^2, 2^2)$, and $\mu = (3, 2, 1)$

$\lambda =$  $\mu =$ 

We have $a^*_\lambda = 6$, $\varepsilon^*_\lambda = 0$, and $r^*_\lambda = 3$. For μ, we have $a^*_\mu = 5$, $\varepsilon^*_\mu = 1$, and $r^*_\mu = 3$. △

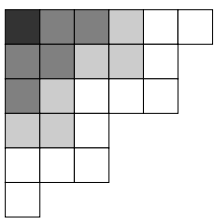
Definition 3.1.9. Let λ be a self-conjugate partition. The (p-)*BG-symbol* of λ is

$$\mathrm{bg}_p(\lambda) := \begin{pmatrix} a^*_0 & a^*_1 & \cdots & a^*_l \\ r^*_0 & r^*_1 & \cdots & r^*_l \end{pmatrix}. \tag{3.1}$$

The *length* of the BG-symbol is l.

△

Example 3.1.10. If $p = 3$, the BG-symbol of the partition $\lambda = (6, 5^2, 3^2, 1)$ from Example 3.1.6, is



$$\mathrm{bg}_3(\lambda) = \begin{pmatrix} 11 & 6 & 5 & 1 \\ 6 & 3 & 3 & 1 \end{pmatrix}.$$

In this diagram, each i-th 3-rim* is shown in a different shade. △

The following lemmas will allow us to prove that two different self-conjugate partitions correspond to different BG-symbols. Lemma 3.1.12 is an analogue of [Mul79a, 2.1]. Its proof is quite technical and the arguments are easier to understand with an example, see Example 3.1.13.

Lemma 3.1.11. *Let λ be a self-conjugate partition. If a^*_λ is an even number, then $p \mid a^*_\lambda$.*

Proof. From the definition (or see Remark 3.1.3), a^*_λ is even if and only if $U_\lambda \cap L_\lambda = \emptyset$. Then the p-rim* of λ does not contain diagonal nodes. From the definition of $\mathrm{Rim}^*_p(\lambda)$, this means that the set U_λ only contains p-segments of length p. And then the same is true for L_λ. Therefore

$$p \mid |U_\lambda \cup L_\lambda| = a^*_\lambda.$$

□

The converse is not true in general, for example, if $p = 3$ and $\lambda = (5,3,2,1,1)$, we have that $a^*_\lambda = 9$.

Lemma 3.1.12. *Let $\tilde{\lambda}$ be a self-conjugate partition, $\varepsilon \in \{0,1\}$ and m, a residue modulo p, such that $m = 0$ if $\varepsilon = 0$. Then, there exists a unique self-conjugate partition λ such that*

(i) $a^*_\lambda \equiv \varepsilon \pmod 2$;

(ii) $r^*_\lambda - \varepsilon^*_\lambda \equiv m \pmod p$ *and*

(iii) $\lambda^{(1)*} = \tilde{\lambda}$.

Moreover, if $\tilde{\lambda} \in \mathrm{BG}_p$, and $p \nmid 2m+1$ when $\varepsilon = 1$, then $\lambda \in \mathrm{BG}_p$.

Proof. Given $\varepsilon \in \{0,1\}$ and m, a residue modulo p, let us see that there is a unique way to add nodes to $\tilde{\lambda}$ to obtain a self-conjugate partition λ such that the added nodes are the p-rim* of λ.

Let us study how nodes (i,j) over the diagonal ($i \le j$) must be added. This will determine all nodes that must be added (if (i,j) is added to $\tilde{\lambda}$, then (j,i) is added as well).

First, the last row i over the diagonal that will contain new nodes (i,j) is uniquely fixed by $\tilde{\lambda}$ and ε. Indeed, let $d = k(\tilde{\lambda})$. If $\varepsilon = 0$, then $i = d$ and $(d, \tilde{\lambda}_d + 1)$ must be added to $\tilde{\lambda}$. If $\varepsilon = 1$, then $i = d+1$ and $(d+1, d+1)$ must be added to $\tilde{\lambda}$.

Now, let (i,j) be the first node that we add (with i fixed as before by $\tilde{\lambda}$ and ε) and $j \in \{\tilde{\lambda}_d + 1, d+1\}$ depending on $\tilde{\lambda}$ and ε. Starting from this node, it is clear that there is a unique way to add nodes such that (i), (ii), and (iii) hold: If the position $(i+1, j)$ just above (i,j) is empty in $\tilde{\lambda}$, we add a node in that position, otherwise we add a node in $(i, j+1)$. We repeat this procedure for adding nodes until we have added m nodes (including (i,j) if $\varepsilon = 0$, not including (i,j) if $\varepsilon = 1$). If the last added node is in row 1 we stop here. If it is added in row $k > 1$, we add a node in row $k-1$ in position $(k-1, \tilde{\lambda}_{k-1} + 1)$ and we restart the procedure to keep adding nodes until having added p nodes. We iterate this procedure, of adding groups of p nodes, until reaching the first row. This way we added nodes over the diagonal. Finally for each node (a,b) added, we add its reflection through the diagonal (b,a). And we obtain a self-conjugate partition λ.

It remains to verify that $\lambda^{(1)*} = \tilde{\lambda}$. If $\varepsilon = 1$, it is straightforward that $\lambda^{(1)*} = \tilde{\lambda}$. Since when removing the nodes of the p-rim* of λ over the diagonal we eventually reach a diagonal node, and then just remove the reflection of the removed nodes. It is clear that in this case we obtain $\tilde{\lambda}$. If $\varepsilon = 0$, the condition $m = 0$ says that a p-segment of λ will eventually reach the row d and this p-segment has exactly p nodes, so that there is no ambiguity when removing p-segments and $\lambda^{(1)*} = \tilde{\lambda}$.

For the last part of the theorem, suppose that $\tilde{\lambda} \in \mathrm{BG}_p$, and let us see that λ obtained as above is also in BG_p. In other words, we are assuming that $\tilde{\lambda}$ does not contain any diagonal (p)-hooks and we want to show that the same is true for λ.

Suppose that λ has a diagonal (p)-hook, say the $(i,i)_\lambda$-th hook, that is $h^\lambda_{ii} = pk$ for some integer $k > 0$. For a partition μ, we set the convention $h^\mu_{ij} = 0$ if $(i,j) \notin [\mu]$.

Since $\tilde{\lambda} \in \mathrm{BG}_p$, then the $(i,i)_\lambda$-th hook is different from the $(i,i)_{\tilde{\lambda}}$-th hook since if they were equal, $\tilde{\lambda}$ would have a (p)-hook, which is not possible. Therefore $(i, \tilde{\lambda}_i + 1) \in [\lambda]$. Since this node is not in $[\tilde{\lambda}]$, by definition, it is on the p-rim* of λ, in particular, it belongs to a p-segment of $\mathrm{Rim}^*_p(\lambda)$ above the diagonal. Consider the two cases: this p-segment starts at row i, or this p-segment starts before row i, that is, this p-segment starts at a row j for $1 \le j < i$.

- If the p-segment containing node $(i, \tilde{\lambda}_i + 1)$ starts at row i, let (i,j) be the first node of this p-segment and (a,b) its last node ($i \le a$). Then $a \le b$ because this segment is above the diagonal.

  Let N be the number of nodes on this p-segment. Then we have:

  $$\begin{aligned} N &= a - i + j - b + 1, \\ h^\lambda_{ii} &= 1 + 2(j-i), \\ h^{\tilde{\lambda}}_{aa} &= 1 + 2(b-a-1), \end{aligned}$$

  where the last identity holds if $b > a$ (since this implies that $(a, b-1) \in \tilde{\lambda}$). Then we have

  $$h^\lambda_{ii} = \begin{cases} 2N - 1 & \text{if } a = b, \\ 2N + h^{\tilde{\lambda}}_{aa} & \text{if } a < b. \end{cases}$$

  If $a = b$, this p-segment is the last segment in U_λ and $\varepsilon^*_\lambda = 1$. So that $N = m+1$ and we get $h^\lambda_{ii} = 2m + 1$. This contradicts $p \nmid 2m+1$. If $a < b$, then the last node of this p-segment, (a,b) is not a diagonal node so that $N = p$ and we get $h^\lambda_{ii} = 2p + h^{\tilde{\lambda}}_{aa}$, which implies $p \mid h^{\tilde{\lambda}}_{aa}$, a contradiction.

- If the p-segment containing node $(i, \tilde{\lambda}_i + 1)$ starts at a row j with $j < i$, then it contains nodes on row $i-1$, in particular $(i-1, \tilde{\lambda}_{i-1} + 1)$. The next node on this p-segment is the node just below: $(i, \tilde{\lambda}_{i-1} + 1)$. Then $\lambda_i = \tilde{\lambda}_{i-1} + 1$. Let us see that $h^{\tilde{\lambda}}_{(i-1,i-1)} = h^\lambda_{ii}$. Since these are diagonal hooks contained in self-conjugate partitions, their lengths are calculated as follows

  $$\begin{aligned} h^{\tilde{\lambda}}_{(i-1,i-1)} &= 2(\tilde{\lambda}_{i-1} - (i-1)) + 1 \\ &= 2\tilde{\lambda}_{i-1} - 2i + 3, \end{aligned}$$

and

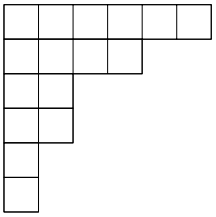
$$\begin{aligned} h^{\lambda}_{(i,i)} &= 2(\lambda_i - i) + 1 \\ &= 2((\tilde{\lambda}_{i-1} + 1) - i) + 1 \\ &= 2\tilde{\lambda}_{i-1} - 2i + 3. \end{aligned}$$

Since $p \mid h^{\lambda}_{(i,i)}$, then $p \mid h^{\tilde{\lambda}}_{(i-1,i-1)}$, a contradiction.

In conclusion, λ does not have any diagonal (p)-hooks, that is, $\lambda \in \mathrm{BG}_p$.

□

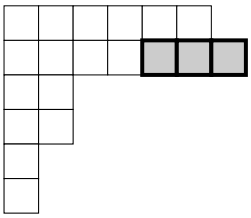
Example 3.1.13. Let $p = 3$. We use the notations of Lemma 3.1.12. Consider the self-conjugate partition $\tilde{\lambda} = (6, 4, 2^2, 1^2)$.

$[\tilde{\lambda}] =$ 

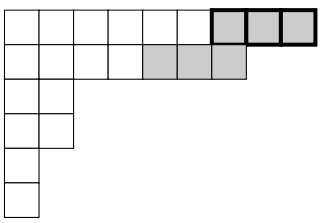
- Let $\varepsilon = 0$, then $m = 0$. Let us see that there is only one self-conjugate partition λ satisfying: a^*_λ is even, $r^*_\lambda - \varepsilon^*_\lambda = |U_\lambda| - \varepsilon^*_\lambda \equiv 0 \pmod 3$ and $\lambda^{(1)*} = \tilde{\lambda}$. We add to $[\tilde{\lambda}]$ the nodes of $\mathrm{Rim}^*_3(\lambda) = U_\lambda \cup L_\lambda$.

  In this case, since $a^*_\lambda \equiv 0 \pmod 2$, then U_λ does not contain diagonal nodes. That is, U_λ consists only on nodes strictly over the diagonal, so that the last row containing nodes from U_λ is row 2, since $k(\tilde{\lambda}) = 2$.

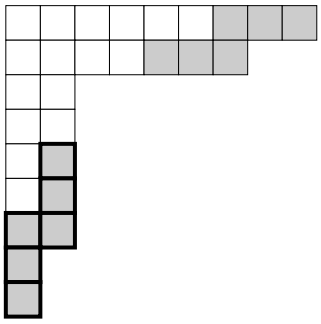
  Since $m = 0$, then every 3-segment of U_λ has 3 nodes. The bottom 3-segment of U_λ is shown in shaded nodes in the following diagram

  Since we have not reached the top of $[\tilde{\lambda}]$, there is at least another 3-segment, which starts at the following upper row :

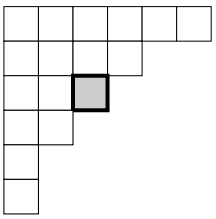
  Now, λ is self-conjugate, then for each upper node (i, j) that we added, we add the node (j, i) (or also because $(i, j) \in U_\lambda$ if and only if $(j, i) \in L$)

$[\lambda] =$ 

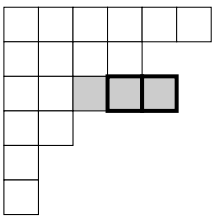
So that $\lambda = (9, 7, 2^5, 1^2)$ is the only possibility for λ self-conjugate such that a^*_λ is even, $r^*_\lambda - \varepsilon^*_\lambda = |U_\lambda| \equiv 0 \pmod 3$ and $\lambda^{(1)*} = \tilde{\lambda}$.

- Let $\varepsilon = 1$, and $m = 2$. Let us see that there is only one possible partition λ satisfying: a^*_λ is odd, $r^*_\lambda - \varepsilon^*_\lambda = |U_\lambda| - \varepsilon^*_\lambda \equiv 2 \pmod 3$ and $\lambda^{(1)*} = \tilde{\lambda}$. We add to $[\tilde{\lambda}]$ the nodes of $\mathrm{Rim}^*_3(\lambda) = U_\lambda \cup L_\lambda$.

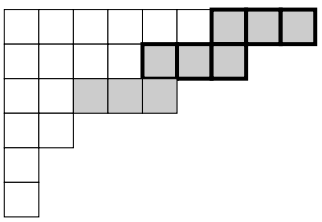
  In this case, $a^*_\lambda \equiv 1 \pmod 2$, then there is a diagonal node in U_λ:



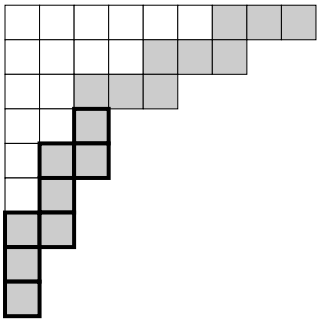
  We add now the rest of the nodes in U_λ. Here $r^*_\lambda - \varepsilon^*_\lambda = |U_\lambda \setminus \{(3,3)\}| \equiv 2 \pmod 3$. That means that the rest of the nodes in U_λ contain one 3-segment of 2 nodes, we add this 3-segment



  and the rest are 3-segments of 3 nodes:



and finally, for making λ self-conjugate:



And we see that $\lambda = (9,7,5,3^2,2^2,1^2)$ is the only possibility for having a^*_λ odd, $r^*_\lambda - \varepsilon^*_\lambda = |U_\lambda| - 1 \equiv 2 \pmod 3$, and $\lambda^{(1)*} = \tilde{\lambda}$.

△

Proposition 3.1.14. *Let p be an odd prime. Two different self-conjugate partitions have different BG-symbols. In other words, the BG-symbol gives rise to an injective map from self-conjugate partitions to the set of two-row positive integer symbols.*

Proof. We proceed by induction on l, the length of the BG-symbol. Let $l = 0$. Let λ be a self-conjugate partition with BG-symbol

$$\mathrm{bg}_p(\lambda) = \binom{a^*_0}{r^*_0}.$$

Notice that having a BG-symbol of length 0 means that $\lambda = \lambda^{(0)*} = \lambda^{(l)*}$ is a hook and its size is a^*_0, that is, there are positive integers u, v such that $\lambda = (u, 1^v)$ and $u + v = a^*_0$. But λ is self-conjugate, then $u - 1 = v$, so that $a^*_0 = 2u - 1$, and $\varepsilon^*_0 = 1$. Then

$$r^*_0 = \frac{a^*_0 + \varepsilon^*_0}{2} = u.$$

Therefore $\lambda = (r^*_0, 1^{r^*_0 - 1})$. This way, λ is determined from its BG-symbol and, from this reasoning, we see that any self-conjugate partition with BG-symbol $\mathrm{bg}_p(\lambda)$ is completely determined by it and is then equal to λ.

Now, fix $l > 0 \in \mathbb{N}$, let λ be a self-conjugate partition with BG-symbol

$$\mathrm{bg}_p(\lambda) = \begin{pmatrix} a^*_0 & a^*_1 & \cdots & a^*_l \\ r^*_0 & r^*_1 & \cdots & r^*_l \end{pmatrix},$$

and let μ be a self-conjugate partition such that $\mathrm{bg}_p(\mu) = \mathrm{bg}_p(\lambda)$.

Then, by definition, the BG-symbol of $\lambda^{(1)*}$ (and also of $\mu^{(1)*}$) is the BG-symbol of λ after removing its first column

$$\mathrm{bg}_p(\lambda^{(1)*}) = \mathrm{bg}_p(\mu^{(1)*}) = \begin{pmatrix} a^*_1 & \cdots & a^*_l \\ r^*_1 & \cdots & r^*_l \end{pmatrix}.$$

By induction, there exists a unique self-conjugate partition $\tilde{\tau}$ such that

$$\mathrm{bg}_p(\tilde{\tau}) = \begin{pmatrix} a_1^* & \cdots & a_l^* \\ r_1^* & \cdots & r_l^* \end{pmatrix}.$$

Then $\tilde{\tau} = \lambda^{(1)*} = \mu^{(1)*}$. Let us see, from Lemma 3.1.12, that $\lambda = \mu$.

Let $\varepsilon = a_0^* \bmod 2$, and $m = r_0^* - \varepsilon_0^* \bmod p$. By Lemma 3.1.11, if $\varepsilon = 0$ then $m = 0$. Therefore, by Lemma 3.1.12 there exists a unique self-conjugate partition τ such that

(i) $a_\tau^* \equiv \varepsilon \pmod 2$;

(ii) $r_\tau^* - \varepsilon_\tau^* \equiv m \pmod p$ and

(iii) $\tau^{(1)*} = \tilde{\tau}$.

But partitions λ and μ are self-conjugate and they satisfy (i) and (ii) since $a_0^* = a_\lambda^* = a_\mu^*$. Moreover $\lambda^{(1)*} = \mu^{(1)*} = \tilde{\tau}$, then by the uniqueness of τ we have that $\tau = \lambda = \mu$.

□

3.2 From BG-partitions to self-Mullineux partitions

As it turns out, the BG-symbol of a BG-partition is a Mullineux symbol of some self-Mullineux partition. Denote by $\mathcal{M}_p$ the set of Mullineux symbols of the self-Mullineux partitions M_p.

Proposition 3.2.1. *Let p be an odd prime and λ a BG-partition. The BG-symbol of λ, $\mathrm{bg}_p(\lambda)$ is the Mullineux symbol of some self-Mullineux partition. That is*

$$\mathrm{bg}_p(\mathrm{BG}_p) \subseteq \mathcal{M}_p.$$

We postpone the proof of this proposition since some technical lemmas are necessary.

Recall, from Proposition 2.2.6 and Remark 2.2.7, that to a Mullineux symbol corresponds a unique p-regular partition. So that the Mullineux symbol determines a bijection between p-regular partitions and their Mullineux symbols. In particular, to a symbol in $\mathcal{M}_p$ corresponds a unique self-Mullineux partition in M_p. As a corollary from Proposition 3.1.14, which says that bg_p is injective, Proposition 3.2.1, which affirms that the image of bg_p is contained in the self-Mullineux symbols, and finally from the fact that the sets BG_p^n and M_p^n have the same number of elements (Proposition 1.3.16), we obtain the following result.

Theorem 3.2.2. *We have that*

$$\mathrm{bg}_p(\mathrm{BG}_p) = \mathcal{M}_p,$$

and the BG-symbol provides an explicit bijection between BG-partitions and self-Mullineux partitions. This bijection is given by associating to a BG-partition λ its BG-symbol $\mathrm{bg}_p(\lambda)$, which corresponds to a unique self-Mullineux partition. This bijection restricts to bijections between BG_p^n and M_p^n for every $n \in \mathbb{N}$.

Remark 3.2.3. If we consider the Mullineux symbol G_p as a bijection from the set of p-regular partitions into its image in the set of two-row arrays of integers, then the bijection in Theorem 4.6.1, from BG_p to M_p is given precisely by $G_p^{-1} \circ \mathrm{bg}_p$. △

Remark 3.2.4. In [AO91], Andrews and Olsson prove a general partition identity, which depends on some integer parameters. A special case of this identity is the fact that the number of (p-)self-Mullineux partitions of a non-negative integer n equals the number of partitions of n with different odd parts, none of them divisible by p, which is in turn equal to the number of BG-partitions.

Now, in [Bes91], C. Bessenrodt shows a combinatorial proof of the Andrews–Olsson identity, which provides, by choosing the right parameters, an explicit bijection between BG_p^n and M_p^n. The bijection from Theorem 4.6.1 is obtained in a more direct way and it is different from Bessenrodt's bijection. In particular, for $p = 5$ and $n = 20$, the partition $(7,6,3,2^2) \in M_5^{20}$ is mapped to partition $(9,3,2,1^6) \in \mathrm{BG}_5^{20}$ under Bessenrodt's bijection, and it is mapped to $(7,5,2^3,1^2) \in \mathrm{BG}_5^{20}$ under bijection from Theorem 4.6.1. △

3.2.1 Some technical lemmas

In this section we prove Proposition 3.2.1. For this proof we need some technical lemmas.

Definition 3.2.5. For a partition $\lambda = (\lambda_1,\ldots)$ we define $k(\lambda)$ as the coordinate of the *maximal* diagonal node of λ:

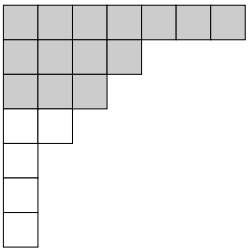
$$k(\lambda) := \max\{i \mid \lambda_i \neq 0 \text{ and } (i,i) \in [\lambda]\}.$$

Which is also equal to $|\{\lambda_i \mid \lambda_i \geq i\}|$. △

Remark 3.2.6. The number $k(\lambda)$ is known in the literature as the size of the side of the *Durfee square* of λ: the biggest square formed by boxes contained in the Young diagram of λ. For more on Durfee squares see for example [AE04]. △

Lemma 3.2.7. *Let p be an odd prime and $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_r)$ a partition in BG_p. Let $k = k(\lambda)$ as in Definition 3.2.5. Then the partition $\mu := (\lambda_1,\lambda_2,\ldots,\lambda_k)$ is p-regular.*

Example 3.2.8. Let $p = 3$ and $\lambda = (7,4,3,2,1^3)$. This partition is a 3-BG-partition. Here $k = 3$, so that $l(\mu) = 3$, and indeed, the partition $\mu = (7,4,3)$ is 3-regular. The following diagram illustrates both the partitions λ and μ (in shaded boxes).



△

Proof of the lemma. Suppose that μ is not p-regular. So that there exists $1 \leq i \leq k$ such that

$$\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}.$$

Since $i+p-1 \leq k$, then $\lambda_{i+p-1} \geq i+p-1$. Then $(i+p-1, i+p-1) \in [\mu] \subseteq [\lambda]$. Let a be the length of the $(i+p-1, i+p-1)$-th hook of λ, that is $a = h^{\lambda}_{(i+p-1,i+p-1)}$. Then, the length of the $(i+p-2, i+p-2)$-th hook of λ, is $h^{\lambda}_{(i+p-2,i+p-2)} = a+2$, since $\lambda_{i+p-1} = \lambda_{i+p-2}$ and λ is self-conjugate. And $h_{(i+p-3,i+p-3)} = a+4$. In general $h^{\lambda}_{(i+p-1-j,i+p-1-j)} = a+2j$ for $j = 0, \ldots, p-1$. That is, the lengths of these hooks are:

$$a, a+2, a+4, \ldots, a+2j, \ldots, a+2(p-1).$$

But since p is odd, this list, modulo p, forms a complete collection of residues. Then, there exists $j \in \{0, \ldots, p-1\}$ such that $p \mid a+2j = h^{\lambda}_{(i+p-1-j,i+p-1-j)}$, and this contradicts the fact that $\lambda \in \mathrm{BG}_p$.

□

In the set of p-BG-partitions, the implication in Lemma 3.1.11 becomes an equivalence:

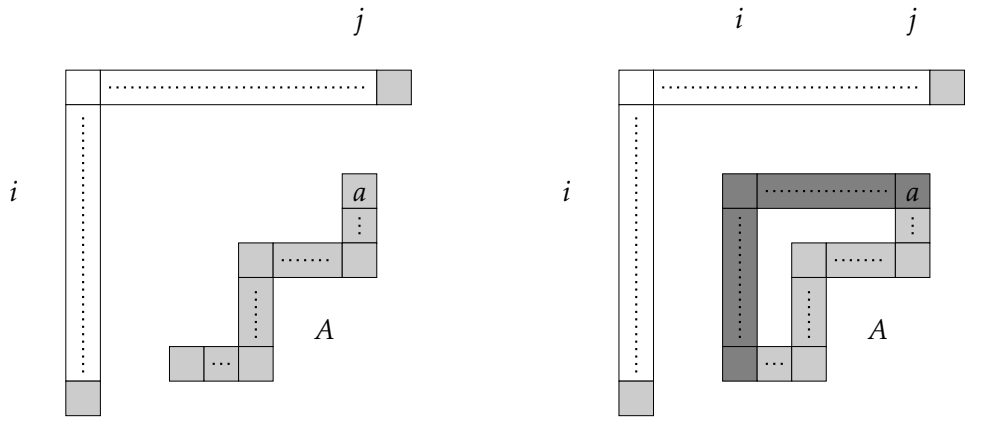
Lemma 3.2.9. *Let $\lambda \in \mathrm{BG}_p$. Then a^*_λ is even if and only if $p \mid a^*_\lambda$.*

Proof. As already noted, the fact that a^*_λ implies $p \mid a^*_\lambda$ is proved in Lemma 3.1.11.

Suppose that $p \mid a^*_\lambda$. If a^*_λ is odd, then $\mathrm{Rim}^*_p(\lambda)$ contains a diagonal node. Then U_λ is formed by p-segments of length p and one last p-segment of length possibly less than p, which, in this case contains the diagonal node. Let us name B the set of nodes in this last p-segment, and let A be the set

$$A = B \cup \{(j,i) \in [\lambda] \mid (i,j) \in B\} \subseteq \mathrm{Rim}^*_p(\lambda).$$

The set A is formed by a symmetrical segment along the rim of $[\lambda]$. See Figure 3.1a.



(a) Segment A.

(b) $(i,i)_\lambda$-th hook in darker shaded boxes.

Figure 3.1

The set $\mathrm{Rim}^*_p(\lambda)$ is formed by the disjoint union of A and some p-segments. Therefore, since $p \mid a^*_\lambda$, we have that $|A| = p$.
Now, let $a = (i,j)$ be the first node of the segment B, that is $i = \min\{r \mid (r,s) \in B\}$ and $j = \max\{s \mid (r,s) \in B\}$. We have that the (i,i)-th hook contains exactly $|A| = p$ nodes. See Figure 3.1b.

This means that λ has a diagonal (p)-hook, which is a contradiction because $\lambda \in \mathrm{BG}_p$.

□

We obtain the following corollary from Remark 3.1.3 and Lemma 3.2.9.

Corollary 3.2.10. *Let $\lambda \in \mathrm{BG}_p$. The following statements are equivalent*

1. $\varepsilon^*_\lambda = 0$.
2. a^*_λ *is even.*
3. $\mathrm{Rim}^*_p(\lambda)$ *does not contain diagonal nodes.*
4. $p \mid a^*_\lambda$.

Consider a partition $\lambda \in M_p$, that is, a fixed point of the Mullineux map. This is a condition that depends only on the columns of the Mullineux symbol of λ. Therefore, the partition $\lambda^{(1)}$ obtained by removing the p-rim of λ is also a fixed point of the Mullineux map, since its Mullineux symbol is obtained by removing the first column of the Mullineux symbol of λ. The following lemma is an analogue property for partitions in BG_p.

Lemma 3.2.11. *If $\lambda \in \mathrm{BG}_p$ then $\lambda^{(1)*} \in \mathrm{BG}_p$. In other words, if λ is a p-BG-partition, then, removing its p-rim* results in a p-BG-partition.*

Proof. Recall (Remark 3.1.5) that if λ is self-conjugate, then so it is for $\lambda^{(1)*}$. In particular, if $\lambda \in \mathrm{BG}_p$, then $\lambda^{(1)*}$ is self-conjugate. It remains to prove that $\lambda^{(1)*}$ does not have any diagonal (p)-hooks.

For simplicity of notations let $\mu = \lambda^{(1)*}$. Suppose that μ has a diagonal (p)-hook, say the $(i,i)_\mu$-th hook, with $h^\mu_{i,i} = pk$ for some $k \in \mathbb{N}$.

We claim that the node $(i, \mu_i + 1)$ is in the p-rim* of λ. Indeed, $(i, \mu_i + 1) \in [\lambda]$ since if $(i, \mu_i + 1) \notin [\lambda]$, then $\mu_i = \lambda_i$ and $h^\lambda_{i,i} = h^\mu_{i,i} = pk$ so that λ has a diagonal (p)-hook, which contradicts the fact that $\lambda \in \mathrm{BG}_p$. Now, since $(i, \mu_i + 1) \in [\lambda] \setminus [\mu]$, then $(i, \mu_i + 1) \in \mathrm{Rim}^*_p(\lambda)$. See Figure 3.2a.

There are now two possible cases: either $(i, \mu_i + 1)$ is the last node of a p-segment of U_λ (the nodes on the p-rim* of λ over the diagonal), or it is not the last node of the p-segment to which it belongs. Let us examine these two cases.

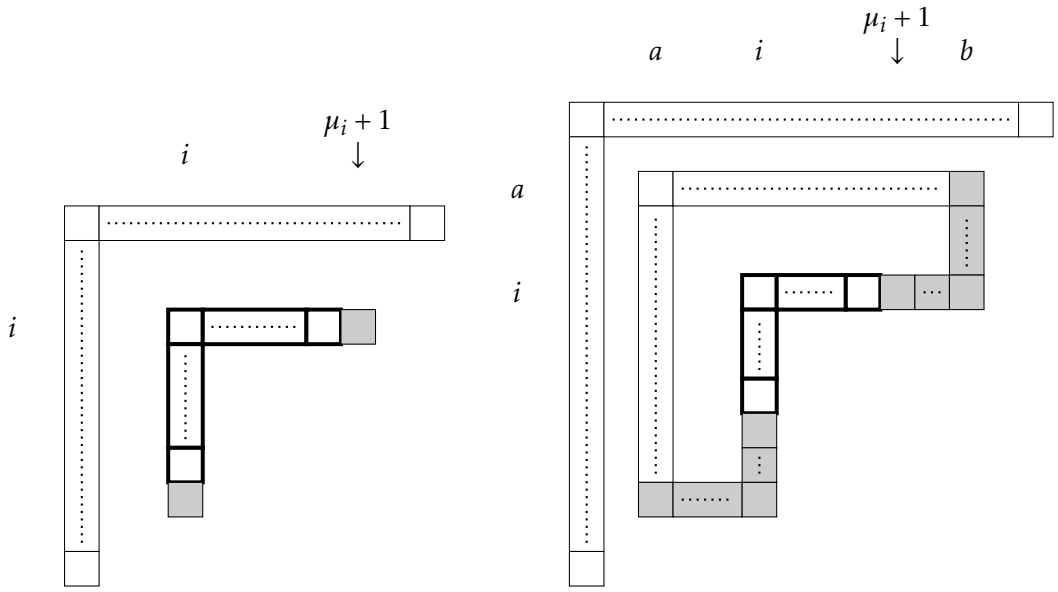
Suppose $(i, \mu_i + 1)$ is the last node of a p-segment of U_λ, and this p-segment starts on a node (a, b). See Figure 3.2b.

Then, the $(a,a)_\lambda$-th hook has length equal to the length of the $(i,i)_\mu$-th hook plus twice the length of the p-segment of $\mathrm{Rim}^*_p(\lambda)$ containing the node $(i, \mu_i + 1)$, that is

$$h^\lambda_{a,a} = p + h^\mu_{i,i} + p = p + pk + p = p(k+2),$$

so that λ contains diagonal (p)-hook, which is impossible.

Suppose now that $(i, \mu_i + 1)$ is not the last node of a p-segment of U_λ. First, notice that the node $(i+1, \mu_i + 2) \notin [\lambda]$. This is true because $(i, \mu_i + 1)$ is in the p-rim* of λ. We claim that $(i+1, \mu_i + 1) \in \mathrm{Rim}^*_p(\lambda) \subseteq [\lambda]$. In Figure 3.3, node $(i+1, \mu_i + 2)$ is illustrated as a cross (meaning it is not in $[\lambda]$) and node $(i+1, \mu_i + 1)$ as a shaded box (as are their opposites with respect to the diagonal). Indeed, $(i+1, \mu_i + 1) \in \mathrm{Rim}^*_p(\lambda)$ because, since



(a) Shaded boxes are in $\mathrm{Rim}_p^*(\lambda)$. (b) Shaded boxes are p-segments in $\mathrm{Rim}_p^*(\lambda)$.

Figure 3.2

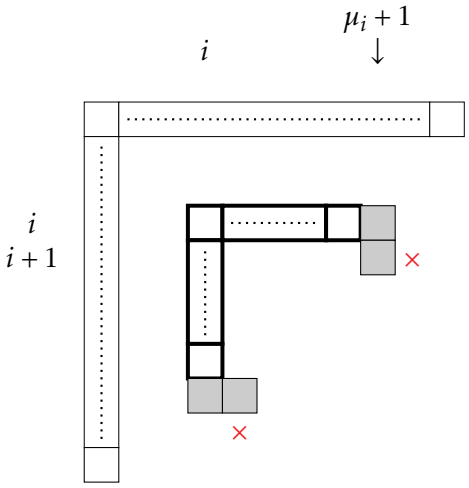


Figure 3.3

$(i, \mu_i + 1)$ is not the last node of a p-segment, then the next node of its p-segment is either to the left or down. But the node to the left of $(i, \mu_i + 1)$, that is, (i, μ_i) is not in the p-rim* of λ since it is in μ, so that the next node of this p-segment is $(i + 1, \mu_i + 1)$, which is then in $\mathrm{Rim}_p^*(\lambda)$.

The fact that $(i + 1, \mu_i + 1) \in \mathrm{Rim}_p^*(\lambda) \subseteq [\lambda]$ and $(i + 1, \mu_i + 2) \notin [\lambda]$ implies that $\lambda_{i+1} = \mu_i + 1$ and therefore the $(i + 1, i + 1)_\lambda$-th hook has length

$$h^\lambda_{(i+1,i+1)} = h^\mu_{(i,i)} = pk,$$

that is, λ has a diagonal (p)-hook, a contradiction.

We conclude that μ does not have any diagonal (p)-hooks and then, $\mu = \lambda^{(1)*} \in \mathrm{BG}_p$. □

3.2.2 Proof of Proposition 3.2.1

Proof of Proposition 3.2.1. Let us first state which properties characterize elements in $\mathcal{M}_p$. That is, if $\lambda \in M_p^n$ which conditions characterize its Mullineux symbol

$$G_p(\lambda) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}.$$

Let $s_i = a_i + \varepsilon_i - r_i$. The partition λ is a fixed point of the Mullineux map if and only if $r_i = s_i$, that is

$$a_i = 2r_i - \varepsilon_i.$$

We also know that λ is the only p-regular partition whose Mullineux symbol satisfies properties (1)–(5) from Proposition 2.2.6. This way, the properties that characterize Mullineux symbols of partitions in M_p are equivalent to the following properties

1. $\varepsilon_i \leq r_i - r_{i+1} < p + \varepsilon_i$ for $0 \leq i < l$,
2. $1 \leq r_l < p + \varepsilon_l$,
3. $\sum_{i=0}^{l} a_i = n$, and
4. $a_i = 2r_i - \varepsilon_i$.

On the other hand, from the definition of ε_i^* and Corollary 3.2.10, we have that

$$\varepsilon_i^* = \begin{cases} 0 \text{ if } p \mid a_i^* \\ 1 \text{ if } p \nmid a_i^* \end{cases}$$

Let $\lambda \in \mathrm{BG}_p^n$. Let us see that its BG-symbol

$$\mathrm{bg}_p(\lambda) = \begin{pmatrix} a_0^* & a_1^* & \cdots & a_l^* \\ r_0^* & r_1^* & \cdots & r_l^* \end{pmatrix}$$

is in $\mathcal{M}_p$ by verifying properties (1)–(4) for a_i^*, ε_i^* and r_i^*.

From the definition of the sequence $a_0^*, \ldots, a_l^*$, it is clear that (3) holds. We have that (4) is satisfied from Remark 3.1.3. Let us first show that (2) holds. Since $\lambda^{(l)*}$ is not the empty partition, $r_l^* \geq 1$. On the other hand, the partition $\lambda^{(l)*}$ is a hook and it is self-conjugate; more precisely $\lambda^{(l)*} = (r_l^*, 1^{a_l^* - r_l^*})$. Then $a_l^* = |\lambda^{(l)*}|$ is odd, so that $\varepsilon_l^* = a_l^* \bmod 2 = 1$. Suppose that $r_l^* \geq p + \varepsilon_l^* = p + 1$. This means that the first p-segment over the diagonal of $\lambda^{(l)*}$ consists of p nodes. But then, there are more nodes remaining in the first row of $[\lambda^{(l)*}]$ that are not in the p-rim*, but this contradicts the maximality of l.

It remains to prove (1). A key element for this task is Lemma 3.2.7, which roughly says that truncating a BG-partition to some particular row results in a p-regular partition. The idea is to use the fact that this truncated partition, being p-regular, satisfies properties from Proposition 2.2.6, which uses numbers from the Mullineux symbol,

and these will give us information about r_i^* and ε_i^*, which are numbers appearing in the BG-symbol.

Let us see that λ satisfies

$$\varepsilon_i^* \leq r_i^* - r_{i+1}^* < p + \varepsilon_i^* \text{ for } 0 \leq i < l.$$

It suffices to prove this for $i = 0$ and then, the property follows recursively by Lemma 3.2.11.
To simplify notation, set:

values associated to λ	values associated to $\lambda^{(1)*}$
$a := a_0^*$	$a' := a_1^*$
$r := r_0^*$	$r' := r_1^*$
$\varepsilon := \varepsilon_0^*$	$\varepsilon' := \varepsilon_1^*$

Let us prove that

$$\varepsilon \leq r - r' < p + \varepsilon.$$

We study the four possible cases for the values of ε and ε', namely

	ε	ε'
(i)	0	0
(ii)	0	1
(ii)	1	0
(iv)	1	1

In each case we will consider some diagram

$$[\underline{\lambda}] := \{(i,j) \in [\lambda] \mid i \leq k(\lambda) \text{ and } j \geq k(\lambda) - x + 1\},$$

for a certain $1 \leq x \leq k(\lambda)$ (which will be chosen depending on the case). That is, $[\underline{\lambda}]$ is obtained from $[\lambda]$ by removing rows below row $k(\lambda)$ and columns up to column $k(\lambda) - x$. In an abuse of notation we will call $\underline{\lambda}$ the partition with Young diagram obtained by shifting the diagram $[\underline{\lambda}]$ to the left by $k(\lambda) - x$ columns. This will allow us to identify nodes of $\underline{\lambda}$ and nodes of λ (for example (i, λ_i), and not $(i, \lambda_i - k(\lambda) + x)$, will be the last node on row i of $\underline{\lambda}$ for $1 \leq i \leq k(\lambda)$).

In each case we denote $\underline{a}$ the number of nodes in $\mathrm{Rim}_p(\underline{\lambda})$, $\underline{r}$ the length of $\underline{\lambda}$ and

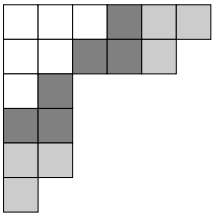
$$\underline{\varepsilon} = \begin{cases} 0 \text{ if } p \mid \underline{a}, \\ 1 \text{ if } p \nmid \underline{a}. \end{cases}$$

And for $\underline{\lambda}^{(1)}$, similar notation with primes: $\underline{a}'$ the number of nodes in $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$, $\underline{r}'$ the length of $\underline{\lambda}^{(1)}$ and

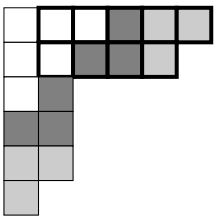
$$\underline{\varepsilon}' = \begin{cases} 0 \text{ if } p \mid \underline{a}', \\ 1 \text{ if } p \nmid \underline{a}'. \end{cases}$$

Let us now consider each of the four cases.

(i) In this case, the fact that both ε and ε' are zero means that neither λ nor $\lambda^{(1)*}$ contain diagonal nodes on their p-rims*. For example as in the partition $(6,5,2^3,1)$ with $p=3$. The Young diagram below on the left represents the 3-rims* of $(6,5,2^3,1)$ and $(6,5,2^3,1)^{(1)*}$ in different shades.



: $\mathrm{Rim}_3^*((6,5,2^3,1))$
: $\mathrm{Rim}_3^*((6,5,2^3,1)^{(1)*})$



Partition $\underline{(6,5,2^3,1)}$ in thicker lines.

Let $x=1$ in this case. The diagram above on the right illustrates $\underline{(6,5,2^3,1)}$. Lemma 3.2.7 ensures that $\underline{\lambda}$ is p-regular, then, from Proposition 2.2.6, we have

$$\underline{r}-\underline{r}'+\underline{\varepsilon}' \leq \underline{a}-\underline{a}' < p+\underline{r}-\underline{r}'+\underline{\varepsilon}'. \tag{3.2}$$

Notice that the nodes in $\mathrm{Rim}_p^*(\lambda)$ over the diagonal of λ are exactly the nodes of $\mathrm{Rim}_p(\underline{\lambda})$, that is, $U_\lambda = \mathrm{Rim}_p(\underline{\lambda})$. Hence $|U_\lambda| = |\mathrm{Rim}_p(\underline{\lambda})|$. That is $r=\underline{a}$. Similarly, $U_{\lambda^{(1)*}} = \mathrm{Rim}_p(\underline{\lambda}^{(1)})$, since $\varepsilon'=0$, meaning that node $(k(\lambda)-1,k(\lambda)-1) \notin \mathrm{Rim}_p^*(\lambda^{(1)*})$ so that this node is not in $\mathrm{Rim}_p(\underline{\lambda}^1)$, either. Hence $r'=\underline{a}'$.

We claim that $\mathrm{Rim}_p(\underline{\lambda})$ and $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$ end at the same row; row $k(\lambda)$. This is not obvious since it could be possible that the p-rim of a partition μ, which always contains nodes in the last row of μ, row $l(\mu)$, contains every node in this last row, and then $\mu^{(1)}$ does not have any nodes in row $l(\mu)$. But in our setting, this is not the case. Indeed, by definition, every node of a partition is in some i-th p-rim of the partition. In particular, the diagonal node $(k(\lambda),k(\lambda))$ is in the j-th p-rim of $\underline{\lambda}$ for some $j>1$ since $\mathrm{Rim}_p(\underline{\lambda})$ and $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$ do not contain diagonal nodes. On the other hand the p-rim of any partition contains nodes in the last row of the partition and since $k(\lambda)$ is the last row of both $\underline{\lambda}$ and $\underline{\lambda}^{(j)}$, then it is also the last row of $\underline{\lambda}^{(1)}$. So that both $\underline{\lambda}$ and $\underline{\lambda}^{(1)}$ contain nodes in row $k(\lambda)$.

Now, the fact that $\mathrm{Rim}_p(\underline{\lambda})$ and $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$ end at the same row means that $l(\underline{\lambda}) = k(\lambda) = l(\underline{\lambda}^{(1)})$, that is $\underline{r}-\underline{r}'=0$.

On the other hand, in this case, we have that $\varepsilon'=0$, which means that $p \mid a'$. But since $a'=2r'-\varepsilon'$, then $p \mid r' = \underline{a}'$ $(p \neq 2)$, which means that $\underline{\varepsilon}'=0$.

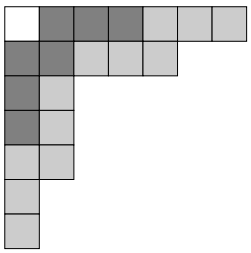
The fact that $\underline{r}-\underline{r}'=0$, together with the fact that $\underline{a}=r$, $\underline{a}'=r'$ and $\underline{\varepsilon}=0$, make Equation 3.2 become

$$0 \leq r-r' < p+0.$$

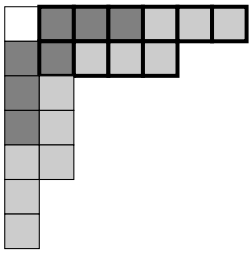
So that $\varepsilon \leq r-r' < p+\varepsilon$, as we wanted to show, since in this case $\varepsilon=0$.

(ii) Suppose that $\varepsilon=0$ and $\varepsilon'=1$. For example as in the partition $(7,5,2^3,1^2)$ with $p=3$. Below, in the left, we represent the 3-rims* of $(7,5,2^3,1^2)$ and

$(7,5,2^3,1^2)^{(1)*}$ in different shades.



: $\mathrm{Rim}_3^*((7,5,2^3,1^2))$
: $\mathrm{Rim}_3^*((7,5,2^3,1^2)^{(1)*})$



Partition $\underline{(7,5,2^3,1^2)}$ in thicker lines.

As in the previous case, let $x = 1$. We illustrate $\underline{(7,5,2^3,1^2)}$ by thicker lines above in the right hand diagram.

Let us see that also in this case we have that $\underline{r} - \underline{r}' = 0$. As before, the nodes in $\mathrm{Rim}_p^*(\lambda)$ over the diagonal of λ, or U_λ, are exactly the nodes of $\mathrm{Rim}_p(\underline{\lambda})$. And we also have that $U_{\lambda^{(1)*}} = \mathrm{Rim}_p(\underline{\lambda}^{(1)})$. So that $r = \underline{a}$ and $r' = \underline{a}'$. On the other hand, since in this case $(k(\lambda),k(\lambda)) \in \mathrm{Rim}_p^*(\lambda^{(1)*})$, then $(k(\lambda),k(\lambda)) \in \mathrm{Rim}_p(\underline{\lambda}^{(1)})$. Furthermore, the fact that $\mathrm{Rim}_p^*(\lambda^{(1)*})$ has a node on row $k(\lambda)$, implies that $\underline{\lambda}^{(1)}$ has a node on row $k(\lambda)$, and then so it is for $\underline{\lambda}$. Therefore $l(\underline{\lambda}) = k(\lambda) = l(\underline{\lambda}^{(1)})$, then $\underline{r} - \underline{r}' = 0$.

Now, consider the two possible cases for $\underline{\varepsilon}'$. If $\underline{\varepsilon}' = 0$, we obtain, as in the previous case

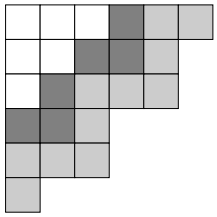
$$0 \leq r - r' < p + 0,$$

which is what we wanted to show. If $\underline{\varepsilon}' = 1$, Equation 3.2 becomes

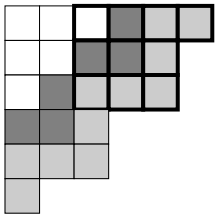
$$1 \leq r - r' < p + 1,$$

In particular $0 \leq r - r' \leq p$. But actually, $r - r' < p$. Indeed, if $r - r' = p$, since $p \mid a = 2r$, then $p \mid r$ and therefore $p \mid r' = \underline{a}'$, which contradicts the fact that $\underline{\varepsilon}' = 1$. In conclusion $0 \leq r - r' < p$, which ends this case.

(iii) Suppose that $\varepsilon = 1$ and $\varepsilon' = 0$. For example as in the partition $(6,5^2,3^2,1)$ with $p = 3$. Below, in the left, we represent the 3-rims* of $(6,5^2,3^2,1)$ and $(6,5^2,3^2,1)^{(1)*}$ in different shades.



: $\mathrm{Rim}_3^*((6,5^2,3^2,1))$
: $\mathrm{Rim}_3^*((6,5^2,3^2,1)^{(1)*})$



Partition $\underline{(6,5^2,3^2,1)}$ in thicker lines.

As before, let $x = 1$. We illustrate $\underline{(6, 5^2, 3^2, 1)}$ in thicker lines, above in the right hand diagram. Let us see that in this case $\underline{r} - \underline{r}' = 1$.

As in the preceding cases, the nodes in $\mathrm{Rim}^*_p(\lambda)$ over the diagonal of λ are exactly the nodes of $\mathrm{Rim}_p(\underline{\lambda})$. This fact implies that $\underline{a} = r$, and since $\varepsilon' = 0$, by the same argument that in case (i), we have that $\underline{a}' = r'$. Now, since $\varepsilon = 1$, the last diagonal node of λ, that is, the node $(k(\lambda), k(\lambda))$ is in $\mathrm{Rim}^*_p(\lambda)$. In particular $(k(\lambda), k(\lambda)) \in \mathrm{Rim}_p(\underline{\lambda})$, and since it is the first node of the last row of $\underline{\lambda}$, that means that all nodes on this last row are in the p-rim of $\underline{\lambda}$. So that this last row $k(\lambda) = \underline{r}$ of $\underline{\lambda}$ does not have any nodes from $\mathrm{Rim}^*_p(\lambda^{(1)*})$ (or $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$). We claim that row $k(\lambda) - 1$ in $\underline{\lambda}$ contains at least one node in $\mathrm{Rim}^*_p(\lambda^{(1)*})$ (or $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$). Indeed, node $(k(\lambda) - 1, k(\lambda) - 1)$ is in $\mathrm{Rim}^*_p(\lambda^{(j)*})$ for a $j > 1$, since it is not in $\mathrm{Rim}^*_p(\lambda^{(1)*})$ (because $\varepsilon' = 0$). If we suppose that row $k(\lambda) - 1$ does not have node in $\mathrm{Rim}^*_p(\lambda^{(1)*})$, we are supposing that to the left of node $(k(\lambda) - 1, k(\lambda) - 1)$ there are only nodes from $\mathrm{Rim}^*_p(\lambda)$. If this is the case, the last node in row $k(\lambda) - 1$ in $\lambda^{(1)*}$ is $(k(\lambda) - 1, k(\lambda) - 1)$, that is: $\lambda^{(1)*}_{k(\lambda)-1} = k(\lambda) - 1$. But the last node on every row (over or on the diagonal) belongs to the p-rim*. In this case, node $(k(\lambda) - 1, k(\lambda) - 1)$ belongs to the p-rim* of $\lambda^{(1)*}$, a contradiction since $\lambda^{(1)*}$ does not have any diagonal nodes on its p-rim*. In conclusion, row $k(\lambda) - 1$ in $\underline{\lambda}$ contains at least one node in $\mathrm{Rim}^*_p(\lambda^{(1)*})$, in particular, row $k(\lambda) - 1$ in $\underline{\lambda}$ contains at least one node in $\underline{\lambda}^{(1)}$, so that $l(\underline{\lambda}^{(1)}) = k(\lambda)$. Therefore $\underline{r} - \underline{r}' = k(\lambda) - l(\underline{\lambda}^{(1)}) = 1$.

On the other hand, since we have that $\varepsilon' = 0$, by the same argument as in case (i), we have that $\underline{\varepsilon}' = 0$.

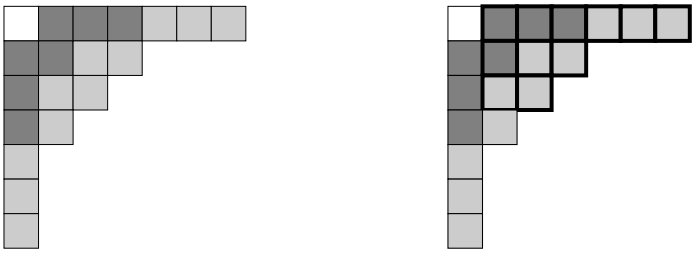
Puting all together in Equation 3.2, we obtain

$$1 \leq r - r' < p + 1.$$

That is, $\varepsilon \leq r - r' < p + \varepsilon$.

(iv) Suppose finally that $\varepsilon = \varepsilon' = 1$. An example is given by partition $(7, 4, 3, 2, 1^3)$ for $p = 3$. Below, in the left, we represent the 3-rims* of $(7, 4, 3, 2, 1^3)$ and $(7, 4, 3, 2, 1^3)^{(1)*}$ in different shades.

Let $x = 2$. Lemma 3.2.7 still assures that $\underline{\lambda}$ is p-regular. And from the way that it is defined, $\underline{\lambda}$ contains both diagonal nodes in $\mathrm{Rim}^*_p(\lambda)$ and $\mathrm{Rim}^*_p(\lambda^{(1)*})$. We illustrate $\underline{(7, 4, 3, 2, 1^3)}$ by thicker lines in the right hand diagram below.



: $\mathrm{Rim}^*_3((7,4,3,2,1^3))$

: $\mathrm{Rim}^*_3((7,4,3,2,1^3)^{(1)*})$

Partition $\underline{(7,4,3,2,1^3)}$ in thicker lines.

Notice that in this case it is not necessarily true that $\underline{a} = r$ and $\underline{a}' = r'$. Since $\underline{\lambda}$ contains the node $(k(\lambda), k(\lambda)-1)$ which is under the diagonal of λ, where the p-rim* does not behave as the p-rim. For the partition $(7,4,3,2,1^3)$, this node is the node $(3,2)$, which in this case is in the p-rim* of $(7,4,3,2,1^3)$. But it could be the case that the node $(k(\lambda), k(\lambda)-1)$ is not in the p-rim* of λ but in the p-rim* of $\lambda^{(1)*}$. This depends on the divisibility of r by p.

Recall that $r = |U_\lambda|$ is the number of nodes in the p-rim* of λ that are above (or on) the diagonal of λ. Let us consider the two cases: $p \mid r$ and $p \nmid r$.

- Suppose that $p \mid r$. As in $\lambda = (7,4,3,2,1^3)$ with $p = 3$ (see the diagram above). In this case every p-segment of the p-rim* of λ contains exactly p nodes. In particular the segment which contains the node $(k(\lambda), k(\lambda))$. And since this node is the last (and p-th) node of this p-segment, then the node to its left $(k(\lambda), k(\lambda)-1)$ is not in the p-rim of $\underline{\lambda}$. And we have $\underline{a} = r$ and $\underline{r} - \underline{r}' = 0$. Moreover, the node $(k(\lambda), k(\lambda)-1)$ is then in the p-rim of $\underline{\lambda}^{(1)}$. So that $\underline{a}' = r' + 1$ (the r' nodes of $\mathrm{Rim}_p^*(\lambda^{(1)*})$ above the diagonal, together with node $(k(\lambda), k(\lambda)-1)$, form $\mathrm{Rim}_p(\underline{\lambda}^{(1)})$). Therefore we have $\underline{a} = r$, $\underline{a}' = r' + 1$ and $\underline{r} - \underline{r}' = 0$. Puting this together in Equation 3.2 we get

$$\underline{\varepsilon}' \leq r - (r'+1) < p + \underline{\varepsilon}',$$

or

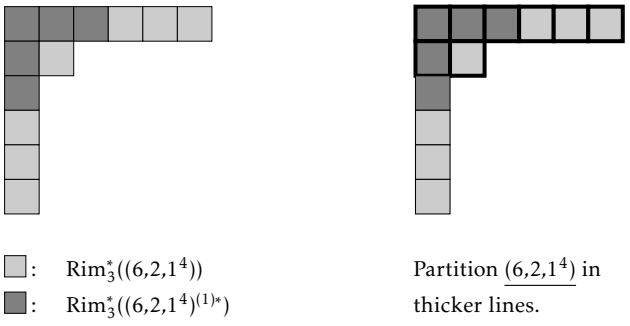
$$\underline{\varepsilon}' + 1 \leq r - r' < p + \underline{\varepsilon}' + 1.$$

But $\underline{\varepsilon}' + 1 \geq 1 = \varepsilon$. Therefore we have

$$1 \leq r - r' < p + \underline{\varepsilon}' + 1.$$

Let us see that $r - r' < p + 1 = p + \varepsilon$. There are two possibilities for $\underline{\varepsilon}'$. Either $\underline{\varepsilon}' = 0$, in which case $r - r' < p + 1$, or $\underline{\varepsilon}' = 1$. If $\underline{\varepsilon}' = 1$, we have $r - r' < p + 2$, so that $r - r' \leq p + 1$. But in fact $r - r' < p + 1$, since if $r - r' = p + 1$, then $r - (r' + 1) = p$. But in this case $p \mid r$, therefore, $p \mid r' + 1 = \underline{a}'$, and this contradicts the fact that $\underline{\varepsilon}' = 1$.

- Suppose that $p \nmid r$. As in $\lambda = (6,2,1^4)$ with $p = 3$.



In this case, the p-segment of $\mathrm{Rim}_p^*(\lambda)$ which contains the node $(k(\lambda), k(\lambda))$ has less than p nodes. This implies that the node to the left of this diagonal node, namely $(k(\lambda), k(\lambda)-1)$, is on the p-rim of $\underline{\lambda}$. Then $\underline{a} = r + 1$ (the r nodes of $\mathrm{Rim}_p^*(\lambda)$ above the diagonal, together with node $(k(\lambda), k(\lambda)-1)$,

form $\mathrm{Rim}_p(\underline{\lambda})$), and we also have that $\underline{r} - \underline{r}' = 1$ and $\underline{a}' = r'$. Equation 3.2 gives

$$1 + \underline{\varepsilon}' \leq (r+1) - r' < p + 1 + \underline{\varepsilon}',$$

or

$$\underline{\varepsilon}' \leq r - r' < p + \underline{\varepsilon}'.$$

But $p + \underline{\varepsilon}' \leq p + 1 = p + \varepsilon$. Then $\underline{\varepsilon}' \leq r - r' < p + \varepsilon$. Let us show that $r - r' \geq 1 = \varepsilon$. There are two possibilities for $\underline{\varepsilon}'$. Either $\underline{\varepsilon}' = 1$, in which case $r - r' \geq 1 = \varepsilon$ or $\underline{\varepsilon}' = 0$. If $\underline{\varepsilon}' = 0$, then $r - r' \geq 0$. But actually $r - r' \geq 1$, since if $r - r' = 0$, from the fact that $p \mid \underline{a}' = r'$ we would have that $p \mid r$, a contradiction.

□

3.3 A proposition on the correspondence and p-blocks

The BG-symbol provides a one-to-one correspondence between the set $\mathrm{BG}_p^n \subseteq \mathrm{Par}(n)$ of p-BG-partitions of n and the set $\mathrm{M}_p^n \subseteq \mathrm{Reg}_p(n)$ of p-self-Mullineux partitions of n. We know that p-blocks of $\mathfrak{S}_n$ define a (set) partition in each of $\mathrm{Par}(n)$ and $\mathrm{Reg}_p(n)$. A natural question is then: does the correspondence agree with the partition into p-blocks ? Since this partition is determined by the p-cores (Theorem 1.4.5), the question says: if $\lambda \in \mathrm{BG}_p^n$ and $\mu \in \mathrm{M}_p^n$ correspond with each other, that is, the BG-symbol of λ is equal to the Mullineux symbol of μ, do λ and μ have the same p-core ? In this section we show that this is the case.

In §2.4 we recalled the definition of the p-residue of a node in $\mathbb{N}^* \times \mathbb{N}^*$. The *$p$-content* $\mathrm{cont}_p(\lambda)$ of a partition λ is the multi-set of p-residues of the nodes in the Young diagram of λ.

Example 3.3.1. If $p = 3$ and $\lambda = (5, 2^2, 1)$, the p-residues of the nodes in the Young diagram of λ are:

0	1	2	0	1
2	0			
1	2			
0				

Then the p-content of λ is the multi-set

$$\mathrm{cont}_3(\lambda) = \{0^4, 1^3, 2^3\},$$

where a power indicates the number of times that an element appears in the multi-set. △

The p-content determines the p-block to which a partition belongs:

Proposition 3.3.2 ([JK81, 2.7.41])**.** *Two partitions λ and μ have the same p-core if and only if they have the same p-content.*

Proposition 3.3.3. *Let $\lambda \in BG_p^n$ and $\mu \in M_p^n$ such that the BG-symbol of λ is equal to the Mullineux symbol of μ. Then λ and μ have the same p-core.*

Proof. The proof is by induction on the length of the symbol.

Suppose that

$$G_p(\mu) = \binom{a}{r} = \mathrm{bg}_p(\lambda)$$

In this case we have that $\lambda = \mu$. Indeed, the unique p-BG-partition with such a bg-symbol is $(r, 1^{r-1})$ (see proof of Proposition 3.1.14). Similarly, the unique p-regular partition with such a Mullineux symbol is $(r, 1^{r-1})$. Since $\lambda = \mu$, they have the same p-core.

Suppose now that

$$G_p(\mu) = \begin{pmatrix} a & a_1 & \cdots & a_z \\ r & r_1 & \cdots & r_z \end{pmatrix} = \mathrm{bg}_p(\lambda).$$

We prove that $\mathrm{cont}_p(\lambda) = \mathrm{cont}_p(\mu)$, and from Proposition 3.3.2 the result follows. From the definition of the symbols, we know that

$$G_p(\mu^{(1)}) = \begin{pmatrix} a_1 & \cdots & a_z \\ r_1 & \cdots & r_z \end{pmatrix} = \mathrm{bg}_p(\lambda^{(1)*}),$$

where the partitions $\lambda^{(1)*}$ and $\mu^{(1)}$ are obtained from λ and μ by respectively removing the p-rim* and p-rim. By induction, $\lambda^{(1)}$ and $\mu^{(1)*}$ have the same p-core. From Proposition 3.3.3, these two partitions have the same multi-set of p-residues:

$$\mathrm{cont}_p(\lambda^{(1)*}) = \mathrm{cont}_p(\mu^{(1)}).$$

Let L and M be the multi-sets with elements from $\mathbb{Z}/p\mathbb{Z}$ such that

$$\mathrm{cont}_p(\mu) = \mathrm{cont}_p(\lambda^{(1)*}) \cup M$$

and

$$\mathrm{cont}_p(\lambda) = \mathrm{cont}_p(\mu^{(1)}) \cup L.$$

Let us see that $M = L$, which will complete the proof.

Notice first the following about p-residues and segments on the rim. If (r, c) is a node in $\mathbb{N}^* \times \mathbb{N}^*$, the p-residue of this node is $(c - r) \mod p$. The residue of the node **above** or to the **right** of (r, c) is $((c-r)+1) \mod p$. The residue of the node **below** or to the **left** of (r, c) is $((c-r)-1) \mod p$. That is to say: the set of p-residues of any path or segment of nodes is a list of residues which are consecutive mod p. In particular, the p-residues of a segment in the p-rim of a partition is a subset of consecutive residues mod p.

Now, it is clear that $|M| = a = |L|$, from the definition of the symbols and of the partitions $\lambda^{(1)*}$ and $\mu^{(1)}$. This number is the length of the p-rim of μ and the length of the p-rim* of λ. Let q and s be the positive integers such that

$$a = pq + s \qquad \text{and} \quad 0 \leq s < p.$$

If $s = 0$, then $p \mid a$. Hence μ is obtained from $\mu^{(1)}$ by adding q segments of length p to form the p-rim of μ. Then

$$M = \{0^q, 1^q, \ldots, (p-1)^q\}.$$

Similarly, λ is obtained from $\lambda^{(1)*}$ by adding q segments of length p to form the p-rim* of λ. Then

$$L = \{0^q, 1^q, \ldots, (p-1)^q\}.$$

Hence, $M = L$ when $s = 0$.

If $s > 0$, $p \nmid a$ so that $\varepsilon_0 = 1 = \varepsilon^*_\lambda$. Recall that for such a symbol we have that $a = 2r-1$, see Definition 3.1.4. Then

$$r = \frac{a+1}{2}. \tag{3.3}$$

Let us see in this case what is the multi-set M. The partition μ is obtained from $\mu^{(1)}$ by adding q segments of length p to form the p-rim of μ, plus one segment of length s. This last segment can be seen as a segment starting at node $(r, 1)$ and then it is a sequence of nodes that are either above or to the right of the preceding node. From the above remark, the set of p-residues of nodes in this last segment is exactly

$$(1-r) \bmod p,\ (2-r) \bmod p,\ \ldots,(s-r) \bmod p.$$

Hence,

$$M = \{0^q, 1^q, \ldots, (p-1)^q\} \cup \{(1-r) \bmod p,\ (2-r) \bmod p,\ \ldots,(s-r) \bmod p\}.$$

We now compute the multi-set L. For this we consider the two cases: q is even or q is odd.

Suppose that q is even. Then s is odd and from Equation 3.3 we have

$$r = p\frac{q}{2} + \frac{s+1}{2},$$

and $\frac{s+1}{2} < s < p$ so that $\frac{s+1}{2} = r \bmod p$. That means that the self-conjugate partition λ has $\frac{q}{2}$ p-segments on the p-rim* over the diagonal (and also under the diagonal), and a "middle" (symmetric with respect to the diagonal) segment on the p-rim* which has exactly $\frac{s+1}{2}$ nodes over the diagonal, one of which is **on** the diagonal. The multi-set of residues of the p-segments is then formed by

$$0^q,\ 1^q,\ \ldots,\ (p-1)^q.$$

Now, for the middle segment: A node on the diagonal has p-residue equal to 0 by definition. Then the p-residues of the nodes in the middle segment are exactly

$$\begin{aligned}\frac{1-s}{2} \bmod p,\ \ldots,-2 \bmod p,\ -1 \bmod p,\ 0 \bmod p,\\ 1 \bmod p,\ 2 \bmod p,\ \ldots,\ \frac{s-1}{2} \bmod p,\end{aligned} \tag{3.4}$$

but $\frac{1-s}{2} = 1 - \frac{s+1}{2} \equiv 1 - r \pmod p$, so that the list (3.4) is a set of s consecutive residues mod p starting at $1 - r \bmod p$, that is

$$(1-r) \bmod p,\ (2-r) \bmod p,\ \ldots,(s-r) \bmod p,$$

which proves that $L = M$, when q is even.

Suppose that q is odd, say $q = 2q' + 1$. Then $a = 2pq' + p + s$ so that the p-rim* of the self-conjugate partition λ is formed of: $2q'$ segments on the rim of length p (q'

segments at each side of the diagonal), and one middle (symmetric) segment of length $p+s$. The p-residues of the $2q'$ segments are

$$0^{2q'},\ 1^{2q'},\ \ldots,\ (p-1)^{2q'}. \tag{3.5}$$

The $p+s$ p-residues of the middle segment are

$$\begin{aligned}\frac{1-p-s}{2} \bmod p,\ \ldots,-2 \bmod p,\ -1 \bmod p,\ 0 \bmod p,\\ 1 \bmod p,\ 2 \bmod p,\ \ldots,\ \frac{p+s-1}{2} \bmod p.\end{aligned} \tag{3.6}$$

Take the first p of these p-residues:

$$\frac{1-p-s}{2} \bmod p,\ \frac{3-p-s}{2} \bmod p,\ldots,\ \frac{1-p-s}{2}+(p-1) \bmod p.$$

which form a complete set of residues mod p. They form, together with the residues on 3.5, the multi-set

$$0^q,\ 1^q,\ \ldots,\ (p-1)^q.$$

The rest of residues in the list (3.6) is the set of s p-residues

$$\frac{1-p-s}{2} \bmod p,\ \frac{3-p-s}{2} \bmod p,\ \ldots,\ \frac{-1-p-s}{2}+(s-1) \bmod p. \tag{3.7}$$

Notice that $\frac{p+s+1}{2} \equiv r \bmod p$. Indeed, $r = \frac{a+1}{2} = pq' + \frac{p+s+1}{2}$. Then, the first p-residue in the list (3.7) is

$$\frac{1-p-s}{2} = 1 - \frac{p+s+1}{2} \equiv 1-r \bmod p.$$

This shows that the list (3.7) is equal to

$$(1-r) \bmod p,\ (2-r) \bmod p,\ \ldots,(s-r) \bmod p,$$

proving that $L = M$. □

3.4 From self-Mullineux partitions to BG-partitions

From Theorem 4.6.1, we know that the BG-symbol induces an algorithm for the correspondence between BG-partitions and self-Mullineux partitions. Then we know that to each Mullineux symbol of a self-Mullineux partition, corresponds a unique BG-partition. Moreover, from the definition of the BG-symbol, and Lemma 3.1.12, we know how to find the BG-partition associated to such a Mullineux symbol under this correspondence. In this section we prove that this inverse algorithm is well defined, that is, we prove that applying it to a Mullineux symbol of a self-Mullineux partition results in a BG-partition. This confirms Theorem 4.6.1 in a combinatorial way without using the fact that $|\mathrm{BG}_p^n| = |M_p^n|$.

Proposition 3.4.1. *Let p be an odd prime and λ a self-Mullineux partition. The Mullineux symbol of λ, $G_p(\lambda)$ is the BG-symbol of some BG-partition. That is*

$$\mathcal{M}_p \subseteq \mathrm{bg}_p(\mathrm{BG}_p).$$

Proof. We give a combinatorial proof of this fact, although it follows also directly from Proposition 3.1.14 and Proposition 3.2.1.

We proceed by induction on l, the length of the Mullineux symbol.

Let $l = 0$ and $S = \begin{pmatrix} a_l \\ r_l \end{pmatrix} \in \mathcal{M}_p$, that is, $S = G_p(\lambda)$ for some $\lambda \in M_p$. Let $\varepsilon_l = 0$ if $p \mid a_l$ and $\varepsilon_l = 1$ otherwise. Since S has exactly one column, then $\lambda = \lambda^{(l)}$ is a hook, that is λ is of the form $\lambda = (x, 1^y)$, with $x \leq p$. On the other hand, since λ is fixed by the Mullineux map, we know that

$$a_l = 2r_l - \varepsilon_l.$$

We claim that $\varepsilon_l = 1$. If $\varepsilon_l = 0$, that is, if $p \mid a_l$, then the p-segments that form $\mathrm{Rim}_p(\lambda) = [\lambda]$ are all of length exactly p. We know that λ is a p-regular hook, this means that λ is formed by exactly one p-segment. If there was more than one p-segment, then $a_l > p$ (so that $a_l = 2kp$ for some $k \geq 1$) it follows that $y \geq p$, and then λ would not be p-regular. Thus $a_l = p = 2r_l$. But this is not possible since p is odd. Then $\varepsilon_l = 1$ and $a_l = 2r_l - 1$.

The partition $\mu = (r_l, 1^{r_l - 1})$ is self-conjugate, and is a hook of length $2r_l - 1 = a_l$. Since $p \nmid a_l$, then $\mu \in \mathrm{BG}_p$. Its BG-symbol is

$$\mathrm{bg}_p(\mu) = \begin{pmatrix} 2r_l - 1 \\ r_l \end{pmatrix} = \begin{pmatrix} a_l \\ r_l \end{pmatrix} = S.$$

In fact $\mu = \lambda$.

Consider now $l > 0$. Let

$$S = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix}$$

be a symbol in $\mathcal{M}_p$ corresponding to a partition λ in M_p. Consider the array

$$\tilde{S} = \begin{pmatrix} a_1 & \cdots & a_l \\ r_1 & \cdots & r_l \end{pmatrix}.$$

By definition, $\tilde{S}$ is the Mullineux symbol of the partition $\lambda^{(1)}$, obtained from λ by removing the nodes on the p-rim. We know that $\lambda^{(1)}$ is fixed by the Mullineux map, given that this only depends on the columns of the symbol. Then $\lambda^{(1)} \in M_p$, and $\tilde{S} \in \mathcal{M}_p$. By induction, there exists a partition $\tilde{\mu} \in \mathrm{BG}_p$ such that

$$\mathrm{bg}_p(\tilde{\mu}) = \tilde{S}.$$

We will apply Lemma 3.1.12. Let $\varepsilon_0 = 0$ if $p \mid a_0$, or $\varepsilon_0 = 1$ otherwise. Let $m = (r_0 - \varepsilon_0) \bmod p$.

Suppose that $\varepsilon_0 = 0$ and let us see that in this case $m = 0$. Since $\varepsilon_0 = 0$, then $p \mid a_0$. But $a_0 = 2r_0 - \varepsilon_0 = 2r_0$, since λ is a fixed point of the Mullineux map. Now, since p is odd, then $p \mid r_0$ so that $m = (r_0 - \varepsilon_0) \bmod p = r_0 \bmod p = 0$.

If $\varepsilon_0 = 1$, we have that $p \nmid a_0$. Therefore $p \nmid 2m+1$ since $2m+1 \equiv 2(r_0 - \varepsilon_0) + 1 \pmod p$ and $2(r_0 - \varepsilon_0) + 1 = 2r_0 - \varepsilon_0 = a_0$.

Lemma 3.1.12 implies that there exists a unique self-conjugate partition $\mu \in \mathrm{BG}_p$ such that

(i) $a^*_\mu \equiv \varepsilon_0 \pmod 2$;

(ii) $r^*_\mu - \varepsilon^*_\mu \equiv m \pmod p$ and

(iii) $\mu^{(1)*} = \tilde{\mu}$.

The condition $\mu^{(1)*} = \tilde{\mu}$ implies that

$$\mathrm{bg}_p(\mu) = \begin{pmatrix} a^*_\mu & \\ & \\ r^*_\mu & \end{pmatrix} \mathrm{bg}_p(\tilde{\mu}) \Bigg) = \begin{pmatrix} a^*_\mu & \\ & \\ r^*_\mu & \end{pmatrix} \tilde{S} \Bigg) = \begin{pmatrix} a^*_\mu & a_1 & \cdots & a_l \\ r^*_\mu & r_1 & \cdots & r_l \end{pmatrix}.$$

Let us see that in fact $\mathrm{bg}_p(\mu) = S$, that is, $a^*_\mu = a_0$ and $r^*_\mu = r_0$. Indeed, from (i), a^*_μ is even if and only if $\varepsilon_0 = 0$, if and only if $p \mid a_\lambda$. But a^*_μ is even if and only if $p \mid a^*_\mu$, by Corollary 3.2.10. This sequence of equivalences says that $\varepsilon_0 = \varepsilon^*_\mu$. Then, by (ii) we have that $r^*_\mu \equiv r_0 \pmod p$.

Since $S \in \mathcal{M}_p$, then, from Proposition 2.2.6, we have, in particular

$$\varepsilon_0 \leq r_0 - r_1 < p + \varepsilon_0. \tag{3.8}$$

On the other hand, since $\mu \in \mathrm{BG}_p$, then $\mathrm{bg}_p(\mu) \in \mathcal{M}_p$, by Proposition 3.2.1, so that, in particular we have

$$\varepsilon^*_\mu \leq r^*_\mu - r_1 < p + \varepsilon^*_\mu. \tag{3.9}$$

Substracting Equation 3.8 from Equation 3.9, we get

$$-p < r^*_\mu - r_0 < p,$$

but since $p \mid r^*_\mu - r_0$ we can conclude that $r^*_\mu - r_0 = 0$, so that $r^*_\mu = r_0$. Therefore $a^*_\mu = 2r^*_\mu - \varepsilon^*_\mu = 2r_0 - \varepsilon_0 = a_0$, and

$$\mathrm{bg}_p(\mu) = \begin{pmatrix} a_0 & a_1 & \cdots & a_l \\ r_0 & r_1 & \cdots & r_l \end{pmatrix} = S.$$

□

4 Stable unitriangular basic sets for blocks of small weights

This chapter is based on [Ber21c]. The organization of this chapter is as follows: In the first section we recall the notions of p-basic sets and p-unitriangular basic sets (UBS) for the symmetric group. Then, in §4.2 we recall the more general notion of p-unitriangular basic sets for unions of p-blocks of $\mathfrak{S}_n$ and we define stable unitriangular basic sets for unions of p-blocks (SUBS). In §4.3 we recall the fact that non self-conjugate blocks or self-conjugate blocks of odd weight have a SUBS with certain underlying set denoted $U_{n,p}^{\gamma}$. Then, in §4.4, we show that the Mullineux map has an easy description for blocks of weight 1, allowing us to show that $U_{n,p}^{\gamma}$ is the underlying set for a different SUBS. The reason for such "parentheses" about weight 1 blocks is that, then in §4.5, in the study of the combinatorics of weight 2 partitions we see that the behavior of the Mullineux map on weight 1 blocks is very similar to the Mullineux map in certain subsets of weight 2 blocks. This allows us to find a SUBS for self-conjugate blocks of weight 2, in §4.6.

4.1 Basic sets and unitriangular basic sets

In this section we recall the definition of a p-basic set and a p-unitriangular basic set for the symmetric group. Definitions are based on [BG10, BGJ20].

For introducing basic sets for $\mathfrak{S}_n$, it is useful to talk in terms of the so-called Grothendieck group of $\mathfrak{S}_n$. Let $2 < p < n$ be a prime. Denote by $\mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n)$ the set of irreducible $\mathbb{F}_p\mathfrak{S}_n$-modules up to equivalence. The *Grothendieck group* of $\mathbb{F}_p\mathfrak{S}_n$ is the $\mathbb{Z}$-module $\mathcal{G}_p(n)$ generated by the symbols $[M]$, for M a finitely generated $\mathbb{F}_p\mathfrak{S}_n$-module together with the relations

$$[M] = \sum_{D \in \mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n)} m_D[D],$$

where m_D is the composition multiplicity of D in M, and $\mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n)$ is the set of p-modular irreducibles of $\mathfrak{S}_n$ up to equivalence. From Theorem 1.3.6 and the Jordan-Hölder theorem we have

Proposition 4.1.1. *The set* $\{[D^\lambda] \mid \lambda \in \mathrm{Reg}_p(n)\}$ *is a* $\mathbb{Z}$*-basis of* $\mathcal{G}_p(n)$.

In the Grothendieck group, we can then rewrite the definition (§1.4.1) of the p-

decomposition numbers $d_{\lambda\mu}$: For every $\lambda \vdash n$

$$[S^\lambda] = \sum_{\mu \in \mathrm{Reg}_p(n)} d_{\lambda\mu}[D^\mu].$$

Moreover, from Theorem 1.4.1, the system of equations formed by all equations as above for λ p-regular, has a unique solution, that is:

Proposition 4.1.2. *The set* $\{[S^\lambda] \mid \lambda \in \mathrm{Reg}_p(n)\}$ *is a* $\mathbb{Z}$*-basis of* $\mathcal{G}_p(n)$.

This motivates the general definition of a subset of Specht modules having such property:

Definition 4.1.3. A *p-basic set* (or basic set) for the symmetric group $\mathfrak{S}_n$ is a subset $B \subset \{[S^\lambda] \mid \lambda \vdash n\}$ such that B is a $\mathbb{Z}$-basis of $\mathcal{G}_p(n)$. △

In these terms, $\{S^\lambda \mid \lambda \vdash n,\ p\text{-regular}\}$ is a p-basic set for $\mathfrak{S}_n$. By abuse of notation we consider the set of partitions $\{\lambda \mid \lambda \in \mathrm{Reg}_p(n)\} = \mathrm{Reg}_p(n)$ instead of the set of Specht modules, as the basic set. With this notations, Proposition 4.1.1 says that, as $\mathbb{Z}$-modules

$$\mathcal{G}_p(n) \simeq \mathbb{Z}\mathrm{Reg}_p(n).$$

Remark 4.1.4. B is a p-basic set of $\mathfrak{S}_n$ if and only if the determinant of the restriction matrix $\mathbf{D}^B_{n,p}$ is 1 or -1 (the matrix is invertible in $\mathbb{Z}$).

The p-basic set $\mathrm{Reg}_p(n)$ has an additional property. The system of equations is easy to solve, since the submatrix is unitriangular (see Theorem 1.4.1). For $\lambda \in \mathrm{Reg}_p(n)$ we have:

$$[S^\lambda] = [D^\lambda] + \sum_{\substack{\mu \in \mathrm{Reg}_p(n) \\ \mu \rhd \lambda}} d_{\lambda\mu}[D^\mu].$$

Hence, the p-basic set $\mathrm{Reg}_p(n)$ not only is a basis for $\mathcal{G}_p(n)$, but it **indexes naturally** the irreducible $\mathbb{F}_p\mathfrak{S}_n$-modules. This motivates the following general definition.

Definition 4.1.5. A *p-unitriangular basic set* (UBS) (or *unitriangular basic set*) for $\mathbb{F}_p\mathfrak{S}_n$ is a set $U \subset \mathrm{Par}(n)$ such that the matrix $\mathbf{D}^U_{n,p}$ (formed by the rows of $\mathbf{D}_{p,n}$ labelled by U and all the columns) has wedge shape. More precisely, a UBS is the datum of a triplet $(U, \leq, \Psi)$ where $U \subset \mathrm{Par}(n)$, $\leq$ is a total order defined on $\mathrm{Par}(n)$ and Ψ is a bijective map:

$$\Psi : U \longrightarrow \mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n),$$

satisfying:

(1) for all $\lambda \in U$ we have $d_{\lambda,\Psi(\lambda)} = 1$, and

(2) for all $M \in \mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n)$ and $\lambda \in U$, we have: if $d_{\lambda,M} \neq 0$ then $\lambda \leq \Psi^{-1}(M)$.

△

Since we are interested in particular in the labelling of simple modules, for this purpose, it is sufficient for the order $\leq$ to be defined on U.

A UBS for $\mathbb{F}_p\mathfrak{S}_n$ is in particular a p-basic set for $\mathbb{F}_p\mathfrak{S}_n$. The interest of UBSs is then that they give a natural parametrization of the irreducible $\mathbb{F}_p\mathfrak{S}_n$-modules. This parametrization is explicit in the sense that there is an explicit bijection between the irreducible $\mathbb{F}_p\mathfrak{S}_n$-modules and the modules in the UBS (which is Ψ).

Example 4.1.6. The set $\mathrm{Reg}_p(n) \subset \mathrm{Par}(n)$, together with the lexicographic order $\leq$ on $\mathrm{Par}(n)$ and the bijection: $\lambda \mapsto D^\lambda$, is a UBS for $\mathbb{F}_p\mathfrak{S}_n$. △

The notion of p-basic set is defined more generally for finite groups, or even symmetric algebras. Such sets are useful for computing p-decomposition matrices. It is an open question whether a p-basic set for a group exists in general[1]. In [BG10], Brunat and Gramain showed that the alternating group $\mathfrak{A}_n$ has a p-basic set. Their strategy is to find a p-basic set $\mathcal{B}_\emptyset$ for $\mathfrak{S}_n$ which satisfies certain properties that make its "restriction to $\mathfrak{A}_n$" be a p-basic set for $\mathfrak{A}_n$. The basic set $\mathcal{B}_\emptyset$ has two properties:

(A) $\mathcal{B}_\emptyset$ is **stable** for conjugation: If $\lambda \in \mathcal{B}_\emptyset$, then $\lambda' \in \mathcal{B}_\emptyset$.

(B) The only self-conjugate partitions in $\mathcal{B}_\emptyset$ are p-BG-partitions: If $\lambda = \lambda' \in \mathcal{B}_\emptyset$, then $\lambda \in \mathrm{BG}_p^n$.

Brunat and Gramain showed that if $\mathcal{B}$ is a UBS for $\mathbb{F}_p\mathfrak{S}_n$ which satisfies the properties (A) and (B), then a UBS for $\mathfrak{A}_n$ can be constructed by restriction [BG10, Lemmas 6.4 and 6.6]. Having a UBS for $\mathfrak{A}_n$ is interesting because it would provide a natural labelling of the irreducible $\mathbb{F}_p\mathfrak{A}_n$-modules. The p-basic set $\mathcal{B}_\emptyset$ does not help in this task, since it is not a UBS for $\mathbb{F}_p\mathfrak{S}_n$ (see [BG10, Remark 6.5]). They conjectured the existence of such a p-basic set for $\mathbb{F}_p\mathfrak{S}_n$, which partly motivated the results in Chapter 3. However, in [BGJ20, §3.2], Brunat, Gramain and Jacon showed that $\mathbb{F}_p\mathfrak{A}_n$ does not always have a UBS (in particular $\mathbb{F}_3\mathfrak{A}_{18}$ and $\mathbb{F}_3\mathfrak{A}_{19}$), proving that the asked p-basic set for $\mathbb{F}_p\mathfrak{S}_n$ does not always exist. Now, the question can be asked "block wise", that is, with notions of p-basic sets and UBSs for p-blocks of $\mathbb{F}_p\mathfrak{S}_n$. This makes the subject of the following sections.

4.2 UBS and SUBS for p-blocks of the symmetric group

We recall the notions of p-basic sets and UBS for p-blocks of $\mathfrak{S}_n$.

Definition 4.2.1. Let $U = \bigcup_{\gamma\in\Gamma} \mathfrak{B}_\gamma$ be a union of p-blocks of $\mathbb{F}_p\mathfrak{S}_n$. Denote by $\mathrm{Irr}(U)$ the set $\bigcup_{\gamma\in\Gamma}\{[D^\lambda] \mid \lambda \in \mathrm{Irr}(\mathfrak{B}_\gamma)\}$. A subset $B \subset U$ is a *p-basic set for U* if B is a basis for the $\mathbb{Z}$-module generated by $\mathrm{Irr}(U)$.

Equivalently: the matrix formed from the rows of $\mathbf{D}_{n,p}$ labelled by B and columns labelled by $\mathrm{Irr}(U)$ is invertible in $\mathbb{Z}$.

If $U = \mathfrak{B}_\gamma$ is formed by only one block then we say that B is a *basic set for the block* $\mathfrak{B}_\gamma$. △

[1] The answer is known to be positive for p-soluble groups, for finite groups of Lie type in the non-defining characteristics under some additional hypotheses, for the finite general linear groups $\mathrm{GL}_2(q)$, $\mathrm{GL}_3(q)$ and $\mathrm{GL}_4(q)$ in the defining characteristic, and also for the symmetric group $\mathfrak{S}_n$. See [BG10, §1]

A p-basic set for $\mathbb{F}_p\mathfrak{S}_n$ is a p-basic set for the union of all the p-blocks of $\mathbb{F}_p\mathfrak{S}_n$.

Definition 4.2.2. Let $\Gamma = \{\gamma \mid \gamma \vdash (n - wp) \text{ is a } p\text{-core}\}$ be some set of p-cores. Let $U = \bigcup_{\gamma\in\Gamma} \mathfrak{B}_\gamma$ be the union of the corresponding blocks of $\mathbb{F}_p\mathfrak{S}_n$. A *$p$-unitriangular basic set (UBS)* (or *unitriangular basic set*) *for* U is a set $B \subset U$ such that the matrix formed by the rows of $\mathbf{D}_{n,p}$ labelled by B and columns labelled by $\mathrm{Irr}(U)$ has wedge shape. Formally, a UBS for U is the datum of a triplet $(B\ ,\ \leq\ ,\ \Psi)$ where $B \subseteq U$, $\leq$ is a total order defined on U and Ψ is a bijection

$$\Psi : B \longrightarrow \mathrm{Irr}(U)$$

satisfying:

(1) for all $\mu \in B$, we have $d_{\mu\Psi(\mu)} = 1$, and

(2) for all $D \in \mathrm{Irr}(U)$ and $\lambda \in U$, if $d_{\lambda D} \neq 0$ then $\lambda \leq \Psi^{-1}(D)$.

Equivalently: the matrix formed from the rows of $\mathbf{D}_{n,p}$ labelled by B and columns labelled by $\mathrm{Irr}(U)$ has wedge shape.

If $U = \mathfrak{B}_\gamma$ is formed by only one block then we say that B is a *UBS for the block* $\mathfrak{B}_\gamma$. △

A UBS for $\mathbb{F}_p\mathfrak{S}_n$ is a UBS for the union of all the p-blocks of $\mathbb{F}_p\mathfrak{S}_n$.

Example 4.2.3. Any UBS $(U, \leq, \Psi)$ for $\mathbb{F}_p\mathfrak{S}_n$ restricts to a UBS $(U_\gamma, \leq_\gamma, \Psi_\gamma)$ for any p-block $\mathfrak{B}_\gamma$ of $\mathbb{F}_p\mathfrak{S}_n$. For example:

Denote $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$ the following subset of $\mathfrak{B}_\gamma$:

$$\mathrm{Reg}_p(\mathfrak{B}_\gamma) := \mathrm{Reg}_p(n) \cap \mathfrak{B}_\gamma.$$

Then $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$ together with the lexicographic order of partitions on $\mathfrak{B}_\gamma$ and the mapping $\lambda \mapsto D^\lambda$, from $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$ to $\mathrm{Irr}_p(\mathfrak{B}_\gamma)$ is a UBS for the block $\mathfrak{B}_\gamma$. △

Having defined the notion of UBS for a block, or union of blocks, we recall that our initial motivation (see discussion after Example 4.1.6) is the quest for an UBS for a union of p-blocks of $\mathfrak{S}_n$, satisfying conditions (A) and (B) (with the motivation that such a SUBS would allow a natural parametrization for certain irreducible $\mathbb{F}_p\mathfrak{A}_n$-modules in that union of blocks). With this in mind we make the following two definitions.

Definition 4.2.4. For a p-block $\mathfrak{B}_\gamma$ of $\mathbb{F}_p\mathfrak{S}_n$, we call its *conjugate block* the block $\mathfrak{B}_{\gamma'}$ consisting of the partitions of n with p-core γ'. If $\gamma = \gamma'$, then $\mathfrak{B}_\gamma = \mathfrak{B}_{\gamma'}$ and we say that $\mathfrak{B}_\gamma$ is a *self-conjugate block*. △

Example 4.2.5. The block of $\mathbb{F}_5\mathfrak{S}_8$ associated to the 5-core $\tau = (3)$ is

$$\mathfrak{B}_\tau = \{(8),\ (4^2),\ (3^2,1^2),\ (3,2,1^3),\ (3,1^5)\}.$$

Its conjugate block is the block $\mathfrak{B}_{\tau'}$ associated to the 5-core $\tau' = (1^3)$, whose partitions are all the conjugates of partitions in $\mathfrak{B}_\tau$:

$$\mathfrak{B}_{\tau'} = \{(6,1^2),\ (5,2,1),\ (4,2^2),\ (2^4),\ (1^8)\}.$$

The block of $\mathbb{F}_5\mathfrak{S}_8$ associated to the 5-core $\gamma = (2,1)$ is a self-conjugate block since $\gamma = \gamma'$.

$$\mathfrak{B}_\gamma = \{(7,1),\ (5,3),\ (3^2,2),\ (2^3,1^2),\ (2,1^6)\}.$$

△

Definition 4.2.6. A UBS $(B, \leq, \Psi)$ for a union U of blocks of $\mathbb{F}_p\mathfrak{S}_n$ is said to be *stable* or a *stable unitriangular basic set (SUBS)* if the following two conditions hold:

(A) B is **stable** for conjugation: If $\lambda \in B$, then $\lambda' \in U$.

(B) The only self-conjugate partitions in U are p-BG-partitions: If $\lambda = \lambda' \in U$, then $\lambda \in \mathrm{BG}_p^n$.

If $U = \mathfrak{B}_\gamma$ and $\mathfrak{B}_\gamma$ is self-conjugate, we say that *B is a SUBS for the block $\mathfrak{B}_\gamma$*. △

As mentioned in the discussion after Example 4.1.6, $\mathbb{F}_p\mathfrak{S}_n$ does not have a SUBS, that is, there is not a SUBS for the union $\bigcup_{\mathfrak{B} \text{ is a } p\text{-block}} \mathfrak{B}$, of all p-blocks of $\mathbb{F}_p\mathfrak{S}_n$.

4.3 A SUBS for blocks of odd weight or not self-conjugate

Given that there is not always a SUBS for $\mathbb{F}_p\mathfrak{S}_n$, it is interesting to have a SUBS for some unions of blocks. By the following proposition, which has been adapted to our notation, this question is already solved for self-conjugate blocks of odd weight and for $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$ where γ is not self-conjugate.

Proposition 4.3.1 ([BGJ20, Theorem 38])**.** *Let $\gamma \vdash (n - wp)$ be a p-core. Let $\mathfrak{B}_\gamma$ be the corresponding block of $\mathbb{F}_p\mathfrak{S}_n$. Consider the following subset of $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$*

$$U_{n,p}^\gamma = \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) < \lambda\} \ \sqcup\ \{\lambda' \mid \lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) = \lambda\},$$

*where $\boldsymbol{m}_p$ denotes the Mullineux map. If w **is odd or if γ is not self-conjugate**, then there exists an order $\preceq_\gamma$ on $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$ and a bijection $\Psi_\gamma : U_{n,p}^\gamma \longrightarrow \mathrm{Irr}(\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'})$ such that $(U_{n,p}^\gamma, \preceq_\gamma, \Psi_\gamma)$ is a SUBS for $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$.*

We make some comments on this proposition. For this, let us define a total order on $\mathrm{Par}(n)$ by setting $\lambda \preceq \mu$ if:

- $\lambda = \mu$,
- or $\lambda = \mu'$ and $\mu \leq \mu'$,
- or if $\mathrm{Max}_{\leq}(\lambda, \lambda') < \mathrm{Max}_{\leq}(\mu, \mu')$,

where $\leq$ denotes the lexicographic order on partitions. We have the following result.

Proposition 4.3.2 ([BGJ20, Proposition 33]). *Let*

$$U_{n,p} := \{\lambda \in \mathrm{Reg}_p(n) \mid \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda' \mid \lambda \in \mathrm{Reg}_p(n) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda \in \mathrm{Reg}_p(n) \mid \boldsymbol{m}_p(\lambda) = \lambda\}. \tag{4.1}$$

Let

$$\begin{aligned} U^1_{n,p} &:= \{\lambda \in \mathrm{Reg}_p(n) \mid \boldsymbol{m}_p(\lambda) < \lambda\}, \\ U^2_{n,p} &:= \{\lambda' \mid \lambda \in \mathrm{Reg}_p(n) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\}, \text{ and} \\ U^3_{n,p} &:= \{\lambda \in \mathrm{Reg}_p(n) \mid \boldsymbol{m}_p(\lambda) = \lambda\}. \end{aligned}$$

And let Ψ be the bijection defined as:

$$\begin{aligned} \Psi : \quad U_{n,p} &\longrightarrow \mathrm{Irr}(\mathbb{F}_p\mathfrak{S}_n) \\ \lambda &\longmapsto \begin{cases} D^\lambda & \text{if } \lambda \in U^1_{n,p} \cup U^3_{n,p}, \\ D^{\boldsymbol{m}_p(\lambda')} & \text{if } \lambda \in U^2_{n,p}. \end{cases} \end{aligned}$$

Then $(U_{n,p}, \trianglelefteq, \Psi)$ is a unitriangular basic set (UBS) for $\mathbb{F}_p\mathfrak{S}_n$.

In general, $U_{n,p}$ is not a SUBS for $\mathbb{F}_p\mathfrak{S}_n$, since it contains the self-Mullineux partitions which could fail to verify properties (A) or (B) in Definition 4.2.6. However, since the restriction $U_{n,p} \cap \mathfrak{B}_\gamma$ of $U_{n,p}$ to each block results into a UBS (see Example 4.2.3), we can still wonder for which unions of blocks such restriction results in a SUBS for the block. Proposition 4.3.1 answers to this by giving a sufficient condition.

The reason why the restriction of the UBS $U_{n,p}$ to self-conjugate blocks of odd weight or blocks which are not self-conjugate is that, for such blocks, the restriction of $U^3_{n,p}$ to the block is empty: The block does not contain self-Mullineux partitions. Let us see why.

First for blocks which are not self-conjugate:

Lemma 4.3.3. *Let $\gamma \vdash (n - pw)$ a p-core such that $\gamma \neq \gamma'$. Let $\mathfrak{B}_\gamma$ be the corresponding block of $\mathbb{F}_p\mathfrak{S}_n$. Then $\mathfrak{B}_\gamma$ does not contain self-Mullineux partitions. Consequently, the UBS which comes from the restriction of the UBS $U_{n,p}$ for $\mathbb{F}_p\mathfrak{S}_n$ (from Proposition 4.3.2) to $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$ is a SUBS for this union of p-blocks.*

Proof. From [Mul79b], we know that the p-core of a partition λ and that of its image $\boldsymbol{m}_p(\lambda)$ under the Mullineux map, are conjugates. This says in particular that self-Mullineux partitions only occur in self-conjugate blocks. Then, if $\mathfrak{B}_\gamma$ is not a self-conjugate block, the set $U_{n,p} \cap (\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'})$ is a SUBS, by restricting the total order $\trianglelefteq$ on $\mathrm{Par}(n)$ to $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$. Indeed, this restriction does not contain any self-Mullineux partitions (any partition from $U^3_{n,p}$), or any self-conjugate partition, and then by construction of $U_{n,p}$, it satisfies conditions (A) and (B) in Definition 4.2.6. □

Denote by $\mathcal{M}^n_p(\gamma)$ and by $\mathrm{BG}^n_p(\gamma)$ the set of self-Mullineux partitions and p-BG-partitions in $\mathfrak{B}_\gamma$.

Proposition 4.3.4 ([BG10, Proposition 6.1]). $|\mathcal{M}^n_p(\gamma)| = |BG^n_p(\gamma)|$.

Now, the following lemma shows why self-conjugate blocks of odd weight do not have any self-Mullineux partitions. It also contains a known fact about the number of self-Mullineux partitions in a block of even weight. A combinatorial proof of this fact can be found in [BO98, Theorem 3.5].

Lemma 4.3.5. *Let $\gamma \vdash (n-pw)$ be a self-conjugate p-core. Let $\mathfrak{B}_\gamma$ be the corresponding block of $\mathbb{F}_p\mathfrak{S}_n$. Then, the set of self-Mullineux partitions in $\mathfrak{B}_\gamma$ is non-empty if and only if w is even. In this case, there are as many self-Mullineux partitions as $(\frac{p-1}{2})$-multipartitions of rank $\frac{w}{2}$.*

Proof. We know that in a fixed block, any partition is completely determined by its p-quotient (§1.1.6). From [BG10, Lemma 3.4], by choosing an appropriate convention, the p-quotient $\overline{\nu}_p = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(p)})$ of $\nu \in \mathrm{BG}_p^n(\gamma)$, which has rank w, is of the form

$$(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(\frac{p-1}{2})}, \emptyset, \nu^{(\frac{p-1}{2})'}, \dots, \nu^{(2)'}, \nu^{(1)'}),$$

where $\nu^{(i)'}$ is the conjugate partition of $\nu^{(i)}$. Hence the rank, w, of this p-quotient is even: it is twice the sum of the ranks of $\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(\frac{p-1}{2})}$. Also, this p-quotient is completely determined by these $\frac{p-1}{2}$ partitions. Thus, $|\mathcal{M}_p^n(\gamma)|$ is equal to the number of $(\frac{p-1}{2})$-multipartitions of $\frac{w}{2}$.

Conversely, if w is even, each $(\frac{p-1}{2})$-multipartition of $\frac{w}{2}$ determines a unique p-quotient

$$(\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(\frac{p-1}{2})}, \emptyset, \nu^{(\frac{p-1}{2})'}, \dots, \nu^{(2)'}, \nu^{(1)'}),$$

which corresponds to a p-BG-partition in the block $\mathfrak{B}_\gamma$. □

Proposition 4.3.1 follows from Lemma 4.3.3 and Lemma 4.3.5.

4.4 About blocks of weight 1

In §4.3 we saw that, in particular, for any block $\mathfrak{B}_\gamma$ of weight 1 of $\mathbb{F}_p\mathfrak{S}_n$, either self-conjugate or not self-conjugate, there is a SUBS for $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$. The underlying set of this SUBS is

$$U_{n,p}^\gamma = \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda' \mid \lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\}. \tag{4.2}$$

In this section we show that the set $U_{n,p}^\gamma$ arises as a SUBS in a different way, that is, with a total order defined differently. The interesting part about this newly defined SUBS for the already well known blocks of weight 1, is that, later in §4.5 we will see that blocks of weight 2 partition into certain subsets ∂_ℓ, and each of these subsets behave similarly to weight 1 blocks with respect to the new order, the Mullineux map and conjugation. We show that the Mullineux map has an easy description for blocks of weight 1. This description allows to characterize the partitions λ in $\mathfrak{B}_\gamma$ such that $\boldsymbol{m}_p(\lambda) < \lambda$. Then, we define a total order, different from $\trianglelefteq_\gamma$ in the restriction of $U_{n,p}^\gamma$ from (4.1). This total order makes the corresponding decomposition matrix lower unitriangular.

Remark 4.4.1. From the form of the decomposition matrix of blocks of weight 1 (see 4.4), it can be seen that the set $U_{n,p}^{\gamma}$ in (4.2) is the underlying set of **any** SUBS for a **self-conjugate** block of weight 1. Indeed: any SUBS consists in $p-1$ partitions of the p partitions in $\mathfrak{B}_\gamma$, which contains a unique self-conjugate partition. Then, the only option for a SUBS to be *stable* is to choose every partition in $\mathfrak{B}_\gamma$ but the self-conjugate one. On the other hand, if the block is not self-conjugate, any choice of $p-1$ partitions in $\mathfrak{B}_\gamma$ (together with their conjugates in $\mathfrak{B}_{\gamma'}$) form a SUBS. In this case, there are $p-1$ possible underlying sets for a SUBS.

4.4.1 Combinatorics of partitions with weight 1

Notation for partitions in a block It can be shown that a block $\mathfrak{B}_\gamma$ of p-weight 1 contains exactly p partitions $\lambda^0, \lambda^1, \ldots, \lambda^{p-1}$ that can be labelled so that

$$\lambda^0 \trianglelefteq \lambda^1 \trianglelefteq \cdots \trianglelefteq \lambda^{p-1}, \tag{4.3}$$

where λ^0 is the unique p-singular partition. Moreover, the decomposition numbers are given as follows. For any partition $\lambda \vdash n$ and for $1 \leq i \leq p-1$, we have

$$d_{\lambda\lambda^i} = \begin{cases} 1, & \text{if } \lambda \in \{\lambda^i, \lambda^{i-1}\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

See [Mat99, Exercise 5.10]. In general, if rows are organized downwards in decreasing lexicographic order, the shape of the matrix $\mathbf{D}_\gamma$ is:

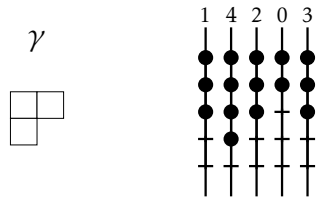
	λ^{p-1}	λ^{p-2}	λ^{p-3}	$\cdots$	λ^2	λ^1
λ^{p-1}	1	·	·	$\cdots$	·	·
λ^{p-2}	1	1	·	$\cdots$	·	·
λ^{p-3}	·	1	1		·	·
⋮	⋮					⋮
λ^2	·	$\cdots$	·		1	·
λ^1	·	$\cdots$	·		1	1
λ^0	·	$\cdots$	·	$\cdots$	·	1

where simple dots are zeros.

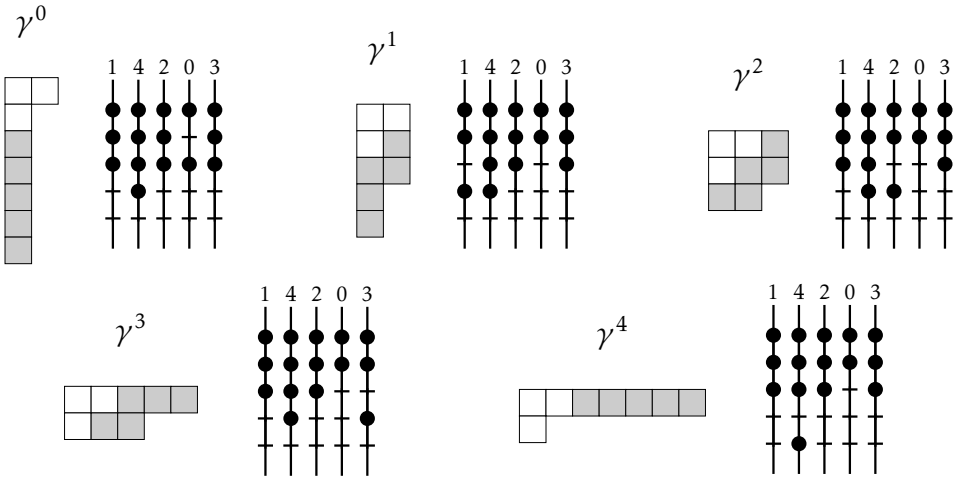
Notice that a consequence of the fact that the dominance order totally orders partitions in $\mathfrak{B}_\gamma$ is that it coincides with the lexicographic order in $\mathfrak{B}_\gamma$. The fact that the dominance order totally orders the partitions in a block of weight 1 can be proven by using the abacus display for partitions. Each partition in $\mathfrak{B}_\gamma$ is obtained from the abacus display of γ by sliding down one bead in some runner, which amounts to adding a p-hook to the Young diagram of γ. For $0 \leq i \leq p-1$ denote by γ^i the partition obtained by sliding down the bead in runner i. From the defined labelling for the runners (§1.1.7), the corresponding p-hook has leg length $(p-1)-i$ (and arm length i), see Remark 1.1.7. Hence, this notation for partitions in $\mathfrak{B}_\gamma$ agrees with the notation chosen above in (4.3). Then:

Throughout this section, unless otherwise specified, $\mathfrak{B}_\gamma$ is a block of p-weight 1, with its partitions denoted $\gamma^0 \trianglelefteq \gamma^1 \trianglelefteq \cdots \trianglelefteq \gamma^{p-1}$.

Example 4.4.2. Let $p = 5$ and $\gamma = (2,1) \vdash (8-5)$. There are in total five partitions in the block $\mathfrak{B}_\gamma$ of $\mathbb{F}_5\mathfrak{S}_8$; four of them are 5-regular. The Young diagram and 5-abacus of γ is:



The five partitions in $\mathfrak{B}_\gamma$ are



where the shaded nodes correspond to the 5-rim-hook added to γ to obtain each partition in the block. We have then

$$\mathfrak{B}_\gamma = \{(2,1^6),\ (2^3,1^2),\ (3^3,2),\ (5,3),\ (7,1)\},$$

where only $(2,1^6)$ is 5-singular. The matrix $\mathbf{D}_{(2,1)}$, with rows organized downwards in decreasing lexicographic order is

		γ^4	γ^3	γ^2	γ^1
γ^4	$7,1$	1			
γ^3	$5,3$	1	1		
γ^2	$3^2,2$	·	1	1	
γ^1	$2^3,1^2$	·	·	1	1
γ^0	$2,1^6$	·	·	·	1

△

Conjugation and the Mullineux map in blocks of weight 1 Since conjugation reverses the dominance order on partitions, it is easy to describe in $\mathfrak{B}_\gamma$, given that it permutes the blocks $\mathfrak{B}_\gamma$ and $\mathfrak{B}_{\gamma'}$. Denote $\tau = \gamma'$ and denote partitions in $\mathfrak{B}_\tau$ as $\tau^0 \trianglelefteq \tau^1 \trianglelefteq \cdots \trianglelefteq \tau^{p-1}$. When conjugating each partition in $\mathfrak{B}_\gamma$, we obtain all partitions in $\mathfrak{B}_\tau$. We have

$$(\gamma^{p-1})' \trianglelefteq \cdots \trianglelefteq (\gamma^1)' \trianglelefteq (\gamma^0)'.$$

Then,

$$(\gamma^i)' = \tau^{(p-1)-i}.$$

Thus, since conjugation produces the p-core to conjugate, in a diagram it is as follows. If γ is not self-conjugate:

$$\mathfrak{B}_\gamma : \quad \gamma^0 \trianglelefteq \gamma^1 \trianglelefteq \cdots \trianglelefteq \gamma^{p-2} \trianglelefteq \gamma^{p-1}$$

$$\mathfrak{B}_\tau : \quad \tau^0 \trianglelefteq \tau^1 \trianglelefteq \cdots \trianglelefteq \tau^{p-2} \trianglelefteq \tau^{p-1}$$

Whereas if γ is self-conjugate, conjugation does the following in $\mathfrak{B}_\gamma$:

$$\gamma^0 \trianglelefteq \gamma^1 \trianglelefteq \cdots \trianglelefteq \gamma^{p-2} \trianglelefteq \gamma^{p-1}$$

The following proposition asserts that the Mullineux map works similarly, but in $\mathrm{Reg}_p(\mathfrak{B}_\gamma) = \{\gamma^1, \ldots, \gamma^{p-1}\}$ and $\mathrm{Reg}_p(\mathfrak{B}_\tau) = \{\tau^1, \ldots, \tau^{p-1}\}$.

Proposition 4.4.3. *For $1 \leq i \leq p-1$ we have*

$$\boldsymbol{m}_p(\gamma^i) = \tau^{p-i},$$

where $\boldsymbol{m}_p$ is the Mullineux map. In terms of arm-length of hooks : If the p-hook of a weight 1 partition γ^i has arm-length i, then the arm-length of the p-hook in $\boldsymbol{m}_p(\gamma^i)$ is $p-i$.

Illustrated as in the diagrams above, the proposition says that the Mullineux map does the following. If γ is not self-conjugate

$$\mathfrak{B}_\gamma : \quad \gamma^0 \trianglelefteq \gamma^1 \trianglelefteq \gamma^2 \trianglelefteq \cdots \trianglelefteq \gamma^{p-2} \trianglelefteq \gamma^{p-1}$$

$$\mathfrak{B}_\tau : \quad \tau^0 \trianglelefteq \tau^1 \trianglelefteq \tau^2 \trianglelefteq \cdots \trianglelefteq \tau^{p-2} \trianglelefteq \tau^{p-1}$$

If γ is self-conjugate:

$$\gamma^0 \trianglelefteq \gamma^1 \trianglelefteq \gamma^2 \trianglelefteq \cdots \trianglelefteq \gamma^{p-2} \trianglelefteq \gamma^{p-1}$$

where the highlighted partitions are the p-singular partitions.

Proof. We prove the equivalent statement: for $1 \leq i \leq p-1$ we have

$$\boldsymbol{m}_p(\gamma^{p-i}) = \tau^i.$$

The proof is by induction on i, and using the form of the decomposition matrix $\mathbf{D}_\gamma$ (or $\mathbf{D}_\tau$) of the block $\mathfrak{B}_\gamma$ (or $\mathfrak{B}_\tau$), described in (4.4).

Let $i = 1$. We show that $\boldsymbol{m}_p(\gamma^{p-1}) = \tau^1$. We know that $d_{\gamma^{p-1},\gamma^{p-1}} = 1$, from the usual UBS $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$. From Proposition 4.6.2, we have that $d_{\gamma^{p-1},\gamma^{p-1}} = d_{(\gamma^{p-1})',\boldsymbol{m}_p(\gamma^{p-1})}$. But

$(\gamma^{p-1})' = \tau^0$. So that $d_{\tau^0 \boldsymbol{m}_p(\gamma^{p-1})} = 1$. Now, from (4.4), the row τ^0 of $\mathbf{D}_\tau$ has exactly one non-zero entry, in the column indexed by the simple module D^{τ^1}. Hence

$$\boldsymbol{m}_p(\gamma^{p-1}) = \tau^1.$$

Let $2 \leq i \leq p-2$, and suppose that $\boldsymbol{m}_p(\gamma^{p-i}) = \tau^i$. We show that $\boldsymbol{m}_p(\gamma^{p-i-1}) = \tau^{(i+1)}$. From (4.4), we know that $d_{\gamma^{p-i-1}\gamma^{p-i-1}} = 1$. Now, from Proposition 4.6.2, $d_{\gamma^{p-i-1}\gamma^{p-i-1}} = d_{(\gamma^{p-i-1})'\boldsymbol{m}_p(\gamma^{p-i-1})}$. Since $(\gamma^{p-i-1})' = \tau^i$, we have that $d_{\tau^i \boldsymbol{m}_p(\gamma^{p-i-1})} = 1$. But the only non-zero entries in row τ^i correspond to the columns τ^{i+1} and τ^i, then $\boldsymbol{m}_p(\gamma^{p-i-1}) \in \{\tau^{i+1}, \tau^i\}$. But, by the induction hypothesis $\tau^i = \boldsymbol{m}_p(\gamma^{p-i})$. Hence, since the Mullineux map is a bijection, $\boldsymbol{m}_p(\gamma^{p-i-1}) = \tau^{i+1}$.

The assertion about the arm-lengths of hooks is justified by Remark 1.1.7. □

4.4.2 SUBS for blocks of weight 1

Having in mind that we will define a SUBS for $\mathfrak{B}_\gamma \cup \mathfrak{B}_{\gamma'}$ with underlying set

$$U_{n,p}^{\gamma} = \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda' \mid \lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\}, \tag{4.5}$$

where $\gamma \vdash n-p$, with the preceding description of the combinatorics and the Mullineux map in $\mathfrak{B}_\gamma$, we are able to characterize the partitions $\lambda \in \mathfrak{B}_\gamma$ such that $\boldsymbol{m}_p(\lambda) < \lambda$, where $<$ is the lexicographic order.

Proposition 4.4.4. *With the same notations, the partitions λ in $\mathfrak{B}_\gamma$ such that $\boldsymbol{m}_p(\lambda) < \lambda$ are exactly*

$$\gamma^{(\frac{p+1}{2})}, \gamma^{(\frac{p+3}{2})}, \ldots, \gamma^{p-1}.$$

Proof. If γ is self-conjugate, it is straightforward from Proposition 4.4.3: Since $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau = \mathfrak{B}_\gamma$ which is equal to the list $\tau^0, \ldots, \tau^{p-1}$. This list is organized increasingly following the lexicographic order, then the second half of the list is formed by the partitions λ such that $\boldsymbol{m}_p(\lambda) < \lambda$.

Suppose that γ is not self-conjugate. Let $1 \leq i \leq p-1$. If $\boldsymbol{m}_p(\gamma^i) < \gamma^i$, then, for every $i \leq j \leq p-1$, we have $\boldsymbol{m}_p(\gamma^j) < \gamma^j$. This holds because $\gamma^i < \gamma^j$ and, on the other hand $\boldsymbol{m}_p(\gamma^i) = \tau^{p-i}$ and $\boldsymbol{m}_p(\gamma^j) = \tau^{p-j}$, so that $\boldsymbol{m}_p(\gamma^j) < \boldsymbol{m}_p(\gamma^i)$. We are saying that any partition λ in $\mathfrak{B}_\gamma$ with $\boldsymbol{m}_p(\lambda) < \lambda$, forces every partition in the block, greater than λ, to have that same property.

Now, since U is a UBS for $\mathfrak{B}_\gamma$, the number of partitions λ such that $\boldsymbol{m}_p(\lambda) < \lambda$ is the same for every block of weight 1. In particular $\mathfrak{B}_\gamma$ and $\mathfrak{B}_\tau$. On the other hand, a SUBS for $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau$ contains $2(p-1)$ partitions. Where $(p-1)$ of this partitions have the property $\boldsymbol{m}_p(\lambda) < \lambda$. Then there must be $(\frac{p-1}{2})$ partition having such property in $\mathfrak{B}_\gamma$ (and in $\mathfrak{B}_\tau$). This, together with the previous paragraph show that such partitions are precisely

$$\gamma^{(\frac{p+1}{2})}, \gamma^{(\frac{p+3}{2})}, \ldots, \gamma^{(p-1)},$$

in $\mathfrak{B}_\gamma$. □

We now set the total order and bijection for the SUBS for $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau$. Let Λ be the following subset of $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau$:

$$\Lambda = \{\lambda \in \mathfrak{B}_\gamma \mid \boldsymbol{m}_p(\lambda) < \lambda\}.$$

Denote partitions in Λ as $\lambda^1, \lambda^2, \ldots, \lambda^r$ in a way such that $\lambda^1 < \lambda^2 < \cdots < \lambda^r$, where $<$ is the lexicographic order on partitions (the partitions λ^i and the number r are completely described by Proposition 4.4.4 and depend on whether γ is self-conjugate or not). Define the following total order $\prec$ in $U_{n,p}^{\gamma}$

$$\lambda^r \succ \lambda^{r-1} \succ \cdots \succ \lambda^1 \succ (\lambda^r)' \succ (\lambda^{r-1})' \succ \cdots \succ (\lambda^1)'.$$

Extend this total order to a total order, equally denoted, in $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau$ such that every partition in $(\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau) \setminus U$ is less than every partition in $U_{n,p}^{\gamma}$.

Define the following bijection.

$$\begin{aligned} \Psi : \quad U_{n,p}^{\gamma} &\longrightarrow \operatorname{Irr}_p(\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau) \\ \lambda &\longmapsto \begin{cases} \lambda & \text{if } \lambda \in \Lambda, \\ \boldsymbol{m}_p(\lambda') & \text{otherwise.} \end{cases} \end{aligned}$$

Proposition 4.4.5. *The total order $\prec$ in $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau$, together with the bijection Ψ and the set $U_{n,p}^{\gamma}$, form a SUBS for $\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau$.*

Proof. We show that the submatrix of $\mathbf{D}_{n,p}$ with rows indexed by $U_{n,p}^{\gamma}$, and columns indexed by $\operatorname{Irr}_p(\mathfrak{B}_\gamma \cup \mathfrak{B}_\tau)$ has wedge shape. Organise the rows and columns of such submatrix as follows:

	λ^r	$\cdots$	λ^1	$\boldsymbol{m}_p(\lambda^r)$	$\cdots$	$\boldsymbol{m}_p(\lambda^1)$
λ^r						
$\vdots$		$\mathbf{D}_1$			$\mathbf{D}_2$	
λ^1						
$(\lambda^r)'$						
$\vdots$		$***$			$\mathbf{D}_3$	
$(\lambda^1)'$						

The sub-matrix $\mathbf{D}_1$ is lower unitriangular. Indeed, from Proposition 4.4.4, the partitions $\lambda^r, \ldots, \lambda^1$ are p-regular. Hence, on one hand $d_{\lambda^i \lambda^i} = 1$ for every $1 \leq i \leq r$. On the other hand if $d_{\lambda^i \lambda^j} \neq 0$ for some $1 \leq i, j \leq r$, then $\lambda_j < \lambda_i$, and, from the definition of $\prec$, we have $\lambda_j \prec \lambda_i$ as well. This shows that $\mathbf{D}_1$ is lower unitriangular. Now, $\mathbf{D}_3 = \mathbf{D}_1$, from Proposition 4.6.2. Then, $\mathbf{D}_3$ is lower unitriangular.

We claim that $\mathbf{D}_2 = (0)_{r \times r}$. Indeed, suppose that $d_{\lambda^i \boldsymbol{m}_p(\lambda^j)} \neq 0$ for some $1 \leq i, j \leq r$ and that $\lambda^i = \gamma^k$ for some $\frac{p+1}{2} \leq k \leq p-1$. We know that row γ^k has only two non-zero entries (one, if $k = p-1$), one in column $\gamma^k = \lambda^i$, and the other in column γ^{k+1}. But $\gamma^{k+1} \neq \boldsymbol{m}_p(\lambda^j)$ for every $1 \leq j \leq r$. Indeed, if γ is self-conjugate, $\gamma^{k+1} = \boldsymbol{m}_p(\gamma^{p-k-1})$, by 4.4.3. But $p-k-1 \leq \frac{p-3}{2}$, then $\gamma^{p-k-1} \notin \Lambda$. A similar reasoning is done if γ is not self-conjugate. Hence $\mathbf{D}_2 = (0)_{r \times r}$, and the submatrix with rows indexed by U is lower unitriangular.

□

4.5 Blocks of weight 2

With SUBS in hand for non self-conjugate blocks or blocks of odd weight, the next step is to study in detail self-conjugate blocks of weight 2. It is the subject of this section.

In this section γ is a p-core and $\mathfrak{B}_\gamma$ is a block with self-conjugate p-core γ and p-weight 2.

In [Ric96], Richards studied blocks of weight 2, in particular the combinatorics of partitions in such blocks and he gave a complete description of the decomposition numbers. In this section we recall some of his definitions and results which we use. Within his definitions there is an object associated to a block called pyramid, it will be one of the main objects in this section. We also use notations from [Fay], where Fayers presents an efficient way to label simple modules.

Remark 4.5.1. Some of the following results and affirmations are still valid if γ is not self-conjugate. △

4.5.1 Combinatorics and notations for partitions in blocks of weight 2

Pyramids

From here we always use the labelling of runners in the abacus as defined in §1.1.7.

Let γ be a p-core. Consider the abacus display for γ. Let $\rho_0 < \rho_1 < \ldots < \rho_{p-1}$ be the positions of the lower beads as in §1.1.7. The *pyramid* of γ is a triangular array $({}_i\gamma_j)_{ij}$ of 0s and 1s, defined as follows: for $0 \leq i \leq j \leq p-1$ let

$${}_i\gamma_j = \begin{cases} 1 & \text{if } \rho_j - \rho_i < p, \\ 0 & \text{if } \rho_j - \rho_i > p. \end{cases}$$

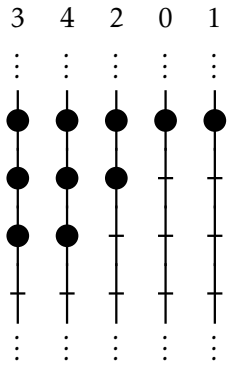
We organise these numbers in a diagram as follows

$$\begin{array}{lccccccccccc}
\underline{\text{row}} & & & & & & & & & & & \\
p-1 & & & & & & & {}_0\gamma_{p-1} & & & & \\
\vdots & & & & & & & \vdots & & & & \\
2 & & & & {}_0\gamma_2 & & {}_1\gamma_3 & \cdots & {}_{p-3}\gamma_{p-1} & & & \\
1 & & & {}_0\gamma_1 & & {}_1\gamma_2 & & {}_2\gamma_3 & \cdots & & {}_{p-2}\gamma_{p-1} & \\
0 & & {}_0\gamma_0 & & {}_1\gamma_1 & & {}_1\gamma_1 & & \cdots & & & {}_{p-1}\gamma_{p-1}
\end{array} \tag{4.6}$$

For short, we write ${}_i0_j$ if ${}_i\gamma_j = 0$ and ${}_i1_j$ if ${}_i\gamma_j = 1$. The definition of the pyramid can be extended for convenience by allowing i and j to be any integers: if $i > j$, then ${}_i\gamma_j = 1$. Otherwise, if $i < 0$ or $j \geq p$ we define ${}_i\gamma_j = 0$. In the pyramid diagram, that

means that the outside upper left and upper right is filled with 0s and the outside lower part is filled with 1s. For $0 \leq k \leq p-1$, we call the *k-th row* of the pyramid as the set or entries ${}_i\gamma_j$ such that $j-i=k$. The *apex* of the pyramid is the entry ${}_0\gamma_{p-1}$.

Example 4.5.2. Let $p=5$ and the 5-core $\gamma=(2,2)$. The 5-abacus for γ is



The positions of the lower beads in each runner are $11, 10, 7, 3$, and 4. Then, organised increasingly, $(\rho_0, \rho_1, \rho_2, \rho_3, p_4) = (3, 4, 7, 10, 11)$, so that the labelling of the runners is $3, 4, 2, 0, 1$ from left to right. The pyramid for γ is

$$
\begin{array}{ccccccccc}
 & & & & {}_0 0_4 & & & & \\
 & & & {}_0 0_3 & & {}_1 0_4 & & & \\
 & & {}_0 1_2 & & {}_1 0_3 & & {}_2 1_4 & & \\
 & {}_0 1_1 & & {}_1 1_2 & & {}_2 1_3 & & {}_3 1_4 & \\
{}_0 1_0 & & {}_1 1_1 & & {}_2 1_2 & & {}_3 1_3 & & {}_4 1_4
\end{array}
$$

△

The definition of the pyramid implies that if an entry is 1, then the two entries just below are 1 as well. Hence, below a 1 there is a whole triangle or pyramid of 1s. Hence if an entry is 0, there are only 0s above it.

Remark 4.5.3. Pyramids are originally defined for p-blocks (p-cores) of any weight. Two different pyramids correspond to different p-cores. But two different p-cores can have the same pyramid. However, there is an equivalence relation that can be defined in the set of p-cores, for which the equivalences classes are certain (Scopes) families of p-cores. Two blocks have the same pyramid if and only if they are *Scopes*-equivalent ([Ric96, Proposition 3.3]).

Moreover, two blocks corresponding to partitions of p-weight w with p-cores in the same family have essentially the same decomposition numbers. See [Sco95] and [Ric96, §3]. △

Notation for partitions in block of weight 2

We recall a notation introduced by Fayers [Fay], for partitions in $\mathfrak{B}_\gamma$. Since $\mathfrak{B}_\gamma$ is of weight 2, a partition λ in $\mathfrak{B}_\gamma$ is obtained from the abacus of γ by moving twice a bead down one position. This can be done in three different ways

- The lowest bead on each of the runners i and j, with $i<j$, is moved one position down. In which case λ is denoted $\langle i, j\rangle$.

- The lowest bead on runner i is moved down two positions. In this case λ is denoted $\langle i \rangle$.
- The lowest bead on runner i is moved down one position and the next bead above it is moved down one position. In this case λ is denoted $\langle i^2 \rangle$.

So that every partition in $\mathfrak{B}_\gamma$ corresponds to exactly one of $\langle i, j \rangle$ for $0 \leq i < j \leq p-1$ or $\langle i \rangle$ or $\langle i^2 \rangle$ for $0 \leq i \leq p-1$. We refer to this notation as $\langle \cdot \rangle$ notation for partitions in $\mathfrak{B}_\gamma$.

Remark 4.5.4 (counting partitions)**.** This shows that there is a total of $\binom{p}{2} + 2p = \frac{p(p+3)}{2}$ partitions in $\mathfrak{B}_\gamma$ (in any block of p-weight 2). Which agrees with Theorem 1.4.8: it is equal to the number of p-partitions of $w = 2$. △

Remark 4.5.5. The $\langle \cdot \rangle$ notation is also well defined in blocks which are not self-conjugate.

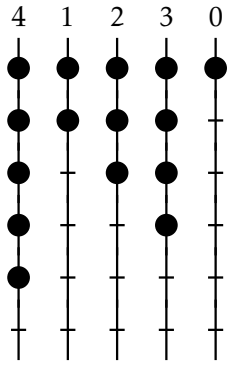
Conjugation in the abacus and notation for self-conjugate partitions

In order to characterize self-conjugate partitions in $\mathfrak{B}_\gamma$ in terms of $\langle \cdot \rangle$ notation, let us study the form of the abacus of γ.

Let us refer to the following transformation of a runner as *reversing*: reflecting the runner with respect to a horizontal axis (turning it upside down), and transforming the beads into empty spaces and viceversa.

Since p is odd, any p-abacus has a runner which is in the middle, we refer to this runner as the *middle runner*. For any other runner, we can associate what we call its *opposite runner* which is the different runner equally spaced to the middle runner.

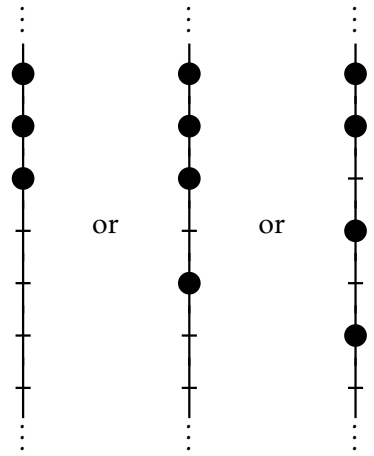
Now, conjugation of partitions can be done in the abacus in two steps: first we switch each runner with its opposite runner, and then we reverse all runners simultaneously (with respect to a same horizontal axis). Thus, the abacus of a self-conjugate p-core is such that the middle runner is its own reverse and opposite runners are mutual reverses. For example, the abacus of the self-conjugate 5-core $\gamma = (6, 5, 3, 2^2, 1)$ is



The chosen labelling for the runners in the abacus has the property that the labels of two runners opposite to each other, add up to $p-1$. Let us see why. Recall the total order $\lessdot$ defined on the set of runners of the abacus (§1.1.7). It is easy to see, following the discussion in the previous paragraph, that conjugation reverses this order, that is, if R and S are two runners of the abacus of a p-core such that $R \lessdot S$, and R' and S' are their images under conjugation, then $S' \lessdot R'$. So if a runner R has label r, then

its image R' under conjugation, has label $p-1-r$. In the abacus of a self-conjugate p-core, this means that the labels $(r_0, r_1, \ldots, r_{p-1})$ must satisfy $r_i = p-1-r_{p-1-i}$. See the example above.

Now, observe that any self-conjugate partition in $\mathfrak{B}_\gamma$ is obtained from the abacus of γ by sliding down two beads, each one in a runner opposite to the other: First, we are saying that such a partition belongs to the first kind of partitions described just before Remark 4.5.5, that is, partitions of the form $\langle i, j\rangle$. Indeed, being self-conjugate, the middle runner is equal to its reverse. A runner on which at most two beads have been moved and which is equal to its reverse is of one of the following forms



but the third option is not possible since it requires more than two movements. The second option requires only one bead movement, but that means that the second (and last) movement occurs in a runner different from the middle runner, which would then not be equal to the reversed opposite runner, since its opposite runner has all of its beads all the way up. This leaves the first option as the only possibility. Thus, none of the two bead movement are done in the middle runner; they are done in a pair of opposite runners. Such a partition is then written $\langle i, j\rangle$ in the $\langle\cdot\rangle$ notation, and more precisely, we have:

Lemma 4.5.6. *In $\langle\cdot\rangle$-notation, the self-conjugate partitions in $\mathfrak{B}_\gamma$ are exactly*

$$\nu_k = \left\langle \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rangle \quad \text{for} \quad 1 \le k \le \frac{p-1}{2}.$$

Example 4.5.7. There are exactly two self-conjugate partitions in the 5-block $\mathfrak{B}_{(2,2)}$, from Example 4.5.2. They are $(6,3,2,1^3) = \langle 1, 3\rangle$ and $(7,2,1^5) = \langle 0, 4\rangle$. △

Notation for p-regular partitions in $\mathfrak{B}_\gamma$

For this subsection, γ is not necessarily self-conjugate.

In [Fay], Fayers introduced a notation $\lceil\cdot\rfloor$ for indexing p-regular partitions in $\mathfrak{B}_\gamma$. We recall this notation and we slightly modify it for our convenience. Recall that, any partition in $\mathfrak{B}_\gamma$ corresponds to one of $\langle i\rangle$, $\langle i^2\rangle$, or $\langle i, j\rangle$ for $0 \le i < j \le p-1$ and $0 \le i \le p-1$. Now, some of these partitions are p-regular, and this depends on γ. Let ${}_i\gamma_j)_{ij}$ be the pyramid of $\mathfrak{B}_\gamma$. For $0 \le i \le j < p$ with $i < p-1$, define

$$\lceil i,j\rfloor := \begin{cases} \langle i+1\rangle & \text{if } i=j \text{ and } {}_{i+1}0_{i+2}\,, \\ \langle i+1,i+2\rangle & \text{if } i=j \text{ and } {}_{i+1}1_{i+2}\,, \\ \langle i+1,j\rangle & \text{if } i\neq j \text{ and } {}_{i+1}0_{j}\,, \\ \langle j^2\rangle & \text{if } i\neq j \text{ and } {}_{i+1}1_{j} \text{ and } {}_{i}0_{j}\,, \\ \langle i\rangle & \text{if } i\neq j \text{ and } {}_{i}1_{j} \text{ and } {}_{i}0_{j+1}, \\ \langle i,j+1\rangle & \text{if } i\neq j \text{ and } {}_{i}1_{j+1}\,. \end{cases}$$

From the abacus and the information encoded in the pyramid, it is easy to see that each p-regular partition corresponds to exactly one of $\lceil i,j\rfloor$ for $0\leq i\leq j<p$ and $i<p-1$. We refer to this notation for p-regular partitions in $\mathfrak{B}_\gamma$ as $\lceil\cdot\rfloor$-notation.

Remark 4.5.8 (counting partitions)**.** This shows that there is a total of $\binom{p-1}{2}+2(p-1)=\frac{p(p+1)-2}{2}$ p-regular partitions partitions in $\mathfrak{B}_\gamma$ (in any block of p-weight 2). Which agrees with Theorem 1.4.9 : it is equal to the number of $(p-1)$-partitions of $w=2$. △

Remark 4.5.9. If $\lceil\cdot\rfloor^{\mathrm{F}}$ denotes Fayers' labelling for p-regular partitions as originally defined in [Fay], then

$$\lceil i,j\rfloor := \begin{cases} \lceil i+1\rfloor^{\mathrm{F}} & \text{if } i=j \text{ and,} \\ \lceil i+1,j\rfloor^{\mathrm{F}} & \text{otherwise.} \end{cases}$$

△

Notice that p-regular partitions are in bijection with the set of **all but one** entry in $({}_i\gamma_j)$ by making

$$\begin{array}{ccc} \mathrm{Reg}_p(\mathfrak{B}_\gamma) & \leftrightsquigarrow & ({}_i\gamma_j) \\ \lceil i,j\rfloor & \longleftrightarrow & {}_i\gamma_j \end{array} \tag{4.7}$$

for $0\leq i\leq j<p$ and $i<p-1$. The reason for using $\lceil\cdot\rfloor$-notation (and shifting the original definition) is the simple description of this correspondence. There is only the entry ${}_{p-1}\gamma_{p-1}$, the last entry to the right of row 0 in the pyramid, which is not associated to a p-regular partition.

Example 4.5.10. Recall the pyramid for the 5-block $\mathfrak{B}_{(2,2)}$ of $\mathbb{F}_5\mathfrak{S}_{14}$, shown in Example 4.5.2. The 5-regular partitions in $\mathfrak{B}_{(2,2)}$ are

$$\mathrm{Reg}_4(\mathfrak{B}_{(2,2)}) = \{(12,2),\ (7^2),\ (6,4^2),\ (3^4,1^2),\ (11,3),\ (7,4,3),\ (6,3^2,1^2),\ (5,3^2,1^3),\\ (3^3,2,1^3),\ (8,3^2),\ (7,3^2,1),\ (6,3,2,1^3),\ (4,3^2,1^4),\ (7,2^2,1^3)\}.$$

In $\lceil\cdot\rfloor$-notation, in the same order:

$$\mathrm{Reg}_4(\mathfrak{B}_{(2,2)}) = \{\lceil 3,3\rfloor,\ \lceil 2,2\rfloor,\ \lceil 1,1\rfloor,\ \lceil 0,0\rfloor,\ \lceil 3,4\rfloor,\ \lceil 2,3\rfloor,\ \lceil 1,3\rfloor,\ \lceil 1,2\rfloor,\\ \lceil 0,1\rfloor,\ \lceil 2,4\rfloor,\ \lceil 1,4\rfloor,\ \lceil 0,3\rfloor,\ \lceil 0,2\rfloor,\lceil 0,4\rfloor\}.$$

△

4.5.2 Richards' ∂ map and the Mullineux map in the pyramid

In this section we explore in detail the correspondence (4.7) between p-regular partitions in $\mathfrak{B}_\gamma$ and entries in the pyramid. We will see how these partitions are distributed in the pyramid: in which positions are the self-Mullineux partitions and what says the position in the pyramid with respect to the dominance order.

Richards' ∂ map

In [Ric96], Richards gives an explicit description of decomposition numbers for blocks of weight 2. For this, he defines a value $\partial\lambda$ associated to every partition λ on such a block. We recall this definition and some of his results which we use later.

As already noted, the core γ of any partition λ in $\mathfrak{B}_\gamma$ is obtained by succesive removal of two rim p-hooks. Let $\partial\lambda$ be the absolute value of the difference of the leg lengths of the two rim p-hooks. Then $0 \leq \partial\lambda = \partial\lambda' \leq p-1$. The value of $\partial\lambda$ is independent of the way in which the rim hooks are removed, see [Ric96, Lemma 4.1]. Then $\partial\lambda$ is well defined.

Denote by ∂_ℓ the set of partitions in $\mathfrak{B}_\gamma$ such that their ∂ value is ℓ, that is,

$$\partial_\ell := \{\lambda \vdash n \mid \lambda \in \mathfrak{B}_\gamma \text{ and } \partial\lambda = \ell\}.$$

We write $\partial_{\gamma,\ell}$ if the block needs to be specified and is not clear from the context. We denote $\partial_\ell^{\mathrm{reg}}$ the subset of p-regular partition in ∂_ℓ, so that $\partial_\ell^{\mathrm{reg}} = \mathrm{Reg}_p(\mathfrak{B}_\gamma) \cap \partial_\ell$. We have

$$\mathfrak{B}_\gamma = \bigsqcup_{\ell=0}^{p-1} \partial_\ell.$$

Example 4.5.11. Following Example 4.5.10, we have, for partitions in the block $\mathfrak{B}_{(2,2)}$ of $\mathbb{F}_5\mathfrak{S}_{14}$:

$$\begin{aligned}
\partial_0 &= \{(2^2,1^{10}),\ (2^7),\ (3^4,1^2),\ (6,4^2),\ (7^2),\ (12,2)\}\\
\partial_1 &= \{(2^3,1^8),\ (3^3,2,1^3),\ (5,3^2,1^3),\ (6,3^2,1^2),\ (7,4,3),\ (11,3)\}\\
\partial_2 &= \{(3^3,1^5),\ (4,3^2,1^4),\ (6,3,2,1^3),\ (7,3^2,1),\ (8,3^2)\}\\
\partial_3 &= \{(6,3,1^5),\ (7,2^2,1^3)\}\\
\partial_4 &= \{(7,2,1^5)\}
\end{aligned}$$

Where highlighted partitions are those which are not 5-regular. The rest of the partitions are 5-regular and, in $\lceil\cdot\rfloor$-notation they are, in the same order:

$$\begin{aligned}
\partial_0^{\mathrm{reg}} &= \{\lceil 0,0\rfloor, \lceil 1,1\rfloor, \lceil 2,2\rfloor, \lceil 3,3\rfloor\}\\
\partial_1^{\mathrm{reg}} &= \{\lceil 0,1\rfloor, \lceil 1,2\rfloor, \lceil 1,3\rfloor, \lceil 2,3\rfloor, \lceil 3,4\rfloor\}\\
\partial_2^{\mathrm{reg}} &= \{\lceil 0,2\rfloor, \lceil 0,3\rfloor, \lceil 1,4\rfloor, \lceil 2,4\rfloor\}\\
\partial_3^{\mathrm{reg}} &= \{\lceil 0,4\rfloor\}
\end{aligned}$$

△

From the abacus and the pyramid, we have some information about particular partitions belonging in some of the sets ∂_ℓ.

Lemma 4.5.12 ([Ric96, §4, p. 398]). *Let $1 \leq k \leq \frac{p-1}{2}$ and let $\nu_k = \left\langle \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rangle$ be a self-conjugate partition in $\mathfrak{B}_\gamma$. Then $\nu_k \in \partial_{2k-1} \cup \partial_{2k}$.*

This is easily seen in the abacus. We include a proof for completeness, which is adapted to notation in this thesis.

Proof. Consider the abacus of a partition λ. Let $1 \leq r \leq p-1$, and let us call *runner-hook* r the hook added to λ by moving down one position the last bead in runner r. Recall, from Remark 1.1.7, that the leg-length of the runner-hook r is the number of beads between the start and final position of the moved bead. Now, If we take the abacus of γ, the leg-length of the runner-hook r is

$$p-1-r.$$

We want to calculate $\partial\nu_k$. For this, we calculate the leg-lengths of two p-hooks successively added to γ to obtain ν_k: the partition $\nu_k = \left\langle \frac{p-1}{2}-k, \frac{p-1}{2}+k \right\rangle$ can be obtained from the abacus of γ by first moving down one bead in runner $\frac{p-1}{2}-k$. From the discussion above the leg-length of this hook is

$$l_1 := p-1-\left(\frac{p-1}{2}-k\right) = \frac{p-1}{2}+k.$$

Now, we have a new abacus in which we have to calculate the leg-length of the runner-hook $\frac{p-1}{2}+k$, runner in which we move the second bead. In the previous abacus for γ, this leg-length would have been $p-1-\left(\frac{p-1}{2}+k\right) = \frac{p-1}{2}-k$, but since we already moved one bead in the abacus for γ, we might have added 1 to this leg-length, since we might have added one bead (the first moved bead) to the set of beads between the start and final position of the second bead. This happens only if the difference of positions of the last beads in runners $\frac{p-1}{2}-k$ and $\frac{p-1}{2}+k$, in the abacus for γ, is less than p. In other words, only if ${}_{(\frac{p-1}{2}-k)}\gamma_{(\frac{p-1}{2}+k)} = 1$. Hence, the leg-length of the runner hook $\frac{p-1}{2}+k$ in this new abacus is

$$l_2 := \begin{cases} \frac{p-1}{2}-k+1 & \text{if } {}_{(\frac{p-1}{2}-k)}\gamma_{(\frac{p-1}{2}+k)} = 1, \\ \frac{p-1}{2}-k & \text{otherwise.} \end{cases}$$

Then

$$\partial\nu_k := \begin{cases} 2k-1 & \text{if } {}_{(\frac{p-1}{2}-k)}\gamma_{(\frac{p-1}{2}+k)} = 1, \\ 2k & \text{otherwise.} \end{cases}$$

□

The family of partitions ∂_0 can be partitioned into two sets ∂_0^+ and ∂_0^- as follows. A partition λ of weight 2 has either two rim p-hooks or one rim p-hook and one rim $2p$-hook (since the p-weight is also equal to the number of rim hooks of length divisible by p). In the first case, Richards showed that the leg lengths of the rim p-hooks are consecutive integers. Define λ to be in ∂_0^+ or ∂_0^- following

$$\lambda \in \partial_0 \text{ has } \begin{cases} \text{two rim } p\text{-hooks} & \begin{cases} \lambda \in \partial_0^+ & \text{if the larger leg is of even length,} \\ \lambda \in \partial_0^- & \text{otherwise.} \end{cases} \\ \text{or} & \\ \text{one rim } p\text{-hook and one rim } 2p\text{-hook} & \begin{cases} \lambda \in \partial_0^+ & \text{if the leg length of the rim } 2p\text{-hook is } \equiv 0 \text{ or } 3 \bmod 4, \\ \lambda \in \partial_0^- & \text{otherwise.} \end{cases} \end{cases}$$

Richards proved the following

Proposition 4.5.13 ([Ric96, Lemma 4.2 and 4.3]). *For $0 \leq i \leq p-1$, the set ∂_ℓ is totally ordered by $\trianglelefteq$. Moreover the partitions of $\mathfrak{B}_\gamma$ which are p-singular are precisely: the smallest partition on each ∂_ℓ for $0 < \ell \leq p-1$ and the smallest partition on each of ∂_0^+ and ∂_0^-.*

Notice the resemblence of the combinatorics of partitions in a set ∂_ℓ with that of partitions in weight 1 blocks shown in §4.4.1.

Remark 4.5.14. Then, there are exactly $(p-1)+2 = p+1$ p-singular partitions in $\mathfrak{B}_\gamma$ (in any block of p-weight 2). Which agrees with the counting of partitions and p-regular partitions in $\mathfrak{B}_\gamma$ in Remark 4.5.4 and Remark 4.5.8. △

Richards's ∂-function on p-regular partitions can be expressed in the $\lceil\cdot\rfloor$ notation. Direct analysis with the abacus of γ and its relation with the leg-lengths of p-hooks of partitions in $\mathfrak{B}_\gamma$ gives:

Proposition 4.5.15 ([Fay, Proposition 4.1]). *Let $({}_i\gamma_j)_{ij}$ be the pyramid of γ. For $0 \leq i \leq j < p$ and $i < p-1$*

$$\partial\lceil i, j\rfloor = j - i - 1 + {}_i\gamma_j.$$

With such an expression for $\partial\lceil i, j\rfloor$, together with the correspondence (4.7), we can say precisely to which positions in the pyramid correspond the partitions $\partial_\ell^{\mathrm{reg}}$ for a given ℓ. This helps for graphically visualizing $\partial_\ell^{\mathrm{reg}}$ in the pyramid:

Corollary 4.5.16. *For $1 \leq l \leq p-1$, the set ∂_ℓ^{reg} is in correspondence with the set of "1" entries in the l-th row on the pyramid and the "0" entries in the $(\ell+1)$-th row. The set ∂_0^{reg} corresponds to the first $p-1$ entries on the 0-th row (all entries in the 0-th row are "1") and the "0" entries in the 1-st row.*

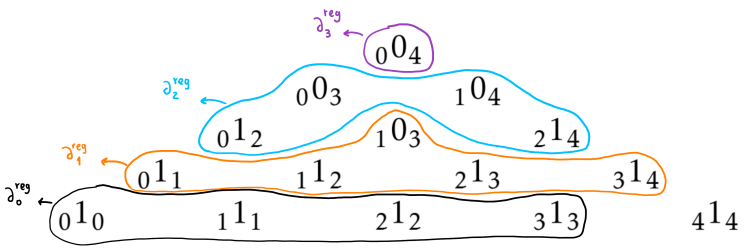
Proof. Let $1 \leq \ell \leq p-1$, and let $\lambda \in \partial_\ell^{\mathrm{reg}}$. Write λ as $\lambda = \lceil i, j\rfloor$ with $0 \leq i \leq j < p$ and $i < p-1$.

Suppose that $i < j$. If ${}_i0_j$, then from Proposition 4.5.15, $j - i = \ell + 1$. So that ${}_i\gamma_j = 0$ is in row $\ell + 1$. Now, if ${}_i1_j$ then from Proposition 4.5.15, $j - i = \ell$. So that ${}_i\gamma_j = 1$ is in row ℓ.

If $i = j$, by definition of the pyramid, ${}_i1_i$ and this entry on the 0-th row. On the other hand, Proposition 4.5.15 implies that $\ell = 0$. And since $0 \leq i < p-1$, this entry is one of the first $p-1$ entries in 0-th row.

□

Example 4.5.17. We continue Example 4.5.11. Associating p-regular partitions in $\mathfrak{B}_{(2,2)}$ with their corresponding entries in the pyramid, we have that, in the pyramid of $\mathfrak{B}_{(2,2)}$, the subsets $\partial_\ell^{\mathrm{reg}}$ are distributed as follows:



△

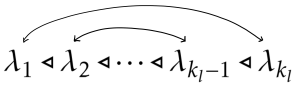
Conjugation and the Mullineux map in the sets ∂_ℓ

We characterize self-conjugate and self-Mullineux partitions in a given set ∂_ℓ, by studying how these involutions behave with respect to $\trianglelefteq$.

Recall, from Proposition 4.5.13 that the sets ∂_ℓ are totally ordered by $\trianglelefteq$. Let $1 \leq \ell \leq p-1$ and denote $k_\ell := |\partial_\ell|$. Denote the partitions in ∂_ℓ as

$$\boxed{\lambda_1} \trianglelefteq \lambda_2 \trianglelefteq \cdots \trianglelefteq \lambda_{k_\ell - 1} \trianglelefteq \lambda_{k_\ell},$$

where the highlighted partition is the unique p-singular partition. Now, from the fact that $\partial\lambda = \partial\lambda'$ for any partition λ, and that conjugation reverses the dominance order ([Mac79, I.1.11]), we have $\lambda'_i = \lambda_{k_\ell - i + 1}$ for any $1 \leq i \leq k_\ell$. Graphically, conjugation does the following in ∂_ℓ for $1 \leq \ell \leq p-1$:



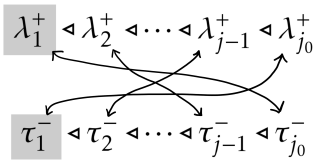
For $\ell = 0$ it is easy to see that, since p is odd, if $\lambda \in \partial_0^+$ then $\lambda' \in \partial_0^-$, that is, conjugation changes the sign of partitions in ∂_0. Then, k_0 is even. Let $k_0 = 2j_0$ and denote partitions in ∂_0^+ as

$$\boxed{\lambda_1^+} \trianglelefteq \lambda_2^+ \trianglelefteq \lambda_3^+ \trianglelefteq \cdots \trianglelefteq \lambda_{j-1}^+ \trianglelefteq \lambda_{j_0}^+,$$

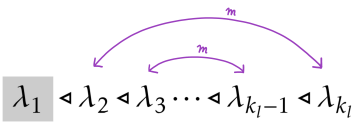
Denote partitions in ∂_0^- as

$$\boxed{\tau_1^-} \trianglelefteq \tau_2^- \trianglelefteq \tau_3^- \trianglelefteq \cdots \trianglelefteq \tau_{j-1}^- \trianglelefteq \tau_{j_0}^-,$$

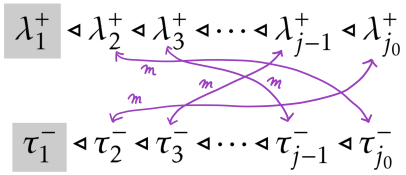
where the highlighted partitions are the only p-singular partitions in ∂_0. Then, since conjugation changes the sign and reverses the dominance order $(\lambda_i^+)' = \tau_{j_0} - i + 1$. Graphically, conjugation does the following in ∂_0:



Let us now see that the Mullineux map behaves in a similar way, but within $\partial_\ell^{\mathrm{reg}}$. Recall that, for a p-regular partition μ, the p-cores of μ and $m_p(\mu)$ are conjugates. Since $\mathfrak{B}_\gamma$ is self-conjugate, then for any $\mu \in \mathrm{Reg}_p(\mathfrak{B}_\gamma)$ the partition $m_p(\mu)$ is also in $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$. Moreover, Richards showed that $(m_p(\mu))'$ is the biggest partition (for $\trianglelefteq$) in $\mathfrak{B}_\gamma$ such that $(m_p(\mu))' \trianglelefteq \mu$ and $\partial(m_p(\mu))' = \partial\mu$, and if $\partial\mu = 0$, it has the same sign as μ ([Ric96, Th. 4.4 and Prop. 2.12]). Then, for $2 \leq i \leq k_\ell$ we have that $(m_p(\lambda_i))' = \lambda_{i-1}$. So that $m_p(\lambda_i) = \lambda'_{i-1} = \lambda_{k_\ell - (i-1)+1} = \lambda_{(k_\ell - i + 1)+1}$. Graphically, the Mullineux map does the following in ∂_ℓ for $1 \leq \ell \leq p-1$:



For $l = 0$, and for $2 \leq i \leq j_0$ we have that $(\boldsymbol{m}_p(\lambda_i^+))' = \lambda_{i-1}^+$. So that $\boldsymbol{m}_p(\lambda_i^+) = (\lambda_{i-1}^+)' = \tau_{j_0-(i-1)+1}^- = \tau_{(j_0-i+1)+1}^-$. Graphically, the Mullineux map does the following in ∂_0:



Remark 4.5.18. Notice the resemblance of the behavior of conjugation and the Mullineux map in each set ∂_ℓ and in a weight 1 block.

The behavior of these two involutions in the sets ∂_l allows to easily deduce in which sets lie their fixed points, that is, the self-conjugate and the self-Mullineux partitions: It all depends on the parities of k_l and j_0, for $1 \leq \ell \leq p-1$ and j_0. The conclusion is summarized in the following lemma.

Lemma 4.5.19. *The set ∂_0 does not contain any self-conjugate or self-Mullineux partition and $|\partial_0|$ is even. For $1 \leq \ell \leq p-1$, the set ∂_ℓ contains* either *exactly one self-conjugate partition or one self-Mullineux partition: If $|\partial_\ell|$ is even, it contains one self-Mullineux partition. If $|\partial_\ell|$ is odd, it contains one self-conjugate partition.*

Proof. From the discussion above describing conjugation in ∂_ℓ and the Mullineux map in $\partial_\ell^{\mathrm{reg}}$, we see that none of these involutions has fixed points in ∂_0. For $\ell \geq 1$, on the other hand, we can see that if $k_\ell = |\partial_\ell|$ is even, then conjugation defines pairs (λ, λ') with $\lambda \neq \lambda'$. In this case $|\partial_\ell^{\mathrm{reg}}| = k_\ell - 1$ is then odd, and the Mullineux map has a unique fixed point: the partition in the middle of the list $\lambda_2, \lambda_3, \ldots, \lambda_{k_\ell}$. The contrary occurs when k_ℓ is odd. □

Example 4.5.20. We continue Example 4.5.17 to illustrate this lemma. The block $\mathfrak{B}_\gamma$ contains exactly two self-conjugate partitions: $(6,3,2,1^3) = \langle 1,3\rangle$ and $(7,2,1^5) = \langle 0,4\rangle$; and two self-Mullineux partitions: $(6,3^1,1^2) = \lceil 1,3\rfloor$ and $(7,2^2,1^3) = \lceil 0,4\rfloor$. The cardinalities of ∂_1, ∂_2, ∂_3 and ∂_4 are respectively 6, 5, 2 and 1. The two self-conjugate partitions are respectively in ∂_2 and ∂_4 and the two self-Mullineux partitions are respectively in ∂_1 and ∂_3. △

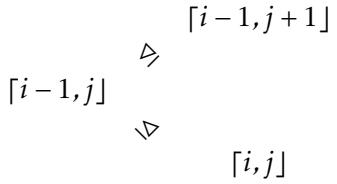
Dominance order in the pyramid

We study how we can compare p-regular partitions in $\mathfrak{B}_\gamma$ with respect to $\trianglelefteq$ depending on their positions in the pyramid of γ. This allows to improve what we already know about the distribution of the sets $\partial_\ell^{\mathrm{reg}}$ in the pyramid (Corollary 4.5.16), and to identify self-Mullineux partitions in the pyramid.

Let $({}_i\gamma_j)_{ij}$ be the pyramid of γ. We identify the entry ${}_i\gamma_j$ with the corresponding p-regular partition $\lceil i,j\rfloor$ in $\mathfrak{B}_\gamma$.

Proposition 4.5.21. *Let $1 \leq i \leq j < p$ and $i < p-1$. Then $\lceil i-1, j\rfloor \trianglelefteq \lceil i-1, j+1\rfloor$ and $\lceil i-1, j\rfloor \trianglelefteq \lceil i, j\rfloor$. In the correspondence with the pyramid array, graphically, we have the*

local configuration



Proof. In [Ric96], Richards introduces a notation for partitions in the block, which depends on the entries of the pyramid. With this notation he obtains a complete description of the dominance order $\trianglelefteq$ in the block (in the appendix §4.7 for Chapter 4 we recall this notation). Every partition in the block corresponds to a pair written $\{s,t\}$ for some $0 \le s < t \le 2p$. Lemma [Ric96, Lemma 4.4] says

$$\{s,t\} \trianglelefteq \{s',t'\} \quad \text{if and only if} \quad s \le s' \quad \text{and} \quad t \le t'. \tag{4.8}$$

We translate Richards' notation in the notation $\lceil\cdot\rfloor$, and using this, our assertion is easily verified. □

This proposition implies that the distribution of the sets $\partial_\ell^{\mathrm{reg}}$ in the pyramid, as we know it from Corollary 4.5.16, does not only correspond to some entries of the pyramid as a set, but this distribution also agrees with the dominance order, which then is increasing from left to right in the pyramid.

Furthermore, in our case, since γ is self-conjugate then, by construction the pyramid is horizontally symmetrical. Hence, each set $\partial_\ell^{\mathrm{reg}}$ occurs in symmetrical positions (is equally distributed with respect to the middle of the pyramid). Then, any self-Mullineux partition $\mu \in \partial_\ell$ (some $1 \le \ell \le p-1$), being in the middle of the list

$$\lambda_2 \trianglelefteq \cdots \trianglelefteq \lambda_\ell,$$

occurs in the middle of the pyramid. And any entry in the middle of the pyramid corresponds, reciprocally to a self-Mullineux partition. We state this fact in the following corollary

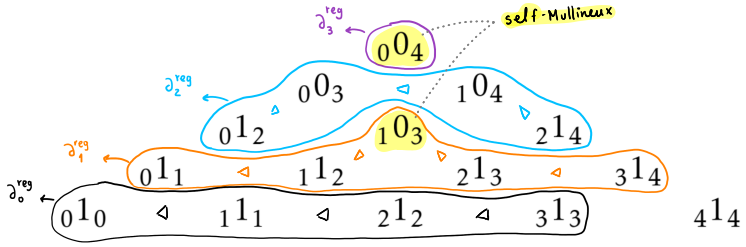
Corollary 4.5.22. *Let $({}_i\gamma_j)_{ij}$ be the pyramid of γ. The self-Mullineux partitions in $\mathfrak{B}_\gamma$ correspond exactly to the entries on the middle column of the pyramid, except for the one on row 0. In $\lceil\cdot\rfloor$-notation, these partition are:*

$$\mu_k = \left\lceil \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rfloor \qquad \text{for} \qquad 1 \le k \le \frac{p-1}{2}.$$

Moreover $\mu_k \in \partial_{2k-1} \cup \partial_{2k}$.

Proof. There is nothing to prove for the first affirmation: Indeed, the indices $i = \frac{p-1}{2} - k$ and $j = \frac{p-1}{2} - k$ for $1 \le k \le \frac{p-1}{2}$ are those exactly in the middle column of the pyramid by construction. Let us see that $\mu_k \in \partial_{2k-1} \cup \partial_{2k}$. Since $\mu_k = \left\lceil \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rfloor$, then μ_k is in row $\left(\frac{p-1}{2} + k\right) - \left(\frac{p-1}{2} - k\right) = 2k$. If ${}_{\frac{p-1}{2}+k}\gamma_{\frac{p-1}{2}-k} = 0$, then by Corollary 4.5.16, $\mu_k \in \partial_{2k-1}^{\mathrm{reg}}$, otherwise $\mu_k \in \partial_{2k}^{\mathrm{reg}}$. □

Example 4.5.23. From the last two results, we can add new information to the diagram in Example 4.5.17, namely (some) dominance order relations, and we can also highlight the entries corresponding to the self-Mullineux partitions in $\mathfrak{B}_{(2,2)}$:



△

Bijection between self-conjugate and self-Mullineux partitions

Here we state the natural correspondance between self-conjugate and self-Mullineux partitions in the block $\mathfrak{B}_\gamma$ given by the chosen notations.

A direct consequence of Lemmas 4.5.6, 4.5.12, 4.5.19 and Corollary 4.5.22 is the following:

Corollary 4.5.24. *There is a natural one-to-one correspondence between the set of self-conjugate partitions and the set of self-Mullineux partitions in $\mathfrak{B}_\gamma$ given by, for $1 \leq k \leq \frac{p-1}{2}$,*

$$\nu_k = \left\langle \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rangle \quad \longleftrightarrow \quad \mu_k = \left\lceil \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rfloor.$$

The partitions μ_k and ν_k are the unique self-Mullineux and self-conjugate partitions, respectively, in the set $\partial_{2k-1} \cup \partial_{2k}$.

Example 4.5.25. We continue Example 4.5.20. We have $\{\nu_1, \mu_1\} \subseteq \partial_1 \cup \partial_2$, where $\nu_1 = \langle 1, 3\rangle = (6, 3, 2, 1^3)$ and $\mu_1 = \lceil 1, 3\rfloor = (6, 3^1, 1^2)$; and $\{\nu_2, \mu_2\} \subseteq \partial_3 \cup \partial_4$, where $\nu_2 = \langle 0, 4\rangle = (7, 2, 1^5)$ and $\mu_2 = \lceil 0, 4\rfloor = (7, 2^2, 1^3)$. △

Remark 4.5.26. In Chapter 3 there is a bijection $\mathrm{BG}_p^n \longleftrightarrow M_p^n$ between the sets of p-BG-partitions of n and the set of p-self-Mullineux partitions of n. Recall that $|\mathcal{M}_p^n \cap \mathfrak{B}| = |\mathrm{BG}_p^n \cap \mathfrak{B}|$ for a p-block $\mathfrak{B}$ (Proposition 4.3.4); the number of p-self-Mullineux partitions in a p-block is equal to the number of p-BG-partitions in the p-block. Now, in a block of weight 2, the set of p-BG-partitions is equal to the set of self-conjugate partitions. On the other hand, from Proposition 3.3.3 the bijection in Chapter 3 restricts to each block, that is, preserves the p-core of a partition. Hence, a question is whether this bijection coincides with the correspondence just defined in Corollary 4.5.24. Tests, which give a positive answer, have been made in GAP ([S+97]) for n up to 56 (p odd $< n$). △

Two lemmas

We state two technical lemmas which are important for the proof of the main theorem (Theorem 4.6.1).

We know "more or less" to which sets ∂_ℓ belong the self-conjugate and self-Mullineux partitions in $\mathfrak{B}_\gamma$; indeed, for $1 \leq \ell \leq \frac{p-1}{2}$, we know that the pair of partitions μ_k and ν_k are in $\partial_{2k-1} \cup \partial_{2k}$, see Corollary 4.5.24. The information encoded in the pyramid of the block $\mathfrak{B}_\gamma$ allows us to be more precise. For this we state the following lemma.

Consider the set $\mu_1, \mu_2, \ldots, \mu_{\frac{p-1}{2}}$ of self-mullineux partitions in $\mathfrak{B}_\gamma$. The entries of the pyramid corresponding to partitions $\mu_1, \mu_2, \ldots, \mu_{\frac{p-1}{2}}$ are are $g_k := {}_i\gamma_j$ with $i = \frac{p-1}{2} - k$ and $j = \frac{p-1}{2} - k$, for $1 \leq k \leq \frac{p-1}{2}$ (Corollary 4.5.22). Such entries are in the middle column of the pyramid. When k runs from 1 to $\frac{p-1}{2}$, these entries run from bottom to top in the pyramid, and because of the definition and properties of the pyramid, the sequence $g_1, g_2, \ldots, g_{\frac{p-1}{2}}$ is of one the forms $1, 1, \ldots, 1, 0, 0, \ldots, 0$, or $1, 1, \ldots, 1$, or else $0, 0, \ldots, 0$.

Definition 4.5.27. In the notation from the preceding paragraph, define $\delta = \delta(\gamma)$ as

$$\delta = \begin{cases} 0 & \text{if } g_k = 0 \text{ for all } 1 \leq k \leq \frac{p-1}{2}\,, \\ \max\{k \mid g_k = 1\} & \text{otherwise.} \end{cases}$$

△

Lemma 4.5.28. *For $k = 1, \ldots, \frac{p-1}{2}$ we have:*

- *If $k \leq \delta$, then $\mu_k \in \partial_{2k}$ and $\nu_k \in \partial_{2k-1}$;*
- *if $k > \delta$, then $\mu_k \in \partial_{2k-1}$ and $\nu_k \in \partial_{2k}$.*

In a table:

$\nu_{\frac{p-1}{2}}$	$\in$	∂_{p-1}
$\mu_{\frac{p-1}{2}}$	$\in$	∂_{p-2}
	$\vdots$	
$\nu_{\delta+1}$	$\in$	$\partial_{2(\delta+1)}$
$\mu_{\delta+1}$	$\in$	$\partial_{2(\delta+1)-1}$
μ_δ	$\in$	$\partial_{2\delta}$
ν_δ	$\in$	$\partial_{2\delta-1}$
	$\vdots$	
μ_2	$\in$	∂_4
ν_2	$\in$	∂_3
μ_1	$\in$	∂_2
ν_1	$\in$	∂_1

Example 4.5.29. We continue Example 4.5.25. The pyramid is in Example 4.5.23. We see that the entries in the middle column corresponding to self-Mullineux partitions are all equal to 0. Hence, here $\delta = 0$, and we have indeed that $\mu_1 \in \partial_1$ and $\mu_2 \in \partial_3$, as this lemma implies. △

Proof of Lemma 4.5.28. The entries $g_1, g_2, \ldots, g_{\frac{p-1}{2}}$ are exactly those in the middle column of the pyramid (except that on row 0 which corresponds to a p-regular partition in ∂_0), from bottom to top and they correspond to the self-Mullineux partitions on the block. These entries are respectively in rows $2, 4, 6, \ldots, p-1$.

Let $1 \leq k \leq \frac{p-1}{2}$. If $k > \delta$, then $g_k = 0$. Hence μ_k which is in row $2k$, belongs to ∂_{2k-1}, by Corollary 4.5.16. Thus, for $k > \delta$, we have $\nu_k \in \partial_{2k}$. If $k \leq \delta$, then $g_k = 1$, and since μ_k is in row $2k$ we know that $\mu_k \in \partial_{2k}$. Then, for $k \leq \delta$ we have $\nu_k \in \partial_{2k-1}$. □

The following is a technical lemma concerning the dominance order in $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$ with respect to positions in the pyramid. This is an adaptation of [Ric96, Lemma 4.4], which is a characterisation of the dominance order in the block to our notation for partitions. This lemma is a key fact in the proof of Theorem 4.6.1.

Let $({}_i\gamma_j)_{ij}$ be the pyramid of the p-core γ. For two entries ${}_i\gamma_j$ and ${}_k\gamma_l$, we say that ${}_i\gamma_j$ *is to the left of* ${}_k\gamma_l$ (or, equivalently ${}_k\gamma_l$ is to the right of ${}_i\gamma_j$) if ${}_i\gamma_j$ is in a column of the pyramid to the left of the column of ${}_k\gamma_l$. In terms of indices, this is equivalent to $i+j<k+l$.

Lemma 4.5.30 ([Ric96, Lemma 4.4])**.** *Let $\lambda,\tau\in\mathrm{Reg}_p(\mathfrak{B}_\gamma)$ such that λ is to the left of τ. Then, $\lambda\trianglelefteq\tau$ or λ and τ are not comparable for the dominance order; written equivalently as $\tau\not\trianglelefteq\lambda$.*

Proof. By definition the pyramid of γ has $2p-1$ columns. Suppose that λ is in column c and τ is in column $c+N$. The proof is by induction on N. The crucial part is the base case. Suppose that $N=1$, and let $\lambda=\lceil i,j\rfloor$ in $\lceil\cdot\rfloor$-notation. Making correspond positions in the pyramid to p-regular partitions the local configuration in column $c+1$ is as follows:

c	$c+1$	
	$\vdots$	
	$\lceil i-2,j+3\rfloor$	A
	$\lceil i-1,j+2\rfloor$	
$\vdots$	$\lceil i,j+1\rfloor$	
$\lambda=\lceil i,j\rfloor$		
$\vdots$	$\lceil i+1,j\rfloor$	
	$\lceil i+2,j-1\rfloor$	B
	$\lceil i+3,j-2\rfloor$	
	$\vdots$	

Now, if $\tau=\lceil i,j+1\rfloor$ or $\tau=\lceil i+1,j\rfloor$ (highlighted in the diagram above), Proposition 4.5.21 says that $\lambda\trianglelefteq\tau$, so that $\tau\not\trianglelefteq\lambda$. Let us split the rest of the partitions (or entries of the pyramid) in this column in two sets A and B, where

$$A=\{\lceil i-k,j+k+1\rfloor \mid k\geq 1\},$$

and

$$B=\{\lceil i+k+1,j-k\rfloor \mid k\geq 1\},$$

where k takes values such that partitions in A and B lie in the pyramid. Consider the two possible cases ${}_i\gamma_j=0$ or ${}_i\gamma_j=1$. In the first case, we necessarily have ${}_i\gamma_{j+1}=0$, as well as all the entries in A. In the second case ${}_{i+1}\gamma_j=1$, as well as all the entries in B. In the pyramid, these configurations look as follows:

Case 1:

c	$c+1$
	0
	⋮
	0
⋮	0
${}_i\gamma_j = 0$	
⋮	*
	⋮
	*

Case 2:

c	$c+1$
	*
	⋮
⋮	*
${}_i\gamma_j = 1$	
⋮	1
	1
	⋮
	1

Let us consider these two cases. In each case, we translate $\lceil\cdot\rfloor$-notation of λ and of τ for each partition τ in column $c+1$, into $\{\cdot\}$-notation to get $\{s,t\}$ and $\{s',t'\}$, respectively and we will see that $\tau \not\trianglelefteq \lambda$ by noticing that $\{s',t'\} \trianglelefteq \{s,t\}$ or they are not comparable, using Proposition 4.7.1 in the appendix §4.7. Consider all possible cases for positions of λ and τ. We get the following values for $\lambda = \{s,t\}$ and $\tau = \{s',t'\}$, where $k \geq 1$:

Case 1:

λ		τ	
s	t	s'	t'
$2i+3$	$2j+1$	$2i-2k+3$	$2j+2k+3$
$2i+2$	$2j+1$	$2i+2k+5$	$\leq 2p$
		$2i+2k+5$	$2j-2k+5$
		$2i+2k+5$	$2j-2k+1$
		$2i+2k+4$	$2j-2k+1$
		$2i+2k+3$	$2j-2k+2$
		$2i+2k+3$	$2j-2k+3$

Case 2:

λ		τ	
s	t	s'	t'
$2i+1$	$2j+3$	$2i+2k+5$	$2j-2k+3$
$2i+2$	$2j+2$	$2i+2k+5$	$\leq 2p$
		$2i+2k+3$	$2j-2k+3$
		$2i-2k+3$	$2j+2k+3$
		$2i-2k+2$	$2j+2k+3$
		$2i-2k+1$	$2j+2k+4$
		$2i-2k+1$	$2j+2k+5$

For each of these possible values for s,s',t,t' we always obtain that $s < s'$ or $t < t'$, then either $\lambda \trianglelefteq \tau$ or λ and τ are not comparable for $\trianglelefteq$, by Proposition 4.7.1. That is $\tau \not\trianglelefteq \lambda$. This concludes the base case.

For the inductive step, suppose that τ is in column $c+N$, with $N > 1$. And suppose that $\tau \trianglelefteq \lambda$. Let us see that there is a contradiction. By Proposition 4.5.21, there is a partition $\tilde{\tau}$ in column $c+(N-1)$ such that $\tilde{\tau} \trianglelefteq \tau$. Since $\tilde{\tau}$ is in column $c+(N-1)$, by induction $\tilde{\tau} \not\trianglelefteq \lambda$. But $\tilde{\tau} \trianglelefteq \tau$ and $\tau \trianglelefteq \lambda$ imply that $\tilde{\tau} \trianglelefteq \lambda$, a contradiction. □

4.6 A SUBS for blocks of weight 2

In this section we will show that the set defined as

$$V^{\gamma}_{n,p} := \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda' \mid \lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\} \sqcup \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \lambda' = \lambda\},$$

is the underlying set of a SUBS for $\mathfrak{B}_\gamma$, a self-conjugate p-block of weight 2 of $\mathbb{F}_p\mathfrak{S}_n$. Denote

$$\begin{aligned} V_\gamma^1 &:= \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) < \lambda\}, \\ V_\gamma^2 &:= \{\lambda' \mid \lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\}, \text{ and} \\ V_\gamma^3 &:= \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \lambda' = \lambda\}. \end{aligned}$$

Then, $V_{n,p}^\gamma = V_\gamma^1 \sqcup V_\gamma^2 \sqcup V_\gamma^3$, and $V_{n,p}^\gamma$ is in one-to-one correspondence with $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$. Indeed, write $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$ as

$$\mathrm{Reg}_p(\mathfrak{B}_\gamma) = W_\gamma^1 \sqcup W_\gamma^2 \sqcup W_\gamma^3,$$

where

$$\begin{aligned} W_\gamma^1 &:= \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \boldsymbol{m}_p(\lambda) < \lambda\}, \\ W_\gamma^2 &:= \{\boldsymbol{m}_p(\lambda) \mid \lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \text{ and } \boldsymbol{m}_p(\lambda) < \lambda\}, \text{ and} \\ W_\gamma^3 &:= \{\lambda \in \mathrm{Reg}_p(\mathfrak{B}_\gamma) \mid \lambda = \boldsymbol{m}_p(\lambda)\}. \end{aligned}$$

Hence, it is straightforward to see that $V_\gamma^1 \sqcup V_\gamma^2$ and $W_\gamma^1 \sqcup W_\gamma^2$ are in bijection, and on the other hand, V_γ^3 and W_γ^3 are in bijection since $V_\gamma^3 = \{\nu_k \mid 1 \le k \le \frac{p-1}{2}\}$ and $W_\gamma^3 = \{\mu_k \mid 1 \le k \le \frac{p-1}{2}\}$. Hence there is a bijection

$$\begin{aligned} \Psi_\gamma : \quad V_{n,p}^\gamma &\longrightarrow \mathrm{Reg}_p(\mathfrak{B}_\gamma) \\ \lambda &\longmapsto \begin{cases} \lambda & \text{if } \lambda \in V_\gamma^1, \\ \boldsymbol{m}_p(\lambda') & \text{if } \lambda \in V_\gamma^2, \\ \mu_k & \text{if } \lambda \in V_\gamma^3 \text{ and } \lambda = \nu_k \text{ for some } 1 \le k \le \frac{p-1}{2}, \end{cases} \end{aligned}$$

which restricts to the bijection from Corollary 4.5.24 on the self-conjugate partitions.

We now define a total order in $\mathfrak{B}_\gamma$. First, label partitions in $V_\gamma^1 = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$, where $t = |V_\gamma^1|$, in such a way that $\lambda_1 > \lambda_2 > \cdots > \lambda_t$ in the lexicographic order. Now, let $\prec$ be a total order in $\mathfrak{B}_\gamma$ such that

$$\begin{aligned} \lambda_1 \succ \lambda_2 \succ \cdots \succ \lambda_t \succ \lambda_1' \succ \lambda_2' \succ \cdots \succ \lambda_t' \succ \nu_1 \succ \nu_2 \succ \cdots \succ \nu_{\delta-1} \\ \succ \nu_\delta \succ \nu_{\frac{e-1}{2}} \succ \nu_{\frac{e-3}{2}} \succ \cdots \succ \nu_{\delta+2} \succ \nu_{\delta+1}, \end{aligned} \tag{4.9}$$

where the number δ is as in Definition 4.5.27, and such that for any other partition $\lambda \in \mathfrak{B}_\gamma \setminus V_{n,p}^\gamma$, we have $\lambda \prec \tau$ for every $\tau \in V_{n,p}^\gamma$. Having defined the bijection Ψ_γ and the total order $\prec$, we can now state our main result:

Theorem 4.6.1. *The set $(V_{n,p}^\gamma, \prec, \Psi_\gamma)$ is a stable unitriangular basic set (SUBS) for the block $\mathfrak{B}_\gamma$.*

We state two important facts about decomposition numbers for the proof:

Proposition 4.6.2 (The Mullineux map [Mat99, §6.4 Rule 11])**.** *Let $\lambda \in \mathrm{Par}(n)$ and $\mu \in \mathrm{Reg}_p(n)$. Then $d_{\lambda\mu} = d_{\lambda'\boldsymbol{m}_p(\mu)}$, where $\boldsymbol{m}_p$ is the Mullineux map.*

Proposition 4.6.3 (Theorem 4.4 [Ric96]). *Let $p \neq 2$ be a prime. Let $\gamma \vdash (n-2p)$ be a p-core and $\mathfrak{B}_\gamma$ the corresponding block of $\mathfrak{S}_n$. Let $\lambda \in \mathfrak{B}_\gamma$ and $\mu \in \mathrm{Reg}_p(\mathfrak{B}_\gamma)$. Then $d_{\lambda\mu} = 1$ if $\lambda = \mu$ or $\lambda = \boldsymbol{m}_p(\mu)'$ or both $\boldsymbol{m}_p(\mu)' \trianglelefteq \lambda \trianglelefteq \mu$ and $\partial\lambda - \partial\mu = \pm 1$; otherwise $d_{\lambda\mu} = 0$.*

Proof of Theorem 4.6.1. The fact that properties (A) and (B) from Definition 4.2.6, hold for $V_{n,p}^\gamma$ are a direct consequence of its definition. Property (A) is true by definition of the set $V_{n,p}^\gamma$. Property (B) holds because, in a p-block of weight 2, every self-conjugate partition is a BG-partition. Let us see this. Recall that, under a certain convention for defining the p-quotient of a partition in a block, if $q_{p,\gamma}(\lambda) = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(p)})$ is the p-quotient of a partition with p-core γ then $q_{p,\gamma'}(\lambda') = (\lambda^{(p)\prime}, \lambda^{(p-1)\prime}, \ldots, \lambda^{(1)\prime})$ is the p-quotient of its conjugate partition λ'. The BG-partitions in a p-block are those self-conjugate partitions ν for which the $(\frac{p+1}{2})$-th partition in the quotient is the empty partition. That is, partitions ν such that the p-quotient is of the form

$$q_{p,\gamma}(\nu) = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\frac{p-1}{2})}, \emptyset, \nu^{(\frac{p-1}{2})\prime}, \ldots, \nu^{(2)\prime}, \lambda^{(1)\prime}).$$

In a block of p-weight 2, the quotient $q_{p,\gamma}(\lambda)$ of a self-conjugate partition λ is a p-multi-partition of total rank 2. Since λ is self-conjugate, $q_{p,\gamma}(\lambda)$ is completely determined by $\lambda^{(1)}, \lambda^{(2)} \ldots, \lambda^{(\frac{p+1}{2})}$, where either $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\frac{p-1}{2})})$ is a multipartition of 1 and $\lambda^{(\frac{p+1}{2})} = \emptyset$, or $\lambda^{(i)} = \emptyset$ for all $1 \leq i \leq \frac{p-1}{2}$ and $\lambda^{(\frac{p+1}{2})}$ is a self-conjugate partition of 2. The second option is not possible, then there exists $1 \leq j < \frac{p+1}{2}$ such that $\nu^{(j)} = (1)$ and $\nu^{(j)} = (1)$ and $\nu^{(i)} = \emptyset$ for $i \neq j$ with $1 \leq i \leq \frac{p+1}{2}$. Hence λ is a BG-partition.

It remains to prove that $(V_{n,p}^\gamma, \prec, \Psi_\gamma)$ is a unitriangular basic set for the block $\mathfrak{B}_\gamma$. For this, consider the square matrix $\widetilde{\mathbf{D}}_\gamma$ formed by the rows of $\mathbf{D}_\gamma$ indexed by $V_{n,p}^\gamma$, arranged according to the total order $\prec$ and columns organized as follows.

	λ_1 $\cdots$ λ_t	$\boldsymbol{m}_p(\lambda_1)$ $\cdots$ $\boldsymbol{m}_p(\lambda_t)$	μ_1 $\cdots$ $\mu_{\delta+1}$
λ_1			
$\vdots$	$\mathbf{D}_1$	$\mathbf{D}_2$	
λ_t			
λ_1'			$\mathbf{D}_5$
$\vdots$	$\mathbf{D}_3$	$\mathbf{D}_4$	
λ_t'			
ν_1			
$\vdots$	$* * *$		$\mathbf{D}_6$
$\nu_{\delta+1}$			

We will show that $\widetilde{\mathbf{D}}_\gamma$ is lower unitriangular. We do it by steps: first, for the square submatrices $\mathbf{D}_1, \ldots, \mathbf{D}_4$, we show that $\mathbf{D}_1 = \mathbf{D}_4$ is lower unitriangular and that $\mathbf{D}_2 = \mathbf{D}_3 = (0)_{t\times t}$. We show as well that $\mathbf{D}_5 = (0)_{2t\times\frac{p-1}{2}}$. Finally we show that $\mathbf{D}_6$ is lower

unitriangular. Having shown this we will have that $(\tilde{V}_\gamma, \prec, \Psi_\gamma)$ is a stable unitriangular basic set for the block $\mathfrak{B}_\gamma$.

$\mathbf{D}_1$ and $\mathbf{D}_4$: Since λ_i is p-regular for $1 \leq i \leq t$, then $d_{\lambda_i\lambda_i} = 1$ for $1 \leq i \leq t$. Let $1 \leq i, j \leq t$. By definition of $\prec$, if $\lambda_i \succ \lambda_j$, then $\lambda_i > \lambda_j$ for the lexicographic order. Hence either $\lambda_i \trianglerighteq \lambda_j$ or λ_i and λ_j are not comparable for $\trianglelefteq$. By Theorem 1.4.3, $d_{\lambda_i\lambda_j} = 0$. This shows that $\mathbf{D}_1$ is lower unitriangular. For $1 \leq i, j \leq t$, by Proposition 4.6.2, $d_{\lambda_i' \boldsymbol{m}_p(\lambda_j)} = d_{\lambda_i\lambda_j}$. Hence $\mathbf{D}_4 = \mathbf{D}_1$ is lower unitriangular.

$\mathbf{D}_2$, $\mathbf{D}_3$, and $\mathbf{D}_5$: For studying $\mathbf{D}_2$, $\mathbf{D}_3$ and $\mathbf{D}_5$, notice that we have the following property: Let $\lambda, \tau \in W_\gamma^1$ such that $\partial\lambda, \partial\tau \geq 1$ and let $\mu \in W_\gamma^3$. Then $\lambda \trianglelefteq \mu$ and $\lambda \trianglelefteq \boldsymbol{m}_p(\tau)$.

We prove this property. Let $1 \leq l \leq p-1$. From §4.5.2 we know that, depending on $|\partial_l^{\mathrm{reg}}|$, the partitions in $\partial_l^{\mathrm{reg}}$ can be listed as either

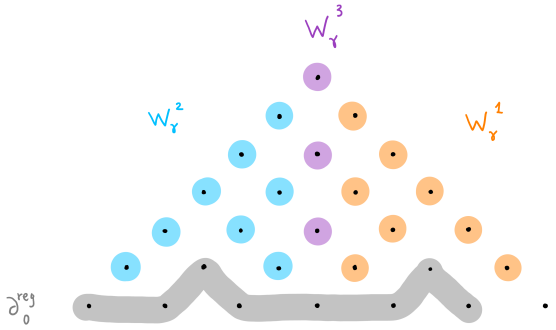
$$\tau_1 \triangleright \tau_2 \triangleright \cdots \triangleright \tau_r \triangleright \mu \triangleright \boldsymbol{m}_p(\tau_r) \triangleright \cdots \triangleright \boldsymbol{m}_p(\tau_2) \triangleright \boldsymbol{m}_p(\tau_1),$$

or

$$\tau_1 \triangleright \tau_2 \triangleright \cdots \triangleright \tau_r \triangleright \boldsymbol{m}_p(\tau_r) \triangleright \cdots \triangleright \boldsymbol{m}_p(\tau_2) \triangleright \boldsymbol{m}_p(\tau_1),$$

where $r = \left\lfloor \frac{|\partial_l^{\mathrm{reg}}|}{2} \right\rfloor$, $\boldsymbol{m}_p$ is the Mullineux map and μ is some self-Mullineux partition.

On the other hand, by Corollary 4.5.16, Corollary 4.5.22 and Proposition 4.5.21, we know that these partitions are distributed in the pyramid from left to right in increasing dominance order, horizontally symmetrical and that the self-Mullineux partitions are in the middle column. These observations allow to identify three zones in the pyramid with the intersection of $\bigcup_{l\geq 1} \partial_l^{\mathrm{reg}}$ with the three subsets W_γ^1, W_γ^2 and W_γ^3 of $\mathrm{Reg}_p(\mathfrak{B}_\gamma)$: W_γ^1 is the left half, W_γ^2 is the right half and W_γ^3 is the column in the middle:



Now, let $\lambda, \tau \in W_\gamma^1$ with $\partial\lambda, \partial\tau \geq 1$ and let $\mu \in W_\gamma^3$. Then μ is to the left of λ in the pyramid, and $\boldsymbol{m}_p(\tau)$ is to the left of λ. By Lemma 4.5.30, $\lambda \trianglelefteq \mu$ and $\lambda \trianglelefteq \boldsymbol{m}_p(\tau)$, which concludes the proof of this property.

Let us see that $\mathbf{D}_2$, $\mathbf{D}_3$ and $\mathbf{D}_5$ are matrices of zeros. Let $\lambda_i, \lambda_j \in V_\gamma^1 = W_\gamma^1$ for some $1 \leq i \leq t$. From the affirmation above, $\lambda_i \trianglelefteq \boldsymbol{m}_p(\lambda_j)$. Then, by Theorem 1.4.3, $d_{\lambda_i \boldsymbol{m}_p(\lambda_j)} = 0$. That is, $\mathbf{D}_2 = (0)_{t\times t}$. On the other hand, by Proposition 4.6.2 $d_{\lambda_i'\lambda_j} = d_{\lambda_i \boldsymbol{m}_p(\lambda_j)} = 0$, so that $\mathbf{D}_3 = \mathbf{D}_2 = (0)_{t\times t}$.

Let $\mu_k \in W_\gamma^3$, for some $1 \leq k \leq \frac{p-1}{2}$. From the assertion above, $\lambda_i \trianglelefteq \mu_k$. Then by Theorem 1.4.3, $d_{\lambda_i\mu_k} = 0$. On the other hand, $d_{\lambda_i'\mu_k} = d_{\lambda_i \boldsymbol{m}_p(\mu_k)} = d_{\lambda_i\mu_k} = 0$. This shows

that $\mathbf{D}_5 = (0)_{2t\times\frac{p-1}{2}}$.

$\underline{\mathbf{D}_6}$: It remains to show that $\mathbf{D}_6$, is lower unitriangular. We first prove that it is lower triangular. Consider the matrix $\widetilde{\mathbf{D}}_6$ defined by taking $\mathbf{D}_6$ and ordering rows as $\nu_1, \nu_2, \ldots, \nu_{\frac{p-1}{2}}$ and columns as $\mu_1, \mu_2, \ldots, \mu_{\frac{p-1}{2}}$. The matrix $\widetilde{\mathbf{D}}_6$ is "*almost*" lower triangular. For making clear what we mean by "*almost*", let us study the precise form of the matrix, with Proposition 4.6.3 and Lemma 4.5.28 (for clarity see Example 4.6.4 after this proof). From Proposition 4.6.3, in $\mathbf{D}_\gamma$ every entry in the column μ_k is zero except for rows μ_k, $\boldsymbol{m}_p(\mu_k)' = \mu_k'$ which are 1, and any λ with $\mu_k' \trianglelefteq \lambda \trianglelefteq \mu_k$ and $\partial\lambda - \partial\mu_k = \pm 1$, which is also 1. Now if $\mu_k \in \partial_l$ then $\mu_k' \in \partial_l$, and for $\widetilde{\mathbf{D}}_6$ we are only interested in rows corresponding to the self-conjugate partitions in $\mathfrak{B}_\gamma$. Self-conjugate partitions and self-Mullineux partitions are never in a same set ∂_l, so that μ_k' is not self-conjugate. Hence we are only left with looking for partitions λ with $\mu_k' \trianglelefteq \lambda \trianglelefteq \mu_k$ and $\partial\lambda - \partial\mu_k = \pm 1$, among self-conjugate partitions $\nu_1, \nu_2, \ldots, \nu_{\frac{p-1}{2}}$. Consider the possible two cases: $\delta = 0$ of $\delta \geq 1$.

If $\delta = 0$, from Lemma 4.5.28, the partitions $\mu_1, \nu_1, \mu_2, \nu_2, \ldots \mu_{\frac{p-1}{2}}, \nu_{\frac{p-1}{2}}$ belong respectively, in that same order, to sets $\partial_1, \partial_2, \partial_3, \partial_4, \ldots, \partial_{\frac{p-3}{2}}, \partial_{\frac{p-1}{2}}$. Then, the first column of $\widetilde{\mathbf{D}}_6$, column μ_1, has possibly a 1 only in row ν_1, since $\partial\nu_1 - \partial\mu_k = 1$, and the rest of entries in this column are equal to 0 since $\partial\nu_i - \partial\mu_k = 1$ for $i \neq 1$. For $1 < k \leq \frac{p-1}{2}$, $\mu_k \in \partial_{2k-1}$ has only two 1; one in row $\nu_{k-1} \in \partial_{2k-2}$ and one in row $\nu_k \in \partial_{2k}$. In this case, then $\widetilde{\mathbf{D}}_6$ takes the following form:

$$\begin{array}{c|ccccc}
 & \mu_1 & \mu_2 & \cdots & \mu_{\frac{p-3}{2}} & \mu_{\frac{p-1}{2}} \\
\nu_1 & * & * & & & \\
\nu_2 & \cdot & * & & & \\
\vdots & & & \ddots & & \\
\nu_{\frac{e-3}{2}} & & & & * & * \\
\nu_{\frac{e-1}{2}} & & & & \cdot & *
\end{array}$$

Where "$*$" is either 0 or 1 (we will see that it is 1) and dots are 0.

If $\delta \geq 1$, the partitions $\nu_1, \mu_1, \nu_2, \mu_2, \ldots, \nu_\delta, \mu_\delta, \mu_{\delta+1}, \nu_{\delta+1}, \ldots, \mu_{\frac{p-1}{2}}, \nu_{\frac{p-1}{2}}$ belong respectively, in that same order, to sets $\partial_1, \partial_2, \partial_3, \partial_4, \ldots, \partial_{\frac{p-3}{2}}, \partial_{\frac{p-1}{2}}$. Hence, for a similar reasoning, starting from column $\mu_{\delta+1}$, the matrix $\widetilde{\mathbf{D}}_6$ is of the same form as in the case $\delta = 0$, but columns μ_k with $1 \leq k \leq \delta$, have a 1 in rows ν_k and ν_{k+1}:

$$\begin{array}{c|ccccccccc}
 & \mu_1 & \mu_2 & \cdots & \mu_\delta & \mu_{\delta+1} & \mu_{\delta+2} & \cdots & \mu_{\frac{p-1}{2}} \\
\nu_1 & * & \cdot & & & & & & \\
\nu_2 & * & * & & & & & & \\
\vdots & & & \ddots & & & & & \\
\nu_\delta & & & & * & & & & \\
\nu_{\delta+1} & & & & & * & * & & \\
\nu_{\delta+2} & & & & & \cdot & * & & \\
\vdots & & & & & & & \ddots & \\
\nu_{\frac{p-1}{2}} & & & & & & & & *
\end{array}$$

In any of the two cases we can see that the total order

$$\nu_1 \succ \nu_2 \succ \cdots \succ \nu_{\delta-1} \succ \nu_\delta \succ \nu_{\frac{p-1}{2}} \succ \nu_{\frac{p-3}{2}} \succ \cdots \succ \nu_{\delta+2} \succ \nu_{\delta+1},$$

which is the order chosen in matrix $\mathbf{D}_6$, makes this matrix lower triangular. It remains to prove that the entries in the diagonal are 1. That is, for every $1 \leq k \leq \frac{p-1}{2}$, we have to prove that $d_{\nu_k\mu_k} = 1$. For this, we use Tables 1 and 2 in [Ric96], adapted by Fayers in [Fay, Proposition 3.1]. Recall that $\nu_k = \left\langle \frac{p-1}{2} - k, \frac{p-1}{2} + k \right\rangle$ and $\mu_k = \lceil \frac{p-1}{2} - k, \frac{p-1}{2} + k \rfloor$. From [Fay, Proposition 3.1], we have in particular that if $\mu = \lceil i-1, j \rfloor$ and $\lambda = \langle i-1, j \rangle$ then $d_{\lambda\mu} = 1$, since for $\mu = \mu_k$ and $\lambda = \nu_k$ we are exactly in that case, then $d_{\nu_k\mu_k} = 1$. Hence $\mathbf{D}_6$ is lower unitriangular and this concludes the proof of Theorem 4.6.1.

□

Example 4.6.4. Let $p = 11$, $n = 36$ and let γ be the 11-core $\gamma = (7, 2, 1^5)$. The self-Mullineux partitions in the block $\mathfrak{B}_\gamma$ of weight 2 of $\mathbb{F}_{11}\mathfrak{S}_{36}$ are

$$\mu_1 = (9, 8, 6, 4, 3, 2^3),\ \mu_2 = (10, 8, 5, 4, 2^4, 1),\ \mu_3 = (12, 8, 4, 2^5, 1^2),$$
$$\mu_4 = (12, 8, 3, 2^5, 1^3),\ \mu_5 = (18, 2^7, 1^4).$$

They belong respectively to $\partial_2, \partial_4, \partial_6, \partial_7, \partial_9$. The self-conjugate partitions in $\mathfrak{B}_\gamma$ are

$$\nu_1 = (8^2, 6, 4, 3^2, 2^2),\ \nu_2 = (9, 8, 5, 4, 3, 2^3, 1),\ \nu_3 = (10, 8, 4^2, 2^4, 1^2),$$
$$\nu_4 = (12, 8, 2^6, 1^4),\ \nu_5 = (18, 2, 1^{16}).$$

They belong respectively to $\partial_1, \partial_3, \partial_5, \partial_8, \partial_{10}$. The matrix $\widetilde{\mathbf{D}}_\gamma$ is

	μ_1	μ_2	μ_3	μ_4	μ_5
ν_1	1				
ν_2	1	1			
ν_3	·	1	1		
ν_4	·	·	·	1	1
ν_5	·	·	·	·	1

△

4.7 Appendix for Chapter 4

In this appendix we recall the system of abbreviations given by Richards in [Ric96] for partitions of p-weight 2.

We fix a p-core γ. Consider the abacus configuration for γ and let $({}_i\gamma_j)$ be the corresponding pyramid. Let $\rho_0 < \rho_1 < \cdots < \rho_{p-1}$ be the positions of the lowest beads on each runner (as in 1.1.7). The abacus configuration for a partition of n with p-core γ and p-weight 2 can be obtained from that for γ in one of three ways:

(a) Moving down one space the beads at positions ρ_i and ρ_j for some $0 \leq i < j \leq p-1$, or

(b) Moving down one space the beads at positions ρ_i and $\rho_i - p$ for some $0 \leq i \leq p-1$, or

(c) Moving down two spaces the bead at position ρ_i for some $0 \leq i \leq p-1$.

In the system of abbreviations, each partition is represented by $\{s,t\}$ for some $0 \leq s < t \leq 2p$, according to the following rules. For partitions of the type (a) we set $s = 2i+1$ and $t = 2j+1$. For partitions of the type (b) we set $s = 2k$ and $t = 2i+1$, where k is the largest such that $0 \leq k \leq i$ and ${}_{k-1}\gamma_i = 0$. Finally, for partitions of the type (c) we set $s = 2i+1$ and $t = 2k+2$, where k is the smallest such that $i \leq k \leq p-1$ and ${}_i\gamma_{k+1}$.

This system of notations has the following property:

Proposition 4.7.1 ([Ric96, Lemma 4.4])**.** *We have*

$$\{s,t\} \trianglelefteq \{s',t'\} \quad \text{if and only if} \quad s \leq s' \quad \text{and} \quad t \leq t'.$$

Hence, the dominance order is easy to describe using this system of notations for partitions.

Appendices

GAP functions

In this appendix we include the code for some functions used when working in this thesis. The functions are written in GAP 3 [S+97] and they require to load the package Specht [Mat]. A file *selfmuBG.g* containing the functions below and a more readable documentation can be found in the Github repository [Ber21b].

The function `EBeta` is the bijection from Theorem 4.6.1. `EBeta` takes an odd prime p and a p-self-Mullineux partitions and it returns the corresponding p-BG-partition. The inverse is `EBetaInverse`. For defining these functions, there is a list of auxiliary functions.

```
DiagramPartition:=function(mu)
# Takes a partition mu and returns the set of nodes
# that form the Young diagram of mu.
local d,i,j;
if mu=[] then
    return [];
else
     d:=[];
     for i in [1..Length(mu)] do
     for j in [1..mu[i]] do
         Add(d,[i,j]);
     od;
     od;
     return Set(d);
fi;
end;

Rim:=function(mu)
# Takes a partition mu and returns the set of nodes
# (i,j) of the rim of mu
local r,c;
r:=[];
for c in DiagramPartition(mu) do
        if not ( [c[1]+1,c[2]+1] in DiagramPartition(mu) ) then
      Add(r,c);
  fi;
```

```
od;
return Set(r);
end;

OrgRim:=function(mu)
# Takes a partition mu and returns a list of the nodes
# of the rim of the diagram of mu, organized from
# "north-east" to "south-west" of the Young diagram.
local r,i,j;
r:=[];
if mu=[] then
        return [];
else
     for i in [1..Length(mu)] do
       j:= mu[i];
      while j > 0 do
       if [i,j] in Rim(mu) then
           Add(r,[i,j]);
      fi;
      j:= j-1;
      od;
      od;
fi;
return r;
end;

HookLength:=function(la,i,j)
# Takes a partition la and a node (i,j) in the Young
# diagram of la and returns the hook-length of the
# (i,j)-th hook
local lap,l;
lap:=ConjugatePartition(la);
l:= Length(la);
if i>=1 and i<=l and j>=1 and j<=la[i] then
        return la[i]-j+lap[j]-i+1;
else
        Print("the node (",i,",",j,") is not in the
        Young diagram of ",la,"\n");
fi;
end;

PartitionDiag:=function(nodes)
# Takes a set of nodes forming a Young diagram and
# returns the associated partition.
local col,n,b,mu,i,j;
mu:=[];
n:= Length(nodes);
if nodes=[] then
```

```
     return [];
else
        i:=1;
        col:=[];
        while i<=n do
                for j in [1..n] do
             if [i,j] in nodes then
             Add(col,j);
              fi;
         od;
         if col=[] then
         return mu;
     else
     Add(mu,Maximum(col));
    fi;
                col:=[];
    i:=i+1;
od;
    return mu;
fi;
end;

DiagBoxRim:=function(mu)
# Takes a self-conjugate partition mu an returns i the
# only diagonal
where (i,i) is the only diagonal node in the rim of mu.
local i;
for i in [1..Length(mu)] do
    if [i,i] in Rim(mu) then
  return i;
  fi;
od;
end;

PRim1:=function(e,mu)
# Set U of the p-rim* of mu.
local n,res,rim,d,i,k,mil,r,l,m,x,b,compte,muprime,
prim,j;
rim:= OrgRim(mu);
prim:= [];
m:= DiagBoxRim(mu);
d:=Position(rim,[m,m]);
if mu=[] then
return [];
elif Length(rim) <= e then
return rim;
else
    i:=1;
```

```
while i <= d do
compte:= e;
j:=i;
while compte <> 0 do
if j <= Length(rim) then
Add(prim,rim[j]);
compte:= compte - 1;
j:=j+1;
else
# Add(prim,rim[j]);
return prim;
fi;

od;
k:=j-1;
if j-1 = Length(rim) then
return prim;
     elif j-1 >= d then
return prim;
else
while rim[k][1] = rim[j-1][1] do
k:= k + 1;
od;
    i:= k;
    compte:= e;
fi;
od;
fi;
return prim;
end;

PRim2:=function(l,e)
# Fix the middle segment.
local r,enlever,b,mil,x;
r:= Length(l) mod e;
if r = 0 then
r := e;
fi;
mil:=l{[Length(l)-r+1..Length(l)]};
x:= mil[1];
enlever:=[];
for b in mil do
if b[2]<x[1] then
        Add(enlever,b);
    fi;
od;
l:=Set(l);
for y in enlever do
```

```
RemoveSet(l,y);
od;
    return l;
end;

PRim3:=function(e,mu)
# The subset U of the p-rim*
local rim;
rim:= PRim1(e,mu);
rim:=PRim2(rim,e);
return rim;
end;

PRim4:=function(e,mu)
# The reflection of subset U of the p-rim*
local rim,srim,j,i,m,b,d,a,k,l,orgsrim,compte,res;
rim:=OrgRim(mu);
srim:=Set(rim);
m:= DiagBoxRim(mu);
d:=Position(rim,[m,m]);
for b in rim do
    if (Position(rim,b) <= d) or b in PRim3(e,mu) then
RemoveSet(srim,b);
fi;
od;
orgsrim:=[];
j:=Length(mu);
while j>0 do
     for i in [1..mu[j]] do
     if [j,i] in srim then
         Add(orgsrim,[j,i]);
     fi;
     od;
     j:=j-1;
od;
res:=[];
a:=1;
compte:=e;
if Length(orgsrim)<=e then
return orgsrim;
else
while a<=Length(orgsrim) do
k:=a;
while compte <> 0 do
if k <= Length(orgsrim) then
Add(res,orgsrim[k]);
compte:= compte-1;
k:=k+1;
```

```
else
return res;
fi;
od;
l:=k-1;
if k-1 = Length(orgsrim) then
return res;
else
while orgsrim[l][2] = orgsrim[k-1][2] do
l:= l + 1;
if l>Length(orgsrim) then
return res;
fi;
od;
a:= l;
compte:= e;
fi;
od;
fi;
return res;
end;

PRim5:=function(e,mu)
# p-rim* for self-conjugate mu
local l;
l:=PRim3(e,mu);
Append(l,PRim4(e,mu));
return l;
end;

ERim:=function(e,mu)
#Takes an odd prime e and a self-conjugate partition
# mu and returns[mu^(1)*,e-rim*] where e-rim is the
# e-rim* of a self-conjugate partition and mu^(1)*
# is the partition obtained from mu after deleting
#the e-rim*.
local diag,erim,b ;
erim:=PRim5(e,mu);
diag:=Set(DiagramPartition(mu));
for b in erim do
RemoveSet(diag,b);
od;
return [PartitionDiag(diag),erim];
end;

AutoSymb:=function(e,mu)
# Takes a prime e and a self-conjugate partition mu
# and returns the first line of the (first line of the)# p-BG-symbol for mu.
```

```
local muprime,symb;
symb:=[];
if mu=[] then
return [];
else
muprime:=mu;
while muprime <> [] do
Add(symb,Length(ERim(e,muprime)[2]));
muprime:=ERim(e,muprime)[1];
od;
return symb;
fi;
end;

EBGsymb:=function(p,mu)
# Takes a prime p and a self-conjugate partition mu
# and returns the p-BG-symbol mu
local i, symb;
symb:=[];
symb[1]:=AutoSymb(p,mu);
symb[2]:=[];
for i in AutoSymb(p,mu) do
    if EuclideanRemainder(i,2)=0 then
    Add(symb[2],i/2);
else
    Add(symb[2],(i+1)/2);
fi;
od;
return symb;
end;

EBetaInverse:=function(e,la)
# Takes an odd prime e, and a e-BG-partition la and
# returns the e-self-Mullineux partition corresponding
# to la under the bijection defined by the bg-symbol.
local n,m,i,l,j ;
n:=Sum(la);
m:=1;
j:=1;
l:=Length(la);
if la<>ConjugatePartition(la) then
return Print("E: this is not a BG-partition. \n");
fi;
while  la[m]>=m do
m:=m+1;
od;
m:=m-1;
while j<=m and HookLength(la,j,j) mod e <> 0 do
```

```
j:=j+1;
od;
j:=j-1;
if j<m then
return Print("E: this is not a BG-partition. \n");
fi;
return PartitionMullineuxSymbol(e,EBGsymb(e,la));
end;

DurfeeNumber:=function(mu)
# takes a partition and returns its durfee number.
local d,i,l;
d:=1;
l:=Length(mu);
if mu=[0] then
        return 0;
else
    while d <= l and d<=mu[d] do
        d:=d+1;
        od;
fi;
return d-1;
end;

EBGPartitions:=function(p,n)
# Takes an odd prime p and an integer n and returns
# the list of p-BG-partitions of n.
local list,d,la,i,inlist;
list:=[];
for la in Partitions(n) do
        if la=ConjugatePartition(la) then
            d:=DurfeeNumber(la);
            i:=1;
            inlist:=true;
            while i <= d and inlist=true do
                if HookLength(la,i,i) mod p = 0 then
                    inlist:=false;
                else i:= i+1;
                fi;
            od;
        if inlist=true then
            Add(list,la);
        fi;
        fi;
od;
return list;
end;
```

```
EBetaSymbol:=function(e,symb)
# Takes a prime e and the Mullineux symbol symb (list
# of rows) of a e-self-Mullineux partition and returns
# the associated e-BG-partition.
local k,i,n,s;
if MullineuxMap(e,PartitionMullineuxSymbol(e,symb))=
PartitionMullineuxSymbol(e,symb) then
        n:=Sum(symb[1]);
        k:=0;
        i:=1;
        while k=0 do
        if EBGsymb(e,EBGPartitions(e,n)[i])=symb then
        k:=1;
        s:=EBGPartitions(e,n)[i];
        else
         i:=i+1;
        fi;
        od;
        return  s;
else
return Print("This is not the symbol of a ",e,"-self-mullineux partition.
        ","\n");
fi;
end;

EBeta:=function(e,mu)
# Takes a prime e and a e-self-Mullineux mu and
# returns the e-BG-partition partition corresponding
# to mu under the bijection defined by the bg-symbol.
local symb ;
if MullineuxMap(e,mu)=mu then
    return EBetaSymbol(e,MullineuxSymbol(e,mu));
else return Print(mu," is not a ",e,"-self-mullineux partition");
fi;
end;
```

Bibliography

[AE04] G. Andrews and K. Eriksson. *Integer partitions*. Cambridge University Press, Cambridge, 2004.

[Alp86] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Modular representations as an introduction to the local representation theory of finite groups.

[AO91] G. Andrews and J. Olsson. Partition identities with an application to group representation theory. *J. Reine Angew. Math.*, 413:198–212, 1991.

[Ber21a] A. Bernal. On self-mullineux and self-conjugate partitions. *Electron. J. Combin.*, 28, 2021.

[Ber21b] A. Bernal. Self-mullineux-SUBS. https://github.com/ana-bernal/self-Mullineux-SUBS, 2021.

[Ber21c] A. Bernal. Unitriangular basic sets for blocks of the symmetric and alternating groups of small weights, 2021. To appear in *Journal of Algebra*. arXiv:2106.07215.

[Bes91] C. Bessenrodt. A combinatorial proof of a refinement of the Andrews-Olsson partition identity. *European J. Combin.*, 12(4):271–276, 1991.

[BG10] O. Brunat and J.-B. Gramain. A basic set for the alternating group. *J. Reine Angew. Math.*, 641:177–202, 2010.

[BGJ20] O. Brunat, J.-B. Gramain, and N. Jacon. On unitriangular basic sets for symmetric and alternating groups, arxiv:2011.00815., 2020. Preprint arXiv:2011.00815.

[BK03] J. Brundan and J. Kujawa. A new proof of the Mullineux conjecture. *J. Algebraic Combin.*, 18(1):13–39, 2003.

[BO94] C. Bessenrodt and J. Olsson. On Mullineux symbols. *J. Combin. Theory Ser. A*, 68(2):340–360, 1994.

[BO98] C. Bessenrodt and J. Olsson. On residue symbols and the Mullineux conjecture. *J. Algebraic Combin.*, 7(3):227–251, 1998.

[Bra47] R. Brauer. On a conjecture by Nakayama. *Trans. Roy. Soc. Canada Sect. III*, 41:11–19, 1947.

[Bre99] D. Bressoud. *Proofs and confirmations*. MAA Spectrum. Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 1999. The story of the alternating sign matrix conjecture.

[Bru98] J. Brundan. Modular branching rules and the Mullineux map for Hecke algebras of type *A*. *Proc. London Math. Soc. (3)*, 77(3):551–581, 1998.

[Cli37] A. H. Clifford. Representations induced in an invariant subgroup. *Ann. of Math. (2)*, 38(3):533–550, 1937.

[CR62] C. Curtis and I. Reiner. Representation theory of finite groups and associative algebras. Pure and Applied Mathematics. 11. New York-London: Interscience Publishers, a division of John Wiley & Sons. xiv, 685 pp, 1962.

[CR08] Joseph Chuang and Raphaël Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math. (2)*, 167(1):245–298, 2008.

[Cur99] C. W. Curtis. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*, volume 15 of *History of Mathematics*. American Mathematical Society, Providence, RI; London Mathematical Society, London, 1999.

[DJ86] R. Dipper and G. James. Representations of Hecke algebras of general linear groups. *Proc. London Math. Soc. (3)*, 52(1):20–52, 1986.

[Fay] M. Fayers. Data for weight 2 blocks of Hecke algebras of type A. http://www.maths.qmul.ac.uk/~mf/papers/wt2data.pdf. Accessed: 2021-05-28.

[Fay08] M. Fayers. Regularisation and the Mullineux map. *Electron. J. Combin.*, 15(1):Research Paper 142, 15, 2008.

[Fay14] M. Fayers. A generalisation of core partitions. *J. Combin. Theory Ser. A*, 127:58–84, 2014.

[Fay21a] M. Fayers. Crystals, regularisation and the Mullineux map, 2021. Preprint arXiv:2105.06302.

[Fay21b] M. Fayers. The Mullineux map. https://groups.oist.jp/representations/seminars, 2021. Talk on OIST Representation Theory Seminar.

[Fei82] W. Feit. *The representation theory of finite groups*, volume 25. Elsevier (North-Holland), Amsterdam, 1982.

[FK97] B. Ford and A. Kleshchev. A proof of the Mullineux conjecture. *Math. Z.*, 226(2):267–308, 1997.

[For97] B. Ford. Irreducible representations of the alternating group in odd characteristic. *Proc. Amer. Math. Soc.*, 125(2):375–380, 1997.

[FRT54] J. S. Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric groups. *Canad. J. Math.*, 6:316–324, 1954.

[Haw71] T. Hawkins. The origins of the theory of group characters. *Arch. History Exact Sci.*, 7(2):142–170, 1971.

[Isa06] M. Isaacs. *Character theory of finite groups*. AMS Chelsea Publishing, Providence, RI, 2006. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423].

[Jac21] N. Jacon. Mullineux involution and crystal isomorphisms, 2021. Preprint arXiv:2107.02641.

[Jam78] G. James. Some combinatorial results involving Young diagrams. *Math. Proc. Cambridge Philos. Soc.*, 83(1):1–10, 1978.

[JK81] G. James and A. Kerber. The representation theory of the symmetric group. *Encyclopedia of Mathematics and its Applications*, 16, 1981.

[Kle95] A. Kleshchev. Branching rules for modular representations of symmetric groups. I. *J. Algebra*, 178(2):493–511, 1995.

[Kle96] A. Kleshchev. Branching rules for modular representations of symmetric groups. III. Some corollaries and a problem of Mullineux. *J. London Math. Soc. (2)*, 54(1):25–38, 1996.

[Mac79] I. Macdonald. *Symmetric functions and Hall polynomials*. The Clarendon Press, Oxford University Press, New York, 1979. Oxford Mathematical Monographs.

[Mat] A. Mathas. Specht 2.4. https://www.maths.usyd.edu.au/u/mathas/specht/. October 1997.

[Mat99] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, volume 15 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[MM90] K. Misra and T. Miwa. Crystal base for the basic representation of $U_q(\mathfrak{sl}(n))$. *Comm. Math. Phys.*, 134(1):79–88, 1990.

[Mul79a] G. Mullineux. Bijections of p-regular partitions and p-modular irreducibles of the symmetric groups. *J. London Math. Soc. (2)*, 20(1):60–66, 1979.

[Mul79b] G. Mullineux. On the p-cores of p-regular diagrams. *J. London Math. Soc. (2)*, 20(2):222–226, 1979.

[Nak41a] T. Nakayama. On some modular properties of irreducible representations of a symmetric group. I. *Jpn. J. Math.*, 18:89–108, 1941.

[Nak41b] T. Nakayama. On some modular properties of irreducible representations of symmetric groups. II. *Jpn. J. Math.*, 17:411–423, 1941.

[Ric96] M. Richards. Some decomposition numbers for Hecke algebras of general linear groups. *Math. Proc. Cambridge Philos. Soc.*, 119(3):383–402, 1996.

[Rob47] G. de B. Robinson. On a conjecture by Nakayama. *Trans. Roy. Soc. Canada Sect. III*, 41:20–25, 1947.

[S+97] M. Schönert et al. *GAP – Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4*. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1997.

[Sag01] B. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[Sco95] J. Scopes. Symmetric group blocks of defect two. *Quart. J. Math. Oxford Ser. (2)*, 46(182):201–234, 1995.

[Xu97] M. Xu. On Mullineux' conjecture in the representation theory of symmetric groups. *Comm. Algebra*, 25(6):1797–1803, 1997.

This thesis was typeset using LaTeX (developed by Leslie Lamport, based on TeX by Donald Knuth). Young diagrams were drawn using the *genyoungtabtikz* package (by Matthew Fayers, inspired by the *youngtab* package by Volker Börchers and Stefan Gieseke).

Ensembles basiques pour les groupes symétriques et alternés, involution de Mullineux

Dans cette thèse nous nous intéressons aux propriétés combinatoires de la théorie des représentations modulaires des groupes symétriques et alternés.

Nous nous concentrons sur le problème de l'étiquetage des représentations irréductibles modulaires des groupes symétriques et alternés. Une façon naturelle d'aborder ce problème est de trouver des ensembles basiques unitriangulaires pour les matrices de décomposition. Une de nos principales motivations est liée à l'application de Mullineux, application qui contrôle la restriction du groupe symétrique au sous-groupe alterné. Dans le but d'avoir un étiquetage particulièrement bien adapté à cette restriction, on cherche des ensembles basiques unitriangulaires stables. De tels ensembles sont stables pour la conjugaison.

Il apparaît lors de notre étude le phénomène remarquable suivant : le nombre de partitions auto-conjuguées avec des longueurs des crochets diagonaux non divisibles par p, appelées BG-partitions, est égal au nombre de points fixes de l'application de Mullineux, ou partitions auto-Mullineux. Nous montrons une correspondance combinatoire explicite entre les deux ensembles de partitions.

Récemment il a été montré qu'il n'existe pas toujours un ensemble basique unitriangulaire pour le groupe alterné. Cependant, ces notions peuvent être définies au niveau des blocs. Nous étudions l'application de Mullineux dans les blocs auto-conjugués de p-poids 2 du groupe symétrique et nous exhibons un ensemble basique unitriangulaire stable pour ces blocs, ce qui implique l'existence d'ensembles basiques unitriangulaires pour certains blocs des groupes alternés.

Mots clés : Combinatoire algébrique, Théorie des représentations, Groupe symétrique, Groupe alterné, involution de Mullineux.

Basic sets for the symmetric and alternating groups, Mullineux involution

This thesis concerns combinatorial properties of the modular representation theory of the symmetric and alternating groups.

We focus on the problem of labelling the modular irreducible representations of the symmetric and alternating groups. A natural way to approach this is through unitriangular basic sets for the decomposition matrices. One of our main motivations is related to the Mullineux map, which controls the restriction from the symmetric group to the alternating subgroup. In order to have a labelling which is particularly well adapted to this restriction, we look for stable unitriangular basic sets. Such sets are stable for conjugation.

In our study, the following remarkable phenomenon is observed : the number of self-conjugate partitions with diagonal hook-lengths not divisible by p, called BG-partitions, is equal to the number of fixed points of the Mullineux map, or self-Mullineux partitions. We give a combinatorial and explicit correspondence between the two sets of partitions.

Recently it has been shown that there is not always a unitriangular basic set for the alternating group. However, these notions can be defined at the level of blocks. In this thesis, we study the Mullineux map in self-conjugate blocks of p-weight 2 of the symmetric group and we construct a stable unitriangular basic set for these blocks, which implies the existence of unitriangular basic sets for some blocks of the alternating groups.

Keywords : Algebraic combinatorics, Representation theory, Symmetric group, Alternating group, Mullineux involution.

Discipline : Mathématiques.
Spécialité : Théorie des représentations.

Université de Reims Champagne-Ardenne
LMR - UMR 9008