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Jessica Martin. Some Principal-Agent problems under societal constraints. General Mathematics [math.GM]. INSA de Toulouse, 2021. English. NNT : 2021ISAT0007 . tel-03511009

HAL Id: tel-03511009

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THÈSE

**En vue de l'obtention du
DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE
Délivré par l'Institut National des Sciences Appliquées de
Toulouse**

**Présentée et soutenue par
Jessica MARTIN**

Le 5 mai 2021

Quelques problèmes Principal-Agent sous contraintes sociétales

Ecole doctorale : **EDMITT - Ecole Doctorale Mathématiques, Informatique et
Télécommunications de Toulouse**

Spécialité : **Mathématiques et Applications**

Unité de recherche :
IMT : Institut de Mathématiques de Toulouse

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To my brother, James.

Remerciements

Cette aventure touchant à sa fin, il est grand temps pour moi de prendre le temps de remercier les personnes qui l'ont rendue possible. Tout naturellement, la première personne que je me dois de remercier est mon directeur de thèse, Stéphane. Tu as su accompagner cette thèse jusqu'au bout avec bienveillance et je t'en remercie. J'aurai beaucoup appris de nos discussions, ainsi que de toutes tes idées et de tes conseils. En particulier je garderai un très bon souvenir de cette collaboration qui aura rendu les confinements successifs un petit peu moins ennuyeux et qui m'aura enfin fait passer le cap des BSDE. Je voudrais aussi remercier tous les membres du jury qui auront pris de leur temps pour me lire et m'écouter. En particulier, merci à Thibaut et à Caroline. J'aurais vu passer un grand nombre de vos travaux ces dernières années et c'est vraiment un honneur pour moi de vous avoir comme rapporteurs. J'ai aussi une pensée particulière pour Anthony, sans qui je me serais sûrement pas engagée dans cette voie. Merci à toi d'avoir été au fil des années aussi encourageant et pédagogue. Merci enfin à René Aid, Clémence Alasseur et Jean-Paul Descamps d'être présents aujourd'hui.

Ces années de thèse auront été effectuées dans la continuité d'années INSA déjà passées en grande partie au GMM. Je tiens donc à remercier l'ensemble des enseignants-chercheurs du département que j'ai pu avoir en cours lors de ces années de m'avoir transmis leur goût des maths, et l'ensemble des membres du GMM pour leur gentillesse sur ces années de doctorat. Deux personnes en particulier auront marqué ce parcours : Hélène et Violaine, merci pour tous les encouragements et le soutien que vous m'aurez apporté. Aussi, Fred, ta bienveillance aura joué un rôle déterminant dans cette aventure de thèse et je t'en suis plus que reconnaissant. J'ai une pensée pour tous les doctorants, dont de façon non-exhaustive, Maria, Edouard, Valentin, Léo, Susely, Firas, Josue, Morgane, Iain, Julie, que j'ai pu côtoyer ces dernières années. Aux "petits" derniers Iain et Julie, bon courage pour ces années à venir, j'espère que l'on ne vous aura pas fait trop peur avec nos histoires... A Morgane, merci pour le calme apaisant que tu as pu apporter au bureau 120 (et pour tous les gâteaux!). J'ai une pensée aussi pour mes frères de thèse, Vincent et Amadou, je vous souhaite tout le meilleur pour la suite. Pour terminer, merci aux doctorants de l'ANR Pacman, et plus particulièrement à Sarah. Enfin, on apprend assez rapidement que qui dit doctorat dit aussi (problèmes) administratif(s)... Ainsi, j'adresse un grand merci à Sandrine du côté INSA et Valérie du côté TSE pour leur gentillesse pendant ces années, et leur grande efficacité pour gérer tout cela.

Je dois beaucoup de ma survie lors de cette thèse à tous ceux qui m'auront sorti ne serait-ce qu'un instant la tête du monde des maths. Ainsi en premier lieu, merci à Tiana pour toute ta patience et ta gentillesse lors de ces cours hebdomadaires ! Arthur, merci pour ta présence depuis tant d'années, pour toutes les soirées et tous les moments qu'on aura partagé... Merci aussi à Thomas pour ta gentillesse et ton humour (en particulier pour le fou rire ayant suivi ce "plat-ventre" Parisien). Aux filles, Pénélope, Lucie et Marine (mais aussi évidemment Mathieu et Maxime), merci de m'avoir appris que *casser des epsilon en quatre* n'est pas une expression usuelle de la langue française... Plus sérieusement, merci pour les week-end Corrèziens, les soirées jeux, et pour vos portes toujours ouvertes. Clément, les renâclements du M2R avec les expéditions (de désespoir) en direction de l'Arsenal et l'omniprésence d'Ornstein Uhlenbeck

semblent tellement loin maintenant. Ces souvenirs m'évoquent une véritable naïveté à l'idée de ce qui allait nous tomber dessus par la suite... Merci d'avoir été là pendant ces années, d'avoir partagé ce bout de chemin. Romain, merci pour toute ta gentillesse, ton sens de l'humour et pour ta capacité à mettre en perspective les galères de la thèse. Aux rescapés de la team Gamma, à savoir Alexis, Alban et Bastien, merci d'être là ! En particulier Alexis merci pour ton appui et tes conseils précieux ces derniers mois. Bastien, merci pour ta joie de vivre contagieuse (et pour avoir raté ton avion ce matin d'Octobre 2017). Alban, merci de m'avoir autant supporté ces deux dernières années. Evidemment, Bastien et Banban tous mes voeux de réussite pour la fin de vos propres thèses, ça va le faire je n'ai pas de doute. Enfin, cette thèse aura été marquée par la présence de deux personnes sans qui mes soirées et fins de semaine auraient été bien plus monotones et certainement moins décadents... Lilian et Loïc, merci pour tous ces moments partagés et bien sûr à très très vite (modulo Covid) pour des week-ends londoniens !

Last but not least, I wish to thank the people who have been here right from the start. Mum and Dad, your unconditional love and support throughout the years has been decisive in getting me over the line (and beyond) today. Thank you for everything. Finally, James, thank you for all your love and support, for being you, and for being one of the funniest writers I know.

Table of Contents

1	Introduction	3
1.1	Des incitations aux problèmes Principal-Agent	3
1.1.1	Les incitations	3
1.1.2	Les problèmes Principal-Agent (P-A)	6
1.2	Quelques techniques utiles pour ces problèmes	12
1.2.1	L'optimisation sur des espaces de Hilbert (cadre discret)	12
1.2.2	Les Equations Différentielles Stochastiques Rétrogrades	14
1.2.3	Le contrôle optimal stochastique (EDP)	18
1.3	Pistes récentes, contributions et perspectives	20
1.3.1	Quelques pistes récentes d'intérêt en Principal-Agent	20
1.3.2	Contributions de cette thèse	24
1.3.3	Perspectives	29
2	Analysis of the Risk-Sharing Principal-Agent problem through the Reverse-Hölder inequality	33
2.1	Setting of this work	33
2.1.1	The Risk-Sharing problem in Economics : a short review	33
2.1.2	The Stochastic Control problem and our contribution	35
2.2	The Risk-Sharing problem	36
2.2.1	The stochastic control formulation	37
2.2.2	The Reverse Hölder inequality and analysis of the Risk-Sharing problem (2.2.2)	38
2.3	A discussion of the result	43
2.4	Proofs	46
2.4.1	Proof of Theorem 2.2.1	46
2.4.2	Proof of Proposition 2.2.3	49
2.4.3	Proof of the optimal contract : Theorem 2.2.2	49
2.4.4	A technical lemma	50
3	A step towards incentives in the gig economy	53
3.1	Contracts in the gig economy	53
3.1.1	A note on the gig economy	53

3.1.2	A Poisson based model	54
3.2	Analysis of a certain Poisson based BSDE	57
3.2.1	Presentation and properties	57
3.2.2	Existence and uniqueness of solutions to (3.2.5)	60
3.3	Dynamic contracting in a Poisson setting	65
3.3.1	Optimal contracts	65
3.3.2	Analysis of incentives in this pay-per-project setting	67
3.4	Proofs	72
3.4.1	Proof of Theorem 3.3.1	72
3.4.2	Proof of Lemma 3.3.1	75
3.4.3	Proof of Theorem 3.3.2	76
4	The Risk-Sharing problem under limited liability constraints in a single-period model	79
4.1	Limited liability and Principal-Agent problems	79
4.1.1	General setting	79
4.1.2	A Risk-Sharing limited liability problem	81
4.2	Results on maximizer existence	83
4.2.1	On maximizer existence on a Hilbert space	83
4.2.2	Preliminary results on the contracting space	84
4.2.3	Existence theorem for Risk-Sharing under limited liability	88
4.2.4	Some comments on the results	89
4.3	Characterizing the optimal contracts	90
4.3.1	One-sided limited liability	92
4.3.2	Double-sided limited liability	93
4.3.3	Some analysis of the effect of limited liability	94
4.3.4	Limited liability in the CARA utility case	95
4.4	Proofs	99
4.4.1	Proof of Theorem 4.2.6	99
4.4.2	Proof of Corollary 4.2.1	100
5	A class of explicit optimal contracts in the face of shutdown	103
5.1	Economic shutdown in recent history and contracts	103
5.2	An extension of a classical model	105
5.2.1	Probability setup	106
5.2.2	Principal-Agent Problem	106
5.3	Optimal Risk-Sharing contract	108
5.3.1	Tackling the Participation Constraint (5.3.2)	109
5.3.2	Dynamic contracting under Risk-Sharing	111
5.4	Contracting under shutdown risk and Moral Hazard	114
5.4.1	Main results	114
5.4.2	Model analysis	117
5.4.3	General deterministic compensators $(\Lambda_t)_{t \in [0, T]}$	127
5.5	Mitigating the effects of agency-free external risk	128
5.5.1	Setting for mitigation	129
5.5.2	The optimal contract	130

5.5.3 A note on BSDEs for mitigation 133

6 A BSDE with default jump and unbounded terminal value arising in a Principal-Agent context 137

6.1 A BSDE for the Agent’s problem in Chapter 5 137

6.1.1 A technical issue arising in the Agent’s problem from Chapter 5 137

6.1.2 Setting in the existing literature and main result for BSDE (6.1.3) . . . 140

6.1.3 The consequences of Theorem 6.1.1 for contracting under a risk of shut-down 141

6.2 Proof of existence of solutions to (6.1.3) 144

6.2.1 Link with solution existence to a well-chosen Brownian BSDE (6.2.5) . 144

6.2.2 Some useful estimates on Y^b of (6.2.5) 147

6.2.3 Construction of a solution to (6.2.5) 149

6.3 Proof of uniqueness of solutions to (6.1.3) 161

Bibliography 165

Publications et travaux en cours liés à cette thèse

Analysis of the Risk-Sharing Principal-Agent problem through the Reverse-Hölder inequality,
en collaboration avec Anthony Réveillac.
Sous révision, voir le Chapitre 2.

A step towards incentives in the gig economy, en collaboration avec Anthony Réveillac.
Travail en cours, voir le Chapitre 3.

The Risk-Sharing problem under limited liability constraints in a single-period model.
A paraître dans Journal of Optimization Theory and Applications, voir le Chapitre 4.

A class of explicit optimal contracts in the face of shutdown, en collaboration avec Stéphane Villeneuve.
Soumis, voir le Chapitre 5.

A BSDE with default jump and unbounded terminal value arising in a Principal-Agent context.
Soumis, voir le Chapitre 6.

Chapter 1

Introduction

1.1 Des incitations aux problèmes Principal-Agent

1.1.1 Les incitations

Les discussions de cette thèse tournent autour des incitations, outil auquel nous sommes tous régulièrement confrontés dans nos sociétés et utilisé pour nous encourager à adopter certains comportements. Dans ce manuscrit, nous nous intéresserons plus particulièrement aux incitations dans un contexte économique, dont certainement la plus universelle est celle de la rémunération reçue en échange d'un travail. Il en existe néanmoins bien d'autres : une prime à la conversion versée par l'Etat pour l'achat d'un véhicule propre, les taxes imposées par l'Etat sur le tabac ou l'alcool pour en décourager la consommation, des bons d'achats distribués à des clients de commerces dans le but de les fidéliser... Un coup d'oeil aux journaux et aux actualités récentes permet de renforcer ce constat de l'omniprésence des incitations économiques dans nos vies. En effet dans les trois derniers mois de 2020, les bailleurs commerciaux se sont vus présentés un crédit d'impôt en échange d'un renoncement à leurs loyers¹. Cette mesure visait à aider les commerçants dans cette période de pandémie. De même au Royaume-Uni des incitations financières ont été mises en place pour encourager les individus à s'isoler en cas de symptômes du coronavirus². Bien qu'utilisées largement à différents niveaux de notre société, il est important de noter que le rôle des incitations fait l'objet de nombreux débats et controverses. Par exemple, certaines études psychologiques tendent à montrer que les individus peuvent développer une dépendance à ces formes de *motivation extrinsèque*³ que sont les incitations pour la performance, au dépend de la *motivation intrinsèque*⁴ que l'on pourrait avoir naturellement pour une tâche ([39], [40]). Toutefois ces résultats sont à nuancer, une autre étude suggère au contraire que les incitations peuvent en fait permettre d'accroître notre motivation intrinsèque ([4]). L'inefficacité des incitations a aussi été étudiée avec un point de vue économique, on peut citer notamment les travaux de Frey et ses collaborateurs sur le sujet. En effet dans [51] il est discuté de l'inefficacité des incitations pour des actions civiques ou morales (de fait des actions que les individus auraient eu tendance à accomplir d'eux-même

¹<https://www.economie.gouv.fr/covid19-soutien-entreprises/report-paiement-loyers>

²<https://www.gov.uk/government/news/new-payment-for-people-self-isolating-in-highest-risk-areas>

³facteurs externes qui poussent à agir tels que les incitations, la contrainte...

⁴facteurs internes qui poussent à agir tel que l'envie, le plaisir, le sens civique

par conviction, comme par exemple le don du sang) mais il est aussi souligné leur intérêt dans de nombreuses autres situations. On peut noter enfin que les questions d'incitation sont au coeur de certains débats politiques. Néanmoins nous nous détacherons de cet aspect politique dans ce travail et partirons du constat que les incitations économiques peuvent avoir un rôle à jouer dans nos sociétés. *Au vu de leur omniprésence et tout en gardant à l'esprit leurs limites et les débats qui leur sont liés, il semble important de développer une bonne compréhension des mécanismes d'incitation en économie et en particulier de comprendre comment les individus ou les entités y réagissent, afin de pouvoir à terme mieux les implémenter.*

En raison de leur rôle clé, la question des incitations intéresse les économistes depuis longtemps. Il semblerait que l'une des premières mentions du terme dans la littérature économique remonte à 1776 lorsque Adam Smith établit dans son livre considéré aujourd'hui comme fondateur, *Recherches sur la nature et les causes de la richesse des nations* [122], le caractère incitatif des salaires. Des développements supplémentaires sur le sujet furent cependant rares sur les années (et même les siècles) à suivre. On peut expliquer cela par la difficulté qu'il peut y avoir à en faire une analyse réaliste. Pour citer Barnard [6] un siècle et demi après Smith: "l'économie des incitations... n'est pas en général développée en théorie économique et n'est certainement pas bien comprise" (traduction effectuée par Laffont et tirée de [85]). Nous pouvons percevoir la complexité de cette dernière en considérant sa forme la plus répandue : le contrat employeur / employé. En effet, lorsqu'un employeur engage un nouvel employé, une question naturelle qui se pose est quelle rémunération lui verser, le but étant d'encourager l'employé à fournir de l'effort pour l'entreprise. C'est donc une question d'incitation, où l'employé est invité de part sa rémunération à effectuer des tâches pour l'employeur. Trop peu rémunéré, celui-ci risque de rompre le contrat ou alors de ne pas produire autant d'efforts qu'il ne le pourrait dans son travail. De même, le rémunérer de façon trop élevée par rapport à ses compétences et à son effort au travail engendrerait un surcoût pour l'employeur. Par ailleurs en analysant le rapport employeur/employé, on se rend compte que cette question d'incitation se pose dans un cadre impacté par plusieurs phénomènes. Tout d'abord, il est impacté par un *problème d'information*. En effet, il est raisonnable de supposer que l'employeur n'a pas d'information continue sur les actions de son employé. Il en voit plutôt les résultats, qui peuvent être aussi impactés par d'autres facteurs (par exemple, l'employé en question peut contribuer à un projet collectif). Il peut donc être difficile (ou plutôt trop coûteux) pour l'employeur de quantifier précisément l'effort effectué par son employé. Le cadre est aussi affecté par un *problème d'intérêt* : les intérêts de l'employeur (dans une entreprise cela relève souvent au moins en partie de la maximisation de profit) peuvent être contraires aux intérêts de l'employé. La rémunération doit permettre d'aligner ces intérêts. Enfin, cette relation employeur/employé n'est pas une relation isolée et s'inscrit dans un *contexte relationnel* plus large : les employés interagissent entre eux, les employeurs ont des concurrents... Toutes ces interactions peuvent alors avoir un effet sur ce que serait une rémunération optimale. Notons que cette complexité n'est pas spécifique au contrat employeur/employé. Par exemple, à plus grande échelle, un Etat peut souhaiter inciter l'industrie à des mesures écologiques et plus particulièrement à diminuer ses émissions de polluants. L'Etat a alors un *problème d'information* : il semblerait naturel que celui-ci cherche à récompenser des entreprises ayant réussi à diminuer leurs émissions mais il serait souvent dur (ou coûteux) de réellement quantifier la diminution liée à chacune d'entre elles. Il y a aussi un problème de *conflit d'intérêt* : diminuer la pollution engendre un nouveau

coût pour les entreprises et cela ne leur est pas profitable. Enfin, encore une fois, les relations Etat/entreprises s'inscrivent dans un *contexte relationnel* bien plus large. Par exemple une entreprise est une source d'emploi qui est bénéfique pour l'Etat. Toutefois l'augmentation des coûts engendrée par des mesures écologiques pourrait avoir des conséquences sur le maintien de sa charge salariale : il y a donc un équilibre/compromis délicat à trouver selon les priorités de l'instance gouvernante. Nous pouvons terminer par un troisième exemple d'incitation où interviennent encore une fois Etat et entreprise. En effet, des accords contractuels peuvent être utilisés dans ce contexte par un Etat (Principal) pour déléguer à un prestataire privé (Agent) la gestion d'un service public. Ceci prend le nom de partenariat public-privé (PPP) et il est crucial pour l'Etat de construire un contrat qui incite le prestataire privé à bien fournir son service. Ce type de contrat est donc sujet à la complexité mentionnée ci-dessus (problèmes d'information et d'intérêt, contexte relationnel délicat). L'enjeu de sa bonne mise en oeuvre est développée dans les travaux de Hajjej et al. sur le sujet tels que [54], [55].

On peut donc retenir que les incitations économiques jouent leur rôle au coeur d'un environnement complexe où s'emmêlent un contexte relationnel parfois riche, des conflits d'intérêt, des problèmes d'information... Pour leur étude nous pouvons considérer plusieurs approches. L'une, plutôt économétrique, consiste en l'analyse de données de relations contractuelles ce qui permettrait - sous réserve d'avoir accès à des données suffisamment riches - de tirer des conclusions sur les bons mécanismes d'incitation à adopter. Un descriptif de divers travaux effectués dans cette veine est donné dans [115] et comprend notamment [59], [86], et [121]. D'autres approches plus théoriques se basent sur des modèles pour les situations contractuelles. Notons que ces deux types d'approches peuvent aller de pair en économie : la première peut permettre de confirmer ou non des résultats établis dans un modèle théorique ou alors de façon inverse, on peut se baser sur des constats pratiques pour formaliser des modèles pour la seconde approche. Cette thèse est consacrée à l'étude de cette seconde piste à base de modèles préétablis de situations contractuelles, et nous ne nous attarderons pas davantage sur l'approche plutôt économétrique. Néanmoins comme suggéré par le descriptif précédent évoquant le cadre complexe dans lequel les incitations économiques sont ancrées, obtenir une telle modélisation ne semble pas évident : il semble assez ardu de formaliser mathématiquement des comportements humains. C'est pourquoi le développement de la théorie des incitations moderne fut un long cheminement et occupa des économistes pendant une bonne partie du XXème siècle. Sa genèse n'a pu avoir lieu qu'après la naissance de la théorie de l'utilité espérée, développée par von Neuman et Morgenstern en 1944 [131], théorie qui propose un premier cadre pour la modélisation du processus de décision d'individus dans un environnement risqué. Notons que ce travail a en fait rendu exploitable une réflexion initiale effectuée par Bernoulli au XIXème siècle dans un article d'abord écrit en latin et dont la traduction anglaise [12] a été publiée dans *Econometrica* en 1954, où il est déjà question de la notion d'utilité (cette notion modélise entre autre le fait que notre rapport à la richesse n'est pas linéaire en celle-ci mais plutôt concave, et la théorie de l'utilité espérée conceptualise le fait que la motivation des individus dépend de la valeur (i.e. utilité) que représente pour lui un résultat et de la probabilité que ce résultat se produise). Se basant sur cette théorie de l'utilité espérée, Vickrey proposa en 1945 dans [130] une formulation économique pour la question d'impôt optimal sur le revenu, et donc la première formulation économique d'une question d'incitation. En particulier, il établit un modèle dans lequel l'Etat cherche à calibrer l'impôt sur le revenu de façon à maximiser l'utilité espérée à l'échelle sociétale, dans une

société dans laquelle chaque individu est incité à maximiser son effort, et avec une contrainte budgétaire. Dans ce modèle, Vickrey arrive à traduire l'optimalité par une équation d'Euler mais reste bloqué pour sa résolution par manque d'outils mathématiques. Il faut attendre le développement du principe du maximum de Pontryagin ([111]) dans un cadre éloigné de l'économie, utilisé habilement dans ce même modèle par Mirrlees en 1971 dans [99], pour que ce problème ouvert pour plus de deux décennies soit enfin résolu. Dans les années suivantes et grâce à de nombreux progrès techniques, les travaux sur les incitations en économie se succédèrent (plus particulièrement dans le cadre de l'assurance). On peut notamment citer d'autres travaux de Mirrlees ([102] et [100]), mais aussi le travail de Hölmstrom dans [57] and celui de Shavell dans [120]. On tend alors vers une *modélisation des incitations qui permet d'intégrer des problèmes d'asymétrie d'information et de conflits d'intérêts*. Une *hypothèse simplificatrice* est néanmoins nécessaire pour cela : seules les *interactions des deux partis du contrat sont modélisées* ignorant ainsi l'effet de leur environnement : on simplifie donc le contexte relationnel souvent complexe évoqué dans les exemples précédents. Par exemple, dans le cas de l'interaction employeur/employé, il s'agit de mettre de côté l'existence de collègues et de concurrents et plus particulièrement l'effet qu'ils peuvent avoir sur le contrat. Cette modélisation a pris le nom de modèle Principal-Agent. C'est aujourd'hui l'un des objets principaux disponibles pour modéliser à échelle microéconomique les questions d'incitation tout en prenant en compte les asymétries d'information et les conflits d'intérêts. L'exploitation de ces modèles Principal-Agent pour des questions d'incitation sont au coeur de ce travail de thèse, et nous allons maintenant nous y intéresser de plus près.

1.1.2 Les problèmes Principal-Agent (P-A)

1.1.2.1 Trois types de problèmes P-A

Un modèle Principal-Agent concerne deux individus : un Principal et un Agent.⁵ Ces individus sont liés par un contrat selon lequel le Principal délègue une tâche à l'Agent. En échange de l'accomplissement de cette tâche, l'Agent touchera une contrepartie. Cela peut modéliser par exemple un employeur (Principal) qui embauche un employé (Agent) qui travaillera pour lui (ce qui permet donc au Principal de bénéficier de la division des tâches) mais aussi avoir une application plus large. En effet, l'exemple développé précédemment sur un Etat souhaitant inciter l'industrie à des mesures écologiques peut se formaliser par ce modèle. L'Etat (Principal) n'a pas la compétence pour directement réduire la pollution émise par les entreprises (Agents) et doit donc déléguer cette tâche. La question clé qui se pose alors dans ces modèles est celle de construire une contrepartie qui incite l'Agent à effectuer cette tâche optimalement (dans un sens à définir). Pour répondre à cette question, il faut faire appel à la notion d'information dont nous avons expliqué l'importance pour les situations contractuelles dans la première partie de cette introduction. En fait celle-ci joue un rôle clé dans les problèmes Principal-Agent et nous pouvons distinguer trois types de problèmes selon la quantité d'information partagée par le Principal et l'Agent. Notons que ces trois types ont tous pour dénominateur commun une *contrainte de participation* modélisant le fait que l'Agent acceptera un contrat seulement si la situation est avantageuse pour lui⁶. La chronologie de la situation

⁵à l'origine on considère un seul Principal et un seul Agent, mais aujourd'hui des modèles multi-Principal et/ou multi-Agent font l'objet d'études, l'on y revient plus tard dans cette introduction

⁶ceci est cohérent par exemple avec le fait qu'un individu peut avoir des prétentions salariales, ou alors ne vouloir qu'accepter que des contrats stables comme des CDI

dans un modèle Principal-Agent est donc toujours la suivante :

1. le Principal propose un contrat à l'Agent
2. si la contrainte de participation de l'Agent est valide, l'Agent accepte le contrat
3. l'Agent effectue alors pour le Principal la tâche engendrée par le contrat
4. l'Agent reçoit une contrepartie pour ses efforts.

En théorie des jeux, un telle situation forme ce que l'on appelle un jeu de Stackelberg où le Principal est le meneur (puisque c'est lui qui lance la possibilité d'un lien contractuel avec l'Agent en proposant le contrat) et l'Agent est le suiveur (voir par exemple [123]). Avec cette structure commune en tête, nous pouvons désormais lister et différencier les trois types de problèmes Principal-Agent.

- Le premier type - communément appelé problème de *Risk-Sharing* (partage de risque) et étudié d'abord par Borch [16], Arrow [5] et Wilson [134] est sûrement le moins réaliste. En effet, il considère que le Principal et l'Agent ont accès à exactement la même information (c'est ce que l'on appelle un *problème d'information complète*). Par exemple, dans le cas d'une relation employeur-employé, cela modélise une situation où le Principal voit absolument tout ce que fait l'Agent. Par conséquent, on estime que le Principal peut dicter à l'Agent comment accomplir la tâche du contrat, et que le Principal propose une contrepartie qui est optimale pour lui-même seulement. Il est intéressant d'étudier cette variante simpliste puisque qu'elle permet d'établir une référence par rapport à laquelle comparer les résultats pour les deux autres types de problèmes qui - de part une modélisation plus complexe - permettent de mieux analyser les questions d'incitations.
- Le second type - communément appelé *Moral Hazard* (aléa moral) et pour lequel des travaux fondateurs sont ceux de Hölmstrom et Milgrom [58], Schattler et Sung [118], Sung [124] et Sannikov [116] est bien plus réaliste. En fait dans cette situation, l'Agent dispose de son libre arbitre et le Principal ne peut pas observer l'action de l'Agent. En particulier, s'il veut agir de façon contraire aux intérêts du Principal, il peut le faire. C'est donc au Principal de fournir une contrepartie qui encourage l'Agent à aligner ses intérêts avec ceux du Principal et la formulation du problème est alors plus élaborée. En particulier, le Principal doit réfléchir à comment construire son contrat en se demandant d'abord comment l'Agent va choisir d'agir étant donné une contrepartie fixée. Un exemple type pour ce cas est la relation employé/employeur, où l'employeur voit seulement le résultat de l'action de l'employé et non pas son action en elle-même. L'information est donc plus partielle par rapport au premier type évoqué ci-dessus.
- Le troisième type - communément appelé *Adverse Selection* (connaissance privée) et étudié notamment dans [7], [30], [74], [76] et [125] modélise une situation dans laquelle l'Agent a une connaissance sur ses caractéristiques (préférences, capacités...) que le Principal n'a pas. Il y a donc de l'information cachée en amont de la signature du contrat. Le Principal est alors amené à proposer un menu de contreparties construit de façon astucieuse pour que l'Agent soit amené à révéler ses caractéristiques. Ce problème peut modéliser le fait que l'employeur ne connaît pas en amont le coût de l'effort de son nouvel employé. Il se pourrait que pour un employé particulier, ce coût soit peu élevé.

Si l'employeur a construit son contrat en se basant sur l'hypothèse d'un coût d'effort plus élevé, alors le contrat établi sera moins efficace que prévu...

Dans cette thèse, nous nous intéressons plus particulièrement aux deux premiers types de problèmes Principal-Agent, à savoir le Risk-Sharing et le Moral Hazard. Le troisième type n'en est pas moins intéressant. Simplement, les questions étudiées lors de cette thèse liées notamment à l'ajout de contraintes à ces problèmes Principal-Agent (plus particulièrement des contraintes sur la valeur du salaire dans le cas du Risk-Sharing), à la modélisation de l'arrêt de l'économie liée aux actualités récentes ou à l'effet de la rémunération à la tâche en lien avec le développement actuel de la *gig economy* nous ont plutôt mené à ces deux premiers types de problèmes. Avant de s'attarder davantage sur ceux-ci, il convient de souligner que ces deux types de problèmes permettent de modéliser des situations contractuelles plus larges que la simple relation employeur/employé comme par exemple des mesures incitatives d'un Etat (Principal) envers des entreprises. Néanmoins quand il s'agira de décrire les problèmes dans ce qui suit (que ce soit l'introduction ou les différents chapitres de cette thèse) nous utiliserons souvent le descriptif imagé d'un employeur et d'un employé.

1.1.2.2 Un formalisme mathématique pour certains problèmes P-A

Afin d'introduire plus précisément les problèmes de Risk-Sharing et de Moral Hazard, nous allons présenter un cadre mathématique Principal-Agent connu. Il s'agit de celui établi par les désormais lauréats Nobel Hölmstrom et Milgrom dans leur travail crucial de 1987 [58] que nous présentons ici avec un formalisme lié à celui développé dans cette thèse. Pour ce faire, nous considérons un unique Principal et un unique Agent dont les préférences regardant le risque sont modélisées par des fonctions d'utilité dites "CARA"⁷ de paramètres d'aversion au risque respectifs $\gamma_P > 0$ et $\gamma_A > 0$ fixés⁸. Ainsi, nous notons U_P et U_A leurs utilités respectives d'expression :

$$U_P(x) = -\exp(-\gamma_P x) \quad \text{and} \quad U_A(x) = -\exp(-\gamma_A x)$$

Les deux individus se lient par le biais d'un contrat sur une période de temps $[0, T]$ où $T > 0$ est un horizon de temps fixé. La chronologie est alors la suivante :

- l'Agent travaille pour le Principal en fournissant une action $(a_s)_{s \in [0, T]}$; cette action lui coûte en tout temps $s \in [0, T]$ $\kappa(a_s)$ où κ est une fonction modélisant son coût de l'effort,
- l'action de l'Agent impacte la richesse du Principal noté X_s^a pour $s \in [0, T]$, richesse qui est aussi sujet à un aléa extérieur,
- à la fin du contrat et donc au temps T , le Principal verse une contrepartie W à l'Agent pour ses efforts.

Hölmstrom et Milgrom proposent de modéliser l'aléa extérieur subi par le Principal par le biais d'un mouvement brownien. On considère donc un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ sur lequel évolue un mouvement brownien classique unidimensionnel $(B_s)_{s \in [0, T]}$ avec sa filtration engendrée (et complétée) $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$. On introduit les notations :

⁷constant absolute risk aversion

⁸notons que les fonctions d'utilités peuvent plus généralement être des fonctions U croissantes, concaves, continues \mathbb{R} avec $\lim_{x \rightarrow -\infty} U(x) = -\infty$; elles incluent la fonction "limite" dite risque-neutre telle que $U(x) = x$

- $\mathbb{H}_2 = \left\{ (a_s)_{s \in [0, T]}, \forall t \in [0, T], a_t \mathcal{F}_t - \text{previsible et } \mathbb{E} \left[\int_0^T a_s^2 ds \right] < +\infty \right\}$
- $\mathcal{W} = \{ W \mathcal{F}_T - \text{mesurable avec } \mathbb{E} [\exp(\alpha W)] < +\infty \forall \alpha \in \mathbb{R} \}$

Ces outils permettent alors de formaliser les deux problèmes Principal-Agent.

Le problème de Risk-Sharing (dans le cadre de Hölmstrom et Milgrom). Dans ce problème le Principal et l'Agent ont accès à la même information, et le Principal peut dicter à l'Agent son action. En fait, le Principal et l'Agent voient sous la même mesure \mathbb{P} le processus de richesse du Principal défini pour une valeur initiale x_0 dans \mathbb{R} et un processus d'action $(a_s)_{s \in [0, T]}$:

$$X_T^a = x_0 + \int_0^T a_s ds + B_T$$

Par conséquent, le contrat optimal pour le Risk-Sharing est tout simplement celui qui est optimal pour le Principal, sous réserve que la contrainte de participation soit valide. On résout donc le problème suivant où l'on voit l'apport de la théorie de l'utilité espérée [131] puisque l'on optimise l'espérance de l'utilité de ce que tire le Principal de la situation (c'est à dire sa richesse en T , moins le salaire payé à l'Agent) :

$$\sup_{(W, a) \in \mathcal{W} \times \mathbb{H}_2} \mathbb{E} [U_P(X_T^a - W)] \quad (1.1.1)$$

sous la contrainte de participation :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right] \geq U_A(y_{PC}), \quad (1.1.2)$$

où $U_A(y_{PC})$ est le niveau moyen d'utilité minimum que l'Agent est prêt à accepter dans un contrat, et $\kappa(x) = \kappa \frac{x^2}{2}$ pour un paramètre $\kappa > 0$ fixé est une fonction modélisant le coût de l'effort pour l'Agent. Bien que ce problème soit peu réaliste pour analyser la plupart des vraies relations contractuelles, il présente un vrai intérêt : en tant que problème de contrôle Markovien, sa résolution souvent aisée permet d'obtenir une intuition sur la forme des relations contractuelles dans un cadre probabiliste et un modèle donné mais aussi d'obtenir une référence à laquelle comparer les autres variantes de problèmes PA dans un même modèle/cadre. La valeur de (1.1.1) formera une borne supérieure de l'utilité espérée que pourra obtenir le Principal dans le cas Moral Hazard : c'est donc un *benchmark* pour le deuxième type de problème P-A.

Le problème de Moral Hazard (dans le cadre de Hölmstrom et Milgrom). Dans cette variante, le Principal et l'Agent n'ont plus accès à la même information et surtout le Principal ne voit pas l'action de l'Agent. Une façon de modéliser ce phénomène est de considérer que le Principal observe le processus de richesse sous une mesure \mathbb{P}^0 , et voit :

$$X_T = x_0 + B_T,$$

alors que l'Agent observe le processus sous une mesure \mathbb{P}^a et voit :

$$X_T^a = x_0 + \int_0^T a_s ds + B_T,$$

où :

$$\left. \frac{d\mathbb{P}^a}{d\mathbb{P}^0} \right|_{\mathcal{F}_t} = \exp \left(\int_0^t a_s dB_s - \frac{1}{2} \int_0^t a_s^2 ds \right),$$

et il convient alors de restreindre les processus d'action considérés à ceux pour lesquels les mesures \mathbb{P}^a et \mathbb{P}^0 sont équivalentes (il faut en fait que le terme exponentiel soit une \mathbb{F} -martingale). Par exemple, il est possible de considérer désormais l'ensemble des actions bornées⁹ \mathcal{E} définit comme :

$$\mathcal{E} = \left\{ (a_s)_{s \in [0, T]} \in \mathbb{H}_2, \quad \forall t \in [0, T] |a_t| < A \right\},$$

pour A une constante positive. Une utilisation de la condition de Novikov¹⁰ donne alors équivalence entre \mathbb{P}^a et \mathbb{P} . Nous avons donc une modélisation de l'action cachée de l'Agent et cela permet de modéliser la relation contractuelle qui est plus subtile que dans le Risk-Sharing. En effet, l'Agent disposant de son libre-arbitre, le Principal a pour un enjeu de lui fournir un contrat l'encourageant à aligner ses intérêts avec ceux du Principal. En particulier, le Principal se pose d'abord la question suivante : étant donné une contrepartie W dans \mathcal{W} , comment l'Agent va t'il réagir ? Il faut donc de résoudre pour tout W dans \mathcal{W} :

$$\sup_{a \in \mathcal{E}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right], \quad (1.1.3)$$

où $\kappa(x) = \kappa \frac{x^2}{2}$ (avec κ une constante positive) est comme précédemment une fonction modélisant le coût de l'effort pour l'Agent. Encore une fois, on constate l'apport de la théorie de l'utilité espérée [131] puisque l'on optimise ici l'espérance de l'utilité de ce que tire l'Agent de la situation (c'est à dire sa contrepartie moins l'effort produit en échange). La résolution de ce problème fournit au Principal un sous-ensemble \mathcal{W}_{IC} de \mathcal{W} de salaires dits *incentive compatible* (c'est à dire compatible avec des incitations) tels qu'il existe un unique processus d'action optimal pour l'Agent. Le Principal dispose alors pour tout W dans \mathcal{W}_{IC} de paires $(W, a^*(W))$ modélisant la réponse de l'Agent à une rémunération W . Il peut alors maximiser sa propre utilité en résolvant :

$$\sup_{W \in \mathcal{W}_{IC}} \mathbb{E}^{\mathbb{P}^{a^*(W)}} \left[U_P(X_T^{a^*(W)} - W) \right], \quad (1.1.4)$$

le tout sous la contrainte :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_s^*(W)) ds \right) \right] \geq U_A(y_{PC}), \quad (1.1.5)$$

où de nouveau $U_A(y_{PC})$ est le niveau moyen d'utilité minimum que l'Agent est prêt à accepter dans un contrat. Notons qu'ici et contrairement à (1.1.1), le problème du Principal (1.1.4) est non Markovien ce qui rend sa résolution parfois plus ardue que dans le cas benchmark.

Les contrats optimaux. L'un des apports majeurs du modèle d'Hölmstrom-Milgrom a été de proposer un cadre dynamique (puisque'il y a une notion de temps dans le modèle) pour

⁹ cette hypothèse qui semble assez restrictive suffit dans ce modèle

¹⁰ on a en effet $\forall a \in \mathcal{E}, \mathbb{E}[\exp(\frac{1}{2} \int_0^T a_s^2 ds)] \leq \mathbb{E}[\exp(\frac{1}{2} \int_0^T A^2 ds)] < +\infty$

lequel les contrats optimaux de Risk-Sharing et de Moral Hazard, c'est à dire les solutions de (1.1.1)-(1.1.2) et (1.1.3)-(1.1.4) sont explicites. Leur preuve peut se trouver dans le papier fondateur [58] et fait appel à la limite d'un cadre discret bien choisi (qui est une approximation du cadre brownien). D'autres références sur ce modèle incluent [37] mais aussi [104] pour le cas particulier du Risk-Sharing. En fait, dans les deux variantes, l'action optimale de l'Agent s'avère être un processus de valeur constante dépendant des paramètres du modèle. La rémunération qui lui est payée en échange est une fonction linéaire du processus de richesse. Si l'on note $a_{RS,t}^*$ et $a_{MH,t}^*$ les actions optimales respectives en tout temps t dans $[0, T]$ et W_{RS}^* et W_{MH}^* les rémunérations optimales respectives pour les problèmes (1.1.1)-(1.1.2) et (1.1.3)-(1.1.4) en fixant $\kappa = 1$ (hypothèse simplificatrice qui a peu d'effets finalement sur l'analyse des optimum) on a :

$$a_{RS,t}^* = 1 \quad \text{et} \quad a_{MH,t}^* = \frac{\gamma_P + 1}{\gamma_P + \gamma_A + 1}, \quad t \in [0, T],$$

et

$$W_{RS}^* = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^{a_{RS}^*} + \beta_{RS}^*; \quad W_{MH}^* = \frac{\gamma_P + 1}{\gamma_P + \gamma_A + 1} X_T^{a_{MH}^*} + \beta_{MH}^*, \quad (1.1.6)$$

où β_{RS}^* et β_{MH}^* sont deux constantes dépendantes des paramètres du modèle, calibrées de façon à ce que la contrainte de participation soit saturée. Le caractère explicite de cet optimum permet une analyse assez fine des conséquences économiques de ces contrats, le lecteur peut trouver une telle analyse dans [37] par exemple. Ce que nous pouvons mentionner ici est que dans les deux cas, une proportion (dépendante des aversions au risque des deux parties) du processus de richesse est transférée à l'Agent via la rémunération. Cela vaut quel que soit le signe du processus de richesse en T . Par conséquent, dans ce modèle l'Agent peut travailler et par la suite devoir de l'argent au Principal puisque le processus de richesse est basé sur un mouvement brownien. Notons aussi que dans les deux cas, l'Agent tire de la situation seulement l'utilité minimale qu'il est prêt à accepter c'est-à-dire $U_A(y_{PC})$... Si cela paraît intuitif dans un cadre Risk-Sharing, on pourrait a priori imaginer que ce ne soit pas le cas sous Moral Hazard puisque l'Agent dispose de son libre-arbitre. Cependant même dans ce dernier cas, le Principal arrive à construire un contrat saturant la contrainte. Enfin, le caractère explicite remarquable (les contrats optimaux sont rarement explicites dans des problèmes Principal-Agent...) trouvé par Hölmstrom et Milgrom inclut des biais dues aux propriétés des utilités CARA qu'il convient d'avoir en tête lorsqu'on procède à l'analyse, comme par exemple les actions constantes qui sont indépendantes du niveau de richesse.

Différentes techniques pour l'analyse. Comme mentionné ci-dessus, ces résultats explicites sont initialement obtenus par Hölmstrom-Milgrom dans [58] en tant que limite d'un cadre discret bien choisi. Cela a été une première étape vers le développement d'un intérêt de la part de nombreux mathématiciens pour les modèles Principal-Agent. Ce modèle a depuis été étendu de nombreuses façons. Quelques travaux à cet effet incluent ceux de Sung et ses collaborateurs ([126] [124] [118] et [119]), celui de Ross [113], mais aussi le travail très complet en temps continu de Cvitanic et Zhang dans [37]. Les méthodes utilisées pour attaquer ces problèmes sont aujourd'hui différentes de celle de Hölmstrom et Milgrom qui n'est pas très portable. Un grand nombre de travaux utilise une approche de type "premier ordre" comme par exemple les travaux de Sung et ses collaborateurs ([124] et [119]). Cependant un tournant méthodique crucial a été apporté par le travail fondateur de Sannikov dans [116]: c'est ce travail qui a

fait le lien entre les problèmes Principal-Agent et un outil tiré des mathématiques financières, les Equations Différentielles Stochastiques Retrogrades (EDSR). Plus précisément, il s'agit de considérer dans le Moral Hazard les salaires W payés à l'Agent comme condition terminale d'une EDSR dont le *driver* est calibré de façon à ce que le problème de l'Agent soit résolu pour un certain processus $a^*(W)$ optimal. Ceci mène alors finalement à des salaires qui sont la valeur terminale d'un processus contrôlé. Le problème du Principal qui suit devient alors un problème de contrôle stochastique assez classique. Nous reviendrons davantage sur cette méthode dans la Section 1.2.2, notant que c'est la méthode la plus appliquée aujourd'hui pour étudier des problèmes Principal-Agent sous Moral Hazard sans doute en partie suite à son extension dans [32]. Nous pouvons aussi mentionner que cette méthode possède sous utilités CARA un analogue pour le Risk-Sharing, consistant à utiliser un théorème de représentation martingale sur la contrainte de participation. Enfin, le lien des problèmes Principal-Agent avec les EDSR a été étendu par Williams dans [133] et Cvitanić, Wan et Zhang dans [34], [35] et [36]. Plus particulièrement, ils arrivent à caractériser pour des fonctions d'utilité générales le problème par le biais d'un système couplé d'Equations Différentielles Stochastiques Retrogrades et Progressives. Cette technique présente l'avantage de s'appliquer dans des cadres généraux mais fournit très rarement un contrat optimal interprétable ce qui limite les analyses économiques... Si nous la mentionnons pour la complétude de ce descriptif, cette dernière approche ne sera pas utilisée dans cette thèse. Dans ce qui suit, nous allons désormais nous arrêter davantage sur certaines des méthodes et techniques mentionnées ci-dessus pour l'analyse de problèmes Principal-Agent sans avoir vocation à être exhaustifs sur cette vaste littérature. Nous nous attarderons plus particulièrement sur des techniques exploitées dans cette thèse. Ceci nous permettra ensuite de mieux décrire les enjeux récents pour les problèmes Principal-Agent et d'introduire les contributions de ce travail.

1.2 Quelques techniques utiles pour ces problèmes

1.2.1 L'optimisation sur des espaces de Hilbert (cadre discret)

Si la formulation de Hölmstrom et Milgrom présentée ci-dessus propose un cadre continu pour l'étude de problèmes d'incitation, nous pouvons tout autant nous intéresser à des modèles discrets. Ceci peut être particulièrement intéressant lorsqu'on souhaite faire évoluer les problèmes considérés (par exemple par l'ajout de contraintes, ou le changement des utilités sous-jacentes). Sous de telles évolutions, la résolution explicite des problèmes et donc l'obtention du contrat optimal peut vite devenir bien plus difficile que dans le modèle d'origine de Hölmstrom et Milgrom. La caractérisation des optima ne peut parfois alors se faire que de façon implicite avec une analyse rendue moins directe. Une simplification alors possible pour obtenir une première intuition sur le comportement des contrats dans un nouveau cadre consiste à enlever la notion du temps continu et de se restreindre à des modèles "jouets" discrets. Si cela peut enlever un niveau de complexité, il faut tout de même faire attention : il existe un célèbre résultat de non-existence dans un cadre discret, formulé par Mirrlees dans [101] en 1999 que nous allons présenter. Pour cela, introduisons un équivalent à une seule période du problème de Moral Hazard présenté ci-dessus et discuté dans [37]. Définissons un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P}^0)$ sur lequel vit une gaussienne centrée réduite B_1 et considérons un Principal et un Agent d'utilités respectives U_P et U_A CARA, c'est à dire $U_P(x) = -\exp(-\gamma_P x)$ et $U_A(x) = -\exp(-\gamma_A x)$ pour $\gamma_P > 0$ et $\gamma_A > 0$ fixés. Le processus de richesse vu par le

Principal sous la mesure \mathbb{P}^0 est :

$$X_1 = x_0 + B_1,$$

où x_0 dans \mathbb{R} est sa valeur initiale. L'action de l'Agent est désormais simplement un réel a , et sous la mesure \mathbb{P}^a sous laquelle l'Agent observe la situation, B_1 a une moyenne de a . La recherche de contrat optimal dans ce cadre consiste alors d'abord en la résolution du problème de l'Agent. Plus précisément il s'agit de résoudre pour tout W dans \mathcal{W} (où par exemple, \mathcal{W} est l'espace de Hilbert des variables aléatoires de carré intégrable $L^2(\Omega)$), et pour κ une fonction donnée qui modélise le coût de l'effort pour l'Agent (on peut par exemple comme au dessus prendre $\kappa(x) = \kappa \frac{x^2}{2}$) :

$$\sup_{a \in \mathbb{R}} \mathbb{E}^{\mathbb{P}^a} [U_A(W - \kappa(a))]. \quad (1.2.1)$$

Cette résolution fournit alors au Principal un sous-ensemble \mathcal{W}_{IC} de \mathcal{W} de salaires tels qu'il existe une unique action optimale pour l'Agent. Il dispose alors pour tout W dans \mathcal{W}_{IC} de paires $(W, a^*(W))$ modélisant la réponse de l'Agent et peut désormais résoudre :

$$\sup_{W \in \mathcal{W}_{IC}} \mathbb{E}^{\mathbb{P}^{a^*(W)}} [U_P(X_1^{a^*(W)} - W)] \quad (1.2.2)$$

sous la contrainte :

$$\mathbb{E}[U_A(W - \kappa(a^*(W)))] \geq U_A(y_{PC}). \quad (1.2.3)$$

C'est pour ce problème que Mirrlees a exhibé dans [101] une non-existence de contrats optimaux. Plus particulièrement, il n'existe pas de contrat (W^*, a^*) résolvant (1.2.1)-(1.2.2)-(1.2.3), et c'est en fait le problème de l'Agent (1.2.1) qui pose problème. Pour résoudre celui-ci il faut restreindre l'espace d'optimisation. Par exemple on peut se restreindre à des contrats linéaires en X_1 (et on retrouve alors le contrat analogue du temps continu, c'est par exemple détaillé dans [37] mais aussi dans la Section 2.3 du Chapitre 2 de cette thèse). Ce cas de non-existence de contrats optimaux généraux souligne l'importance de la question de l'analyse d'existence de solutions. Ceci est parfois possible en parallèle de leur caractérisation mais peut aussi nécessiter une approche plus générale et variationnelle. Cette approche variationnelle consiste à se munir d'une topologie bien choisie pour laquelle la fonction valeur est semi-continue supérieurement et l'espace d'optimisation est compacte (par exemple, lorsque l'on considère des salaires dans $L^2(\Omega)$ on peut naturellement considérer la topologie faible afin d'exploiter la compacité-faible des boules bornées de $L^2(\Omega)$). Plusieurs travaux importants d'existence existent, dont ceux de Page [106], de Kadan et al. [75] et celui de Carlier et al. [25] (se plaçant dans le cadre spécifique du troisième type de Principal-Agent, sous sélection averse, cadre qui diffère donc de celui de cette thèse).

Evidemment, une deuxième question se pose alors (question qui n'est d'ailleurs pas propre au cadre discret bien que les techniques pour y répondre listées en suivant sont plutôt utilisées en discret) : si un minimiseur existe, qui est-il ? Plus précisément, quelle est la forme du salaire à payer à l'Agent et comment doit-il travailler en contrepartie ? C'est bien cette question qui permet d'analyser la structure des contrats optimaux, et donc à terme de les mettre en application. Une approche typique qui peut être utilisée pour ces problèmes d'optimisation contraints dans des cadres probabilisés discrets est une approche de type premier ordre. Elle fait appel à un Lagrangien ainsi qu'à un théorème de Kuhn-Tucker généralisé tel que présenté

dans [90]. Par exemple, considérons le problème de Risk-Sharing discret analogue à (1.1.1)-(1.1.2) en posant sur le même espace de probabilité que pour le problème de Moral-Hazard discret ci-dessus :

$$X_1 = x_0 + a + B_1,$$

où x_0 dans \mathbb{R} est une valeur fixée, B dans $L^2(\Omega)$ et a est un réel modélisant l'action de l'Agent. Le problème de Risk-Sharing discret consiste alors à résoudre :

$$\sup_{(W,a) \in \mathcal{W} \times \mathbb{R}} \mathbb{E}[U_P(X_1 - W)] \quad (1.2.4)$$

sous la contrainte :

$$\mathbb{E}[U_A(W - \kappa(a))] \geq y_{PC}, \quad (1.2.5)$$

où U_P , U_A et κ sont comme dans l'exemple précédent. Admettons qu'une solution à ce problème existe, celui-ci peut être caractérisé par le biais du Lagrangien $L(W, a, \lambda) = \mathbb{E}[U_P(X_1 - W) + \lambda U_A(W - \kappa(a))]$ où λ est un multiplicateur de Lagrange (ceci est d'ailleurs détaillé dans [37]). L'approche Lagrangienne fournit une caractérisation de l'optimum via des conditions du premier ordre, et des conditions dits de relâchement. En particulier, l'une des conditions satisfaites à l'optimum est :

$$\frac{U'_P(X_1 - W)}{U'_A(W - \kappa(a))} = \lambda. \quad (1.2.6)$$

Autrement dit *le ratio des utilités marginales à l'optimum est constant* : c'est ce que l'on appelle la règle de Borch pour le Risk-Sharing ([16]). Dans le cas que l'on considère, où les utilités sont CARA, cette égalité est impliquée par la linéarité de W en X_1 . Dans les autres cas, elle est peu exploitable et rares sont les cadres où l'on peut espérer ainsi obtenir une solution pleinement explicite. Néanmoins, les contrats implicites qui découlent caractérisés par une série d'équations se prêtent parfois plus facilement à certaines analyses que ce que l'on pourrait avoir en temps continu. Cela permet alors d'obtenir de premières intuitions sur les conséquences d'un nouveau cadre. On peut enfin évoquer l'existence de quelques techniques numériques pour l'approximation de minimiseurs sur des espaces de Hilbert, qui pourraient aussi permettre d'obtenir de l'information sur les contrats dans des cas non explicites. Par exemple, Barty et ses collaborateurs ont développé dans [8] et [9] une approche par gradient stochastique permettant d'obtenir une approximation de minimiseurs sur $L^2(\Omega)$.

1.2.2 Les Equations Différentielles Stochastiques Rétrogrades

Revenons maintenant aux deux problèmes Principal-Agent (1.1.1)-(1.1.2) et (1.1.3)-(1.1.4)-(1.1.5) formalisés dans le cadre continu présenté précédemment. La principale différence entre les deux types de problème est l'étape supplémentaire de la résolution du problème de l'Agent dans le cas du Moral Hazard. Celle-ci permet d'intégrer le libre arbitre de l'Agent à la situation. La maximisation de l'utilité du Principal qui suit ensuite ressemble de très près à l'unique problème d'optimisation dans le cas du Risk-Sharing (à la différence bien sûr que dans le cas du Moral Hazard, le Principal maximise seulement son utilité en W et non plus (W, a)). Comme évoqué ci-dessus suite aux développements sur le modèle de Hölmstrom et Milgrom, le problème de l'Agent est typiquement résolu aujourd'hui dans ce cadre continu en utilisant des outils tirés plus largement des mathématiques financières (et notamment du travail de Rouge et El Karoui dans un tout autre contexte dans [114]) : les Equations Différentielles

Stochastiques Rétrogrades combinées à un principe d'optimalité martingale. L'utilisation de ces outils pour le problème de l'Agent a été effectué par Sannikov dans son papier crucial [116] mais aussi par Cvitanić et al. dans [32]. Nous pouvons introduire cette approche en présentant tout d'abord le principe d'optimalité martingale pour le Problème (1.1.3) que l'on rappelle consiste à résoudre pour W dans \mathcal{W} :

$$\sup_{a \in \mathcal{E}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right].$$

Le principe d'optimalité martingale s'énonce alors comme suit.

Principe d'Optimalité Martingale pour le Problème (1.1.3). Etant donné W dans \mathcal{W} , supposons que l'on arrive à construire une famille de processus stochastiques $R^a(W) = (R_t^a)_{t \in [0, T]}$ indexée par a dans \mathcal{E} telle que :

1. $R_T^a = U_A \left(W - \int_0^T \kappa(a_s) ds \right)$ pour tout a dans \mathcal{E} ,
2. R^a est une \mathbb{P}^a surmartingale pour tout a dans \mathcal{E} ,
3. R_0^a est indépendant de a ,
4. il existe un unique a^* dans \mathcal{E} telle que R^{a^*} est une \mathbb{P}^{a^*} -martingale.

Alors on obtient :

$$\forall a \in \mathcal{E}, \quad \mathbb{E}^{\mathbb{P}^a} [R_T^a] \leq R_0^a \stackrel{2.}{=} R_0^{a^*} \stackrel{3.}{=} \mathbb{E}^{\mathbb{P}^{a^*}} [R_T^{a^*}],$$

ce qui se réécrit en utilisant le point 1. ci-dessus :

$$\forall a \in \mathcal{E}, \quad \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right] \leq \mathbb{E}^{\mathbb{P}^{a^*}} \left[U_A \left(W - \int_0^T \kappa(a_s^*) ds \right) \right]$$

ce qui implique que :

$$\sup_{a \in \mathcal{E}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right] = \mathbb{E}^{\mathbb{P}^{a^*}} \left[U_A \left(W - \int_0^T \kappa(a_s^*) ds \right) \right],$$

et le problème de l'Agent est alors résolu, donnant la paire $(W, a^*(W))$. W est selon la terminologie introduite précédemment *incentive compatible*. Ce principe présente donc une méthode de résolution du problème mais implique la construction de la famille $(R_t^a)_{t \in [0, T]}$ qui est à priori peu guidée. Une approche maintenant bien connue pour sa construction fait appel à une Equation Différentielle Stochastique Rétrograde (ou EDSR), appelée comme telle puisque c'est une EDS pour laquelle la valeur terminale et non initiale est connue. Plus précisément dans le cadre du Problème (1.1.3), on peut considérer une paire (Y, Z) de processus dans¹¹ $\mathcal{S}^2 \times \mathbb{H}^2$ satisfaisant l'EDSR¹² :

$$\begin{cases} Y_T = W \\ Y_t = W - \int_0^T Z_s dB_s + \int_0^T f(s, Y_s, Z_s) ds \end{cases} \quad (1.2.7)$$

¹¹où \mathcal{S}^2 est l'ensemble des processus Y qui sont \mathbb{F} -adaptés et tels que $\mathbb{E}[\sup_{t \in [0, T]} Y_t^2] < +\infty$.

¹²et on suppose tout d'abord que cet objet a un sens, c'est à dire que (Y, Z) existent

pour f une fonction (appelée driver) à préciser. On pose alors :

$$R_t^a = U_A \left(Y_t - \int_0^t \kappa(a_s) ds \right), \quad t \in [0, T]$$

et on calibre la fonction f via le calcul stochastique de façon à ce que le Principe d'Optimalité des Martingales tienne. Dans le cas précis du Problème (1.1.3) on obtient un driver ne dépendant que de s et de Z_s dont l'expression est :

$$f^*(s, Z_s) = \frac{\gamma_A}{2} Z_s^2 + \kappa(a^*(Z_s)),$$

où $a^*(Z_s) = \frac{Z_s}{\kappa}$. Pour que ce raisonnement ait un sens, il faut alors que l'EDSR donné par (1.2.7) avec la fonction f^* calibrée possède une unique solution. A cet effet et dans ce cas particulier, l'on peut faire appel au travail fondateur de Briand et Hu sur les EDSR browniennes à driver quadratique ([20]) pour obtenir l'existence, en notant que l'on a W tel que $\mathbb{E}[\exp(\alpha W)] < +\infty$ pour tout α réel. L'unicité suit alors d'un raisonnement classique basé sur le fait qu'à un certain changement de probabilité près, Y est une martingale locale. Ainsi par ce raisonnement, on peut conclure sur le problème (1.1.3) de l'Agent dans le Moral Hazard. En effet, on sait désormais que pour tout W tel que $\mathbb{E}[\exp(\alpha W)] < +\infty$ pour tout α réel et de la forme :

$$W = y + \int_0^T Z_s dB_s + \int_0^T f^*(s, Z_s) ds, \quad (1.2.8)$$

pour Z dans \mathbb{H}_2 , et y une valeur initiale réelle, l'action optimale de l'Agent vaut $a^*(Z_s) = \frac{Z_s}{\kappa}$. On peut aussi montrer que la paire $(W, a^*(W))$ satisfait la contrainte de participation (1.1.5) si et seulement si $y \geq y_{PC}$. Le problème du Principal (1.2.2) devient alors un problème de contrôle où l'on optimisera en y et Z . De part cette description, on peut soulever un inconvénient de cette méthode astucieuse : on se retrouve à restreindre les rémunérations dans \mathcal{W} à ceux qui ont des moments exponentiels et qui sont de la forme progressive (1.2.8) de l'EDSR calibrée (des rémunérations qui sont alors contrôlées par y et Z). On peut se demander si l'on a peut-être laissé de côté des contrats clés par ce biais. Néanmoins, c'est sûrement l'une des méthodes les plus utilisées et qui permet malgré cela de gérer ces problèmes à priori difficile en gardant une généralité assez conséquente.

Tout ce descriptif méthodique était basé sur le postulat que l'équation (1.2.7) possède une solution une fois le driver calibré, ce qui s'est trouvé être le cas par les résultats de [20]. En fait à chaque fois que l'on utilise cette méthode il est crucial de pouvoir garantir l'existence et l'unicité de solutions à l'objet donné par (1.2.7). Pour ce faire, on tombe souvent comme ici dans le cadre d'EDSR connues et déjà utilisées puisque la littérature à ce sujet est désormais vaste notamment dans le cadre brownien. En effet, cette théorie a attiré une large attention depuis le papier fondateur de Pardoux et Peng dans un cadre Lipschitz ([108]) de 1990. On peut citer entre autres les travaux de Darling et Pardoux [38] pour un cadre monotone, de Kobylanski [82] (croissance quadratique et condition terminale bornée), Briand et Hu [20] (croissance quadratique et condition terminale à moment exponentiel) mais aussi le travail de El Karoui, Peng et Quenez [78] qui introduit la notion de comparaison d'EDSR. Pour une revue plus exhaustive de cette littérature à base brownienne, on peut se référer au cours de

Bouchard sur le sujet (voir [18]). Evidemment, des EDSRs peuvent aussi être introduites lorsqu'on dépasse ce cadre probabilisé brownien et de nombreux résultats existent là aussi dont [45] dans un cadre où l'on considère à la fois un mouvement brownien et un processus de défaut, et dans un cadre à saut [103]. Il arrive toutefois aussi que l'on se retrouve hors du cadre connu (ou plutôt juste à la frontière de celui-ci), et il apparaît alors nécessaire de coupler l'analyse d'un problème Principal-Agent à l'analyse d'existence et d'unicité de solutions à une EDSR associée. Pour cela, on peut parfois construire manuellement sa solution en se basant sur des résultats existants qu'il paraît possible d'étendre en levant par exemple des hypothèses ou alors il faut revenir aux bases établies par [108] : trouver des estimées permettant d'utiliser une itération de Picard sur l'espace de solutions et montrer que l'on obtient pour cette itération un point fixe.

Afin de faire le lien avec la partie suivante, nous pouvons terminer cette partie par l'introduction d'un autre objet mathématique, la représentation martingale. Nous l'avons mentionnée brièvement comme outil analogue aux EDSR pour le Risk-Sharing, surtout dans cas d'utilités CARA. Cet objet est étroitement relié aux EDSR. Par exemple, dans (1.2.7) ci-dessus et dans le cas où f est nulle, c'est à dire le cas où l'on cherche (Y, Z) dans $\mathcal{S}^2 \times \mathbb{H}^2$ tels que :

$$\begin{cases} Y_T = W \\ Y_t = W - \int_0^t Z_s dB_s \end{cases} \quad (1.2.9)$$

l'on obtient $Y_t = \mathbb{E}[W|\mathcal{F}_t]$ et Z est donné de façon unique via le théorème de représentation martingale brownien :

$$\mathbb{E}[W|\mathcal{F}_t] = \mathbb{E}[W] + \int_0^t Z_s dB_s.$$

De tels théorèmes peuvent être utilisés pour obtenir une forme paramétrée (comme la forme donnée dans (1.2.8)) des rémunérations satisfaisant la contrainte de participation dans le problème du Risk-Sharing où l'on n'a en particulier pas de problème de maximisation pour l'Agent. En effet étant donné (W, a) on peut poser :

$$\chi_t = \mathbb{E} \left[U_A \left(W - \int_t^T \kappa(a_s) ds \right) \middle| \mathcal{F}_t \right],$$

et (W, a) satisfait la contrainte de participation si et seulement si $\chi_0 \geq U_A(y_{PC})$. On peut alors écrire que $\chi_t = \mathcal{M}_t \mathcal{L}_t$, où :

$$\mathcal{M}_t = \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \middle| \mathcal{F}_t \right] \quad \text{et} \quad \mathcal{L}_t = \exp(-\gamma_A \int_0^t \kappa(a_s) ds),$$

et l'on note que par construction $(\mathcal{M}_t)_{t \in [0, T]}$ est une martingale. Sous réserve de conditions d'intégrabilité, cette martingale peut alors se représenter. En appliquant du contrôle stochastique à χ_t avec cette représentation, et en exploitant le lien avec la contrainte de participation, on arrive dans un cadre brownien comme celui de Hölmstrom-Milgrom à des rémunérations nécessairement de la forme :

$$W = y + \int_0^T Z_s dB_s + \int_0^T \tilde{f}^*(s, Z_s, a_s) ds, \quad (1.2.10)$$

pour $\tilde{f}^*(s, Z_s, a_s) = \frac{\gamma A}{2} Z_s^2 + \kappa(a_s)$, qui sont alors paramétrées par (y, a, Z) avec $y \geq y_{PC}$. Ainsi, le problème d'optimisation pour le Risk-Sharing deviendra ici aussi un problème de contrôle où on optimisera en y, a et Z .

Notons en guise de conclusion que s'il existe de nombreux résultats de représentation martingale et d'existence de solutions à des EDSR dans un cadre continu, ce n'est pas le cas dans un cadre discret. Par exemple, si on reprend le problème discret (1.2.1) dans la Section 1.2.1 pour B_1 gaussien, on ne peut pas procéder de façon analogue au raisonnement en temps continu puisque le théorème de représentation Gaussien n'existe pas, et l'équivalent discret d'une EDSR non plus... C'est une des raisons pour laquelle les techniques évoquées dans cette partie, et les techniques de la section précédentes sont aussi différentes pour des problèmes aussi analogues.

1.2.3 Le contrôle optimal stochastique (EDP)

Nous pouvons maintenant nous concentrer sur la dernière étape de résolution des problèmes Principal-Agent continus. Nous nous tournons vers cette dernière étape avec en tête :

- pour le Moral Hazard, des rémunérations W de la forme de (1.2.8), contrôlées par y et Z , et donnant lieu à $a^*(W)$ pour l'Agent
- pour le Risk-Sharing, des rémunérations W de la forme de (1.2.10), contrôlées par y, a et Z

Il reste alors à maximiser l'utilité du Principal. A cet effet, nous pouvons utiliser une approche introduite plus généralement par Bellman dans les années 50 dans [10], basée sur le principe de la programmation dynamique. Il s'agit d'obtenir des conditions d'optimalité par le biais d'une certaine équation, appelée Equation aux Dérivées Partielles d'Hamilton-Jacobi-Bellman (EDP d'HJB).

Pour introduire cette approche par HJB considérons plus particulièrement le problème du Risk-Sharing dans le cadre d'Hölmstrom et Milgrom. Comme mentionné ci-dessus, nous pouvons désormais nous restreindre aux rémunérations ayant des moments exponentiels s'écrivant de la forme :

$$W = y + \int_0^T Z_s dB_s + \int_0^T \tilde{f}^*(s, Z_s, a_s) ds,$$

pour $\tilde{f}^*(s, Z_s, a_s) = \frac{\gamma A}{2} Z_s^2 + \kappa(a_s)$, avec Z dans \mathbb{H}_2 , a dans \mathcal{E} et $y \geq y_{PC}$. Pour ces rémunérations, on veut résoudre :

$$V_P^{RS} = \sup_{(W,a) \in \mathcal{W} \times \mathbb{H}_2} \mathbb{E} [U_P(X_T^a - W)]$$

On définit alors pour (y, a, Z) dans un ensemble Γ le processus contrôlé :

$$W_{0,t}^{y,a,Z} = y + \int_0^t Z_s dB_s + \int_0^t \tilde{f}^*(s, Z_s, a_s) ds$$

où :

$$\Gamma = \left\{ (y, a, Z) \in \mathbb{R} \times \mathcal{E} \times \mathbb{H}_2, y \geq y_{PC}, \mathbb{E} \left[\exp \left(\alpha W_{0,T}^{y,Z} \right) \right] < +\infty \forall \alpha \in \mathbb{R} \right\}.$$

Avec ces informations en tête¹³, nous pouvons attaquer la résolution du problème du Principal (1.1.4). En introduisant la notation $X_{0,T}^{x,a} = x + \int_0^T a_s ds + B_T$ (ceci souligne la dépendance en x , 0 et T du processus de richesse) la fonction valeur qui nous est d'intérêt se réécrit :

$$V_P^{RS} = \sup_{(y,a,Z) \in \Gamma} \mathbb{E} \left[U_P \left(X_{0,T}^{x,a} - W_{0,T}^{y,a,Z} \right) \right].$$

Ce problème est un problème de contrôle contraint dont nous pouvons écrire une variante moins contrainte :

$$\sup_{y \geq y_{PC}} V(0, x, y),$$

pour une fonction valeur V définie comme :

$$V(0, x, y) = \sup_{(a,Z) \in \mathbb{H}_2 \times \mathbb{H}_2} \mathbb{E} \left[U_P \left(X_{0,T}^{x,a} - W_{0,T}^{y,a,Z} \right) \right],$$

notant que puisque $\Gamma \subset \mathbb{R} \times \mathbb{H}_2 \times \mathbb{H}_2$,

$$V_P^{RS} \leq \sup_{y \geq y_{PC}} V(0, x, y).$$

C'est en fait la fonction valeur V que nous pouvons, par la théorie classique du contrôle stochastique, espérer relier à une équation d'Hamilton-Jacobi-Bellmann donnée ici par :

$$\partial_t v(t, x, y) + \sup_{a,Z} \left\{ \partial_x v(t, x, y) a + \partial_y v(t, x, y) \left(\frac{\gamma_A}{2} Z^2 + \kappa(a) \right) + \partial_{yy} v(t, x, y) \frac{Z^2}{2} + \frac{1}{2} \partial_{xx} v(t, x, y) + \partial_{xy} v(t, x, y) Z \right\},$$

associée à la condition terminale $v(T, x, y) = U_P(x - y)$. Dans ce cas précis, grâce notamment aux propriétés de séparabilité des utilités exponentielles sous-jacentes au problème, cette EDP possède une solution régulière. En effet, on peut poser l'Ansatz $v(t, x, y) = U_P(x - y) \Phi_0(t)$ et constater par le calcul que l'équation se simplifie pour ne devenir qu'une équation en Φ_0 , résoluble, et liée à $a^* = \frac{1}{\kappa}$ et $Z^* = \frac{\gamma_P}{\gamma_P + \gamma_A}$. On vérifie alors que la fonction v définie alors comme $v(t, x, y) = U_P(x - y) \Phi_0(t)$ mais comportant aussi l'expression :

$$v(t, x, y) = \mathbb{E} \left[U_P \left(X_{t,T}^{x,a^*} - W_{t,T}^{y,a^*,Z^*} \right) \right]$$

est solution de l'équation d'Hamilton-Jacobi-Bellmann. Ensuite, par la régularité de v ici, on peut utiliser un théorème de vérification classique pour établir que $v = V$. On déduit enfin le résultat de l'optimisation en y grâce à la monotonie de V :

$$\sup_{y \geq y_{PC}} V(0, x, y) = \mathbb{E} \left[U_P \left(X_T^{x,a^*} - W_T^{y_{PC},a^*,Z^*} \right) \right]$$

et enfin puisque (y_{PC}, a^*, Z^*) appartient en fait à Γ (ces contrôles sont constants), nous obtenons que $\sup_{y \geq y_{PC}} V(0, x, y) \leq V_P^{RS}$. Combiné avec l'inégalité inverse montrée précédemment ceci permet de conclure sur l'optimalité du triplé (y_{PC}, a^*, Z^*) pour le problème de contrôle menant donc à :

$$V_P^{RS} = \mathbb{E} \left[U_P \left(X_{0,T}^{x,a^*} - W_{0,T}^{y_{PC},a^*,Z^*} \right) \right].$$

¹³il est dur de spécifier quels Z garantissent l'intégrabilité nécessaire dans Γ mais crucialement l'on peut noter que pour des Z bornés cela tient

Si tout se passe bien pour aboutir sur le problème du Principal dans ce cas précis, nous pouvons noter qu'il est initialement question ci-dessus d'une équation aux dérivées partielles du second ordre et hautement non-linéaire : il y a peu de chances dans le cas général d'obtenir une résolution explicite ou même des garanties sur l'existence de solutions ... En fait, même des petits changements sur ce modèle comme par exemple l'introduction de contraintes peuvent rendre la résolution de la HJB liée bien moins triviale. Dans certains cas, on peut utiliser la notion de solution de viscosité pour de telles équations, notion initialement introduite par Crandall et Lions dans [29], pour récupérer des solutions. Enfin, même quand l'existence d'un optimum régulier serait établie, obtenir des informations sur celui-ci peut être particulièrement difficile. Il est possible dans ces situations de se tourner vers l'utilisation ou le développement de méthodes numériques pour résoudre le problème. Une méthode numérique bien établie pour cela (sous certaines conditions) est l'algorithme de Howard ([60]), des avancées récentes incluent [79]. Le domaine de recherche tourné vers le développement de nouvelles méthodes numériques pour la résolution de tels problèmes de contrôle stochastique connaît actuellement un intérêt particulier, avec l'émergence d'approches basées sur les réseaux de neurones. La littérature à cet effet commence aujourd'hui à devenir vaste ; des travaux récents incluent [66] et [105].

1.3 Pistes récentes, contributions et perspectives

1.3.1 Quelques pistes récentes d'intérêt en Principal-Agent

Le descriptif de la section précédente met en lumière l'existence d'une artillerie mathématique bien fournie aujourd'hui permettant l'analyse d'un certain nombre de problèmes Principal-Agent, même s'il est parfois nécessaire d'étendre un petit peu les limites de celle-ci. Nous allons maintenant revenir aux problèmes Principal-Agent en eux-mêmes et les questions appliquées, liées aux incitations, qu'ils permettent d'analyser. En fait nous possédons désormais une bonne compréhension des implications économiques de contrats dans un certain nombre de contextes. Les contrats utilisés en réalité sont souvent linéaires. Ceci est cohérent avec les rémunérations données dans (1.1.6) solutions des problèmes un Principal/un Agent dans un cadre CARA et Gaussien. La robustesse de telles rémunérations a fait l'objet d'études comme par exemple [28]. Néanmoins, nous pouvons formaliser de nombreux variants de ce cadre - parfois en lien avec des actualités très récentes - dans lesquels des questions se sont posés récemment ou se posent même encore. Dans la suite, nous allons en lister quelques-uns afin d'essayer de décrire le paysage de la recherche actuelle en Principal-Agent, en gardant en tête que ces différentes pistes peuvent bien souvent se recouper.

Le cas multi-Agent. Si le formalisme Principal-Agent a d'abord modélisé les interactions entre un seul Principal et un seul Agent, il paraît plus que raisonnable de vouloir étendre cela à un cadre multi-Agent. Cette extension permettrait entre autre d'améliorer légèrement le réalisme de ces modèles et de s'éloigner de cas de deux individus modélisés comme isolés de tout contexte relationnel. On pourrait alors formaliser des situations contractuelles à l'échelle d'une population toute entière, comme par exemple un fournisseur d'internet qui fournit un débit à des millions de clients, ou un gouvernement qui souhaite inciter de nombreuses firmes à adopter un comportement plus éco-responsable... Plusieurs travaux ont été effectués dans cette optique, dont dans le cadre continu celui de Koo et al. dans [83] et très récemment en

2018 et 2019 les travaux d'Elie et Possamaï dans [48] et Elie, Mastrolia et Possamaï dans [47]. Dans [48] il est question de N Agents en compétition dans un cadre Hölmstrom-Milgrom, et il est montré que des méthodes classiques d'analyse de problèmes Principal-Agent peuvent être étendues à un tel contexte. Le travail de [47] étend [48] à un cadre où le nombre d'Agents devient très très grand. Mathématiquement l'on tend alors vers un jeu à champ moyen et l'analyse fait appel à la théorie introduite sur le sujet par Lasry et Lions ([87], [88], [89]) mais aussi Huang, Caines et Malhamé ([63], [64]), et présentée dans [26], [27]. Enfin, une extension récente dans cette lignée de travaux consiste à ajouter des hiérarchies dans les situations contractuelles. Initialement introduits par Sung dans [126] dans un modèle discret, la question est étendue par Hubert dans [65] pour un modèle continu. Plus précisément, un Principal délègue à un manager qui délègue lui-même à une multitude d'Agents et ce travail permet l'analyse de contrats optimaux dans ce contexte.

Le cas multi-Principal. On peut également se poser la question de ce que deviennent les contrats dans une situation multi-Principal. En effet, si l'on reprend l'exemple qui illustre le cas multi-Agent, un client d'un fournisseur d'internet a accès à de nombreuses entreprises pour ce service proposant de nombreuses offres. Les fournisseurs sont donc en concurrence ce qui impacte sûrement les offres qu'ils proposent. Il est ainsi légitime de se poser la question de ce qu'il se passe contractuellement lorsqu'un Agent est confronté à un certain nombre de Principals. Si cette interrogation est pertinente, elle est aussi difficile à analyser de façon satisfaisante. Dans [61], Hu et al. proposent un modèle en temps continu dans lequel l'Agent est à tout temps dans une relation contractuelle avec un seul parmi N Principals. L'Agent influence les temps auxquels il change de Principal en agissant sur l'intensité d'un processus de Poisson. Il se trouve alors que le problème d'optimisation du Principal devient difficile à résoudre (puisque'elle devient ce que l'on appelle inconsistente en temps). On note néanmoins que cela est facilité dans le cas limite de ce problème où l'on a une infinité de Principals, par le biais de la théorie des jeux à champs moyens mentionnée ci-dessus pour le multi-Agent. De façon un petit peu différente, un problème N -Principals pour un seul Agent est aussi traité par Mastrolia et Ren dans [95] avec la particularité que l'Agent ne change pas de Principal au cours du temps. A tout temps donné, il est lié contractuellement à chacun des Principals. Ceci modélise de façon large la situation assez courante dans laquelle quelqu'un/un organisme travaille en parallèle sur plusieurs projets pour plusieurs personnes/organismes différents... L'enjeu est alors pour chacun des Principals de proposer une compensation encourageant l'Agent à prioriser son propre projet. Le modèle proposé par les auteurs permet une analyse concrète des contrats dans cette situation par des outils classiques. Pour terminer cette petite partie nous pouvons mentionner la question de la limited commitment (engagement limité), qui pourrait être lié aux problèmes multi-Principal. En fait, dans les modèles simples de PA évoqués précédemment, le Principal et l'Agent sont engagés sans possibilité de rupture avant la fin de la période contractuellement. En réalité on peut imaginer introduire la possibilité pour l'Agent de rompre le contrat avant son terme, celui-ci ayant reçu une meilleure proposition ailleurs peut-être auprès d'un autre Principal (et donc on serait en quelque sorte dans une situation multi-Principal). On peut citer le travail de Miao et Yuzhe qui aborde ces questions [98] même si au mieux de nos connaissances, ce type de question est encore très ouverte aujourd'hui.

L'incorporation de phénomènes sociétaux. Nous pouvons regrouper un ensemble de con-

tributions assez variées en Principal-Agent sous la thématique plutôt large d'incorporation de phénomènes sociétaux. Nous en listons quelques-uns dans ce qui suit.

- Une première piste de cette thématique est considérée par Hajjej et al., notamment dans [54] et [55] où il est question de *contrats de partenariat public-privé (PPP)*. Il s'agit de contrats à long terme, où la construction/gestion d'un bien public est externalisée à un consortium (partie privée) en échange d'une rente. Dans le monde occidental, cette forme de gestion se répand, en partie en raison de la taille de la dette publique de certains Etats mais ces contrats ont leur part d'inconvénients surtout lorsqu'ils sont implémentés sans régulation contractuelle suffisante¹⁴. Notons que malgré ces inconvénients, l'intérêt des PPP dans le monde occidental a été souligné lors des actualités récentes : ils ont pu être utilisés en 2020 lors de la lutte contre la pandémie de Coronavirus ([132]). Dans [54] et [55] un formalisme Principal-Agent est utilisé par les auteurs pour analyser ces PPP et essayer de mieux les comprendre.
- Une deuxième piste est celle de l'ajout de contraintes au modèle, notamment en ce qui concerne la rémunération. En effet, en jetant un coup d'oeil à (1.1.6), on se rend compte que la rémunération optimale linéaire peut prendre des valeurs sur toute la droite réelle, ce qui paraît finalement peu réaliste (l'Agent peut travailler en échange d'une rémunération négative ; le Principal peut payer à l'Agent bien plus que la valeur de sa firme)... C'est ainsi qu'une piste de recherche sur l'ajout de contraintes au modèle à vu le jour, permettant d'analyser l'effet de salaires minimaux, l'effet de "bandes" de salaires etc. On parle alors de problème de *Limited Liability*, ou responsabilité limitée. La Section 4.1.1 du Chapitre 4 de cette thèse décrit de nombreux travaux à ce sujet, mais nous pouvons noter que plusieurs questions restent, notamment l'effet de la limited liability en temps continu. (Notons que le cas spécifique de la limited liability en temps continu et sous "contrôle de volatilité" - extension aux modèles Principal-Agent introduite initialement par Cadenillas et al. dans [22] - du processus de richesse est traité dans [37]. Les questions restent quand l'on ne contrôle pas la volatilité ce qui est le cadre contractuel présenté tout au long de cette introduction et analysé dans cette thèse).
- La notion de contrainte évoquée ci-dessus permet de faire le lien avec une troisième piste : *la gestion de nos ressources*. En tant que société, nous sommes de plus en plus conscients de la *finitude* de nos ressources, et il paraît important d'inclure cela à nos modèles. Pour illustrer cela, considérons un Principal qui gère une entreprise de pêche et un Agent qui est un pêcheur travaillant pour cette entreprise. Le processus de richesse pourrait être le chiffre d'affaires de l'entreprise qui dépendrait évidemment de la quantité de poissons pêchés par l'Agent. A l'échelle de cette entreprise, la quantité de cette ressource semble infini et c'est ce qui est considéré dans les modèles Principal-Agent. Néanmoins, il serait socialement responsable de modéliser sa finitude.¹⁵ Par exemple, on pourrait borner la quantité de ressource pêchée par intervalle de temps

¹⁴on peut notamment évoquer à cet effet l'utilisation massive au 19ème siècle de PPP pour la gestion des systèmes de régulation d'eau en Asie, Afrique et Amérique du Sud et ayant finalement mené à des monopoles abusifs faute de régulation suffisante

¹⁵la question de la surpêche n'est pas nouvelle en économie, on peut évoquer des travaux tels que [68] en 2002

donné. Ceci devrait avoir un impact sur les contrats puisque l'Agent devrait adapter son comportement à cette contrainte. Il semblerait que l'ajout de telles contraintes au modèle soit une piste ouverte en Principal-Agent.

- Nous pouvons naturellement mentionner la ressource plus spécifique (et produite) qu'est *l'électricité*. Cette énergie a la particularité de nécessiter en tout temps un parfait équilibre entre production et consommation. Cela a un effet sur la construction de contrats des fournisseurs d'électricité et les problèmes P-A ont été utilisés récemment pour modéliser ce phénomène. On peut citer par exemple le travail d'Aïd et al. dans [1], Alasseur et al. dans [3] mais aussi Elie et al. dans [46]. D'ailleurs la fragilité de l'équilibre production/consommation et l'enjeu de la bonne gestion de cette ressource ont été mis en lumière lors de la pandémie de Covid-19 qui a fait par moments planer un risque de coupures d'électricité en France (il y a de nombreux articles de journaux à ce sujet, nous pouvons en citer un relatif à Toulouse : [44]).
- Enfin, il n'est plus à débattre que le début du 21ème siècle a eu un impact majeur sur nos économies de part *l'essor de la digitalisation* suite aux progrès techniques des dernières décennies. Nous avons désormais accès à de plus en plus de plateformes digitales proposant de mettre en relation des clients avec des services, que ce soit pour la livraison de repas (par exemple Deliveroo), des services VTC (par exemple Uber), mais aussi des tâches techniques tels que le développement de code (par exemple Malt). On peut regrouper ces modèles de fonctionnement basés sur le digital sous le nom de *gig economy*, et noter que ces modèles contribuent à chambouler la notion de travail dans nos sociétés. Ceci est par exemple discuté par Jean Tirole dans [128]. Selon lui (extrait adapté de [128] dans [129]) : "*Technology, of course, is already making independent work a lot easier. It puts workers into contact with customers and helps them run a back office.* " Comprendre les contrats dans cette nouvelle économie digitale semble être ainsi une piste importante pour l'avenir. On peut citer un premier travail effectué à cet effet avec le formalisme Principal-Agent dans [15] qui ouvre la porte à de futures recherches.

Au delà de l'aléa brownien. Pour terminer, nous pouvons souligner que tous les modèles Principal-Agent en temps continu mentionnés ci-dessus font appel à un cadre probabiliste brownien. Ce processus stochastique possède de nombreux avantages puisque ses propriétés sont bien connues, et il modélise de façon raisonnable le PnL (profit and loss / profit et pertes) que peut subir un organisme ou une entreprise lors l'exercice de son activité. Néanmoins, on peut vouloir étendre ce cadre brownien afin de mieux correspondre à certains contextes appliqués, ou mieux modéliser certains phénomènes. C'est ainsi qu'un petit nombre de travaux s'est intéressé à un cadre où intervient aussi un processus de Poisson permettant d'introduire des événements survenants de façon ponctuelle. Un premier travail a été celui de Biais et al. dans [14] où un processus de Poisson sert - dans un cadre complètement risque neutre - à modéliser des pertes qui peuvent subvenir lorsque l'Agent gère un projet potentiellement rentable. Dans un tout autre contexte qui est celui d'une banque (Agent) qui gère des emprunts pour des investisseurs (Principal), Pagès et Possamai ont considéré dans [107] un tel processus pour modéliser la possibilité de défaut des emprunteurs. La banque peut réduire le risque de défaut par le biais d'une surveillance (couteuse) des emprunteurs : les investisseurs veulent donc inciter la banque à mettre en place un certain niveau de surveillance par un contrat bien calibré. Un contexte bancaire est aussi abordé en utilisant une modélisation

mixte brownienne/sauts par El Euch et al. dans [50]. Plus précisément, les auteurs mélangent un brownien (non contrôlé) et un processus à saut (contrôlé) pour l'analyse d'incitations pour la tenue de marché financier. Enfin, l'extension au delà du brownien a aussi été utilisée avec succès pour des problèmes Principal-Agent dans un contexte de prévention. Par exemple Capponi et Frei utilisent dans [23] un processus de Poisson pour modéliser une situation où un Agent fournit des efforts pour contribuer au processus de richesse du Principal, processus qui est aussi affecté par des accidents coûteux (ce sont ces accidents qui sont modélisés par des sauts) et l'Agent peut choisir d'agir pour prévenir de tels accidents. La question du contrat optimal mais aussi de l'action optimale pour l'Agent dans ce cadre (action qui est donc un mélange d'effort et de prévention) est alors étudiée par les auteurs dans un cadre CARA. Aussi, Bensalem et al. utilisent eux des sauts contrôlés dans [11] pour modéliser des problématiques de prévention en assurance. Ces différents travaux illustrent la cohérence de dépasser le cadre brownien, puisque cela permet de rendre les modèles Principal-Agent un petit peu plus réalistes en incluant différents phénomènes. Ainsi, cette piste de recherche pour les problèmes Principal-Agent semble être encore assez porteuse aujourd'hui.

1.3.2 Contributions de cette thèse

Avec ces différentes pistes récentes, nous pouvons nous tourner vers un descriptif des contributions de cette thèse. Celles-ci s'insèrent plutôt au sein des deux dernières pistes récentes évoquées, à savoir l'incorporation de phénomènes sociétaux aux problèmes Principal-Agent ainsi que le dépassement du cadre brownien.

1.3.2.1 Une approche alternative pour le Risk-Sharing sous utilités CARA

Cette première contribution est une contribution que l'on peut décrire comme méthodique pour un problème spécifique : celui du Risk-Sharing sous utilités CARA. Comme décrit ci-dessus, la résolution de ce problème fait classiquement appel en temps discret à une approche Lagrangienne. Cette approche Lagrangienne est parfois aussi utilisée en temps continu. Dans tous les cas, elle repose fortement sur la linéarité classiquement considérée du processus de richesse en l'action de l'Agent. Une approche alternative en temps continu est alors l'utilisation d'un théorème de représentation martingale permettant d'obtenir une forme contrôlée de toute rémunération satisfaisant la contrainte de participation. Il faut alors résoudre le problème de contrôle du Principal pour terminer. Dans ce travail en collaboration avec Anthony Réveillac et présenté dans le Chapitre 2, nous proposons une approche alternative pour ces problèmes qui unifie problèmes en temps discret et problèmes en temps continu pour des processus de richesse très divers, permettant donc d'obtenir un champ large de contrats de référence dans le cas CARA. Cette approche exploite une inégalité mathématique, l'inégalité de Reverse-Hölder, pour obtenir dans le Théorème 2.2.1 une décomposition optimale de l'utilité du Principal qui inclue la contrainte de participation de l'Agent. Par une étape supplémentaire, cette décomposition permet d'obtenir dans le Théorème 5.5.1 la solution du problème. En outre, une célèbre règle pour le Risk-Sharing appelée la règle de Borch et établie par Karl Borch dans [16] apparaît ici comme condition pour l'atteinte par l'utilité du Principal de cette décomposition optimale. Par conséquent, on récupère la rémunération linéaire classique. Si cette contribution est surtout méthodique, avec un cadre unifié pour certains problèmes de Risk-Sharing, elle nous amène aussi dans le Corollaire 2.3.1 vers une interprétation plus générale de l'action

de l'Agent. En effet, dans le cas classique où l'action contribue linéairement au processus de richesse du Principal, il est connu que l'action optimale de l'Agent est reliée à une transformée de Legendre-Fenchel de la fonction coût de l'effort κ . Dans ce Corollaire l'on obtient plus généralement que l'action optimale maximise en temps continu :

$$\sup_{a \in \mathbb{H}_2} \mathbb{E} \left[U_A \left(\frac{\gamma_P}{\gamma_P + \gamma_A} X_T^a - \int_0^T \frac{\gamma_P}{\gamma_P + \gamma_A} \kappa(a_s) ds \right) \right], \quad (1.3.1)$$

où X_T^a est le processus de richesse considéré dans les deux cadres. Un résultat analogue tient dans les cadres discrets. Cela suggère que le Risk-Sharing dans un monde CARA peut-être considéré comme optimal en un certain sens pour l'Agent même si l'utilité de ce que celui-ci tire de la situation n'est elle pas maximisée. En effet, l'Agent tire :

$$\mathbb{E} \left[U_A \left(X_T^a - \int_0^T \kappa(a_s) ds \right) \right],$$

de la situation alors que dans (1.3.1) un ratio $\frac{\gamma_P}{\gamma_P + \gamma_A}$ est présent. Nous pouvons noter que la seule existence de solution au problème (1.3.1) engendre existence d'optimum pour le problème du Risk-Sharing. Cette approche est utilisée dans le Chapitre 3 qui suit, où le modèle est basé sur un processus de Poisson.

1.3.2.2 Vers une modélisation pour la gig-economy : le cas de la rémunération à la tâche

La première contribution méthodique est exploitée dans la deuxième contribution de cette thèse qui forme le Chapitre 3. Ce Chapitre a pour but de proposer un (petit) pas vers une analyse des contrats dans le cadre de la gig-economy. Evidemment, cette nouvelle économie amène plusieurs niveaux de complexité pour les contrats : les contrats proposés sont typiquement courts (voir sans engagement), avec une rémunération à la tâche, et bien souvent un travailleur dans cette économie peut jongler entre différentes formes d'activité (pour modéliser cette dernière caractéristique on pourrait imaginer faire appel à [95] ou aux problèmes d'engagement limité). Dans ce travail en collaboration avec Anthony Réveillac, nous proposons d'utiliser le cadre de type Hölmstrom-Milgrom pour apporter spécifiquement une analyse de l'effet d'une rémunération à la tâche sur les contrats optimaux. Pour cela, nous considérons un processus de richesse qui est un processus de Poisson, où l'action de l'Agent influe sur l'intensité (qui peut varier en temps) des sauts de celui-ci. Ceci peut modéliser par exemple un Agent qui fournit de l'effort pour accomplir des tâches à valeur directe pour le Principal (c'est ce qu'il se passe pour des chauffeurs de VTC utilisant des plateformes digitales). Sans surprise au vue des résultats du Chapitre 2 le Risk-Sharing mène à un contrat linéaire classique (en fait chacun des sauts du Poisson N_T est partagé avec l'Agent par un coefficient $\frac{\gamma_P}{\gamma_P + \gamma_A}$), et l'action optimale le maximiseur de (1.3.1) adapté à ce cadre. Notons plus particulièrement que l'action varie désormais en temps, en lien avec l'intensité sous-jacente du Poisson. Dans le cas du Moral Hazard nous utilisons l'approche EDSR pour attaquer le problème de l'Agent. L'EDSR qui découle du problème possède un driver dépendant seulement du contrôle et qui est Lipschitz en celui-ci. Pour son analyse, les techniques classiques s'appliquent et sont détaillées pour la complétude du travail. La fonction valeur du Principal est ensuite maximisée sous certaines hypothèses. Nous constatons désormais que sous Moral

Hazard chacun des sauts du processus de Poisson est partagé avec l'Agent avec un coefficient $(K_t^*)_{t \in [0, T]}$ dépendent du temps : la part aléatoire de la rémunération est alors de la forme $\sum_{i=1}^{N_T} K_{t_i}^*$ où t_i est l'instant avéré du i ème saut, contrastant avec $\frac{\gamma_P}{\gamma_P + \gamma_A} N_T$ pour le Risk-Sharing. En particulier, le ratio de partage dépend de la valeur de l'intensité sous-jacente. Plus l'intensité est élevée, plus ce partage semble être important. L'action optimale de l'Agent dépend aussi de K^* . En fait de part cette rémunération, l'Agent est incité à davantage fournir d'efforts lorsque l'intensité sous-jacente du marché est favorable, et à moins en fournir lorsque le marché est plus calme. Ceci suggère - dans ce cadre très simpliste où seules les interactions entre deux entités sont considérées - une nécessité de tenir compte en tout temps de l'intensité du marché pour les rémunérations sur les plateformes digitales et semble être cohérent avec le phénomène de surge pricing pratiqué par certaines plateformes. Evidemment, ce travail présente de nombreuses limites et il serait intéressant à l'avenir de creuser ceux-ci. Enfin, les dernières analyses du contrat de Moral Hazard dans ce Chapitre mènent à un constat important : malgré le processus de richesse du Principal positif, la rémunération versée à l'Agent peut être négative avec probabilité positive dans un certain nombre de cadres. Ceci semble appuyer le fait que la positivité des rémunérations n'est pas du tout naturelle pour le Principal (ici la négativité du brownien n'est pas en cause). Une interprétation économique parfois donnée à ce phénomène est un mécanisme de loyer mais nous pouvons naturellement vouloir empêcher cela. Ceci souligne l'enjeu de l'étude de problèmes de Limited Liability comme fait dans le Chapitre suivant.

1.3.2.3 Le problème de Risk-Sharing discret sous contrainte de limited liability

L'ajout de contraintes dans les problèmes Principal-Agent est un problème de vaste intérêt. En particulier, il serait intéressant d'ajouter une contrainte de positivité des contreparties ou même plus généralement d'ajouter des bornes réelles sur celles-ci. Ceci n'est pas un problème trivial, en particulier en temps continu dans le cadre que nous considérons qui est sans contrôle de volatilité. Dans cette thèse, il a donc été question d'analyser l'effet d'ajout de contraintes constantes à un modèle discret, dans le cas de référence du Risk-Sharing, afin d'obtenir des intuitions sur l'effet de cette contrainte. Cette contribution, qui se trouve dans le Chapitre 4 comporte deux aspects. Tout d'abord, une analyse variationnelle de l'existence de contrats optimaux a été effectuée pour des fonction d'utilités et des processus de richesse divers. La difficulté pour l'analyse d'existence découle du caractère aléatoire de l'un des objets sur lesquels on cherche à maximiser qu'est le salaire. Ces salaires vivant dans un espace de Hilbert ($L^2(\Omega)$) nous considérons naturellement la topologie faible pour cette analyse. Il s'agit alors d'exploiter la faible-compacité des boules bornées pour la norme induite de $L^2(\Omega)$ qui découle du théorème de Banach-Alaoglu afin d'obtenir des conditions d'existence. A cet effet, des conditions suffisantes pour l'existence de contrats optimaux sont présentées dans le Théorème 4.2.6 et son Corollaire CARA 4.2.1 (en effet, étendre le Théorème 4.2.6 au cas CARA nécessite un petit peu de travail dans le cas où l'on a seulement une borne inférieure sur la contrepartie et utilise une approche de type pénalisation). Par la suite, la caractérisation de ces contrats optimaux est effectuée dans un cadre général. Cela mène dans la Section 4.3.3 à une analyse de l'effet de ces bornes. En particulier, on retrouve une structure particulière pour la rémunération, structure qui est donnée par une règle de Borch (voir (1.2.6) et [16]). En fait cette règle de Borch tient ici en partie, pour certaines réalisations de l'aléa, et donne une forme optionnelle à la rémunération. On trouve aussi que dans le cas où le salaire

est simplement minoré, la contrainte de participation reste saturée et l'action de l'Agent est supérieure à l'action demandée sous Risk-Sharing classique. Ceci semble assez naturel : il faut compenser le fait que l'Agent prenne moins de risque en lui demandant plus d'efforts. Enfin, le cas précis de la caractérisation sous utilités CARA est étudiée. La caractérisation est alors quasiment pleinement explicite. Par exemple, dans le cas où on impose seulement la contrainte de positivité, le salaire optimal est de la forme :

$$W^* = \left(\frac{\gamma_P}{\gamma_P + \gamma_A} X^{a^*} + \beta^* \right)_+,$$

où β^* est calibré de façon à saturer la contrainte de participation. Ainsi la positivité est assurée par un salaire nul pour un certain nombre de réalisations. Une forme translatée de cette rémunération est optimale pour des bornes inférieures réelles m autres que 0. Ceci s'interprète alors de façon pertinente pour les applications :

- pour un certain nombre de réalisations de l'aléa (en particulier, des contextes économiques moins favorables), l'Agent est payé la valeur m : c'est une part fixe
- dans certaines cadres (en particulier, des contextes économiques plus favorables), l'Agent reçoit un salaire au dessus de m , augmentée de façon proportionnelle aux résultats du Principal : l'Agent perçoit alors aussi une part variable.

Notons enfin que ce cas CARA se prête à des interprétations numériques pour obtenir les valeurs de a^* et β^* : nous le faisons pour la contrainte de positivité afin de clôturer le Chapitre.

1.3.2.4 Comment inciter sous connaissance d'un risque d'arrêt de l'économie ?

Le mois de mars 2020 (et l'année 2020 de façon générale) marqueront sans trop de doute l'histoire des économies mondiales pour des années à venir. En France, deux moments ont sûrement été particulièrement marquants : les soirs du samedi 14 mars et lundi 16 mars 2020. Suite à ces annonces, des pans entiers de l'économie du pays ont connu une mise à l'arrêt parfois quasiment immédiate. En effet, pour citer Edouard Philippe le soir du 14 mars 2020 : *"j'ai donc décidé, jusqu'à nouvel ordre, la fermeture à compter de ce soir minuit de tous les lieux recevant du public non indispensables à la vie du pays. Il s'agit notamment des restaurants, cafés, cinémas, discothèques"*. Ils ne commenceront à réouvrir que le 2 juin. Ce phénomène d'arrêt aussi subite de l'économie fut vécue par de nombreux autres pays dans le monde et fut sans précédent. Une question qui peut se poser alors est celle de l'effet d'une telle possibilité sur les incitations. Plus particulièrement, puisque les risques de pandémie sont souvent exclus de clauses d'assurance, et puisque les mesures économiques exceptionnelles mises en place aujourd'hui par les Etats ne sont sûrement pas tenables sur le long terme, on peut se demander comme tenir compte de ce risque de façon interne aux contrats. Ce type de modèle permettrait aussi d'analyser les contrats dans des milieux de nature risqués tels que les start-ups. L'analyse de cela en collaboration avec Stéphane Villeneuve constitue une contribution de cette thèse et en forme le Chapitre 5. En particulier, nous avons étendu le modèle d'Hölmstrom-Milgrom en ajoutant un processus de défaut (à intensité déterministe) au cadre stochastique. Lorsque le défaut survient (et s'il survient), il a pour conséquence d'arrêter complètement le processus de richesse. Dans ce cadre, nous utilisons les approches

classiques détaillées précédemment pour analyser les incitations optimales. Il s'avère assez remarquablement que le contrat optimal obtenu reste interprétable et même explicite, que ce soit pour le cas de référence du Risk Sharing ou le cas plus réaliste du Moral Hazard. Son expression est l'objet des Théorèmes 5.3.1 et 5.4.1 et on y observe que l'Agent travaille comme dans le modèle classique, s'arrêtant au moment du défaut. La rémunération optimale a sous Moral Hazard l'expression suivante :

$$W^* = y_{PC} + Z^* B_{T \wedge \tau}^* + \left(\frac{\gamma_A}{2} Z^{*2} + \kappa(a^*(Z^*)) \right) (T \wedge \tau) + K_0^* - \left(\frac{c_1}{\gamma_P + \gamma_A} + \frac{\lambda}{\gamma_A} \right) (T \wedge \tau).$$

où K_0^* et c_1 sont des constantes dépendant des paramètres du modèle, et Z^* et $a^*(Z^*)$ sont les mêmes constantes que dans le modèle classique. On observe qu'une partie de la rémunération :

$$y_{PC} + Z^* B_{T \wedge \tau}^* + \left(\frac{\gamma_A}{2} Z^{*2} + \kappa(a^*(Z^*)) \right) (T \wedge \tau),$$

est une extension de l'optimum d'Hölmstrom-Milgrom arrêté plus tôt si un arrêt survient. Cette partie classique se somme avec une partie linéaire pour le temps de défaut qui possède une expression alternative :

$$K_0^* - \left(\frac{c_1}{\gamma_P + \gamma_A} + \frac{\lambda}{\gamma_A} \right) (T \wedge \tau) = K_{\tau-}^* \mathbf{1}_{\tau \leq T} + \int_0^T \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K_s^*) - 1] (1 - N_s) ds,$$

où K_t^* est une fonction déterministe du temps à signe constant (et dépendant des paramètres du modèle, en particulier des aversions au risque) sur $[0, T]$ et $(N_t)_{t \in [0, T]}$ est un processus de défaut, c'est-à-dire pour $t \in [0, T]$, $N_t = \mathbf{1}_{\tau \leq t}$ où τ est le temps de défaut. En particulier, on voit dans cette seconde expression que cette nouvelle composante du salaire comporte deux phénomènes :

- le transfert d'un montant $K_{\tau-}^*$ au moment du défaut, si le défaut survient
- un cumul continu (de signe opposé au transfert) et jusque $T \wedge \tau$ d'un montant

Cette seconde partie du salaire peut donc s'interpréter comme un phénomène d'auto-assurance ou de spéculation selon les aversions au risque des différentes entités : soit l'Agent sacrifie un peu de sa rémunération en échange d'un paiement au moment du défaut éventuel, soit l'Agent reçoit une rémunération supplémentaire avec la possibilité de devoir de l'argent si le défaut survient. Nous étendons ensuite (de façon simple) ce modèle pour inclure une possibilité pour le Principal d'investir au moment du défaut afin de continuer à produire (possiblement de façon moins efficace qu'avant). On tire la conclusion cohérente qu'il s'avère qu'il n'est pas toujours optimal pour le Principal d'investir, puisqu'il faut que les gains futurs puissent espérer compenser le coût de l'investissement.

1.3.2.5 Existence et unicité de solutions pour une EDSR dans un cas brownien/défaut

La dernière contribution de cette thèse - présente dans le Chapitre 6 - va de pair avec l'analyse de l'effet d'un risque d'arrêt de l'économie du Chapitre 5. Il s'agit en fait de l'étude de l'existence et de l'unicité de solutions pour l'EDSR qui découle de l'analyse du problème de l'Agent apparaissant sous Moral Hazard avec le risque de défaut. En effet, il se trouve que celle-ci se situe (au mieux de nos connaissances) à la frontière de la littérature connue. Son

driver comporte deux composantes séparées liées au brownien et au processus de défaut où la partie relative au brownien est quadratique et la partie relative au défaut résulte d'une transformation exponentielle classique. En fait, ce driver rentre dans le cadre des EDSR étudiés par Jeanblanc et al. dans [71]. Cependant leur résultat d'analyse et d'existence ne s'applique pas puisque leur condition terminale est supposée bornée (ceci leur permet de surmonter la difficulté technique liée à leur intensité de défaut qui est non déterministe). Ce Chapitre vise alors à étendre le résultat de Jeanblanc et al. à un cadre où la condition terminale présente seulement des moments exponentiels mais lorsque l'intensité de défaut sous-jacent est déterministe. En utilisant le constat de Kharroubi et al. dans [80] on observe que l'existence de solutions dans un tel cadre mixte peut être obtenue en prouvant l'existence de solutions à une EDSR seulement brownienne bien choisie. L'approche utilisée pour obtenir cette existence est alors la méthode développée dans le travail fondateur de Briand et Hu présenté dans [20] pour un cadre brownien quadratique. Plus particulièrement on considère d'abord une condition terminale qui est une approximation bornée de la condition terminale non bornée souhaitée. L'existence de solutions dans ce cas ci est alors donnée par [71]. En suivant la méthode de Briand et Hu de [20] et en utilisant des résultats de Kobylanski dans [82], on peut alors construire une solution pour la condition terminale souhaitée : il s'agit d'effectuer un passage à la limite du cas borné exploitant des estimées sur les solutions. Enfin, l'unicité est obtenue par une méthode classique en exploitant le fait que le driver ne dépend pas de la valeur du processus défini par l'EDSR : ceci contraste avec des travaux plus généraux tels que [21] ou [41] qui alors font appel à des propriétés de convexité du driver. Ici le processus est - à un changement de mesure près - une martingale locale et l'exploitation de ce constat permet enfin d'obtenir l'unicité souhaitée.

1.3.3 Perspectives

Nous clôturons ce chapitre introductif par un descriptif de différentes questions qui pourraient être abordées dans la continuité de ce travail.

Tout d'abord, nous pouvons revenir sur la question de la Limited Liability. D'abord suscitée par le manque de positivité de la rémunération dans le modèle d'Hölmstrom-Milgrom présenté dans la Section 1.1.2.2 de cette introduction (en particulier, comme tout au long de cette thèse, dans un modèle sans contrôle de volatilité), l'enjeu pour sa mise en oeuvre est confirmée dans le Chapitre 3 de ce travail. En effet, l'une des conclusions de ce Chapitre semble être qu'il n'est absolument pas naturel pour le Principal d'assurer une telle positivité. Cela peut s'interpréter comme une compatibilité difficile de contraintes presque-sûres et de problèmes de maximisation d'utilité espérée. Quelques réponses concernant l'effet de la contrainte de positivité pour la rémunération (et plus généralement de contraintes réelles) sont apportées dans le Chapitre 4 pour le cas particulier du Risk-Sharing dans un modèle discret. Notamment, la rémunération optimale prend une forme de type "call", et dans le cas de la contrainte de positivité, l'Agent travaille davantage afin de compenser la baisse de sa prise de risque. L'extension de ceci à un modèle en temps continu ainsi que l'analyse du cas du Moral Hazard serait pertinent et est fait en partie par Possamaï, Réveillac et Villeneuve dans [112] même si des questions semblent rester ouvertes. Une piste similaire qui pourrait être intéressante serait l'ajout de contraintes en probabilité. En effet, au lieu d'exiger par exemple $W \geq 0 \mathbb{P} - a.s.$, on pourrait exiger $\mathbb{P}(W \geq 0) \geq \alpha$, où α serait un paramètre donné. Ceci

rappelle quelques travaux sur la surréplication tel que celui de Föllmer et. al. dans [53], mais aussi les travaux de Bouchard, Elie et Touzi dans cette veine dont [19]. Enfin, dans une veine similaire, l'ajout de contraintes modélisant une certaine responsabilité sociétale paraît aujourd'hui, à l'heure où la pression pour une RSE (Responsabilité Sociétale des Entreprises) ne fait que croître, plus que nécessaire. Nous l'avons mentionné un petit peu dans les pistes ci-dessus, on pourrait par exemple souhaiter aujourd'hui des contraintes de type $X_T^a \leq m$ ou $X_t^a \leq m \forall t \in [0, T]$ pour m un réel. Cela affecterait sûrement l'action de l'Agent, et nous pouvons supposer au vu des difficultés qu'engendrent la Limited Liability que l'analyse de ces questions ne serait pas triviale.

Lorsqu'on a commencé à travailler sur les incitations sous risque d'arrêt de l'économie en mars 2020, on aurait pu espérer que ce serait un petit peu moins d'actualité directe lors de l'écriture de ce manuscrit. Malheureusement, le Covid a encore aujourd'hui des effets sur notre économie. En France les bars et les restaurants ont par exemple pu réouvrir courant 2020 mais connaissent en janvier 2021 de nouveau une fermeture (tout comme le milieu du spectacle, les cinémas, les salles de sport...). Aussi, nous pouvons attendre des arrêts ponctuels plus localisés de nos économies même dans un monde post-pandémie (à cause par exemple de catastrophes naturelles qui ont tendance à mettre de façon localisée et temporaire des économies à l'arrêt). Il semblerait donc pertinent de s'intéresser à des extensions de ce travail telles que le passage d'une intensité de défaut déterministe à une intensité stochastique. Celui-ci ne serait pas si direct, par exemple le résultat sur l'existence de solutions à l'EDSR liée pour le problème de l'Agent manque actuellement.¹⁶ Une autre extension de ce travail serait de permettre le contrôle de l'intensité de défaut par l'Agent. Ceci peut être très cohérent lorsqu'on considère le modèle du Chapitre 5 pour analyser les contrats dans l'environnement risqué de la start-up, et ajoute des questions techniques. Par exemple, si l'action de l'Agent contrôle le drift du brownien mais aussi l'intensité de défaut, construire le changement de probabilité qui permet de passer de la mesure de l'Agent à la mesure du Principal ne semble désormais pas si direct. Enfin, on pourrait envisager d'introduire une différence de croyance pour le Principal et l'Agent concernant l'intensité de défaut et se tourner alors vers un problème de type Adverse Selection. Pour terminer, ce type de modèle permet aussi d'inclure une possibilité de burn-out (l'un des syndromes psychologiques les plus courants de nos jours selon [97]) de l'Agent dans les situations contractuelles, notamment si le défaut est contrôlé et n'agit que sur la partie "drift" et non brownienne du processus de richesse du Principal. Ceci serait encore plus réaliste dans un cadre multi-Agent, où chaque Agent contribuerait de façon collaborative à la richesse tout en possédant son propre risque de burnout (burnout qui aurait pour impact une surcharge de travail pour les Agents restants) : on peut imaginer qu'un équilibre s'établirait afin de minimiser ce risque.

Bien sûr, la digitalisation de nos économies est un autre phénomène d'actualité (pour certains aspects en lien avec le Covid d'ailleurs), chamboulant nos rapports au travail notamment via le développement de la *gig economy*. Le Chapitre 3 propose un premier petit modèle pour l'analyse par un formalisme Principal-Agent d'une des caractéristiques de cette économie (à

¹⁶D'ailleurs, d'un point de vue technique, l'extension de résultats du Chapitre 6 sur l'EDSR à un cas où l'intensité est stochastique (en fait, lever l'hypothèse de bornitude de [71]) pourrait être intéressant à creuser, ainsi que l'extension du résultat d'existence et d'unicité à un cas où le driver dépend de la valeur du processus.

savoir la modélisation de la rémunération à la tâche) mais de nombreuses questions restent. Comprendre l'effet des contrats dans ce nouveau monde permettrait d'améliorer leur implémentation et peut-être aider à la mise en place de mesures protégeant les travailleurs. D'abord, les ensembles d'optimisation ad hoc pourraient être peaufinés. Le cadre probabiliste considéré pourrait être étendu ce qui serait un premier pas vers une amélioration du réalisme. Mais le modèle pourrait aussi être complexifié : un des propres de cette économie plus digitale est la normalisation de la parallélisation des activités, dont la modélisation pourrait faire appel à un cadre multi-Principal où l'Agent peut librement jongler entre les Principals (par exemple dans la veine de [95]). De même, on pourrait vouloir viser un modèle multi-Agent afin de modéliser par exemple la compétition entre individus dans une telle situation. Enfin comme mentionné plus tôt dans l'introduction, on pourrait inscrire les développements concernant l'engagement limité (c'est à dire, la possibilité pour le Principal ou l'Agent de rompre leur contrat avant la date T) comme pertinent pour ce cadre. De telles questions suscitent notamment des problèmes techniques intéressants liés à la perte de consistance en temps du problème de contrôle sous-jacent qu'il y a enjeu à creuser et comprendre. Une piste pour cela serait une modification de la contrainte de participation devenant dynamique et par exemple de la forme :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \middle| \mathcal{F}_t \right] \geq R_t, \quad \forall t \in [0, T]$$

pour un processus $(R_t)_{t \in [0, T]}$ donné (voir par exemple [98]).

Chapter 2

Analysis of the Risk-Sharing Principal-Agent problem through the Reverse-Hölder inequality

This Chapter is the result of a collaboration with Anthony Réveillac and is presented here as a slightly modified version of the related and submitted pre-print [93]

2.1 Setting of this work

2.1.1 The Risk-Sharing problem in Economics : a short review

As established in the introduction to this thesis (see Chapter 1), many economic situations in the areas of optimal contracting or incentive policy design can now be gathered under the so-called Principal-Agent formulation where an Agent is asked to perform an action on behalf of a Principal in exchange for some compensation. We may note that the focal point of many existing works (eg. [58], [118], [119], [124], [116]...) is one particular type of Principal-Agent problem, the Moral Hazard case, and the amount of interest for this problem is on par with its realism. Indeed and as developed on in Section 1.1.2.1 of Chapter 1, the Moral Hazard problem allows for optimal contracting in a situation where the Principal cannot control and monitor the Agent's actions. This matches reasonably with real-world applications : an investor who delegates his portfolio management to a banker cannot expect to control the banker's actions, a group of shareholders is not able to monitor every doing of the CEO of its company...

In reality it is often enlightening to compare the optimal contract under Moral Hazard with the optimal Risk-Sharing rule. This rule comes about in the full information situation, when the Principal dictates the Agent's actions and guarantees a minimal level of utility - called reservation utility - to him. Specific literature on this problem, which is called the first-best Principal-Agent problem, includes important works such as [37], [33] and [104]. However, it is overall more scarce than the literature on the Moral Hazard problem and there are at least two reasons behind this. The first important reason is that the first-best problem is also a so-called Risk-Sharing problem. Risk-Sharing problems are documented in several

wider fields of economics and insurance, in another context than Principal-Agent problems and more general literature exists using for example risk-measure approaches. We refer the reader to Section 1 of [49] for a review of the existing literature on the Risk-Sharing question. Another reason for a scarce literature is that the first-best Principal-Agent problem is less realistic than other variations on the Principal-Agent problem such as Moral Hazard, since it does not fit strictly speaking in an incentive policy design framework. This lack of realism means that the problem is less subject to extensions than the Moral Hazard problem or other variations on the Principal-Agent problem. One may nevertheless note that in our digitalized economies where machines are being used increasingly, optimal contracting in a full Risk-Sharing setting - thus without Moral Hazard - is more and more relevant in itself.

When analyzing Principal-Agent problems of any form two main questions arise. Does a solution to the set problem actually exist? And if so, can we obtain some form of characterization for it? As touched on in the introduction to this thesis, the existence question is fundamental as without any such guarantees trying to characterize a solution is pointless. For this reason, specifically obtaining existence results has aroused a considerable amount of interest across the past few decades and for different types of Principal-Agent problems. In 1987, Page considered the Moral Hazard problem in [106] and was able to prove existence. To do so, he used a set of assumptions including a compact set for the agent's actions, bounded wages, and a compact set for monetary outcomes. The same year and in their seminal paper [58], Hölmstrom and Milgrom proved existence in a single period Moral Hazard problem with a discrete (and thus bounded) wealth process, a compact set of feasible actions, and CARA utilities. They then build a continuous time model that approaches the discrete one. Shortly after, Schättler and Sung extended the continuous time model of [58] and established in [118] a set of necessary and sufficient conditions for existence of solutions in a continuous time setting. One such condition is that the agent's actions belong to a compact set but they also suppose that the wage be bounded below before proving that their hypothesis can be slightly lifted in order to include a few more general wages such as linear ones. A more recent work is that of Jewitt, Kadan and Swinkels in [74] where one key assumption is that the wealth process is supposed bounded. Many works thus consider quite restrictive hypothesis to obtain existence. However, a few papers aim to provide more general existence conditions. These include the work of Kadan, Reny and Swinkels in [75] for the Moral Hazard and Adverse Selection setting using only a bounded wage assumption. The Adverse Selection case in continuous-time is also tackled by Carlier in [24] using calculus of variations and h-convexity. This work is further extended by Carlier and Zhang in [25] using calculus of variations once again along with an important assumption to obtain the required compactness. As a consequence to this assumption, the authors consider mainly Lipschitz utilities in their examples.

We are thus led to several conclusions. A first conclusion is that *the first-best problem is a benchmark problem for Principal-Agent analysis yet specific literature on it is scarce*. Furthering the literature on this problem and on its possible extensions will help gain better understanding of Principal-Agent problems as a whole. Another conclusion is that *a key setting for Principal-Agent problems involves CARA utility and a discrete or gaussian wealth process as an extension of the Hölmstrom-Milgrom model*. This model is for example discussed in [104] by Muller who characterizes the solutions to the problem in a continuous time Risk-Sharing setting. Finally, *establishing solution existence can be both key and non-trivial for*

*Principal-Agent problems before going ahead with any form of characterization, in particular for models where optima can only be characterized implicitly*¹. In particular note that due to the stochastic nature of the problems, the underlying contract spaces are in most cases non-compact. Additional assumptions on models are then needed to obtain compactness, particularly in discrete time settings where results from the field of optimal stochastic control cannot be used, to be able to apply standard existence theorems from the field of optimization.

In this Chapter *we aim to bring to light a result that is at the crossroads of these three conclusions* and we are able to do so in the specific mathematical formalism of the introductory Section 1.1.2.2. For completeness, we recall this formalism in the following before summarizing our contribution. In the Section 2.2.1 we then further specify the model before tackling the problem.

2.1.2 The Stochastic Control problem and our contribution

The Stochastic Control problem that we consider is that of the Risk-Sharing in Section 1.1.2.2 of the introductory Chapter 1. We recall it for completeness.

We consider a Principal who owns a firm (or a portfolio) whose wealth is subject to uncertainty and an Agent to whom a wage is offered in exchange for a participation to the firm (and we also refer the reader to [37] and [33] for more details). Their respective utilities U_P and U_A are CARA utilities defined as :

$$U_P(x) = -\exp(-\gamma_P x) \quad \text{and} \quad U_A(x) = -\exp(-\gamma_A x),$$

where $\gamma_P > 0$ and $\gamma_A > 0$ are two fixed risk aversion coefficients. In order to reduce his exposure to uncertainty, the Principal hires the Agent at time $t = 0$ in a take-it-or-leave-it contract in which (if accepted) the Agent is asked to produce an effort $a := (a_t)_{t \in [0, T]}$ in continuous time in return for the payment of a wage W at some maturity time $T > 0$. The wealth of the Principal then becomes the stochastic process $X^a := (X_t^a)_{t \in [0, T]}$ with :

$$X_t^a = x_0 + \int_0^t a_s ds + B_t, \quad t \in [0, T],$$

where x_0 is the initial wealth at time $t = 0$ and $B := (B_t)_{t \in [0, T]}$ is a one-dimensional Brownian motion modeling the stochastic exposure of the Principal. To model the cost of effort for the Agent, we introduce a function κ defined on \mathbb{R}_+ and chosen to be strictly convex, continuous and non-decreasing. A simple example of such a function is the quadratic cost function which, for a fixed constant $\kappa > 0$, is defined for any x in \mathbb{R}_+ as $\kappa(x) = \kappa \frac{|x|^2}{2}$.

An important remark is that the take-it-or-leave-it contract (which from now on will be modeled by the pair (wage, action) = (W, a)) will be accepted by the Agent as soon as a Participation Constraint (PC) condition (or reservation utility constraint) given below is satisfied :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right] \geq U_A(y_{PC}), \quad (2.1.1)$$

¹the importance of this is of course underlined by the non-existence result of Mirrlees in [101], as described in Chapter 1 of this thesis

where $y_{PC} \geq 0$ represents the level of requirement for the Agent to accept the contract.

In this setting, we are interested in the first-best problem which simply writes as :

$$\sup_{(W,a) \text{ subject to (2.1.1)}} \mathbb{E}[U_P(X_T^a - W)]. \quad (2.1.2)$$

One classical way of proving existence of a solution to such a problem in a discrete-time model is to use a variational approach where one finds a topology of the underlying contract space that ensures upper semi-continuity, concavity and coercivity of the Principal's expected utility, whilst rendering the contract space convex and closed. The sticking point here is often coercivity and more particularly coercivity in W . This may be ensured in some cases when the stochastic exposure (a discrete-time version of B in our model) is bounded. However it is difficult to do so under more general assumptions.

Contribution of this Chapter. In the following we provide an alternative proof of existence and uniqueness of solutions to Problem (2.1.2) with next to no further assumptions. To do so, we show that optimality is given through an optimal decomposition of the Principal's utility as a product of the Agent's utility and an extra quantity that can be optimized to give the Agent's optimal action. As a consequence, we obtain an *all-in-one proof of existence, uniqueness and characterization of the optimum*. We believe that the strength of this method lies in the *generality of the settings* that it allows us to consider. Indeed, even though we write the model in continuous-time, we are able to deal with through the same framework *both the discrete and continuous time case, actions that belong to \mathbb{R}^+ or even to any compact subset of \mathbb{R}^+ , a general effort cost function κ , and more importantly any form of random variable / random process that has exponential moments* (this last assumption is a purely technical as we are optimizing exponential expected utilities). Furthermore, the proof does not rely on any a priori intuition on the form of the optimal contract. We thus provide an existence and uniqueness result for the first-best Principal Agent problem under CARA utilities in more general settings than those generally considered in the literature. This is summarized in Corollary 2.3.1 which also provides *new insight into the Agent's optimal action*. As a by-product to this approach we also obtain through our decomposition of the Principal's expected utility a revisit of the well-known Borch rule for Risk-Sharing.

The remainder of the Chapter is structured as follows. First in Section 2.2 we detail our model and tackle the Risk-Sharing problem as described in the introduction. Then in Section 2.3 we discuss the strengths of this approach. The lengthier proofs are finally collected in Section 2.4.

2.2 The Risk-Sharing problem

We first further specify the definition of the utility maximization problem (2.1.2). Then we exploit a key result : the Reverse Hölder inequality. As its name suggests it is closely linked to the more well-known Hölder inequality.

2.2.1 The stochastic control formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a stochastic process $B := (B_t)_{t \in [0, T]}$ is defined with its natural and completed filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$. In this Section, we consider the case of Brownian motion (this appears specifically through its exponential moments in Proposition 2.2.3 and Theorem 2.2.2). However, as will become apparent in Corollary 2.3.1, we may consider more general processes B . The only requirement for this is that :

$$\sup_{t \in [0, T]} \mathbb{E}[\exp(qB_t)] < +\infty, \quad \forall q \in \mathbb{R}^*,$$

which is in line with the use of CARA utilities, and is verified for example for Brownian motion. We denote by $\mathbb{E}[\cdot]$ the expectation with respect to the probability measure \mathbb{P} .

The Agent will be asked to perform an action a continuously in time, according to the performances of the firm. Hence we introduce the set \mathcal{P} of \mathbb{F} -predictable stochastic processes $a = (a_t)_{t \in [0, T]}$ and the set of actions is given as :

$$\mathbb{H}_2 := \left\{ a = (a_t)_{t \in [0, T]} \in \mathcal{P}, \text{ s.t. } \mathbb{E} \left[\exp \left(q \int_0^T |a_t|^2 dt \right) \right] < +\infty, \quad \forall q > 0 \right\}.$$

Given a in \mathbb{H}_2 , the wealth of the principal at any intermediate time t between 0 and the maturity T is given by :

$$X_t^a = x_0 + \int_0^t a_s ds + B_t, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad (2.2.1)$$

where $x_0 \in \mathbb{R}$ is a fixed real number. For any action a in \mathbb{H}_2 , we set $\mathbb{F}^{X^a} := (\mathcal{F}_t^{X^a})_{t \in [0, T]}$ the natural filtration generated by X^a . In particular, we are interested in the set of $\mathcal{F}_T^{X^a}$ -measurable random variables which provides the natural set for the wage W paid by the Principal to the Agent. More precisely, we set ² :

$$\mathcal{W} := \left\{ \mathcal{F}_T^{X^a} \text{-measurable random variables } W, \quad \mathbb{E}[\exp(qW)] < +\infty, \quad \forall q \in \mathbb{R}^* \right\}.$$

The fact that we ask for so-called finite exponential moments of any (positive, respectively negative) order for the action (respectively for the wage) is purely technical. As we will see, the optimal contract will satisfy these technical assumptions.

The cost of effort for the Agent is modeled by a convex, continuous and non-decreasing function κ defined on \mathbb{R}_+ . We can now further specify the Risk-Sharing problem (2.1.1)-(2.1.2) with our probabilistic setting. Indeed, fixing γ_P , γ_A and y_{PC} it writes as :

$$\sup_{(W, a) \in C^{adm}} \mathbb{E}[U_P(X_T^a - W)], \quad (2.2.2)$$

where :

$$C^{adm} := \{(W, a) \in \mathcal{C} \times \mathbb{H}_2, (2.1.1) \text{ is in force}\}$$

² $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$

is the set of admissible contracts satisfying the participation constraint (2.1.1). As touched on in Section 1.2.2, this continuous time problem may be dealt with using tools from the field of optimal stochastic control. Indeed, one may exploit Equation (2.1.1) to obtain a parametrization of all the wages satisfying the Participation Constraint for a given action process $(a_t)_{t \in [0, T]}$. This allows us to rewrite Problem (2.2.2) as a standard optimal control problem and one way of then proving existence involves using verification results as long as the required hypotheses are verified.

Our contribution to this problem is to perform an optimal decomposition (as in Theorem 2.2.1) of the Principal's utility in terms of the one of the Agent; optimality that will coincide with the well-known Economics law in that framework (the Borch rule). The main ingredient (on the mathematical side) relies on the Reverse Hölder inequality that we recall below.

2.2.2 The Reverse Hölder inequality and analysis of the Risk-Sharing problem (2.2.2)

Proposition 2.2.1 (Reverse Hölder inequality). *Let $p \in (1, +\infty)$. Let F and G be two random variables such that $G \neq 0$, \mathbb{P} -a.s.. Then :*

(i) *The Reverse Hölder inequality holds, that is,*

$$\mathbb{E}[|F \times G|] \geq \mathbb{E} \left[|F|^{\frac{1}{p}} \right]^p \times \mathbb{E} \left[|G|^{\frac{-1}{p-1}} \right]^{-p+1}.$$

(ii) *In addition, the inequality is an equality, that is,*

$$\mathbb{E}[|F \times G|] = \mathbb{E} \left[|F|^{\frac{1}{p}} \right]^p \times \mathbb{E} \left[|G|^{\frac{-1}{p-1}} \right]^{-p+1}$$

if and only if there exists some constant (that is non-random) $\alpha \geq 0$ such that $|F| = \alpha |G|^{-\frac{p}{p-1}}$.

In order to proceed with our analysis of Problem (2.2.2), we define the set of all admissible wages for a given action a in \mathbb{H}_2 :

$$\mathcal{W}(a) = \left\{ W \in L^2(\Omega), \quad \text{such that } (W, a) \text{ in } C^{adm} \right\}.$$

With these notations in mind, we rewrite Problem (2.2.2) as :

$$\sup_{(W, a) \in C^{adm}} \mathbb{E} [U_P(X_T^a - W)], \quad (2.2.3)$$

or alternatively :

$$\sup_{W \in \mathcal{W}(a)} \sup_{a \in \mathbb{H}_2} \mathbb{E} [U_P(X_T^a - W)]. \quad (2.2.4)$$

To tackle this problem, we use the inequality of Proposition 2.2.1 and the multiplicative nature of the Principal's utility in order to obtain an attainable upper bound for the Principal's utility.

Theorem 2.2.1. *Let a in \mathbb{H}_2 . We have :*

(i) *For any W in $\mathcal{W}(a)$,*

$$\begin{aligned} & \mathbb{E}[U_P(X_T^a - W)] \\ & \leq \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}}. \end{aligned} \quad (2.2.5)$$

(ii) *For W in $\mathcal{W}(a)$. The following conditions are equivalent :*

(ii')

$$\begin{aligned} & \mathbb{E}[U_P(X_T^a - W)] \\ & = \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}} \end{aligned} \quad (2.2.6)$$

(ii'') *(a, W) satisfies the Borch rule, that is, there exists α in \mathbb{R}_+^* such that :*

$$\frac{U'_P(X_T^a - W)}{U'_A \left(W - \int_0^T \kappa(a_t) dt \right)} = \alpha, \quad \mathbb{P} - a.s.. \quad (2.2.7)$$

(ii''') *The wage W is of the form :*

$$W = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^a + \frac{\gamma_A}{\gamma_P + \gamma_A} \int_0^T \kappa(a_t) dt + \beta, \quad \text{with } \beta \in \mathbb{R},$$

calibrated such that $(W, a) \in \mathcal{C}^{adm}$.

Proof. See Section 2.4.1. □

We may make a couple of remarks on this decomposition as follows.

- The Borch rule for Risk-Sharing, derived by Karl Borch in [16] and [17], appears here in (2.2.7) as a condition for equality between the Principal's expected utility and its decomposition. This sheds a new light onto the rule in the CARA utility case. Indeed, it only allows for contracts that enable an isolation of the effect of the Agent's expected utility in the Principal's expected utility. We will see further on that the Borch rule is a necessary optimality condition for the Risk-Sharing problem just like it is when using the Lagrangian method. However, at this stage it only provides a structural form for the contract (W, a) . This form is obviously valid for an infinite number of contracts: for any action process a , there exists a minimal value for β above which any wage W defined by (ii''') leads to a participation constraint satisfying contract.

- When analyzing the first-best problem it seems intuitive that the Principal and the Agent's utilities should be of opposite effect: the Principal should want to maximize his utility whilst minimizing that of the Agent. This is encompassed in (2.2.5). Indeed due to the negative power $-\frac{\gamma_P}{\gamma_A}$, the upper bound increases as the Agent's utility decreases.

Now that we have a decomposition of the Principal's expected utility and a condition for it to hold exactly (Borch rule), we turn to further exploiting this decomposition and the appearance of the Agent's expected utility in order to obtain a bound that is free of W and a . We do this in two stages in the following two Propositions.

Proposition 2.2.2. *Let a in \mathbb{H}_2 . For any W in $\mathcal{W}(a)$,*

$$\begin{aligned} & \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}} \\ & \leq U_P(-y_{PC}) \times \inf_{\tilde{a} \in \mathbb{H}_2} \mathbb{E} \left[\left| U_P \left(X_T^{\tilde{a}} - \int_0^T \kappa(\tilde{a}_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \end{aligned}$$

Proof. Apply the participation constraint (2.1.1) to the right hand term and optimize in a in the left hand term. \square

It thus remains to perform the optimization in a . To do so, we introduce the following two notations :

$$\tilde{\kappa}(p) := \sup_{x \in \mathbb{R}_+} (px - \kappa(x)), \text{ for any } p \geq 0,$$

and

$$\kappa^*(p) := \operatorname{argsup}_{x \in \mathbb{R}_+} (px - \kappa(x)), \text{ for any } p \geq 0.$$

$\tilde{\kappa}$ is the Legendre transform of κ , and κ^* is its related argument. These are well defined due to the convexity of κ . We use these two notations in the following Proposition to perform the optimization in a and obtain our upper bound.

Proposition 2.2.3. *For any a in \mathbb{H}_2 it holds that :*

$$\begin{aligned} \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} & \geq \mathbb{E} \left[\left| U_P \left(X_T^{a^*} - \int_0^T \kappa(a_t^*) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \\ & = \exp(-\gamma_P x_0) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right), \end{aligned}$$

where $a_t^* := \kappa^*(1)$ for any t in $[0, T]$. We thus have for any (W, a) in C^{adm} :

$$\begin{aligned} & \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}} \\ & \leq U_P(x_0 - y_{PC}) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right). \end{aligned}$$

Proof. See Section 2.4.2. □

The combination of Theorem 2.2.1, Proposition 2.2.2 and Proposition 2.2.3 allows us to exploit a decomposition of the Principal's expected utility, the Participation Constraint and an optimization in a in order to upper bound the Principal's value function. The upper bound that we obtain is free of W and a and is key for our existence proof. Indeed, we are now able to show that this upper bound is attained for an admissible contract, which is in fact unique. This gives us our main result, which is simultaneously an existence, uniqueness, and characterization result for solutions to the first-best problem and is the object of the following key Theorem.

Theorem 2.2.2 (Existence, Uniqueness and Characterization). *(i) Consider (W, a) in any contract in C^{adm} , then it holds that :*

$$\mathbb{E}[U_P(X_T^a - W)] \leq U_P(x_0 - y_{PC}) \exp\left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)}\right)\right).$$

(ii) Now let (W^, a^*) be such that :*

$$a_t^* = \kappa^*(1), \forall t \in [0, T]$$

and

$$W^* = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^{a^*} + \beta^*,$$

$$\beta^* := \frac{T\gamma_A |\gamma_P|^2}{2|\gamma_A + \gamma_P|^2} + y_{PC} + T\kappa(\kappa^*(1)) - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + T\kappa^*(1)).$$

Then (W^, a^*) is the only contract verifying (2.2.7) and saturating the participation constraint.*

(iii) Furthermore :

$$\mathbb{E}\left[U_P\left(X_T^{a^*} - W^*\right)\right] = U_P(x_0 - y_{PC}) \exp\left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)}\right)\right).$$

Thus (W^, a^*) is the unique contract attaining the upper bound. It follows that for any (W, a) in C^{adm} :*

$$\mathbb{E}[U_P(X_T^a - W)] \leq \mathbb{E}\left[U_P\left(X_T^{a^*} - W^*\right)\right],$$

and (W^, a^*) is the optimal contract for the first-best Principal-Agent problem.*

Proof. See Section 2.4.3. □

A note on the risk-neutral limit

The analysis provided throughout this Chapter concerns a Principal and an Agent who are both risk averse with the risk aversion modeled through the CARA utility functions. In fact, the key to this work is the exponential properties of the CARA utilities. However, an important case in the literature is that of a risk neutral Principal who wishes to employ a risk

averse Agent. More precisely, if we for example set ourselves in the discrete-time setting, the Risk-Sharing problem (2.2.3) becomes :

$$\sup_{(W,a) \in \mathcal{C}^{adm}} \mathbb{E}[X_T^a - W], \quad (2.2.8)$$

where we use the same notations as previously (in particular the Participation Constraint is in force for the Agent with utility function $U_A(x) = -\exp(-\gamma_A x)$). Since our Reverse Hölder approach relies on the structure of functions U_P and U_A , we cannot carry it directly in the risk neutral case. However, as it is well-known, the risk neutral framework can be seen as a limit case with formally $\gamma_P = 0$ by rescaling the mapping U_P to become $\tilde{U}_P(x) := -\frac{\exp(-\gamma_P x) - 1}{\gamma_P}$ and by letting γ_P go to 0. Hence, we can use our approach with \tilde{U}_P and U_A to obtain existence of an optimum and its characterization in the risk-neutral case.

Consider a contract (W, a) that satisfies the (PC). Then by Lemma 2.4.1 (in Section 2.4.4),

$$\begin{aligned} \mathbb{E}[X_T^a - W] &= \mathbb{E} \left[\lim_{\gamma_P \rightarrow 0} \tilde{U}_P(X_T^a - W) \right] \\ &= \lim_{\gamma_P \rightarrow 0} \mathbb{E} \left[\tilde{U}_P(X_T^a - W) \right] \\ &= \lim_{\gamma_P \rightarrow 0} \gamma_P^{-1} (\mathbb{E}[U_P(X_T^a - W)] + 1) \\ &\leq \lim_{\gamma_P \rightarrow 0} \gamma_P^{-1} (\mathbb{E}[U_P(X_T^{a^*} - W^*)] + 1), \end{aligned}$$

according to (ii) of Theorem 2.2.2. Using the explicit computation of the upper bound's value, we have that

$$\begin{aligned} \mathbb{E}[X_T^a - W] &= \mathbb{E} \left[\lim_{\gamma_P \rightarrow 0} \tilde{U}_P(X_T^a - W) \right] \\ &\leq \lim_{\gamma_P \rightarrow 0} \gamma_P^{-1} \left(U_P(x_0 - y_{PC}) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right) + 1 \right) \\ &= \lim_{\gamma_P \rightarrow 0} \frac{\exp \left(\frac{\gamma_A \gamma_P^2 T}{2(\gamma_A + \gamma_P)} \right) \exp(-\gamma_P(x_0 - y_{PC} + T\tilde{\kappa}(1))) - 1}{\gamma_P} \\ &= x_0 - y_{PC} + T\tilde{\kappa}(1). \end{aligned}$$

So we have given the upper bound $x_0 - y_{PC} + T\tilde{\kappa}(1)$ to the value problem of the Risk Neutral Principal. An explicit computation gives that this upper bound can be attained by choosing the contract (W_{RN}^*, a_{RN}^*) with

$$a_{RN}^*(t) = \kappa^*(1) \text{ and } W_{RN}^* = y_{PC} + T\kappa(\kappa^*(1)),$$

which is formally the optimal contract found in Theorem 2.2.2 with $\gamma_P = 0$. The optimal parameters have economic meaning : the Principal is neutral to risk and is thus willing to give a fixed wage to his Agent regardless of the performance of the output process. We note that in this case, the Risk-Sharing structure of the problem disappears and the Principal carries all of the risk. We thus provide an extension of the Reverse-Hölder framework to the (known) risk-neutral case.

2.3 A discussion of the result

A unifying framework for the Risk-Sharing problem. As mentioned in Chapter 1 the Risk-Sharing problem is sometimes discussed in a single-period framework where the action of the Agent is simply modeled by a real number a . More precisely, the Principal's wealth at time $t = 0$ is supposed to be worth some value x_0 in \mathbb{R} . At time $t = 1$ (therefore after one period during which the Agent has "performed" his action a) the wealth is worth :

$$X^a = x_0 + a + B,$$

where B is a random variable with bounded exponential moments. Classically a Lagrangian method is performed to compute the optimal parameters in such a setting and the Borch rule (2.2.7) translates the first-order optimality condition. The Reverse-Hölder method can also be applied in this single period setting and in doing so we recover the optimal contract (W^*, a^*) given by :

$$a^* = \kappa^*(1),$$

and :

$$W^* = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^{a^*} + \beta^*,$$

where :

$$\beta^* := y_{PC} + \kappa(\kappa^*(1)) - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + \kappa^*(1)) - \frac{1}{\gamma_A} \ln \left(\mathbb{E} \left[\exp \left(\frac{-\gamma_P \gamma_A}{\gamma_P + \gamma_A} B \right) \right] \right),$$

provided the expectation is finite. In the continuous-time model that we have studied, the infinite-dimensional feature of the space of actions a does not allow one to perform a Lagrangian technique directly whilst the Reverse-Hölder approach applies. Hence, our approach generalizes the Lagrangian one in the CARA setting.

In a continuous-time setting the Risk-Sharing problem may be tackled by first using classical results such as Martingale Representation to obtain a parametrized wage process that encompasses the participation constraint. This wage process may then be injected into the Principal's value function and classical stochastic control techniques used in an attempt to compute an optimum. Of course in the Brownian case that we have considered, using this more standard approach also yields the result and resembles in some ways the Lagrangian technique for a single period setting. In fact, Theorem 2.2.1 shows that the equality condition in the Reverse-Hölder inequality is equivalent to the first order optimality condition for the Lagrangian technique. Relation (2.2.6) that we recall below :

$$\begin{aligned} & \mathbb{E} [U_P(X_T^a - W)] \\ &= \mathbb{E} \left[\left[U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right]^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}} \end{aligned}$$

contains this optimality by expressing the Principal utility in terms of the one of the Agent which appears as the second term the right hand side. Hence, optimality already imposes the form of the wage according to (ii'') in Theorem 2.2.1 where the premium β is chosen so that

the Participation Constrained is bound. At this stage it is interesting to note that only the optimal action remains to be determined by minimizing the quantity:

$$\mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}}, \quad (2.3.1)$$

or of course equivalently maximizing :

$$\mathbb{E} \left[U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}},$$

The latter can be seen as some sort of Risk-Sharing problem in which the Principal coincides with the Agent, as the Principal receives X_T^a to which he subtracts the cost of effort $\kappa(a)$. The exponents that renormalize the risk aversion coefficient translate this duplicity of the Principal who somehow plays both the role of the Principal and of the Agent. This feature is a direct consequence of the fact that the Principal monitors the Agent's action. Note that this term rewrites as follows (purely due to the exponential utilities) :

$$\mathbb{E} \left[U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} = \mathbb{E} \left[U_A \left(X_T^a - \int_0^T \kappa(a_t) dt \right)^{\frac{\gamma_P}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_P + \gamma_A}{\gamma_A}},$$

signifying that under CARA utility, the Risk-Sharing contract does maximize a utility for the Agent which is normalized through the ratio $\frac{\gamma_P}{\gamma_P + \gamma_A}$.

Finally, as a consequence to the above discussion, we wish to emphasize that the Reverse-Hölder technique allows one to also tackle the Risk-Sharing problem for more intricate production processes. As mentioned above, the minimization in a of 2.3.1 is the main step to using this approach and such a remark yields the following Corollary to Theorem 2.2.2.

Corollary 2.3.1. *(to Theorem 2.2.2) Let $(X_t^a)_{t \in [0, T]}$ be some production process that has exponential moments and that is somehow dependent on the action process $(a_t)_{t \in [0, T]}$. Suppose that there exists some process $(a_t^*)_{t \in [0, T]}$ in \mathbb{H}_2 such that :*

$$\mathbb{E} \left[U_P \left(X_T^{a^*} - \int_0^T \kappa(a_t^*) dt \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] = \sup_{a \in \mathbb{H}_2} \mathbb{E} \left[U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right],$$

then there exists a solution to the first-best Principal-Agent problem :

$$\sup_{(W, a) \in C^{adm}} \mathbb{E} [U_P (X_T^a - W)],$$

where the optimal action process is $(a_t^*)_{t \in [0, T]}$ and the optimal wage is of the form :

$$W^* = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^{a^*} + \frac{\gamma_A}{\gamma_P + \gamma_A} \int_0^T \kappa(a_t^*) dt + \beta,$$

with β calibrated to saturate the participation constraint. Uniqueness of $(a_t^*)_{t \in [0, T]}$ provides uniqueness of the optimal contract.

Note that it suffices for $(a_t^*)_{t \in [0, T]}$ to be defined implicitly for this result to hold. Such a method may therefore allow the resolution of the Risk-Sharing problem for alternative settings such as a continuous-time action-controlled jump process setting, where a Lagrangian approach would falter in providing an optimum : this is in fact applied in Chapter 3 of this thesis for a Poisson process whose intensity is impacted by the Agent. This Corollary also further extends the alternative interpretation of the Agent's optimal action in a Risk-Sharing setting already mentioned above. Classically in a Brownian setting, this optimal action is the Legendre transform of κ . In the Corollary we see that in cases where the Agent's action does not linearly affect the production process, the optimal action still has some meaning as the maximizer of a certain utility.

Consequences for the Moral Hazard problem. The introduction to this Chapter included some mention of a different Principal-Agent problem involving Moral Hazard. This problem is further introduced in Chapter 1 of this thesis where we mention that we may write it as follows :

$$\sup_{(W, a) \in \mathcal{C} \times \mathbb{H}_2} \mathbb{E} [U_P (X_T^a - W)],$$

such that :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right] \geq U_A(y_{PC}),$$

and :

$$a \in \operatorname{argmax}_{a' \in \mathbb{H}_2} \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a'_t) dt \right) \right]. \quad (2.3.2)$$

At first glance this problem resembles the Risk-Sharing problem : it still involves maximization of the Principal's utility with a constraint in expectation. However the addition of (2.3.2), modeling the Agent acting to maximize his own utility for any given wage, substantially increases the problem's complexity. The problem is sometimes simplified by assuming that the Agent chooses the distribution of the production process in a way that is unobservable by the Principal. For example in continuous-time, it may be assumed that the Principal observes :

$$X_T = x_0 + B_T,$$

under some measure \mathbb{P}^0 whilst the Agent observes :

$$X_T^a = x_0 + B_T + \int_0^T a_s ds$$

under \mathbb{P}^a where $\frac{d\mathbb{P}^a}{d\mathbb{P}^0} = \mathcal{E}(-\int_0^T a_s dB_s)_T$. As a consequence, the expected values above have to be considered under the relevant probability measures.

Despite the addition of (2.3.2), the problem still involves expected utility maximization under a constraint for the Agent's expected utility. Therefore we may naturally question whether the Reverse-Hölder approach described above could be tailored to this new case. Unfortunately we suspect not. A key pointer in this direction is the fact that in the Risk-Sharing problem, the Reverse-Hölder approach provides a unified framework allowing us to tackle both the single-period case and the continuous-time case through the same key steps. Should

Reverse-Hölder apply for Moral Hazard, we would expect similar unification yet Mirrlees' has proven non-existence for optimal contracts in the single period Moral Hazard problem (see [101]). In fact, as mentioned in Chapter 1 solving the Moral Hazard problem in a single period problem is often simply not possible for arbitrary contracts...

A discrete Moral Hazard optimum may nevertheless be derived through a pre-supposed restriction to linear contracts. In the following we consider such contracts. We show that an optimal decomposition of the Principal's utility still exists in this case but is not linked to Reverse-Hölder. More precisely let us consider

$$X = x_0 + B,$$

where $B \sim \mathcal{N}(0, 1)$ under \mathbb{P}^0 and $B \sim \mathcal{N}(0, a)$ under \mathbb{P}^a and restrict wages to those of the form :

$$W^* = \alpha X + \beta, \quad (\alpha, \beta) \in \mathbb{R}^2.$$

Following the reasoning of Chapter 2.2 of [37], the Agent chooses to work $a^* = \frac{\alpha}{\kappa}$ and the Principal's expected utility for the Moral Hazard problem verifies :

$$\mathbb{E}^{\mathbb{P}^a} [U_P(X^a - W)] \leq U_A(y_{PC})^{-\frac{\gamma_P}{\gamma_A}} \mathbb{E}^{\mathbb{P}^{a^*}} \left[\exp \left(-\gamma_P \left(B + \kappa(a^*) - \frac{\gamma_A}{2} \kappa^2 a^{*2} - a^* \kappa B \right) \right) \right]$$

with equality for the optimal contract :

$$\mathbb{E}^{\mathbb{P}^{a^*}} [U_P(X^{a^*} - W)] \leq U_A(y_{PC})^{-\frac{\gamma_P}{\gamma_A}} \mathbb{E}^{\mathbb{P}^{a^*}} \left[\exp \left(-\gamma_P \left(B + \kappa(a^*) - \frac{\gamma_A}{2} \kappa^2 a^{*2} - a^* \kappa B \right) \right) \right]$$

Therefore in this setting an optimal decomposition of the Principal's utility does exist under a priori linearity assumptions on the contract. This contrasts with Risk-Sharing where the decomposition is a consequence of the inequality, that also guarantees optimality of the family of linear contracts. In particular we note that the condition $a^* = \frac{\alpha}{\kappa}$ in Moral Hazard prevents the use of the inequality by linking the optimal action to the contracts slope. The independent nature of these two parameters is key to the Reverse-Hölder resolution of this problem in the Risk-Sharing case.

2.4 Proofs

In this Section we collect the proofs of the technical results we made use of to proceed with our analysis.

2.4.1 Proof of Theorem 2.2.1

Proof. We fix a in \mathbb{H}_2 and prove each item of the Theorem.

- (i) For the first result, we express the (expected) utility of the Principal in terms of the one

of the Agent. We have :

$$\begin{aligned}
& \mathbb{E} [U_P (X_T^a - W)] \\
&= \mathbb{E} \left[U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \times \exp \left(\gamma_P \left(W - \int_0^T \kappa(a_t) dt \right) \right) \right] \\
&= \mathbb{E} \left[U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \times \left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|^{-\frac{\gamma_P}{\gamma_A}} \right]. \tag{2.4.1}
\end{aligned}$$

We wish to extract the Agent's utility from this expression and obtain at least an inequality. To do so, we need some kind of Hölder inequality. However the classical Hölder inequality cannot be applied for two reasons : first the exponent $-\frac{\gamma_P}{\gamma_A}$ of the utility of the Agent is negative; and then the negativity of the mapping U_P calls for the use of a Hölder inequality in the reverse direction. These two features are taken into account in the so-called *Reverse Hölder inequality* which can be seen as a counterpart to the classical Hölder inequality and given in Proposition 2.2.1. In particular, we wish to use Item (i). More precisely, let :

$$F := U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right), \quad G := \left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|^{-\frac{\gamma_P}{\gamma_A}}. \tag{2.4.2}$$

Note naturally that these two random variables depend on the contract (W, a) under interest.

We wish to apply Reverse-Hölder to F and G with some exponent p that we calibrate so that $|G|^{-\frac{1}{p-1}} = \left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|$; which immediately gives $p = 1 + \frac{\gamma_P}{\gamma_A} = \frac{\gamma_A + \gamma_P}{\gamma_A} > 1$. We thus immediately obtain:

$$\begin{aligned}
\mathbb{E} \left[|F|^{\frac{1}{p}} \right] &= \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_P + \gamma_A}} \right], \\
\mathbb{E} \left[|G|^{\frac{-1}{p-1}} \right] &= -\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right].
\end{aligned}$$

Applying (i) of Proposition 2.2.1 to F and G with this particular choice of p in Relation (2.4.1) gives our result :

$$\begin{aligned}
& \mathbb{E} [U_P (X_T^a - W)] \\
&= -\mathbb{E} [|FG|] \\
&\leq \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \times \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right]^{-\frac{\gamma_P}{\gamma_A}}. \tag{2.4.3}
\end{aligned}$$

(ii) We first prove that (ii') is equivalent to (ii''). This involves finding an equality condition for (2.4.3). Through (ii) of Proposition 2.2.1, Inequality (2.4.3) is an equality if and only

the contract (a, W) is such that there exists a positive constant α such that the random variables F and G defined in (2.4.2) enjoys :

$$|F| = \alpha |G|^{-\frac{p}{p-1}}.$$

By definition of F , G and $p = \frac{\gamma_A + \gamma_P}{\gamma_A}$ this condition reads as :

$$\frac{\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|}{\left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_P + \gamma_A}{\gamma_A}}} = \alpha$$

Thus using the exponential form of our utilities we obtain the condition :

$$\frac{|U_P(X_T^a - W)|}{\left| U_A \left(W - \int_0^T \kappa(a_t) dt \right) \right|} = \alpha,$$

which is equivalent to

$$\frac{U'_P(X_T^a - W)}{U'_A \left(W - \int_0^T \kappa(a_t) dt \right)} = \frac{\gamma_P}{\gamma_A} \alpha.$$

Setting α to $\frac{\gamma_P}{\gamma_A} \alpha$ we obtain our result.

We now prove that (ii'') is equivalent to (ii'''). When (ii'') holds, (W, a) satisfies (2.1.1) and we have the following series of implications where α is a positive constant :

$$\begin{aligned} & \frac{U'_P(X_T^a - W)}{U'_A \left(W - \int_0^T \kappa(a_t) dt \right)} = \alpha \\ \Rightarrow & (\gamma_P + \gamma_A)W - \gamma_P X_T^a - \gamma_A \int_0^T \kappa(a_t) dt = \ln \left(\alpha \frac{\gamma_A}{\gamma_P} \right) \\ \Rightarrow & W = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^a + \frac{\gamma_A}{\gamma_P + \gamma_A} \int_0^T \kappa(a_t) dt + \ln \left(\alpha \frac{\gamma_A}{\gamma_P} \right), \\ \Rightarrow & W = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^a + \frac{\gamma_A}{\gamma_P + \gamma_A} \int_0^T \kappa(a_t) dt + \beta, \end{aligned}$$

where $\beta = \ln \left(\alpha \frac{\gamma_A}{\gamma_P} \right)$.

Conversely, let us suppose that (ii''') holds. Then (W, a) where $W = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^a + \frac{\gamma_A}{\gamma_P + \gamma_A} \int_0^T \kappa(a_t) dt + \beta$ satisfies (2.1.1) and we have that :

$$\frac{U'_P(X_T^a - W)}{U'_A \left(W - \int_0^T \kappa(a_t) dt \right)} = \frac{\gamma_P}{\gamma_A} \exp((\gamma_P + \gamma_A)\beta) \in \mathbb{R}_+^*.$$

□

2.4.2 Proof of Proposition 2.2.3

Let $a \in \mathbb{H}_2$. We have

$$\begin{aligned}
& \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \\
&= \mathbb{E} \left[\exp \left(-\gamma_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \\
&= \mathbb{E} \left[\exp \left(-\gamma_P \left(x_0 + \int_0^T (a_t - \kappa(a_t)) dt + B_T \right) \right)^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \\
&= \exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} x_0 \right) \mathbb{E} \left[\exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} B_T - \frac{T}{2} \frac{|\gamma_P \gamma_A|^2}{|\gamma_A + \gamma_P|^2} \right) \exp \left(\int_0^T \Phi(a_t) dt \right) \right],
\end{aligned}$$

where

$$c \mapsto \Phi(c) := -\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} (c - \kappa(c)) + \frac{|\gamma_P \gamma_A|^2}{2|\gamma_A + \gamma_P|^2}.$$

Note that the mapping Φ is convex on \mathbb{R}_+ , and letting $a^* := \kappa^*(1)$,

$$\Phi(c) \geq \Phi(a^*) = \frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right), \quad \forall c \geq 0.$$

So,

$$\begin{aligned}
& \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_P + \gamma_A}} \right] \\
& \geq \exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} x_0 \right) \exp \left(\frac{T \gamma_P \gamma_A}{\gamma_A + \gamma_P} \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right),
\end{aligned}$$

as $\mathbb{E} \left[\exp \left(-\frac{\gamma_P \gamma_A}{\gamma_A + \gamma_P} B_T - \frac{T}{2} \frac{|\gamma_P \gamma_A|^2}{|\gamma_A + \gamma_P|^2} \right) \right] = 1$. We thus deduce our result :

$$\begin{aligned}
& \mathbb{E} \left[\left| U_P \left(X_T^a - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_P + \gamma_A}} \right]^{\frac{\gamma_A + \gamma_P}{\gamma_A}} \\
& \geq \exp(-\gamma_P x_0) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right),
\end{aligned}$$

as the last relation in the statement follows using the Participation Constraint (2.1.1).

2.4.3 Proof of the optimal contract : Theorem 2.2.2

We consider the contract (W^*, a^*) defined by setting :

$$a_t^* := \kappa^*(1)$$

and

$$W^* = \frac{\gamma_P}{\gamma_P + \gamma_A} X_T^{a^*} + \beta^*,$$

$$\beta^* := \frac{T\gamma_A|\gamma_P|^2}{2|\gamma_A + \gamma_P|^2} + y_{PC} + T\kappa(\kappa^*(1)) - \frac{\gamma_P}{\gamma_A + \gamma_P} (x_0 + T\kappa^*(1)).$$

We first study the participation constraint to verify the admissibility of such a contract. We have that :

$$\begin{aligned} \mathbb{E} \left[U_A \left(W^* - \int_0^T \kappa(a_s^*) ds \right) \right] &= \mathbb{E} \left[U_A \left(\frac{\gamma_P}{\gamma_P + \gamma_A} X_T^{a^*} + \beta^* - T\kappa^*(1) \right) \right] \\ &= \mathbb{E} \left[U_A \left(\frac{\gamma_P}{\gamma_P + \gamma_A} (x_0 + \kappa^*(1) + B_T) + \beta^* - T\kappa^*(1) \right) \right] \\ &= \mathbb{E} \left[U_A \left(\frac{\gamma_P}{\gamma_P + \gamma_A} B_T + \frac{T\gamma_A|\gamma_P|^2}{2|\gamma_A + \gamma_P|^2} + y_{PC} \right) \right] = U_A(y_{PC}). \end{aligned}$$

Thus W^* belongs to $\mathcal{W}(a^*)$. According to Item (ii) of Theorem 2.2.1, the contract satisfies the Borch Rule. Furthermore, it is of the form $(W^*, \kappa^*(1))$ where W^* saturates the Participation Constraint. It follows that the equality conditions to reach the upper bound of the Principal's Expected Utility are verified and we have that for any a in \mathbb{H}_2 and any W in $\mathcal{W}^*(a)$,

$$\begin{aligned} \mathbb{E}[U_P(X_T^a - W)] \\ \leq \mathbb{E} \left[U_P \left(X_T^{a^*} - W^* \right) \right] = U_P(x_0 - y_{PC}) \exp \left(\gamma_P T \left(-\tilde{\kappa}(1) + \frac{\gamma_P \gamma_A}{2(\gamma_A + \gamma_P)} \right) \right). \end{aligned}$$

We deduce that (W^*, a^*) is the optimal contract for the first-best Principal-Agent problem.

2.4.4 A technical lemma

Lemma 2.4.1. *Let (W, a) be an admissible contract in \mathcal{C} . The sequence of random variables $(\tilde{U}_P(X_T^a - W))_{0 < \gamma_P < 1}$ is uniformly integrable. And so :*

$$\mathbb{E}[X_T^a - W] = \mathbb{E} \left[\lim_{\gamma_P \rightarrow 0} \tilde{U}_P(X_T^a - W) \right] = \lim_{\gamma_P \rightarrow 0} \mathbb{E} \left[\tilde{U}_P(X_T^a - W) \right].$$

Proof. The second part of the statement is a consequence of the uniform integrability (UI) and of the fact that the identity mapping is the limit (as γ_P goes to 0) of \tilde{U}_P . So we focus on the UI property and apply de la Vallée-Poussin criterion. We have :

$$\begin{aligned} &\sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\tilde{U}_P(X_T^a - W)|^2 \right] \\ &= \gamma_P^{-2} \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\exp(-\gamma_P(X_T^a - W)) - 1|^2 \right] \\ &= \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\bar{X}|^2 \exp(-\gamma_P \bar{X}) \right], \end{aligned}$$

where \bar{X} is a random point between 0 and $X_T^a - W$ (using mean value theorem). By Cauchy-Schwarz's inequality we have,

$$\begin{aligned} & \sup_{0 < \gamma_P < 1} \mathbb{E} \left[|\tilde{U}_P(X_T^a - W)|^2 \right] \\ & \leq \mathbb{E} \left[|\bar{X}|^4 \right]^{1/2} \sup_{0 < \gamma_P < 1} \mathbb{E} \left[\exp(-4\gamma_P \bar{X}) \right]^{1/2}. \end{aligned}$$

As $|\bar{X}| \leq |X_T^a - W|$, \mathbb{P} -a.s., we have that $\mathbb{E} \left[|\bar{X}|^4 \right] < +\infty$. Regarding the second term,

$$\begin{aligned} & \sup_{0 < \gamma_P < 1} \mathbb{E} \left[\exp(-4\gamma_P \bar{X}) \right] \\ & \leq \sup_{0 < \gamma_P < 1} \left(\mathbb{P} \left[\bar{X} \geq 0 \right] + \mathbb{E} \left[\exp(-4\gamma_P \bar{X}) \mathbf{1}_{\bar{X} < 0} \right] \right) \\ & \leq 1 + \mathbb{E} \left[\exp(-4R\bar{X}) \mathbf{1}_{\bar{X} < 0} \right] \\ & \leq 1 + \mathbb{E} \left[\exp(4R|X_T^a - W|) \right] < +\infty. \end{aligned}$$

□

Chapter 3

A step towards incentives in the gig economy

3.1 Contracts in the gig economy

3.1.1 A note on the gig economy

The digitalization of the economy that has taken place since the start of the 21st century has led to the development of many digital platforms that connect customers with workers. Based on new business models, these platforms are characterized by easy access to on-demand labour and form a so-called "gig economy" in which independent workers are hired for short-term commitments, generally paid for the completion of a specific task or project. A now ubiquitous job in the gig-economy is that of a driver : passenger driver using a ride-sharing platform, freelance delivery driver... Another example is a freelance programmer as many platforms now allow companies to find and compensate freelance programmers to complete specific tasks for them.

The development of this new and parallel economy has led to a shift in the way that we define "workers". Previously, a worker tended to have a long term contract, set working hours and benefits as well as a fixed salary. Nowadays, gig workers can choose to work when they want, for the hours that they want in exchange for task based compensation. Of course, this new form of employment has been the source of much controversy : the high levels of flexibility, variety and autonomy (see [135]) brought about by this employment model are counterbalanced by a lack of stability, few benefits, little workplace protection (see [43]). In fact in the USA, a country where more than 1 in 10 workers rely on gig work for their primary income¹, various states have been considering implementing new regulations in order to increase worker rights in the gig economy. However it seems that this form of punctual work may be here to stay : on November 3rd 2020, voters in California were given the opportunity to change the status of app-based drivers from independent contractors to employees and thus provide them with health, retirement and disability benefits. In the end the vote turned out

¹according to the 2017 Contingent Worker Supplement conducted by the Census Bureau for the Bureau of Labor and Statistics

in favor of the independent contractor status for such workers. This message, as well as the continuing growth of many digital platforms, suggests that developing understanding of the gig economy may be a relevant path of academic research.

There are in fact many particularities of contracts in the gig-economy as opposed to "normal" contracting agreements. One such specificity is that a gig worker can easily (providing he has the right skills) pop from one platform to another and therefore choose different work at different times. Modeling this may call for a multi-Principal type problem. Another particularity is that pay is generally task-based, with the worker being paid a portion of the generated wealth of his completed task. The latter remark leads to this focus of this work : we attempt to model optimal incentives that are purely task-based, where the Principal considers solely the wealth generated by the Agent as a basis for the Agent's compensation. To model this and as an adaptation of classical Principal-Agent models, we consider that the Agent sets out to work for the Principal over a pre-decided period of time (for example, for a couple of hours). During that time, the wealth that the Agent generates is modeled through a jump process, jumping by some fixed positive value each time a task is completed. This process has some time-dependent intensity, which fits with applications : the demand on ride-sharing apps for example is time-dependent, and therefore the wealth that the Agent may be able to generate through this platform will be time-dependent too. In particular, we consider that the wealth process has some market intensity, intensity that the Agent has access to at no cost by "coasting". He may also have an effect on this market intensity through his actions : through positive actions, the intensity increases, through negative actions the intensity decreases. Of course if there is no demand, the Agent will not be able to generate any wealth despite being available to work and despite providing effort... In line with standard Principal-Agent problems, the Agent receives some compensation at the end of the contracting period. The question that we aim to answer in this model is that of optimal compensation. *More specifically in a model where an Agent can generate wealth for the Principal by completing tasks, and in particular choose or not to have an effect on the intensity of the task completion, how should the Principal compensate the Agent ? Will the Agent then choose to coast, or will he choose to positively affect the market intensity ?* We wish to underline that this model has many limits : it does not model the possibility for an Agent to change platform or competition between Agents and assumes that the Agent does not stop early. It solely attempts to model task-based income (where each task yields the same value) and there is for example no consideration for task difficulty : it is supposed that over the short contracting period, the tasks are homogeneous in that sense. Let us now further specify this model.

3.1.2 A Poisson based model

Probabilistic setting and model. In this work we consider a single Agent and single Principal tied through a full commitment contract over a time period $[0, T]$ where $T > 0$ is fixed. This time period - assumed to be short - could for example model one ride-sharing driving session. With this in mind, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}^0)$ carrying :

- N a Poisson process with deterministic, finite and positive intensity process $(\lambda_s)_{s \in [0, T]}$. In particular we denote $\bar{\lambda} = \sup_{t \in [0, T]} \lambda_t$, and $\underline{\lambda} = \inf_{t \in [0, T]} \lambda_t$. We suppose $\underline{\lambda} > 0$.

We denote as $\mathbb{G} = (\mathcal{F}_t)_{t \in [0, T]}$ the filtration generated by the Poisson process N . In this setting, we introduce the notation X for the revenue process :

$$X_T = \sum_{i=1}^{N_T} 1,$$

that will model the revenue that the Agent's tasks yield for the Principal over the contracting period (in this very simplistic model, each task has an identical value of 1 - it may quite trivially extend to a setting where each task has another positive identical value). Finally, we also introduce the following sets :

- \mathcal{S}^2 the set of \mathbb{G} -adapted RCLL processes Y such that $\mathbb{E} \left[\sup_{t \in [0, T]} Y_t^2 \right] < +\infty$
- \mathbb{H}^2 the set of \mathbb{G} -predictable processes Y such that $\mathbb{E} \left[\int_0^T Y_t^2 dt \right] < +\infty$.

As mentioned previously, we suppose that the Agent can, if he chooses to, exert some effort in order to influence the intensity of the underlying Poisson process. Therefore the Agent's efforts can either increase or decrease the frequency of any tasks that are completed (this may model several phenomena from speed of task completion of tasks to the changes in the Agent's reputation on his demand). Of course, if the Agent chooses to, he can also simply coast, providing minimal effort, and have access to the "market" intensity $(\lambda_s)_{s \in [0, T]}$. We suppose that the Principal can suffer from Moral Hazard and cannot see the Agent's actions : the Agent and the Principal will therefore observe the production process through a different measure. In order to model this, we let \mathcal{P} denote the set of \mathbb{G} -predictable stochastic processes and define the set :

$$\mathcal{A} := \left\{ (a_t)_{t \in [0, T]} \in \mathcal{P}, \text{ s. t. } -1 < a_t \leq M \quad \forall t \in [0, T] \right\},$$

for some M in \mathbb{R}^+ chosen to be as big as needed. ² We suppose that the Agent observes the process N under a measure \mathbb{P}^a defined through the transformation :

$$\frac{d\mathbb{P}^a}{d\mathbb{P}^0} = L_t \quad ; \quad dL_t = L_{t-} a_t \{dN_t - \lambda_t dt\},$$

where $(a_t)_{t \in [0, T]}$ belongs to \mathcal{A} . Of course due to the boundedness of $(a_t)_{t \in [0, T]}$ in \mathcal{A} , \mathbb{P}^a and \mathbb{P}^0 are two equivalent probability measures and L_t has the closed form :

$$L_t = \exp \left(- \int_0^t a_s \lambda_s ds + \int_0^t \log(1 + a_s) dN_s \right).$$

The effect of this change in measure is a change in the intensity of the Poisson process : under \mathbb{P}^a , the intensity of the Poisson process N becomes $(\lambda_s(1 + a_s))_{s \in [0, T]}$. In line with classical Principal-Agent problems, we suppose that performing $(a_s)_{s \in [0, T]}$ costs the Agent some effort modeled by the quadratic cost function $\kappa(a) = \frac{\kappa a^2}{2}$. In particular across the contracting period, the Agent's effort cost is :

$$\int_0^T \kappa(a_s) ds.$$

²with for example $M \geq \sup \left(\frac{\bar{\lambda}}{\kappa \frac{\gamma_A \gamma_P}{\gamma_A + \gamma_P}}, \frac{\bar{\lambda} c}{\kappa \gamma_A} \right)$, where the reason for this upper bound will become apparent in the section on incentives

As a reward for his efforts, we suppose that the Agent receives some wage W at time T , with W belonging to the set \mathcal{W} defined as follows :

$$\mathcal{W} := \left\{ \mathcal{G}_T - \text{measurable random variables } W, \mathbb{E} [W^2] < +\infty \right\}.$$

Finally we consider that the Principal and the Agent both tract exponential utility from their underlying wealth, modeled through the utility functions :

$$U_P(x) := -\exp(-\gamma_P x) \quad \text{and} \quad U_A(x) := -\exp(-\gamma_A x), \quad x \in \mathbb{R},$$

where $\gamma_P > 0$ and $\gamma_A > 0$ are two fixed risk aversion parameters. We may now model contracting in this setting.

The Moral Hazard problem. We consider a classical formulation of the Moral Hazard problem adapted to this setting. This first involves solving the Agent's problem by computing for each W in \mathcal{W} and when possible :

$$\sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right], \quad (3.1.1)$$

yielding a set of so called "incentive compatible" wages W and their associated actions $\mathcal{E}(W)$. We then turn to the Principal's problem which writes as solving :

$$\sup_{W \in \mathcal{W}} \sup_{a \in \mathcal{E}(W)} \mathbb{E}^{\mathbb{P}^{a^*}} [U_P(X_T - W)]. \quad (3.1.2)$$

It is supposed that the Agent may have outside options and therefore will only engage in contracting settings that are guaranteed to provide him with a certain level of expected utility. Therefore, we ensure that the following constraint holds at the optimum :

$$\mathbb{E}^{\mathbb{P}^{a^*}} \left[U_A \left(W^* - \int_0^T \kappa(a_s^*) ds \right) \right] \geq U_A(y_{PC}), \quad (3.1.3)$$

for some fixed real y_{PC} that characterizes the minimal level of expected utility that the Agent wants.

Of course, as is standard in contracting problems, we may compare the Moral Hazard optimum to the Risk-Sharing benchmark where the Principal and the Agent both observe the production process under the same measure \mathbb{P}^a . In particular the Principal observes the Agent's action process and dictates his action whilst simply ensuring that a Participation Constraint is in place. We write the contracting problem as follows.

The "Risk-Sharing" problem. The Risk-Sharing problem writes as solving the following constrained optimization problem :

$$\sup_{(W,a) \in \mathcal{W} \times \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} [U_P(X_T - W)], \quad (3.1.4)$$

such that :

$$\mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right] \geq U_A(y_{PC}), \quad (3.1.5)$$

for some fixed y_{PC} .

Findings related to incentives in a pay-per-project setting. In line with classical optimal incentives using expected utility maximization, we find that the Agent should be paid a proportion of the generated income. Under Risk-Sharing this proportion is the classical ratio $\frac{\gamma_P}{\gamma_P + \gamma_A}$. However, under Moral Hazard, this proportion varies over time. In particular, if N_T is the number of tasks completed over the contracting period, $(t_1 \dots t_{N_T})$ the times at which the tasks are realized and $Y_i, i \in \{1 \dots N_T\}$ is the value of each of these tasks, the random part of the Agent's compensation writes as :

$$\sum_{i=1}^{N_T} K_{t_i}^*,$$

where $K_{t_i}^*$ is the proportion of the wealth of the i 'th tasks paid to the Agent. This proportion depends on the underlying parameters of the model, with one key finding being that it increases when the underlying intensity increases, whilst also being upper bounded. This phenomena seems to align with real phenomena such as Uber's surge pricing : the revenue received by Uber drivers in periods of high demand is higher than under low demand. Such a tool is generally used by ride sharing platforms in order to be able to match demand and supply when demand is high, and such considerations are beyond the scope of this simple model. It is therefore interesting to note that such a phenomena seems to appear in a one-on-one Principal-Agent model. This has an effect on the action that the Agent will produce : the Agent's action increases as the market intensity increases, so the Agent produces more effort when demand is high. Finally we note that wage positivity is not guaranteed, despite the positivity of the Principal's wealth process which brings some insight into the difficulty that arises in order to enforce wage positivity in Principal-Agent problems... These findings are detailed and discussed in Section 3.3.2.

Structure of this work. The work is structured as follows. First we introduce a certain useful Backward Stochastic Differential Equation. This tool is then used to tackle an incentive problem in such a pay-per-project setting resulting in a derivation of the optimal contract for the Principal-Agent problems at hand. Finally, we provide some analysis of some consequences of such a contract bearing in mind the limits of the model.

3.2 Analysis of a certain Poisson based BSDE

3.2.1 Presentation and properties

We turn to introducing a key object for this work : a certain Backward Stochastic Differential Equation (BSDE). To this end we first define the centered process M closely related to X :

$$X_t = \sum_{i=1}^{N_t} 1, \quad M_t := N_t - \int_0^t \lambda_s ds,$$

which is a \mathbb{G} -martingale under \mathbb{P}^0 . Given $\rho > 0$ some fixed constant, we define the function f^ρ as follows :

$$f^\rho(s, k) := -\frac{\lambda_s c}{\rho}(e^{-\rho k} - 1 + \rho k) + \sup_{a > -1} \left\{ -\kappa(a) - \frac{\lambda_s c}{\rho} a(e^{-\rho k} - 1) \right\}. \quad (3.2.1)$$

With this function in mind, we consider pairs (J, K) of processes in $\mathcal{S}^2 \times \mathbb{H}^2$ solution to the BSDE :

$$J_t = J_T - \int_t^T K_s dM_s + \int_t^T f^\rho(s, K_s) ds, \quad t \in [0, T], \quad J_T := W, \quad (3.2.2)$$

where W is some \mathcal{G}_T -measurable random variable such that $\mathbb{E}[W^2] < +\infty$. Such processes will be key to the dynamic contracting part of this work and in order to be able to exploit them, we first need to establish their existence and uniqueness. This seems reasonable to expect given the form of the driver f^ρ (which is close to something that could be dealt with through a Cole-Hopf transform, and is also free of any dependency on J). The following calculations yield a remarkable and very useful property of the driver.

Lemma 3.2.1. *The driver f^ρ is Lipschitz in k .*

Proof. We first rewrite the driver function as follows :

$$\begin{aligned} f^\rho(s, k) &= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \sup_{a > -1} \left\{ -\frac{\kappa}{2}|a|^2 - \frac{\lambda_s}{\rho} a(e^{-\rho k} - 1) \right\} \\ &= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) - \frac{\kappa}{2} \inf_{a > -1} \left\{ |a|^2 + \frac{2\lambda_s}{\kappa\rho} a(e^{-\rho k} - 1) \right\} \\ &= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \right|^2 - \frac{\kappa}{2} \inf_{a > -1} \left| a + \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) \right|^2. \end{aligned}$$

In addition,

$$\inf_{a > -1} \left| a + \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) \right|^2 = 0, \quad \text{attained for } a^*(k) = -\frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1), \quad \text{if } \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) < 1,$$

and

$$\inf_{a > -1} \left| a + \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) \right|^2 = \left| \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) - 1 \right|^2, \quad \text{and is not attained,} \quad \text{if } \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) \geq 1.$$

Now set :

$$\bar{k}_s := -\frac{1}{\rho} \log \left(1 + \frac{\rho\kappa}{\lambda_s} \right)$$

so that

$$[k > \bar{k}_s] \Leftrightarrow \left[\frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) < 1 \right].$$

Now let s in $[0, T]$, then using the above calculation we can distinguish between two cases to help analyze the driver f^ρ .

If $k > \bar{k}_s$ then :

$$\begin{aligned}
f^\rho(s, k) &= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \right|^2 - \frac{\kappa}{2} \inf_{a > -1} \left| a + \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) \right|^2 \\
&= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \right|^2 - \frac{\kappa}{2} \times 0 \\
&= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{(\lambda_s)^2}{\rho^2} \frac{1}{2\kappa} |e^{-\rho k} - 1|^2 \\
&= -k\lambda_s + \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \left(\frac{\lambda_s}{2\rho\kappa}(e^{-\rho k} - 1) - 1 \right),
\end{aligned}$$

which is Lipschitz in k as $e^{-\rho k}$ is bounded. We denote as C_1 the related Lipschitz constant.

If $k \leq \bar{k}_s$ then :

$$\begin{aligned}
f^\rho(s, k) &= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \right|^2 - \frac{\kappa}{2} \inf_{a > -1} \left| a + \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) \right|^2 \\
&= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \right|^2 - \frac{\kappa}{2} \left| \frac{\lambda_s}{\rho\kappa}(e^{-\rho k} - 1) - 1 \right|^2 \\
&= -\frac{\lambda_s}{\rho}(e^{-\rho k} - 1 + \rho k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) \right|^2 - \frac{1}{2\kappa} \left| \frac{\lambda_s}{\rho}(e^{-\rho k} - 1) - \kappa \right|^2 \\
&= -\lambda_s k - \frac{\kappa}{2},
\end{aligned}$$

which is of course Lipschitz in k , we denote as C_1 the related Lipschitz constant. Note in addition that : $\lim_{k \rightarrow \bar{k}_s; k > \bar{k}_s} f(s, k) = f(s, \bar{k}_s)$ so if $k_1 \leq \bar{k}_s$ and $k_2 > \bar{k}_s$, then :

$$\begin{aligned}
|f(s, k_1) - f(s, k_2)| &= | -\lambda_s c(k_1 - \bar{k}_s) + f(s, \bar{k}_s) - f(s, k_2) | \\
&\leq C_1 |k_1 - \bar{k}_s| + C_2 |k_2 - \bar{k}_s| \\
&\leq C |k_1 - k_2| \quad \text{for some } C \text{ in } \mathbb{R}.
\end{aligned}$$

So f^ρ is a Lipschitz driver in k . □

The Lipschitz property of the driver (combined with the fact that $(\lambda_s)_{s \in [0, T]}$ is deterministic) allows for very standard techniques to be used to derive existence and uniqueness of solutions to the equation. Furthermore, for a certain terminal condition that will appear in contracting analysis, the solution to the equation is fully explicit. This is the object of the following Lemma.

Lemma 3.2.2. *Given a terminal condition W such that $\mathbb{E}[W^2] < +\infty$, there exists a unique pair of processes (J, K) in $\mathcal{S}^2 \times \mathbb{H}^2$ solution to the BSDE (3.2.5). In the particular case where $W = N_T$, the solution is explicit and given by the pair (J, K) defined as :*

$$J_t := N_t + \int_t^T \beta_s ds, \quad K_t = 1,$$

where $(\beta_s)_{s \in [0, T]}$ is defined below :

$$\beta_s := \frac{\lambda_s}{\rho}(e^{-\rho} - 1) \left(-1 + \frac{\lambda_s}{2\kappa\rho}(e^{-\rho} - 1) \right). \quad (3.2.3)$$

Proof. As the underlying driver is Lipschitz, the existence and uniqueness result is quite classical and can be obtained through standard techniques involving a priori estimates and a fixed point theorem. For completeness, an outline of such reasoning for Lipschitz drivers is provided in the Appendix of this document. We turn to the specific case where the solution is explicit, noting that :

$$f(s, 1) = -\lambda_s + \beta_s. \quad (3.2.4)$$

Then :

$$N_T - \int_t^T 1 dN_s + \int_t^T \beta_s ds = N_T - \int_t^T 1 dM_s + \int_t^T (\beta_s - \lambda_s) ds$$

and this process is adapted so :

$$\begin{aligned} & N_T - \int_t^T 1 dM_s + \int_t^T (\beta_s - \lambda_s) ds \\ &= \mathbb{E}^{\mathbb{P}^0} \left[N_T - \int_t^T 1 dM_s + \int_t^T (\beta_s - \lambda_s) ds \middle| \mathcal{G}_t \right] \\ &= \mathbb{E}^{\mathbb{P}^0} \left[N_T + \int_t^T (\beta_s - \lambda_s) ds \middle| \mathcal{G}_t \right] \text{ (we remove the martingale increment)} \\ &= \mathbb{E}^{\mathbb{P}^0} \left[N_T - \int_0^T \lambda_s ds \middle| \mathcal{G}_t \right] + \int_0^T \lambda_s ds + \int_t^T (\beta_s - \lambda_s) ds \\ &= N_t - \int_0^t \lambda_s ds + \int_0^T \lambda_s ds + \int_t^T (\beta_s - \lambda_s) ds \\ &= N_t + \int_t^T \beta_s ds = N_t + \int_t^T \beta_s ds = J_t, \end{aligned}$$

and

$$J_T = N_T.$$

Both J and K satisfy the dynamics of the BSDE, and of course $J \times K$ belongs to $\mathcal{S}^2 \times \mathbb{H}^2$. Therefore the pair (J, K) is an explicit solution to the BSDE (and is the unique one through the uniqueness result). \square

3.2.2 Existence and uniqueness of solutions to (3.2.5)

For completeness of this work, the following illustrates the classical method that can be used to tackle the existence of a pair of processes (J, K) in $\mathcal{S}^2 \times \mathbb{H}^2$ solution to :

$$J_t = J_T - \int_t^T K_s dM_s + \int_t^T f(s, K_s) ds, \quad t \in [0, T], \quad J_T := W, \quad (3.2.5)$$

where W is some \mathcal{G}_T -measurable random variable such that $\mathbb{E}[W^2] < +\infty$ and where $\forall (k_1, k_2) \in \mathbb{R}^2$, $|f(s, k_1) - f(s, k_2)| \leq C|k_1 - k_2|$ for some fixed Lipschitz constant C . This reasoning therefore completes the proof of Lemma 3.2.2

3.2.2.1 An estimate

First we prove some a-priori estimates. For $\beta > 0$ and given ϕ in \mathbb{H}^2 , we introduce the weighted norms :

$$\|\phi\|_\beta^2 = \mathbb{E} \left[\int_0^T e^{\beta s} \phi_s^2 ds \right] \quad \text{and} \quad \|\phi\|_{\beta, \lambda}^2 = \mathbb{E} \left[\int_0^T e^{\beta s} \phi_s^2 \lambda_s ds \right].$$

Now let W_1 and W_2 be two \mathcal{G}_T -measurable and square integrable terminal conditions. Let f^1 and f^2 be two drivers satisfying the Lipschitz property given above. Let us suppose that there exists a solution (J^1, K^1) and (J^2, K^2) to (3.2.5) for these two respective settings. Set $\bar{W} = W_1 - W_2$, $\bar{J} = J^1 - J^2$ and $\bar{K} = K^1 - K^2$. Also set $\bar{f}_s = f^1(s, K_s^1) - f^2(s, K_s^2)$. Then :

$$\bar{J}_t = \bar{W} - \int_t^T \bar{K}_s dM_s + \int_t^T \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds, \quad t \in [0, T],$$

and therefore, recalling that $\tilde{X}_T = \sum_{i=1}^{N_T} Y_i - \int_0^T \lambda_s c ds$:

$$\begin{aligned} (\bar{J}_t)^2 &= \bar{W}^2 + 2 \int_t^T \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \\ &\quad + 2 \int_t^T \bar{J}_s \bar{K}_s \lambda_s ds - \int_t^T (\bar{J}_s^2 - \bar{J}_{s-}^2) dN_s \\ &= \bar{W}^2 + 2 \int_t^T \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \\ &\quad + 2 \int_t^T \bar{J}_s \bar{K}_s \lambda_s ds - \int_t^T ((\bar{J}_{s-} + \bar{K}_s)^2 - \bar{J}_{s-}^2) dN_s \\ &= \bar{W}^2 + 2 \int_t^T \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \\ &\quad + 2 \int_t^T \bar{J}_s \bar{K}_s \lambda_s ds - \int_t^T (\bar{K}_s^2 + 2\bar{J}_{s-} \bar{K}_s) dN_s \\ &= \bar{W}^2 + 2 \int_t^T \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \\ &\quad + 2 \int_t^T \bar{J}_s \bar{K}_s \lambda_s ds - \int_t^T (\bar{K}_s^2 + 2\bar{J}_{s-} \bar{K}_s) dM_s \\ &\quad - \int_t^T (\bar{K}_s^2 + 2\bar{J}_{s-} \bar{K}_s) \lambda_s ds, \quad \text{as } dM_s = dN_s - \lambda_s ds. \end{aligned}$$

As a consequence :

$$\begin{aligned} e^{\beta t} (\bar{J}_t)^2 &= e^{\beta T} \bar{W}^2 - \int_t^T \beta e^{\beta s} \bar{J}_s^2 ds + 2 \int_t^T e^{\beta s} \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \\ &\quad + 2 \int_t^T e^{\beta s} \bar{J}_s \bar{K}_s \lambda_s ds - \int_t^T e^{\beta s} (\bar{K}_s^2 + 2\bar{J}_{t-} \bar{K}_s) dM_s \\ &\quad - \int_t^T e^{\beta s} (\bar{K}_s^2 + 2\bar{J}_{t-} \bar{K}_s) \lambda_s ds \end{aligned}$$

and reordering these terms yields :

$$\begin{aligned} & e^{\beta t} \bar{J}_t^2 + \int_t^T \beta e^{\beta s} \bar{J}_s^2 ds + \int_t^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \\ &= e^{\beta T} \bar{W}^2 + 2 \int_t^T e^{\beta s} \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds - \int_t^T e^{\beta s} (\bar{K}_s^2 + 2\bar{J}_t - \bar{K}_s) dM_s. \end{aligned}$$

We wish to take conditional expectation with respect to \mathcal{G}_t to obtain our estimates. Doing so we obtain that :

$$\begin{aligned} & \mathbb{E} \left[e^{\beta t} \bar{J}_t^2 \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\int_t^T \beta e^{\beta s} \bar{J}_s^2 ds \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\int_t^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \middle| \mathcal{G}_t \right] \\ &= \mathbb{E} \left[e^{\beta T} \bar{W}^2 \middle| \mathcal{G}_t \right] + \mathbb{E} \left[2 \int_t^T e^{\beta s} \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \middle| \mathcal{G}_t \right]. \end{aligned}$$

In order to exploit the Lipschitz property of f^1 , note that :

$$\begin{aligned} f^1(s, K_s^1) - f^2(s, K_s^2) &= f^1(s, K_s^1) - f^1(s, K_s^2) + \bar{f}_s \\ &\leq C |K_s^1 - K_s^2| + |\bar{f}_s| = \tilde{C} \sqrt{\lambda_s} |K_s^1 - K_s^2| + |\bar{f}_s|, \end{aligned}$$

and so for any $\epsilon > 0$:

$$\begin{aligned} & \mathbb{E} \left[2 \int_t^T e^{\beta s} \bar{J}_s \left(f^1(s, K_s^1) - f^2(s, K_s^2) \right) ds \middle| \mathcal{G}_t \right] \\ &\leq \mathbb{E} \left[\int_t^T e^{\beta s} 2 |\bar{J}_s| \left(\tilde{C} \sqrt{\lambda_s} |K_s^1 - K_s^2| + |\bar{f}_s| \right) ds \middle| \mathcal{G}_t \right] \\ &\leq \mathbb{E} \left[\int_t^T e^{\beta s} \frac{|\bar{J}_s|^2}{\epsilon^2} + e^{\beta s} \epsilon^2 \left(\tilde{C} \sqrt{\lambda_s} |K_s^1 - K_s^2| + |\bar{f}_s| \right)^2 ds \middle| \mathcal{G}_t \right] \quad \text{through Young's inequality} \\ &= \mathbb{E} \left[\int_t^T e^{\beta s} \frac{|\bar{J}_s|^2}{\epsilon^2} ds \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\int_t^T e^{\beta s} \epsilon^2 \left(\tilde{C} \sqrt{\lambda_s} |K_s^1 - K_s^2| + |\bar{f}_s| \right)^2 ds \middle| \mathcal{G}_t \right] \\ &\leq \mathbb{E} \left[\int_t^T e^{\beta s} |\bar{J}_s|^2 ds \middle| \mathcal{G}_t \right] \left(\frac{1}{\epsilon^2} + 2\tilde{C} \right) + \mathbb{E} \left[\int_t^T e^{\beta s} \epsilon^2 \left(\tilde{C} \sqrt{\lambda_s} |K_s^1 - K_s^2| + |\bar{f}_s| \right)^2 ds \middle| \mathcal{G}_t \right] \\ &\leq \mathbb{E} \left[\int_t^T e^{\beta s} |\bar{J}_s|^2 ds \middle| \mathcal{G}_t \right] \left(\frac{1}{\epsilon^2} + 2\tilde{C} \right) + 3\epsilon^2 \tilde{C}^2 \mathbb{E} \left[\int_t^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \middle| \mathcal{G}_t \right] + 3\epsilon^2 \mathbb{E} \left[\int_t^T e^{\beta s} |\bar{f}_s|^2 ds \middle| \mathcal{G}_t \right] \end{aligned}$$

So finally :

$$\begin{aligned}
& e^{\beta t} \bar{J}_t^2 + \beta \mathbb{E} \left[\int_t^T e^{\beta s} \bar{J}_s^2 ds \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\int_t^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \middle| \mathcal{G}_t \right] \\
& \leq \mathbb{E} \left[e^{\beta T} \bar{W}^2 \middle| \mathcal{G}_t \right] + \mathbb{E} \left[\int_t^T e^{\beta s} |\bar{J}_s|^2 ds \middle| \mathcal{G}_t \right] \left(\frac{1}{\epsilon^2} + 2\tilde{C} \right) + 3\epsilon^2 \tilde{C}^2 \mathbb{E} \left[\int_t^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \middle| \mathcal{G}_t \right] \\
& + 3\epsilon^2 \mathbb{E} \left[\int_t^T e^{\beta s} |\bar{f}_s|^2 ds \middle| \mathcal{G}_t \right]
\end{aligned}$$

We may then calibrate $\beta > 0$ and $\epsilon > 0$ such that :

$$3\epsilon^2 \tilde{C}^2 \leq 1 \quad \text{and} \quad \frac{1}{\epsilon^2} + 2\tilde{C} \leq \beta$$

which yields :

$$e^{\beta t} \bar{J}_t^2 \leq \mathbb{E} \left[e^{\beta T} \bar{W}^2 \middle| \mathcal{G}_t \right] + 3\epsilon^2 \mathbb{E} \left[\int_t^T e^{\beta s} |\bar{f}_s|^2 ds \middle| \mathcal{G}_t \right] \quad \text{a.s.}$$

Integrating with respect to time and taking the expected value we then get :

$$\mathbb{E} \left[\int_0^T e^{\beta t} \bar{J}_t^2 ds \right] \leq T e^{\beta T} \mathbb{E}[W^2] + T 3\epsilon^2 \mathbb{E} \left[\int_0^T e^{\beta s} |\bar{f}_s|^2 ds \right] \quad (3.2.6)$$

Finally, also setting ϵ such that :

$$3\epsilon^2 \tilde{C}^2 < 1,$$

we have :

$$\mathbb{E} \left[\int_0^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \right] (1 - 3\epsilon^2 \tilde{C}^2) \leq \mathbb{E} \left[e^{\beta T} \bar{W}^2 \right] + 3\epsilon^2 \mathbb{E} \left[\int_0^T e^{\beta s} |\bar{f}_s|^2 ds \right]$$

which implies that :

$$\mathbb{E} \left[\int_0^T e^{\beta s} \bar{K}_s^2 \lambda_s ds \right] \leq \frac{1}{(1 - 3\epsilon^2 \tilde{C}^2)} \left(e^{\beta T} \mathbb{E} \left[\bar{W}^2 \right] + 3\epsilon^2 \mathbb{E} \left[\int_0^T e^{\beta s} |\bar{f}_s|^2 ds \right] \right) \quad (3.2.7)$$

Summing (3.2.6) and (3.2.7), we finally conclude that :

$$\|\bar{J}\|_\beta^2 + \|\bar{K}\|_{\beta, \lambda}^2 \leq \left(e^{\beta T} \mathbb{E}[\bar{W}^2] + 3\epsilon^2 \mathbb{E} \left[\int_0^T e^{\beta s} |\bar{f}_s|^2 ds \right] \right) \left(T + \frac{1}{1 - 3\epsilon^2 \tilde{C}^2} \right) \quad (3.2.8)$$

3.2.2.2 Existence and uniqueness of solutions

First let us note that in the particular case where g is only time-dependent (and thus not dependent on K), the result holds from a Martingale Representation Theorem for Poisson processes (such as Proposition 2.2.6 of [70]) along with some standard calculations.

With this in mind, we are able to use the estimate for the slightly more general case where we consider f such that $|f(s, k_1) - f(s, k_2)| \leq C|k_1 - k_2|$. As λ_s is deterministic (and

lower bounded), this may rewrite as $|f(s, k_1) - f(s, k_2)| \leq \tilde{C}\sqrt{\lambda_s}|k_1 - k_2|$ as used in the estimates above. We consider the space \mathbb{H}_β^2 to be $\mathbb{H}^2 \times \mathbb{H}^2$ equipped with the weighted norm $\|J, K\|_\beta^2 = \|J\|_\beta^2 + \|K\|_{\lambda, \beta}^2$.

Given the terminal condition W , first, we fix some point (Y^1, Z^1) in \mathbb{H}_β^2 and we define a mapping m from \mathbb{H}_β^2 onto itself, by setting $(J^1, K^1) := m(Y^1, Z^1)$ the solution to the BSDE (3.2.5) with the driver function $g^1(s) = f(s, Z_s^1)$ (this driver function is only time-dependent and therefore existence will be obtained through the martingale representation approach mentioned above). We wish to prove that m is a contraction on \mathbb{H}_β^2 using the above estimate. To do so, we consider another point (Y^2, Z^2) in \mathbb{H}_β^2 and let $(J^2, K^2) := m(Y^2, Z^2)$ (i.e. the solution to the BSDE with the driver $g_s^2 = f(s, Z_s^2)$). Then through the estimate above, as soon as $\frac{1}{\epsilon^2} < \beta$:

$$\|\bar{J}\|_\beta^2 + \|\bar{K}\|_{\beta, \lambda}^2 \leq \mathbb{E} \left[\int_0^T e^{\beta s} |f(s, Z_s^1) - f(s, Z_s^2)|^2 ds \right] 3\epsilon^2 (T+1)$$

but as f is Lipschitz in Z :

$$\begin{aligned} \|\bar{J}\|_\beta^2 + \|\bar{K}\|_{\beta, \lambda}^2 &\leq \mathbb{E} \left[\int_0^T e^{\beta s} |f(s, Z_s^1) - f(s, Z_s^2)|^2 ds \right] 3\epsilon^2 (T+1) \\ &\leq \mathbb{E} \left[\int_0^T e^{\beta s} \tilde{C}^2 \lambda_s |\bar{Z}_s|^2 ds \right] 3\epsilon^2 (T+1) \\ &= \|\bar{Z}\|_{\beta, \lambda}^2 3\tilde{C}^2 \epsilon^2 (T+1) \\ &\leq (\|\bar{Z}\|_{\beta, \lambda}^2 + \|\bar{Y}\|_\beta^2) 3\tilde{C}^2 \epsilon^2 (T+1) \end{aligned}$$

and we wish to calibrate ϵ such that :

$$3\tilde{C}^2 \epsilon^2 (T+1) < 1$$

in order to obtain a fixed point. In fact we may set :

$$\epsilon = \frac{1}{2\tilde{C}} \sqrt{\frac{1}{T+1}}$$

then :

$$3\tilde{C}^2 \epsilon^2 (T+1) = \frac{3}{4} < 1.$$

We also set $\beta = \frac{1}{\epsilon^2} + 1$, which ensures that $\beta > \frac{1}{\epsilon^2}$ as required above. So choosing such a ϵ and β ,

$$\|\bar{J}\|_\beta^2 + \|\bar{K}\|_{\beta, \lambda}^2 \leq \frac{3}{4} (\|\bar{Z}\|_{\beta, \lambda}^2 + \|\bar{Y}\|_\beta^2)$$

and m is a contraction from \mathbb{H}_β^2 to \mathbb{H}_β^2 . Therefore m admits a unique fixed point (J, K) in the Banach space \mathbb{H}_β^2 which is the unique solution to (3.2.5).

3.3 Dynamic contracting in a Poisson setting

We now proceed to tackling our dynamic contracting problems, that is attempt to compute wage/action pairs solving the benchmark problem (3.1.4)-(3.1.5) and the Moral Hazard problem (3.1.1)-(3.1.2)-(3.1.3). We present this work as follows : first we summarize our findings before providing some analysis of the optimal contracts. The proofs are collected in the following dedicated Section.

3.3.1 Optimal contracts

Benchmark contract. First, we have the following result on the benchmark contract, solution to the Risk-Sharing problem (3.1.4)-(3.1.5) where we recover a quite classical linear contract for Risk-Sharing under CARA utility (linear in the wealth process, with a coefficient $\frac{\gamma_P}{\gamma_P + \gamma_A}$).

Theorem 3.3.1. *Let $\bar{\rho} = \frac{\gamma_P \gamma_A}{\gamma_P + \gamma_A}$. The optimal contract in the Risk-Sharing setting, solution to the Risk-Sharing problem (3.1.4)-(3.1.5), is given by :*

$$a_s^* = \frac{\lambda_s}{\bar{\rho} \kappa} (1 - e^{-\bar{\rho}}), \quad s \in [0, T],$$

and :

$$W^* = y_{PC} + \frac{\gamma_P}{\gamma_P + \gamma_A} N_T + \int_0^T \kappa(a_s^*) ds + \frac{1}{\gamma_A} \int_0^T \lambda_s (e^{-\bar{\rho}} - 1) (1 + a_s^*) ds.$$

Proof. See Section 3.4.1. □

We may note that with Chapter 2 in mind, this result is rather unsurprising. Despite the slightly alternative wealth process for the Principal, the structure of the optimal compensation remains purely linear. The optimal action is not the Legendre transform of κ anymore but is related to an optimization problem, given in Corollary 2.3.1.

Incentives under Moral Hazard. When it comes to contracting under Moral Hazard, we first tackle the Agent's problem with the underlying aim of linking compensation packages paid to the Agent and his resulting action. In order words, we wish to solve for W in \mathcal{W} and when possible :

$$\sup_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right], \quad (3.3.1)$$

In particular, we consider controlled wage processes of the form :

$$W_t^{y_0, K} = y_0 + \int_0^t K_s dM_s - \int_0^t f^{\gamma_A}(s, K_s) ds, \quad t \in [0, T],$$

where (y_0, K) belongs to the set Γ defined as :

$$\Gamma = \left\{ (y, K) \in \mathbb{R} \times \mathbb{H}_2, \quad K_s > -\frac{1}{\gamma_A} \log \left(1 + \frac{\gamma_A \kappa}{\lambda_s} \right), \quad s \in [0, T], \quad Z^a \text{ and } \tilde{Z}^a \text{ true-martingale} \right\}$$

where Z^a and \tilde{Z}^a are defined for each action process in \mathcal{A} as :

$$Z_t^a = \exp \left(\int_{(0, t]} -\gamma_A K_s dN_s - \int_0^t \lambda_s (e^{-\gamma_A K_s} - 1) (1 + a_s) ds \right), \quad t \in [0, T], \quad (3.3.2)$$

and :

$$\tilde{Z}_t^{a^*(K_s)} = \exp \left(\int_{(0,t]} -\gamma_P(1 - K_s) dN_s - \int_0^t \lambda_s (e^{-\gamma_P(1-K_s)} - 1) (1 + a^*(K_s)) ds \right), \quad t \in [0, T].$$

The martingale property of Z^a will be useful to prove Lemma 3.3.1 whilst that of \tilde{Z}^a will be useful for Theorem 3.3.2. Whilst we will see that the lower bound assumption on K_s is necessary for the well-posedness of the problem, the exponential martingale related assumptions on processes $(K_t)_{t \in [0, T]}$ defining Γ are undeniably ad hoc. We may note however that for processes that are bounded, these exponential martingale related assumptions hold (see for example [70]). Therefore in optimizing on Γ , we are in particular optimizing on bounded processes (and perhaps some other ones). Such boundedness is quite reasonable : lower boundedness is required for well-posedness (see the discussions related to the following Lemma), upper boundedness seems natural as the Principal has no reason to wish to give the Agent too much of his positive wealth process. We have the following result.

Lemma 3.3.1. *Let us consider a wage W in \mathcal{W} such that $W = W_T^{y_0, K}$ for some $(y_0, K) \in \Gamma$. Then W is incentive compatible yielding the Agent's optimal response :*

$$a^*(K_t) = \frac{\lambda_t}{\gamma_A \kappa} (1 - e^{-\gamma_A K_t}), t \in [0, T].$$

Proof. See Section 3.4.2. □

The result of Lemma 3.3.1 brings to light the reason behind the restricted values of K in Γ : the lower bound on the values ensures that a wage $W = W_T^{y_0, K}$ actually yields an admissible optimal action on behalf of the Agent. In particular, any process K where at some time point s in $[0, T]$ we have :

$$K_s \leq -\frac{1}{\gamma_A} \log \left(1 + \frac{\gamma_A \kappa}{\lambda_s} \right),$$

would induce the Agent to "kill" the Principal's wealth process by working $a_s(K_s) = -1$ which is not admissible (such an action process would not belong to \mathcal{A}) and is not a phenomena that would benefit the Principal. In fact it probably would not benefit the Agent either as the Principal would not possess any wealth to share.

We may also link the controlled wage process to the participation constraint as follows. As is quite standard in such Principal-Agent problems, the Agent's utility tracted through $(W_T^{y_0, K}, a^*(K))$ solely depends on $y_0 \dots$

Remark 3.3.1. *Given (y_0, K) in Γ , a wage W in \mathcal{W} such that $W = W_T^{y_0, K}$ satisfies the participation constraint if and only if $y_0 \geq y_{PC}$. This can be seen through the construction of the family R^a in the proof.*

Therefore going forward, we will solve the Principal's problem for wages in W in \mathcal{W} such that $W = W_T^{y_0, K}$, where (y_0, K) belongs to Γ and $y_0 \geq y_{PC}$. Therefore we introduce a new control set that is a subset of Γ :

$$\bar{\Gamma} = \{(y, K) \in \Gamma, y \geq y_{PC}\}.$$

Finally in order to solve the problem, we introduce the key function :

$$g(s, k) = -\frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1) \right|^2 - \frac{1}{\gamma_P} \lambda_s (e^{-\gamma_P(1-k)} - 1) \left(1 + \frac{\lambda_s}{\gamma_A \kappa} (1 - e^{-\gamma_A k})\right).$$

Remark 3.3.2. *The function g has a unique maximizer in k on $(-\frac{1}{\gamma_A} \log(1 + \frac{\gamma_A \kappa}{\lambda_s}), +\infty)$ for each $s \in [0, T]$ and for each fixed set of parameters $(\kappa, \gamma_P, \gamma_A)$. Indeed, for each $s \in [0, T]$ and for each fixed set of parameters $(\kappa, \gamma_P, \gamma_A)$:*

- $k \mapsto g(s, k)$ is a continuous function (it even belongs to $C^\infty(\mathbb{R})$)
- $k \mapsto g(s, k)$ is increasing on $(-\frac{1}{\gamma_A} \log(1 + \frac{\gamma_A \kappa}{\lambda_s}), 0]$ (checked numerically)
- $k \mapsto g(s, k)$ is strictly concave on \mathbb{R}^+ (checked numerically)
- $g(s, 0) = -\frac{1}{\gamma_P} \lambda_s (e^{-\gamma_P} - 1) \in \mathbb{R}$
- $\lim_{k \rightarrow +\infty} g(s, k) = -\infty$

We now are in a position to give an important theorem.

Theorem 3.3.2. *Let $(K_s^*)_{s \in [0, T]}$ be the deterministic process that maximizes for each s in $[0, T]$ $g(s, k)$ in k for k in $(-\frac{1}{\gamma_A} \log(1 + \frac{\gamma_A \kappa}{\lambda_s}), +\infty)$. Then the dynamic contracting problem :*

$$\sup_{(y, K) \in \bar{\Gamma}} \mathbb{E} \left[U_P(X_T - W_T^{y, K}) \right]$$

is solved for $(y, K) = (y_{PC}, K^)$. The optimal contract in this setting is therefore :*

$$W_T^{y_{PC}, K^*} = y_{PC} + \int_0^T K_s^* dM_s - \int_0^T f^{\gamma_A}(s, K_s^*) ds$$

inducing the optimal action :

$$a^*(K_t^*) = \frac{\lambda_t}{\gamma_A \kappa} (1 - e^{-\gamma_A K_t^*}) \quad t \in [0, T].$$

Proof. (of Theorem 3.3.2)

See Section 3.4.3 □

3.3.2 Analysis of incentives in this pay-per-project setting

The optimal contract for the Risk-Sharing problem in this pay-per-project setting and given in Theorem 3.3.1 is without much surprise an adaptation of the Hölmstrom-Milgrom contract where a proportion $\frac{\gamma_P}{\gamma_P + \gamma_A}$ of the Principal's wealth process is given to the Agent in exchange for his (time-dependent here) action. As per Corollary 2.3.1, this optimal action is not the Legendre transform of κ but rather solves the following problem :

$$\sup_{a \in \mathbb{H}_2} \mathbb{E}^{\mathbb{P}^a} \left[U_A \left(\frac{\gamma_P}{\gamma_P + \gamma_A} X_T - \frac{\gamma_P}{\gamma_P + \gamma_A} \int_0^T \kappa(a_t) dt \right) \right].$$

Its expression is given by $a_s^* = \frac{\lambda_s}{\bar{\rho}\kappa}(1 - e^{-\bar{\rho}})$, $s \in [0, T]$, where we recall that $\bar{\rho} = \frac{\gamma_P\gamma_A}{\gamma_P + \gamma_A}$. The time-dependent action increases the intensity of the underlying Poisson process : it is optimal for the Principal to ask that the Agent provides some effort in order to tract some intensity that is above that basic market intensity. The behavior of such a_s^* in the cost of effort remains classical : as κ increases, the action decreases. We note however that this action depends on the risk aversions, which is a small change from the Brownian setting of Section 1.1.2.2 in the introductory Chapter where $a_s^* = \frac{1}{\kappa}$, $s \in [0, T]$.

A departure from the Hölmstrom-Milgrom contract arises in the Moral Hazard setting in Theorem 3.3.2, where the Principal has to calibrate K_s^* and obtain some response from the Agent. This response is of the form :

$$a^*(K_s^*) = \frac{\lambda_s}{\gamma_A\kappa}(1 - e^{-\gamma_A K_s^*}),$$

and is therefore increasing in K_s^* . We note that K_s^* is time-dependent. Indeed at any given time $s \in [0, T]$, K_s^* maximises in k the function :

$$g(s, k) = -\frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1) \right|^2 - \frac{1}{\gamma_P} \lambda_s (e^{-\gamma_P(1-k)} - 1) \left(1 + \frac{\lambda_s}{\gamma_A\kappa}(1 - e^{-\gamma_A k})\right),$$

and is thus dependent on the value of λ_s . This is different from standard wages paid under Moral Hazard where the Agent receives a proportion $\frac{\gamma_P + 1}{\gamma_P + \gamma_A + 1}$ (when $\kappa = 1$) of the underlying income process. Here instead of receiving such a fixed proportion (and some constants calibrated such that the participation constraint is saturated), the Agent receives :

$$W_T^{y_{PC}, K^*} = y_{PC} + \int_0^T K_s^* dM_s - \int_0^t f^{\gamma_A}(s, K_s^*) ds.$$

Let $(t_1 \dots t_{N_T})$ be the different realized jump times. Then such a wage rewrites as :

$$\begin{aligned} W_T^{y_{PC}, K^*} &= y_{PC} + \sum_{i=1}^{N_T} K_{t_i}^* - \int_0^T K_s^* \lambda_s ds - \int_0^t f^{\gamma_A}(s, K_s^*) ds \\ &= y_{PC} + \sum_{i=1}^{N_T} K_{t_i}^* + \int_0^T \left(\frac{\lambda_s}{\gamma_A}(e^{-\gamma_A K_s^*} - 1) - \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A}(e^{-\gamma_A K_s^*} - 1) \right|^2 \right) ds \end{aligned}$$

Some comparative statics.

As the expression for K^* is implicit, we may visualize the relation between K_s^* and the different model parameters (λ_s , the risk-aversions...). This is done in Figures 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6.

Through these figures and some other tests, we observe that :

- K^* belongs to $[0, 1]$: it is a ratio which is in line with the findings of other more classical Moral Hazard problems
- K^* is increasing in λ ; a^* increases in λ
- K^* and a^* are increasing in γ_P and decreasing in γ_A ; again this is in line with the findings of other more classical Moral Hazard problems

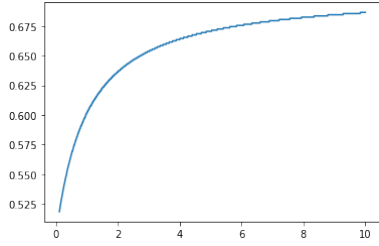


Figure 3.1 – K_s^* as a function of the value of λ_s , for $\gamma_A = \gamma_P = \kappa = 1$.

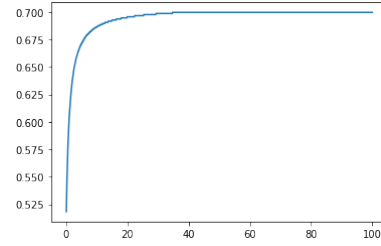


Figure 3.2 – K_s^* as a function of the value of λ_s , for $\gamma_A = \gamma_P = \kappa = 1$.

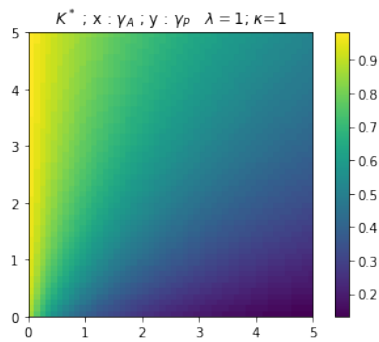


Figure 3.3 – K_s^* as a function of the value of γ_P & γ_A ; $\lambda_s = \kappa = 1$.

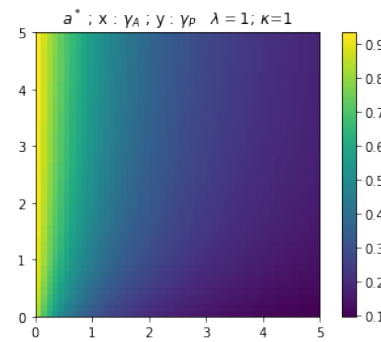


Figure 3.4 – a_s^* as a function of the value of γ_P & γ_A ; $\lambda_s = \kappa = 1$.

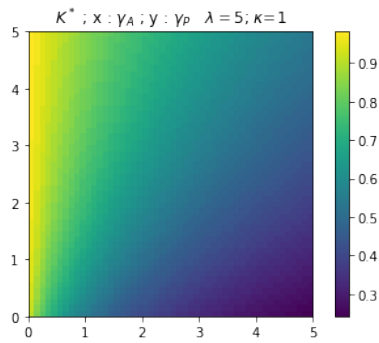


Figure 3.5 – K_s^* as a function of the value of γ_P & γ_A ; $\lambda_s = 5$; $\kappa = 1$.

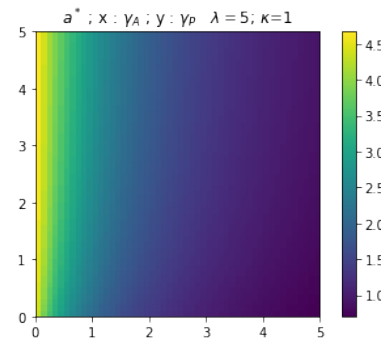


Figure 3.6 – a_s^* as a function of the value of γ_P & γ_A ; $\lambda_s = 5$; $\kappa = 1$.

From an economic perspective, this seems to suggest that when the underlying intensity λ_s is high, the value of K_s^* is high encouraging the Agent to make the most of the situation. In any case however K_s^* remains a proportion. When the intensity λ_s is low, the Agent is less incentivized to work although his action does remain positive. We could just as well intuitively think that the contrary may happen and that the Agent would be increased to work more when market intensity is low and less when market intensity is high thus reaching an overall balance. Perhaps some intuition into the actual link between λ_s and K_s^* may be that

when λ_s is low, the cost for the Agent to increase the realized intensity by a certain amount through his actions is higher than when λ_s is already high.

This phenomena would suggest that for app-based digital platforms, high underlying demand should lead to increased activity from the Agent in exchange for a higher proportion of the income process. This is quite realistic and in line with phenomena such as surge pricing. We may note however that surge pricing is often used by some app-based digital platforms to improve the match between demand and supply by encouraging people to work at times of high demand. Of course the model here does not include such demand and supply considerations yet a surge pricing type phenomena still seems to occur. We may wonder whether it would still occur in settings where the task value increases when intensity is high (thus better modeling surge pricing - eg. carshare prices are generally higher when demand is higher).

We note that the optimal value for K^* as a function of the underlying intensity seems to have a saturation value. This is visible in Figure 3.2 and 3.4. It therefore seems that beyond a certain level of underlying intensity, the Principal does not gain so much by providing a higher level K^* . In particular, the limit value as the intensity goes to infinity seems to be some ratio lower than 1. This makes sense as the Principal does not want to eventually give away all of her firm as the intensity increases : the ratio reaches a ceiling value and therefore so does the Agent's action.

The dependency of K^* in the risk-aversion parameters seems quite interpretable : it seems increasing in γ_P and decreasing in γ_A . This is also the behavior of the "classical" (in the Brownian case where $\kappa = 1$) ratio $\frac{\gamma_P+1}{\gamma_P+\gamma_A+1}$. Naturally, we compare K^* to $\frac{\gamma_P+1}{\gamma_P+\gamma_A+1}$: some graphs of this comparison of given in Figures 3.7, 3.8, 3.9, 3.10, 3.11 and 3.12.

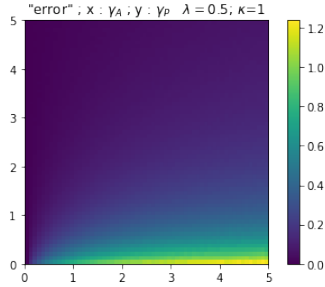


Figure 3.7 – Error term as a function of the value of γ_P & γ_A ; $\lambda_s = 0.5$; $\kappa = 1$.

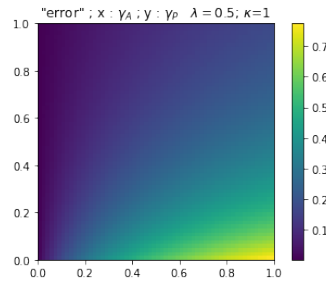


Figure 3.8 – Error term as a function of the value of γ_P & γ_A ; $\lambda_s = 0.5$; $\kappa = 1$.

where we plot an "error" term worth $\frac{\gamma_P+1}{\gamma_P+\gamma_A+1} - K^*$ as a function γ_A (on the x-axis) and γ_P (on the y-axis) for various values of λ (0.5, 1 and 3) and for $\kappa = 1$ fixed. On each row the right hand figure is a "zoom" of the left hand figure. We note that the difference between $\frac{\gamma_P+1}{\gamma_P+\gamma_A+1}$ and K^* generally seems to increase as γ_A increases and be close to 0 for low values of γ_A . The error also increases as λ decreases suggesting that for a Poisson process with a lower intensity value, K^* needs to be quite different from $\frac{\gamma_P+1}{\gamma_P+\gamma_A+1}$ for optimality. When the intensity is higher and the Poisson process is therefore likely to jump more, $\frac{\gamma_P+1}{\gamma_P+\gamma_A+1}$ is close to optimal.

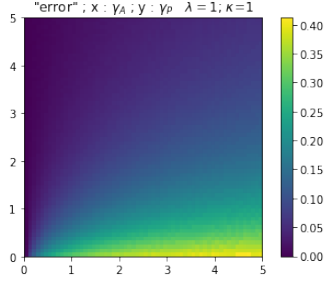


Figure 3.9 – Error term as a function of the value of γ_P & γ_A ; $\lambda_s = 0.5$; $\kappa = 1$.

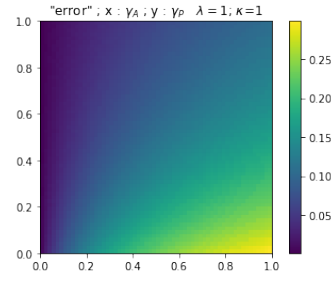


Figure 3.10 – Error term as a function of the value of γ_P & γ_A ; $\lambda_s = 0.5$; $\kappa = 1$.

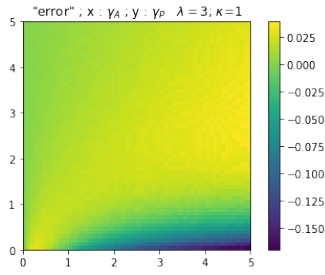


Figure 3.11 – Error term as a function of the value of γ_P & γ_A ; $\lambda_s = 0.5$; $\kappa = 1$.

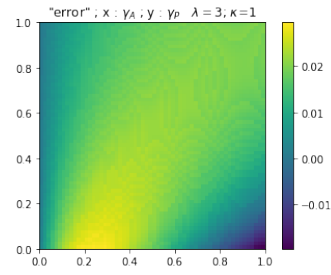


Figure 3.12 – Error term as a function of the value of γ_P & γ_A ; $\lambda_s = 0.5$; $\kappa = 1$.

A note on the sign of the wage. We may note that through the positivity of the process $(K_s^*)_{s \in [0, T]}$, the integral term in the wage worth :

$$\int_0^T \left(\frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) - \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) \right|^2 \right) ds$$

is negative-valued. Yet it is not guaranteed that the underlying Poisson process will actually jump or jump much (particularly when $(\lambda_s)_{s \in [0, T]}$ is low) : by construction N_T may take values over \mathbb{N}^* , each with positive probability. Therefore in particular in settings where y_{PC} is much smaller than $\left| \int_0^T \left(\frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) - \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) \right|^2 \right) ds \right|$, the Agent may well end up with a negative wage (noting that K_s^* is free of y_{PC}), we may calibrate for any set of model parameters values of y_{PC} for which this may hold)... In fact in cases where :

$$y_{PC} + \int_0^T \left(\frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) - \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) \right|^2 \right) < 0,$$

we must have :

$$N_T > \left| y_{PC} + \int_0^T \left(\frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) - \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A} (e^{-\gamma_A K_s^*} - 1) \right|^2 \right) \right|$$

for there to be a chance of wage positivity (as $\sum_{i=1}^{N_T} K_{t_i}^* \leq N_T$). This is despite the wealth process for the Principal being positive.

This remark brings some insight into the difficulty that arises when attempting to enforce wage positivity in Principal-Agent models (therefore said to enforce limited liability) which is a natural extension of Principal-Agent problems, particularly in the Brownian case that directly extends the Hölmstrom-Milgrom formulation of the introductory chapter of this thesis. Indeed, we note here that even in a setting where the Principal's wealth process is necessarily positive, it is not optimal for the Principal to pay the Agent a wage that is always guaranteed to be positive... Due to the dynamic form of the contract arising through the Agent's problem, the Agent's wage includes some other terms along with the share of the wealth process, that may in some cases drag the value of the compensation below zero. A classical economic interpretation of this negativity is some form of rent paid by the Agent. However, we may wish to obtain models without such a rent phenomena.

Such limited liability problems were the object of some focus during this thesis, in particular some time was spent trying to add the constraint $W \geq 0$ $\mathbb{P} - a.s.$ to Problem (1.1.1)-(1.1.2) and this work finally resulted in the discussions of Chapter 4 in a single period model. In dynamic settings, trying to enforce both the participation constraint (1.1.2) and a wage positivity constraint of the form $W \geq 0$ $\mathbb{P} - a.s.$ seemed quite difficult with the cause of difficulty seeming at the time to be the underlying presence of Brownian motion (and its possible negativity). The results in this setting however seem to highlight that the effect of Brownian motion may not be the sole cause of difficulty. As the wage may be negative even here (for a positive production process), it seems that wage positivity is really not natural for the Principal... In particular, it may be that trying to combine both the participation constraint and limited liability is counterproductive, and that in a dynamic setting solely considering limited liability may be a path to consider.

3.4 Proofs

3.4.1 Proof of Theorem 3.3.1

The approach that we use in order to prove this Theorem is that derived in [93]. It is also derived in the previous Chapter in this thesis. In particular we exploit the following result.

Lemma 3.4.1 (from [93] - see also Corollary 2.3.1 of this thesis.). *Suppose that there exists some process $(a_s^*)_{s \in [0, T]}$ solution to :*

$$\inf_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[\left| U_P \left(N_T - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right],$$

then there exists an optimal contract for problem (3.1.4)-(3.1.5) given by $(a_s^)_{s \in [0, T]}$ and :*

$$W^* = \frac{\gamma_P}{\gamma_P + \gamma_A} N_T + \int_0^T \frac{\gamma_A}{\gamma_P + \gamma_A} \kappa(a_s^*) ds + C$$

with C calibrated such that the participation constraint (3.1.5) is bound.

With this result in mind, the contracting problem (3.1.4)-(3.1.5) reduces to finding some process $(a_s^*)_{s \in [0, T]}$ such that :

$$\inf_{a \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^a} \left[\left| U_P \left(N_T - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] = \mathbb{E}^{\mathbb{P}^{a^*}} \left[\left| U_P \left(N_T - \int_0^T \kappa(a_t^*) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right], \quad (3.4.1)$$

and we turn to doing this using classical techniques. In particular, we use a Martingale Optimality Principle and construct a family $(R_t^a)_{t \in [0, T]}$ satisfying the following four properties :

- $R_T^a = \left| U_P \left(N_T - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}}$
- $R_0^a = R_0^b$, for any (a, b) in $(\mathcal{A})^2$.
- R^a is a \mathbb{P}^a -submartingale for any a in \mathcal{A} .
- R^{a^*} is a \mathbb{P}^{a^*} -martingale for some a^* in \mathcal{A} .

Indeed, given such a family, we would have for any a in \mathcal{A} :

$$\mathbb{E}^{\mathbb{P}^a} [R_T^a] \geq \mathbb{E}^{\mathbb{P}^a} [R_0^a] = \mathbb{E}^{\mathbb{P}^{a^*}} [R_0^{a^*}] = \mathbb{E}^{\mathbb{P}^{a^*}} [R_T^{a^*}], \quad (3.4.2)$$

yielding the solution to Problem (3.4.1). We construct such a family in the following Lemma, where we use the notation $\bar{\rho} = \frac{\gamma_P \gamma_A}{\gamma_P + \gamma_A}$ which yields :

$$\left| U_P \left(N_T - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} = \exp \left(\bar{\rho} \int_0^T \kappa(a_s) ds - \bar{\rho} N_T \right).$$

Lemma 3.4.2. *Let (J, K) in $\mathcal{S}^2 \times \mathbb{H}^2$ be the unique solution to the BSDE :*

$$J_t = J_T - \int_t^T K_s dM_s + \int_t^T f^{\bar{\rho}}(s, K_s) ds, \quad t \in [0, T], \quad J_T := N_T, \quad (3.4.3)$$

and set $R_t^a = \exp \left(\bar{\rho} \int_0^t \kappa(a_s) ds \right) \exp(-\bar{\rho} J_t)$, $t \in [0, T]$. Then $(R_t^a)_{t \in [0, T]}$ satisfies the four MOP properties above and the minimizer for Problem (3.4.1) is the action process belonging to \mathcal{A} :

$$a_t^* = \frac{\lambda_t}{\bar{\rho} \kappa} (1 - e^{-\bar{\rho}}), \quad t \in [0, T].$$

Proof. The BSDE (3.4.3) is precisely of the form of those analyzed in Lemma 3.2.2. It even has the following explicit solution :

$$J_t := X_t + \int_t^T \beta_s ds, \quad K_t = 1,$$

where $(\beta_s)_{s \in [0, T]}$ is given in (3.2.3). Now, setting $R_t^a = \exp \left(\bar{\rho} \int_0^t \kappa(a_s) ds \right) \exp(-\bar{\rho} J_t)$, $t \in [0, T]$, we immediately have that :

- $R_0^a = R_0^b$, for any (a, b) in $(\mathcal{A})^2$.

- $R_T^a = \left| U_P \left(N_T - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}}$

We tackle the last two points. First we show that :

- R^a is a \mathbb{P}^a submartingale for any a in \mathcal{A} .

For a in \mathcal{A} , we introduce the process $(M_s^a)_{s \in [0, T]}$ where $M_s^a = M_s - \lambda_s a_s$, $s \in [0, T]$. Then :

$$\begin{aligned} J_t &= J_0 + \int_0^t K_s dM_s - \int_0^t f^{\bar{\rho}}(s, K_s) ds \\ &= J_0 + \int_0^t K_s dM_s^a - \int_0^t \left(f^{\bar{\rho}}(s, K_s) - \lambda_s K_s a_s \right) ds \\ &= J_0 + \int_0^t dM_s^a - \int_0^t \left(f^{\bar{\rho}}(s, 1) - \lambda_s a_s \right) ds, \quad \text{as } K = 1. \end{aligned}$$

Now let φ and Z^a be defined as follows :

$$\varphi_s := -1 + e^{-\bar{\rho}}, \quad s \in [0, T] \quad (\log(1 + \varphi_s) = -\bar{\rho}), \quad dZ_t^a = Z_{t-}^a \varphi_t dM_t^a, \quad Z_0^a = 1.$$

The process Z^a has the explicit expression :

$$\begin{aligned} Z_t^a &= \exp \left(\int_{(0, t]} \log(1 + \varphi_s) dN_s - \int_0^t \lambda_s \varphi_s (1 + a_s) ds \right), \\ &= \exp \left(\int_{(0, t]} -\bar{\rho} dN_s - \int_0^t \lambda_s \varphi_s (1 + a_s) ds \right) \\ &= \exp \left(\int_{(0, t]} -\bar{\rho} dM_s^a - \int_0^t \lambda_s \varphi_s (1 + a_s) ds - \int_0^T \bar{\rho} \lambda_s (1 + a_s) ds \right) \end{aligned}$$

and Z^a is a \mathbb{P}^a -martingale through [70] for example (as $\log(1 + \varphi_s)$ is in fact a constant). We use the expression for J and for Z^a to rewrite R^a as follows :

$$\begin{aligned} R_t^a &:= \exp \left(\bar{\rho} \int_0^t \kappa(a_s) ds \right) \exp(-\bar{\rho} J_t) \\ &= \exp(-\bar{\rho} J_0) \exp \left(-\bar{\rho} \int_0^t dM_s \right) \exp \left(\bar{\rho} \int_0^t \left[\kappa(a_s) + f^{\bar{\rho}}(s, 1) \right] ds \right) \\ &= \exp(-\bar{\rho} J_0) \exp \left(-\bar{\rho} \int_0^t dM_s^a \right) \exp \left(\bar{\rho} \int_0^t \left[\kappa(a_s) - \lambda_s a_s + f^{\bar{\rho}}(s, 1) \right] ds \right) \\ &= \exp(-\bar{\rho} J_0) Z_t^a \exp \left(\int_0^t \left[\bar{\rho} \kappa(a_s) + \bar{\rho} f^{\bar{\rho}}(s, 1) - \bar{\rho} \lambda_s a_s + \lambda_s (1 + a_s) (\varphi_s - \bar{\rho}) \right] ds \right) \\ &= \exp(-\bar{\rho} J_0) Z_t^a \exp \left(\int_0^t \left[\bar{\rho} \kappa(a_s) + \bar{\rho} f^{\bar{\rho}}(s, 1) + \lambda_s (1 + a_s) \varphi_s + \bar{\rho} \lambda_s \right] ds \right). \end{aligned}$$

This process is a \mathbb{P}^a -submartingale if and only if :

$$\left[\bar{\rho} \kappa(a_s) + \bar{\rho} f^{\bar{\rho}}(s, 1) + \lambda(1 + a_s) \varphi_s + \bar{\rho} \lambda_s \right] \geq 0, \quad \text{for } \mathbb{P} \otimes dt \text{ a.e.}(s, \omega),$$

which holds true by definition of $f^{\bar{\rho}}$ that we recall :

$$f^{\bar{\rho}}(s, 1) := -\frac{\lambda_s}{\bar{\rho}} (e^{-\bar{\rho}} - 1 + \bar{\rho}) + \sup_{a > -1} \left\{ -\kappa(a) - \frac{\lambda_s}{\bar{\rho}} a (e^{-\bar{\rho}} - 1) \right\}.$$

It remains to prove that R^a constructed as above satisfies the final point :

- R^{a^*} is a \mathbb{P}^{a^*} martingale for some a^* in \mathcal{A}

To show this note that choosing $a_s^* = \frac{\lambda_s}{\rho\kappa}(1 - e^{-\bar{\rho}})$, $s \in [0, T]$ belonging to \mathcal{A} as :

$$\forall s \in [0, T], a_s^* \geq 0 \quad \text{and} \quad a_s^* \leq \frac{\bar{\lambda}_s}{\rho\kappa},$$

yields $R_t^{a^*} = \exp(-\bar{\rho}J_0) Z_t^{a^*}$. As a consequence, R^{a^*} is a martingale. \square

This result allows us to apply Lemma 3.4.1 for Problem (3.4.1) in the vein of (3.4.2). Indeed for any a in \mathcal{A} we have :

$$\mathbb{E}^{\mathbb{P}^a} \left[\left| U_P \left(N_T - \int_0^T \kappa(a_t) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right] \geq \mathbb{E}^{\mathbb{P}^{a^*}} \left[\left| U_P \left(N_T - \int_0^T \kappa(a_t^*) dt \right) \right|^{\frac{\gamma_A}{\gamma_A + \gamma_P}} \right].$$

Finally setting

$$W^* = \frac{\gamma_P}{\gamma_P + \gamma_A} N_T + \int_0^T \frac{\gamma_A}{\gamma_P + \gamma_A} \kappa(a_s^*) ds + C \quad ; \quad a_s^* = \frac{\lambda_s}{\rho\kappa}(1 - e^{-\bar{\rho}}), \quad s \in [0, T],$$

with

$$C = y_{PC} + \frac{\gamma_P}{\gamma_P + \gamma_A} \int_0^T \kappa(a_s^*) ds + \frac{1}{\gamma_A} \log \left(\mathbb{E}^{\mathbb{P}^{a^*}} \left[\exp \left(-\frac{\gamma_A \gamma_P}{\gamma_P + \gamma_A} N_T \right) \right] \right)$$

saturates the participation constraint. Indeed given such a pair (W^*, a^*) :

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^{a^*}} \left[U_A \left(W^* - \int_0^T \kappa(a_s^*) ds \right) \right] \\ &= U_A(y_{PC}) \mathbb{E}^{\mathbb{P}^{a^*}} \left[\exp \left(-\gamma_A \frac{\gamma_P}{\gamma_P + \gamma_A} N_T \right) \right] \exp \left(-\log \left(\mathbb{E}^{\mathbb{P}^{a^*}} \left[\exp \left(-\frac{\gamma_A \gamma_P}{\gamma_P + \gamma_A} N_T \right) \right] \right) \right) \\ &= U_A(y_{PC}). \end{aligned}$$

and so Lemma 3.4.1 the allows us to conclude on Problem (3.4.1). The final result (and in particular the expression for W^*) then follows from the fact that :

$$\mathbb{E}^{\mathbb{P}^{a^*}} \left[\exp \left(-\frac{\gamma_A \gamma_P}{\gamma_P + \gamma_A} N_T \right) \right] = \exp \left(\int_0^T \lambda_s (e^{-\bar{\rho}} - 1) (1 + a_s^*) ds \right).$$

3.4.2 Proof of Lemma 3.3.1

Lemma 3.3.1 is an applied consequence of the following result.

Lemma 3.4.3. *Given W in \mathcal{W} such that $W = W_T^{y_0, K}$ for some $(y_0, K) \in \Gamma$, let (J, K) in $\mathcal{S}^2 \times \mathbb{H}^2$ be the unique solution to the BSDE :*

$$J_t = J_T - \int_t^T K_s dM_s + \int_t^T f^{\gamma_A}(s, K_s) ds, \quad t \in [0, T], \quad J_T := W, \quad (3.4.4)$$

and set $R_t^a = \exp \left(\gamma_A \int_0^t \kappa(a_s) ds \right) \exp(-\gamma_A J_t)$, $t \in [0, T]$. Then $(R_t^a)_{t \in [0, T]}$ satisfies the four following properties :

- $R_T^a = U_A \left(W - \int_0^T \kappa(a_t) dt \right)$
- $R_0^a = R_0^b$, for any (a, b) in $(\mathcal{A})^2$.
- R^a is a \mathbb{P}^a super-martingale for any a in \mathcal{A} .
- R^{a^*} is a \mathbb{P}^{a^*} martingale for some a^* in \mathcal{A} .

In particular as $\frac{\lambda_t}{\gamma_A \kappa} (1 - e^{-\gamma_A K_t}) > -1$:

$$a^*(K_t) = \frac{\lambda_t}{\gamma_A \kappa} (1 - e^{-\gamma_A K_t}), t \in [0, T].$$

and $R_T^{a^*}$ solves Problem (3.3.1).

Proof. The proof of this result very much mirrors the proof of Lemma 3.4.2 to which we refer the reader for the most part : the first two properties hold by construction of R^a and the third property by the form of f^{γ_A} . We detail the choice of a^* . As we have :

$$f^{\gamma_A}(s, k) = -\frac{\lambda_s}{\gamma_A} (e^{-\gamma_A k} - 1 + \gamma_A k) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A} (e^{-\gamma_A k} - 1) \right|^2 - \frac{\kappa}{2} \inf_{a > -1} \left| a + \frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1) \right|^2,$$

we need to detail with the optimization problem in a , noting that here contrary to the Risk-Sharing setting, a specific value for k is not known. Note that :

$$\inf_{a > -1} \left| a + \frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1) \right|^2 = 0, \quad \text{and } a^*(k) = -\frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1), \quad \text{if } \frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1) \leq 1,$$

and

$$\inf_{a > -1} \left| a + \frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1) \right|^2 = \left| \frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1) - 1 \right|^2, \quad \text{and } a^*(k) = -1 \quad \text{if } \frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A k} - 1) > 1.$$

Using our restricted values for K_s (as $K_s > -\frac{1}{\gamma_A} \log \left(1 + \frac{\gamma_A \kappa}{\lambda_s} \right)$, $s \in [0, T]$), we always fall into the first case where :

$$a^*(K_s) = -\frac{\lambda_s}{\gamma_A \kappa} (e^{-\gamma_A K_s} - 1) = \frac{\lambda_s}{\gamma_A \kappa} (1 - e^{-\gamma_A K_s}), s \in [0, T].$$

In particular, both $a^*(K_s) > -1$ and $a^*(K_s) \leq M$ so $a^*(K_s)$ belongs to \mathcal{A} . We finally obtain as in Lemma 3.4.2 that R^{a^*} is a martingale. So we find that the family of processes R^a satisfy all the requirements and obtain the conclusion. \square

3.4.3 Proof of Theorem 3.3.2

Let (y_0, K) be some point in $\bar{\Gamma}$. Recall that :

$$f^{\gamma_A}(s, k) := -\frac{\lambda_s}{\gamma_A} (e^{-\gamma_A k} - 1 + \gamma_A k) + \sup_{a > -1} \left\{ -\kappa(a) - \frac{\lambda_s}{\gamma_A} a (e^{-\gamma_A k} - 1) \right\}.$$

In particular, if $k > -\frac{1}{\gamma_A} \log\left(1 + \frac{\gamma_A \kappa}{\lambda_s}\right)$ then :

$$f^{\gamma_A}(s, k) = -\frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1 + \gamma_A k) + \frac{1}{2\kappa} \left(\frac{\lambda_s}{\kappa}(e^{-\gamma_A k} - 1) \right)^2.$$

With this in mind, let us compute the Principal's value function at (y_0, K) bearing in mind that the related action for a wage $W_T^{y_0, K}$ is $a^*(K)$ defined as :

$$a^*(K_s) = -\frac{\lambda_s}{\gamma_A \kappa}(e^{-\gamma_A K_s} - 1)$$

We obtain the Principal's value function :

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[U_P(N_T - W_T^{y_0, K}) \right] \\ &= \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[U_P \left(N_T - y_0 - \int_0^T K_s dM_s + \int_0^T f^{\gamma_A}(s, K_s) ds \right) \right] \\ &= \exp(\gamma_P y_0) \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[U_P \left(N_T - \int_0^T K_s dM_s + \int_0^T f^{\gamma_A}(s, K_s) ds \right) \right] \\ &= \exp(\gamma_P y_0) \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[U_P \left(\int_0^T (1 - K_s) dN_s + \int_0^T \left\{ -\frac{\lambda_s}{\gamma_A}(e^{-\gamma_A K_s} - 1) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A}(e^{-\gamma_A K_s} - 1) \right|^2 ds \right\} \right) \right]. \end{aligned}$$

Now recall that :

$$\tilde{Z}_t^a = \exp \left(\int_{(0, t]} -\gamma_P(1 - K_s) dN_s - \int_0^t \lambda_s(e^{-\gamma_P(1 - K_s)} - 1)(1 + a^*(K_s)) ds \right)$$

and :

$$g(s, k) = -\frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1) + \frac{1}{2\kappa} \left| \frac{\lambda_s}{\gamma_A}(e^{-\gamma_A k} - 1) \right|^2 - \frac{1}{\gamma_P} \lambda_s(e^{-\gamma_P(1 - k)} - 1) \left(1 + \frac{\lambda_s}{\gamma_A \kappa}(1 - e^{-\gamma_A k}) \right).$$

Let $(K_s^*)_{s \in [0, T]}$ defined as the maximizer for each s on $[0, T]$ and for k over $(-\frac{1}{\gamma_A} \log\left(1 + \frac{\gamma_A \kappa}{\lambda_s}\right), +\infty)$.

We may further rewrite and upper bound the Principal's utility :

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[U_P(X_T - W_T^{y_0, K}) \right] \\ &= \exp(\gamma_P y_0) \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[Z_T^{a^*(K)} U_P \left(\int_0^T g(s, K_s) ds \right) \right] \\ &\leq \exp(\gamma_P y_0) \mathbb{E}^{\mathbb{P}^{a^*(K)}} \left[Z_T^{a^*(K)} \right] U_P \left(\int_0^T g(s, K_s^*) ds \right) \quad \text{as } K^* \text{ is deterministic} \\ &= \exp(\gamma_P y_0) U_P \left(\int_0^T g(s, K_s^*) ds \right) \quad \text{as } Z^{a^*(K)} \text{ is a } \mathbb{P}^{a^*(K)}\text{-martingale and } Z_0^{a^*(K)} = 1 \\ &= \exp(\gamma_P y_0) \mathbb{E}^{\mathbb{P}^{a^*(K^*)}} \left[Z_T^{a^*(K^*)} U_P \left(\int_0^T g(s, K_s^*) ds \right) \right] \\ &= \exp(\gamma_P y_0) \mathbb{E}^{\mathbb{P}^{a^*(K^*)}} \left[Z_T^{a^*(K^*)} U_P \left(\int_0^T g(s, K_s^*) ds \right) \right]. \\ &\leq \exp(\gamma_P y_{PC}) \mathbb{E}^{\mathbb{P}^{a^*(K^*)}} \left[Z_T^{a^*(K^*)} U_P \left(\int_0^T g(s, K_s^*) ds \right) \right]. \end{aligned}$$

This final upper bound is attained by setting :

$$W_T^{y_{PC}, K^*} = y_{PC} + \int_0^T K_s^* dM_s - \int_0^T f^{\gamma_A}(s, K_s^*) ds,$$

yielding the conclusion combined with the observation that as K_s^* is deterministic (and bounded on $[0, T]$), we have :

- $\mathbb{E} \left[\left(W_T^{y_{PC}, K^*} \right)^2 \right] < +\infty$
- $Z_t^{a^*(K^*)}$ is a true martingale for all $t \in [0, T]$.
- $\tilde{Z}_t^{a^*(K^*)}$ is a true martingale for all $t \in [0, T]$.

In particular the last two statements hold through the discussions on stochastic exponentiels in a Poisson setting in [70].

The Risk-Sharing problem under limited liability constraints in a single-period model

This Chapter is an extended version of an article accepted to the Journal of Optimization Theory and Optimization, which can be found at [91].

4.1 Limited liability and Principal-Agent problems

4.1.1 General setting

Many studies of the Principal-Agent problem allow the Principal to pay the Agent an unbounded wage. For instance the optimal wage in the standard Risk-Sharing problem and the standard Moral Hazard problems presented in Section 1.1.2.2 of Chapter 1 may be negative with positive probability due to the underlying Brownian motion. Of course, the use of Brownian motion is a modeling choice and one could aim to overcome wage negativity by considering a positive-valued random process instead of a Brownian motion to model the impact of external events. In practice however, allowing such negativity does seem to fit quite well with real applications as companies can make losses : one only has to look at some of the relatively recent annual incomes of Tesla. However, one can reasonably suppose that employees at Tesla earn positive wages and this somewhat contrasts with the optimal wages of Section 1.1.2.2. Indeed in both the Risk-Sharing and the Moral Hazard settings, one of the takehome messages is that the Agent may have to work on behalf of the Principal before also owing him money. The concluding analysis of Chapter 3 with contracting in a Poisson setting underlines the importance of the wage positivity question. A natural extension to Principal-Agent problems therefore involves analyzing the effect of adding exogenous bounds on the wage paid by the Principal. This would prevent situations such as the one just discussed where negative pay is given in exchange for positive effort. Additionally, this would allow the Principal to have some control over the wages paid to the Agent through an upper bound. Adding such bounds is said to introduce limited liability to the problem : limited liability for the Agent when the wage is bounded below and for the Principal when the wage

is bounded above.

Limited liability in the Principal-Agent problem has been introduced before and under a variety of forms, mostly in a Moral Hazard setting. In [57] Hölmstrom introduced wage bounds as a by-product of an existence proof for solutions to a Principal-Agent problem. Sappington then analyzed the case of a risk-neutral Principal and a risk-averse Agent, computing the optimal wage for an Agent who chooses his action after the state of nature is realized [117]. The particular and important case of limited liability and debt contracts was studied by Innes in [67] and extended in [96] and [42], with a possibility for negotiation. The risk-neutral Agent case was studied by Park [109] and Kim [81]. More recently in [74], Jewitt et al. provided a proof of existence and uniqueness of an optimal action and wage for a risk-neutral Principal and risk-averse Agent under general limited liability bounds and Moral Hazard. The paper provides a characterization of the optimal wage, which is of option form, but not of the optimal action. In fact an open question brought up by the authors in this paper is the effect of a lower bound on the Agent's action : this is unclear due to the lack of a closed-form expression for the optimum yet it is a vital economic question. Some answers are provided by Kadan et al. in [77]. Indeed, in the setting of [74], the paper provides sufficient conditions under which an increase in a lower wage bound increases the Agent's action. The effect of introducing limited liability into the Moral Hazard problem has therefore been quite thoroughly studied in a number of settings with some key open questions remaining. We can note in particular that the interesting question of enforcing wage positivity for the Moral Hazard problem in a Hölmstrom-Milgrom model as introduced in Section 1.1.2.2 has not yet been tackled in the literature. Quite naturally, one may hope to be able to turn to analysis of limited liability in the Risk-Sharing problem in order to gain some further insight into the effect of limited liability on Principal-Agent problems. However it appears that such relevant literature for the Risk-Sharing problem is actually much more scarce. Some questions are answered in [81] where Kim considers the case of a risk-neutral Principal and risk-averse Agent with a bounded output process. For any fixed action a , the author is able to characterize a class of wages that satisfy a set of relevant constraints. Thus rather than fully solving the limited liability problem, Kim proves that for every fixed action there exists a class of wages that satisfy all of the constraints necessary to be an optimum. Limited liability in a Risk-Sharing setting is also discussed by Cvitanic and Zhang in [37]. They derive an implicit characterization of the optimal wage for a Risk-Sharing problem in a continuous setting with a lower bound constraint and for an Agent who has the crucial ability to control volatility.

The aim of this Chapter is to complement [37] and [81] and bring some insight into the effect of limited liability on Principal-Agent problems where there is no volatility control. In line with the discussions of Section 1.2.1 of Chapter 1 underlining some benefits of studying single-period problems to gain insight into changes in contracting models, and the short literature review above highlighting a lack of Risk-Sharing related literature, analysis is done on the benchmark Risk-Sharing problem in a single period setting. It allows for general utility functions and thus extends beyond the realms of CARA utility and risk-neutral utility functions whose expressions simplify the reasoning required to analyze the optimal wage and action (for example in the CARA case one could try to find necessary and sufficient optimality conditions through a Lagrangian taking advantage of explicit calculations). As a consequence, in this work the analysis of maximizer existence is split from that of its

characterization. Existence is tackled first using a variational approach. Once existence is established, maximizer characterization is approached using Luenberger's generalized K.K.T. theorem [90]. Note that a characterization of both the optimal wage and action is obtained, which contrasts with [81] and even [74]. We are therefore able to provide analysis of the effect of limited liability on optimal contracts. Finally, remark that the setting considered is different from (and complementary to) that of Page's important work on solution existence [106]. Indeed Risk-Sharing only concerns itself with the Principal's optimization problem, and this work includes analysis of wages that are unbounded above. Also remark that this work differs from the key existence proofs for Principal-Agent problems provided in [75]. Indeed they consider the Principal's problem across the subset of so-called incentive compatible contracts. These contracts verify some optimality properties for the Agent and such optimality helps for topological analysis.

4.1.2 A Risk-Sharing limited liability problem

Let $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space with $L^2(\Omega)$ the set of square integrable random variables. As is usual throughout this thesis, we consider a single Principal who employs a single Agent to perform a task in exchange for some compensation. The two parties are modeled through their utility functions, U_P and U_A , that verify $U'_P > 0, U''_P \leq 0, U'_A > 0$ and $U''_A \leq 0$, and are continuous on \mathbb{R} with $\lim_{x \rightarrow -\infty} U(x) = -\infty$. As the setting is a single-period one, the Agent's action is modeled as some real number a which affects the Principal's production process, $X^a := x_0 + a + B$, where B belongs to $L^2(\Omega)$. Performing the action costs the Agent some effort modeled through $\kappa(a) := K \frac{a^2}{2}$ for $K > 0$ some fixed cost parameter. As a reward for his effort the Agent is paid a wage denoted W and belonging to $L^2(\Omega)$. In such a context, one may analyze the Risk-Sharing Principal-Agent problem or "first-best" problem¹ :

$$\sup_{(W,a) \in L^2(\Omega) \times \mathbb{R}} \mathbb{E} [U_P(X^a - W)], \quad (4.1.1)$$

such that the Agent's participation constraint is satisfied :

$$\mathbb{E} [U_A(W - \kappa(a))] \geq U_A(y_{PC}), \quad y_{PC} \in \mathbb{R}^+ \text{ (fixed)}. \quad (4.1.2)$$

The aim of this work is to add limited liability constraints :

$$m \leq W \quad \mathbb{P} - a.s., \quad (4.1.3)$$

or :

$$m \leq W \leq M \quad \mathbb{P} - a.s., \quad (4.1.4)$$

where m and M are two fixed positive parameters with $m \leq y_{PC} \leq M$. Such an assumption ensures that the constant contract $(y_{PC}, 0)$ satisfies both the participation constraint (4.1.2) and the relevant limited liability constraint. It also has economic meaning as the upper bound for y_{PC} ensures that the Principal is not over investing in human capital. In this setting, two

¹Once again, this "one-sided" optimization for the Principal may, with good reason, seem unfair. This problem is in fact a benchmark problem that is often used as a measure of comparison with other Principal-Agent problems such as Moral Hazard where optimization for the Agent also comes into play. One may also note that in our increasingly digitalized economies with an ever growing use of machines, analysis of optimal contracting without Moral Hazard (and thus in a Risk-Sharing setting) is increasingly relevant in itself too.

variants of the limited liability problem are considered : (4.1.1)-(4.1.2)-(4.1.3) where limited liability for the Agent is enforced, and (4.1.1)-(4.1.2)-(4.1.4) where limited liability for the Principal is also enforced.

Remark 4.1.1. *A very relevant real life application of constraint (4.1.4) is the effect of pay bands on optimal contracting. Indeed, pay band based compensation schemes are widely used amongst organizations and the results of this Chapter show how a classical linear wage (involving a fixed part and a performance related part) and a classical action may be optimally tweaked in order to fit into a pay band system.*

The following utility/process compatibility property is assumed from now on.

Assumption 4.1.1. *The constant contract $(y_{PC}, 0)$ provides the Principal with a finite utility :*

$$\mathbb{E} \left[|U_P(X^0 - y_{PC})| \right] < +\infty.$$

In other words, the random variable B and the Principal's utility function are compatible.

As is apparent through the Principal-Agent litterature but also in this thesis, the most widely studied cases for Principal-Agent problems involve either a risk-neutral Principal with a risk-averse Agent, or a risk-averse Principal with a risk-averse Agent each under CARA utilities. In both cases, the forms of the utility functions U_P and U_A simplify the reasoning required to analyze the optimal wage and action. In the CARA case one may find necessary and sufficient optimality conditions through a Lagrangian taking advantage of explicit calculations. Moreover in the case of a risk-averse Agent with a risk-neutral Principal, solving the lower bounded Limited Liability problem for example is trivial. Indeed the contract defined by :

$$W^* = y_{PC} + \kappa(a^*) \quad \text{and} \quad a^* = \frac{1}{K},$$

is optimal. One goal of this work is to allow for more general utility functions U_P/U_A such as :²

- A C^2 extension of the logarithmic utility :

$$U(x) = \log(x)\mathbf{1}_{x \geq 1} - \frac{1}{2}(x^2 - 4x + 3)\mathbf{1}_{x < 1}. \quad (4.1.5)$$

- A partially IARA and C^2 variant of the CARA utility :

$$U(x) = -e^{-x}\mathbf{1}_{x \geq 0} - \left(\frac{1}{2}x^2 - x + 1 \right) \mathbf{1}_{x < 0}, \quad \text{for } \gamma \text{ fixed.} \quad (4.1.6)$$

This utility partially disrupts the constant risk aversion of the CARA utility.³

- A power type utility :

$$U(x) = 2x - \sqrt{x^2 + 1}. \quad (4.1.7)$$

Therefore, unless otherwise specified, the utility functions do not have a specific form and simply verify the properties listed at the beginning of this Section. Of course, the CARA

²these are variants on classical utility functions whose validity has been extensively studied.

³ $x \mapsto \frac{U''(x)}{U'(x)}$ is increasing up to 0 and is worth 1 beyond 0. On \mathbb{R}^- it thus exposes "IARA" (increasing absolute risk aversion).

utility case will be specifically discussed especially for analysis at the end of this Chapter.

The rest of the Chapter is structured as follows. Section 4.2 deals with maximizer existence, culminating in the main results that are Theorem 4.2.6 and its Corollary 4.2.1. In Section 4.3, characterization of the related optima is presented in general settings as well as some analysis of the effect of limited liability and some discussion of the specific CARA case. For ease of reading the proofs of the main results are relegated to Section 4.4.

4.2 Results on maximizer existence

4.2.1 On maximizer existence on a Hilbert space

The limited liability problems at hand involve optimizing across two subsets of the Hilbert space :

$$E := L^2(\Omega) \times \mathbb{R}.$$

Let us denote :

$$\left\langle \begin{pmatrix} W_1 \\ a_1 \end{pmatrix}; \begin{pmatrix} W_2 \\ a_2 \end{pmatrix} \right\rangle_E := \mathbb{E}[W_1 W_2] + a_1 a_2, \quad \text{and} \quad \|(W, a)\|_E := \sqrt{\mathbb{E}[W^2] + |a|^2}.$$

and d_E the induced distance on E . On the real line, maximizer or minimizer existence is obtained through the Weierstrass theorem, stating that a continuous function on a closed interval of \mathbb{R} attains its bounds. In infinite dimensional spaces similar results exist and will be used in the following, after recalling some theory taken from [84] adapted to this setting.

Definition 4.2.1. *A functional $f : E \rightarrow \bar{\mathbb{R}}$ is upper semi continuous at (W, a) in E if for any sequence $(W_n, a_n)_{n \in \mathbb{N}}$ such that $(W_n, a_n) \xrightarrow{n \rightarrow +\infty} (W, a)$:*

$$f(W, a) \geq \limsup_{n \rightarrow +\infty} f(W_n, a_n). \quad (4.2.1)$$

It is weakly upper semi continuous if $(W_n, a_n) \xrightarrow{n \rightarrow +\infty} (W, a)$ implies (4.2.1).

With this in mind, a key result for maximizer existence is the Generalized Weierstrass theorem.

Theorem 4.2.1 (Generalized Weierstrass theorem, see Chapter 7 of [84]). *Let X be a weakly compact subset of E and $f : X \rightarrow \bar{\mathbb{R}}$ be a weakly upper semicontinuous functional. Then :*

$$\exists (W_0, a_0) \in X, \quad \text{such that} \quad f(W_0, a_0) = \sup_{(W, a) \in X} f(W, a).$$

There are two key ingredients in this theorem : compactness of the underlying space and upper semi continuity of the functional f . As E is a Hilbert space, we may turn to the Banach-Alaoglu theorem for a (weak-)compactness result.

Theorem 4.2.2 (Banach-Alaoglu theorem, see Chapter 7 of [84]). *The closed unit ball of a Hilbert space is weakly compact.*

This result establishes weak-topology as a natural candidate to prove maximizer existence, reason for which Theorem 4.2.1 was given above using weak-topology. It remains to find conditions that ensure weak upper semi continuity of functionals. In fact, it can sometimes be established manually. It can also sometimes be useful to be able to deduce weak upper semi-continuity from strong upper semi-continuity. This can be done through a consequence to Mazur's theorem that we first state.

Theorem 4.2.3 (Mazur's theorem, see Chapter 7 of [84].). *A convex subset of a Hilbert space is strongly closed if and only if it is weakly closed.*

Theorem 4.2.4 (Consequence of Mazur's theorem, eg. see Chapter 7 of [84].). *Let $f : E \rightarrow \bar{\mathbb{R}}$ be a concave function. Then f is upper semi continuous in the strong topology on E if and only if f is upper semi continuous in the weak topology on E .*

The ingredients mentioned so far lead to the following result, which is a consequence of the Generalized Weierstrass theorem (Theorem 4.2.1) on our Hilbert space and which will be applied to our Limited Liability problem. It combines Theorems 7.3.4. and 7.3.7 of [84].

Theorem 4.2.5 (eg. see Chapter 7 of [84].). *Suppose that $f : M \subseteq \bar{\mathbb{R}}$ is weakly upper semi continuous over a weakly closed subset M of E . Also suppose that f is proper on M . Then if either of the following two conditions holds :*

- M is bounded
- f is norm-coercive, meaning that $\|(W, a)\|_E \rightarrow +\infty \Rightarrow f(W, a) \rightarrow -\infty$,

then :

$$\exists (W_0, a_0) \in M, \quad \text{such that} \quad f(W_0, a_0) = \sup_{(W, a) \in M} f(W, a).$$

4.2.2 Preliminary results on the contracting space

In order to exploit Theorem 4.2.5, some key preliminary results are first derived involving the useful subsets of E that will sometimes be referred to as contracting sets :

$$C_m^M := \left\{ (W, a) \in E, \quad m \leq W \leq M \text{ } \mathbb{P}\text{-a.s.}, \quad \mathbb{E}[U_A(W - \kappa(a))] \geq U_A(y_{PC}) \right\},$$

and

$$C_m := \left\{ (W, a) \in E, \quad m \leq W \text{ } \mathbb{P}\text{-a.s.}, \quad \mathbb{E}[U_A(W - \kappa(a))] \geq U_A(y_{PC}) \right\}.$$

Both sets contain the constant contract $(y_{PC}, 0)$ and are thus non-empty. The limited liability optimization problems that we are interested in may be rewritten as follows :

$$\sup_{(W, a) \in C_m^M} \mathbb{E}[U_P(X^a - W)], \tag{4.2.2}$$

and

$$\sup_{(W, a) \in C_m} \mathbb{E}[U_P(X^a - W)]. \tag{4.2.3}$$

Remark 4.2.1. For any (W, a) in C_m^M or C_m , using Jensen's inequality (as U_A is concave) and as U_A is increasing :

$$\mathbb{E}[U_A(W - \kappa(a))] \geq U_A(y_{PC}) \Rightarrow \mathbb{E}[W] \geq y_{PC} + \kappa(a).$$

Therefore (again through Jensen's inequality for U_P and as U_P is increasing) :

$$\mathbb{E}[U_P(X^a - W)] \leq U_P(x_0 + \mathbb{E}[B] + a - \mathbb{E}[W]) \leq U_P(x_0 - y_{PC} + \mathbb{E}[B] + a^* - \kappa(a^*)),$$

where $a^* = \frac{1}{K} = \operatorname{argsup}_{x \in \mathbb{R}} x - \kappa(x)$, and the Principal's utility is upper bounded across both constraint sets.

Remark 4.2.2. For any $(W, a) \in E$, $\mathbb{E}[U_A(W - \kappa(a))] < +\infty$. Indeed, as W belongs to $L^2(\Omega)$, $\mathbb{E}[W] - \kappa(a)$ is some finite number. Applying Jensen's inequality, as U_A is concave :

$$\mathbb{E}[U_A(W) - \kappa(a)] \leq U_A(\mathbb{E}[W] - \kappa(a)) < +\infty.$$

As the participation constraint also lower bounds the Agent's utility, one may deduce that for any (W, a) in C_m^M or C_m , $\mathbb{E}[U_A(W - \kappa(a))]$ exists.

A few of the preliminary results below hold for both C_m and C_m^M . The notation \mathcal{C} is used when one subset can be trivially substituted for the other.

Remark 4.2.3. For any (W, a) in \mathcal{C} where a belongs to \mathbb{R}_+ , $(W, |a|)$ also belongs to \mathcal{C} through the symmetry of the quadratic cost function κ . Furthermore,

$$U_P(X^a - W) \leq U_P(X^{|a|} - W) \quad \mathbb{P} - a.s.$$

thus an optimal action (if it exists) will be non-negative.

This remark is sound as it makes sense for the Principal to align the effect of the Agent's action with positive variations of the production process. It also means that one can consider that the Agent's actions take their values in \mathbb{R}^+ .

Lemma 4.2.1. The Principal's value function is coercive in a across \mathcal{C} .

Proof. Consider a sequence $(W_n, a_n)_{n \in \mathbb{N}}$ in \mathcal{C} such that $a_n \xrightarrow{n \rightarrow +\infty} +\infty$. Applying Jensen's inequality to the Participation Constraint (4.1.2), $\mathbb{E}[W_n] \geq y_{PC} + \kappa(a_n)$. Now applying Jensen to the Principal's value function, coercivity is obtained :

$$\mathbb{E}[U_P(X^{a_n} - W_n)] \leq U_P(x_0 + a_n + \mathbb{E}[B] - y_{PC} - \kappa(a_n)) \xrightarrow{n \rightarrow +\infty} -\infty.$$

□

Remark 4.2.4. By construction, U'_P and U'_A are positive and decreasing mappings. Therefore, $0 \leq U'_A(W - \kappa(a)) \leq U'_A(m - \kappa(a))$, and thus $U'_A(W - \kappa(a))$ belongs to L^1 over \mathcal{C} . In the following we also assume that U_P is differentiable over \mathcal{C} , with U'_P integrable.

Lemma 4.2.2. \mathcal{C} is a convex subset of E , and weakly closed.

Proof. Weakly closed property. Let $(W_n, a_n) \xrightarrow[n \rightarrow +\infty]{} (W, a) \in E$. First one may prove that $W \geq m$ $\mathbb{P} - a.s.$ (the reasoning trivially extends to the upper bound in C_m^M). To do so suppose that $\mathbb{P}(W < m) > 0$. The weak-convergence of (W_n, a_n) implies in particular that :

$$\mathbb{E}[W_n \Phi] \xrightarrow[n \rightarrow +\infty]{} \mathbb{E}[W \Phi] \quad \forall \Phi \in L^2(\Omega),$$

and thus :

$$\mathbb{E}[(W_n - m)\Phi] \xrightarrow[n \rightarrow +\infty]{} \mathbb{E}[(W - m)\Phi] \quad \forall \Phi \in L^2(\Omega). \quad (4.2.4)$$

Now set $\Phi = \mathbf{1}_{W < m}$. Then Φ belongs to $L^2(\Omega)$ and using Equation (4.2.4) :

$$\mathbb{E}[(W_n - m)\mathbf{1}_{W < m}] \xrightarrow[n \rightarrow +\infty]{} \mathbb{E}[(W - m)\mathbf{1}_{W < m}].$$

By construction, $\mathbb{E}[(W_n - m)\mathbf{1}_{W < m}] \geq 0$ and $\mathbb{E}[(W - m)\mathbf{1}_{W < m}] \leq 0$ and so the convergence cannot hold. Thus $W \geq m$ $\mathbb{P} - a.s.$

Now it remains to deal with the participation constraint. As $(W_n, a_n) \xrightarrow[n \rightarrow +\infty]{} (W, a) \in E$ and κ is continuous on \mathbb{R} , it holds that $(W_n, \kappa(a_n)) \xrightarrow[n \rightarrow +\infty]{} (W, \kappa(a))$ and in particular for all Z in $L^2(\Omega)$:

$$\mathbb{E}[Z(W_n - W - \kappa(a_n) + \kappa(a))] \xrightarrow[n \rightarrow +\infty]{} 0.$$

As $U_A'' \leq 0$ (meaning that U_A is concave), the following inequality holds :

$$\mathbb{E}[U_A(W_n - \kappa(a_n))] \leq \mathbb{E}[U_A(W - \kappa(a))] + \mathbb{E}[U_A'(W - \kappa(a))(W_n - W - \kappa(a_n) + \kappa(a))],$$

and thus as soon as U_A' belongs to $L^2(\Omega)$, we have :

$$\limsup_{n \rightarrow +\infty} \mathbb{E}[U_A(W_n - \kappa(a_n))] \leq \mathbb{E}[U_A(W - \kappa(a))].$$

In particular, as $U_A(y_{PC}) \leq \mathbb{E}[U_A(W_n - \kappa(a_n))]$ it follows that $U_A(y_{PC}) \leq \mathbb{E}[U_A(W - \kappa(a))]$. Therefore \mathcal{C} is a weakly closed subset of E .

Convexity Let (W_1, a_1) and (W_2, a_2) belong to \mathcal{C} and let $\lambda > 0$. Then :

$$m \leq \lambda W_1 + (1 - \lambda)W_2 \quad \mathbb{P} - a.s..$$

This trivially extends to the upper bound in C_m^M . It remains to deal with the Participation Constraint :

$$\begin{aligned} & \mathbb{E}[U_A(\lambda W_1 + (1 - \lambda)W_2 - \kappa(\lambda a_1 + (1 - \lambda)a_2))] \\ & \geq \mathbb{E}[U_A(\lambda W_1 + (1 - \lambda)W_2 - \lambda \kappa(a_1) + (1 - \lambda)\kappa(a_2))] \\ & \geq \lambda \mathbb{E}[U_A(W_1 - \kappa(a_1))] + (1 - \lambda)\mathbb{E}[U_A(W_2 - \kappa(a_2))] \geq U_A(y_{PC}). \end{aligned}$$

Thus $(\lambda W_1 + (1 - \lambda)W_2, \lambda a_1 + (1 - \lambda)a_2)$ belongs to \mathcal{C} and convexity holds. \square

Note that the proof of Lemma 4.2.2 includes proof that the Agent's expected utility is weakly u.s.c. on \mathcal{C} through construction of the constraint sets.

Remark 4.2.5. *Mazur's theorem may be applied to deduce from Lemma 4.2.2 that \mathcal{C} is also strongly closed.*

Lemma 4.2.3. *The Principal's value function is concave.*

Proof. Let (W_1, a_1) and (W_2, a_2) belong to \mathcal{C} and let $\lambda > 0$. Then :

$$\begin{aligned} & \mathbb{E}[U_P(X^{\lambda a_1 + (1-\lambda)a_2} - \lambda W_1 - (1-\lambda)W_2)] \\ &= \mathbb{E}[U_P(\lambda X^{a_1} - \lambda W_1 + (1-\lambda)X^{a_2} + (1-\lambda)W_2)] \\ &\geq \lambda \mathbb{E}[U_P(X^{a_1} - W_1)] + (1-\lambda) \mathbb{E}[U_P(X^{a_2} - W_2)]. \end{aligned}$$

□

The following classical result establishes upper semi-continuity of the Principal's value function across \mathcal{C} .

Lemma 4.2.4. *The Principal's value function is weakly u.s.c.*

Proof. Let (W_n, a_n) be a sequence in \mathcal{C} that converges weakly to (W, a) \mathcal{C} . As U_P is concave,

$$\mathbb{E}[U_P(X^{a_n} - W_n)] \leq \mathbb{E}[U_P(X^a - W)] + \mathbb{E}[U'_P(X^a - W)(a_n - W_n - a + W)]$$

But weak-convergence yields :

$$\mathbb{E}[U'_P(X^a - W)(a_n - W_n - a + W)] \xrightarrow{n \rightarrow +\infty} 0$$

and thus :

$$\limsup_{n \rightarrow +\infty} \mathbb{E}[U_P(X^{a_n} - W_n)] \leq \mathbb{E}[U_P(X^a - W)].$$

□

A final result is related to the coercivity in W of the Principal's value function. This is needed in order to guarantee existence in the lower bound case.

Lemma 4.2.5. *Suppose that $\text{supp}(B) \subset [-\infty, b)$, for $b \in \mathbb{R}$. Suppose that $U_P''' \geq 0$. Then for any $0 \leq a \leq a_{max}$ where a_{max} some positive constant, the Principal's value function is coercive in W across C_m .*

Proof. Let $(W_n, a_n)_n$ be a sequence in C_m such that $0 \leq a_n \leq a_{max}$ and $\mathbb{E}[W_n^2] \rightarrow +\infty$. One may consider two cases.

- If $\mathbb{E}[W_n] \rightarrow +\infty$, then through Jensen's inequality applied to the Principal's value function :

$$\mathbb{E}[U_P(X^{a_n} - W_n)] \leq U_P(x_0 + a_n + \mathbb{E}[B] - \mathbb{E}[W_n]) \xrightarrow{n \rightarrow +\infty} -\infty.$$

- If not, suppose $\mathbb{E}[W_n] \leq K$ with K some real constant. Through a Taylor expansion :

$$\begin{aligned} \mathbb{E}[U_P(X^{a_n} - W_n)] &= \mathbb{E}[U_P(X^0 - y_{PC})] + \mathbb{E}[U'_P(X^0 - y)(a_n - W_n + y_{PC})] \\ &\quad + \mathbb{E}[U''_P(X^{a_\epsilon} - W_\epsilon)(a_n - W_n + y)^2], \end{aligned}$$

where (W_ϵ, a_ϵ) is some convex combination of (W_n, a_n) and $(y_{PC}, 0)$ (and belongs to C_m through Lemma 4.2.2). Now as $U'_P > 0$, it holds that :

$$\mathbb{E}[U'_P(X^0 - y_{PC})(a_n - W_n + y_{PC})] \leq \mathbb{E}[U'_P(X^0 - y_{PC})(a_{max} - m + y_{PC})].$$

It remains to deal with the second order term. First note that :

$$U''_P(X^{a_\epsilon} - W_\epsilon) \leq U''_P(X^{a_{max}} - m) \leq \sup_{B \leq b} U''_P(X^{a_{max}} - m) \quad \mathbb{P} - a.s.,$$

where $S := \sup_{B \leq b} U''_P(X^{a_{max}} - m)$ is a strictly negative real number and exists through the boundedness of B and the fact that $U'''_P \geq 0$. Therefore :

$$\begin{aligned} \mathbb{E}[U''_P(X^{a_\epsilon} - W_\epsilon)(a_n - W_n + y_{PC})^2] &\leq S \left((a_n + y_{PC})^2 + \mathbb{E}[W_n]^2 - 2\mathbb{E}[W_n](a_n + y_{PC}) \right) \\ &\leq S \left((a_{max} + y_{PC})^2 + \mathbb{E}[W_n]^2 \right) \xrightarrow{n \rightarrow +\infty} -\infty, \end{aligned}$$

and the conclusion is obtained : $\mathbb{E}[U_P(X^{a_n} - W_n)] \xrightarrow{n \rightarrow +\infty} -\infty$. □

Remark 4.2.6. *The hypothesis $U''' \geq 0$ is satisfied for example by DARA (decreasing absolute risk aversion) utility functions. Empirical analysis of economic data mostly validates the DARA hypothesis for practical applications (see [52]).*

Before turning to the main theorem of this Chapter, we may summarize the preliminary results for Problems (4.2.2) and (4.2.3) obtained so far :

- the Principal's value function is weakly u.s.c.
- the contracting space is a convex and weakly closed subset of E
- under the assumptions of Lemma 4.2.5, the Principal's value function is norm-coercive on the contracting space

We may also note that C_m^M only contains bounded wages. These properties stand us in good stead to use Theorem 4.2.5 for the Risk-Sharing limited liability problems, as done in the following subsection.

4.2.3 Existence theorem for Risk-Sharing under limited liability

The following main result establishes existence of maximizers for the Limited Liability problems at hand.

Theorem 4.2.6. *We have the following results for Problems (4.2.2) and (4.2.3) :*

1. *Problem (4.2.2) has a maximizer.*
2. *Problem (4.2.3) has a maximizer in the two following cases :*
 - *supp(B) \subset $[-\infty, b)$, $b \in \mathbb{R}$ and $U'''_P \geq 0$.*
 - *there exists a positive mapping f such that :*

(a) $\mathbb{E}[W^2] \rightarrow +\infty \implies \mathbb{E}[f(W)] = +\infty$, and :

$$\mathbb{E}[U_P(X^a - W)] - \epsilon \mathbb{E}[f(W)]$$

has a unique maximizer (W^ϵ, a^ϵ) on C_m .

(b) there exists $K > 0$ such that $\mathbb{E}[W_\epsilon^2] \leq K$ for all $\epsilon \in (0, 1)$.

Proof. The proof can be found in Section 4.4.1 of this Chapter. The first result as well as the first case of the second result hold from Theorem 4.2.5 using the preliminary results above. The second case of the second result requires a slightly different and perturbation based approach, as detailed in the proof. \square

Remark 4.2.7. *When the Principal's value function is strictly concave, the existence results of Theorem 4.2.6 are also uniqueness results.*

We may note that existence for a CARA Principal under lower limited liability and with an unbounded B is not directly guaranteed through Theorem 4.2.6, in particular it does not seem to be guaranteed by the first case of the second result. Of course although one aim of this work was to attempt to establish results for more general utility functions, being able to include the CARA case seemed relevant especially with analysis of the effect of bounds in mind. As such, the following Corollary uses the second condition of Part 2 of Theorem 4.2.6 to extend to a CARA setting

Corollary 4.2.1 (Existence result for lower limited liability and CARA utilities). *Suppose that $U_P(x) := -e^{-\gamma_P x}$ and $U_A(x) := -e^{-\gamma_A x}$, for $\gamma_P > 0$ and $\gamma_A > 0$, then there exists a unique solution to Problem (4.2.3).*

Proof. The existence proof can be found in Section 4.4.2 of this Chapter. Uniqueness follows from strict concavity. \square

4.2.4 Some comments on the results

We make the following comments on the main Theorem.

- It is apparent from Theorem 4.2.6 that existence of a maximizer in the solely lower bounded case requires heavier assumptions than the double bounded case. However let $\mathcal{B}_K = \{W \in L^2(\Omega), \mathbb{E}[W^2] \leq K^2\}$. Then $\sup_{(W,a) \in C_{m,K}} \mathbb{E}[U_P(X^a - W)]$ admits a solution.
- Theorem 4.2.6 uses assumptions on the Principal's utility function in order to ensure existence of solutions to the limited liability problem in the lower bounded case. However it does not use any assumptions on the Agent's utility function beyond it being increasing and concave. It does not use any assumptions on the Principal's utility function in the double bounded case either. This allows for substantial freedom in modeling the utilities of the various underlying entities.

These examples illustrate the main existence theorem for different utilities.

Example 4.2.1. *In the following examples, supposing that the Agent has any concave and increasing utility function, different limited liability existence results are provided.*

1. **Logarithmic utility.** Let a Principal have extended logarithmic utility, as defined in (4.1.5). Suppose that the underlying uncertainty B has a uniform distribution on $[-5; 5]$. Set $m = 0$ and $M = 2$. Then the contracting problem $\sup_{(W,a) \in C_0^2} \mathbb{E}[U_P(X^a - W)]$ admits a maximizer.
2. **CARA utility.** Let a Principal have CARA utility. Suppose that the underlying uncertainty B has a bounded distribution. Set $m = 0$. Then the slightly altered contracting problem $\sup_{(W,a) \in C_0} \mathbb{E}[U_P(X^a - W)]$ admits a maximizer.

The reasoning leading to the existence results was based on a wealth process X^a of the form $X^a = x_0 + a + B$ that directly affects the Principal who tracts $\mathbb{E}[U_P(X^a - W)]$. Whilst this is a modeling choice that fits with classical Principal-Agent literature, we may wish to extend beyond this "linear perturbation" setting. This is detailed below.

Modifying the effect of a on X^a . Given a differentiable mapping f that is concave and such that :

$$\lim_{|x| \rightarrow +\infty} f(x) - \kappa(x) = -\infty, \quad (4.2.5)$$

and setting : $X^a = x_0 + f(a) + B$, the existence results of Theorem 4.2.6 still hold. Indeed, such a modification only affects the Principal's expected utility. First note that the Principal's expected utility remains upper bounded : Remark 4.2.1 still holds with a^* now the maximizer of $f - \kappa$. The crucial ingredients for existence and related to the Principal's utility are Lemma 4.2.1 on coercivity in a and Lemma 4.2.3 on concaveness (note that the two other results, Lemma 4.2.4 and Lemma 4.2.5 follow as a consequence to Lemma 4.2.3). Coercivity in a extends through 4.2.5 and as f is concave, the Principal's expected utility remains concave in (W, a) . Thus the existence result may be extended to such a setting and the optimum will be characterizable through the results of Section 4.3.

Modifying the effect of X^a on the Principal's utility. Echoing this natural extension, one may wish to model a setting where the wealth of the Principal is given by $g(X^a) - W$ (for example, the Principal may only earn part of X^a), where g is a differentiable mapping. As such a function affects B , we require that :

$$\mathbb{E}[U_P(g(X^0) - y_{PC})] = \mathbb{E}[U_P(g(x_0 + B) - W)] < +\infty.$$

Once again, such an extension holds as soon as g is concave and : $\lim_{|x| \rightarrow +\infty} g(x) - \kappa(x) = -\infty$, and the optimum will be characterizable through the results of Section 4.3.

4.3 Characterizing the optimal contracts

With the existence of solutions in mind, one may turn to their characterization. The theory used here for characterizing optimizers on convex subsets of Hilbert spaces (as it is proven in Lemma 4.2.2 that the contracting sets are convex) is described in Luenberger's monograph [90]. Our aim in the following is to use this theory to characterize the limited liability optima in situations where its existence is ensured and existence is supposed as a preliminary as well as the Gâteaux-differentiability of the expected utility functions. We introduce the notations

$\tilde{U}_P = -U_P$, $\tilde{U}_A = -U_A$ and the positive cone :

$$\mathcal{P}_W := \left\{ W \in L^2(\Omega), \quad W \geq 0 \quad \mathbb{P}\text{-a.s.} \right\}.$$

Characterization of the optimum exploits necessary optimality conditions taken from [90]. In particular, we may define as follows the Lagrangians that will be key to characterizing the optimal contracts.

Definition 4.3.1. *Consider the mapping :*

$$\mathcal{L}_m^M : E \times \mathbb{R}^+ \times \mathcal{P}_W \times \mathcal{P}_W \mapsto \mathbb{R},$$

defined as :

$$\mathcal{L}_m^M(W, a, \lambda, Z, Y) := F(a, W) + G_{PC}(a, W) + H_{LL}(W, Y, Z),$$

where :

$$\begin{cases} F(a, W) := \mathbb{E} \left[\tilde{U}_P (X^a - W) \right], \\ G_{PC}(a, W, \lambda) := \lambda \left(\mathbb{E} \left[\tilde{U}_A (W - \kappa(a)) \right] - \tilde{U}_A(y) \right), \\ H_{LL}(W, Y, Z) := \mathbb{E}[Z(m - W)] + \mathbb{E}[Y(W - M)]. \end{cases}$$

Then \mathcal{L}_m^M is the Lagrangian for the double-sided limited liability problem. The Lagrangian for the one-sided limited liability problem is obtained by formally setting Y to 0 in \mathcal{L} .

Through [90], the maximizers for Problems (4.2.2) and (4.2.3) satisfy stationarity and slackness conditions given in the following Proposition.

Proposition 4.3.1. *Let (W^*, a^*) be a solution to Problem (4.2.2) and (λ^*, Z^*, Y^*) in $\mathbb{R}^+ \times \mathcal{P}_W \times \mathcal{P}_W$ be Lagrange multipliers. Then :*

1. $\mathbb{E} \left[\tilde{U}'_P (X^{a^*} - W^*) \right] - \lambda^* \kappa'(a) \mathbb{E} \left[\tilde{U}'_A (W^* - \kappa(a^*)) \right] = 0$
2. $-\tilde{U}'_P (X^{a^*} - W^*) + \lambda^* \tilde{U}'_A (W^* - \kappa(a^*)) - Z^* + Y^* = 0$
3. $\lambda^* (\mathbb{E} [U_A(W^* - \kappa(a^*))]) - U_A(y_{PC}) = 0$
4. $\mathbb{E}[Z^*(W^* - m)] = 0 \quad \text{and} \quad \mathbb{E}[Y^*(M - W^*)] = 0.$

The optima $(W^*, a^*, \lambda^*, Z^*)$ for Problem (4.2.3) satisfies the same optimality conditions where Y is formally set to 0.

These conditions are used in the following to obtain information on the optimal contracts.

4.3.1 One-sided limited liability

Consider a setting for which Problem (4.2.3) admits a solution (W^*, a^*) . Let (λ^*, Z^*) in $\mathbb{R}^+ \times \mathcal{P}_W$ be the two related Lagrange multipliers. According to Proposition 4.3.1, the following four equations are satisfied :

$$\begin{cases} \mathbb{E} [\tilde{U}'_P (X^{a^*} - W^*)] - \lambda^* \kappa'(a) \mathbb{E} [\tilde{U}'_A (W^* - \kappa(a^*))] = 0 \\ -\tilde{U}'_P (X^{a^*} - W^*) + \lambda^* \tilde{U}'_A (W^* - \kappa(a^*)) - Z^* = 0 \\ \lambda^* (\mathbb{E} [U_A (W^* - \kappa(a^*))] - U_A (y_{PC})) = 0 \\ \mathbb{E} [Z^* (W^* - m)] = 0. \end{cases} \quad (4.3.1)$$

Using these necessary optimality conditions, we first derive some preliminary results on the optima before further characterizing it.

Lemma 4.3.1. *The optimal λ^* satisfies $\lambda^* > 0$, meaning that the Participation Constraint binds at the optimum.*

Proof. Using the second equality of (4.3.1) and as $Z^* \in \mathcal{P}_W$, it holds that :

$$-\tilde{U}'_P (X^{a^*} - W^*) + \lambda^* \tilde{U}'_A (W^* - \kappa(a^*)) \geq 0,$$

implying that : $\lambda^* \geq \frac{\tilde{U}'_P (X^{a^*} - W^*)}{\tilde{U}'_A (W^* - \kappa(a^*))} > 0$. □

Lemma 4.3.2. *The optimal Z^* and W^* satisfy :*

$$Z^* (W^* - m) = 0 \quad \mathbb{P} - a.s. \quad \text{and} \quad \mathbb{P} (W^* = m) < 1.$$

Proof. By definition it holds that $Z^* \geq 0$ $\mathbb{P} - a.s.$ and $W^* - m \geq 0$ $\mathbb{P} - a.s.$. Using the final equality of (4.3.1) :

$$\mathbb{E} [Z^* (W^* - m)] = 0 \quad \implies \quad Z^* (W^* - m) = 0 \quad \mathbb{P} - a.s.$$

For the second assertion, the first equality of (4.3.1) states that :

$$\mathbb{E} [\tilde{U}_A (W^* - \kappa(a^*))] = \tilde{U}_A (y_{PC}).$$

Now suppose for contradiction that $W^* = m$ $\mathbb{P} - a.s.$. Then a^* must verify $y = m - \kappa(a^*)$ for any $y \geq 0$. This is impossible as for any x in \mathbb{R}_+ , $\kappa(x) \geq 0$, and a contradiction is reached. □

We have the following information on the optima for the limited liability problem.

Proposition 4.3.2 (General characterization for one-sided limited liability). *Let $(W^*, a^*, \lambda^*, Z^*)$ be a solution to Problem (4.2.3) along with the Lagrange multipliers. Then :*

- (W^*, a^*) saturates the participation constraint :

$$\mathbb{E} [U_A (W^* - \kappa(a^*))] = U_A (y_{PC}).$$

- (Borch rule for one-sided limited liability). On the event $\{Z^* = 0\}$, the ratio of marginal utilities of the Principal and the Agent is constant :

$$\lambda^* = \frac{\tilde{U}'_P(X^{a^*} - W^*)}{\tilde{U}'_A(W^* - \kappa(a^*))} \mathbf{1}_{Z^*=0}.$$

- The optimal action in the limited liability problem is greater than the optimal action in the standard risk-sharing problem, meaning that $a^* \geq \frac{1}{K}$.

Proof. - The first statement is a direct consequence to Lemma 4.3.1.

- Lemma 4.3.2 implies that $\mathbb{P}(Z^* = 0) > 0$, so setting oneself on the event $\{Z^* = 0\}$ and using the second equation of Lemma 4.3.1 it holds that

$$-\tilde{U}'_P(X^{a^*} - W^*) + \lambda^* \tilde{U}'_A(W^* - \kappa(a^*)) = 0,$$

and the result follows.

- By definition, Z^* is a positive random variable and therefore $\mathbb{E}[Z^*] \geq 0$. Using Equation 2 of (4.3.1) we have :

$$\begin{aligned} \mathbb{E}[Z^*] &= -\mathbb{E}[\tilde{U}'_P(X^{a^*} - W^*)] + \lambda^* \mathbb{E}[\tilde{U}'_A(W^* - \kappa(a^*))] \\ &= \lambda^* \mathbb{E}[\tilde{U}'_A(W^* - \kappa(a^*))](\kappa'(a^*) - 1), \end{aligned}$$

where the second equality is obtained through Equation 1 of (4.3.1). But as $\lambda^* > 0$ and $\mathbb{E}[\tilde{U}'_A(W^* - \kappa(a^*))] < 0$, it must hold that :

$$\kappa'(a^*) - 1 \geq 0$$

which as $a^* \geq 0$ equates to $a^* \geq \frac{1}{K}$ where $\frac{1}{K}$ is the optimal action for the standard risk-sharing problem (see for example the discrete model discussed in Chapter 2) \square

4.3.2 Double-sided limited liability

This subsection is structured in a similar manner to the previous one : the necessary optimality conditions are listed, some preliminary information is obtained, and finally a characterization of the optimum is provided.

Once again suppose that Problem (4.2.2) admits a solution (W^*, a^*) . Let (λ^*, Z^*, Y^*) in $\mathbb{R}^+ \times \mathcal{P}_W \times \mathcal{P}_W$ be the three related Lagrange multipliers. The following four equations are satisfied :

$$\begin{cases} \mathbb{E} \left[\tilde{U}'_P(X^{a^*} - W^*) \right] - \lambda^* \kappa'(a^*) \mathbb{E} \left[\tilde{U}'_A(W^* - \kappa(a^*)) \right] = 0 \\ -\tilde{U}'_P(X^{a^*} - W^*) + \lambda^* \tilde{U}'_A(W^* - \kappa(a^*)) - Z^* + Y^* = 0 \\ \lambda^* (\mathbb{E} [U_A(W^* - \kappa(a^*))] - U_A(y_{PC})) = 0 \\ \mathbb{E}[Z^*(W^* - m)] = 0 \quad \text{and} \quad \mathbb{E}[Y^*(M - W^*)] = 0. \end{cases} \quad (4.3.2)$$

Lemma 4.3.3. *Let ω be such that $Z^*(\omega) = Y^*(\omega)$, then $Z^*(\omega) = 0$ and $Y^*(\omega) = 0$.*

Proof. Let ω be such that $Z^*(\omega) = Y^*(\omega) = c$ for some non-negative c . For reasons mirroring those of Lemma 4.3.2 it must hold that :

$$Z^*(\omega)W^*(\omega) = Z^*(\omega)m \quad \text{and} \quad Y^*(\omega)W^*(\omega) = Y^*(\omega)M,$$

which rewrites as $cW^*(\omega) = cm$ and $cW^*(\omega) = cM$. As $m \neq M$, this is only possible for $c = 0$. \square

Proposition 4.3.3 (General characterization for double-sided limited liability). *Let $(W^*, a^*, \lambda^*, Z^*, Y^*)$ be a solution to Problem (4.2.2) along with the related Lagrange multipliers. Then $(W^*, a^*, \lambda^*, Z^*, Y^*)$ satisfies a Borch rule for double-sided limited liability. Indeed on the event $\{Z^* = Y^*\}$, the ratio of marginal utilities of the Principal and the Agent is constant :*

$$\lambda^* = \frac{\tilde{U}'_P(X^{a^*} - W^*)}{\tilde{U}'_A(W^* - \kappa(a^*))} \mathbf{1}_{Z^*=0} \quad \text{and} \quad Y^*=0.$$

Proof. The proof of Proposition 4.3.2 may be adapted to this setting to show here that :

$$\lambda^* = \frac{\tilde{U}'_P(X^{a^*} - W^*)}{\tilde{U}'_A(W^* - \kappa(a^*))} \mathbf{1}_{Z^*=Y^*},$$

and the final result is obtained through Lemma 4.3.3. \square

4.3.3 Some analysis of the effect of limited liability

This characterization leads us to some applied analysis of the effect of limited liability on optimal contracting. First, under one-sided limited liability just like in the standard Risk Sharing problem, the optimal contract must bind the participation constraint : this means that setting a lower bound for the wage does not have any effect on the Agent's expected utility. So whilst the lower bound can be considered to improve the Agent's situation by ensuring that the Agent's losses are limited, he is actually on average no better off than without the constraint. Furthermore, the Agent's action is greater or equal to that of the optimal action in the Risk Sharing problem, and the action is therefore indeed able to compensate for the minimum wage constraint. What the Agent gains in diminished risk, he seems to lose in having to provide greater effort and one may wonder whether such a tradeoff is always worth it for the Agent. Also, the limited liability constraint leads to a Borch rule on the ratio of marginal utilities (see [16], [37] but also Section ?? of Chapter 2) holding only on part of the underlying probability space and characterizing the optimal wage as an "option" type wage. Indeed, introducing the limited liability constraint restricts the Borch rule to hold only on the specific event $\{Z^* = 0\}$ rather than everywhere. Note that on $\{Z^* = 0\}$, W^* may be greater than or equal to m . However, the complement of $\{Z^* = 0\}$ is the event $\{Z^* > 0\}$ on which one must have : $W = m$ $\mathbb{P} - a.s.$. So under limited liability constraints and working ω by ω , either a Borch rule holds or the optimal wage is worth m . So it follows that optimal wage must be of option form, i.e. :

$$W^* = m\mathbf{1}_{\tilde{W} < m} + \tilde{W}\mathbf{1}_{\tilde{W} \geq m},$$

with \tilde{W} some random variable satisfying the Borch rule when valued above m . This is coherent with the "option" forms of the wages found in [74], [81], [37] in different settings for

limited liability.

Much of the interpretation of the one-sided limited liability characterization extends to double-sided limited liability. However, one may make some specific remarks. First note that in this result there is no statement regarding the value of a^* . In fact, the analysis related to a^* is slightly more subtle here. Indeed :

$$\begin{aligned}\mathbb{E}[Z^* - Y^*] &= -\mathbb{E}\left[\tilde{U}'_P\left(X^{a^*} - W^*\right)\right] + \lambda^*\mathbb{E}\left[\tilde{U}'_A\left(W^* - \kappa(a^*)\right)\right] \\ &= -\lambda^*\kappa'(a^*)\mathbb{E}\left[\tilde{U}'_A\left(W^* - \kappa(a^*)\right)\right] + \lambda^*\mathbb{E}\left[U'_A\left(W^* - \kappa(a^*)\right)\right] \\ &= -\lambda^*\mathbb{E}\left[\tilde{U}'_A\left(W^* - \kappa(a^*)\right)\right]\left(\kappa'(a^*) - 1\right),\end{aligned}$$

where Z^* and Y^* are two non-negative random variables, and the sign of the right hand side term is given by the sign of $\kappa'(a^*) - 1$. In other words, under double-sided limited liability constraints, the Agent's action can be either above or below that of the standard Risk-Sharing problem. This depends on the sign of $\mathbb{E}[Z^* - Y^*]$ which will be linked to the values of m and M . Also there is no statement about the Participation Constraint either, as one does not know the sign of $Z^* - Y^*$ to apply the same reasoning as in the one-sided case. In fact information on the probability of Z^* being equal to Y^* is lacking. Finally one can once again specify the form of the optimal wage W^* :

$$W^* = m\mathbf{1}_{\tilde{W} < m} + \tilde{W}\mathbf{1}_{m \leq \tilde{W} \leq M} + M\mathbf{1}_{\tilde{W} > M},$$

with \tilde{W} some random variable satisfying a Borch rule when valued between m and M .

4.3.4 Limited liability in the CARA utility case

In the following we turn to the very tractable CARA case with $U_P(x) := -e^{-\gamma_P x}$ and $U_A(x) := -e^{-\gamma_A x}$, for $\gamma_P > 0$ and $\gamma_A > 0$ two fixed risk aversion coefficients. In such a setting Theorems 4.2.6 and Corollary 4.2.1 guarantee existence of both a unique one-sided and unique double-sided limited liability optimum. We may therefore use the previously described approach for contract characterization, and take advantage of the pleasing properties of the CARA utilities to further the characterization. In fact in the following Proposition exploitable expressions of these optima is provided.

Proposition 4.3.4 (Limited liability contracts in a CARA setting). *1. In both cases the participation constraint is saturated at the optimum.*

2. Let $(W^, a^*, \lambda^*, Z^*)$ be the optimum for one-sided limited liability. Then set :*

$$\tilde{W} := \frac{\gamma_P}{\gamma_P + \gamma_A}X^{a^*} + \frac{\gamma_A}{\gamma_P + \gamma_A}\kappa(a^*) + \frac{1}{\gamma_P + \gamma_A}\ln\left(\frac{\gamma_A\lambda^*}{\gamma_P}\right).$$

The optimal wage W^ is of the form :*

$$W^* = m\mathbf{1}_{\tilde{W} < m} + \tilde{W}\mathbf{1}_{\tilde{W} \geq m}$$

and the optimal action a^ is of the form :*

$$a^* = \kappa'^{-1}\left(1 + e^{\gamma_A y_{PC}} \frac{\mathbb{E}[Z^*]}{\lambda^*\gamma_A}\right).$$

3. Let $(W^*, a^*, \lambda^*, Z^*, Y^*)$ be the optimum for double-sided limited liability. Then set:

$$\tilde{W} := \frac{\gamma_P}{\gamma_P + \gamma_A} X^{a^*} + \frac{\gamma_A}{\gamma_P + \gamma_A} \kappa(a^*) + \frac{1}{\gamma_P + \gamma_A} \ln \left(\frac{\gamma_A \lambda^*}{\gamma_P} \right).$$

The optimal wage W^* is of the form :

$$W^* = m \mathbf{1}_{\tilde{W} < m} + \tilde{W} \mathbf{1}_{m \leq \tilde{W} \leq M} + M \mathbf{1}_{\tilde{W} > M}$$

and the optimal action a^* is of the form :

$$a^* = \kappa'^{-1} \left(1 + e^{\gamma_A y_{PC}} \frac{\mathbb{E}[Z^* - Y^*]}{\lambda^* \gamma_A} \right).$$

Note that it holds that :

$$\mathbb{E}[Z^* - Y^*] \geq \lambda^* \gamma_A e^{-\gamma_A y_{PC}} (\kappa'(0) - 1) \geq -\lambda^* \gamma_A e^{-\gamma_A y_{PC}},$$

thus ensuring that a^* is non-negative.

Proof. This Proposition is a direct consequence of the previously listed necessary optimality conditions written with CARA utilities. To see that the Participation Constraint must be bound, use the first optimality condition. In both cases :

$$-\gamma_P \mathbb{E} \left[e^{-\gamma_P (X^{a^*} - W^*)} \right] + \lambda^* \gamma_A \kappa'(a^*) \mathbb{E} \left[e^{-\gamma_A (W^* - \kappa(a^*))} \right] = 0,$$

then either $a^* = 0$ and :

$$\mathbb{E} \left[e^{-\gamma_P (X^{a^*} - W^*)} \right] = 0,$$

which is absurd, or $a^* > 0$:

$$\lambda^* = \frac{\gamma_P \mathbb{E} \left[e^{-\gamma_P (X^{a^*} - W^*)} \right]}{\gamma_A \kappa'(a^*) \mathbb{E} \left[e^{-\gamma_A (W^* - \kappa(a^*))} \right]} > 0.$$

It follows that $\lambda^* > 0$ and the Participation Constraint is saturated. The characterizations follow. \square

Some further analysis

In both cases the optimal wage is familiar : it is a truncated variant of the form of optimal wage in the Risk-Sharing case (see [37]). The optimal action is related to that of the classical risk sharing case too : in the classical Risk-Sharing case, $a^* = \kappa'^{-1}(1)$. Unfortunately, the optimal wage and actions are still not given under a fully closed form as they depend on the underlying Lagrange multipliers... We may nevertheless obtain further information on the CARA optimum by performing numerical analysis based on the following remarks :

- As the Principal's value function is strictly concave, the optimum is unique in both cases.

- Let $\bar{W}_\beta^a = \frac{\gamma_P}{\gamma_P + \gamma_A} X^a + \beta$ for some $\beta \in \mathbb{R}$ and :

$$W_\beta^a = m \mathbf{1}_{\bar{W} < m} + \bar{W}_\beta^a \mathbf{1}_{\bar{W} \geq m}.$$

Pairs (W_β^a, a) that satisfy the participation constraint form a subset of the contracting space C_m , and the limited liability optimum belongs to this subset (due to the form of the wage). In fact the one-sided limited liability contract is the unique pair (W_β^a, a) that maximizes the Principal's utility whilst saturating the Participation Constraint.

- The remark above extends to the double-sided case.

From these remarks, we obtain another interpretation of the optimal limited liability contract. Using it, we may characterize the limited liability wages by calibrating two reals a and β . This lends itself quite nicely to numerical computations of the optimum in various cases. In the following, we use this to derive some numerical analysis of the optimal contract in the case where $B \sim \mathcal{N}(0, 1)$ and we simply ask for wage positivity meaning that the wage is of the form :

$$\left(\frac{\gamma_P}{\gamma_P + \gamma_A} X^a + \beta \right)_+.$$

We are able to make the following remarks. First, suppose that one of the following statements holds :

(A1) *the value of y_{PC} is greater than or equal to that of x_0 ,*

(A2) *the value of γ_P is of similar order to or smaller than that of γ_A ,*

meaning that either the Principal is less risk averse than the Agent or the Agent has high expectations with respect to the situation. Then we have the following observations on the optimum.

1. *The value of $\mathbb{E}[Z^*]$ is close to 0,*

where we recover the value of $\mathbb{E}[Z^*]$ at the optimum through the expression :

$$\mathbb{E}[Z^*] = -\mathbb{E}[\tilde{U}'_P(X^{a^*} - W^*)] + \lambda^* \mathbb{E}[\tilde{U}'_A(W^* - \kappa(a^*))].$$

This remark can be touched on analytically. Indeed, Z^* is positive only when $X^{a^*} \leq \frac{-\beta_{a^*}}{\rho}$ which equates to

$$B \leq -x_0 - a^* - \frac{\beta_{a^*}}{\rho}.$$

In practice the right-hand term here is often negative, and this inequality is verified quite rarely. When it is verified, the values of Z^* generally remain close to 0 so the expectation of the random variable is non-negative (as required) but low.

2. *The optimal action is close to $\frac{1}{K}$,*

and this can be expected from the expression of a^* , as $\mathbb{E}[Z^*]$ generally close to 0 and not compensated by λ^* . When this is the case, the optimal intercept β also tends to the optimal β from the Risk-Sharing problem.

3. *The Principal's expected utility is then drastically reduced,*

as opposed to without limited liability. This is also unsurprising as the Agent's expected utility remains the same through the binding participation constraint, whilst the form of wage is favorable to the Agent and not to the Principal. In fact as $a_{LL}^* \approx a_{RS}^*$ and $\beta_{LL}^* \approx \beta_{RS}^*$, it very roughly holds that (numerically observed, not a mathematical equality) :

$$\mathbb{E} \left[U_P \left(X^{a_{LL}^*} - W_{LL}^* \right) \right] \approx \mathbb{E} \left[U_P \left(X^{a_{RS}^*} - W_{RS}^* \right) \mathbb{1}_{W_{RS}^* \geq 0} \right]$$

This leads to the conclusion that a CARA Principal who is at most as risk-averse as the Agent he is contracting with will enforce limited liability by taking the positive part of a wage that is very close to the Risk-Sharing wage and ask the Agent to provide very similar effort to the Risk-Sharing effort. The Agent's expected utility stays the same as under Risk-Sharing, through the saturated Participation Constraint. The Principal's utility is however reduced. Therefore in such a setting, the lower limited liability bound is quite beneficial to the Agent who takes less risk than without the bound but does not have to provide much extra effort in exchange.

Now suppose that (A1) and (A2) do not hold but that one of the following holds :

(A3) *the value of y_{PC} is less than that of x_0 ,*

(A4) *the value of γ_P is significantly greater than that of γ_A ,*

which is the opposite situation to above. Then we may make the following two observations.

1. *The Agent's action is significantly greater than $\frac{1}{K}$.*

2. *The intercept of the linear part of the wage is no longer close to the Risk-Sharing intercept.*

The conclusions that we may make for a CARA Principal who is much more risk-averse than the Agent⁴ are quite different to the previous conclusions. In this case the Agent's expected utility remains that same as in the Risk-Sharing case. However the Principal asks more of the Agent by asking him to provide a substantially higher effort. The intercept of the linear part of the wage is then calibrated to saturate the Limited Liability constraint. Therefore, when a Principal is quite averse to risk and the Agent wishes to take less risk than in a standard setting (thus transferring some risk to the Principal), the Principal does not simply give the Agent the positive part of a standard Risk-Sharing wage in exchange for a Risk-Sharing effort. In fact the Principal asks more of the Agent, and adjusts the wage intercept too. This setting therefore contrasts quite a bit with the setting analyzed above.

⁴It is important to underline that it is often considered that a Principal should be less risk-averse than his Agent (for example a big company is probably less risk-averse than most of its employees). However one can imagine contracting settings (for example involving much smaller companies) where the opposite may be the case and where limited liability may still be relevant for an Agent.

One question that this analysis seems to bring up is the relevance of the wage positivity constraint in some settings (basically, when the Principal is very risk-averse). Indeed, whilst enforcing it protects the Agent from bad outcomes from B , the Agent can be asked to provide much more effort to make up for this constraint. However on average the Agent's takeaway utility remains the same as under full Risk-Sharing. Therefore, when an Agent enters into a contracting agreement with a risk-averse Principal, asking for limited liability may not be profitable to the Agent : he has a choice to make between asking for limited liability and having to provide a greater effort, or providing less effort in exchange for the standard Risk-Sharing wage. In any case, a compromise between risk and effort is apparent.

We may note that the optimal wage paid under a minimal wage constraint $W \geq m$ \mathbb{P} -*a.s.* with m different to 0 is a sort of translation of the "call" form wage paid under wage positivity. Such a wage seems to be in line with applications. Indeed, the Agent is guaranteed a fixed part m and in settings where the economic outlook is favorable, the Agent receives extra pay (a sort of bonus) that is computed based on the firms income. Of course once again due to the lower risk taken by the Agent in such a setting, he provides more effort compared to in the classical Risk-Sharing setting.

4.4 Proofs

The following section collects a couple of the lengthier proofs from this analysis.

4.4.1 Proof of Theorem 4.2.6

Let us note in any case that the Principal's value function is proper on the contracting space : it is upper bounded through Remark 4.2.1. Furthermore, there exists an admissible pair $a = 0, W = y_{PC}$ such that the Principal's value function is finite. We turn to proving the different statements.

1. Through Lemma 4.2.1, the Principal's value function is coercive in a . Through positivity of any optimizer, optimization in a may be restricted to some interval $[0, a_{max}]$. Then C_m^M is both bounded in W (by construction) and in a . Through Lemma 4.2.2, C_m^M is weakly closed. Finally, through Lemma 4.2.4, the Principal's value function is weakly upper semi-continuous. The result follows through Theorem 4.2.5.
2. (first case). Through Lemma 4.2.1, the Principal's value function is coercive in a . Through Lemma 4.2.5, whose assumptions are satisfied, the value function is coercive in W . It is thus jointly coercive in W and a . Indeed, let (W_n, a_n) in C_m be such that $\|(W_n, a_n)\|_E \xrightarrow{n \rightarrow +\infty} +\infty$, then it holds that that:
 - either $a_n \xrightarrow{n \rightarrow +\infty} +\infty$ and the Principal's value function is coercive through coercivity in a ,
 - or $0 \leq a_n \leq a_{max}$ where a_{max} is some positive constant, and $\mathbb{E}[W_n^2] \xrightarrow{n \rightarrow +\infty} +\infty$.
 The Principal's value function is then coercive through coercivity in W . Now through Lemma 4.2.2, C_m^M is weakly closed. Finally, through Lemma 4.2.4 and Lemma 4.2.3, the Principal's value function is weakly upper semi-continuous. The result follows through Theorem 4.2.5.

2. (second case) The aim is to consider the perturbed solutions (W_ϵ, a_ϵ) and to analyze their behavior as ϵ goes to 0. As such consider some sequence $(\epsilon_n)_{n \in \mathbb{N}}$ with $\forall n \in \mathbb{N}, 0 < \epsilon_n < 1$ and $\lim_{n \rightarrow +\infty} \epsilon_n = 0$. The reasoning from Lemma 4.2.1 and Remark 4.2.3 still holds and thus there exists some a_{max} such that $a_{\epsilon_n} \in [0, a_{max}]$, for any $n \in \mathbb{N}$. Also by assumption, $\mathbb{E}[W_{\epsilon_n}^2] \leq K$, for any $n \in \mathbb{N}$. Using the Banach-Alaoglu theorem, one can find a subsequence of ϵ such that :

$$\epsilon_n \xrightarrow{n \rightarrow +\infty} 0, \quad W_{\epsilon_n} \rightharpoonup W^*, \quad a_{\epsilon_n} \xrightarrow{n \rightarrow +\infty} a^*,$$

where (W^*, a^*) belongs to C_m through Lemma 4.2.2. Finally, recall that $(W, a) \mapsto \mathbb{E}[U_P(X^a - W)]$ is weakly usc. Thus for any $(W, a) \in C_m$, it holds that :

$$\begin{aligned} \mathbb{E}[U_P(X^{a^*} - W^*)] &\geq \limsup_{n \rightarrow +\infty} \mathbb{E}[U_P(X^{a_{\epsilon_n}} - W_{\epsilon_n})] \\ &\geq \limsup_{n \rightarrow +\infty} \mathbb{E}[U_P(X^{a_{\epsilon_n}} - W_{\epsilon_n})] - \epsilon_n \mathbb{E}[f(W_{\epsilon_n})] \quad \text{as } f \text{ positive} \\ &\geq \limsup_{n \rightarrow +\infty} \mathbb{E}[U_P(X^a - W)] - \epsilon_n \mathbb{E}[f(W)] \quad \text{as } (W_{\epsilon_n}, a_{\epsilon_n}) \text{ maximizer} \\ &= \mathbb{E}[U_P(X^a - W)], \end{aligned}$$

(W^*, a^*) is therefore an optimal contract for Problem (4.2.3).

4.4.2 Proof of Corollary 4.2.1

This subsection is dedicated to proving Corollary 4.2.1 and is made up of a series of preliminary results.

Lemma 4.4.1. *Set $U_P(x) := -e^{-\gamma_P x}$ and $U_A(x) := -e^{-\gamma_A x}$. Then there exists a solution to the following problem :*

$$\sup_{(W, a) \in C_m} F_\epsilon(W, a), \quad (4.4.1)$$

where $F_\epsilon(W, a) = \mathbb{E}[U_P(X^a - W)] + \epsilon \mathbb{E}[U_P(-W)]$.

Proof. Through Lemma 4.2.2, C_m is weakly closed. Through Lemma 4.2.4 and Lemma 4.2.3, the Principal's value function is weakly upper semi-continuous. Lemma 4.2.1 shows F_ϵ is coercive in a . Finally, coercivity in W is obtained through the construction of F_ϵ . Indeed for any x on \mathbb{R}^+ $e^x - x^2 \geq 0$. Therefore, consider a sequence $(W_n, a_n)_n$ in C_m with $a_n \in [0, a_{max}]$ for any n in \mathbb{N} . As $W \geq m \mathbb{P} - a.s.$ (with $m \geq 0$) :

$$\lim_{n \rightarrow +\infty} \mathbb{E}[W_n^2] = +\infty \Rightarrow \lim_{n \rightarrow +\infty} \mathbb{E}[(\gamma_P W_n)^2] = +\infty \Rightarrow \lim_{n \rightarrow +\infty} \mathbb{E}[e^{\gamma_P W_n}] = +\infty. \quad (4.4.2)$$

As a consequence to the Participation Constraint, $\mathbb{E}[W_n] \geq y_{PC} + \kappa(a_n)$. Through Jensen's inequality (and denoting $\tilde{a} := \frac{1}{K}$) :

$$\begin{aligned} F_\epsilon(W_n, a_n) &= -\mathbb{E}\left[e^{-\gamma_P(X^{a_n} - W_n)}\right] - \epsilon \mathbb{E}\left[e^{\gamma_P W_n}\right] \\ &\leq -e^{-\gamma_P(x_0 + \mathbb{E}[B] - y_{PC} + a_n - \kappa(a_n))} - \epsilon \mathbb{E}\left[e^{\gamma_P W_n}\right] \\ &\leq -e^{-\gamma_P(x_0 + \mathbb{E}[B] - y_{PC} + \tilde{a} - \kappa(\tilde{a}))} - \epsilon \mathbb{E}\left[e^{\gamma_P W_n}\right]. \end{aligned}$$

Therefore using (4.4.2), $\lim_{n \rightarrow +\infty} \mathbb{E} [W_n^2] \Rightarrow \lim_{n \rightarrow +\infty} F_\epsilon(W_n, a_n) = -\infty$. Using coercivity in a , this yields : $\|(W_n, a_n)\|_E \rightarrow +\infty \Rightarrow \lim_{n \rightarrow +\infty} F_\epsilon(W_n, a_n) = +\infty$. Existence of maximizers is then obtained through Theorem 4.2.5. \square

Denote as (W_ϵ, a_ϵ) the optimal contract for perturbed Problem (4.4.1). As the perturbed value function is strongly convex, the contract is unique. The following Lemma is key and upper bounds the L^2 norm of the perturbed wage.

Lemma 4.4.2. *Let (W_ϵ, a_ϵ) be an ϵ -solution to Problem (4.4.1) for a fixed $0 < \epsilon < 1$. Then $\mathbb{E} [W_\epsilon^2] \leq C_{max}$, for C_{max} some constant.*

Proof. Consider any $0 < \epsilon < 1$. The unique ϵ -solution (W_ϵ, a_ϵ) can be characterized by applying a generalized K.K.T. as detailed in Section 4.3. As such there exists some $\lambda_\epsilon \geq 0$ and Z_ϵ in \mathcal{V} where $\mathcal{V} := \{ X \in L^2(\Omega), \quad X \geq 0 \quad \mathbb{P}\text{-a.s.} \}$, such that :

$$\begin{cases} \gamma_P e^{-\gamma_P(X^{a_\epsilon} - W_\epsilon)} + \epsilon \gamma_P e^{\gamma_P W_\epsilon} - Z_\epsilon - \lambda_\epsilon \gamma_A e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))} = 0 & (1) \\ -\gamma_P \mathbb{E}[e^{-\gamma_P(X^{a_\epsilon} - W_\epsilon)}] + a_\epsilon K \lambda_\epsilon \mathbb{E}[e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}] = 0 & (2) \\ \mathbb{E}[Z_\epsilon(W_\epsilon - m)] = 0 \quad \text{and} \quad \lambda_\epsilon \left(\mathbb{E}[e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}] - e^{-\gamma_A y} \right) = 0. & (3) \end{cases}$$

First suppose that $a_\epsilon = 0$ and/or $\lambda_\epsilon = 0$. Then in (2), $\mathbb{E}[e^{-\gamma_P(X^{a_\epsilon} - W_\epsilon)}] = 0$, which is absurd. In particular it must hold that $\mathbb{E}[e^{-\gamma_P(X^{a_\epsilon} - W_\epsilon)}] \geq R$ where $-R < 0$ is the corresponding Risk-Sharing optimum (see for example [93]). Therefore :

$$\gamma_P \mathbb{E}[e^{-\gamma_P(X^{a_\epsilon} - W_\epsilon)}] = a_\epsilon K \lambda_\epsilon \mathbb{E}[e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}],$$

implying that for any fixed $\epsilon > 0$, $a_\epsilon \lambda_\epsilon = \frac{\gamma_P \mathbb{E}[e^{-\gamma_P(X^{a_\epsilon} - W_\epsilon)}]}{K \mathbb{E}[e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}]} \geq \frac{R}{K e^{-\gamma_A y}} > 0$. Recall that by construction $\lambda_\epsilon \geq 0$. It follows that for any $\epsilon > 0$, $a_\epsilon > 0$ and $\lambda_\epsilon > 0$ and as a consequence the Participation Constraint binds. Through (1) and as $(y, 0)$ belongs to C_m it holds that : $0 \leq \lambda_\epsilon \leq \frac{\gamma_P \mathbb{E}[e^{-\gamma_P(X^0 - y)} + e^{\gamma_P y_{PC}}]}{\gamma_A e^{-\gamma_A y_{PC}}} := C$, and it follows that $a_\epsilon \geq \frac{R}{K C e^{-\gamma_A y_{PC}}} > 0$. Now, as both Z_ϵ and $W_\epsilon - m$ are positive :

$$\mathbb{E}[Z_\epsilon(W_\epsilon - m)] = 0 \Rightarrow Z_\epsilon(W_\epsilon - m) = 0 \quad \mathbb{P}\text{-a.s.}$$

Therefore as soon as $Z_\epsilon > 0$, it must hold that $W_\epsilon = m$ and as a consequence, $0 \leq Z_\epsilon \leq Z_\epsilon \mathbf{1}_{W_\epsilon = m} \quad \mathbb{P}\text{-a.s.}$ Using (1) :

$$\begin{aligned} 0 \leq Z_\epsilon &\leq Z_\epsilon \mathbf{1}_{W_\epsilon = m} \leq \gamma_P e^{-\gamma_P(X^{a_\epsilon} - m)} + \epsilon \gamma_P e^{\gamma_P m} \\ &\leq \gamma_P e^{-\gamma_P(X^0 - m)} + \gamma_P e^{\gamma_P m}, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

and finally :

$$\begin{aligned} 0 \leq m \leq W_\epsilon &= \frac{1}{\gamma_P} \ln \left(\frac{Z_\epsilon + \lambda_\epsilon e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}}{\gamma_P(\epsilon + e^{-\gamma_P X^{a_\epsilon}})} \right) \\ &\leq \frac{1}{\gamma_P} \ln \left(\frac{Z_\epsilon + \lambda_\epsilon e^{-\gamma_A(W_\epsilon - \kappa(a_\epsilon))}}{\gamma_P e^{-\gamma_P X^{a_\epsilon}}} \right) \\ &\leq \frac{1}{\gamma_P} \ln \left(\frac{\gamma_P e^{-\gamma_P(X^0 - m)} + \gamma_P e^{\gamma_P m} + C e^{-\gamma_A(m - \kappa(a_{max}))}}{\gamma_P e^{-\gamma_P X^{a_{max}}}} \right), \end{aligned}$$

and for any $0 < \epsilon < 1$ it holds that :

$$\mathbb{E} [W_\epsilon^2] \leq \frac{1}{\gamma_P^2} \mathbb{E} \left[\ln \left(\frac{\gamma_P e^{-\gamma_P(X^0 - m)} + \gamma_P e^{\gamma_P m} + C e^{-\gamma_A(m - \kappa(a_{max}))}}{\gamma_P e^{-\gamma_P X^{a_{max}}}} \right)^2 \right] = C_{max}.$$

□

With these tools in mind the proof of Corollary 4.2.1 is quite direct. Indeed, the assumptions of the second part of Theorem 4.2.6 (point 2.) are satisfied and the result follows. The perturbation method used in this reasoning resembles that of the Tikhonov method for ill-posed problems ([127]), albeit for a different perturbation term. In fact (W_ϵ, a_ϵ) makes $\mathbb{E}[\exp(-\gamma_P(X^{a_\epsilon} - W_\epsilon))]$ as small as possible without $\mathbb{E}[\exp(\gamma_P W_\epsilon)]$ becoming too big, and thus without $\mathbb{E}[W_\epsilon^2]$ becoming too big. Lemma 4.4.2 formalizes this by showing that the L^2 -norm of W_ϵ is upper-bounded for ϵ smaller than 1

Chapter 5

A class of explicit optimal contracts in the face of shutdown

This Chapter is largely based on a recent work conducted with Stéphane Villeneuve. It is a slightly modified version of the preprint [94].

5.1 Economic shutdown in recent history and contracts

Without seeking to oppose public health and economic growth, there is no doubt that the management of the Covid crisis had serious consequences on entire sectors of the economy. The first few months of 2020 will go down in world history as a period of time characterized by massive layoffs, forced closures of non-essential companies, disruption of cross-border transportation whilst populations were subject to lockdown and/or social distancing measures and hospitals and the medical world struggled to get a grasp on the Sars-Cov-2 pandemic. Whilst the immediate priority was saving lives, decongesting hospitals and preventing the spread of the disease, many extraordinary economic support measures were taken to help businesses and individuals stay afloat during these unprecedented times and in the hope of tempering the economic crisis that would follow. Although the world has lived through many crises over the past centuries, from several Panics in the 1800s and the Great Depression of the 1930s to the more recent Financial Crisis of 2008, never before has the global economy as a whole come to such a standstill due to an external event. Such large shutdown risks do not only materialize during pandemics but throughout many major other large events. The massive bushfires that affected Australia towards the end of 2019, temporarily halting agriculture, construction activity and tourism in some areas of the country are another recent example. As we begin to see a glimpse of hope for a way out through vaccination, the focus is turning to building the world of tomorrow with the idea that we must learn to live with such risks. This Chapter tries to make its contribution by focusing on a simple microeconomic issue. In a world subject to moral hazard, how can we agree to an incentive contract whose obligations could be made impossible or at least very difficult because of the occurrence of a risk of the nature of the Covid19 pandemic? Including such a shutdown risk-sharing in contracts seems crucial going forward for at least two reasons. First, it is not certain that public authority will be able to continue to take significant economic support measures to insure the partners

of a contract if the frequency of such global risks were to increase. On the other hand, the private insurance market does not offer protection against the risk of a pandemic which makes pooling too difficult. It therefore seems likely that we will have to turn to an organized form of risk sharing between the contractors.

Economic theory has a well-developed set of tools to analyze incentive and risk-sharing problems using expected-utility theory. Most of the now abundant literature related to dynamic contracting through a Principal-Agent model has, so far, mostly been based on continuously governed (eg. Brownian motion) output-processes. This was the case of the foundational work of Holmström and Milgrom [58] and many of its many extensions such as those of Schattler and Sung in [118] and [119], and the more recent contributions of Sannikov in [116], and Cvitanic et al [36] and [32]. However some relatively recent works have introduced jump processes into continuous time contracting. Biais et al. were the first to do so in [14] where they study optimal contracting between an insurance company and a manager whose effort can reduce the occurrence of an underlying accident. In a similar vein, work by Capponi and Frei [23] also used a jump-diffusion driven outcome process in order to include the possibility for accidents to negatively affect revenue. Here, we extend the classical Holmström and Milgrom [58] framework to include a shutdown risk. We do not claim that this model with CARA preferences is general enough to come up with robust economic facts, but it has the remarkable advantage of being explicitly creditworthy, which allows us to find an explicit optimal contract that disentangles the incentives from external risk-sharing and allows us to understand the sensitivity of the optimal contract to the different exogenous parameters of the model. Our work uses a jump-diffusion process too and presents some structural similarities with [23] : both models consider a risk averse principal and agent with exponential utility and reach an explicit characterization of the optimal wage. However, Capponi and Frei combine the continuous part of the diffusion and the accident jump process additively and they are able to allow prevention through intensity control of the jump process. This makes sense as many accidents are preventable through known measures. Our framework uses a different form of jump diffusion to enable the shutdown event to completely stop revenue generation in a continuous-time setting. This is done by building on a standard continuous Brownian-motion based output process. The main novelty is the multiplicative effect of the jump risk : upon the arrival of the risk, the whole of the output process comes to a halt. As an extension, we allow the halt to no longer be a complete fatality : production may continue at a degraded level through an investment by the principal. From a methodological viewpoint, our reasoning uses a now standard method in dynamic contracting based on [116] and [32] which consists in transforming both the first-best and second-best problems into classical Markovian control problems. The solution to these control problems can be characterized through a Hamilton-Jacobi-Bellman equation. Quite remarkably, this equation has, in our context, an explicit solution that is closely linked to a so-called Bernoulli ODE which facilitates many extensions.

To the best of our knowledge, this Chapter is the first to explicitly introduce a default in a dynamic Principal-agent framework, in both a first-best (also called full Risk-Sharing) and second-best (also called Moral Hazard) setting. A key feature of our study is that the shape of the optimal contract is linear. More precisely, the agent's compensation is the sum of two functions: the first is linear with respect to the output and serves to give the incentives, while the second is linear with respect to the effective duration of the contract and serves to share

the default risk. While the linear incentive part of the contract is in line with the existing literature on continuous-time Principal-Agent problems without default under exponential utilities, the risk-sharing part deserves some clarification. The contract exposes both agents to a risk of exogenous interruption but it has two different regimes that are determined by an explicit relation between the risk-aversions and the agent's effort cost. Under the first regime, the agent is more sensitive to the risk of default than the principal. In this case, the principal deposits on the date 0 a positive amount onto an escrow account whose balance will then decrease over time at a constant rate. It is crucial to observe that the later the default arrives, the more the amount in the escrow account decreases to a point where it may even become negative. If the default occurs, the principal transfers the remaining balance to the agent. Under the second regime, the principal is more sensitive to the risk of default. In this case, the principal deposits a negative amount into the escrow account, which now grows at a constant rate and symmetrical reasoning applies. This linearity contrasts with the optimum obtained in [23] as the additive contribution of their jump process to revenue generation leads to a sub-linear wage. This result is coherent with the work by Hoffman and Pfeil [56] which proves that, in line with the empirical studies by Bertrand and Mullainathan [13], the agent must be rewarded or punished for a risk that is beyond his control.

Finally, this Chapter also explicitly characterizes the optimal contract when a possibility for shutdown risk mitigation exists at a cost. Such a possibility is coherent with agency-free external risk: prevention is not possible, at least on a short-term or medium-term time scale. At best the principal may be able to invest to mitigate its effects. Crucially we find that in many circumstances, investing is not optimal for the principal. When it is, it is only optimal up until some cutoff time related to a balance between the cost of investment, the agent's rents and possible remaining gain.

The rest of the Chapter is structured as follows. In Section 5.2, we present the model and the Principal-Agent problems that we consider. In Section 5.3, we analyse the first-best case where the Principal observes the Agent's effort. Then in Section 5.4, we give our main results and analysis. In Section 5.5, we extend our model to include a possibility for mitigation upon a halt.

5.2 An extension of a classical model

The model is inherited from the classical work of Holmström and Milgrom [58] developed in Chapter 1. A Principal contracts with an Agent to manage a project she owns. The Agent influences the project's profitability by exerting an unobservable effort. For a fixed effort policy, the output process is still random and the idiosyncratic uncertainty is modeled by a Brownian motion. We assume that the contract matures at time $T > 0$ and both Principal and Agent are risk-averse with CARA preferences. The departure from the classical model is as follows : we assume the project is facing some external risk that could partially or totally interrupt the production at some random time τ . The probability distribution of τ is assumed to be independent of the Brownian motion that drives the uncertainty of the output process and also independent of the Agent's actions. Finally, we assume that the contract offers a transfer W at time T from the Principal to the Agent that is a functional of the output process.

5.2.1 Probability setup

Let $T > 0$ be some fixed time horizon. The key to modeling our Principal-Agent problems under an agency-free external risk of default is the simultaneous presence over the interval $[0, T]$ of a continuous random process and a jump process as well as the ability to extend the standard mathematical techniques used for dynamic contracting to this mixed setting. Thus, we shall deal with two kinds of information : the information from the output process, denoted as $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and the information from the default time, i.e. the knowledge of the time where the default occurred in the past, if the default has appeared. This construction is not new and occurs frequently in mathematical finance¹.

The complete probability space that we consider will be denoted as $(\Omega, \mathcal{G}, \mathbb{P}^0)$, with two independent stochastic processes :

- B a standard one-dimensional \mathbb{F} -Brownian motion,
- N the right-continuous single-jump process defined as $N_t = \mathbf{1}_{\tau \leq t}$, t in $[0, T]$ where τ is some positive random variable independent of B that models the default time.

N will also be referred to as the default indicator process. We therefore use the standard approach of progressive enlargement of filtration by considering $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ the smallest complete right-continuous extension of \mathbb{F} that makes τ a \mathbb{G} -stopping time. Because τ is independent of B , B is a \mathbb{G} -Brownian motion under \mathbb{P}^0 according to Proposition 1.21 p 11 in [2]. We also suppose that there exists a bounded deterministic compensator of N , $\Lambda_t = \int_0^t \lambda(s) ds$ for some bounded function $\lambda(\cdot)$ called the intensity implying that:

$$M_t = N_t - \int_0^t \lambda(s)(1 - N_s) ds, \quad t \in [0, T]$$

is a \mathbb{G} -compensated martingale. Note that through knowledge of λ , the Principal and Agent can compute at time 0 the probability of default happening over the contracting period $[0, T]$. Indeed :

$$\mathbb{P}(\tau \leq T) = 1 - \exp(-\Lambda_T).$$

We first suppose for computational ease that the intensity λ is a constant. We will see in Section 5.4.3 that our results may easily be lifted to more general deterministic compensators.

Remark 5.2.1. *Here we will suppose that the compensator of N is common knowledge to both the Principal and the Agent. We could imagine settings where the Principal and Agent's beliefs regarding the risk of default may differ : this natural extension of our work would call for analysis of the dynamic contracting problem under hidden information which is left for future research.*

5.2.2 Principal-Agent Problem

We suppose that the Agent agrees to work for the Principal over a time period $[0, T]$ and provide up to the default time a costly action $(a_t)_{t \in [0, T]}$ belonging to \mathcal{A} , where \mathcal{A} denotes the set of admissible \mathbb{F} -predictable strategies that will be specified later on. The Principal-Agent

¹We refer the curious reader to the two important references [2] and [72].

problem models the realistic setting where the Principal cannot observe the Agent's effort. As such the Agent chooses his action in order to maximize his own utility. The Principal must offer a wage based on the information driven by the output process up to the default time that incentivizes the Agent to work efficiently and contribute positively to the output process. Mathematically, the unobservability of the Agent's behaviour is modeled through a change of measure. Under \mathbb{P}^0 , we assume that the project's profitability evolves as

$$X_t := x_0 + \int_0^t (1 - N_s) dB_s.$$

Thus, \mathbb{P}^0 corresponds to the probability distribution of the profitability when the Agent makes no effort over $[0, T]$. When the Agent makes an effort $a = (a_t)_t$, we shall assume that the project's profitability evolves as

$$X_t := x_0 + \int_0^t a_s(1 - N_s) ds + \int_0^t (1 - N_s) dB_s^a,$$

where B^a is a \mathbb{F} -Brownian motion under a measure \mathbb{P}^a . The Agent fully observes the decomposition of the production process under a measure \mathbb{P}^a whilst the Principal only observes the realization of X_t . In order for the model to be consistent, the probabilities \mathbb{P}^0 and \mathbb{P}^a must be equivalent for all $(a_t)_{t \in [0, T \wedge \tau]}$ belonging to \mathcal{A} . Therefore, we introduce the following set of actions

$$\mathcal{B} = \left\{ a = (a_t)_t : \mathbb{F}\text{-predictable and taking values in } [-A, A] \text{ for some } A > 0 \right\}.$$

The action process in \mathcal{B} are uniformly bounded by some fixed constant $A > 0$ that will be assumed as large as necessary. For $a \in \mathcal{B}$, we define \mathbb{P}^a as

$$\frac{d\mathbb{P}^a}{d\mathbb{P}^0} |_{\mathcal{G}_T} = \exp \left(\int_0^T a_s(1 - N_s) dB_s - \frac{1}{2} \int_0^T |a_s|^2 (1 - N_s) ds \right) := L_T.$$

Because $\mathbb{E}^0(L_T) = 1$, $(B_t^a)_{t \in [0, T]}$ with $B_t^a = B_t - \int_0^t a_s(1 - N_s) ds, t \in [0, T]$ is a \mathbb{G} -Brownian motion under \mathbb{P}^a according to Proposition 3.6 c) p 55 in [2]. It is key to note that if halt occurs, i.e. if $\tau \leq T$, then the production process is halted before T meaning that : $X_{t \wedge \tau}^a = X_t^a, t \in [0, T]$. Let us then observe that an action $a = (a_t)_t$ of \mathcal{B} can be extended to a \mathbb{G} -predictable process $(\tilde{a}_t)_{t \in [0, T]}$ by setting $\tilde{a}_t = a_t \mathbb{1}_{t \leq \tau}$.

The cost of effort for the Agent is modeled through a quadratic cost function : $\kappa(a) := \kappa \frac{a^2}{2}$, for $\kappa > 0$ some fixed parameter. As a reward for the Agent's effort, the Principal pays him a wage W at time T . W is assumed to be a $\mathcal{G}_{T \wedge \tau}$ random variable which means that the payment at time T in case of an early default is known at time τ . The Principal and the Agent are considered to be risk averse and this risk aversion is modeled through two CARA utility functions :

$$U_P(x) := -\exp(-\gamma_P x) \text{ and } U_A(x) := -\exp(-\gamma_A x),$$

where $\gamma_P > 0$ and $\gamma_A > 0$ are two fixed constants modeling the Principal's and the Agent's risk aversion.

In this setting and for any given wage W , the Agent maximizes his own utility and solves :

$$V_0^A(W) = \sup_{a \in \mathcal{B}} \mathbb{E}^a \left[U_A \left(W - \int_0^T \kappa(a_s(1 - N_s)) ds \right) \right]. \quad (5.2.1)$$

A wage W is said to be incentive compatible if there exists an action policy $a^*(W) \in \mathcal{B}$ that maximises (5.2.1) and thus satisfies

$$V_0^A(W) = \mathbb{E}^{a^*(W)} \left[U_A \left(W - \int_0^T \kappa(a_s^*(W)(1 - N_s)) ds \right) \right].$$

When the Principal is able to offer an incentive compatible wage W , she knows what the Agent's best reply will be. As such the Principal establishes a set $\mathcal{A}^*(W) \subset \mathcal{B}$ of best replies for the Agent for any incentive compatible W . Therefore, the first task is to characterize the set of incentive-compatible wages \mathcal{W}_{IC} . Only then may the Principal consider maximizing his own utility by solving :

$$\sup_{W \in \mathcal{W}_{IC}} \sup_{a^* \in \mathcal{A}^*(W)} \mathbb{E}^{a^*(W)} \left[U_P \left(X_T^{a^*(W)} - W \right) \right] \quad (5.2.2)$$

under the participation constraint

$$\mathbb{E}^{a^*(W)} \left[U_A \left(W - \int_0^T \kappa(a_s^*(W)(1 - N_s)) ds \right) \right] \geq U_A(y_{PC}), \quad (5.2.3)$$

where y_{PC} is a monetary reservation utility for the Agent.

Remark 5.2.2. *We would like to observe that although our work was initially motivated by observing the economic consequences of a pandemic, our model is very much suited to contracting for other risky settings. One such setting is that of start ups. Indeed, the rate of failure amongst such new ventures is massive : empirical studies estimate the rate of failure to be between 50% and 90%. The risks leading to failure are not all controllable, sometimes survival of a start up just comes down to luck : being in the right place at the right time, being the one of several similar start ups who manages to breakthrough etc. It is therefore relevant, when it comes to designing optimal contracting policies for new ventures, to bear this risk of failure in mind. Our contracting model allows that either in its simple constant intensity form, or using a deterministic intensity that could for example model an improved survival rate as time goes on.*

Of course a high risk of default is not only something faced by start ups. In fact some industries as a whole, such as the real estate industry, also have a high underlying risk of default for many reasons. Companies that are tied to such industries without perhaps being able to control the cause of default may wish to be able to adjust their contracts accordingly which is possible through this model.

5.3 Optimal Risk-Sharing contract

We begin with analysis of the first-best benchmark (the full Risk-Sharing problem) which leads to a simple optimal sharing rule. Of course this problem is not the most realistic when

it comes to modeling dynamic contracting situations. However it provides a benchmark to which we can compare the more realistic Moral Hazard situation. Indeed, the Principal's utility in the full Risk-Sharing problem is the best that the Principal will ever be able to obtain in a contracting situation as he may observe (and it is thus assumed that he may dictate) the Agent's action.

To write the first-best problem, we assume that both the Principal and the Agent observe the variations of the same production process $(X_t^a)_{t \in [0, T]}$ under \mathbb{P}^0 :

$$X_t^a := x_0 + \int_0^t a_s(1 - N_s)ds + \int_0^t (1 - N_s)dB_s. \quad t \in [0, T] \quad (5.3.1)$$

The Agent is guaranteed a minimum value of expected utility through the participation constraint :

$$\mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_s(1 - N_s))ds \right) \right] \geq U_A(y_{PC}), \quad (5.3.2)$$

but has no further say on the wage or action. Consider the admissible set :

$$\mathcal{A}_{PC} := \left\{ (W, a) \text{ such that } W \text{ is } \mathcal{G}_{T \wedge \tau} \text{ measurable with } \mathbb{E}[\exp(-2\gamma_A W)] < +\infty, \right. \\ \left. (a_t)_t \in \mathcal{B}, \text{ and (5.3.2) is satisfied} \right\}.$$

The full Risk-Sharing problem involves maximizing the Principal's utility across \mathcal{A}_{PC} :

$$\sup_{(W, a) \in \mathcal{A}_{PC}} \mathbb{E}[U_P(X_T^a - W)]. \quad (5.3.3)$$

5.3.1 Tackling the Participation Constraint (5.3.2)

A first step to optimal contracting in this first-best setting involves answering the following question: can we characterize the set \mathcal{A}_{PC} ? Following the standard route, we will first establish a necessary condition. For a given pair $(W, a) \in \mathcal{A}_{PC}$, let us introduce the Agent's continuation utility $(U_t^{(W, a)})_t$ as follows:

$$U_t^{(W, a)} := \mathbb{E}_t \left[U_A \left(W - \int_t^T \kappa(a_s(1 - N_s))ds \right) \right],$$

where we use the shorthand notation : $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{G}_t]$. We may write the Agent's continuation value process as the product :

$$U_t^{(W, a)} = \mathcal{M}_t^{(W, a)} \mathcal{D}_t^{(W, a)},$$

where :

$$\mathcal{M}_t^{(W, a)} := \mathbb{E}_t \left[U_A \left(W - \int_0^T \kappa(a_s(1 - N_s))ds \right) \right] \quad \text{and} \quad \mathcal{D}_t^{(W, a)} := \exp \left(-\gamma_A \int_0^t \kappa(a_s(1 - N_s))ds \right).$$

Observe that for any admissible pair $(W, a) \in \mathcal{A}_{PC}$, the process $\mathcal{M} = (\mathcal{M}_t^{(W,a)})_t$ is a \mathbb{G} -square integrable martingale. According to the Martingale Representation Theorem for \mathbb{G} -martingales (see [2], Theorem 3.12 p. 60), there exists some predictable pair (z_s, l_s) in $\mathbb{H}^2 \times \mathbb{H}^2$, where \mathbb{H}^2 is the set of \mathbb{F} -predictable processes Z with $\mathbb{E} \left[\int_0^T |Z_t|^2 dt \right] < +\infty$, such that :

$$\mathcal{M}_t^{(W,a)} := \mathcal{M}_0^{(W,a)} + \int_0^t z_s(1 - N_s)dB_s + \int_0^t l_s(1 - N_s)dM_s.$$

Integration by parts yields the dynamic of U , noting that \mathcal{D} has finite variation :

$$dU_t^{(W,a)} = -\gamma_A \kappa(a_t(1 - N_s))U_t^{(W,a)} dt + \mathcal{D}_t^{(W,a)} z_t(1 - N_s)dB_t + \mathcal{D}_t^{(W,a)} l_t(1 - N_s)dM_t.$$

Setting $Z_t^{(W,a)} := \mathcal{D}_t^{(W,a)} z_t \in \mathbb{H}^2$ and $K_t^{(W,a)} := \mathcal{D}_t^{(W,a)} l_t \in \mathbb{H}^2$, we obtain:

$$dU_t^{(W,a)} = -\gamma_A \kappa(a_t(1 - N_s))U_t^{(W,a)} dt + Z_t^{(W,a)}(1 - N_s)dB_t + K_t^{(W,a)}(1 - N_s)dM_t.$$

By construction, we have that $U_T^{(W,a)} = U_A(W)$. It follows that $(U_t^{(W,a)}, Z_t^{(W,a)}, K_t^{(W,a)})$ is a solution to the BSDE:

$$-dU_t^{(W,a)} = -Z_t^{(W,a)}(1 - N_s)dB_t - K_t^{(W,a)}(1 - N_s)dM_t + \gamma_A \kappa(a_t(1 - N_s))U_t^{(W,a)} dt, \quad (5.3.4)$$

with $U_T^{(W,a)} = U_A(W)$. Therefore, (5.3.2) is satisfied if and only if $U_0^{(W,a)} \geq U_A(y_{PC})$.

Remark 5.3.1. Let \mathbb{S}^2 be the set of \mathbb{G} -adapted RCLL processes U such that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |U_t|^2 \right] < +\infty.$$

Through Proposition 2.6 of [45], the solution to (5.3.4) is unique in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$. Indeed, the driver $g(\omega, U) = \gamma_A \kappa(a_t(1 - N_t))U$ is uniformly Lipschitz in U because $(a_t)_t$ is bounded and the terminal condition is in L^2 .

To sum up, we have the following necessary condition for admissibility.

Lemma 5.3.1. *If $(W, a) \in \mathcal{A}_{PC}$ then there exists a unique solution $(U_t^{(W,a)}, Z_t^{(W,a)}, K_t^{(W,a)})$ in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$ to the BSDE (5.3.4) such that $U_0^{(W,a)} \geq U_A(y_{PC})$.*

To obtain a sufficient condition, we introduce, for $\pi = (y_0, a, \beta, H) \in \mathbb{R} \times \mathcal{B} \times \mathbb{H}^2 \times \mathbb{H}^2$, the wage process $(W_t^\pi)_t$ defined as

$$\begin{aligned} W_t^\pi := & y_0 + \int_0^t \beta_s(1 - N_s)dB_s + \int_0^t H_s(1 - N_s)dM_s + \int_0^t \left\{ \frac{\gamma_A}{2} \beta_s^2(1 - N_s) + \kappa(a_s(1 - N_s)) \right. \\ & \left. - \frac{\lambda}{\gamma_A} [\exp(-\gamma_A H_s) - 1 + \gamma_A H_s](1 - N_s) \right\} ds, \end{aligned} \quad (5.3.5)$$

and consider the set

$$\Gamma := \left\{ (y_0, a, \beta, H) \in \mathbb{R} \times \mathcal{B} \times \mathbb{H}^2 \times \mathbb{H}^2 \text{ such that } y_0 \geq y_{PC} \text{ and } \mathbb{E} [\exp(-2\gamma_A W_T^\pi)] < +\infty. \right\}.$$

We have the following result.

Lemma 5.3.2. *For any $\pi \in \Gamma$, the pair (W_T^π, a) belongs to \mathcal{A}_{PC} .*

Proof. We apply Itô's formula to the process $Y_t^\pi = U_A(W_t^\pi)$ to obtain

$$dY_t^\pi = -\gamma_A Y_t^\pi \beta_t (1 - N_t) dB_t + Y_t^\pi \left(e^{-\gamma_A H_t} - 1 \right) (1 - N_t) dM_t - \gamma_A \kappa(a_t(1 - N_t)) Y_t^\pi dt.$$

Moreover, because $\pi \in \Gamma$, $Y_T^\pi = U_A(W_T^\pi)$ is square-integrable. Remark 5.3.1 yields the triplet $(Y_t^\pi, -\gamma_A Y_t^\pi \beta_t, Y_t^\pi (e^{-\gamma_A H_t} - 1))$ is the unique solution in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$ to BSDE (5.3.4) with terminal condition $U_A(W_T^\pi)$ when $\pi \in \Gamma$. Therefore,

$$Y_0^\pi = U_A(y_0) = \mathbb{E} \left[U_A \left(W_T^\pi - \int_0^T \kappa(a_s(1 - N_s)) ds \right) \right] \geq U_A(y_{PC}),$$

and thus (5.3.2) is satisfied. \square

Remark 5.3.2. *The admissible contracts are essentially the terminal values of the controlled processes (5.3.5) for $\pi \in \Gamma$. The difficulty is that we do not know how to characterize the β and H processes that guarantee that π belongs to Γ . Nevertheless, it is easy to check by a standard application of the Gronwall lemma that if β and H are bounded then $\pi \in \Gamma$. This last observation will prove to be crucial in the explicit resolution of our problem.*

5.3.2 Dynamic contracting under Risk-Sharing

Using Lemma 5.3.2, the full Risk-Sharing problem under default writes as the Markovian control problem :

$$V_P^{FB} := \sup_{\pi=(y_0, a, Z, K) \in \Gamma} \mathbb{E} \left[U_P \left(X_T^{(x_0, a)} - W_T^\pi \right) \right], \quad (5.3.6)$$

where $X_t^{(x_0, a)}$ is given by :

$$dX_s^{(x_0, a)} = a_s(1 - N_s) ds + (1 - N_s) dB_s,$$

with $X_0^{(x_0, a)} = x_0$ and the wage process is given by :

$$dW_s^\pi = Z_s(1 - N_s) dB_s + K_s(1 - N_s) dM_s + \left\{ \frac{\gamma_A}{2} Z_s^2(1 - N_s) + \kappa(a_s(1 - N_s)) + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K_s) - 1 + \gamma_A K_s](1 - N_s) \right\} ds,$$

with $W_0^\pi = y_0$.

We have the following key theorem for the first-best problem.

Theorem 5.3.1. *Let $a_t^* = \frac{1}{\kappa}$, $Z_t^* = \frac{\gamma_P}{\gamma_P + \gamma_A}$, and let :*

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

where :

$$\Phi_0(t) := \left(\frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}}, \quad (5.3.7)$$

with :

$$c_1 := \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2 := \lambda \frac{\gamma_P + \gamma_A}{\gamma_A}.$$

Then $\pi^* = (y_{PC}, a^*, Z^*, K^*) \in \Gamma$ parameterizes the optimal contract $(W_T^{\pi^*}, a^*)$ for the first-best problem.

The rest of this subsection is dedicated to the proof of this Theorem. We first make the following observation. As X remains constant after τ , the Principal has no further decision to make after the default time. Thus, its value function is constant and equal to $U_P(x - y)$ on the interval $[\tau, T]$. We now focus on the control part of the problem (i.e. computation of the optimal control triplet $\tilde{\pi} = (a, Z, K)$ for a given pair (x_0, y_0)). To do so, we follow the dynamic programming approach developed in [110], Section 4 to define the value function

$$V(0, x_0, y_0) = \sup_{\tilde{\pi} \in \tilde{\Gamma}} \mathbb{E} \left[U_P(X_T^a - W_T^{\tilde{\pi}})(1 - N_T) + \int_0^T U_P(X_t^a - W_t^{\tilde{\pi}}) \lambda e^{-\lambda t} dt \right], \quad (5.3.8)$$

where

$$\tilde{\Gamma} = \left\{ \tilde{\pi} \in \mathcal{B} \times \mathbb{H}^2 \times \mathbb{H}^2 \right\},$$

Because $\Gamma \subset \mathbb{R} \times \tilde{\Gamma}$, we have

$$V_P^{FB} \leq \sup_{y_0 \geq y_{PC}} V(0, x_0, y_0).$$

According to stochastic control theory, the Hamilton-Jacobi-Bellman equation associated to the stochastic control problem (5.3.8) is the following (see [137]):

$$\begin{aligned} & \partial_t v(t, x, y) + \sup_{a, Z, K} \left\{ \partial_x v(t, x, y) a + \partial_y v(t, x, y) \left[\frac{\gamma_A}{2} Z^2 + \kappa(a) + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K) - 1] \right] \right. \\ & \left. + \lambda [U_P(x - y - K) - v_0(t, x, y)] + \partial_{yy} v(t, x, y) \frac{Z^2}{2} + \frac{1}{2} \partial_{xx} v(t, x, y) + \partial_{xy} v(t, x, y) Z \right\} = 0, \end{aligned} \quad (5.3.9)$$

with the boundary condition :

$$v(T, x, y) = U_P(x - y).$$

It happens that the HJB equation (5.3.9) is explicitly solvable by exploiting the separability property of the exponential utility function.

Lemma 5.3.3. *The function $v(t, x, y) = U_P(x - y)\Phi_0(t)$ with :*

$$\Phi_0(t) = \left(\frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}},$$

where :

$$c_1 = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2 = \lambda \frac{\gamma_P + \gamma_A}{\gamma_A},$$

solves (in the classical sense) the HJB partial differential equation (5.3.9).

Furthermore $a_t^* = \frac{1}{\kappa}$, $Z_t^* = \frac{\gamma_P}{\gamma_P + \gamma_A}$ and $K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t))$ are the optimal controls.

Proof. We search for a solution to Equation (5.3.9) for a v of the form :

$$v(t, x, y) = U_P(x - y)\Phi_0(t),$$

with Φ_0 a positive mapping. Such a v satisfies (5.3.9) if and only if $\Phi_0(t)$ solves the PDE :

$$\begin{aligned} \Phi_0'(t) + \inf_{a, Z, K} \left\{ -\gamma_P \Phi_0(t)a + \gamma_P \Phi_0(t) \left(\frac{\gamma_A}{2} Z^2 + \kappa(a) + \frac{\lambda}{\gamma_A} \{\exp(-\gamma_A K) - 1\} \right) \right. \\ \left. + \gamma_P^2 \Phi_0(t) \frac{Z^2}{2} + \frac{\gamma_P^2}{2} \Phi_0(t) - \gamma_P^2 \Phi_0(t)Z + \lambda (\exp(\gamma_P K) - \Phi_0(t)) \right\} = 0, \end{aligned}$$

with the boundary condition $\Phi_0(T) = 1$. As Φ_0 is a positive mapping, the infimum is well defined. We derive the following first order conditions that must be satisfied by the optimal controls :

$$\begin{cases} \gamma_P \Phi_0(t) = \gamma_P \kappa a \Phi_0(t) \\ \gamma_P \Phi_0(t) Z (\gamma_A + \gamma_P) = \gamma_P^2 \Phi_0(t) \\ \gamma_P \Phi_0(t) \lambda \exp(-\gamma_A K) = \gamma_P \lambda \exp(\gamma_P K), \end{cases}$$

equating to :

$$a^* = \frac{1}{\kappa}, \quad Z^* = \frac{\gamma_P}{\gamma_P + \gamma_A}, \quad K^* = \frac{\log(\Phi_0(t))}{\gamma_P + \gamma_A}.$$

It follows that :

$$\begin{aligned} & \inf_{a, Z, K} \left\{ -\gamma_P \Phi_0(t)a + \gamma_P \Phi_0(t) \left(\frac{\gamma_A}{2} Z^2 + \kappa(a) + \frac{\lambda}{\gamma_A} \{\exp(-\gamma_A K) - 1\} \right) \right. \\ & \left. + \gamma_P^2 \Phi_0(t) \frac{Z^2}{2} + \frac{\gamma_P^2}{2} \Phi_0(t) - \gamma_P^2 \Phi_0(t)Z + \lambda (\exp(\gamma_P K) - \Phi_0(t)) \right\} \\ & = -\gamma_P \Phi_0(t)a^* + \gamma_P \Phi_0(t) \left(\frac{\gamma_A}{2} Z^{*2} + \kappa(a^*) + \frac{\lambda}{\gamma_A} \{\exp(-\gamma_A K^*) - 1\} \right) \\ & + \gamma_P^2 \Phi_0(t) \frac{Z^{*2}}{2} + \frac{\gamma_P^2}{2} \Phi_0(t) - \gamma_P^2 \Phi_0(t)Z^* + \lambda (\exp(\gamma_P K^*) - \Phi_0(t)) \\ & = \underbrace{\Phi_0(t) \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A)}}_{\text{terms with } Z^*} \underbrace{- \Phi_0(t) \frac{\gamma_P}{2\kappa}}_{\text{terms with } a^*} \underbrace{- \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \Phi_0(t) + \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \Phi_0(t) \frac{\gamma_P}{\gamma_P + \gamma_A}}_{\text{terms with } K^*}. \end{aligned}$$

We may inject this expression back into the PDE on Φ_0 . Doing so yields the following Bernoulli equation :

$$\Phi_0'(t) + c_1 \Phi_0(t) + c_2 \Phi_0(t) \frac{\gamma_P}{\gamma_P + \gamma_A} = 0, \quad \Phi_0(T) = 1,$$

where

$$c_1 = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2 = \lambda \frac{\gamma_P + \gamma_A}{\gamma_A}.$$

The unique solution to this equation is (see for instance [136]) :

$$\Phi_0(t) = \left(\frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}},$$

and the result follows. \square

Proof of Theorem 5.3.1. The value function $v(t, x, y) = U_P(x - y)\Phi_0(t)$ is a classical solution to the HJB equation (5.3.9). A standard verification theorem yields that $v = V$. Through Lemma 5.3.3, the optimal controls for the full Risk-Sharing problem are :

$$a_t^* = \frac{1}{\kappa}, Z_t^* = \frac{\gamma_P}{\gamma_P + \gamma_A} \text{ and } K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

with Φ_0 as defined in Lemma 5.3.3. These controls are free of y and it follows that :

$$V(0, x_0, y_0) = E \left[U_P \left(X_T^{(x_0, a^*)} - W_T^{(y_0, a^*, Z^*, K^*)} \right) \right],$$

is a decreasing function of y_0 . Thus we obtain

$$\sup_{y_0 \geq y_{PC}} V(0, x_0, y_0) = E \left[U_P \left(X_T^{(x_0, a^*)} - W_T^{(y_{PC}, a^*, Z^*, K^*)} \right) \right].$$

Finally, we observe that the optimal controls are bounded and thus Remark (5.3.2) yields $\pi^* = (y_{PC}, a^*, Z^*, K^*) \in \Gamma$. As a consequence,

$$\sup_{y_0 \geq y_{PC}} V(0, x_0, y_0) \leq \mathbb{E} \left[U_P \left(X_T^{(x_0, a)} - W_T^{(y_0, a^*, Z^*, K^*)} \right) \right] \leq V_P^{FB}.$$

Because the reverse inequality holds, the final result follows. \square

5.4 Contracting under shutdown risk and Moral Hazard

5.4.1 Main results

The following is dedicated to our main result for the Moral Hazard problem. We shall state our main theorem with the explicit optimal contract before turning to some analysis of the effect of the shutdown on dynamic contracting. In the case of moral hazard, one is forced to make a stronger assumption about the nature of a contract. This stronger hypothesis will naturally appear to justify the martingale optimality principle. In our setting, a contract is a $\mathbb{G}_{T \wedge \tau}$ measurable random variable W such that for every $\beta \in \mathbb{R}$, we have

$$\mathbb{E} [\exp(\beta W)] < +\infty.$$

A first step to optimal contracting involves answering the preliminary question: can we characterize incentive compatible wages and if so what is the related optimal action for the Agent? The characterization of incentive compatible contracts relies on the martingale optimality principle (see [62] and [114]) that we recall below.

Lemma 5.4.1 (Martingale Optimality Principle). *Given a contract W , consider a family of stochastic processes $R^a(W) := (R_t^a)_{t \in [0, T]}$ indexed by a in \mathcal{B} that satisfies :*

1. $R_T^a = U_A(W - \int_0^T \kappa(a_s(1 - N_s))ds)$ for any a in \mathcal{B}
2. R^a is a \mathbb{P}^a -supermartingale for any a in \mathcal{B}
3. R_0^a is independent of a .

4. There exists a^* in \mathcal{B} such that R^{a^*} is a \mathbb{P}^{a^*} -martingale.

Then,

$$R_0^{a^*} = \mathbb{E}^{a^*} \left[U_A(W - \int_0^T \kappa(a_s^*) ds) \right] \geq \mathbb{E}^a \left[U_A(W - \int_0^T \kappa(a_s(1 - N_s)) ds) \right],$$

meaning that a^* is the optimal Agent's action in response to the contract W .

We will construct such a family following the standard route. Consider a given contract W , we define the family $R^a(W) := (R_t^a)_{t \in [0, T]}$ by

$$R_t^a := -\exp \left(-\gamma_A \left(Y_t(W) - \int_0^t \kappa(a_s(1 - N_s)) ds \right) \right),$$

where $(Y(W), Z(W), K(W))$ in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$ is the unique solution of the following BSDE under \mathbb{P}^0

$$Y_t(W) = W - \int_t^T f(Z_s(W), K_s(W))(1 - N_s) ds - \int_t^T Z_s(W)(1 - N_s) dB_s - \int_t^T K_s(W)(1 - N_s) dM_s, \quad (5.4.1)$$

with

$$f(z, k) := \frac{1}{2} \gamma_A z^2 + \lambda k + \frac{\lambda}{\gamma_A} (e^{-\gamma_A k} - 1) + \inf_{a \in \mathcal{B}} \{ \kappa(a) - az \}.$$

Remark 5.4.1. *The theoretical justification of the well-posedness of the BSDE (5.4.1) deserves some comments. The first results were obtained in [80] and [71] when the contract W is assumed to be bounded. The necessary extension in our model when W admits an exponential moment has been treated recently in the paper [92] which is also Chapter 6 of this thesis.*

By construction, $R_T^a = U_A(W - \int_0^T \kappa(a_s(1 - N_s)) ds)$ for any a in \mathcal{B} . Moreover, $R_0^a = Y_0(W)$ is independent of the Agent's action a . We compute the variations of R^a and obtain :

$$\begin{aligned} &= -\gamma_A R_s^a Z_s(1 - N_s) dB_s + R_s^a (e^{-\gamma_A K_s} - 1)(1 - N_s) dM_s \\ &+ R_s^a \gamma_A \left\{ \frac{1}{2} \gamma_A Z_s^2 - f(Z_s, K_s) + \kappa(a_s(1 - N_s)) + \lambda K_s + \frac{\lambda}{\gamma_A} (e^{-\gamma_A K_s} - 1) \right\} (1 - N_s) ds. \\ &= -\gamma_A R_s^a Z_s(1 - N_s) dB_s^a + R_s^a (e^{-\gamma_A K_s} - 1)(1 - N_s) dM_s \\ &+ R_s^a \gamma_A \left\{ \frac{1}{2} \gamma_A Z_s^2 - f(Z_s, K_s) + \kappa(a_s(1 - N_s)) + \lambda K_s + \frac{\lambda}{\gamma_A} (e^{-\gamma_A K_s} - 1) - a_s Z_s \right\} (1 - N_s) ds. \end{aligned}$$

Thus R^a is a \mathbb{P}^a -super-martingale for every a in \mathcal{B} , the function

$$a^*(z) = -A \mathbb{1}_{z \leq -\frac{A}{\kappa}} + \frac{z}{\kappa} \mathbb{1}_{-\frac{A}{\kappa} \leq z \leq \frac{A}{\kappa}} + A \mathbb{1}_{z \geq \frac{A}{\kappa}}$$

is a unique minimizer for f and R^{a^*} is a \mathbb{P}^{a^*} -martingale. As a consequence, every contract W is incentive compatible which a unique best reply $a^*(Z(W))$. Finally, a contract W satisfies the participation constraint if and only if $Y_0(W) \geq y_{PC}$.

Relying on the idea of Sannikov [116] and its recent theoretical justification by Cvitanic, Possamai and Touzi [32], we will consider the Agent promised wage $Y(W)$ as a state variable

to embed the Principal's problem into the class of Markovian problems, by considering the sensitivities of the Agent's promised wage $Z(W)$ and $K(W)$ as control variables. For $\pi = (y_0, Z, K) \in [y_{PC}; +\infty) \times \mathbb{H}^2 \times \mathbb{H}^2$, we define under \mathbb{P}^0 , the control process called the Agent continuation value

$$W_t^{(y_0, Z, K)} = y_0 + \int_0^t Z_s(1 - N_s)dB_s + \int_0^t K_s(1 - N_s)dM_s + \int_0^t f(Z_s, K_s)(1 - N_s) ds. \quad (5.4.2)$$

Under $\mathbb{P}^* := \mathbb{P}^{(a^*(Z))}$, we thus have

$$W_t^{(y_0, Z, K)} = y_0 + \int_0^t Z_s(1 - N_s)dB_s^* + \int_0^t K_s(1 - N_s)dM_s \quad (5.4.3)$$

$$+ \int_0^t \left\{ \frac{\gamma_A}{2} Z_s^2 + \kappa(a^*(Z_s)) + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K_s) - 1 + \gamma_A K_s] \right\} (1 - N_s) ds. \quad (5.4.4)$$

Now, we consider the set

$$\zeta = \left\{ \pi = (Z, K) \in \mathbb{H}^2 \times \mathbb{H}^2 \text{ such that } \forall \beta \in \mathbb{R}, \mathbb{E} \left[\exp(\beta W_T^{(y, Z, K)}) \right] < +\infty \text{ for } y \in \mathbb{R} \right\}.$$

By construction, $W_T^{(y, \pi)}$ is a contract that satisfies the participation constraint for every $\pi \in \zeta$ and $y \geq y_{PC}$. Moreover, by the well-posedness of the BSDE (5.4.1), every contract W that satisfies the participation constraint can be written $W_T^{(Y_0(W), Z(W), K(W))}$ with $\pi(W) = (Z(W), K(W)) \in \zeta$. Therefore, the problem of the Principal can now be rewritten as the following optimisation problem

$$V_P := \sup_{y \geq y_{pc}} v(0, x, y),$$

where

$$v(0, x, y) = \sup_{\pi \in \zeta} \mathbb{E}^* [U_P(X_{T \wedge \tau} - W_{T \wedge \tau}^\pi)] \quad (5.4.5)$$

To characterize the optimal contract, we will proceed analogously as in the full risk sharing case by constructing a smooth solution to the HJB equation associated to the Markov control problem (5.4.5) given by

$$\begin{aligned} 0 = & \partial_t v(t, x, y) + \inf_{Z, K} \left\{ \partial_x v(t, x, y) \frac{Z}{\kappa} + \partial_y v(t, x, y) \left[\frac{\gamma_A}{2} Z^2 + \kappa(a^*(Z)) + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K) - 1] \right] \right. \\ & \left. + \lambda [U_P(x - y - K) - v(t, x, y)] + \partial_{yy} v(t, x, y) \frac{Z^2}{2} + \frac{1}{2} \partial_{xx} v(t, x, y) + \partial_{xy} v(t, x, y) Z \right\}, \end{aligned} \quad (5.4.6)$$

Lemma 5.4.2. *Assume the constant A in the definition of the set of admissible efforts \mathcal{B} satisfies*

$$A > \frac{\gamma_P + \kappa^{-1}}{\kappa(\gamma_P + \gamma_A) + 1}.$$

Then, the function $U_P(x - y)\Phi_0(t)$, with

$$\Phi_0(t) = \left(\frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}},$$

where

$$c_1 = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \frac{\gamma_P \kappa^{-1} (\gamma_P + \kappa^{-1})}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2 = \lambda \frac{\gamma_P + \gamma_A}{\gamma_A}.$$

solves in the classical sense the HJB equation (5.5.3). In particular $Z_t^* = \frac{\gamma_P + \kappa^{-1}}{\gamma_P + \gamma_A + \kappa^{-1}}$ and

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

Proof. Because the assumption on A implies $a^*(z) = z/\kappa$, the proof of this lemma is a direct adaptation of the proof of Lemma 5.3.3 to which we refer the reader. \square

We are in a position to prove the main result of this section

Theorem 5.4.1. *We have the following explicit characterizations of the optimal contracts.*

Let A as in the Lemma 5.4.2 and let $Z_t^* = \frac{\gamma_P + \kappa^{-1}}{\gamma_P + \gamma_A + \kappa^{-1}}$ and $K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t))$, where Φ_0 is defined as in (5.3.7) with the constants :

$$c_1 := \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \frac{\gamma_P \kappa^{-1} (\gamma_P + \kappa^{-1})}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2 := \lambda \frac{\gamma_P + \gamma_A}{\gamma_A}.$$

Then (y_{PC}, Z^*, K^*) parametrizes the optimal wage for the Moral Hazard problem. The Agent performs the optimal action $\frac{Z^*}{\kappa}$.

Proof. Because the function $U_P(x - y)\Phi_0(t)$ is a classical solution to the HJB equation (5.5.3) and the optimal controls are bounded and free of y , we proceed analogously as in the proof of Theorem 5.3.1. Finally, we have to prove that the optimal wage $W^* = Y_T^{(y_{PC}, Z^*, K^*)}$ admits exponential moments to close the loop. According to (5.4.2), we have

$$W^* = y_{PC} + Z^* B_{T \wedge \tau}^* + \frac{1}{2} \left(\gamma_A + \frac{1}{\kappa} \right) (Z^*)^2 (T \wedge \tau) + K_{\tau-}^* \mathbf{1}_{\tau \leq T} + \int_0^T \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K_s^*) - 1] (1 - N_s) ds.$$

Because $(B_t^*)_t$ is a Brownian motion and K_t^* is deterministic, it is straightforward to check that W^* admits exponential moments. \square

5.4.2 Model analysis

The optimal contract includes two components. One is linear in the output with an incentivizing slope that is similar to the classical optimal contract found in [58]. This is necessary to implement a desirable level of effort. The other is unrelated to the incentives but linked to the shutdown risk sharing. It is key to observe that this second term is nonzero even if the shutdown risk does not materialize before the termination of the contract.

The characterization of the optimal contracts in Theorem 5.4.1 sparks an immediate observation: the two parties only need to be committed to the contracting agreement up until $T \wedge \tau$. Therefore in this simple model, using an expected-utility related reasoning and without considering mechanisms such as employment law, the occurrence of the agency-free external risk, halting production, leads to early contract terminations. This is in line with what actually

happened during the Covid pandemic. Indeed in the USA and in eight weeks of the pandemic, 36.5 million people applied for unemployment insurance. In more protective economies, mass redundancies were only prevented through the instauration of furlough type schemes allowing private employees' wages to temporarily be paid by governments. This phenomena makes fundamental sense : a Principal whose output process is completely halted cannot enforce the Agent to work hard because she has no revenue to provide the incentives. Let's focus on the second term:

$$K_{\tau-}^* \mathbf{1}_{\tau \leq T} + \int_0^T \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K_s^*) - 1] (1 - N_s) ds. \quad (5.4.7)$$

Understanding the effect of these extra terms is crucial to fully understand the sharing of agency-free shutdown risk. First, we study the sign of the control K^* .

Lemma 5.4.3. *Let c_1 and c_2 be the relevant constants given in Theorem 5.4.1 then the optimal control $(K_t^*)_{t \in [0, T]}$ can be expressed as :*

$$K_t^* = \frac{1}{\gamma_A} \log \left(\mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) ((T - t) \wedge \tau) \right) \right] \right) \quad t \in [0, T]. \quad (5.4.8)$$

Proof. We have that :

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

with

$$\Phi_0(t) = \left(\frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}}.$$

The aim here is to link this expression for Φ_0 to that of an expected value. As such, we consider the following expected value that decomposes as shown :

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) ((T - t) \wedge \tau) \right) \right] \\ &= \mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) (T - t) \right) \mathbf{1}_{\tau > T-t} \right] + \mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) \tau \right) \mathbf{1}_{\tau \leq T-t} \right]. \end{aligned}$$

Using $c_2 = \frac{\gamma_P + \gamma_A}{\gamma_A} \lambda$, the first term of the expected value rewrites as follows :

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) (T - t) \right) \mathbf{1}_{\tau > T-t} \right] &= \exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) (T - t) \right) \exp(-\lambda(T - t)) \\ &= \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right). \end{aligned}$$

It remains to compute the second term. We obtain :

$$\begin{aligned}
\mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) \tau \right) \mathbf{1}_{\tau \leq T-t} \right] &= \int_0^{T-t} \lambda \exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) s \right) \exp(-\lambda s) ds \\
&= \int_0^{T-t} \lambda \exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) s - \lambda s \right) ds \\
&= \int_0^{T-t} \lambda \exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} c_1 s \right) ds \\
&= \left[\frac{\lambda}{c_1} \frac{\gamma_P + \gamma_A}{\gamma_A} \exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} c_1 s \right) \right]_0^{T-t} \\
&= \left[\frac{c_2}{c_1} \exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} c_1 s \right) \right]_0^{T-t} \\
&= \frac{c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T-t) \right) - \frac{c_2}{c_1}.
\end{aligned}$$

Combining both terms we reach the final expression :

$$\mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) ((T-t) \wedge \tau) \right) \right] = \frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T-t) \right) - \frac{c_2}{c_1}.$$

Therefore recalling the initial expression for the function Φ_0 :

$$\Phi_0(t) := \left(\frac{c_1 + c_2}{c_1} \exp \left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T-t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}},$$

we identify that :

$$\Phi_0(t) = \left(\mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) ((T-t) \wedge \tau) \right) \right] \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}}.$$

As a consequence, we may also rewrite K_t^* . Indeed :

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

and with the new expression for Φ_0 we obtain the result :

$$K_t^* = \frac{1}{\gamma_A} \log \left(\mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1 + c_2) ((T-t) \wedge \tau) \right) \right] \right).$$

□

Remark 5.4.2. *We have the same expression for the optimal control K^* in the first-best case, using for c_1 and c_2 the relevant constants given in Theorem 5.3.1.*

We thus have the following Corollary.

Corollary 5.4.1. *The shutdown risk-sharing component of the optimal wage is linear in the default time. More precisely, the optimal wage is*

$$W^* = y_{PC} + Z^* B_{T \wedge \tau}^* + \left(\frac{\gamma_A}{2} Z^{*2} + \kappa(a^*) \right) (T \wedge \tau) + K_0^* - \left(\frac{c_1}{\gamma_P + \gamma_A} + \frac{\lambda}{\gamma_A} \right) (T \wedge \tau).$$

where $a^* = \frac{1}{\kappa}$ under Risk-Sharing and $a^* = \frac{Z^*}{\kappa}$ under Moral Hazard.

Proof. Because $K_T^* = 0$, the optimal wage can be written $W^* = y_{PC} + Z^* B_{T \wedge \tau}^* + f(T \wedge \tau)$, with

$$f(t) = K_t^* + \int_0^t \frac{\lambda}{\gamma_A} (\exp(-\gamma_A K_s^*) - 1) ds, t \in [0, T].$$

Let us define

$$g(t) = \frac{c_1 + c_2}{c_1} \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - \frac{c_2}{c_1}.$$

We have $g'(t) = -(c_1 + c_2) \frac{\gamma_A}{\gamma_P + \gamma_A} \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right)$.

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} K_t^* &= \frac{1}{\gamma_A} \frac{g'(t)}{g(t)} \\ &= \frac{1}{\gamma_P + \gamma_A} \left\{ \frac{-(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right)}{\frac{c_1 + c_2}{c_1} \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - \frac{c_2}{c_1}} \right\} \\ &= \frac{c_1}{\gamma_P + \gamma_A} \left\{ \frac{-(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right)}{(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - c_2} \right\}. \end{aligned}$$

Also

$$\frac{\partial}{\partial t} \int_0^t \frac{\lambda}{\gamma_A} (\exp(-\gamma_A K_s^*) - 1) ds = \frac{\lambda}{\gamma_A} (\exp(-\gamma_A K_t^*) - 1)$$

and so :

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^t \frac{\lambda}{\gamma_A} (\exp(-\gamma_A K_s^*) - 1) ds &= \frac{\lambda}{\gamma_A} (\exp(-\gamma_A K_t^*) - 1) \\ &= \frac{\lambda}{\gamma_A} \left(\frac{1}{g(t)} - 1 \right) \quad \text{well-defined as } g(t) > 0 \text{ on } [0, T] \\ &= \frac{\lambda}{\gamma_A} \left\{ \frac{1}{\frac{c_1 + c_2}{c_1} \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - \frac{c_2}{c_1}} - 1 \right\} \\ &= \frac{c_1 \lambda}{\gamma_A} \left\{ \frac{1}{(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - c_2} \right\} - \frac{\lambda}{\gamma_A} \end{aligned}$$

Finally, we have :

$$\begin{aligned}
f'(t) &= \frac{\partial}{\partial t} K_t^* + \frac{\partial}{\partial t} \int_0^t \frac{\lambda}{\gamma_A} (\exp(-\gamma_A K_s^*) - 1) ds \\
&= \frac{c_1}{\gamma_P + \gamma_A} \left\{ \frac{-(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right)}{(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - c_2} \right\} \\
&\quad + \frac{c_1 \lambda}{\gamma_A} \left\{ \frac{1}{(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - c_2} \right\} - \frac{\lambda}{\gamma_A} \\
&= \frac{-c_1}{\gamma_P + \gamma_A} \frac{1}{(c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - c_2} \left\{ (c_1 + c_2) \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t)\right) - \lambda \frac{\gamma_A}{\gamma_P + \gamma_A} \right\} \\
&\quad - \frac{\lambda}{\gamma_A} \\
&= -\frac{c_1}{\gamma_P + \gamma_A} - \frac{\lambda}{\gamma_A} \quad \text{as } c_2 = \frac{\lambda \gamma_A}{\gamma_P + \gamma_A} \\
&= -\left(\frac{c_1}{\gamma_P + \gamma_A} + \frac{\lambda}{\gamma_A} \right).
\end{aligned}$$

Hence, f is linear with $f(0) = K_0^*$ and thus it follows that :

$$f(t) = K_0^* - \left(\frac{c_1}{\gamma_P + \gamma_A} + \frac{\lambda}{\gamma_A} \right) t, \quad t \in [0, T],$$

□

Remark 5.4.3. *A little algebra gives*

$$f(t) = K_0^* - \left(\frac{c_1}{\gamma_P + \gamma_A} + \frac{\lambda}{\gamma_A} \right) t = K_0^* - \frac{1}{\gamma_P + \gamma_A} (c_1 + c_2) t$$

The slope of f is of opposite sign to the sign of $c_1 + c_2$, which will prove to be crucial in the next definition of insurance and speculation regimes.

Quite compellingly this alternative form for K^* leads to easy analysis of the sign of the control, given in the following lemma.

Lemma 5.4.4. *The sign of K^* over the contracting period $[0, T]$ is constant and entirely determined by the model's risk aversions γ_P and γ_A , and the Agent's effort cost κ . Indeed :*

- under Risk-Sharing the sign of K^* is equal to the sign of $\gamma_P \gamma_A - \gamma_P \kappa^{-1} - \gamma_A \kappa^{-1}$,
- under Moral Hazard the sign of K^* is equal to the sign of $\gamma_P \gamma_A - \gamma_P \kappa^{-1} - (\kappa^{-1})^2$.

Also K_t^* varies monotonously in time, with $K_T^* = 0$.

Proof. From the expression (5.4.8), we easily deduce that :

- If $c_1 + c_2 = 0$ then $K_t^* = 0$ for every $t \in [0, T]$,

- if $c_1 + c_2 > 0$ then $K_t^* > 0$ for $t \leq \tau$ and the function $t \rightarrow K_t^*$ decreases,
- if $c_1 + c_2 < 0$ then $K_t^* < 0$ for $t \leq \tau$ and the function $t \rightarrow K_t^*$ increases.

Replacing c_1 and c_2 by their relevant expressions in each case leads to the result. \square

$K_0^* > 0$ is the extra compensation asked at the signature of the contract by an agent who is more sensitive to the shutdown risk than the principal. We can easily visualize the sign of the control K_0^* as a function of γ_P , γ_A and κ . Below we plot the sign depending on the risk-aversions and fixing $\kappa = 1$ and $\kappa = 2$. The first two plots (Figure 5.9 and 5.2) correspond to the full Risk-Sharing case whilst Figures 5.3 and 5.4 show the sign of K^* in the Moral Hazard case. The x -axis holds the values of γ_A and the y -axis the values of γ_P . Both risk-aversion constants are valued between 0 and 10 and the origin is in the bottom left corner. Blue encodes a negative sign and green a positive sign.

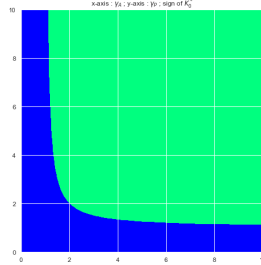


Figure 5.1 – Sign of K_0^* depending on γ_P and γ_A for $\kappa = 1$.

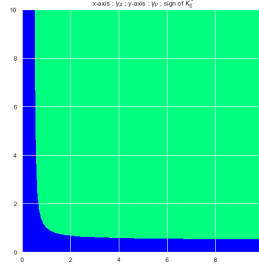


Figure 5.2 – Sign of K_0^* depending on γ_P and γ_A for $\kappa = 2$.

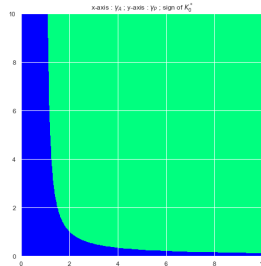


Figure 5.3 – Sign of K_0^* depending on γ_P and γ_A for $\kappa = 1$.

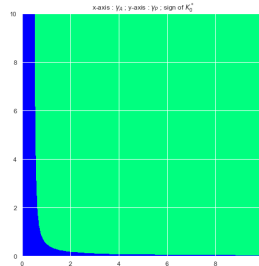


Figure 5.4 – Sign of K_0^* depending on γ_P and γ_A for $\kappa = 2$.

We observe that in most situations, the sign of K_0^* is positive. A negative sign occurs when either the principal or the agent are close to being risk-neutral (symmetrically so in the full Risk-Sharing case but asymmetrically so in the Moral Hazard case : the sign switches from negative to positive at a much lower level of risk-aversion for the principal than the agent). Also note that increasing the agent's effort coefficient κ decreases the level of risk-aversion for which K_0^* goes from positive to negative. Note that it is known by both the Principal and

the Agent at time 0 whether the contract will fall into either regime.

Our key result shows that the risk of shutdown adds an extra linear term to the optimal compensation that, up to the underlying constants, has the same structure under both full Risk Sharing and under Moral Hazard. Even if the optimal contract separates the role of incentives from that of shutdown risk sharing, the amount of the *insurance deposit* K_0^* depends strongly on the magnitude of the moral hazard problem. Therefore, we may naturally quantify the effect of Moral Hazard on the risk-sharing part. With this question in mind we compare the values of K_0^* under Risk-Sharing and Moral Hazard for different parameter values. This may be observed in the Figures 5.5 to 5.9 below where we represent the values of K_0^* under Moral Hazard (red) and Risk-Sharing (blue) as a function of one of the underlying parameters ($\gamma_P, \gamma_A, \lambda, \kappa, T$) whilst fixing the remaining 4.

These figures lead to a crucial observation : under any given set of parameters, the value of K_0^* under Moral Hazard is always greater than the value of K_0^* under Risk Sharing. Note that in some settings, given some parameters the signs of the two cases are not always the same as the sign is that of $c_1 + c_2$ which depends on the values of γ_P, γ_A and κ in a different manner in Risk-Sharing and Moral Hazard. Of course though, as soon as K_0^* for Risk-Sharing is positive, K_0^* for Moral Hazard is positive too. We may note that although λ and T do not impact the sign, variations in their values have an impact on the magnitude $|K_0^*|$: the higher the risk of early termination of the contract, the lower the amount of insurance requested by the agent and the longer the duration of the contract, the higher the amount of insurance requested by the agent.

Figure 5.5 – Values of K_0^* depending on γ_P

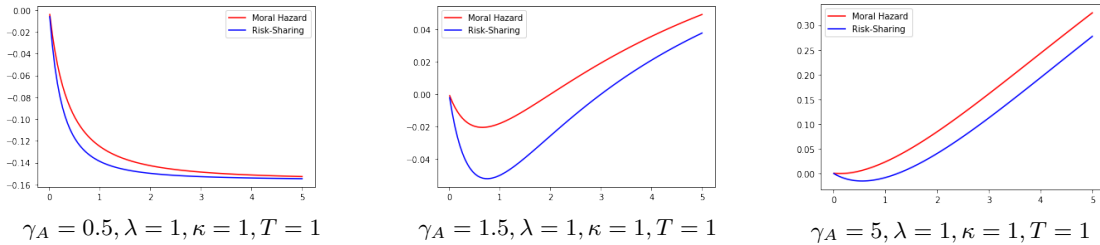


Figure 5.6 – Values of K_0^* depending on γ_A

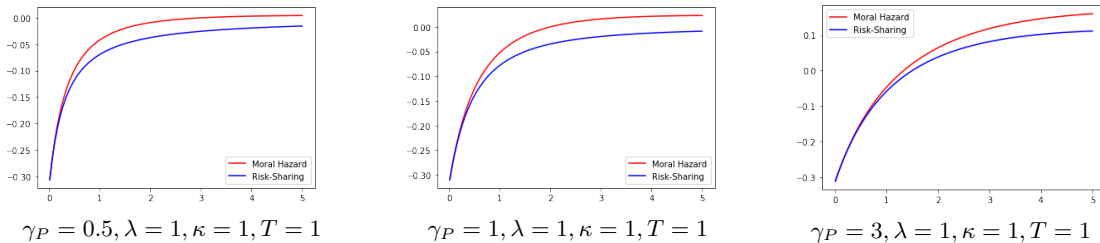


Figure 5.7 – Values of K_0^* depending on λ

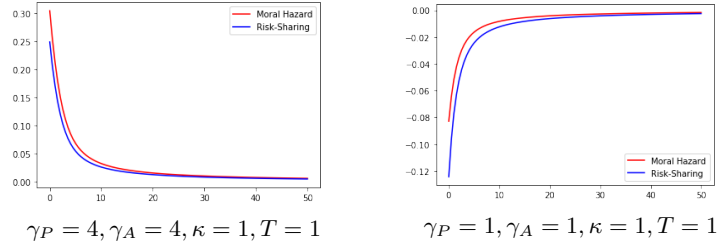


Figure 5.8 – Values of K_0^* depending on κ

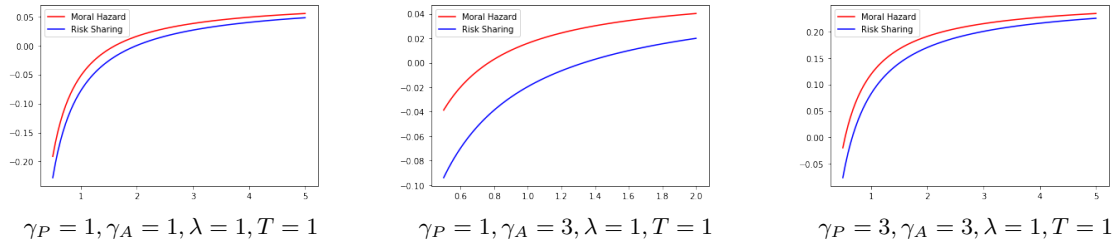
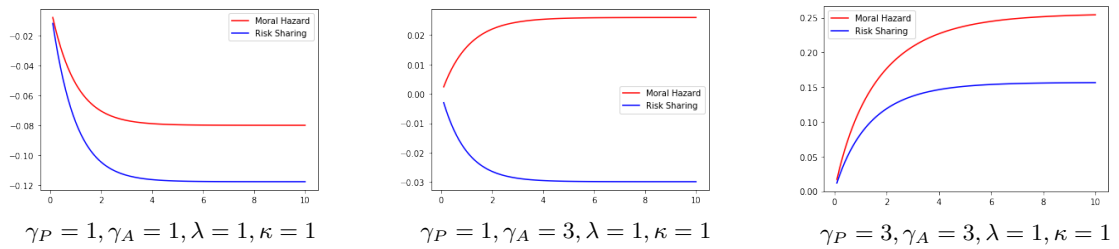


Figure 5.9 – Values of K_0^* depending on T



Additionally, the parameter λ does have a quantifiable effect of the wage as it affects its expected value :

$$\mathbb{E}(f(T \wedge \tau)) = K_0^* - \frac{c_1 + c_2}{\gamma_P + \gamma_A} \left(\frac{1 - e^{-\lambda T}}{T} \right).$$

This expected value is represented below as a function of γ_A and γ_P for different values of λ (with $T = 1$ and $\kappa = 1$ fixed). The first three figures (Figures 5.10, 5.11 and 5.12) concern the full Risk-Sharing case whilst the second set of figures (Figures 5.13, 5.14 and 5.15) concern the Moral Hazard setting.

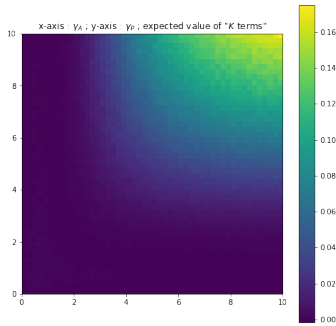


Figure 5.10 – Expected value depending on γ_P and γ_A for $\lambda = 0.5$.

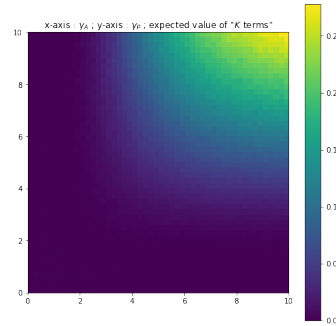


Figure 5.11 – Expected value depending on γ_P and γ_A for $\lambda = 1$.

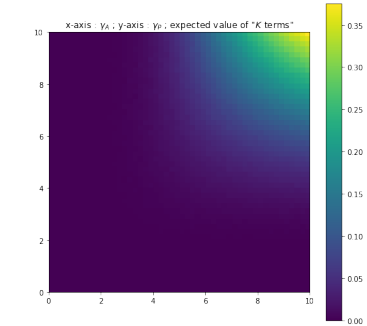


Figure 5.12 – Expected value depending on γ_P and γ_A for $\lambda = 5$.

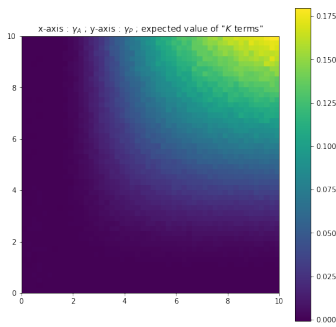


Figure 5.13 – Expected value depending on γ_P and γ_A for $\lambda = 0.5$.

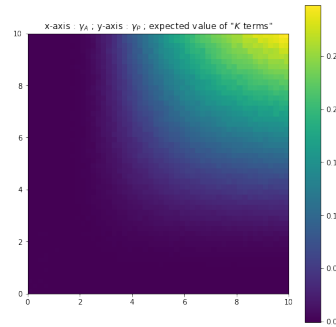


Figure 5.14 – Expected value depending on γ_P and γ_A for $\lambda = 1$.

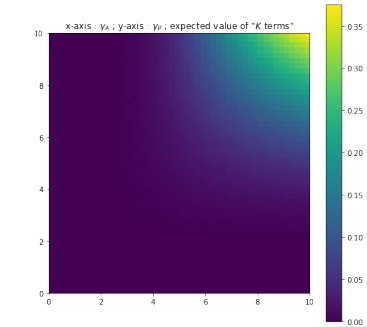


Figure 5.15 – Expected value depending on γ_P and γ_A for $\lambda = 5$.

The value seems to be in many cases very close to 0. As such, the Agent earns, on average, a very similar wage to a "stopped" Holmstrom-Milgrom wage. However when the Principal and the Agent are particularly risk-averse, the expected value increases quite notably and the Agent gains slightly more on average. This is in line with the papers by Hoffman and Pfeil [56] and Bertrand and Mullainathan [13] which show the Agent must be rewarded for a risk that is beyond his control. Note that the simulations show that for fixed levels of risk-aversion the expected value increases as λ increases. For example for $\gamma_P = \gamma_A = 7$, in Figure 5.13 the

expected value is approximately worth 0.075 and in Figure 5.14 it is approximately equal to 0.15. As such the risk-averse Agent is increasingly rewarded for the uncontrollable risk.

Finally, we can make a few further comments related to the underlying expected utilities in this new contracting setting. First, the Agent's expected utility is the same under both full Risk-Sharing and Moral Hazard. Indeed he walks away with his participation constraint :

$$\mathbb{E} \left[U_A \left(W^* - \int_0^T \kappa(a_s^*) ds \right) \right] = U_A(y_{PC}),$$

where (W^*, a^*) designates the Risk-Sharing or Moral Hazard optimal contract under shutdown risk. Such a result also holds in the same setting without shutdown risk : the Agent tracts average the same expected utility under full Risk-Sharing or Moral Hazard, with or without an underlying agency-free external risk.

When it comes to the Principal we may wonder how his expected utility may be affected by the possibility of a halt in production. In particular we may question what the Principal loses in not being able to observe the Agent's action under a likelihood of agency-free external risk ? To answer this we denote as $V_0^{RS}(0, x, y)$ the Principal's expected utility under full Risk-Sharing and $V_0^{MH}(0, x, y)$ under Moral Hazard. We have that :

$$V_0^{RS}(0, x, y) = U_P(x - y)\Phi_0^{RS}(0) \quad \text{and} \quad V_0^{MH}(0, x, y) = U_P(x - y)\Phi_0^{MH}(0)$$

where Φ_0^{RS} and Φ_0^{MH} are the related functions from Theorem 5.3.1 and Theorem 5.4.1. With these expressions we have that :

$$\frac{V_0^{MH}(0, x, y)}{V_0^{RS}(0, x, y)} = \frac{\Phi_0^{MH}(0)}{\Phi_0^{RS}(0)}.$$

The Moral Hazard problem involves optimizing across a more restricted set of contracts. Therefore we know that :

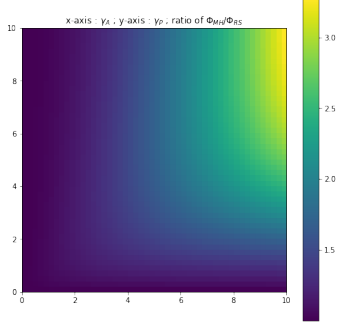
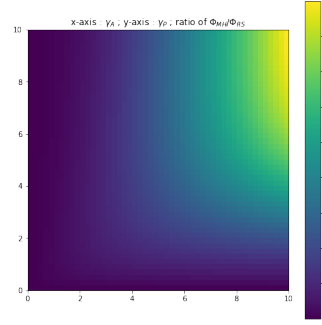
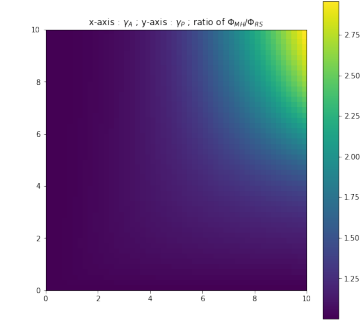
$$V_0^{MH}(0, x, y) \leq V_0^{RS}(0, x, y),$$

and as $U_P(x - y) < 0$:

$$\Phi_0^{RS}(0) \leq \Phi_0^{MH}(0).$$

We may question whether this inequality leads to a big gap between the expected utilities and we answer this by plotting the ratio $\frac{\Phi_0^{MH}(0)}{\Phi_0^{RS}(0)}$ for different values of λ and with $T = 1$ and $\kappa = 1$ fixed.

We first observe a standard result : for low levels of risk-aversion the ratio is close to 1 and the Principal does not lose much by not observing the Agent's actions. As the values of risk-aversion increase the Principal loses out more and more by not being in a first best setting. This classical result comes with an observation that is specific to the presence of shutdown risk : as λ increases (and therefore as the chance of shutdown risk occurring before T increases), the ratio stays close to 1 for higher and higher levels of risk-aversion. For example in Figure 5.16 when $\gamma_P = \gamma_A = 6$ we observe that $\frac{\Phi_0^{MH}(0)}{\Phi_0^{RS}(0)} \approx 2$ yet in Figure 5.18 for the same levels of risk-aversion we have that $\frac{\Phi_0^{MH}(0)}{\Phi_0^{RS}(0)} \approx 1$. So a high possibility of some production halt occurring reduces the gap between the full Risk-Sharing contract and the Moral Hazard contract. Such a phenomena may be due to the fact that a high possibility of a production

Figure 5.16 – Ratio depending on γ_P and γ_A for $\lambda = 0.5$.Figure 5.17 – Ratio depending on γ_P and γ_A for $\lambda = 1$.Figure 5.18 – Ratio depending on γ_P and γ_A for $\lambda = 5$.

halt at some point in the time interval means that the wage process evolves on a time period that is on average shorter before stopping. There is thus less time for a significant gap to appear between the full Risk-Sharing case and the Moral Hazard case.

As this analysis comes to a close we finish this section by discussing a possibility for extension with more general deterministic compensators.

5.4.3 General deterministic compensators $(\Lambda_t)_{t \in [0, T]}$

Throughout this Chapter, we have considered a constant compensator λ for the jump process. This choice allows for clearer calculations but it is key to note that our results extend to the case where λ is no longer constant such as :

$$\Lambda_t = \int_0^t \lambda_s ds,$$

with $(\lambda_s)_{s \in [0, T]}$ some deterministic positive mapping such that $\Lambda_T < +\infty$. The proofs for the optimal contracting are simply a direct extension of the proofs of the previous sections. Of course due to the independence between B and N , a time dependent compensator does not induce any change to the Holmstrom-Milgrom part of the wages. Only the part related to K^* is affected. We provide the details in the following.

The full Risk-Sharing problem

The optimal wage for the Risk-Sharing problem in such a setting is of the form :

$$W_t = y^* + \int_0^t Z_s^*(1 - N_s) dB_s + \int_0^t K_s^*(1 - N_s) dM_s \\ + \int_0^t \left\{ \frac{\gamma_A}{2} Z_s^{*2} + \kappa(a_s^*(1 - N_s)) + \frac{\lambda_s}{\gamma_A} [\exp(-\gamma_A K_s^*) - 1 + \gamma_A K_s^*] \right\} (1 - N_s) ds,$$

where :

$$y^* = y_{PC}, a_t^* = \frac{1}{\kappa}, Z_t^* = \frac{\gamma_P}{\gamma_P + \gamma_A} \quad \text{and} \quad K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

with $\Phi_0(t)$ solution to the Bernoulli equation :

$$\Phi'_0(t) + c_1(t)\Phi_0(t) + c_2(t)\Phi_0(t)^{\frac{\gamma_P}{\gamma_P + \gamma_A}} = 0, \quad \Phi_0(T) = 1,$$

where

$$c_1(t) = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda_t \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2(t) = \lambda_t \frac{\gamma_P + \gamma_A}{\gamma_A}.$$

The Moral Hazard problem

The optimal wage in the Moral-Hazard problem is again of the form :

$$\begin{aligned} W_t &= y^* + \int_0^t Z_s^*(1 - N_s)dB_s^* + \int_0^t K_s^*(1 - N_s)dM_s \\ &= + \int_0^t \left\{ \frac{1}{2}\gamma_A Z_s^{*2} + \frac{Z_s^{*2}}{2\kappa} + \lambda_s K_s^* + \frac{\lambda_s}{\gamma_A}(e^{-\gamma_A K_s^*} - 1) \right\} (1 - N_s)ds, \end{aligned}$$

where :

$$y^* = y_{PC}, Z_t^* = \frac{\gamma_P + \kappa^{-1}}{\gamma_P + \gamma_A + \kappa^{-1}} \quad \text{and} \quad K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

with $\Phi_0(t)$ solution to the Bernoulli equation :

$$\Phi'_0(t) + c_1(t)\Phi_0(t) + c_2(t)\Phi_0(t)^{\frac{\gamma_P}{\gamma_P + \gamma_A}} = 0, \quad \Phi_0(T) = 1,$$

with :

$$c_1(t) = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \frac{\gamma_P \kappa^{-1}(\gamma_P + \kappa^{-1})}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \lambda_t \frac{\gamma_P + \gamma_A}{\gamma_A} \quad \text{and} \quad c_2(t) = \lambda_t \frac{\gamma_P + \gamma_A}{\gamma_A}.$$

Remark 5.4.4. *Lemma 5.4.3 still holds in this context :*

$$K_t^* = \frac{1}{\gamma_A} \log \left(\mathbb{E} \left[\exp \left(\frac{\gamma_A}{\gamma_P + \gamma_A} (c_1(t) + c_2(t)) ((T - t) \wedge \tau) \right) \right] \right) \quad t \in [0, T].$$

The sign of K^ is thus once again constant across $[0, T]$ and given by the criteria of Lemma 5.4.4. Therefore the speculation and insurance regimes for the Agent still hold under a time dependent compensator.*

5.5 Mitigating the effects of agency-free external risk

This Chapter has so far modeled the occurrence of a halt as a complete fatality suffered by both parties in the contracting agreement. Yet the recent crisis has highlighted the ability of humans and businesses to react and adapt when faced with adversity. We now include such phenomena in the contracting setting by allowing the Principal to invest upon a halt in order to continue some form of (possibly disrupted) production. This is quite a natural and realistic variant on our initial model. Indeed when faced with a period of lockdown, companies may for example invest in teleworking infrastructure so that a number of employees whose jobs are doable remotely can continue to work. Similarly, jobs that require some form of presence could continue if companies invest in protective equipment and adapt their organization. We may wonder how such a mechanism may affect optimal contracting.

5.5.1 Setting for mitigation

Mathematically, we consider that the production process evolves as previously up until $\tau \wedge T$. If a halt happens at some time $\tau \leq T$, we allow the Principal to invest an amount $i > 0$ to continue production at a degraded level $\theta \in (0, 1)$. It is assumed that the investment decision is at the Principal's convenience. It is modeled by a control D which is a \mathbb{G}_τ -measurable random variable with values in $\{0, 1\}$. The θ parameter is firm-specific and reflects the effectiveness of the post-shutdown reorganization.

Under the initial probability measure \mathbb{P}^0 , the output process X_t evolves as

$$X_t = x_0 + \int_0^t ((1 - N_s) + N_s \mathbb{1}_{D=1}) dB_s.$$

We recall from [73] Lemma 4.4. the decomposition of a \mathbb{G} -adapted process ϕ . There exist a \mathbb{F} -adapted process ϕ^0 and a family of processes $(\phi_t^1(u), u \leq t \leq T)$ that are $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable such that

$$\phi_t = \phi_t^0 \mathbb{1}_{t < \tau} + \phi_t^1(\tau) \mathbb{1}_{t \geq \tau}.$$

Contract: A contract W is a \mathcal{G}_T -measurable random variable satisfying $\mathbb{E}(\exp(-2\gamma_A W)) < +\infty$ of the form

$$W = W^0 \mathbb{1}_{T < \tau} + (W_T^{1,1}(\tau) \mathbb{1}_{D=1} + W_T^{1,0}(\tau) \mathbb{1}_{D=0}) \mathbb{1}_{\tau \leq T}.$$

where W^0 is \mathcal{F}_T -measurable and $W_t^{1,1}(u)$ and $W_t^{1,0}(u)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ measurable. We will assume that $W_T^{1,0}(\tau) = W_{T \wedge \tau}^{1,0}(\tau)$ since in the absence of investment, it is no longer necessary to give incentives after τ .

Effort process: In this setting, the Agent will adapt his effort to the occurrence of the shutdown risk. This is mathematically modeled by a \mathbb{G} -adapted process $(a_t)_t$ in the form

$$a_t = a_t^0 \mathbb{1}_{t < \tau} + a_t^1(\tau) \mathbb{1}_{t \geq \tau}$$

where a^0 and a^1 are respectively \mathbb{F} -adapted and $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable. Furthermore, we assume that the effort processes are bounded by some constant A . We then define \mathbb{P}^a as $\frac{d\mathbb{P}^a}{d\mathbb{P}^0} |_{\mathcal{G}_T} = L_T^\theta$, with

$$L_T^\theta = \exp \left(\int_0^T a_s^0 (1 - N_s) + \theta a_s^1(\tau) N_s \mathbb{1}_{D=1} dB_s - \frac{1}{2} \int_0^T (a_s^0)^2 (1 - N_s) + \theta^2 (a_s^1(\tau))^2 N_s \mathbb{1}_{D=1} ds \right)$$

Because the processes a^0 and a^1 are bounded, $(B_t^a)_{t \in [0, T]}$ with

$$B_t^a = B_t - \int_0^t (a_s^0 (1 - N_s) + \theta a_s^1(\tau) N_s \mathbb{1}_{D=1}) ds, t \in [0, T]$$

is a \mathbb{G} -Brownian motion under \mathbb{P}^a . Under \mathbb{P}^a , the output process evolves as

$$X_t = x_0 + \int_0^t (a_s^0 (1 - N_s) + \theta a_s^1(\tau) N_s \mathbb{1}_{D=1}) ds + \int_0^t ((1 - N_s) + N_s \mathbb{1}_{D=1}) dB_s.$$

5.5.2 The optimal contract

We first make the following observation. After τ , if the default time occurs before the maturity of the contract, the Principal has a binary decision to take. If she decides to not invest, she gets the value $V^{1,0}(x, y) = U_P(x - y)$ where x is the level of input and y is the Agent continuation value. On the other hand, if she decides to invest, she will face for $t \geq \tau$ the moral hazard problem of Holmstrom and Milgrom for which we know the optimal contract and the associated value function

$$V^{1,1}(t, x, y) = U_P(x - y)\Phi_1(t, \theta)$$

where $\Phi_1(t, \theta) = \exp(-\gamma_P C_{inv}(T - t))$ and $C_{inv} := \frac{(\gamma_P + \frac{\theta^2}{\kappa})^2}{2(\gamma_P + \gamma_A + \frac{\theta^2}{\kappa})} - \frac{\gamma_P}{2}$.

Because the Principal has to pay a sunk cost $i > 0$ to invest, she will decide optimally to invest if and only if at τ for a given (x, y) , she observes

$$V^{1,1}(\tau, x, y) \geq V^{1,0}(x, y),$$

or equivalently $C_{inv}(T - \tau) > i$. Hence, if $C_{inv} > 0$, the optimal control will be $D^* = \mathbb{1}_{\{\tau < T - \frac{i}{C_{inv}}\}}$. To sum up, we have

Lemma 5.5.1. 1. *Investment for mitigation is never optimal upon a halt if :*

$$C_{inv} < 0 \quad \text{or} \quad i > TC_{inv}.$$

2. *Now suppose that :*

$$C_{inv} > 0 \quad \text{and} \quad i < TC_{inv}.$$

Mitigation is optimal up until the cutoff time t_{max} defined as :

$$t_{max} := T - \frac{i}{C_{inv}}.$$

Note that $i < TC_{inv}$ guarantees that $t_{max} \geq 0$.

We are in a position to solve the before-default Principal problem. Proceeding analogously as in Section 5.4, the before-default value function is given by the Markovian control problem

$$V_P = \sup_{y \geq y_{PC}} V_0(0, x, y),$$

with

$$V(0, x_0, y_0) = \sup_{\pi=(Z,K) \in \zeta} \mathbb{E} \left[U_P(X_T^\pi - W_T^\pi)(1 - N_T) + \int_0^T \max(V^{1,0}(X_t^\pi, W_t^\pi), V^{1,1}(t, X_t^\pi, W_t^\pi)) \lambda e^{-\lambda t} dt \right], \quad (5.5.1)$$

and

$$\begin{aligned} dX_t &= a^*(Z_t)(1 - N_t) dt + (1 - N_t) dB_t^*, \\ dW_t^\pi &= Z_s(1 - N_s)dB_s^* + K_s(1 - N_s)dM_s \\ &\quad + \left\{ \frac{\gamma_A}{2} Z_s^2 + \kappa(a^*(Z_s)) + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K_s) - 1 + \gamma_A K_s] \right\} (1 - N_s) ds. \end{aligned} \quad (5.5.2)$$

Remark 5.5.1. *To be perfectly complete, we develop in Section 5.5.3 the martingale optimality principle which makes it possible to obtain the dynamics (5.5.2).*

Theorem 5.5.1. *We have the following explicit characterizations of the optimal contracts. Assume the constant A in the definition of the set of admissible efforts \mathcal{B} satisfies*

$$A > \frac{\gamma_P + \kappa^{-1}}{\kappa(\gamma_P + \gamma_A) + 1}.$$

Let $Z_t^* = \frac{\gamma_P + \kappa^{-1}}{\gamma_P + \gamma_A + \kappa^{-1}}$ and

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log \left(\frac{\Phi_0(t)}{\min \{1, \exp(\gamma_P i) \Phi_1(t, \theta)\}} \right)$$

with Φ_0 as defined above with :

$$c_1 := \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \frac{\gamma_P \kappa^{-1} (\gamma_P + \kappa^{-1})}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A}$$

and

$$c_2(t) = \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \min \left\{ 1, \exp(\gamma_P i) \Phi_1(t, \theta) \right\}^{\frac{\gamma_A}{\gamma_P + \gamma_A}}$$

Then (y_{PC}, Z^*, K^*) parametrizes the optimal wage for the Moral Hazard problem with a possibility for mitigation. The Agent performs the optimal action $\frac{Z^*}{\kappa}$ before τ and $\frac{\theta Z^*}{\kappa}$ after τ when $\tau < T - i/C_{inv}$.

Proof. The reasoning used to compute the optimal Moral Hazard contract very much parallels the reasoning used above in Section 5.4. As a consequence, we are much more brief in the following proof.

The Hamilton-Jacobi-Bellman equation associated to the value function V_0 is the following :

$$\begin{aligned} 0 = \partial_t v_0(t, x, y) + \inf_{Z, K} \left\{ \partial_x v_0(t, x, y) \frac{Z}{\kappa} + \partial_y v_0(t, x, y) \left[\frac{\gamma_A}{2} Z^2 + \frac{Z^2}{2\kappa} + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K) - 1] \right] \right. \\ \left. + \lambda [v_1(t, x, y + K) - v_0(t, x, y)] + \partial_{yy} v_0(t, x, y) \frac{Z^2}{2} + \frac{1}{2} \partial_{xx} v_0(t, x, y) + \partial_{xy} v_0(t, x, y) Z \right\}, \end{aligned} \quad (5.5.3)$$

with the boundary condition :

$$v_0(T, x, y) = U_P(x - y)$$

where

$$v_1(t, x, y) = U_P(x - y) \min \left\{ 1, \exp(\gamma_P i) \Phi_1(t, \theta) \right\}.$$

Lemma 5.5.2. *Assume $A > \frac{\gamma_P + \kappa^{-1}}{\gamma_P + \gamma_A + \kappa^{-1}}$. The function $v_0(t, x, y) = U_P(x - y) \Phi_0(t)$, with*

$$\Phi_0(t) := \exp(-c_1 t) \left\{ \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} T\right) + \frac{\gamma_A}{\gamma_P + \gamma_A} \int_t^T c_2(s) \exp\left(\frac{\gamma_A}{\gamma_P + \gamma_A} c_1 s\right) ds \right\}^{\frac{\gamma_P + \gamma_A}{A}}$$

where

$$c_1 = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \frac{\gamma_P \kappa^{-1} (\gamma_P + \kappa^{-1})}{2(\gamma_P + \gamma_A + \kappa^{-1})} - \lambda \frac{\gamma_P + \gamma_A}{\gamma_A}$$

and

$$c_2(t) = \lambda \frac{\gamma_P + \gamma_A}{\gamma_A} \min \left\{ 1, \exp(\gamma_P i) \Phi_1(t, \theta) \right\}^{\frac{\gamma_A}{\gamma_P + \gamma_A}}.$$

solves in the classical sense the HJB equation (5.5.3). In particular $Z_t^* = \frac{\gamma_P + \kappa^{-1}}{\gamma_P + \gamma_A + \kappa^{-1}}$ and

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log \left(\frac{\Phi_0(t)}{\min \left\{ 1, \exp(\gamma_P i) \Phi_1(t, \theta) \right\}} \right).$$

Proof. The proof of this lemma is a direct adaptation of the proof of Lemma 5.4.2 to which we refer the reader. \square

The proof of the final result relies on the regularity of v_0 and a standard verification result. Because the controls are free of y , we deduce that $V_P = V_0(0, x, y_{PC})$. \square

The main change brought about by investment involves the halt related control K^* . Indeed the optimal Z^* in the Moral Hazard case are simply the optimal "Holmström-Milgrom" controls for the related production process. At first glance, the optimal control K^* seems to be quite different from that of Theorem 5.4.1. However one may verify that when we are in a setting where investment is never optimal (through the criteria of Lemma 5.5.1), the expression for K^* simplifies to exactly that of Theorem 5.4.1. The key to deduce this is that in such a setting, $\min \left\{ 1, \exp(\gamma_P i) \Phi_1(t, \theta) \right\} = 1$. We may therefore focus our analysis on the effects on investment when investing may be optimal (i.e. when $C_{inv} > 0$ and $i < TC_{inv}$). In such a setting, K^* has two phases :

- before t_{max} , K^* is adjusted to account for the possibility of risk mitigation
- after t_{max} , K^* has the same values as without mitigation. Indeed :

$$\min \left\{ 1, \exp(\gamma_P i) \Phi_1(t, \theta) \right\} = 1 \quad \text{for } t \geq t_{max}.$$

We are able to analyze the effect of different parameters and to do so represent the deterministic part of K^* as a function of time in the following figures.

We fix parameters $\gamma_P = \kappa = T = 1, \gamma_A = 0.5$: again this allows for mitigation to be optimal before some t_{max} .

We immediately observe that with mitigation, the value of K^* before t_{max} and is higher than without mitigation : the possibility for mitigation shrinks the opportunities for speculation (see Figures 5.19 to 5.21) and increasingly so as the probability of a halt increases. In fact the sign of K^* may now change over the duration of the contracting period : see Figure 5.21. Quite naturally, t_{max} varies with θ and i . Indeed it decreases as i increases or θ decreases : as the cost of investment increases and/or the level of degradation in continued production increases, more time is needed for investment for continued production to be worth it.

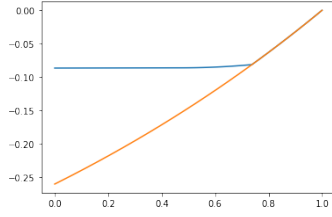


Figure 5.19 – $\lambda = 0.5$, $i = 0.1$, $\theta = 0.9$

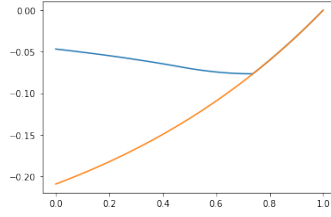


Figure 5.20 – $\lambda = 1$, $i = 0.1$, $\theta = 0.9$.

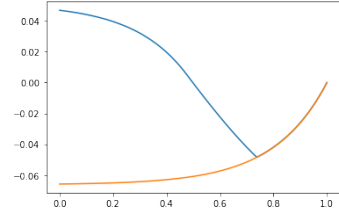


Figure 5.21 – $\lambda = 5$, $i = 0.1$, $\theta = 0.9$.

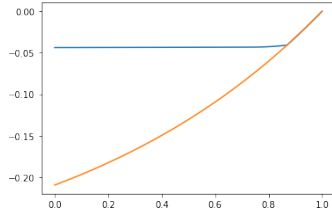


Figure 5.22 – $i = 0.05$, $\lambda = 1$, $\theta = 0.9$

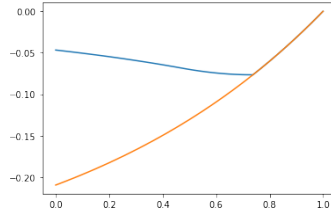


Figure 5.23 – $i = 0.1$, $\lambda = 1$, $\theta = 0.9$.

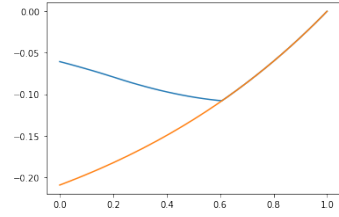


Figure 5.24 – $i = 0.15$, $\lambda = 1$, $\theta = 0.9$.

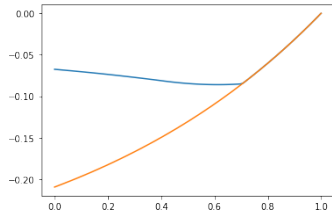


Figure 5.25 – $i = 0.1$, $\lambda = 1$, $\theta = 0.85$

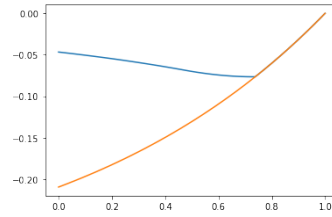


Figure 5.26 – $i = 0.1$, $\lambda = 1$, $\theta = 0.9$.

5.5.3 A note on BSDEs for mitigation

We sketch the martingale optimality principle arising from the Agent’s problem in the investment setting. We set $\mathbb{H}_{\mathbb{G}}^2$ is the set of \mathbb{G} -adapted processes Z with $\mathbb{E}[\int_0^T Z_s^2 ds] < +\infty$ and $\mathbb{S}_{\mathbb{G}}^2$ is the set of \mathbb{G} -predictable processes Y with cadlag paths such that Z with $\mathbb{E}[\sup_{t \in [0, T]} Y_t^2] < +\infty$. For a \mathbb{G}_τ -measurable random variable D with values in $\{0, 1\}$ that models the investment decision, we consider a contract W as a \mathbb{G}_T measurable r.v. which can be decomposed under the form :

$$W = W^0 \mathbf{1}_{T < \tau} + \left(W_T^{1,1}(\tau) \mathbf{1}_{D=1} + W_T^{1,0}(\tau) \mathbf{1}_{D=0} \right) \mathbf{1}_{\tau \leq T},$$

where W^0 is \mathcal{F}_T -measurable, and $W_t^{1,1}(u)$ and $W_t^{1,0}(u)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ measurable, with in particular $W_T^{1,0}(\tau) = W_{T \wedge \tau}^{1,0}(\tau)$. Given a contract W , the Agent faces the following control problem,

$$\sup_{a \in \mathcal{B}} \mathbb{E}^{\mathbb{P}^a} \mathbb{E} \left[U_A \left(W - \int_0^T \kappa(a_s) ds \right) \right].$$

Remember that an effort process is now a \mathbb{G} -adapted process $(a_t)_{t \in [0, T]}$ and consequently has the form :

$$a_t = a_t^0 \mathbf{1}_{t < \tau} + a_t^1(\tau) \mathbf{1}_{t \geq \tau}$$

where a^0 is \mathbb{F} -adapted and a^1 is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ measurable and where both are assumed bounded by some constant A . By convention, we still denote by \mathcal{B} the set of such effort.

Lemma 5.5.3. *Suppose that there exists some unique triplet (Y, Z, K) in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ such that :*

$$Y_t = W - \int_t^T Z_s((1 - N_s) + N_s \mathbf{1}_{D=1}) dB_s - \int_t^T K_s(1 - N_s) dM_s - \int_t^T f(s, Z_s, K_s) ds,$$

where

$$\begin{aligned} f(s, Z_s, K_s) &= \left(\lambda K_s + \frac{\lambda}{\gamma_A} (e^{-\gamma_A K_s} - 1) \right) (1 - N_s) \\ &\quad + \frac{1}{2} \gamma_A Z_s^2 ((1 - N_s) + N_s \mathbf{1}_{D=1}) + \inf_{a \in \mathcal{B}} \{ \kappa(a_s) - a_s Z_s (1 - N_s) - \theta a_s Z_s \mathbf{1}_{\tau \leq s, D=1} \}, \end{aligned}$$

then

$$R_t^a = U_A \left(Y_t - \int_0^t \kappa(a_s) ds \right)$$

satisfies a Martingale Optimality Principle for the Agent's problem in this setting.

Proof. By construction, $R_T^a = U_A \left(W - \int_0^T \kappa(a_s) ds \right)$ and R_0^a is independent of the Agent's action a . As in Section 5.4, we compute the variations of R^a to obtain:

$$\begin{aligned} dR_s^a &= -\gamma_A R_s^a Z_s ((1 - N_s) + N_s \mathbf{1}_{D=1}) dB_s^a + R_s^a (e^{-\gamma_A K_s} - 1) (1 - N_s) dM_s \\ &\quad + R_s^a \gamma_A \left(\frac{1}{2} \gamma_A Z_s^2 ((1 - N_s) + N_s \mathbf{1}_{D=1}) - f(s, Z_s, K_s) + \kappa(a_s) + \left(\lambda K_s + \frac{\lambda}{\gamma_A} (e^{-\gamma_A K_s} - 1) \right) (1 - N_s) \right) \\ &\quad + R_s^a \gamma_A (-a_s Z_s (1 - N_s) + \theta a_s N_s \mathbf{1}_{D=1}) \\ &= -\gamma_A R_s^a Z_s ((1 - N_s) + N_s \mathbf{1}_{D=1}) dB_s^a + R_s^a (e^{-\gamma_A K_s} - 1) (1 - N_s) dM_s \\ &\quad + R_s^a \gamma_A \left(\frac{1}{2} \gamma_A Z_s^2 ((1 - N_s) + N_s \mathbf{1}_{D=1}) - f(s, Z_s, K_s) + \kappa(a_s^0 (1 - N_s)) + \kappa(\theta a_s^1 N_s \mathbf{1}_{D=1}) \right) \\ &\quad + R_s^a \gamma_A \left(\left(\lambda K_s + \frac{\lambda}{\gamma_A} (e^{-\gamma_A K_s} - 1) \right) (1 - N_s) \right) \\ &\quad + R_s^a \gamma_A \left(-a_s^0 Z_s (1 - N_s) + \theta a_s^1 N_s \mathbf{1}_{D=1} \right) \end{aligned}$$

and therefore \mathbb{R}^a is a super-martingale for every a in \mathcal{B} . Setting :

$$a_s^{0*}(z) = -A \mathbf{1}_{\frac{z}{\kappa} < -A} + A \mathbf{1}_{\frac{z}{\kappa} > A} + \frac{z}{\kappa} \mathbf{1}_{-A \leq \frac{z}{\kappa} \leq A}$$

and

$$a_s^{1*}(z) = -A \mathbf{1}_{\frac{\theta z}{\kappa} \leq -A} + A \mathbf{1}_{\frac{\theta z}{\kappa} > A} + \frac{\theta z}{\kappa} \mathbf{1}_{-A \leq \frac{\theta z}{\kappa} \leq A},$$

then

$$a_t^* = a_t^{0*} \mathbf{1}_{t < \tau} + a_t^{1*} \mathbf{1}_{t \geq \tau}.$$

We get that \mathbb{R}^{a^*} is a \mathbb{P}^{a^*} -martingale and the Agent's response given W is then a^* . □

It remains to show that there actually exists a unique solution to (Y, Z, K) in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ to :

$$Y_t = W - \int_t^T Z_s((1 - N_s) + N_s \mathbf{1}_{D=1}) dB_s - \int_t^T K_s(1 - N_s) dM_s - \int_t^T f(s, Z_s, K_s) ds,$$

where

$$\begin{aligned} f(s, Z_s, K_s) &= \left(\lambda K_s + \frac{\lambda}{\gamma_A} (e^{-\gamma_A K_s} - 1) \right) (1 - N_s) \\ &+ \frac{1}{2} \gamma_A Z_s^2 ((1 - N_s) + N_s \mathbf{1}_{D=1}) + \inf_{a \in \mathcal{B}} \{ \kappa(a_s) - a_s Z_s (1 - N_s) - \theta a_s Z_s \mathbf{1}_{\tau \leq s, D=1} \}. \end{aligned}$$

We give a sketch of the approach for this. First note for any s in $[0, T]$ fixed, and for any $t \in [s, T]$, conditionally to $\{\tau \leq T\} \cap \{D = 1\}$ there exists a unique pair $(Y^i, Z^i) \in \mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ solution to the BSDE :

$$Y_t^i(s) = W_T^{1,1}(s) - \int_t^T f^1(Z_s^i(s)) ds - \int_t^T Z_s^i(s) dB_s, \quad (5.5.4)$$

where $f^1(z) = \frac{1}{2} \gamma_A z^2 + \inf_{a \in \mathcal{B}} (\kappa(a) - \theta a z)$ and where the notation $(Y^i(s), Z^i(s))$ is used to emphasize the dependency in s of the terminal condition and its effect on the solution. This existence result simply follows from the fact that for each s , (5.5.4) is now simply a Brownian BSDE that fits into the classical quadratic setting of Briand and Hu. We may then set :

$$\tilde{W} = Y_\tau^i(\tau) \mathbf{1}_{\tau \leq T} \mathbf{1}_{D=1} + W_{T \wedge \tau}^{1,0}(\tau) \mathbf{1}_{\tau \leq T} \mathbf{1}_{D=0} + W^0 \mathbf{1}_{T < \tau},$$

which is a $\mathbb{G}_{T \wedge \tau}$ measurable random-variable and will have exponential moments for any α . Indeed, the first term comes from the solution of a quadratic Brownian BSDE for which the terminal condition has any exponential moment. Thus the solution to the BSDE will have any exponential moment. The two other terms have exponential moments through the assumption on W . We set :

$$f^2(z, k) = \frac{1}{2} \gamma_A z^2 + \lambda k + \frac{\lambda}{\gamma_A} (e^{-\gamma_A k} - 1) + \inf_{a \in \mathcal{B}} (\kappa(a) - a Z).$$

This fits right into the setting of the recent work [92] (also in Chapter 6 of this thesis) on a default BSDE for Principal Agent problems. In particular, there exists a unique triplet $(\tilde{Y}, \tilde{Z}, \tilde{K})$ in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ such that :

$$\tilde{Y}_t = \tilde{W} - \int_{t \wedge \tau}^{T \wedge \tau} \tilde{Z}_s dB_s - \int_{t \wedge \tau}^{T \wedge \tau} \tilde{K}_s dM_s - \int_{t \wedge \tau}^{T \wedge \tau} f^2(\tilde{Z}_s, \tilde{K}_s) ds.$$

Finally, setting :

- $Y_t = \tilde{Y}_t(1 - N_t) + Y_t^i(\tau) N_t \mathbf{1}_{D=1}$
- $Z_t = \tilde{Z}_t(1 - N_t) + Z_t^i(\tau) N_t \mathbf{1}_{D=1}$
- $K_t = \tilde{K}_t(1 - N_t)$

and noting that :

$$f(s, z, k) = f^1(z)N_s\mathbf{1}_{D=1} + f^2(z, k)(1 - N_s),$$

we obtain that (Y, Z, K) is a solution to :

$$Y_t = W - \int_t^T Z_s((1 - N_s) + N_s\mathbf{1}_{D=1})dB_s - \int_t^T K_s(1 - N_s)dM_s - \int_t^T f(s, Z_s, K_s)ds. \quad (5.5.5)$$

Finally, uniqueness holds through a classical reasoning, noting that up to a change in probability, Y is a local martingale that has continuous paths on a certain left-hand neighbourhood of T .

Chapter 6

A BSDE with default jump and unbounded terminal value arising in a Principal-Agent context

This Chapter is a slightly modified version of a submitted preprint which can be found at [92].

6.1 A BSDE for the Agent's problem in Chapter 5

6.1.1 A technical issue arising in the Agent's problem from Chapter 5

This Chapter is devoted to the proof of a technical result arising in Chapter 5 which is presented and used in Section 5.4 for analysis of the Agent's problem in Moral Hazard under a risk of economic shutdown. We recall that such a problem asks what the Agent's optimal action may be given a compensation package. More precisely, given a compensation package W satisfying relevant integrability and measurability properties, the goal is to find a in \mathcal{A} where \mathcal{A} models the set of possible actions that the Agent may perform (and will be further specified later) solving :

$$\sup_{a \in \mathcal{A}} \mathbb{E} \left[U \left(W - \int_0^T \kappa(a_s) ds \right) \right], \quad (6.1.1)$$

where $[0, T]$ with $T > 0$ models a contracting period, κ is an effort function, U a utility function which models the Agent's preference. This maximization problem therefore aims to model the need for a balance between a compensation package and the amount of effort that it would require from the Agent.

Such a problem falls into the setting of Section 1.2.2 of the introductory Chapter, where it is mentioned that it may be tackled using tools from mathematical finance in the vein of the work of Sannikov in [116], but also Cvitanic, Possamai and Touzi in [32] and [31]. This now classical approach involves establishing a link between each compensation package W and the Agent's related optimal effort $(a_t)_{t \in [0, T]}$ through a Backward Stochastic Differential Equation

(BSDE). The crux of the problem then becomes the existence and uniqueness of solutions to the derived BSDE. Sometimes existence and uniqueness of solutions to the underlying BSDE may be tackled using existing literature (in the Brownian literature for example one may call for results such as those of [20] or [108]) or quickly recovered using standard techniques : in Chapter 3 of this thesis the compound-Poisson based BSDE that arises for the Agent's problem is based on a Lipschitz driver and existence therefore follows quite quickly. When it comes to the Agent's problem in Chapter 4, we are faced with a probabilistic setting involving both a Brownian motion and a default process. In such a setting some existence and uniqueness results exist for BSDEs such as those derived in [71], [45] and [80]. However to the best of our knowledge at the time of writing this thesis, the result needed for our specific Principal-Agent problem falls just outside the boundaries of this existing literature. The goal of the following is to overcome such a limitation. To this end, the rest of this Chapter is structured as follows. First, we present the BSDE for which we need an existence and uniqueness result. Then we discuss the current literature on such BSDEs before stating the result in Theorem 6.1.1 and linking it to the Principal-Agent problem. The remainder of the Chapter is then dedicated to the main result.

The BSDE of interest

To present the BSDE, we first fix the setting under which we wish to consider Problem (6.1.1). Of course this setting very much resembles that of Chapter 5 and we recall it for completeness of this Chapter. We denote as $T > 0$ a fixed time horizon and consider $(\Omega, \mathcal{G}, \mathbb{P})$ a complete probability space on which two independent stochastic processes evolve :

- $(B_t)_{t \in [0, T]}$ a standard unidimensional Brownian motion,
- $(N_t)_{t \in [0, T]}$ a single-jump process of the form $N_t = \mathbf{1}_{\tau \leq t}$, $t \in [0, T]$ where τ is some random variable modeling a default time.

We denote as $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ the natural completed filtration generated by the Brownian motion and as $\mathbb{G} = \{\mathcal{G}_t, t \in [0, T]\}$ the enlarged filtration generated by $(\mathcal{F}_{t \wedge \tau})_{t \in [0, T]}$ and $(N_t)_{t \in [0, T]}$. Crucially, as B is independent of τ , B remains a \mathbb{G} -Brownian motion (see for example [2]).

It is assumed throughout that the process N admits an absolutely continuous and deterministic compensator, denoted Λ where $\Lambda_t = \int_0^t \lambda_s ds$ for some integrable mapping $s \mapsto \lambda_s$. This implies that :

$$M_t := N_t - \int_0^t \lambda_s (1 - N_s) ds, \quad t \in [0, T],$$

is a \mathbb{G} -compensated martingale. Note that as the compensator is deterministic, there exists some constant L satisfying $L = \sup_{t \in [0, T]} \lambda_t$. We suppose also that $l = \inf_{t \in [0, T]} \lambda_t > 0$.

In relation with \mathbb{F} and \mathbb{G} , we introduce the following spaces :

- $\mathcal{P}(\mathbb{F})$ the set of \mathbb{F} -predictable processes valued in \mathbb{R}
- $\mathcal{P}(\mathbb{G})$ the set of \mathbb{G} -predictable processes valued in \mathbb{R}
- $\mathbb{H}_{\mathbb{F}}^2 = \left\{ Z \in \mathcal{P}(\mathbb{F}), \mathbb{E} \left[\int_0^T |Z_t|^2 dt \right] < +\infty \right\}$

- $\mathbb{H}_{\mathbb{G}}^2 = \left\{ Z \in \mathcal{P}(\mathbb{G}), \mathbb{E} \left[\int_0^T |Z_t|^2 dt \right] < +\infty \right\}$
- $\mathbb{S}_{\mathbb{F}}^2 = \left\{ Y \in \mathcal{P}(\mathbb{F}), \text{ with continuous paths and } \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^2 \right] < +\infty \right\}$
- $\mathbb{S}_{\mathbb{G}}^2 = \left\{ Y \in \mathcal{P}(\mathbb{G}), \text{ with cadlag paths and } \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t|^2 \right] < +\infty \right\}$
- $\mathbb{S}_{\mathbb{F}}^{\infty} = \left\{ Y \in \mathcal{P}(\mathbb{F}), \text{ with continuous paths and } \sup_{t \in [0, T]} |Y_t| < +\infty \right\}$

These spaces allow us to specify the variant of Problem (6.1.1) of interest. Given a wage modeled by an unbounded and $\mathcal{G}_{T \wedge \tau}$ -measurable random variable W such that $\mathbb{E}[\exp(\alpha W)] < +\infty$ for all α in \mathbb{R} , we wish to solve :

$$\sup_{a \in \mathcal{E}} \mathbb{E}^{\mathbb{P}^a} \left[U \left(W - \int_0^T \frac{a_s^2}{2} (1 - N_s) ds \right) \right], \quad (6.1.2)$$

where $U(x) = -\exp(-\gamma x)$ for some fixed $\gamma > 0$ is a classical exponential utility function and

$$\mathcal{E} = \left\{ (a_t)_{t \in [0, T]} \text{ in } \mathbb{H}_{\mathbb{G}}^2 \text{ with } a_t \in [-A, A] \quad \forall t \in [0, T]; a_t = 0 \quad \forall t \geq \tau. \right\},$$

and finally :

$$\frac{d\mathbb{P}^a}{d\mathbb{P}} = \exp \left(\int_0^T a_s (1 - N_s) dB_s - \frac{1}{2} \int_0^T a_s^2 (1 - N_s) ds \right).$$

As any process a in \mathcal{E} is bounded, the probability measures \mathbb{P}^a and \mathbb{P} are equivalent and the process $(B_{t \wedge \tau}^a)_{t \in [0, T]}$ with $B_{t \wedge \tau}^a = B_{t \wedge \tau} - \int_0^{t \wedge \tau} a_s ds$, $t \in [0, T]$ is a \mathbb{P}^a -Brownian motion.

As developed Remark 5.4.1 of Section 5.4 of Chapter 5, Problem (6.1.2) may be linked through a Martingale Optimality Principle (given in Lemma 5.4.1) to the solutions (Y, Z, K) in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ (if they exist) to the following BSDE :

$$Y_t = W - \int_{t \wedge \tau}^{T \wedge \tau} Z_s dB_s - \int_{t \wedge \tau}^{T \wedge \tau} K_s dN_s + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Z_s, K_s) ds, \quad (6.1.3)$$

where :

$$f(t, z, k) = -\gamma \frac{z^2}{2} + g(z) - \frac{\lambda_t}{\gamma} (e^{-\gamma k} - 1), \quad (6.1.4)$$

where g is some continuous mapping defined as :

$$g(z) := - \inf_{a \in [-A, A]} \left(\frac{a^2}{2} - az \right) = \sup_{a \in [-A, A]} \left(-\frac{a^2}{2} + az \right).$$

Note that this mapping is positive. Indeed $a \mapsto -\frac{a^2}{2} + az$ is worth 0 at 0, and 0 belongs to $[-A, A]$. Also note that $g(z) \leq \frac{z^2}{2}$. As a consequence, in the following, we will study the BSDE (6.1.3) with g in (6.1.4) assumed to satisfy the slightly more general assumption :

$$0 \leq g(z) \leq C \left(1 + \frac{z^2}{2} \right).$$

for some $C > 0$.

Having introduced the BSDE (6.1.3), the following definition specifies what we will consider as its solutions.

Definition 6.1.1. *A triplet (Y, Z, K) in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ is a solution of BSDE (6.1.3) if relation (6.1.3) is satisfied for every t in $[0, t \wedge T]$, with :*

- $Y_t = Y_{T \wedge \tau}$ for $t \geq T \wedge \tau$,
- $Z_t = 0$ and $K_t = 0$ for t on $(T \wedge \tau, T)$.

and

$$\mathbb{E} \left[\int_0^{T \wedge \tau} |f(t, Z_t, K_t)| dt + \left(\int_0^{T \wedge \tau} |Z_t|^2 dt \right)^{\frac{1}{2}} \right] < +\infty. \quad (6.1.5)$$

6.1.2 Setting in the existing literature and main result for BSDE (6.1.3)

The backward stochastic differential equation defined through the dynamics (6.1.3) differs slightly from those considered in the existing vast literature on BSDEs of which an overview is given in Section 1.2.2 of Chapter 1. Of course, the initial literature on BSDEs such as the founding work of Pardoux and Peng [108] or its extension [38] focused on the solely Brownian case and the extra default related term in (6.1.3) is a departure from such literature. It is nevertheless important to note that the equation does bear some resemblance to some well-known Brownian BSDEs. Indeed, the Brownian term has quadratic growth and such setting was studied by Briand and Hu in [20]. When it comes to BSDEs in a mixed Brownian / default setting, several results already exist. In [45], the authors develop some results for a BSDE with a Lipschitz driver and a square integrable terminal condition. In [71], a very similar BSDE to (6.1.3) is tackled with a more general compensator but with the specificity of a bounded terminal condition. This condition on the terminal condition is not restrictive for their financial setting but not quite satisfactory for our dynamic contracting setting : to counterbalance this, we have a deterministic compensator. Our setting therefore falls between these two works, with some similarities and differences with both cases. On the one hand, we consider an unbounded terminal condition, which therefore presents similarities with [45]. The main departure in this case is the non-Lipschitz driver f . On the other hand, we study a driver that is very similar to that in [71] but cannot apply their results due to our non-bounded terminal condition.

Due to the form of the driver, one may be tempted at first glance to deal with existence for (6.1.3) through a Cole-Hopf type transform. In fact, in the setting where $g = 0$, this is possible. Indeed in such a case, the BSDE solved by a triplet (Y, Z, K) writes as :

$$Y_t = W - \int_{t \wedge \tau}^{T \wedge \tau} Z_s dB_s - \int_{t \wedge \tau}^{T \wedge \tau} K_s dN_s + \int_{t \wedge \tau}^{T \wedge \tau} \left\{ -\gamma \frac{Z_s^2}{2} - \frac{\lambda_s}{\gamma} (e^{-\gamma K_s} - 1) \right\} ds,$$

and setting $\mathcal{Z} = -\gamma e^{-\gamma Y} Z$ and $\mathcal{K} = e^{-\gamma Y} (1 - e^{-\gamma K})$:

$$e^{-\gamma Y_t} = e^{-\gamma W} - \int_{t \wedge \tau}^{T \wedge \tau} \mathcal{Z}_s dB_s - \int_{t \wedge \tau}^{T \wedge \tau} \mathcal{K}_s dM_s. \quad (6.1.6)$$

Now let some $t \leq \tau$ \mathbb{P} -a.s., then there exists a sequence of localization times $(\tau_n)_{n \in \mathbb{N}}$ such that $e^{-\gamma Y_{t \wedge \tau_n}} = \mathbb{E}[e^{-\gamma W} | \mathcal{G}_t]$ and therefore $e^{-\gamma Y_t} = \mathbb{E}[e^{-\gamma W} | \mathcal{G}_t]$. A closed form expression for Y is reached :

$$Y_t = -\frac{1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W} | \mathcal{G}_t] \right)$$

We can then recover \mathcal{Z} and \mathcal{K} (and thus Z and K) through a martingale representation theorem in a default jump setting (see for example [45]). Of course for $t > \tau$, we have the closed form : $Y_t = Y_\tau, Z_t = 0, K_t = 0$. This approach does not yield such a satisfactory result when we consider a more general g as it results in (6.1.6) taking on an extra driver term. More precisely performing a Cole-Hopf transform with $x \mapsto e^{-\gamma x}$ leads to the appearance of a term related to g, Z and Y . Performing such a transform with $x \mapsto e^{(C-\gamma)x}$ (as $0 \leq g(z) \leq C(1 + \frac{z^2}{2})$) leads to the appearance of terms related to K . In both of these cases, we cannot conclude as above and to the best of our knowledge and calculations, a "simplifying" transformation is not tractable. However this "Cole-Hopf failure" is very much reminiscent of the problem tackled by Briand and Hu in [20] in the solely Brownian case and the following work can in fact be seen as an extension of [20] to a setting that includes a default process (noting that the driver here is simpler than theirs in the sense that it does not depend on Y). The result obtained through this work falls between the results of [45] and [71] and writes as follows.

Theorem 6.1.1 (Main result). *Suppose that for any $\alpha \in \mathbb{R}$:*

$$\mathbb{E}[\exp(\alpha W)] < +\infty \tag{6.1.7}$$

then there exists a unique triplet (Y, Z, K) in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ solution to the BSDE (6.1.3).

Proof. Existence is dealt with in Section 6.2 whilst uniqueness is proven in Section 6.3. \square

Before turning to the proof of this result, we present its link with Problem (6.1.2).

6.1.3 The consequences of Theorem 6.1.1 for contracting under a risk of shut-down

In the following we complete Section 5.4.1 of Chapter 5 and show how the solutions to our BSDE are useful for Problem (6.1.2) with a slight restriction on the random variable W . In particular, we consider random variables W that we can write as :

$$W = y_0 + \int_0^{T \wedge \tau} Z_t dB_t + \int_0^{T \wedge \tau} K_t dN_t - \int_0^{T \wedge \tau} f(t, Z_t, K_t) dt, \tag{6.1.8}$$

with $(Z, K) \in \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ bounded (with $Z_t \mathbf{1}_{t > \tau} = K_t \mathbf{1}_{t > \tau} = 0$), and y_0 some real. This restriction is useful to fit into the setting of Theorem 6.1.1 as discussed in the following remark.

Remark 6.1.1. *Under such assumptions, W satisfies $\mathbb{E}[\exp(\alpha W)] < +\infty$. Indeed given such a W (characterized by Z, K and y_0), let :*

$$Y_t = y_0 + \int_0^{t \wedge \tau} Z_t dB_t + \int_0^{t \wedge \tau} K_t dN_t - \int_0^{t \wedge \tau} f(t, Z_t, K_t) dt, \quad W = Y_T.$$

Then $\forall t$ in $[0, T]$, there exists a sequence of localization times $(\tau_n)_{n \in \mathbb{N}}$ such that :

$$\mathbb{E}[\exp(\alpha Y_{t \wedge \tau_n})] = \exp(\alpha y_0) + \mathbb{E} \left[\int_0^{\tau_n \wedge T \wedge \tau} \exp(\alpha Y_{s-}) h(Z_s, K_s) ds \right],$$

for h some continuous function of (Z, K) . As (Z, K) are assumed bounded, there exists some positive deterministic and bounded function β_s such that :

$$\begin{aligned} \mathbb{E}[\exp(\alpha Y_{t \wedge \tau_n})] &\leq \exp(\alpha y_0) + \mathbb{E} \left[\int_0^{\tau_n \wedge T \wedge \tau} \exp(\alpha Y_{s-}) \beta_s ds \right] \\ &\leq \exp(\alpha y_0) + \mathbb{E} \left[\int_0^T \exp(\alpha Y_{s-}) \beta_s ds \right] \\ &= \exp(\alpha y_0) + \int_0^T \mathbb{E} [\exp(\alpha Y_{s-})] \beta_s ds \end{aligned}$$

But through Fatou's lemma, $\mathbb{E}[\exp(\alpha Y_t)] \leq \lim_{n \rightarrow +\infty} \mathbb{E}[\exp(\alpha Y_{t \wedge \tau_n})]$, and we deduce :

$$\mathbb{E}[\exp(\alpha Y_t)] \leq \exp(\alpha y_0) + \int_0^T \mathbb{E} [\exp(\alpha Y_{s-})] \beta_s ds.$$

We conclude through Gronwall's lemma that $\mathbb{E}[\exp(\alpha Y_t)]$ is bounded for any $t \in [0, T]$. In particular, $\mathbb{E}[\exp(\alpha Y_T)] = \mathbb{E}[\exp(\alpha W)] < +\infty$, and Theorem 6.1.1 holds for W . In particular given the processes (Z, K) that are used to define W in (6.1.8), setting :

$$Y_t = W - \int_{t \wedge \tau}^{T \wedge \tau} Z_t dB_t - \int_{t \wedge \tau}^{T \wedge \tau} K_t dM_t + \int_{t \wedge \tau}^{T \wedge \tau} (f(t, Z_t, K_t) - \lambda_t K_t) dt$$

provides the unique solution triplet (Y, Z, K) to (6.1.3) for W .

With this crucial integrability property in mind we have the following result linking Theorem 6.1.1 with Problem (6.1.2).

Corollary 6.1.1. *Given W of the form (6.1.8) with Z and K bounded and fixed, set Y in $\mathbb{S}_{\mathbb{G}}^2$ such that :*

$$Y_t = W - \int_{t \wedge \tau}^{T \wedge \tau} Z_t dB_t - \int_{t \wedge \tau}^{T \wedge \tau} K_t dM_t + \int_{t \wedge \tau}^{T \wedge \tau} (f(t, Z_t, K_t) - \lambda_t K_t) dt.$$

Then given $a \in \mathcal{E}$, the process $(R_t^a)_{t \in [0, T]}$ defined as :

$$R_t^a = U \left(Y_t - \int_0^{t \wedge \tau} \frac{a_s^2}{2} ds \right),$$

satisfies a Martingale Optimality Principle and the optimal process $(a_t)_{t \in [0, T]}$ solution to (6.1.2) is given through :

$$a_t^* = \arg \inf_{a \in [-A, A]} \left(\frac{a^2}{2} - a Z_t \right).$$

Proof. First note that R_0 is free of a and :

$$R_T^a = U \left(W - \int_0^{T \wedge \tau} \frac{a_s^2}{2} ds \right)$$

Now we compute the variations of R^a and obtain¹ :

$$\begin{aligned} dR_t^a &= -\gamma R_t^a Z_t dB_t + R_t^a (e^{-\gamma K_t} - 1) dM_t \\ &\quad + R_t^a \gamma \left(\frac{\gamma}{2} Z_t^2 + f(t, Z_t, K_t) - \lambda_t K_t + \frac{a_t^2}{2} + \lambda_t K_t + \frac{\lambda_t}{\gamma} (e^{-\gamma K_t} - 1) \right) dt \\ &= -\gamma R_t^a Z_t dB_t^a + R_t^a (e^{-\gamma K_t} - 1) dM_t \\ &\quad + R_t^a \gamma \left(\frac{\gamma}{2} Z_t^2 + f(t, Z_t, K_t) + \frac{a_t^2}{2} + \frac{\lambda_t}{\gamma} (e^{-\gamma K_t} - 1) - a_t Z_t \right) dt \end{aligned}$$

where we recall that :

$$f(t, z, k) = -\gamma \frac{z^2}{2} - \inf_{a \in [-A, A]} \left(\frac{a^2}{2} - az \right) - \frac{\lambda_t}{\gamma} (e^{-\gamma k} - 1).$$

Therefore :

$$dR_t^a = -\gamma R_t^a Z_t dB_t^a + R_t^a (e^{-\gamma K_t} - 1) dM_t + R_t^a \gamma \left(\frac{a_t^2}{2} - a_t Z_t - \inf_{a \in [-A, A]} \left(\frac{a_t^2}{2} - a_t Z_t \right) \right) dt.$$

But by construction :

$$\frac{a_t^2}{2} - a_t Z_t - \inf_{a \in [-A, A]} \left(\frac{a^2}{2} - az \right) \geq 0$$

and as R^a is negative :

$$R_t^a \gamma \left(\frac{a_t^2}{2} - a_t Z_t - \inf_{a \in [-A, A]} \left(\frac{a_t^2}{2} - a_t Z_t \right) \right) \leq 0 \quad \mathbb{P} - a.s.$$

Therefore for any a , R^a is a supermartingale. Furthermore setting :

$$a_t^* = \arg \inf_{a \in [-A, A]} \left(\frac{a^2}{2} - a Z_t \right),$$

we have :

$$dR_t^{a^*} = -\gamma R_t^{a^*} Z_t dB_t^{a^*} + R_t^{a^*} (e^{-\gamma K_t} - 1) dM_t \quad ; \quad R_0^{a^*} = -1.$$

Therefore :

$$R_t^{a^*} = -\mathcal{E} \left(\int_0^t -\gamma Z_s dB_s^{a^*} \right) \mathcal{E} \left(\int_0^t (e^{-\gamma K_s} - 1) dM_s \right).$$

Through the boundedness of Z and Novikov's criterion, the first stochastic exponential is a true \mathbb{P}^{a^*} -martingale. Now let $X_t = \int_0^t (e^{-\gamma K_s} - 1) dM_s$. Through the boundedness of K , this is a true martingale. Furthermore, $\Delta_\tau X = -1 + e^{-\gamma K_\tau} \geq -1 + \delta$ for some $\delta > 0$ and

¹note that these variations also hold after τ as in that case by definition $a = Z = K = 0$.

$|\Delta_\tau X| \leq 1 + e^{-\gamma K\tau} \leq c$ for some $c > 0$ (both due to the boundedness of K). Therefore through Theorem 2 of [69], the second stochastic exponential is a \mathbb{P}^{α^*} -martingale too.

We conclude that for any a in \mathcal{E} :

$$\mathbb{E}^{\mathbb{P}^\alpha} [R_T^a] \leq \mathbb{E}^{\mathbb{P}^\alpha} [R_0^a] = \mathbb{E}^{\mathbb{P}^{\alpha^*}} [R_0^{\alpha^*}] = \mathbb{E}^{\mathbb{P}^{\alpha^*}} [R_T^{\alpha^*}],$$

yielding the conclusion. \square

Remark 6.1.2. *The boundedness that we assume for (Z, K) in (6.1.8) is used in the above proof in order to ensure that the stochastic exponentials that define R^{α^*} are true martingales and in Remark 6.1.1 in order to ensure exponential integrability of W . This boundedness assumption can therefore if needed be extended to any assumption that ensures that the two exponentials are true martingales and that W has exponential integrability.*

6.2 Proof of existence of solutions to (6.1.3)

The following section is dedicated to proving the existence part of Theorem 6.1.7 through the following steps :

1. reducing the problem to proving existence of solutions to a Brownian BSDE, as done in [71],
2. adapting the approach of [20] to deal with the Brownian BSDE.

6.2.1 Link with solution existence to a well-chosen Brownian BSDE (6.2.5)

Completing the first step calls for the following result of [71].

Lemma 6.2.1 (Slight extension of Lemma 3.1 of [71]). *Let Z be any $\mathcal{G}_{T \wedge \tau}$ -measurable random variable such that for any α in \mathbb{R} :*

$$\mathbb{E}[\exp(\alpha Z)] < +\infty.$$

Then there exists some \mathcal{F}_T -measurable random variable Z^b and some \mathbb{F} -predictable process Z^a such that :

$$Z = Z^b \mathbf{1}_{T < \tau} + Z^a \mathbf{1}_{\tau \leq T},$$

where for any α in \mathbb{R} ,

$$\mathbb{E}[\exp(\alpha Z^b)] < +\infty \quad \text{and} \quad \mathbb{E}[\exp(\alpha Z_t^a)] < +\infty, \quad \text{a.e. on } [0, T],$$

and :

$$\int_0^T \mathbb{E}[\exp(\alpha Z_s^a)] \lambda_s ds < +\infty.$$

Proof. The proof of the decomposition can be found in the proof of Lemma 3.1 of [71]. The integrability properties come from the fact that the decomposition is as follows :

$$Z = Z \mathbf{1}_{T < \tau} + Z \mathbf{1}_{\tau \leq T} := Z^b \mathbf{1}_{T < \tau} + Z^a \mathbf{1}_{\tau \leq T},$$

and the integrability properties of Z carry to Z^b and Z^a . Indeed :

$$\begin{aligned}\mathbb{E}[\exp(\alpha Z)] &= \mathbb{E}[\exp(\alpha Z)\mathbf{1}_{T < \tau}] + \mathbb{E}[\exp(\alpha Z)\mathbf{1}_{\tau \leq T}] \\ &= \mathbb{E}[\exp(\alpha Z^b)\mathbf{1}_{T < \tau}] + \mathbb{E}[\exp(\alpha Z^a)\mathbf{1}_{\tau \leq T}] \\ &= \mathbb{E}[\exp(\alpha Z^b)]\mathbb{P}[T < \tau] + \mathbb{E}\left[\int_0^T \exp(\alpha Z_s^a)\lambda_s \exp\left(-\int_0^s \lambda_u du\right) ds\right] \\ &= \mathbb{E}[\exp(\alpha Z^b)]\mathbb{P}[T < \tau] + \int_0^T \mathbb{E}[\exp(\alpha Z_s^a)]\lambda_s \exp\left(-\int_0^s \lambda_u du\right) ds\end{aligned}$$

which is the sum of two positive terms. Therefore $\mathbb{E}[\exp(\alpha Z)] < +\infty$ implies that :

$$\mathbb{E}[\exp(\alpha Z^b)]\mathbb{P}[T < \tau] < +\infty,$$

implying that $\mathbb{E}[\exp(\alpha Z^b)] < +\infty$, and :

$$\int_0^T \mathbb{E}[\exp(\alpha Z_s^a)]\lambda_s \exp\left(-\int_0^s \lambda_u du\right) ds < +\infty,$$

which implies that $\mathbb{E}[\exp(\alpha Z_t^a)] < +\infty$, a.e. on $[0, T]$ and also as $\exp(-\int_0^s \lambda_u du) \geq \exp(-LT) > 0$:

$$\int_0^T \mathbb{E}[\exp(\alpha Z_s^a)]\lambda_s ds < +\infty,$$

yielding the conclusion. \square

This lemma may be applied to the terminal condition W which is $\mathcal{G}_{T \wedge \tau}$ -measurable. There thus exists some \mathcal{F}_T -measurable W^b and some \mathbb{F} -predictable process W^a such that :

$$W = W^b\mathbf{1}_{T < \tau} + W^a\mathbf{1}_{\tau \leq T}. \quad (6.2.1)$$

In particular W^a and W^b satisfy :

$$\mathbb{E}[\exp(\alpha W^b)] < +\infty \quad \text{and} \quad \mathbb{E}[\exp(\alpha W_t^a)] < +\infty, \text{ a.e. on } [0, T], \quad (6.2.2)$$

and :

$$\int_0^T \mathbb{E}[\exp(\alpha W_s^a)]\lambda_s ds < +\infty. \quad (6.2.3)$$

As a consequence we may derive the following further integrability properties for $(W_t^a)_{t \in [0, T]}$ and W^b .

Lemma 6.2.2. *Let W^a and W^b satisfy (6.2.2). Then :*

- for any $\gamma > 0$, $\mathbb{E}[\exp(\gamma|W^b|)] < +\infty$ and $\mathbb{E}[\exp(\gamma|W_t^a|)] < +\infty$, a.e. on $[0, T]$
- $\int_0^T \mathbb{E}[\exp(\gamma|W_s^a|)]\lambda_s ds < +\infty$.

Sketch of proof. The proofs all follow the same reasoning and are consequences of the above equations. For example for the first property :

$$\begin{aligned}\mathbb{E}[\exp(\gamma|W^b|)] &= \mathbb{E}[\exp(\gamma W^b)\mathbf{1}_{W^b \geq 0}] + \mathbb{E}[\exp(-\gamma W^b)\mathbf{1}_{W^b < 0}] \\ &\leq \mathbb{E}[\exp(\gamma W^b)] + \mathbb{E}[\exp(-\gamma W^b)] < +\infty,\end{aligned}$$

using (6.2.2) for $\alpha = \gamma$ and $\alpha = -\gamma$. \square

Now in order to link BSDE (6.1.3) to a Brownian BSDE, we define solutions to a BSDE in the solely Brownian case.

Definition 6.2.1. *Given $(W_t^a)_{t \in [0, T]}$ and W^b as defined in (6.2.1). Given f as in (6.1.4), we say that a pair (Y^b, Z^b) in $\mathbb{S}_{\mathbb{F}}^2 \times \mathbb{H}_{\mathbb{F}}^2$ is solution to the Brownian BSDE :*

$$Y_t^b = W^b + \int_t^T f(s, Z_s^b, W_s^a - Y_s^b) ds - \int_t^T Z_s^b dB_s \quad (6.2.4)$$

if the relation is satisfied for all t in $[0, T]$ and :

$$\mathbb{E} \left[\int_0^T |f(s, Z_s^b, W_s^a - Y_s^b)| ds + \left(\int_0^T |Z_s^b|^2 ds \right)^{\frac{1}{2}} \right] < +\infty.$$

With this definition in mind, we state the following proposition which is a direct adaptation of Theorem 4.1 of [80].

Proposition 6.2.1. *Let $(W_t^a)_{t \in [0, T]}$ and W^b be as defined in (6.2.1). Assume that the Brownian BSDE :*

$$Y_t^b = W^b + \int_t^T f(s, Z_s^b, W_s^a - Y_s^b) ds - \int_t^T Z_s^b dB_s \quad (6.2.5)$$

has a solution (Y^b, Z^b) in $\mathbb{S}_{\mathbb{F}}^2 \times \mathbb{H}_{\mathbb{F}}^2$. Then (Y, Z, K) defined as :

$$\begin{cases} Y_t = Y_t^b \mathbf{1}_{t < \tau} + W_\tau^a \mathbf{1}_{t \geq \tau} \\ Z_t = Z_t^b \mathbf{1}_{t \leq \tau} \\ K_t = (W_t^a - Y_t^b) \mathbf{1}_{t \leq \tau} \end{cases}$$

is a solution of (6.1.3) and (Y, Z, K) belong to $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$.

Proof. The proof follows that of Theorem 4.1 of [80]. Steps 1. and 2. of the proof are unchanged. It remains to deal with Step 3. which involves proving the three integrability properties. This is done below.

- By construction of Y and as $\forall t \in [0, T] \quad \mathbf{1}_{\tau \leq t} \leq \mathbf{1}_{\tau \leq T} \quad \mathbb{P} - a.s.$ we have :

$$|Y_t| \leq |Y_t^b| + |W_\tau^a| \mathbf{1}_{\tau \leq T}, \quad \forall t \in [0, T],$$

and so as $x^2 + 1 \geq |x|, x \in \mathbb{R}$:

$$|Y_t|^2 \leq 2 + |Y_t^b|^2 + |W_\tau^a|^2 \mathbf{1}_{\tau \leq T}, \quad \forall t \in [0, T],$$

where by assumption Y^b belongs to $\mathbb{S}_{\mathbb{F}}^2$. Furthermore :

$$\mathbb{E} \left[|W_\tau^a|^2 \mathbf{1}_{\tau \leq T} \right] \leq \mathbb{E} [\exp(W_\tau^a) \mathbf{1}_{\tau \leq T}] + \mathbb{E} [\exp(-W_\tau^a) \mathbf{1}_{\tau \leq T}] < +\infty,$$

(see the proof of Lemma 6.2.1) and it follows that Y belongs to $\mathbb{S}_{\mathbb{G}}^2$.

- Z belongs to $\mathbb{H}_{\mathbb{G}}^2$ as Z^b belongs to $\mathbb{H}_{\mathbb{F}}^2$.

- By construction again :

$$|K_t| \leq |W_t^a| + |Y_t^b|, \quad \forall t \in [0, T] \quad \text{so} \quad |K_t|^2 \leq 2 + |W_t^a|^2 + |Y_t^b|^2$$

But Y^b belongs to $\mathbb{S}_{\mathbb{F}}^2$ by assumption and thus also to $\mathbb{H}_{\mathbb{G}}^2$. Also as for all α , $\int_0^T \mathbb{E} [\exp(\alpha W_s^a)] \lambda_s ds < +\infty$ and $\inf_{s \in [0, T]} \lambda_s > 0$, we have $\int_0^T \mathbb{E} [\exp(\alpha W_s^a)] ds = \mathbb{E} \left[\int_0^T \exp(\alpha W_s^a) ds \right] < +\infty$ so

:

$$\mathbb{E} \left[\int_0^T |W_t^a|^2 dt \right] \leq \mathbb{E} \left[\int_0^T \exp(W_t^a) dt \right] + \mathbb{E} \left[\int_0^T \exp(-W_t^a) dt \right] < +\infty.$$

Therefore K belongs to $\mathbb{H}_{\mathbb{G}}^2$.

□

As a consequence, proving existence of solutions to (6.1.3) may be reduced to proving existence of solutions to the Brownian BSDE (6.2.5).

6.2.2 Some useful estimates on Y^b of (6.2.5)

In the following we derive an estimate on the values of Y^b that will be useful to prove existence of solutions to the BSDE (6.2.5).

Lemma 6.2.3 (Bounds on Y^b). *Suppose that W^b is bounded. Then if (Y^b, Z^b) is a solution to (6.2.5) in $\mathbb{S}_{\mathbb{F}}^\infty \times \mathbb{H}_{\mathbb{F}}^2$ with :*

$$\int_0^\cdot Z^b dB_s \quad \text{a true martingale} \tag{6.2.6}$$

then :

$$Y_t^b \leq \log \left(\mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right),$$

and :

$$Y_t^b \geq \frac{-1}{\gamma} \log \left(\mathbb{E} [e^{-\gamma W^b} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \right).$$

Proof. **Proof of upper bound.** As (Y^b, Z^b) solves (6.2.5), the pair satisfies :

$$Y_t^b = W^b + \int_t^T f(s, Z_s^b, W_s^a - Y_s^b) ds - \int_t^T Z_s^b dB_s$$

where :

$$f(t, z, k) = -\gamma \frac{z^2}{2} + g(z) - \frac{\lambda_t}{\gamma} (e^{-\gamma k} - 1).$$

As $0 \leq g(z) \leq C(1 + \frac{z^2}{2})$, we can compare (Y^b, Z^b) to (\bar{Y}, \bar{Z}) defined as the unique solution to the following quadratic BSDE with a bounded terminal condition (see for example [82]) :

$$\bar{Y}_t = W^b + \int_t^T \left\{ C + C \frac{\bar{Z}_s^2}{2} + \frac{\lambda_s}{\gamma} \right\} ds - \int_t^T \bar{Z}_s dB_s$$

In particular, \bar{Y} satisfies the closed-form expression :

$$e^{\bar{Y}_t} = \mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right],$$

and we may conclude through a comparison theorem on Y^b and \bar{Y} that $Y_t^b \leq \bar{Y}_t$ $\mathbb{P} - a.s.$ over $[0, T]$ and thus :

$$Y_t^b \leq \log \left(\mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right) \quad \mathbb{P} - a.s..$$

Proof of lower bound. Again, as (Y^b, Z^b) solves (6.2.5), the pair satisfies :

$$Y_t^b = W^b + \int_t^T \left\{ -\gamma \frac{Z_s^{b2}}{2} + g(Z_s^b) - \frac{\lambda_s}{\gamma} (e^{-\gamma(W_s^a - Y_s^b)} - 1) \right\} ds - \int_t^T Z_s^b dB_s.$$

Applying Ito's formula :

$$\begin{aligned} e^{-\gamma W^b} &= e^{-\gamma Y_t^b} - \int_t^T \gamma e^{-\gamma Y_t^b} \left\{ \gamma \frac{Z_s^{b2}}{2} - g(Z_s^b) + \frac{\lambda_s}{\gamma} (e^{-\gamma(W_s^a - Y_s^b)} - 1) \right\} \\ &\quad - \int_t^T \gamma e^{-\gamma Y_s^b} Z_s^b dB_s + \frac{\gamma^2}{2} \int_t^T e^{-\gamma Y_s^b} \frac{Z_s^{b2}}{2} ds \\ &= e^{-\gamma Y_t^b} - \int_t^T \gamma e^{-\gamma Y_t^b} \left\{ -g(Z_s^b) + \frac{\lambda_s}{\gamma} (e^{-\gamma(W_s^a - Y_s^b)} - 1) \right\} \\ &\quad - \int_t^T \gamma e^{-\gamma Y_s^b} Z_s^b dB_s \end{aligned}$$

Therefore :

$$e^{-\gamma Y_t^b} = e^{-\gamma W^b} + \int_t^T e^{-\gamma Y_t^b} \left\{ -g(Z_s^b) + \frac{\lambda_s}{\gamma} (e^{-\gamma(W_s^a - Y_s^b)} - 1) \right\} + \int_t^T \gamma e^{-\gamma Y_s^b} Z_s^b dB_s,$$

implying that (using the fact that g is positive) :

$$e^{-\gamma Y_t^b} \leq e^{-\gamma W^b} + \int_t^T \frac{\lambda_s}{\gamma} e^{-\gamma W_s^a} ds + \int_t^T \gamma e^{-\gamma Y_s^b} Z_s^b dB_s,$$

and as $\int_0^T Z^b dB_s$ a true martingale and Y is bounded taking conditional expectation yields :

$$Y_t^b \geq \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W^b} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \right) \quad \mathbb{P} - a.s..$$

□

6.2.3 Construction of a solution to (6.2.5)

We now turn to constructing a solution to (6.2.5) and do so following the method of [20]. In particular we use the previously proven estimate to prove the following result.

Lemma 6.2.4. *Let $(W_s^a)_{s \in [0, T]}$ and W^b be as defined in (6.2.1). Then the Brownian BSDE :*

$$Y_t^b = W^b + \int_t^T f(s, Z_s^b, W_s^a - Y_s^b) ds - \int_t^T Z_s^b dB_s$$

has a solution (Y^b, Z^b) in $\mathbb{S}_{\mathbb{F}}^2 \times \mathbb{H}_{\mathbb{F}}^2$.

Proof. We consider the case where W is non-negative. This positivity assumption may be lifted with the steps of this proof remaining the same : we refer the reader to the end of Section 4 of [20] for more details on this.

For ease of reading as the proof is long, we first list the 6 steps of the proof (these are the 6 steps of [20]) :

1. Construction of an approximating BSDE with bounded terminal value.
2. Lifting the boundedness assumption using a well-chosen stopping time.
3. Removing the stopping time in order to recover the dynamics of (6.2.5) for a process (Y^b, Z^b) .
4. Proof that the constructed Y^b belongs to $\mathbb{S}_{\mathbb{F}}^2$.
5. Proof that the constructed Z^b belongs to $\mathbb{H}_{\mathbb{F}}^2$.
6. Proof that the driver related integrability property of Definition 6.2.1 holds for (Y^b, Z^b) .

1. Solutions to a bounded approximating BSDE. For any n in \mathbb{N}^* , we set $W^{b^n} = W^b \wedge n$ and $W^{a^n} = W^a \wedge n$. In such a setting, Theorem 4.12 of [71] applies. In particular, the BSDE (6.2.5) has a unique solution (Y^{b^n}, Z^{b^n}) belonging to $\mathbb{S}_{\mathbb{F}}^\infty \times \mathbb{H}_{\mathbb{F}}^2$:

$$Y_t^{b^n} = W^{b^n} + \int_t^T f(s, Z_s^{b^n}, W_s^{a^n} - Y_s^{b^n}) ds - \int_t^T Z_s^{b^n} dB_s \quad (6.2.7)$$

Furthermore $\int_0^T Z_s^{b^n} dB_s$ is a true martingale. As a consequence, the assumptions of Lemma 6.2.3 are satisfied for (Y^{b^n}, Z^{b^n}) . In particular we obtain :

$$Y_t^{b^n} \leq \log \left(\mathbb{E} \left[e^{W^{b^n} + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right), \quad (6.2.8)$$

$$Y_t^{b^n} \geq \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W^{b^n}} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^{a^n}} ds \middle| \mathcal{F}_t \right] \right). \quad (6.2.9)$$

Now as Y^{b^n} is bounded as well as W^{b^n} and $(W_s^{a^n})_{s \in [t, T]}$, and as

$$-\frac{\lambda_s}{\gamma} e^{-\gamma W_s^{a^n}} \leq -\frac{\lambda_s}{\gamma} e^{-\gamma W_s^{a^{n+1}}} \quad \mathbb{P} - a.s.,$$

a comparison theorem for quadratic BSDEs applies (eg. [82], Theorem 2.6) and yields that :

$$\forall t \in [0, T], \quad Y_t^{b^n} \leq Y_t^{b^{n+1}}.$$

As a consequence we may define : $Y^b = \sup_n Y^{b^n}$. Our goal now is to show that the estimates (6.2.8) and (6.2.9) proven for Y^{b^n} for each n extend to Y^b . First note that :

$$\forall t \in [0, T], \quad Y_t^b \leq \lim_{n \rightarrow +\infty} \log \left(\mathbb{E} \left[e^{W^{b^n} + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right).$$

We deal with the term inside the logarithm. By construction :

$$\forall n \in \mathbb{N}, \quad e^{W^{b^n} + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \leq e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds},$$

where $\mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \right] < +\infty$. Furthermore :

$$\lim_{n \rightarrow +\infty} e^{W^{b^n} + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} = e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \quad \mathbb{P} - a.s.$$

Using the conditional dominated convergence theorem :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[e^{W^{b^n} + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] = \mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \quad \mathbb{P} - a.s.,$$

and thus :

$$\lim_{n \rightarrow +\infty} \log \left(\mathbb{E} \left[e^{W^{b^n} + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right) = \log \left(\mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right) \quad \mathbb{P} - a.s..$$

We reach the upper bound :

$$\forall t \in [0, T], \quad Y_t^b \leq \log \left(\mathbb{E} \left[e^{W^b + \int_t^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right) \quad \mathbb{P} - a.s.. \quad (6.2.10)$$

We proceed in a similar manner to deal with the lower bound. First, note that :

$$\forall t \in [0, T], \quad Y_t^b \geq \lim_{n \rightarrow +\infty} \left\{ \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W^{b^n}} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^{a^n}} ds | \mathcal{F}_t \right] \right) \right\}$$

As :

$$e^{-\gamma W^{b^n}} \leq e^{\gamma |W^b|} \quad \mathbb{P} - a.s.$$

where by assumption, $\mathbb{E}[e^{\gamma |W^b|}] < +\infty$ but also $e^{W^{b^n}} \rightarrow e^{W^b} \quad \mathbb{P} - a.s.$, we may use conditional dominated convergence to obtain :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[e^{-\gamma W^{b^n}} | \mathcal{F}_t \right] = \mathbb{E} \left[e^{-\gamma W^b} | \mathcal{F}_t \right] \quad \mathbb{P} - a.s..$$

It remains to deal with $\mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^{a^n}} ds | \mathcal{F}_t \right]$. But through monotonicity :

$$\lim_{n \rightarrow +\infty} \int_t^T \lambda_s e^{-\gamma W_s^{a^n}} ds = \int_t^T \lambda_s e^{-\gamma W_s^a} ds \quad \mathbb{P} - a.s.$$

and :

$$\int_t^T \lambda_s e^{-\gamma W_s^a} ds \leq \int_t^T \lambda_s e^{\gamma |W_s^a|} ds \quad \mathbb{P} - a.s.$$

where :

$$\mathbb{E} \left[\int_t^T \lambda_s e^{\gamma |W_s^a|} ds \right] < +\infty.$$

Therefore conditional dominated convergence yields :

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^{a^n}} ds | \mathcal{F}_t \right] = \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \quad \mathbb{P} - a.s.$$

and we reach the lower bound :

$$Y_t^b \geq \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W^b} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \right). \quad (6.2.11)$$

Using these two bounds on Y^b , we may compute $\lim_{t \rightarrow T} Y_t^b$. Indeed, on the one hand through (6.2.10), we have :

$$\forall t \in [0, T], Y_t^b \leq \int_t^T \left\{ C + \frac{\lambda_s}{\gamma} \right\} ds + \log \left(\mathbb{E} \left[e^{W^b} | \mathcal{F}_t \right] \right),$$

implying that :

$$\lim_{t \rightarrow T} Y_t^b \leq \lim_{t \rightarrow T} \left[\int_t^T \left\{ C + \frac{\lambda_s}{\gamma} \right\} ds + \log \left(\mathbb{E} \left[e^{W^b} | \mathcal{F}_t \right] \right) \right]$$

The integral part of the upper bound is fully deterministic and goes to 0 when t goes to T . It remains to deal with $\lim_{t \rightarrow T} \log \left(\mathbb{E} \left[e^{W^b} | \mathcal{F}_t \right] \right)$. As $\mathbb{E}[|e^{W^b}|] < +\infty$, Lévy's zero one law applies and yields $\lim_{t \rightarrow T} \mathbb{E}[e^{W^b} | \mathcal{F}_t] = e^{W^b} \mathbb{P} - a.s.$, which in turn through continuity of the logarithm mapping on \mathbb{R}^{+*} yields $\lim_{t \rightarrow T} \log \left(\mathbb{E} \left[e^{W^b} | \mathcal{F}_t \right] \right) = W^b \mathbb{P} - a.s.$. We deduce that $\lim_{t \rightarrow T} Y_t^b \leq W^b \mathbb{P} - a.s.$. Similarly, on the other hand and through (6.2.11), we have :

$$\lim_{t \rightarrow T} Y_t^b \geq \lim_{t \rightarrow T} \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W^b} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \right).$$

And for any $S < T$, using the zero-one law and as $\mathbb{E} \left[e^{-\gamma W^b} + \int_t^T \lambda_s e^{-\gamma W_s^a} ds \right] < +\infty$:

$$\begin{aligned} & \lim_{t \rightarrow T} \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{-\gamma W^b} | \mathcal{F}_t] + \mathbb{E} \left[\int_t^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \right) \\ & \geq \lim_{t \rightarrow T} \frac{-1}{\gamma} \log \left(\mathbb{E} \left[e^{-\gamma W^b} + \int_S^T \lambda_s e^{-\gamma W_s^a} ds | \mathcal{F}_t \right] \right) \\ & = \frac{-1}{\gamma} \log \left(e^{-\gamma W^b} + \int_S^T \lambda_s e^{-\gamma W_s^a} ds \right) \rightarrow W^b \quad \mathbb{P} - a.s. \text{ when } S \rightarrow T \end{aligned}$$

Therefore, we finally have : $\lim_{t \rightarrow T} Y_t^b \geq W^b \mathbb{P} - a.s.$, and so :

$$\lim_{t \rightarrow T} Y_t^b = W^b = Y_T^b. \mathbb{P} - a.s..$$

2. Lifting the boundedness assumption through an estimate on the process.

We now aim to use the bounds computed above in order to lift the assumed boundedness in n on the terminal condition, in order to show that the process Y^b that we construct as the limit of Y^{b^n} solves the required BSDE. To do so, we define :

$$\phi_t^U(W^b) = \log \left(\mathbb{E} \left[e^{|W^b| + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right)$$

and

$$\phi_t^B(W^b, W^a) = \frac{-1}{\gamma} \log \left(\mathbb{E}[e^{\gamma|W^b|} | \mathcal{F}_t] + \mathbb{E} \left[\int_0^T \lambda_s e^{\gamma|W_s^a|} ds \middle| \mathcal{F}_t \right] \right)$$

Note that these functions are closely related to the bounds in (6.2.10) and (6.2.11). In fact, for any n in \mathbb{N} and through (6.2.8) and (6.2.9), we have :

$$\forall t \in [0, T], \phi_t^B(W^b, W^a) \leq Y_t^{b^n} \leq \phi_t^U(W^b) \mathbb{P} - a.s. \quad (6.2.12)$$

We also define the stopping time :

$$\tau_k = \inf \left\{ t \in [0, T], \phi_t^U(W^b) \geq k \text{ or } \phi_t^B(W^b, W^a) \leq -k \right\} \wedge T.$$

As (Y^{b^n}, Z^{b^n}) solve :

$$Y_t^{b^n} = W^{b^n} + \int_t^T f(s, Z_s^{b^n}, W_s^{a^n} - Y_s^{b^n}) ds - \int_t^T Z_s^{b^n} dB_s,$$

the stopped processes $(Y_{t,k}^n, Z_{t,k}^n) = (Y_{t \wedge \tau_k}^{b^n}, Z_{t \wedge \tau_k}^{b^n})$ satisfy the dynamic :

$$Y_{t,k}^n = Y_{T,k}^n + \int_t^T \mathbf{1}_{s \leq \tau_k} f(s, Z_{s,k}^n, W_s^{a^n} - Y_{s,k}^n) ds - \int_t^T Z_{s,k}^n dB_s,$$

Our aim is to keep k fixed and pass to the limit in n in the above equation. We introduce the notations :

$$\bar{f}_s^n(Z_s, Y_s) = -\frac{Z_s^2}{2}(\gamma - 1) + g(Z_s) - \frac{\lambda_s}{\gamma}(e^{-\gamma(W_s^{a^n} - Y_s)} - 1)$$

and

$$\bar{f}_s(Z_s, Y_s) = -\frac{Z_s^2}{2}(\gamma - 1) + g(Z_s) - \frac{\lambda_s}{\gamma}(e^{-\gamma(W_s^a - Y_s)} - 1).$$

Therefore we wish to take n to infinity in the BSDE :

$$Y_{t,k}^n = Y_{T,k}^n + \int_{t \wedge \tau_k}^{T \wedge \tau_k} \bar{f}_s^n(Z_{s,k}^n, Y_{s,k}^n) ds - \int_{t \wedge \tau_k}^{T \wedge \tau_k} Z_{s,k}^n dB_s. \quad (6.2.13)$$

To do so, we note the following.

- Given (z, y) , \bar{f}_n and \bar{f} only differ through the terms $e^{-\gamma(W_s^{a^n} - y)}$ and $e^{-\gamma(W_s^a - y)}$. In fact, by construction of $W_s^{a^n}$, we have pointwise convergence :

$$e^{-\gamma(W_s^{a^n} - y)} \xrightarrow{\mathbb{P}\text{-a.s.}} e^{-\gamma(W_s^a - y)}.$$

Furthermore, \bar{f}^n and \bar{f} are continuous in (z, y) and \bar{f}^n is a monotonically decreasing sequence. Therefore through Dini's theorem applied to $e^{-\gamma(W_s^{a^n} - y)}$ the driver $(\bar{f}^n)_{n \in \mathbb{N}}$ converges uniformly in (z, y) on compact sets to \bar{f} and therefore locally uniformly.

- By construction for any n in \mathbb{N} , $|Y_{T,k}^n| < k$ and :

$$\lim_{n \rightarrow +\infty} Y_{T,k}^n = \sup_n Y_{T,k}^n, \quad \sup_n |Y_{T,k}^n| < k.$$

- There exists some $\alpha > 0$, $\bar{C} > 0$ such that for any n in \mathbb{N} , for any y in $[-k, k]$, for any z in \mathbb{R} :

$$|\bar{f}_s^n(y, z)| \leq \alpha + \bar{C} \frac{z^2}{2} + \frac{L}{\gamma} (e^{\gamma k + \gamma |W_s^a|} + 1),$$

where $L = \sup_{s \in [0, T]} \lambda_s$. and $\mathbb{E} \left[\int_0^T e^{\gamma |W_s^a|} ds \right] < +\infty$ through Lemma 6.2.2 (and the lower-boundedness of $(\lambda_s)_{s \in [0, T]}$).

- For each n , the BSDE (6.2.13) has a solution $(Y_{t,k}^n, Z_{t,k}^n)$ in $\mathbb{S}_{\mathbb{F}}^\infty \times \mathbb{H}_{\mathbb{F}}^2$ and $(Y_{t,k}^n)_{n \in \mathbb{N}}$ is monotonic in n , with $\sup_n |Y_{t,k}^n| < k < +\infty$.

- τ_k is an increasing sequence of stopping times satisfying : $\tau_k \leq T$ a.s..

Therefore the assumptions of Proposition 2.4 of [82] are satisfied and there exists a pair of processes $(Y_{t,k}, Z_{t,k})$ in $\mathbb{S}_{\mathbb{F}}^\infty \times \mathbb{H}_{\mathbb{F}}^2$ such that :

$$\lim_{n \rightarrow +\infty} Y_{t,k}^n = Y_{t,k} \quad \text{uniformly (and increasingly) on } [0, T],$$

and :

$$(Z_{t,k}^n) \text{ converges to } Z_{t,k} \text{ in } \mathbb{H}_{\mathbb{F}}^2,$$

and $(Y_{t,k}, Z_{t,k})$ solves :

$$Y_{t,k} = Y_{T,k} + \int_{t \wedge \tau_k}^{T \wedge \tau_k} \bar{f}_s(Z_{s,k}, Y_{s,k}) ds - \int_{t \wedge \tau_k}^{T \wedge \tau_k} Z_{s,k} dB_s$$

or equivalently :

$$Y_{t,k} = Y_{T,k} + \int_t^T \mathbf{1}_{s \leq \tau_k} \bar{f}_s(Z_{s,k}, Y_{s,k}) ds - \int_t^T Z_{s,k} dB_s, \quad (6.2.14)$$

and the paths of $Y_{t,k}$ are continuous on $[0, \tau_k]$.

3. Removing the effect of k . It remains to justify sending k to infinity in (6.2.14) and to show that solutions to (6.2.5) are recovered.

We deal with the limit of Y_k first. Recall that we defined over $[0, T]$ the process Y^b as $Y_t^b = \sup_n Y_t^{b^n}$ where Y^{b^n} satisfies the BSDE (6.2.7). Equally, $Y_{t,k} = \sup_{n \in \mathbb{N}} Y_{t \wedge \tau_k}^{b^n}$.

This yields :

$$Y_{t \wedge \tau_k}^b = Y_{t,k}.$$

In particular, $Y_t^b = \lim_{k \rightarrow +\infty} Y_{t \wedge \tau_k}^b = \lim_{k \rightarrow +\infty} Y_{t,k}$. But $Y_{\cdot,k}$ has continuous paths on $[0, \tau_k]$ for each k , and as $\tau_k \leq \tau_{k+1}$ and $\tau_k \xrightarrow[k \rightarrow +\infty]{} T$, the process Y^b is continuous on $[0, T)$.

Furthermore we previously proved that $\lim_{t \rightarrow T} Y_t^b = W^b$ \mathbb{P} -a.s., and by construction $\lim_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} Y_{T \wedge \tau_k}^{b^n} = W^b$ \mathbb{P} -a.s.. Therefore Y^b is continuous on $[0, T]$.

Now concerning $Z_{\cdot,k}$, we define a process Z^b as follows :

$$Z_t^b = Z_{t,k} \quad \text{if } t \in (0, \tau_k).$$

Recall that $Z_{\cdot,k}$ belongs to $\mathbb{H}_{\mathbb{F}}^2$. This new process allows us to rewrite (6.2.14) as :

$$Y_{t \wedge \tau_k}^b = Y_{T \wedge \tau_k}^b + \int_{t \wedge \tau_k}^{T \wedge \tau_k} \bar{f}_s(Z_s^b, Y_s^b) ds - \int_{t \wedge \tau_k}^{T \wedge \tau_k} Z_s^b dB_s,$$

where we have for all k ,

$$\begin{aligned} \mathbb{P} \left(\int_0^T |Z_s^b|^2 ds = \infty \right) &= \mathbb{P} \left(\int_0^T |Z_s^b|^2 ds = \infty, \tau_k = T \right) + \mathbb{P} \left(\int_0^T |Z_s^b|^2 ds = \infty, \tau_k < T \right) \\ &\leq \mathbb{P} \left(\int_0^{\tau_k} |Z_{s,k}|^2 ds = \infty \right) + \mathbb{P}(\tau_k < T), \end{aligned}$$

But $\mathbb{P}(\int_0^{\tau_k} |Z_{s,k}|^2 ds = \infty) = 0$ for any k as Z_k belongs to $\mathbb{H}_{\mathbb{F}}^2$. Therefore for any k :

$$\mathbb{P} \left(\int_0^T |Z_s^b|^2 ds = \infty \right) \leq \mathbb{P}(\tau_k < T)$$

but $\lim_{k \rightarrow +\infty} \mathbb{P}(\tau_k < T) = 0$. Therefore $\mathbb{P}(\int_0^T |Z_s^b|^2 ds = \infty) = 0$.

Taking k to ∞ , the pair (Y^b, Z^b) satisfy :

$$Y_t^b = Y_T^b + \int_t^T \bar{f}_s(Z_s^b, Y_s^b) ds - \int_t^T Z_s^b dB_s,$$

and (Y^b, Z^b) satisfies the required dynamics. As mentioned at the start of the construction, this lifts to a non-signed terminal condition W . The key is to consider the doubly truncated approximation $W^{b^{n,p}} = W^+ \wedge n - W^- \wedge p$.

For (Y^b, Z^b) to be a solution to (6.2.5) in the sense of Definition (6.2.1), it remains to prove some integrability properties, notably that $Y^b \in \mathbb{S}_{\mathbb{F}}^2$, $Z^b \in \mathbb{H}_{\mathbb{F}}^2$ and :

$$\mathbb{E} \left[\int_0^T |\bar{f}_s(Z_s^b, Y_s^b)| ds + \left(\int_0^T |Z_s^b|^2 ds \right)^{\frac{1}{2}} \right] < +\infty.$$

4. Proof that Y^b belongs to $\mathbb{S}_{\mathbb{P}}^2$. First note that through (6.2.12) we have the following bounds on Y^b :

$$\forall t \in [0, T], Y_t^b \leq \log \left(\mathbb{E} \left[e^{|W^b| + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right)$$

and :

$$\forall t \in [0, T], Y_t^b \geq \frac{-1}{\gamma} \log \left(\mathbb{E} \left[e^{\gamma |W^b| + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds} \middle| \mathcal{F}_t \right] \right).$$

Now let $\alpha \geq 1$ fixed, then through the upper bound above :

$$e^{\alpha Y_t^b} \leq \mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right]^\alpha \quad \mathbb{P} - a.s..$$

and :

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} e^{\alpha Y_t^b} \right] &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right]^\alpha \right] \\ &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right], \\ &\leq \mathbb{E} \left[\left(\sup_{t \in [0, T]} \mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right)^2 \right], \end{aligned}$$

as $x \geq 1 \Rightarrow x \leq x^2$ and as the exponential term is greater than 1. But $\mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right]$ is a continuous martingale such that :

$$\mathbb{E} \left[\left(\mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right)^2 \right] \leq \mathbb{E} \left[e^{2\alpha |W^b| + \int_0^T 2\alpha (C + \frac{\lambda_s}{\gamma}) ds} \right] < +\infty$$

and using Doob's inequality :

$$\mathbb{E} \left[\left(\sup_{t \in [0, T]} \mathbb{E} \left[e^{\alpha |W^b| + \int_0^T \alpha (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_t \right] \right)^2 \right] \leq 4 \mathbb{E} \left[e^{2\alpha |W^b| + \int_0^T 2\alpha (C + \frac{\lambda_s}{\gamma}) ds} \right] < +\infty.$$

Therefore $\mathbb{E} \left[\sup_{t \in [0, T]} e^{\alpha Y_t^b} \right] < +\infty$, for $\alpha \geq 1$. Similarly let $\beta \geq \gamma$ fixed :

$$e^{-\beta Y_t^b} \leq \mathbb{E} \left[e^{\gamma |W^b| + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds} \middle| \mathcal{F}_t \right]^{\frac{\beta}{\gamma}} \quad \mathbb{P} - a.s..$$

and :

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} e^{-\beta Y_t^b} \right] &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \mathbb{E} \left[e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \middle| \mathcal{F}_t \right]^{\frac{\beta}{\gamma}} \right] \\ &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{\beta}{\gamma}} \middle| \mathcal{F}_t \right] \right] \\ &\leq \mathbb{E} \left[\left(\sup_{t \in [0, T]} \mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{\beta}{\gamma}} \middle| \mathcal{F}_t \right] \right)^2 \right], \end{aligned}$$

where again, $\mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{\beta}{\gamma}} \middle| \mathcal{F}_t \right]$ is a continuous martingale such that :

$$\mathbb{E} \left[\mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{\beta}{\gamma}} \middle| \mathcal{F}_t \right]^2 \right] \leq \mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{2\beta}{\gamma}} \right]$$

and using the Minkowski inequality :

$$\mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{2\beta}{\gamma}} \right]^{\frac{\gamma}{2\beta}} \leq \mathbb{E} \left[e^{2\beta |W^b|} \right]^{\frac{\gamma}{2\beta}} + \mathbb{E} \left[\left(\int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{2\beta}{\gamma}} \right]^{\frac{\gamma}{2\beta}},$$

where :

$$\mathbb{E} \left[e^{2\beta |W^b|} \right]^{\frac{\gamma}{2\beta}} < +\infty$$

and through Jensen :

$$\mathbb{E} \left[\left(\int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{2\beta}{\gamma}} \right]^{\frac{\gamma}{2\beta}} \leq \left(T^{\frac{2\beta}{\gamma} - 1} \right)^{\frac{\gamma}{2\beta}} \mathbb{E} \left[\int_0^T \lambda_s^{\frac{2\beta}{\gamma}} e^{2\beta |W_s^a|} ds \right]^{\frac{\gamma}{2\beta}} < +\infty.$$

Therefore :

$$\mathbb{E} \left[\mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{\beta}{\gamma}} \middle| \mathcal{F}_t \right]^2 \right] < +\infty,$$

and using Doob's inequality again :

$$\mathbb{E} \left[\left(\sup_{t \in [0, T]} \mathbb{E} \left[\left(e^{\gamma |W^b|} + \int_0^T \lambda_s e^{\gamma |W_s^a|} ds \right)^{\frac{\beta}{\gamma}} \middle| \mathcal{F}_t \right] \right)^2 \right] < +\infty.$$

Therefore $\mathbb{E} \left[\sup_{t \in [0, T]} e^{-\beta Y_t^b} \right] < +\infty$ for $\beta \geq \gamma$.

Now let $\nu \geq \max(1, \gamma)$. Then :

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} e^{\nu |Y_t^b|} \right] &\leq \mathbb{E} \left[\sup_{t \in [0, T]} e^{-\nu Y_t^b} + e^{\nu Y_t^b} \right] \\ &\leq \mathbb{E} \left[\sup_{t \in [0, T]} e^{-\nu Y_t^b} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} e^{\nu Y_t^b} \right] < +\infty \text{ (using the two } \neq \text{ cases proven above).} \end{aligned}$$

As $e^{\nu|x|} \geq \nu^2 x^2$ on \mathbb{R} , this yields $\mathbb{E} \left[\sup_{t \in [0, T]} Y_t^{b2} \right] < +\infty$, and therefore Y^b belongs to $\mathbb{S}_{\mathbb{F}}^2$.

5. Proof that Z^b belongs to $\mathbb{H}_{\mathbb{F}}^2$.

To show the property for Z^b , recall that (Y^b, Z^b) satisfy :

$$Y_t^b = W^b + \int_t^T f(s, Z_s^b, W_s^a - Y_s^b) ds - \int_t^T Z_s^b dB_s$$

where :

$$f(t, z, k) = -\gamma \frac{z^2}{2} + g(z) - \frac{\lambda_t}{\gamma} (e^{-\gamma k} - 1).$$

We give the calculations below for the case where $0 \leq g(z) \leq \frac{z^2}{2}$, which generalize without a problem to $0 \leq g(z) \leq C(1 + \frac{z^2}{2})$.² First we compute some bounds on the driver and obtain :

$$f(s, Z_s^b, W_s^a - Y_s^b) \leq \frac{Z_s^{b2}}{2} + \frac{\lambda_s}{\gamma},$$

and :

$$\begin{aligned} -f(s, Z_s^b, W_s^a - Y_s^b) &\leq \gamma \frac{Z_s^{b2}}{2} + \frac{\lambda_s}{\gamma} e^{-\gamma(W_s^a - Y_s^b)} \\ &\leq \gamma \frac{Z_s^{b2}}{2} + \frac{\lambda_s}{\gamma} e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \end{aligned}$$

Therefore :

$$\begin{aligned} |f(s, Z_s^b, W_s^a - Y_s^b)| &\leq \frac{Z_s^{b2}}{2} (\gamma + 1) + \frac{\lambda_s}{\gamma} \left(1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right) \\ &\leq \frac{Z_s^{b2}}{2} (\gamma + 1) + \frac{\lambda_s}{\gamma} (1 + g(W^a, W^b)), \end{aligned}$$

where $g(W^a, W^b) = e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma$. We use this to deduce the integrability properties for Z^b , and as in [20] do so using the mapping :

$$u(x) = \frac{1}{(\gamma + 1)^2} (e^{(\gamma+1)x} - 1 - (\gamma + 1)x).$$

²In fact, it suffices to set $u(x) = \frac{1}{(\gamma+C)^2} (e^{(\gamma+C)x} - 1 - (\gamma + C)x)$ in the reasoning below to deal with the more general upper bound.

Note that $x \mapsto u(|x|)$ belongs to $\mathcal{C}^2(\mathbb{R})^3$ and applying Ito's lemma we have for any $t \in [0, T]$:

$$\begin{aligned} u(|Y_0^b|) &= u(|Y_t^b|) + \int_0^t \left(u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) f(s, Z_s^b, W_s^a - Y_s^b) - \frac{1}{2} u''(|Y_s^b|) |Z_s^b|^2 \right) ds \\ &\quad - \int_0^t u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) Z_s^b dB_s. \end{aligned}$$

But :

$$\begin{aligned} \operatorname{sgn}(Y_s^b) f(s, Z_s^b, W_s^a - Y_s^b) &= -f(s, Z_s^b, W_s^a - Y_s^b) \mathbf{1}_{Y_s^b < 0} + f(s, Z_s^b, W_s^a - Y_s^b) \mathbf{1}_{Y_s^b \geq 0} \\ &\leq |f(s, Z_s^b, W_s^a - Y_s^b)|. \end{aligned}$$

Furthermore :

- u maps \mathbb{R}^+ to itself
- $u'(x) = \frac{1}{\gamma+1} (e^{(\gamma+1)x} - 1) \geq 0$ on \mathbb{R}^+
- $u'(x)(\gamma+1) - u''(x) = e^{(\gamma+1)x} - 1 - e^{(\gamma+1)x} = -1$.

Therefore :

$$\begin{aligned} u(|Y_0^b|) &\leq u(|Y_t^b|) + \int_0^t \left(u'(|Y_s^b|) \left(\frac{Z_s^{b2}}{2} (\gamma+1) + \frac{\lambda_s}{\gamma} (1 + g(W^a, W^b)) \right) - \frac{1}{2} u''(|Y_s^b|) |Z_s^b|^2 \right) ds \\ &\quad - \int_0^t u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) Z_s^b dB_s. \\ &= u(|Y_t^b|) + \int_0^t \left\{ -\frac{Z_s^{b2}}{2} + u'(|Y_s^b|) \frac{\lambda_s}{\gamma} (1 + g(W^a, W^b)) \right\} ds \\ &\quad - \int_0^t u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) Z_s^b dB_s. \\ &\leq u(|Y_t^b|) + \int_0^t \left\{ -\frac{Z_s^{b2}}{2} + \frac{e^{(\gamma+1)|Y_s^b|}}{\gamma+1} \frac{\lambda_s}{\gamma} (1 + g(W^a, W^b)) \right\} ds \\ &\quad - \int_0^t u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) Z_s^b dB_s \end{aligned}$$

and we obtain :

$$\begin{aligned} \int_0^T \frac{Z_s^{b2}}{2} ds &\leq u(|Y_T^b|) + \int_0^T \frac{e^{(\gamma+1)|Y_s^b|}}{\gamma+1} \frac{\lambda_s}{\gamma} (1 + g(W^a, W^b)) ds \\ &\quad - \int_0^t u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) Z_s^b dB_s \\ &\leq \frac{1}{(\gamma+1)^2} e^{(\gamma+1)|Y_T^b|} + \int_0^T \frac{e^{(\gamma+1)|Y_s^b|}}{\gamma+1} \frac{\lambda_s}{\gamma} (1 + g(W^a, W^b)) ds \\ &\quad - \int_0^t u'(|Y_s^b|) \operatorname{sgn}(Y_s^b) Z_s^b dB_s. \end{aligned}$$

³this is despite the presence of the absolute value function, due to the pleasant behavior of u around 0

Therefore there exists a localization sequence $(\tau_n)_{n \in \mathbb{N}}$ such that :

$$\mathbb{E} \left[\int_0^{T \wedge \tau_n} \frac{Z_s^{b^2}}{2} ds \right] \leq \sup_{t \in [0, T]} \mathbb{E} \left[\frac{1}{(\gamma + 1)^2} e^{(\gamma+1)|Y_t^b|} \right] + \mathbb{E} \left[\int_0^T \frac{e^{(\gamma+1)|Y_s^b|}}{\gamma + 1} \frac{\lambda_s}{\gamma} \left(1 + g(W^a, W^b) \right) ds \right].$$

We have proven above that :

$$\sup_{t \in [0, T]} \mathbb{E} \left[\frac{1}{(\gamma + 1)^2} e^{(\gamma+1)|Y_t^b|} \right] < +\infty,$$

and we also have that :

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \frac{e^{(\gamma+1)|Y_s^b|}}{\gamma + 1} \frac{\lambda_s}{\gamma} \left(1 + g(W^a, W^b) \right) ds \right] \\ & \leq \mathbb{E} \left[\int_0^T \frac{e^{2(\gamma+1)|Y_s^b|}}{(\gamma + 1)^2} ds \right] + \mathbb{E} \left[\int_0^T \frac{\lambda_s^2}{\gamma^2} \left(1 + g(W^a, W^b) \right)^2 ds \right]. \end{aligned}$$

But :

$$\mathbb{E} \left[\int_0^T \frac{e^{2(\gamma+1)|Y_s^b|}}{(\gamma + 1)^2} ds \right] < +\infty,$$

and :

$$\begin{aligned} \mathbb{E} \left[\int_0^T \frac{\lambda_s^2}{\gamma^2} \left(1 + g(W^a, W^b) \right)^2 ds \right] &= \mathbb{E} \left[\int_0^T \frac{\lambda_s^2}{\gamma^2} \left(1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right)^2 ds \right] \\ &= T \int_0^T \frac{\lambda_s^2}{\gamma^2} ds + \int_0^T 2 \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right] ds \\ &\quad + \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[e^{-2\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^{2\gamma} \right] ds. \end{aligned}$$

We may conclude by reasoning with Jensen and different values of γ . For example :

$$\begin{aligned} & \int_0^T 2 \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right] ds \\ &= \int_0^T 2 \frac{\lambda_s^2}{\gamma^2} ds + 2 \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right] ds \end{aligned}$$

and for the second expectation term above (and using $x^2 + y^2 \geq 2xy$) we have :

$$\begin{aligned} & 2 \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right] ds \\ & \leq \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[e^{-2\gamma W_s^a} \right] ds + \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[\mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^{2\gamma} \right] ds, \end{aligned}$$

but the first term is finite through the boundedness of $s \mapsto \lambda_s$ and through Lemma 6.2.2. To deal with the second term, we first suppose that $2\gamma < 1$, then :

$$\begin{aligned} & \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[\mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^{2\gamma} \right] ds \\ & \leq \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[e^{2\gamma W^b + 2\gamma \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \right] ds \\ & = \mathbb{E} \left[e^{2\gamma W^b + 2\gamma \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \right]^{2\gamma} \int_0^T \frac{\lambda_s^2}{\gamma^2} ds < +\infty, \end{aligned}$$

and when $2\gamma \geq 1$:

$$\begin{aligned} & \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[\mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^{2\gamma} \right] ds \\ & \leq \int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[e^{2\gamma W^b + 2\gamma \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \right] ds \\ & = \mathbb{E} \left[e^{2\gamma W^b + 2\gamma \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \right] \int_0^T \frac{\lambda_s^2}{\gamma^2} ds < +\infty. \end{aligned}$$

Therefore $\int_0^T \frac{\lambda_s^2}{\gamma^2} \mathbb{E} \left[1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right] ds < +\infty$. We may conclude on the third term in the same way and finally reach :

$$\mathbb{E} \left[\int_0^T \left(1 + g(W^a, W^b) \right)^2 ds \right] < +\infty,$$

and thus :

$$\mathbb{E} \left[\int_0^{T \wedge \tau_n} \frac{Z_s^b{}^2}{2} ds \right] < M,$$

where $M < +\infty$. Through Fatou's lemma,

$$\mathbb{E} \left[\int_0^T \frac{Z_s^b{}^2}{2} ds \right] \leq \liminf_{n \rightarrow +\infty} \mathbb{E} \left[\int_0^{T \wedge \tau_n} \frac{Z_s^b{}^2}{2} ds \right] < M,$$

and we obtain $Z^b \in \mathbb{H}_{\mathbb{F}}^2$.

6. Proof of driver related property. We finally prove that :

$$\mathbb{E} \left[\int_0^T |f(s, Z_s^b, W_s^a - Y_s^b)| ds + \left(\int_0^T |Z_s^b|^2 ds \right)^{\frac{1}{2}} \right] < +\infty.$$

First note that as Z^b belongs to $\mathbb{H}_{\mathbb{F}}^2$, and $x \mapsto x^{\frac{1}{2}}$ is concave on \mathbb{R}^+ :

$$\mathbb{E} \left[\left(\int_0^T |Z_s^b|^2 ds \right)^{\frac{1}{2}} \right] \leq \mathbb{E} \left[\left(\int_0^T |Z_s^b|^2 ds \right) \right]^{\frac{1}{2}} < +\infty.$$

It remains to deal with :

$$\mathbb{E} \left[\int_0^T |f(s, Z_s^b, W_s^a - Y_s^b)| ds \right]$$

where⁴ :

$$|f(s, Z_s^b, W_s^a - Y_s^b)| \leq \frac{Z_s^{b2}}{2}(\gamma + 1) + \frac{\lambda_s}{\gamma} \left(1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right),$$

But again as Z^b belongs to $\mathbb{H}_{\mathbb{F}}^2$,

$$\mathbb{E} \left[\int_0^T \frac{Z_s^{b2}}{2}(\gamma + 1) ds \right] < +\infty,$$

and the remaining term is :

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \frac{\lambda_s}{\gamma} \left(1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right) ds \right] \\ &= \int_0^T \frac{\lambda_s}{\gamma} ds + \int_0^T \frac{\lambda_s}{\gamma} \mathbb{E} \left[\mathbb{E} \left[e^{W^b - W_s^a + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right] ds, \end{aligned}$$

and reasoning similarly to above in Part 5, with Jensen and $\gamma > 1/\gamma \leq 1$, and the integrability properties of the wage terms, we reach :

$$\mathbb{E} \left[\int_0^T \frac{\lambda_s}{\gamma} \left(1 + e^{-\gamma W_s^a} \mathbb{E} \left[e^{W^b + \int_0^T (C + \frac{\lambda_s}{\gamma}) ds} \middle| \mathcal{F}_s \right]^\gamma \right) ds \right] < +\infty$$

and finally :

$$\mathbb{E} \left[\int_0^T |f(s, Z_s^b, W_s^a - Y_s^b)| ds + \left(\int_0^T |Z_s^b|^2 ds \right)^{\frac{1}{2}} \right] < +\infty.$$

We conclude that the constructed pair of processes (Y^b, Z^b) is solution to (6.1.3) in the sense of Definition 6.2.1 and the proof is therefore complete. \square

6.3 Proof of uniqueness of solutions to (6.1.3)

We have the following uniqueness result for BSDE (6.1.3). We may note that the non dependence of our driver in y is key to this uniqueness. Without such dependence, the question is harder to deal with and may call for approaches such as those of [21] or [41] using driver properties.

Lemma 6.3.1. *There exists at most one triplet (Y, Z, K) in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ solution to BSDE (6.1.3) with terminal condition W .*

⁴again, we detail the case where $0 \leq g(z) \leq \frac{z^2}{2}$, which generalizes without a problem to $0 \leq g(z) \leq C(1 + \frac{z^2}{2})$.

Proof. Let (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) - both in $\mathbb{S}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2 \times \mathbb{H}_{\mathbb{G}}^2$ - be two solutions to the BSDE (6.1.3). Denote :

$$\hat{Y} = Y^1 - Y^2, \hat{Z} = Z^1 - Z^2, \hat{K} = K^1 - K^2.$$

Then :

$$\hat{Y}_t = - \int_{t \wedge \tau}^{T \wedge \tau} \hat{K}_s dM_s - \int_{t \wedge \tau}^{T \wedge \tau} \hat{Z}_s dB_s + \int_{t \wedge \tau}^{T \wedge \tau} \left\{ h(s, Z_s^1, K_s^1) - h(s, Z_s^2, K_s^2) \right\} ds \quad (6.3.1)$$

$$(6.3.2)$$

where

$$h(t, z, k) = -\gamma \frac{z^2}{2} + g(z) - \frac{\lambda_t}{\gamma} (e^{-\gamma k} - 1 + \gamma k).$$

Through linearization in K , denoting as $K_s^{1,2}$ a point between K_s^1 and K_s^2 , we have :

$$\begin{aligned} & h(s, Z_s^1, K_s^1) - h(s, Z_s^2, K_s^2) \\ &= -\gamma \frac{(Z_s^1)^2 - (Z_s^2)^2}{2} + g(Z_s^1) - g(Z_s^2) + \frac{\lambda_s}{\gamma} \left(\gamma(K_s^2 - K_s^1) + e^{-\gamma K_s^2} - e^{-\gamma K_s^1} \right) \\ &= -\gamma \frac{(Z_s^1)^2 - (Z_s^2)^2}{2} + g(Z_s^1) - g(Z_s^2) - \lambda_s \hat{K}_s + \lambda_s e^{-\gamma K_s^{1,2}} \hat{K}_s \\ &= -\gamma \frac{(Z_s^1)^2 - (Z_s^2)^2}{2} + g(Z_s^1) - g(Z_s^2) + \lambda_s m(K_s^1, K_s^2) \hat{K}_s \end{aligned}$$

where :

$$m(K_s^1, K_s^2) = -1 + e^{-\gamma K_s^{1,2}}.$$

It remains to deal with the terms in Z , for $0 \leq g(z) \leq C \left(1 + \frac{z^2}{2}\right)$ and we obtain :

$$-\gamma \frac{(Z_s^1)^2 - (Z_s^2)^2}{2} + g(Z_s^1) - g(Z_s^2) = l(Z_s^1, Z_s^2) \hat{Z}_s,$$

where :

$$l(Z_s^1, Z_s^2) = \begin{cases} \frac{-\gamma}{2} (Z_s^1 + Z_s^2) + \frac{1}{Z} \{g(Z_s^1) - g(Z_s^2)\} & \text{if } \hat{Z} \neq 0 \\ \frac{-\gamma}{2} (Z_s^1 + Z_s^2) & \text{if } \hat{Z} = 0 \end{cases}$$

With these notations we obtain :

$$h(s, Z_s^1, K_s^1) - h(s, Z_s^2, K_s^2) = \lambda_s m(K_s^1, K_s^2) \hat{K}_s + l(Z_s^1, Z_s^2) \hat{Z}_s,$$

and (6.3.1) rewrites as :

$$\hat{Y}_t = \int_{t \wedge \tau}^{T \wedge \tau} \left\{ \lambda_s m(K_s^1, K_s^2) \hat{K}_s + l(Z_s^1, Z_s^2) \hat{Z}_s \right\} ds - \int_{t \wedge \tau}^{T \wedge \tau} \hat{K}_s dM_s - \int_{t \wedge \tau}^{T \wedge \tau} \hat{Z}_s dB_s.$$

Now set :

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E} \left(\int_0^T l(Z_s^1, Z_s^2) dB_s \right) \mathcal{E} \left(\int_0^T m(K_s^1, K_s^2) dM_s \right) = L_T.$$

The above change in probability will be key to concluding and we can first give an outline of the main idea in the case where \hat{K} and \hat{Z} are bounded (and therefore so are $l(Z_s^1, Z_s^2)$ and $m(K_s^1, K_s^2)$). Then \mathbb{P} and \mathbb{Q} are equivalent probability measures and under \mathbb{Q} :

- $dB_s^l = dB_s - l(Z_s^1, Z_s^2)ds$ is Brownian motion increment,
- $dM_s^m = dM_s - m(K_s^1, K_s^2)\lambda_s ds$ is a local martingale increment,

and we have :

$$\hat{Y}_t = - \int_{t \wedge \tau}^{T \wedge \tau} \hat{K}_s dM_s^m - \int_{t \wedge \tau}^{T \wedge \tau} \hat{Z}_s dB_s^l.$$

As \hat{K} and \hat{Z} are bounded, the two stochastic integrals are two martingales and taking conditional expectation on both sides, we obtain :

$$\hat{Y}_t = \mathbb{E}^{\mathbb{Q}}[\hat{Y}_t | \mathcal{G}_{t \wedge \tau}] = 0.$$

We conclude that $\hat{Y} = 0$ \mathbb{Q} -*a.s.* but also \mathbb{P} -*a.s.* through equivalence of the two probability measures.

Of course we now need to extend this idea to a more general setting and do so by exploiting the following facts :

- \hat{Y} jumps at most once at time τ
- \hat{Y} is continuous before and after τ .
- $\mathbb{P}(\tau = T) = 0$

As a consequence, $\forall \omega \in \Omega, t \mapsto \hat{Y}_t(\omega)$ is left-continuous on a neighbourhood of T , and as $\hat{Y}_T = 0$ \mathbb{P} -*a.s.*, we have $\forall \omega \in \Omega$:

$$\forall \epsilon > 0, \exists t_0(\omega) > 0, \quad \forall t \in [T - t_0(\omega), T], \quad |\hat{Y}_t| < \epsilon \quad \mathbb{P} - a.s.. \quad (6.3.3)$$

With this in mind, and we consider some fixed time $t \in [0, T)$, some fixed ϵ , and set for any $n \in \mathbb{N}$:

$$\tau_n = \inf \left\{ s \geq t, |m_s| + |\hat{Z}_s| + |\hat{K}_s| + |l_s| \geq n \right\} \wedge T,$$

where :

$$l_s = l(Z_s^1, Z_s^2) \quad \text{and} \quad m_s = m(K_s^1, K_s^2).$$

Therefore \mathbb{P} and \mathbb{Q} are equivalent probability measures up until τ_n and

$$\hat{Y}_{t \wedge \tau_n} = \hat{Y}_{\tau_n} - \int_{t \wedge \tau \wedge \tau_n}^{T \wedge \tau \wedge \tau_n} \hat{K}_s dM_s^m - \int_{t \wedge \tau \wedge \tau_n}^{T \wedge \tau \wedge \tau_n} \hat{Z}_s dB_s^l.$$

Where the two integrals are true \mathbb{Q} -martingales. Now by construction, $\forall n \in \mathbb{N}, \tau_n \leq \tau_{n+1}$ \mathbb{P} -*a.s.* and $\lim_{n \rightarrow +\infty} \tau_n = T$ and therefore :

$$\forall \omega \in \Omega, \exists n_0(\omega) \in \mathbb{N}, \forall n \geq n_0(\omega), \tau_n \in [T - t_0(\omega), T] \quad \mathbb{P} - a.s.,$$

and through (6.3.3) :

$$|\hat{Y}_{\tau_n}| \leq \epsilon \quad \mathbb{Q} - a.s..$$

We deduce that :

$$\hat{Y}_{t \wedge \tau_n} = \mathbb{E}^{\mathbb{Q}} \left[\hat{Y}_{\tau_n} | \mathcal{G}_t \right], \quad \text{where} \quad -\epsilon \leq \hat{Y}_{\tau_n} \leq \epsilon, \quad \mathbb{Q} - a.s..$$

So $|\hat{Y}_{t \wedge \tau_n}| < \epsilon$ \mathbb{Q} -a.s. but also \mathbb{P} -a.s. and taking n to $+\infty$, τ_n goes to T and this yields $|\hat{Y}_t| \leq \epsilon$ \mathbb{P} -a.s.. This reasoning may be repeated for any $\epsilon > 0$ and therefore $\hat{Y}_t = 0$ \mathbb{P} -a.s.. As a consequence $Y^1 = Y^2$ \mathbb{P} -a.s.. Finally as Y^1 is a special semi-martingale, its decomposition is unique and we further deduce that $Z^1 = Z^2$ and $K^1 = K^2$ (yielding uniqueness of the solution triplet).

□

Remark 6.3.1. *The reasoning in the proof of uniqueness may also be applied to derive a simple "comparison" type result on the solutions to our BSDE. Given (Y^1, Z^1, K^1) the unique solution to BSDE (6.1.3) with terminal condition W^1 and (Y^2, Z^2, K^2) the unique solution to BSDE (6.1.3) with terminal condition W^2 . Suppose that $W^1 \geq W^2$ \mathbb{P} -a.s.. Then through an adaptation of the reasoning above, $\forall t \in [0, T]$, $Y_t^1 \geq Y_t^2$ \mathbb{P} -a.s..*

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Quelques problèmes Principal-Agent sous contraintes sociétales

Dans cette thèse, nous nous intéressons à quelques problèmes d'incitation sous contraintes sociétales. Nous utilisons pour cela le formalisme Principal-Agent et nous nous basons sur le modèle fondateur d'Hölmstrom-Milgrom. Dans un premier temps, nous regardons les problèmes d'information complète dans ce modèle et établissons un cadre alternatif, unifiant et à application large pour leur résolution. En plus de mettre en lumière une nouvelle signification de la règle de Borch en économie, cette approche étend l'interprétation de l'action optimale de l'Agent. Cette méthode est ensuite appliquée pour une partie d'un premier travail d'incitation où nous proposons d'analyser l'effet de la rémunération à la tâche, caractérisant les contrats dans la gig economy. La richesse du Principal est ici modélisée par un processus de Poisson dont l'Agent peut affecter l'intensité. Les contrats optimaux sont alors explicites, comportant une modification de la linéarité classique attendue sous information partielle dans ce modèle. Nous notons que malgré la richesse positive du Principal, la rémunération payée à l'Agent peut être négative soulignant l'enjeu de l'ajout de contraintes de positivité dans ces problèmes. C'est sur ce type de problème d'incitation sous contraintes que nous nous penchons alors dans un modèle discret. Nous établissons l'existence de solutions par approche variationnelle sous utilités assez générales. Ces solutions sont ensuite caractérisées par des méthodes du premier ordre permettant une analyse de l'effet des contraintes sur les contrats. Notamment nous obtenons des contreparties de forme optionnelle. Enfin, nous nous tournons vers un problème d'actualité : inciter sous risque d'arrêt de l'économie. Nous étendons pour cela le modèle d'Hölmstrom-Milgrom par l'ajout d'un processus de défaut et nous obtenons assez remarquablement une rémunération explicite, linéaire pour le temps de défaut. Nous étendons ensuite (simplement) ce modèle pour inclure une possibilité de continuité de service malgré l'arrêt, sous réserve d'un investissement suffisant par le Principal. Pour terminer cette thèse, nous prouvons un résultat nécessaire à ce dernier problème : l'existence et l'unicité de solutions pour une certaine équation différentielle stochastique rétrograde Brownienne avec défaut.

Some Principal-Agent problems under societal constraints

In this thesis, we analyse several incentive problems under societal constraints. To this end we use the Principal-Agent formalism and base ourselves on the founding model of Hölmstrom and Milgrom. We first look at full information problems in this model and establish an alternative and unifying framework their resolution with broad application. As well as shining a new light on the meaning of the Borch rule in economy, this approach extends the interpretation of the Agent's optimal action. This method is then applied in the full information part of a first incentive problem where we analyze the effect of task-based pay, characterizing contracts in the gig economy. The Principal's wealth is modeled through a Poisson process of which the Agent can change the intensity. The optimal contracts are explicit, with a slight modification of the classical linearity expected under partial information in this model. We note that despite the Principal's positive wealth, the Agent's wage can be negative which underlines the stakes related to adding positivity constraints to such problems. We therefore then turn to such constrained incentive problems in a discrete model. We establish solution existence through a variational approach under relatively general utilities. These solutions are then characterized through first order methods, allowing some analysis of the effect of constraints on contracts. In particular, we obtain optional type wages. Finally, we turn to a problem related to current affairs : incentives under a risk of economic shutdown. To do so, we extend the Hölmstrom-Milgrom model by adding a default process and we obtain an explicit wage that is linear in the default time. This model is then extended to include a possibility of continued activity despite the shutdown, conditioned on sufficient investment from the Principal. We conclude this thesis with the proof of a necessary result for this last incentive problem : existence and uniqueness of solutions to a certain Brownian backward stochastic differential equation with default.