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Modélisation, analyse et contrôle de certaines interactions vagues-corps rigides

Gaston Vergara

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École doctorale
**Mathématiques
et informatique**

Université
de **BORDEAUX**

THÈSE DE DOCTORAT

présentée par

GASTÓN VERGARA-HERMOSILLA

sous la direction de FRANCK SUEUR et MARIUS TUCSNAK

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE

SPÉCIALITÉ : MATHÉMATIQUES APPLIQUÉES

Modelling, analysis and control of some water waves-rigid body interactions

Soutenue le 20 Octobre 2021 à l'Institut de Mathématiques de Bordeaux
devant le jury composé de :

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Title : Analysis and controllability of some fluid-structure models

Abstract : This PhD dissertation deals with the study of some fluid-structure interaction models in shallow water regimes. Particularly, the manuscript brings contributions on interactions between a fluid with fixed and floating rigid bodies. In a first part, we start by studying the mathematical modelling and simulations of an oscillating water column. In this device, the waves come from shore, find a bottom step, and then reach a chamber to change the air volume to drive a turbine to produce electricity. In this first approach, constant atmospheric pressure is assumed on the free surface of the fluid inside the chamber. Waves governed by the one-dimensional shallow water equations in the presence of this device are essentially reformulated as two transmission problems: the first one is associated with a step in front of the structure, and the second one is related to the wave structure interaction. On the other hand, by considering the notion of Riemann invariants to obtain the discretized version of the transmission conditions, we implement the Lax-Friedrichs scheme to get numerical solutions of the model. Furthermore, by using the concept of nodal profile controllability, we address the problem of boundary controllability for the introduced model. Finally, we close the first part of the thesis by proposing a second nonlinear model of an oscillating water column device involving a transmission condition that describes a time-dependent air pressure flow inside the chamber, which is obtained by considering the free surface Bernoulli's equation and properties relative to the shallow water regime. In a second part of the manuscript, we study some models describing the vertical motion of a solid floating at the free surface of a viscous shallow fluid. The rigid structure involved is assumed to be controlled by a vertical force exerted via an actuator. We start by proving the well-posedness of the model, which is obtained by considering adequate function spaces and convenient operators between them. Furthermore, by obtaining an explicit form of the transfer function associated, we prove some results on the input-output stability of the system. Finally, we study a Cummins type integro-differential equation describing the motion of a partially immersed structure floating in a viscous fluid in an unbounded domain. Relying on the stability results of Matignon for fractional systems, the explicit solutions of the system are established, leading to an explicit knowledge of the long-time behaviour of them.

Keywords : Nonlinear shallow water equations - Viscous shallow water equations
- Fluid-structure models - Oscillating water column device - Boundary controllability
- Cummins-type equations

Titre : Analyse et contrôlabilité de certains modèles fluide-structure

Résumé : Cette thèse porte sur l'étude de certains modèles d'interaction fluide-structure dans des régimes d'eau peu profonde. En particulier, le manuscrit apporte des contributions sur les interactions entre un fluide et des corps rigides fixes et flottants. Dans une première partie, nous commençons par étudier la modélisation mathématique et les simulations d'une colonne d'eau oscillante. Dans ce dispositif, les vagues viennent du rivage, heurtent une marche basse, puis atteignent une chambre pour changer le volume d'air et entraîner une turbine qui produit de l'électricité. Dans cette première approche, une pression atmosphérique constante est supposée sur la surface libre du fluide à l'intérieur de la chambre. Les vagues régies par les équations unidimensionnelles des eaux peu profondes en présence de ce dispositif sont essentiellement reformulées sous la forme de deux problèmes de transmission : le premier est associé à une marche devant la structure, et le second est lié à l'interaction vagues-structure. D'autre part, en considérant la notion d'invariants de Riemann pour obtenir la version discrétisée des conditions de transmission, nous implémentons le schéma de Lax-Friedrichs pour obtenir des solutions numériques du modèle. De plus, en utilisant le concept de contrôlabilité du profil nodal, nous abordons le problème de la contrôlabilité frontière du système. Enfin, nous clôturons la première partie de la thèse en proposant un deuxième modèle non linéaire d'un dispositif à colonne d'eau oscillante impliquant une condition de transmission qui décrit un flux de pression d'air dépendant du temps à l'intérieur de la chambre, qui est obtenu en considérant l'équation de Bernoulli à surface libre et les propriétés relatives au régime des eaux peu profondes. Dans la deuxième partie, nous étudions quelques modèles décrivant le mouvement vertical d'un solide flottant à la surface libre d'un fluide visqueux peu profond. Cette structure rigide est supposée être contrôlée par une force verticale exercée via un actionneur. Nous commençons par prouver que le modèle est bien posé en considérant des espaces fonctionnels adéquats et des opérateurs convenables entre eux. En obtenant une forme explicite de la fonction de transfert associée, nous prouvons quelques résultats sur la stabilité input-output du système. Enfin, nous étudions une équation intégro-différentielle de type Cummins décrivant le mouvement d'une structure partiellement immergée flottant dans un fluide visqueux sur un domaine non-borné. Sur la base des résultats de stabilité de Matignon pour les systèmes fractionnaires, on trouve des solutions explicites du système, conduisant à une description précise de leur comportement en temps long.

Mots-clés : Équations non linéaires en eau peu profonde - Équations visqueuses en eau peu profonde - Modèles fluide-structure - Colonne d'eau oscillante - Contrôlabilité frontière - Équations de type Cummins

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mathématiques en Europe.

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Authors' comments

All publications within this thesis are identical to the original paper as published or submitted, with the following exceptions:

- The numbering of the figures and equations of each paper has been changed to include the number of the chapter.
- The positions of the figures may differ from the final submitted or published version of the papers to improve readability.
- The figures included in this manuscript that mention the physical configuration can differ from those originally considered in the articles in their colour.
- The referencing and typesetting have been altered so that there is a consistent format throughout the entire thesis.
- The Chapter 2 is based on the Sections 1, 2 and 3 of [79]. Similarly, the Chapter 3 is based on the Sections 1, 2 and 3 of [9].

On the other hand, we remark the fact that the notations are local to each chapter.

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Introduction

Introduction (Français)

Dans cette brève introduction, nous présentons les principales motivations et résultats qui seront abordés dans la thèse. Pour une discussion plus en détail sur le sujet, nous renvoyons au chapitre suivant.

Dans ce manuscrit, nous étudions quelques aspects de l'analyse et de la contrôlabilité de certains modèles fluide-structure. En général, les problèmes d'interaction fluide-structure peuvent être décrits par un ensemble d'équations différentielles et de conditions au bord, qui décrivent la dépendance mutuelle entre la dynamique du fluide et de la structure. Dans cette thèse nous nous concentrons sur certains systèmes du EDP's décrivant la présence d'un corps rigide partiellement immergé en régime d'eau peu profonde, qui sont généralement appliquée pour modéliser, par exemple, un convertisseur d'énergie houlomotrice ou un navire. Ces interactions fluide-structure ont été intensément étudiées dans différents scénarios au cours des derniers siècles. Par exemple, l'une des premières approches à ce sujet est due à Froude, qui publia en 1864 un article sur le roulement des navires [26]. L'une des premières approches pour traiter le cas d'un fluide non-visqueux a été proposée par John dans deux articles célèbres [38, 39]. Dans ses recherches, il développe une théorie linéaire, où les variations temporelles de la surface mouillée du corps immergé sont négligées et le mouvement du corps solide est supposé être de faible amplitude. En complément, nous renvoyons aux articles de Lannes et al. [35, 48, 46] pour la description d'une approche non linéaire. Dans [46], Lannes étudie un système vague-structure qui considère deux problèmes de frontières libres; le premier consiste à décrire l'évolution dans le temps de la surface lorsqu'elle est en contact avec l'air, et un second qui provient de la description de l'évolution dans le temps de la portion de frontière du

solide en contact avec le fluide. La pression exercée par le fluide sur le fond de la structure flottante, qui est une inconnue du système, est vue comme le multiplicateur de Lagrange associé à la contrainte que sous la surface de l'objet, la surface du fluide doit coïncider avec le fond de la structure fixe et partiellement immergée.

Par contre, le cas d'un corps rigide partiellement immergé dans un fluide visqueux en régimes d'eau peu profonde, à notre connaissance, a été beaucoup moins étudié dans la littérature, du moins, d'un point de vue mathématique. Cette lacune a été partiellement comblée par le papier récents [57] de Maity, San Martín, Tucsnak et Takahashi, qui présentent un système d'EDP modélisant le mouvement couplé d'un fluide visqueux à surface libre. Leur approche suppose que l'écoulement obéit à un régime d'eau peu profonde (modélisé par les équations visqueuses de Saint-Venant dans une dimension spatiale) et en utiliser un formalisme Hamiltonien.

Dans l'esprit des travaux récents sur ce domaine dus à Tucsnak et al. et Lannes et al., ce manuscrit, present des recherches sur deux approches différentes du problème d'interaction fluide-structure, en considérant les interactions entre des objets rigides et des fluides visqueux ou non-visqueux. Ensuite, nous présenterons brièvement l'organisation et les principales contributions de la thèse.

Ce manuscrit est composé de deux parties. La première traite de la modélisation, de la simulation et de la contrôlabilité frontière de structures partiellement immergées dans un régime d'eau peu profonde. Plus précisément :

- Dans le Chapitre 1 nous présentons le modèle mathématique et les simulations d'une colonne d'eau oscillante sur une configuration simplifiée. Dans ce dispositif, les vagues modélises par les équations d'eaux peu profondes non linéaires unidimensionnelles arrivent du large, rencontrent une marche dans le fond puis arrivent dans une chambre pour changer le volume d'air et activer la turbine. Le système est reformulé en deux problèmes de transmission : l'un est lié au mouvement des vagues sur la topographie en gradins et l'autre est lié à l'interaction vagues-structure à l'entrée de la chambre. Nous utilisons enfin les équations caractéristiques des invariants de Riemann pour obtenir les conditions de transmission discrétisées et nous implémentons le schéma de Lax-Friedrichs pour obtenir des solutions numériques.

Le chapitre 1 est basé sur :

- *Modelling and simulation of a wave energy converter.*

En collaboration avec Edoardo Bocchi et Jiao He.

Publié dans ESAIM: Proceedings and Surveys, 70, pp. 68-83, 2021.

<https://doi.org/10.1051/proc/202107005>

- Dans le Chapitre 2, nous abordons le problème de la contrôlabilité frontière pour le système d'eau peu profonde non linéaire unidimensionnel, décrivant l'écoulement de surface libre de l'eau ainsi que l'écoulement sous une structure

fixe et partiellement immergée. Le système d'équations différentielles considéré peut être interprété comme un modèle simplifié d'un type particulier de convertisseur d'énergie houlomotrice appelé colonne d'eau oscillante. Les exigences physiques conduisent naturellement au problème de la contrôlabilité exacte dans une région prescrite. En particulier, nous utilisons le concept de contrôlabilité du profil nodal dans lequel à un point donné (le nœud) des profils temporels pour les états doivent être accessibles par des contrôles frontière. En réécrivant le système dans un système hyperbolique avec des conditions aux limites non locales, nous nous établissons d'abord les solutions classiques semi-globales du système, puis obtenons la contrôlabilité locale et le profil nodal en utilisant une méthode constructive. De plus, sur la base de ce processus constructif, nous fournissons un concept algorithmique pour calculer la fonction de contrôle frontière requise pour générer une solution qui résout ces problèmes de contrôle.

Le chapitre 2 est basé sur :

- *Boundary controllability of a system modelling a partially immersed obstacle.*

En collaboration avec Günter Leugering et Yue Wang.

Publié dans ESAIM: Control, Optimisation and Calculus of Variations, paper No. 80, 27, 2021.

<https://doi.org/10.1051/cocv/2021076>

- Dans le Chapitre 3 nous proposons un nouveau modèle mathématique non linéaire d'une colonne d'eau oscillante. Les équations unidimensionnelles des eaux peu profondes en présence de ce dispositif sont essentiellement reformulées sous forme de deux problèmes de transmission : le premier est associé à une discontinuité devant le dispositif et le second est lié à l'interaction entre les vagues et une structure fixe et partiellement immergée. En tirant parti de l'équation de Bernoulli à surface libre, nous fermons le système en dérivant une condition de transmission qui implique une pression d'air dépendante du temps à l'intérieur de la chambre de l'appareil, au lieu d'une pression atmosphérique constante. Nous montrons alors que le deuxième problème de transmission peut être réduit à un problème de valeur limite initiale hyperbolique quasi-linéaire avec une condition aux limites semi-linéaire déterminée par une EDO qui dépend de la trace de la solution de l'EDP au bord.

Le chapitre 3 est basé sur :

- *Well-posedness of a nonlinear shallow water model for an oscillating water column with time-dependent air pressure.*

En collaboration avec Edoardo Bocchi et Jiao He.

Soumis.

<https://arxiv.org/pdf/2104.11570>

Dans la seconde partie du manuscrit, nous abordons l'étude d'un modèle décrivant la dynamique d'une structure rigide flottant sur un fluide visqueux en régime d'eau peu profonde. En particulier

- Dans le Chapitre 4, nous étudions la version linéarisée du modèle d'EDP introduit par Maity et al. [57] pour décrire le mouvement vertical d'un solide flottant à la surface libre d'un fluide visqueux peu profond. Le solide est contrôlé par une force verticale exercée via un actionneur. Cette force est l'entrée du système, tandis que la sortie est la distance du solide au fond. La première nouveauté que nous apportons est que nous prouvons que les équations en jeu définissent un système linéaire bien posé. Ceci est fait en considérant des espaces de fonction adéquats et des opérateurs entre eux. Une autre contribution de ce travail est d'établir la stabilité entrée-sortie du système. Pour cela, nous donnons une forme explicite de la fonction de transfert et nous montrons qu'elle se situe dans l'espace de Hardy H^∞ du demi-plan droit.

Le chapitre 4 est basé sur :

- *Well-posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid.*

En collaboration avec Denis Matignon et Marius Tucsnak.

Publié dans IFAC-PapersOnLine 53 (2), pp. 7491-7496, 2020.

<https://doi.org/10.1016/j.ifacol.2020.12.1311>

- Dans le Chapitre 5, nous étudions l'équation intégro-différentielle de type Cummins pour les domaines non-bornés, qui survient lorsque le système d'EDP introduit par Maity et al. [57] qui décrit l'interaction des ondes de surface avec un solide flottant, en tenant compte de la viscosité du fluide, est linéarisé autour des conditions d'équilibre. Une preuve de la stabilité entrée-sortie du système est donnée, grâce à une représentation diffusive de l'opérateur fractionnaire généralisé $\sqrt{1 + \mu s}$. De plus, en s'appuyant sur les résultats de stabilité de Matignon pour les systèmes fractionnaires [59], des solutions explicites sont établies à la fois dans les domaines fréquentiel et temporel, conduisant à une connaissance explicite du vitesse de décroissance de la solution. Enfin, des simulations numériques illustrent de la transition entre différents vitesses de décroissance en fonction de la viscosité μ .

Le chapitre 5 est basé sur :

- *Asymptotic behaviour of a system modelling rigid structures floating in a viscous fluid.*

En collaboration avec Denis Matignon et Marius Tucsnak.

Publié dans IFAC-PapersOnLine 54 (9), pp. 205-212, 2021.

<https://doi.org/10.1016/j.ifacol.2021.06.146>

Introduction

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Shallow water regimes

Modelling free surface flows is a classical research domain with important applications in science and engineering, such as studying the atmosphere, design of ocean wave energy devices, coastal exploitation, and ocean circulation, for instance. The main issue of a free surface flow consists of providing an analytical description of the flow conditions and determining the free surface position by imposing dynamic and kinematic boundary conditions on the free surface. The description of the already mentioned flow conditions are achieved by determining the velocity vector field and the scalar field corresponding to the intensity of the pressure. A usual manner to get a complete description of this type of flow problems in a fluid mechanics framework is by considering the classical Navier-Stokes equations under assumptions relative to the incompressibility, irrotationality, homogeneity and viscosity of the fluid. However, from a computational point of view, the numerical resolution of the Navier-Stokes equations for a free surface flow is known to be dramatically expensive. For instance, in the three-dimensional case, the complete resolution typically involves numerical complexities on implementing the discretization methods and the meshing procedures. A usual alternative to deal with these difficulties is by deriving asymptotic models coming from the governing equations. In the literature, it is possible to find several of this kind of reduced models. The variety is determined by different aspects, such as the regimes where these are valid and the approximation orders. These different regimes are obtained by introducing two dimensionless parameters, the so called *nonlinearity* parameter and the *shallowness* parameter, which

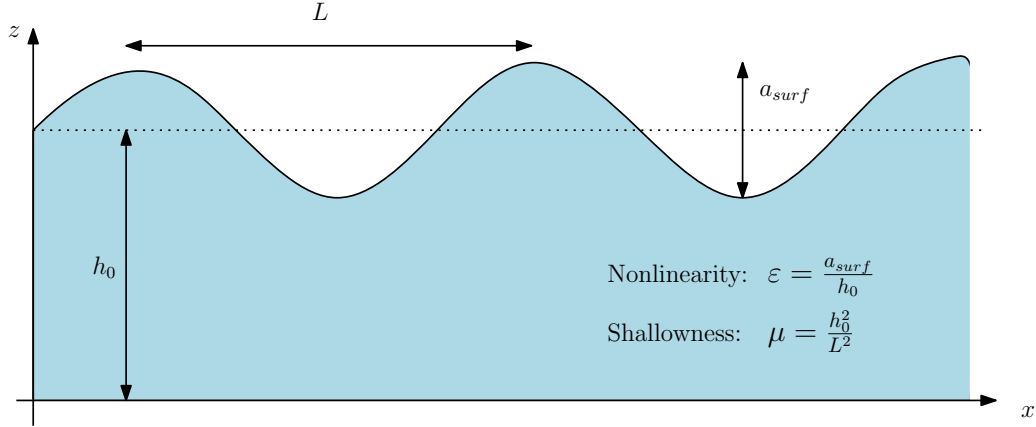


Figure 1 – Characteristic scales for the water waves problem.

are defined respectively by

$$\varepsilon = \frac{a_{surf}}{h_0}, \quad \mu = \frac{h_0^2}{L^2}, \quad (1)$$

where we consider the length scales of the water waves given by

1. The typical wavelength L ,
2. The order of the free surface amplitude a_{surf} ,
3. The typical water depth h_0 .

More specifically, by considering different assumptions on these dimensionless parameters, one obtained different regimes; for instance, assuming that the nonlinearity parameter is small, one gets a regime with small amplitude waves. A graphical sketch of these length scales, including the nonlinearity and shallowness parameters, is presented in Figure 1.

In this manuscript we focus our attention on water waves models in shallow water regimes. More precisely, we will study some mathematical problems that involved the nonlinear shallow water equations, also called Saint-Venant equations, in its viscous and inviscid version. These kind of systems can be deduced from dimensionless forms of the Navier-Stokes equations in the viscous case [30], and from the Craig-Sulem-Zakharov (CSZ) equations in the inviscid case [46], by assuming the shallowness parameter small, *i.e.* $\mu \ll 1$, and different assumptions on the nonlinearity parameter. For instance, by considering a small amplitude $\varepsilon \sim \mu$ into the dimensionless CSZ equations, we get the Boussinesq system which is a system that describes water waves with a (weakly nonlinear) second order approximation, if we consider a medium amplitude $\varepsilon \sim \sqrt{\mu}$, we obtain the Serre equations with $O(\mu^2/\varepsilon)$, and if we assume a large amplitude $\varepsilon \sim 1$, we can get the Green-Naghdi equations

$O(\mu^2)$ and nonlinear shallow water equations $O(\mu)$. This last set of equations was derived originally in 1871 by Barré de Saint-Venant in [17, 18], and conform a PDE system that describes the free surface flow of an incompressible fluid affected by gravity in cases where the horizontal scale is assumed to be greater than the vertical scale. In the literature of inviscid shallow water flows, we highlight the recently paper review of Lannes [47], where he presents a careful and unified derivation of the classical asymptotic models, such as the nonlinear shallow water, the Boussinesq equations and the Serre–Green–Naghdi (SGN) equations. These models are treated by considering an asymptotic analysis of the water waves equations in a horizontal discharge formulation. Hence, by removing the terms of order $O(\mu)$ in the dimensionless CSZ equations, and assuming by simplicity that $\varepsilon = 1$, Lannes obtains the nonlinear shallow water equations in the horizontal discharge formulation, which in one (space) dimension read as

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{1}{h} q^2 \right) + h \partial_x \zeta = 0, \end{cases} \quad (2)$$

where ζ denotes the surface elevation, $h = h_0 + \zeta$, with h_0 is the height of the fluid in equilibrium respect to a flat bottom, and q is the horizontal discharge. The equation (2) will be very relevant in the first part of this manuscript, where we deal with fluid-structure interactions in shallow water, as it will allow us to manipulate the transmission conditions in the traces of the contact areas between the fluid and the rigid object immersed in it.

On the other hand, Lannes in [47] presents less known approaches on shallow water flows, such as multi-layer models, which appears as an alternative to the Boussinesq or SGN models for numerical implementations, and the Isobe–Kakinuma model, which consists of a system of Euler–Lagrange equations for an approximate Lagrangian, derived by approximating the velocity potential in the Lagrangian appropriately. For details on the multi-layer models, we recommend to the reader see the seminal work of Bai and Cheung [3], and the article of Casulli and Stelling [13], while in the case of the Isobe–Kakinuma model, we recommend seeing the seminal papers of Isobe [37], and Kakinuma [40], and the recent article of Kazakova and Richards [42].

In the literature on viscous Saint-Venant equations, we can mention different approaches, derivations and well-posedness results. For instance, in 1985 Kloeden in [43], by using the energy methods of Matsumura and Nishida, he proves the global temporal existence of classical solutions in the dissipative shallow water equations on a spatially periodic horizontal domain. In 1991 Bernardi and Pironneau in [15] by studying the vertical averages of the incompressible Navier-Stokes equations from the point of view of numerical analysis present results on the existence in cases when the Reynolds number is small and convergence of algorithms. In 1995, Orenca

in [64] deals with a model for the evolution of two-dimensional viscous flows and prove results relative to the existence of solutions under constraints of smallness for the initial data. In 1996, under the assumption that the external force field derives from potential, and by adapting the methods of Matsumura and Nishida to prove the existence of solutions of a system of equations modelling the motion of a compressible, viscous, heat-conducting fluid, Sundbye in [76], prove the existence of global-in-time strong solutions of a viscous shallow system for small enough initial velocities, initial heights, and small external force fields. In 2001, Gerbeau and Perthame in [30], derive one of the most commonly used Saint-Venant system that considers small friction, viscosity and Coriolis-Boussinesq factor departing from the Navier-Stokes system with a free moving boundary, which is validated by considering numerical comparisons between their model and direct Navier-Stokes simulations. More precisely, the authors recover a vertical distribution of the velocity field, which leads to a model that includes viscosity, which is given by

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0 \\ \frac{\partial hu}{\partial t} + \frac{\partial hu^2}{\partial x} + \frac{g}{2} \frac{\partial h^2}{\partial x} = -\frac{\kappa u}{1 + \frac{\kappa h}{3\mu}} + 4\mu \frac{\partial}{\partial x} \left(h \frac{\partial u}{\partial x} \right), \end{cases} \quad (3)$$

where $\mu > 0$ and $\kappa \geq 0$ denote the viscosity and a friction of the fluid, respectively, u denote the horizontal component of the velocity field and h is the fluid height respect to the flat bottom. In 2005, Rodríguez and Taboada-Vázquez in [74], by introducing a small adimensional parameter related to the depth of the fluid domain and studying the asymptotic behaviour of it in the case when tending to zero, they derive a shallow water model which includes a novel diffusion term. In 2009, Bresch published an interesting and complete chapter in [10], where he reviews the results from derivations of shallow-water systems, which depend on several mathematical and physical aspects, where we mention, for example, apriori conditions, oscillating topography and boundary effects and the compressible-incompressible limit around a constant or inconstant height profile.

Fluid-structure interactions in shallow water

In this section, we present some recently introduced models on the dynamics of fluids in the presence of a partially immersed rigid body in shallow water, which typically are applied to describe, for example, a wave energy converter or a ship. The wave-structure interaction problem describes the motion of a mechanical system formed by a fluid delimited by a free surface and a partially immersed solid body. This wave-structure system involves two free boundary problems; the first one describes the evolution in time of the surface when it is in contact with the air, and a second one describes the evolution in time of the portion of the boundary of the solid in

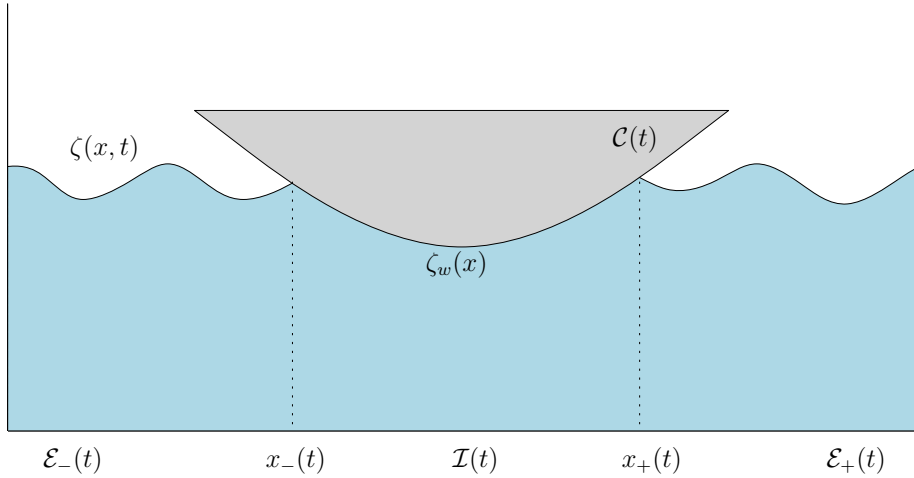


Figure 2 – Configuration.

contact with the fluid, i.e. describes the dependence in time of the “wetted surface” $\partial_w \mathcal{C}(t)$.

One of the first approaches to deal with this problem was proposed by John in two famous articles [38, 39]. In his works, time variations of the wetted surface of the immersed body are neglected, and the motion of the solid body is assumed to be with a small amplitude. Nowadays, this approach is still used in different fields of science and engineering, for instance, to treat simulations of wave-structures interactions. However, despite John’s approach take into account nonlinear effects, the numerical resolution of the Laplace equation for the velocity potential, on time-dependent fluid domains that at the contact line contains wedges or corners, produces an expensive computational cost (see for instance [12]). Recently, Lannes proposed a general approach to describe interactions between water waves and partially immersed floating structures in [46]. In his work, Lannes assumes that the projection of the immersed part of the object is an interval $\mathcal{I}(t) = (x_-(t), x_+(t))$, with $x_-(t) < x_+(t)$, called *interior domain*. Its complement interval with respect to \mathbb{R} is called *exterior domain*, and is explicitly defined by

$$\mathcal{E}(t) = \mathcal{E}_-(t) \cup \mathcal{E}_+(t), \quad \text{where } \mathcal{E}_-(t) = (-\infty, x_-(t)), \text{ and } \mathcal{E}_+(t) = (x_+(t), \infty). \quad (4)$$

In the following, we focus on describing the water waves in a shallow water regime with a first-order approximation in the presence of a partially immersed fixed body. For a graphical sketch of the configuration considered see Figure 2. By denoting by $q_i(t)$ the horizontal discharge defined on the interior domain $\mathcal{I}(t)$, the *nonlinear shallow water equations with an immersed structure* [46], reads as

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{1}{h} q^2 \right) + h \partial_x \zeta = 0, \end{cases} \quad \text{on } \mathcal{E}(t), \quad (5)$$

with

$$\begin{cases} \partial_x q_i = 0, \\ \partial_t q_i + \partial_x \left(\frac{1}{h_w} q_i^2 \right) + gh \partial_x \zeta_w = -\frac{h}{\rho} \partial_x \underline{P}_i, \end{cases} \quad \text{on } \mathcal{I}(t). \quad (6)$$

The equations (5)-(6), are complemented with the following two equations; the first one relative to the continuity of the pressure at the free surface \underline{P} , which is given by

$$\underline{P}_i(t, x) = P_{atm} \quad \text{on } \{x_-(t), x_+(t)\}, \quad (7)$$

and, the second one about the continuity of the free surface elevation ζ , which reads as

$$\zeta(t, x) = \zeta_w(x) \quad \text{on } \{x_-(t), x_+(t)\}. \quad (8)$$

In order to close the system, a third coupling condition about the continuity of the horizontal discharge is considered. This condition is a natural consequence of the smoothness of the vector field U on the fluid domain, which follows by considering the constraints of incompressibility and irrotationality of the fluid and some elliptic estimates (see [46] for details). This coupling condition reads

$$q(t, x) = q_i(t) \quad \text{on } \{x_-(t), x_+(t)\}. \quad (9)$$

In summary, the wave-structure interaction problem, in this case, consists in to find (ζ, q) , (q_i, \underline{P}_i) and $x_-(t)$, $x_+(t)$ such that equations (5)-(9) follow.

We remark the fact that the coupling conditions (7)-(8) defined at the contact lines follow if and only if, in the neighbourhoods of $\{x_-(t), x_+(t)\}$, the boundaries of the partially immersed body are not vertical. Hence, in the presence of a partially immersed object with vertical walls, the coupling conditions (7)-(8) need to be correctly modified. A natural manner to deal with it is by relaxing the continuity of the free surface elevation (8). Consequently, the continuity of the pressure at the free surface \underline{P} needs to be replaced by an adequate and general expression. This generalization has been treated by Lannes in [46], in the cases of a partially immersed body with vertical walls floating freely and with prescribed motion in a shallow regime. Similarly, Bocchi deals with a generalization of an immersed body with vertical walls floating freely on a two-dimensional spatial domain in [6]. In the first part of this manuscript, we deal with a new generalization of the coupling conditions (7)-(8) on the configuration of a partially immersed and fixed object, which is motivated by some applications of this topic in ocean wave energy, see Chapters 1, 2 and 3 for more details.

On the other hand, a model of a rigid structure with vertical walls floating in a viscous fluid in a shallow water regime was proposed recently in the literature by Maity, San Martín, Takahashi and Tucsnak in [57]. In order to describe their model, let us introduce some notations. We denote by ρ the density of the fluid, which is supposed to be constant, by g the constant of gravity acceleration, and by

μ and M the positive constants relative respectively, to the viscosity of the fluid and the mass of the rigid body. Furthermore, we denote by $v(x, t)$, $h(x, t)$ and $p(x, t)$ denote respectively, the velocity and the height of the free surface of the fluid and the pressure force exerted by the fluid on the solid, by $h_S(t)$ the height from the bottom of the cylinder with respect to the horizontal bottom of the fluid container. All these functions are defined on times $t \geq 0$ and on positions $x \in [0, \ell]$, for $\ell > 0$. An important constrain that defines the model that we introduce below is that the body moves only vertically. Hence, similarly to the modelling process described in the section below, the spatial domain is split into two disjoint subdomains defined by the projection of the cylinder on the flat bottom (a, b) , with $a, b \in [0, \ell]$, which is denoted by \mathcal{I} , and its complement $\mathcal{E} = (0, \ell) \setminus \overline{\mathcal{I}}$. For a graphical sketch of the configuration considered see Figure 3.

$$[f]_a := f(a^+) - f(a^-), \quad [f]_b := f(b^+) - f(b^-).$$

With this notation, the model of a rigid structure floating in a viscous fluid is

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (t > 0, x \in \mathcal{I} \cup \mathcal{E}), \quad (10)$$

$$\frac{\partial}{\partial t} \left(\frac{q}{h} \right) + \frac{\partial}{\partial x} \left(\frac{q^2}{2h^2} + gh + p \right) - \mu \frac{\partial}{\partial x} \left(\frac{1}{h} \frac{\partial q}{\partial x} \right) = 0 \quad (t > 0, x \in \mathcal{I} \cup \mathcal{E}), \quad (11)$$

$$M\ddot{h}_S(t) = -Mg + \int_a^b p(x, t) dx \quad (t > 0), \quad (12)$$

with

$$p(x, t) = 0 \quad (t \geq 0, x \in \mathcal{E}), \quad (13)$$

$$h(x, t) = h_S(t) \quad (t \geq 0, x \in \mathcal{I}), \quad (14)$$

and with transmission conditions given by

$$[q(\cdot, t)]_a = [q(\cdot, t)]_b = 0 \quad (t > 0), \quad (15)$$

$$\left[p(\cdot, t) + \frac{q^2(a, t)}{2h^2(\cdot, t)} + gh(\cdot, t) - \frac{\mu}{h} \frac{\partial q}{\partial x}(\cdot, t) \right]_a = 0 \quad (t \geq 0), \quad (16)$$

$$\left[p(\cdot, t) + \frac{q^2(b, t)}{2h^2(\cdot, t)} + gh(\cdot, t) - \frac{\mu}{h} \frac{\partial q}{\partial x}(\cdot, t) \right]_b = 0 \quad (t \geq 0). \quad (17)$$

How is usual, the system above is complemented with the following initial and boundary conditions

$$q(0, t) = q(\ell, t) = 0 \quad (t > 0), \quad (18)$$

$$h(0, x) = h_0(x) \quad (x \in \mathcal{E}), \quad (19)$$

$$q(0, x) = q_0(x) \quad (x \in [0, \ell]), \quad (20)$$

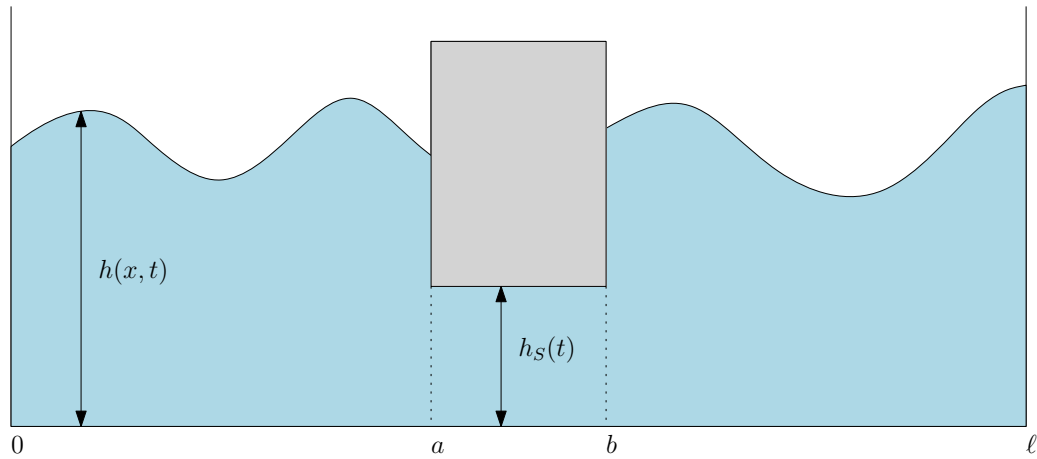


Figure 3 – Configuration.

$$h_S(0) = h_{S0}. \quad (21)$$

In the second part of the manuscript, we will study some stability issues related to two different linearized versions of the fluid-structure system (10)-(21). For more details, see Chapter 4 and 5.

Plan and contributions of the thesis

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The contributions of this thesis are motivated by applications on ocean wave energy of some recently introduced fluid-structure models. More specifically, we study some aspects relative to the modelling, analysis and controllability of some mathematical systems in the spirit of the works of Lannes [46], and Tucsnak et al. [57]. The results presented in this manuscript are presented in two parts, which will be described in the following sections.

Contributions of the first part

In a first part of the manuscript, we deal with the modelling, simulation and boundary controllability of partially immersed structures in a shallow water regime. The contribution of these topics are presented in the following 3 subsection.

Contributions of Chapter 1

In Chapter 1 we study the mathematical configuration of a particular kind of wave energy converter. More precisely, we derive a mathematical model and present simulations of an oscillating water column. In this device, waves governed by the one-dimensional nonlinear shallow water equations arrive from offshore, encounter a step in the bottom and then arrive into a chamber to change the volume of the air to activate a turbine. For simplicity, in this first chapter, we assume that the surface pressure inside the chamber is given by the constant atmospheric pressure. The system proposed is formulated as two transmission problems: one is related to the wave motion over the stepped topography (located at $x = 0$), and the other is related to the wave-structure interaction at the chamber entrance. Hence, by considering as spacial domain the interval $(-l, l_1)$, with $l, l_1 > 0$, the subintervals

$$\mathcal{E}_0 = (-l, 0), \quad \mathcal{E}_1 = (0, l_0 - r), \quad \mathcal{I} = (l_0 - r, l_0 + r), \quad \mathcal{E}_2 = (l_0 + r, l_1),$$

and denoting by $\zeta(t, x)$ the free surface elevation, $h(t, x)$ the fluid height, by ρ the fluid density, by \underline{P} the surface pressure of the fluid and by $q(t, x)$ the horizontal discharge, the first transmission problem defined reads as

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \quad h = h_s + \zeta \text{ in } \mathcal{E}_0, \quad h = h_0 + \zeta \text{ in } \mathcal{E}_1, \quad (22)$$

with transmission conditions at $x = 0$ given by

$$\zeta|_{x=0^-} = \zeta|_{x=0^+}, \quad q|_{x=0^-} = q|_{x=0^+}. \quad (23)$$

The second transmission problem in $\mathcal{E}_1 \cup \mathcal{E}_2$ reads as

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \quad h = h_0 + \zeta, \quad (24)$$

with transmission conditions at $x = l_0 \pm r$ given by

$$q|_{x=l_0-r} = q|_{x=l_0+r}, \quad -\alpha \frac{d}{dt} q_i = \left(\frac{q^2}{2h^2} + g\zeta \right) \Big|_{x=l_0-r} - \left(\frac{q^2}{2h^2} + g\zeta \right) \Big|_{x=l_0+r}, \quad (25)$$

where $\alpha = \frac{2r}{h_w}$, $h_w = h_0 + \zeta_w$ and ζ_w constant (see Subsection 1.1 for the rigorous statements of these quantities). To derive the transmission conditions relative to the first transmission problem, we consider the continuity of the surface elevation

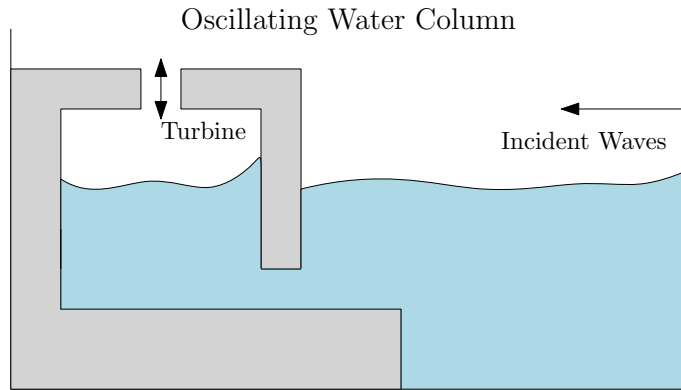


Figure 4 – Sketch of an oscillating water column.

and continuity of the horizontal discharge at the step located at $x = 0$, whereas the transmission conditions involved in the second transmission problem are obtained from the continuity of the horizontal discharge at the walls of the device (see Figure 4 for a graphical idea of this device) and under the assumption that the total fluid-structure energy at time t is equal to the integral between 0 and t of the sum between the energy flux at the entry of the domain and the difference of the energy fluxes at the step. As we have already said, we close the chapter by using the characteristic equations of Riemann invariants to obtain the discretized version of the transmission conditions and hence, we implement a finite difference scheme with a Lax-Friedrichs numerical flux to get numerical simulations.

Chapter 1 is based on:

- *Modelling and simulation of a wave energy converter.*

In collaboration with Edoardo Bocchi et Jiao He.

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Contributions of Chapter 2

In Chapter 2 we address the problem of boundary controllability for a nonlinear model describing the interaction between waves in a shallow water regime with a partially immersed and fixed structure. More precisely, we consider the transmission problems introduced in Chapter 1 on an equivalent physical configuration (see Figure 2.1 for a graphical sketch of this idea) with a boundary control related to the surface elevation. In order to deal with it, the nonlinear shallow water equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \quad \text{for } x \in \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2, \quad (26)$$

with

$$\mathcal{E}_0 = (-l, 0), \quad \mathcal{E}_1 = (0, l_0 - r), \quad \mathcal{E}_2 = (l_0 + r, l_1), \quad (27)$$

are coupled with boundary conditions given by

$$x = -l : \quad \zeta_0 = f(t), \quad (28)$$

$$x = l_1 : \quad q_2 = 0, \quad (29)$$

where $f(t)$ denotes a prescribed boundary function or a boundary control to be determined. In addition, we couple the system above with the transmission conditions developed in [8], which read as

$$x = 0 : \quad \begin{cases} \zeta_0(t, 0) = \zeta_1(t, 0), \\ q_0(t, 0) = q_1(t, 0), \end{cases} \quad (30)$$

$$x = l_0 \pm r : \quad \begin{cases} q_2(t, l_0 + r) = q_1(t, l_0 - r) = q_w(t), \\ \left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right] \Big|_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right] \Big|_{x=l_0-r} = -\alpha \frac{d}{dt} q_w(t), \end{cases} \quad (31)$$

where $q_w(t)$ is the horizontal discharge on the domain $(l_0 - r, l_0 + r)$, the subscript i making mention to the restriction of the respective function to the spacial domain \mathcal{E}_i and $\alpha = \frac{2r}{h_w}$.

The aim of the exact controllability for system (26) is looking for a boundary control $f(t)$ acting at $x = -l$ though the boundary condition (28), such that the prescribed final data $(\zeta_{iT}(x), q_{iT}(x))$, is attained in the entire shallow water regime at a given time T :

$$U_i(T, x) := (\zeta_i, q_i)(T, x) = (\zeta_{iT}(x), q_{iT}(x)), \quad x \in \mathcal{E}_i, \quad i = 0, 1, 2, \quad (32)$$

where the U_i is generated by the control f as a solution of system (26)-(29) with the initial condition:

$$(\zeta_i, q_i)(0, x) = (\zeta_{i0}(x), q_{i0}(x)), \quad x \in \mathcal{E}_i, \quad i = 0, 1, 2. \quad (33)$$

To ensure the existence and uniqueness of a piecewise C^1 semi-global solution for this problem, we give below some assumptions on the initial data and final data: for each $i = 0, 1, 2$,

- (S1) $\zeta_{i0}(x), q_{i0}(x)$ are C^1 functions with small norm $\|(\zeta_{i0}, q_{i0})\|_{C^1[\bar{\mathcal{E}}_i] \times C^1[\bar{\mathcal{E}}_i]}$,
- (S2) $\zeta_{i0}(x), q_{i0}(x)$ satisfy the C^1 compatibility at the points

$$(t, x) \in \{(0, -l), (0, l_1)\},$$

and the piecewise C^1 compatibility at the adjoint points

$$(t, x) \in \{(0, 0), (0, l_0 - r), (0, l_0 + r)\}.$$

Accordingly,

(S1') $\zeta_{iT}(x), q_{iT}(x)$ are C^1 functions with small norm $\|(\zeta_{iT}, q_{iT})\|_{C^1[\bar{\mathcal{E}}_i] \times C^1[\bar{\mathcal{E}}_i]}$,

(S2') $\zeta_{iT}(x), q_{iT}(x)$ satisfy the C^1 compatibility at the points

$$(t, x) \in \{(T, -l), (T, l_1)\},$$

and the piecewise C^1 compatibility at the adjoint points

$$(t, x) \in \{(T, 0), (T, l_0 - r), (T, l_0 + r)\}.$$

To study this control problem, we start by rewriting the governing equations as into a hyperbolic system with nonlocal boundary conditions, and we establish the semi-global classical solutions of the system, *i.e.* the classical solutions on the time interval $[0, T_0]$, where $T_0 > 0$ is a preassigned and possibly quite large number. This preliminary result read as

Theorem .1. *Let $T > 0$ be given. For any given initial data (ζ_{i0}, q_{i0}) with small norm $\|(\zeta_{i0}(\cdot), q_{i0}(\cdot))\|_{C^1[\mathcal{E}_i] \times C^1[\mathcal{E}_i]}$ ($i = 0, 1, 2$) and boundary function $f(t)$ with small norm $\|f(\cdot)\|_{C^1[0, T]}$, satisfying the conditions (S1)-(S2), the forward mixed initial-boundary value problem of the shallow water system (26) on the connected water regime with the initial condition (33), the boundary conditions (28)-(29) and the interface conditions (30)-(31) admits a unique semi-global piecewise C^1 solution $U_i = (\zeta_i(t, x), q_i(t, x))$, ($i = 0, 1, 2$) with small norm*

$$\sum_{i=0}^2 \|(\zeta_i(\cdot, \cdot), q_i(\cdot, \cdot))\|_{C^1[\mathcal{R}_i(T)] \times C^1[\mathcal{R}_i(T)]},$$

on the domain

$$\mathcal{R}(T) = \bigcup_{i=0}^2 \mathcal{R}_i(T) = \bigcup_{i=0}^2 \{(t, x) \mid 0 \leq t \leq T, x \in \mathcal{E}_i\}.$$

We remark the fact that the backward case is analogous. The proof of theorem above is based on Riemann invariants associated with the nonlinear shallow water equations and on the results of Li and Xi in [55], about mixed initial-boundary value problem with non-local boundary condition. See Section 2.3.1 for the rigorous statements of the proof.

The following result states the existence of solutions for the boundary control problem, which can be constructed explicitly (see Subsection 2.3.2 for the rigorous statements of this constructive method.)

Theorem .2. *(Exact Controllability). Let*

$$T > 2 \left(\frac{l}{\sqrt{g\bar{h}_0}} + \frac{l_1 - 2r}{\sqrt{g\bar{h}_1}} \right). \quad (34)$$

For any given initial data (ζ_{i0}, q_{i0}) and final data (ζ_{iT}, q_{iT}) satisfying the assumptions (S1)-(S2) and (S1')-(S2'), respectively, there exists a boundary control $f(t)$ with small norm $\|f(t)\|_{C^1[0,T]}$, such that mixed initial-boundary value problem for equation (26) with the initial condition (33), the boundary conditions (28)-(29) and the transmission conditions (30)-(31) admits a unique piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t, x)$, $(i = 0, 1, 2)$ with small piecewise $C^1 \times C^1$ norm on the domain

$$\mathcal{R}_i(T) = \{(t, x) | 0 \leq t \leq T, x \in \mathcal{E}_i\},$$

which exactly satisfies the desired final condition (32).

We note that in the theorem above, we just considered the steady-state at rest, however, we can easily establish the corresponding local exact controllability in the neighbourhood of a stationary subsonic continuously differentiable. On the other hand, a second controllability problem is considered by using the concept of nodal profile controllability. This kind of exact boundary controllability was recently introduced by Gugat, Herty and Schleper, in [33]. The nodal profile controllability does not ask the solution to exactly attain any given final state at a suitable time $t = T$ by means of boundary controls. Instead it asks the state to exactly fit any given profile on a node after a suitable time $t = T$ utilizing boundary controls.

For system (26), we consider a given demand in fluid height and horizontal discharge at the end $x = l_1$ by:

$$U_2(t, l_1) := (\zeta_2, q_2)(t, l_1) = (\bar{h}_2 + \zeta_B(t), q_B(t)), \quad t \in [T, \bar{T}]. \quad (35)$$

It is worth to mention that the given nodal profile function $U_B := (\bar{h}_2 + \zeta_B(t), q_B(t))$ should be compatible with the boundary condition (29) in the time interval $[0, \bar{T}]$ at the node $x = l_1$, which implies that the demand q_B must be set as 0. While, the other one, $\zeta_B(t)$, can be chosen as any given C^1 function of time after a finite time T .

In the following, we present our result on the exact controllability of nodal profile:

Theorem .3. (*Exact Controllability of Nodal Profile*). Let

$$T > \left(\frac{l}{\sqrt{g\bar{h}_0}} + \frac{l_1 - 2r}{\sqrt{g\bar{h}_1}} \right) \quad (36)$$

and let \bar{T} be an arbitrarily given number satisfying $\bar{T} > T$. Then for any given initial data (ζ_{i0}, q_{i0}) satisfying assumptions (S1)-(S2), and for any given demand of the surface elevation $\zeta_B(t)$ with small norm $\|\zeta_B\|_{C^1[T, \bar{T}]}$, there exists a boundary control $f(t) \in C^1[0, \bar{T}]$, such that the mixed initial-boundary value problem for equation (26) with initial condition (33), boundary conditions (28)-(29) and transmission

conditions (30)-(31) admits a unique piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t, x)$ ($i = 0, 1, 2$) with small $C^1 \times C^1$ norm on the domain

$$\mathcal{R}_i(\bar{T}) = \{(t, x) | 0 \leq t \leq \bar{T}, x \in \mathcal{E}_i\},$$

which exactly satisfies the given nodal profile condition at the end $x = l_1$:

$$\zeta_2(t, l_1) = \zeta_B(t), \quad \forall t \in [T, \bar{T}]. \quad (37)$$

Chapter 2 is based on:

- *Boundary controllability of a system modelling a partially immersed obstacle.*
In collaboration with Günter Leugering and Yue Wang.
Published in ESAIM: Control, Optimisation and Calculus of Variations, paper No. 80, 27, 2021.
<https://doi.org/10.1051/cocv/2021076>

Contributions Chapter 3

In Chapter 3 we propose a nonlinear model given by 1D nonlinear shallow water equations in the presence of an oscillating water column device. The one-dimensional shallow water equations in the presence of this device are essentially reformulated as two transmission problems: the first one is associated with a step, located at $x = 0$, in front of the device and the second one is related to the interaction between waves and a fixed partially-immersed structure with vertical walls, located at $x = l_0 \pm r$. In order to present here these transmission problems, let us introduce the main notations involved in them. We divide the spatial domain $(-\infty, l_1)$ into the exterior domain, and the interior domain is given respectively by

$$\mathcal{I} := (l_0 - r, l_0 + r) \quad \text{and} \quad \mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$$

where $\mathcal{E}^- := (-\infty, 0)$ and $\mathcal{E}^+ = \mathcal{E}_l^+ \cup \mathcal{E}_r^+ := (0, l_0 - r) \cup (l_0 + r, l_1)$. On the other hand we denote by $\zeta(t, x)$ the free surface elevation, by ζ_w the height of bottom of the partially-immersed structure, by $h(t, x)$ the fluid height, by $q(t, x)$ the horizontal discharge, by $\underline{P}(t, x)$ the surface pressure of the fluid, by $P_{\text{ch}}(t)$ the time-dependent variation of P_{air} inside the OWC chamber, by P_{atm} the constant atmospheric pressure, and by h_s, h_0 the fluid heights at rest in \mathcal{E}^- and \mathcal{E}^+ , respectively. Hence, the transmission problem derived in this chapter reads as

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 & \text{in } (0, T) \times \mathcal{E}^- \cup \mathcal{E}_l^+ \cup \mathcal{E}_r^+, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_s + \zeta} \right) + g(h_s + \zeta) \partial_x \zeta = 0 & \text{in } (0, T) \times \mathcal{E}^-, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = 0 & \text{in } (0, T) \times \mathcal{E}_l^+ \cup \mathcal{E}_r^+, \\ \underline{P} = P_{\text{atm}} & \text{in } (0, T) \times \mathcal{E}^- \cup \mathcal{E}_l^+, \\ \underline{P} = P_{\text{atm}} + P_{\text{ch}}(t) & \text{in } (0, T) \times \mathcal{E}_r^+, \end{cases}$$

with transmission conditions over the step

$$\zeta|_{x=0^-} = \zeta|_{x=0^+}, \quad q|_{x=0^-} = q|_{x=0^+},$$

and transmission conditions across the structure side-walls

$$\llbracket q \rrbracket = 0, \quad \langle q \rangle = q_i,$$

where q_i and P_{ch} satisfies

$$\begin{cases} \frac{dq_i}{dt} = -\frac{1}{\alpha} \left[g\zeta + \frac{q^2}{2(h_0 + \zeta)^2} \right] - \frac{1}{\alpha\rho} P_{\text{ch}}, \\ \frac{dP_{\text{ch}}}{dt} = -\gamma_1 P_{\text{ch}} + \gamma_2 q_i, \end{cases}$$

with some constants $\alpha, \gamma_1, \gamma_2$ and

$$\llbracket q \rrbracket = q|_{x=l_0+r} - q|_{x=l_0-r} \quad \text{and} \quad \langle q \rangle = \frac{1}{2} (q|_{x=l_0-r} + q|_{x=l_0+r}).$$

The boundary condition at $x = l_1$ is given by the wall boundary condition $q = 0$ for $x = l_1$.

To the author is knowledge, this is the first nonlinear model for the interaction between shallow water waves and an OWC involving a time-dependent air pressure inside the chamber of the device, instead of a constant atmospheric pressure as in [8]. It turns out that, since the air pressure is time-dependent and space-independent, there is no direct influence of the new air pressure on the fluid equations. It is only involved in the transmission conditions that couple the fluid equations before and after the structure side-walls. This is established by using a simple but new idea: considering an irrotational fluid, we take advantage of free surface Bernoulli's equation for the fluid velocity potential. Indeed, evaluating it at the surface elevation and writing the equation at the precision of the nonlinear shallow water regime, we are able to derive a condition where the difference between the surface pressure inside and outside the OWC chamber appears. This represents a substantial improvement,

and a generalisation of the previous nonlinear model derived in [8], as it recovers the same transmission condition in the case of a constant air pressure inside the chamber.

By taking advantage of the free surface Bernoulli's equation, we close the system by deriving a transmission condition that involves a time-dependent air pressure inside the chamber of the device instead of a constant atmospheric pressure. We then show that the second transmission problem can be reduced to a quasilinear hyperbolic initial boundary value problem with a semilinear boundary condition determined by an ODE depending on the trace of the solution to the PDE at the boundary.

Chapter 3 is based on:

- *Well-posedness of a nonlinear shallow water model for an oscillating water column with time-dependent air pressure.*

In collaboration with Edoardo Bocchi and Jiao He.

Submitted.

<https://arxiv.org/pdf/2104.11570>

Contributions of the second part

In the second part of the manuscript, we deal with the study of a model describing the dynamic of a rigid structure floating in a viscous fluid in a shallow water regime. In order to present the main contributions of this part, let us start by introducing some general notations. Given $t > 0$, we denote by $h(t, x)$ the height of the free surface of the fluid, by $q(t, x)$ the flux of viscous fluid in the direction x and by $h_S(t)$ the distance from the bottom of the rigid body to the bottom of the fluid, supposed to be horizontal. With this notations, the contributions of this second part of the thesis are described in the following two subsections.

Contributions of Chapter 4

In Chapter 4 we study the linearized version of the PDE model introduced by Maity et al. [57] for describes the vertical motion of a solid floating at the free surface of a shallow viscous fluid. In this linear model, the motion of the solid is assumed to be controlled by a vertical force exerted via an actuator, which defines the input of the system, whereas the output is defined as the distance from the solid to the bottom. Hence, the model reads as

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x \in \mathcal{E}), \quad (38)$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0, \quad (x \in \mathcal{E}), \quad (39)$$

$$h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) = p(t, a^+) + h_S(t) - \mu \frac{\partial q}{\partial x}(t, a^+), \quad (40)$$

$$h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) = p(t, b^-) + h_S(t) - \mu \frac{\partial q}{\partial x}(t, b^-), \quad (41)$$

$$\dot{h}_S(t) + \frac{\partial q}{\partial x} = 0 \quad (x \in \mathcal{I}), \quad (42)$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \quad (x \in \mathcal{I}), \quad (43)$$

$$\ddot{h}_S(t) = \int_a^b p(t, x) dx + u(t) \quad (t > 0), \quad (44)$$

where p is a Lagrange multiplier, similar to a pressure term (which is obtained in the Hamiltonian modelling process), u is the input function, whereas the output is

$$y(t) = h_S(t) \quad (t \geq 0). \quad (45)$$

The first novelty we bring in is that we prove that the governing equations define a well-posed linear system in the sense of Weiss (for details see [77] or [84]). This is done by considering adequate function spaces and convenient operators between them.

In fact, by considering

$$X := \mathbb{C} \times H^1(\mathcal{E}) \times L^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C}, \quad (46)$$

we present the following reformulation of the linear system.

Theorem .4. *Equations (38)-(45) can be recast as*

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz, \end{aligned} \quad (47)$$

where the components of the state trajectory $z(t)$ are $h_S(t)$, $h(t, \cdot)$, $q(t, \cdot)$, $q(t, a)$ and $q(t, b)$, B is in $\mathcal{L}(\mathbb{C}, X)$, C is in $\mathcal{L}(X, \mathbb{C})$ and A is a generator of an analytic semigroup on X .

Then, by using the classical definition of well-posed linear systems (see, for instance, [77]), the above theorem implies the following result

Corollary .5. *Equations (38)-(45) define a well-posed linear system with state space X defined in (46) and input and output spaces $U = Y = \mathbb{C}$.*

Another contribution of this work is establishing that the system is input-output stable. In order to present this result, we remember that a well-posed linear system of the form (47) is said input-output stable if equations (47) define, for $z(0) = 0$ a bounded map $u \mapsto y$ from $L^2([0, \infty); U)$ to $L^2([0, \infty); Y)$. Considering this our second main result can be stated as:

Theorem .6. *The system described by (38)-(45) is input-output stable.*

The proof of the theorem below is based on founding the explicit form of the transfer function and showing that it lies in the Hardy space H^∞ of the right-half plane. In fact by proving that the resolvent set $\rho(A)$ contains $\mathbb{C}_0 = \{s \in \mathbb{C} : \Re(s) > 0\}$, it follows that the transfer function

$$G(s) = C(sI - A)^{-1}B,$$

of the system (38)-(45) is defined for every $s \in \mathbb{C}_0$. Moreover, by computing the Laplace transform of the solution of (38)-(45) with zero initial data, we obtain the following result

Proposition .7. *The transfer function of the system (38)-(45) is given by*

$$G(s) := \frac{1}{\left(1 + \frac{l^3}{12}\right) s^2 + \frac{l^2}{2} s \sqrt{1 + \mu s} + \mu l s + l} \quad (s \in \mathbb{C}_0). \quad (48)$$

Hence, the Theorem .6 follows by consider the following Lemma, which is proved by using basic arguments of complex analysis and roots of polynomials.

Lemma .8. *Let F be the function defined by*

$$F(s) = \left(1 + \frac{l^3}{12}\right) s^2 + \frac{l^2}{2} s \sqrt{1 + \mu s} + \mu l s + l, \quad (49)$$

and let $\mathbb{C}_0 = \{s \in \mathbb{C} : \Re(s) > 0\}$. Then there exists a neighborhood \mathcal{O} of $\overline{\mathbb{C}_0}$ such that F is holomorphic on \mathcal{O} . Moreover, F does not vanish on $\overline{\mathbb{C}_0}$.

Chapter 4 is based on:

- *Well-posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid.*

In collaboration with Denis Matignon and Marius Tucsnak.

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<https://doi.org/10.1016/j.ifacol.2020.12.1311>

Contributions of Chapter 5

In Chapter 5 we study an integro-differential equation of Cummins type that arises when the nonlinear PDE system introduced by Maity et al. [57] is linearized around equilibrium in an unbounded domain. More specifically, by considering some assumptions on the pressure term involved in the linear system described above and using the Laplace transform on it, we derive an integro-differential equation which is stated in the following proposition (for a rigorous statement of this result see Proposition 5.2.2).

Proposition .9. *The vertical movement of a floating object, in an unbounded viscous fluid that is initially at rest, is described by the following integro-differential equation*

$$\left(1 + \frac{l^3}{12}\right) \ddot{h}_S(t) = -\frac{l^2}{2} \int_0^t F(\sigma) \dot{h}_S(t - \sigma) d\sigma - l \left(h_S(t) + \mu \dot{h}_S(t)\right), \quad (50)$$

with initial conditions

$$h_S(0) = h_0, \quad \dot{h}_S(0) = 0,$$

and where l is a constant parameter associated with the floating body and F is the causal distribution, such that $\widehat{F}(s) = \sqrt{1 + \mu s}$ in $\Re(s) > -1/\mu$.

Relying on Matignon stability results for fractional systems [59], explicit solutions of equation (50) are established both in the frequency and the time domains, leading to an explicit knowledge of the decay rate of the solution. A key step in this procedure is by studying the distributions on the complex plane \mathbb{C} of the roots of a real polynomial associated with the equation (50), which we called *Viscous polynomial*. This polynomial reads as

$$P_T(\lambda) = \left(1 + \frac{l^3}{12}\right) \lambda^4 + l^2 \sqrt{\mu} \lambda^3 + \left(l\mu - \frac{2}{\mu} \left(1 + \frac{l^3}{12}\right)\right) \lambda^2 - \frac{l^2}{\sqrt{\mu}} \lambda + \frac{1}{\mu^2} \left(1 + \frac{l^3}{12}\right). \quad (51)$$

Thus, by computing the set of roots $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of $P_T(\lambda)$ and considering the following modification of the two parametric Mittag-Leffler function [31],

$$\mathcal{E}_\alpha(\lambda, t) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\int_0^\infty t^{\alpha k + \alpha} e^{-t} dt},$$

the explicit solution of the equation (50) in the time domain is presented in the following theorem (for a rigorous statement of this result see Theorem 5.4.6).

Theorem .10. *The solution of the equation (50) is given by*

$$h_S(t) = \exp\left(-\frac{1}{\mu} t\right) \left(\sum_{i=1}^4 \Theta_i \mathcal{E}_{\frac{1}{2}}(\lambda_i, t)\right), \quad (52)$$

with constants $\Theta_i := r_i h_0 + \tilde{r}_i \dot{h}_0$.

As we already mentioned, thanks to this explicit solution and by considering the approaches of Matignon on the long-time behaviour of the Mittag-Leffler functions in [59, 61], we obtain the following stability result.

Theorem .11. *For the solution (52) of the equation (50), for a given value of the viscosity μ , two cases may occur, depending of the location of the four roots λ_i of the viscous polynomial P_T :*

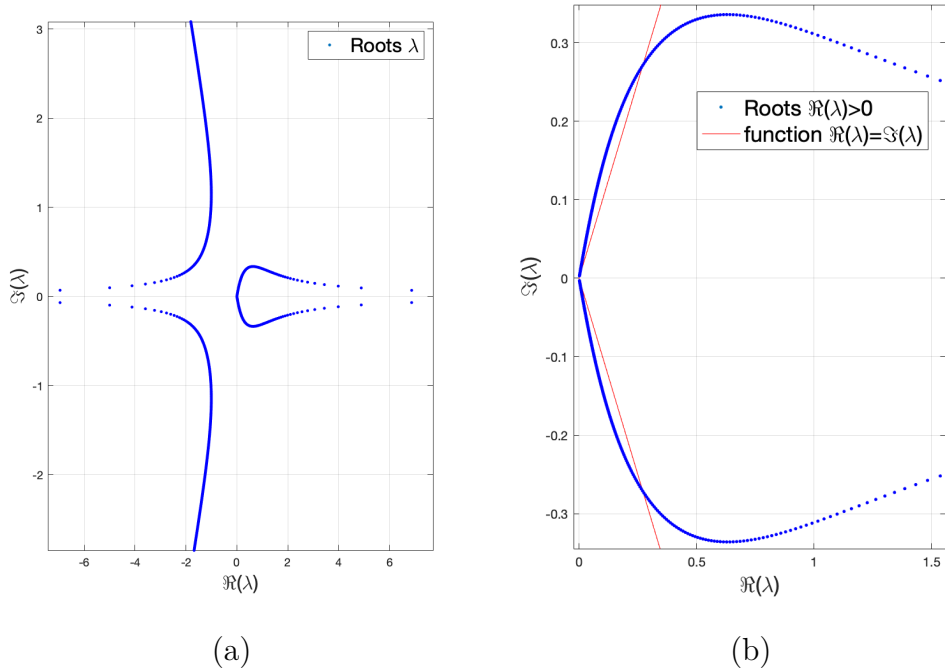


Figure 5 – Evolution of the four roots λ_i in the complex plane \mathbb{C} , as a function of μ . (a): global picture with 4 trajectories. (b): zoom in the right-half plane $\Re(\sigma) > 0$, 2 trajectories crossing the segment $\Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

- either there is at least one root with $\Re(\lambda_j) > |\Im(\lambda_j)|$ then the asymptotics is of exponential type, with rate $\delta(\mu) := \frac{1}{\mu} - \Re(\lambda^2) > 0$

$$h_S(t) \sim \sum_j C_j \exp\left(\left(\lambda_j^2 - \frac{1}{\mu}\right) t\right), \quad (53)$$

- or all the four roots lie in $|\arg(\lambda)| > \frac{\pi}{4}$, then the asymptotics is of mixed type,

$$h_S(t) \sim C t^{-\frac{3}{2}} \exp\left(-\frac{1}{\mu} t\right). \quad (54)$$

We close the chapter by providing numerical evidence that both situations stated by Theorem .11 may occur in practice. In fact, in figure 5 the trajectory of the four roots λ_i is drawn as a function of μ in the σ -plane: two roots belong to the left half-plane and will have no counterpart in the Laplace plane; the two other roots belong to the right half-plane and will give rise to a pole in the Laplace plane; picture (b) provides a zoom on these two, which cross the segments $\Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

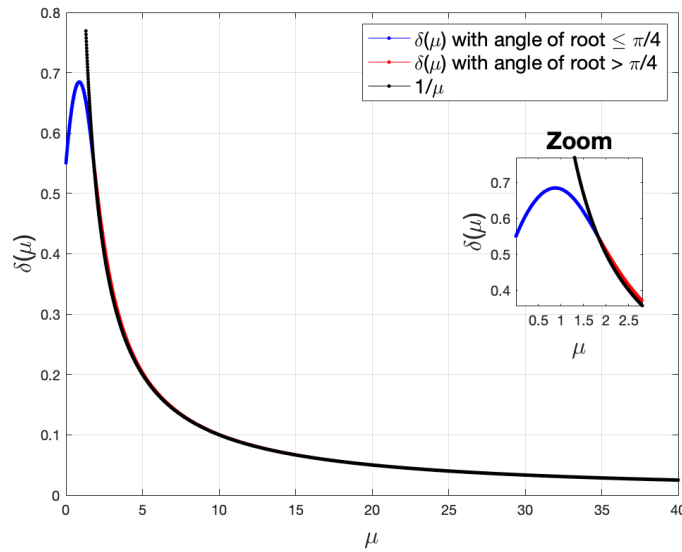


Figure 6 – Damping rate $\delta(\mu) = \Re(\lambda^2) - \frac{1}{\mu}$ as a function of the viscosity term μ .

On the other hand, we illustrate in Figure 6 the transition between the two possible regimes, as the viscosity μ of the fluid increases. Note that above the critical value μ^c , $\delta(\mu) = 1/\mu$, meaning that we are in the mixed type regime. Indeed, the two roots $\lambda_{1,2}$ now fulfill $|\arg(\lambda_{1,2})| > \frac{\pi}{4}$.

Chapter 5 is based on:

- *Asymptotic behaviour of a system modelling rigid structures floating in a viscous fluid.*

In collaboration with Denis Matignon and Marius Tucsnak.

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<https://doi.org/10.1016/j.ifacol.2021.06.146>

Perspectives

In this section, we present some conclusions and perspectives, related to the contributions presented in this manuscript, which we believe merit in-depth research.

As we said above, the research aimed in this thesis presents contributions on the modelling, analysis and control of some water waves-rigid body interactions. Hence, in the first part, we developed the modelling, simulation and boundary controllability of partially immersed structures in a shallow water regime, which has been motivated by a particular kind of wave energy converter (WEC), the so-called oscillating water column (OWC). These contributions constitute a significant and novelty point of

view that allows understanding the behaviour of this waves-structure interaction, and in consequence, to handle a general description of the phenomenons involved in the OWC device. More precisely, these issues are reached by describing the features of the interaction between this kind of WEC and the water waves by taking into account in the modelling process nonlinear effects of the fluid, by developing explicit transmission conditions and by studying a problem of the exact controllability of nodal profile for one of the nonlinear models proposed. These results could allow engineers and oceanographers, to have a better comprehension of the hydrodynamic involved in the OWC device, which may guarantee an effective implementation and maximize the ratio of the produced energy to the energy used in the implementation process.

In what follows, we present some perspectives and future directions of research related to the first part of this dissertation. In particular, we note that the following issues remain open:

- To study the modelling of an OWC device taking into account waves that includes dispersive effects, wich are neglected by the nonlinear shallow water equations. This could be studied by considering waves governed by the one dimensional nonlinear Boussinesq equations:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \left(1 - \frac{h_0^2}{3} \partial_x^2\right) \partial_t q + \partial_x \left(\frac{1}{2} g h^2 + \frac{1}{h} q^2\right) = 0, \quad (h = h_0 + \zeta). \end{cases}$$

Another possible contribution of this research's perspective, is the study of new methods for the numerical implementation of this dispersive OWC model. This idea can be based on a reformulation of the system and the resolution of the dispersive boundary layer that appears at the boundary when non-homogeneous boundary conditions are considered [48].

- To study the configuration of the OWC device taking into consideration the viscosity of the fluid in the modelling process. This could be done supposing that the flow obeys a shallow water regime (modelled by the viscous Saint-Venant equations in one space dimension) and by using a Hamiltonian formalism inspired in the approach introduced in [67] for the inviscid case with the methodology used in [29] in the case of the Navier-Stokes-Fourier system. Another possible contribution of this idea is the study of the well-posedness of the obtained PDEs/ODEs system in function spaces similar to the standard ones for strong solutions of viscous shallow water equations.
- To study the modelling, analysis and control of the OWC device in a two-dimensional setting.

- To study the boundary controllability of nodal profile for the nonlinear system introduced in the Chapter 3.
- By considering the PDE-based OWC models proposed in the first part of the manuscript, to design ad hoc optimal control strategies to implement the system for wave energy extraction purposes, and compare its efficiency with the classical models utilized by ocean wave energy researchers.

In the second part of the manuscript, we studied two different linearized versions of a PDE system introduced by Maity et al. in [57], which describe the dynamics of a rigid structure with vertical walls floating in a viscous fluid in a shallow water regime. Our contributions in this second part, begin by providing results on the well-posedness in the sense of Weiss (see [77]) and on the input-output stability of a first linear model, where the motion of the solid is assumed to be controlled by a vertical force exerted via an actuator, which defines an input for the system, whereas the output is defined as the distance from the solid to the fluid bottom. On the other hand, we derived via a Laplace transform argument on a second linearized version of the model of Maity et al., an integro-differential equation of Cummins type that describes the vertical movement of a floating object, in an unbounded viscous fluid that is initially at rest. We closed our contributions, relying on classical results on fractional differential equation systems, by establishing explicit solutions for the Cummins type equation, which lead us present results about the asymptotic behaviour of the solutions. Thus, and similarly to the motivations mentioned above, the contributions recently described can be utilized as novelty and significant tools to improve the performance of *heaving buoys* devices, which correspond to a particular kind of WECs called *wave-activated bodies*, which the main characteristics are that their motion is created by the wave and move essentially vertically [2].

In the following, we present some perspectives and future directions of research related to the second part of this manuscript. In particular, we remark that the following issues remain open:

- To study the modelling of the case of partially immersed structures with non-vertical walls floating in viscous fluids in a shallow water regime. Two powerful strategies to attack this problem is by using the Hamiltonian argument introduced in [67] and the approach developed by Maity et al. in [57].
- To extend the modelling and analysis process considered by Maity et al. in [57], to two-dimensional configurations in cases where the partially immersed object has vertical and non-vertical walls.
- By considering the fluid-structure nonlinear model introduced by Maity et al. in [57], to derive an integro-differential equation of Cummins type describing the motion of a body in the return to equilibrium problem including nonlinear

effects. Another possible contribution of this idea is the study of the asymptotic behaviour of the ideally obtained Cummins type equation.

- By considering the integro-differential equation of Cummins type studied in the second part of the manuscript to describe the motion of a heaving buoys device, to design and implement an optimal control strategy for wave energy extraction, and compare the productivity obtained with the classical models considered in the ocean wave energy community.

Modelling and control of partially immersed structures

CHAPTER 1

Modelling a simplified model of an oscillating water column device

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*This work is in collaboration with Edoardo Bocchi and Jiao He.
It is based on [8].*

1.1 Introduction

1.1.1 General setting

This chapter is devoted to model and simulate an on-shore oscillating water column (OWC), which is a particular type of wave energy converter (WEC) that transforms the energy of waves reaching the shore into electric energy. The structure is installed at the shore in such a way that the water partially fills a chamber, which is connected with the outside through a hole where a turbine is placed (see Figure 1.1). Incoming waves collide with the exterior part of the immersed wall and, after the collision, one part of the wave is reflected while the other part passes below the fixed partially immersed wall and enters the chamber. This increases the water volume inside the chamber and consequently, it creates an airflow that activates the turbine by passing through it and the same occurs when the volume of water reduces inside the chamber. The perpetuation of the incoming waves makes the water inside the chamber oscillate and act as a liquid piston, whose oscillations create electric energy. In this work the wave energy converter is deployed with a stepped bottom, which means that incoming waves encounter a step in the bottom topography just before reaching the structure. The influence of such step in the OWC device will be discussed later in Section 1.4. The present research is essentially motivated by a series of works by Rezanejad and collaborators on the experimental and numerical study of nearshore OWCs, in particular, we refer to Rezanejad and Soares [72], where the authors used a linear potential theory to do simulations and showed the improvement of the efficiency when a step is added. Our goal is to numerically study this type of WEC considering as the governing equations for this wave-structure interaction the nonlinear shallow water equations derived by Lannes in [46], whose local well-posedness was obtained by Iguchi and Lannes in [35] in the one-dimensional case and by Bocchi in [6] in the two-dimensional axisymmetric case. In the Boussinesq regime and for a fixed partially immersed solid, similar equations were studied by Bresch, Lannes and Métivier in [11] and in the shallow water viscous case by Maity, San Martín, Takahashi and Tucsnak in [57] and by Vergara-Hermosilla, Matignon, and Tucsnak in [80].

We consider an incompressible, irrotational, inviscid and homogeneous fluid in a shallow water regime, which occurs in the region where the OWC is installed. Following [46], the motion of the fluid is governed by the 1D nonlinear shallow water equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P} \end{cases} \quad \text{for } x \in (-l, l_1), \quad (1.1)$$

where $\zeta(t, x)$ is free surface elevation, $h(t, x)$ is the fluid height, ρ is the fluid density, \underline{P} is the surface pressure of the fluid and $q(t, x)$ is the horizontal discharge defined

1.1.2 Organization of the chapter

In Section 1.2, we derive the model used in the numerical simulations following [6, 11, 35]. In particular, we show that the equations (1.1) can be reformulated as two transmission problems, one related to the step in the bottom topography and one related to the wave-structure interaction at the entrance of the chamber. Furthermore, the equations in the exterior domain are written as two transport equations on Riemann invariants. In Section 1.3, we discretize the equations in conservative form using the Lax-Friedrichs scheme and use the Riemann invariants to derive the discretization of the entry condition and boundary conditions. In Section 1.4, we give several computations showing the numerical solutions of the model and compare the OWC device with and without stepped bottom. At the end of this section, we show the accuracy of the numerical scheme to validate our computations and we discuss the absorbed power and the efficiency of the OWC.

Notations

We divide the domain of the problem $(-l, l_1)$ into two parts. The interval $\mathcal{I} = (l_0 - r, l_0 + r)$ is called *interior domain*, which is the projection onto the line of the wetted part of the structure, and its complement $\mathcal{E} = (-l, l_1) \setminus \overline{\mathcal{I}}$, called *exterior domain*, which is the union of three intervals $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ with

$$\mathcal{E}_0 = (-l, 0), \quad \mathcal{E}_1 = (0, l_0 - r) \quad \text{and} \quad \mathcal{E}_2 = (l_0 + r, l_1),$$

where l_1 is the position of the end of the chamber and l_0 and r are respectively the position of the center and the half length of the partially immersed structure. From the nature of the problem, $l_1 > l_0 > r$. Moreover, the boundary of \mathcal{I} is formed by the *contact points* $\{l_0 \pm r\}$, which are the projections on the real line of the triple contact points between fluid, solid and air. For any function f defined in the real line, its restrictions on the interior domain and the exterior domain are respectively denoted by

$$f_i := f|_{\mathcal{I}} \quad \text{and} \quad f_e := f|_{\mathcal{E}}.$$

1.2 Presentation of the model

1.2.1 Governing equations

In this section, we present the mathematical model that describes the oscillating water column process considered in this work. The model can be essentially divided in three parts: the wave motion over a discontinuous topography represented by the step, the wave-structure interaction at the entrance of the chamber and the wave motion in the chamber. In the exterior domain \mathcal{E} , where the fluid is in contact

with the air, the surface pressure \underline{P}_e is constrained and is assumed to be equal to the constant atmospheric pressure P_{atm} , while the surface elevation ζ_e is not known. Contrarily, in the interior domain \mathcal{I} , that is the region under the partially immersed structure, the surface elevation ζ_i is constrained to coincide with the parametrization of the wetted surface, which is assumed to be the graph of some function ζ_w . The surface pressure \underline{P}_i is unknown and it turns out to be a Lagrange multiplier associated with the constraint on ζ_i . For more details on this approach for the study of wave-structure interaction, we refer to [46]. In this work we consider a partially immersed fixed structure with vertical side walls, the parametrization ζ_w is a constant both in time and space. Summing up, we have an opposite behaviour for the surface elevation and the surface pressure under the structure and elsewhere, that is

$$\zeta_i = \zeta_w, \quad \underline{P}_i \text{ is unknown} \quad \text{and} \quad \zeta_e \text{ is unknown,} \quad \underline{P}_e = P_{\text{atm}}.$$

For the exterior domain, we distinguish the region before the step, denoted by \mathcal{E}_0 and the region after the step, denoted by $\mathcal{E}_1 \cup \mathcal{E}_2$. The fluid heights are defined respectively by

$$h_e = h_s + \zeta_e \quad \text{in} \quad \mathcal{E}_0, \quad h_e = h_0 + \zeta_e \quad \text{in} \quad \mathcal{E}_1 \cup \mathcal{E}_2,$$

where h_s and h_0 are the fluid heights at rest before the step and after the step respectively. Denoting by s the height of the step, we have $h_s = h_0 + s$.

Therefore the nonlinear shallow water equations (1.1) can be written as the following three systems:

1. for $x \in \mathcal{E}_0$,

$$\begin{cases} \partial_t \zeta_e + \partial_x q_e = 0, \\ \partial_t q_e + \partial_x \left(\frac{q_e^2}{h_e} \right) + gh_e \partial_x \zeta_e = 0 \end{cases} \quad \text{and} \quad h_e = h_s + \zeta_e, \quad (1.7)$$

2. for $x \in \mathcal{E}_1 \cup \mathcal{E}_2$

$$\begin{cases} \partial_t \zeta_e + \partial_x q_e = 0, \\ \partial_t q_e + \partial_x \left(\frac{q_e^2}{h_e} \right) + gh_e \partial_x \zeta_e = 0 \end{cases} \quad \text{and} \quad h_e = h_0 + \zeta_e, \quad (1.8)$$

3. for $x \in \mathcal{I}$,

$$\begin{cases} \partial_x q_i = 0, \\ \partial_t q_i = -\frac{h_w}{\rho} \partial_x \underline{P}_i \end{cases} \quad \text{and} \quad h_w = h_0 + \zeta_w. \quad (1.9)$$

1.2.2 Derivation of the transmission conditions

The following section is devoted to showing that the motion over the stepped bottom and the wave-structure interaction can be reduced to two transmission problems for the nonlinear shallow water equations. To do that, we derive the transmission conditions relating the different parts of the model, respectively at the step in the bottom topography and at the side walls of the partially immersed structure.

At the topography step

We consider the problem before the entrance of the chamber not as one shallow water system with a discontinuous topography but rather as a transmission problem between two shallow water systems with flat bottoms where the fluid heights are respectively $h_s + \zeta_e$ and $h_0 + \zeta_e$.

The first transmission condition is given by the continuity of the surface elevation at the step, namely

$$\zeta_e|_{x=0^-} = \zeta_e|_{x=0^+}, \quad (1.10)$$

where the traces at $x = 0^-$ and at $x = 0^+$ are the traces at $x = 0$ of the unknowns before the step and after the step respectively.

The second transmission condition is given by the continuity of the horizontal discharge at the step, namely

$$q_e|_{x=0^-} = q_e|_{x=0^+}. \quad (1.11)$$

At the structure side-walls

The transmission conditions at the side-walls of the partially immersed structure are derived from the continuity of the horizontal discharge at the side-walls and the assumption that the total fluid-structure energy is equal to the integral in time of the energy flux at the entry of the domain. The continuity of the horizontal discharge at $x = l_0 \pm r$ together with the fact that $\partial_x q_i = 0$ gives the first transmission condition between the \mathcal{E}_1 and \mathcal{E}_2 , which reads

$$\llbracket q_e \rrbracket := q_e|_{x=l_0+r} - q_e|_{x=l_0-r} = 0. \quad (1.12)$$

Let us now derive the second transmission condition at $x = l_0 \pm r$. To do that, we show the local conservation of the fluid energy in the exterior domain and in the interior domain as in [11].

Exterior domain. Considering the nonlinear shallow water equations in \mathcal{E} , multiplying the first equation in (1.7)-(1.8) by $\rho g \zeta_e$ and the second equation by $\rho \frac{q_e}{h_e}$, and

considering the fact that $\partial_t h_e = -\partial_x q_e$, we obtain

$$\begin{cases} \partial_t \left(\rho \frac{q_e^2}{2h_e} \right) + \rho g \zeta_e \partial_x q_e = 0, \\ \partial_t \left(\rho \frac{q_e^2}{2h_e} \right) - \rho \frac{q_e^2}{2h_e^2} \partial_x q_e + \rho \frac{q_e}{h_e} \partial_x \left(\frac{q_e^2}{h_e} \right) + \rho g q_e \partial_x \zeta_e = 0. \end{cases} \quad (1.13)$$

Adding both equations in (1.13), we obtain

$$\partial_t \left(\rho g \frac{\zeta_e^2}{2} + \rho \frac{q_e^2}{2h_e} \right) + \rho g \zeta_e \partial_x q_e + \rho g q_e \partial_x \zeta_e - \rho \frac{q_e^2}{2h_e^2} \partial_x q_e + \rho \frac{q_e}{h_e} \partial_x \left(\frac{q_e^2}{h_e} \right) = 0.$$

We compute that

$$g\zeta \partial_x q + gq \partial_x \zeta = \partial_x (g\zeta q) \quad \text{and} \quad -\frac{q^2}{2h^2} \partial_x q + \frac{q}{h} \partial_x \left(\frac{q^2}{h} \right) = \partial_x \left(\frac{q^3}{2h^2} \right),$$

and, denoting by \mathbf{e}_{ext} and by \mathbf{f}_{ext} respectively the local fluid energy and the local flux

$$\mathbf{e}_{ext} = \rho \frac{q_e^2}{2h_e} + g\rho \frac{\zeta_e^2}{2} \quad \text{and} \quad \mathbf{f}_{ext} = \rho \frac{q_e^3}{2h_e^2} + g\rho \zeta_e q_e,$$

we obtain the local conservation of the fluid energy in the exterior domain

$$\partial_t \mathbf{e}_{ext} + \partial_x \mathbf{f}_{ext} = 0. \quad (1.14)$$

Interior domain. Let us remark that from the first equation in (1.9) one gets that $q_i \equiv q_i(t)$ in the interior domain. Multiplying the second equation in (1.9) by $\frac{q_i}{h_w}$, we obtain

$$\partial_t \left(\rho \frac{q_i^2}{2h_w} \right) + \partial_x (q_i P_i) = 0,$$

and, denoting by \mathbf{e}_{int} and by \mathbf{f}_{int} respectively the local fluid energy and the local flux

$$\mathbf{e}_{int} = \rho \frac{q_i^2}{2h_w} + \rho g \frac{\zeta_w^2}{2} \quad \text{and} \quad \mathbf{f}_{int} = q_i P_i,$$

we obtain the local conservation of the fluid energy in the interior domain,

$$\partial_t \mathbf{e}_{int} + \partial_x \mathbf{f}_{int} = 0. \quad (1.15)$$

Inspired in [11], now we assume that the total fluid-structure energy at time t is equal to the integral between 0 and t of the sum between the energy flux at the entry of the domain and the difference of the energy fluxes at the step, *i.e.*

$$E_{\text{fluid}} + E_{\text{solid}} = \int_0^t \left(\mathbf{f}_{ext}|_{x=-l} + \mathbf{f}_{ext}|_{x=0^+} - \mathbf{f}_{ext}|_{x=0^-} \right),$$

with the fluid energy defined by

$$E_{\text{fluid}} = \int_{\mathcal{I}} \mathbf{e}_{\text{int}} + \int_{\mathcal{E}} \mathbf{e}_{\text{ext}}.$$

This assumption is an adaptation to a bounded domain case of the conservation of total fluid-structure energy assumed in [11]. We remark that the difference of the energy fluxes at the step $\mathbf{f}_{\text{ext}}|_{x=0^+} - \mathbf{f}_{\text{ext}}|_{x=0^-}$ does not vanish due to the discontinuity of the fluid height at $x = 0$ in the presence of the step. The fact that the structure is fixed ($\frac{d}{dt}E_{\text{solid}} = 0$) yields

$$\frac{d}{dt}E_{\text{fluid}} = \int_{\mathcal{I}} \partial_t \mathbf{e}_{\text{int}} + \int_{\mathcal{E}} \partial_t \mathbf{e}_{\text{ext}} = \mathbf{f}_{\text{ext}}|_{x=-l} + \mathbf{f}_{\text{ext}}|_{x=0^+} - \mathbf{f}_{\text{ext}}|_{x=0^-}.$$

From (1.14) and (1.15) we have

$$-\int_{\mathcal{I}} \partial_x \mathbf{f}_{\text{int}} - \int_{\mathcal{E}} \partial_x \mathbf{f}_{\text{ext}} = \mathbf{f}_{\text{ext}}|_{x=-l} + \mathbf{f}_{\text{ext}}|_{x=0^+} - \mathbf{f}_{\text{ext}}|_{x=0^-}.$$

Using the boundary conditions (1.2) and (1.3) we get

$$\llbracket \mathbf{f}_{\text{int}} \rrbracket = \llbracket \mathbf{f}_{\text{ext}} \rrbracket,$$

where the brackets $\llbracket \cdot \rrbracket$ are defined as in (1.12). By definition of the fluxes it follows

$$\llbracket q_i P_i \rrbracket = \rho \left[\left[\frac{q_e^3}{2h_e^2} + g\zeta_e q_e \right] \right]$$

and from (1.2) and (1.12) we obtain

$$\llbracket P_i \rrbracket = \rho \left[\left[\frac{q_e^2}{2h_e^2} + g\zeta_e \right] \right].$$

Integrating on $(l_0 - r, l_0 + r)$, the second equation in (1.9) yields

$$-\frac{\rho 2r}{h_w} \frac{d}{dt} q_i = \llbracket P_i \rrbracket.$$

Combining the last two equalities, we get the following transmission condition

$$-\frac{2r}{h_w} \frac{d}{dt} q_i = \left[\left[\frac{q_e^2}{2h_e^2} + g\zeta_e \right] \right]. \quad (1.16)$$

1.2.3 Reformulation as two transmission problems

Coupling the governing equations (1.7)-(1.9) with the conditions derived in the previous section, we have therefore reduced the problem of the OWC essentially to two transmission problems. The first one in $\mathcal{E}_0 \cup \mathcal{E}_1$ reads:

$$\begin{cases} \partial_t \zeta_e + \partial_x q_e = 0, \\ \partial_t q_e + \partial_x \left(\frac{q_e^2}{h_e} \right) + gh_e \partial_x \zeta_e = 0, \end{cases} \quad h_e = h_s + \zeta_e \quad \text{in } \mathcal{E}_0, \quad h_e = h_0 + \zeta_e \quad \text{in } \mathcal{E}_1, \quad (1.17)$$

with transmission conditions at $x = 0$

$$\zeta_e|_{x=0^-} = \zeta_e|_{x=0^+}, \quad q_e|_{x=0^-} = q_e|_{x=0^+}. \quad (1.18)$$

The second transmission problem in $\mathcal{E}_1 \cup \mathcal{E}_2$ reads:

$$\begin{cases} \partial_t \zeta_e + \partial_x q_e = 0, \\ \partial_t q_e + \partial_x \left(\frac{q_e^2}{h_e} \right) + gh_e \partial_x \zeta_e = 0, \end{cases} \quad h_e = h_0 + \zeta_e, \quad (1.19)$$

with transmission conditions at $x = l_0 \pm r$

$$\llbracket q \rrbracket = 0, \quad -\alpha \frac{d}{dt} q_i = \left[\left[\frac{q_e^2}{2h_e^2} + g\zeta_e \right] \right], \quad (1.20)$$

where $\alpha = \frac{2r}{h_w}$ and $h_w = h_0 + \zeta_w$.

1.2.4 Riemann invariants

Let us now rewrite the nonlinear shallow water equations (1.7) and (1.8) in the exterior domain \mathcal{E} in a compact form by introducing the couple $U = (\zeta_e, q_e)^T$:

$$\partial_t U + A(U) \partial_x U = 0, \quad (1.21)$$

where

$$A(U) = \begin{pmatrix} 0 & 1 \\ gh_e - \frac{q_e^2}{h_e^2} & \frac{2q_e}{h_e} \end{pmatrix}.$$

The eigenvalues $\lambda_+(U)$ and $-\lambda_-(U)$ of the matrix $A(U)$ and the associated eigenvectors $e_+(U)$ and $e_-(U)$ are given by

$$\lambda_+(U) = \frac{q_e}{h_e} + \sqrt{gh_e}, \quad -\lambda_-(U) = \frac{q_e}{h_e} - \sqrt{gh_e},$$

$$e_+(U) = \left(\sqrt{gh_e} - \frac{q_e}{h_e}, 1 \right)^T, \quad e_-(U) = \left(-\sqrt{gh_e} - \frac{q_e}{h_e}, 1 \right)^T.$$

Notice that $\lambda_+ > 0$ and $\lambda_- > 0$. Taking the scalar product of (1.21) and eigenvectors, we obtain

$$\partial_t \left(2\sqrt{gh_e} \pm \frac{q_e}{h_e} \right) \pm \left(\sqrt{gh_e} \pm \frac{q_e}{h_e} \right) \partial_x \left(2\sqrt{gh_e} \pm \frac{q_e}{h_e} \right) = 0.$$

Let us introduce the right and the left Riemann invariant R and L associated to the nonlinear shallow water equations, respectively

$$R(U) := 2 \left(\sqrt{gh_e} - \sqrt{gh_0} \right) + \frac{q_e}{h_e}, \quad L(U) := 2 \left(\sqrt{gh_e} - \sqrt{gh_0} \right) - \frac{q_e}{h_e}. \quad (1.22)$$

Hence we can write the 1D nonlinear shallow water equations in the exterior domain as the two following transport equations on R and L :

$$\partial_t R(U) + \lambda_+(U) \partial_x R(U) = 0, \quad \partial_t L(U) - \lambda_-(U) \partial_x L(U) = 0. \quad (1.23)$$

We will see that these two transport equations of Riemann invariants are helpful when we solve our model by numerical method. More details about Riemann invariants of the nonlinear shallow water equations can be found in [48].

1.3 Discretization of the model

We have reformulated in the previous section the mathematical model of the oscillating water column as two transmission problems. This section is devoted to discretize the nonlinear shallow water equations (1.7)-(1.9) at the level of the numerical scheme. More precisely, we will use the Lax-Friedrichs scheme to solve our main equations and use Riemann invariants to address the entry conditions and all boundary conditions.

Numerical notations

We use the following notations throughout this subsection:

- in our system, the whole numerical domain $[-l, l_0]$ is composed of four parts: $[-l, 0]$, $[0, l_0 - r]$, $[l_0 - r, l_0 + r]$ and $[l_0 + r, l_1]$. Each interval is divided into cells $(\mathcal{A}_i)_{1 \leq i \leq n_x}$ with $\mathcal{A}_i = [x_{i-1}, x_i]_{1 \leq i \leq n_x}$ of size δ_x . More precisely, we have

$$\begin{aligned} x_0 &= -l, \dots, x_i = -l + i\delta_x, \dots, x_{n_1, x} = 0; \\ x_{n_1, x+1} &= \delta_x, \dots, x_{n_1, x+i} = i\delta_x, \dots, x_{n_1, x+n_2, x} = l_0 - r; \\ x_{n_1, x+n_2, x+1} &= l_0 - r + \delta_x, \dots, x_{n_1, x+n_2, x+i} = l_0 - r + i\delta_x, \dots, x_{n_1, x+n_2, x+n_3, x} = l_0 + r; \\ x_{n_1, x+n_2, x+n_3, x+1} &= l_0 + r + \delta_x, \dots, x_{n_1, x+n_2, x+n_3, x+i} \\ &= l_0 + r + i\delta_x, \dots, x_{n_1, x+n_2, x+n_3, x+n_4, x} = l_1, \end{aligned}$$

with $l = n_{1,x}\delta_x$, $l_0 - r = n_{2,x}\delta_x$, $2r = n_{3,x}\delta_x$ and $l_1 - (l_0 + r) = n_{4,x}\delta_x$;

- we denote by δ_t the time step. According to the CFL condition, time step δ_t can be specified by δ_x ;
- for any quantity U , we denote by U_i^m its value at the position x_i at time $t^m = m\delta_t$. For instance, the variables ζ_i^m denotes the value of the free surface elevation ζ at the position x_i at time $t^m = m\delta_t$.

1.3.1 Discretization of the equation

The finite difference method is a standard discretization approach for partial differential equations, especially those that arise from conservation laws. We first rewrite equation (1.21) as the following conservative form :

$$\partial_t U + \partial_x(F(U)) = 0, \quad (1.24)$$

with

$$F(U) = \left(q_e, \frac{1}{2}g(h_e^2 - h_0^2) + \frac{q_e^2}{h_e} \right)^T.$$

By means of a finite difference approach, equation (1.24) can be discretized as

$$\frac{U_i^{m+1} - U_i^m}{\delta_t} + \frac{F_{i+1/2}^m - F_{i-1/2}^m}{\delta_x} = 0,$$

where the flux F is discretized with cell centres indexed as i and cell edge fluxes indexed as $i \pm 1/2$. The choice of $F_{i\pm 1/2}^m$ depends on the numerical scheme. We consider here the well-known Lax–Friedrichs scheme proposed by Lax [49] to get the discrete flux

$$F_{i-1/2}^m = \frac{1}{2} (F_i^m + F_{i-1}^m) - \frac{\delta_x}{2\delta_t} (U_i^m - U_{i-1}^m), \quad (1.25)$$

where $i \geq 1$ and $F_i^m = F(U_i^m)$.

1.3.2 Discretization of the entry condition

At the entrance of our system, the surface elevation is given by a prescribed function f depending only on time,

$$\zeta^m|_{x=-l} = f^m := f(t^m).$$

In order to express the entry condition for the horizontal discharge, let us first recall that from (1.22) one has

$$q_e = \frac{h_e}{2} (R - L), \quad R + L = 4 \left(\sqrt{gh_e} - \sqrt{gh_0} \right),$$

where R and L are respectively the right and the left Riemann invariant associated to the nonlinear shallow water equations. We get

$$q_e = h_e \left(2 \left(\sqrt{gh_e} - \sqrt{gh_0} \right) - L \right).$$

Hence, the value of q_e at $x = -l$ is given by

$$q_e|_{x=-l} = (h_0 + f(t)) \left(2 \left(\sqrt{g(h_0 + f(t))} - \sqrt{gh_0} \right) - L|_{x=-l} \right).$$

On the right-hand side of the relation above, $L|_{x=-l}$ is unknown. First we have to determine $L|_{x=-l}$ in order to determine $q_e|_{x=-l}$. This can be achieved by the transport equation for L in (1.23). After discretizing it as in [58], we get

$$\frac{L_0^m - L_0^{m-1}}{\delta_t} - \lambda_- \frac{L_1^{m-1} - L_0^{m-1}}{\delta_x} = 0, \quad (1.26)$$

where L_0^m is the value of L at $x = -l$ at time t^m and λ_- is computed as a linear interpolation between $\lambda_{-,0}$ and $\lambda_{-,1}$ following [48], namely

$$\lambda_- = \beta \lambda_{-,0} + (1 - \beta) \lambda_{-,1}$$

with $0 \leq \beta \leq 1$ such that $\lambda_- \delta_t = \beta \delta_x$. Moreover, we can compute λ_- as

$$\lambda_- = \frac{\lambda_{-,1}}{1 + \frac{\delta_t}{\delta_x} \lambda_{-,1} - \frac{\delta_t}{\delta_x} \lambda_{-,0}}.$$

Thus, we have

$$L_0^m = \left(1 - \lambda_- \frac{\delta_t}{\delta_x} \right) L_0^{m-1} + \lambda_- \frac{\delta_t}{\delta_x} L_1^{m-1}, \quad (1.27)$$

which gives L_0^m in terms of its values at the previous time step and in terms of interior points.

1.3.3 Discretization of the boundary conditions

Since our system is composed by four parts, it remains three boundary conditions should be taken into consideration besides the entry condition at $x = -l$. When wave arrives from the offshore, it will encounter a step in the bottom and then arrive into a chamber, and finally arrive to the wall (see the configuration 1.1). More precisely, the first boundary condition is at the discontinuity of the topography located at $x = 0$ and the second is at the partially immersed structure side-walls located at $x = l_0 \pm r$. The last boundary condition is at the end of the chamber, located at $x = l_1$.

At the topography step

Let us first consider the shallow water wave equations with discontinuous topography, namely, it is a system with depth h_s on $\mathbb{R}_- = \{x < 0\}$ and depth h_0 on $\mathbb{R}_+ = \{x > 0\}$. Our equation turns out to be

$$\partial_t U + \partial_x (F(U)) = 0,$$

with

$$F(U) = \begin{cases} \left(q_e, \frac{1}{2}g \left((h_s + \zeta_e)^2 - h_s^2 \right) + \frac{q_e^2}{h_s + \zeta_e} \right)^T, & \text{in } (0, T) \times \mathbb{R}_-, \\ \left(q_e, \frac{1}{2}g \left((h_0 + \zeta_e)^2 - h_0^2 \right) + \frac{q_e^2}{h_0 + \zeta_e} \right)^T, & \text{in } (0, T) \times \mathbb{R}_+. \end{cases}$$

From transmission conditions (1.10) and (1.11), we have the continuity of the surface elevation ζ_e and of the horizontal discharge q_e at $x = 0$:

$$\zeta_e^l|_{x=0} = \zeta_e^r|_{x=0}, \quad q_e^l|_{x=0} = q_e^r|_{x=0}. \quad (1.28)$$

Let us denote the right Riemann invariant in the domain \mathbb{R}_- by R^l and the left Riemann invariant in the domain \mathbb{R}_+ by L^r . We then find two expressions of q_e describing $q_e^l|_{x=0}$ and $q_e^r|_{x=0}$, respectively,

$$\begin{cases} q_e^l|_{x=0} = (h_s + \zeta_e^l|_{x=0}) \left(R^l|_{x=0} - 2 \left(\sqrt{g(h_s + \zeta_e^l|_{x=0})} - \sqrt{gh_s} \right) \right), \\ q_e^r|_{x=0} = (h_0 + \zeta_e^r|_{x=0}) \left(2 \left(\sqrt{g(h_0 + \zeta_e^r|_{x=0})} - \sqrt{gh_0} \right) - L^r|_{x=0} \right). \end{cases} \quad (1.29)$$

According to the relations (1.28), we observe that (1.29) is a system of two nonlinear equations on the two unknowns $\zeta_e^l|_{x=0}$ (respectively $\zeta_e^r|_{x=0}$) and $q_e^l|_{x=0}$ (respectively $q_e^r|_{x=0}$). We write it in the compact form

$$F(x_1, x_2) = 0, \quad (1.30)$$

where $x_1 = \zeta_e^l|_{x=0}$, $x_2 = q_e^l|_{x=0}$ and the vector $F = (F_1, F_2)$ is given by

$$\begin{aligned} F_1 &= (h_s + x_1) \left(R^l|_{x=0} - 2 \left(\sqrt{g(h_s + x_1)} - \sqrt{gh_s} \right) \right) - x_2, \\ F_2 &= (h_0 + x_1) \left(2 \left(\sqrt{g(h_0 + x_1)} - \sqrt{gh_0} \right) - L^r|_{x=0} \right) - x_2. \end{aligned}$$

In the case $h_s = h_0$ (without step) we can derive from (1.29) a third degree equation on $\sqrt{h_0 + \zeta_e^l|_{x=0}}$ and take the unique solution that gives $\zeta_e^l|_{x=0} = 0$ when $R^l|_{x=0}, L^r|_{x=0} = 0$ (we refer to [46] for this case). Here, since $h_s \neq h_0$, we use

MATLAB nonlinear system solver *fsolve* with initial point $(0, 0)$ to solve (1.30). Before doing that, we have to determine the values of the two Riemann invariants $R^l|_{x=0}$ and $L^r|_{x=0}$. The transport equations for R^l and L^r are the following:

$$\partial_t R^l + \lambda_+^l(U) \partial_x R^l = 0, \quad \partial_t L^r - \lambda_-^r(U) \partial_x L^r = 0, \quad (1.31)$$

where the corresponding eigenvalue λ_+^l in the domain \mathbb{R}_- is given by

$$\lambda_+^l(U) = \frac{q_e}{h_s + \zeta_e} + \sqrt{g(h_s + \zeta_e)}, \quad (1.32)$$

and the corresponding eigenvalue $-\lambda_-^r$ in the domain \mathbb{R}_+ is given by

$$-\lambda_-^r(U) = \frac{q_e}{h_0 + \zeta_e} - \sqrt{g(h_0 + \zeta_e)}. \quad (1.33)$$

Let us emphasize that we use here the same interpolation for λ_+ and λ_- as in [58]. After discretization of equations (1.31), we get

$$\begin{aligned} \frac{(R^l)_{n_1,x}^m - (R^l)_{n_1,x}^{m-1}}{\delta_t} + \lambda_+^l \frac{(R^l)_{n_1,x}^{m-1} - (R^l)_{n_1,x-1}^{m-1}}{\delta_x} &= 0, \\ \frac{(L^r)_{n_1,x}^m - (L^r)_{n_1,x}^{m-1}}{\delta_t} - \lambda_-^r \frac{(L^r)_{n_1,x+1}^{m-1} - (L^r)_{n_1,x}^{m-1}}{\delta_x} &= 0, \end{aligned}$$

where λ_+^l , λ_-^r are as in (1.32)-(1.33) and we recall that $(R^l)_{n_1,x}^m$ is the value of R^l at $x_{n_1,x}$ and t^m (see Notations 1.3). Hence, we have

$$\begin{aligned} (R^l)_{n_1,x}^m &= \left(1 - \lambda_+^l \frac{\delta_t}{\delta_x}\right) (R^l)_{n_1,x}^{m-1} + \lambda_+^l \frac{\delta_t}{\delta_x} (R^l)_{n_1,x-1}^{m-1}, \\ (L^r)_{n_1,x}^m &= \left(1 - \lambda_-^r \frac{\delta_t}{\delta_x}\right) (L^r)_{n_1,x}^{m-1} + \lambda_-^r \frac{\delta_t}{\delta_x} (L^r)_{n_1,x+1}^{m-1}, \end{aligned} \quad (1.34)$$

which give $(R^l)_{n_1,x}^m$ and $(L^r)_{n_1,x}^m$ in terms of their values at the previous time step and in terms of interior points.

Gathering the relations (1.28), (1.29) and (1.34), we can solve $\zeta_e^l|_{x=0}$ (respectively $\zeta_e^r|_{x=0}$) and $q_e^l|_{x=0}$ (respectively $q_e^r|_{x=0}$), which give us the boundary conditions at the step.

At the structure side-walls

Compared with the derivation of the boundary conditions near the step, the idea to derive the boundary condition near the fixed partially immersed structure is almost

the same. There are two differences between them. The first one is that, since the depth is always h_0 , the eq. (1.29) becomes

$$\begin{aligned} q_e^l|_{x=l_0-r} &= (h_0 + \zeta_e^l|_{x=l_0-r}) \left(R^l|_{x=l_0-r} - 2 \left(\sqrt{g(h_0 + \zeta_e^l|_{x=l_0-r})} - \sqrt{gh_0} \right) \right), \\ q_e^r|_{x=l_0+r} &= (h_0 + \zeta_e^r|_{x=l_0+r}) \left(2 \left(\sqrt{g(h_0 + \zeta_e^r|_{x=l_0+r})} - \sqrt{gh_0} \right) - L^r|_{x=l_0+r} \right), \end{aligned} \quad (1.35)$$

where we denote the horizontal discharge in the exterior domain on the left-hand side of the object by q_e^l and on the right-hand side of the object by q_e^r . Let us recall that q_i is the horizontal discharge in the interior domain \mathcal{I} . From the first transmission condition in (1.20), we know that

$$q_e^l|_{x=l_0-r} = q_i = q_e^r|_{x=l_0+r}.$$

The second difference is that, unlike in the previous subsection, we do not have the continuity condition of ζ_e at the structure side-walls. Nevertheless, we consider the discretization of the second transmission condition in (1.20), hence we get

$$-\alpha \frac{(q_e)_{l_0-r}^m - (q_e)_{l_0-r}^{m-1}}{\delta t} = \frac{((q_e^l)_{l_0+r}^{m-1})^2}{2(h_0 + (\zeta_e^l)_{l_0+r}^{m-1})^2} + g(\zeta_e^l)_{l_0+r}^{m-1} - \frac{((q_e^r)_{l_0-r}^{m-1})^2}{2(h_0 + (\zeta_e^r)_{l_0-r}^{m-1})^2} - g(\zeta_e^r)_{l_0-r}^{m-1}.$$

where for the sake of clarity $(q_e)_{l_0-r}^m = (q_e)_{n_1, x+n_2, x}^m$ and $(q_e)_{l_0+r}^m = (q_e)_{n_1, x+n_2, x+n_3, x}^m$ (analogously for $(\zeta_e)_{l_0-r}^m$ and $(\zeta_e)_{l_0+r}^m$). Then, q_e at $x = l_0 - r$ is expressed as

$$\begin{aligned} (q_e)_{l_0-r}^m &= (q_e)_{l_0-r}^{m-1} - \frac{\delta t}{\alpha} \left(\frac{((q_e^l)_{l_0+r}^{m-1})^2}{2(h_0 + (\zeta_e^l)_{l_0+r}^{m-1})^2} - \frac{((q_e^r)_{l_0-r}^{m-1})^2}{2(h_0 + (\zeta_e^r)_{l_0-r}^{m-1})^2} \right) \\ &\quad - \frac{\delta t}{\alpha} g \left((\zeta_e^l)_{l_0+r}^{m-1} - (\zeta_e^r)_{l_0-r}^{m-1} \right), \end{aligned} \quad (1.36)$$

which gives $(q_e)_{l_0-r}^m$ in terms of its values at the previous time step and in terms of interior points. Now we can solve $(q_e)_{l_0-r}^m$ immediately. Once the value of $(q_e)_{l_0-r}^m$ is obtained, we can find the values of $\zeta_e^l|_{x=l_0-r}$ and $\zeta_e^r|_{x=l_0+r}$ by using equations (1.35) and the transport equations for the Riemann invariants as the strategy in Section 1.3.3.

At the end of the chamber

The corresponding boundary condition at the end of the chamber, located at $x = l_1$, is given by

$$q_e|_{x=l_1} = 0.$$

Hence, recalling the definition of the right-going Riemann invariant R , we recover the surface elevation ζ_e at $x = l_1$, namely

$$\zeta_e|_{x=l_1} = \frac{1}{g} \left(\frac{R|_{x=l_1}}{2} + \sqrt{gh_0} \right)^2 - h_0.$$

1.4 Numerical validations

In this section, we use the scheme introduced in Section 1.3 to simulate our model. For the fluid, we always consider the density of water $\rho = 1000 \text{ kg/m}^3$ and the gravitational acceleration $g = 9.81 \text{ m/s}^2$. The entry of the domain is set at $x = -l = -30 \text{ m}$ and the prescribed function f is given by

$$f(t) = \sin\left(\frac{2\pi}{T}t\right),$$

where $T = 1.5 \text{ s}$ is the period. Using the notations as before, we consider $l_0 = 11 \text{ m}$, $r = 1 \text{ m}$ and $l_1 = 17 \text{ m}$ and the fluid height at rest before the step $h_s = 15 \text{ m}$. We compute the solution by using the Lax-Friedrichs scheme in the exterior domain $[-30, 10] \cup [12, 17]$, with a refined mesh with $N_x = 2300$ and a time step $\delta_t = \frac{0.7}{\sqrt{gh_s}}\delta_x$ with space step $\delta_x = 0.02 \text{ m}$. Here, the CFL number is 0.7, which is commonly used to prescribe the terms of the finite-difference approximation of a PDE (see for instance [65]). In the interior domain, the solution can be computed using the transmission conditions (1.20) with $h_w = h_0 + \zeta_w$ and $\zeta_w = -7.5 \text{ m}$.

1.4.1 Numerical solutions

In real applications, an OWC device can be deployed on a stepped sea bottom in order to improve its performance. It is important then to have a good understanding of the impact of a step in the topography. Here, we test and compare the case without step $s = 0 \text{ m}$ ($h_0 = 15 \text{ m}$) to the case with a step of height $s = 5 \text{ m}$ ($h_0 = 10 \text{ m}$) considering the previous physical parameters. The numerical solutions are plotted in Figure 1.2 at times $t = 1.7 \text{ s}$, $t = 3.3 \text{ s}$ and $t = 5 \text{ s}$. The plots (a), (c), (e) show the solutions without stepped bottom, while the plots (b), (d), (f) show the solutions with stepped bottom.

We find that, before the waves encounter the step, there is no significant difference between the OWC model without stepped bottom and with stepped bottom (see (a) and (b)). But when the waves encounter the step in the bottom and arrive into the chamber, we can see that, the waves in the OWC model without stepped bottom move significantly faster than the waves in the OWC model with stepped bottom. In particular, at $t = 3.3 \text{ s}$ the waves in the OWC model without stepped bottom has already arrived to the chamber and will begin to change the water level in the chamber, while the waves in the OWC model with stepped bottom have not reached yet and the water will rise inside the chamber later (see (c) and (d)). As the step at bottom is a sort of obstacle for the incoming wave, this phenomenon is reasonable.

As one may expect, the incoming wave split into two parts when it touches the left wall of the partially immersed structure. One part enters the chamber and changes the volume of the air that makes the turbine rotate. The other part is reflected and becomes an outgoing wave, as we can see in Figure 1.2. At $t = 5 \text{ s}$, the reflected wave

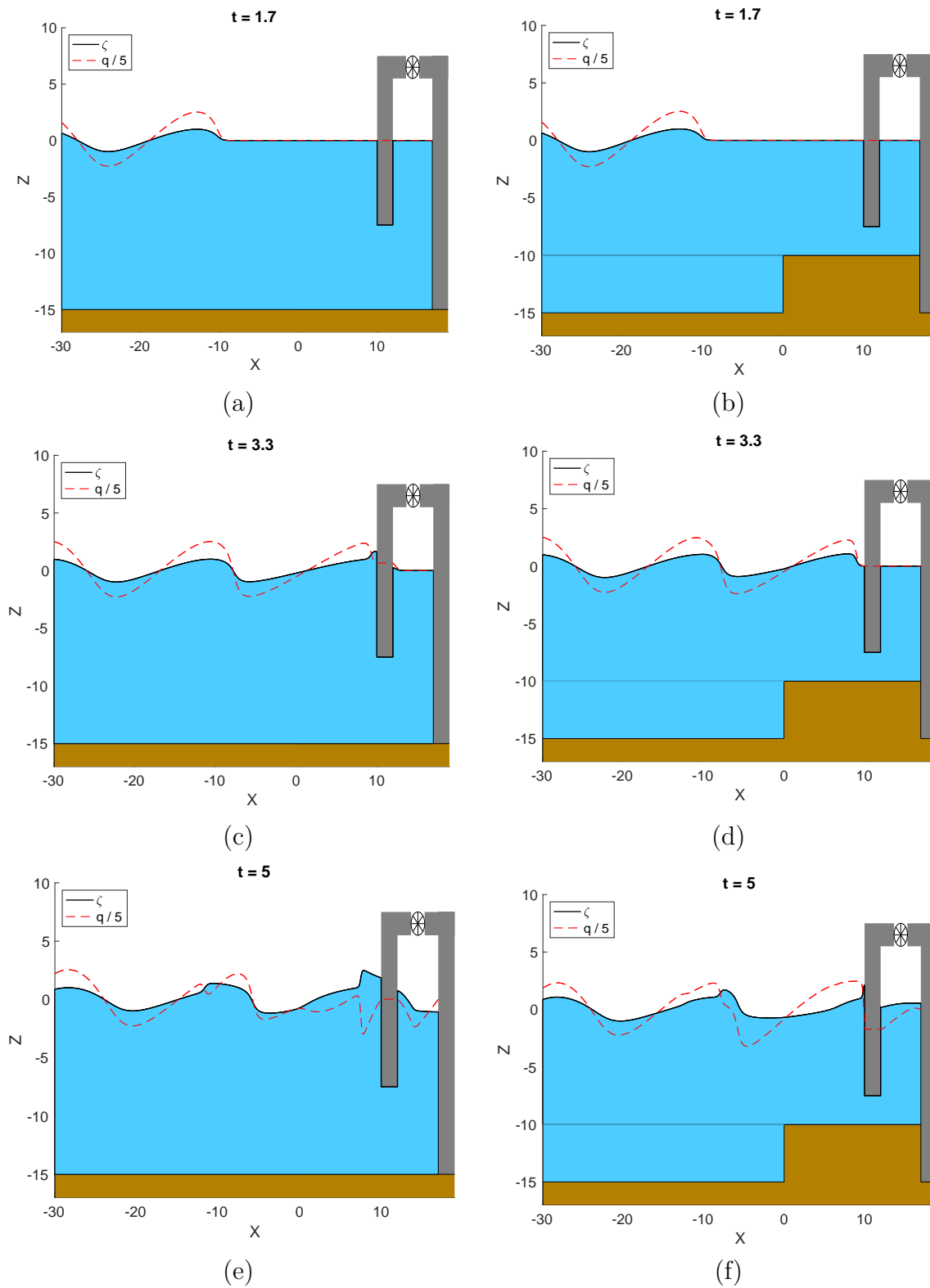


Figure 1.2 – Comparisons between the numerical results without step (left) and with step (right) at times $t = 1.7$ s, $t = 3.3$ s and $t = 5$ s.

in the OWC model without stepped bottom already reaches $x = -10$ m, while the reflected wave in the OWC model with stepped bottom has not reached $x = -10$ m (see (e) and (f)). This shows that the reflected waves in the OWC model with stepped bottom move slower than the waves in the OWC model without stepped bottom.

This difference can be explained by the fact that more incident wave energy is converted when a step is added. In other words, the OWC with stepped bottom would be more efficient than the one without stepped bottom, which is in agreement with the result by Rezanejad and Soares in [72].

1.4.2 Accuracy analysis

In numerical validations, accuracy analysis is of importance. As we can see in Figure 1.1, the configuration of OWC device is essentially constituted from three parts: the domain before the step in the sea bottom, the domain after the step and the chamber. We implement our algorithms by gathering together the three parts. It is worth mentioning that one compact algorithm is also actionable.

In order to make it possible to verify our algorithm, we do the following accuracy analysis. Under the same initial wave and physical parameters, we compare the free surface elevation ζ_e of the classical nonlinear shallow water wave model with our model without discontinuous topography. Figure 1.3 shows that there is no significant difference between the two cases. Moreover, we also find that the error is of order 10^{-3} (see Figure 1.4), which is acceptable since the Lax-Friedrichs method is first-order accurate in space.

1.4.3 Absorbed power and efficiency

Designing a WEC of high efficiency is nowadays a hot topic in all regions and countries over the world. In this regard, we present in this section the method to calculate the absorbed power as well as the efficiency of the OWC considered in this work.

The primary efficiency η_{Reg} of the device is defined by the ratio of the absorbed power from the waves to the incident wave power. From the seminal work of Evans in [22], we know that in the linear time-harmonic theory the volume flux $Q(t) = \text{Re}\{qe^{-i\omega t}\}$ is assumed linearly proportional to the pressure in the chamber $P(t) = \text{Re}\{pe^{-i\omega t}\}$. Using this assumption, the average power absorbed from regular waves over one wave period, denoted by P_{Reg} , is given by

$$P_{Reg} = \frac{1}{2}\lambda|p|^2, \quad (1.37)$$

where p is the time independent and λ is a positive constant associated with linear air turbine characteristics. On the other hand, following [72] in experiments the average

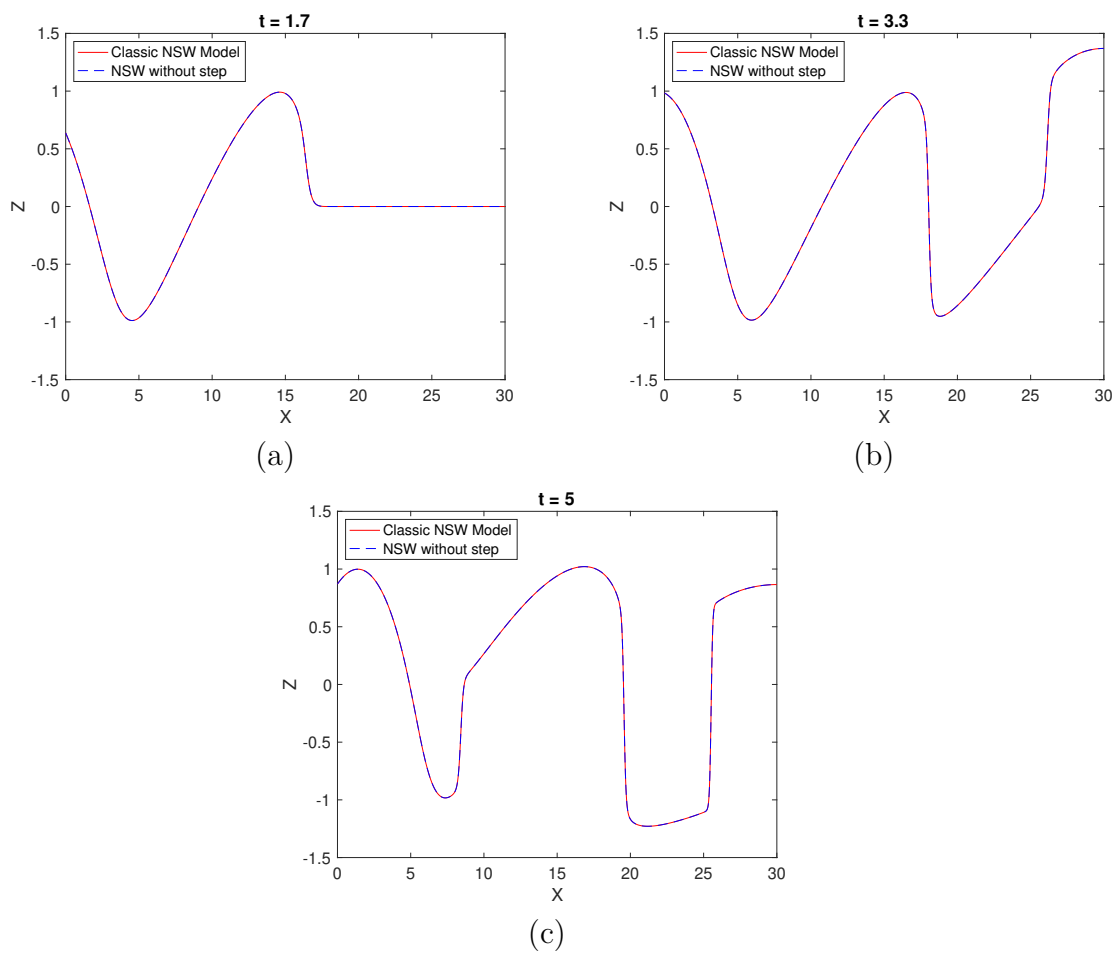


Figure 1.3 – Comparison between classical NSW model and our model without step in different times considering $\delta_x = L/1500$ m and $L = 30$ m.

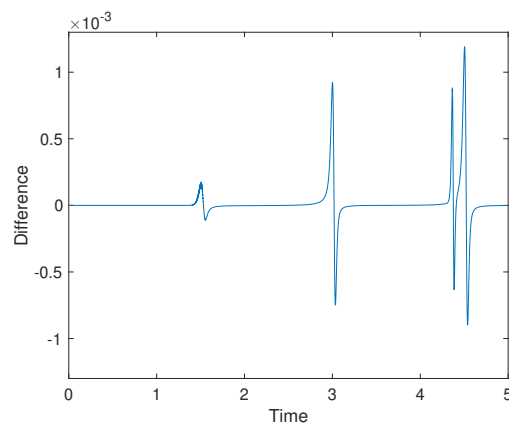


Figure 1.4 – Difference between classical NSW model and our NSW model without step considering $\delta_x = L/1500$ m and $L = 30$ m.

power absorbed from regular waves can be determined by:

$$P_{Reg} = \frac{1}{T} \int_0^T PQ dt, \quad (1.38)$$

where T is the duration of the test. The incident wave power P_{inc} is defined as the product of total energy per wave period E_{inc} and the group velocity c_g (see [19]):

$$P_{inc} = E_{inc} c_g,$$

with

$$E_{inc} = \frac{1}{2} \rho g L A^2, \quad c_g = \frac{\omega}{2k} \left(1 + \frac{2kh_s}{\sinh(2kh_s)} \right),$$

and the dispersion relation given by

$$\omega^2 = gk \tanh(kh_s),$$

where ω is the frequency, k is the wave number, h_s is the fluid height at rest before the step, ρ is the density of the fluid, g is the gravitational acceleration and L is the projected width of the WEC perpendicular to the incident wave direction, A is the amplitude of the wave. In the shallow water regime $kh_s \ll 1$ and the group velocity reduces to $c_g = \sqrt{gh_s}$. Thus, the primary efficiency of the device in regular wave is given by

$$\eta_{Reg} = \frac{P_{Reg}}{P_{inc}}.$$

We notice that in both (1.37) and (1.38) the absorbed power (hence the primary efficiency) strongly depends on the air pressure in the chamber. In our model, it is considered to be a constant, namely the atmospheric pressure P_{atm} . However, when the waves arrive into the chamber and change the volume of the air, the air pressure in the chamber will certainly change as well. In this case, the pressure will no more be a constant, but depends on time. Hence, to study more rigorously the absorbed power and the primary efficiency of the OWC, this fact must be taken into account in the model. This will be addressed in our future work. Analogously, the improvement of the efficiency of an OWC device deployed on a stepped sea bottom can be also investigated with a better knowledge of the air pressure in the chamber. From the results in Section 1.4.1, we can expect that significant improvements in the efficiency can be achieved by adding a step at the bottom of the sea.

CHAPTER 2

Boundary controllability for a system modelling an immersed structure

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*This work is in collaboration with Günter Leugering and Yue Wang.
It is based on [79].*

2.1 Introduction

2.1.1 Context

Free surface interactions with fixed or floating structures have been intensively studied by the mathematical community in the last years; modelling, well-posedness, numerical simulations, etc. Recently, Lannes proposed in [46] a new formulation of

the water-waves problem in order to take into account the presence of a floating body. More precisely, in his work Lannes implemented a method for the full water waves equations and for reduced asymptotic models, such as the Boussinesq and the nonlinear shallow-water equations, where the pressure exerted by the fluid on the partially immersed structure appears as a Lagrange multiplier associated to the constraint that under the floating structure, the surface of the fluid coincides with the bottom of the structure. In the case of the nonlinear shallow water equations, the resulting fluid-structure model with vertical lateral walls case has been studied in [46] and in the more general case of non vertical walls in [35]. An extension to a system modelling a floating structure on a viscous shallow water regime has been recently studied by Maity et al. in [57], and by Matignon, Tucsnak and Vergara-Hermosilla in [80] and [81].

Bresch, Lannes and Métivier in [11] treat on the derivation and mathematical analysis of a fluid-structure interaction problem with a configuration where the motion of the fluid is governed by the Boussinesq system, which is a dispersive perturbation of the hyperbolic nonlinear shallow water equations, and in the presence of a fixed partially immersed obstacle. They showed that the fluid-structure interaction problem can be reduced to a transmission problem.

Similarly and motivated for mathematical modeling and simulations of a specific type of wave energy converting device, the so-called oscillating water column (OWC) device, Bocchi, He and Vergara-Hermosilla discuss in [8] the interaction between an inviscid fluid with the partially submerged fixed wall structure of a OWC device, considering the nonlinear shallow water equations to describe the fluid motion, and obtaining explicit transmission conditions for the system and respective reduced transmission problems. Hence, by considering their results, in this work we deep on a particular kind of boundary controllability on the transmission problems studied in [8] on an equivalent physical configuration. More precisely, we deal with the exact boundary controllability of nodal profile on a system modelling a structure partially immersed in a fluid governed by the nonlinear shallow water equations [46], considering a discontinuity in the height of the fluid bottom and the transmission conditions developed in [8]. This physical situation is presented in a graphical sketch in Figure 1.

This kind of boundary controllability was motivated by practical applications on gas networks and introduced recently in the literature by Gugat, Herty and Schleper, in [33]. Their new approach was almost immediately generalized by Li to general 1-D first order quasilinear hyperbolic systems with general nonlinear boundary conditions in [52].

As is well known, the usual exact boundary controllability that asks on the solution to the system under certain boundary controls to satisfy a given final state at a suitably large time $t = T$, however, the exact boundary controllability of nodal profile, requires that the value of solution satisfies the given profiles on one or more nodes for $t \geq T$ by using boundary controls. This approach can be established

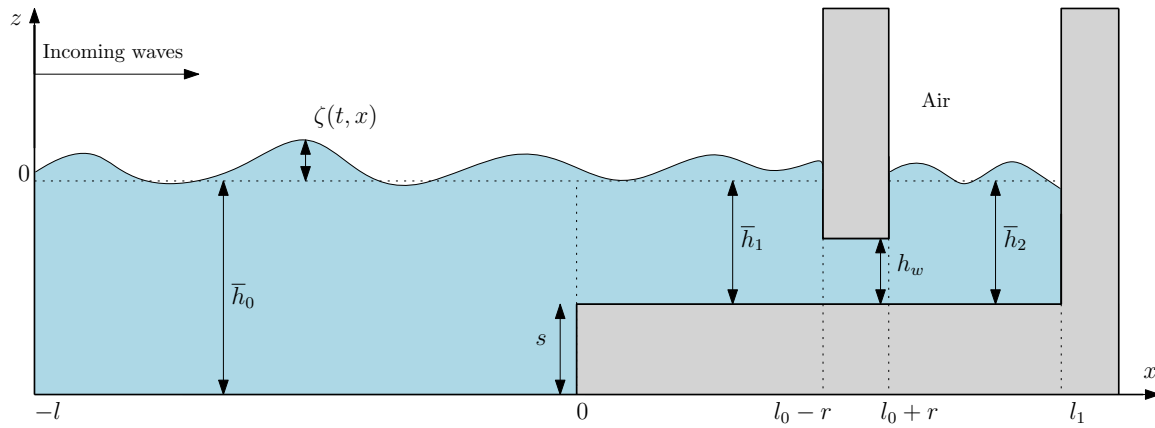


Figure 2.1 – Configuration.

by means of a constructive method with modular structure, by using the following three ingredients: existence and uniqueness of semi-global classical solution to the mixed initial-boundary value problem, exchanging the role of the the space variable x and time variable t , and the uniqueness of classical solution to the one-sided mixed initial-boundary value problem. This approach was synthesized in a systematic way in the book of Li et al. [53], where they also deal with the local exact boundary controllability of nodal profile on a tree-like network with general topology. For more details, see [32], [82] and [83].

2.1.2 General settings

Mathematically speaking, in this work we consider an incompressible, irrotational, inviscid and homogeneous fluid in a shallow water regime. The motion of the fluid is governed by the 1D nonlinear shallow water equations [46], which are given by

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \quad (2.1)$$

where $\zeta(t, x)$ is free surface elevation, $h(t, x)$ is the fluid height given by $h_b + \zeta$ with h_b denoting the height of the bottom of the fluid, $q(t, x)$ is the horizontal discharge defined by

$$q(t, x) := \int_{-h_b}^{\zeta(t, x)} u(t, x, z) dz, \quad (2.2)$$

with $u(t, x, z)$ denoting the horizontal component of the fluid velocity vector field. In the following, we consider the interval $(-l, l_1)$ as spacial domain, and we assume

that the height from the bottom h_b is given by

$$h_b = \begin{cases} h_0, & x \in (-l, 0), \\ h_1, & x \in [0, l_1), \end{cases}$$

where h_1 and h_0 are real and positive constants such that $h_1 = h_0 + s$, with $s > 0$. Furthermore, we assume that

$$\left| \frac{q}{h} \right| < \sqrt{gh}. \quad (2.3)$$

For our analysis, we divide the domain of the problem $(-l, l_1)$ into two parts: the interior domain $\mathcal{I} = (l_0 - r, l_0 + r)$, and its complement $\mathcal{E} = (-l, l_1) \setminus \overline{\mathcal{I}}$, called exterior domain, and which is the union of three intervals $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ with

$$\mathcal{E}_0 = (-l, 0), \quad \mathcal{E}_1 = (0, l_0 - r), \quad \mathcal{E}_2 = (l_0 + r, l_1). \quad (2.4)$$

Let

$$\zeta(t, x) = \begin{cases} \zeta_i(t, x), & x \in \mathcal{E}_i, i = 0, 1, 2, \\ \zeta_w \in \mathbb{R}, & x \in \mathcal{I}. \end{cases} \quad (2.5)$$

where ζ_w is the parameterization of the bottom of the solid. With this, the fluid heights respect to the flat bottom are defined by

$$h(t, x) = \begin{cases} h_i(t, x) := \bar{h}_i + \zeta_i(t, x), & x \in \mathcal{E}_i, i = 0, 1, 2, \\ h_w \in \mathbb{R}, & x \in \mathcal{I}, \end{cases} \quad (2.6)$$

where \bar{h}_i ($i = 0, 1, 2$) are the fluid height at rest with $\bar{h}_1 = \bar{h}_2 = h_1$, $\bar{h}_0 = h_0$ and $h_w = \bar{h}_1 + \zeta_w$. We note that from (2.6), the horizontal discharge on \mathcal{I} defines just a time-dependent function, and therefore the horizontal discharge can be written as

$$q(t, x) = \begin{cases} q_i(t, x), & x \in \mathcal{E}_i, i = 0, 1, 2, \\ q_w(t), & x \in \mathcal{I}, \end{cases} \quad (2.7)$$

where $\partial_t q_w(t) = -\frac{h_w}{\rho} \partial_x P_w$ with the fluid density ρ and an unknown surface pressure P_w . Thus, the coupled system of nonlinear shallow water equations for $x \in \mathcal{E}_i$ can be written as

$$\begin{cases} \partial_t \zeta_i + \partial_x q_i = 0, \\ \partial_t q_i + \partial_x \left(\frac{q_i^2}{h_i} \right) + gh_i \partial_x \zeta_i = 0, \end{cases} \quad i = 0, 1, 2, \quad (2.8)$$

with the boundary conditions given as

$$x = -l : \quad \zeta_0 = f(t), \quad (2.9)$$

$$x = l_1 : \quad q_2 = 0, \quad (2.10)$$

where $f(t)$ denotes a prescribed boundary function or a boundary control to be determined.

In addition, we consider the transmission conditions developed in [8], which read as follows:

$$x = 0 : \quad \begin{cases} \zeta_0(t, 0) = \zeta_1(t, 0), \\ q_0(t, 0) = q_1(t, 0), \end{cases} \quad (2.11)$$

$$x = l_0 \pm r : \quad \begin{cases} q_2(t, l_0 + r) = q_1(t, l_0 - r) = q_w(t), \\ \left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right] \Big|_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right] \Big|_{x=l_0-r} = -\alpha \frac{d}{dt} q_w(t), \end{cases} \quad (2.12)$$

where $\alpha = \frac{2r}{h_w}$.

The initial data of the system is given at rest,

$$\zeta(0, x) = \begin{cases} 0, & x \in \mathcal{E}, \\ \zeta_w, & x \in \mathcal{I}, \end{cases} \quad \text{and} \quad q(0, x) = 0. \quad (2.13)$$

2.1.3 Outline of the chapter

In Section 2 we present the main results about exact controllability of final data given on the whole or a part of fluid domain at a given finite time $T > 0$ (see in Subsection 2.1), and exact controllability of nodal profile for a given demand at one end $x = l_1$ (See in Subsection 2.2). In Subsection 3.1 we prove some results relative to the well-posedness result of semi-global C^1 solutions, and finally in Subsection 3.2 we prove the main theorems of the paper by considering a constructive method with modular structure.

2.2 Main results

Now we come to our results about boundary controllability controllability for the system (2.8)-(2.13). In order to ensure the well-posedness of quasilinear system (2.8) with (2.9)-(2.12), we consider the exact controllability problem in the neighbourhood of the equilibrium state. In the following, Theorem 2.2.1 and Theorem 2.2.4 show the local exact controllability and the exact controllability of nodal profile near the rest state given as (2.13), respectively. The corresponding results near other equilibrium state can be obtained in a similar way, how we explain with some words in Remark 2.2.2.

2.2.1 Exact controllability with one boundary control.

The aim of exact controllability for system (2.8) is looking for a boundary control $f(t)$ acting at $x = -l$ though the boundary condition (2.9), such that the prescribed

final data $(\zeta_{iT}(x), q_{iT}(x))$ is attained in the entire water regime at a given time T :

$$U_i(T, x) := (\zeta_i, q_i)(T, x) = (\zeta_{iT}(x), q_{iT}(x)), \quad x \in \mathcal{E}_i, \quad i = 0, 1, 2, \quad (2.14)$$

where the U_i is generated by the control f as a solution of system (2.8)-(2.12) with the initial condition:

$$(\zeta_i, q_i)(0, x) = (\zeta_{i0}(x), q_{i0}(x)), \quad x \in \mathcal{E}_i, \quad i = 0, 1, 2. \quad (2.15)$$

In order to ensure the existence and uniqueness of piece-wise C^1 semi-global solution for this problem, here we give some assumptions on the initial data and final data in advance: for each $i = 0, 1, 2$,

- (S1) $\zeta_{i0}(x), q_{i0}(x)$ are C^1 functions with small norm $\|(\zeta_{i0}, q_{i0})\|_{C^1[\bar{\mathcal{E}}_i] \times C^1[\bar{\mathcal{E}}_i]}$,
- (S2) $\zeta_{i0}(x), q_{i0}(x)$ satisfy the C^1 compatibility at the points

$$(t, x) \in \{(0, -l), (0, l_1)\},$$

and the piecewise C^1 compatibility at the adjoint points

$$(t, x) \in \{(0, 0), (0, l_0 - r), (0, l_0 + r)\}.$$

Accordingly,

- (S1') $\zeta_{iT}(x), q_{iT}(x)$ are C^1 functions with small norm $\|(\zeta_{iT}, q_{iT})\|_{C^1[\bar{\mathcal{E}}_i] \times C^1[\bar{\mathcal{E}}_i]}$,
- (S2') $\zeta_{iT}(x), q_{iT}(x)$ satisfy the C^1 compatibility at the points

$$(t, x) \in \{(T, -l), (T, l_1)\},$$

and the piecewise C^1 compatibility at the adjoint points

$$(t, x) \in \{(T, 0), (T, l_0 - r), (T, l_0 + r)\}.$$

The following result states the existence of such a control, which can be constructed explicitly in Section 3.

Theorem 2.2.1. (*Exact Controllability*). *Let*

$$T > 2 \left(\frac{l}{\sqrt{g\bar{h}_0}} + \frac{l_1 - 2r}{\sqrt{g\bar{h}_1}} \right). \quad (2.16)$$

For any given initial data (ζ_{i0}, q_{i0}) and final data (ζ_{iT}, q_{iT}) satisfying the assumptions (S1)-(S2) and (S1')-(S2'), respectively, there exists a boundary control $f(t)$ with small norm $\|f(t)\|_{C^1[0, T]}$, such that mixed initial-boundary value problem for equation (2.8) with the initial condition (2.15), the boundary conditions (2.9)-(2.10) and the transmission conditions (2.11)-(2.12) admits a unique piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t, x)$, ($i = 0, 1, 2$) with small piecewise $C^1 \times C^1$ norm on the domain $\mathcal{R}_i(T) = \{(t, x) | 0 \leq t \leq T, x \in \mathcal{E}_i\}$, which exactly satisfies the desired final condition (2.14).

In the above result, we consider only the steady-state at rest, but we can easily establish the corresponding local exact controllability in the neighborhood of a stationary subsonic continuously differentiable state.

Remark 2.2.2. *The exact controllability result given in Theorem 2.2.1 still holds for initial states in a C^1 -neighborhood of a stationary subsonic continuously differentiable state (q_{is}, ζ_{is}) , $i = 0, 1, 2$. For $x \in \mathcal{E}_i$, $i = 0, 1, 2$, the stationary solutions $(q_{is}(x), \zeta_{is}(x))$ are given by*

$$q_{is}(x) = \text{const}, \quad (2.17)$$

$$\frac{d}{dx} \left(\frac{q_{is}^2}{\bar{h}_i + \zeta_{is}} \right) + g(\bar{h}_i + \zeta_{is}) \frac{d}{dx} (\zeta_{is}) = 0. \quad (2.18)$$

Remark 2.2.3. *The result given in Theorem 2.2.1 illustrates that after a finite time, the control given at one end $x = -l$ can effect the state functions ζ and q in the entire space horizon $x \in \mathcal{E}$. In fact, this result can be generalized to a special case, where the final condition (2.14) is replaced by a demand given on a part of the horizon:*

$$U_2(T, x) := (\zeta_2, q_2)(T, x) = (\zeta_{2T}(x), q_{2T}(x)), \quad x \in \mathcal{E}_2, \quad (2.19)$$

where $\zeta_{2T}(x), q_{2T}(x)$ are regarded as given final state functions in the “indoor water regime” \mathcal{E}_2 . Now, the infimum controllability time (2.16) should be modified by a smaller lower bound:

$$T > \frac{l}{\sqrt{g\bar{h}_0}} + \frac{2l_1 - l_0 - 3r}{\sqrt{g\bar{h}_1}}. \quad (2.20)$$

A limit case, with a given demand only at the end node $x = l_1$, suggests a controllability problem of nodal profile, which will be shown in details in the following subsection.

2.2.2 Exact controllability of nodal profile: a given demand at the end $x = l_1$.

Stimulated by some practical applications, Gugat et al. [33] and Li [52] proposed in 2010 another kind of exact boundary controllability, called the nodal profile control. This kind of controllability does not ask the solution to exactly attain any given final state at a suitable time $t = T$ by means of boundary controls, instead it asks the state to exactly fit any given profile function w.r.t time on a node after a suitable time T .

For equation (2.8), we consider a given demand in fluid height and horizontal discharge at the end $x = l_1$ by

$$U_2(t, l_1) := (\zeta_2, q_2)(t, l_1) = (\bar{h}_2 + \zeta_B(t), q_B(t)), \quad t \in [T, \bar{T}]. \quad (2.21)$$

It's worth to mention that the given nodal profile function $U_B := (\bar{h}_2 + \zeta_B(t), q_B(t))$ should be compatible with the boundary condition (2.10) in the time interval $[0, \bar{T}]$ at the node $x = l_1$, which implies that the demand q_B must be set as 0. While, the other one, $\zeta_B(t)$, can be chosen as any given C^1 function of time after a finite time T .

In the following, we give a positive answer to the exact controllability of nodal profile:

Theorem 2.2.4. (*Exact Controllability of Nodal Profile*). *Let*

$$T > \left(\frac{l}{\sqrt{g\bar{h}_0}} + \frac{l_1 - 2r}{\sqrt{g\bar{h}_1}} \right) \quad (2.22)$$

and let \bar{T} be an arbitrarily given number satisfying $\bar{T} > T$. Then for any given initial data (ζ_{i0}, q_{i0}) satisfying assumptions (S1)-(S2), and for any given demand of the surface elevation $\zeta_B(t)$ with small norm $\|\zeta_B\|_{C^1[T, \bar{T}]}$, there exists a boundary control $f(t) \in C^1[0, \bar{T}]$, such that the mixed initial-boundary value problem for equation (2.8) with initial condition (2.15), boundary conditions (2.9)-(2.10) and transmission conditions (2.11)-(2.12) admits a unique piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t, x)$ ($i = 0, 1, 2$) with small $C^1 \times C^1$ norm on the domains $\mathcal{R}_i(\bar{T}) = \{(t, x) | 0 \leq t \leq \bar{T}, x \in \mathcal{E}_i\}$, which exactly satisfies the given nodal profile condition at the end $x = l_1$:

$$\zeta_2(t, l_1) = \zeta_B(t), \quad \forall t \in [T, \bar{T}]. \quad (2.23)$$

Remark 2.2.5. *In the above results, we consider only the steady state at rest, but we can easily replace it with any other steady state that is compatible with the boundary and transmission conditions (2.10)-(2.12) and establish local exact controllability of nodal profile near a given subsonic continuously differentiable stationary state $(q_{is}(x), \zeta_{is}(x))$, $x \in \mathcal{E}_i$, $i = 0, 1, 2$: There exists a C^1 -neighborhood of the stationary state such that for all initial data in this neighborhood that satisfies the C^1 -compatibility conditions and for any given smooth subsonic desired fluid height $\bar{h}_2 + \zeta_B(t, l_1)$ and discharge profile $q_B(t, l_1)$ at the end $x = l_1$ that is in a sufficiently small C^1 -neighborhood of the boundary data corresponding to the stationary state we can construct a continuously differentiable control $f = f(t)$ such that the demand is fulfilled exactly for all $t \in [T, \bar{T}]$. Moreover, this control generates a continuously differentiable system state in the entire domain.*

2.3 Proofs

2.3.1 Existence and uniqueness of semi-global C^1 solution

We reduce the system in the exterior domain \mathcal{E} to a compact form by introducing the couple $U_i = (\zeta_i, q_i)^T, i = 0, 1, 2$:

$$\partial_t U_i + A_i(U_i) \partial_x U_i = 0, \quad x \in \mathcal{E}_i, \quad (2.24)$$

with

$$A_i(U_i) = \begin{pmatrix} 0 & 1 \\ gh_i - \frac{q_i^2}{h_i^2} & \frac{2q_i}{h_i} \end{pmatrix}, \quad (2.25)$$

which has two distinct eigenvalues:

$$\lambda_i^- = \frac{q_i}{h_i} - \sqrt{gh_i} < 0 < \lambda_i^+ = \frac{q_i}{h_i} + \sqrt{gh_i} \quad (2.26)$$

and the corresponding left eigenvectors can be taken as

$$l_i^- = \left(-\sqrt{gh_i} - \frac{q_i}{h_i}, 1 \right)^T, \quad l_i^+ = \left(\sqrt{gh_i} - \frac{q_i}{h_i}, 1 \right)^T. \quad (2.27)$$

In this part, by recalling that \bar{h}_i denote the height of the fluid at rest in the domain \mathcal{E}_i , we introduce the Riemann invariants to the nonlinear shallow water equations, which read as

$$\begin{cases} L_i = 2 \left(\sqrt{gh_i(t, x)} - \sqrt{g\bar{h}_i} \right) - \frac{q_i}{h_i}, \\ R_i = 2 \left(\sqrt{gh_i(t, x)} - \sqrt{g\bar{h}_i} \right) + \frac{q_i}{h_i}, \end{cases} \quad \text{for } i = 0, 1, 2. \quad (2.28)$$

Then, (2.28) can be equivalently rewritten in

$$\begin{cases} \sqrt{gh_i} = \frac{R_i + L_i}{4} + \sqrt{g\bar{h}_i}, \\ \frac{q_i}{h_i} = \frac{R_i - L_i}{2}, \end{cases} \quad \text{for } i = 0, 1, 2, \quad (2.29)$$

and then, we obtain

$$\begin{cases} \zeta_i = \frac{\left(\frac{R_i + L_i}{4} + \sqrt{g\bar{h}_i} \right)^2}{g} - \bar{h}_i, \\ q_i = \frac{R_i - L_i}{2} \frac{\left(\frac{R_i + L_i}{4} + \sqrt{g\bar{h}_i} \right)^2}{g}, \end{cases} \quad \text{for } i = 0, 1, 2. \quad (2.30)$$

Thus, the 1D nonlinear shallow water equations in \mathcal{E} can be rewritten as the following form

$$\begin{cases} \partial_t L_i + \lambda_i^- \partial_x L_i = 0, \\ \partial_t R_i + \lambda_i^+ \partial_x R_i = 0, \end{cases} \quad \text{for } i = 0, 1, 2. \quad (2.31)$$

Obviously, for each $i = 0, 1, 2$, $L_i(t, x)$ is the Riemann invariant corresponding to the negative eigenvalue λ_i^- , while $R_i(t, x)$ is the Riemann invariant corresponding to the positive eigenvalue λ_i^+ . λ_i^- (resp. λ_i^+) is the entering (resp. departing) characteristic on the right-side boundary, while λ_i^- (resp. λ_i^+) is the departing (resp. entering) characteristic on the left-side boundary. In order to guarantee the well-posedness of the mixed initial-boundary value problem on this coupled network, the boundary conditions on each boundary must satisfy (see [51]):

1. the number of the boundary conditions must be equal to that of the entering characteristics;
2. the boundary conditions can be written in the form that the Riemann invariants corresponding to the entering characteristics can be explicitly expressed by all other Riemann invariants (corresponding to the departing characteristics).

We can see that the boundary conditions (2.9) and (2.10) can be rewritten in terms of the Riemann invariants as

$$x = -l : \quad R_0 = 2 \left(\sqrt{g(f + \bar{h}_0)} - \sqrt{g\bar{h}_0} \right) - \frac{R_0 - L_0}{2}, \quad (2.32)$$

and

$$x = l_1 : \quad R_2 - L_2 = 0, \quad (2.33)$$

which imply that

$$x = -l : \quad L_0 = 3R_0 - 4 \left(\sqrt{g(f + \bar{h}_0)} - \sqrt{g\bar{h}_0} \right) \quad (2.34)$$

and

$$x = l_1 : \quad R_2 = L_2, \quad (2.35)$$

where R_0 and L_2 are the Riemann invariants corresponding to entering characteristic on $x = -l$ and $x = l_1$, respectively. We note that

$$\|f\|_{C^1[0,T]} \text{ small} \iff \left\| \sqrt{g(f + \bar{h}_0)} - \sqrt{g\bar{h}_0} \right\|_{C^1[0,T]} \text{ small}. \quad (2.36)$$

At $x = 0$, the Riemann invariants corresponding to entering characteristic are R_0 and L_1 . Let

$$\begin{cases} F_1 = \zeta_0 - \zeta_1, \\ F_2 = q_0 - q_1. \end{cases} \quad (2.37)$$

By a direct calculation and considering (2.29)-(2.30), we have

$$\frac{\partial F_1}{\partial R_0} = \frac{\partial \zeta_0}{\partial R_0} = \frac{1}{g} \cdot 2 \left(\frac{L_0 + R_0}{4} + \sqrt{gh_0} \right) \cdot \frac{1}{4} = \frac{\sqrt{gh_0}}{2g}. \quad (2.38)$$

Similarly,

$$\begin{aligned} \frac{\partial F_1}{\partial L_1} &= -\frac{\partial \zeta_1}{\partial L_1} = -\frac{\sqrt{gh_1}}{2g}, \\ \frac{\partial F_2}{\partial R_0} &= \frac{\partial q_0}{\partial R_0} = \frac{h_0}{2} + \frac{q_0}{2h_0g} \sqrt{gh_0} = \frac{\sqrt{h_0}}{2\sqrt{g}} \left(\sqrt{gh_0} + \frac{q_0}{h_0} \right) > 0, \\ \frac{\partial F_2}{\partial L_1} &= -\frac{\partial q_1}{\partial L_1} = \frac{h_1}{2} + \frac{q_1}{2h_1g} \sqrt{gh_1} = \frac{\sqrt{h_1}}{2\sqrt{g}} \left(\sqrt{gh_1} + \frac{q_1}{h_1} \right) > 0. \end{aligned} \quad (2.39)$$

Thus,

$$\begin{aligned} \left| \frac{\partial(F_1, F_2)}{\partial(R_0, L_1)} \right|_{(R_0, L_1)=(0,0)} &= \left[\frac{\sqrt{h_0 h_1}}{4g} \left(\sqrt{gh_1} + \frac{q_1}{h_1} \right) + \frac{\sqrt{h_0 h_1}}{4g} \left(\sqrt{gh_0} + \frac{q_0}{h_0} \right) \right]_{(\zeta_0, \zeta_1, q_0, q_1)=(0,0,0,0)} \\ &= \frac{\sqrt{h_0 h_1}}{4g} \left(\sqrt{gh_1} + \sqrt{gh_0} \right) \neq 0, \end{aligned} \quad (2.40)$$

then by implicit function theorem, in a neighborhood of $(R_0, L_1) = (0, 0)$, the transmission conditions (2.11) can be equivalently rewritten as

$$\begin{cases} R_0 = \hat{F}_1(R_1, L_0), \\ L_1 = \hat{F}_2(R_1, L_0) \end{cases} \quad (2.41)$$

with $\hat{F}_1(0, 0) \equiv \hat{F}_2(0, 0) \equiv 0$.

Similarly, at $x = l_0 \pm r$, the Riemann invariants corresponding to entering characteristic are R_1 and L_2 . Using the initial data and integral from 0 to t , the second transmission condition in (2.12) can be replaced by a nonlocal condition as follows

$$\begin{aligned} -\alpha q_w(t) &= -\alpha q_1(0, l_0 - r) + \int_0^t \left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right]_{x=l_0-r} d\tau \\ &= \int_0^t \left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right]_{x=l_0-r} d\tau. \end{aligned} \quad (2.42)$$

Then, by the first transmission condition in (2.12), we rewrite (2.12) into

$$\begin{cases} q_1(t, l_0 - r) = q_2(t, l_0 + r), \\ -\alpha q_1(t, l_0 - r) = \int_0^t \left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right]_{x=l_0-r} d\tau. \end{cases} \quad (2.43)$$

Now, let us introduce the two boundary functions G_1 and G_2 as follow:

$$\begin{cases} G_1 := q_1(t, l_0 - r) - q_2(t, l_0 + r), \\ G_2 := \alpha q_1(t, l_0 - r) + \int_0^t \left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right]_{x=l_0-r} d\tau. \end{cases} \quad (2.44)$$

In fact, by substituting (2.30) into (2.44), G_1 and G_2 can be also regarded as functions with respect to Riemann invariants R_1, L_1, R_2, L_2 . For simplicity, we denote $R_1(t, l_0 - r), L_1(t, l_0 - r)$ and $R_2(t, l_0 + r), L_2(t, l_0 + r)$ by R_1, L_1 and R_2, L_2 , respectively, in the following expression:

$$\begin{aligned} G_1 &= G_1(R_1, L_1, R_2, L_2) \\ &= \frac{R_1 - L_1}{2} \frac{\left(\frac{R_1 + L_1}{4} + \sqrt{g\bar{h}_1} \right)^2}{g} - \frac{R_2 - L_2}{2} \frac{\left(\frac{R_2 + L_2}{4} + \sqrt{g\bar{h}_2} \right)^2}{g}, \end{aligned} \quad (2.45)$$

$$\begin{aligned} G_2 &= G_2 \left(R_1, L_1, \int_0^t G(R_1, L_1, R_2, L_2) d\tau \right) \\ &= \alpha \left(\frac{R_1 - L_1}{2} \frac{\left(\frac{R_1 + L_1}{4} + \sqrt{g\bar{h}_1} \right)^2}{g} \right) + \int_0^t G(R_1, L_1, R_2, L_2) d\tau, \end{aligned} \quad (2.46)$$

where G is the function determined by $\left[\frac{q_2^2}{2h_2^2} + g\zeta_2 \right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1 \right]_{x=l_0-r}$, and $\int_0^t G(R_1, L_1, R_2, L_2) d\tau$ is regarded as a new variable of G_2 .

Thus, if the Jacobian

$$\left| \frac{\partial(G_1, G_2)}{\partial(R_1, L_2)} \right|_{(R_1, L_2)=(0,0)} \neq 0, \quad (2.47)$$

then by implicit function Theorem, (2.43) can be solved as

$$\begin{cases} R_1(t) = \hat{G}_1 \left(L_1, R_2, \int_0^t G(L_1, R_1, L_2, R_2) d\tau \right), \\ L_2(t) = \hat{G}_2 \left(L_1, R_2, \int_0^t G(L_1, R_1, L_2, R_2) d\tau \right). \end{cases} \quad (2.48)$$

Indeed, by calculation, we obtain

$$\begin{aligned}
\frac{\partial G_1}{\partial R_1} &= \frac{\partial q_1}{\partial R_1} = \frac{\sqrt{h_1}}{2\sqrt{g}} \left(\sqrt{gh_1} + \frac{q_1}{h_1} \right) > 0, \\
\frac{\partial G_1}{\partial L_2} &= -\frac{\partial q_2}{\partial L_2} = \frac{\sqrt{h_2}}{2\sqrt{g}} \left(\sqrt{gh_2} + \frac{q_2}{h_2} \right) > 0, \\
\frac{\partial G_2}{\partial R_1} &= \alpha \frac{\partial q_1}{\partial R_1} = \alpha \frac{\sqrt{h_1}}{2\sqrt{g}} \left(\sqrt{gh_1} + \frac{q_1}{h_1} \right) > 0, \\
\frac{\partial G_2}{\partial R_2} &= 0.
\end{aligned} \tag{2.49}$$

Therefore,

$$\left| \frac{\partial(G_1, G_2)}{\partial(R_1, L_2)} \right| = -\alpha \frac{\sqrt{h_1 h_2}}{4g} \left(\sqrt{gh_1} + \frac{q_1}{h_1} \right) \left(\sqrt{gh_2} + \frac{q_2}{h_2} \right) < 0, \tag{2.50}$$

which implies (2.47).

Using the results and methods in [51, Chapter 4], we get the existence and uniqueness of piecewise local C^1 solution to the mixed initial-boundary value problem (2.8) and (2.9)-(2.12). And then, by considering a method similar to [50], [54], and the generalized result for a kind of non-local boundary condition as Lemma 4.1 in [55], we can obtain the corresponding results on existence and uniqueness of piecewise semi-global C^1 solution.

Theorem 2.3.1. *Let $T > 0$ be given. For any given initial data (ζ_{i0}, q_{i0}) with small norm $\|\zeta_{i0}(\cdot), q_{i0}(\cdot)\|_{C^1[\mathcal{E}_i] \times C^1[\mathcal{E}_i]}$ ($i = 0, 1, 2$) and boundary function $f(t)$ with small norm $\|f(\cdot)\|_{C^1[0, T]}$, satisfying the conditions (S1)-(S2), the forward mixed initial-boundary value problem of the shallow water system (2.8) on the connected water regime with the initial condition (2.15), the boundary conditions (2.9)-(2.10) and the interface conditions (2.11)-(2.12) admits a unique semi-global piecewise C^1 solution $U_i = (\zeta_i(t, x), q_i(t, x))$ ($i = 0, 1, 2$) with small norm*

$$\sum_{i=0}^2 \|(\zeta_i(\cdot, \cdot), q_i(\cdot, \cdot))\|_{C^1[\mathcal{R}_i(T)] \times C^1[\mathcal{R}_i(T)]},$$

on the domain

$$\mathcal{R}(T) = \bigcup_{i=0}^2 \mathcal{R}_i(T) = \bigcup_{i=0}^2 \{(t, x) \mid 0 \leq t \leq T, x \in \mathcal{E}_i\}.$$

Similarly, for the backward mixed initial-boundary value problem, we have

Theorem 2.3.2. *Let $T > 0$ be given. For any given initial data (ζ_{i0}, q_{i0}) with small norm $\|\zeta_{i0}(\cdot), q_{i0}(\cdot)\|_{C^1[\mathcal{E}_i] \times C^1[\mathcal{E}_i]}$ ($i = 0, 1, 2$) and boundary function $f(t)$ with small norm $\|f(\cdot)\|_{C^1[0, T]}$, satisfying the conditions (S1')-(S2'), the backward mixed initial-boundary value problem of shallow water system (2.8) with the final condition (2.14), the boundary conditions (2.9)-(2.10) and the interface conditions (2.11)-(2.12) admits a unique semi-global piecewise C^1 solution $U_i = (\zeta_i(t, x), q_i(t, x))$, ($i = 0, 1, 2$) with small norm*

$$\sum_{i=0}^2 \|(\zeta_i(\cdot, \cdot), q_i(\cdot, \cdot))\|_{C^1[\mathcal{R}_i(T)] \times C^1[\mathcal{R}_i(T)]},$$

on the domain

$$\mathcal{R}(T) = \bigcup_{i=0}^2 \mathcal{R}_i(T) = \bigcup_{i=0}^2 \{(t, x) \mid 0 \leq t \leq T, x \in \mathcal{E}_i\}.$$

2.3.2 Proof of exact controllability. Constructive method.

In order to prove Theorem 2.2.1, by means of the constructive method with modular structure, it suffices to prove the following.

Lemma 2.3.3. *Let $T > 0$ be defined by (2.16). For any given initial data (ζ_{i0}, q_{i0}) and final data (ζ_{iT}, q_{iT}) with small norm $\|(\zeta_{i0}, q_{i0})\|_{C^1[\bar{\mathcal{E}}_i]^2}$ and $\|(\zeta_{iT}, q_{iT})\|_{C^1[\bar{\mathcal{E}}_i]^2}$, the nonlinear system (2.8)-(2.12) admits a piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t, x)$, ($i = 0, 1, 2$) with small $C^1 \times C^1$ norm on the domain $\mathcal{R}_i(T)$, which satisfies simultaneously the initial condition (2.15) and the final condition (2.14).*

Proof. By (2.16) and $\bar{h}_1 = \bar{h}_2$, there exists an $\epsilon_0 > 0$ so small (see [50] for details) such that

$$\frac{1}{2}T > \sup_{|\zeta_0|+|q_0| \leq \epsilon_0} \frac{l}{\sqrt{g\bar{h}_0}} + \sup_{|\zeta_1|+|q_1| \leq \epsilon_0} \frac{l_0 - r}{\sqrt{g\bar{h}_1}} + \sup_{|\zeta_2|+|q_2| \leq \epsilon_0} \frac{l_1 - l_0 - r}{\sqrt{g\bar{h}_2}}. \quad (2.51)$$

Let

$$T_1 = \sup_{|\zeta_0|+|q_0| \leq \epsilon_0} \frac{l}{\sqrt{g\bar{h}_0}}, T_2 = \sup_{|\zeta_1|+|q_1| \leq \epsilon_0} \frac{l_0 - r}{\sqrt{g\bar{h}_1}}, T_3 = \sup_{|\zeta_2|+|q_2| \leq \epsilon_0} \frac{l_1 - l_0 - r}{\sqrt{g\bar{h}_2}}. \quad (2.52)$$

Step 1:

On the domains

$$\mathcal{R}_f = \{(t, x) \mid 0 \leq t \leq T_1 + T_2 + T_3, x \in \mathcal{E}\},$$

$$\text{and } \mathcal{R}_b = \{(t, x) \mid 0 \leq t \leq T - (T_1 + T_2 + T_3), x \in \mathcal{E}\},$$

we proceed by solving the corresponding forward and backward mixed problem (2.8) with the initial data (2.15) (or the final data (2.14)), boundary condition (2.10)

at $x = l_1$, transmission conditions (2.11)-(2.12) and artificial boundary condition (2.9) at $x = -l$ (in which f can be taken as any C^1 function with small C^1 norm). By Theorem 2.3.1 and Theorem 2.3.2, there exist unique piecewise C^1 solutions $U_{if} = (\zeta_{if}, q_{if})(t, x)$ and $U_{ib} = (\zeta_{ib}, q_{ib})(t, x)$, ($i = 0, 1, 2$), respectively. Furthermore, we get the trace $\zeta_{2f}(t, l_1), q_{2f}(t, l_1)$ and $\zeta_{2b}(t, l_1), q_{2b}(t, l_1)$ at $x = l_1$.

Step 2:

We will solve the following leftward problem from $x = l_1$ to $x = l_0 + r$ to get the solution $U_2 = U_2(t, x)$ in the domain $\{0 \leq t \leq T, x \in \mathcal{E}_2\}$:

$$\begin{cases} \partial_t U_2 + A_2(U_2) \partial_x U_2 = 0, \\ x = l_1 : U_2 = (\bar{\zeta}_2(t), 0), & 0 \leq t \leq T, \\ t = 0 : q_2(0, x) = 0, & x \in \mathcal{E}_2, \\ t = T : \zeta_2(T, x) = \zeta_{2T}(t), & x \in \mathcal{E}_2, \end{cases} \quad (2.53)$$

where $\bar{\zeta}_2(t)$ is a C^1 function satisfying

$$\bar{\zeta}_2(t) = \begin{cases} \zeta_{2f}(t, l_1), & 0 \leq t \leq T_1 + T_2 + T_3, \\ \zeta_{2b}(t, l_1), & T - T_1 - T_2 - T_3 \leq t \leq T, \end{cases} \quad (2.54)$$

and $\zeta_{2T}(t)$ is given by the final data (2.14).

The problem (2.53) is well-posed (by exchanging the role of t and x in hyperbolic system) and admits a solution $U_2 = (\zeta_2, q_2)(t, x)$ on the domain

$$\mathcal{R}_2 := \{(t, x) \mid 0 \leq t \leq T, x \in \mathcal{E}_2\}.$$

Then, we get the value of trace $\zeta_2(t, l_0 + r)$ and $q_2(t, l_0 + r)$ at $x = l_0 + r$.

Step 3:

With the known functions $\zeta_2(t, l_0 + r)$ and $q_2(t, l_0 + r)$, we try to determine the $\zeta_1(t, l_0 - r)$ and $q_1(t, l_0 - r)$ by the transmission conditions (2.12) at $x = l_0 \pm r$.

At first, by the continuity condition, we have

$$q_1(t, l_0 - r) = q_2(t, l_0 + r) \quad (2.55)$$

and

$$\frac{d}{dt} q_w(t) = \frac{d}{dt} q_2(t, l_0 + r). \quad (2.56)$$

Substituting the value of $q_1(t, l_0 - r)$, $q_2(t, l_0 + r)$, $\frac{d}{dt} q_w(t)$ and $\zeta_2(t, l_0 + r)$ into the second formula of (2.12), it becomes an equation for $\zeta_1(t, l_0 - r)$ that

$$F(\zeta_1(t, l_0 - r)) = 0 \quad (2.57)$$

with

$$\frac{\partial F}{\partial \zeta_1} = -g - \frac{q_1^2}{h_1^3} = -\frac{1}{h_1} \left(gh_1 + \frac{q_1^2}{h_1^2} \right) \neq 0, \quad t \in [0, T], \quad (2.58)$$

provided with the hyperbolicity near the rest state. Thus, we can solve (2.57) near the rest state and uniquely determine $\zeta_1(t, l_0 - r)$ by implicit function Theorem.

We denote the values of $\zeta_1(t, l_0 - r)$ and $q_1(t, l_0 - r)$ obtained in above as $\bar{\zeta}_1(t)$ and $\bar{q}_1(t)$. Hence, we can verify that

- $\bar{\zeta}_1(t)$ and $\bar{q}_1(t)$ are both C^1 functions,
- $U_1 = (\bar{\zeta}_1(t), \bar{q}_1(t))$ is compatible with the value of $U_{1f}(t, l_0 - r)$ and $U_{1b}(t, l_0 - r)$ on the corresponding intervals.

Step 4:

Solve the following leftward problem from $x = l_0 - r$ to $x = 0$ to get the solution $U_1 = U_1(t, x)$ in the domain $\{0 \leq t \leq T, x \in \mathcal{E}_1\}$:

$$\begin{cases} \partial_t U_1 + A_1(U_1) \partial_x U_1 = 0, \\ x = l_0 - r : U_1 = (\bar{\zeta}_1(t), \bar{q}_1(t)), \quad 0 \leq t \leq T, \\ t = 0 : q_1(0, x) = 0, \quad x \in \mathcal{E}_1, \\ t = T : \zeta_1(T, x) = \zeta_{1T}(t), \quad x \in \mathcal{E}_1, \end{cases} \quad (2.59)$$

where $(\bar{\zeta}_1(t), \bar{q}_1(t))$ is obtained from Step 3 and $\zeta_{2T}(t)$ is given by the final data (2.14).

The problem (2.59) is well-posed and admits a solution $U_1 = (\zeta_1, q_1)(t, x)$ on the domain $\mathcal{R}_1 := \{(t, x) | 0 \leq t \leq T, x \in \mathcal{E}_1\}$. Then, we get the values of trace $\zeta_1(t, 0)$ and $q_1(t, 0)$.

Step 5:

By the transmission conditions (2.11) at $x = 0$, it is easy to get

$$(\bar{\zeta}_0, \bar{q}_0) := (\zeta_0(t, 0), q_0(t, 0)) = (\zeta_1(t, 0), q_1(t, 0)). \quad (2.60)$$

Similarly, we solve the leftward problem from $x = 0$ to $x = -l$ to get the solution $U_0 = U_0(t, x)$ in the domain $\{0 \leq t \leq T, x \in \mathcal{E}_0\}$:

$$\begin{cases} \partial_t U_0 + A_1(U_0) \partial_x U_0 = 0, \\ x = 0 : U_0 = (\bar{\zeta}_0(t), \bar{q}_0(t)), \quad 0 \leq t \leq T, \\ t = 0 : q_0(0, x) = 0, \quad x \in \mathcal{E}_0, \\ t = T : \zeta_0(T, x) = \zeta_{1T}(t), \quad x \in \mathcal{E}_0. \end{cases} \quad (2.61)$$

Then we take the boundary control $f(t)$ as the trace $\zeta_0(t, -l)$. We can verify that the solutions $U_i = U_i(t, x)$, $i = 0, 1, 2$ constructed in the steps 2, 4, 5 satisfy all requirements of the Lemma. \square

2.3.3 Proof of Theorem 2.2.4.

Proof. The prescribed boundary data $\zeta_B(t)$ and a fixed control function $f(t)$ in (2.8) would generate an overdetermined initial-boundary value problem. To prove Theorem 2.2.4, we will find a piecewise C^1 solution $(\zeta_i(t, x), q_i(t, x))$ for system (2.8) on the domain $\mathcal{R}(\bar{T})$, which satisfies simultaneously the initial condition (2.15), the interface conditions (2.11)-(2.12), the null flux boundary condition (2.10), and the given nodal profile (2.23). Substituting this solution into the boundary condition (2.9), we obtain the desired boundary control $f(t)$.

We construct a solution to the control problem using the following steps explained in more detail below, where T_1 , T_2 and T_3 are defined as in (2.52).

Step 1: We construct the solution $U_{if} = (\zeta_{if}, q_{if})(t, x)$ on the domain

$$\mathcal{R}_f = \{(t, x) \mid 0 \leq t \leq T_1 + T_2 + T_3, x \in \mathcal{E}\}$$

by solving system (2.8) with the initial data (2.15), null flux boundary condition (2.10) at $x = l_1$, transmission conditions (2.11)-(2.12) and the boundary condition $\zeta_0(t, -l) = F(t)$, where F is an artificial given function satisfying the C^1 -compatibility condition at $(t, x) = (0, -l)$.

Step 2: From the forward solution U_{if} , we denote the trace at $x = l_1$ as

$$(\zeta_2(t, l_1), q_2(t, l_1)) = (\zeta_{Bf}(t), 0),$$

thus ζ_{Bf} is a C^1 function in the time interval $[0, T_1 + T_2 + T_3]$ and $q_2(t, l_1) \equiv 0$ is determined by the boundary condition. Thus, we can find a C^1 function $\bar{\zeta}_2(t) \in C^1[0, \bar{T}]$ such that

$$\bar{\zeta}_2(t) = \begin{cases} \zeta_{Bf}(t, l_1), & 0 \leq t \leq T_1 + T_2 + T_3, \\ \zeta_B(t), & T \leq t \leq \bar{T}, \end{cases} \quad (2.62)$$

where ζ_B is the given demand.

Step 3: We first solve the leftward problem (2.53) in the domain \mathcal{R}_2 to get the solution $U_2 = (\zeta_2, q_2)(t, x)$ and then do the same construction procedure shown as the Step 3, Step 4 and Step 5 in the proof of Lemma 2.3.3 to get solution $U_i = (\zeta_i, q_i)(t, x)$, $i = 1, 0$ in the domain \mathcal{R}_1 and \mathcal{R}_0 , respectively.

Step 4: Using the trace $\zeta_0(t, -l)$ from the solution $U_0 = (\zeta_0, q_0)(t, x)$, we finally compute the control function $f(t) = \zeta_0(t, -l)$.

Step 5: Using the uniqueness theorem, we verify that: at time $t = 0$, $U_i(0, x) \equiv U_{if}(0, x)$, $x \in \mathcal{E}_i$. Thus, the solution U_i ($i = 1, 2, 3$) is the piecewise C^1 solution to solve the overdetermined initial-boundary value problem. □

CHAPTER 3

Modelling of an oscillating water column with time-dependent air pressure

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*This work is in collaboration with Edoardo Bocchi and Jiao He.
It is based on [9].*

3.1 Introduction

3.1.1 General settings.

This article is devoted to the modelling and the mathematical analysis of a particular wave energy converter (WEC) called oscillating water column (OWC). In this

device, incoming waves arrive from the offshore, collide against a partially-immersed fixed structure. The incident wave rebounds but part of the water passes below the structure and enters a partially-closed chamber, whose boundaries are the partially-immersed structure at the left, a solid wall at the right and a solid wall with a hole at the top. The water volume inside the chamber increases and compress air at the upper end of the chamber, forcing it through the hole where a turbine is installed and creates electrical energy. Viceversa, when the water volume decreases, the air outside the chamber enters, activates the turbine and increases the air volume inside the chamber. The name OWC comes from the fact that it behaves like an oscillating liquid piston, a water column, that compresses air inside the chamber. Therefore, this device allows to convert the energy (both kinetic and potential) associated with a moving wave into useful energy. For more details on the transformation of wave energy to electric energy in this type of WEC we refer to [66]. OWCs are one example of a large variety of WECs that can be found in hydrodynamical engineering. For their classification and description, we refer the interested readers to [2].

Among all these devices, floating structures and their interaction with water waves have been particularly studied in the last years. In [46] Lannes derived a model for the interaction between waves and floating structures taking into account nonlinear effects, which have been neglected in previous analytical approaches in the literature (see for instance [38, 39] where floating structures first were modelled). He derived the model in the general multidimensional case considering a depth-averaged formulation of the water waves equations and then the shallow water asymptotic models for the fluid motion given by the nonlinear shallow water equations and the Boussinesq equations. In [35] Iguchi and Lannes proved the local well-posedness of the one-dimensional nonlinear shallow water equations in the presence of a freely moving floating structure with non-vertical side-walls. In [6] Bocchi showed local well-posedness of the nonlinear shallow water equations in the two-dimensional axisymmetric without swirl case for a floating object moving only vertically and with vertical side-walls. In [11] Bresch, Lannes and Métivier considered the case when the structure is fixed with vertical walls and the fluid equations are governed by the one-dimensional Boussinesq equations. Local well-posedness was obtained in the same time scale as in the absence of an object, that is $O(1/\varepsilon)$ where ε is the nonlinearity parameter. Recently, Beck and Lannes in [4] extended the previous analysis to the case of a floating structure with vertical or non-vertical side-walls having only a vertical motion, for which a shorter time scale $O(\varepsilon^{-1/2})$ is obtained. In [57] Maity, San Martín, Takahashi and Tucsnak considered one-dimensional viscous shallow water equations and a solid with vertical side-walls moving vertically. In this viscous case, they showed local well-posedness for every initial data and global if the initial data are close to an equilibrium state. Furthermore, a particular configuration has been investigated, called the return to equilibrium, where the floating structure is dropped from a non-equilibrium position with zero initial velocity into the fluid initially at rest and let evolve towards its equilibrium state. This problem can be

easily done experimentally in wave tanks and is used to determine important characteristics of floating objects. Engineers assume that the solid motion is governed by a linear integro-differential equation, the Cummins equation, that was empirically derived by Cummins in [16] and the experimental data coming from this configuration are used to determine the coefficients of this equation. A nonlinear Cummins equation in the one-dimensional case was derived by Lannes in [46] and a nonlinear integro-differential Cummins equation was derived in the two dimensional axisymmetric without swirl case by Bocchi in [7]. Recently, in [4] the authors derived in the one-dimensional case an abstract general Cummins equation that takes an explicit simple form in the nonlinear non-dispersive and the linear dispersive cases. More recently, Vergara-Hermosilla, Matignon and Tucsnak treated a viscous version of the Cummins equation in the one-dimensional case in [81].

In the last decades oscillating water columns have been widely investigated both experimentally and numerically in the hydrodynamical engineering literature for the sake of understanding how to increase the performance of these wave energy converters in order to facilitate a real installation. For instance, we refer to [20, 23, 24, 56, 71, 73, 72] and references therein. In particular, in [71] Rezanejad, Bhattacharjee and Soares showed that the inclusion of an artificial step at the sea bottom may lead to a significantly increased efficiency, that is the capacity of power absorption of a wave energy converter. All these works were essentially based on the linear water wave theory introduced by Evans in [21, 22], in which the wave motion is assumed time-harmonic. Motivated by the lack of a nonlinear analysis for this type of wave energy converter, we modelled and simulated an OWC in a first paper [8] taking into account the nonlinear effects following the series of works in the case of floating structures presented before. We included in the model a discontinuous topography by considering a step in the sea bottom in front of the device, inspired by [71] and because it represents an interesting mathematical problem itself. In this first and simpler approach, a constant air pressure were considered inside the chamber, although it does not seem realistic since the variations of the fluid volume cause variations of the air volume inside the chamber. Recently, Vergara-Hermosilla, Leugering and Wang treated in [79] the exact boundary controllability of this simplified OWC model introduced in [8].

This chapter is a direct continuation of a first approach established by the authors in [8] and its aim is derive a nonlinear model that describes the interaction between waves and the OWC by taking into account time-variations of the air pressure inside the chamber.

3.1.2 Organization of the chapter

The outline of the chapter is as follows. We present in Section 3.2 the nonlinear mathematical model of an oscillating water column in the shallow water regime. In Subsection 3.2.1 we first introduce the different domains involved in the model and

present the one-dimensional nonlinear shallow water equations in the presence of a partially-immersed structure. After showing the duality property of constraints and unknowns, we split the equations into three different systems corresponding to the region before the step, after the step and under the structure. Moreover, boundary conditions are given to complete the model. Subsection 3.2.2 is devoted to the air pressure dynamics. We assume that the air pressure is equal to the constant atmospheric pressure outside the chamber and we consider it as a time-dependent variation of the atmospheric pressure inside the chamber. We explicitly show the evolution equation of the air pressure variation and rewrite it in terms of the horizontal discharge under the partially-immersed structure.

In Section 3.3 we essentially reformulate the model as two transmission problems. In Subsection 3.3.1 we derive the transmission problem associated with the step in front of the OWC and the corresponding transmission conditions are given by the continuity of the surface elevation and the horizontal discharge over the step. In Subsection 3.3.2 we derive the transmission problem related to the wave-structure interaction at the side-walls of the partially-immersed object. The continuity of the horizontal discharge at the side-walls gives one transmission condition. However, due to the lack of continuity for the surface elevation at the side-walls, one additional condition is necessary to close the system and guarantee the well-posedness of this problem. Therefore in Subsection 3.3.3 we take advantage of free-surface Bernoulli's equation to derive a second transmission condition that takes into account the influence of the time-dependent variation of the air pressure inside the chamber. We show in Subsection 3.3.4 that the second transmission problem, that is the one across the structure side-walls, can be reduced to a 4×4 initial boundary value problem with a semilinear boundary condition. In Subsection 3.3.5 we show that the interaction of waves with the solid wall at the end of the OWC chamber can be described by a 2×2 IBVP with a wall boundary condition for the horizontal discharge.

3.2 Derivation of the model

3.2.1 Fluid equations

We consider an incompressible, irrotational, inviscid and homogeneous fluid that interacts with an on-shore oscillating water column device in a shallow water regime. This means that characteristic fluid height is small with respect to the characteristic horizontal scale in the longitudinal direction. Let us denote by $\zeta(t, x)$ the free surface elevation, which is assumed to be a graph, and by $-h_s$ and $-h_0$ (with $h_s, h_0 > 0$) the parametrization of the flat bottom before and after the step, respectively. The step is located at $x = 0$ with size $s = h_s - h_0 > 0$. The two-dimensional fluid domain is $\Omega(t) = \Omega^l(t) \cup \Omega^r(t)$ where the fluid domain before the step is

$$\Omega^l(t) = \{(x, z) \in (-\infty, 0) \times \mathbb{R} : -h_s < z < \zeta(t, x)\}$$

and the fluid domain after the step is

$$\Omega^r(t) = \{(x, z) \in (0, l_1) \times \mathbb{R} : -h_0 < z < \zeta(t, x)\}.$$

The partially-immersed structure is centered at $x = l_0$, with horizontal length $2r$ and vertical walls located at $x = l_0 \pm r$. Its presence permits to divide the horizontal projection of the fluid domain into two domains: the *exterior* domain $(-\infty, l_0 - r) \cup (l_0 + r, l_1)$, where the fluid is not in contact with the structure, and the *interior* domain $(l_0 - r, l_0 + r)$, where the contact occurs. We denote them by \mathcal{E} and \mathcal{I} , respectively. We divide the exterior domain into $\mathcal{E}^- = (-\infty, 0)$, the subset of \mathcal{E} before the step, and $\mathcal{E}^+ = (0, l_0 - r) \cup (l_0 + r, l_1)$, the subset of \mathcal{E} after the step. Furthermore, later in the analysis we will need to distinguish the part of \mathcal{E}^+ outside the chamber and inside the chamber. Hence we denote by \mathcal{E}_l^+ and \mathcal{E}_r^+ the subsets $(0, l_0 - r)$ and $(l_0 + r, l_1)$, respectively.

Due to the difference between the fluid heights before and after the step, the horizontal discharge $q(t, x)$ is defined by

$$q(t, x) = \begin{cases} \int_{-h_s}^{\zeta(t, x)} u(t, x, z) dz & \text{for } (t, x) \in (0, T) \times \mathcal{E}^-, \\ \int_{-h_0}^{\zeta(t, x)} u(t, x, z) dz & \text{for } (t, x) \in (0, T) \times \mathcal{E}^+ \cup \mathcal{I}, \end{cases}$$

where $u(t, x, z)$ is the horizontal component of the fluid velocity. It follows that $q = h\bar{u}$ where $\bar{u}(t, x)$ is the vertically averaged horizontal fluid velocity and $h(t, x)$ is the fluid height, equal to $h_s + \zeta$ before the step and to $h_0 + \zeta$ after the step. After integrating over the fluid height the horizontal component of the free surface Euler equations, adimensionalizing the equations and truncating at precision $O(\mu)$, where μ is the shallowness parameter, one can obtain the nonlinear shallow water equations in the presence of a structure. We refer to [46, 47] for the derivation of the equations in the multi-dimensional case and [6] in the two-dimensional axisymmetric with no swirl case. Here we consider the one-dimensional nonlinear shallow water equations in the presence of a partially-immersed structure:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} \right) + gh \partial_x \zeta = -\frac{h}{\rho} \partial_x \underline{P}, \end{cases} \quad \text{in } (0, T) \times (-\infty, l_1), \quad (3.1)$$

where $\underline{P}(t, x)$ is the surface pressure of the fluid, g is the gravitational constant and ρ is the constant fluid density. Here we are not interested neither in viscous nor capillary effects (in the scale considered for OWCs they are negligible). Therefore, we do not consider any influence of the viscosity on the momentum equation and of the surface tension on the surface pressure, assuming continuity with the air (or

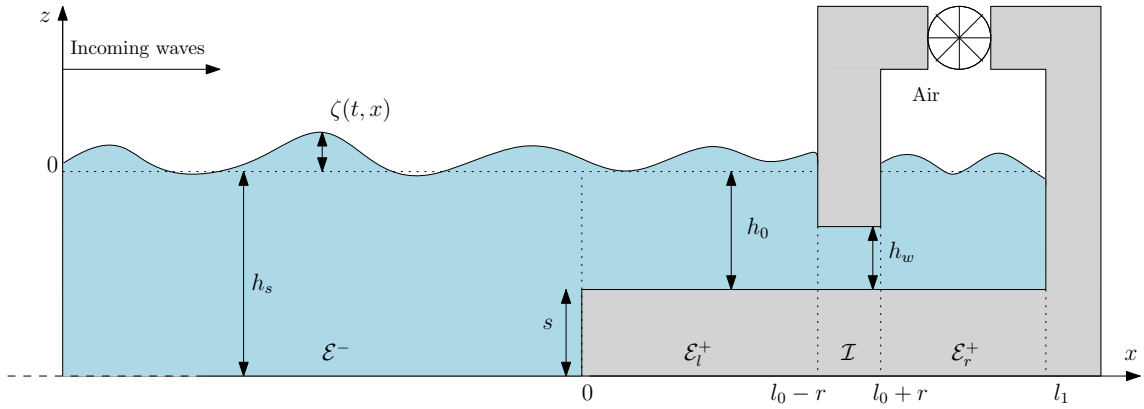


Figure 3.1 – Configuration of the oscillating water column device.

an upper-fluid whose density is negligible with respect to the lower-fluid density) pressure outside the fluid domain. In general, it is assumed that the air pressure is equal to the constant (both in time and space) atmospheric pressure. In a first and simpler approach, the authors modelled the oscillating water column device in [8] with a constant air pressure, both outside and inside the chamber. A novelty of this work is that we consider an air pressure function which is not constant through all the domain. Indeed, while outside the chamber it is reasonable to consider a constant air pressure, inside the chamber the motion of the waves produce variations of the air pressure and this fact must be taken into consideration to describe more precisely the behaviour of a wave energy converter of this type.

Let us now talk about the partially-immersed structure. We assume that the bottom of the structure can be parametrized as graph of a function ζ_w and for the sake of simplicity we consider a solid with a flat bottom, yielding $\zeta_w = \zeta_w(t)$. We remark that the same theory holds in the case of objects with non-flat bottom. The fact that in an oscillating water column device the partially-immersed structure is fixed implies that ζ_w is a constant of the problem both in space and time. Dealing with floating structures leads to consider a time-dependent function ζ_w related to the velocity of the moving object (see [6, 46] for nonlinear shallow water equations, [4] for Boussinesq equations).

Constraints and unknowns The interaction between floating or fixed structures and water waves, inherits a duality property. On the one hand, in the exterior domain the surface pressure is constrained to be equal the air pressure while the surface elevation is free, i.e.

$$\begin{cases} \underline{P}(t, x) = P_{\text{air}}(t, x), \\ \zeta(t, x) \text{ is unknown,} \end{cases} \quad \text{for } (t, x) \in (0, T) \times \mathcal{E}, \quad (3.2)$$

where $P_{\text{air}}(t, x)$ is the known air pressure function. On the other hand, in the interior domain the surface elevation matches the bottom of the solid while the surface pressure is free, i.e.

$$\begin{cases} \zeta(t, x) = \zeta_w, \\ \underline{P}(t, x) \text{ is unknown,} \end{cases} \quad \text{for } (t, x) \in (0, T) \times \mathcal{I}. \quad (3.3)$$

It has been shown in [46] that the \underline{P} in the interior domain can be seen as a Lagrange multiplier associated with the contact constraint $\zeta(t, x) = \zeta_w$ (it holds also for the water waves equations in the presence of a floating structure). Injecting the constraints (3.2)-(3.3) and making explicit the difference between the fluid height before the step and after the step, the equation (3.1) can be split into the following three systems

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_s + \zeta} \right) + g(h_s + \zeta) \partial_x \zeta = -\frac{h_s + \zeta}{\rho} \partial_x P_{\text{air}}, \\ \underline{P} = P_{\text{air}}, \end{cases} \quad \text{in } (0, T) \times \mathcal{E}^-, \quad (3.4)$$

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = -\frac{h_0 + \zeta}{\rho} \partial_x P_{\text{air}}, \\ \underline{P} = P_{\text{air}}, \end{cases} \quad \text{in } (0, T) \times \mathcal{E}^+, \quad (3.5)$$

and

$$\begin{cases} q = q_i(t), \\ \frac{dq_i}{dt} = -\frac{h_w}{\rho} \partial_x \underline{P}, \\ \zeta = \zeta_w, \end{cases} \quad \text{in } (0, T) \times \mathcal{I}, \quad (3.6)$$

where q_i is a time-dependent function that coincides with the horizontal discharge in the interior domain.

Boundary conditions Let us discuss here the boundary conditions that couple with (3.5)-(3.6). It is reasonable to assume that free surface elevation is continuous at the step if this is far enough from the crest of the waves. Moreover, considering the slip (or impermeability) condition at the step for the velocity field, the elliptic regularity for the potential in the interior of the fluid domain and the continuity of the surface elevation, one gets the continuity of the horizontal discharge at the step (see [46]). Therefore we have

$$\zeta|_{x=0^-} = \zeta|_{x=0^+}, \quad q|_{x=0^-} = q|_{x=0^+}.$$

Concerning the vertical walls of the partially-immersed structure, for the same reason as for the step we get the continuity of the horizontal discharge. Of course, since the structure have vertical walls, the continuity of the surface elevation at the solid walls and of the surface pressure fails (this would not be the case for instance in the case of a boat, see [35, 46]), we therefore have

$$\zeta|_{x=(l_0 \pm r) \pm} \neq \zeta_w, \quad P|_{x=(l_0 \mp r) \pm} \neq P_{\text{air}}, \quad q|_{x=(l_0 \pm r)^+} = q|_{x=(l_0 \pm r)^-}. \quad (3.7)$$

We will see in the next section how to supply the lack of continuity for the both the pressure and the surface elevation at the structure walls and derive a condition which will close the system. Finally, at the end the chamber we consider a solid wall condition, that is

$$q|_{x=l_1} = 0. \quad (3.8)$$

Remark 3.2.1. *In the case of a bounded fluid domain, as in [8, 48], with a left boundary at $x = -l$ one should add the boundary condition*

$$\zeta|_{x=-l} = f,$$

where $f = f(t)$ is a given time-dependent entry function. This is necessary dealing with numerical applications and f can be determined from experimental data. Indeed, during experiments in wave tanks it is usual to create waves with a lateral piston that permits to know the exact entry value of the surface elevation at any given time. Moreover, in [48] the authors showed that the knowledge of the entry value of the surface elevation allows to get the entry value of the horizontal discharge using the existence of Riemann invariants for the 1d nonlinear shallow water equations.

3.2.2 Air pressure dynamics

In this subsection we focus on the air pressure, which is not in general a constant function. In particular, we distinguish the cases of the air outside the chamber and inside the chamber. On the one hand, in $\mathcal{E}^- \cup \mathcal{E}_l^+$ the variations of the air pressure are negligible and it can be considered equal to the constant atmospheric pressure. *i.e.*

$$P_{\text{air}}(t, x) = P_{\text{atm}} \quad \text{for } (t, x) \in (0, T) \times \mathcal{E}^- \cup \mathcal{E}_l^+, \quad (3.9)$$

On the other hand, in \mathcal{E}_r^+ , where the air is partially trapped inside the chamber and pushed by the waves motion, a constant air pressure is no more realistic. However, if we consider the air behaving as an incompressible fluid (there is a hole on the top of the chamber) we can reasonably assume that air pressure inside the chamber is uniform. Therefore we deal with a time-dependent air pressure function and in particular we write it as a variation of the atmospheric pressure, *i.e.*

$$P_{\text{air}}(t, x) = P_{\text{atm}} + P_{\text{ch}}(t) \quad \text{for } (t, x) \in (0, T) \times \mathcal{E}_r^+, \quad (3.10)$$

where $P_{\text{ch}}(t)$ is the time-dependent variation. With this type of hypothesis on the air pressure inside the chamber, it is possible to find in ocean engineering literature an evolution equation governing the dynamics the pressure variation $P_{\text{ch}}(t)$. For instance, we refer to [20, 24]. It is derived for oscillating water column with Wells turbines [70], for which the relation between the pressure drop and the velocity of the air in the resistance layer is linear. Assuming this characteristics of the device, we have that P_{ch} satisfies the following linear ODE:

$$\frac{dP_{\text{ch}}}{dt} + \frac{\gamma P_{\text{atm}}}{h_{\text{ch}}K} P_{\text{ch}} = \frac{\gamma P_{\text{atm}}}{h_{\text{ch}}} \frac{d\bar{\zeta}}{dt}, \quad (3.11)$$

where γ is the polytropic expansion index for air ($\gamma = 1.4$), h_{ch} is the height of the chamber and K is a resistance parameter. Despite these known parameters of the device, the spatially averaged free surface elevation $\bar{\zeta}$ over \mathcal{E}_r^+ is unknown. In general in ocean engineering and marine energy literature, authors determine this value from experimental data calculated by gauges located inside the chamber. In our analytic approach, we rewrite it in terms of the horizontal discharge at the entrance of the chamber, at $x = l_0 + r$. Indeed, using the continuity equation in (3.5) we have

$$\frac{d\bar{\zeta}}{dt} = \frac{d}{dt} \left(\frac{1}{|\mathcal{E}_r^+|} \int_{\mathcal{E}_r^+} \zeta(t, x) dx \right) = \frac{1}{|\mathcal{E}_r^+|} \int_{\mathcal{E}_r^+} \partial_t \zeta(t, x) dx = \frac{q_{|x=(l_0+r)^+}}{|\mathcal{E}_r^+|} = \frac{q_i}{|\mathcal{E}_r^+|}, \quad (3.12)$$

where in the last two equalities we have used the wall condition (3.8) and the continuity condition (3.7) for the horizontal discharge. Therefore (3.11) reads

$$\frac{dP_{\text{ch}}}{dt} = -\gamma_1 P_{\text{ch}} + \gamma_2 q_i,$$

where γ_1 and γ_2 are constants depending on the device parameters as in (3.11). This equation in fact shows that the dynamics of the air pressure variation inside the chamber is determined by the value of the horizontal discharge under the partially-immersed structure.

3.3 Reformulation of the model as transmission problems

This section is devoted to the reformulation of the model that we have previously derived. More precisely, we show that (3.5)-(3.6) can be written as one transmission problem over the step and one transmission problem across the structure side-walls and we reduce them to two 4×4 initial boundary value problems (IBVP). Moreover, at the end of the OWC chamber, the interaction of waves with the solid wall can be recast as a 2×2 IBVP.

3.3.1 Transmission problem over the step

The first transmission problem is associated with the discontinuity in the bottom topography, located at $x = 0$, in front of the OWC device. In the region $\mathcal{E}^- \cup \mathcal{E}_l^+$, the air pressure is assumed to be the constant atmospheric pressure (see (3.9)), so together with (3.4)-(3.5), the equations in the domain near the step are written as follows:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 & \text{in } (0, T) \times \mathcal{E}^- \cup \mathcal{E}_l^+, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_s + \zeta} \right) + g(h_s + \zeta) \partial_x \zeta = 0 & \text{in } (0, T) \times \mathcal{E}^-, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = 0 & \text{in } (0, T) \times \mathcal{E}_l^+, \\ \underline{P} = P_{\text{atm}} & \text{in } (0, T) \times \mathcal{E}^- \cup \mathcal{E}_l^+, \\ \zeta = \zeta_0, \quad q = q_0 & \text{in } \mathcal{E}^- \cup \mathcal{E}_l^+ \end{cases} \quad (3.13)$$

with transmission conditions

$$\zeta|_{x=0^-} = \zeta|_{x=0^+}, \quad q|_{x=0^-} = q|_{x=0^+}.$$

where ζ_0 and q_0 are the initial data for the surface elevation and the horizontal discharge in $\mathcal{E}^- \cup \mathcal{E}_l^+$.

A rigorous study of such kind of transmission problem for 1d nonlinear shallow water equations with a discontinuous topography has been already treated by Iguchi and Lannes in [35] (see Section 6.1 therein) in the full line. More precisely, they showed that this problem can be written as a 4×4 quasilinear hyperbolic IBVP on the half line and they obtained a local well-posedness result. For this reason, we omit the well-posedness theory for the transmission problem that involve the step localized at $x = 0$ since, without loss of generality, it can be reduced to the same problem as in [35].

3.3.2 Transmission problem across the structure side-walls

The second transmission problem is associated with the wave-structure interaction at the vertical side-walls of the partially-immersed object. Recall that the air pressure is assumed to be the constant atmospheric pressure in \mathcal{E}_l^+ and a time-dependent variation of the atmospheric pressure in \mathcal{E}_r^+ (see (3.9)-(3.10)), so together with (3.4)-(3.5), the equations in the domain $\mathcal{E}_l^+ \cup \mathcal{E}_r^+$ are written as follows:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = 0, \\ \underline{P} = P_{\text{atm}}, \end{cases} \quad \text{in } (0, T) \times \mathcal{E}_l^+, \quad (3.14)$$

and

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = 0, & \text{in } (0, T) \times \mathcal{E}_r^+, \\ \underline{P} = P_{\text{atm}} + P_{\text{ch}}(t), \end{cases} \quad (3.15)$$

coupled with the transmission condition given by the continuity of q across the side-walls

$$q|_{x=(l_0-r)^-} = q|_{x=(l_0+r)^+}. \quad (3.16)$$

In the interior domain one has

$$\begin{cases} q = q_i(t), \\ \frac{dq_i}{dt} = -\frac{h_w}{\rho} \partial_x \underline{P}, & \text{in } (0, T) \times \mathcal{I}. \\ \zeta = \zeta_w, \end{cases} \quad (3.17)$$

Notice that the source term vanishes in (3.15) since it is the spatial derivative of the time-dependent air pressure inside the chamber.

We will see later that, after doing a change of variable, the 2×2 transmission problem (3.14)-(3.16) can be recast as a 4×4 hyperbolic quasilinear initial boundary value problem (IBVP). It is known that a necessary condition to ensure the well-posedness of this type of problems is that the number of boundary conditions must be equal to the number of positive eigenvalues of the system (see [5]). In our case we will have two positive eigenvalues, the positive eigenvalue of $A(U)$ in \mathcal{E}_r^+ and the opposite of the negative eigenvalue of $A(U)$ in \mathcal{E}_l^+ . Unfortunately, the continuity of q across the side-walls only give us one transmission condition and an additional transmission condition is indispensable. This will be derived in the next subsection.

3.3.3 Derivation of the second transmission condition

In the case of a boat as in [35], it turns out to be the situation of non-vertical lateral walls and the second transmission is determined by the continuity of the surface elevation at the contact points where the waves, the air and the solid meet. Contrarily in the presence of vertical side-walls, which is the case considered in this paper, this condition ceases to hold. Fortunately, from (3.17) we know that the horizontal discharge q in the interior domain is equal to q_i that depends only on time. Therefore the second transmission condition reads $q|_{x=(l_0 \pm r)^\pm} = q_i$ or equivalently

$$\langle q \rangle = q_i \quad \text{with} \quad \langle q \rangle = \frac{1}{2} \left(q|_{x=(l_0-r)^-} + q|_{x=(l_0+r)^+} \right).$$

When the air pressure is assumed to be constant both outside and inside the chamber, the fluid-structure system can be assumed to be isolated, yielding that the total fluid-structure energy is a conserved quantity. Then, using local conservation of energy derived from the equations, one obtains an evolution equation on q_i depending on the traces of the ζ and q at both side walls. This has been done in [8] for the nonlinear shallow water equations and in [11] for the Boussinesq system. This approach cannot be used in our case because there is no direct influence of the air pressure variation on the momentum equations in (3.14)-(3.15). As mentioned previously, one novelty of this article is the derivation of a more general evolution equation that takes into account the influence of the time-dependent variation of the air pressure inside the chamber. To do this, we use free surface Bernoulli's equation coupled in the presence of a partially-immersed structure

$$\begin{cases} \partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 + \frac{1}{\rho} P + gz = C(t) & \text{in } \Omega(t), \\ \nabla \Phi \cdot N = 0 & \text{at fluid-structure interface,} \end{cases} \quad (3.18)$$

where $\Phi(t, x, z) : (0, T) \times (-\infty, l_1) \times \mathbb{R} \rightarrow \mathbb{R}$ is a scalar velocity potential such that the velocity field $\mathbf{U}(t, x, z) = \nabla \Phi(t, x, z)$ in $\Omega(t)$, $C(t)$ is a time-dependent constant and N is the normal vector to the fluid-structure interface pointing inwards. This equation is derived from free surface Euler's equation in the presence of a structure using the irrotationality condition for the fluid velocity vector field. Let us introduce the notation $\underline{f}(t, x) = f(t, x, \zeta(t, x))$. By evaluating (3.18) at the points $(l_0 - r - \varepsilon, \zeta(t, l_0 - r - \varepsilon))$ and $(l_0 + r + \varepsilon, \zeta(t, l_0 + r + \varepsilon))$ for some small $\varepsilon > 0$, we thus get

$$\begin{aligned} & \frac{1}{2} |\underline{\nabla \Phi}|_{x=l_0-r-\varepsilon}|^2 + \frac{1}{\rho} P|_{x=l_0-r-\varepsilon} + g\zeta|_{x=l_0-r-\varepsilon} \\ &= \frac{1}{2} |\underline{\nabla \Phi}|_{x=l_0+r+\varepsilon}|^2 + \frac{1}{\rho} P|_{x=l_0+r+\varepsilon} + g\zeta|_{x=l_0+r+\varepsilon} + \underline{\partial_t \Phi}|_{x=l_0+r+\varepsilon} - \underline{\partial_t \Phi}|_{x=l_0-r-\varepsilon}. \end{aligned} \quad (3.19)$$

We write the difference of $\underline{\partial_t \Phi}$ as

$$\begin{aligned} \underline{\partial_t \Phi}|_{x=l_0+r+\varepsilon} - \underline{\partial_t \Phi}|_{x=l_0-r-\varepsilon} &= \int_{l_0-r-\varepsilon}^{(l_0-r)^-} \partial_x \underline{\partial_t \Phi} dx - \int_{\zeta(t, (l_0-r)^+)}^{\zeta(t, (l_0-r)^-)} \partial_t \partial_z \Phi dz \\ &+ \int_{(l_0-r)^+}^{(l_0+r)^-} \partial_x \underline{\partial_t \Phi} dx + \int_{\zeta(t, (l_0+r)^-)}^{\zeta(t, (l_0+r)^+)} \partial_t \partial_z \Phi dz + \int_{(l_0+r)^+}^{l_0+r+\varepsilon} \partial_x \underline{\partial_t \Phi} dx. \end{aligned} \quad (3.20)$$

We notice first that, applying the chain rule inside the third integral in (3.20), we have

$$\int_{(l_0-r)^+}^{(l_0+r)^-} \partial_x \underline{\partial_t \Phi} dx = \int_{(l_0-r)^+}^{(l_0+r)^-} \left(\underline{\partial_t \partial_x \Phi} + \underline{\partial_t \partial_z \Phi} \partial_x \zeta \right) dx = \int_{(l_0-r)^+}^{(l_0+r)^-} \underline{\partial_t \partial_x \Phi} dx$$

where the second term in the integral vanishes due to the impermeability condition in (3.18) at the bottom of the solid parameterized by ζ_w . Indeed, it reads

$$0 = \underline{\nabla\Phi} \cdot N_w = -\underline{\partial_z\Phi}$$

with $N_w = (-\partial_x\zeta_w, -1)^T = (0, -1)^T$ since the bottom of the solid is flat (ζ_w does not depend on the spatial variable) and we have $\underline{\partial_t\partial_z\Phi} = \underline{\partial_z\Phi}$ in the interior domain since the structure is fixed (ζ_w does not depend on the time variable).

In the absence of the solid, it is known that the horizontal and the vertical component of the fluid velocity, respectively u and w , satisfy the following identities:

$$u(x, z) = \partial_x\Phi(x, z) = \bar{u}(x) + O(\mu), \quad w(x, z) = \partial_z\Phi(x, z) = O(\mu), \quad (3.21)$$

where $\bar{u}(x)$ is the vertically averaged horizontal velocity and μ is the shallowness parameter (see [45]). Then, under the nonlinear shallow water approximation, *i.e.* $\mu \ll 1$ and precision $O(\mu)$, it yields $u \sim \bar{u}$ and $w \sim 0$. To the authors' knowledge this property has not been proven yet in the presence of a partially-immersed object and it seems not direct to obtain such a result. Therefore, in order to derive the transmission condition, we consider (3.21) as an assumption on the fluid velocity in the presence of a structure. Under this hypothesis and the nonlinear shallow water approximation, the second and the fourth integrals in (3.20) vanish. Then, taking the limit as $\varepsilon \rightarrow 0$ of (3.19) and by smoothness of the fluid velocity in the intervals $(l_0 - r - \varepsilon, l_0 - r)$ and $(l_0 + r, l_0 + r + \varepsilon)$ and the fact that $|\underline{\nabla\Phi}|^2 \sim \bar{u}^2$ in the nonlinear shallow water approximation, we obtain

$$\begin{aligned} \frac{1}{2}\bar{u}^2|_{x=(l_0-r)^-} + \frac{1}{\rho}P|_{x=(l_0-r)^-} + g\zeta|_{x=(l_0-r)^-} \\ = \frac{1}{2}\bar{u}^2|_{x=(l_0+r)^+} + \frac{1}{\rho}P|_{x=(l_0+r)^+} + g\zeta|_{x=(l_0+r)^+} \\ + \int_{(l_0-r)^+}^{(l_0+r)^-} \partial_t \bar{u} dx. \end{aligned}$$

Writing the averaged horizontal velocity in terms of the horizontal discharge q , that is $\bar{u} = \frac{q}{h}$, we get

$$\begin{aligned} \frac{1}{2} \frac{q^2}{(h_0 + \zeta)^2} \Big|_{x=(l_0-r)^-} + \frac{1}{\rho} P \Big|_{x=(l_0-r)^-} + g\zeta \Big|_{x=(l_0-r)^-} \\ = \frac{1}{2} \frac{q^2}{(h_0 + \zeta)^2} \Big|_{x=(l_0+r)^+} + \frac{1}{\rho} P \Big|_{x=(l_0+r)^+} + g\zeta \Big|_{x=(l_0+r)^+} + \int_{(l_0-r)^+}^{(l_0+r)^-} \partial_t \left(\frac{q}{h_0 + \zeta} \right) dx, \end{aligned}$$

where we have used the fact that $h = h_0 + \zeta$ under the structure and next to its side-walls. Finally, from the first and the third equations in (3.17) we obtain the ODE

$$-\frac{2r}{h_w} \frac{d^2 q_i}{dt} = \left[g\zeta + \frac{q^2}{2(h_0 + \zeta)^2} \right] + \frac{[[P]]}{\rho}$$

where $[[f]] = f|_{x=(t_0+r)^+} - f|_{x=(t_0-r)^-}$ is the jump between the side-walls of a function f defined on \mathcal{E}^+ . Recall that $2r$ is the length of the partially-immersed structure. Using the constraints on the surface pressure in (3.14)-(3.15), we explicitly write the jump as

$$[[P]] = P_{\text{atm}} + P_{\text{ch}}(t) - P_{\text{atm}}$$

and we conclude that

$$-\alpha \frac{dq_i}{dt} = \left[g\zeta + \frac{q^2}{2(h_0 + \zeta)^2} \right] + \frac{P_{\text{ch}}(t)}{\rho} \quad (3.22)$$

with $\alpha = \frac{2r}{h_w}$.

Remark 3.3.1. *The ODE (3.22) is a generalization of the one derived in [8] by the authors. Indeed, considering the air pressure equal to the constant atmospheric pressure also inside the chamber, one has $P_{\text{ch}} \equiv 0$ and the same equation as in [8] is recovered.*

Then the transmission problem (3.14)-(3.16) reads

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 & \text{in } (0, T) \times \mathcal{E}_l^+ \cup \mathcal{E}_r^+, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = 0 & \text{in } (0, T) \times \mathcal{E}_l^+ \cup \mathcal{E}_r^+, \\ \underline{P} = P_{\text{atm}} & \text{in } (0, T) \times \mathcal{E}_l^+, \\ \underline{P} = P_{\text{atm}} + P_{\text{ch}}(t) & \text{in } (0, T) \times \mathcal{E}_r^+, \\ \zeta = \zeta_0, \quad q = q_0 & \text{in } \mathcal{E}_l^+ \cup \mathcal{E}_r^+, \end{cases} \quad (3.23)$$

with transmission conditions

$$[[q]] = 0, \quad \langle q \rangle = q_i \quad (3.24)$$

and q_i, P_{ch} satisfying

$$\begin{cases} \frac{dq_i}{dt} = -\frac{1}{\alpha} \left[g\zeta + \frac{q^2}{2(h_0 + \zeta)^2} \right] - \frac{P_{\text{ch}}}{\alpha\rho} \\ \frac{dP_{\text{ch}}}{dt} = -\gamma_1 P_{\text{ch}} + \gamma_2 q_i \\ q_i(0) = q_{i,0}, \quad P_{\text{ch}}(0) = P_{\text{ch},0}. \end{cases} \quad (3.25)$$

where ζ_0 and q_0 are the initial data for the surface elevation and horizontal discharge in $\mathcal{E}_l^+ \cup \mathcal{E}_r^+$, $q_{i,0}$ and $P_{\text{ch},0}$ are the initial data for the horizontal discharge in \mathcal{I} structure and for the air pressure variation inside the chamber. We recall that the second equation in (3.25) is borrowed from (3.12).

3.3.4 Reduction of the transmission problem across the structure to an IBVP

In this subsection we show how the 2×2 transmission problem (3.23)-(3.25) can be reduced to a 4×4 one-dimensional quasilinear IBVP with a semilinear boundary condition. Since we are interested in the analysis of the wave-structure interaction at the side-walls of the partially-immersed structure, without loss of generality we consider the transmission problem on the full line and we ignore the presence of the step and of the solid wall at the end of the chamber, which have been both already treated in [35] as mentioned previously. Furthermore, for the sake of simplicity we consider the transmission problem over the boundaries $x = 0^\pm$. This change does not affect the mathematical structure of the transmission problem and the same analysis can be developed considering the physical boundaries $x = l_0 \pm r$ determined by the object side-walls.

Therefore we rewrite (3.23)-(3.25) in the full line under the compact form

$$\begin{cases} \partial_t U + A(U) \partial_x U = 0 & \text{in } (0, T) \times \mathbb{R}_- \cup \mathbb{R}_+, \\ U|_{t=0} = U_0(x) & \text{in } \mathbb{R}_- \cup \mathbb{R}_+, \\ M^+ U|_{x=0^+} - M^- U|_{x=0^-} = V(G(t)) & \text{in } (0, T). \end{cases} \quad (3.26)$$

with $U = (\zeta, q)^T$, $U_0 = (\zeta_0, q_0)^T$,

$$A(U) = \begin{pmatrix} 0 & 1 \\ g(h_0 + \zeta) - \frac{q^2}{(h_0 + \zeta)^2} & \frac{2q}{h_0 + \zeta} \end{pmatrix}, \quad M^\pm = \begin{pmatrix} 0 & 1 \\ 0 & \pm \frac{1}{2} \end{pmatrix},$$

and $G = (q_i, P_{\text{ch}})^T$ satisfying the evolution equation

$$\begin{cases} \dot{G} = \Theta(G, U|_{x=0^-}, U|_{x=0^+}), \\ G(0) = G_0, \end{cases} \quad (3.27)$$

Writing $U = (U_1, U_2)^T$, $G = (G_1, G_2)^T$ and $\Theta = (\Theta_1, \Theta_2)^T$, we have that $V(G) = (0, G_1)^T$ and

$$\begin{aligned} \Theta_1 \left(G, U|_{x=0^-}, U|_{x=0^+} \right) &= \\ & - \frac{1}{\alpha} \left[\left(gU_1 + \frac{U_2^2}{2(h_0 + U_1)^2} \right) \Big|_{x=0^+} - \left(gU_1 + \frac{U_2^2}{2(h_0 + U_1)^2} \right) \Big|_{x=0^-} \right] - \frac{G_2}{\alpha\rho}, \\ \Theta_2 \left(G, U|_{x=0^-}, U|_{x=0^+} \right) &= -\gamma_1 G_2 + \gamma_2 G_1. \end{aligned}$$

Equation (3.27) has the same form of the kinematic-type evolution equation considered in [35] where the authors dealt with a free boundary transmission problem. Here, although we consider a fixed boundary transmission problem, the same situation occurs: the derivative of G has the same regularity as the trace of the solution at the boundary. The boundary condition is semilinear, in the sense that the evolution equation (3.27) is nonlinear only on the trace of the solution at the boundary and not on its derivatives. This would be the case when considering a boat-type structure, which turns out to be a free boundary hyperbolic problem. A kinematic-type evolution equation for the moving contact points $x_{\pm}(t)$ can be derived after time-differentiating the boundary condition $U(t, x_{\pm}(t)) = U_i(t, x_{\pm}(t))$, where U_i is a known function. In the nonlinear equation obtained, there are terms involving traces of derivatives $\partial U|_{x=0^{\pm}}$ and the boundary condition is fully nonlinear because there is a loss of one derivative in the estimates (see [35]). Here we deal with a less singular evolution equation.

Let us now recast (3.26)-(3.27) as an IBVP by introducing a change of variable $x' = -x$ in \mathbb{R}_- and writing

$$\begin{aligned} u^+(t, x) &= U(t, x), & u^-(t, x) &= U(t, -x), \\ u_0^+(x) &= U_0(x), & u_0^-(x) &= U_0(-x), \end{aligned}$$

Thus, the system (3.26) is equivalent to the following 4×4 quasilinear hyperbolic system in $\Omega_T := (0, T) \times \mathbb{R}_+$

$$\begin{cases} \partial_t u + \mathcal{A}(u) \partial_x u = 0 & \text{in } \Omega_T, \\ u|_{t=0} = u_0(x) & \text{in } \mathbb{R}_+, \\ \mathcal{M} u|_{x=0} = V(G(t)) & \text{in } (0, T), \end{cases} \quad (3.28)$$

where $u = (u^-, u^+)^T$, $u_0 = (u_0^-, u_0^+)^T$ are \mathbb{R}^4 -valued functions and

$$\mathcal{A}(u) = \text{diag} \left(-A(u^-), A(u^+) \right), \quad \mathcal{M} = \begin{pmatrix} -M^- & M^+ \end{pmatrix}$$

are respectively 4×4 and 2×4 matrices. Moreover, the ODE (3.27) reads

$$\begin{cases} \dot{G} = \Theta \left(G, u|_{x=0} \right), \\ G(0) = G_0. \end{cases} \quad (3.29)$$

In order to study the wellposedness of our proposed model in this chapter (which is not included in this manuscript), we present the IBVP associated with the presence of a solid wall at the end of the OWC chamber.

3.3.5 IBVP at the end of the OWC chamber

We have already seen that the presence of a solid wall at the end of the chamber leads to consider the wall boundary condition for the horizontal discharge (3.8). Therefore, the problem associated with the interaction of waves with this physical boundary can be recast as the following IBVP:

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 & \text{in } (0, T) \times \mathcal{E}_r^+, \\ \partial_t q + \partial_x \left(\frac{q^2}{h_0 + \zeta} \right) + g(h_0 + \zeta) \partial_x \zeta = 0 & \text{in } (0, T) \times \mathcal{E}_r^+, \\ \underline{P} = P_{\text{atm}} + P_{\text{ch}}(t) & \text{in } (0, T) \times \mathcal{E}_r^+, \\ \zeta = \zeta_0, \quad q = q_0 & \text{in } \mathcal{E}_r^+, \end{cases} \quad (3.30)$$

with boundary condition

$$q = 0 \quad \text{in } x = l_1, \quad (3.31)$$

where P_{ch} is the solution to (3.23)-(3.25), and ζ_0 and q_0 are the initial data for the surface elevation and the horizontal discharge in \mathcal{E}_r^+ . Without loss of generality (3.30)-(3.31) can be seen as a particular case of the interaction between waves and a moving lateral piston treated in [35] (see Section 4 therein). Indeed, when the lateral piston is fixed the same situation as at the end of the chamber of the oscillating water column occurs. While the authors introduced a Lagrangian diffeomorphism in equations in a fixed domain, here we already have it. Therefore one can directly use the theory in [35] to derive a well-posedness result for (3.23)-(3.31) which we will omit in the present chapter.

Some remarks on viscous fluid-structure models

CHAPTER 4

Well-posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid

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*This work is in collaboration with Denis Matignon and Marius Tucsnak.
It is based on [80].*

4.1 Introduction

In this work, we consider an infinite-dimensional system describing the vertical motion of a solid floating at the free surface of a viscous fluid with finite depth and a flat bottom. This system is motivated by the growing interest of wave energy extractors that float on the sea and extract energy by activating a hydraulic pump, which in turn, drives a hydraulic motor connected to a generator. In such an arrangement, the torque on the generator can be controlled, leading to a controllable vertical force on the floating object, see for instance [44] or [66]. The input of the considered system is the force acting on the solid by an actuator, whereas the output is the distance from the solid to the sea bottom. The novelty brought by this work is twofold:

- The viscous effects are taken in consideration from the beginning of the modelling process, by adapting a method describing viscous free boundary value flows which has been introduced in [57].
- We give an explicit form of the transfer function, allowing, in particular, to establish the input-output stability of the system. In a future work we aim using this explicit form to implement simple feedback laws.

The leading assumptions on the fluid are that it is one dimensional and unbounded in the horizontal direction, that the flow can be described within the shallow water approximation (this mean that the horizontal length scale of motion L is much greater than the perpendicular fluid depth D , i.e. $D/L \ll 1$) and that the viscosity effects cannot be neglected. Concerning the solid, we assume that it has vertical walls, that it can move only vertically and that it is subject to a vertical control force. The output signal is the distance from the bottom of the solid to the sea bottom. More precisely, we consider the model introduced in [57], with the particularity that the fluid is supposed to be infinite in the horizontal direction, denoting $\mathcal{I} := (a, b)$ the projection on the fluid bottom of the solid domain and setting $\mathcal{E} := \mathbb{R} \setminus [a, b]$. The floating solid is supposed, without loss of generality, to have mass $\mathcal{M} = 1$ and it is constrained to move only in the vertical direction. Given $t > 0$, we denote by $h(t, x)$ the height of the free surface of the fluid, by $q(t, x)$ the flux of viscous fluid in the direction x and by $h_S(t)$ the distance from the bottom of the rigid body to the bottom of the fluid, supposed to be horizontal, as described in Fig. 4.1. We denote

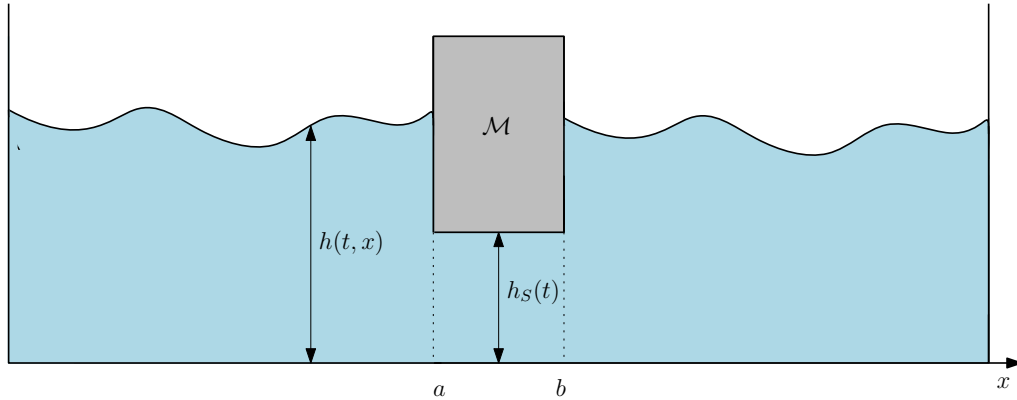


Figure 4.1 – Graphical sketch of the model. The function $h(t, x)$ denote the height of the free surface of the fluid, and $h_S(t)$ is the function which describes the distance from the bottom of the rigid body to the bottom of the fluid.

by \bar{h} and \bar{h}_S the equilibrium height for the fluid and the solid, respectively. Then, following [57], we have

$$\bar{h} = \bar{h}_S + \frac{1}{b - a},$$

and for simplicity, we assume that

$$\bar{h} = 1, \quad g = 1, \quad \bar{p} = \frac{1}{b-a}.$$

Hence, by linearizing the nonlinear fluid-structure model introduced by Maity et al. in [57], around the trajectory (in the equilibrium positions described above) $(h_S, h, q, p) = (\bar{h}_S, \bar{h}, 0, \bar{p})$, we obtain the linear systems given by

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x \in \mathcal{E}), \quad (4.1.1)$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0, \quad (x \in \mathcal{E}), \quad (4.1.2)$$

$$h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) = p(t, a^+) + h_S(t) - \mu \frac{\partial q}{\partial x}(t, a^+), \quad (4.1.3)$$

$$h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) = p(t, b^-) + h_S(t) - \mu \frac{\partial q}{\partial x}(t, b^-), \quad (4.1.4)$$

$$\dot{h}_S(t) + \frac{\partial q}{\partial x} = 0 \quad (x \in \mathcal{I}), \quad (4.1.5)$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \quad (x \in \mathcal{I}), \quad (4.1.6)$$

$$\ddot{h}_S(t) = \int_a^b p(t, x) dx + u(t) \quad (t > 0), \quad (4.1.7)$$

where p is a Lagrange multiplier, similar to a pressure term (which is obtained in the Hamiltonian modelling process), u is the input function whereas the output function is defined as

$$y(t) = h_S(t) \quad (t \geq 0). \quad (4.1.8)$$

Our first main result is the following reformulation of the system, where we consider the state space defined by

$$X := \mathbb{C} \times H^1(\mathcal{E}) \times L^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C}. \quad (4.1.9)$$

Theorem 4.1.1. *Equations (4.1.1)-(4.1.8) can be recast as*

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Cz, \end{aligned} \quad (4.1.10)$$

where the components of the vector $z(t)$ are $h_S(t)$, $h(t, \cdot)$, $q(t, \cdot)$, $q(t, a)$ and $q(t, b)$, B is in $\mathcal{L}(\mathbb{C}, X)$, C is in $\mathcal{L}(X, \mathbb{C})$ and A is a generator of an analytic semigroup on X .

Using the classical definition of well-posed linear systems in the sense of Weiss (for details see [77] or [84]), the above theorem implies the following result:

Corollary 4.1.2. *Equations (4.1.1)-(4.1.8) define a well-posed linear system with state space X defined in (4.1.9) and input and output spaces $U = Y = \mathbb{C}$.*

Remark 4.1.3. *From the above results it follows, in particular, that for every*

$$z_0 = \begin{bmatrix} h_{S,0} \\ h_0 \\ q_0 \\ q_{a,0} \\ q_{b,0} \end{bmatrix} \in X$$

and every $u \in L^2[0, \infty)$, the initial value problem formed of (4.1.1)-(4.1.8) and the initial condition $z(0) = z_0$ admits a unique solution

$$z(t, x) = \begin{bmatrix} h_S(t) \\ h(t, x) \\ q(t, x) \\ q(t, a) \\ q(t, b) \end{bmatrix},$$

in $C([0, \infty); X)$. Moreover, it is easily checked that \tilde{z} defined by

$$\tilde{z}(t, x) = \begin{bmatrix} h_S(t) \\ h(t, a + b - x) \\ -q(t, a + b - x) \\ -q(t, b) \\ -q(t, a) \end{bmatrix},$$

satisfies (4.1.1)-(4.1.8). Moreover, if we assume that

$$q_0(x) = -q_0(a + b - x), \quad h_0(x) = h_0(a + b - x) \quad (x \in \mathcal{E}), \quad (4.1.11)$$

then $\tilde{z}(0, \cdot) = z_0$, thus \tilde{z} satisfies the same initial value problem as z . Using the uniqueness of solutions of this initial value problem we deduce that $\tilde{z} = z$. This means, in particular, that for initial data satisfying (4.1.11) we have

$$q(t, a) = -q(t, b), \quad h(t, a) = h(t, b) \quad (t \geq 0).$$

Continuing with our results, we remember that a well-posed linear system of the form (4.1.10) is said input-output stable if equations (47) define, for $z(0) = 0$, a bounded map $u \mapsto y$ from $L^2([0, \infty); U)$ to $L^2([0, \infty); Y)$. Considering this our second main result can be stated as:

Theorem 4.1.4. *The system described by (4.1.1)-(4.1.8) is input-output stable.*

The remaining part of this chapter is organized as follows. In Section 4.2 we prove Theorem 4.1.1. Finally, Section 4.3 is devoted to the proof of our second main result, asserting the input-output stability of the considered system.

4.2 Proof of Theorem 4.1.1

For $t \geq 0$, we set $q_a(t) := q(t, a)$ and $q_b(t) := q(t, b)$. Since (4.1.5) implies that q is a linear function of x on \mathcal{I} , for every $t \geq 0$ and $x \in \mathcal{I}$,

$$\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b - a}, \quad (4.2.1)$$

$$q(t, x) = q_a(t) \left(\frac{x - b}{a - b} \right) + q_b(t) \left(\frac{x - a}{b - a} \right), \quad (4.2.2)$$

$$\frac{\partial q}{\partial x}(t, x) = \frac{q_b(t) - q_a(t)}{b - a}. \quad (4.2.3)$$

We differentiate (4.1.6) with respect to x and use (4.1.3)-(4.1.5) to arrive at

$$\begin{aligned} \frac{\partial^2 p}{\partial x^2}(t, x) &= \ddot{h}_S(t) \quad (x \in \mathcal{I}), \\ p(t, a^+) &= p_a(t), \quad p(t, b^-) = p_b(t), \end{aligned} \quad (4.2.4)$$

where

$$p_a(t) := h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) - h_S(t) - \mu \dot{h}_S(t), \quad (4.2.5)$$

$$p_b(t) := h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) - h_S(t) - \mu \dot{h}_S(t). \quad (4.2.6)$$

Moreover, the first equation in (4.2.4) implies that, for every $t \geq 0$, $p(t, x)$ is a second order polynomial in x . Hence, by recalling that for all polynomial $g(x)$ of degree 2, defined on a real interval $[c, d]$, the following formulas follow:

$$\begin{aligned} \int_c^d g(x) dx &= g(c)(d - c) + g'(c) \frac{(d - c)^2}{3} + g'(d) \frac{(d - c)^2}{6}, \\ \int_c^d g(x) dx &= g(d)(d - c) - g'(c) \frac{(d - c)^2}{6} - g'(d) \frac{(d - c)^2}{3}, \end{aligned}$$

we conclude that

$$\begin{aligned} \int_a^b p(t, x) dx &= p(t, a)l - \dot{q}_a(t) \frac{l^2}{3} - \dot{q}_b(t) \frac{l^2}{6} \\ &= p(t, b)l + \dot{q}_a(t) \frac{l^2}{6} + \dot{q}_b(t) \frac{l^2}{3}, \end{aligned}$$

where we set $l := b - a$. Combining this with (4.1.7) and (4.2.1) we deduce that

$$\begin{aligned} \left[1 + \frac{l^3}{3} \right] \dot{q}_a(t) - \left[1 - \frac{l^3}{6} \right] \dot{q}_b(t) &= p(t, a)l^2 + lu(t), \\ - \left[1 - \frac{l^3}{6} \right] \dot{q}_a(t) + \left[1 + \frac{l^3}{3} \right] \dot{q}_b(t) &= -p(t, b)l^2 - lu(t). \end{aligned}$$

Inverting the above linear system, we get

$$\begin{bmatrix} \dot{q}_a(t) \\ \dot{q}_b(t) \end{bmatrix} = M \begin{bmatrix} p(t, a) \\ -p(t, b) \end{bmatrix} + \frac{1}{l} M \begin{bmatrix} u(t) \\ -u(t) \end{bmatrix}, \quad (4.2.7)$$

where M is the matrix given by

$$M := \frac{1}{l(1 + \frac{l^3}{12})} \begin{pmatrix} 1 + \frac{l^3}{3} & 1 - \frac{l^3}{6} \\ 1 - \frac{l^3}{6} & 1 + \frac{l^3}{3} \end{pmatrix}. \quad (4.2.8)$$

Considering equation (4.1.3)-(4.1.4) together with (4.2.3) we deduce that

$$p(t, a) = h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) - h_S(t) + \mu \frac{q_b - q_a}{b - a}, \quad (4.2.9)$$

and

$$p(t, b) = h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) - h_S(t) + \mu \frac{q_b - q_a}{b - a}. \quad (4.2.10)$$

Therefore, by using the approach developed above, the system (4.1.1)-(4.1.7) writes in the equivalent form¹

$$\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b - a} \quad (t \geq 0), \quad (4.2.11)$$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (x \in \mathcal{E}), \quad (4.2.12)$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0 \quad (x \in \mathcal{E}), \quad (4.2.13)$$

$$\begin{bmatrix} \dot{q}_a(t) \\ \dot{q}_b(t) \end{bmatrix} = \frac{M}{b - a} \begin{bmatrix} u(t) \\ -u(t) \end{bmatrix} + M \begin{bmatrix} h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) - h_S(t) + \mu \frac{q_b - q_a}{b - a} \\ -h(t, b^+) + \mu \frac{\partial q}{\partial x}(t, b^+) + h_S(t) - \mu \frac{q_b - q_a}{b - a} \end{bmatrix}. \quad (4.2.14)$$

Let X be defined by (4.1.9), set

$$W := \mathbb{C} \times H^1(\mathcal{E}) \times H^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C},$$

and denote by $z := [h_S \ h \ q \ q_a \ q_b]^T$ a generic element of X . Consider the operator $A : \mathcal{D}(A) \rightarrow X$ defined by

$$\mathcal{D}(A) := \{z \in W \mid q(t, a) = q_a(t), q(t, b) = q_b(t)\}, \quad (4.2.15)$$

¹where by considering the equations (4.2.7), (4.2.9) and (4.2.10) we have reformulated the equations (4.1.3)-(4.1.4) and (4.1.7).

$$Az := \begin{bmatrix} -\frac{q(b)-q(a)}{b-a} \\ -\frac{dq}{dx} \\ -\frac{dh}{dx} + \mu \frac{d^2q}{dx^2} \\ R_1z \\ R_2z \end{bmatrix}, \quad (4.2.16)$$

where

$$\begin{bmatrix} R_1z \\ R_2z \end{bmatrix} := M \begin{bmatrix} h(a^-) - \mu \frac{dq}{dx}(a^-) - h_S + \mu \frac{q_b - q_a}{b-a} \\ -h(b^+) - \mu \frac{dq}{dx}(b^+) - h_S + \mu \frac{q_b - q_a}{b-a} \end{bmatrix}.$$

In the situation when \mathcal{E} is supposed to be bounded (which means that the fluid is contained in a container), it has been proved in [57, Section 6] that the corresponding version of A defined in (4.2.15)-(4.2.16) generates an analytic semigroup. This proof can be transposed with obvious modifications to our case so that the operator A generates an analytic semigroup on X . We set

$$Bu := [0, 0, 0, \frac{lu}{2(1 + \frac{l^3}{12})}, -\frac{lu}{2(1 + \frac{l^3}{12})}]^T \text{ and } Cz := h_S, \quad (4.2.17)$$

and we observe that $B \in \mathcal{L}(\mathbb{C}, X)$ and $C \in \mathcal{L}(X, \mathbb{C})$. Hence the proof of Theorem 4.1.1 is completed.

4.3 Proof of Theorem 4.1.4

It has been shown in [57] that in the case of a bounded container, the linearized system describing the motion of the floating body is exponentially stable. It is not difficult to check that in our case we have that 0 lies in the spectrum of A , thus the system is no longer exponentially stable. However, we have the following result where \mathbb{C}_0 denotes the open right-half plane

$$\mathbb{C}_0 := \{s \in \mathbb{C} : \Re s > 0\}. \quad (4.3.1)$$

Proposition 4.3.1. *The resolvent set $\rho(A)$ contains \mathbb{C}_0 .*

Proof. Let $\lambda \in \mathbb{C}_0$ and $F = [f_1, f_2, f_3, f_4, f_5]^T$ in X . The equation $(\lambda I - A)z = F$ for $z \in \mathcal{D}(A)$ reads

$$\lambda h_S + \frac{q(b) - q(a)}{b - a} = f_1, \quad (4.3.2)$$

$$\lambda h(x) + \frac{dq}{\partial x} = f_2(x) \quad (x \in \mathcal{E}), \quad (4.3.3)$$

$$\lambda q(x) + \frac{dh}{dx} - \mu \frac{d^2q}{dx^2} = f_3(x) \quad (x \in \mathcal{E}), \quad (4.3.4)$$

$$M \begin{bmatrix} h(a^-) - \mu \frac{dq}{dx}(a^-) - h_S + \mu \frac{q(b)-q(a)}{b-a} \\ -h(b^+) + \mu \frac{dq}{dx}(b^+) + h_S - \mu \frac{q(b)-q(a)}{b-a} \end{bmatrix} = \begin{bmatrix} f_4 \\ f_5 \end{bmatrix} - \lambda \begin{bmatrix} q_a \\ q_b \end{bmatrix}, \quad (4.3.5)$$

$$\lim_{x \rightarrow -\infty} q(x) = \lim_{x \rightarrow \infty} q(x) = 0, \quad (4.3.6)$$

$$q(a) = q_a, \quad q(b) = q_b. \quad (4.3.7)$$

From (4.3.5) and (4.2.14), it follows that

$$\mu \frac{q(b) - q(a)}{l} + h(a^-) - h_S - \mu \frac{dq}{dx}(a^-) = \left[1 + \frac{l^3}{3}\right] \frac{(\lambda q_{a,0} - f_4)}{l^2} + \left[1 - \frac{l^3}{6}\right] \frac{(f_5 - \lambda q_{b,0})}{l^2}, \quad (4.3.8)$$

$$-\mu \frac{q(b) - q(a)}{l} - h(b^+) + h_S + \mu \frac{dq}{dx}(b^+) = \left[1 - \frac{l^3}{6}\right] \frac{(f_4 - \lambda q_{a,0})}{l^2} + \left[1 + \frac{l^3}{3}\right] \frac{(\lambda q_{b,0} - f_5)}{l^2}. \quad (4.3.9)$$

We next transform (4.3.2)-(4.3.7) into a boundary value problem for q by eliminating h , h_S , q_a , q_b from the above mentioned equations. First, from (4.3.3) and (4.3.4), we deduce

$$\lambda q - \left(\mu + \frac{1}{\lambda}\right) \frac{d^2q}{dx^2} = \phi_1 \quad (x \in \mathcal{E}), \quad (4.3.10)$$

where

$$\phi_1 := f_3 - \frac{1}{\lambda} \frac{df_2}{dx} \in L^2(\mathcal{E}).$$

Next, using (4.3.2), (4.3.3) and (4.3.7) in (4.3.8) and (4.3.9) it follows that

$$\left(\mu + \frac{1}{\lambda}\right) \frac{dq}{dx}(a^-) = \left(\mu + \frac{1}{\lambda} + \frac{\lambda}{l}\right) \frac{q(b) - q(a)}{l} - \frac{\lambda l}{6} (2q(a) + q(b)) + \phi_2 \quad (4.3.11)$$

$$\left(\mu + \frac{1}{\lambda}\right) \frac{dq}{dx}(b^+) = \left(\mu + \frac{1}{\lambda} + \frac{\lambda}{l}\right) \frac{q(b) - q(a)}{l} + \frac{\lambda l}{6} (q(a) + 2q(b)) + \phi_3 \quad (4.3.12)$$

with

$$\begin{aligned} \phi_2 &:= \left(\frac{f_2(a^-)}{\lambda} - \frac{f_1}{\lambda}\right) + \left[1 + \frac{l^3}{3}\right] \frac{f_4}{l^2} - \left[1 - \frac{l^3}{6}\right] \frac{f_5}{l^2}, \\ \phi_3 &:= \left(\frac{f_2(b^+)}{\lambda} - \frac{f_1}{\lambda}\right) + \left[1 - \frac{l^3}{6}\right] \frac{f_4}{l^2} - \left[1 + \frac{l^3}{3}\right] \frac{f_5}{l^2}. \end{aligned}$$

We first prove the existence of a weak solution to (4.3.10)-(4.3.12) with the condition (4.3.6) at infinity. We thus set $V := H^1(\mathcal{E})$. The weak formulation of (4.3.10)-(4.3.12) with the condition (4.3.6) is to find $q \in V$ such that

$$B_\lambda(q, \psi) = \int_{\mathcal{E}} \phi_1 \bar{\psi} dx + \phi_2 \bar{\psi}(a) - \phi_3 \bar{\psi}(b) \quad (\psi \in V), \quad (4.3.13)$$

where

$$\begin{aligned}
B_\lambda(q, \psi) := & \int_{\mathcal{E}} \left[\lambda q \bar{\psi} + \left(\mu + \frac{1}{\lambda} \right) \frac{dq}{dx} \frac{d\bar{\psi}}{dx} \right] dx \\
& - \left[\left(\frac{1}{\lambda} + \mu + \frac{\lambda}{l} \right) \frac{q(b) - q(a)}{l} - \frac{\lambda l}{6} [2q(a) + q(b)] \right] \bar{\psi}(a) \\
& + \left[\left(\frac{1}{\lambda} + \mu + \frac{\lambda}{l} \right) \frac{q(b) - q(a)}{l} + \frac{\lambda l}{6} [q(a) + 2q(b)] \right] \bar{\psi}(b). \quad (4.3.14)
\end{aligned}$$

For $q, \psi \in V$,

$$\begin{aligned}
B_\lambda(q, \psi) = & \int_{\mathcal{E}} \left[\lambda q \bar{\psi} + \left(\mu + \frac{1}{\lambda} \right) \frac{dq}{dx} \frac{d\bar{\psi}}{dx} \right] dx \\
& + \left(\frac{1}{\lambda} + \mu + \frac{\lambda}{l} \right) \frac{(q(b) - q(a)) (\bar{\psi}(b) - \bar{\psi}(a))}{l} + \\
& \frac{\lambda l}{6} [q(a) \bar{\psi}(a) + q(b) \bar{\psi}(b) + (q(b) + q(a)) (\bar{\psi}(b) + \bar{\psi}(a))]. \quad (4.3.15)
\end{aligned}$$

From the above formula, it follows that for any $\lambda \in \mathbb{C}_0$ we can consider positive constants $C = C(\lambda)$, $\alpha = \min\{\operatorname{Re} \lambda, \operatorname{Re} 1/\lambda\}$ such that

$$|B_\lambda(q, \psi)| \leq C \|q\|_V \|\psi\|_V, \operatorname{Re} B_\lambda(q, q) \geq \alpha \|q\|_V^2 \quad (4.3.16)$$

where $q, \psi \in V$, thus B_λ is a bounded and coercive form on V . Moreover, the right hand side of (4.3.13) clearly defines a bounded linear functional on V . Thus, the conclusion follows by the complex version of the Lax-Milgram Lemma (see, for instance, Lemma 5.4 on [1]). \square

4.3.1 Transfer function

From Proposition 4.3.1 it follows that the transfer function

$$G(s) = C(sI - A)^{-1}B,$$

of the system (4.1.1)-(4.1.8) is defined for every $s \in \mathbb{C}_0$. In this subsection we compute this transfer function and we show that it lies in the Hardy space $H^\infty(\mathbb{C}_0)$ and thus obtain the main ingredient of the proof of Theorem 4.1.4. In other terms, we compute the Laplace transform of the solution of (4.1.1)-(4.1.8) with zero initial data. More precisely, we have:

Proposition 4.3.2. *The transfer function of the system (4.1.1)-(4.1.8) is given by*

$$G(s) := \frac{1}{\left(1 + \frac{l^3}{12}\right) s^2 + \frac{l^2}{2} s \sqrt{1 + \mu s} + \mu l s + l} \quad (s \in \mathbb{C}_0). \quad (4.3.17)$$

Proof. We first express $h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-)$ in terms of h_S and \dot{h}_S . To this end, for $x \in \mathcal{I}$, we first note that

$$q(t, b) - q(t, a) = -l\dot{h}_S(t). \quad (4.3.18)$$

Moreover, using Remark 4.1.3 we obtain

$$h(t, a^-) = h(t, b^+), \quad -q(t, a^-) = q(t, b^+), \quad (4.3.19)$$

thus

$$q(t, a) = \frac{l}{2}\dot{h}_S, \quad q(t, b) = -\frac{l}{2}\dot{h}_S. \quad (4.3.20)$$

From equations (4.2.11)-(4.2.12) it follows that for $x \in (-\infty, a]$ we have

$$\begin{aligned} \frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} - \mu \frac{\partial^3 q}{\partial t \partial x^2} &= 0, \\ q(t, x) &\rightarrow 0 \text{ as } x \rightarrow -\infty, \quad q(t, a) = \frac{b-a}{2}\dot{h}_S(t), \\ q(0, x) &= \frac{\partial q}{\partial t}(0, x) = 0. \end{aligned} \quad (4.3.21)$$

For $f \in L^1[0, \infty]$, we denote by \hat{f} the Laplace transform of f . Applying the Laplace transform to both sides of (4.3.21), we obtain

$$\begin{aligned} s^2 \hat{q} - (1 + s\mu) \frac{\partial^2 \hat{q}}{\partial x^2} &= 0 \\ \hat{q}(s, x) &\rightarrow 0 \text{ as } x \rightarrow -\infty, \quad \hat{q}(s, a) = \frac{b-a}{2} \hat{h}_S, \quad \text{Re}(s) > 0. \end{aligned} \quad (4.3.22)$$

Hence we can conclude that

$$\hat{q}(s, x) = \frac{b-a}{2} e^{\frac{-sa}{\sqrt{1+s\mu}}} e^{\frac{sx}{\sqrt{1+s\mu}}} \hat{h}_S(s) \quad (4.3.23)$$

and

$$\begin{aligned} \hat{h}(s, a^-) - \mu \frac{\partial \hat{q}}{\partial x}(s, a^-) &= -\frac{l}{2} \left(\frac{1}{s} + \mu \right) \frac{s}{\sqrt{1+s\mu}} \hat{h}_S(s) \\ &= -\frac{l}{2} (\sqrt{1+\mu s}) \hat{h}_S(s). \end{aligned} \quad (4.3.24)$$

In a similar way, we obtain

$$\hat{h}(s, b^+) - \mu \frac{\partial \hat{q}}{\partial x}(s, b^+) = -\frac{l}{2} (\sqrt{1+\mu s}) \hat{h}_S(s). \quad (4.3.25)$$

Moreover, applying the Laplace transform to (4.3.20) we obtain

$$\hat{q}(s, a) = \frac{l}{2} s \hat{h}_S, \quad \hat{q}(s, b) = -\frac{l}{2} s \hat{h}_S. \quad (4.3.26)$$

Finally, applying Laplace transform to (4.2.14), we obtain

$$\begin{bmatrix} s\widehat{q}_a(s) \\ s\widehat{q}_b(s) \end{bmatrix} = \frac{M}{b-a} \begin{bmatrix} \widehat{u}(s) \\ -\widehat{u}(s) \end{bmatrix} + M \begin{bmatrix} \widehat{h}(s, a^-) - \mu \frac{\partial \widehat{q}}{\partial x}(s, a^-) - \widehat{h}_S(s) + \mu \frac{\widehat{q}_b - \widehat{q}_a}{b-a} \\ -\widehat{h}(s, b^+) + \mu \frac{\partial \widehat{q}}{\partial x}(s, b^+) + \widehat{h}_S(s) - \mu \frac{\widehat{q}_b - \widehat{q}_a}{b-a} \end{bmatrix}. \quad (4.3.27)$$

Hence, using (4.3.24) and (4.3.26) in (4.3.27), we conclude

$$\left(1 + \frac{l^3}{12}\right) s^2 \widehat{h}_S = l \left[-\frac{ls\sqrt{1+\mu s}}{2} - 1 - \mu s \right] \widehat{h}_S + \widehat{u}.$$

The above relation implies that

$$\widehat{h}_S(s) = G(s)\widehat{u}(s),$$

where G is given by (4.3.17), which ends the proof. \square

Lemma 4.3.3. *Let F be the function defined by*

$$F(s) = \left(1 + \frac{l^3}{12}\right) s^2 + \frac{l^2}{2} s\sqrt{1+\mu s} + \mu ls + l, \quad (4.3.28)$$

and let \mathbb{C}_0 be the open right-half plane, as defined in (4.3.1). Then there exists a neighborhood \mathcal{O} of $\overline{\mathbb{C}_0}$ such that F is holomorphic on \mathcal{O} . Moreover, F does not vanish on $\overline{\mathbb{C}_0}$.

Proof. The fact that F is holomorphic on a neighborhood of $\overline{\mathbb{C}_0}$ follows from the corresponding property of each term in the right-hand side of (4.3.28) (including the one involving the square-root, for which we take the principal determination).

Let $s \in \overline{\mathbb{C}_0}$. We set $z := \sqrt{1+\mu s}$. Since $z^2 = 1+\mu s$ we have

$$\operatorname{Re}(z^2 - 1) \geq 0. \quad (4.3.29)$$

In particular $\operatorname{Re}(z^2) > 0$, which implies that $\arg(z^2) \in (-\pi/2, \pi/2)$. Consequently, we have

$$\arg(z) \in (-\pi/4, \pi/4). \quad (4.3.30)$$

As $s = \frac{z^2-1}{\mu}$, $F(s) = 0$ is equivalent to

$$\left(1 + \frac{l^3}{12}\right) \left(\frac{z^2-1}{\mu}\right)^2 + \frac{l^2}{2} \left(\frac{z^2-1}{\mu}\right) z + lz^2 = 0. \quad (4.3.31)$$

Multiplying (4.3.31) by $\frac{\mu^2}{1+l^3}$, we obtain

$$(z^2 - 1) + 2 \frac{l^2 \mu}{4 \left(1 + \frac{l^3}{12}\right)} (z^2 - 1) z + \frac{l^4 \mu^2 z^2}{16 \left(1 + \frac{l^3}{12}\right)^2} + \frac{l \mu^2 z^2}{16 \left(1 + \frac{l^3}{12}\right)^2} \left(16 + \frac{l^3}{3}\right) = 0,$$

and then

$$\left(z^2 - 1 + \frac{l^2 \mu z}{4\left(1 + \frac{l^3}{12}\right)}\right)^2 + \frac{l\mu^2 z^2}{16\left(1 + \frac{l^3}{12}\right)^2} \left(16 + \frac{l^3}{3}\right) = 0. \quad (4.3.32)$$

Hence equation (4.3.31) is equivalent to $P(z)Q(z) = 0$, where

$$P(z) := z^2 - 1 + \frac{l^2 \mu z}{4\left(1 + \frac{l^3}{12}\right)} + iz \frac{\mu\sqrt{l}}{4\left(1 + \frac{l^3}{12}\right)} \sqrt{16 + \frac{l^3}{3}} \quad (4.3.33)$$

$$Q(z) := z^2 - 1 + \frac{l^2 \mu z}{4\left(1 + \frac{l^3}{12}\right)} - iz \frac{\mu\sqrt{l}}{4\left(1 + \frac{l^3}{12}\right)} \sqrt{16 + \frac{l^3}{3}}. \quad (4.3.34)$$

Let us prove that $P(z) \neq 0$ and $Q(z) \neq 0$. To this aim we write $z = x + iy$ with $x > 0$ (due to (4.3.29) and (4.3.30)) and $y \in \mathbb{R}$. We note that:

$$\operatorname{Re}(P(z)) = \mu \operatorname{Re}(s) + \frac{l^2 \mu x}{4\left(1 + \frac{l^3}{12}\right)} - y \frac{\mu\sqrt{l}}{4\left(1 + \frac{l^3}{12}\right)} \sqrt{16 + \frac{l^3}{3}},$$

$$\operatorname{Im}(P(z)) = y \left(2x + \frac{l^2 \mu}{4\left(1 + \frac{l^3}{12}\right)}\right) + x \frac{\mu\sqrt{l}}{4\left(1 + \frac{l^3}{12}\right)} \sqrt{16 + \frac{l^3}{3}},$$

$$\operatorname{Re}(Q(z)) = \mu \operatorname{Re}(s) + \frac{l^2 \mu x}{4\left(1 + \frac{l^3}{12}\right)} + y \frac{\mu\sqrt{l}}{4\left(1 + \frac{l^3}{12}\right)} \sqrt{16 + \frac{l^3}{3}},$$

$$\operatorname{Im}(Q(z)) = y \left(2x + \frac{l^2 \mu}{4\left(1 + \frac{l^3}{12}\right)}\right) - x \frac{\mu\sqrt{l}}{4\left(1 + \frac{l^3}{12}\right)} \sqrt{16 + \frac{l^3}{3}}.$$

If $\operatorname{Im}(P(z)) = 0$, since l, μ, x are positive, then $y < 0$. But since $\operatorname{Re}(z^2 - 1) \geq 0$, this implies that $\operatorname{Re}(P(z)) > 0$. Therefore $P(z) \neq 0$.

If $\operatorname{Re}(Q(z)) = 0$, since $\operatorname{Re}(z^2 - 1) \geq 0$ and $x > 0$ are positive we conclude that $y < 0$. This implies that $\operatorname{Im}(Q(z)) < 0$. Therefore $Q(z) \neq 0$.

Thus $F(s) \neq 0$ on $\overline{\mathbb{C}_0}$, which concludes the proof of the lemma. \square

4.3.2 Proof of Theorem 4.1.4

By Lemma 4.3.3 we know that the function F defined in (4.3.28) is not vanishing on $\overline{\mathbb{C}_0}$. Moreover, since

$$\lim_{|s| \rightarrow \infty} |F(s)| = +\infty,$$

we have that the map $s \mapsto |F(s)|$ is bounded from below on $\overline{\mathbb{C}_0}$. We conclude that the transfer function G defined in (4.3.17) is such that

$$\sup_{s \in \mathbb{C}_0} |G(s)| < \infty.$$

By the Paley-Wiener theorem (see, for instance, [75, Chapter 19.2]), this implies that (4.1.1)-(4.1.8) is input-output stable, so that the proof of the theorem is complete.

4.4 Further work

Another issues that we do not mention in this work, but that are interesting for our future research are the following:

- We will study the sectorial properties of the resolvent of the operator A defined in eq. (4.2.16), in order to establish results related to the existence and uniqueness of strong solutions and strong stability of the system.
- We will consider as output the vertical velocity, instead of the height, and we will study if it would be a well-posed system in some suitable state space. Moreover, we will explore the stability properties of this new system and its applications to energy wave extractors.

CHAPTER 5

Asymptotic behaviour of a system modelling rigid structures floating in a viscous fluid

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*This work is in collaboration with Denis Matignon and Marius Tucsnak.
It is based on [81].*

5.1 Introduction

In this work we consider the return to the equilibrium problem of a model describing the vertical motion of a solid floating at the free surface of a viscous fluid with finite depth and flat bottom. This problem concerns a particular configuration of a

system of coupling the free surface motion of a fluid and a partially immersed floating structure.

More precisely, it consists of situation where a partially submerged solid body is released at zero velocity from a non-equilibrium position.

The return to equilibrium problem (also called *free decay test*) consists in describing the large time behavior of the oscillation amplitude of the solid. The interest of this problem is that it can easily be used numerically or experimentally, and is useful to determine important characteristics of floating objects, from an engineering point of view.

For inviscid fluids filling an unbounded domain, the motion of the solid is often described in the literature by a linear integro-differential equation, known as the Cummins equation, which has been obtained empirically by [16]. In his paper the Cummins equation for vertical displacements of a floating structure reads as

$$(M + a_\infty) \ddot{h}_S(t) = c h_S(t) + K * \dot{h}_S(t), \quad (5.1.1)$$

where $h_S(t)$ denotes the displacement of the structure from the equilibrium position, M denotes the mass of the structure, a_∞ denotes the added mass at infinite frequency, c is the hydrostatic stiffness, and $K(t)$ denotes the radiation force impulse response function. An equation with similar characteristics but including non-linear effects has been developed in [46].

As far as we know, the only work using a viscous model for the fluid is [57], where an equation of Cummins type is obtained, even in cases in which the fluid could be bounded by vertical walls; here, we are interested in describing the model of Cummins type in an unbounded viscous domain. More precisely, we study the correct version of this model for vertical displacements of a floating structure, which now reads:

$$\left(1 + \frac{(b-a)^3}{12}\right) \ddot{h}_S(t) = -\frac{(b-a)^2}{2} F * \dot{h}_S(t) - \mu(b-a) \dot{h}_S(t) - (b-a) h_S(t), \quad (5.1.2)$$

where μ is the viscosity coefficient of the fluid, $(b-a)$ is the width of the interval $\mathcal{I} = (a, b)$ obtained by projecting the floating object (supposed symmetric around the axis $x = \frac{1}{2}(a+b)$) on the flat horizontal bottom, and $\mathcal{E} = \mathbb{R} \setminus [a, b]$ denotes the viscous fluid domain. Moreover, F is the causal distribution with Laplace transform $\hat{F}(s) = \sqrt{1 + \mu s}$.

The novelties brought in by this work are:

- the correct form of an equation of Cummins type for an unbounded viscous domain,
- the proof of stability of the system, and its asymptotic behaviour at infinity, using a diffusive representation,

- an explicit form of the solutions of the system in the time domain, and the large time behaviour of these, recovering another stability proof,
- numerical evidence of the transition between the different decay rates of the system as a function of the viscosity coefficient μ of the fluid.

This work is a companion to [80], where this system with force as input, and distance from the solid to the sea bottom as output was first recast as a linear well-posed system and second proved to be input-to-output stable.

The outline of the chapter is as follows: in Section 5.2, the physical model and its linearization around a steady state are recalled; in Section 5.3, an equivalent diffusive representation of the system is provided, which helps prove stability and even compute refined asymptotics in some cases; in Section 5.4, the analytical solution of the system is provided thanks to Mittag-Leffler special functions, the asymptotic behaviour are provided in full generality, helping to recover the previous stability property; finally a conclusion is drawn and future works are investigated in Section 5.5.

5.2 Recalls on the linearized physical model

In this Section the floating solid is supposed, without loss of generality, to have mass $\mathcal{M} = 1$ and it is constrained to move only in the vertical direction. Given $t > 0$, we denote by $h(t, x)$ the height of the free surface of the fluid, by $q(t, x)$ the flux of viscous fluid in the direction x and by $h_S(t)$ the distance from the bottom of the rigid body to the bottom of the fluid, supposed to be horizontal, as described in Fig. 5.1.

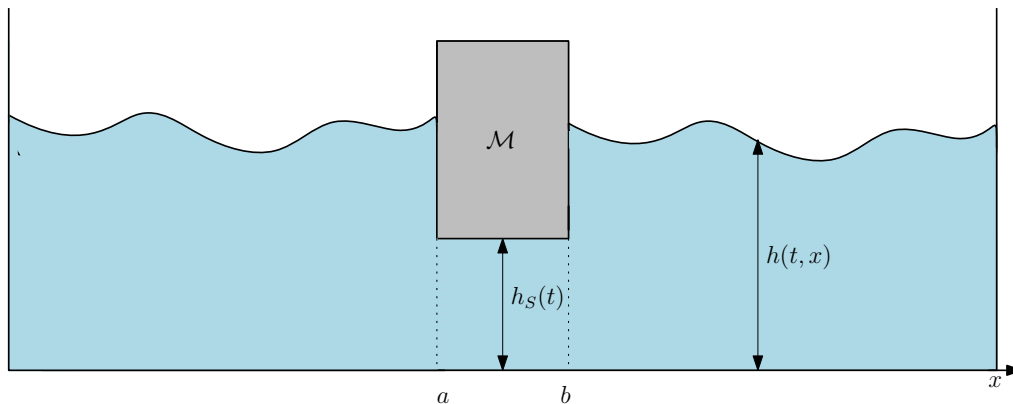


Figure 5.1 – Graphical sketch of the model. The function $h(t, x)$ denote the height of the free surface of the fluid and $h_S(t)$ is the function which describes the distance from the bottom of the rigid body to the bottom of the fluid.

We consider the model introduced in [57], with the particularity that the fluid is supposed to be infinite in the horizontal direction, denoting $\mathcal{I} := (a, b)$ the projection on the fluid bottom of the solid domain and setting $\mathcal{E} := \mathbb{R} \setminus [a, b]$. Then, following [57], we have

$$\bar{h} = \bar{h}_S + \frac{1}{b-a},$$

and for simplicity, we assume that

$$\bar{h} = 1, \quad g = 1, \quad \bar{p} = \frac{1}{b-a}.$$

Hence, by linearizing the nonlinear fluid-structure model introduced by Maity et al. in [57], around the trajectory (in the equilibrium positions described above) $(h_S, h, q, p) = (\bar{h}_S, \bar{h}, 0, \bar{p})$ we obtain the equations

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad (x \in \mathcal{E}), \quad (5.2.1)$$

$$\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = 0, \quad (x \in \mathcal{E}), \quad (5.2.2)$$

$$h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) = p(t, a^+) + h_S(t) - \mu \frac{\partial q}{\partial x}(t, a^+), \quad (5.2.3)$$

$$h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) = p(t, b^-) + h_S(t) - \mu \frac{\partial q}{\partial x}(t, b^-), \quad (5.2.4)$$

$$\dot{h}_S(t) + \frac{\partial q}{\partial x} = 0 \quad (x \in \mathcal{I}), \quad (5.2.5)$$

$$\frac{\partial q}{\partial t} + \frac{\partial p}{\partial x} = 0 \quad (x \in \mathcal{I}), \quad (5.2.6)$$

$$\ddot{h}_S(t) = \int_a^b p(t, x) dx \quad (t > 0), \quad (5.2.7)$$

where p is a Lagrange multiplier, similar to a pressure term (which is obtained in the Hamiltonian modelling process),

Remark 5.2.1. *In particular, for initial data satisfying*

$$q_0(x) = -q_0(a+b-x), \quad h_0(x) = h_0(a+b-x) \quad (x \in \mathcal{E}), \quad (5.2.8)$$

we have

$$q(t, a) = -q(t, b), \quad h(t, a) = h(t, b) \quad (t \geq 0).$$

By rewriting the pressure term p in system (5.2.1)-(5.2.7) as $p = p_1 + p_2$, where p_1 and p_2 solve

$$\frac{\partial^2 p_1}{\partial x^2} = \ddot{h}_S, \quad p_1(t, a) = p_1(t, b) = 0, \quad (5.2.9)$$

$$\frac{\partial^2 p_2}{\partial x^2} = 0, \quad p_2(t, a) = p_a(t), \quad p_2(t, b) = p_b(t), \quad (5.2.10)$$

respectively, with

$$p_a(t) := h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) - h_S(t) - \mu \dot{h}_S(t), \quad (5.2.11)$$

$$p_b(t) := h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) - h_S(t) - \mu \dot{h}_S(t), \quad (5.2.12)$$

then, by solving equations (5.2.9) and (5.2.10), it follows that

$$p_1(t, x) = \ddot{h}_S \left(\frac{x^2}{2} - \frac{b+a}{2}x + \frac{ab}{2} \right), \quad (5.2.13)$$

and

$$p_2(t, x) = p_a(t) + (p_b(t) - p_a(t)) \frac{x-a}{l}, \quad (5.2.14)$$

where $l := b - a$. Substituting these values of p_1 and p_2 in (5.2.7), we obtain

$$\begin{aligned} \left(1 + \frac{l^3}{12}\right) \ddot{h}_S(t) &= p_a(t)l + (p_b(t) - p_a(t)) \frac{l}{2} \\ &= \frac{l}{2} (p_a(t) + p_b(t)). \end{aligned}$$

Considering the values of p_a and p_b from (5.2.11) and (5.2.12) respectively, the equation above can be rewritten as

$$\begin{aligned} \left(1 + \frac{l^3}{12}\right) \ddot{h}_S(t) &= -l \left(h_S(t) + \mu \dot{h}_S(t) \right) \\ &\quad + \frac{l}{2} \left(h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-) + h(t, b^+) - \mu \frac{\partial q}{\partial x}(t, b^+) \right). \end{aligned} \quad (5.2.15)$$

We first express $h(t, a^-) - \mu \frac{\partial q}{\partial x}(t, a^-)$ in terms of h_S and \dot{h}_S . To this end, for $x \in \mathcal{I}$, we first note that

$$q(t, b) - q(t, a) = -l \dot{h}_S(t). \quad (5.2.16)$$

Moreover, using Remark 5.2.1 we obtain

$$h(t, a^-) = h(t, b^+), \quad -q(t, a^-) = q(t, b^+), \quad (5.2.17)$$

thus

$$q(t, a) = \frac{l}{2} \dot{h}_S, \quad q(t, b) = -\frac{l}{2} \dot{h}_S. \quad (5.2.18)$$

For $t \geq 0$, we set $q_a(t) := q(t, a)$ and $q_b(t) := q(t, b)$. Since (5.2.5) implies that q is a linear function of x on \mathcal{I} , for every $t \geq 0$ and $x \in \mathcal{I}$,

$$\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b - a} \quad (t \geq 0). \quad (5.2.19)$$

From equations (5.2.19) and (5.2.1) it follows that for $x \in (-\infty, a]$ we have

$$\begin{aligned} \frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} - \mu \frac{\partial^3 q}{\partial t \partial x^2} &= 0, \\ q(t, x) &\rightarrow 0 \text{ as } x \rightarrow -\infty, \quad q(t, a) = \frac{b-a}{2} \dot{h}_S(t), \\ q(0, x) &= \frac{\partial q}{\partial t}(0, x) = 0. \end{aligned} \quad (5.2.20)$$

For $f \in L^1[0, \infty]$, let \hat{f} the Laplace transform of f . Applying the Laplace transform to both sides of (5.2.20), we obtain

$$\begin{aligned} s^2 \hat{q} - (1 + s\mu) \frac{\partial^2 \hat{q}}{\partial x^2} &= 0, \\ \hat{q}(s, x) &\rightarrow 0 \text{ as } x \rightarrow -\infty, \quad \hat{q}(s, a) = \frac{b-a}{2} \hat{h}_S, \quad \Re(s) > 0. \end{aligned} \quad (5.2.21)$$

Hence we can conclude that

$$\hat{q}(s, x) = \frac{b-a}{2} e^{\frac{-sa}{\sqrt{1+s\mu}}} e^{\frac{sx}{\sqrt{1+s\mu}}} \hat{h}_S(s), \quad (5.2.22)$$

and

$$\begin{aligned} \hat{h}(s, a^-) - \mu \frac{\partial \hat{q}}{\partial x}(s, a^-) &= -\frac{l}{2} \left(\frac{1}{s} + \mu \right) \frac{s}{\sqrt{1+s\mu}} \hat{h}_S(s) \\ &= -\frac{l}{2} (\sqrt{1+\mu s}) \hat{h}_S(s). \end{aligned} \quad (5.2.23)$$

In a similar way, we obtain

$$\hat{h}(s, b^+) - \mu \frac{\partial \hat{q}}{\partial x}(s, b^+) = -\frac{l}{2} (\sqrt{1+\mu s}) \hat{h}_S(s). \quad (5.2.24)$$

By considering Remark 5.2.1 and the inverse of Laplace transform of equations (5.2.23) and (5.2.24), we obtain the following result:

Proposition 5.2.2. *The vertical movement of a floating object, in an unbounded viscous fluid that is initially at rest, is described by the following integro-differential equation*

$$\left(1 + \frac{l^3}{12}\right) \ddot{h}_S(t) = -\frac{l^2}{2} \int_0^t F(\sigma) \dot{h}_S(t-\sigma) d\sigma - l (h_S(t) + \mu \dot{h}_S(t)). \quad (5.2.25)$$

with initial conditions

$$h_S(0) = h_0, \quad \dot{h}_S(0) = 0,$$

and where F is the causal distribution, such that $\hat{F}(s) = \sqrt{1+\mu s}$ in $\Re(s) > -1/\mu$.

5.3 Diffusive representation, stability proof and asymptotic behaviour

The main idea of this section is to get rid of the F term. First since its Laplace transform is not bounded in any right-half plane, it does not correspond to a causal function, but rather a causal distribution: indeed, when $\mu \rightarrow \infty$, the term \sqrt{s} appears, which is related to the fractional derivative of order $1/2$, see e.g. [62] and references therein. On the contrary, $1/\sqrt{s}$ is bounded and corresponds to the fractional integration of order $1/2$, this is the reason why we shall be interested rather in

$$\hat{G}(s) := \frac{\hat{F}(s) - 1}{s} = \frac{\mu}{1 + \sqrt{1 + \mu s}}, \text{ for } \Re(s) > -1/\mu. \quad (5.3.1)$$

This extra transfer function is of so-called *diffusive type*, and enjoys nice properties, see e.g. [63]: it is a completely monotone function, i.e. $G(t) := \int_0^\infty g(\xi) \exp(-\xi t) d\xi$ for some appropriate positive and real-valued weight function g to be computed, or equivalently $\hat{G}(s) := \int_0^\infty g(\xi) (s + \xi)^{-1} d\xi$, for $\Re(s) > 0$. Following e.g. [61], we can compute g explicitly as:

$$g(\xi) := \lim_{\epsilon \rightarrow 0^+} \frac{1}{2i\pi} (\hat{G}(-\xi - i\epsilon) - \hat{G}(-\xi + i\epsilon)), \quad (5.3.2)$$

$$= \frac{1}{\pi} \frac{\sqrt{\mu\xi - 1}}{\mu\xi}, \text{ for } \xi > 1/\mu. \quad (5.3.3)$$

This weight is indeed real valued, positive, and fulfills the well-posedness condition

$$\int_{\mu^{-1}}^\infty \frac{g(\xi)}{1 + \xi} d\xi < \infty \quad (5.3.4)$$

that is required for the functional setting to make sense.

5.3.1 Extended diffusive representation

For the G transfer function above with input $v := \dot{h}$ and output $y := G*v$, a diffusive realization is of the form:

$$\partial_t \phi(t, \xi) = -\xi \phi(t, \xi) + v(t), \quad \phi(0, \xi) = 0 \quad (5.3.5)$$

$$y(t) = \int_{\mu^{-1}}^\infty g(\xi) \phi(t, \xi) d\xi. \quad (5.3.6)$$

The formal proof is straightforward and relies on the fact that $\partial_t(e^{-\xi t} * v) = -\xi(e^{-\xi t} * v) + v(t)$.

Now for the F transfer function with input v and new output $z := F * v$, since $\hat{F}(s) = 1 + s\hat{G}(s)$, the following extended diffusive realisation can be proposed:

$$\partial_t \varphi(t, \xi) = -\xi \varphi(t, \xi) + v(t), \quad \varphi(0, \xi) = 0 \quad (5.3.7)$$

$$z(t) = \int_{\mu^{-1}}^{\infty} g(\xi) \partial_t \varphi(t, \xi) d\xi + 1 v(t). \quad (5.3.8)$$

Indeed, defining as energy $\mathcal{E}_\varphi(t) := \frac{1}{2} \int_{\mu^{-1}}^{\infty} \xi g(\xi) |\varphi(t, \xi)|^2 d\xi$, one can easily compute the following balance:

$$\frac{d}{dt} \mathcal{E}_\varphi(t) = +v(t) z(t) - 1 (v(t))^2 - \int_{\mu^{-1}}^{\infty} g(\xi) |\partial_t \varphi(t, \xi)|^2 d\xi. \quad (5.3.9)$$

This latter energy balance will play a key role when analyzing the stability of the coupled system. Note that the whole rigorous functional analytic setting needed to address this problem is fully detailed in [63], both for standard and extended diffusive realizations.

5.3.2 Energy balance and new stability proof

Consider the original system (5.2.15), set $\dot{h} := v$ and $z := F * \dot{h}$, it can then be viewed as a coupled system

$$\left\{ \begin{array}{l} \left(1 + \frac{l^3}{12}\right) \ddot{h}_S + z(t) + l\dot{h}_S + l\mu h_S = 0 \\ v(t) = \dot{h}_S(t) \\ \partial_t \varphi(t, \xi) = -\xi \varphi(t, \xi) + v(t); \quad \varphi(0, \xi) = 0 \\ z(t) = \int_{\mu^{-1}}^{\infty} g(\xi) \partial_t \varphi(t, \xi) d\xi + v(t). \end{array} \right. \quad (5.3.10)$$

The mechanical energy of the oscillator is defined by

$$E(t) := \frac{1}{2} \left(1 + \frac{l^3}{12}\right) (\dot{h}_S)^2(t) + \frac{1}{2} l\mu (h_S)^2(t).$$

Its energy balance reads

$$\frac{d}{dt} E(t) = -l(\dot{h}_S)^2(t) - \dot{h}_S(t) (F * \dot{h}_S)(t);$$

while the first term is indeed negative, the second has no definite sign; however it reads $-v(t) z(t)$ and compensates exactly with $v(t) z(t)$ in (5.3.9).

This is the reason why we shall define a global energy functional $\mathcal{E}(t) := E(t) + \mathcal{E}_\varphi(t)$ for the augmented system with state variables (h_S, ω, φ) in the state space

$\mathbb{R} \times \mathbb{R} \times \widetilde{H}$, where $\widetilde{H} = \left\{ \varphi \in L_{\text{loc}}^2(\mathbb{R}^+, dg), \int_0^\infty \xi |\varphi|^2 dg(\xi) < \infty \right\}$. Indeed, the global energy balance reads, at least formally:

$$\frac{d}{dt} \mathcal{E}(t) = -(1+l)(\dot{h}_S)^2(t) - \int_{\mu^{-1}}^\infty g(\xi) |\partial_t \varphi(t, \xi)|^2 d\xi \leq 0.$$

This decay of the global energy is the starting point to the following asymptotic stability result.

Proposition 5.3.1. *For all $(h_{S,0}, \omega_0) \in \mathbb{C}^2$, the solution of the coupled system (5.3.10), with initial condition $(h_{S,0}, \omega_0, 0)$, satisfies*

$$(h_S, \dot{h}_S, \varphi)(t) \rightarrow_{t \rightarrow \infty} 0 \quad \text{in } \mathbb{C}^2 \times \widetilde{H}.$$

Proof. Indeed, since the weight $g(\xi)$ is positive and satisfies the well-posedness condition (5.3.4), Theorem 3.7 in [63] applies directly to our problem. \square

5.3.3 Asymptotic behaviour (special case)

Thanks to the diffusive representation of \hat{F} , involving a branch cut on $(-\infty, -\frac{1}{\mu}]$ on \mathbb{R}^- , following e.g. [61], it is known thanks to the Watson lemma that the branchpoint at $s = -\frac{1}{\mu}$ with local behaviour $\frac{\sqrt{\mu}}{\pi} \sqrt{\xi - \frac{1}{\mu}}$ translates into $\frac{\sqrt{\mu}}{\pi} \Gamma(\frac{3}{2}) e^{-\frac{t}{\mu}} t^{-3/2}$ as $t \rightarrow +\infty$ by inverse Laplace transform.

But as usual, apart from the branchcut, other singularities of the transfer function like poles s_k can appear, giving rise to $r_k e^{s_k t}$ terms in the time domain. At this stage however, we are not in a position to state whether or not $\Re(s_k) \leq -\frac{1}{\mu}$, so our result is only a partial one.

Proposition 5.3.2. *If all the poles s_k of the transfer function lie in the left halfplane $\Re(s) < -\frac{1}{\mu}$, then the asymptotic behaviour of the solution h_S of the system (5.3.10) reads*

$$h_S(t) \sim K e^{-\frac{t}{\mu}} t^{-3/2}, \quad \text{as } t \rightarrow +\infty.$$

Hence, there is a need to inspect the location of the poles more thoroughly in order to analyze the asymptotic behaviour of the solution in the general case, i.e. whatever the location of those poles.

5.4 Analytical solution and asymptotic behaviour

For simplicity in this subsection, we use the following notations: $l = b - a$, $A = 1 + \frac{l^3}{12}$, $B = \frac{l^2}{2}$ and $C = l\mu$; all are positive constants. In Section 5.4.1, the case of the inviscid fluid $\mu = 0$ is recalled, while in Section 5.4.2, the general case of the viscous fluid $\mu > 0$ is examined. Finally in Section 5.4.3, numerical evidence is provided of the possible transition between different asymptotic regimes, as the viscosity μ increases.

5.4.1 Case $\mu = 0$

If we consider $\mu = 0$ in (5.2.15), the model reduces to an ODE:

$$\begin{cases} A\ddot{h}_S + B\dot{h}_S + lh_S = 0, \\ h_S(0) = h_0, \quad \dot{h}_S(0) = \dot{h}_0. \end{cases} \quad (5.4.1)$$

This model has the form of a simple mechanical oscillator, free of external forces, which we shall call *free oscillation*.

Applying Laplace transform to the equation (5.4.1), and after simplifications, we get

$$[As^2 + Bs + l] \hat{h}_S(s) = [As + B] h_0 + A\dot{h}_0. \quad (5.4.2)$$

By calculating the inverse of the Laplace transform of the rational function appearing implicitly in (5.4.2), we obtain that the solution for the model (5.4.1) is given by

$$h_S(t) = (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)) e^{-\delta t}, \quad (5.4.3)$$

when $B^2 < 4Al$, where

$$\delta = \frac{B}{2A}, \quad \omega_0 = \sqrt{\frac{l}{A}}, \quad \omega_d = \sqrt{\omega_0^2 - \delta^2} = \frac{\sqrt{4Al - B^2}}{2A}, \quad (5.4.4)$$

are the damping coefficient, the undamped natural angular frequency and the damped angular frequency, respectively. The constants C_1 and C_2 , are given by

$$C_1 = h_0, \quad C_2 = \frac{\dot{h}_0 + h_0\delta}{\omega_d} = \frac{h_0B + 2\dot{h}_0A}{\sqrt{4Al - B^2}}. \quad (5.4.5)$$

Remark 5.4.1. If $(b - a) > \sqrt[3]{6}$, the free oscillation is overdamped, that is, if $\delta > \omega_0$, then ω_d is imaginary. In this situation, $B^2 > 4Al$, the general solution for the model (5.4.1) is a linear combination of two real, decaying exponential functions, with explicit form given by

$$h_S(t) = (C_1 \cosh(\bar{\omega}_d t) + \bar{C}_2 \sinh(\bar{\omega}_d t)) e^{-\delta t}, \quad (5.4.6)$$

where $\bar{C}_2 = \frac{h_0B + 2\dot{h}_0A}{\sqrt{|4Al - B^2|}}$ and $\bar{\omega}_d = \frac{\sqrt{|4Al - B^2|}}{2A}$.

5.4.2 Case $\mu > 0$

Considering $\mu > 0$ and applying Laplace transform to the equation (5.2.15), setting $\bar{B} := \sqrt{\mu}B$, and $\varepsilon := \frac{1}{\mu}$, we obtain after simplifications

$$[As^2 + \bar{B}s\sqrt{s + \varepsilon} + Cs + l] \hat{h}_S(s) = [As + \bar{B}\sqrt{s + \varepsilon} + C] h_0 + A\dot{h}_0, \quad (5.4.7)$$

Remark 5.4.2. *In the case $\varepsilon = 0$, which corresponds to an infinitely viscous fluid, the above equation is a Fractional Differential Equation (FDE) of order $1/2$.*

When $\varepsilon > 0$, this is a Generalized Fractional Differential Equation (GFDE), originally studied in [60]; to tackle this, we proceed in 4 steps:

1. perform a change of variables in order to work with polynomials,
2. decompose the rational functions of interest into simple elements,
3. apply the inverse Laplace transform, using Mittag-Leffler special functions of fractional calculus,
4. make use of the adapted algebraic stability criterion to get the asymptotic behaviour of the solution, and conclude to stability.

Change of variables

Let us denote $\sigma := \sqrt{s + \varepsilon}$, then the pseudo polynomials appearing in (5.4.7) can be equivalently transformed thanks to the algebraic relation $s = -\varepsilon + \sigma^2$.

$$\begin{aligned} n_0(s) &:= As + \bar{B}\sqrt{s + \varepsilon} + C \\ &= A\sigma^2 + \bar{B}\sigma + (C - \varepsilon A) := N_0(\sigma), \end{aligned} \quad (5.4.8)$$

$$\begin{aligned} d(s) &:= As^2 + \bar{B}s\sqrt{s + \varepsilon} + Cs + l, \\ &= A\sigma^4 + \bar{B}\sigma^3 + (C - 2\varepsilon A)\sigma^2 - \varepsilon\bar{B}\sigma + \varepsilon^2 A, \\ &:= P_T(\sigma). \end{aligned} \quad (5.4.9)$$

The viscous polynomial P_T is real valued, of degree 4, and has 4 complex roots, called λ_i , which can be found analytically in Appendix B, $P_T(\sigma) = A \prod_{i=1}^4 (\sigma - \lambda_i)$; alternatively, they can be computed numerically as in Section 5.4.3 to study their parametric dependence w.r.t. μ .

Remark 5.4.3. *One has to be careful with this change of variables. Indeed, as is usual with multivalued complex functions, a cut has to be performed first on the branch cut $(-\infty, -\varepsilon]$, then $\forall s \in \mathbb{C} \setminus (-\infty, -\varepsilon]$, $\exists! \sigma \in \mathbb{C}_0^+$, defined by $\sigma := \sqrt{s + \varepsilon}$, that is with positive real part. But care must be taken that a complex number σ with negative real part has no counterpart s in the Laplace plane $\mathbb{C} \setminus (-\infty, -\varepsilon]$ given by this relation.*

Decomposition into simple elements

From (5.4.7), we get

$$\begin{aligned}\hat{h}_S(s) &= \frac{n_0(s)}{d(s)} h_0 + \frac{A}{d(s)} \dot{h}_0, \\ &= \frac{N_0(\sigma)}{P_T(\sigma)} h_0 + \frac{A}{P_T(\sigma)} \dot{h}_0, \\ &= \left(\sum_{i=1}^4 \frac{r_i}{\sigma - \lambda_i} \right) h_0 + \left(\sum_{i=1}^4 \frac{\tilde{r}_i}{\sigma - \lambda_i} \right) \dot{h}_0.\end{aligned}$$

Each r_i and \tilde{r}_i are to the residues of the rational function of interest at the pole λ_i : they correspond either to the response to initial displacement h_0 , or to the response to initial velocity \dot{h}_0 . Their algebraic expression can be found in Appendix C.

Time-domain solution

The key issue here is to identify $\mathcal{L}^{-1} \left(\frac{1}{\sqrt{s + \varepsilon} - \lambda} \right)$ in some right-half plane to be determined later, for $\varepsilon \geq 0$ and $\lambda \in \mathbb{C}$. The easiest way to proceed is to use the shift theorem for Laplace transform, and identify the eigenfunctions of the fractional derivative operators, which are Mittag-Leffler functions.

Definition 5.4.4. Let us denote $\mathcal{E}_\alpha(\lambda, t)$ the function for which

$$\mathcal{L}[\mathcal{E}_\alpha(\lambda, \cdot)](s) = \frac{1}{s^\alpha - \lambda}, \quad \text{for } \Re(s) > a_\lambda. \quad (5.4.10)$$

This special function is related to the so-called two parametric Mittag-Leffler functions,

$$\mathcal{E}_\alpha(\lambda, t) := t^{\alpha-1} E_{\alpha, \alpha}(z = \lambda t^\alpha),$$

where we have used

Definition 5.4.5. The two-parametric Mittag-Leffler function is the complex-valued function defined by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (5.4.11)$$

where $\alpha > 0, \beta \in \mathbb{C}$ and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Euler Gamma function.

See for instance [68] or [62] for many useful properties of these functions.

Thanks to the shift theorem for Laplace transforms, we are now in a position to identify the useful elementary functions,

$$\mathcal{L}^{-1} \left(\frac{1}{\sqrt{s + \varepsilon} - \lambda} \right) = \exp(-\varepsilon t) \mathcal{E}_{\frac{1}{2}}(\lambda, t),$$

and state the following result in the time domain:

Theorem 5.4.6. *The solution of the GFDE (5.4.7) is given by*

$$h_S(t) = \exp(-\varepsilon t) \left(\sum_{i=1}^4 \Theta_i \mathcal{E}_{\frac{1}{2}}(\lambda_i, t) \right), \quad (5.4.12)$$

with constants $\Theta_i := r_i h_0 + \tilde{r}_i \dot{h}_0$.

Thanks to this explicit solution, we are now in a position to examine the asymptotic behaviour more in depth.

Asymptotic behaviour (general case)

Indeed, let us recall the following seminal results about the long time behaviour of the Mittag-Leffler functions:

Theorem 5.4.7 (Matignon 1996, [59]). *We have the following asymptotic equivalents for $\mathcal{E}_\alpha(\lambda, t)$ as t reaches $+\infty$:*

- for $|\arg(\lambda)| \leq \alpha \frac{\pi}{2}$,

$$\mathcal{E}_\alpha(\lambda, t) \sim \frac{1}{\alpha} \lambda^{\frac{1}{\alpha}-1} e^{\lambda^{\frac{1}{\alpha}} t}, \quad (5.4.13)$$

- for $|\arg(\lambda)| > \alpha \frac{\pi}{2}$,

$$\mathcal{E}_\alpha(\lambda, t) \sim \frac{\alpha}{\Gamma(1-\alpha)} \lambda^{-2} t^{-1-\alpha}, \quad (5.4.14)$$

which belongs to $L^r([1, +\infty), \mathbb{R})$, for all $r \geq 1$.

Recently, some higher order asymptotics have been provided to all sorts of Mittag-Leffler functions, see [69, Section 1.4].

For our purpose, the following asymptotics are needed:

Theorem 5.4.8 (Matignon 1998, [61]). *We have the following asymptotic equivalents for $\exp(-\varepsilon t) \mathcal{E}_{1/2}(\lambda, t)$ as t reaches $+\infty$:*

- for $|\arg(\lambda)| \leq \frac{\pi}{4}$,

$$e^{-\varepsilon t} \mathcal{E}_{1/2}(\lambda, t) \sim 2 \lambda \exp((\lambda^2 - \varepsilon) t), \quad (5.4.15)$$

- for $|\arg(\lambda)| > \frac{\pi}{4}$,

$$e^{-\varepsilon t} \mathcal{E}_{1/2}(\lambda, t) \sim \frac{1}{2\Gamma(1/2)} \lambda^{-2} t^{-\frac{3}{2}} \exp(-\varepsilon t). \quad (5.4.16)$$

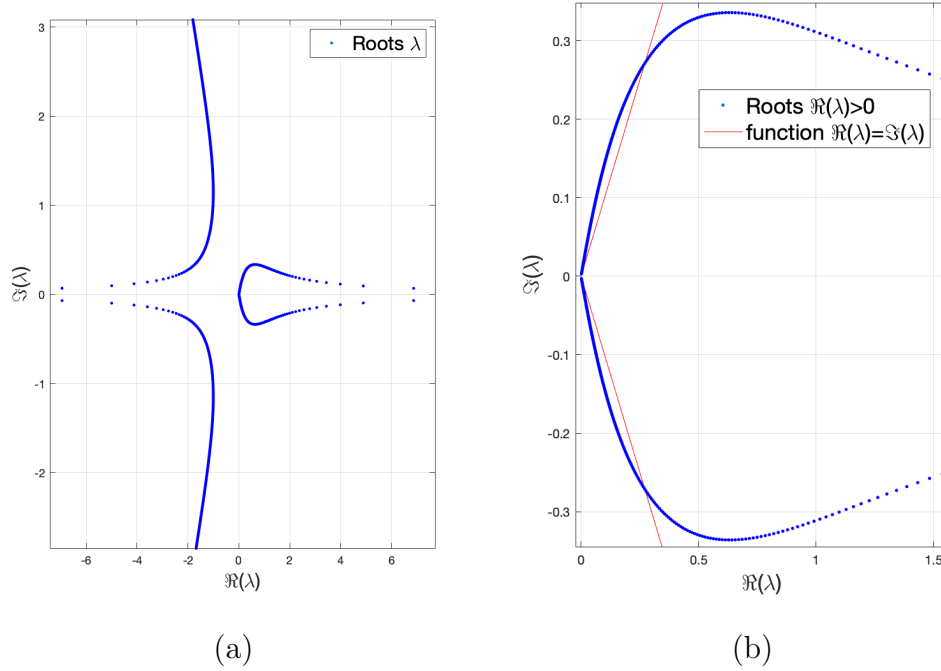


Figure 5.2 – Evolution of the four roots λ_i in the σ -plane, as a function of μ . (a): global picture with 4 trajectories. (b): zoom in the right-half plane $\Re(\sigma) > 0$, 2 trajectories crossing the segment $\Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

Indeed, with (5.4.16), the case of Proposition 5.3.2 is recovered as a special case, which occurs if and only if *all* the roots λ_i fulfill $|\arg(\lambda_i)| > \frac{\pi}{4}$.

Otherwise, if *but one* λ_0 lies in the sector $|\arg(\lambda)| < \frac{\pi}{4}$, then a very different asymptotic behaviour is to be found, namely a purely exponentially decaying one, with decay rate $\delta := \varepsilon - \Re(\lambda^2) > 0$ (it must be positive indeed, since asymptotic stability has already been proved in Proposition 5.3.1). To be more specific from a geometric viewpoint, by decomposing λ into its real and imaginary parts, the new zone of interests lies between the sector $\Re(\lambda) > |\Im(\lambda)|$ and the hyperbola $\Re(\lambda)^2 < \varepsilon + \Im(\lambda)^2$.

We are now in a position to state the general stability theorem:

Theorem 5.4.9. *For the solution (5.4.12) of the GFDE (5.4.7), for a given value of the viscosity μ , two cases may occur, depending of the location of the four roots λ_i of the viscous polynomial P_T :*

- *either there is at least one root with $\Re(\lambda_j) > |\Im(\lambda_j)|$ then the asymptotics is of exponential type, with rate $\delta(\mu) := \frac{1}{\mu} - \Re(\lambda^2) > 0$*

$$h_S(t) \sim \sum_j C_j \exp\left(\left(\lambda_j^2 - \frac{1}{\mu}\right) t\right), \quad (5.4.17)$$

- or all the four roots lie in $|\arg(\lambda)| > \frac{\pi}{4}$, then the asymptotics is of mixed type,

$$h_S(t) \sim C t^{-\frac{3}{2}} \exp\left(-\frac{1}{\mu} t\right). \quad (5.4.18)$$

Proof. Using the explicit solution (5.4.12), and the asymptotic results of Theorem 5.4.8 for one root λ_i , upon selecting between these roots, we obtain the desired asymptotic result. \square

5.4.3 Evolution of the asymptotic behaviour with viscosity

The goal of this last part is to provide numerical evidence that both situations stated by Theorem 5.4.9 may occur in practise. In particular, we shall illustrate the transition between the two possible regimes, as the viscosity μ of the fluid increases.

In Figure 5.2 the trajectory of the four roots λ_i is drawn as a function of μ in the σ -plane: two roots belong to the left half-plane and will have no counterpart in the Laplace plane; the two other roots belong to the right half-plane and will give rise to a pole in the Laplace plane; picture (b) provides a zoom on these two, which cross the segments $\Re(\lambda) = |\Im(\lambda)|$ for a critical value μ^c of the viscosity.

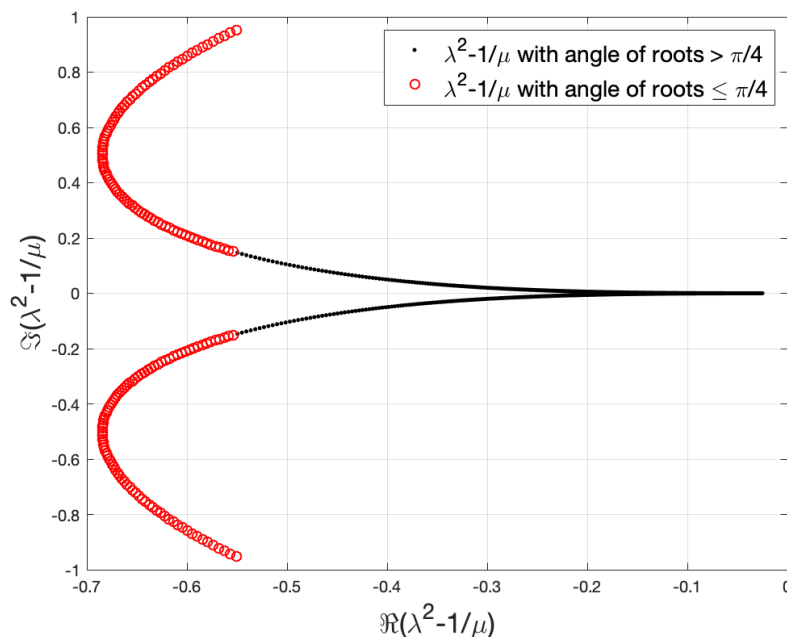


Figure 5.3 – Plot of the poles $s_j = \lambda_j^2 - \frac{1}{\mu}$ in the Laplace plane.

Figure 5.3 shows the 2 conjugate poles $s_j = \lambda_j^2 - \frac{1}{\mu}$ in the Laplace corresponding to the 2 roots $\lambda_{1,2}$: starting from the case $\mu = 0$, and increasing μ , there is some more damping up to some value μ^* , then the damping reduces monotonically towards 0.

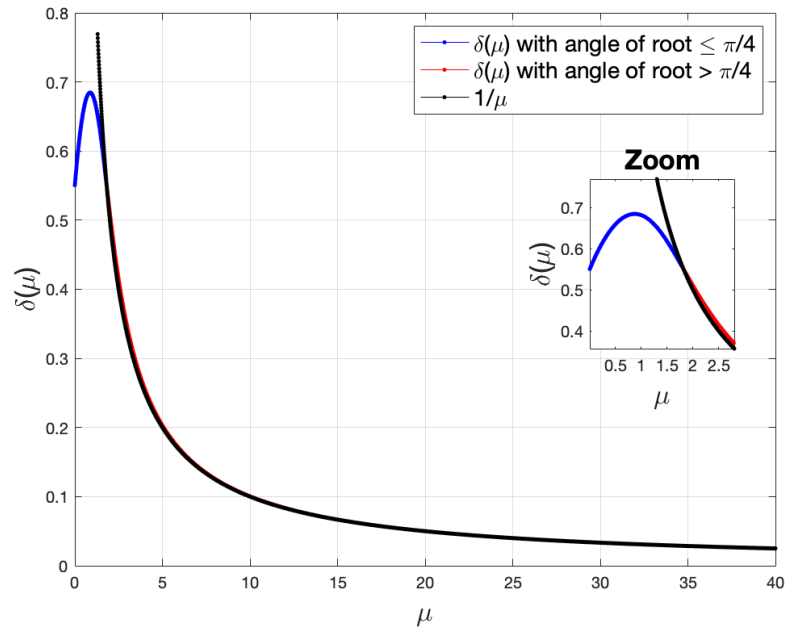


Figure 5.4 – Damping rate $\delta(\mu) = \Re(\lambda^2) - \frac{1}{\mu}$ as a function of the viscosity term μ

Figure 5.4 shows the damping rate $\delta(\mu)$ as a function of μ , as can be forecast from Figure 5.3. Note that above the critical value μ^c , $\delta(\mu) = 1/\mu$, meaning that we are in the mixed type regime. Indeed, the two roots $\lambda_{1,2}$ now fulfill $|\arg(\lambda_{1,2})| > \frac{\pi}{4}$

5.5 Conclusion

In this work, we have given an analytic solution and computed the refined asymptotic estimates of solution the return to the equilibrium problem of a model describing the vertical motion of a solid floating at the free surface of a viscous fluid with finite depth and flat bottom. Moreover, numerical evidence has been provided of the possible transition between different asymptotic regimes when the viscosity of the fluid increases.

Appendix

APPENDIX A

On a dual to the properties of Hurwitz polynomials

This appendix is based on [78].

A.1 Introduction

In this appendix we present the first part, of a series of three works, on a new approach about the classification of the roots of real polynomials in one variable in the right half complex plane. This new idea arises from the need to obtain simple explicit criteria for the area of the complex plane not covered by the theory of Hurwitz polynomials (also known as stable polynomials). In fact, our results are natural extensions of the classical Theorems of Routh-Hurwitz and Hermite-Biehler for the complement zone; $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$.

In the literature we highlight as main references for the study of roots of real polynomials on the left half complex plane and its applications to system theory in a general framework the books of Gantmacher [27] and [28], and the book of Iooss and Joseph [36]. Chappellat, Mansour and Brattacharyya present classic stability criteria with elementary demonstrations in their article [14] while new and interesting ideas about the demonstration of these results have been developed by Holtz in [34]. For a generalization to real polynomials in several variables we mention the work of Fettweis in [25]. The approach introduced in this work consist of a systematic use of the linear transformation $z \mapsto -z$, on the properties that define the Hurwitz polynomials, which leads us to use and explore the notion originally introduced by Vergara-Hermosilla et al. in [81] about anti-Hurwitz polynomials. This notion

can be recast as a dual result to the main necessary and sufficient conditions on stable polynomials. What is more, our Theorems and Propositions also depend on the coefficients of the polynomial in question which makes it more manipulable for applications in science and engineering. With this preamble,, we are in a position of establish the main result, which read as:

Proposition A.1. *Let $f(X) = a_0X^n + a_1X^{n-1} + \dots + a_{n-1}X + a_n \in \mathbb{R}[X]$ of degree ≥ 3 . Then $f(X)$ is an anti-Hurwitz polynomial, if and only if it satisfies the conditions:*

1. $(-1)^i a_i > 0$, for all $i \in \{0, \dots, n\}$.
2. $(-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i > 0$, for all $i \in \{1, \dots, n\}$.

The outline of the appendix is organized as follow. In Section A.2 we state the main definitions and properties that describe the Hurwitz polynomials emphasizing the Hurwitz matrix and the Theorem of Routh-Hurwitz. In Section A.3 we define the anti-Hurwitz polynomials, demonstrate our first main result, and establish explicit criteria for real polynomials of less than or equal order 4 and derivatives.

A.2 Hurwitz polynomials

For $n \in \mathbb{N}$ we denote by \mathcal{P}_n the set of all degree n polynomials with real coefficients.

Definition A.1. *A polynomial $f(X) \in \mathbb{R}[X]$ is Hurwitz if the real part of all its complex roots is negative i.e., $\Re(u) < 0$ for any $u \in \mathbb{C}$ satisfying $f(u) = 0$.*

Let \mathcal{H} denote the set of all Hurwitz polynomials, and we set $\mathcal{H}_n = \mathcal{H} \cap \mathcal{P}_n$. The set of all Hurwitz polynomials in \mathcal{H}_n with positive coefficients is denoted by \mathcal{H}_n^+ .

Theorem A.2 (Stodola condition). *If a polynomial $f(X) \in \mathbb{R}[X]$ is Hurwitz, then all its coefficients are of the same sign.*

Proof. The roots of a real polynomial are symmetric with respect to the real line. For $f(X)$, we can write

$$f(X) = a_0 \prod_k (X - s_k) \prod_j (X - \alpha_j - i\beta_j) \prod_j (X - \alpha_j + i\beta_j), \quad (\text{A.1})$$

where each s_k are real roots, and $\alpha_j \pm i\beta_j$ are complex roots of $f(X)$ with nonzero imaginary part. Note that s_j, α_j are negative. Since the expressions $(X - s_k)$ and $X^2 - 2\alpha_j X + (\alpha_j^2 + \beta_j^2)$ have positive coefficients, their product has the same property. \square

Let $f(X) = a_0X^n + \dots + a_{n-1}X + a_n \in \mathcal{P}_n$ be a polynomial. The *Hurwitz matrix* of a polynomial, denoted as $H(f(X))$, is the square matrix of size n defined as follows:

$$\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & a_3 & \cdots & 0 & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & a_n & 0 \\ 0 & 0 & 0 & \cdots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{pmatrix} \quad (\text{A.2})$$

For every $k \in \{1, \dots, n\}$, let $H_k(f(X))$ denote the square matrix of size k obtained from the first k rows and columns of $H(f(X))$, and we set:

$$\Delta_k = \det(H_k(f(X))), \quad (\text{A.3})$$

where $\det(H_k)$ denotes the determinant of the square matrix H_k .

Theorem A.3 (Routh-Hurwitz). *A polynomial $f(X) = a_0X^n + \dots + a_{n-1}X + a_n \in \mathcal{P}_n$ with $a_0 > 0$ is Hurwitz if and only if $\Delta_k > 0$ for all $k \in \{1, \dots, n\}$.*

For a proof of this result see for instance [27],[34] or [36].

A.3 Anti-Hurwitz Polynomials

In this section we establish the definition of anti-Hurwitz polynomials and a dual criterion to the Theorem of Routh-Hurwitz. To this end, we introduce the following definition.

Definition A.1. *A polynomial $f(X) \in \mathcal{P}_n$ is said to be anti-Hurwitz if the real part of all its complex roots is positive, i.e., $\Re(u) > 0$ for all $u \in \mathbb{C}$ satisfying $f(u) = 0$.*

Lemma A.2. *A polynomial $f(X) \in \mathcal{P}_n$ is anti-Hurwitz if and only if $f(-X)$ is Hurwitz.*

Proof. Let $f(X)$ be an anti-Hurwitz polynomial and u a complex root of $f(-X)$. Then $f(-u) = 0$ and $\Re(-u) > 0$, i.e., $\Re(u) < 0$. Therefore $f(-X)$ is Hurwitz. On the other hand, if $f(-X)$ is a Hurwitz polynomial and u a complex root of $f(X)$, then $f(u) = f(-(-u)) = 0$. In this case, $\Re(-u) < 0$, i.e., $\Re(u) > 0$. Hence, $f(X)$ is anti-Hurwitz. \square

Lemma A.3. *Let $f(X) \in \mathbb{R}[X]$ be a polynomial of degree n and Δ_i the determinant of the Hurwitz submatrix $H_i(f(X))$, for $1 \leq i \leq n$. Then we have*

$$\Delta_i^- = (-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i, \quad (\text{A.1})$$

where Δ_i^- is the determinant of i -th Hurwitz submatrix $H_i(f(-X))$.

Proof. The matrix for $H_i(f(-X))$ is written as

$$\begin{pmatrix} -a_1 & -a_3 & -a_5 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\ 0 & -a_1 & -a_3 & \cdots & 0 & 0 & 0 \\ 0 & a_0 & a_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -a_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & a_n & 0 \\ 0 & 0 & 0 & \cdots & -a_{n-3} & -a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{pmatrix}. \quad (\text{A.2})$$

Comparing it with the matrix of $H_i(f(X))$, we immediately see that

$$\Delta_i^- = (-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i. \quad (\text{A.3})$$

□

Proposition A.4. *Let $f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n \in \mathbb{R}[X]$ of degree ≥ 3 . Then $f(X)$ is an anti-Hurwitz polynomial, if and only if it satisfies the conditions:*

1. $(-1)^i a_i > 0$, for all $i \in \{0, \dots, n\}$.
2. $(-1)^{\lfloor \frac{i+1}{2} \rfloor} \Delta_i > 0$, for all $i \in \{1, \dots, n\}$.

Proof. By lemma (A.2), we know that $f(X)$ is an anti-Hurwitz polynomial if and only if $f(-X)$ is a Hurwitz polynomial. In this case, the coefficient of X^i in $f(-X)$ is $(-1)^i a_{n-i}$. Without loss of generality, we may suppose that $a_0 > 0$. Now the Stodola Condition (A.2) and Theorem (A.3), gives us that $(-1)^i a_{n-i} > 0$, for $i \in \{0, 1, \dots, n\}$ and $\Delta_i^- > 0$. Hence, we conclude by Lemma (A.3). □

In the following we establish simple criteria on the property of anti-Hurwitz, applicable to real polynomials of less than or equal order 4 and derivatives of polynomials. To this end we consider a polynomial $p(X) \in \mathbb{R}[X]$ and the necessary and sufficient conditions developed in the Proposition A.4. The criteria read as:

- The polynomial $p(X) = X^2 + a_1X + a_2$ is an anti-Hurwitz polynomial, if and only if

$$-a_1, a_2 > 0. \quad (\text{A.4})$$

- The polynomial $p(X) = X^3 + a_1X^2 + a_2X + a_3$ is an anti-Hurwitz polynomial, if and only if

$$-a_1, a_2, -a_3 > 0 \quad \text{and} \quad a_2 - a_1a_2 > 0. \quad (\text{A.5})$$

- The polynomial $p(X) = X^4 + a_1X^3 + a_2X^2 + a_3X + a_4$ is an anti-Hurwitz polynomial, if and only if

$$-a_1, a_2, -a_3, a_4 > 0 \quad \text{and} \quad a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0. \quad (\text{A.6})$$

- Let $p(X)$ be an anti-Hurwitz polynomial of degree n and let $P'(X)$ denote the first-order derivative of $p(X)$ with respect to X . Then $-p'(X)$ is again an anti-Hurwitz polynomial.

APPENDIX B

Analysis of the viscous polynomial

In this appendix we develop explicit formulas for the roots and its distribution in the complex plane of the so-called Viscous polynomial in the variable λ , which is given by

$$P_T(\lambda) = \left(1 + \frac{l^3}{12}\right) \lambda^4 + l^2 \sqrt{\mu} \lambda^3 + \left(l\mu - \frac{2}{\mu} \left(1 + \frac{l^3}{12}\right)\right) \lambda^2 - \frac{l^2}{\sqrt{\mu}} \lambda + \frac{1}{\mu^2} \left(1 + \frac{l^3}{12}\right), \quad (\text{B.1})$$

where l and μ are real and positive numbers.

B.1 Roots

By combine terms in the Viscous polynomial (B.1), we obtain an equivalent form given by

$$P_T(\lambda) = \left(1 + \frac{l^3}{12}\right) \left(\lambda^2 - \frac{1}{\mu}\right)^2 + l\sqrt{\mu}\lambda \left(l\lambda^2 + \sqrt{\mu}\lambda - \frac{l}{\mu}\right). \quad (\text{B.1})$$

Multiplying this by $1/\lambda^2$, denoting by $y = \lambda - \frac{1}{\lambda\mu}$ and suppose that $\lambda\mu \neq 0$, we conclude that solve the equation $P_T(\lambda) = 0$ is equivalent to solve

$$Q(y) = \left(1 + \frac{l^3}{12}\right) y^2 + l^2 \sqrt{\mu} y + l\mu = 0. \quad (\text{B.2})$$

This means that we can actually compute the roots via a nested sequence of two quadratic equations. In fact, the roots of $Q(y) = 0$ are given by

$$y_{1,2} = \frac{-l^2 \sqrt{\mu} \pm \sqrt{l\mu(l^3 - (1 + l^3/12))}}{2(1 + l^3/12)}, \quad (\text{B.3})$$

then, like $y_{1,2} = \frac{\lambda^2 \mu - 1}{\lambda \mu}$, we see that the roots of the equation (B.1) follows of solve

$$\mu \lambda^2 - y_{1,2} \mu \lambda - 1 = 0. \quad (\text{B.4})$$

Therefore, the explicit roots of eq. (B.1) are given by

$$\lambda_{1,2,3,4} = \frac{y_{1,2} \mu \pm \sqrt{y_{1,2}^2 \mu^2 + 4\mu}}{2\mu} \in \mathbb{C}. \quad (\text{B.5})$$

Remark B.1. *If $l \geq \sqrt[3]{6}$ then all the roots of Viscous polynomial are reals. In fact, the discriminant of the polynomial (B.2) is given by $\Delta_Q = l^4 \mu - 4l\mu(1 + (l^3/12))$. We see that Δ_Q is positive if and only if $l \geq \sqrt[3]{6}$. If the roots of equation (B.2) are reals, then the discriminant of equation (B.4) is positive, and hence the result follow.*

B.2 Distribution of roots

In this section, we study how the roots of the Viscous polynomial (B.1) are distributed on \mathbb{C} . To this end, we introduce the notion of anti-Hurwitz polynomial. Moreover, we denote by $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ the set of roots of the polynomial $P_T(\lambda)$ defined in eq. (5.4.9) and we consider the following set

$$\mathcal{L}_{1/2} = \{z \in \mathbb{C} : z \neq 0, |\arg z| \leq \pi/2\}.$$

Definition B.1. *A real polynomial $f(X)$ in the complex variable X is said to be Hurwitz if the real part of all its roots is negative, that is $\Re(u) < 0$ for all $u \in \mathbb{C}$ such that $f(u) = 0$.*

The following result attributed to A. Stodola (see for instance pp. 81 in [41]), is a well-known necessary condition for a real polynomial to be Hurwitz.

Theorem B.2 (Stodola condition). *If a polynomial with real coefficients is Hurwitz, then all its coefficients are of the same sign.*

Remark B.3. *Since there are different signs in the coefficients of the Viscous polynomial $p_T(\lambda)$, we conclude that it is not Hurwitz, i.e. $\Lambda \cap \mathcal{L}_{1/2} \neq \emptyset$.*

Definition B.4. *A real polynomial $f(X)$ in the complex variable X is a anti-Hurwitz polynomial if and only if, the real part of all its complex roots is positive, that is; $\Re(u) > 0$ for all $u \in \mathbb{C}$ such that $f(u) = 0$.*

Lemma B.5. *A real polynomial $f(X)$ is anti-Hurwitz if and only if $f(-X)$ is Hurwitz.*

Proof. If $f(X)$ is anti-Hurwitz and u is a complex root of $f(-X)$, since $f(-u) = 0$, we conclude that $\Re(-u) > 0$. Hence $\Re(u) < 0$ and therefore $f(-X)$ is Hurwitz. Similarly, if $f(-X)$ is Hurwitz and u is a root of $f(X)$, since $f(u) = f(-(-u)) = 0$ we conclude that $\Re(-u) < 0$. Hence $\Re(u) > 0$ and then $f(X)$ is an anti-Hurwitz polynomial. \square

Remark B.6. *Since there are different signs in the coefficients of the polynomial $p_T(-\lambda)$, by Lemma B.5 we conclude that $p_T(\lambda)$ is not anti-Hurwitz, i.e. $\Lambda \cap \mathbb{C} \setminus \mathcal{L}_{1/2} \neq \emptyset$.*

APPENDIX C

Residues

In this appendix, our aim is show the explicit form of each r_i and each \tilde{r}_i involved in Chapter 5, which correspond to the partial-fraction decomposition of the rational functions present in the equation (5.4.9). To this end, if Λ is the set of the four roots of the viscous polynomial P_T , with $\lambda_i \in \Lambda$ and $d(s) = A \prod_{i=1}^4 (\sqrt{s + \varepsilon} - \lambda_i)$, then

$$\frac{A}{P_T(\sigma)} = \sum_{i=1}^4 \frac{\tilde{r}_i}{(\sigma - \lambda_i)},$$

where

$$\tilde{r}_1 = - \left(\lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 + \lambda_1^2 \lambda_4 - \lambda_1^3 - \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_2 \lambda_4 - \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 \right)^{-1}, \quad (\text{C.1})$$

$$\tilde{r}_2 = -A \left(\lambda_1 \lambda_2^2 + \lambda_2^2 \lambda_3 + \lambda_2^2 \lambda_4 - \lambda_2^3 - \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 - \lambda_2 \lambda_3 \lambda_4 \right)^{-1}, \quad (\text{C.2})$$

$$\tilde{r}_3 = -A \left(\lambda_1 \lambda_3^2 + \lambda_2 \lambda_3^2 + \lambda_3^2 \lambda_4 - \lambda_3^3 - \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 - \lambda_1 \lambda_3 \lambda_4 - \lambda_2 \lambda_3 \lambda_4 \right)^{-1}, \quad (\text{C.3})$$

$$\tilde{r}_4 = -A \left(\lambda_1 \lambda_4^2 + \lambda_2 \lambda_4^2 + \lambda_3 \lambda_4^2 - \lambda_4^3 + \lambda_1 \lambda_2 \lambda_3 - \lambda_1 \lambda_2 \lambda_4 - \lambda_1 \lambda_3 \lambda_4 - \lambda_2 \lambda_3 \lambda_4 \right)^{-1}. \quad (\text{C.4})$$

Moreover,

$$\frac{A\sigma^2 + B\mu^{1/2}\sigma + (C - A/\mu)}{P_T(\sigma)} = \sum_{i=1}^4 \frac{r_i}{(\sigma - \lambda_i)},$$

where

$$r_1 = (\mu A)^{-1}(C\mu - A + \lambda_1^2 A\mu + \lambda_1 B\mu^{3/2})\tilde{r}_1^{-1}, \quad (\text{C.5})$$

$$r_2 = A\mu^{-1}(C\mu - A + \lambda_2^2 A\mu + \lambda_2 B\mu^{3/2})\tilde{r}_2^{-1}, \quad (\text{C.6})$$

$$r_3 = A\mu^{-1}(C\mu - A + \lambda_3^2 A\mu + \lambda_3 B\mu^{3/2})\tilde{r}_3^{-1}, \quad (\text{C.7})$$

$$r_4 = A\mu^{-1}(C\mu - A + \lambda_4^2 A\mu + \lambda_4 B\mu^{3/2})\tilde{r}_4^{-1}. \quad (\text{C.8})$$

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