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THÈSE

Pour obtenir le diplôme de doctorat

Spécialité SCIENCES ECONOMIQUES

Préparée au sein de l'Université de Caen Normandie

Essais sur la répartition des gains dans un jeu coopératif : indice d'interaction, valeur de Shapley-Owen, valeur de Myerson

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**Thèse soutenue le 30/11/2021
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UNIVERSITÉ
CAEN
NORMANDIE



**Essais sur la répartition des gains dans un jeu
coopératif : indice d'interaction, valeur de
Shapley-Owen, valeur de Myerson**

Thèse de Doctorat : Sciences Economiques

De

Rodrigue TIDO TAKENG

Sous la direction de :

Sébastien COURTIN

30 novembre 2021

Par la présente note, je déclare qu'en dehors des auteurs et documents cités dans cette thèse,
le reste du travail est l'œuvre de l'esprit.

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✠ Résumé Général ✠

UN jeu coopératif à utilité transférable (jeu TU) représente toute situation où des agents en interaction ont la possibilité de communiquer librement et de passer des accords contraignants de coopération pour un gain collectif à redistribuer. Il existe une littérature très riche sur les jeux TU et leurs extensions. Dans cette thèse, nous proposons plusieurs généralisations de différents résultats de cette littérature. L'un des problèmes fondamentaux des jeux TU est celui de la répartition des gains issus de la coopération entre les agents. La contribution marginale pure (l'effet marginal en terme de gain lorsque chaque agent rejoint l'ensemble des autres agents) semble être la méthode naturelle permettant de répartir les gains entre les agents. Casajus and Huettner [2018] critiquent cette approche et illustrent leurs propos *via* une décomposition de la contribution marginale pure à partir de la valeur de Shapley [1953]. De manière similaire Chantreuil et al. [2019] proposent une décomposition de la valeur de Shapley centrée sur la contribution marginale pure dans le cadre de la théorie des inégalités. Ces deux résultats sont les points de départ de mes travaux de thèse.

Dans le chapitre 1, nous généralisons ces différentes approches en utilisant les indices d'interaction (mesures permettant de capter l'interaction qui existe entre les agents). Par l'approche axiomatique, on définit une famille d'indices d'interaction décomposables. Un exemple illustratif en aide à la décision multi-critère est proposé.

Dans le chapitre 2, nous étendons l'approche utilisée par Chantreuil et al. [2019] sur des jeux TU avec structure de coalitions. On obtient ainsi trois décompositions de la valeur de Shapley-Owen (Owen [1977]) ; ce qui permet par la suite d'en déduire trois décompositions des indices d'inégalités. Un exemple d'application en théorie des inégalités est proposé.

Notre troisième chapitre est consacré à l'axiomatisation de la valeur de Shapley-Owen (Owen [1977]). Casajus [2021] définit les axiomes du premier ordre comme étant les axiomes énoncés à partir de la contribution marginale des agents. Il définit un axiome du second ordre

comme étant un axiome énoncé à partir d'une double contribution marginale (contribution marginale d'un agent i à la contribution marginale d'un autre agent j quand i quitte le jeu). Suivant cette approche, les axiomes utilisés par Young [1985] pour l'axiomatisation de la valeur de Shapley ([Shapley, 1953]) sont des axiomes du premier ordre. Casajus [2021] propose donc un résultat similaire avec les axiomes du second ordre. Nous montrons qu'il existe une relation entre les axiomes de Casajus [2021] et ceux de Young [1985]. On déduit que l'axiomatisation de Casajus [2021] est une seconde version de celle de Young [1985]. Nous proposons par la suite une seconde version de l'axiomatisation de la valeur de Shapley-Owen proposée par Khmelnitskaya and Yanovskaya [2007].

Dans notre quatrième et dernier chapitre, nous nous intéressons au problème de l'évaluation de la productivité d'un travailleur et de l'organisation des travailleurs dans un environnement de production incertain. Pongou and Tondji [2018] ont proposé quelques solutions sur l'évaluation de la productivité d'un travailleur dans un tel environnement. Nous ajoutons à ce modèle une structure de communication, c'est-à-dire que les travailleurs peuvent s'organiser suivant un réseau, et nous évaluons la productivité de chaque travailleur.

Mots clés : Jeux TU, théorie des inégalités, production et organisation, indices d'interaction, axiomatisation, décomposition

✧ General Abstract ✧

A cooperative game with transferable utility (TU game) represents any situation where interacting agents have the possibility to communicate freely and make cooperative agreements to share a collective gain. There is a very rich literature on TU games and their extensions. In this thesis, we generalize several results of this literature. One of the fundamental problems of TU games is the sharing of the gains from cooperation between agents. The pure marginal contribution method (the marginal effect in terms of gain when each agent joins the other agents) seems to be the natural way to share such gains among agents. Casajus and Huettner [2018] criticize this approach and illustrate their point of view via a decomposition of the pure marginal contribution from the [Shapley, 1953] value. Similarly Chantreuil et al. [2019] suggest a decomposition of the Shapley value centered on the pure marginal contribution applied to the analysis of inequality. These two results are the starting point of the work in my thesis.

In Chapter 1, we generalize these different approaches using interaction indices. By the axiomatic approach, we define a family of decomposable interaction indices, and build an example in the field of multi-criteria aid for decision.

In the second chapter, we extend the approach used by Chantreuil et al. [2019] on TU games enriched with a coalition structure. We thus obtain three decompositions of the Shapley-Owen value (Owen [1977]). This subsequently allows us to deduce three decompositions of the inequality indices. We then apply our work to inequality theory.

Our third chapter is devoted to the axiomatization of the Shapley-Owen value (Owen [1977]). Casajus [2021] defines first-order axioms as axioms based upon the marginal contribution. He defines a second-order axiom as an axiom based upon the double marginal contribution (marginal contribution of an agent i to the marginal contribution of another agent j when i leaves the game). Following this approach, the axioms used by Young [1985] for the axiomatization of the

Shapley [1953] value are first-order axioms. Casajus [2021] therefore offers a similar result with second-order axioms. We show that there is a relation between the axioms of Casajus [2021] and those of Young [1985]. We deduce that the axiomatization of Casajus [2021] is a second version of that of Young [1985]. Moreover, we propose a second version of the Shapley-Owen value axiomatization proposed by Khmelnitskaya and Yanovskaya [2007].

In our fourth and last chapter, we focus on the problem of evaluating the productivity of a worker and the organization of workers in an uncertain production environment. Pongou and Tondji [2018] proposed some solutions concerning the evaluation of the productivity of a worker in such an environment. We add to this model a communication structure, that is, where workers can organize themselves into a network, and we evaluate the productivity of each worker.

Key words : TU games, inequality theory, production and organization, interaction indice, axiomatization, decomposition

✧ Introduction Générale ✧

0.1. Théorie des jeux coopératifs

DANS la vie courante, plusieurs situations mettent en interaction des agents (le terme agent est pris ici au sens large. Un agent peut être un travailleur, un parti politique, un votant, une entreprise, un pays, etc) qui sont chacun soucieux de leur bien être individuel (le bien être d'un travailleur peut être son salaire, celui d'un parti politique peut être le pouvoir, etc.). L'action optimale d'un individu dépend des anticipations qu'il forme sur la décision d'un autre. L'objet de la théorie de jeux est d'analyser et de formaliser de telles situations. La théorie de jeux est un outil d'analyse et de modélisation en biologie, informatique, sport et en sciences sociales (économie, sciences politiques, relations internationales, etc). Elle est subdivisée en deux grandes familles : la théorie de jeux coopératifs et la théorie de jeux non coopératifs. Dans un jeu coopératif, les individus ont la possibilité de passer entre eux des accords contraignants et conditionnés par un gain prévu par les règles du jeu. Dans un jeu non coopératif, les individus peuvent coopérer mais les accords passés ne sont pas contraignants. Toutes les options stratégiques sont proposées et les promesses n'engagent que ceux qui y croient.

Cette thèse s'inscrit dans le cadre de la théorie des jeux coopératifs, qui fait l'objet d'une vaste littérature. Sans toutefois être exhaustif, on peut citer Von Neumann and Morgenstern [1944], Shapley [1953], Harsanyi [1959b], Shubik [1962], Owen [1972, 1975], Aumann and Dreze [1974], Dubey et al. [1981], Peleg and Sudhölter [2007], etc. Un jeu coopératif peut être formalisé par un jeu à utilité transférable (jeu TU). Un jeu TU est la donnée du résultat de chaque coopération possible, c'est-à-dire, la donnée d'un ensemble fini d'éléments appelés joueurs et d'une fonction caractéristique qui détermine le gain de chaque coalition (ensemble de joueurs). En politique un joueur peut être un parti politique, un votant, un électeur ; en relations

internationales un joueur peut être un territoire, un pays, une organisation, un continent ; en biologie (plus précisément l'épidémiologie) un joueur peut être un facteur de risque ; en théorie des inégalités un joueur peut être une source d'inégalité ; en production et organisation industrielle un joueur peut être un travailleur ; en aide à la décision multi-critère un joueur peut être un critère.

0.2. Répartition des gains

La problématique de la théorie des jeux coopératifs est celle de la formation des coalitions et de la répartition des gains (ou des coûts) issus de la coopération. On peut citer par exemple le problème d'allocation des coûts pour la construction des biens publics (point d'adduction d'eau potable, source de production d'électricité,...) au voisinage des communes. Une solution consiste à associer à chaque jeu TU un ensemble de partages possibles découlant chacun d'un regroupement possible. C'est-à-dire un ensemble de techniques de répartition des gains (ou des coûts) en fonction de la manière dont les joueurs se sont regroupés. Si les joueurs décident de se regrouper tous ensemble, un partage peut être par exemple la répartition égalitaire. Deux concepts de solutions ont été proposés dans la littérature : les solutions ponctuelles (l'ensemble de partages est unique pour tout jeu TU) et les solutions ensemblistes (l'ensemble de partages n'est pas unique pour au moins un jeu TU). Plusieurs auteurs à l'instar de Gillies [1953], Aumann and Dreze [1974] ont travaillé sur des solutions ensemblistes.

Dans cette thèse si, nous abordons brièvement les solutions ensemblistes ; nous nous intéressons beaucoup plus aux solutions ponctuelles et ses extensions.

0.2.1. Solution ponctuelle

Une solution ponctuelle est encore appelée valeur. Une valeur peut être par exemple une façon de répartir un gâteau entre des individus. Sous l'hypothèse que les joueurs se regroupent suivant la grande coalition (l'ensemble de tous les joueurs), plusieurs valeurs ont été définies dans la littérature. On peut citer entre autres la valeur de Shapley [1953], la valeur de Banzhaf (Owen,1975), la valeur de solidarité (Nowak and Radzik [1994]), le partage égalitaire et la solution naïve (Casajus and Huettner [2018]) aussi appelé contribution marginale pure (Chantreuil et al. [2019]). La valeur de Shapley et celle de Banzhaf sont les plus connues de la littérature et sont considérées comme des valeurs classiques. Ces deux valeurs sont calculées à partir de la contribution marginale de chaque joueur, c'est-à-dire l'effet observé en terme de gain lorsque

ledit joueur rejoint une coalition donnée. La valeur de Shapley d'un joueur dans un jeu TU est l'espérance mathématique de ses contributions marginales. La valeur de Banzhaf d'un joueur dans un jeu TU est la moyenne arithmétique de ses contributions marginales.

0.2.2. Indices d'interaction

Plusieurs généralisations de la valeur de Shapley et de Banzhaf ont été proposées dans une littérature dont les applications sont beaucoup plus orientées vers l'informatique (aide à la décision multi-critère, intelligence artificielle, réseau, etc), la statistique et l'analyse de données, l'économie (théorie des inégalités, etc) ou encore le sport. Intéressons nous par exemple à l'actualité sportive. Considérons l'équipe du Paris Saint-Germain (PSG) de la saison sportive 2021-2022. On peut se poser les questions suivantes : quelle peut être l'importance de Messi (Me) au sein du PSG ? Comment mesurer l'importance de Messi (Me) et de Mbappé (Mb) ensemble ou encore l'importance de Messi (Me), de Mbappé (Mb) et de Neymar (Ne) tous les trois au sein du PSG ?

Soit v une fonction caractéristique évaluant la performance de chaque groupe de joueurs. Par exemple, $v(\{Me\})$ est la performance individuelle que Messi apporte au sein du PSG. Si $v(\{Me, Mb\}) \geq v(\{Me\}) + v(\{Mb\})$ alors on dira que qu'il existe une relation positive entre Messi et Mbappé. Ces deux attaquants peuvent travailler ensemble sur le plan offensif. Si $v(\{Me, Mb\}) \leq v(\{Me\}) + v(\{Mb\})$ alors on dira plutôt qu'il existe une relation négative entre ces deux joueurs. Ils n'ont pas intérêt à travailler ensemble. Par contre, si $v(\{Me, Mb\}) = v(\{Me\}) + v(\{Mb\})$ alors faire jouer ces deux joueurs ensemble ou séparément ne change rien. On dira qu'il n'existe pas de relation entre ces deux joueurs. Cette relation entre ces deux joueurs est appelée interaction. L'interaction entre Messi et Mbappé peut être conceptualisée par l'expression suivante : $v(\{Me, Mb\}) - v(\{Me\}) - v(\{Mb\})$. La présence d'un autre joueur du PSG (Neymar par exemple) modifie nécessairement l'interaction entre Messi et Mbappé. Owen [1972] propose donc une définition plus générale de l'interaction comme suit : $[v(T \cup \{Me, Mb\}) - v(T \cup \{Me\})] - [v(T \cup \{Mb\}) - v(T)]$ où T est un groupe quelconque des autres joueurs.

Murofushi and Soneda [1993a] interprètent cette expression comme étant l'interaction entre Messi et Mbappé en présence des autres joueurs du groupe T . Grabisch et al. [2000b] appellent cette expression l'interaction marginale entre Messi et Mbappé en présence de T . Selon Casajus [2021], cette expression décrit de combien doit varier la contribution marginale de Mbappé à

un groupe de joueurs T lorsque Messi décide de rejoindre ce groupe. Une généralisation de la notion d'interaction entre plusieurs joueurs a par ailleurs été suggérée par Grabisch [1997]. La notion d'interaction peut être contextualisée dans plusieurs autres situations. L'exemple avec l'équipe du PSG est juste une illustration pour mieux appréhender la notion d'interaction.

L'importance d'un ou de plusieurs joueurs peut être captée à partir d'un indice d'interaction. Un indice d'interaction associe à chaque coalition d'un jeu TU un nombre réel correspondant à l'importance de cette coalition dans ledit jeu TU. Autrement dit, un indice d'interaction mesure le phénomène d'interaction entre les joueurs. Nous pouvons constater qu'une valeur ne permet pas de déterminer l'importance d'une coalition contenant plus de deux joueurs. Ces valeurs ont donc été généralisées pour donner lieu aux indices d'interaction. Selon Owen [1972], un indice mesurant l'interaction entre deux joueurs doit dépendre de l'interaction marginale entre ces deux joueurs en présence des autres. Il propose donc un indice mesurant l'interaction entre deux joueurs. Vingt neuf ans plus tard, Grabisch [1997] étend cet indice aux coalitions contenant plus de deux joueurs. Cet indice sera appelé l'indice d'interaction de Shapley. Il coïncide avec la valeur de Shapley lorsque la coalition se réduit à un singleton, c'est-à-dire, lorsque l'on n'étudie que l'importance d'un seul joueur. D'autres indices généralisant la valeur de Shapley et celle de Banzhaf tels que le "Chaining interaction index" et l'indice d'interaction de Banzhaf vont par la suite être proposés respectivement par Marichal and Roubens [1999] and Roubens [1996]. De nombreux auteurs, à l'instar de Grabisch and Roubens [1999a] et Katsushige et al. [2006], ont généralisé les semi-valeurs sur les coalitions. Ces semi-valeurs généralisées sont appelées "Cardinal-probabilistic interaction indices" (Grabisch and Roubens [1999a]) car un "Cardinal-probabilistic interaction index" d'une coalition donnée s'écrit comme une somme pondérée des interactions entre les joueurs de ladite coalition ; cette famille de poids est une distribution de probabilité dépendant uniquement du cardinal de la coalition et du nombre total de joueurs. L'indice d'interaction de Shapley, l'indice d'interaction de Banzhaf et le "chaining interaction index" sont quelques exemples de "Cardinal-probabilistic interaction indices" classiques.

0.2.3. Axiomatisation

Compte tenu de l'existence d'une multitude de valeurs dans la littérature, il se pose la question du choix d'une "bonne" valeur dans un processus de répartition du gain. L'approche communément adoptée dans la littérature est l'approche normative. Suivant cette approche des axiomes (ou propriétés) sont définis et ensuite les valeurs sont passées en revue à l'autre de ces

axiomes. Ceci conduit au problème d'axiomatisation. Axiomatiser ou caractériser une valeur, c'est déterminer un ensemble d'axiomes ou de propriétés que seule cette valeur vérifie. Plusieurs auteurs se sont intéressés à l'axiomatisation de la valeur de Shapley. On peut citer entre autres Shapley [1953], Shubik [1962], Roth [1977], Myerson [1980] et Young [1985]. L'axiomatisation de la valeur de Banzhaf a aussi intéressé un bon nombre d'auteurs dans la littérature. On peut citer notamment Lehrer [1988], Haller [1994] et Feltkamp [1995]. La valeur de Shapley et celle de Banzhaf appartiennent à une famille plus générale appelée semi-valeur (Dubey et al. [1981]). La semi-valeur d'un joueur dans un jeu TU est la somme pondérée des contributions marginales dudit joueur, la famille de poids utilisée pour la pondération étant une distribution de probabilité. De nombreux auteurs, à l'instar de Grabisch and Roubens [1999a] et Katsushige et al. [2006], se sont intéressés à l'axiomatisation des indices d'interaction.

0.3. Modèles plus généraux que celui des jeux TU

Dans plusieurs situations pratiques, l'ensemble de coalitions possibles est limité par une structure (de coalitions, de communication, technique, etc). C'est le cas par exemple des relations internationales entre pays qui appartiennent à des regroupements tels que l'Union Européenne, l'Alliance des pays de l'Amérique du Nord, etc. Certains pays vont coopérer plus facilement ensemble que d'autres. Il y'a donc une structure de coalitions entre ces pays. On peut aussi voir le cas d'un parlement où les députés appartiennent à des groupes parlementaires différents.

0.3.1. Les jeux TU avec structure de coalitions

On constate que dans la classe des jeux TU, il n'existe pas de restriction de communication entre les joueurs. Chaque joueur peut coopérer avec n'importe quel autre joueur. Aumann and Dreze [1974] introduisent les structures de coalitions comme étant une partition de l'ensemble de joueurs. Ces structures de coalitions décrivent donc les différentes manières de se regrouper, de communiquer. Owen [1977] suggère une extension de la valeur de Shapley dans la classe des jeux TU avec structure de coalitions appelée la valeur de Shapley-Owen et ensuite propose une axiomatisation. Plusieurs autres caractérisations vont par la suite être proposées (Hart and Kurz [1983b], Hamiache [1999, 2001], Khmelnitskaya and Yanovskaya [2007], etc). L'extension de la valeur de Banzhaf dans la classe des jeux TU avec structure de coalitions est aussi suggérée par Owen [1981] et appelée valeur de Banzhaf-Owen. Une modification de cette valeur va être

proposée par Alonso-Meijide and Fiestras-Janeiro [2002b] et ensuite plusieurs axiomatisation vont être données. Les semi-valeurs vont elles aussi être étendues dans la classe des jeux TU avec structure de coalitions par Albizuri and Zarzuelo [2004] et Albizuri [2009].

0.3.2. Les jeux TU avec structure de communication

De manière générale, la structure de coalition n'est pas une relation de communication transitive. Autrement dit, il existe des situations dans lesquelles un joueur i peut être en communication avec un joueur j et j en communication avec un joueur k alors que i et k ne sont pas en communication. C'est le cas des coopérations bilatérales entre pays appartenant chacun à un regroupement différent.

Myerson [1977] introduit une structure de communication à partir d'un graphe indirect dont les sommets sont vus comme des joueurs et les liens décrivent l'ensemble des communications bilatérales possible entre les joueurs. Myerson [1977] associe au jeux TU cette structure de communication et obtient donc un nouveau modèle appelé jeux TU avec structure de communication. Dans ce modèle, il propose une extension de la valeur de Shapley appelé valeur de Myerson. Il définit la valeur de Myerson comme étant la valeur de Shapley d'un jeu modifié prenant en compte les communications entre les joueurs. Une axiomatisation de cette valeur est proposée par la suite à l'aide de deux axiomes intuitifs. Plusieurs auteurs vont par la suite proposer d'autres axiomatisations de la valeur de Myerson. On peut citer entre autre Myerson [1980], Borm et al. [1992], Béal et al. [2015], etc. La valeur de Banzhaf va être elle aussi étendue à des jeux TU avec structure de communication par Owen [1986]. Plusieurs axiomatisations vont être proposées par Alonso-Meijide and Fiestras-Janeiro [2006].

De manière générale, la valeur de Myerson n'est pas individuellement rationnelle, c'est-à-dire qu'il existe des situations où la valeur de Myerson d'un joueur n'est pas supérieure à son gain individuel. Amer and Carreras [1995] proposent une révision du jeu modifié défini par Myerson [1977] et obtiennent une nouvelle valeur appelée valeur de Myerson individuellement rationnelle ou valeur de Myerson^R. Cette valeur est une extension de la valeur de Myerson et les deux coïncident lorsque le jeu TU est sur-additif (il est plus profitable de se regrouper collectivement que d'être séparé).

0.3.3. Les jeux TU avec plusieurs niveaux de participation (fuzzy cooperative game theory)

Dans la classe des jeux TU, chaque joueur rejoint une coalition avec un niveau de participation égal à 1 et participe avec un niveau 0 lorsqu'il est absent. Aubin [1981] montre que dans la réalité, certains joueurs ne participent pas pleinement. Ce qui l'amène à introduire plus tard les jeux TU avec plusieurs niveaux de participation. Chaque niveau de participation appartient à un intervalle borné $[0, 1]$. Ces niveaux de participation sont des variables continues. Plusieurs auteurs tels que Butnariu [1980], Tsurumi et al. [2001], Jiménez-Losada et al. [2010], Gallego et al. [2014] s'intéressent à cette littérature. La valeur de Shapley et celle de Banzhaf sont définies dans ce nouveau modèle et par la suite des axiomatisations sont proposées. Une structure de communication est définie sur ce modèle et une extension de la valeur de Myerson est proposée par Genjiu et al. [2017].

Les jeux TU sont appliqués à plusieurs situations de la vie.

0.4. Quelques applications des jeux TU

0.4.1. Inégalités

Inégalité signifie tout simplement le caractère de tout ce qui n'est pas égal. Une inégalité peut être observée au niveau des revenus au sein d'une population, au niveau des territoires, du sexe, etc. L'inégalité ne doit pas toujours être liée à la notion d'injustice. Le revenu d'un individu peut être un patrimoine, une combinaison de plusieurs sources (travail, allocation sociale, capital,...), etc. L'inégalité totale du revenu est captée à l'aide d'un outil appelé indice d'inégalité. Plusieurs indices ont été définis dans la littérature. On peut citer entre autre la variance (Fisher [1918]), l'indice de Gini [1921], l'indice d'Atkinson [1970], l'indice de Theil [1967], etc. L'indice le plus utilisé dans la littérature est celui de Gini [1921] qui associe à chaque distribution de revenu un réel compris entre 0 et 1. 0 signifie que la population est égalitaire, plus l'inégalité est proche de 1 plus la société est inégalitaire et 1 signifie qu'un seul individu détient toute la richesse.

La réduction des inégalités demeure une question d'actualité : comment aider un décideur public à construire des politiques publiques réduisant les inégalités de revenus ? Considérons une société où la rémunération de chaque individu provient de trois sources : revenu du travail, allocation sociale et revenu du capital. Dans une telle société supposons que les individus les plus riches ont un grand revenu du capital, un grand revenu du travail et une faible allocation

sociale. Les individus de classe modeste ont un revenu de travail moyen, un revenu du capital moyen et une allocation sociale moyenne. Les individus pauvres ont un faible revenu du travail, un faible revenu du capital et une allocation sociale moyenne. Le décideur public a plusieurs leviers sous la main. Il peut décider imposer le revenu du capital et d'augmenter l'allocation sociale, ou encore imposer le revenu du capital et d'augmenter le revenu du travail, etc. Il devient primordial pour le décideur de connaître la part d'inégalité attribuable à chaque source ou groupe de sources de revenus. D'où la nécessité de décomposer les indices d'inégalité.

Trois approches sont proposées dans la littérature pour la décomposition des indices d'inégalité. La première approche consiste soit à des transformations mathématiques pour aboutir à une décomposition, soit à l'axiomatisation *via* des propriétés normatives parmi lesquelles certaines propriétés donnent une idée intuitive d'une bonne décomposition. Plusieurs auteurs à l'instar de Dagum [1997], Chameni [2006b], Mornet et al. [2013] et Shorrocks [1982a], ont travaillé suivant cette approche. La deuxième approche consiste à décomposer les indices à l'aide des outils statistiques, il s'agit ici de l'économétrie. Certains auteurs comme Lerman and Yitzhaki [1985] et Firpo et al. [2018] ont travaillé dans ce sens là. La dernière approche consiste à l'utilisation des outils des jeux TU. Chantreuil et al. [2019] utilisent cette dernière approche pour décomposer les indices d'inégalité car elle est valable pour tout indice d'inégalité. Un jeu d'inégalité est ainsi défini. Dans cette littérature les joueurs sont les sources d'inégalités. Pour un indice d'inégalité donné, la fonction caractéristique attribue l'inégalité produite par un groupe de sources lorsque les autres sources sont neutralisées. Un jeu d'inégalité diffère d'un autre à partir de la méthodologie utilisée pour neutraliser une ou plusieurs sources d'inégalité. Par exemple dans un jeu nul un groupe de sources est neutralisé en affectant un revenu nul à ces sources. Dans un jeu égalitaire un groupe de sources est neutralisé en redistribuant de façon égalitaire leur revenus aux autres sources.

L'importance d'une source est la part d'inégalité qu'on peut attribuer à cette source dans un jeu d'inégalité. Une méthode naturelle permettant de mesurer l'importance d'une source est de supprimer cette source de l'ensemble des sources et de mesurer son effet. On obtient ainsi la contribution marginale pure de ladite source dans un jeu d'inégalité. Il s'avère que cette façon de calculer l'importance d'une source ne prend pas en compte l'inégalité globale. Ceci est dû à l'existence des interactions entre les sources de revenu. La décomposition de la valeur de Shapley proposée par Chantreuil et al. [2019] leur permet d'en déduire une décomposition des indices d'inégalité car la valeur de Shapley est efficiente (c'est-à-dire que la somme des parts

des inégalités attribuables à chaque source est égale à l'inégalité totale). Maintenant supposons que la distribution de revenu est hiérarchisée. Par exemple imaginons une distribution de revenu provenant de trois sources principales appelées sources primaires : le revenu du travail, l'allocation sociale et le revenu du capital. On peut décider d'aller à un niveau encore plus fin. Le revenu du travail peut être subdivisé en deux sources secondaires : le salaire et la prime. L'allocation sociale peut être subdivisée en deux : l'allocation familiale et l'allocation de logement. Et enfin le revenu du capital peut être lui aussi subdivisé en deux : le revenu du capital financier et le revenu du capital immobilier. On parle donc de revenu hiérarchisé car chaque source secondaire est liée à une source primaire.

Il devient important pour nous de décomposer les indices d'inégalité à partir de la valeur de Shapley-Owen car elle prend en compte la structure hiérarchique.

0.4.2. Processus de décision

Shapley and Shubik [1954] utilisent un jeu TU comme un outil important pour étudier la problématique de la mesure du pouvoir réel d'un individu dans un processus de décision. Ils définissent donc un jeu TU particulier appelé jeu simple. Il s'agit des votes "pour" ou "contre". Dans ce vote, on dispose d'une assemblée qui va régulièrement prendre des décisions. Lorsqu'un projet est soumis à cette assemblée, chaque membre a le choix de voter pour ou contre et la décision collective est soit le rejet, soit l'adoption. Plusieurs généralisations de cette classe de jeux ont été proposées dans la littérature. On peut citer entre autres : les jeux simples avec abstention (il s'agit des votes "pour" ou "abstention" ou "contre" et la décision collective est soit le rejet, soit l'adoption) ; les jeux à plusieurs niveaux d'approbation (dans ce modèle, on a plusieurs alternatives ordonnées à l'entrée et plusieurs alternatives ordonnées à la sortie. C'est l'exemple du parlement Belge : un joueur peut "être présent et voter pour" ou "être présent et s'abstenir" ou "s'absenter" ou "être présent et voter contre". La décision collective est soit l'adoption, soit le rejet) ; et les jeux multi-types (plusieurs options s'offrent aux joueurs et ces options là ne sont pas nécessairement ordonnées. Chaque joueur choisit une et une seule option. La décision collective est un nombre réel qui agrège les votes individuels). Plusieurs auteurs à l'instar de Shapley and Shubik [1954], Felsenthal and Machover [1997], Freixas and Zwicker [2003], Freixas [2005], Tchantcho et al. [2008], Laruelle and Valenciano [2012], Pongou et al. [2014] et Courtin et al. [2017] ont travaillé sur les jeux simples et ces différentes généralisations.

0.4.3. Economie industrielle

Une approche différente à celle proposée par Shapley and Shubik [1954] est suggérée dans une autre littérature beaucoup plus orientée vers la production et l'organisation industrielle. Aguiar et al. [2016] définissent un modèle appelé firme comme étant un triplet formé d'un ensemble de travailleurs/inputs, d'un ensemble fini des différentes actions ou niveaux de participation qu'un travailleur peut choisir et d'une fonction de production qui associe à chaque profil d'actions un réel/output. L'ensemble des actions des travailleurs peut être le nombre d'heures de travail, les différents niveaux d'effort, les différents types de métiers ou d'emplois, etc. Cet ensemble n'est pas nécessairement un ensemble ordonné. Ces différentes actions ne sont pas forcément des entiers naturels. Dans ce cadre, une firme est une extension des jeux TU. Si tous les travailleurs décident de se regrouper ensemble alors pour chaque profil d'action donné, la production totale est connue. La part de production attribuable à chaque travailleur est déterminée à l'aide d'une extension de la valeur de Shapley définie dans ce nouveau modèle. Cette valeur est appelée "Shapley wage function". Aguiar et al. [2016] proposent deux axiomatisations de cette valeur en utilisant l'approche de Young [1985] et celle de Shubik [1962].

Dans le modèle proposé par Aguiar et al. [2016], un profil est réalisé avec certitude. Les travailleurs ont un même ensemble de niveaux de participation. Pongou and Tondji [2018] proposent un nouveau modèle généralisant celui des firmes, jeux à plusieurs niveaux d'approbation et jeux multi-types. Dans ce modèle les profils de niveau de participation sont incertains et chaque travailleur a son ensemble de niveaux de participation. Un profil de distribution de probabilité est ainsi défini. Un environnement de production incertain est la donnée d'un ensemble d'inputs (travailleurs, facteurs de risques, etc.), d'un profil d'ensemble de niveaux de participation, d'un profil de distribution de probabilité et d'une fonction de production. Dans cette littérature, on suppose que les travailleurs se regroupent et une production totale espérée est obtenue. La valeur de Shapley va être définie dans ce modèle et une axiomatisation suivant l'approche de Young [1985] est proposée. Cette valeur est appelée "a priori Shapley value". Si la production exacte est connue alors cette dernière valeur est révisée et est appelée "Bayesian Shapley value". Une valeur est appelée procédure de valuation dans ce modèle car on s'intéresse au problème d'évaluation de la part de production attribuable à chaque input. Le problème de répartitions de gain est appelé problème de valuation.

L'objectif de cette thèse est de proposer à la fois des généralisations des travaux sur les

valeurs faits dans le cadre des jeux TU et des généralisations des travaux faits dans le cadre de modèles plus généraux (modèles plus larges que celui des jeux TU).

0.5. Les différents chapitres de cette thèse

0.5.1. Chapitre 1

Dans un premier temps, on va d'abord s'intéresser à la généralisation (pour des coalitions à plus de deux joueurs) de certains résultats proposés par Casajus and Huettner [2018] et Chantreuil et al. [2019] dans la classe des jeux TU.

Dans la littérature des jeux TU, selon Casajus and Huettner [2018] une façon naïve de déterminer la productivité d'un joueur sera d'écartier ce joueur de l'ensemble de joueurs, puis de mesurer sa contribution marginale. Une valeur calculant la productivité d'un joueur de cette manière sera appelée solution naïve (Casajus and Huettner [2018]) ou contribution marginale pure (Chantreuil et al. [2019]). Il s'avère que la somme de toutes les productivités des joueurs ne correspond pas à la productivité totale. De plus, cette façon de calculer la productivité d'un joueur ne prend pas en compte les autres joueurs. Une partie de la productivité doit être affectée aux autres joueurs car la présence d'interactions entre les joueurs démontre à suffisance la contribution des autres joueurs. Afin d'illustrer clairement le problème posé par l'utilisation de la solution naïve, Casajus and Huettner [2018] proposent une décomposition de la solution naïve à partir de la valeur de Shapley. Ils montrent que la solution naïve d'un joueur peut se décomposer en deux parties : la partie directe appelée "le décomposeur" mesure la productivité dudit joueur à l'aide de la valeur de Shapley ; et la partie indirecte mesure à l'aide du "décomposeur", la contribution que ce joueur apporte aux autres joueurs quand il se retire du jeu. Chantreuil et al. [2019] proposent un résultat similaire. Ils montrent que la valeur de Shapley d'un joueur peut s'écrire comme étant la différence entre la contribution marginale pure (solution naïve) dudit joueur et une somme pondérée de ses interactions marginales.

Le chapitre 1 de notre thèse est consacré à la décomposition des indices d'interaction (Courtin et al. [2020]). Une axiomatisation permettant de déterminer la famille des indices d'interaction décomposables suivant l'approche de Chantreuil et al. [2019] est proposée. Cette axiomatisation permet ainsi d'en déduire une décomposition des "Cardinal-probabilistic indices". Nous proposons aussi une décomposition de quelques "Cardinal-probabilistic indices" suivant l'approche de Casajus and Huettner [2018]. Par la suite, une illustration sur un exemple en aide

à la décision multi-critère est proposée suivant nos différentes approches.

Pour diverses limites observées dans la classe des jeux TU, il faut considérer des jeux TU sur des modèles plus généraux. Les trois chapitres suivants de cette thèse sont consacrés à cette étude.

0.5.2. Chapitre 2

Nous avons motivé plus haut la nécessité de décomposer les indices d'inégalité à partir de la valeur de Shapley-Owen. Ce chapitre est consacré à cette étude là (Tido Takeng et al. [2020]).

Dans ce chapitre, l'approche axiomatique nous permet de déterminer les différentes familles des valeurs avec hiérarchisation décomposables. Trois décompositions de la valeur de Shapley-Owen sont proposées. A partir de l'efficacité de la valeur de Shapley-Owen, nous déduisons donc trois décompositions des indices d'inégalité. Par la suite, une illustration sur un exemple fictif dans le cadre des inégalités est proposée suivant nos différentes approches.

0.5.3. Chapitre 3

La valeur de Shapley-Owen d'un joueur dépend de ses contributions marginales. Nous montrons au Chapitre 2 qu'elle peut s'exprimer en fonction de ses interactions marginales. Plusieurs axiomatisations de la valeur de Shapley-Owen utilisent des axiomes énoncés à partir de la contribution marginale. Il est donc intéressant pour nous de proposer des axiomes similaires en utilisant plutôt l'interaction marginale.

Ce chapitre (Casajus and Tido Takeng [2021]) est consacré à l'axiomatisation de la valeur de Shapley-Owen. Casajus [2021] appelle axiome du premier ordre tout axiome énoncé à partir de la contribution marginale. Les axiomes énoncés à partir de l'interaction marginale sont appelés axiomes du second ordre. Selon sa méthodologie, Young [1985] axiomatise la valeur de Shapley à partir des axiomes du premier ordre. Casajus [2021] propose donc une axiomatisation de la valeur de Shapley à partir des axiomes du second ordre. Dans ce chapitre nous montrons que l'ensemble des axiomes de Casajus [2021] entraîne (implique) l'ensemble des axiomes de Young [1985]. On déduit donc que le résultat de Casajus [2021] est une seconde version du résultat de Young [1985]. Nous proposons par la suite une axiomatisation de la valeur de Shapley-Owen à partir des axiomes du second ordre et montrons que ce résultat est une seconde version du résultat de Khmelnitskaya and Yanovskaya [2007].

0.5.4. Chapitre 4

Pongou and Tondji [2018] proposent une solution au problème de valuation des inputs dans un environnement de production incertain. En utilisant la méthodologie proposée par Myerson [1977] et celle de Genjiu et al. [2017] dans ce chapitre, nous considérons un modèle d'environnement de production incertain et nous supposons que les travailleurs sont regroupés suivant un réseau (ou graphe) appelé structure de communication. Ainsi, nous montrons comment déterminer la productivité d'un travailleur. Suivant deux approches, nous proposons une extension des procédures de valuation "a priori Shapley value" et "Bayesian Shapley value", et nous obtenons deux nouvelles procédures de valuation que nous nommons "a priori Myerson value" avec sa forme révisée appelé "a priori Myerson^R value"; et "Bayesian Myerson value" avec sa forme révisée appelée "Bayesian Myerson^R value". Des axiomatisations de ces nouvelles procédures vont être proposées. Nous introduisons par la suite les solutions ensemblistes dans ce nouveau modèle.

Dans la littérature, ce chapitre (Tido Takeng [2021]) s'inscrit donc dans le cadre de la production et de l'organisation (industrielle, etc). Les résultats de ce chapitre peuvent aussi être interprétés comme des extensions/applications des travaux de Myerson [1977] sur un environnement de production incertain avec structure de communication.

Decomposition of interaction indices : alternative interpretations of cardinal-probabilistic interaction indices

Résumé :

Dans la théorie des jeux coopératifs, le concept d'indice d'interaction est une extension du concept de valeur qui prend en compte les interactions entre les joueurs. Dans cet article (chapitre), nous nous focalisons sur les "cardinal-probabilistic interaction indices" qui sont des généralisations des semi-values. Nous proposons deux types de décomposition. Suivant la première décomposition, un "cardinal-probabilistic interaction index" d'une coalition donnée est la différence entre l'indice d'interaction externe (encore appelé co-möbuis) de ladite coalition et une somme pondérée des impacts individuelles des autres joueurs sur l'indice d'interaction de la coalition considérée. La seconde décomposition est basée sur la notion du "décomposeur" qui subdivise l'indice d'interaction d'une coalition en deux parties : la partie directe est le "décomposeur" qui mesure l'interaction de la dite coalition et la partie indirecte indique comment les autres joueurs affectent individuellement l'interaction entre les joueurs de ladite coalition. Nous présentons deux illustrations : la première est faite dans le cadre des jeux coopératifs et la deuxième est faite dans le contexte de l'aide à la décision multi-critère.

Mots clés : Théorie des jeux, aide à la décision multi-critère, "cardinal-probabilistic interaction indices", indice d'interaction externe

JEL Codes : C71 ; D60 ; C44.

Abstract :

In cooperative game theory, the concept of interaction index is an extension of the concept of value that takes into account interactions among players. In this paper (chapter), we focus on cardinal-probabilistic interaction indices which are generalizations of the class of semivalues. We provide two types of decompositions. With the first one, a cardinal-probabilistic interaction index for a given coalition equals the difference between its external interaction index (or co-Möbius transform) and a weighted sum of the individual impact of the remaining players on the interaction index of the considered coalition. The second decomposition, based on the notion of the "decomposer", splits an interaction index into a direct part, the decomposer, which measures the interaction in the coalition considered, and an indirect part, which indicates how all remaining players individually affect the interaction of the coalition considered. We propose alternative characterizations of the cardinal-probabilistic interaction indices. We then present two illustrations, one in the context of cooperative game theory, the other in multicriteria aid for decisions.

Key words : Game theory, multicriteria aid for decisions, cardinal-probabilistic interaction indices, external interaction index

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1.1. Introduction

A large literature on cooperative games with transferable utility (TU-game), described as a situation within which players can obtain a pay-off by cooperating, is devoted to the definition and the characterization of a solution. A solution measures the player's importance in the game, *i.e.* the average added value that a player brings to all possible coalitions. The most well-known solutions are those of Shapley (Shapley [1953]) and Banzhaf (Banzhaf [1965]). As pointed out by Grabisch and Roubens [1999a], a solution does not exhibit any information on the existing cooperation between players or their interactions. Consider two players i and j , and a value function v which assigns, to every subset of players in a game, a real number. We investigate the sign of the relation $v(ij) - (v(i) + v(j))$. If the sign is positive, it is natural to say that players i and j interact positively because considering the pair of players (ij) leads us to a situation within which the value of (ij) is more important. In a symmetric way, if the sign is negative, it seems reasonable to say that players i and j interact negatively because considering the pair of players (ij) leads us to a situation within which the value is lower. If the difference equals zero, one can assume that there is no interaction.

However, the presence of a third or fourth player should potentially modify the interaction between players i and j . Therefore, one needs to consider a more elaborate definition, one that

takes into account what happens when i and j are in the presence of T , where T is a subset of players not containing players i and j . According to Kojadinovic [2005], Owen [1972] was the first to define a measure that evaluates the interaction between two players in the presence of T .

Grabisch [1997] suggested going even further by considering higher-order interactions, that is interactions between more than two players. He introduced the Shapley interaction index, which generalizes the Shapley solution. The notion of higher-order interactions has also been considered by Marichal and Roubens [1999] and Roubens [1996] who proposed respectively the Chaining interaction index and the Banzhaf interaction index. The latter is an extension of the Banzhaf solution, while the former is another extension of the Shapley solution. These three interactions indices belong to the class of cardinal-probabilistic interaction indices introduced by Grabisch and Roubens [1997].⁴ Cardinal-probabilistic interaction indices generalize the concept of cardinal-probabilistic values also known as semivalues (see Dubey et al. [1981]). In this paper, we present alternative interpretations of the cardinal-probabilistic interaction indices. In order to do that, two disjoint decomposition approaches are put forward.

The first one splits into two parts a cardinal-probabilistic interaction index for a given coalition S : the first part is the external interaction index for this coalition S ; the second part is a weighted sum of the individual impact of the remaining players on the interaction index of the coalition considered. The external interaction index, also called co-Möbius transform (Katsushige et al. [2006]), is a higher-order extension of the marginal contribution of a player to the grand coalition.

The second decomposition, based on Casajus and Huettner [2018] approach, decomposes interaction indices into a direct part and an indirect part : the direct part measures the interaction of a given coalition in the game, while the indirect part indicates how the remaining players individually affect the interaction index of the coalition considered.

These two decompositions will allow us to present a new characterization of the class of cardinal-probabilistic interaction indices. The rest of the paper is a numerical illustration of the proposed methodology, using hypothetical data.

The paper is structured as follows : in Section 2, we propose some basic notions and notations. Section 3 presents the first decomposition approach based on the external interaction index. Section 4 introduces the Casajus and Huettner decomposition approach and presents a new characterization of the well-known interaction indices. Section 5 is an illustration with

4. Cardinal-probabilistic interaction indices are derived from the class of probabilistic interaction indices by additionally imposing the symmetry axiom to this class.

1.2. Preliminaries

1.2.1. Game with transferable utility

Let (N, v) be a game with transferable utility (TU game), where $N = \{1, 2, \dots, n\}$ is the set of players, and let v be a characteristic function which assigns a real number $v(S)$ to every coalition $S \subseteq N$. We assume that $v(\emptyset) = 0$. The cardinality of sets S, T, \dots will be denoted by corresponding lower cases s, t, \dots . Following standard practice, we often refer to the game v instead of the game (N, v) .⁵ We denote by $TU(N)$ the set of cooperative games with transferable utility on N and denote by 2^N the set of all subsets of N .

A game $v \in TU(N)$ is monotonic if $v(S) \leq v(T)$ whenever $S \subset T$. A permutation on the set of players N is a bijective map π from N into itself. The pair $(N, \pi v)$ is also a TU game where $\pi v(S) := v(\pi^{-1}S)$ for all $S \subseteq N$. A player $i \in N$ is said to be dummy in a game $v \in TU(N)$ if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq N \setminus i$. For each game $v \in TU(N)$ and $S \subseteq N$, we denote by v^S , the game v restricted to S , given by $v^S(T) := v(T), \forall T \subseteq S$. We call this game a *Restricted game*. A *Reduced game*, $v_{\cup T}^S$, is the game $v \in TU(N)$ restricted to S in the presence of $T \subseteq N \setminus S$, i.e. $v_{\cup T}^S(A) := v(A \cup T) - v(T), \forall A \subseteq S$. The two previous games have been introduced by Grabisch and Roubens [1999a].

In combinatorics, the set function $m(v, \cdot) : 2^N \rightarrow \mathbb{R}$ is called the Möbius transform of v and is given by $m(v, S) := \sum_{L \subseteq S} (-1)^{s-l} v(L), \forall S \subseteq N$. Katsushige et al. [2006] and Grabisch et al. [2000a] propose an equivalent representation of a TU game, in which any game $v \in TU(N)$ can be uniquely expressed in terms of $\{m(v, S)\}_{S \subseteq N}$ by $v(T) = \sum_{S \subseteq T} m(v, S), \forall T \subseteq N$. They also show that $\sum_{T \supseteq S} m(v, T) = \sum_{L \subseteq S} (-1)^l v(N \setminus L), \forall S \subseteq N$. The second part of this equation is called the co-Möbius transform of v applied to S , denoted by $m^*(v, S)$.

1.2.2. Interaction indices

Consider first the following example.

Example 1.2.1. *Three cities (a, b and c) are in contract with a company to be connected to a nearby power source. The project costs are 8, 9 and 12 (million euros) for a, b and c,*

5. In order to avoid heavy notation, we adopt the following conventions : we will omit braces for singletons, e.g., by writing $v(i), S \setminus i$ instead of $v(\{i\}), S \setminus \{i\}$. Similarly, for pairs, we will write ij instead of $\{i, j\}$.

respectively. For geographical reasons, the company proposes a reduced cost for joint contracts between a and b (17), a and c (18), b and c (22), and a , b and c (25).

We can represent this example by using a TU game, in which the players are the three cities, $N = \{a, b, c\}$, and in which the value of each coalition is given by the following table :

S	a	b	c	ab	ac	bc	abc
$v(S)$	-8	-9	-12	-17	-18	-22	-25

We observe that a cooperation between a and c has a positive impact since the joint cost is reduced, i.e. $v(ac) > v(a) + v(c)$. We will say that a and c interact positively. In contrast, $v(bc) < v(b) + v(c)$, meaning that the cooperation between b and c increases the joint cost. In this case b and c interact negatively. Finally, We can say that there are no interactions between a and b since $v(ab) = v(a) + v(b)$. For i, j in N , the sign of the relation $v(ij) - v(i) - v(j)$ informs us on the interaction. One can take the reasoning further by investigating the interaction between i and j in the presence of the other cities T . Thus, depending on the way i and j interact in the presence of T , the expression $v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T)$ will be positive or negative. It will be zero if and only if i and j do not interact in the presence of T . For instance, for $i = a$, $j = b$ and $T = c$, we have $v(abc) - v(ac) - v(bc) + v(c) = 3$ meaning that there is a positive interaction.

This example raises the question on how to measure the interaction among players. An interaction index of a game $v \in TU(N)$ is a function Ψ which assigns a vector of real values representing the interaction of each coalition $S \in 2^N \setminus \emptyset$.

$$\begin{aligned} \Psi : TU(N) \times 2^N \setminus \emptyset &\longrightarrow \mathbb{R} \\ (v, S) &\longmapsto \Psi(v, S) \end{aligned}$$

A positive (negative) sign can be interpreted as a positive (respectively negative) simultaneous interaction among all players in S . Note that when $S = i$, an interaction index is simply a one-point solution concept for a TU game.

In this paper, we focus on two classes of interaction indices : probabilistic and cardinal-probabilistic interaction indices. One needs to introduce additional notations and definitions in order to present these two classes. Given a game $v \in TU(N)$, for any $S \in 2^N \setminus \emptyset$ and for all $T \subseteq N \setminus S$, the discrete derivative with respect to the set S (see Katsushige et al. [2006]) is defined by $\Delta_S v(T) = \sum_{L \subseteq S} (-1)^{s-l} v(L \cup T)$, and with convention $\Delta_\emptyset v(T) := v(T)$.⁶ We can interpre-

6. Note that $\Delta_S v(\emptyset) = m(v, S)$ and $m^*(v, S) = \Delta_S v(N \setminus S)$

ted $\Delta_S v(T)$ as the simultaneous interaction among players in S in the presence of T . When $\Delta_S v(T) > 0$ (respectively < 0), it seems sensible to consider that there exists a positive (respectively negative) simultaneous interaction among the players in S in the presence of T . However, $\Delta_S v(T) = 0$ should obviously not be interpreted as the absence of interactions among players in S in the presence of T but as the absence of simultaneous interactions among the players in S in the presence of T . Note that, if $S = i$, the first order derivative, $\Delta_i v(T) = v(T \cup i) - v(T)$, is the marginal contribution of a player i to a coalition T . And if $S = ij$, then the second order derivative (introduced by Owen [1972]), $\Delta_{ij} v(T) = v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T)$ can be regarded as the difference between the marginal contributions $\Delta_j v(T \cup i) = v(T \cup ij) - v(T \cup i)$ and $\Delta_j v(T) = v(T \cup j) - v(T)$. Following Kojadinovic [2005], Grabisch and Roubens [1999a], this expression is called the marginal interaction between i and j in the presence of a coalition T .

A probabilistic interaction index (Grabisch and Roubens [1999b], Grabisch and Roubens [1997]) Ψ^p of a coalition $S \subseteq N$ in a game $v \in TU(N)$ is of the form

$$\Psi^p(v, S) = \sum_{T \subseteq N \setminus S} P_N^S(T) \Delta_S v(T)$$

where, for any $S \subseteq N$, the family of coefficients $\{P_N^S(T)\}_{T \subseteq N \setminus S}$ forms a probability distribution on $\mathcal{P}(N \setminus S)$. If $S \not\subseteq N$, we naturally set $\Psi^p(v, S) = 0$.

A cardinal-probabilistic interaction index (Grabisch and Roubens [1999b], Grabisch and Roubens [1997]) is a probabilistic interaction index such that, additionally, for any $S \subseteq N$, the coefficients $P_N^S(T)$ ($T \subseteq N \setminus S$) depend only on the cardinalities of the coalitions S , T , and N , i.e., for any $s \in \{0, \dots, n\}$, there exists a family of nonnegative real numbers $\{P_n^s(t)\}_{t=0, \dots, n-s}$ fulfilling $\sum_{t=0}^{n-s} \binom{n-s}{t} P_n^s(t) = 1$ such that, for any $S \subseteq N$ and any $T \subseteq N \setminus S$, we have $P_N^S(T) = P_n^s(t)$. One can notice that for $S = i$ a cardinal-probabilistic interaction index is simply called semivalue (see Dubey et al. [1981]).

Well-known examples of cardinal-probabilistic interaction indices are the Shapley interaction index denoted by Ψ_{Sh} with $P_n^s(t) = \frac{t!(n-t-s)!}{(n-s+1)!}$ (Grabisch [1997]), the Chaining interaction index denoted by Ψ_{Ch} with $P_n^s(t) = \frac{s(s+t-1)!(n-t-s)!}{n!}$ (Marichal and Roubens [1999]) and the Banzhaf interaction index denoted by Ψ_{Bz} with $P_n^s(t) = \frac{1}{2^{n-s}}$ (Roubens [1996]). The co-Möbius transform of a game $v \in TU(N)$ is also a cardinal-probabilistic interaction index with $P_n^s(t) = 1$ if $t = n - s$ and 0 otherwise. Katsushige et al. [2006] and Grabisch et al.

[2000a] call this interaction index the external interaction index (Ψ_{ext}).⁷ Note that for $S = i$, $\Psi_{ext}(v, i) = v(N) - v(N \setminus i)$, *i.e.* the pure marginal contribution of player i to the grand coalition. Furthermore, the pure marginal contribution of i to j 's pure marginal contribution to the grand coalition is given by $\Psi_{ext}(v, ij) = [v(N) - v(N \setminus j)] - [v(N \setminus i) - v(N \setminus ij)]$.

Let us now introduce some properties considered by Grabisch and Roubens [1999a].

Definition 1.2.1 (Axioms)

- i) Linearity :** An interaction index Ψ satisfies the Linearity property if for two games $v, w \in TU(N)$, $n \geq 1$, for any $S \in 2^N \setminus \emptyset$, $\alpha \in \mathbb{R}$, we have, $\Psi(\alpha v + w, S) = \alpha \Psi(v, S) + \Psi(w, S)$.
- ii) Dummy :** An interaction index Ψ satisfies the Dummy property if for all game $v \in TU(N)$, $n \geq 1$, for any dummy player $i \in N$ in the game v , then $\Psi(v, i) = v(i)$ and for any $S \subseteq N \setminus i$, $s \geq 1$, $\Psi(v, S \cup i) = 0$.
- iii) Symmetry :** An interaction index Ψ satisfies the Symmetry property if for all permutation π , for all TU game $v \in TU(N)$, $n \geq 1$, and for all $S \subseteq N$, $s \geq 1$, then $\Psi(\pi v, \pi(S)) = \Psi(v, S)$.
- iv) Recursivity :** An interaction index Ψ verifies the Recursivity property if for a game $v \in TU(N)$ ($n \geq 2$), for all $S \subseteq N$ ($s \geq 2$) and for all $j \in S$

$$\Psi(v, S) = \Psi(v_{\cup_j}^{N \setminus j}, S \setminus j) - \Psi(v^{N \setminus j}, S \setminus j)$$

The linearity axiom implies that interaction indices are linear combinations of the basic information related to the game : the worth of each coalition of players. The dummy axiom states that a dummy player has a value equal to their worth and that they do not interact with any outside coalition. The symmetry axiom indicates that the names of the players play no role in determining the outcome of an interaction index. According to the recursivity axiom, the interaction at a given level is linked to the difference in interactions defined at the previous level. For example, for two players i and j , the axiom says that their interaction should depend on the interaction when one of the players is removed (measured by the interaction in the corresponding

7. More precisely, $\Psi_{ext}(v, S) := m^*(v, S) = \Delta_S v(N \setminus S) \quad \forall S \subseteq N$.

reduced and restricted games).

Note that Shapley and Banzhaf interaction indices satisfy the recursivity axiom (Grabisch and Roubens [1999a]), but not the Chaining interaction index. These three interaction indices satisfy the linearity, dummy and symmetry axioms.

1.3. External interaction decomposition approach

In this section, we introduce a new class of interaction indices and then provide an axiomatic characterization. Moreover, we present an alternative interpretation of well-known interaction indices and apply this new approach to Example 1.

One of natural ways to measure the interaction among players of coalition $S \subseteq N$ in a TU game is to find the external interaction index of this coalition. The external interaction index is problematic since the players of the coalition S interact with the other players. One could argue that every player contributes to the external interaction index of S , and therefore this contribution should be partly attributed to the others. Chantreuil et al. [2019] and Casajus and Huettner [2018] answer this problem for the case of one-point solutions. In order to tackle this problem in general, let us consider the following definition.

Definition 1.3.1

The class of Additive Interaction Decomposable (AID) indices

An interaction index $\Psi : TU(N) \times 2^N \setminus \emptyset \rightarrow \mathbb{R}$ is an AID index if for all game $v \in TU(N)$ and for any coalition $S \subseteq N$ there exists a set of real numbers $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ such that,

$$\Psi(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2}$$

This definition states that an interaction index Ψ is an AID index if it can be split into two parts. The first one is the external interaction index for a given coalition S , and the second one is a weighted sum of half of the simultaneous interactions between the players of the coalition S and a single player j in the presence of all other coalitions T . Naturally, the case where $f_n^s(t) = 0$, leads to the external interaction index.

Remark 1.3.1. *If the set of real numbers $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ fulfills the following conditions :*

i) $f_n^s(t) \geq 0$, for all $t = 0, \dots, n - s - 1$; ii) $\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} f_n^s(t) = 1$, then, $\Psi(v, S)$ can

be interpreted as the difference between the external interaction index for a given coalition S and the sum over the players $j \in N \setminus S$, of the mathematical expectation on $2^{N \setminus (S \cup j)}$ of the discrete derivative $\frac{\Delta_{S \cup j} v(T)}{2}$ with respect to the probability distribution $\{ {}^j f_N^S(T) \}_{T \subseteq N \setminus (S \cup j)}$ where ${}^j f_N^S(T) = f_n^s(t)$.

The next result provides an axiomatic characterization of AID interaction indices on the class of TU games.

Theorem 1.3.1. *The following properties of an interaction index Ψ are equivalent :*

- i) Ψ satisfies the Linearity, Dummy and Symmetry axioms
- ii) Grabisch and Roubens [1999a] family interaction indices : For any game $v \in TU(N)$, $n \geq 1$, and $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{P_n^s(t)\}_{t=0, \dots, n-s}$, such that

$$\Psi(v, S) = \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T),$$

and for any $S \not\subseteq N$, we have $\Psi(v, S) = 0$.

- iii) Ψ is AID.

Every cardinal-probabilistic index satisfies the Linearity, Dummy and Symmetry axioms. From Theorem 1.3.1, we can deduce a new rewriting (or interpretation) of cardinal-probabilistic interaction indices.

Corollary 1.3.1. *Every cardinal-probabilistic interaction index Ψ^p applied to a game $v \in TU(N)$ is AID. Moreover, for any non-empty coalition $S \subseteq N$,*

$$\Psi^p(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2}$$

where $f_n^s(t)$ is given by

$$f_n^s(t) = \begin{cases} \frac{2P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{2[t \times f_n^s(t-1) + P_n^s(t)]}{n-t-s} & \text{otherwise} \end{cases}$$

and the set $\{P_n^s(t)\}_{t=0, \dots, n-s}$ consists of nonnegative constants associated to the cardinal-probabilistic interaction index.

From Corollary 1.3.1, we deduce that the Shapley, the Chaining and the Banzhaf interaction indices are AID.⁸

Corollary 1.3.2. *The real numbers of the Shapley interaction index as an AID index are given by*

$$f_n^s(t) = \frac{2(t+1)!(n-t-s-1)!}{(n-s+1)!} \quad \text{for all } t = 0, \dots, n-s-1;$$

The real numbers of the Chaining interaction index as an AID index are given by

$$f_n^s(t) = \frac{2(t+s)!(n-t-s-1)!}{n!} \quad \text{for all } t = 0, \dots, n-s-1$$

And the real numbers of the Banzhaf interaction index as an AID index are given by

$$f_n^s(t) = \begin{cases} \frac{1}{(n-s)2^{n-s-1}} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s-1}} & \text{otherwise.} \end{cases}$$

Note that the real numbers of the Shapley and Banzhaf interaction indices are in agreement with the conditions put forward in Remark 1, which is not the case of the Chaining interaction index.

Proposition 1.3.1

For a given game $v \in TU(N)$, $n \geq 1$ and $S \subseteq N$, $s \geq 1$,

$$\begin{aligned} & - \text{ If } f_n^s(t) = \frac{2(t+1)!(n-t-s-1)!}{(n-s+1)!} \text{ then } \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} f_n^s(t) = 1. \\ & - \text{ If } f_n^s(t) = \begin{cases} \frac{1}{(n-s)2^{n-s-1}} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s-1}} & \text{otherwise} \end{cases} \text{ then } \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} f_n^s(t) = 1. \end{aligned}$$

The next proposition gives the necessary and sufficient condition for which an AID interaction index is a cardinal-probabilistic interaction index.

⁸ One can notice that when $S = i$, according to Corollary 1.3.1, every semivalue is AID, especially the Shapley solution (Chantreuil et al. [2019]).

Proposition 1.3.2

Let Ψ be an AID interaction index with the associated family of real numbers $\{f_n^s(t)\}_{t=0,\dots,n-s-1}$. For any coalition $S \subseteq N$, we have

$$\Psi(v, S) = \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T)$$

where

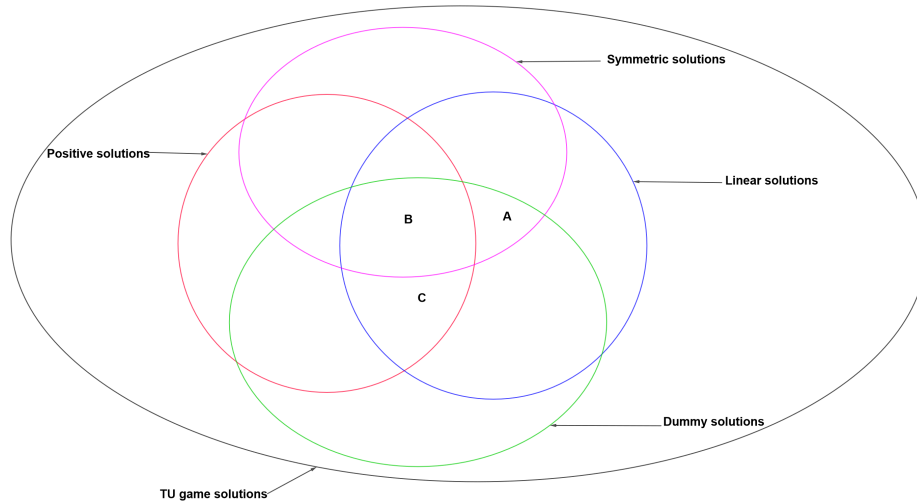
$$P_n^s(t) = \left[\frac{(n-s-t)f_n^s(t) - t f_n^s(t-1)}{2} \right] q_n^s(t) + \frac{(n-s)f_n^s(0)r_n^s(t)}{2} \\ + \left[1 - \frac{(n-s)f_n^s(n-s-1)}{2} \right] k_n^s(t)$$

and

$$q_n^s(t) = \begin{cases} 1 & \text{if } 0 < t < n-s \\ 0 & \text{otherwise} \end{cases} ; \quad k_n^s(t) = \begin{cases} 1 & \text{if } t = n-s \\ 0 & \text{otherwise} \end{cases} ; \quad r_n^s(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Ψ is a cardinal-probabilistic interaction index if and only if $f_n^s(n-s-1) \leq \frac{2}{n-s}$, $f_n^s(0) \geq 0$ and $f_n^s(t) \geq \frac{t}{n-t-s} f_n^s(t-1)$ with $0 < t < n-s-1$.

Note that the class of AID indices is not a subclass of the class of probabilistic interaction indices and vice versa. In fact, let's consider the following graph restricted to one-point solution concept for TU game :



The class of semivalues (cardinal-probabilistic interaction indices with $S = i$) is the domain B , the class of AID solutions (AID interaction indices with $S = i$) is the domain $A \cup B$ and the class of probabilistic values (probabilistic interaction indices with $S = i$) is the domain $B \cup C$.⁹

We illustrate the first decomposition proposed in this section with the Shapley interaction index.

Example 1.3.1. *Example 1.2.1 continued.*

Cities (S)	a	b	c	ab	ac	bc	abc
$\Psi_{Sh}(v, S)$	-6	-8.5	-10.5	1.5	3.5	0.5	3
$\Psi_{ext}(v, S)$	-3	-7	-8	3	5	2	3
$\sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2}$	3	1.5	2.5	1.5	1.5	1.5	0

If the three cities decide to join together, they will pay an amount of 25 million euros. Consider the simple case where we examine separately the contribution of each city to the overall project (columns 2, 3 and 4). By using the Shapley interaction index ($\Psi_{Sh}(v, S)$), city a will pay 6 million euros, city b will pay 8.5 million euros and city c will pay 10.5 million euros. We can notice that if we sum up the direct marginal contributions of the three cities ($\Psi_{ext}(v, S)$), the amount is equal to 18 millions ($3+7+8$), meaning that the marginalist view can not be sufficient to explain the overall repartition. This is due to its inability to take into account the links between the cities, measured by the pairwise interaction. The amount that will be paid by city a for instance must be decomposed into two parts : the first one is its pure marginal contribution to the grand

9. **Positivity** : A solution Ψ satisfies the Positivity property if for all monotonic game $v \in TU(N)$ and $i \in N$, then $\Psi(v, i) \geq 0$.

coalition $N \setminus \{i\}$; the second part is the sum of its weighted pairwise interactions (3). The same reasoning can be applied when considering cities $\{a,b\}$ (column 5), $\{a,c\}$ (column 6),...

1.4. Casajus and Huettner decomposition approach

In this section, we are going to decompose an interaction index by using another interaction index called the decomposer.

1.4.1. Decomposability

Given an interaction index Ψ on the game $v \in TU(N)$, an auxiliary game is a game v^Ψ where $v^\Psi(S) = \sum_{i \in S} \Psi(v^S, i)$ for all $S \subseteq N$.

We can now introduce the different notions of decomposability under investigation in this section. Casajus and Huettner [2018] argue that every player's presence is necessary for generating the pay-off of any given player and, therefore, this pay-off should be partly attributed to the others. They propose to decompose an interaction index by another interaction index (called a CH-decomposer) into a direct part and an indirect part, where the indirect part indicates how much a player contributes to the pay-off of another player according to the CH-decomposer. They first define the notion of a CH-decomposable interaction index with respect to the existence of a CH-decomposer of such an interaction index.

Definition 1.4.1 (CH-decomposability (Casajus and Huettner [2018]))

An interaction index Ψ is CH-decomposable if there exists an interaction index φ such that, for all game $v \in TU(N)$ and $i \in N$,

$$\Psi(v, i) = \varphi(v, i) + \sum_{j \in N \setminus i} [\varphi(v, j) - \varphi(v^{N \setminus i}, j)]$$

φ is then called the CH-decomposer of Ψ .

The expression $\varphi(v, i)$ reflects player i 's direct contribution subsumed under the interaction index φ , while the expression $\sum_{j \in N \setminus i} [\varphi(v, j) - \varphi(v^{N \setminus i}, j)]$ reflects player i 's indirect contribu-

tions.

Casajus and Huettner [2018] clarifies the existence and the structure of such a CH-decomposer in the following proposition.

Proposition 1.4.1

(Casajus and Huettner [2018]) If an interaction index Ψ is CH-decomposable, then its unique CH-decomposer φ is given by : For all game $v \in TU(N)$ and for all $i \in N$,

$$\varphi(v, i) = \sum_{\substack{T \subseteq N \\ i \in T}} \frac{m(v^\Psi, T)}{t^2}$$

The dividends $m(v^\Psi, T)$ are the Möbius transform of V^Ψ applied to T .

In order to extend this decomposability notion to higher-order interactions, one needs to introduce a rather similar notion of decomposability — the Individual-decomposability (I-decomposability).

Definition 1.4.2 (I-decomposability)

An interaction index Ψ is I-decomposable if there exists an interaction index φ such that, for all game $v \in TU(N)$ and $i \in N$,

$$\Psi(v, i) = \varphi(v, i) + \sum_{j \in N \setminus i} [\varphi(v, i) - \varphi(v^{N \setminus j}, i)]$$

φ is then called the I-decomposer of Ψ .

An interaction index is I-decomposable if it can be split into a direct part and an indirect part, where the direct part $\varphi(v, i)$ is the I-decomposer and the indirect part $\sum_{j \in N \setminus i} [\varphi(v, i) - \varphi(v^{N \setminus j}, i)]$ indicates how much the other players contribute to the pay-off of a given player i according to the I-decomposer. If the I-decomposer of Ψ exists, then it provides a kind of foundation for Ψ . The difference between the two previous notions of decomposability concerns the second part of the equations. However, there is a link between CH-decomposability and I-decomposability as shown by the following proposition.

Proposition 1.4.2

If an interaction index is CH-decomposable then it is I-decomposable. Moreover, the I-decomposer is unique and coincides with the unique CH-decomposer that is itself CH-decomposable. The uniqueness holds only for individual interaction terms.

Considering higher-order interactions, we can extend the I-decomposability to Coalition-decomposability (C-decomposability) as follows.

Definition 1.4.3 (C-decomposability)

An interaction index Ψ is C-decomposable if there exists an interaction index φ such that, for all game $v \in TU(N)$, $S \in 2^N \setminus \emptyset$,

$$\Psi(v, S) = \varphi(v, S) + \sum_{j \in N \setminus S} [\varphi(v, S) - \varphi(v^{N \setminus j}, S)]$$

φ is called the C-decomposer of Ψ .

For a given coalition S , the simultaneous interaction of the players of coalition S is split into two parts. The direct part is the simultaneous interaction of the players of coalition S measured by the C-decomposer. The indirect part indicates the extent to which the other players contribute individually to the interactions of coalition S measured by the C-decomposer.

The next results present some features of the C-decomposability and of the C-decomposer.

Proposition 1.4.3

If an interaction index Ψ is C-decomposable, then the C-decomposer of Ψ is unique.

Proposition A1 clarifies the uniqueness of the C-decomposer.

The next theorem shows that the recursivity property links CH-decomposability and C-decomposability.

Theorem 1.4.1. *Let Ψ be an interaction index satisfying the Recursive axiom. If the interaction index Ψ is CH-decomposable, then it is also C-decomposable, and the C-decomposer is unique. This C-decomposer denoted φ is given by : for all game $v \in TU(N)$, for any coalition $S \in 2^N \setminus \emptyset$ and any player $i \in S$,*

$$\varphi(v, S) = \begin{cases} \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ |T|=\beta}} \varphi(v_{\cup((S \setminus i) \setminus T)}, i) & \text{if } s \geq 2 \\ \varphi(v, i) & \text{if } s = 1 \end{cases}$$

φ is given by Proposition 1.4.1 when the coalition contains only one player.

1.4.2. Cardinal-probabilistic interaction index

We now study the interaction indices taken into consideration in this paper under the prism of the second decomposability approach.

Firstly, we show that Shapley and Banzhaf interaction indices are C-decomposable and that the corresponding C-decomposer is unique ; and the Chaining interaction index is not C-decomposable.

Proposition 1.4.4

The Shapley and the Banzhaf interaction indices are C-decomposable and the C-decomposer of each of these indices is unique.

Proposition 1.4.5

The Chaining interaction index is not C-decomposable.

Secondly, we show that a cardinal-probabilistic interaction index is a unique C-decomposer of a given AID index.

Proposition 1.4.6

Let Ψ^p be a cardinal-probabilistic interaction index. For all game $v \in TU(N)$, Ψ^p is the unique C-decomposer of the AID index χ given as follows : for any coalition $S \in 2^N \setminus \emptyset$,

$$\chi(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} \left[\frac{f_n^s(t)}{2} - P_{t+1}^s(n) \right] \Delta_{S \cup j} v(T)$$

Where the coefficients $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ are given by Corollary 1.3.1 and the coefficients $\{P_{t+1}^s(n)\}_{t=0, \dots, n-s-1}$ are the nonnegative real numbers of the cardinal-probabilistic interaction index Ψ^p .

The main implications of this result are the following.

Remark 1.4.1. (Implications)

- i) *The Shapley interaction index is the unique C-decomposer of the external interaction index.*¹⁰
- ii) *The Banzhaf interaction index is the unique C-decomposer of the interaction index χ_B defined as follows : for all game $v \in TU(N)$ and for any coalition $S \in 2^N \setminus \emptyset$,*

$$\chi_B(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} \left[\frac{f_n^s(t)}{2} - \frac{1}{2^{n-s}} \right] \Delta_{S \cup j} v(T)$$

where $f_n^s(t)$ is given in Corollary 1.3.2.

- iii) *The Chaining interaction index is the unique C-decomposer of the interaction index χ_C defined as follows : for all game $v \in TU(N)$ and for any coalition $S \in 2^N \setminus \emptyset$,*

$$\chi_C(v, S) = \Psi_{ext}(v, S) + \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} \frac{(s-1)(t+s)!(n-t-s-1)!}{n!} \Delta_{S \cup j} v(T)$$

Let us conclude by an illustration of the C-decomposability of the Shapley interaction index.

Example 1.4.1. *Example 1.2.1 continued.*

10. When $S = i$, we obtain Theorem 3 of Casajus and Huettner [2018], page 39.

Cities (S)	a	b	c	ab	ac	bc	abc
$\Psi_{Sh}(v, S)$	-6	-8.5	-10.5	1.5	3.5	0.5	3
$\varphi(v, S)$	$-\frac{47}{6}$	$-\frac{115}{12}$	$-\frac{145}{12}$	$\frac{3}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	3
$\sum_{j \in N \setminus S} [\varphi(v, S) - \varphi(v^{N \setminus j}, S)]$	$\frac{11}{6}$	$\frac{13}{12}$	$\frac{19}{12}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0

Using the Shapley interaction index, city a will pay 6 million euros. The amount that will be paid by city a can be decomposed into two parts : the first part is the amount of city a measured by an interaction index φ ($-\frac{47}{6}$). The second part is the contributions of cities b and c to the amount given in the first part ($\frac{11}{6}$).

1.5. Another application : An illustration with MCDA

Under the viewpoint of cooperative game theory, everyone agrees that interaction phenomena do exist in real situations. The notion of interaction has been studied in various other fields, such as fuzzy systems, aggregation function theory, statistics and data analysis (Marichal and Mathonet [2008]). The Cardinal-probabilistic interaction index has always been central in the modelling and analysis of preferences. The decompositions studied in this paper allow us to propose other interpretations of the importance of criteria.

1.5.1. Notations and basic notions

MCDA

We consider a multicriteria decision aiding (MCDA) problem with n criteria. N is the index set of criteria (players are criteria in the context of MCDA). Let $X = X_1 \times X_2 \times \dots \times X_n$ be the set of alternatives, with $n \geq 2$. An alternative $x = (x_1, \dots, x_n)$ is identified to an element of X . \succeq is a preference relation on X which is complete (i.e., $x \succeq y$ or $y \succeq x$, for all $x, y \in X$) and transitive (i.e., $x \succeq y$ and $y \succeq z \implies x \succeq z$, for all $x, y, z \in X$). For $T \subseteq N$, the notation $(x_T, y_{-T}) \in X$ is the compound alternative taking value x_i if $i \in T$ and value y_i otherwise.

In the context of MCDA, the characteristic function is called a capacity or a fuzzy measure if $\mu(N) = 1$. A monotonic TU game in the context of MCDA will be denoted by (N, μ) . A 2-additive capacity (Mayag et al. [2011]) is a capacity μ such that the Möbius transform of μ satisfies the following conditions : i) for all $S \subseteq N$ such that $s > 2$, $m(\mu, S) = 0$; ii) there exists $T \subseteq N$ such that $t = 2$ and $m(\mu, T) \neq 0$.

The various criteria are recoded numerically using, for each $i \in N$, a function u_i from X_i into \mathbb{R} . We denote by U the overall utility function. For all $x \in X$, $U(x) = (u_1(x_1), \dots, u_n(x_n))$.

Interaction

Debreu [1960], Keeney and Raiffa [1976] introduce the notion of "preferentially independent" to discuss the notion of interaction in the context of MCDA.

Definition 1.5.1

Preferentially independent

i) $T \subseteq N$ is preferentially independent of its complement $N \setminus T$ if for every $x, y, z, z' \in X$

$$(x_T, z_{-T}) \succeq (y_T, z_{-T}) \iff (x_T, z'_{-T}) \succeq (y_T, z'_{-T})$$

ii) The criteria X_1, \dots, X_n are (mutually) preferentially independent if every $T \subseteq N$ is preferentially independent of its complement.

Debreu [1960], Keeney and Raiffa [1976] show that when the criteria are not mutually preferentially independent, there is interaction among the criteria, while there is no interaction if mutual preference independence holds.

$$\text{interaction} \iff \text{not}(\text{mutual preferential independence}).$$

Following Marichal [2002], for an alternative $x = (x_1, \dots, x_n) \in X$, the Choquet integral (Choquet [1953]) w.r.t. a 2-additive capacity μ , called the 2-additive Choquet integral for short, is given by :¹¹

$$C_\mu(u_1(x_1), \dots, u_n(x_n)) := \sum_{i \in N} u_i(x_i) [\mu(i) + \sum_{j \in N \setminus i} m(\mu, ij)] - \frac{1}{2} \sum_{\{i,j\} \subseteq N} m(\mu, ij) |u_i(x_i) - u_j(x_j)|$$

Mayag and Bouyssou [2020] show that the sign of $m(\mu, ij)$ is not always stable. If $m(\mu, ij) > 0$ then there is a positive interaction between criteria i and j . If $m(\mu, ij) < 0$ then there is a negative interaction between i and j . Note that $m(\mu, ij)$ could also be zero, which means there would be no interaction. This sign depends on the choice of the capacity μ .

11. We recall that $\Delta_{ij}\mu(\emptyset) = m(\mu, ij) = \mu(ij) - \mu(i) - \mu(j)$ is the interaction between the two criteria i and j .

1.5.2. Numerical illustration

We consider the following classical example in MCDA.

Example 1.5.1 (Grabisch and Labreuche [2010]). *An assessment of students with the help of three criteria - Mathematics (M), Physics (P) and Language Skills (L) - is given (the set of criteria is $N = \{M, P, L\}$). Assuming an evaluation scale from 0 to 20, consider four students a , b , c and d with the following marks (i.e. $X = \{a, b, c, d\}$):*

	Mathematics (M)	Physics (P)	Language skills (L)
<i>Student a</i>	8	18	12
<i>Student b</i>	8	12	18
<i>Student c</i>	16	18	12
<i>Student d</i>	16	12	18

To select the best students, the director expresses the following preferences

i) *For a student who is bad at Mathematics, Physics is more important than Language Skills :*

$$\implies a \succ b \quad (1.1)$$

ii) *For a student who is good at Mathematics, Language Skills are more important than Physics :*

$$\implies d \succ c \quad (1.2)$$

Let $T = \{P, L\}$ be a subset of N . We have $a = (8, 18, 12) = (a_{-T}, a_T)$, $c = (16, 18, 12) = (c_{-T}, a_T)$, $b = (8, 12, 18) = (b_{-T}, b_T)$ and $d = (16, 12, 18) = (d_{-T}, b_T)$.

If $T = \{P, L\}$ is preferentially independent of its complement $N \setminus T = \{M\}$, then (1.1) $\implies c \succ d$, which contradicts (1.2). So the preference information $a \succ b$ and $d \succ c$ is not representable by a model verifying mutual preferential independence. Therefore, there exists an interaction between criteria. More precisely, M and P have a negative interaction ; P and L (and similarly M and L) have a positive interaction.

<i>Subjects</i> (S)	M	P	L	M, P	M, L	P, L	M, P, L
$\mu(S)$	0.3	0.3	0.2	0.4	0.7	0.7	1

<i>Alternative</i> x	a	b	c	d
$C_\mu(x)$	12.6	12	14.2	15.2

$\{i, j\}$	$\{M, P\}$	$\{M, L\}$	$\{P, L\}$
$m(\mu, ij)$	-0.2	0.2	0.2

These preferences are representable by the previous capacity ($d \succ c \succ a \succ b$).

We are now going to focus on studying the measure of the interaction between the criteria. Defining the importance of each subset of N using a capacity can be seen as a way of measuring the existing interaction between the criteria. In other words, how can our previous TU game results be extended to the context of MCDA ? We will then answer the two following questions :

1) What sort of link are we able to exhibit between **the importance, the higher-order interaction terms** and **the higher-order pure marginal contribution** ?

If the interaction index Ψ is AID then the link between these three notions can be given by the following relation :

$$\Psi(\mu, S) = \Psi_{ext}(\mu, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} \mu(T)}{2}$$

where $\Psi(\mu, S)$ measures the importance of criteria belonging to S .

$\Psi_{ext}(\mu, S)$ is the higher-order pure marginal contribution of criteria of S to the grand coalition and $\sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} \mu(T)}{2}$ is the higher-order interaction of criteria belonging to S with the other criteria.

If we measure the importance with the Shapley interaction index (Ψ_{Sh}) (or the Banzhaf interaction index (Ψ_{Bz}) or the Chaining interaction index (Ψ_{Ch})), we get the following results for our example :

Subjects (S)	M	P	L	M, P	M, L	P, L	M, P, L
$\Psi_{Sh}(\mu, S) = \Psi_{Bz}(\mu, S) = \Psi_{Ch}(\mu, S)$	0.3	0.3	0.4	-0.2	0.2	0.2	0
$\Psi_{ext}(\mu, S)$	0.7	0.7	0.8	-0.2	0.2	0.2	0
$\sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} \mu(T)}{2}$	0.4	0.4	0.4	0	0	0	0

One can notice that the importance of mathematics, physics and languages skills are respectively 30%, 30% and 40%. The importance of mathematics equals the pure marginal contribution of mathematics to the grand coalition $\{M, P, L\}$ minus a weighted sum of all its pairwise interactions. The importance of mathematics and physics equals the pure marginal contribution

of mathematics to physics' pure marginal contribution to the grand coalition minus a weighted sum of the individual impact of languages skills. The rest of the interpretations is done in the same way.

II) What sort of link are we able to exhibit between **the importance, the decomposer and the contribution** ?

If the interaction index Ψ is C-decomposable then the degree of interaction between the criteria belonging to S can be given by the following relation :

$$\Psi(\mu, S) = \varphi(\mu, S) + \sum_{j \in N \setminus S} [\varphi(\mu, S) - \varphi(\mu^{N \setminus j}, S)]$$

$\varphi(\mu, S)$ is the simultaneous interaction of criteria belonging to S measured by φ (the decomposer). $\sum_{j \in N \setminus S} [\varphi(\mu, S) - \varphi(\mu^{N \setminus j}, S)]$ indicates the extent to which the other criteria **contribute** individually to the interactions of criteria belonging to S measured by φ . If we suppose that Ψ is the Shapley interaction index (or the Banzhaf interaction index (Ψ_{Bz}) or the Chaining interaction index (Ψ_{Ch})) and φ is the C-decomposer of Ψ , then we have the following results for our example :

Subjects (S)	M	P	L	M, P	M, L	P, L	M, P, L
$\Psi_{Sh}(\mu, S) = \Psi_{Bz}(\mu, S) = \Psi_{Ch}(\mu, S)$	0.3	0.3	0.4	-0.2	0.2	0.2	0
$\varphi(\mu, S)$	0.3	0.3	0.3	-0.15	0.2	0.15	0
$\sum_{j \in N \setminus S} [\varphi(\mu, S) - \varphi(\mu^{N \setminus j}, S)]$	0	0	0.1	-0.05	0	0.05	0

One can notice that the importance of mathematics is decomposable into two parts : the first part is the importance of mathematics measured by another interaction index φ called the I-decomposer and the second part is the sum of individual contribution of physics and languages skills to the I-decomposer. Furthermore, the importance of mathematics and physics is decomposable into two parts : the first part, called the C-decomposer, measures the importance of mathematics and physics by another interaction index φ and the second part indicates the extent to which languages skills contribute to the importance given by the C-decomposer. The C-decomposer provides a kind of foundation for the interaction index Ψ .

1.6. Conclusion

This paper is devoted to the study of the cardinal-probabilistic interaction indices, especially the Shapley, the Chaining and the Banzhaf interaction indices. Owing to alternative decompositions of the cardinal-probabilistic interaction indices, we provide a numerical illustration of the proposed methodology, by using hypothetical data.

There are several ways in which the interaction index could be explored in further research. The most interesting direction is the extension of the interaction indices to games with a priori relationships between players. In a game with a coalition structure, it is supposed that players organize themselves into disjoint coalitions which are defined a priori. The most well-known solutions are the Owen-Shapley (Owen [1977]) and the Owen-Banzhaf solutions (Owen [1981]). A full characterization of the interaction indices in this context would be worthwhile pursuing.

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Appendix : Proofs

Proof of Theorem 1.3.1. It suffices to show that (ii) is equivalent to (iii), since it has been proved in Grabisch and Roubens [1999a] that (i) and (ii) are equivalent.

(iii) \implies (ii) For any game $v \in TU(N)$, $n \geq 1$, if Ψ is AID, then, for any $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{f_n^s(t)\}_{t=0, \dots, n-s-1}$ such that,

$$\Psi(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2}$$

We know that,

$$\Delta_S v(T) = \sum_{L \subseteq S} (-1)^{s-l} v(L \cup T)$$

For any player $i \in S$, $\Delta_S v(T)$ can be expressed recursively as follows :

$$\Delta_S v(T) = \Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)$$

$$\begin{aligned}
\Psi(v, S) &= \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2} \\
&= \Delta_S v(N \setminus S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times \left[\frac{\Delta_{S \setminus i} v(T \cup ij)}{2} - \frac{\Delta_{S \setminus i} v(T \cup j)}{2} \right. \\
&\quad \left. - \frac{\Delta_{S \setminus i} v(T \cup i)}{2} + \frac{\Delta_{S \setminus i} v(T)}{2} \right] \\
&= \Delta_S v(N \setminus S) - a + c + b - d
\end{aligned}$$

where

$$\begin{aligned}
a &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup ij)}{2} = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \in T}} f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} \\
&= \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} \sum_{j \in T} f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} = \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} \\
b &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \notin T}} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} \\
&= \sum_{T \subseteq N \setminus S} \sum_{j \notin T \cup S} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} = \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} \\
c &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus (S \cup j)} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup j)}{2} = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \in T}} f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T)}{2} \\
&= \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} \sum_{j \in T} f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T)}{2} = \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T)}{2} \\
d &= \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T)}{2} = \sum_{j \in N \setminus S} \sum_{\substack{T \subseteq N \setminus S \\ j \notin T}} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T)}{2} \\
&= \sum_{T \subseteq N \setminus S} \sum_{j \notin T \cup S} f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T)}{2} = \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T)}{2}
\end{aligned}$$

Using the expressions of a , b , c and d , $\Psi(v, S)$ can be written as :

$$\begin{aligned}
\Psi(v, S) &= \Delta_S v(N \setminus S) - \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} + \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \times \frac{\Delta_{S \setminus i} v(T)}{2} \\
&+ \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T \cup i)}{2} - \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \times \frac{\Delta_{S \setminus i} v(T)}{2} \\
&= \Delta_S v(N \setminus S) - \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} t f_n^s(t-1) \left[\frac{\Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)}{2} \right] \\
&+ \sum_{T \subseteq N \setminus S} (n-s-t) f_n^s(t) \left[\frac{\Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)}{2} \right] \\
&= \sum_{\substack{T \subseteq N \setminus S \\ T \neq \emptyset}} [(n-s-t) f_n^s(t) - t f_n^s(t-1)] \frac{\Delta_S v(T)}{2} + (n-s) f_n^s(0) \times \frac{\Delta_S v(\emptyset)}{2} \\
&+ [1 - (n-s) \times \frac{f_n^s(n-s-1)}{2}] \Delta_S v(N \setminus S) \\
&= \sum_{T \subseteq N \setminus S} [(n-s-t) f_n^s(t) - t f_n^s(t-1)] q_n^s(t) \frac{\Delta_S v(T)}{2} + \sum_{T \subseteq N \setminus S} (n-s) f_n^s(0) r_n^s(t) \frac{\Delta_S v(T)}{2} \\
&+ \sum_{T \subseteq N \setminus S} [1 - (n-s) \times \frac{f_n^s(n-s-1)}{2}] k_n^s(t) \Delta_S v(T) \\
&= \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T)
\end{aligned}$$

where

$$q_n^s(t) = \begin{cases} 1 & \text{if } 0 < t < n-s \\ 0 & \text{otherwise} \end{cases} ; k_n^s(t) = \begin{cases} 1 & \text{if } t = n-s \\ 0 & \text{otherwise} \end{cases} ; r_n^s(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$P_n^s(t) = \left[\frac{(n-s-t) f_n^s(t) - t f_n^s(t-1)}{2} \right] q_n^s(t) + \frac{(n-s) f_n^s(0) r_n^s(t)}{2} + \left[1 - \frac{(n-s) f_n^s(n-s-1)}{2} \right] k_n^s(t)$$

Finally, for all $S \not\subseteq N$, for all $T \subseteq N \setminus S$, we have $\Delta_S v(T) = 0$ (Katsushige et al. [2006] page 76). We can deduce that $\Psi(v, S) = 0$.

(iii) \iff (ii) We suppose that, for any game $v \in TU(N)$, $n \geq 1$, and $S \subseteq N$, $s \geq 1$, there exists a collection of real constants $\{P_n^s(t)\}_{t=0, \dots, n-s}$, such that,

$$\Psi(v, S) = \sum_{T \subseteq N \setminus S} P_n^s(t) \Delta_S v(T)$$

We set,

$$f_n^s(t) = \begin{cases} \frac{2 \times P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{2(1 - P_n^s(n-s))}{n-s} & \text{if } t = n - s - 1 \\ \frac{2(t \times f_n^s(t-1) + P_n^s(t))}{n-t-s} & \text{otherwise} \end{cases}$$

$P_n^s(t)$ can be expressed as follows :

$$P_n^s(t) = \left[\frac{(n-s-t)f_n^s(t) - t f_n^s(t-1)}{2} \right] q_n^s(t) + \frac{(n-s)f_n^s(0)r_n^s(t)}{2} + \left[1 - \frac{(n-s)f_n^s(n-s-1)}{2} \right] k_n^s(t)$$

Finally, using the reverse of the first part of this proof, we can deduce that Ψ is an AID interaction index.

$f_n^s(t)$ is simply given as follows

$$f_n^s(t) = \begin{cases} \frac{2 \times P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{2(t \times f_n^s(t-1) + P_n^s(t))}{n-t-s} & \text{otherwise} \end{cases}$$

since by linearity, everything is determined based on dirac games ($e_K(S) = 1$ if $S = K$ and 0 otherwise).

$$\Psi(e_N, S) = P_n^s(n-s) = 1 - \frac{(n-s)f_n^s(n-s-1)}{2}$$

and then,

$$f_n^s(n-s-1) = \frac{2(1 - P_n^s(n-s))}{n-s}$$

We conclude that, an interaction index is AID if and only if it satisfies the three axioms : Linearity, Dummy and Symmetry. \square

Proof of Proposition 1.3.1.

i) The Shapley interaction index is AID and the associated coefficient is

$f_n^s(t) = \frac{2^{(t+1)!(n-t-s-1)!}}{(n-s+1)!}$, with $0 \leq t \leq n-s-1$. Moreover,

$$\begin{aligned} \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} f_n^s(t) &= 2 \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} \frac{(t+1)!(n-t-s-1)!}{(n-s+1)!} \\ &= \frac{2}{(n-s+1)(n-s)} \sum_{t=0}^{n-s-1} (t+1) \\ &= \frac{2}{(n-s+1)(n-s)} \left[n-s + \frac{(n-s)(n-s-1)}{2} \right] = 1 \end{aligned}$$

ii) The Banzhaf interaction index is AID and the associated coefficient is

$$f_n^s(t) = \begin{cases} \frac{1}{(n-s)2^{n-s-1}} & \text{if } t = 0 \\ \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s-1}} & \text{otherwise} \end{cases}$$

Moreover,

$$\begin{aligned} &\sum_{t=0}^{n-s-1} \binom{n-s-1}{t} f_n^s(t) \\ &= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \binom{n-s-1}{t} \frac{t \times f_n^s(t-1) \times 2^{n-s} + 1}{(n-t-s)2^{n-s-1}} \\ &= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + 2 \sum_{t=1}^{n-s-1} \binom{n-s-1}{t-1} f_n^s(t-1) \\ &= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + 2 \sum_{t=0}^{n-s-2} \binom{n-s-1}{t} f_n^s(t) \\ &= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-2} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} \\ &\quad + 2 \sum_{t=0}^{n-s-3} \binom{n-s-1}{t} f_n^s(t) \\ &= \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-1} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^{n-s-2} \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} \\ &\quad + \dots + \frac{1}{(n-s)2^{n-s-1}} + \sum_{t=1}^1 \frac{\binom{n-s-1}{t}}{(n-t-s)2^{n-s-1}} + \frac{1}{(n-s)2^{n-s-1}} \\ &= \binom{n-s-1}{0} \frac{1}{2^{n-s-1}} + \binom{n-s-1}{1} \frac{1}{2^{n-s-1}} + \binom{n-s-1}{2} \frac{1}{2^{n-s-1}} + \dots + \binom{n-s-1}{n-s-1} \frac{1}{2^{n-s-1}} \\ &= \frac{1}{2^{n-s-1}} \sum_{t=0}^{n-s-1} \binom{n-s-1}{t} \\ &= 1 \end{aligned}$$

□

Proof of Proposition 1.3.2. This result follows from the proof of Theorem 1.3.1. \square

Proof of Proposition 4.6.2. Given a game $v \in TU(N)$ with $n \geq 2$, i and j two players in N : It is easier to show by induction on n that if an interaction index is I-decomposable, then its I-decomposer φ' is unique. Proposition 2 (page 39) of Casajus and Huettner [2018] states that if an interaction index is CH-decomposable, then there exists a unique CH-decomposer φ that is itself CH-decomposable. Furthermore, Theorem 4 (page 40) of Casajus and Huettner [2018] shows that

$$[\varphi(v, i) - \varphi(v^{N \setminus j}, i)] = [\varphi(v, j) - \varphi(v^{N \setminus i}, j)]$$

Hence, $\varphi = \varphi'$. This implies that the unique CH-decomposable CH-decomposer of any CH-decomposable interaction index is an I-decomposer of this interaction index and therefore coincides with the unique I-decomposer. \square

Proof of Proposition A1. Let φ and φ' be two C-decomposers of Ψ . Given a game $v \in TU(N)$ and for all $S \in 2^N \setminus \emptyset$ a coalition, we will show by induction on the cardinality of n that $\varphi(v, S) = \varphi'(v, S)$.

Induction basis : given a game $v \in TU(N)$ and for all $S \in 2^N \setminus \emptyset$, if $n = 1$ then, $\Psi(v, S) = \varphi(v, S) = \varphi'(v, S)$.

Induction hypothesis : given a game $v \in TU(N)$ and for all $S \in 2^N \setminus \emptyset$ with $2 \leq n \leq k$ ($k \in \mathbb{N}$), we suppose that $\varphi(v, S) = \varphi'(v, S)$

Induction step : given a game $v \in TU(N)$ with $n = k + 1$ and for all $S \in 2^N \setminus \emptyset$,

$$\begin{aligned} \Psi(v, S) &= \varphi(v, S) + \sum_{j \in N \setminus S} [\varphi(v, S) - \varphi(v^{N \setminus j}, S)] \\ &= \varphi'(v, S) + \sum_{j \in N \setminus S} [\varphi'(v^N, S) - \varphi'(v^{N \setminus j}, S)] \end{aligned}$$

$$(n - s + 1)\varphi(v, S) - \sum_{j \in N \setminus S} \varphi(v^{N \setminus j}, S) = (n - s + 1)\varphi'(v, S) - \sum_{j \in N \setminus S} \varphi'(v^{N \setminus j}, S)$$

$$(n - s + 1)\varphi(v, S) = (n - s + 1)\varphi'(v, S)$$

$$\varphi(v, S) = \varphi'(v, S)$$

□

Lemma 1.6.1

An interaction index Ψ satisfies the Recursive axiom if and only if Ψ is defined as follows.

For every game $v \in TU(N)$ with $n \geq 2$, $S \subseteq N$ a coalition of N with $2 \leq s \leq n$ and any player $i \in S$, we have

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=\beta}} \Psi(v_{\cup((S \setminus i) \setminus T)}^{N \setminus (S \setminus i)}, i)$$

Proof of Lemma 1.6.1. \Leftarrow) We will show that the interaction index Ψ satisfies the Recursive axiom.

For all game $v \in TU(N)$ with $n \geq 2$, $S \subseteq N$ a coalition of N with $2 \leq s \leq n$ and any player $i \in S$. W.l.o.g. we suppose that $S := \{i_1, i_2, \dots, i_s\}$ and $i = i_1$. We have,

$$\Psi(v_{\cup i_s}^{N \setminus i_s}, S \setminus i_s) = \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup(S \setminus i_1) \setminus T}^{N \setminus (S \setminus i_1)}, i_1)$$

and

$$\Psi(v^{N \setminus i_s}, S \setminus i_s) = \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1)$$

Furthermore,

$$\begin{aligned}
\Psi(v, S) &= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta; i_s \notin T}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{\beta=1}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta; i_s \in T}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{\beta=1}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta-1}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) + \sum_{\beta=0}^{s-2} (-1)^{\beta+1} \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) - \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\
&= \Psi(v_{\cup i_s}^{N \setminus i_s}, S \setminus i_s) - \Psi(v^{N \setminus i_s}, S \setminus i_s)
\end{aligned}$$

Ψ satisfies the Recursive axiom.

\implies) We suppose that an interaction index Ψ satisfies the Recursive axiom. Let us show by induction on the cardinality of the coalition that Ψ is defined as follows :

For all game $v \in TU(N)$ with $n \geq 2$ and $S \subseteq N$ a coalition of N ($2 \leq s \leq n$). W.l.o.g. we suppose that $\{i_1, i_2, \dots, i_s\}$.

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

Induction basis : for all game $v \in TU(N)$ and S any coalition of N containing two players.

Let i_1 and i_2 be these two players. Since Ψ satisfies the Recursive axiom, we have

$$\Psi(v, S) = \Psi(v_{\cup i_2}^{N \setminus i_2}, i_1) - \Psi(v^{N \setminus i_2}, i_1) = \sum_{\beta=0}^{2-1} (-1)^\beta \sum_{\substack{T \subseteq i_2 \\ t=\beta}} \Psi(v_{\cup(i_2 \setminus T)}^{N \setminus i_2}, i_1)$$

Induction hypothesis : for all game $v \in TU(N)$ with $n \geq 2$, let $k \in \mathbb{N}$ be any integer such that $k \leq n$ and let $S = \{i_1, i_2, \dots, i_s\}$ be any coalition of N with $2 < s \leq k$. We suppose that

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

Induction step : for all game $v \in TU(N)$ with $n \geq 2$, $n > k$; $S := \{i_1, i_2, \dots, i_s\}$ any coalition of N with $s = k + 1$. Let us show that,

$$\Psi(v, S) = \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1)$$

$$\begin{aligned} \Psi(v, S) &= \Psi(v_{\cup i_s}^{N \setminus i_s}, S \setminus i_s) - \Psi(v^{N \setminus i_s}, S \setminus i_s) \\ &= \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) - \sum_{\beta=0}^{s-2} (-1)^\beta \sum_{\substack{T \subseteq (S \setminus i_1) \setminus i_s \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus (T \cup i_s))}^{N \setminus (S \setminus i_1)}, i_1) \\ &= \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i_1 \\ t=\beta}} \Psi(v_{\cup((S \setminus i_1) \setminus T)}^{N \setminus (S \setminus i_1)}, i_1) \end{aligned}$$

□

Proof of Theorem 1.4.1. Let Ψ be a CH-decomposable interaction index. From Proposition 4.6.2, the I-decomposer is unique and coincides with the unique CH-decomposable CH-decomposer. Using Proposition 1.4.1, the I-decomposer is the interaction index φ given by : for all game $v \in TU(N)$ and for all $j \in N$,

$$\varphi(v, j) = \sum_{\substack{T \subseteq N \\ j \in T}} \frac{m(v^\Psi, T)}{t^2} \quad (\text{A1})$$

Let us show by induction that the unique C-decomposer of the interaction index Ψ is the interaction index Ψ_d defined as follows :

For all game $v \in TU(N)$, for all coalition $S \in 2^N \setminus \emptyset$ and for any player $i \in S$,

$$\Psi_d(v, S) = \begin{cases} \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=\beta}} \varphi(v_{\cup((S \setminus i) \setminus T)}^{N \setminus (S \setminus i)}, i) & \text{if } s \geq 2 \\ \varphi(v, i) & \text{if } s = 1 \end{cases}$$

From Lemma 1.6.1, Ψ_d satisfies the Recursive axiom when the coalition contains more than one player.

Induction basis : for all game $v \in TU(N)$, $S \in 2^N \setminus \emptyset$, if $s = 1$ (that is $S = i$) then Ψ is

I-decomposable from the hypothesis and the I-decomposer is the interaction index φ .

$$\Psi_d(v, i) = \varphi(v, i)$$

Induction hypothesis : for all game $v \in TU(N)$, let $k \in \mathbb{N}$ be any integer such that $k \leq n$ and let S be any coalition of N with $2 \leq s \leq k$. We suppose that :

$$\Psi(v, S) = \Psi_d(v, S) + \sum_{j \in N \setminus S} [\Psi_d(v, S) - \Psi_d(v^{N \setminus j}, S)]$$

Induction step : for all game $v \in TU(N)$, $n > k$ and $S \in 2^N$ such that $s = k + 1$. Let us show that :

$$\Psi(v, S) = \Psi_d(v, S) + \sum_{j \in N \setminus S} [\Psi_d(v, S) - \Psi_d(v^{N \setminus j}, S)]$$

Let $i \in S$. Since Ψ satisfies the Recursive axiom, we have

$$\begin{aligned} \Psi(v, S) &= \Psi(v_{\cup i}^{N \setminus i}, S \setminus i) - \Psi(v^{N \setminus i}, S \setminus i) \\ &= \Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) + \sum_{j \in N \setminus S} [\Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) - \Psi_d(v_{\cup i}^{N \setminus ij}, S \setminus i)] \\ &\quad - \Psi_d(v^{N \setminus i}, S \setminus i) - \sum_{j \in N \setminus S} [\Psi_d(v^{N \setminus i}, S \setminus i) - \Psi_d(v^{N \setminus ij}, S \setminus i)] \\ &= \Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) - \Psi_d(v^{N \setminus i}, S \setminus i) + \sum_{j \in N \setminus S} \left[\left(\Psi_d(v_{\cup i}^{N \setminus i}, S \setminus i) \right. \right. \\ &\quad \left. \left. - \Psi_d(v^{N \setminus i}, S \setminus i) \right) - \left(\Psi_d(v_{\cup i}^{N \setminus ij}, S \setminus i) - \Psi_d(v^{N \setminus ij}, S \setminus i) \right) \right] \\ &= \Psi_d(v, S) + \sum_{j \in N \setminus S} [\Psi_d(v, S) - \Psi_d(v^{N \setminus j}, S)] \end{aligned}$$

Since Ψ_d satisfies the Recursive axiom. Ψ_d is a C-decomposer of the interaction index Ψ . Ψ is C-decomposable and from Proposition A1, the C-decomposer is unique. \square

Proof of Proposition 1.4.5. Let $v \in TU(N)$ ($n \geq 2$) be a game on N and $S \in 2^N$ with $s \geq 2$.
Let $j \in S$,

$\Psi_{Ch}(v, S)$

$$\begin{aligned}
&= \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_S v(T) \\
&= \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T \cup j) - \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T) \\
&= \sum_{\substack{T \subseteq N \setminus (S \setminus j) \\ j \in T}} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) - \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T) \\
&= \sum_{T \subseteq N \setminus (S \setminus j)} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) \\
&\quad - \sum_{\substack{T \subseteq N \setminus (S \setminus j) \\ j \notin S}} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) - \sum_{T \subseteq N \setminus S} \frac{s(s+t-1)!(n-t-s)!}{n!} \Delta_{S \setminus j} v(T) \\
&= \sum_{T \subseteq N \setminus (S \setminus j)} \frac{s(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) \\
&\quad - \sum_{T \subseteq N \setminus S} \frac{s(s+t-2)!(n-t-s)!(n-t-s+1+s+t-1)}{n!} \Delta_{S \setminus j} v(T) \\
&= s \sum_{T \subseteq N \setminus (S \setminus j)} \frac{(s+t-2)!(n-t-s+1)!}{n!} \Delta_{S \setminus j} v(T) - s \sum_{T \subseteq N \setminus S} \frac{(s+t-2)!(n-t-s)!}{(n-1)!} \Delta_{S \setminus j} v(T) \\
&= \frac{s}{s-1} \left[\Psi_{Ch}(v, S \setminus j) - \Psi_{Ch}(v^{N \setminus j}, S \setminus j) \right]
\end{aligned}$$

For any $i \in S$, we repeatedly apply this process and we obtain,

$$\Psi_{Ch}(v, S) = \left(\prod_{l=0}^{s-2} \frac{s-l}{s-l-1} \right) \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=s-\beta-1}} \Psi_{Ch}(v^{N \setminus ((S \setminus i) \setminus T)}, i)$$

Finally, by induction on the cardinality of S , we obtain

$$\Psi_{Ch}(v, S) = s\Psi_{dc}(v, S) + \sum_{j \in N \setminus S} [\Psi_{dc}(v, S) - \Psi_{dc}(v^{N \setminus j}, S)]$$

where

$$\Psi_{dc}(v, S) = \begin{cases} \left(\prod_{l=0}^{s-2} \frac{s-l}{s-l-1} \right) \sum_{\beta=0}^{s-1} (-1)^\beta \sum_{\substack{T \subseteq S \setminus i \\ t=s-\beta-1}} \varphi(v^{N \setminus ((S \setminus i) \setminus T)}, i) & \text{if } s \geq 2 \\ \varphi(v, i) & \text{if } s = 1 \end{cases}$$

φ is the unique I-decomposer of the Shapley solution and is given by Equation (A4). The Chaining interaction index is not C-decomposable. \square

Proof of Proposition 2.3.4. Let Ψ^p be a cardinal-probabilistic interaction index applied to a game $v \in TU(N)$. From Corollary 1.3.1, for any coalition $S \subseteq N$,

$$\Psi^p(v, S) = \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2}$$

where $f_n^s(t)$ is given by

$$f_n^s(t) = \begin{cases} \frac{2 \times P_n^s(0)}{n-s} & \text{if } t = 0 \\ \frac{2(t \times f_n^s(t-1) + P_n^s(t))}{n-t-s} & \text{otherwise} \end{cases}$$

and the set $\{P_n^s(t)\}_{t=0, \dots, n-s}$ are nonnegative constants associated to the cardinal-probabilistic interaction index. According to Katsushige et al. [2006], the coefficients $P_n^s(t)$ obey the recurrence relation :

$$P_n^s(t) + P_n^s(t+1) = P_{n-1}^s(t)$$

$$\begin{aligned} & \Psi^p(V, S) + \sum_{j \in N \setminus S} [\Psi^p(v, S) - \Psi^p(v^{N \setminus j}, S)] \\ &= \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} f_n^s(t) \times \frac{\Delta_{S \cup j} v(T)}{2} + \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} P_n^s(t+1) \Delta_{S \cup j} v(T) \\ &= \Psi_{ext}(v, S) - \sum_{j \in N \setminus S} \sum_{T \subseteq N \setminus S \cup j} \left[\frac{f_n^s(t)}{2} - P_n^s(t+1) \right] \Delta_{S \cup j} v(T) \\ &= \chi(v, S) \end{aligned}$$

From Proposition A1, the C-decomposer is unique. □

Decompositions of inequality measures from the perspective of the Shapley-Owen value

Résumé :

Cet article (chapitre) propose trois nouvelles décompositions des mesures d'inégalité tirées du cadre de la théorie des jeux coopératifs. Il permet de prendre en considération l'impact des interactions des joueurs plutôt que les contributions des joueurs aux inégalités. Ces approches innovantes sont particulièrement adaptées à l'étude des inégalités de revenus lorsque le revenu a une structure hiérarchique : le revenu est composé de plusieurs sources primaires, avec la particularité que chacune d'elles est également composée de sources secondaires. Nous revisitons la valeur de Shapley-Owen qui quantifie l'importance de chacune de ces sources secondaires dans l'inégalité globale de revenu. Notre contribution principale est de décomposer cette importance en deux parties : la contribution marginale pure de la source considérée et une somme pondérée des interactions par paires. Nous proposons ensuite une caractérisation de chacune des valeurs avec structure hiérarchique décomposables additivement *via* l'interaction (AID). Nous donnons une application de ces décompositions dans le contexte de la théorie des inégalités.

Mots clés : Théorie de jeux, structure hiérarchique, caractérisation, AID, mesures d'inégalité, valeur de Shapley-Owen.

JEL Codes : C71, D63.

Abstract :

This article (chapter) proposes three new decompositions of inequality measures, drawn from the framework of cooperative game theory. It allows the impact of players' interactions rather than players' contributions to inequality to be taken into consideration. These innovative approaches are especially suited for the study of income inequality when the income has a hierarchical structure : the income is composed of several primary sources, with the particularity that each of them is also composed of secondary sources. We revisit the Shapley-Owen value that quantifies the importance of each of these secondary sources in the overall income inequality. Our main contribution is to decompose this importance into two parts : the *pure marginal contribution* of the considered source and a weighted sum of pairwise interactions. We then provide an axiomatic characterization of each additive interaction decomposable (AID) coalitional value considered in this paper. We give an application of these decompositions in the context of inequality theory.

Key words : Game theory, Hierarchical structures, Axiomatic characterization, AID, Inequality measures, Shapley-Owen value.

JEL Codes : C71, D63.

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2.1. Introduction

IN the economic literature, there are two main approaches to decomposing inequality measures. The first one, the decomposition by sub-population, consists of splitting the observed inequality between the groups composing the society. For instance, it may be useful to know whether the income inequality is more or less strong within the group of men than it is in the group of women, and to estimate the share of inequality attributed to the income differences between men and women. When this decomposition is envisaged in the literature, the inequality index often used is the Theil index as its decomposition, first proposed by Theil [1967], allows to split inequality into two components : within-group inequality and between-group inequality. However, there are other measures which are decomposable into sub-populations, such as the Gini index with the decomposition proposed by Dagum [1997], a decomposition that has been extended by Chameni [2006a,b], Mussard et al. [2006], Ebert [2010] and Mornet et al. [2013]. The other dominant approach decomposes inequality measures into a set of factors (e.g. income sources). For instance, with a view on public policies it might be worthwhile evaluating the importance of each income source of income inequality (e.g. labor, capital, social transfers...), especially to appreciate the effectiveness of social transfers to reduce inequality. This decomposition has been notably axiomatized by Shorrocks [1982b], but only the decomposition of (the square of) the coefficient of variation respects these axioms. Some authors have departed

from these axioms to propose a decomposition of relative inequality measures. For example, Lerman and Yitzhaki [1985] proposed a factor decomposition of the Gini index. Moreover, the multiple decomposition is an approach that takes into account both sub-population effects and factor effects on inequality (e.g. Jenkins and Van Kerm [2006]), and sometimes an additional dimension (e.g. time with Mussard and Savard [2012] or Mussini [2013]).

Furthermore, all inequality decomposition measures do not use the same methodologies. There are various processes to decompose an inequality index, some are based on mathematical arrangements (all the previously cited authors, with the exception of Lerman and Yitzhaki [1985]), some use statistical tools (e.g. Lerman and Yitzhaki [1985], Firpo et al. [2018]), while others originate from game theory (Chantreuil and Trannoy [2013], Shorrocks [2013]). In the present article, our innovative decompositions are based on the last category, i.e. cooperative game theory. In the continuity of Auvray and Trannoy [1992], Chantreuil and Trannoy [2013] and Shorrocks [2013] proposed the Shapley decomposition. This methodology allows to identify and rank the factors that are sources of inequality - in other words, to evaluate the importance of each inequality factor, which is determined by the Shapley value (?). It should also be noted that this decomposition works with a wide set of inequality measures. Next, this decomposition was extended by Chantreuil et al. [2019]. They pointed out that the importance of a factor in an income inequality does not match with the variation of inequality that would be observed if the income differences associated with it were dropped. This variation corresponds in absolute terms to the pure marginal contribution (PMC) of the source, that is the inequality difference between a situation where all factors are a source of inequality and a situation where one considered factor is no longer a source of differences between individuals. As highlighted by Chantreuil et al. [2019], the importance of the source is equal to its PMC minus a weighted sum of pairwise interactions. We know that a pairwise interaction between two factors is equal to the inequality created by both of them in the presence of another set of factors T , minus the inequality created by each of them in the presence of T , plus the inequality generated by T (see definition 2.3.1). The pairwise interactions highlight the nature of the relation of each pair of factors in the inequality game (complementary, substitutable, independent) and the intensity of this link. In other words, it indicates how much two sources of inequality might favor the same individuals or, on the contrary, how much one source can compensate the inequality created by the other. Thus, the inequality variation observed after such a change would be equal to the importance of the considered attribute only if it had no interaction with the other income sources

considered in the decomposition. Recently, Courtin et al. [2020] worked on the decomposition of interaction indices. Interaction indices can be viewed as values, which not only assign a payoff to every source but also to every coalition or group of sources (For more information see Courtin et al. [2020]).

However, the decomposition of the Shapley value proposed by Chantreuil et al. [2019] is not well suited when the income sources have a hierarchical structure. For instance, when the income can be divided into three sources (e.g. capital income, labor income and social transfers), it is valuable to know the contribution of all “secondary” sources (e.g. financial and estate capital, wage and bonus, family and housing transfers) to their respective “primary” sources in order to better appreciate the source of inequality. In such cases, Shorrocks [2013] and Chantreuil and Trannoy [2013] proposed to use the Shapley-Owen value (Owen [1977]), notably because the Shapley-Owen value of one primary source is independent from the desegregation of the other primary sources, which is not the case with the Shapley value. Following the previous example, this means that the contribution of labor income to the overall income inequality won't change according to the number of secondary sources considered in the capital income.

Our contribution. In this paper, we propose three new decompositions of inequality measures which split into two parts the importance of each source of inequality : its PMC and a term of pairwise interactions (Propositions 2.4.1, 4.4.1 and 2.4.3). However, unlike Chantreuil et al. [2019], we do not decompose an inequality index with the Shapley value but do so more globally by using the Shapley-Owen value⁴ (Owen [1977]), following the propositions of Shorrocks [2013] and Chantreuil and Trannoy [2013]. The Shapley-Owen value belongs to the class of coalitional semivalues introduced and axiomatically characterized by Albizuri and Zarzuelo [2004] while Dubey et al. [1981] defined semivalues. Coalitional semivalues link the classic, well-known coalitional values (Shapley-Owen value (Owen [1977]) and Banzhaf-Owen value (Owen [1981])) since most of the properties satisfied by the Shapley-Owen value are found to hold also for the Banzhaf-Owen value and most of the coalitional semivalues. From a theoretical point of view, coalitional semivalues can be considered as an alternative or complement to the Shapley-Owen and Banzhaf-Owen values. We propose three decompositions of the Shapley-Owen value and we show that the second decomposition remains valid for the class of coalitional semivalues. Following each decomposition, we provide an axiomatic characterization of additive interaction decomposable (AID) coalitional values. The rest of the paper is a numerical

4. Which is well known as a generalization of the Shapley Value.

illustration of the proposed methodology, using hypothetical data.

Organization of the paper. The next section is mainly devoted to notations and recalls needed to obtain our three decompositions. Three innovative Shapley-Owen decompositions and axiomatic characterizations of AID coalitional values are proposed in section three, while the fourth section provides an application of our decompositions to analyze the inequality of a given hierarchical income distribution. Lastly, section five concludes the paper, and all the proofs are presented in the Appendix.

2.2. Preliminaries

Terminology and notation. In this paper, sets will be normally represented by upper-case italic letters, while their cardinalities will be denoted by the corresponding lower-case italic letters. We denote by the countably infinite set \mathfrak{U} , the *universe of all players* and by \mathcal{N} the set of all finite subsets of \mathfrak{U} . For any set $E \subseteq \mathfrak{U}$, the set of all subsets of E is denoted by 2^E . Any non-empty subset of \mathfrak{U} is called *coalition* of \mathfrak{U} , while any finite partition of \mathfrak{U} will be called *coalitional structure* over \mathfrak{U} . We have determined such a coalitional structure throughout the document and it will be denoted by $\mathfrak{B} := \{B_1, \dots, B_h\}$, $h \geq 1$. Furthermore, for any coalition $T \in \mathcal{N}$, the set $\mathfrak{B}^T := \{X \cap T : X \cap T \neq \emptyset, X \in \mathfrak{B}\}$ is the coalitional structure over T induced by \mathfrak{B} .

We will usually consider the finite set of players⁵ $N := \{i_1, i_2, \dots, i_n\} \in \mathcal{N}$, $n \geq 1$. Thus, we set $\mathfrak{B}^N := \{S_\ell, \ell \in M\}$ as the coalitional structure over N induced by \mathfrak{B} , where $M := \{1, \dots, m\}$ with $1 \leq m \leq n$. Each element of \mathfrak{B}^N will be called a *structural coalition*, and the *representative* in M of any player $i \in N$ is the unique element $m_i \in M$ such that $i \in S_{m_i}$. Consequently, by setting $M_T := \{\ell \in M : S_\ell \cap T \neq \emptyset\}$, for any $T \in 2^N$, $T \neq \emptyset$, the coalitional structure $\mathfrak{B}^T = \{S_\ell \cap T : \ell \in M_T\}$. In order to avoid heavy notation, for all $T \in 2^N$ and $P \in 2^{M_T}$, we posit $\mathbf{P}_T := \bigcup_{\ell \in P} (S_\ell \cap T)$, in particular $\mathbf{P}_N = \bigcup_{\ell \in P} S_\ell$ and $\mathbf{P}_{N \setminus i} = \bigcup_{\ell \in P} (S_\ell \cap (N \setminus i))$, $i \in N$. \mathbf{P}_T represents the union of all the members of the structural coalitions in P that also are in T .

Let $\pi : \mathfrak{U} \rightarrow \mathfrak{U}$ be a mapping, and $\pi\mathfrak{B} := \{\pi X : X \in \mathfrak{B}\}$.⁶ Let us define the set $\mathfrak{B}_\pi := \pi\mathfrak{B} \cup \{\mathfrak{U} \setminus \pi\mathfrak{U}\}$ if $\mathfrak{U} \setminus \pi\mathfrak{U} \neq \emptyset$, and $\mathfrak{B}_\pi := \pi\mathfrak{B}$ otherwise. Next, \mathfrak{B}_π is a coalitional

5. For instance, the players can be the income sources supposed finite.

6. Notice that, while \mathfrak{B} is a coalitional structure over \mathfrak{U} , $\pi\mathfrak{B}$ might not be a coalitional structure over \mathfrak{U} .

structure over \mathfrak{U} if and only if, $\pi B_r \cap \pi B_\ell = \emptyset$ whenever $r \neq \ell$.⁷

For the sake of convenience, we will omit braces for singletons, e.g. by writing $v(i)$ (resp. $u \setminus i$) instead of $v(\{i\})$ (resp. $u \setminus \{i\}$); similarly for pairs, ij instead of $\{i, j\}$. For $N \in \mathcal{N}$, $L \subseteq N$ and for any $j \in N$, we set,

$$Coal [\mathfrak{B}^L, j] := \{T \subsetneq N / T := \mathbf{P}_L \cup K; K \subseteq (S_{m_j} \cap L) \setminus j; P \subseteq M_L \setminus \{m_j\}\}. \quad (\text{A1})$$

Some members in L who are in the same structural coalition with j (i.e. K) join the union of all the members of the structural coalitions over L in P (i.e. \mathbf{P}_L), and this union forms a coalition T . $Coal [\mathfrak{B}^L, j]$ then represents the set of such T .

2.2.1. Transferable utility games

Any pair (\mathfrak{U}, v) where v is a function on $2^{\mathfrak{U}}$ satisfying $v(\emptyset) = 0$, and which assigns to every coalition $S \subseteq \mathfrak{U}$ a real number $v(S)$, seen as the worth of S , is called a *TU game* — *Transferable Utility game*. A subset $L \subseteq \mathfrak{U}$ is a carrier of v , if $v(T \cap L) = v(T)$ for all $T \subseteq \mathfrak{U}$. For $N \in \mathcal{N}$, we denote by Γ the set of all games on \mathfrak{U} with finite carriers and by $\Gamma(N)$ the subspace of Γ of games with carrier N . Following usual practice, (\mathfrak{U}, v) will be shortened by simply writing v . In addition, for each $v \in \Gamma$ and any $S \subseteq \mathfrak{U}$, we denote by v^S , the *restricted game* of v upon S , given by $v^S(T) := v(T)$, for all $T \subseteq S$; and we denoted by $v_{\cup L}^S$ with $L \subseteq N \setminus S$, the *reduced game* of v upon S in the presence of L , defined by $v_{\cup L}^S(T) := v(L \cup T) - v(T)$, for all $T \subseteq S$. Let $\pi : \mathfrak{U} \rightarrow \mathfrak{U}$ be a mapping, the pair $(N, \pi v)$ is also a TU game where $\pi v(S) := v(\pi^{-1}S)$ for all $S \subseteq N$. A player $i \in \mathfrak{U}$ is dummy in a game $v \in \Gamma$ if $v(S \cup i) - v(S) = v(i)$ for all $S \subseteq \mathfrak{U} \setminus i$.

Now, given any coalition $T \in 2^{\mathfrak{U}}$; (resp. $T \in 2^N$, $N \in \mathcal{N}$), the *unanimity game* on T denoted by u_T , is defined in such a way that, for all $L \subset \mathfrak{U}$ (resp. $L \subseteq N$), $u_T(L)$ is equal to 1 if $T \subseteq L$, and 0 otherwise. Furthermore, as pointed out by ?, the set of unanimity games forms a basis of the space Γ (resp. $\Gamma(N)$). In fact, for any $v \in \Gamma$, $v = \sum_{\substack{T \in 2^{\mathfrak{U}} \\ T \neq \emptyset}} \lambda_T(v) \cdot u_T$, where $\lambda_T(v) := v(T) - \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} \lambda_S(v)$, for all $T \in 2^{\mathfrak{U}} \setminus \emptyset$ (resp. for any $v \in \Gamma(N)$, $v = \sum_{\substack{T \in 2^N \\ T \neq \emptyset}} \lambda_T(v) \cdot u_T$, where $\lambda_T(v) := v(T) - \sum_{\substack{S \subseteq T \\ S \neq \emptyset}} \lambda_S(v)$, for all $T \in 2^N \setminus \emptyset$) — see *Harsanyi dividends* (Harsanyi [1959a]).

A *value* is a function $\varphi : \Gamma \rightarrow \mathbb{R}$ that assigns to every player i in a game $v \in \Gamma$ their pros-

7. In that case, \mathfrak{B}_π^N is the coalitional structure induced by \mathfrak{B}_π over N , and $\pi S_j \cap \pi S_k = \emptyset$ whenever $j \neq k$.

pect $\varphi_i(v)$ of playing the game. Semivalues were introduced and axiomatically characterized by Dubey et al. [1981], who also defined a value φ^S as a semivalue when for any $N \in \mathcal{N}$, there exists a family of nonnegative real numbers $\{q(n, t)\}_{t=0}^{n-1}$, fulfilling $\sum_{t=0}^{n-1} \binom{n-1}{t} q(n, t) = 1$ such that for all $i \in N$ and all $v \in \Gamma(N)$, $\varphi_i^S(v) = \sum_{T \subseteq N \setminus i} q(n, t) [v(T \cup i) - v(T)]$. For instance, the Shapley value (Shapley, 1953), denoted by φ^{Sh} where $q(n, t) = \frac{t!(n-t-1)!}{n!}$, and, the Banzhaf value (Owen, 1975), denoted by φ^{Bz} where $q(n, t) = \frac{1}{2^{n-1}}$, are well-known examples of semivalues.

2.2.2. Games with a coalitional structure

A *game with a coalitional structure* is defined as a triplet $(\mathfrak{U}, v, \mathfrak{B})$ where $v \in \Gamma$ and \mathfrak{B} is a coalitional structure over \mathfrak{U} . Henceforth, we will write (v, \mathfrak{B}) instead of the game $(\mathfrak{U}, v, \mathfrak{B})$. We denote by G the set of all finite games enriched with a coalitional structure and $G(N)$ a subspace of G of games with carrier N . For any $N \in \mathcal{N}$, $M := \{1, 2, \dots, m\}$ is the set of representatives of structural coalitions in \mathfrak{B}^N . The *quotient game* $(M, v^q) \in \Gamma(M)$ is the *restricted game* upon M played by the structural coalitions, or rather, by the set M of their representatives as follows : $v^q(P) = v(\mathbf{P}_N)$, for all $P \subseteq M$. This article is mainly based on the concept of *coalitional value*, and by coalitional value we mean any function Ψ that assigns to every player i in a game $(v, \mathfrak{B}) \in G$ their prospect $\Psi_i(v, \mathfrak{B}) \in \mathbb{R}$ of playing the game. We denote by Ψ^N the restriction of Ψ to $G(N)$.

Among the most well-known coalitional values in cooperative games theory, we consider the Shapley-Owen value (Owen [1977]) and the Banzhaf-Owen value (Owen [1981]). We define these two coalitional values as follows :

Definition 2.2.1 (Shapley-Owen value (Owen [1977]))

For any $N \in \mathcal{N}$ and any $(v, \mathfrak{B}) \in G(N)$, the Shapley-Owen value denoted by Ψ^{Sh} is the coalitional value defined for all $i \in N$ as follows,

$$\Psi_i^{Sh}(v, \mathfrak{B}) := \sum_{P \subseteq M \setminus m_i} \sum_{K \subseteq S_{m_i} \setminus i} \frac{p!(m-p-1)!k!(s_{m_i}-k-1)!}{m!s_{m_i}!} [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)].$$

We recall that $\mathfrak{B}^N := \{B_\ell \cap N : B_\ell \cap N \neq \emptyset, B_\ell \in \mathfrak{B}\} \equiv \{S_\ell : \ell \in M\}$ with $M := \{1, 2, \dots, m\}$.^a

a. M being the set of representatives of the structural coalitions in \mathfrak{B}^N .

Definition 2.2.2 (Banzhaf-Owen value (Owen [1981]))

For any $N \in \mathcal{N}$ and any $(v, \mathfrak{B}) \in G(N)$, the Banzhaf-Owen value denoted by Ψ^{Bz} is the coalitional value defined for all $i \in N$ as follows,

$$\Psi_i^{Bz}(v, \mathfrak{B}) := \sum_{P \subseteq M \setminus m_i} \sum_{K \subseteq S_{m_i} \setminus i} \frac{1}{2^{m-1}} \frac{1}{2^{s_{m_i}-1}} [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)].$$

Where $\mathfrak{B}^N := \{S_\ell : \ell \in M\}$ with $M := \{1, 2, \dots, m\}$.

The notion of coalitional semivalue was introduced and axiomatically characterized by Albizuri and Zarzuelo [2004].

Definition 2.2.3 (Coalitional semivalue ⁸)

Let \mathfrak{B} be a coalitional structure over \mathfrak{U} . For any $N \in \mathcal{N}$ and $i \in N$, Ψ^{CS} is a coalitional semivalue, if there are two families of nonnegative real numbers $\{a(m, p)\}_{p=0}^{m-1}$ and $\{b(s_{m_i}, k)\}_{k=0}^{s_{m_i}-1}$ fulfilling the two conditions $\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) = 1$ and $\sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} b(s_{m_i}, k) = 1$, such that :

$$\begin{aligned} \Psi_i^{CS}(v, \mathfrak{B}) &:= \sum_{P \subseteq M \setminus m_i} \sum_{K \subseteq S_{m_i} \setminus i} a(m, p) \cdot b(s_{m_i}, k) [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)] \\ &= \sum_{T \in Coal[\mathfrak{B}^N, i]} a(m, p) \cdot b(s_{m_i}, k) [v(T \cup i) - v(T)]. \end{aligned}$$

for all $v \in \Gamma(N)$.

The coalitional semivalue of a given player i in a game with a coalitional structure $(v, \mathfrak{B}) \in G(N)$ is equal to the mathematical expectation on $Coal[\mathfrak{B}^N, i]$ of the marginal contribution $v(T \cup i) - v(T)$ of player i to a coalition $T \in Coal[\mathfrak{B}^N, i]$ with respect to the probability distribution $\{Q_T\}_{T \in Coal[\mathfrak{B}^N, i]}$ where $Q_T = a(m, p) \cdot b(s_{m_i}, k)$. We recall that m, p, s_{m_i} and k are determined by T (see equation (A1)).

In particular, the Shapley-Owen value and Banzhaf-Owen value are well-known examples of coalitional semivalues.

We shall consider the following nine properties as *axioms* for a coalitional value Ψ on $(v, \mathfrak{B}) \in G$.

- A1.** (*Efficiency or Carrier (i)*) For all game $(v, \mathfrak{B}) \in G(N)$, $\sum_{i \in N} \Psi_i(v, \mathfrak{B}) = v(N)$.
- A2.** (*Carrier (ii)*) For all $(v, \mathfrak{B}), (v, \mathfrak{B}') \in G$, if there exists a carrier N of v such that $\mathfrak{B}^N = (\mathfrak{B}')^N$, then $\Psi_i(v, \mathfrak{B}) = \Psi_i(v, \mathfrak{B}')$, for all player i .
- A3.** (*Linearity*) For all constants $c_1, c_2 \in \mathbb{R}$ and games $(v_1, \mathfrak{B}), (v_2, \mathfrak{B}) \in G$, we have $\Psi_i(c_1 \cdot v_1 + c_2 \cdot v_2, \mathfrak{B}) = c_1 \Psi_i(v_1, \mathfrak{B}) + c_2 \Psi_i(v_2, \mathfrak{B})$, for all player i .
- A4.** (*Rearrangement*) For all game $(v, \mathfrak{B}) \in G$, let $\pi : \mathfrak{U} \rightarrow \mathfrak{U}$ be a mapping such that $\pi B_q \cap \pi B_r = \emptyset$ whenever $q \neq r$. Then $\Psi_{\pi i}(\pi v, \mathfrak{B}_\pi) = \Psi_i(v, \mathfrak{B})$ for all $i \in B_p$.
- A5.** (*Coalitional partnership*) Let $(v, \mathfrak{B}) \in G(N)$ such that B_p is formed by veto players⁹ in v . Let $\pi : \mathfrak{U} \rightarrow \mathfrak{U}$ be a mapping such that $\pi B_q \cap \pi B_r = \emptyset$, and $|\pi B_p| = 1$. Then

9. A player i is veto in a game v if $v(S) = 0$ whenever $i \notin S$.

$$\Psi_i(v, \mathfrak{B}) = \Psi_i(\Psi_{\pi B_p}(\pi v, \mathfrak{B}_\pi) \cdot u_{B_p}, \mathfrak{B}), \text{ for all } i \in B_p.$$

A6. (Monotonicity) Let $(v, \mathfrak{B}) \in G$ such that v is monotonic,¹⁰ for all $i \in \mathcal{U}$, we have,
 $\Psi_i(v, \mathfrak{B}) \geq 0$.

A7. (Projection) Let $(v, \mathfrak{B}) \in G$. If v is an additive game,¹¹ then $\Psi_i(v, \mathfrak{B}) = v(i)$, for all player i .

A8. (Dummy player) If player i is a dummy in a game $(v, \mathfrak{B}) \in G$, then $\Psi_i(v, \mathfrak{B}) = v(i)$.

A9. (Quotient game) For any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $k \in M$, the following holds,
 $\sum_{i \in S_k} \Psi_i^N(v, \mathfrak{B}^N) = \Psi_k^M(v^q, [M])$, where $[M] := \{\{1\}, \{2\}, \dots, \{m\}\}$.

Albizuri and Zarzuelo [2004] stated and interpreted the two axioms **A4**—**A5**. Moreover, they used the six axioms **A2**—**A7** and provided an axiomatic characterization of coalitional semi-values. The axioms **A1**—**A3** and **A6**—**A9** are standard in the litterature (see Alonso-Meijide et al. [2007],...). Axiom **A8** is stronger than axiom **A7**. If the mapping π is one-to-one, then **A4** becomes the Anonymity axiom.

In the proof of Theorem 7 proposed by Albizuri and Zarzuelo [2004], when we remove the monotonicity and replace the projection by the dummy player property, we obtain the following result :

Theorem 2.2.1. *A coalitional value Ψ satisfies carrier (ii), linearity, rearrangement, dummy player property and coalitional partnership, if and only if, for any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $i \in N$, there exist two families of **real numbers** $\{a(m, p)\}_{p=0}^{m-1}$ and $\{b(s_{m_i}, k)\}_{k=0}^{s_{m_i}-1}$ fulfilling*

$$\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) = 1 \text{ and } \sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} b(s_{m_i}, k) = 1 \text{ such that,}$$

$$\Psi_i(v, \mathfrak{B}) := \sum_{P \subseteq M \setminus m_i} \sum_{K \subseteq S_{m_i} \setminus i} a(m, p) \cdot b(s_{m_i}, k) [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)]. \quad (\text{A2})$$

where $\mathfrak{B}^N := \{S_\ell : \ell \in M\}$ with $M := \{1, 2, \dots, m\}$.

2.3. Interaction and Decomposition

10. A game v is monotonic if $v(L) \leq v(T)$ whenever $L \subset T$.

11. A game $v \in \Gamma(N)$ is an additive game if for every player $i \in N$ there exists $a_i \in \mathbb{R}$ such that $v(T) = \sum_{i \in N} a_i$ for every $T \subseteq N$.

2.3.1. Notion of interaction

To introduce the notion of interaction considered in this article, we use the approach proposed by Kojadinovic [2005], Grabisch and Roubens [1999a], Katsushige et al. [2006]. A simple way to address this phenomenon consists of considering two players i and j and a characteristic function v , such that if the relation $v(ij) > v(i) + v(j)$ is satisfied then, it seems natural to say that players i and j interact positively. Indeed, considering the pair of players (ij) leads us to a situation in which the value of (ij) is more important. In a symmetrical way, if the relation $v(ij) < v(i) + v(j)$ is satisfied, it seems natural to state that players i and j interact negatively because considering the pair of attributes (ij) leads us to a situation in which the pay-off is lower. Finally, if the relation $v(ij) = v(i) + v(j)$ is satisfied, we may affirm that players i and j do not interact. Therefore, the interaction between players i and j depends on the difference $v(ij) - v(i) - v(j)$. However, the presence of a third or a fourth player should potentially modify the interaction between players i and j . To take into account this phenomenon, Murofushi and Soneda [1993b] introduce what Kojadinovic [2005] calls the interaction between players i and j in the presence of T , where T is a subset of players not containing players i and j . This interaction is given by the following definition.

Definition 2.3.1 (player pairwise interaction)

For any $N \in \mathcal{N}$, let $v \in \Gamma(N)$ and $i, j \in N$, with $i \neq j$. The interaction between players i and j in the presence of others in $T \subseteq N \setminus ij$, is denoted by $Int_v^N(i, j, T)$, and defined as follows, $Int_v^N(i, j, T) := v(ij \cup T) - v(i \cup T) - v(j \cup T) + v(T)$.

Similarly, we define the pairwise interaction between structural coalitions as follows.

Definition 2.3.2 (structural coalitions pairwise interaction)

For any $N \in \mathcal{N}$, let $(v, \mathfrak{B}) \in G(N)$ and $h, \ell \in M$, with $h \neq \ell$. The interaction between the structural coalitions S_h and S_ℓ in the presence of other structural coalitions in $\mathbf{P}_N := \bigcup_{\ell \in P} S_\ell$ with $P \subseteq M \setminus \{\ell, h\}$, is denoted by $Int_v^{\mathfrak{B}^N}(S_h, S_\ell, \mathbf{P}_N)$, and defined as follows, $Int_v^{\mathfrak{B}^N}(S_h, S_\ell, \mathbf{P}_N) := v(S_h \cup S_\ell \cup \mathbf{P}_N) - v(S_h \cup \mathbf{P}_N) - v(S_\ell \cup \mathbf{P}_N) + v(\mathbf{P}_N)$.

Any player i can also interact with each structural coalitional S_ℓ , $\ell \neq m_i$. We define this pairwise interaction as follows :

Definition 2.3.3 (*player and structural coalition pairwise interaction*)

For any $N \in \mathcal{N}$, let $(v, \mathfrak{B}) \in G(N)$, $i \in N$, and $\ell \in M \setminus m_i$. The interaction between player i and structural coalition S_ℓ in the presence of others in $T := K \cup \mathbf{P}_N$ with $P \subseteq M \setminus \{\ell, m_i\}$ and $K \subseteq S_{m_i} \setminus i$, is denoted by $\overline{Int}_v^{\mathfrak{B}^N}(i, S_\ell, T)$, and defined as follows $\overline{Int}_v^{\mathfrak{B}^N}(i, S_\ell, T) := v(i \cup S_\ell \cup T) - v(S_\ell \cup T) - v(i \cup T) + v(T)$.

Given any $i \in N$, i 's marginal contributions can be represented by the dividends of coalitions they belong to, as presented in the following well-known result.

Proposition 2.3.1

For any $N \in \mathcal{N}$ and for all $v, w \in \Gamma(N)$ and $i \in N$, the statements below imply each other.

- (i) For all coalition $T \subseteq N \setminus i$, $v(T \cup i) - v(T) = w(T \cup i) - w(T)$.
- (ii) For all coalition $T \subseteq N \setminus i$, $\lambda_{T \cup i}(v) = \lambda_{T \cup i}(w)$.

A corresponding version of the previous equivalence result, in terms of interactions, is provided in this paper as follows.

Proposition 2.3.2

For any $N \in \mathcal{N}$ and for all $v, w \in \Gamma(N)$ and $i, j \in N$, the statements below imply each other.

- (i) For all coalition $T \subseteq N \setminus ij$, $\overline{Int}_v^N(i, j, T) = \overline{Int}_w^N(i, j, T)$.
- (ii) For all coalition $T \subseteq N \setminus ij$, $\lambda_{T \cup ij}(v) = \lambda_{T \cup ij}(w)$.

2.3.2. Decompositions of the Shapley-Owen value

In the class of TU games, a natural way to find the payoff of a player i is to look at its effect when it joins all other players. Following Casajus and Huettner [2018], this approach is

problematic because one could argue that all every player contributes to the payoff of i and therefore this contribution should be partly attributed to the others. Chantreuil et al. [2019] proposed a similar approach by using the interaction notion.

Following Chantreuil et al. [2019], we propose three approaches for the decomposition of the Shapley-Owen value. The first subsection is the structural coalition pairwise approach, the second subsection is the pairwise interaction between a player and a structural coalition approach, and the third subsection is the player's pairwise interaction approach. We provide an axiomatic characterization of each additive interaction decomposable coalitional value.

One crucial concept in this paper is the *pure marginal contribution*, defined as follows.

Definition 2.3.4 (*pure marginal contribution (PMC)*)

For any $N \in \mathcal{N}$ and for all $v \in \Gamma(N)$, the pure marginal contribution PMC_i of a player i is defined by,^a $PMC_i(v) := v(N) - v(N \setminus i)$; while the pure marginal contribution PMC_T of a coalition T is defined by,^b $PMC_T(v) := v(N) - v(N \setminus T)$.

a. Casajus and Huettner [2018] called the naive solution of player i .

b. Marichal et al. [2007] called the external generalized value of coalition T .

It is worth noting that our pure marginal contributions are not solely defined for a single player, as is done in the literature, but rather for any coalition of N . Now, we can provide our first decomposition of the Shapley-Owen value, as given in the next subsection.

First decomposition approach of the Shapley-Owen value

The notion of AID was used for the first time by Courtin et al. [2020] in the context of interaction indices. We propose a similar definition for the games enriched with a coalitional structure.

Definition 2.3.5 (*basic Additive Interaction Decomposable (AID)*)

For any $N \in \mathcal{N}$, let $(v, \mathfrak{B}) \in G(N)$ be a game with coalitional structure. Any coalitional value Ψ is AID if there exists a family of real numbers $\{f(m, p)\}_{p=0}^{m-2}$ such that, for any $S_\ell \in \mathfrak{B}^N$,

$$\sum_{i \in S_\ell} \Psi_i(v, \mathfrak{B}) := PMC_{S_\ell}(v) - \sum_{r \in M \setminus \ell} \sum_{P \subseteq M \setminus \{\ell, r\}} f(m, p) \times \frac{Int_v^{\mathfrak{B}^N}(S_\ell, S_r, \mathbf{P}_N)}{2}. \quad (\text{A3})$$

This definition states that a coalitional value of any game with a coalitional structure (v, \mathfrak{B}) is AID if the sum of the coalitional values of players within a structural coalition $S_\ell \in \mathfrak{B}^N$ can be expressed as the difference between the pure marginal contribution $PMC_{S_\ell}(v)$ of the structural coalition S_ℓ and a weighted sum of the half of pairwise interactions between the structural coalition S_ℓ and each of the other structural coalitions in the presence of the other structural coalitions¹².

The next result provides an axiomatic characterization of an AID coalitional value.

Theorem 2.3.1 (*Axiomatization of an AID coalitional value*). *A coalitional value Ψ is AID, if and only if, it satisfies the linearity, dummy player property, anonymity and quotient game property.*

Corollary 2.3.1 (*First decomposition of a coalitional semivalue*). *Every coalitional semivalue Ψ^{CS} that satisfies the quotient game property is AID. Furthermore, for any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $S_\ell \in \mathfrak{B}^N$,*

$$\sum_{i \in S_\ell} \Psi_i^{CS}(v, \mathfrak{B}) := PMC_{S_\ell}(v) - \sum_{r \in M \setminus \ell} \sum_{P \subseteq M \setminus \{\ell, r\}} f(m, p) \cdot \frac{Int_v^{\mathfrak{B}^N}(S_\ell, S_r, \mathbf{P}_N)}{2}.$$

where

$$f(m, p) = \begin{cases} \frac{2 \times a(m, 0)}{m-1} & \text{if } p = 0 \\ \frac{2[p \times f(m, p-1) + a(m, p)]}{m-p-1} & \text{otherwise.} \end{cases}$$

12. If the family of real numbers $\{f^r(M, P)\}_{P \subseteq M \setminus \{\ell, r\}}$ with $f^r(M, P) = f(m, p)$, is a probability distribution over $2^{M \setminus \{\ell, r\}}$ then the second part of equation (A3) can be interpreted as the sum over the other structural coalition S_r of the mathematical expectation on $2^{M \setminus \{\ell, r\}}$ of the pairwise interactions $\frac{Int_v^{\mathfrak{B}^N}(S_\ell, S_r, \mathbf{P}_N)}{2}$ with respect to the probability distribution $\{f^r(M, P)\}_{P \subseteq M \setminus \{\ell, r\}}$.

and the set $\{a(m, p)\}_{p=0, \dots, m-1}$ are nonnegative constants associated to the coalitional semivalue.

From corollary 2.3.1, we can deduce the first decomposition of the Shapley-Owen value.

Corollary 2.3.2 (First decomposition of the Shapley-Owen value). *The Shapley-Owen is AID. Furthermore, for any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $S_\ell \in \mathfrak{B}^N$,*

$$\sum_{i \in S_\ell} \Psi_i^{Sh}(v, \mathfrak{B}) := PMC_{S_\ell}(v) - \sum_{r \in M \setminus \ell} \sum_{P \subseteq M \setminus \{\ell, r\}} \frac{2(p+1)!(m-p-2)!}{m!} \times \frac{Int_v^{\mathfrak{B}^N}(S_\ell, S_r, \mathbf{P}_N)}{2}.$$

Second decomposition approach of the Shapley-Owen value

In the first decomposition of coalitional values, if we remove the quotient game property and replace the anonymity property by rearrangement then, by adding some specific axioms, we obtain another decomposition. Let us first define the second decomposition.

Definition 2.3.6 (Additive Interaction Decomposable star (AID*))

For any $N \in \mathcal{N}$, let $(v, \mathfrak{B}) \in G(N)$ be a game and $i \in N$ be a player. A coalitional value Ψ is AID* if there exist two families of real numbers $\{g(s_{m_i}, k)\}_{k=0}^{s_{m_i}-1}$ and $\{f(m, p)\}_{p=0}^{m-2}$ such that $\sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} g(s_{m_i}, k) = 1$ and

$$\Psi_i(v, \mathfrak{B}) := \varphi_i(v_i) - \sum_{\ell \in M \setminus m_i} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_\ell}, i]} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{Int_v^{\mathfrak{B}^N}(i, S_\ell, T)}{2},$$

with $v_i = v_{\cup(N \setminus S_{m_i})}$ and $\varphi_i(v_i) := \sum_{K \subseteq S_{m_i} \setminus i} g(s_{m_i}, k) [v_i(i \cup K) - v_i(K)]$.

This definition states that a coalitional value of any game with a coalitional structure (v, \mathfrak{B}) is AID* if the coalitional value for any given player $i \in N$ can be expressed as the difference between its weighted sum of marginal contributions $\varphi_i(v_i)$ and a weighted sum of the half of pairwise interactions of i with each structural coalition S_ℓ , $\ell \neq m_i$ in the presence of the other players in $T \in Coal[\mathfrak{B}^{N \setminus S_\ell}, i]$.

Remark 2.3.1. *If we have $\sum_{p=0}^{m-2} \sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} \times \binom{m-2}{p} \times f(m, p) \cdot g(s_{m_i}, k) = 1$, $g(s_{m_i}, k) \geq 0$ with $k = 0, \dots, s_{m_i}$ and $f(m, p) \geq 0$ with $p = 0, \dots, m-2$ then $\Psi_i(v, \mathfrak{B})$ can also be interpreted as the difference between the weighted sum of marginal contributions of i and the sum*

over $\ell \in M \setminus m_i$ of the mathematical expectation on $\text{Coal} [\mathfrak{B}^{N \setminus S_\ell}, i]$ of the pairwise interaction $\frac{\text{Int}_v^{\mathfrak{B}^N}(i, S_\ell, T)}{2}$ with respect to the probability distribution given by $\{F(M, P) \times G(S_{m_i}, K)\}_{\substack{P \subseteq M \setminus \{\ell, m_i\} \\ K \subseteq S_{m_i} \setminus i}}$ where $F(M, P) := f(m, p)$ and $G(S_{m_i}, K) := g(s_{m_i}, k)$, with $p = 0, \dots, m-2$ while $k = 0, \dots, s_{m_i}-1$

The next result provides an axiomatic characterization of AID* coalitional value.

Theorem 2.3.2 (Axiomatization of AID* coalitional value). A coalitional value Ψ is AID*, if and only if, it satisfies the linearity, carrier (ii), rearrangement, dummy player property and coalitional partnership.

Our second decomposition can be applied to any coalitional semivalue as shown below.

Corollary 2.3.3 (General decomposition of coalitional semivalue).

Any coalitional semivalue Ψ^{CS} satisfies AID*. In addition, For any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $i \in N$,

$$\Psi_i^{CS}(v, \mathfrak{B}) = \varphi_i^S(v_i) - \sum_{\ell \in M \setminus m_i} \sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus S_\ell}, i]} f(m, p) \cdot b(s_{m_i}, k) \cdot \frac{\text{Int}_v^{\mathfrak{B}^N}(i, S_\ell, T)}{2},$$

with $\varphi_i^S(v_i) := \sum_{R \subseteq S_{m_i} \setminus i} b(s_{m_i}, r) [v_i(i \cup R) - v_i(R)]$, and

$$f(m, p) := \begin{cases} \frac{2 \times a(m, 0)}{m-1} & \text{if } p = 0 \\ \frac{2[p \times f(m, p-1) + a(m, p)]}{(m-p-1)} & \text{if } 0 < p \leq m-2, \end{cases}$$

where $a(m, p)$ and $b(s_{m_i}, k)$ the coefficients associated to Ψ as a coalitional semivalue (p and k are given by $T \in \text{Coal} [\mathfrak{B}^{N \setminus S_\ell}, i]$ see equation(A1)).

Remark 2.3.2. Courtin et al. [2020] show that $\varphi_i^S(v_i)$ can be decomposed as follows :

$$\varphi_i^S(v_i) = \text{PMC}_i(v) - \sum_{j \in S_{m_i} \setminus i} \sum_{R \subseteq S_{m_i} \setminus ij} h(s_{m_i}, r) \cdot \frac{\text{Int}_v^N(i, j, R)}{2},$$

with $v_i = v_{\cup(N \setminus S_{m_i})}$, and $h(s_{m_i}, r) = \begin{cases} \frac{2 \times g(s_{m_i}, 0)}{s_{m_i}-1} & \text{if } r = 0 \\ \frac{2[r \times h(s_{m_i}, r-1) + g(s_{m_i}, r)]}{(s_{m_i}-r-1)} & \text{otherwise.} \end{cases}$

In particular, the second decomposition of the Shapley-Owen value is given by the following result.

Corollary 2.3.4 (Second decomposition of the Shapley-Owen value).

For any $N \in \mathcal{N}$, let $(v, \mathfrak{B}) \in G(N)$ be a game N enriched with a coalitional structure, the Shapley value Ψ^{Sh} can be decomposable as follows. For any $i \in N$,

$$\begin{aligned} \Psi_i^{Sh}(v, \mathfrak{B}) &= PMC_i(v) - \sum_{j \in S_{m_i} \setminus i} \sum_{R \subseteq S_{m_i} \setminus ij} INT_{v_i}^{intra}(i, j, R) \\ &\quad - \sum_{\ell \in M \setminus m_i} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_\ell}, i]} INT_v^{extra}(i, S_\ell, T), \end{aligned}$$

where $v_i = v_{\cup(N \setminus S_{m_i})}^{S_{m_i}}$, $INT_{v_i}^{intra}(i, j, R) := \frac{(r+1)!(s_{m_i}-r-2)!}{m!} Int_{v_i}^N(i, j, R)$ and

$INT_v^{extra}(i, S_\ell, T) := \frac{(p+1)!(m-p-2)!}{m!} \frac{k!(s_{m_i}-k-1)!}{s_{m_i}!} Int_v^{\mathfrak{B}^N}(i, S_\ell, T)$ (p and k are given by $T \in Coal[\mathfrak{B}^{N \setminus S_\ell}, i]$ see equation (A1)).

Hence, this second decomposition separates the PMC of each player from their interactions with other players, while differentiating between their pairwise interactions with each player within their own structural coalition and their pairwise interactions with every other structural coalition.

Corollary 2.3.5. For any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $i \in N$, the Shapley-Owen value of player i in (v, \mathfrak{B}) can be interpreted as in Remark 2.3.1.

Two structural coalitions can be symmetric each other with respect to the AID^* measure of a coalitional value Ψ . Hence, the sum of all interactions of each player belonging to a structural coalition S_ℓ with an entire structural coalition S_r , $r \neq \ell$, is equal to the sum of all interactions of each player in S_r with the entire structural coalition S_ℓ . This result is formally given below

Proposition 2.3.3

For any $N \in \mathcal{N}$ and $(v, \mathfrak{B}) \in G(N)$, every structural coalition S_r and S_ℓ are symmetric with respect to the AID^* of Ψ^{Sh} , i.e.,

$$\sum_{i \in S_\ell} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_r}, i]} INT_v^{extra}(i, S_r, T) = \sum_{i \in S_r} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_\ell}, i]} INT_v^{extra}(i, S_\ell, T).$$

Third decomposition approach of the Shapley-Owen value

When we maintain the axioms of the second decomposition of coalitional values and add the efficiency property, we obtain another decomposition of the Shapley-Owen value. A coalitional

value is AID^{**} , if it is AID^* and satisfies the efficiency. The second decomposition is weaker than the third one (AID^{**}). The Shapley-Owen value is the only AID^{**} coalitional value¹³.

Theorem 2.3.3 (Third decomposition of the Shapley-Owen value).

For any $N \in \mathcal{N}$ and $(v, \mathfrak{B}) \in G(N)$, the Shapley-Owen value is an AID^{**} coalitional value. For all player $i \in N$,

$$\Psi_i^{Sh}(v, \mathfrak{B}) = PMC_i(v) - \sum_{j \in N \setminus i} \sum_{T \in Coal[\mathfrak{B}^{N \setminus i}, j]} INT_v^{OW}(i, j, T), \quad (A4)$$

where $INT_v^{OW}(i, j, T) := \frac{Q_T \times Int_v^N(i, j, T)}{2}$, with Q_T given as follows,

$$Q_T := \begin{cases} 2 \times \frac{(p+1)!(m-p-2)!}{m!} \times \frac{k!(s_{m_j}-k-1)!}{s_{m_j}!} & \text{if } S_{m_i} \setminus i = \emptyset \\ 2 \times \frac{p!(m-p-1)!}{m!} \times \frac{(k+1)!(s_{m_j}-k-2)!}{s_{m_j}!} & \text{if } S_{m_i} = S_{m_j} \\ 2 \times \frac{p!(m-p-1)!}{m!} \times \frac{k!(s_{m_j}-k-1)!}{s_{m_j}!} & \text{if } \emptyset \neq (S_{m_i} \setminus i) \subset T \\ 0 & \text{if } \emptyset \neq (S_{m_i} \setminus i) \subset (N \setminus T). \end{cases}$$

(p and k are given by $T \in Coal[\mathfrak{B}^{N \setminus i}, j]$ see equation(A1))

Moreover, $\sum_{T \in Coal[\mathfrak{B}^{N \setminus i}, j]} Q_T = 1$.

This theorem states that the Shapley-Owen value for a given $i \in N$ in a game with a coalitional structure (v, \mathfrak{B}) can be expressed as the difference between its pure marginal contribution $PMC_i(v)$ and a weighted sum of pairwise interactions between player i and each of the other players in $N \setminus i$.¹⁴

Equation (A4) can be rewritten as follows :

$$\begin{aligned} & \Psi_i^{Sh}(v, \mathfrak{B}) \\ = & PMC_i(v) - \sum_{j \in S_{m_i} \setminus i} \sum_{\substack{T \in Coal[\mathfrak{B}^N, i] \\ j \notin T}} \frac{p!(m-p-1)!}{m!} \cdot \frac{(k+1)!(s_{m_j}-k-2)!}{s_{m_j}!} \cdot Int_v^N(i, j, T) \\ & - \sum_{\ell \in M \setminus m_i} \sum_{j \in S_\ell} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_{m_i}}, i]} \frac{(p+1)!(m-p-2)!}{m!} \cdot \frac{k!(s_{m_j}-k-1)!}{s_{m_j}!} \cdot Int_v^N(i, j, T \cup (S_{m_i} \setminus i)) \end{aligned}$$

13. Albizuri and Zarzuelo [2004] (Theorem 5) showed that the Shapley-Owen value is the only coalitional value on $(v, \mathfrak{B}) \in G(N)$ that satisfies efficiency, rearrangement and additivity.

14. The second part of the decomposition can also be interpreted as a weighted sum over the players $j \in N \setminus i$, of the mathematical expectation on $Coal[\mathfrak{B}^{N \setminus i}, j]$ of the pairwise interactions $\frac{Int_v^N(i, j, T)}{2}$ with respect to the probability distribution $\{Q_T\}_{T \in Coal[\mathfrak{B}^{N \setminus i}, j]}$.

Following the definition of AID^{**} , we can also provide the property below.

Proposition 2.3.4

For any $N \in \mathcal{N}$ and $(v, \mathfrak{B}) \in G(N)$, two players i and j belonging to the same structural coalition, are symmetric with respect to the AID^{**} of Ψ^{Sh} , i.e.,

$$\sum_{T \in Coal[\mathfrak{B}^{N \setminus i, j}]} INT_v^{OW}(i, j, T) = \sum_{T \in Coal[\mathfrak{B}^{N \setminus j, i}]} INT_v^{OW}(j, i, T).$$

Remark 2.3.3. *The first, second and third decompositions of the Shapley-Owen value generalize the main result of Chantreuil et al. [2019].*

2.4. Inequality index decomposition

2.4.1. An inequality game

This section presents the notion of inequality game and our three decompositions of inequality measures. We first recall that $N := \{i_1, i_2, \dots, i_n\}$ with $n \geq 2$, which will here represent the set of secondary sources that will be defined further. Likewise, $\Omega := \{1, \dots, \omega\}$, with $\omega \geq 2$, will represent the set of individuals, with generic individuals denoted by a, b , while income components are denoted by i, j . We define a situation as an $\omega \times n$ -matrix $x := [x_a^i]$, where $x_a^i \geq 0$ represents the amount of each income i , received by individual a . Thus, row a indicates the amount of income components received by individual a , while column i indicates the distribution of component i between all ω individuals (e.g. Table 2.1). Let us denote by $\mathcal{X} := \{x | x_a^i \geq 0, \text{ for all } a \in \Omega, i \in N\}$ the set of all situations. With respect to a given situation x , the distribution of income by sources i is given by $x^i := (x_1^i, \dots, x_\omega^i)^\tau$, where τ produces the transposed vector. The aggregate distribution is $X := \sum_{i \in N} x^i$ while the corresponding mean income is given by $\mu(X) := \frac{1}{\omega} \sum_{a \in \Omega} \sum_{i \in N} x_a^i$. Likewise, for all $T \subset N$, the aggregate distribution for the set of sources not included in T is $X^{-T} := \sum_{i \notin T} x^i$ and then $\mu(X^{-T}) := \frac{1}{\omega} \sum_{a \in \Omega} \sum_{i \notin T} x_a^i$ is the corresponding mean income.

The *inequality index* of a distribution of income X is measured by a function $I : \mathbb{R}^\omega \rightarrow [0, 1]$ such that $I(X) = 0$ for an equal income distribution. Thus, for a given inequality index

I , the distribution of income by components helps us to build a distribution of income among subsets of components, namely an application $Y : 2^N \times \mathbb{R}_+ \rightarrow \mathbb{R}^\omega$, such that $Y(\emptyset, \cdot) = 0$ and for all $T \subseteq N$, $Y(T, \lambda) = (\sum_{i \in T} x_1^i + \lambda, \dots, \sum_{i \in T} x_\omega^i + \lambda)$, with $\lambda \in \mathbb{R}_+$. This leads to the definition of a new TU game on N denoted by $v_{I,\lambda}$, and defined by $v_{I,\lambda}(N) := I(X)$ and for all $T \subseteq N$, $v_{I,\lambda}(T) = I(Y(T, \lambda))$ — we called it the λ -inequality game associated to I ; and the parameter λ indicates the way in which the sources not included in any subset T of N must be considered. For instance, they are *nullified* if $\lambda = 0$ or *equalized* if $\lambda = \mu(X^{-T})$.¹⁵

Below are the three decompositions of inequality measures of the paper.

Proposition 2.4.1 (1st decomposition of inequality measures)

Let I be any inequality measure, $\lambda \in \mathbb{R}_+$, \mathfrak{B}^N be a coalitional structure over N and $v_{I,\lambda} \in \Gamma(N)$, with $v_{I,\lambda}$ being the λ -inequality game associated to I . Then, for all situation x ,

$$I(X) = \sum_{\ell \in M} PMC_{S_\ell}(v_{I,\lambda}) - \sum_{\ell \in M} \mathbb{INT}_{v_{I,\lambda}}^{coalition}(S_\ell), \quad \text{with } X := \sum_{i \in N} x^i, \quad (\text{A5})$$

and $\mathbb{INT}_{v_{I,\lambda}}^{coalition}(S_\ell) := \sum_{r \in M \setminus \ell} \sum_{P \subseteq M \setminus \{\ell, r\}} \frac{(p+1)!(m-p-2)!}{m!} \times Int_{v_{I,\lambda}}^{\mathfrak{B}^N}(S_\ell, S_r, \mathbf{P}_N)$.

Proposition 2.4.2 (2nd decomposition of inequality measures)

Let I be any inequality measure, $\lambda \in \mathbb{R}_+$, \mathfrak{B}^N be a coalitional structure over N and $v_{I,\lambda} \in \Gamma(N)$, with $v_{I,\lambda}$ being the λ -inequality game associated to I . Then, for all situation x ,

$$I(X) = \sum_{i \in N} PMC_i(v_{I,\lambda}) - \sum_{i \in N} \mathbb{INT}_{v_{I,\lambda}}^{intra}(i) - \sum_{i \in N} \mathbb{INT}_{v_{I,\lambda}}^{extra}(i), \quad (\text{A6})$$

with $X := \sum_{i \in N} x^i$ and the functions $\mathbb{INT}_{v_{I,\lambda}}^{intra}(i) := \sum_{j \in S_{m \setminus i}} \sum_{K \subseteq S_{m \setminus i} \setminus j} INT_{(v_{I,\lambda})_i}^{intra}(i, j, K)$, and $\mathbb{INT}_{v_{I,\lambda}}^{extra}(i) := \sum_{\ell \in M \setminus m_i} \sum_{T \in Coal[\mathfrak{B}^N \setminus S_\ell, i]} INT_{v_{I,\lambda}}^{extra}(i, S_\ell, T)$.

15. See Chantreuil and Trannoy [2011] for the presentation of other games.

Proposition 2.4.3 (*3rd decomposition of inequality measures*)

Let I be any inequality measure, $\lambda \in \mathbb{R}_+$, \mathfrak{B}^N be a coalitional structure over N and $v_{I,\lambda} \in \Gamma(N)$, with $v_{I,\lambda}$ being the λ -inequality game associated to I . Then, for all situation x ,

$$I(X) = \sum_{i \in N} PMC_i(v_{I,\lambda}) - \sum_{i \in N} \mathbb{INT}_{v_{I,\lambda}}^{OW}(i), \quad (\text{A7})$$

where $X := \sum_{i \in N} x^i$, and $\mathbb{INT}_{v_{I,\lambda}}^{OW}(i) := \sum_{j \in N \setminus i} \sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus \{i,j\}}]} INT_{v_{I,\lambda}}^{OW}(i, j, T)$.

Proposition 2.4.3 states that all inequality measures for a hierarchical income distribution can be decomposed into two terms, the sum of the pure marginal contributions of all income components minus the sum of the sums of all pairwise interactions of all income components with all other income components.

The interpretation of Proposition 4.4.1 is different than the interpretation of Proposition 2.4.3. In Proposition 2.4.3, the interaction is between pairs of income components while in Proposition 4.4.1 the interaction term has two components : a pairwise interaction term between players of the same coalition and a pairwise interaction term between one income component and each coalition of income components. Thus, proposition 2.4.3 is suited to highlight the interactions between each pair of income sources. However, one might suppose that one income source interacts with every other income source only within its structural coalition, and with the other structural coalitions as a whole. In such a case, proposition 4.4.1 should be chosen to make the decomposition as it allows to differentiate between intra and extra interactions. Proposition 2.4.1 is similar to the Chantreuil et al. [2019] decomposition, with the difference that we do not consider the contribution of one player (with its PMC and its pairwise interaction term), but rather the contribution of one structural coalition (with the structural coalition PMC and the structural coalitions pairwise interaction).

One may notice that the decompositions of the inequality index given by formulae (A5), (A6) and (A7) are similar to the decomposition proposed by Chantreuil et al. [2019], either in the case where the coalitional structure is composed solely of subsets containing a single player (i.e. $m = n$) or when the coalitional structure is only composed of one set, which is actually the whole set N containing all the players (i.e. $m = 1$ while $n \geq 2$).

Furthermore, unlike Chantreuil et al. [2019], and for the sake of clarity, we only consider the equalized inequality game in the next subsection, where the inequality coming from the sources not included in T is equalized. Therefore, we only take into account the case where $\lambda = \mu(X^{-T})$. Thus, the sources not included in the considered coalition T are equalized so as to drop income differences for these sources. This choice has two main advantages. First, we keep the same amount of total income distributed in the considered economy in all the coalitions. This way, differences in inequality between coalitions do not come from an “income size effect”. Second, the interpretation is more straightforward. Indeed, the contribution to the inequality of an equalizing source is null, and if a source increases the total income inequality it should have a positive contribution, and vice-versa.

By using axiomatic methods, we shown that the second decomposition is weaker than the third one. The last decomposition is the best for the decompositions of inequality with hierarchical structure because the purpose of a hierarchical structure is to go to an even finer level (primary source). The last decomposition isolates the pure marginal contribution to pairwise interactions between primary sources.

2.4.2. Decompositions of a hierarchical income distribution

A hierarchical income distribution is defined (table 2.1) in order to illustrate the revisited Owen decomposition of an inequality index. Three primary income sources are considered : labor, social transfers and capital. Each of these sources is subdivided into two secondary sources, respectively : wage and bonus, family allowances and housing subsidies, financial capital and estate capital. We also observe in table 2.1 that the total income of individual 3 is higher than the total income of individual 2, itself higher than individual 1’s income. All secondary sources lead to an individual ranking by income, with the exception of family allowances, which are lump-sum transfers, and housing subsidies, which are higher for the poorer people.

TABLE 2.1 – Hierarchical income distribution

Individual	Labor		Social transfers		Capital	
	Wage	Bonus	Family	Housing	Financial	Estate
1	20	10	10	20	20	30
2	35	15	10	10	50	50
3	45	25	10	0	100	50
Total	100	50	30	30	170	130

First, in order to highlight the interest of using the Shapley-Owen value rather than the Sha-

pley value, we compare the Gini decomposition into income sources with both values between two cases. In the first one, we decompose the inequality measure according to the hierarchical income distribution as given in table 2.1. In the second one, we do the same exercise with the difference that the two distributions associated to capital incomes are merged. In table 2.2, the contributions of the secondary sources not associated to capital incomes are unchanged when using the Shapley-Owen values, as is also the case of the contribution of the capital income (0.122 in each case). However, with the Shapley value, the change in aggregation of the capital income modifies all contributions (not only its own contribution ; more surprisingly, even the ones that did not belong to the capital income source changed) ¹⁶.

TABLE 2.2 – Shapley-Owen and Shapley values sensitivity to aggregation

Value	Case	Labor income		Social Transfers		Capital income		
		Wage	Bonus	Family	Housing	Financial	Estate	Sum
Shapley	N°1	0.02920	0.01720	-	- 0.01440	0.10110	0.02370	0.12480
	N°2	0.02723	0.01634	-	- 0.01198	0.12527		0.12527
Shapley-Owen	N°1	0.02720	0.01630	-	- 0.00870	0.09910	0.02290	0.12200
	N°2	0.02720	0.01630	-	- 0.00870	0.12200		0.12200

Note : The inequality measure is the Gini index. The “Sum” column corresponds to the sum of the financial and estate capital income contributions, in case they are divided.

First decomposition

For brevity’s sake, the first decomposition based on proposition 2.4.1 is not given. Nevertheless, it highlights the possibility to consider structural coalitions as players ¹⁷ when studying the sources of inequality.

Second decomposition

In Table 2.3, a revisited Shapley-Owen decomposition (based on proposition 4.4.1) is proposed in order to point out the difference between the PMC of an attribute (indicating the variation of inequality that would be observed if the inequality it creates were to be disregarded) and two terms of pairwise interactions (indicating the nature and the intensity of the relationship between this source and the other sources of its structural coalition or any other structural

16. These observations are also verified with a null-game, where $\lambda = 0$.

17. We consider the Quotient game where structural coalitions are viewed as players (Alonso-Meijide et al. [2007]).

coalition). There are three kinds of relationships : negative, positive and null. In the first case, a negative relationship means that both sources generate, on average, more inequality together than separately. On the other hand, a positive relationship indicates that both sources create, on average, less inequality together than separately. In specific cases, such as a lump-sum source in an equalized inequality game with the *Gini index* as a characteristic function, a source may be independent of others. Results are given in table 2.3, where the inequality is measured by the Gini index ¹⁸.

According to table 2.3, the financial capital income is the most important source of inequality. Moreover, if the financial capital income were no longer a source of inequality, the inequality would decrease by 66.67%, which makes this source the one with the greatest impact on inequality. In addition, as expected, the housing subsidies have a negative effect on inequality. If this source were equalized, then the inequality would increase by 16.67%.

Next, it appears that the pure marginal contributions are higher than their corresponding contributions in absolute terms. Thus, this indicates that removing inequality from those income sources (except the family social transfer source) would reduce (in case of the housing source) the inequality or increase it to an higher level than their corresponding contributions. This also indicates that the sum of interactions has the same sign as the contribution for each income source. Economically speaking, the inequality associated with labor and capital income sources accumulates with the inequality created by other structural coalitions of income sources.

TABLE 2.3 – A revisited Owen decomposition - Second Decomposition

	Labor income		Social transfers		Capital income	
	Wage	Bonus	Family	Housing	Financial	Estate
Ψ^{Sh}	0.0272	0.0163	-	- 0.0087	0.0991	0.0229
	17.36%	10.42%	0.00%	-5.56%	63.19%	14.58%
PMC	0.0327	0.0196	-	- 0.0261	0.1046	0.0261
	20.83%	12.50%	0.00%	-16.67%	66.67%	16.67%
$\mathbb{INT}_{v_{I,\lambda}}^{intra}$	-0.0000	-0.0000	-	-	0.0000	0.0000
	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
$\mathbb{INT}_{v_{I,\lambda}}^{extra}$	0.0054	0.0033	-	- 0.0174	0.0054	0.0033
	3.47%	2.08%	0.00%	-11.11%	3.47%	2.08%

Note : All percentages are calculated relatively to the total inequality. Besides, “-” indicates a perfect null value, while “0.0000” indicates a value close to zero.

If we take a look at the details of the extra pairwise interactions in table 2.4, it appears, as

18. The Gini index, noted G , for n individuals ordered from the lowest income to the highest income is denoted by : $G = \frac{2}{n} \sum_{i=1}^n (p_i - L_i)$, where p_i and L_i are the respective cumulative share of population and the cumulative share of income.

expected, that labor and capital income sources interact negatively with social transfer sources, as they do not benefit the same individuals. But these negative interactions are not high enough to offset the positive interactions between the two.

TABLE 2.4 – The weighted sum of pairwise interactions between i and S_ℓ

i/S_ℓ	L1	L2	T1	T2	K1	K2	Total
L1			-0.0054		0.0109		0.0054
L2			-0.0033		0.0065		0.0033
T1	-				-		-
T2	-0.0087				-0.0087		-0.0174
K1	0.0109		-0.0054				0.0054
K2	0.0065		-0.0033				0.0033

Note : “-” indicates a perfect null value, while an empty cell indicates that there is no interaction term.

Third decomposition

The third decomposition is proposed in table 2.5, based on proposition 2.4.3. The decomposition is similar to the second decomposition but the interactions are between the secondary sources. In section 3, we show that the second decomposition of the Shapley-Owen value is weaker than the third decomposition.

TABLE 2.5 – A revisited Owen decomposition - First Decomposition

	Labor income		Social transfers		Capital income	
	Wage	Bonus	Family	Housing	Financial	Estate
Ψ^{Sh}	0.0272	0.0163	-	- 0.0087	0.0991	0.0229
	17.36%	10.42%	0.00%	-5.56%	63.19%	14.58%
PMC	0.0327	0.0196	-	- 0.0261	0.1046	0.0261
	20.83%	12.50%	0.00%	-16.67%	66.67%	16.67%
$\text{INT}_{v_i, \lambda}^{\text{OW}}$	0.0054	0.0033	-	- 0.0174	0.0054	0.0033
	3.47%	2.08%	0.00%	-11.11%	3.47%	2.08%

Note : The wage contribution is equal to 0.0272, which represents 17.36% of the total inequality measured by the Gini index (0.156863). All percentages are calculated relatively to the total inequality.

Furthermore, owing to the third decomposition proposed in this article, it is possible to decompose the interaction of any source with the other sources. For the sake of clarity, in the next tables, we denote the labor income sources by $L1$ and $L2$ for *wage* and *bonus* respectively, the social transfers by $T1$ and $T2$ for *family* and *housing* transfers respectively, and the capital income sources by $K1$ and $K2$ for *financial* and *estate* incomes respectively. Table 2.3 provides all pairwise interactions. Table 2.6 shows that the pairwise interactions between $L1$ and $L2$ and

between $L1$ and $K2$ are positive and relatively high. Thus, $L1$ has an amplifier link with these two sources as, on average, these two pairwise interactions create more inequality when they are considered together than alone. By contrast, the interaction between $L1$ and $T2$ is negative, in the sense that $T2$ reduces the inequality created by $L1$.

TABLE 2.6 – The weighted sum of pairwise interactions between i and j

i/j	L1	L2	T1	T2	K1	K2	Total
L1		0.0033	-	- 0.0022	0.0000	0.0044	0.0054
L2	0.0033		-	- 0.0000	0.0000	- 0.0000	0.0033
T1	-	-		-	- 0.0000	0.0000	-
T2	- 0.0054	- 0.0033	-		- 0.0054	- 0.0033	- 0.0174
K1	0.0022	0.0022	- 0.0000	- 0.0022		0.0033	0.0054
K2	0.0000	- 0.0000	-	0.0000	0.0033		0.0033

Note : “-” indicates a perfect null value, while “0.0000” indicates a value close to zero. As postulated previously, the pairwise interaction between i and j , in presence of the other sources, is equal to

$$\sum_{T \in Coal[\mathfrak{B}^{-i,j}]} INT_{v_i, \lambda}^{OW}(i, j, T).$$

The interpretation of the third decomposition may differ from the second decomposition as it has a different conception of interaction. For instance, according to the third one, the interaction term of the income source $L2$ is only due to its positive interaction with $L1$, the other source in its coalition. However, the second decomposition highlighted that $L2$ has a negative interaction with the T coalition and a positive one with the K coalition, which is more straightforward as $L2$ benefits the same individuals as the K sources (see table 2.4).

One can notice that the interaction between two sources that do not belong to the same structural coalition is not necessarily symmetric. For instance, in table 2.6, the interaction between $i = L1$ and $j = T2$ is equal to -0.0022 , but the other way round ($i = T2$ and $j = L1$) it is equal to -0.0054 . This asymmetry is explained by the fact that the sets of coalitions T' , taken into account to measure the interaction, are not the same in both cases. As pointed out previously, each coalition taken into consideration to compute the interaction between i and j can include all the other sources of the structural coalition of i . Thus, as by definition the structural coalition for two players that do not belong to the same group is not the same, the coalitions used for the interaction computation are different. Therefore, unlike a classic Shapley decomposition, the Shapley-Owen decomposition shows that the influence of a source i on a source j is not necessarily the same as the influence of j on i . For instance, we may say in this given case that the presence of $L1$ increases the importance of $T2$ by 0.0054 points, while the presence of $T2$ increases the importance of $L1$ by 0.0022 points.

Furthermore, by way of additional information, we compare this revisited Owen decomposition to the revisited Shapley decomposition proposed by Chantreuil et al. [2019]. The results of this decomposition are given in table 2.7. The PMC of each source of income inequality is the same in all cases ; the differences between the two decompositions lie in the interaction term. Indeed, the Shapley decomposition disregards the hierarchical structure of income ; it considers each source independently from the structural coalition it is included in. For instance, in the Shapley decomposition, the interaction between the wage source and the family social transfer source can solely be computed in the presence of the financial capital income source. This cannot be the case for the Shapley-Owen decomposition which would consider all the capital income sources in such an example. Therefore, the differences lie in another approach of the interactions between the income sources.

TABLE 2.7 – A revisited Shapley decomposition

	Labor income		Social transfers		Capital income	
	Wage	Bonus	Family	Housing	Financial	Estate
φ^{Sh}	0.0292 18.61%	0.0172 10.97%	- 0.00%	- 0.0144 -9.17%	0.1011 64.44%	0.0237 15.14%
PMC	0.0327 20.83%	0.0196 12.50%	- 0.00%	- 0.0261 -16.67%	0.1046 66.67%	0.0261 16.67%
$\mathbb{INT}_{v_I, \lambda}^{SH}$	0.0035 2.22%	0.0024 1.53%	- 0.00%	-0.0118 -7.50%	0.0035 2.22%	0.0024 1.53%

Note : here, the Shapley value $\varphi^{Sh} := PMC - \mathbb{INT}_{v_I, \lambda}^{SH}$ measures the importance of each source. For instance, the importance of the wage source according to the Shapley value is equal to 0.0292, which represents 18.61% of the total inequality measured by the Gini index (0.156863).

2.5. Conclusion

In this article, our main results are the new inequality index decompositions based on the Shapley-Owen value, which are the generalizations of the Shapley decomposition proposed by Chantreuil et al. [2019]. These decompositions are especially suited when there is a hierarchical income distribution. In such a case, we show that the importance of an income source can be re-written as its pure marginal contribution, minus a term of pairwise interactions. Thus, we show why the impact of removing wage differences associated to one source does not match its importance to inequality. The difference between the two is explained by the interaction that a source can have with the other sources. Indeed, the importance of a source of inequality depends on the other sources. Our tool can identify the nature of the link between each pair of sources

and compute the intensity of this link. Moreover, even though we apply our new decompositions to the famous Gini index, it can also be used with a wide range of inequality indices. We believe that these new decompositions can be very useful to policymakers in plainly understanding the role of each income source in inequality and the evolution of their importance over time. Therefore, they may help policymakers to appreciate the consequences of an equalizing public policy on inequality.

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Appendix : Proofs

Proof of Theorem 4.4.1. \implies) Following the proof of Proposition 8 given by Albizuri and Zarzuelo [2004], the coalitional value Ψ given by equation (A2) satisfies the linearity, carrier (ii), rearrangement and coalitional partnership. The conditions $\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) = 1$ and $\sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} b(s_{m_i}, k) = 1$ guarantee that the coalitional semivalue satisfies the dummy player property.

\impliedby) The dummy player property is stronger than the projection property. In the proof of Proposition 11 advanced by Albizuri and Zarzuelo [2004], Proposition 11 is necessity and they used Lemma 9. So, the corner stone of their result is Lemma 9. In Proposition 11, when we replace the projection by the dummy player property and maintain their proof, Equation (28) of Lemma 9 holds without using the monotonicity since all players outside of N , even R , are null players. Therefore, the dummy player property alone would imply Equation (28). In this case, when we follow the rest of the proof, the monotonicity property will be useful only to show that the families of real $\rho^{N,t,s}$ are nonnegative. If we dropped the monotonicity property, the coefficients $\rho^{N,t,s}$ would be the real numbers. Consequently, the coefficients $a(m, p)$ and $b(s_{m_i}, k)$ given in Proposition 11 should be the real numbers. \square

Proof of Proposition 4.6.2. This proof will be obtained by induction. For any $N \in \mathcal{N}$, $v, w \in \Gamma(N)$, $i, j \in N$, $i \neq j$, and $T \subseteq N \setminus ij$.

Induction basis : for $t = 0$, we obtain $[v(ij) - v(j)] - [v(j) - v(\emptyset)] = \lambda_{ij}(v)$, and $[w(ij) - w(j)] - [w(j) - w(\emptyset)] = \lambda_{ij}(w)$ at the same time, which proves the claim.

Induction hypothesis : let the result hold for all $T \subseteq N \setminus ij$ such that $0 \leq t \leq k < n - 2$.

Induction step : consider T being such that $t = k + 1$, then we directly obtain,

$$[v(T \cup ij) - v(T \cup j)] - [v(T \cup i) - v(T)] = \sum_{S \subseteq T} \lambda_{S \cup ij}(v) = \lambda_{T \cup ij}(v) + \sum_{S \subsetneq T} \lambda_{S \cup ij}(v).$$

Analogously, $[w(T \cup ij) - w(T \cup j)] - [w(T \cup i) - w(T)] = \lambda_{T \cup ij}(w) + \sum_{S \subsetneq T} \lambda_{S \cup ij}(w)$.

In view of the induction hypothesis, this proves the claim. \square

Proof of Theorem 4.3.3. For any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $k \in M$, if a coalitional value Ψ satisfies the quotient game property then we have, $\sum_{i \in S_k} \Psi_i^N(v, \mathfrak{B}^N) = \Psi_k^M(v^q, [M])$. Following Courtin et al. [2020] (Theorem 2), a value is AID^{**}, if and only, it satisfies linearity, anonymity (rearrangement when the mapping π is one-to-one) and dummy player property. \square

Proof of Corollary 2.3.1. We use Theorem 4.3.3 stated in this document and Corollary 2 proposed by Courtin et al. [2020]. \square

Proof of Theorems 4.3.5 Following Theorem 4.4.1, it will suffice for us to prove that a coalitional value is AID^{*}, if and only if, for any $N \in \mathcal{N}$, $(v, \mathfrak{B}) \in G(N)$ and $i \in N$, there exist two families of **real numbers** $\{a(m, p)\}_{p=0, \dots, m-1}$ and $\{b(s_{m_i}, k)\}_{k=0, \dots, s_{m_i}-1}$ fulfilling

$$\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) = 1 \text{ and } \sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} b(s_{m_i}, k) = 1 \text{ such that,}$$

$$\Psi_i(v, \mathfrak{B}) := \sum_{P \subseteq (M \setminus m_i)} \sum_{K \subseteq (S_{m_i} \setminus i)} a(m, p) \cdot b(s_{m_i}, k) [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)].$$

\implies) Let an AID^{*} coalitional value Ψ . From Definition 2.3.6, for any $N \in \mathcal{N}$ and $(v, \mathfrak{B}) \in G(N)$, there exist two families of real numbers $\{g(s_{m_i}, k)\}_{k=0}^{s_{m_i}-1}$ and $\{f(m, p)\}_{p=0}^{m-2}$ such that,

$$\sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} g(s_{m_i}, k) = 1 \text{ and}$$

$$\begin{aligned}
\Psi_i(v, \mathfrak{B}) &= \sum_{K \subseteq S_{m_i} \setminus i} g(s_{m_i}, k) [v(i \cup K \cup N \setminus S_{m_i}) - v(K \cup N \setminus S_{m_i})] \\
&\quad - \sum_{\ell \in M \setminus m_i} \sum_{P \subseteq M \setminus \{\ell, m_i\}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{\overline{\text{Int}}_v^{\mathfrak{B}^N}(i, S_\ell, T)}{2} \\
&= \sum_{K \subseteq S_{m_i} \setminus i} g(s_{m_i}, k) [v(i \cup K \cup N \setminus S_{m_i}) - v(K \cup N \setminus S_{m_i}) - [a - b - c + d]], \quad (\text{A1})
\end{aligned}$$

$$\begin{aligned}
\text{with } a &:= \sum_{\ell \in M \setminus m_i} \sum_{P \subseteq M \setminus \{\ell, m_i\}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup S_\ell \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{\ell \in M \setminus m_i} \sum_{\substack{P \subseteq M \setminus m_i \\ \ell \in P}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p - 1) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{\substack{P \subseteq M \setminus m_i \\ P \neq \emptyset}} \sum_{\ell \in P} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p - 1) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{\substack{P \subseteq M \setminus m_i \\ P \neq \emptyset}} \sum_{K \subseteq S_{m_i} \setminus i} p \times f(m, p - 1) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2},
\end{aligned}$$

$$\begin{aligned}
b &:= \sum_{\ell \in M \setminus m_i} \sum_{P \subseteq M \setminus \{\ell, m_i\}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{\ell \in M \setminus m_i} \sum_{\substack{P \subseteq M \setminus m_i \\ \ell \notin P}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{P \subseteq M \setminus m_i} \sum_{\ell \notin m_i \cup P} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{P \subseteq M \setminus m_i} \sum_{K \subseteq S_{m_i} \setminus i} (m - p - 1) f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(i \cup \mathbf{P}_N \cup K)}{2},
\end{aligned}$$

$$\begin{aligned}
\text{while } c &:= \sum_{\ell \in M \setminus m_i} \sum_{P \subseteq M \setminus \{\ell, m_i\}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(S_\ell \cup \mathbf{P}_N \cup K)}{2} \\
&= \sum_{\substack{P \subseteq M \setminus m_i \\ P \neq \emptyset}} \sum_{K \subseteq S_{m_i} \setminus i} p \cdot f(m, p - 1) \cdot g(s_{m_i}, k) \cdot \frac{v(\mathbf{P}_N \cup K)}{2}
\end{aligned}$$

$$\begin{aligned}
\text{and } d &:= \sum_{\ell \in M \setminus m_i} \sum_{P \subseteq M \setminus \{\ell, m_i\}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(\mathbf{P}_N \cup K)}{2} \\
&= \sum_{P \subseteq M \setminus m_i} \sum_{K \subseteq S_{m_i} \setminus i} (m - p - 1) \cdot f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{v(\mathbf{P}_N \cup K)}{2}.
\end{aligned}$$

Equation (A1) becomes,

$$\Psi_i(v, \mathfrak{B}) = \sum_{P \subseteq (M \setminus m_i)} \sum_{K \subseteq (S_{m_i} \setminus i)} a(m, p) \cdot b(s_{m_i}, k) [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)],$$

where $b(s_{m_i}, k) = g(s_{m_i}, k)$, and

$$\begin{aligned}
a(m, p) &= \frac{[(m - 1 - p)f(m, p) - pf(m, p - 1)]e(m, p)}{2} + \frac{(m - 1)f(m, 0) \cdot r(m, p)}{2} \\
&\quad + \left[1 - \frac{(m - 1)f(m, m - 2)}{2}\right] \ell(m, p),
\end{aligned}$$

$$\text{with } e(m, p) = \begin{cases} 1 & \text{if } 0 < p < m - 1 \\ 0 & \text{otherwise} \end{cases}; \quad \ell(m, p) = \begin{cases} 1 & \text{if } p = m - 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$r(m, p) = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) = 1$. In fact,

$$\begin{aligned}
\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) &= \frac{(m - 1)f(m, 0)}{2} + 1 - \frac{(m - 1)f(m, m - 2)}{2} \\
&\quad + \sum_{p=1}^{m-2} \frac{(m - p - 1) \binom{m-1}{p} f(m, p)}{2} - \sum_{p=1}^{m-2} \frac{p \binom{m-1}{p} f(m, p - 1)}{2} \\
&= 1 + \sum_{p=0}^{m-2} \frac{(m - p - 1) \binom{m-1}{p} f(m, p)}{2} - \sum_{p=1}^{m-1} \frac{p \binom{m-1}{p} f(m, p - 1)}{2} \\
&= 1 + \sum_{p=0}^{m-2} \frac{(m - p - 1) \binom{m-1}{p} f(m, p)}{2} - \sum_{p=0}^{m-2} \frac{(p + 1) \binom{m-1}{p+1} f(m, p)}{2} \\
&= 1.
\end{aligned}$$

\Leftarrow) We suppose that for any $N \in \mathcal{N}$, (v, \mathfrak{B}) and $i \in N$, there exist two families of **real**

numbers $\{a(m, p)\}_{p=0, \dots, m-1}$ and $\{b(s_{m_i}, k)\}_{k=0, \dots, s_{m_i}-1}$ fulfilling $\sum_{p=0}^{m-1} \binom{m-1}{p} a(m, p) = 1$

and $\sum_{k=0}^{s_{m_i}-1} \binom{s_{m_i}-1}{k} b(s_{m_i}, k) = 1$ such that,

$$\Psi_i(v, \mathfrak{B}) := \sum_{P \subseteq (M \setminus m_i)} \sum_{K \subseteq (S_{m_i} \setminus i)} a(m, p) \cdot b(s_{m_i}, k) [v(\mathbf{P}_N \cup K \cup i) - v(\mathbf{P}_N \cup K)].$$

If we set $g(s_{m_i}, k) = b(s_{m_i}, k)$ and

$$f(m, p) = \begin{cases} \frac{2 \times a(m, 0)}{m-1} & \text{if } p = 0 \\ \frac{2(1-a(m-1))}{m-1} & \text{if } t = m - 2 \\ \frac{2[p \times f(m, p-1) + a(m, p)]}{m-p-1} & \text{otherwise,} \end{cases}$$

then

$$a(m, p) = \frac{[(m-1-p)f(m, p) - pf(m, p-1)]e(m, p)}{2} + \frac{(m-1)f(m, 0) \cdot r(m, p)}{2} + \left[1 - \frac{(m-1)f(m, m-2)}{2}\right] \ell(m, p),$$

and

$$\begin{aligned} \Psi_i(v, \mathfrak{B}) &:= \sum_{K \subseteq S_{m_i} \setminus i} g(s_{m_i}, k) [v(i \cup K \cup N \setminus S_{m_i}) - v(K \cup N \setminus S_{m_i})] \\ &\quad - \sum_{\ell \in M \setminus m_i} \sum_{P \subseteq M \setminus \{\ell, m_i\}} \sum_{K \subseteq S_{m_i} \setminus i} f(m, p) \cdot g(s_{m_i}, k) \cdot \frac{\overline{\text{Int}}_v^{\mathfrak{B}^N}(i, S_\ell, T)}{2}. \end{aligned}$$

Albizuri and Zarzuelo [2004] show that every coalitional semivalue satisfies the linearity property, and we know that by linearity everything is determined on dirac games ($\delta_T(S) = 1$ if $S = T$ and 0 otherwise).

$$\Psi_i(\delta_N, \mathfrak{B}) = a(m, m-1) \cdot b(s_{m_i}, s_{m_i}-1) = b(s_{m_i}, s_{m_i}-1) - \frac{(m-1)b(s_{m_i}, s_{m_i}-1) \cdot f(m, m-2)}{2}.$$

Then,

$$f(m, m-2) = \frac{2(1-a(m, m-1))}{(m-1)}.$$

The families of real numbers $\{g(s_{m_i}, k)\}_{k=0, \dots, s_{m_i}-1}$ exist (coefficients of value φ). It suffices

to set

$$f(m, p) = \begin{cases} \frac{2 \times a(m, 0)}{m-1} & \text{if } p = 0 \\ \frac{2(p \times f(m, p-1) + a(m, p))}{m-p-1} & \text{otherwise,} \end{cases}$$

and we conclude that Ψ is AID*. \square

Proof outline of Corollary 2.3.5.

$$\sum_{k=0}^{s_{m_i}-1} \sum_{p=0}^{m-2} \binom{s_{m_i}-1}{k} \times \binom{m-2}{p} \times f(m, p) \times \frac{k!(s_{m_i}-k-1)!}{s_{m_i}!} = \sum_{p=0}^{m-2} \binom{m-2}{p} \times f(m, p) = 1. \quad \square$$

Proof outline of Proposition 2.3.3.

- (i) $\sum_{i \in S_\ell} [\Psi_i^{Sh}(v^N, \mathfrak{B}^N) - \Psi_i^{Sh}(v^{N \setminus S_r}, \mathfrak{B}^{N \setminus S_r})] = \sum_{i \in S_\ell} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_r}, i]} INT_v^{\text{extra}}(i, S_r, T).$
 $\sum_{i \in S_r} [\Psi_i^{Sh}(v^N, \mathfrak{B}^N) - \Psi_i^{Sh}(v^{N \setminus S_\ell}, \mathfrak{B}^{N \setminus S_\ell})] = \sum_{i \in S_r} \sum_{T \in Coal[\mathfrak{B}^{N \setminus S_\ell}, i]} INT_v^{\text{extra}}(i, S_\ell, T).$
- (ii) The Shapley value satisfies the Balanced contributions property i.e., for all $v \in \Gamma(N)$,
 $i, j \in N$, $\varphi_j^{Sh}(v^N) - \varphi_j^{Sh}(v^{N \setminus i}) = \varphi_i^{Sh}(v^N) - \varphi_i^{Sh}(v^{N \setminus j}).$
- (iii) The Owen value satisfies the *quotient game property* (see Alonso-Meijide et al. [2007]).

The statements (ii) and (iii) imply that the Owen value satisfies the Balanced contribution across the structural coalitions property. More formally, that is to say that :

$$\sum_{i \in S_\ell} [\Psi_i^{Sh}(v^N, \mathfrak{B}^N) - \Psi_i^{Sh}(v^{N \setminus S_r}, \mathfrak{B}^{N \setminus S_r})] = \sum_{i \in S_r} [\Psi_i^{Sh}(v^N, \mathfrak{B}^N) - \Psi_i^{Sh}(v^{N \setminus S_\ell}, \mathfrak{B}^{N \setminus S_\ell})].$$

\square

Proof of Theorem 2.3.3. First part : Let Ψ be a coalitional value and for all $v \in \Gamma(N)$ and for any $\mathfrak{B}^N := \{S_1, S_2, \dots, S_m\}$ a coalition structure over N . If Ψ satisfies the efficiency then, we have

$$\begin{aligned} v(N) - v(N \setminus i) &= \sum_{j \in N} \Psi_j^N(v^N, \mathfrak{B}^N) - \sum_{j \in N \setminus i} \Psi_j^N(v^{N \setminus i}, \mathfrak{B}^{N \setminus i}) \\ &= \Psi_i^N(v^N, \mathfrak{B}^N) + \sum_{j \in S_{m_i} \setminus i} [\Psi_j^N(v^N, \mathfrak{B}^N) - \Psi_j^N(v^{N \setminus i}, \mathfrak{B}^{N \setminus i})] \\ &\quad - \sum_{\ell \in M \setminus m_i} \sum_{j \in S_\ell} [\Psi_j^N(v^N, \mathfrak{B}^N) - \Psi_j^N(v^{N \setminus i}, \mathfrak{B}^{N \setminus i})] \end{aligned}$$

Moreover, if Ψ satisfies the linearity, rearrangement, projection and coalitional partnership, then, Ψ is the Shapley-Owen value. As a result of lengthy calculations, if $j \in S_{m_i}$ then we

have,

$$\begin{aligned} & \Psi_j^N(v^N, \mathfrak{B}^N) - \Psi_j^N(v^{N \setminus i}, \mathfrak{B}^{N \setminus i}) = \\ & \sum_{j \in S_{m_i} \setminus i} \sum_{\substack{T \in \text{Coal}[\mathfrak{B}^N, i] \\ j \notin T}} \frac{p!(m-p-1)!}{m!} \cdot \frac{(k+1)!(s_{m_j} - k - 2)!}{s_{m_j}!} \cdot \text{Int}_v^N(i, j, T) \end{aligned}$$

and if $j \notin S_{m_i}$ then we have,

$$\begin{aligned} & \Psi_j^N(v^N, \mathfrak{B}^N) - \Psi_j^N(v^{N \setminus i}, \mathfrak{B}^{N \setminus i}) = \\ & \sum_{\ell \in M \setminus m_i} \sum_{j \in S_\ell} \sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus S_{m_i}}, i]} \frac{(p+1)!(m-p-2)!}{m!} \times \frac{k!(s_{m_j} - k - 1)!}{s_{m_j}!} \cdot \text{Int}_v^N(i, j, T \cup (S_{m_i} \setminus i)) \end{aligned}$$

Finally, we have

$$\begin{aligned} & \Psi_i^N(v^N, \mathfrak{B}^N) = \\ & v(N) - v(N \setminus i) - \sum_{j \in S_{m_i} \setminus i} \sum_{\substack{T \in \text{Coal}[\mathfrak{B}^N, i] \\ j \notin T}} \frac{p!(m-p-1)!}{m!} \cdot \frac{(k+1)!(s_{m_j} - k - 2)!}{s_{m_j}!} \text{Int}_v^N(i, j, T) \\ & - \sum_{\ell \in M \setminus m_i} \sum_{j \in S_\ell} \sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus S_{m_i}}, i]} \frac{(p+1)!(m-p-2)!}{m!} \times \frac{k!(s_{m_j} - k - 1)!}{s_{m_j}!} \text{Int}_v^N(i, j, T \cup (S_{m_i} \setminus i)) \end{aligned}$$

Second part : Let us show that, $\sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus i}, j]} Q_T = 1$.

First case : $\frac{Q_T}{2} = \alpha_1$ if $S_{m_i} \setminus i = \emptyset$.

Thus, $\sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus i}, j]} Q_T = \sum_{p=0}^{m-2} \sum_{k=0}^{s_{m_j}-1} 2 \binom{m-2}{p} \frac{(p+1)!(m-p-2)!}{m!} \times \binom{s_{m_j}-1}{k} \frac{k!(s_{m_j}-k-1)!}{s_{m_j}!}$.

It yields that $\sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus i}, j]} Q_T = \sum_{p=0}^{m-2} \sum_{k=0}^{s_{m_j}-1} \frac{2(p+1)}{m(m-1)} \times \frac{1}{s_{m_j}} = 2 \sum_{p=0}^{m-2} \frac{p+1}{m(m-1)}$.

Hence, $\sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus i}, j]} Q_T = \frac{2}{m(m-1)} \left[\frac{(m-1)(m-2)}{2} + (m-1) \right] = 1$. As desired.

Second case : $\frac{Q_T}{2} = \alpha_2$ if $S_{m_i} = S_{m_j}$. As previously, $\sum_{T \in \text{Coal}[\mathfrak{B}^{N \setminus i}, j]} Q_T = 1$.

Third case : $\frac{Q_T}{2} = \begin{cases} \alpha_3 & \text{if } (S_{m_i} \setminus i) \subset T \\ 0 & \text{if } (S_{m_i} \setminus i) \subset (N \setminus T) \end{cases}$, and then, $\sum_{\substack{T \in \text{Coal}[\mathfrak{B}^{N \setminus i}, j] \\ (S_{m_i} \setminus i) \subset T}} Q_T = 1$.

□

Proof outline of Proposition 2.3.4. If a coalitional value Ψ satisfies the balanced contributions within a structural coalition property (Alonso-Meijide et al. [2007]), i.e., for all $v \in \Gamma(N)$, for any $\mathfrak{B}^N := \{S_1, S_2, \dots, S_m\}$ a coalition structure over N , for all $\ell \in M$, $i, j \in S_\ell$, $\Psi_j^N(v^N, \mathfrak{B}^N) - \Psi_j^N(v^{N \setminus i}, \mathfrak{B}^{N \setminus i}) = \Psi_i^N(v^N, \mathfrak{B}^N) - \Psi_i^N(v^{N \setminus j}, \mathfrak{B}^{N \setminus j})$. Thus, the result follows since the Owen value satisfies this property. □

Proof of Propositions 2.4.1, 4.4.1 & 2.4.3. Due to the efficiency of the Shapley-Owen value, these propositions are deduced directly from Corollary 2.3.2, Corollary 2.3.4 and Theorem 2.3.3 respectively. □

Second-order productivity, second-order payoffs, and the Shapley-Owen value

Résumé :

Dans la classe des jeux coopératifs à utilité transférable (jeux TU) avec structure de coalitions (jeux CS), nous introduisons les concepts de productivités de second ordre des blocs (coalitions formées) et de paies de second ordre des blocs pour des valeurs sur les jeux CS en tant que généralisations des productivités de second ordre des joueurs dans les jeux TU et des paies de second ordre des joueurs pour les valeurs sur les jeux TU (Casajus, *Discrete Applied Mathematics*, 304, 212–219, 2021). Les productivités de second ordre des joueurs sont conceptualisées comme des contributions marginales de second ordre, c'est-à-dire la manière dont un joueur affecte les contributions marginales d'un autre joueur aux coalitions ne contenant aucun d'eux en rejoignant ces coalitions. Les paies de second ordre des joueurs sont conceptualisés comme l'effet d'un joueur quittant le jeu lors de la paie d'un autre joueur. De manière analogue, les productivités de second ordre des blocs sont conceptualisées comme étant leurs productivités de second ordre dans le jeu entre blocs ; les paies de second ordre des blocs sont conceptualisées comme étant leurs paies de second ordre dans le jeu entre les blocs. Nous montrons que la valeur de Shapley-Owen est l'unique valeur efficace pour les jeux CS qui reflète les productivités de second ordre des joueurs et des blocs en termes de leurs paies de second ordre.

Mots clés : Jeu TU, valeur de Shapley, valeur de Shapley-Owen, contributions marginales du second ordre, paiements de second ordre

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Abstract :

We introduce the concepts of the components' second-order productivities in cooperative games with transferable utility (TU games) with a coalition structure (CS games) and of the components' second-order payoffs for one-point solutions for CS games as generalizations of the players' second-order productivities in TU games and of the players' second-order payoffs for one-point solutions for TU games (Casajus, *Discrete Applied Mathematics*, 304, 212–219, 2021). The players' second-order productivities are conceptualized as second-order marginal contributions, that is, how one player affects another player's marginal contributions to coalitions containing neither of them by entering these coalitions. The players' second-order payoffs are conceptualized as the effect of one player leaving the game on the payoff of another player. Analogously, the components' second-order productivities are conceptualized as their second-order productivities in the game between components; the components' second-order payoffs are conceptualized as their second-order payoffs in the game between components. We show that the Shapley-Owen value is the unique efficient one-point solution for CS games that reflects the players' and the components' second-order productivities in terms of their second-order payoffs.

Key words : TU game, Shapley value, Shapley-Owen value, second-order marginal contributions, second-order payoffs

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3.1. Introduction

A cooperative game with transferable utility for a finite player set (TU game or simply game) is given by a coalition function that assigns a worth to any coalition (subset of the player set), where the empty coalition obtains zero. (One-point) solutions for TU games assign a payoff to any player in any TU game. The Shapley value [Shapley, 1953] probably is the most eminent one-point solution concept for TU games. And its characterization by Young [1985] probably is its most important one.

Young [1985] characterizes the Shapley value by three properties of solutions : efficiency, symmetry, and marginality or strong monotonicity. Efficiency : the players' payoffs sum up to the worth generated by the grand coalition. Symmetry : equally productive⁴ players obtain the same payoff. Marginality : a player's payoff only depends on her own productivity. Strong monotonicity : whenever a player's productivity in a game weakly increases so does her payoff. Note that strong monotonicity implies marginality. This result indicates that the Shapley value is *the* efficient solution that reflects the players' productivities by their payoffs.

The organization of players into groups can be modelled by coalition structures—partitions of the player set. Games enriched with a coalition structure are addressed as CS games and the corresponding solutions as CS solutions. Owen [1977] generalizes of the Shapley value into an efficient CS solution where the components of the coalition structure are treated like

4. In this paper, a player's productivity in a game refers to her influence on the generation of worth as expressed by her marginal contributions to coalitions not containing her, that is, the differences between the worth generated after she entered such a coalition and the worth generated before she entered.

players.⁵ Khmelnitskaya and Yanovskaya [2007] provide a characterization of the Owen value that breathes the spirit of Young's [1985] characterization of the Shapley value.⁶ This characterization uses four properties : efficiency, marginality, symmetry within components, and symmetry across components. Symmetry across components : components that are equally productive in the game between components obtain the same sum of payoffs of their members.

Recently, Casajus [2021] suggests a second-order version of Young's [1985] characterization of the Shapley value. This characterization is based on the notions of the players' second-order productivities and second-order payoffs. A player's second-order productivity with respect to another player reflects how the former affects the latter player's marginal contribution to coalitions containing neither of them by entering these coalitions ; a player's second-order payoff with respect to another player reflects how the former affects the latter player's payoff by leaving the game. The Shapley value is the unique efficient solution that reflects the players' second-order productivities in terms of their second-order payoffs. More precisely, it is the unique solution that satisfies efficiency and second-order versions of symmetry and marginality. Second-order symmetry : players who are equally second-order productive with respect to a third player obtain the same second-order payoff with respect to this third player. Second-order marginality : a player's second-order payoff with respect to another player only depends on her own second-order productivity with respect to this other player.

In this paper, we suggest a second-order version of Khmelnitskaya and Yanovskaya's [2007] characterization of the Shapley-Owen value. In particular, we show that the Shapley-Owen value is the unique CS solution that satisfies efficiency, second-order marginality, and second-order versions of symmetry within components and symmetry across components. (Theorem 3.4.2). Second-order symmetry within components is just the restriction of second-order symmetry to players within the same component. Second-order symmetry across components : components that are equally second-order productive in the game between components obtain the same sum of second-order payoffs of their members. This result is partly based on three facts. Second-order marginality implies marginality (Proposition 3.4.1). Efficiency and second-order symmetry within components imply symmetry within components (Proposition 3.4.2). Efficiency and second-order symmetry across components imply symmetry across components (Proposi-

5. Alternative efficient CS solutions have been suggested by Kamijo [2009] and Alonso-Meijide et al. [2014], for example. Alternative non-efficient CS solutions have been suggested by Aumann and Dreze [1974], Owen [1982], and Alonso-Meijide and Fiestras-Janeiro [2002a], for example.

6. Alternative characterizations of the Owen value have been suggested by Owen [1977] himself, Hart and Kurz [1983a], Calvo et al. [1996], Hamiache [2001], Albizuri [2008], and Casajus [2010], for example.

tion 3.4.3).

The remainder of this paper is organized as follows. In Section 2, we provide basic definitions and notation. In Section 3, we survey the characterizations of the Shapley value by Young [1985] and by Casajus [2021]. In Section 4, we first survey the characterization of the Shapley-Owen value by Khmelnitskaya and Yanovskaya [2007]. Then, we provide our second-order approach to the Shapley-Owen value. Some remarks conclude the paper.

3.2. Basic definitions and notation

Let the universe of players \mathfrak{U} be a countably infinite set, and let \mathcal{N} denote the set of all finite subsets of \mathfrak{U} . The cardinalities of $S, T, N \in \mathcal{N}$ are denoted by $s, t, \text{ and } n$, respectively. A (finite TU) game for the player set $N \in \mathcal{N}$ is given by a **coalition function** $v : 2^N \rightarrow \mathbb{R}$, $v(\emptyset) = 0$, where 2^N denotes the power set of N . Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S . The set of all games for N is denoted by $\mathbb{V}(N)$; the set of all games is denoted by $\mathbb{V} := \bigcup_{N \in \mathcal{N}} \mathbb{V}(N)$.

For $N \in \mathcal{N}$, $T \subseteq N$, and $v \in \mathbb{V}(N)$, the **subgame** $v|_T \in \mathbb{V}(T)$ is given by $v|_T(S) = v(S)$ for all $S \subseteq T$; for $i \in N$ and $S \subseteq N$, we occasionally write v_{-i} and v_{-S} instead of $v|_{N \setminus \{i\}}$ and $v|_{N \setminus S}$, respectively. For $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T^N \in \mathbb{V}$ given by $u_T^N(S) = 1$ if $T \subseteq S$ and $u_T^N(S) = 0$ otherwise is called a **unanimity game**. Any $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N: T \neq \emptyset} \lambda_T(v) \cdot u_T^N, \quad (\text{A1})$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends [Harsanyi, 1959a] and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subset T: S \neq \emptyset} \lambda_S(v). \quad (\text{A2})$$

Players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}(N)$ if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$.

A **rank order** of $N \in \mathcal{N}$ is a bijection $\rho : N \rightarrow \{1, 2, \dots, |N|\}$ with the interpretation that

i is the $\rho(i)$ th player in ρ ; the set of rank orders of N is denoted by $R(N)$. The set of players before i in ρ is denoted by $B_i(\rho) := \{\ell \in N : \rho(\ell) < \rho(i)\}$. The **marginal contribution** of i in ρ and $v \in \mathbb{V}(N)$ is denoted by

$$MC_i^v(\rho) := v(B_i(\rho) \cup \{i\}) - v(B_i(\rho)). \quad (\text{A3})$$

A **solution** for \mathbb{V} is an operator that assigns to any $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$ a payoff $\varphi_i(v)$. The **Shapley value** [Shapley, 1953] for \mathbb{V} , Sh , is given by

$$\text{Sh}_i(v) := \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{t} = \sum_{S \subseteq N \setminus \{i\}} \frac{v(S \cup \{i\}) - v(S)}{n \cdot \binom{n-1}{s}} = \sum_{\rho \in R} \frac{1}{|R(N)|} \cdot MC_i^v(\rho) \quad (\text{A4})$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

For $N \in \mathcal{N}$, let $\mathfrak{P}(N)$ denote the set of all partitions (**coalition structures**) of N ; the component of player $i \in N$ in $\mathcal{P} \in \mathfrak{P}(N)$ is denoted by $\mathcal{P}(i)$. For $N \in \mathcal{N}$, $\mathcal{P} \in \mathfrak{P}(N)$, $T \subseteq N$ and $i \in N$, let $\mathcal{P}(T) \subseteq \mathcal{P}$ be given by $\mathcal{P}(T) := \{P \in \mathcal{P} \mid T \cap P \neq \emptyset\}$, let $\mathcal{P}|_T \in \mathfrak{P}(T)$ be given by $\mathcal{P}|_T := \{T \cap P \mid P \in \mathcal{P}(T)\}$, let $\mathcal{P}_{-T} \in \mathfrak{P}(N \setminus T)$ be given by $\mathcal{P}_{-T} := \mathcal{P}|_{N \setminus T}$, and let $\mathcal{P}_{-i} \in \mathfrak{P}(N \setminus \{i\})$ be given by $\mathcal{P}_{-i} := \mathcal{P}_{-\{i\}}$.

A **CS game** for $N \in \mathcal{N}$ is a pair (v, \mathcal{P}) , where $v \in \mathbb{V}(N)$ and $\mathcal{P} \in \mathfrak{P}(N)$. Let $\mathbb{VP}(N)$ denote the set of all CS games for N and let $\mathbb{VP} := \cup_{N \in \mathcal{N}} \mathbb{VP}(N)$ denote the set of all CS games.

A **(CS) solution** for \mathbb{VP} is an operator φ that assigns to any $N \in \mathcal{N}$, $i \in N$, and $(v, \mathcal{P}) \in \mathbb{VP}(N)$ a payoff $\varphi_i(v, \mathcal{P})$; for $P \in \mathcal{P}$, we set $\varphi_P(v, \mathcal{P}) = \sum_{i \in P} \varphi_i(v, \mathcal{P})$. For $N \in \mathcal{N}$ and $\mathcal{P} \in \mathfrak{P}(N)$, the set of all rank orders that respect \mathcal{P} is denoted by

$$R(N, \mathcal{P}) := \{\rho \in R(N) \mid \text{for all } P \in \mathcal{P} \text{ and } i, j \in P : |\rho(i) - \rho(j)| < |P|\},$$

that is, in any such rank order, the players from any component follow each other without players from other components between them. The **Shapley-Owen value** [Owen, 1977] for \mathbb{VP} ,

Ow, is the CS solution given by

$$\text{Ow}_i(v, \mathcal{P}) := \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{|\mathcal{P}(i) \cap T| \cdot |\mathcal{P}(T)|} \quad (\text{A5a})$$

$$= \sum_{\mathcal{C} \subseteq \mathcal{P} \setminus \{\mathcal{P}(i)\}} \sum_{S \subseteq \mathcal{P}(i) \setminus \{i\}} \frac{v\left(S \cup \{i\} \cup \bigcup_{C \in \mathcal{C}} C\right) - v\left(S \cup \bigcup_{C \in \mathcal{C}} C\right)}{|\mathcal{P}(i)| \cdot \binom{|\mathcal{P}(i)|-1}{s} \cdot |\mathcal{P}| \cdot \binom{|\mathcal{P}|-1}{|\mathcal{C}|}} \quad (\text{A5b})$$

$$= \sum_{\rho \in R(N, \mathcal{P})} \frac{1}{|R|} \cdot MC_i^v(\rho) \quad (\text{A5c})$$

for all $N \in \mathcal{N}$, $i \in N$, and $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$.

Fix an injection $\iota : \mathcal{N} \rightarrow \mathfrak{U}$, $N \mapsto \iota_N$ for $N \in \mathcal{N}$. For any $N \in \mathcal{N}$, $\mathcal{P} \in \mathcal{P}(N)$, and $v \in \mathbb{V}(N)$, set $[\mathcal{P}] := \{\iota_P \mid P \in \mathcal{P}\}$ and let $v_{\mathcal{P}} \in \mathbb{V}([\mathcal{P}])$ be given by

$$v_{\mathcal{P}}([\mathcal{C}]) := v\left(\bigcup_{C \in \mathcal{C}} C\right) \quad \text{for all } \mathcal{C} \subseteq \mathcal{P}. \quad (\text{A6})$$

The TU game $v_{\mathcal{P}}$ is called the **game between components** or **intermediate game** for the CS game (v, \mathcal{P}) . For $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$, and $P \in \mathcal{P}$, we have

$$\text{Ow}_P(v, \mathcal{P}) = \text{Sh}_{\iota_P}(v_{\mathcal{P}}). \quad (\text{A7})$$

3.3. The Shapley value⁷

The marginal contributions of a player $i \in N$, $N \in \mathcal{N}$ in the game $v \in \mathbb{V}(N)$ given as

$$v(S \cup \{i\}) - v(S), \quad S \subseteq N \setminus \{i\} \quad (\text{A8})$$

indicate her (individual) productivity or contribution to the generation of worth in the game v . The right-hand formula of the Shapley value in (A4) indicates that the players' Shapley value payoffs reflects their productivities in games as expressed by their own marginal contributions. Young [1985] shows that the Shapley value is the unique efficient such solution.

Efficiency, E. For all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$, we have $\sum_{\ell \in N} \varphi_{\ell}(v) = v(N)$.

Symmetry, S. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are symmetric in v , we have $\varphi_i(v) = \varphi_j(v)$.

Marginality, M. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) = \varphi_i(w)$.

Theorem 3.3.1 (Young, 1985). *The Shapley value is the unique solution for \mathbb{V} that satisfies efficiency (E), symmetry (S), and marginality (M).*⁸

Symmetry and marginality can be paraphrased as follows. Symmetry : players who are equally productive in a game should obtain the same payoff. Marginality : a player who is equally productive in two games should obtain the same payoff in these games. Therefore, a solution that is intended to reflect the players' productivities should satisfy these properties.

Later on, Casajus [2021] introduces the notions of the players' second-order productivities and second-order payoffs. Second-order productivities are conceptualized as second-order marginal contributions : the second-order marginal contributions of player $i \in N$, $N \in \mathcal{N}$ with respect to player $j \in N \setminus \{i\}$ in a game $v \in \mathbb{V}(N)$ are given as

$$[v(S \cup \{i, j\}) - v(S \cup \{i\})] - [v(S \cup \{j\}) - v(S)], \quad S \subseteq N \setminus \{i, j\}. \quad (\text{A9})$$

These describe how player i affects the productivity of player j .⁹ The second-order payoff of player $i \in N$, $N \in \mathcal{N}$ with respect to player $j \in N \setminus \{i\}$ in a game $v \in \mathbb{V}(N)$ is given by

$$\varphi_j(v) - \varphi_j(v_{-i}).$$

It describes how player i affects the payoff of player j .¹⁰

Based on these notions, Casajus [2021] motivates natural second-order versions of symmetry and marginality. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j, k \in N$, $i \neq j \neq k \neq i$, players i and j are called **second-order symmetric with respect to player k** if

$$\begin{aligned} & [v(T \cup \{i, k\}) - v(T \cup \{i\})] - [v(T \cup \{k\}) - v(T)] \\ & = [v(T \cup \{j, k\}) - v(T \cup \{j\})] - [v(T \cup \{k\}) - v(T)] \end{aligned}$$

for all $T \subseteq N \setminus \{i, j, k\}$.

8. Originally, Young [1985] invokes anonymity (called symmetry by him) instead of symmetry (in our parlance). Although anonymity is stronger than symmetry, it is well-known and easy to check that anonymity can be replaced with symmetry in his characterization. Moreover, his characterization works on fixed player sets.

9. The second-order marginal contributions of player i to player j in the game v equal player j 's contributions to player i . Often, these are referred to as the second-order derivative of v with respect to i and j .

10. Second-order (and higher-order) Casajus and Huettner [2018, Definition 9] introduce second-order (and higher-order) payoffs as second-order (and higher-order) contributions.

Second-order symmetry, 2S. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and $i, j, k \in N$, $i \neq j \neq k \neq i$ such that players i and j are second-order symmetric with respect to player k , we have

$$\varphi_k(v) - \varphi_k(v_{-i}) = \varphi_k(v) - \varphi_k(v_{-j}).$$

Second-order marginality, 2M. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$ such that

$$\begin{aligned} & [v(T \cup \{i, j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)] \\ &= [w(T \cup \{i, j\}) - w(T \cup \{i\})] - [w(T \cup \{j\}) - w(T)] \end{aligned}$$

for all $T \subseteq N \setminus \{i, j\}$, we have

$$\varphi_j(v) - \varphi_j(v_{-i}) = \varphi_j(w) - \varphi_j(w_{-i}).$$

Second-order symmetry and second-order marginality can be paraphrased as follows. Second-order symmetry : players who are equally second-order productive with respect to a third player in a game should be assigned the same second-order payoff with respect to the latter. Second-order marginality : a player who is equally second-productive with respect to a another player in two games should be assigned the same second-order payoff with respect to the latter in these games. Therefore, it seems to be plausible that a solution the second-order payoffs of which are intended to reflect the players' second-order productivities satisfies these properties.

It turns out that the Shapley value reflects the players' second-order productivities in terms of their second-order payoffs in the same vein as it reflects the players' (first-order) productivities in terms of their (first-order) payoffs.

Theorem 3.3.2 (Casajus, 2021). *The Shapley value is the unique solution for \mathbb{V} that satisfies efficiency (E), second-order symmetry (2S), and second-order marginality (2M).*

The proof of this theorem uses the fact that second-order marginality implies marginality, the proof of which is rather short.

Proposition 3.3.1 (Casajus, 2021)

If a solution for \mathbb{V} satisfies second-order marginality (2M), then it satisfies marginality (M).

Nevertheless, the proof of Theorem 3.3.2 is much more involved than the proof of Theorem 3.3.1 due to use of second-order symmetry instead of symmetry.

On the one hand, second-order symmetry does not imply symmetry [Casajus, 2021, Remark 3]. On the other hand, the counterexamples in Casajus [2021, Remark 3] fail efficiency. As our first result, we show that the proof of Theorem 3.3.2 can be simplified substantially by providing a rather short proof that efficiency and second-order symmetry imply symmetry.

Proposition 3.3.2

If a solution for \mathbb{V} satisfies second-order symmetry (**2S**) and efficiency (**E**), then it satisfies symmetry (**S**).

Proof. Let the solution φ satisfy **2S** and **E**. For $|N| = 1$, nothing is to show. Let now $|N| > 1$. Let (*) $i, j \in N, i \neq j, N \in \mathcal{N}$ be symmetric in $v \in \mathbb{V}(N)$. Fix $h \in \mathcal{U} \setminus N$, set $M := N \cup \{h\}$, and let $w \in \mathbb{V}(M)$ be given by

$$w = \sum_{T \subseteq N: T \neq \emptyset} \lambda_T(v) \cdot u_T^M + \sum_{T \subseteq N \setminus \{i, j\}} [\lambda_{T \cup \{i\}}(v) \cdot u_{T \cup \{h\}}^M + \lambda_{T \cup \{i, j\}}(v) \cdot u_{T \cup \{i, h\}}^M + \lambda_{T \cup \{i, j\}}(v) \cdot u_{T \cup \{j, h\}}^M],$$

that is, a player h is added to v such that (**) i and j remain symmetric in w , (***) h is symmetric to both i and j in w , and (****) $w_{-h} = v$.

Since i and j are symmetric in w , they are second-order symmetric with respect to any $k \in M \setminus \{i, j\}$ in w . Hence, we have

$$\varphi_k(w_{-i}) \stackrel{\mathbf{2S}}{=} \varphi_k(w_{-j}) \quad \text{for all } k \in M \setminus \{i, j\}. \quad (\text{A10})$$

Now, we obtain

$$\begin{aligned} \varphi_j(w_{-i}) &\stackrel{\mathbf{E}}{=} w_{-i}(M \setminus \{i\}) - \sum_{k \in N \setminus \{i, j\}} \varphi_k(w_{-i}) \\ &\stackrel{(**), (\text{A10})}{=} w_{-j}(M \setminus \{j\}) - \sum_{k \in N \setminus \{i, j\}} \varphi_k(w_{-j}) \\ &\stackrel{\mathbf{E}}{=} \varphi_i(w_{-j}). \end{aligned}$$

In view of (***), we analogously obtain

$$\varphi_j(w_{-h}) = \varphi_h(w_{-j}) \quad \text{and} \quad \varphi_i(w_{-h}) = \varphi_h(w_{-i}). \quad (\text{A11})$$

Finally, we have

$$\varphi_i(v) \stackrel{(\text{****})}{=} \varphi_i(w_{-h}) \stackrel{(\text{A11})}{=} \varphi_h(w_{-i}) \stackrel{(\text{A10})}{=} \varphi_h(w_{-j}) \stackrel{(\text{A11})}{=} \varphi_j(w_{-h}) \stackrel{(\text{****})}{=} \varphi_j(v),$$

which concludes the proof. \square

3.4. The Shapley-Owen value

In this section, we first survey the characterization of the Owen value by Khmelnitskaya and Yanovskaya [2007]. Then, we provide a second-order version of this characterization similar to the second-order characterization of the Shapley value by Casajus [2021] as surveyed in Section 3.3.

3.4.1. The (first-order) characterization by Khmelnitskaya and Yanovskaya (2007)

Khmelnitskaya and Yanovskaya [2007] generalize the characterization of the Shapley value due to Young [1985]. This characterization indicates that the Shapley-Owen value is the unique efficient CS solution that reflects both the players' and the components' (first-order) productivities in terms of the players' (first-order) payoffs.¹¹

Efficiency, E. For all $N \in \mathcal{N}$ and $(v, \mathcal{P}) \in \mathbb{VP}(N)$, we have $\sum_{\ell \in N} \varphi_\ell(v, \mathcal{P}) = v(N)$.

Marginality, M. For all $N \in \mathcal{N}$, $(v, \mathcal{P}), (w, \mathcal{P}) \in \mathbb{VP}(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v, \mathcal{P}) = \varphi_i(w, \mathcal{P})$.

Symmetry within components, SwC. For all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{VP}(N)$, $P \in \mathcal{P}$, and $i, j \in P$ such that i and j are symmetric in v , we have $\varphi_i(v) = \varphi_j(v)$.

For all $N \in \mathcal{N}$ and $(v, \mathcal{P}) \in \mathbb{VP}(N)$, the components $P, Q \in \mathcal{P}$ are called symmetric in

11. Recently, Hu [2021, Theorem 3.2] kind of rediscovered this characterization. Instead of marginality, he uses coalitional strategic equivalence [Chun, 1989]. Nowadays, however, it is well understood that coalitional strategic equivalence is equivalent to marginality [see, for example, Casajus, 2011, Footnote 3]. Coalitional strategic equivalence : For all $N \in \mathcal{N}$, $T \subseteq N$, $T \neq \emptyset$, $i \in N \setminus T$, $\xi \in \mathbb{R}$, and $v \in \mathbb{V}(N)$, we have $\varphi_i(v) = \varphi_i(v + \xi \cdot u_T^N)$.

(v, \mathcal{P}) , if

$$v \left(P \cup \bigcup_{C \in \mathcal{C}} C \right) - v \left(\bigcup_{C \in \mathcal{C}} C \right) = v \left(Q \cup \bigcup_{C \in \mathcal{C}} C \right) - v \left(\bigcup_{C \in \mathcal{C}} C \right)$$

for all $\mathcal{C} \subseteq \mathcal{P} \setminus \{P, Q\}$, that is, if and only if the representatives of P and Q are symmetric in the intermediate game $v_{\mathcal{P}}$.

Symmetry across components, SaC. For all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$, and $P, Q \in \mathcal{P}$ such that P and Q are symmetric in (v, \mathcal{P}) , we have $\varphi_P(v, \mathcal{P}) = \varphi_Q(v, \mathcal{P})$.

Efficiency and marginality are just the CS versions of the properties for (TU) solutions with the same name and with the same interpretation. Symmetry within components is a natural relaxation of symmetry within the CS framework. Symmetry across components treats the components as players : equally productive components should obtain the same payoff as expressed by the sum of their members' payoffs. Moreover, both symmetry within components and symmetry across component can be viewed as generalizations of symmetry. Whereas the former is equivalent to symmetry for the trivial coalition structure $\{N\}$, the latter is so for the atomistic coalition structure $\{\{i\} \mid i \in N\}$.

Theorem 3.4.1 (Khmelnitskaya and Yanovskaya, 2007). *The Shapley-Owen value is the unique CS solution for $\mathbb{V}\mathbb{P}$ that satisfies efficiency (E) symmetry within components (SwC), symmetry across components (SaC), and marginality (M).*

3.4.2. A second-order characterization

In this subsection, we simultaneously translate the second-order characterization of the Shapley value to CS solutions and the (first-order) characterization of the Shapley-Owen value to the second-order framework.

Second-order marginality, 2M. For all $N \in \mathcal{N}$, $(v, \mathcal{P}), (w, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$, and $i, j \in N$, $i \neq j$ such that

$$\begin{aligned} & [v(T \cup \{i, j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)] \\ & = [w(T \cup \{i, j\}) - w(T \cup \{i\})] - [w(T \cup \{j\}) - w(T)] \end{aligned}$$

for all $T \subseteq N \setminus \{i, j\}$, we have

$$\varphi_j(v, \mathcal{P}) - \varphi_j(v_{-i}, \mathcal{P}_{-i}) = \varphi_j(w, \mathcal{P}) - \varphi_j(w_{-i}, \mathcal{P}_{-i}).$$

In essence, this property is just a restatement of second-order marginality for TU games, where the coalition structure is fixed but can be ignored otherwise. Therefore, the proof of Proposition 3.3.1 runs through smoothly within the framework of CS games and we obtain

Proposition 3.4.1

If a solution for $\forall\mathcal{P}$ satisfies second-order marginality (**2M**), then it satisfies marginality (**M**).

Second-order symmetry within components, 2SwC. For all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \forall\mathcal{P}(N)$, $P \in \mathcal{P}$, $i, j \in P$, and $k \in N \setminus P$ such that i and j are second-order symmetric with respect to k in v , we have

$$\varphi_k(v, \mathcal{P}) - \varphi_k(v_{-i}, \mathcal{P}_{-i}) = \varphi_k(v, \mathcal{P}) - \varphi_k(v_{-j}, \mathcal{P}_{-j}).$$

This property restricts second-order symmetry for TU games to players within the same component. Yet, the coalition structure can be ignored regarding the third player to whom the second-order marginal contributions and the second-order payoffs are related. The proof of Proposition 3.3.2 essentially runs through smoothly with second-order symmetry within components instead of symmetry within components and symmetry within components instead of symmetry : one just has to put player h into the component containing players i and j . Hence, we obtain

Proposition 3.4.2

If a solution for \forall satisfies strong second-order symmetry within components (**2SwC**) and efficiency (**E**), then it satisfies symmetry within components (**SwC**).

In order to obtain a second-order version of symmetry across components, we first provide the notion of second-order symmetry of components. For all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \forall\mathcal{P}(N)$, and $A, B, C \in \mathcal{P}$ pairwise different, components A and B are called **second-order symmetric with**

respect to component C in (v, \mathcal{P}) if

$$\begin{aligned} & \left[v \left(C \cup A \cup \bigcup_{D \in \mathcal{D}} D \right) - v \left(A \cup \bigcup_{D \in \mathcal{D}} D \right) \right] - \left[v \left(C \cup \bigcup_{D \in \mathcal{D}} D \right) - v \left(\bigcup_{D \in \mathcal{D}} D \right) \right] \\ &= \left[v \left(C \cup B \cup \bigcup_{D \in \mathcal{D}} D \right) - v \left(B \cup \bigcup_{D \in \mathcal{D}} D \right) \right] - \left[v \left(C \cup \bigcup_{D \in \mathcal{D}} D \right) - v \left(\bigcup_{D \in \mathcal{D}} D \right) \right] \end{aligned}$$

for all $\mathcal{D} \subseteq \mathcal{P} \setminus \{A, B, C\}$.

Remark 3.4.1. Note that the components A and B are second-order symmetric with respect to component C in (v, \mathcal{P}) if and only if their representatives ι_A and ι_B are second-order symmetric with respect to the representative ι_C of component C in the intermediate game $v_{\mathcal{P}}$.

Second-order symmetric components are equally second-order productive with respect to a third component. Therefore, if a CS solution is intended to reflect the components' second-order productivities in terms of their second-order payoffs, it seems to be plausible that the second-order payoffs of second-order symmetric components are the same.

Second-order symmetry across components, 2SaC. For all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$, and $A, B, C \in \mathcal{P}$ pairwise different such that A and B are second-order symmetric with respect to C in v , we have

$$\varphi_C(v, \mathcal{P}) - \varphi_C(v_{-A}, \mathcal{P}_{-A}) = \varphi_C(v, \mathcal{P}) - \varphi_C(v_{-B}, \mathcal{P}_{-B}).$$

Using the general idea of the proof of Proposition 3.3.2 one shows that second-order symmetry across components and efficiency imply symmetry across components.

Proposition 3.4.3

If a solution for $\mathbb{V}\mathbb{P}$ satisfies second-order symmetry across components (**2SaC**) and efficiency (**E**), then it satisfies symmetry across components (**SaC**).

Proof. Let the CS solution φ satisfy **2SaC** and **E**. If $|\mathcal{P}| = 1$, then nothing is to show. Let now $N \in \mathcal{N}$ and $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$ be such that $|\mathcal{P}| > 1$. Moreover, let $(*) P, Q \in \mathcal{P}$, $P \neq Q$ be symmetric in v . Fix $h \in \mathcal{U} \setminus N$, set $M := N \cup \{h\}$ and $\mathcal{Q} := \mathcal{P} \cup \{\{h\}\}$, and let $w \in \mathbb{V}(M)$

be given by

$$\begin{aligned}
w &= \sum_{T \subseteq N: T \neq \emptyset} \lambda_T(v) \cdot u_T^M \\
&+ \sum_{T \subseteq N \setminus (P \cup Q)} \sum_{S \subseteq P: S \neq \emptyset} \lambda_{T \cup S}(v) \cdot u_{T \cup \{h\}}^M \\
&+ \sum_{T \subseteq N \setminus (P \cup Q)} \sum_{S \subseteq P \cup Q: S \cap P \neq \emptyset, S \cap Q \neq \emptyset} \lambda_{T \cup \{i,j\}}(v) \cdot u_{T \cup (S \setminus Q) \cup \{h\}}^M \\
&+ \sum_{T \subseteq N \setminus (P \cup Q)} \sum_{S \subseteq P \cup Q: S \cap P \neq \emptyset, S \cap Q \neq \emptyset} \lambda_{T \cup \{i,j\}}(v) \cdot u_{T \cup (S \setminus P) \cup \{h\}}^M,
\end{aligned}$$

that is, $\{h\}$ is added to (v, \mathcal{P}) such that (**) P and Q remain symmetric in (w, \mathcal{Q}) , (***) $\{h\}$ is symmetric to both P and Q in w , and (****) $(w_{-\{h\}}, \mathcal{Q}_{-\{h\}}) = (v, \mathcal{P})$.

Since P and Q are symmetric in (w, \mathcal{Q}) , they are second-order symmetric with respect to any $R \in \mathcal{Q} \setminus \{P, Q\}$ in (w, \mathcal{Q}) . Hence, we have

$$\varphi_R(w_{-P}, \mathcal{Q}_{-P}) \stackrel{2\text{SaC}}{=} \varphi_R(w_{-Q}, \mathcal{Q}_{-Q}) \quad \text{for all } R \in \mathcal{Q} \setminus \{P, Q\}. \quad (\text{A12})$$

Now, we obtain

$$\begin{aligned}
\varphi_Q(w_{-P}, \mathcal{Q}_{-P}) &\stackrel{\mathbf{E}}{=} w_{-P}(M \setminus P) - \sum_{R \in \mathcal{Q} \setminus \{P, Q\}} \varphi_R(w_{-P}, \mathcal{Q}_{-P}) \\
&\stackrel{(**), (\text{A12})}{=} w_{-Q}(M \setminus Q) - \sum_{R \in \mathcal{Q} \setminus \{P, Q\}} \varphi_R(w_{-Q}) \\
&\stackrel{\mathbf{E}}{=} \varphi_P(w_{-Q}, \mathcal{Q}_{-Q}).
\end{aligned}$$

In view of (***), we analogously obtain

$$\varphi_Q(w_{-\{h\}}, \mathcal{Q}_{-\{h\}}) = \varphi_{\{h\}}(w_{-Q}, \mathcal{Q}_{-Q}) \quad (\text{A13})$$

and

$$\varphi_P(w_{-\{h\}}, \mathcal{Q}_{-\{h\}}) = \varphi_{\{h\}}(w_{-P}, \mathcal{Q}_{-P}). \quad (\text{A14})$$

Finally, we have

$$\begin{aligned}
\varphi_P(v, \mathcal{P}) &\stackrel{(\ast\ast\ast\ast)}{=} \varphi_P(w_{-h}, \mathcal{Q}_{-\{h\}}) \\
&\stackrel{(A14)}{=} \varphi_{\{h\}}(w_{-P}, \mathcal{Q}_{-P}) \\
&\stackrel{(A12)}{=} \varphi_{\{h\}}(w_{-Q}, \mathcal{Q}_{-Q}) \\
&\stackrel{(A13)}{=} \varphi_Q(w_{-\{h\}}, \mathcal{Q}_{-\{h\}}) \stackrel{(\ast\ast\ast\ast)}{=} \varphi_Q(v, \mathcal{P}),
\end{aligned}$$

which concludes the proof. \square

Propositions 3.4.1, 3.4.2, and 3.4.3, allow us to “transfer” Theorem 3.4.1 to the second-order framework. We obtain

Theorem 3.4.2. *The Shapley-Owen value is the unique CS solution for $\mathbb{V}\mathbb{P}$ that satisfies efficiency (E), second-order marginality (2M), second-order symmetry within components (2SwC), and second-order symmetry across components (2SaC).*

Proof. It is well-known that the Shapley-Owen value satisfies E. Straightforward but tedious calculations using (A5b) show the following formulas for the second-order Shapley-Owen value payoffs in terms of second-order marginal contributions. Let $N \in \mathcal{N}$, $i, k \in N$, $i \neq k$, and $(v, \mathcal{P}) \in \mathbb{V}(N)$. If $k \in \mathcal{P}(i)$, then

$$\begin{aligned}
&\text{Ow}_k(v, \mathcal{P}) - \text{Ow}_k(v_{-i}, \mathcal{P}_{-i}) \\
&= \sum_{\mathcal{C} \subseteq \mathcal{P} \setminus \{\mathcal{P}(k)\}} \sum_{S \subseteq \mathcal{P}(i) \setminus \{i, k\}} \left[\frac{v \left(S \cup \{i, k\} \cup \bigcup_{C \in \mathcal{C}} C \right) - v \left(S \cup \{i\} \cup \bigcup_{C \in \mathcal{C}} C \right)}{|\mathcal{P}(k)| \cdot \binom{|\mathcal{P}(k)|-1}{s+1} \cdot |\mathcal{P}| \cdot \binom{|\mathcal{P}|-1}{|C|}} \dots \right. \\
&\quad \left. - \frac{v \left(S \cup \{k\} \cup \bigcup_{C \in \mathcal{C}} C \right) - v \left(S \cup \bigcup_{C \in \mathcal{C}} C \right)}{|\mathcal{P}(k)| \cdot \binom{|\mathcal{P}(k)|-1}{s+1} \cdot |\mathcal{P}| \cdot \binom{|\mathcal{P}|-1}{|C|}} \right]. \tag{A15}
\end{aligned}$$

If $k \in N \setminus \mathcal{P}(i)$, then

$$\begin{aligned}
& \text{Ow}_k(v, \mathcal{P}) - \text{Ow}_k(v_{-i}, \mathcal{P}_{-i}) \\
&= \sum_{C \subseteq \mathcal{P} \setminus \{\mathcal{P}(k), \mathcal{P}(i)\}} \sum_{S \subseteq \mathcal{P}(k) \setminus \{k\}} \left[\frac{v \left(S \cup \{k\} \cup \mathcal{P}(i) \cup \bigcup_{C \in \mathcal{C}} C \right) - v \left(S \cup \mathcal{P}(i) \cup \bigcup_{C \in \mathcal{C}} C \right)}{|\mathcal{P}(k)| \cdot \binom{|\mathcal{P}(k)|-1}{s} \cdot |\mathcal{P}| \cdot \binom{|\mathcal{P}|-1}{|\mathcal{C}|+1}} \dots \right. \\
&\quad \left. - \frac{v \left(S \cup \{k\} \cup (\mathcal{P}(i) \setminus \{i\}) \cup \bigcup_{C \in \mathcal{C}} C \right) - v \left(S \cup (\mathcal{P}(i) \setminus \{i\}) \cup \bigcup_{C \in \mathcal{C}} C \right)}{|\mathcal{P}(k)| \cdot \binom{|\mathcal{P}(k)|-1}{s} \cdot |\mathcal{P}| \cdot \binom{|\mathcal{P}|-1}{|\mathcal{C}|+1}} \right].
\end{aligned} \tag{A16}$$

From (A15) and (A16) it is immediate that the Shapley-Owen value satisfies **2M** and **2SwC**. By Remark 3.4.1 and in view of the well-known fact that the Shapley-Owen values satisfies **IG**, it also satisfies **2SaC**.

Let the CS solution φ satisfy **E**, **2M**, **2SwC**, and **2SaC**. By Propositions 3.4.1, 3.4.2, and 3.4.3, the CS solution φ satisfies **M**, **SwC**, and **SaC**. By Theorem 3.4.1, we have $\varphi = \text{Ow}$. \square

Remark 3.4.2. *The characterization in Theorem 3.4.2 is non-redundant for $|N| > 1$. The zero CS solution, Z , given by $Z_i(v, \mathcal{P}) := 0$ for all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$, and $i \in N$ satisfies all properties but efficiency. The component egalitarian CS solution, CE , given by*

$$\text{CE}_i(v, \mathcal{P}) := \frac{v(N)}{|\mathcal{P}(i)| \cdot |\mathcal{P}|}$$

for all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{V}\mathbb{P}(N)$, and $i \in N$ satisfies all properties but marginality. Fix a bijection $\varrho : \mathcal{U} \rightarrow \mathbb{N}$. For any $N \in \mathcal{N}$ and $\mathcal{P} \in \mathfrak{P}(N)$, let

$$\begin{aligned}
& R(N, \mathcal{P}, \varrho) \\
&:= \{ \rho \in R(N, \mathcal{P}) \mid \text{for all } P \in \mathcal{P} \text{ and } i, j \in P : \rho(i) > \rho(j) \text{ if and only if } \varrho(i) > \varrho(j) \}.
\end{aligned}$$

The ϱ -Owen value, Ow^ϱ , given by

$$\text{Ow}_i^\varrho(v, \mathcal{P}) := \sum_{\rho \in R(N, \mathcal{P}, \varrho)} MC_i^v(\rho)$$

for all $N \in \mathcal{N}$, $(v, \mathcal{P}) \in \mathbb{VP}(N)$, and $i \in N$ satisfies all properties but second-order symmetry within components. The Shapley value for CS games ignoring the coalition structure satisfies all properties but second-order symmetry across components.

3.5. Concluding remarks

In this paper, we suggest a characterization of the Shapley-Owen value indicating that the latter is the unique efficient CS solution that reflects the players' and components' second-order productivities in terms of their second-order payoffs. The natural question now arises whether this may hold true for higher-order productivities and higher-order payoffs. In view of the results of Casajus [2020, Appendix A], the Shapley-Owen value should satisfy the corresponding higher-order properties, whereas not being the unique efficient CS solution to do so.

Winter [1989] generalizes the Shapley-Owen value to games enriched with a level structure, that is, a finite sequence of coalition structures becoming successively finer. Khmelnitskaya and Yanovskaya [2007, Theorem 2] indicate how their characterization can be extended to this level structure value. We leave it to the reader to provide the obvious extension of our characterization of the Shapley-Owen value to that level structure value.

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Uncertain production environment and communication structure : the Bayesian Myerson value

Résumé :

Pongou et Tondji dans leur article (Pongou et Tondji, *Games and Economic Behavior*, 108, 206–224, 2018), décrivent un environnement de production incertain comme étant toute situation où l'action d'un "input" est incertaine. Chaque "input" a un ensemble fini d'actions, et l'incertitude est formalisée comme une distribution de probabilité sur cet ensemble. Ces "inputs" peuvent être des travailleurs dans une entreprise, des sommets dans une économie de réseaux, des titres sur un marché financier, etc. Ensuite, ils examinent le problème de la répartition de la productivité entre les inputs dans cet environnement. En utilisant l'approche axiomatique, ils proposent une solution appelée "a priori Shapley value". Dans cet article/chapitre, nous examinons quelques applications de la valeur de Myerson (Myerson, *Mathematics of Operations Research*, 2, 225–229, 1977) dans le cadre d'un environnement de production incertain. En définissant quelques axiomes intuitifs, nous résolvons le problème de la répartition de la productivité entre les "inputs" dans un environnement de production incertain avec structure de communication. Selon la structure d'information, on obtient une nouvelle valeur que nous nommons "a priori Myerson value" et sa révision individuellement rationnelle nommée "a priori Myerson^R value"; ou "Bayesian Myerson value" et sa révision individuellement rationnelle nommée "Bayesian Myerson^R value". De plus, nous introduisons la notion de cœur dans ce nouvel environnement.

Mots clés : Environnement de production incertain, structure de communication, "a priori Myerson value", "Bayesian Myerson value", cœur

Abstract :

Pongou and Tondji (Pongou and Tondji, Games and Economic Behavior, 108, 206-224, 2018) described an uncertain production environment as a situation where input supply is uncertain. Each input has a finite set of actions, and uncertainty is formalized as a probability distribution over this set. These inputs can be workers in a firm, vertices in a networked economy, securities in a financial market, etc. Then they examined the problem of valuing inputs in that environment. By using axiomatic methods, they provided a solution called the *a priori* Shapley value. Knowing the output level, they provided a solution called the Bayesian Shapley value. In this paper/chapter, we examine some applications of the Myerson value (Myerson, Mathematics of Operations Research, 2, 225–229, 1977) in the field of uncertain production environment. By defining some intuitive axioms, we solve the problem of valuing inputs in Pongou and Tondji's (2018) environment enriched by a communication structure. Depending on the information structure, this leads to the *a priori* Myerson value and its individual rational revision called the *a priori* Myerson^R value, or the Bayesian Myerson value and its individual rational revision called the Bayesian Myerson^R value. Furthermore, we introduce the notion of the core in this environment.

Key words : Uncertain production environment, communication structure, *a priori* Myerson value, Bayesian Myerson value, core

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4.1. Introduction

AN uncertain production environment (Pongou and Tondji [2018]) for a finite set of inputs is given by a production function that assigns to each input profile a measurable output. An input supply is uncertain. Each input takes its values from a finite set of actions, and uncertainty is modeled as a probability distribution over this set. For instance, if an input is a worker then the set of actions could be the set of hours of labor or effort levels that a worker can supply. The set of actions for each input might represent a different set of objects. This set is not necessarily an ordered set. It can be the set of job types that are available in the firm, where the jobs are not necessarily ranked.

An uncertain production environment generalizes the transferable-utility environment also known as a cooperative game with transferable utility (TU game). A TU game is described as a situation in which inputs (called players in the framework of TU games) can obtain worth through cooperation. The payoff is distributed among the inputs by using one-point solutions.

The Shapley [1953] value is probably the best-known efficient one-point solution concept for TU games.

Pongou and Tondji [2018] consider the problem of valuing inputs in a production environment in which input supply is uncertain. They suggest an extension of the Shapley value as a solution to the valuation problem and called this solution “the a priori Shapley value”. Knowing the output level, they provided a new solution called “the Bayesian Shapley value”. An axiomatic characterization of these two solutions was proposed by using the Young [1985] approach.

An important problem observed in industrial organization is valuing inputs under supply in a production environment, which depends on the networked economy considered. In an uncertain production environment there are no constraints on communication possibilities among inputs. There are some situations where the communications between inputs are restricted and not all groups of inputs can cooperate. Let us consider the following valuation problem taken from Pongou and Tondji [2018] and adapted to our context.

Example 4.1.1 (Firm and suppliers). *An economy consists of a firm and three suppliers A, B and C who respectively supply some kind of products on their own x_A , x_B and x_C to the firm, with $x_A \in \{0, 4, 9\}$, $x_B \in \{0, 9, 16\}$ and $x_C \in \{0, 9\}$. Supplier A provides 0 units of commodity x_A with probability 0.25, 4 units of this commodity with probability 0.5, and 9 units with probability 0.25. Supplier B provides 0 units of commodity x_B with probability 0.3, 9 units of this commodity with probability 0.4, and 16 units with probability 0.3. Supplier C provides 0 units of commodity x_C with probability 0.25, and 9 units with probability 0.75. The firm’s technology is represented by a production function f . We also assume that the price of each output produced is equal to 1, so that for an input profile $x = (x_A, x_B, x_C)$, $f(x)$ also represents the revenue of the output generated by x . Now suppliers want to provide products that are more valuable. They form groups (strategic alliances) in order to benefit from synergies in production such that they can achieve their goal.*

They can organize themselves in many ways and their choice depends on which organization is best for them. For instance, the suppliers can organize themselves as shown in the following diagram :

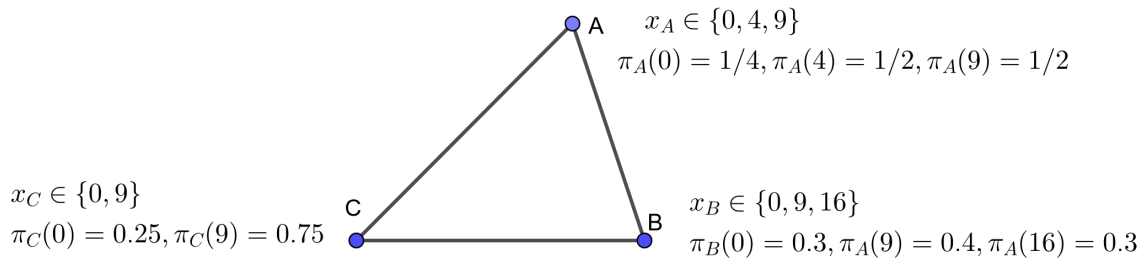


FIGURE 4.1 – Communication between suppliers

Suppose that due to geographical reasons, suppliers A and C form groups (strategic alliances) in order to benefit from synergies in production. We have the following graph :

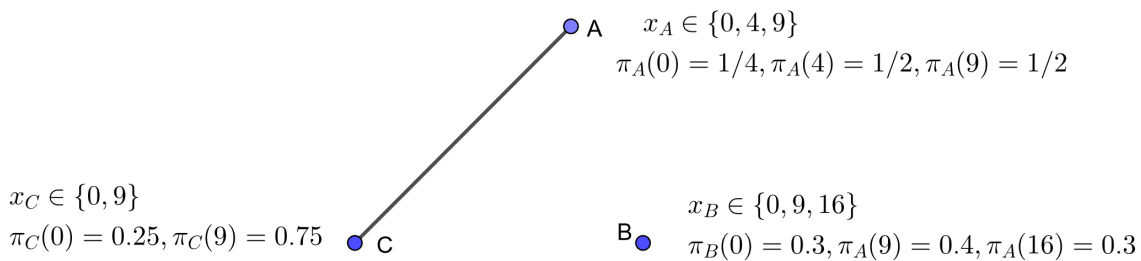


FIGURE 4.2 – Communication between suppliers

In this paper we want to answer the following valuation problems set by Pongou and Tondji [2018].

- a) What value can we attach to each input *a priori* ?
- b) Assume an output level is realized. What value can we attach to each input *a posteriori* ?

If the suppliers are organized as shown Figure 4.1, then our valuation problem can be solved by using the Pongou and Tondji [2018] approach. But if our suppliers are organized as shown Figure 4.2, then our valuation problem cannot be solved by using the Pongou and Tondji [2018] approach since there is a communication restriction among the suppliers.

Myerson [1977] introduced the communication structure which can be described by an undirected graph with vertices as inputs and links as communication relationships. Myerson defined a communication game as a combination of a TU game and a communication structure. The Myerson value for a communication situation is therefore the Shapley value of its communication game. Besides Myerson [1977], more axiomatizations can be found in the literature (Myerson [1980], Borm et al. [1992], Béal et al. [2015], etc).

The Myerson value is generally not individually rational, that is, there exists a situation where the Myerson value of an input (payoff) is less than its individual worth. Amer and Car-

Myerson [1995] revised the way to find the worth of a coalition in a communication game and then introduced an individual rational Myerson value for a communication game called the Myerson^R value. The Myerson^R value is an extension of the Myerson value. The two values coincide when the TU game is superadditive.

In TU games, we can observe that there are two participation levels : 0 and 1 (the set of actions for each input is 0 and 1). If we suppose that inputs have the possibility to cooperate with different participation levels, Aubin [1981] proposed the concept of a fuzzy coalition considering the uncertainty of inputs' participation levels and introduced fuzzy cooperative games. All the inputs have the same set of participation levels and each participation level belongs to the bounded set $[0, 1]$ (inputs participation levels are continuous variable). To calculate the worth of a fuzzy coalition (input profile of participation levels) from a game it is necessary to consider a specific partition by levels of this fuzzy set. Two specific partitions by levels of fuzzy coalitions studied in the literature are the proportional model (Butnariu [1980]) and the Choquet model (Tsurumi et al. [2001]). Genjiu et al. [2017] consider cooperative games on communication structure with fuzzy coalition. The Myerson value and its individual rational revision are defined, axiomatically characterized and discussed based on the Choquet integral form and proportional form respectively. The core of fuzzy cooperative games has been studied by Aubin [1981], Tijs et al. [2004], Yu and Zhang [2009], and Genjiu et al. [2017].

In this paper, we study the problem of valuing inputs in an uncertain production environment with a communication structure. By using the Myerson [1977] and Genjiu et al. [2017] approaches, we generalize the solution proposed by Pongou and Tondji [2018] (the *a priori* Shapley value). We uniquely characterize this solution and call it the *a priori* Myerson value. Its individual rational revision, called the *a priori* Myerson^R value, is also uniquely characterized. Two intuitive axioms are used for the *a priori* Myerson value (resp. the *a priori* Myerson^R value) : probabilistic component efficiency and probabilistic fairness (resp. optimal-probabilistic component efficiency and probabilistic fairness). Knowing the output level, we generalize the Bayesian Shapley value proposed by Pongou and Tondji [2018] and obtain a new solution called the Bayesian Myerson value. We uniquely characterize this solution by using two intuitive axioms : component efficiency *a posteriori* and fairness *a posteriori*. Its individual revision, called the Bayesian Myerson^R, is also characterized by using two axioms : optimal-component efficiency *a posteriori* and fairness *a posteriori*. The notions of imputation and core are introduced in this new model.

Many authors have studied the generalization of the Shapley value to games in which players have multiple options. Among them we have Freixas [2005], Courtin et al. [2016], and Hsiao and Raghavan [1993]. The Uncertain production environment introduced by Pongou and Tondji [2018] generalizes multiple options. We depart from Pongou and Tondji's paper [2018] by incorporating communication structure. By using the Myerson [1977] approach, we extend the Shapley value in that environment and propose an axiomatic characterization with two intuitive axioms. We also introduce the notion of core in this paper.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries on uncertain production environments and communication games. We then introduce a new model called the uncertain production environment with communication structure. Section 3 (resp. Section 4) studies the valuation problem *a priori* (resp. *a posteriori*) in a production environment with communication structure. We close this section with the study of the non-emptiness of the core. Section 5 is devoted to discussions where we show how to adapt our results in the field of fuzzy cooperative game theory and obtain a powerful result. Section 6 concludes, and all the proofs are presented in the Appendix.

4.2. Notations and preliminaries

4.2.1. Uncertain production environment

Pongou and Tondji [2018] introduced an **uncertain production environment** as a tuple $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ where $N = \{1, 2, \dots, n\}$ is a finite set of inputs of cardinality n , $T_i = \{0, 1, 2, \dots, t_i\}$ is a finite set of values an input i can take and f is a production function which maps an input profile $x = (x_1, \dots, x_n) \in T = \prod_{i=1}^n T_i$ to a real number output $f(x)$, and $\pi = (\pi_i)_{i \in N}$ is a profile of probability distributions of inputs. More formally, for each input i , we have :

$$\begin{aligned} \pi_i : T_i &\longrightarrow [0, 1] \\ x_k &\mapsto \pi_i(x_k) \quad \text{with } \sum_{x_k \in T_i} \pi_i(x_k) = 1. \end{aligned}$$

If all inputs are inactive, there will be no output, which means that $f(0, 0, \dots, 0) = 0$. An input level $x_k \in T_i$ may be the number of hours supplied by a worker in a firm, the unobserved ability of a worker, the level of effort that a student puts into a class project, etc. Note that the elements of each input set T_i need not be interpreted as numbers. For the rest, we suppose that the elements of $T_i = \{0, 1, 2, \dots, t_i\}$ can simply be viewed as labels (each input supply is a

discrete variable).

Pongou and Tondji [2018] assume that the input variables are independently distributed. The probability of the realization of any profile $x \in T$, denoted $\pi(x)$, is $\pi(x) = \prod_{i=1}^n \pi_i(x_i)$ where x_i is the input supply of i .

Pongou and Tondji [2018] provided a solution for the valuing input problem in an uncertain production environment. They first consider the following notation and definition.

Let x be an input profile. We denote by $|x| = |\{i \in N : x_i > 0\}|$ the number of inputs which are active at x . The binary relations \preceq and \triangleleft on the set of all the possible input profiles of an uncertain production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ is defined as follows : for any input profiles $a, x \in T$, $[a \preceq x]$ if and only if $[a_i \neq x_i \implies a_i = 0]$; and $[a \triangleleft x]$ if and only if $[a \preceq x \text{ and } a \neq x]$. Moreover, for $i \in N$ and $a, x \in T$, the notation $[a \triangleleft_0^i x]$ means that $[a \triangleleft x]$ and $a_i = 0$. For example, $(1, 0, 1, 2, 0, \dots, 10) \triangleleft (1, 1, 1, 2, 3, \dots, 10)$.

Let $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ be an uncertain environment production. Two input factors i and j are said to be **probabilistically symmetric** if : (1) $\pi_i = \pi_j$; and (2) for any input profile x such that $x_i = 0$ or $x_j = 0$, $f(x) = f(\tau_{ij}(x))$, where $\tau_{ij}(x) = x - x_i e_i + x_j e_i - x_j e_j + x_i e_j$, with $e_k = (0, \dots, 0, 1, 0 \dots, 0)$, $k \in \{i, j\}$, being the n -component unit vector whose k^{th} component is one and all the other components are zero.

Pongou and Tondji [2018] defined a valuation procedure as a solution concept for solving valuation problems.

A **valuation procedure** is a function ϕ which maps any uncertain production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ to a vector $\phi(\mathcal{P})$, where $\phi_i(\mathcal{P})$ is a real number representing the value input i . We will simply denote $\phi_i(\mathcal{P})$ by $\phi_i(f)$.

Pongou and Tondji [2018] suggested an example of valuation procedure called **the a priori Shapley value**.

Definition 4.2.1

For any uncertain production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$, **the a priori Shapley value** of the input i , and denoted $ASV_i(f)$, is given as follows :

$$ASV_i(f) = \sum_{x \in T} \pi(x) \left\{ \sum_{a \triangleleft_0^i x} \frac{|a|!(|x| - |a| - 1)!}{|x|!} [f(a + x_i e_i) - f(a)] \right\}$$

Pongou and Tondji [2018] examine the applications of the best-known Shapley [1953]. They interpret the *a priori* Shapley value as an extension of the generalization of the Shapley value proposed by Aguiar et al. [2016] as presented below.

If a production environment \mathcal{P} is such that an input profile x is realized with certainty (i.e., $\pi(x) = 1$), then for each input i , the *a priori* Shapley value becomes **the Shapley wage function** (Aguiar et al. [2016]) denoted SWF and given as follows :

$$ASV_i(f) = SWF_i(f, x) = \sum_{a \prec_i^0 x} \frac{|a|!(|x| - |a| - 1)!}{|x|!} [f(a + x_i e_i) - f(a)].$$

The *a priori* Shapley value of an input i is simply the weighted average of its Shapley wage over the set of input profiles. The *a priori* Shapley value is an extension of the Shapley [1953] value . In fact, the transferable-utility environment (TU game) in which the classical Shapley value was first defined can be modeled as a tuple $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$, where $T_i = \{0, 1\}$ for each i , f is a characteristic function defined on T , and $\pi = (\pi_i)_{i \in N}$ with $\pi_i(0) = 0$ and $\pi_i(1) = 1$ for each input i (meaning that the grand coalition forms). More interpretations of the *a priori* Shapley value can be found in Pongou and Tondji [2018].

Pongou and Tondji [2018] generalize the characterization of the Shapley value due to Young [1985] as follows.

Theorem 4.2.1. *The a priori Shapley value is the unique valuation procedure that satisfies probabilistic efficiency, probabilistic symmetry and marginality.*

Following Pongou and Tondji's [2018] interpretation, the probabilistic symmetry axiom states that probabilistically interchangeable inputs should be identically valued *a priori*. The probabilistic efficiency axiom says that the expected output of the production environment should be entirely shared among the different inputs. The marginality axiom states that, if there is a change in the technology that increases the marginal productivity of an input, then the valuation procedure should value that input more under the new technology.

Following the framework of fuzzy cooperative games (Genjiu et al. [2017]), each **crisp coalition**² $S \subseteq N$ corresponds to an input profile e^S with $e_i^S = 1$ if $i \in S$ and otherwise $e_i^S = 0$. Thus, fuzzy cooperative games generalize the TU game³. The **associated crisp game**

2. Crisp coalition means coalition (group of players) in TU games.

3. We can observe that, in fuzzy cooperative game each active input of the crisp coalition supply for the highest

w of f is defined as $w(S) = f(e^S)$ for each $S \subseteq N$. For each $S \subseteq N$, the **crisp subgame** of w induced by S is denoted w^S and given by $w^S(R) := w(R)$, for all $R \subseteq S$. The **restriction of a crisp game** (Genjiu et al. [2017]) w to S , denoted by w_S , is defined by $w_S(R) := w(S \cap R)$ for all $R \subseteq N$. If $w(S \cup R) + w(S \cap R) \geq (>)w(S) + w(R)$ for all $S, R \subseteq N$, then the crisp game w is (strictly) convex. In particular, the crisp game w is (strictly) superadditive when $w(S \cap R) = 0$.

We shall consider the superadditive extension of w (Amer and Carreras [1995]), denoted by \bar{w} and defined as follows :

$$\bar{w}(S) = \max_{\mathbf{D} \in \mathcal{PI}(S)} \sum_{R \in \mathbf{D}} w(R), \text{ for any } S \subseteq N$$

where $\mathcal{PI}(S)$ represents the set of all partitions⁴ of S . $\bar{w} = w$ when w is superadditive.

We denote by $SH(w)$ the Shapley value for a crisp game w .

Following Genjiu et al. [2017], the **carrier** (or support) of x is defined by $supp(x) = \{i \in N : x_i > 0\}$. Let $x \in T$, denote $Q(x) = \{x_i | x_i > 0, i \in N\}$ and $q(x) = |Q(x)|$. We write the non-zero elements in $Q(x)$ with the increasing order as $h_1 < h_2 < \dots < h_{q(x)}$ and denote by $h_0 = 0$.

A key issue for a fuzzy cooperative game is how to determine its characteristic function by extensions of the classical case. An uncertain production environment is an extension of a TU game. We can construct different production functions following the different approaches proposed in the framework of fuzzy cooperative games. Butnariu [1980] introduced the following notion :

A production function f is said to have **proportional form** if for each profile $x \in T$,

$$f(x) := \sum_{k=1}^{q(x)} h_k w^{supp(x)}(x^{h_k}) = \sum_{k=1}^{q(x)} h_k w(x^{h_k})$$

where $x^{h_k} = \{i \in N : x_i = h_k\}$ for any $k = 1, 2, \dots, q(x)$.

Tsurumi et al. [2001] also introduced the following notion :

A production function f is said to have the **Choquet integral form** (Choquet [1953]) if for each profile $x \in T$,

$$f(x) := \sum_{k=1}^{q(x)} [h_k - h_{k-1}] w^{supp(x)}([x]_{h_k}) = \sum_{k=1}^{q(x)} [h_k - h_{k-1}] w([x]_{h_k})$$

where $[x]_{h_k} = \{i \in N : x_i \geq h_k\}$ for any $k = 1, 2, \dots, q(x)$.

participation level (one) rather than the uncertain production environment where each active player of the crisp coalition supply for the lowest participation level (one).

4. A partition of S is a sequence (S_1, S_2, \dots, S_m) such that $\bigcup_{k=1}^m S_k = S$ et $S_k \cap S_l = \emptyset$ with $k, l \in \{1, \dots, m\}$.

Following Pongou and Tondji [2018], we imagine a firm in which the manager observes the level of output Z , but cannot observe the inputs (or ability) of workers. The realized output has to be shared among the workers. What share of this output should the manager give each worker? In answering this question, Pongou and Tondji [2018] assume that the manager is Bayesian and that she has *prior* beliefs about the distribution of inputs (that is, she knows π). Pongou and Tondji [2018] used the following notation.

Notation 4.2.1. Let $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ be an uncertain production environment. We denote by $\mathcal{R}_f = \{Z \in \mathbb{R} : \exists x \in T, f(x) = Z\}$ the set of realizable outputs. For any $Z \in \mathcal{R}_f$, define by $Z_f = \{x \in T : f(x) = Z\}$ the set of input profiles that produce Z . For any $Z \in \mathcal{R}_f$, we denote also by $\mathcal{P}_Z = (N, (T_i)_{i \in N}, f, P(\cdot|Z))$ the posteriori production environment associated to Z , where $P(\cdot|Z)$ is the posterior probability distribution over input profiles and is defined as follows :

$$P(a|Z) = \begin{cases} \frac{\pi(a)}{\sum_{x \in Z_f} \pi(x)} & \text{if } a \in Z_f \\ 0 & \text{otherwise} \end{cases}$$

The *posterior* probability distributions, also called the Bayesian Shapley value by Pongou and Tondji [2018], is defined as follows :

Definition 4.2.2

The **Bayesian Shapley value** is the function BSV defined, for any uncertain production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ and any realizable output $Z \in \mathcal{R}_f$, by :

$$BSV_i(\mathcal{P}_Z) := \frac{\sum_{x \in Z_f} \pi(x) SW F_i(f, x)}{\sum_{x \in Z_f} \pi(x)}, \quad i \in N.$$

This expression can be simply written as follows :

$$BSV_i(\mathcal{P}_Z) = \sum_{x \in T} P(x|Z) SW F_i(f, x), \quad i \in N.$$

Pongou and Tondji [2018] define the notion of symmetry *a posteriori* and axiomatically characterize the Bayesian Shapley value by using the Young [1985] approach.

Definition 4.2.3

Let $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ be an uncertain environment production and $Z \in \mathcal{R}_f$ a realizable output. Two input factors i and j are said to be symmetric *a posteriori* if for any input profile $x \in Z_f$ such that $x_i = 0$ or $x_j = 0$, $f(x) = f(\tau_{ij}(x))$.

Theorem 4.2.2. *The Bayesian Shapley value is the unique valuation procedure that satisfies efficiency a posteriori, symmetry a posteriori and marginality a posteriori.*

Following the Pongou and Tondji [2018] interpretation, the axiom of symmetry *a posteriori* states that inputs that are interchangeable *a posteriori* should be identically valued. The axiom of efficiency *a posteriori* states that any realized output Z of an uncertain production environment should be entirely shared among the different inputs. The axiom of marginality *a posteriori* states that, if there is a change in the technology that increases the marginal productivity of an input in the *posterior* production environment of an uncertain production environment, then the valuation procedure should give a greater value to that input under the new technology.

Several applications of uncertain production environments to real-life situations can be found in Pongou and Tondji [2018].

We consider the following example :

Example 4.2.1 (Example 4.1.1 continued). *We suppose that the suppliers can organize themselves as shown by figure 4.1.*

We consider a crisp game w (crisp game of f) where $w(S)$ is the outcome earned by the working together of suppliers in S : $w(A) = w(B) = w(C) = 10$, $w(A, B) = 20$, $w(A, C) = 23$, $w(B, C) = 21$ and $w(A, B, C) = 33$.

TABLE 4.1 – Value of $f(x)$ and $SWF(f, x)$ when f is in Choquet integral form

x	$f(x)$	$SWF(f, x)$	x	$f(x)$	$SWF(f, x)$	x	$f(x)$	$SWF(f, x)$
$(0,0,0)$	0	$(0,0,0)$	$(0,0,9)$	90	$(0,0,90)$	$(0,9,0)$	90	$(0,90,0)$
$(0,9,9)$	189	$(0, \frac{189}{2}, \frac{189}{2})$	$(0,16,0)$	160	$(0,160,0)$	$(0,16,9)$	259	$(0, \frac{329}{2}, \frac{189}{2})$
$(4,0,0)$	40	$(40,0,0)$	$(4,0,9)$	142	$(\frac{92}{2}, 0, \frac{192}{2})$	$(4,9,0)$	130	$(\frac{80}{2}, \frac{180}{2}, 0)$
$(4,9,9)$	237	$(\frac{268}{6}, \frac{559}{6}, \frac{595}{6})$	$(4,16,0)$	200	$(\frac{80}{2}, \frac{320}{2}, 0)$	$(4,16,9)$	307	$(\frac{268}{6}, \frac{979}{6}, \frac{595}{6})$
$(9,0,0)$	90	$(90,0,0)$	$(9,0,9)$	207	$(\frac{207}{2}, 0, \frac{207}{2})$	$(9,9,0)$	180	$(\frac{180}{2}, \frac{180}{2}, 0)$
$(9,9,9)$	297	$(\frac{603}{6}, \frac{549}{6}, \frac{630}{6})$	$(9,16,0)$	150	$(\frac{80}{2}, \frac{220}{2}, 0)$	$(9,16,9)$	367	$(\frac{503}{6}, \frac{869}{6}, \frac{830}{6})$

Using the information in table 4.1, we can derive the a priori Shapley value of each supplier as follows : $ASV_A(f) = \frac{64728}{960}$, $ASV_B(f) = \frac{98943}{960}$ and $ASV_C(f) = \frac{93363}{960}$.

We suppose that the realizable output $Z = 90$. We have

$$\begin{aligned} BMV(\gamma_Z) &= \frac{\pi((9, 0, 0)) \cdot (90, 0, 0) + \pi((0, 0, 9)) \cdot (0, 0, 90) + \pi((0, 9, 0)) \cdot (0, 90, 0)}{\pi((9, 0, 0)) + \pi((0, 0, 9)) + \pi((0, 9, 0))} \\ &= \left(\frac{540}{19}, \frac{360}{19}, \frac{810}{19} \right). \end{aligned}$$

TABLE 4.2 – Value of $f(x)$ and $SWF(f, x)$ when f is on proportional form

x	$f(x)$	$SWF(f, x)$	x	$f(x)$	$SWF(f, x)$	x	$f(x)$	$SWF(f, x)$
$(0,0,0)$	0	$(0,0,0)$	$(0,0,9)$	90	$(0,0,90)$	$(0,9,0)$	90	$(0,90,0)$
$(0,9,9)$	189	$(0, \frac{189}{2}, \frac{189}{2})$	$(0,16,0)$	160	$(0,160,0)$	$(0,16,9)$	250	$(0, \frac{320}{2}, \frac{180}{2})$
$(4,0,0)$	40	$(40,0,0)$	$(4,0,9)$	130	$(\frac{80}{2}, 0, \frac{180}{2})$	$(4,9,0)$	130	$(\frac{80}{2}, \frac{180}{2}, 0)$
$(4,9,9)$	229	$(\frac{240}{6}, \frac{567}{6}, \frac{567}{6})$	$(4,16,0)$	200	$(\frac{80}{2}, \frac{320}{2}, 0)$	$(4,16,9)$	290	$(\frac{240}{6}, \frac{960}{6}, \frac{540}{6})$
$(9,0,0)$	90	$(90,0,0)$	$(9,0,9)$	207	$(\frac{207}{2}, 0, \frac{207}{2})$	$(9,9,0)$	180	$(\frac{180}{2}, \frac{180}{2}, 0)$
$(9,9,9)$	297	$(\frac{603}{6}, \frac{549}{6}, \frac{630}{6})$	$(9,16,0)$	250	$(\frac{180}{2}, \frac{320}{2}, 0)$	$(9,16,9)$	367	$(\frac{621}{6}, \frac{960}{6}, \frac{621}{6})$

Using the information in table 4.2, the a priori Shapley value of each supplier can be derived as follows : $ASV_A(f) = \frac{66828}{960}$, $ASV_B(f) = \frac{101988}{960}$ and $ASV_C(f) = \frac{87048}{960}$.

We suppose that the realizable output $Z = 90$. Then $BSV_A(\mathcal{P}_Z) = \frac{540}{19}$, $BSV_B(\mathcal{P}_Z) = \frac{360}{19}$ and $BSV_C(\mathcal{P}_Z) = \frac{810}{19}$.

Now, we suppose that f is any production function. For instance we suppose that f is given by : $f(x) = 10(2x_1^{1/2} + 2x_2^{1/2} + 3x_3^{1/2})^2$. After filling the values of $f(x)$ and $SWF(f, x)$ for any input x in a new table, by using the approach proposed in Table 4.1 or Table 4.2, we derive the a priori Shapley value of each supplier and obtain the following result : $ASV_A(f) = \frac{13400}{16}$, $ASV_B(f) = \frac{17040}{16}$ and $ASV_C(f) = \frac{24030}{16}$. If we suppose that the realizable output is $Z = 2250$ then $f(0, 9, 9) = f(9, 0, 9) = 2250$. Hence, $BSV_A(\mathcal{P}_Z) = 540$, $BSV_B(\mathcal{P}_Z) = 360$ and $BSV_C(\mathcal{P}_Z) = 1350$.

We can observe that the suppliers do not supply inputs with the same probability and the effect of each input on the production of the output is different. Each supplier is therefore expected to have a different effect on the output of the firm.

4.2.2. Uncertain production environment with a communication structure

If we allow the suppliers (workers, etc) to organize themselves then a communication structure (Myerson [1977]) will obtain. A **communication structure** is an undirected graph (N, L) where $L \subseteq \{ij : i \neq j; i, j \in N\}$ represents a set of possible bilateral relations or a collection of possible communication links. The graph (N, L) is **complete** if there is total cooperation among all the inputs of N , i.e, for any $i, j \in N, i \neq j, ij \in L$. A **subgraph** of (N, L) is another graph $(S, L|_S)$ satisfying that $S \subseteq N$ and $L|_S = \{ij \in L : i, j \in S\}$. A **path** of graph (N, L) is a sequence of inputs $(i_1, i_2, \dots, i_k), k \in N, k > 1$ from N such that $i_\ell i_{\ell+1} \in L$ for all $\ell \in \{1, \dots, k-1\}$. A **cycle** in graph (N, L) is a path (i_1, i_2, \dots, i_k) where $k \geq 4$ and $i_k = i_1$. A graph is called **cycle-complete** if its restriction on the vertices of any cycle is a complete graph. Inputs $i, j \in N$ are called connected in (N, L) if there is a path (i_1, i_2, \dots, i_k) such that $i_1 = i$ and $i_k = j$. Following Béal et al. [2015], connectedness induces a partition $\mathcal{C}(N, L)$ of N , the set of **components** of (N, L) , such that $H \in \mathcal{C}(N, L), i, j \in H, k \in N \setminus H, i \neq j$ implies that i and j are connected and that i and k are not connected in (N, L) .

Myerson [1977] defined a solution concept for a TU game with communication structure as the Shapley value of the modified crisp game denoted by w/L and given by :

$$(w/L)(S) = \sum_{R \in \mathcal{C}(S, L|_S)} w(R) \quad \text{for any } S \subseteq N.$$

Following Amer and Carreras [1995], individual rationality cannot be ensured for the Myerson value. Amer and Carreras [1995] introduced a new modified crisp game \bar{w}/L and Genjiu et al. [2017] called it **the superadditive cover of crisp game** w/L . The crisp game \bar{w}/L is given by :

$$(\bar{w}/L)(S) = \max_{\mathbf{D} \in \mathcal{PT}(S, L)} \sum_{R \in \mathbf{D}} w(R), \quad \text{for any } S \subseteq N$$

where $\mathcal{PT}(S, L)$ represents the set of all partitions of S into connected subsets in graph (N, L) . Following Theorem 2.1 in Amer and Carreras [1995] $\bar{w}/L = w/L$ when w is superadditive. The superadditive cover allows inputs (workers) in a crisp coalition to organize themselves and establish an optimal partition.

Amer and Carreras [1995] defined a new solution concept called the Myerson^R value as the Shapley value of the new modified crisp game.

An **uncertain production environment with a communication structure** is a tuple $\gamma = (N, (T_i)_{i \in N}, f, \pi, L)$ where $(N, (T_i)_{i \in N}, f, \pi, L)$ is an uncertain production environment and $L \subseteq \{ij : i \neq j; i, j \in N\}$. We denote by UPC_N the set of uncertain production environments

with a communication structure over N .

The production environment with a communication structure fits many real-life situations as shown example 4.1.1. Most of the examples proposed in fuzzy cooperative game theory with communication structure can be adapted in this context by supposing that each input i is a discrete variable taking a value from a finite set T_i and uncertainty is formalized as a probability distribution over this set.

For instance, let us consider the following example taken from Xianghui et al. [2017] and adapted to our context.

Example 4.2.2. Consider a river with a number of tributaries, along which four villages A, B, C and D are distributed. These villages now decide to develop the river for irrigation and tourism in order to obtain a certain economic benefit. Each village contributes both human and material resources as part of the cooperation. In the figure below (Figure 4.3), we have the participation level x_i of each village i and the probability distribution over the set of participation levels of each village i . Let the villages be represented by the nodes of a graph and rivers between them by links.

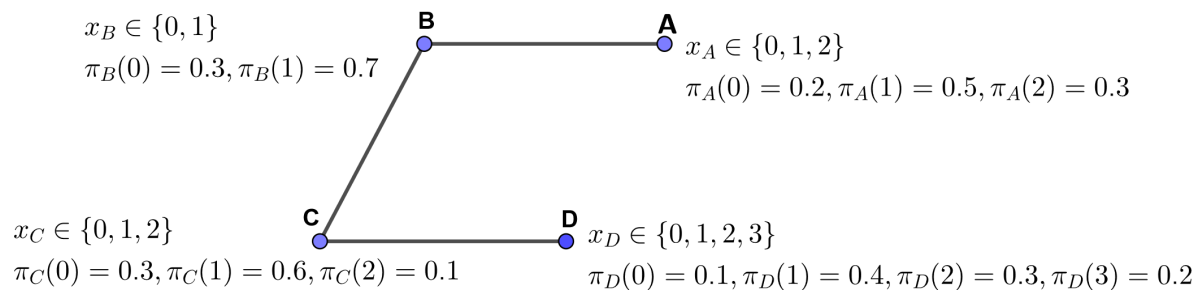


FIGURE 4.3 – Villages and river with a number of tributaries

The company's (in charge of this project) technology is represented by a Choquet integral production function f . The value $w(S)$ for any $S \subseteq \{A, B, C, D\}$ being the worth gained by common cooperation of villages in S : $w(\{A\}) = w(\{B\}) = w(\{C\}) = w(\{D\}) = 20$, $w(\{B, C\}) = 60$, $w(\{A, B\}) = 100$, $w(\{C, D\}) = 50$, $w(\{A, B, C\}) = 100$, $w(\{B, C, D\}) = 120$ and $w(\{A, B, C, D\}) = 200$.

- What is the expected payoff of each village ?
- If the output level is known, how do we find the payoff of each village ?

This problem can be modeled by an uncertain environment production with communication structure (N, T, π, f, L) where N is the set of villages, T is a profile of the set of participation levels and L is a communication structure.

In the next two sections we present some alternative solutions to our valuation problems, where we suppose that there exists a communication structure among the inputs.

4.3. Valuing a priori inputs in a production environment with communication structure

One solution can be proposed to the first question of our valuation problems : the *a priori* Myerson value. This solution is described below and its individual rational revision, called the *a priori* Myerson^R value, is also given. Both of them are uniquely characterized by using intuitive axioms. This section ends with the study of the non-emptiness of the core.

4.3.1. The *a priori* Myerson value

Let's consider a production environment $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$. We introduce a new production environment $\mathcal{P}^\gamma = (N, (T_i)_{i \in N}, f^\gamma, \pi)$ where f^γ is a **graph production function** such that for all $x \in T$, $f^\gamma(x)$ reflects the worth obtained by the input profile x with the production function f on the production environment γ . The ***a priori* Myerson value** for γ , denoted by $AMV(\gamma)$, is defined as the priori Shapley value of uncertain production environments \mathcal{P}^γ , i.e.,

$$AMV(\gamma) = ASV(\mathcal{P}^\gamma)$$

We now introduce two properties of valuation procedure over UPC_N : Probabilistic component efficiency and Probabilistic fairness.

P1. Probabilistic component efficiency : for all $\gamma \in \text{UPC}_N$ and $S \in \mathcal{C}(N, L)$,

$$\sum_{i \in S} \phi_i(\gamma) = \sum_{x \in T} \pi(x) f^\gamma \left(\sum_{i \in S} x_i e_i \right).$$

P2. Probabilistic fairness : for all $\gamma \in \text{UPC}_N$ and $ij \in L$, $\phi_i(\gamma) - \phi_i(\gamma_{-ij}) = \phi_j(\gamma) - \phi_j(\gamma_{-ij})$ where $\gamma_{-ij} = (N, (T_i)_{i \in N}, f, \pi, L \setminus ij)$.

Probabilistic component efficiency allows the inputs to organize themselves in components and the expected output produced by the members of a given component should be entirely shared among them. Probabilistic fairness requires that removing a link from the graph changes the values of the inputs forming this link by the same amount.

We will use these two properties for the axiomatization of the *a priori* Myerson value by using two approaches.

First approach : We use the Genjiu et al. [2017] approach, i.e., we suppose that the production

function has a particular form and we create a link with the Myerson [1977] result.

Let the production function f be based on the Choquet integral, i.e.,

$$f(x) = \sum_{k=1}^{q(x)} [h_k - h_{k-1}] w^{supp(x)}([x]_{h_k}) \quad \text{for each } x \in T.$$

Following Pongou and Tondji [2018], ASV satisfies the additivity property. ASV also satisfies the homogeneity property (i.e. for all $\alpha \in \mathbb{R}$, $ASV(\alpha f) = \alpha ASV(f)$). So ASV satisfies the linearity property and it follows that :

$$ASV(f) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH((w^{supp(x)})_{[x]_{h_k} \cap supp(x)}) \right\}.$$

For each profile $x \in T$, we define a graph production function in Choquet integral form by

$$f^\gamma(x) = \sum_{k=1}^{q(x)} [h_k - h_{k-1}] (w^{supp(x)} / L|_{supp(x)})([x]_{h_k})$$

By the definition of AMV we have,

$$AMV(\gamma) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH((w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) \right\}$$

where $(w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}$ is the restriction of $w^{supp(x)} / L|_{supp(x)}$ to $[x]_{h_k} \cap supp(x)$.

Now, if the production function f is in proportional form, it follows that

$$AMV(\gamma) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} h_k SH((w^{supp(x)} / L|_{supp(x)})_{x^{h_k} \cap supp(x)}) \right\}.$$

We then provide an axiomatic characterization of AMV based in the Choquet integral form or proportional form.

Theorem 4.3.1. *The a priori Myerson value is the unique valuation procedure over UPC_N that satisfies probabilistic component efficiency and probabilistic fairness based on Choquet integral form or proportional form.*

Second approach : we directly extend the Myerson [1977] approach by defining a particular case of the graph production function called the **included graph production function**, denoted f_I^γ and given by :

$$f_I^\gamma(x) = \sum_{S \in \mathcal{C}(\text{Supp}(x), L|_{\text{supp}(x)})} f\left(\sum_{i \in S} x_i e_i\right), \text{ for all } x \in T.$$

If f is in Choquet integral form, then

$$f_I^\gamma(x) = \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \sum_{S \in \mathcal{C}(\text{Supp}(x), L|_{\text{supp}(x)})} w^{\text{supp}(x)}([x]_{h_k} \cap S) \quad (\text{A1})$$

Genjiu et al. [2017] (in Remark 1) interpret equation (A1) as follows : the graph production function implies that all players can cooperate in a component regardless of their participation levels. They also remark that the included graph production function is different from the graph production function used in the first approach (f^γ when it is in Choquet integral form). In fact, $f_I^\gamma \geq f^\gamma$ if the crisp game w is superadditive, and $f_I^\gamma = f^\gamma$ if γ is complete.

We then derive an axiomatic characterization of AMV when f^γ is the included graph production function.

Theorem 4.3.2. *The a priori Myerson value is the unique valuation procedure over UPC_N that satisfies probabilistic component efficiency and probabilistic fairness based on the included graph form.*

Let us apply this solution to the first question in our previous valuation problem.

Example 4.3.1 (Example 4.1.1 continued). *We suppose that the suppliers can organize themselves as shown by figure 4.2.*

We also suppose that the firm's technology is represented by a production function f where the crisp game $w \in TU(N)$ is such that $w(A) = w(B) = w(C) = 10$ and $w(A, C) = 25$.

TABLE 4.3 – Value of $f^\gamma(x)$ and $SWF(f^\gamma, x)$ when f is in Choquet integral form

x	$f^\gamma(x)$	$SWF(f^\gamma, x)$	x	$f^\gamma(x)$	$SWF(f^\gamma, x)$	x	$f^\gamma(x)$	$SWF(f^\gamma, x)$
$(0,0,0)$	0	$(0,0,0)$	$(0,0,9)$	90	$(0,0,90)$	$(0,9,0)$	90	$(0,90,0)$
$(0,9,9)$	180	$(0,90,90)$	$(0,16,0)$	160	$(0,160,0)$	$(0,16,9)$	250	$(0,160,90)$
$(4,0,0)$	40	$(40,0,0)$	$(4,0,9)$	150	$(50,0,100)$	$(4,9,0)$	130	$(40,90,0)$
$(4,9,9)$	240	$(50,90,100)$	$(4,16,0)$	200	$(40,160,0)$	$(4,16,9)$	310	$(50,160,100)$
$(9,0,0)$	90	$(90,0,0)$	$(9,0,9)$	225	$(\frac{225}{2}, 0, \frac{225}{2})$	$(9,9,0)$	180	$(90,90,0)$
$(9,9,9)$	315	$(\frac{675}{6}, 90, \frac{675}{6})$	$(9,16,0)$	250	$(90,160,0)$	$(9,16,9)$	385	$(\frac{675}{6}, 160, \frac{675}{6})$

Using the information in table 4.3, it follows that the a priori Myerson value of each input can be derived as follows : $AMV_A(\gamma) = \frac{74100}{960}$, $AMV_B(\gamma) = \frac{100800}{960}$ and $AMV_C(\gamma) = \frac{92700}{960}$.

TABLE 4.4 – Value of $f^\gamma(x)$ and $SWF(f^\gamma, x)$ when f is in proportional form

x	$f^\gamma(x)$	$SWF(f^\gamma, x)$	x	$f^\gamma(x)$	$SWF(f^\gamma, x)$	x	$f^\gamma(x)$	$SWF(f^\gamma, x)$
(0,0,0)	0	(0,0,0)	(0,0,9)	90	(0,0,90)	(0,9,0)	90	(0,90,0)
(0,9,9)	180	(0,90,90)	(0,16,0)	160	(0,160,0)	(0,16,9)	250	(0,160,90)
(4,0,0)	40	(40,0,0)	(4,0,9)	130	(40,0,90)	(4,9,0)	130	(40,90,0)
(4,9,9)	220	(40,90,90)	(4,16,0)	200	(40,160,0)	(4,16,9)	290	(40,160,90)
(9,0,0)	90	(90,0,0)	(9,0,9)	225	($\frac{225}{2}, 0, \frac{225}{2}$)	(9,9,0)	180	(90,90,0)
(9,9,9)	315	($\frac{675}{6}, 90, \frac{675}{6}$)	(9,16,0)	250	(90,160,0)	(9,16,9)	385	($\frac{675}{6}, 160, \frac{675}{6}$)

Using the information in table 4.4, we derive the a priori Myerson value of the following inputs : $AMV_A(\gamma) = \frac{70500}{960}$, $AMV_B(\gamma) = \frac{100800}{960}$ and $AMV_C(\gamma) = \frac{89100}{960}$.

Second approach based on included graph : now we suppose that f is any production function. For instance, we suppose that f is given by : $f(x) = 10(2x_1^{1/2} + 2x_2^{1/2} + 3x_3^{1/2})^2$. After filling the values of $f_I^\gamma(x)$ and $SWF(f_I^\gamma, x)$ for any input x in a new table, by using the approach proposed in Table 4.3 or Table 4.4, we derive the a priori Myerson value of each supplier and obtain the following result : $AMV_A(\gamma) = ASV_A(f_I^\gamma) = \frac{9560}{16}$, $AMV_B(\gamma) = ASV_B(f_I^\gamma) = \frac{6720}{16}$ and $AMV_C(\gamma) = ASV_C(f_I^\gamma) = \frac{17550}{16}$.

We can observe that in the presence of a communication structure the production output changes and each supplier is expected to have a different effect on the output of the firm.

4.3.2. The a priori Myerson^R value

Let $\gamma = (N, (T_i)_{i \in N}, f, \pi, L)$ be a production environment. The **modified production function** \bar{f} is a production function where the associated crisp game is \bar{w} . Also, let $\bar{\mathcal{P}}^\gamma = (N, (T_i)_{i \in N}, \bar{f}^\gamma, \pi)$ be another production environment where \bar{f}^γ is the **modified graph production function** such that for all $x \in T$, $\bar{f}^\gamma(x)$ reflects the worth obtained by the input profile x with the production function \bar{f} on the production environment γ . If the crisp game w is super-additive and f is in the Choquet integral form or proportional form, then $\bar{f}^\gamma = f^\gamma$. The a priori Myerson^R value, denoted by $AMV^R(\gamma)$, is defined as the a priori Shapley value for $\bar{\mathcal{P}}^\gamma$, i.e.,

$$AMV^R(\gamma) = ASV(\bar{\mathcal{P}}^\gamma).$$

We revise the probabilistic component efficiency axiom as follows :

P3. Optimal-probabilistic component efficiency : for all $\gamma \in \text{UPC}_N$ and $S \in \mathcal{C}(N, L)$,

$$\sum_{i \in S} \phi_i(\gamma) = \max_{\mathbf{D} \in \mathcal{PI}(S, L)} \sum_{R \in \mathbf{D}} \sum_{x \in T} \pi(x) f^\gamma \left(\sum_{i \in R} x_i e_i \right).$$

Our interpretation is not so different from that suggested by Amer and Carreras [1995]. Optimal-probabilistic component efficiency allows the inputs to organize themselves in components but also to establish an optimal partition within each of these. The optimal expected output produced by the members of a given component should be entirely shared among them.

We will use this property and the probabilistic fairness for the axiomatization of the a priori Myerson value by using two approaches.

First approach : We use the Genjiu et al. [2017] approach, i.e., we suppose that the production function has a particular form and create a link with the Amer and Carreras [1995] result.

We define the production function \bar{f}^γ in the Choquet integral form by

$$\bar{f}^\gamma(x) = \sum_{k=1}^{q(x)} [h_k - h_{k-1}] (\bar{w}^{supp(x)} / L|_{supp(x)}) ([x]_{h_k}), \text{ for each } x \in T.$$

If \bar{f}^γ is in the Choquet integral form, then it follows immediately that the a priori Myerson^R value can be written as follows :

$$AMV^{\mathbb{R}}(\gamma) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH \left((\bar{w}^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)} \right) \right\}.$$

Now, if the production function \bar{f}^γ in proportional form, it follows that

$$AMV^{\mathbb{R}}(\gamma) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} h_k SH \left((\bar{w}^{supp(x)} / L|_{supp(x)})_{x^{h_k} \cap supp(x)} \right) \right\}.$$

We propose an axiomatic characterization of the a priori Myerson^R value based on the Choquet integral form or proportional form.

Theorem 4.3.3. *The a priori Myerson^R value is the unique valuation procedure over UPC_N that satisfies optimal-probabilistic component efficiency and probabilistic fairness based on the Choquet integral form or proportional form.*

Second approach : we directly extend the Amer and Carreras [1995] approach by defining a

particular case of the graph production function called **superadditive cover of included graph production function**, denoted \bar{f}_I^γ and given by :

$$\bar{f}_I^\gamma(x) = \max_{\mathbf{D} \in \mathcal{PT}(Supp(x), L)} \sum_{R \in \mathbf{D}} f\left(\sum_{i \in R} x_i e_i\right), \text{ for all } x \in T.$$

We derive the following result :

Theorem 4.3.4. *The a priori Myerson^R value is the unique valuation procedure over UPC_N that satisfies probabilistic component efficiency and probabilistic fairness based on the included graph form.*

4.3.3. The a priori imputation and the a priori core

Let's consider a production environment $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$. A valuation procedure ϕ on γ is expected to be **individually rational** if $\phi_i(\gamma) \geq \sum_{x \in T} \pi(x) \cdot x_i \cdot f(e_i)$ for each $i \in N$. ϕ is expected to be **collectively rational** if $\sum_{i \in S} \phi_i(\gamma) \geq \sum_{x \in T} \pi(x) f\left(\sum_{i \in S} x_i e_i\right)$ for all $S \subseteq N$.

In the framework of fuzzy cooperative games, Tsurumi et al. [2001] proposed a definition of an imputation, and Yu and Zhang [2009] proposed a definition of core. In this litterature the input profile is realized with certainty. We revised these two definitions in the context of a production environment where the input supply is uncertain.

Definition 4.3.1

The **a priori imputation** of an uncertain production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$, denoted by $I(\mathcal{P})$ or simply $I(f)$, is defined as follows :

$$I(f) = \left\{ y \in \mathbb{R}_+^n \mid \sum_{i \in N} y_i = \sum_{x \in T} \pi(x) f(x); y_i = 0 \text{ if } x_i = 0 \forall x \in T; y_i \geq \sum_{x \in T} \pi(x) \cdot x_i \cdot f(e_i) \forall i \in N \right\}.$$

The a priori imputation of an uncertain production environment is the set of valuation procedures that are probabilistically efficient and expected to be individually rational.

Definition 4.3.2

The *a priori* core of the production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$, denoted by $C(\mathcal{P})$ or simply $C(f)$ is defined as follows :

$$C(f) = \left\{ y \in \mathbb{R}_+^n \mid \sum_{i \in N} y_i = \sum_{x \in T} \pi(x) f(x), \sum_{i \in S} y_i \geq \sum_{x \in T} \pi(x) f\left(\sum_{i \in S} x_i e_i\right) \text{ for all } S \subseteq N \right\}.$$

The *a priori* core of an uncertain production environment is the set of valuation procedures that are probabilistic efficient and expected to be collectively rational.

We show that the *a priori* Myerson^R value belongs to the *a priori* imputation set and we also study the non-emptiness of the *a priori* core.

Theorem 4.3.5. *For a given production environment $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ where f is in Choquet integral form or proportional form, it holds that*

- i) $AMV^R(\gamma) \in I(\bar{f}^\gamma)$;
- ii) $AMV(\gamma) \in C(f^\gamma)$, if the graph is cycle-complete and the associated crisp game of f is convex.

4.4. Valuing inputs *a posteriori* in a production environment with communication structure

Knowing the output level, we propose one solution for the second question of our valuation problems : the Bayesian Myerson value. This solution is described below and its individual rational revision, called the Bayesian Myerson^R value, is also given. Both of them are uniquely characterized by using intuitive axioms. This section ends with the study of the non-emptiness of the core.

4.4.1. The Bayesian Myerson value

When we suppose that there exists a communication structure among the inputs, the Bayesian Shapley value proposed by Pongou and Tondji [2018] can be reformulated as follows :

Definition 4.4.1

The Bayesian Myerson value is the function BMV defined, for any uncertain production environment with a communication structure $\gamma = (N, (T_i)_{i \in N}, f, \pi, L)$ and any realizable output $Z \in \mathcal{R}_{f^\gamma}$, by :

$$BMV_i(\gamma_Z) = BSV_i(\mathcal{P}_Z^\gamma), \forall i \in N,$$

where $\gamma_Z = (N, (T_i)_{i \in N}, f, P^\gamma(\cdot|Z), L)$, $\mathcal{P}_Z^\gamma = (N, (T_i)_{i \in N}, f^\gamma, P^\gamma(\cdot|Z))$ and

$$P^\gamma(h|Z) = \begin{cases} \frac{\pi(h)}{\sum_{x \in Z_{f^\gamma}} \pi(x)} & \text{if } h \in Z_{f^\gamma} \\ 0 & \text{otherwise} \end{cases}.$$

We provide an axiomatic characterization of the Bayesian Myerson value by using the following properties :

P4. Component efficiency a posteriori : for all $\gamma \in \mathbf{UPC}_N$, $S \in \mathcal{C}(N, L)$, and for any

$$Z \in \mathcal{R}_{f^\gamma}, \sum_{i \in S} \phi_i(\gamma_Z) = \sum_{x \in T} P^\gamma(x|Z) f^\gamma\left(\sum_{i \in S} x_i e_i\right).$$

P5. Fairness a posteriori : for all $\gamma \in \mathbf{UPC}_N$, $ij \in L$; and for any $Z \in \mathcal{R}_{f^\gamma}$ and $Z' \in \mathcal{R}_{f^{\gamma-ij}}$ such that $Z_{f^\gamma} = Z'_{f^{\gamma-ij}}$, $\phi_i(\gamma_Z) - \phi_i((\gamma-ij)_{Z'}) = \phi_j(\gamma_Z) - \phi_j((\gamma-ij)_{Z'})$.

Component efficiency a posteriori allows the inputs to organize themselves in components and for any realized output of an UPC, the part produced by the members of a given component should be entirely shared among these members. Fairness a posteriori requires that removing a link from the graph changes the values of the inputs forming this link by the same amount when the set of input profiles producing the realized output is not affected.

Theorem 4.4.1. *The Bayesian Myerson value is the unique valuation procedure over \mathbf{UPC}_N that satisfies the component efficiency a posteriori and fairness a posteriori based on Choquet integral form or on proportional form or on included graph form.*

Under some conditions, the average of the Bayesian Myerson value denoted, U is the a priori Shapley value as shown by the following proposition.

Proposition 4.4.1

For a given $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ and any $ij \in L$, if $\mathcal{R}_{f\gamma} = \mathcal{R}_{f\gamma-ij}$ and $Z_{f\gamma} = Z_{f\gamma-ij}$ for all $Z \in \mathcal{R}_{f\gamma}$ then based on Choquet integral form or on proportional form or on included graph form, we have :

$$U_i(\gamma) := \sum_{Z \in \mathcal{R}_{f\gamma}} P^\gamma(Z) BMV_i(\gamma_Z) = AMV_i(\gamma)$$

when $P^\gamma(Z) = \sum_{x \in Z_{f\gamma}} \pi(x)$.

Let us apply this solution to the second question of our previous valuation problem.

Example 4.4.1 (Example 4.1.1 continued). *We suppose that the suppliers can organize themselves as shown by figure 4.2.*

We also suppose that the firm's technology is represented by a production function f where the crisp game $w \in TU(N)$ is such that $w(A) = w(B) = w(C) = 10$ and $w(A, C) = 25$.

Using our first approach, we suppose that f is in Choquet integral form or proportional form and the realizable output $Z = 250$.

$$BMV(\gamma_Z) = \frac{\pi((9, 16, 0)) \cdot (90, 160, 0) + \pi((0, 16, 9)) \cdot (0, 160, 90)}{\pi((9, 16, 0)) + \pi((0, 16, 9))} = (36, 160, 54).$$

It follows that $BMV_A(\gamma_Z) = 36$, $BMV_B(\gamma_Z) = 160$ and $BMV_C(\gamma_Z) = 54$.

We suppose that f is given by : $f(x) = 10(2x_1^{1/2} + 2x_2^{1/2} + 3x_3^{1/2})^2$ and the realizable output is $Z = 360$. By using our second approach based on included graph form, we have $f_I^\gamma(9, 0, 0) = f_I^\gamma(0, 9, 0) = 360$. After computing the Bayesian Myerson value, we have $BMV_A(\gamma_Z) = 216$, $BMV_B(\gamma_Z) = 144$ and $BMV_C(\gamma_Z) = 0$.

4.4.2. The Bayesian Myerson^R value

Knowing the output level, the individual rational revision of the Bayesian Myerson value, called the Bayesian Myerson^R, can be defined as follows :

Definition 4.4.2

The Bayesian Myerson^R value is the function BMV^R defined, for any uncertain production environment with a communication structure $\gamma = (N, (T_i)_{i \in N}, f, \pi, L)$ and any realizable output $Z \in \mathcal{R}_{\bar{f}^\gamma}$, by :

$$BMV_i^R(\tilde{\gamma}_Z) = BSV_i(\bar{P}_Z^\gamma), \forall i \in N,$$

where $\tilde{\gamma}_Z = (N, (T_i)_{i \in N}, f, \bar{P}^\gamma(\cdot|Z), L)$, $\bar{P}_Z^\gamma = (N, (T_i)_{i \in N}, \bar{f}^\gamma, \bar{P}^\gamma(\cdot|Z))$ and

$$\bar{P}^\gamma(h|Z) = \begin{cases} \frac{\pi(h)}{\sum_{x \in Z_{\bar{f}^\gamma}} \pi(x)} & \text{if } h \in Z_{\bar{f}^\gamma} \\ 0 & \text{otherwise} \end{cases}$$

We revise the component efficiency a *posteriori* axiom as follows :

P6. Optimal-component efficiency a *posteriori* : for all $\gamma \in \text{UPC}_N$, $S \in \mathcal{C}(N, L)$, and for

$$\text{any } Z \in \mathcal{R}_{\bar{f}^\gamma}, \sum_{i \in S} \phi_i(\tilde{\gamma}_Z) = \max_{D \in \text{PI}(S, L)} \sum_{R \in D} \sum_{x \in T} \bar{P}^\gamma(x|Z) \bar{f}^\gamma\left(\sum_{i \in R} x_i e_i\right).$$

Inputs of a given component have the possibility to establish an optimal partition within this component. Their optimal realized output should be shared among them.

The theorem below is an axiomatic characterization of the Bayesian Myerson^R value.

Theorem 4.4.2. *The Bayesian Myerson^R value is the unique valuation procedure over UPC_N that satisfies the optimal-component efficiency a *posteriori* and fairness a *posteriori* based on Choquet integral form or on proportional form or on included graph form.*

Under some conditions, the average of the Bayesian Myerson^R value denoted U is the a *priori* Shapley^R value as shown by the following proposition.

Proposition 4.4.2

For a given $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ and any $ij \in L$, if $\mathcal{R}_{\bar{f}^\gamma} = \mathcal{R}_{\bar{f}^{\gamma-ij}}$ and $Z_{\bar{f}^\gamma} = Z_{\bar{f}^{\gamma-ij}}$ for all $Z \in \mathcal{R}_{\bar{f}^\gamma}$ then based on Choquet integral form or on proportional form or on included graph form, we have :

$$U_i(\gamma) := \sum_{Z \in \mathcal{R}_{\bar{f}^\gamma}} \bar{P}^\gamma(Z) BMV_i(\tilde{\gamma}_Z) = AMV_i^{\mathbb{R}}(\gamma)$$

$$\text{when } \bar{P}^\gamma(Z) = \sum_{x \in Z_{\bar{f}^\gamma}} \pi(x).$$

4.4.3. The imputation a posteriori and the core a posteriori

Let us consider a production environment $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ and a realizable output $Z \in \mathcal{R}_f$. A valuation procedure ϕ on γ_Z is said to be **individually rational** if $\phi_i(\gamma_Z) \geq \sum_{x \in T} P(x|Z) \cdot x_i \cdot f(e_i)$ for each $i \in N$. ϕ is **collectively rational** if $\sum_{i \in S} \phi_i(\gamma_Z) \geq \sum_{x \in T} P(x|Z) \cdot f(\sum_{i \in S} x_i e_i)$ for all $S \subseteq N$.

Knowing the output level in an uncertain production environment, we revise the definition of a priori imputation and a priori core as follows :

Definition 4.4.3

The **imputation a posteriori** of an uncertain production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ with a realizable output $Z \in \mathcal{R}_f$, denoted by $I(\mathcal{P}_Z)$ or simply $I(f, Z)$, is defined as follows :

$$I(f, Z) = \left\{ y \in \mathbb{R}_+^n \mid \sum_{i \in N} y_i = Z; y_i = 0 \text{ if } x_i = 0 \forall x \in Z_f; y_i \geq \sum_{x \in T} P(x|Z) \cdot x_i \cdot f(e_i) \forall i \in N \right\}.$$

The imputation a posteriori of an uncertain production environment is the set of valuation procedures that are efficient a posteriori and individually rational.

Definition 4.4.4

The **core a posteriori** of a production environment $\mathcal{P} = (N, (T_i)_{i \in N}, f, \pi)$ with a realizable output $Z \in \mathcal{R}_f$, denoted by $C(\mathcal{P}_Z)$ or simply $C(f, Z)$, is defined as follows :

$$C(f, Z) = \left\{ y \in \mathbb{R}_+^n \mid \sum_{i \in N} y_i = Z, \sum_{i \in S} y_i \geq \sum_{x \in T} P(x|Z) \cdot f\left(\sum_{i \in S} x_i e_i\right) \text{ for all } S \subseteq N \right\}.$$

The core a posteriori of an uncertain production environment is the set of valuation procedures that are efficient a posteriori and collectively rational.

We show that the Bayesian Myerson^R value belongs to the imputation a posteriori set and we also study the non-emptiness of the core a posteriori.

Theorem 4.4.3. For a given production environment $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ with a realizable output $Z \in \mathcal{R}_{f^\gamma}$ where f is in Choquet integral form or proportional form, it holds that

- i) $BMV^{\mathbb{R}}(\tilde{\gamma}_Z) \in I(\bar{f}^\gamma, Z)$;
- ii) $BMV(\gamma_Z) \in C(f^\gamma, Z)$, if the graph is cycle-complete and the associated crisp game of f is convex.

4.5. Discussion : fuzzy cooperative game theory

Following the framework of fuzzy cooperative game theory, most of the authors assume that the fuzzy game⁵ is in proportional form or Choquet integral form or multilinear extension form, etc., and then derive the Shapley value in this environment. Genjiu et al. [2017] studied the Myerson value for the subclass of fuzzy cooperative games with communication structure where the fuzzy game is in proportional form or Choquet integral form. In this environment, we can define the Myerson value (resp. the Myerson^R) as the “generalized Shapley value”⁶ of the included graph game⁷ (resp. supperadditive cover of included graph game). By adapting the result of Theorem 4.3.2 (resp. Theorem 4.3.4) in this environment, we obtain an axiomatic

5. Fuzzy games are called production functions in this paper.

6. The generalized Shapley value is the simplified expression of the Shapley value proposed by Li and Zhang [2009] for fuzzy cooperative games. In this paper, it is known as the Shapley wage function.

7. See Definition 3 proposed by Genjiu et al. [2017]. The included graph game is called the included graph production function in this paper.

characterization of the Myerson value (resp. Myerson^R value) to the class of fuzzy cooperative games without assuming that the fuzzy game has a particular form (proportional form or Choquet integral form, etc.).

Theorem 4.5.1. ⁸

- i) *The Myerson value is the unique allocation rule for fuzzy cooperative games with communication structure that satisfies f-component efficiency and f-fairness.*
- ii) *The Myerson^R value is the unique allocation rule for fuzzy cooperative games with communication structure that satisfies optimal-probabilistic component efficiency and f-fairness.*

4.6. Conclusion

In this paper, we solve the problem of valuing inputs in the field of uncertain production environments with communication structure. By using the axiomatization approach, we show that the *a priori* Myerson value is a solution to our valuation problem. We also gave a suggestion for its individual rational revision, called the *a priori* Myerson^R value. When the output level is known, we defined and axiomatically characterized the Bayesian Myerson value as a solution of our valuation problem. Its individual rational revision called, the Bayesian Myerson^R value, was also suggested. We study the non-emptiness of the core based on the Choquet integral form or proportional form approach. It would also be good to study the non-emptiness of the core based on the included graph form. The Myerson value focuses on the inputs (vertices), while the position value defined by Meessen [1988] emphasizes the role of the links. In the present paper, we used the Myerson approach. Another important point is the study of our valuation problem by using the position value approach (Meessen [1988], Xianghui et al. [2017]). We leave these important remarks for further research.

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Appendix : Proofs

8. These three axioms are stated in Genjiu et al. [2017]. When an input profile is realized with certainty and $T_i = [0, 1]$ for all $i \in N$ (inputs participation levels are continuous variable), the probabilistic component efficiency, optimal f-component efficiency and probabilistic fairness properties become respectively the f-component efficiency, optimal f-component efficiency and f-fairness properties in fuzzy cooperative game theory.

Before proving the results of this paper, we need the following two propositions :

Proposition 4.6.1 (Prop. 3.1 in Genjiu et al. [2017])

Let (N, L) be a communication structure and w be a crisp game. For any $S \in \mathcal{C}(N, L)$, $R \subseteq N$, we have

$$\sum_{i \in S} SH_i((w/L)_R) = (w/L)(R \cap S).$$

where $(w/L)_R$ is the restriction of w/L to R .

Proposition 4.6.2 (Prop. 3.3 in Genjiu et al. [2017])

Let $\gamma \in \mathbf{UPC}_N$ be a production environment. For each $x \in T$ and $R \subseteq N$, the graph production function f^γ in Choquet integral form satisfies

$$f^\gamma\left(\sum_{i \in R} x_i e_i\right) = \sum_{k=1}^{q(x)} [h_k - h_{k-1}] (w/L)([x]_{h_k} \cap R).$$

These propositions are helpful for the following proof :

Proof of Theorem 4.3.1. \implies **Sufficiency** : the *a priori* Myerson value satisfies the probabilistic component efficiency and probabilistic fairness properties.

i) For all $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \mathbf{UPC}_N$ with f in Choquet integral form and $S \in$

$\mathcal{C}(N, L)$,

$$\begin{aligned}
\sum_{i \in S} AMV_i(\gamma) &= \sum_{i \in S} \left\{ \sum_{x \in T} \pi(x) SWF_i(f^\gamma, x) \right\} \\
&= \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \sum_{i \in S} SH_i((w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) \right\} \\
&= \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \sum_{i \in S \cap supp(x)} SH_i((w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) \right\} \\
&\stackrel{\text{Prop. 4.6.1}}{=} \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \cdot (w^{supp(x)} / L|_{supp(x)})_{([x]_{h_k} \cap S \cap supp(x))} \right\} \\
&= \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \cdot (w / L|_{supp(x)})_{([x]_{h_k} \cap S \cap supp(x))} \right\} \\
&\stackrel{\text{Prop. 4.6.2}}{=} \sum_{x \in T} \pi(x) f^\gamma \left(\sum_{i \in supp(x) \cap S} x_i e_i \right) \\
&= \sum_{x \in T} \pi(x) f^\gamma \left(\sum_{i \in S} x_i e_i \right)
\end{aligned}$$

ii) For all $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \mathbf{UPC}_N$ with f in choquet integral and $ij \in L$,

$$\begin{aligned}
AMV_i(\gamma) &= \sum_{x \in T} \pi(x) \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH_i((w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) \\
AMV_i(\gamma_{-ij}) &= \sum_{x \in T} \pi(x) \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH_i((w^{supp(x)} / (L|_{supp(x)} \setminus ij))_{[x]_{h_k} \cap supp(x)})
\end{aligned}$$

For any $x \in T$ and $k \in \{1, 2, \dots, q(x)\}$, if $\{i, j\} \subseteq supp(x)$ and $\{i, j\} \subseteq [x]_{h_k}$, it follows from the fairness of the Myerson value that

$$\begin{aligned}
SH_i((w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) - SH_i((w^{supp(x)} / (L|_{supp(x)} \setminus ij))_{[x]_{h_k} \cap supp(x)}) &= \\
SH_j((w^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) - SH_j((w^{supp(x)} / (L|_{supp(x)} \setminus ij))_{[x]_{h_k} \cap supp(x)}) &
\end{aligned}$$

Otherwise, we have $L|_{supp(x)} = L|_{supp(x)} \setminus ij$.

Hence, $AMV_i(\gamma) - AMV_i(\gamma_{-ij}) = AMV_j(\gamma) - AMV_j(\gamma_{-ij})$.

\Leftarrow) **Necessity** : We suppose that ϕ^1 and ϕ^2 are two valuation procedures satisfying the probabilistic component efficiency and the probabilistic fairness properties. Now let us show by induction on the number of links that $\phi^1 = \phi^2$.

If $L = \emptyset$, we can deduce from the probabilistic component efficiency that for any $i \in N$, $\phi_i^1(\gamma) = \phi_i^2(\gamma) = \sum_{x \in T} \pi(x) \cdot f^\gamma(x_i e_i) = \sum_{x \in T} \pi(x) \cdot x_i \cdot w(i)$.

If $L \neq \emptyset$, let γ be the minimal graph in links with $\phi^1(\gamma, x) \neq \phi^2(\gamma, x)$, which implies that

$\phi^1(\gamma') = \phi^2(\gamma)$ for all $\gamma' \neq \gamma$ with $|L'| < |L|$. Therefore, $\phi^1(\gamma_{-ij}) = \phi^2(\gamma_{-ij})$ for all $ij \in L$. Meanwhile, the probabilistic fairness property yields that

$$\phi_i^1(\gamma) - \phi_j^1(\gamma) = \phi_i^1(\gamma_{-ij}) - \phi_j^1(\gamma_{-ij}) = \phi_i^2(\gamma_{-ij}) - \phi_j^2(\gamma_{-ij}) = \phi_i^2(\gamma) - \phi_j^2(\gamma) \quad (\text{A1})$$

It follows that $\phi_i^1(\gamma) - \phi_i^2(\gamma) = \phi_j^1(\gamma) - \phi_j^2(\gamma)$ for any $ij \in L$.

Let us choose $S \in \mathcal{C}(N, L)$. If $S = \{i\}$ then by probabilistic component efficiency, we have $\phi_i^1(\gamma) = \sum_{x \in T} \pi(x) f^\gamma(x_i e_i) = \phi_i^2(\gamma)$.

If $|S| \geq 2$ then $\sum_{i \in S} \phi_i^1(\gamma) = \sum_{x \in T} \pi(x) f^\gamma(\sum_{i \in S} x_i e_i) = \sum_{i \in S} \phi_i^2(\gamma)$ by probabilistic component efficiency. It is clear that $\sum_{i \in S} [\phi_i^1(\gamma) - \phi_i^2(\gamma)] = \sum_{i \in S} \phi_i^1(\gamma) - \sum_{i \in S} \phi_i^2(\gamma) = 0$.

Since $\sum_{i \in S} [\phi_i^1(\gamma) - \phi_i^2(\gamma)] \stackrel{(\text{A1})}{=} |S| [\phi_i^1(\gamma) - \phi_i^2(\gamma)]$, we conclude that $\phi_i^1(\gamma) = \phi_i^2(\gamma)$. \square

Proof of Theorem 4.3.2. \implies **Sufficiency** : The a priori Myerson value defined from the included graph production function satisfies the probabilistic component efficiency and probabilistic fairness properties.

i) For all $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ and $R \in \mathcal{C}(N, L)$, we define a new production function f^R as follows :

$$f^R(x) = \sum_{S \in \mathcal{C}(\text{Supp}(x) \cap R, L|_{\text{Supp}(x) \cap R})} f(\sum_{i \in S} x_i e_i), \text{ for all } x \in T.$$

It is clear that

$$\mathcal{C}(\text{Supp}(x), L|_{\text{Supp}(x)}) = \bigcup_{R \in \mathcal{C}(N, L)} \mathcal{C}(\text{Supp}(x) \cap R, L|_{\text{Supp}(x) \cap R}).$$

Therefore $f_I^\gamma = \sum_{R \in \mathcal{C}(N, L)} f^R$. Moreover, for any $H \in \mathcal{C}(N, L)$,

$$\begin{aligned}
f_I^\gamma\left(\sum_{i \in H} x_i e_i\right) &= \sum_{R \in \mathcal{C}(N, L)} f^R\left(\sum_{i \in H} x_i e_i\right) \\
&= \sum_{R \in \mathcal{C}(N, L)} \sum_{S \in \mathcal{C}(\text{supp}(x) \cap R \cap H, L|_{\text{supp}(x) \cap R \cap H})} f\left(\sum_{i \in S} x_i e_i\right) \\
&= \sum_{S \in \mathcal{C}(\text{supp}(x) \cap H, L|_{\text{supp}(x) \cap H})} f\left(\sum_{i \in S} x_i e_i\right) \\
&= f^H\left(\sum_{i \in H} x_i e_i\right)
\end{aligned}$$

Following Aguiar et al. [2016], the Shapley wage function satisfies the additivity property. Hence, for any $x \in T$,

$$SWF(f_I^\gamma, x) = \sum_{R \in \mathcal{C}(N, L)} SWF(f^R, x).$$

For any $i \notin R$, $SWF_i(f^R, x) = 0$ since $f^R(a + x_i e_i) = f^R(a)$, for all $a \triangleleft_0^i x$.

For any $H \in \mathcal{C}(N, L)$, we have,

$$\begin{aligned}
\sum_{i \in H} AMV_i(\gamma) &= \sum_{i \in H} \sum_{x \in T} \pi(x) SWF_i(f_I^\gamma, x) \\
&= \sum_{x \in T} \pi(x) \sum_{R \in \mathcal{C}(N, L)} \sum_{i \in H} SWF(f^R, x) \\
&= \sum_{x \in T} \pi(x) \sum_{i \in H} SWF_i(f^H, x) = \sum_{x \in T} \pi(x) \sum_{i \in N} SWF_i(f^H, x) \\
&= \sum_{x \in T} \pi(x) f^H(x) \quad (\text{Probabilistic efficiency of ASV}) \\
&= \sum_{x \in T} \pi(x) f_I^\gamma\left(\sum_{i \in H} x_i e_i\right).
\end{aligned}$$

The *a priori* Myerson value defined from the included graph production function satisfies the probabilistic component efficiency.

ii) For all $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \mathbf{UPC}_N$ and $ij \in L$, we set $\tilde{f} = f^\gamma - f^{\gamma-ij}$.

Inputs i and j are probabilistically symmetric in the environment production $(N, (T_i)_{i \in N}, \tilde{f}, \pi)$.

Pongou and Tondji [2018] show that the *a priori* Shapley value satisfies the probabilistic symmetry, i.e., $ASV_i(\tilde{f}) = ASV_j(\tilde{f})$.

Since the a priori Shapley value satisfies the linearity property, it follows that

$$ASV_i(f^\gamma) - ASV_i(f^{\gamma-ij}) = ASV_j(f^\gamma) - ASV_j(f^{\gamma-ij}).$$

That is,

$$AMV_i(\gamma) - AMV_i(\gamma-ij) = AMV_j(\gamma) - AMV_j(\gamma-ij).$$

\Leftarrow) **Necessity** : The proof of uniqueness is similar to the proof of Theorem 4.3.1. We substitute f^γ by f_I^γ and we have the result. □

We consider the approach used by Genjiu et al. [2017] (Theorem 4.4) for the following proof :

Proof of Theorem 4.3.3. \implies) **Sufficiency** : The a priori Myerson^R value satisfies the optimal-probabilistic component efficiency and probabilistic fairness properties.

i) We prove AMV^R satisfies the optimal-probabilistic component efficiency.

For all $\gamma \in \text{UPC}_N$ and $S \in \mathcal{C}(N, L)$, by using the first part of the proof of Theorem 4.3.1, we have

$$\begin{aligned} \max_{\mathbf{D} \in \mathcal{PT}(S, L)} \sum_{R \in \mathbf{D}} \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \cdot (w^{\text{supp}(x)} / L|_{\text{supp}(x)})([x]_{h_k} \cap R \cap \text{supp}(x)) \right\} \\ \stackrel{\text{Prop. 4.6.2}}{=} \max_{\mathbf{D} \in \mathcal{PT}(S, L)} \sum_{R \in \mathbf{D}} \sum_{x \in T} \pi(x) f^\gamma \left(\sum_{i \in R} x_i e_i \right). \end{aligned}$$

Using the definition of AMV^R , we have

$$\begin{aligned} \sum_{i \in S} AMV_i^R(\gamma) &= \sum_{i \in S} ASV_i(\bar{\mathcal{P}}^\gamma) \\ &= \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \sum_{i \in S} SH_i((\bar{w}^{\text{supp}(x)} / L|_{\text{supp}(x)})_{[x]_{h_k} \cap \text{supp}(x)}) \right\} \\ &= \max_{\mathbf{D} \in \mathcal{PT}(S, L)} \sum_{R \in \mathbf{D}} \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] \cdot (w^{\text{supp}(x)} / L|_{\text{supp}(x)})([x]_{h_k} \cap R \cap \text{supp}(x)) \right\} \end{aligned}$$

Hence, AMV^R satisfies the optimal-probabilistic component efficiency and the last

equality is true for the following reason :

$$\begin{aligned}
& \sum_{i \in S} SH_i((\bar{w}^{supp(x)} / L|_{supp(x)})_{[x]_{h_k} \cap supp(x)}) \\
\stackrel{\text{Prop. 4.6.1}}{=} & (\bar{w}^{supp(x)} / L|_{supp(x)})_{([x]_{h_k} \cap S)} = \max_{\mathbf{D} \in \mathcal{PT}([x]_{h_k} \cap S, L|_{supp(x)})} \sum_{R \in \mathbf{Q}} w^{supp(x)}(R) \\
= & \max_{\mathbf{D} \in \mathcal{PT}([x]_{h_k} \cap S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
& \max_{\mathbf{D} \in \mathcal{PT}(S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{supp(x)} / L|_{supp(x)})_{([x]_{h_k} \cap R \cap supp(x))} = \\
& \max_{\mathbf{D} \in \mathcal{PT}(S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R \cap supp(x)).
\end{aligned}$$

We only need to prove that for every $k \in \{1, 2, \dots, q(s)\}$,

$$\begin{aligned}
& \max_{\mathbf{D} \in \mathcal{PT}(S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R \cap supp(x)) = \\
& \max_{\mathbf{D} \in \mathcal{PT}([x]_{h_k} \cap S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R).
\end{aligned}$$

(a) Let $\mathbf{D}_0 = \{R_1, R_2, \dots, R_d\}$ be the optimal partition of $(S, L|_{supp(x)})$ for

$$\max_{\mathbf{D} \in \mathcal{PT}(S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R \cap supp(x)),$$

i.e.,

$$\begin{aligned}
& \max_{\mathbf{D} \in \mathcal{PT}(S, L|_{supp(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R \cap supp(x)) \\
& = \sum_{R \in \mathbf{D}_0} (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R \cap supp(x)) \\
& = \sum_{i=1}^d (w^{[x]_{h_k} \cap supp(x)} / L|_{[x]_{h_k} \cap supp(x)})(R_i \cap supp(x)).
\end{aligned}$$

Since the set of components of $[x]_{h_k} \cap supp(x) \cap R_i$ ($i \in \{1, 2, \dots, d\}$) constitutes a

partition of subgraph $(S \cap [x]_{h_k} \cap \text{supp}(x), L|_{\text{supp}(x)})$, we obtain that

$$\begin{aligned} & \max_{\mathbf{D} \in \mathcal{PI}(S \cap [x]_{h_k} \cap \text{supp}(x), L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)) \\ & \geq \sum_{i=1}^d (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})([x]_{h_k} \cap \text{supp}(x) \cap R_i) \\ & = \max_{\mathbf{D} \in \mathcal{PI}(S, L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)). \end{aligned}$$

(b) Assume that $\mathbf{D}' = \{R'_1, R'_2, \dots, R'_a\}$ is the optimal partition of $(S \cap [x]_{h_k}, L|_{\text{supp}(x)})$ for

$$\max_{\mathbf{D} \in \mathcal{PI}(S \cap [x]_{h_k}, L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)).$$

Then there exists correspondingly a partition

$\mathbf{D} = \{R'_1, R'_2, \dots, R'_a, \dots, R'_b\} \in \mathcal{PI}(S, L|_{\text{supp}(x)})$. Thus,

$$\begin{aligned} & \max_{\mathbf{D} \in \mathcal{PI}(S \cap [x]_{h_k} \cap \text{supp}(x), L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)) \\ & = \sum_{i=1}^a (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R'_i \cap \text{supp}(x)) \\ & \leq \sum_{i=1}^b (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R'_i \cap \text{supp}(x)) \\ & \leq \max_{\mathbf{D} \in \mathcal{PI}(S, L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)). \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} & \max_{\mathbf{D} \in \mathcal{PI}(S, L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)) \\ & = \max_{\mathbf{D} \in \mathcal{PI}(S \cap [x]_{h_k} \cap \text{supp}(x), L|_{\text{supp}(x)})} \sum_{R \in \mathbf{D}} (w^{[x]_{h_k} \cap \text{supp}(x)} / L|_{[x]_{h_k} \cap \text{supp}(x)})(R \cap \text{supp}(x)). \end{aligned}$$

ii) Next we check that $AMV^{\mathbb{R}}$ satisfies probabilistic fairness. For any $ij \in L$, by the definition of $AMV^{\mathbb{R}}$ we have

$$AMV_i^{\mathbb{R}}(\gamma) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH_i((\bar{w}^{\text{supp}(x)} / L|_{\text{supp}(x)})|_{[x]_{h_k} \cap \text{supp}(x)}) \right\},$$

and

$$AMV_i^{\mathbb{R}}(\gamma_{-ij}) = \sum_{x \in T} \pi(x) \left\{ \sum_{k=1}^{q(x)} [h_k - h_{k-1}] SH_i((\bar{w}^{supp(x)} / (L \setminus \{ij\}))|_{supp(x)})_{[x]_{h_k} \cap supp(x)} \right\}.$$

By the fairness of the crisp Myerson^R value, we have

$$AMV_i^{\mathbb{R}}(\gamma) - AMV_i^{\mathbb{R}}(\gamma_{-ij}) = AMV_j^{\mathbb{R}}(\gamma) - AMV_j^{\mathbb{R}}(\gamma_{-ij}).$$

⇐) **Necessity** : The proof of uniqueness is similar to the proof of Theorem 4.3.1. □

Proof of Theorem 4.3.4. Both the uniqueness and the existence can be justified as in the proof of Theorem 4.3.2, providing only slight and obvious modifications, e.g. in defining the production function f^R . □

Proof of Theorem 4.3.5. Let $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \mathbf{UPC}_N$ be a production environment where f is in choquet integral form or proportional form. Let w be the crisp game of f .

i) Let $x \in T$ be an input profile. For each $i \in N$, $SWF_i(\bar{f}^\gamma, x) = 0$ when $x_i = 0$. From

Proposition 4.2 proposed by Genjui et al. [2017], when we substitute their crisp game v by the crisp subgame $w^{supp(x)}$, it follows that $SWF_i(\bar{f}^\gamma, x) \geq x_i \cdot w(\{i\})$ for each $i \in N$.

From Proposition 4.3 (Genjui et al. [2017]) it follows that $\sum_{i \in N} SWF_i(\bar{f}^\gamma, x) = \bar{f}^\gamma(x)$.

We deduce that : firstly, for each $i \in N$, $AMV_i^{\mathbb{R}}(\gamma) = 0$ if $x_i = 0$ for all $x \in T$.

Secondly, $AMV_i^{\mathbb{R}}(\gamma) \geq \sum_{x \in T} \pi(x) \cdot x_i \cdot \bar{f}^\gamma(e_i)$ for each $i \in N$.

Thirdly, $\sum_{i \in N} AMV_i^{\mathbb{R}}(\gamma) = \sum_{x \in T} \pi(x) \bar{f}^\gamma(x)$.

We conclude that $AMV^{\mathbb{R}}(\gamma) \in I(\bar{f}^\gamma)$.

ii) Let $x \in T$ be an input profile. From Theorem 5.1 proposed by Genjui et al. [2017],

when we substitute their crisp game v by the crisp subgame $w^{supp(x)}$, it follows that

$\sum_{i \in S} SWF_i(f^\gamma, x) \geq f^\gamma(\sum_{i \in S} x_i e_i)$ for all $S \subseteq N$. Hence, $\sum_{i \in S} AMV_i(\gamma) \geq \sum_{x \in T} \pi(x) f^\gamma(\sum_{i \in S} x_i e_i)$

for all $S \subseteq N$. Together with the first part of this proof (i), we conclude that $AMV(\gamma) \in$

$C(f^\gamma)$. □

Proof of Theorem 4.4.1. The proof follows the same reasoning as that of Theorem 4.3.1. \square

Proof of Proposition 4.4.1. Consider a production environment $\gamma = (N, (T_i)_{i \in N}, f, \pi, L)$. In order to establish the proof, it suffices to show that the valuation procedure U satisfies the axioms of probabilistic component efficiency, and probabilistic fairness.

i) Firstly, let us show that U satisfies the probabilistic component efficiency axiom. Summing up $U_i(\gamma)$ yields :

$$\sum_{i \in S} U_i(\gamma) = \sum_{Z \in \mathcal{R}_{f^\gamma}} P^\gamma(Z) \sum_{i \in S} BMV_i(\gamma_Z).$$

Now, for $Z \in \mathcal{R}_{f^\gamma}$, we have :

$$\begin{aligned} \sum_{i \in S} BMV_i(\gamma_Z) &= \sum_{i \in S} \left(\sum_{x \in T} P^\gamma(x|Z) SWF_i(f^\gamma, x) \right) \\ &= \sum_{x \in T} P^\gamma(x|Z) \sum_{i \in S} SWF_i(f^\gamma, x) \\ &= \sum_{x \in T} P^\gamma(x|Z) \cdot f^\gamma \left(\sum_{i \in S} x_i e_i \right). \end{aligned}$$

Hence, it follows that : $\sum_{i \in S} U_i(\gamma) = \sum_{Z \in \mathcal{R}_{f^\gamma}} \sum_{x \in Z_{f^\gamma}} \pi(x) f^\gamma \left(\sum_{i \in S} x_i e_i \right)$, since $P^\gamma(Z) = \sum_{x \in Z_{f^\gamma}} \pi(x)$. For any $x \in T$, there exists $Z \in \mathcal{R}_{f^\gamma}$ such that $f^\gamma(x) = Z$. Also, for any $Z \in \mathcal{R}_{f^\gamma}$, there exists $x \in T$ such that $f^\gamma(x) = Z$. Therefore,

$$\sum_{Z \in \mathcal{R}_{f^\gamma}} \sum_{x \in Z_{f^\gamma}} \pi(x) f^\gamma \left(\sum_{i \in S} x_i e_i \right) = \sum_{x \in T} \pi(x) f^\gamma \left(\sum_{i \in S} x_i e_i \right).$$

ii) Secondly, we show that U satisfies the probabilistic fairness axiom. For any $ij \in L$,

$$U_i(\gamma) = \sum_{Z \in \mathcal{R}_{f^\gamma}} P^\gamma(Z) BMV_i(\gamma_Z) \text{ and } U_i(\gamma_{-ij}) = \sum_{Z' \in \mathcal{R}_{f^{\gamma_{-ij}}}} P^{\gamma_{-ij}}(Z') BMV_i((\gamma_{-ij})_{Z'}).$$

If $\mathcal{R}_{f^\gamma} = \mathcal{R}_{f^{\gamma_{-ij}}}$ and $Z_{f^\gamma} = Z_{f^{\gamma_{-ij}}}$ for all $Z \in \mathcal{R}_{f^\gamma}$ then

$$U_i(\gamma) - U_i(\gamma_{-ij}) = U_j(\gamma) - U_j(\gamma_{-ij}).$$

\square

Proof of Theorem 4.4.2. The proof follows the same reasoning as that of Theorem 4.3.3. \square

Proof of Proposition 4.4.2. The proof follows the same reasoning as that of Proposition 4.4.1.

□

Proof of Theorem 4.4.3. Let $\gamma = (N, (T_i)_{i \in N}, f, \pi, L) \in \text{UPC}_N$ be a production environment with a realizable output $Z \in \mathcal{R}_{f^\gamma}$. We suppose that f is in choquet integral form or proportional form. Let w be the crisp game of f .

- i)** By using the first part (i) of the proof of Theorem 4.3.5, we deduce that $BMV^{\mathbb{R}}(\tilde{\gamma}_Z) \in I(\bar{f}^\gamma, Z)$.
- ii)** From the proof of Theorem 4.3.5, we deduce that $\sum_{i \in S} BMV(\gamma_Z) \geq \sum_{x \in T} P^\gamma(x|Z) \cdot f^\gamma(\sum_{i \in S} x_i e_i)$ for all $S \subseteq N$. Together with the first part of this proof (i), we conclude that $BMV(\gamma_Z) \in C(f^\gamma, Z)$.

□

✠ Conclusion Générale ✠

PARVENU au terme de nos travaux de thèse, il était question d'une part de généraliser quelques résultats proposés par Casajus and Huettner [2018] et Chantreuil et al. [2019] publiés récemment dans la littérature des jeux TU ; d'autre part de proposer quelques extensions des travaux de Chantreuil et al. [2019] et Casajus [2021] sur des jeux coopératifs avec structure de coalitions ; et enfin de généraliser quelques résultats de Pongou and Tondji [2018] publiés dans la littérature des environnements de production.

Le premier chapitre était consacré à la décomposition des indices d'interaction. Deux approches ont été abordées. Il s'agit notamment de l'approche proposée par Chantreuil et al. [2019] et celle proposée par Casajus and Huettner [2018]. Il s'en suit que les "Cardinal-probabilistic interaction indices" sont décomposables suivant l'approche de Chantreuil et al. [2019] et quelques-uns le sont suivant l'approche de Casajus and Huettner [2018]. Un exemple illustratif en aide à la décision multi-critère a été donné.

Le second chapitre était consacré à la décomposition des indices d'inégalités à partir de la valeur de Shapley-Owen suivant l'approche de Chantreuil et al. [2019]. Trois décompositions de la valeur de Shapley-Owen ont été proposées. Ces décompositions ont permis par la suite d'obtenir trois décompositions des indices d'inégalité. Un exemple d'application a été proposé dans le cadre de la théorie des inégalités pour illustrer notre démarche.

Une seconde version de l'axiomatisation de la valeur de Shapley-Owen suivant l'approche de Khmelnitskaya and Yanovskaya [2007] a été proposée au Chapitre 3. Dans ce chapitre, on a montré aussi que l'axiomatisation de la valeur de Shapley faite par Casajus [2021] était une seconde version de celle proposée par Young [1985].

Dans le quatrième et dernier chapitre, nous supposons que les travailleurs sont organisés suivant un réseau dans un environnement de production incertain et nous montrons comment déterminer la productivité d'un travailleur. Une procédure de valuation et sa forme révisée ont

été proposées comme solution à l'évaluation de la productivité des travailleurs. Des axiomatisations ont été proposées suivant deux approches. Des exemples illustratifs ont aussi été donnés. Les solutions ensemblistes ont été introduites dans ce nouvel environnement.

Toute généralisation ou extension raisonnable d'un résultat doit produire un résultat équivalent lorsqu'on se restreint à la base. A partir de notre Chapitre 1, il s'en suit que la décomposition de Chantreuil et al. [2019] (resp. Casajus and Huettner [2018]) coïncide exactement avec la nôtre lorsqu'on se restreint aux valeurs suivant la première approche (resp. deuxième approche). Le Chapitre 2 nous permet d'en déduire que la décomposition des indices d'inégalité proposée par Chantreuil et al. [2019] coïncide exactement avec la nôtre lorsqu'on se restreint au TU. Au Chapitre 3, lorsqu'on se restreint au jeu TU la seconde version de l'axiomatisation de la valeur de Shapley proposé par Casajus [2021] coïncide avec la nôtre. Enfin au Chapitre 4, le concept de solution proposé par Pongou and Tondji [2018] pour le problème d'évaluation de la productivité d'un travailleur dans un environnement de production incertain coïncide avec le notre lorsque le graphe est complet (c'est-à-dire, lorsque toutes les communications sont possibles entre les travailleurs). La motivation principale de cette thèse est basée sur un challenge de recherche orienté vers la résolution des problèmes dans un cadre plus général. Notre approche consiste à généraliser ou étendre des résultats qui existe déjà dans la littérature mais dans une classe restreinte.

La thématique abordée dans cette thèse demeure un champ de recherche très vaste car tout nouveau résultat proposé dans la littérature suscite une idée de généralisation ou d'extension. Parmi une multitude de travaux proposés dans la littérature, nous en avons sélectionné quelques uns. Il sera intéressant pour nous d'élargir notre champ de recherche à des travaux sur les solutions ensemblistes et sur les jeux non coopératifs, et de s'intéresser d'avantage à l'utilisation de la théorie de jeux pour la résolution de certains problèmes de société, comme nous l'avons fait avec les inégalités.

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