Beta-expansions and fractal properties of beta-expansions

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Résumé

Cette thèse est consacrée aux propriétés métriques et topologiques des ensembles liés aux bêta-développements introduits par Rényi.

Dans une première partie, on étudie les propriétés topologiques des ensembles irréguliers obtenus à partir de propriétés de longueurs d'intervalles fondamentaux des bêta-développements. Plus précisément, pour $x \in [0,1)$, notons $I_n(x)$ l'intervalle fondamental d'ordre n contenant x. Nous prouvons que l'ensemble extrêmement irrégulier contenant des points dont la limite supérieure $\frac{-\log_\beta |I_n(x)|}{n}$ est égale à $1+\lambda(\beta)$ est résiduel, où $\lambda(\beta)$ est une constante dépendant de β. Notre objet d'étude $\frac{\log_\beta |I_n(x)|}{n}$ n'est pas une moyenne ergodic de Birkhoff, notre recherche est donc différente de celle d'Olsen sur les ensembles extrêmement irréguliers.

Dans une deuxième partie, on étudie la taille des ensembles exceptionnels constitués des points dont les limites supérieure et inférieure de $\frac{r_n(x,\beta)}{\varphi(n)}$ sont des valeurs différentes, où $r_n(x,\beta)$ est la fonction run-length définie par la longueur maximale du nombre de zéros consécutifs parmi les n premiers chiffres du bêta-développement de $x \in [0,1)$ et φ est une suite positive croissante. Nous prouvons que les ensembles extrêmement divergents dont les limites supérieure et inférieure de $\frac{r_n(x,\beta)}{\varphi(n)}$ sont respectivement 0 et infini, sont vides ou de dimension de Hausdorff plaine et résiduel selon du taux de croissance de la fonction φ. Notre résultat généralise les résultats de Li et Wu pour le cas $\beta = 2$. Comme la longeur des intervalles fondamentaux est plus difficile à estimer, notre construction des ensembles de Cantor est plus delicate que celle de Li et Wu.

Dans une troisième partie, nous montrons que la limite supérieure (respectivement inférieure) de $\frac{r_n(x,\beta)}{n}$ est liée à l'approximation diophantienne classique (respectivement l'approximation diophantienne uniforme). Nous calculons la dimension de Hausdorff des ensembles de niveau contenant des points dont les limites supérieure et inférieure de $\frac{r_n(x,\beta)}{n}$ sont prescrites. De plus, nous étudions aussi les propriétés topologiques de ces ensembles de niveau. Les mêmes problèmes dans l'espace des paramètres sont également examinés.

Enfin, nous étudions les propriétés de récurrence uniforme de l'orbite d'un point $x \in [0,1)$ sous la bêta-transformation vers lui-même de manière uniforme. La mesure de Lebesgue et la dimension de Hausdorff des ensembles avec le taux de récurrence uniforme prescrit sont obtenues. Notre résultat est une généralisation de cului de Shen et Wang sur la récurrence asymptotique des orbites des bêta-transformations.

Mot-clés : Dimension de Hausdorff, bêta-développements, résiduel, approximation diophantienne, approximation diophantienne uniforme, fonction run-length, récurrence uniforme.

Abstract

This thesis is devoted to the metric and topological properties of the sets related to beta-expansions introduced by Rényi.

The first part is to investigate the topological property of the irregular set on the lengths of basic intervals in beta-expansions. More precisely, for each $x \in [0, 1)$, denote by $I_n(x)$ the basic interval of order n containing x. We prove that the extremely irregular set containing points x in [0, 1) whose upper limit of $\frac{-\log_\beta |I_n(x)|}{n}$ equals to $1 + \lambda(\beta)$ is residual, where $\lambda(\beta)$ is a constant depending on β. Our object of study $\frac{-\log_\beta |I_n(x)|}{n}$ is not an ergodic average of Birkhoff, so our research is different from that of Olsen on extremely irregular sets.

In the second part, we study the size of the exceptional set consisting of points whose limit superior and limit inferior of $\frac{r_n(x,\beta)}{\varphi(n)}$ are different prescribed values, where $r_n(x, \beta)$ is the run-length function defined by the maximal length of consecutive zeros amongst the first n digits in the beta-expansion of $x \in [0, 1)$ and φ is an increasing function. It is proved that the extremely divergence set whose limit superior and limit inferior of $\frac{r_n(x,\beta)}{\varphi(n)}$ are 0 and infinity respectively, is either empty or of full Hausdorff dimension and residual in [0, 1] according to the increasing rate of φ. Our result generalizes the results of Li and Wu Li and Wu for the case $\beta = 2$. As the length of fundamental intervals is more difficult to estimate, our construction of Cantor sets is more difficult than that of Li and Wu.

In the third part, we prove that the limit superior (respectively limit inferior) of $\frac{r_n(x,\beta)}{n}$ is linked to the classical Diophantine approximation (respectively uniform Diophantine approximation). We calculate the Hausdorff dimension of the sets of points whose limit superior and limit inferior of $\frac{r_n(x,\beta)}{n}$ are prescribed. Furthermore, we also study the topological properties of these sets. The same problems in the parameter space are also examined.

Finally, we study the uniform recurrence properties of the orbit of a point $x \in [0, 1)$ under the beta-transformation to the point itself in a uniform way. The Lesbesgue measure and Hausdorff dimension of the set of points with prescribed uniform recurrence rate are obtained. Our result is a generalization of Shen and Wang on the sets of points with asymptotic recurrence property.

Key words : Hausdorff dimension, β-expansions, residual, Diophantine approximation, uniform Diophantine approximation, run-length function, uniform recurrence

Table des matières

Chapitre 1 Introduction (version française)

1.1 Bêta-transformation et bêta-développements

Soit $\beta > 1$ un réel. La β-transformation sur $[0,1)$ est définie par

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor = \{\beta x\},$$

où $\lfloor \xi \rfloor$ signifie la partie entière de ξ. En 1957, Rényi [68] a montré que, tout nombre réel $x \in [0,1)$ peut être développé de manière unique comme une série

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + ... + \frac{\varepsilon_n(x,\beta)}{\beta^n} + \cdots ,$$

où $\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor$ pour tout $n \in \mathbb{N}$. Nous appellons $\varepsilon_n(x,\beta)$ le n-ième chiffre de x et $\varepsilon(x,\beta) := (\varepsilon_1(x,\beta), \ldots, \varepsilon_n(x,\beta), \ldots)$ le bêta-développement de x.

Pour un mot admissible $\omega = (\omega_1, \ldots, \omega_n)$, c'est-à-dire, un préfixe du bêta développement d'un certain $x \in [0,1)$ (voir Section 3.2.2 pour plus de détails). L'intervalle fondamental d'ordre n associé à ω, noté $I_n(\omega)$, est défini par

$$I_n(\omega) = I_n(\omega_1, ..., \omega_n) := \{x \in [0,1) : \varepsilon_1(x,\beta) = \omega_1, ..., \varepsilon_n(x,\beta) = \omega_n\}. \tag{1.1.1}$$

L'intervalle fondamental d'ordre n contenant x s'écrit comme $I_n(x)$. Nous pouvons vérifier que l'intervalle fondamental $I_n(x)$ est un intervalle fermé à gauche et ouvert à droite (voir Lemme 3.2.5). On note la longueur de $I_n(x)$ par $|I_n(x)|$.

Bien que 1 n'est pas dans l'ensemble de définition de T_β, on peut étendre la définition de β-transformation à $x = 1$. Soit $T_\beta(1) = \beta - \lfloor \beta \rfloor$. On peut écrire 1 comme une série :

$$1 = \frac{\varepsilon_1(1,\beta)}{\beta} + \cdots + \frac{\varepsilon_n(1,\beta)}{\beta^n} + \cdots ,$$

où $\varepsilon_n(1,\beta) = \lfloor \beta T_\beta^{n-1}(1) \rfloor$. Spécialement, si le β-développement de 1 est finie, c'est-à-dire, qu'il existe un entier $m \geq 1$ tel que $\varepsilon_m(1,\beta) > 0$ et $\varepsilon_k(1,\beta) = 0$ pour tout $k > m$, β est appelé un nombre de Parry simple. Dans ce cas, on note

$$\varepsilon^*(\beta) = (\varepsilon_1^*, \varepsilon_2^*, \ldots) = (\varepsilon_1(1,\beta), \varepsilon_2(1,\beta), \ldots, \varepsilon_m(1,\beta) - 1)^\infty,$$

où $\omega^\infty = (\omega, \omega, \dots)$. Si la série de 1 n'est pas finie, notons $\varepsilon^*(\beta) = \varepsilon(1,\beta)$. Dans les deux cas, on a

$$1 = \frac{\varepsilon_1^*}{\beta} + \cdots + \frac{\varepsilon_n^*}{\beta^n} + \cdots ,$$

La suite $\varepsilon^*(\beta)$ est en conséquence appelée le β-développement infini de 1.

Le bêta-développement de l'unité 1 joue un rôle important non seulement dans la recherche des propriétés dynamiques de l'orbite de 1, mais aussi, dans l'estimation de la longueur de $I_n(x)$ ([20]). Pour chaque entier $n \geq 1$, on note par $t_n = t_n(\beta)$ la longueur maximale de zéros consécutifs après le n-ième chiffre du β-développement infini de 1. C'est-à-dire,

$$t_n = t_n(\beta) := \max\{k \geq 1 : \varepsilon_{n+1}^* = \varepsilon_{n+2}^* = ... = \varepsilon_{n+k}^* = 0\}. \tag{1.1.2}$$

Si l'entier k n'existe pas, posons $t_n = 0$. Soit

$$\lambda(\beta) = \limsup_{n\to+\infty} \frac{\Gamma_n(\beta)}{n},$$

où

$$\Gamma_n = \Gamma_n(\beta) := \max_{1\leq k\leq n} t_k(\beta). \tag{1.1.3}$$

L'estimation sur les longueurs des intervalles fondamentaux est un outil important dans la recherche sur les propriétés fractales des β-développements : les propriétés diophantiennes des orbites [20], le problème shrinking target [72], les propriétés de récurrence d'un β-développement [74] etc. Fan et Wang [35] ont établi une relation entre la longueur de $I_n(x)$ et le β-développement infini de 1 et ont donné une méthode afin de calculer la longueur $I_n(x)$, voir Théorème 3.2.5 pour plus de détails. De plus, ils ont introduit et étudié des quantités qui décrivent la décroissance de la longueur de $I_n(x)$. Plus précisémment, pour tout $x \in [0,1)$, on définit la densité inférieure et la densité supérieure de x respectivement par,

$$\underline{D}(x) = \liminf_{n\to+\infty} \frac{-\log_\beta |I_n(x)|}{n} \quad \text{and} \quad \overline{D}(x) = \limsup_{n\to+\infty} \frac{-\log_\beta |I_n(x)|}{n}.$$

Il est remarqué dans [49] que pour tout $x \in [0,1)$, nous avons

$$\underline{D}(x) = 1, \quad 1 \leq \overline{D}(x) \leq 1 + \lambda(\beta), \tag{1.1.4}$$

où $\lambda(\beta)$ est une constante dependant de β. En appliquant le théorème de Shannon-McMillan-Breiman [79] à la mesure de Parry [65], on obtient $\underline{D}(x) = \overline{D}(x) = 1$ Lebesgue presque partout sur $[0,1)$. Un cas spécifique est $\lambda(\beta) = 0$, où on obtient $\underline{D}(x) = \overline{D}(x) = 1$. Ceci implique que la limite de $\frac{-\log_\beta |I_n(x)|}{n}$ existe pour tout $x \in [0,1)$. L'ensemble de ces β avec $\lambda(\beta) = 0$ dans $(1, +\infty)$ est de mesure de Lebesgue pleine [70].

Nous nous intéressons, pour tout cas $\lambda(\beta) > 0$, aux ensembles exceptionnels des points avec la densité supérieure prescrite. Pour tout $1 < \delta < 1 + \lambda(\beta)$, on définit

$$D_\delta = \{x \in [0,1) : \overline{D}(x) = \delta\}.$$

Les points dans D_δ avec $1 < \delta \leq 1 + \lambda(\beta)$ sont appelés δ-irréguliers. Fan et Wang [35] ont montré que

$$\dim_H D_\delta = \frac{\lambda(\beta) + 1 - \delta}{\delta \cdot \lambda(\beta)}, \tag{1.1.5}$$

pour tout $1 < \delta \leq 1 + \lambda(\beta)$.

Il est naturel de poser la question sur la taille des ensembles D_δ d'un point de vue topologique. Étant motivés par ce point, au chapitre 3, nous établissons le théorème suivant qui illustre que l'ensemble extrêmement irrégulier

$$D := D_{1+\lambda(\beta)} = \{x \in [0,1) : \overline{D}(x) = 1 + \lambda(\beta)\}$$

est résiduel (grand du point de vue topologique) pour tout $\lambda(\beta) > 0$. Ce que nous devrons remarquer, est que non seulement la mesure de Lebesgue est nulle sur D mais aussi la dimension de Hausdorff. Ce résultat est quelque peu similaire à celui de Olsen [62] sur les nombres extrêmement non-normaux.

Théorème 1.1.1. *Soit $\beta > 1$ avec $\lambda(\beta) > 0$. Alors l'ensemble D est résiduel, c'est-à-dire que $[0,1]\backslash D$ est de première catégorie.*

En effet, il existe des ensembles irréguliers de mesure de Lebesgue nulle, mais résiduels, ce qui implique que de tels ensembles peuvent être volumineux au sens de la topologie. Par exemple, il est prouvé que les ensembles contenant certains types de points irréguliers associés aux développements en base entières sont résiduels [2, 44, 62]. Il est montré par Baek et Olsen [5] que l'ensemble des points extrêment non-normaux d'un ensemble auto-similaire est résiduel. Madritsch [58] a étendu et généralisé les résultats en [5] aux partitions de Markov. De plus, les nombres non-normaux dans un système dynamique satisfaisant la propriété de spécification sont résiduels (voir Madritsch et Petrykiewicz [59]). Cependant, dans la recherche des nombres non-normaux, les fréquences des chiffres et des blocs ont été étudiées. Dans notre cas, la densité supérieure $\overline{D}(x)$ ne peut pas être exprimée sous forme de fréquences.

1.2 Fonction run-length

Pour tout nombre réel $x \in [0,1)$, et entier naturel $n \geq 1$, la fonction run-length notée par $r_n(x,\beta)$ est la longueur maximale du nombre de zéros consécutifs parmi

$(\varepsilon_1(x,\beta),\ldots,\varepsilon_n(x,\beta))$, c'est-à-dire,

$$r_n(x,\beta)=\max\{j\geq 1:\varepsilon_{i+1}(x,\beta)=...=\varepsilon_{i+j}(x,\beta)=0 \text{ pour certain } 0\leq i\leq n-j\}.$$

Posons $r_n(x,\beta)=0$ lorsqu'un tel j n'existe pas. Pour la base $\beta=2$, Erdös et Rényi [26] ont montré que Lebesgue presque partout sur $[0,1)$,

$$\lim_{n\to+\infty}\frac{r_n(x,2)}{\log_2 n}=1. \tag{1.2.1}$$

La taille des ensembles déterminés par la fonction $r_n(x,2)$ a été beaucoup étudiée. Ma, Wen et Wen [56] ont prouvé que l'ensemble des points violants (1.2.1) est de dimension de Hausdorff pleine. Par la suite, Li et Wu [51, 52] ont remplacé la fonction $n\mapsto\log_2 n$ dans (1.2.1) par une fonction strictement croissante $\varphi:\mathbb{N}\to\mathbb{R}^+$ avec $\lim\limits_{n\to+\infty}\varphi(n)=+\infty$, et ont introduit l'ensemble extrêmement exceptionnel par rapport à φ :

$$E_{\max}=\left\{x\in[0,1):\liminf_{n\to+\infty}\frac{r_n(x,2)}{\varphi(n)}=0,\ \limsup_{n\to+\infty}\frac{r_n(x,2)}{\varphi(n)}=+\infty\right\}.$$

Ils ont montré que la dimension de Hausdorff de $E_{\max}$ est 1 et que $E_{\max}$ est résiduel sur $[0,1]$ lorsque la fonction φ est strictement croissante et satisfaisant $\lim\limits_{n\to+\infty}\frac{n}{\varphi(n^{1+\alpha})}=+\infty$ pour tout $0<\alpha\leq 1$ (voir [51] pour plus de détails). Après, dans [52], ils ont montré que $E_{\max}$ est de dimension de Hausdorff 1 et que $E_{\max}$ est résiduel sous la condition $\lim\limits_{n\to+\infty}\frac{n}{\varphi(n)}=+\infty$.

Naturellement, il est intéressant de déterminer la taille des ensembles ci-dessus quand la base 2 est substituée par un nombre réel général $\beta>1$. En effet, Tong, Yu et Zhao [77] ont abouti au même résultat que [26] : pour Lebesgue presque tout $x\in[0,1)$, on a

$$\lim_{n\to+\infty}\frac{r_n(x,\beta)}{\log_\beta n}=1.$$

Ainsi, l'ensemble

$$E=\left\{x\in[0,1):\liminf_{n\to+\infty}\frac{r_n(x,\beta)}{\log_\beta n}<\limsup_{n\to+\infty}\frac{r_n(x,\beta)}{\log_\beta n}\right\}$$

est de mesure de Lebesgue nulle. En prenant une fonction $\varphi:\mathbb{N}\to\mathbb{R}^+$ strictement croissant et satisfaisant $\lim\limits_{n\to+\infty}\varphi(n)=+\infty$ au lieu de la fonction $n\mapsto\log_\beta n$, on définit l'ensemble exceptionel extrêmement divergents par

$$E_{\max}^{\varphi}=\left\{x\in[0,1):\liminf_{n\to+\infty}\frac{r_n(x,\beta)}{\varphi(n)}=0,\ \limsup_{n\to+\infty}\frac{r_n(x,\beta)}{\varphi(n)}=+\infty\right\}. \tag{1.2.2}$$

Nous étendons les résultats de Li et Wu ([51, 52]) en généralisant la base 2 à tout nombre réel $\beta>1$ dans les théorèmes suivants.

Théorème 1.2.1. *Soit* $\varphi : \mathbb{N} \to \mathbb{R}^+$ *une fonction strictement croissante avec* $\lim\limits_{n\to+\infty} \varphi(n) = +\infty$. *Soit* $E_{\max}^{\varphi}$ *l'ensemble défini par* (1.2.2), *alors*
(1) Si $\limsup\limits_{n\to+\infty} \frac{n}{\varphi(n)} < +\infty$, *on a* $E_{\max}^{\varphi} = \emptyset$;
(2) Si $\limsup\limits_{n\to+\infty} \frac{n}{\varphi(n)} = +\infty$, *on a* $\dim_{\mathrm{H}} E_{\max}^{\varphi} = 1$.

Remarque 1.2.1. *Le résultat de (1) dans le théorème 1.2.1 est trivial puisque si* $\limsup\limits_{n\to+\infty} \frac{n}{\varphi(n)} < +\infty$, *d'après le fait* $r_n(x,\beta) \leq n$, *pour tout* $n \in \mathbb{N}$ *et tout* $x \in [0,1)$, *on a*

$$\limsup_{n\to+\infty} \frac{r_n(x,\beta)}{\varphi(n)} = \limsup_{n\to+\infty} \frac{r_n(x,\beta)}{n} \cdot \frac{n}{\varphi(n)} < +\infty.$$

Ainsi, $E_{\max}^{\varphi} = \emptyset$. *Donc il suffit de montrer (2) dans le théorème 1.2.1.*

Nous nous intéressons aussi à la taille de l'ensemble $E_{\max}^{\varphi}$ au sens topologique. Nous établissons le théorème suivant.

Théorème 1.2.2. *Soit* $\varphi : \mathbb{N} \to \mathbb{R}^+$ *une fonction strictement croissante avec* $\lim\limits_{n\to+\infty} \varphi(n) = +\infty$. *Soit* $E_{\max}^{\varphi}$ *l'ensemble défini en* (1.2.2). *Alors,* $E_{\max}^{\varphi}$ *est résiduel lorsque* $\limsup\limits_{n\to+\infty} \frac{n}{\varphi(n)} = +\infty$.

En fixant $\varphi(n) = \log_\beta n$ dans la formule (1.2.2) de la définition de l'ensemble $E_{\max}^{\varphi}$ et en combinant les résultats de Tong, Yu et Zhao [77], on arrive immédiatement le corollaire suivant. Ce résultat montre qu'un ensemble peut être très petit au sens de la topologie mais être volumineux du point de vue de la théorie de la mesure et de la dimension.

Corollaire 1.2.1. *L'ensemble* $\{x : \lim\limits_{n\to+\infty} \frac{r_n(x,\beta)}{\log_\beta n} = 1\}$ *est à la fois de mesure de Lebesgue pleine et de première catégorie.*

Un point important à noter ici est que lorsque β est un entier, on retrouve le résultat de Li et Wu (voir [51, 52]) pour la dimension de Hausdorff et la catégorie de $E_{\max}$. En effet, les intervalles fondamentaux d'ordre n étant de longueur égale à β^{-n}, tous les mots admissibles peuvent être concaténés les uns avec les autres. Lorsque β n'est pas un entier, les comportements dynamiques seront beaucoup plus compliqués pour différentes classes de $\beta > 1$; par exemple, une classe est l'ensemble de $\beta > 1$ pour lequel le sous-décalage correspondant satisfait à la propriété de spécification, l'autre est le complément de cet ensemble. Pour $\beta > 1$ dans la première classe, l'admissibilité de la concaténation de mots admissibles peut être garantie en ajoutant un nombre fini de zéros uniformément. Dans ces cas, la longueure des intervalles fondamentaux d'ordre n est équivalente à β^{-n}. Cependant, pour $\beta > 1$ dans la deuxième classe, les propriétés ci-dessus des mots admissibles et des intervalles fondamentaux disparaissent, ce qui pose de nombreuses difficultés pour la construction des ensembles de Cantor et des ensembles denses contenus dans $E_{\max}^{\varphi}$ et pour la vérification de la condition du principe modifié de distribution de masse (voir Lemme 3.1.8). Il y a beaucoup de résults liés à la fonction run-length, voir [37, 53, 54, 73] et dans les références données dans ces articles.

1.3 Fonction run-length en lien avec l'approximation diophantienne

Dans la section 1.2, on voit que alors l'ensemble des points extrêmement divergents

$$E_{\max}^{\varphi} = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = 0, \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = +\infty \right\}$$

est vide si $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} < +\infty$, et est de dimension de Hausdorff 1 et résiduel dans $[0,1)$ sinon. Pour tout $0 \le a \le b \le +\infty$, on définit

$$E_{a,b}^{\varphi} = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = a, \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = b \right\}.$$

De fait, l'aticle [34] nous fournit un résultat plus général sur l'ensemble $E_{a,b}^{\varphi}$. En effet, cet ensemble est de dimension de Hausdorff 1 lorsque $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} < +\infty$. Cela nous amène à considérer le problème de la taille de l'ensemble $E_{a,b}^{\varphi}$ lorsque la fonction φ est d'ordre n (i.e., $\lim\limits_{n\to\infty} \frac{\varphi(n)}{n} = c$ pour certain c avec $0 < c < \infty$).

Pour tout $0 \le a \le b \le 1$, nous allons étudier la taille de l'ensemble

$$E_{a,b} := E_{a,b}(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\}. \tag{1.3.1}$$

En effet, le comportement asymptotique de $\frac{r_n(x,\beta)}{n}$ est en lien direct avec l'approximation diophantienne des β-développements. Pour tout $x \in [0,1)$, Bugeaud et Liao [18] ont définit l'exposant $v_\beta(x)$ comme le supremum des nombres réels v pour lesquels l'équation

$$T_\beta^n x \le \beta^{-nv}$$

admet une infinité de solutions $n \in \mathbb{N}$. Ils ont également définit l'exposant $\hat{v}_\beta(x)$ comme le supremum des nombres réels $\hat{v}$ pour lesquels, pour tout $N \gg 1$, il existe une solution n avec $1 \le n \le N$, telle que

$$T_\beta^n x \le \beta^{-N\hat{v}}.$$

Nous verrons (Lemmes 6.1.1 et 6.1.2) que pour tout $0 < a < 1,\ 0 < b < 1$,

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a \iff \hat{v}_\beta(x) = \frac{a}{1-a} \quad \text{et} \quad \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \iff v_\beta(x) = \frac{b}{1-b}.$$

Grace à ces équivalences, nous obtenons le résultat suivant.

Théorème 1.3.1. *L'ensemble $E_{0,0}$ est de mesure de Lebesgue pleine. Si $\frac{b}{1+b} < a \leq 1,\ 0 < b \leq 1$, alors $E_{a,b} = \emptyset$. Sinon, on a*

$$\dim_{\mathrm{H}} E_{a,b} = 1 - \frac{b^2(1-a)}{b-a}.$$

Soit $0 \leq a \leq 1$ et $0 \leq b \leq 1$. Nous étudons les ensembles de niveau suivants

$$E_a := E_a(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a \right\}$$

et

$$F_b := F_b(\beta) = \left\{ x \in [0,1) : \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\}. \tag{1.3.2}$$

En utilisant le Théorème 1.3.1, nous obtenons le résultat suivant concernant la dimension de Hausdorff de E_a et F_b.

Corollaire 1.3.1. *(1) Lorsque $0 \leq a \leq \frac{1}{2}$, on a*

$$\dim_{\mathrm{H}} E_a = (1-2a)^2.$$

Sinon, $E_a = \emptyset$.
(2) Pour tout $0 \leq b \leq 1$, on a

$$\dim_{\mathrm{H}} F_b = 1 - b.$$

On remarque que l'assertion (2) du Théorème 2.3.2 a aussi été obtenu dans [53, Theorem 1.1] (voir [80] pour le cas $\beta = 2$).

Similaire aux résultats de [51, 52], l'ensemble des points extrêmements divergents est résiduel, et est donc grand au sens de la topologie.

Théorème 1.3.2. *L'ensemble $E_{0,1}$ est résiduel dans $[0,1]$.*

Il est important de noter que $E_{0,1}$ est négligeable au sens de la mesure de Lebesgue et de la dimension de Hausdorff. Cependant, les ensembles considérés dans [34, 51, 52] sont de dimension de Hausdorff 1. Soit $\varphi : \mathbb{N} \to \mathbb{R}^+$ une fonction strictement croissante avec $\lim_{n\to+\infty} \varphi(n) = +\infty$ et $\liminf_{n\to\infty} \frac{\varphi(n)}{n} = 0$. Comme l'intersection de deux ensembles résiduels est un ensemble résiduel, en combinant les Théorèmes 1.3.2 and 1.2.1, on peut déduire que l'ensemble plus petit

$$E := E(\varphi,\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = 0,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = 1 \right\}$$

est aussi résiduel dans $[0,1]$.

Le β-développement de 1 caractérise complètement tous les mots admissibles dans le β-système dynamique (voir Théorème 3.2.4 dans la section 3 pour plus de détails).

Nous étudions aussi la function run-length $r_n(\beta)$ des β-développement de 1 lorsque β varie dans l'espace de paramètres $\{\beta \in \mathbb{R} : \beta > 1\}$, i.e.,

$$r_n(\beta) = \max\{1 \le j \le n : \varepsilon_{i+1}(1,\beta) = \cdots = \varepsilon_{i+j}(1,\beta) = 0 \text{ pour certain } 0 \le i \le n-j\}.$$

Il existe des résultats concernant $r_n(\beta)$ qui sont similaires à ceux obtenus pour $r_n(x,\beta)$. Dans [43], Hu, Tong et Yu ont prouvé que pour Lebesgue presque tout $1 < \beta < 2$, on a

$$\lim_{n\to\infty} \frac{r_n(\beta)}{\log_\beta n} = 1.$$

Cao et Chen [21] ont montré que pour toute fonction $\varphi : \mathbb{N} \to \mathbb{R}^+$ strictement croissante avec $\lim_{n\to+\infty} \varphi(n) = +\infty$ et $\liminf_{n\to+\infty} \frac{\varphi(n)}{n} = 0$, pour tout $0 \le a \le b \le +\infty$, l'ensemble

$$\left\{\beta \in (1,2) : \liminf_{n\to\infty} \frac{r_n(\beta)}{\varphi(n)} = a,\ \limsup_{n\to\infty} \frac{r_n(\beta)}{\varphi(n)} = b\right\}$$

est de dimension de Hausdorff pleine. On remarque que les résultats dans [21] et [43] peuvent facilement être généralisés à l'espace de paramètres $\{\beta \in \mathbb{R} : \beta > 1\}$. Par soucis de simplicité, nous considérons l'espace de paramètres $(1,2)$. Pour tout $0 \le a \le b \le 1$, soit

$$E_{a,b}^P = \left\{\beta \in [1,2] : \liminf_{n\to\infty} \frac{r_n(\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(\beta)}{n} = b\right\}. \tag{1.3.3}$$

Nous avons le théorème suivant.

Théorème 1.3.3. *L'ensemble $E_{0,0}^P$ est de mesure de Lebesgue pleine. Si $\frac{b}{1+b} < a \le 1$, $0 < b \le 1$, alors $E_{a,b}^p = \emptyset$. Sinon on a :*

$$\dim_{\mathrm{H}} E_{a,b}^P = 1 - \frac{b^2(1-a)}{b-a}.$$

De la même façon, pour tout $0 \le a \le 1$ et $0 \le b \le 1$, définissons

$$E_a^P = \left\{\beta \in [1,2] : \liminf_{n\to+\infty} \frac{r_n(\beta)}{n} = a\right\},$$

et

$$F_b^P = \left\{\beta \in [1,2] : \liminf_{n\to+\infty} \frac{r_n(\beta)}{n} = b\right\}.$$

Nous avons le résultat suivant.

Corollaire 1.3.2. *(1) Quand $0 \le a \le \frac{1}{2}$, on a*

$$\dim_{\mathrm{H}} E_a^P = (1-2a)^2.$$

Sinon $E_a^P = \emptyset$.
(2) Pour tout $0 \le b \le 1$, on a

$$\dim_{\mathrm{H}} F_b^P = 1 - b.$$

De plus, de la même façon qu'au théorème 1.3.2, on a le résultat suivant.

Théorème 1.3.4. *L'ensemble $F_{0,1}^P$ est résiduel dans [1,2].*

1.4 Propriétés de récurrence uniforme

Le théorème de récurrence de Poincaré constitue l'un des résultats les plus fondamentaux en systèmes dynamiques. Ce théorème établit que tout ensemble de mesure strictement positive, pour une mesure invariante, est récurrent. Soit $(X, \mathcal{B}, \mu, T)$ un système dynamique mesuré, avec μ une meusre borélienne finie. Notons d la métrique sur X. Le théorème de récurrence de Poincaré implique notamment que l'orbite d'un point typique est asymptotiquement récurrente. Plus précisément, on a

$$\liminf_{n\to+\infty} d(T^n(x), x) = 0$$

pour μ presque tout $x \in X$. Boshernitzan [14] a étudié la vitesse à laquelle une orbite est asymptotiquement récurrente. Il a démontré que si, pour $\alpha \geq 0$, la mesure de Hausdorff, $\mathcal{H}^\alpha$ est σ-finie sur X, (i.e, X peut être écrit comme une union dénombrable d'ensembles X_i avec $\mathcal{H}^\alpha(X_i) < +\infty$, pour tout $i \in \mathbb{N}$), alors on a

$$\liminf_{n\to+\infty} n^{\frac{1}{\alpha}} d(T^n x, x) < +\infty \text{ pour } \mu\text{-presque tout } x \in X.$$

Si $\mathcal{H}^\alpha(X) = 0$, alors

$$\liminf_{n\to+\infty} n^{\frac{1}{\alpha}} d(T^n x, x) = 0 \text{ pour } \mu\text{-presque tout } x \in X.$$

Plus tard, de nombreux comportements asymptotiques ont également été étudiés. Beaucoup de recherches sont motivés par le théorème de récurrence de Poincaré, tels que, les temps de premier retour [7], le lemme de Borel-Cantelli dynamique [23], les temps d'attentes [36], les problèmes de shrinking targets [39, 40, 72] etc.

A contrario du point de vue précédent (asymptotique), le théorème de Dirichlet fournit un point de vue différent sur l'approximation dynamique : le point de vue de l'approximation uniforme. Le théorème de Dirichlet établit que pour tout nombre irrationnel positif θ, pour tout nombre entier $N \geq 1$, on peut trouver un entier $1 \leq n \leq N$, de sorte que

$$\|n\theta\| < N^{-1} \tag{1.4.1}$$

où $\|\cdot\|$ désigne la distance à l'entier le plus proche. L'uniformité repose ici sur le fait que l'équation (1.4.1) a une solution pour tout entier N assez grand. Remarquons que $\|T_\theta^n x - x\| = \|n\theta\|$, où $T_\theta : \mathbb{R}/\mathbb{Z} :\to \mathbb{R}/\mathbb{Z}$, $T_\theta x \mapsto x + \theta$. Le théorème de Dirichlet peut ainsi être vu comme une propriété de récurrence uniforme du système dynamique $(\mathbb{R}/\mathbb{Z}, T_\theta)$. Certains résultats concernant l'approximation uniforme peuvent être trouvés dans [17, 18, 19, 47, 48] et dans les références données dans ces articles.

Le théorème de Dirichlet incite à étudier les points de $[0,1)$ qui sont uniforméments récurrents pour le système dynampique $([0,1),T_\beta)$. On décrira la taille de ces ensembles du point de vue de la mesure et du point de vue de la dimension. On considère les exposants d'approximations suivants, l'un étant relié à le récurrence asymptotique et l'autre à le récurrence uniforme par les orbites de T_β :

Définition 1.4.1. *Soit $\beta > 1$. Pour tout $x \in [0,1)$, on définit $r_\beta(x)$ comme le supremum des réels r tels que l'équation :*

$$|T_\beta^n x - x| < (\beta^n)^{-r}$$

admet une infinité de solutions $n \in \mathbb{N}$. On définit également $\hat{r}_\beta(x)$ comme le supremum des réels $\hat{r}$ tels que pour tout $N \in \mathbb{N}$ assez grand, l'équation :

$$|T_\beta^n x - x| < (\beta^N)^{-\hat{r}}$$

a une solution $n \in \mathbb{N}$, avec $1 \leq n \leq N$.

Les exposants $r_\beta(x)$ et $\hat{r}_\beta(x)$ sont les analogues des exposants introduits dans [3] (voir aussi [17, 18]). L'exposant r_β est relié aux propriétés de récurrence asymptotique et l'exposant $r_\beta(x)$ et $\hat{r}_\beta(x)$, à celles de récurrence uniforme. Remarquons que, par définition, $0 \leq r_\beta(x) \leq +\infty$ et $0 \leq \hat{r}_\beta(x) \leq +\infty$. On peut également vérifier que $\hat{r}_\beta(x) \leq r_\beta(x)$. L'exposant asymptotique r_β a été étudié par Tan et Wang [74], qui ont montré que

$$\dim_{\mathrm{H}} \{x \in [0,1) : r_\beta(x) \geq r\} = \frac{1}{1+r}.$$

On établit des résultats similaires pour l'exposant $\hat{r}_\beta$. Où $\frac{1}{+\infty} = 0$.

Théorème 1.4.1. *L'ensemble $\{x \in [0,1) : r_\beta(x) = 0\}$ est de mesure de Lebesgue pleine. Si $\hat{r} > 1$, l'ensemble $\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r}\}$ est dénombrable. Si $0 \leq \hat{r} \leq 1$, on a*

$$\dim_{\mathrm{H}} \{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r}\} = \dim_{\mathrm{H}} \{x \in [0,1) : \hat{r}_\beta(x) = \hat{r}\} = \left(\frac{1-\hat{r}}{1+\hat{r}}\right)^2.$$

Le théorème 1.4.1 est en fait une conséquence d'un résultat plus général qui donne la dimension de Hausdorff de l'ensemble des points $x \in [0,1)$ à r_β et $\hat{r}_\beta$ fixés. Plus précisément pour $0 \leq r \leq +\infty$ et $0 \leq \hat{r} \leq +\infty$, notons

$$R_\beta(r,\hat{r}) := \{x \in [0,1) : r_\beta(x) = r, \hat{r}_\beta(x) = \hat{r}\}.$$

Nous avons le théorème suivant.

Théorème 1.4.2. *L'ensemble* $R_\beta(0,0)$ *est de mesure de Lebesgue pleine. Si* $0 \leq \frac{r}{1+r} < \hat{r} \leq +\infty$*, l'ensemble* $R_\beta(r,\hat{r})$ *est dénombrable. Si* $\hat{r} \leq \frac{r}{1+r}$ *et* $0 < r \leq +\infty$*, on a*

$$\dim_{\mathrm{H}} R_\beta(r,\hat{r}) = \frac{r-(1+r)\hat{r}}{(1+r)(r+\hat{r})}.$$

Où $\frac{+\infty}{(+\infty)(+\infty)} = 0$.

En utilisant le Théorèm 1.4.1, nous obtenons le résult suivant concernant l'exposant asympotique r_β.

Théorème 1.4.3. *Pour tout* $0 \leq r \leq +\infty$*, on a*

$$\dim_{\mathrm{H}}\{x \in [0,1) : r_\beta(x) = r\} = \frac{1}{1+r}.$$

Chapitre 2 Introduction (English version)

2.1 Beta-transformation and beta-expansions

Let $\beta > 1$ be a real number. The *β-transformation* on $[0,1)$ is defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor = \{\beta x\},$$

where $\lfloor \xi \rfloor$ means the integer part of ξ. In 1957, Rényi showed (see [68]) that, every real number $x \in [0,1)$ can be uniquely expanded as a series

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \cdots + \frac{\varepsilon_n(x,\beta)}{\beta^n} + \cdots ,$$

where $\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor$ for all $n \geq 1$. We call $\varepsilon_n(x,\beta)$ *the n-th digit of x* and $\varepsilon(x,\beta) := (\varepsilon_1(x,\beta), \ldots, \varepsilon_n(x,\beta), \ldots)$ the *β-expansion of x*.

For an admissible word $\omega = (\omega_1, \ldots, \omega_n)$, that is, a prefix of the digit sequence for some $x \in [0,1)$ (see Section 2.2.2 for details). The *basic interval of order n associated to ω*, denoted by $I_n(\omega)$, is defined by

$$I_n(\omega) = I_n(\omega_1, \ldots, \omega_n) := \{x \in [0,1) : \varepsilon_1(x,\beta) = \omega_1, \ldots, \varepsilon_n(x,\beta) = \omega_n\}. \tag{2.1.1}$$

The basic interval of order n containing x is written as $I_n(x)$. We can check that the basic interval $I_n(x)$ a left-closed and right-open interval (see Lemma 3.2.5). We denote the length of $I_n(x)$ by $|I_n(x)|$. In some papers, the basic interval $I_n(\omega)$ is also called the cylinder of order n associated to ω.

Although 1 is not in the domain of T_β, we extend the definition of the β-transformation to $x = 1$. Let $T_\beta(1) = \beta - \lfloor \beta \rfloor$, we have

$$1 = \frac{\varepsilon_1(1,\beta)}{\beta} + \cdots + \frac{\varepsilon_n(1,\beta)}{\beta^n} + \cdots ,$$

where $\varepsilon_n(1,\beta) = \lfloor \beta T_\beta^{n-1}(1) \rfloor$. Specially, if the β-expansion of 1 is finite, that is, there is an integer $m \geq 1$ such that $\varepsilon_m(1,\beta) > 0$ and $\varepsilon_k(1,\beta) = 0$ for all $k > m$, β is called a *simple Parry number*. In this case, we set

$$\varepsilon^*(\beta) := (\varepsilon_1^*, \varepsilon_2^*, \ldots) = (\varepsilon_1(1,\beta), \varepsilon_2(1,\beta), \ldots, \varepsilon_m(1,\beta) - 1)^\infty,$$

where $\omega^\infty = (\omega, \omega, \ldots)$. If the β-expansion of 1 is not finite, set $\varepsilon^*(\beta) = \varepsilon(1, \beta)$. In both cases, we have
$$1 = \frac{\varepsilon_1^*}{\beta} + \cdots + \frac{\varepsilon_n^*}{\beta^n} + \cdots .$$
The sequence $\varepsilon^*(\beta)$ is consequently called *the infinite β-expansion of* 1. The β-expansion of the unit 1 plays an important role not only in researching the dynamical properties of the orbit of 1, but also in estimating the length of $I_n(x)$ ([20]). For each integer $n \geq 1$, denote by $t_n = t_n(\beta)$ the maximal length of consecutive zeros after the n-th digit of the β-expansion of 1. That is,

$$t_n = t_n(\beta) := \max\{k \geq 0 : \varepsilon_{n+1}^* = \varepsilon_{n+2}^* = \cdots = \varepsilon_{n+k}^* = 0\}. \tag{2.1.2}$$

If such k does not exist, let $t_n = 0$. Let

$$\lambda(\beta) = \limsup_{n\to\infty} \frac{\Gamma_n(\beta)}{n},$$

where

$$\Gamma_n = \Gamma_n(\beta) := \max_{1\leq k\leq n} t_k(\beta). \tag{2.1.3}$$

Remark 2.1.1. By the definition of t_n, we have $I_n(\varepsilon_1^*, \ldots, \varepsilon_n^*) = I_{n+t_n}(\varepsilon_1^*, \ldots, \varepsilon_n^*, 0^{t_n})$.

The estimation on the lengths of basic intervals is a useful tool to research on the fractals in β-expansion such as the the Diophantine problem [20], the shrinking target problem [72], the recurrence properties [74] and so on. Fan and Wang [35] established a relationship between the length of $I_n(x)$ and the β-expansion of 1 and gave a way to calculate the length of $I_n(x)$, see Theorem 3.2.9 for more details. Furthermore, they introduced and studied the quantities which describe the growth of the length of $I_n(x)$. More precisely, for any $x \in [0, 1)$, define the lower and upper density at x for β-expansion respectively as follows :

$$\underline{D}(x) = \liminf_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n} \quad \text{and} \quad \overline{D}(x) = \limsup_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n}.$$

It is known [49] that for any $x \in [0, 1)$, we have

$$\underline{D}(x) = 1,\ 1 \leq \overline{D}(x) \leq 1 + \lambda(\beta). \tag{2.1.4}$$

Applying Shannon-McMillan-Breiman Theorem [79] to Parry measure [65], we obtain $\underline{D}(x) = \overline{D}(x) = 1$ for Lebesgue almost all $x \in [0, 1)$. A special case is $\lambda(\beta) = 0$. When $\lambda(\beta) = 0$, we have $\underline{D}(x) = \overline{D}(x) = 1$ which indicates that the limit of $\frac{-\log_\beta |I_n(x)|}{n}$ exists for all $x \in [0, 1)$. The set of such β with $\lambda(\beta) = 0$ in $(1, +\infty)$ is of full Lebesgue measure

[50]. Then we turn to focus on the exceptional set with respect to the upper density for the case $\lambda(\beta) > 0$. For any $1 < \delta \leq 1 + \lambda(\beta)$, define

$$D_\delta = \{x \in [0,1) : \overline{D}(x) = \delta\},$$

which is a Lebesgue null set. The points in D_δ with $1 < \delta \leq 1 + \lambda(\beta)$ are called *δ-irregular.* Fan and Wang [35] showed that

$$\dim_{\mathrm{H}} D_\delta = \frac{\lambda(\beta) + 1 - \delta}{\delta \cdot \lambda(\beta)}, \tag{2.1.5}$$

for every $1 < \delta \leq 1 + \lambda(\beta)$.

It is natural to ask what the size of the sets D_δ are in the topological viewpoint. Motivated by this, in Chapter 3, we establish the following theorem which illustrates that the extremely irregular set $D := D_{1+\lambda(\beta)} = \{x \in [0,1) : \overline{D}(x) = 1 + \lambda(\beta)\}$ is residual (i.e., is large from a topological viewpoint) for every $\lambda(\beta) > 0$. What we should notice is that not only the Lebesgue measure of D is 0, but also its Hausdorff dimension is 0. This result is somewhat similar to Olsen's work [62] on the extremely non-normal number.

Theorem 2.1.2. *Let $\beta > 1$ with $\lambda(\beta) > 0$. Then the set D is residual, in other words, $[0,1] \setminus D$ is of the first category.*

In fact, there are some irregular sets with zero measure, but residual, which implies that such sets can be large in the sense of topology. For instance, the sets containing some kinds of irregular points associated with integer expansion are proved to be of residue [2, 44, 62]. It is shown in Baek and Olsen [5] that the set of extremely non-normal points is residual. Madritsch [58] extended and generalized the results in [5] to non-normal numbers with respect to Markov partitions. Also, the non-normal numbers in dynamical system fulfilling the specification property are residual (see Madritsch and Petrykiewicz [59]). However, in the research of non-normal number, the frequencies of digits and blocks were investigated. In our case, the upper density $\overline{D}(x)$ cannot be expressed as some frequencies.

2.2 Run-length function

For every real number $x \in [0,1)$, and every integer $n \geq 1$, the *run-length function*, denoted by $r_n(x,\beta)$, is the maximal length of consecutive zeros amongst the first n-digits of the β-expansion of x which is deonted by $(\varepsilon_1(x,\beta), \ldots, \varepsilon_n(x,\beta))$, that is,

$$r_n(x,\beta) = \max\{j \geq 1 : \varepsilon_{i+1}(x,\beta) = \cdots = \varepsilon_{i+j}(x,\beta) = 0 \text{ for some } 0 \leq i \leq n-j\}.$$

We set $r_n(x,\beta)=0$ if such j does not exist. For the base $\beta=2$, Erdös and Rényi [26] showed that for Lebesgue almost all $x\in[0,1)$,

$$\lim_{n\to\infty}\frac{r_n(x,2)}{\log_2 n}=1. \tag{2.2.1}$$

The size of sets about the function $r_n(x,2)$ has raised much attention. Ma, Wen and Wen [56] proved that the set of points violating (2.2.1) is of full dimension. Subsequently, Li and Wu [51, 52] replaced the function $n\mapsto\log_2 n$ in (2.2.1) by a monotonically increasing function $\varphi:\mathbb{N}\to\mathbb{R}^+$ with $\lim\limits_{n\to\infty}\varphi(n)=+\infty$, and they introduced the extremely exceptional set related to φ which contains worst divergence points as

$$E_{\max}=\left\{x\in[0,1):\liminf_{n\to\infty}\frac{r_n(x,2)}{\varphi(n)}=0,\ \limsup_{n\to\infty}\frac{r_n(x,2)}{\varphi(n)}=+\infty\right\}.$$

They first got a weaker conclusion that the Hausdorff dimension of $E_{\max}$ is 1 and $E_{\max}$ is residual in $[0,1]$ when the monotonically increasing function $\varphi(n)$ satisfying $\lim\limits_{n\to\infty}\frac{n}{\varphi(n^{1+\alpha})}=+\infty$ for some $0<\alpha\le 1$, see [51] for more details. After that, in [52], they showed that $E_{\max}$ has Hausdorff dimension 1 and is of residue under the condition $\lim\limits_{n\to\infty}\frac{n}{\varphi(n)}=+\infty$. If we let $\varphi(n)=\log_2 n$ in $E_{\max}$, the result that $\dim_{\mathrm{H}} E_{\max}=1$ in [51] is somewhat surprising since this set is much smaller than the set which was studied by Ma, Wen and Wen [56]. Naturally, it is of interest to consider whether the above properties will be true if 2 is substituted for a general real number $\beta>1$. As a matter of fact, Tong, Yu and Zhao [77] gave a similar result as Erdörs and Rényi [26] that for Lebesgue almost all $x\in[0,1)$, we have

$$\lim_{n\to\infty}\frac{r_n(x,\beta)}{\log_\beta n}=1.$$

Thus, the set

$$E=\left\{x\in[0,1):\liminf_{n\to\infty}\frac{r_n(x,\beta)}{\log_\beta n}<\limsup_{n\to\infty}\frac{r_n(x,\beta)}{\log_\beta n}\right\}$$

has null Lebesgue measure. By using the monotonically increasing function $\varphi:\mathbb{N}\to\mathbb{R}^+$ with $\lim\limits_{n\to\infty}\varphi(n)=+\infty$ instead of the function $n\mapsto\log_\beta n$, we take the exceptional set containing the worst divergence points as follows into consideration, that is,

$$E^{\varphi}_{\max}=\left\{x\in[0,1):\liminf_{n\to\infty}\frac{r_n(x,\beta)}{\varphi(n)}=0,\ \limsup_{n\to\infty}\frac{r_n(x,\beta)}{\varphi(n)}=+\infty\right\}. \tag{2.2.2}$$

We extend Li and Wu's results (see [51, 52]) by generalizing the base 2 into every real number $\beta>1$, which can be expressed as the following theorems.

Theorem 2.2.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be an increasing function with $\lim\limits_{n\to\infty} \varphi(n) = +\infty$. Let $E_{\max}^{\varphi}$ be the set defined as (2.2.2), then*
(1) If $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} < +\infty$, we have $E_{\max}^{\varphi} = \emptyset$;
(2) If $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} = +\infty$, we have $\dim_{\mathrm{H}} E_{\max}^{\varphi} = 1$.

Remark 2.2.2. The result of (1) in Theorem 2.2.1 is obvious since if $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} < +\infty$, the fact that $r_n(x,\beta) \leq n$ for all $n \geq 1$ and $x \in [0,1)$ gives that

$$\limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} \cdot \frac{n}{\varphi(n)} \leq \limsup_{n\to\infty} \frac{n}{\varphi(n)} < +\infty.$$

Thus, $E_{\max}^{\varphi} = \emptyset$. So we only need to show (2) in Theorem 2.2.1 in this paper.

It occurs naturally to know how large the set $E_{\max}^{\varphi}$ is in the topological sense. Hence, we establish the following theorem.

Theorem 2.2.3. *Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be an increasing function with $\lim\limits_{n\to\infty} \varphi(n) = +\infty$. Let $E_{\max}^{\varphi}$ be the set defined as (2.2.2), we have $E_{\max}^{\varphi}$ is residual when $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} = +\infty$.*

Setting $\varphi(n) = \log_\beta n$ in the set $E_{\max}^{\varphi}$ given as (2.2.2) and combining the results of Tong, Yu and Zhao [77], we have the following corollary. This result gives an example that a set can be very small in the sense of topology but be large from the measure-theoretical and dimensional points of view.

Corollary 2.2.4. *The set $\{x : \lim\limits_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = 1\}$ is both of full measure and of the first category.*

An important point should be noticed here is that when β is an integer, we find that Li and Wu's results (see [51, 52]) for the Hausdorff dimension and category of $E_{\max}$ can be extended. This is because the basic intervals of order n equal to β^{-n} which implies that all of the admissible words can be concatenated with each other. When β is not an integer, the dynamical behaviors will be much more complicated for different classes of $\beta > 1$, for example, one class is the set of $\beta > 1$ for which the corresponding subshift satisfies specification property, the other one is the complement of such set. For $\beta > 1$ in the first class, the admissibility of concatenating admissible words can be ensured by adding uniformly finitely many zeros and the basic intervals of order n are equivalent to β^{-n}. However, for $\beta > 1$ in the second class, the above properties of admissible words and basic intervals vanish, which causes lots of difficulties in constructing Cantor sets and dense sets contained in $E_{\max}^{\varphi}$ and checking the condition of modified mass distribution principle (see Lemma 3.1.8). There are many results related to run-length function, see [37, 53, 54, 73] and the references given there.

2.3 Run-length function related to Diophantine approximation

In section 2.3, we can see that the set with extremely worst divergence points

$$E_{\max}^{\varphi} = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = 0,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = +\infty \right\}$$

is either empty or of full Hausdorff dimension and residual in $[0,1)$ according to $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} < +\infty$ or not. For all $0 \leq a \leq b \leq +\infty$, let

$$E_{a,b}^{\varphi} = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = a,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = b \right\}.$$

In fact, [34] provides a more general results on the set $E_{a,b}^{\varphi}$ that $\dim_{\mathrm{H}} E_{a,b}^{\varphi} = 1$ where the strictly increasing function φ satisfies that $\liminf\limits_{n\to\infty} \frac{\varphi(n)}{n} = 0$ and $\frac{\varphi(n)}{n}$ is non-increasing. It naturally occurs to a problem that what the size of the set $E_{a,b}^{\varphi}$ when the function φ has the same order of n (i.e., $\lim\limits_{n\to\infty} \frac{\varphi(n)}{n} = c$ for some $0 < c < \infty$).

For all $0 \leq a \leq b \leq 1$, we are going to consider the size of the set

$$E_{a,b} := E_{a,b}(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\}. \tag{2.3.1}$$

Actually, the asymptotic behavior of $\frac{r_n(x,\beta)}{n}$ is directly related to the Diophantine approximation of β-expansions. For all $x \in [0,1)$, Bugeaud and Liao [18] defined the exponent $v_\beta(x)$ to be the supremum of the real numbers v for which the equation

$$T_\beta^n x \leq \beta^{-nv}$$

has infinitely many positive solutions with integer n. They also defined the exponent $\hat{v}_\beta(x)$ to be the supremum of the real numbers $\hat{v}$ for which, for all $N \gg 1$, there is a solution with $1 \leq n \leq N$, such that

$$T_\beta^n x \leq \beta^{-N\hat{v}}.$$

We will see (Lemmas 6.1.1 and 6.1.2) that for all $0 < a < 1,\ 0 < b < 1$,

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a \ \Leftrightarrow\ \hat{v}_\beta(x) = \frac{a}{1-a} \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \ \Leftrightarrow\ v_\beta(x) = \frac{b}{1-b}.$$

By this relationship, we provide the following result.

Theorem 2.3.1. *The set $E_{0,0}$ has full Lebesgue measure. If $\frac{b}{1+b} < a \leq 1,\ 0 < b \leq 1$, then $E_{a,b} = \emptyset$. Otherwise, we have*

$$\dim_{\mathrm{H}} E_{a,b} = 1 - \frac{b^2(1-a)}{b-a}.$$

Let $0 \le a \le 1$ and $0 \le b \le 1$. We can further study the level sets

$$E_a := E_a(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a \right\}$$

and

$$F_b := F_b(\beta) = \left\{ x \in [0,1) : \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\}. \tag{2.3.2}$$

Using Theorem 2.3.1, we obtain the following results of the Hausdorff dimensions of E_a and F_b.

Corollary 2.3.2. *(1) When $0 \le a \le \frac{1}{2}$, we have*

$$\dim_{\mathrm{H}} E_a = (1-2a)^2.$$

Otherwise, $E_a = \emptyset$.
(2) For all $0 \le b \le 1$, we have

$$\dim_{\mathrm{H}} F_b = 1 - b.$$

We remak that the statement (2) of Corollary 2.3.2 was also been obtained in [53, Theorem 1.1] (see [80] for the case $\beta = 2$).

Similar to the results of Li and Wu [51, 52], the set of extremely divergent points is residual, and thus is large in the sense of topology.

Theorem 2.3.3. *The set $E_{0,1}$ is residual in* $[0,1]$.

It is worth noting that the set $E_{0,1}$ is negligible with respect to the Lebesgue measure and Hausdorff dimension. However, the sets considered in [34, 51, 52] have Hausdorff dimension 1. Let φ be a monotonically increasing function with $\lim_{n\to\infty} \varphi(n) = +\infty$ and $\liminf_{n\to\infty} \frac{\varphi(n)}{n} = 0$. Since the intersection of two residual sets is still residual, by combining Theorems 2.3.3 and 2.2.3, we deduce that the smaller set

$$E := E(\varphi,\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = 0,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = 1 \right\}$$

is also residual in $[0,1]$.

The β-expansion of 1 completely characterizes all of the admissible words in the β-dynamical system (see Theorem 3.2.4 in Section 3 for more details). We also study the run-length function $r_n(\beta)$ of the β-expansion of 1 as β varies in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$, i.e.,

$$r_n(\beta) = \max\{1 \le j \le n : \varepsilon_{i+1}(1,\beta) = \cdots = \varepsilon_{i+j}(1,\beta) = 0 \text{ for some } 0 \le i \le n-j\}.$$

There are some results on $r_n(\beta)$ which are similar to those of $r_n(x,\beta)$. In [43], Hu, Tong and Yu proved that for Lebesgue almost all $1<\beta<2$, we have

$$\lim_{n\to\infty}\frac{r_n(\beta)}{\log_\beta n}=1.$$

Cao and Chen [21] showed that for any φ which is a monotonically increasing function with $\lim\limits_{n\to\infty}\varphi(n)=+\infty$ and $\liminf\limits_{n\to\infty}\frac{\varphi(n)}{n}=0$, for all $0\le a\le b\le+\infty$, the set

$$\left\{\beta\in(1,2):\liminf_{n\to\infty}\frac{r_n(\beta)}{\varphi(n)}=a,\ \limsup_{n\to\infty}\frac{r_n(\beta)}{\varphi(n)}=b\right\}$$

is of full Hausdorff dimension. Remark that the results of [21] and [43] can be easily generalized to the whole parameter space $\{\beta\in\mathbb{R}:\beta>1\}$. For simplicity, in this paper, we will also consider the parameter space $(1,2)$. For all $0\le a\le b\le 1$, let

$$E^P_{a,b}=\left\{\beta\in(1,2):\liminf_{n\to\infty}\frac{r_n(\beta)}{n}=a,\ \limsup_{n\to\infty}\frac{r_n(\beta)}{n}=b\right\}. \tag{2.3.3}$$

We have the following theorem.

Theorem 2.3.4. *The set $E^P_{0,0}$ has full Lebesgue measure. If $\frac{b}{1+b}<a\le 1,\ 0<b\le 1$, then $E^P_{a,b}=\emptyset$. Otherwise, we have*

$$\dim_{\mathrm{H}} E^P_{a,b}=1-\frac{b^2(1-a)}{b-a}.$$

Similarly, for every $0\le a\le 1$ and $0\le b\le 1$, we consider the set

$$E^P_a=\left\{\beta\in(1,2):\liminf_{n\to\infty}\frac{r_n(\beta)}{n}=a\right\},$$

and

$$F^P_b=\left\{\beta\in(1,2):\limsup_{n\to\infty}\frac{r_n(\beta)}{n}=b\right\}.$$

Corollary 2.3.5. *(1) When $0\le a\le\frac{1}{2}$, we have*

$$\dim_{\mathrm{H}} E^P_a=(1-2a)^2.$$

Otherwise, $E^P_a=\emptyset$.

(2) For every $0\le b\le 1$, we have

$$\dim_{\mathrm{H}} F^P_b=1-b.$$

In addition, similar to Theorem 2.3.3, we have the following theorem.

Theorem 2.3.6. *The set $E^P_{0,1}$ is residual in $[1,2]$.*

2.4 Uniform Recurrence Properties

Poincaré Recurrence Theorem is one of the most fundamental results in dynamical system. It is stated that there is a non-trivial recurrence to a measurable set with positive measure in any measure-theoretical dynamical system. Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system with a finite Borel measure μ. Let d be a metric on X. The well-known Poincaré Recurrence Theorem shows that typically the orbit of a point asymptotically approaches to the point itself. More precisely,

$$\liminf_{n\to\infty} d(T^n x, x) = 0$$

for μ-almost all $x \in X$. Boshernitzan [14] described the speed of such asymptotic recurrence. In fact, he proved that if there is some $\alpha > 0$ such that the α-dimensional Hausdorff measure $\mathcal{H}^\alpha$ is σ-finite on X (i.e., X can be written as a countable union of subsets X_i with $\mathcal{H}^\alpha(X_i) < \infty$ for all $i = 1, 2, \ldots$), then

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) < \infty \text{ for } \mu\text{-almost all } x \in X.$$

Moreover, if $\mathcal{H}^\alpha(X) = 0$, then

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) = 0 \text{ for } \mu\text{-almost all } x \in X.$$

There are also many other studies on the asymptotic behavior of the orbits motivated by Poincaré Recurrence Theorem including the first return time [7], dynamical Borel-Cantelli Lemma [23], waiting time [36], shrinking target problems [39, 40, 72] and so on.

Different to the asymptotic way of approximation, the famous Dirichlet Theorem provides another point of view of the study on the approximation of the orbits : a uniform way. The Dirichlet Theorem states that for any positive irrational real number θ, for all real number $N \geq 1$, there is an integer n with $1 \leq n \leq N$ satisfying

$$\|n\theta\| < N^{-1}, \tag{2.4.1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. The uniformity lies in that (2.4.1) has an integer solution for any sufficiently large N. Note that $\|n\theta\| = \|T_\theta^n x - x\|$ where $T_\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is defined by $T_\theta x = x + \theta$. Thus, the Dirichlet Theorem can be explained as that under the dynamical system $(\mathbb{R}/\mathbb{Z}, T_\theta)$, all points x uniformly return to the point itself with the speed $\frac{1}{N}$. Motivated by the Dirichlet Theorem, some results of the uniform approximation properties have already appeared in [17, 18, 19, 47, 48].

In our paper, we want to investigate the uniform recurrence property of a point to itself in the beta-dynamical system. We aims at giving the sizes (Lebesgue measure and Huasdorff dimension) of the sets of points with prescribed uniform recurrence rate.

In our paper, we consider the following two exponents of recurrence, one is for asymptotic recurrence, and the other is for uniform recurrence.

Definition 2.4.1. Let $\beta > 1$. For all $x \in [0, 1)$, define

$$r_\beta(x) := \sup\{0 \leq r \leq +\infty : |T_\beta^n x - x| < (\beta^n)^{-r} \text{ for infinitely many } n \in \mathbb{N}\}$$

and

$$\hat{r}_\beta(x) := \sup\{0 \leq \hat{r} \leq +\infty : \text{ for all } N \gg 1, \text{ there is } n \in [1, N], \text{ s.t. } |T_\beta^n x - x| < (\beta^N)^{-\hat{r}}\}.$$

The exponents $r_\beta(x)$ and $\hat{r}_\beta(x)$ are analogous to the exponents introduced in [3], see also [17, 18]. By the definitions of $r_\beta(x)$ and $\hat{r}_\beta(x)$, it can be checked that $\hat{r}_\beta(x) \leq r_\beta(x)$ for all $x \in [0, 1)$. Actually, applying Philipp's result [67], we can deduce that the set $\{x : r_\beta(x) = 0\}$ is of full Lebesgue measure (see Section 7.1.1). The asymptotic exponent $r_\beta(x)$ has been studied by Tan and Wang [74] who showed that for all $0 \leq r \leq +\infty$,

$$\dim_{\mathrm{H}} \{x \in [0, 1) : r_\beta(x) \geq r\} = \frac{1}{1 + r}, \tag{2.4.2}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension and $\frac{1}{+\infty} = 0$. We refer the readers to Falconer [27] for more properties of Hausdorrf dimension. Our main result is as follows.

Theorem 2.4.2. *Let $\beta > 1$. The set $\{x \in [0, 1) : \hat{r}_\beta(x) = 0\}$ is of full Lebesgue measure. When $\hat{r} > 1$, the set $\{x \in [0, 1) : \hat{r}_\beta(x) \geq \hat{r}\}$ is countable. When $0 \leq \hat{r} \leq 1$, we have*

$$\dim_{\mathrm{H}}\{x \in [0, 1) : \hat{r}_\beta(x) \geq \hat{r}\} = \dim_{\mathrm{H}}\{x \in [0, 1) : \hat{r}_\beta(x) = \hat{r}\} = \left(\frac{1 - \hat{r}}{1 + \hat{r}}\right)^2.$$

Actually, Theorem 2.4.2 follows from the following more general result which gives the Hausdorff dimension of the set of points whose exponents $r_\beta(x)$ and $\hat{r}_\beta(x)$ are both prescribed. For all $0 \leq r \leq +\infty,\ 0 \leq \hat{r} \leq +\infty$, let

$$R_\beta(\hat{r}, r) := \{x \in [0, 1) : \hat{r}_\beta(x) = \hat{r},\ r_\beta(x) = r\}.$$

Theorem 2.4.3. *Let $\beta > 1$. The set $R_\beta(0, 0)$ is of full Lebesgue measure. When $0 \leq \frac{r}{1+r} < \hat{r} \leq +\infty$, the set $R_\beta(\hat{r}, r)$ is countable. When $0 \leq \hat{r} \leq \frac{r}{1+r},\ 0 < r \leq +\infty$, we have*

$$\dim_{\mathrm{H}} R_\beta(\hat{r}, r) = \frac{r - (1 + r)\hat{r}}{(1 + r)(r - \hat{r})}.$$

Where $\frac{+\infty}{(+\infty)(+\infty)} = 0$

By Theorem 2.4.3, the following new result related to the asymptotic exponent $r_\beta(x)$ is immediate.

Corollary 2.4.4. *Let $\beta > 1$. For all $0 \leq r \leq +\infty$, we have*

$$\dim_{\mathrm{H}} \{x \in [0, 1) : r_\beta(x) = r\} = \frac{1}{1+r}.$$

Chapitre 3 Preliminaries

3.1 Measure and dimension

The notion of fractal dimensions (Hausdorff dimension, boxing dimension, packing dimension and so on) are of great importance in the study of fractal geometry (see Falconer [27]). Roughly speaking, dimension determines the size of space that a set occupies near to each of its points. In this section we first recall briefly the definitions of Hausdorff dimension, boxing dimension, packing dimension and give some properties of these dimensions. In the final of this section, we will give a classical technique to calculate the Hausdorff dimension of a set. For more information of dimensions and measures, we refer the readers to see Falconer [28, 27, 29, 30], Falconer and Howroyd [31], Federer [32, 33], Howroyd [42], Mattila [60], Rogers [69] and the references therein.

3.1.1 Hausdorff measure and Hausdorff dimension

Hausdorff dimension has the advantage of being defined for any set since it is based on the Hausdorff measure which is relatively easy to operate. We first give the definition of Hausdorff measure and then give the definition of Hausdorff dimension.

Let $U \subseteq \mathbb{R}^n$ be non-empty where $\mathbb{R}^n$ is the n-dimensional Euclidean space. The diameter of U is the largest distance between any pair of points in U, that is

$$|U| = \sup\{|x-y| : x, y \in U\}.$$

Let s be a non-negative real number and $E \subseteq \mathbb{R}^n$. For each $\delta > 0$, the *δ-covering of E*, written as $\{U_i\}_{i=1}^{\infty}$, is a countable (or finite) family of sets of diameter less than δ which covers E, that is, $E \subseteq \bigcup_{i=1}^{\infty} U_i$ and $|U_i| \leq \delta$. Let

$$\mathcal{H}_\delta^s(E) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \bigcup_{i=1}^{\infty} U_i \text{ is the } \delta-\text{covering of } E\right\}.$$

Consequently, $\mathcal{H}_\delta^s(E)$ is the minimum of the sum of the sth powers of the diameters of the sets covering E whose diameter is less than δ. A quick calculation shows that the infimum $\mathcal{H}_\delta^s(E)$ increases when δ tends to 0. The *s-dimensional Hausdorff measure of F* is defined by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(E).$$

This limit exists for any $E \subseteq \mathbb{R}^n$ and the limit can be (and generally is) 0 or ∞. We can check that $\mathcal{H}^s$ is a measure. Hausdorff measure satisfies the propeties as follows.

Proposition 3.1.1 (Falconer [27]). *Let E and F be two subsets of $\mathbb{R}^n$.*
(1) We have $\mathcal{H}^s(\emptyset) = 0$.
(2) If $E \subseteq F$, then $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$.
(3) Let $\{E_i\}_{i=1}^{\infty}$ be a countable sequence of subsets in $\mathbb{R}^n$, then

$$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(E_i).$$

(4) If E is a Borel subset of $\mathbb{R}^n$, there is a constant c such that

$$\mathcal{H}^s(E) = \frac{\mathrm{vol}^n(E)}{c}$$

where $\mathrm{vol}^n(E)$ is the n-dimensional Lebesgue measure (i.e., the usual n-dimensional volume).
(5) (Scaling property) Let f be a similarity function with scale factor $\lambda > 0$. Then

$$\mathcal{H}^s(f(E)) = \lambda^s \mathcal{H}^s(E).$$

(6) Let f be the function mapping from E to $\mathbb{R}^n$ such that

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}, \ \forall\ x, y \in E,$$

for some constants $c > 0$ and $\alpha > 0$. It holds that for all s,

$$\mathcal{H}^{\frac{s}{\alpha}}(f(E)) \leq c^{\frac{s}{\alpha}} \mathcal{H}^s(E).$$

(7) (Translation and rotation invariant) Let f be an isometry (i.e., $|f(x) - f(y)| = |x - y|$), then

$$\mathcal{H}^s(f(E)) = \mathcal{H}^s(E).$$

Hence,

$$\mathcal{H}^s(E + z) = \mathcal{H}^s(E)$$

where $E + z = \{y + z : y \in E\}$.
(8) (Criticality) Let $0 \leq s < t < \infty$. We have the following results.
(I) If $\mathcal{H}^s(E) < +\infty$, then $\mathcal{H}^t(E) = 0$.
(II) If $\mathcal{H}^t(E) > 0$, then $\mathcal{H}^s(E) = +\infty$.

For all $E \subseteq \mathbb{R}^n$, the criticality of Hausdorff measure (Proposition 3.1.1) shows that there exits a critical value of s such that $\mathcal{H}^s(E)$ "jumps" from infinity to 0. We call this critical value, denote by $\dim_{\mathrm{H}}(E)$, the *Hausdorff dimension of* E. Namely,

$$\begin{aligned}\dim_{\mathrm{H}} E &= \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\} \\ &= \inf\{s : \mathcal{H}^s(E) < \infty\} = \sup\{s : \mathcal{H}^s(E) > 0\}.\end{aligned}$$

Now we display some properties of Hausdorff dimension which is important to the calculation of Hausdorff dimension.

Proposition 3.1.2 (Falconer [27]). *Let* $E, F \subseteq \mathbb{R}^n$.
(1) (Monotonicity) If $E \subseteq F$*, we have* $\dim_{\mathrm{H}} E \leq \dim_{\mathrm{H}} F$.
(2) (Countable stability) Let $\{E_i\}_{i=1}^{\infty}$ *be a countable collection of sets, then*

$$\dim_{\mathrm{H}}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup\{\dim_{\mathrm{H}} E_i : 1 \leq i \leq \infty\}.$$

(3) (Countable sets) If E *is countable then* $\dim_{\mathrm{H}} E = 0$.
(4) (Open sets) If E *is open, then* $\dim_{\mathrm{H}} E = n$.
(5) (Smooth sets) If E *is a continuously differentiable* m*-dimensional submanifold (i.e.* m*-dimensional surface) of* $\mathbb{R}^n$*, then* $\dim_{\mathrm{H}} E = m$.
(6) (Transformation property) Let $E \subseteq \mathbb{R}^n$*. Assume that* $f : E \to \mathbb{R}^m$ *be the function verifies*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}, \ \forall\ x, y \in E$$

for constants $c > 0$ *and* $\alpha > 0$*. Then* $\dim_{\mathrm{H}} f(E) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} E$*. Furthermore, if* f *is a Lipschitz transformation satisfying that there is a constant* c*, such that*

$$|f(x) - f(y)| \leq c|x - y|$$

for any $x, y \in E$*, then* $\dim_{\mathrm{H}} f(E) \leq \dim_{\mathrm{H}} E$*. If* f *is a bi-Lipschitz transformation, that is,*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|, \ \forall\ x, y \in E$$

for some $0 < c_1 \leq c_2 < \infty$*, then* $\dim_{\mathrm{H}} f(E) = \dim_{\mathrm{H}} E$.

3.1.2 Box-counting dimension and packing dimension

Box-counting, also called Minkowski dimension or Minkowski – Bouligand dimension, is introduced by Bouligand [15] in 1929. It is one of the most popular dimensions using in fractal geometry. It is used widely since it is relatively easy to calculate mathematically and estimate empirically.

For all $r > 0$ and any non-empty bounded set $E \subseteq \mathbb{R}^N$. Suppose that $N_r(E)$ is the smallest number of balls with radius r in Euclidean metric which can cover E. The *upper and lower box-couting dimension of* E, written as $\overline{\dim}_{\mathrm{B}}E$ and $\underline{\dim}_{\mathrm{B}}E$ respectively, is given by

$$\overline{\dim}_{\mathrm{B}}E = \limsup_{r\to 0} \frac{\log N_r(E)}{-\log r},$$

and

$$\underline{\dim}_{\mathrm{B}}E = \liminf_{r\to 0} \frac{\log N_r(E)}{-\log r}.$$

If $\overline{\dim}_{\mathrm{B}}E = \underline{\dim}_{\mathrm{B}}E$, the common value, denoted by $\dim_{\mathrm{B}}E$, is called the *box-counting dimension of* E. There are some equivalent definitions of box-countable dimensions, for instance, $N_r(E)$ can be defined as the greatest number of disjoint closed balls of radius r with center in E, see [27] for more details. Ir can be checked that, for all $E \subseteq \mathbb{R}^n$, we have

$$\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}}E \leq \overline{\dim}_{\mathrm{B}}E.$$

Box-counting dimension verifies the following properties.

Proposition 3.1.3 (Falconer [27])**.** *Let* $E, F \subseteq \mathbb{R}^n$*.*
(1) (Monotonicity) If $E \subseteq F$*, then*

$$\underline{\dim}_{\mathrm{B}}E \leq \underline{\dim}_{\mathrm{B}}F, \quad \overline{\dim}_{\mathrm{B}}E \leq \overline{\dim}_{\mathrm{B}}F.$$

(2) (Smooth sets) If E *is a smooth* m*-dimensional submanifold of* $\mathbb{R}^n$*, then*

$$\dim_{\mathrm{B}} F = m.$$

(3) (Finite stability) The upper box-counting dimension is finitely stable, that is

$$\overline{\dim}_{\mathrm{B}}(E \cup F) = \max\left\{\overline{\dim}_{\mathrm{B}}E,\ \overline{\dim}_{\mathrm{B}}F\right\}.$$

What should be noted here is that the corresponding identity does not hold for lower box-counting dimension and both the upper and lower box-counting dimensions do not have countable stability.
(4) (Transformation property) Both $\underline{\dim}_{\mathrm{B}}$ *and* $\overline{\dim}_{\mathrm{B}}$ *are bi-Lipschitz invariant. That is, if* f *is a bi-Lipschitz, then*

$$\underline{\dim}_{\mathrm{B}}f(E) = \underline{\dim}_{\mathrm{B}}E,\ \overline{\dim}_{\mathrm{B}}f(E) = \overline{\dim}_{\mathrm{B}}E.$$

(5) Denote the closure of E *(i.e., the smallest closed subset of* $\mathbb{R}^n$ *containing* E*) by* $\overline{E}$*. We have*

$$\underline{\dim}_{\mathrm{B}}\overline{E} = \underline{\dim}_{\mathrm{B}}E,\ \overline{\dim}_{\mathrm{B}}\overline{E} = \overline{\dim}_{\mathrm{B}}E.$$

3.1.3 Packing measure and packing dimension

Packing measure and packing dimension on $(\mathbb{R}^n, |\cdot|)$ were first introduced by Tricot [78] and then studied by Taylor and Tricot [75] in the early 1980s which play a dual role to Hausdorff measure and Hausdorff dimension. Falconer [27] and Mattila [60] contain systematic accounts between the Hausdorff measure and packing measure, Hausdorff dimension and packing dimension.

Let $E \subseteq \mathbb{R}^n$ and $\delta > 0$. The δ-packing of set E is a family of disjoint open balls of radius smaller than δ with center in E. For all $s \geq 0$, let

$$\mathcal{P}^s_\delta(E) = \sup\left\{\sum_{i\geq 1} |U_i|^s : \{U_i\}_{i=1}^\infty \ is\ a\ \delta - packing\ of\ E.\right\}.$$

Note that $\mathcal{P}^s_\delta(E)$ decreases when δ goes to 0, the limit of $\mathcal{P}^s_\delta(E)$ exists, let

$$\mathcal{P}^s_0(F) = \lim_{\delta\to 0} \mathcal{P}^s_\delta(F).$$

We can see that $\mathcal{P}^s_0$ is not a measure since σ-additive property does not hold. We define the *s-packing measure of E* by

$$\mathcal{P}^s(E) = \inf\left\{\sum_{i=1}^\infty \mathcal{P}^s_0(E_i) : E \subseteq \bigcup_{i=1}^\infty E_i\right\}.$$

Thus, the *packing dimension of E* is defined by

$$\begin{aligned}\dim_{\mathrm{P}} E &= \inf\{s : \mathcal{P}^s(E) = 0\} = \sup\{s : \mathcal{P}^s(E) = \infty\}\\ &= \inf\{s : \mathcal{P}^s(E) < \infty\} = \sup\{s : \mathcal{P}^s(E) > 0\}.\end{aligned}$$

Similar to Hausdorff dimension, packing dimension also has the following elementary properties.

Proposition 3.1.4 (Falconer [27]). *Let $E, F \subseteq \mathbb{R}^n$.*
(1) (Monotonicity) If $E \subseteq F$, we have $\dim_{\mathrm{P}} E \leq \dim_{\mathrm{P}} F$.
(2) (Countable stability) Let $\{E_i\}_{i=1}^\infty$ be a countable collection of sets, then

$$\dim_{\mathrm{P}}\left(\bigcup_{i=1}^\infty E_i\right) = \sup\{\dim_{\mathrm{P}} E_i : 1 \leq i \leq \infty\}.$$

(3) (Countable sets) If E is countable then $\dim_{\mathrm{P}} E = 0$.
(4) (Open sets) If E is open, then $\dim_{\mathrm{P}} E = n$.
(5) (Smooth sets) If E is a continuously differentiable m-dimensional submanifold (i.e. m-dimensional surface) of $\mathbb{R}^n$, then $\dim_{\mathrm{P}} E = m$.
(6) (Transformation property) If f is a Lipschitz transformation, then $\dim_{\mathrm{P}} f(E) \leq \dim_{\mathrm{P}} E$. *If f is a bi-Lipschitz transformation, then* $\dim_{\mathrm{P}} f(E) = \dim_{\mathrm{P}} E$.

The relationship of Hausdorff dimension, boxing dimension and packing dimension is as follows.

Proposition 3.1.5 (Falconer [27]). *For all $F \subseteq \mathbb{R}^N$, we have*

$$\dim_{\mathrm{H}} F \leq \dim_{\mathrm{P}} F \leq \overline{\dim}_{\mathrm{B}} F.$$

We also have the following basic facts on the boxing dimension and packing dimension.

Proposition 3.1.6 (Falconer [27]). *(1) If a set $F \subset \mathbb{R}^n$ is of second category, then $\dim_{\mathrm{P}} F = n$.*
(2) If a set $F \subset \mathbb{R}^n$ is dense, then $\dim_{\mathrm{P}} F = n$.

3.1.4 Calculation of Hausdorff dimension

As is discussed in Section 2.1, Hausdorff dimension characterize the size of the set in $\mathbb{R}^n$. In this section, we provide some basic techniques to calculate Hausdorff dimension. Specially, we give a typical result which is a useful tool to estimate the lower bound of Hausdorff dimension. For the upper bound of Hausdorff dimension, it is enough to find a specific covering $\{U_i\}_{i=1}^{\infty}$ and evaluate $\sum\limits_{i=1}^{\infty} |U_i|^s$, however, for a lower bound we must show that the sum of $|U_i|^s$ is greater than the sum of sth power of the diameter of all δ-coverings. In particular, when working with Hausdorff dimension as opposed to box dimension, consideration must be given to covers where some of $|U_i|$ are very small and others have relatively large diameter and hence this prohibits sweeping estimates for $\sum\limits_{i=1}^{\infty} |U_i|^s$ such as those available for upper bounds. Now we give a useful theorem to estimate the lower bound.

Theorem 3.1.7 (Falconer [27]). *(Mass distribution principle) Let $E \subseteq \mathbb{R}^n$. Let μ be a mass distribution on E and suppose that for some s, there exists real numbers $c > 0$ and $\epsilon > 0$ satisfying that*

$$\mu(U) \leq c|U|^s$$

for all sets U with radius less than ϵ. Then

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{c} \text{ and } \dim_{\mathrm{H}} E \geq s.$$

Furthermore, Bugeaud and Wang [20] provided the following modified mass distribution principle which is of great importance in estimating the lower bound of the Hausdorff dimension of sets considered in the β dynamical system. For convenience, denote by I_n for all basic intervals of order n.

Theorem 3.1.8 (Bugeaud and Wang [20]). *Let μ be a Borel measure and E be a Borel measurable set with $\mu(E) > 0$. Assume that there is a constant $c > 0$ and an integer $N \geq 1$ such that for all $n \geq N$ and each basic interval I_n, the inequality $\mu(I_n) \leq c|I_n|^s$ is valid. Then,* $\dim_{\mathrm{H}} E \geq s$.

3.2 β-dynamical system

The representation of real numbers is one of the most important tool in the research of Number Theory and Dynamical Systems. The well-known representations of numbers include decimal expansions, binary expansions, continued fraction and so on. The beta-expansion is what we are concerned about in this thesis. We now give some fundamental result on the the β-dynamical system. It can be found in [12, 18, 20, 35] and the references therein.

3.2.1 Equivalence of β-dynamical system

In section 2.1, we have already know the β-transformation given by Rényi. Another typical β-transformation is given by

$$T'_\beta x = \beta x - \lceil \beta x \rceil + 1 \ (0 < x \leq 1),$$

where $\lceil x \rceil$ stands for the smallest integer no less than x. Similarly, by the iteration of T'_β, every $x \in (0, 1]$ can be written as :

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \cdots + \frac{\varepsilon_n(x,\beta) + T'^n_\beta x}{\beta^n} = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x,\beta)}{\beta^n},$$

where, for each $n \geq 1$,

$$\varepsilon_n(x,\beta) = \lceil \beta T'^{n-1}_\beta x \rceil - 1.$$

The transformation T'_β guarantees that every $x \in (0, 1]$ has an infinite series expansion, i.e., $\varepsilon_n(x,\beta) \neq 0$ for infinitely many $n \in \mathbb{N}$. This is because $T'_\beta(x)$ is strictly larger than 0. As a mater of fact, the β-expansions under the above two transformation coincide except at the points with a finite expansion under the algorithm T_β. As a consequence, the results will be consistent when we consider the Lebesgue measure, Hausdorf dimension or topological properties under the two transformations. The transformation T_β generates the beta-dynamical system $([0, 1), T_\beta)$.

It is proved by Rényi [68] that the dynamical system $([0, 1), T_\beta)$ admits $\log \beta$ as its topological entropy as the following theorem. Here and subsequently, we denote by $\sharp$ the cardinality of a finite set.

Theorem 3.2.1 (Rényi [68])**.** *For any $n \geq 1$, we have*

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1}, \quad \lim_{n\to\infty} \frac{\sharp\Sigma_\beta^n}{n} = \log\beta.$$

Moreover, Rényi [68] showed that there is an invariant measure in the β-dynamical system as follows.

Theorem 3.2.2 (Rényi [68])**.** *There exists a unique invariant ergodic measure ν_β which is equivalent to the Lebesgue measure. More precisely, there is a constant $c_\beta := 1 - \frac{1}{\beta}$ such that, for any Borel measurable set B, we have*

$$c_\beta \nu_\beta(B) \leq \mathcal{L}(B) \leq \frac{1}{c_\beta}\nu_\beta(B). \tag{3.2.1}$$

3.2.2 Admissibility

The definition of β-expansion gives the fact that every digit $\varepsilon_n(x,\beta)$ lies in the set $\mathcal{A} = \{0, 1, \cdots, \lceil\beta\rceil - 1\}$. We can check that not every word in $\mathcal{A}^n$ is the β-expansion of some $x \in [0,1)$. For example, when $\beta = \frac{\sqrt{5}+1}{2}$ which is the golden mean, there is not $x \in [0,1)$ such that its β-expansion is $(1,1,0^\infty) \in \mathcal{A}^{\mathbb{N}}$. This leads to the definition of admissible words and sequences.

Definition 3.2.3. A word $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}^n$ is called *admissible* with respect to β if there exists an $x \in [0,1)$ such that the β-expansion of x begins with $(\varepsilon_1, \ldots, \varepsilon_n)$. Similarly, an infinite sequence $(\varepsilon_1, \ldots, \varepsilon_n, \ldots) \in \mathcal{A}^{\mathbb{N}}$ is called *admissible* with respect to β if there is a real number $x \in [0,1)$ whose β-expansion is $(\varepsilon_1, \ldots, \varepsilon_n, \ldots)$.

For convenience, denote by Σ_β^n the set of all β-admissible words of length n, i.e.,

$$\Sigma_\beta^n = \{(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}^n : \exists\ x \in [0,1),\ \text{s.t. } \varepsilon_j(x,\beta) = \varepsilon_j,\ \forall\ 1 \leq j \leq n\}.$$

Denote by Σ_β^* the set of all β-admissible words of finite length, i.e., $\Sigma_\beta^* = \bigcup\limits_{n=0}^{\infty} \Sigma_\beta^n$. The set of β-admissible sequences is denoted by Σ_β , i.e.,

$$\Sigma_\beta = \{(\varepsilon_1, \varepsilon_2, \ldots) \in \mathcal{A}^{\mathbb{N}} : \exists\ x \in [0,1),\ \text{s.t. } \varepsilon(x,\beta) = (\varepsilon_1, \varepsilon_2, \ldots)\}.$$

In order to characterize the admissible words and sequence. We endow the space $\mathcal{A}^N$ with the *lexicographical order* $<_{\text{lex}}$:

$$(\omega_1, \omega_2, \ldots) <_{\text{lex}} (\omega_1', \omega_2', \ldots)$$

if $\omega_1 < \omega_1'$ or there exists an integer $j > 1$, such that, for all $1 \leq k < j$, $\omega_k = \omega_k'$ but $\omega_j < \omega_j'$. The symbol $\leq_{\text{lex}}$ means $=$ or $<_{\text{lex}}$. Moreover, for all $n, m \geq 1$, $(\omega_1, \ldots, \omega_n) <_{\text{lex}} (\omega_1', \ldots, \omega_m')$ stands for $(\omega_1, \ldots, \omega_n, 0^\infty) <_{\text{lex}} (\omega_1', \ldots, \omega_m', 0^\infty)$.

The following theorem due to Parry [65] yields that the β-dynamical system is totally determined by the infinite β-expansion of 1. For all $\omega = (\omega_1, \omega_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$, let σ be the shift transformation such that $\sigma\omega = (\omega_2, \omega_3, \ldots)$.

Theorem 3.2.4 (Parry [65])**.** *(1) For every $n \geq 1$, $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$ if and only if $(\omega_{j+1}, \ldots, \omega_n) \leq_{\text{lex}} (\varepsilon_1^*, \ldots, \varepsilon_{n-j}^*)$ for all $0 \leq j < n$.*
(2) For any $k \geq 1$, $\sigma^k \varepsilon(1, \beta) <_{\text{lex}} \varepsilon(1, \beta)$.
(3) For all $1 < \beta_1 < \beta_2$, it holds that $\varepsilon^*(1, \beta_1) <_{\text{lex}} \varepsilon^*(1, \beta_2)$. Consequently, for every $n \geq 1$, we have

$$\Sigma_{\beta_1}^n \subseteq \Sigma_{\beta_2}^n \quad \text{and} \quad \Sigma_{\beta_1} \subseteq \Sigma_{\beta_2}.$$

3.2.3 Basic intervals

Recall the definition of basic interval defined by (2.1.1), it is shown in Fan and Wang [35] that the basic interval is left-closed and right-open as follows.

Lemma 3.2.5. *Let $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$ with $n \geq 1$. We have $I(\omega_1, \ldots, \omega_n)$ is a left-closed and right-open interval with $\frac{\omega_1}{\beta} + \cdots + \frac{\omega_n}{\beta^n}$ as its left endpoint.*

We can check that $|I_n(\omega)| \leq \beta^{-n}$ for all $\omega \in \Sigma_\beta^n$ $(n \geq 1)$, now we give the definition of full basic intervals which plays a vital part of the estimation of the length of the basic interval with order n as follows.

Definition 3.2.6. Let $N \in \mathbb{N}$. For any $\omega \in \Sigma_\beta^n$ $(n \geq 1)$. The basic interval $I_n(\omega)$ is said to be *full* if $|I_n(\omega)| = \beta^{-n}$. Moreover, the word ω is also called a full word if $I_n(\omega)$ is full.

Fan and Wang [35] gives the characterization of full intervals as follow.

Theorem 3.2.7 (Fan and Wang [35])**.** *Let $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$ with $n \geq 1$. Then the followings are equivalent :*
(1) The basic interval $I_n(\omega_1, \ldots, \omega_n)$ is full.
(2) It holds that $T_\beta^n I_n(\omega_1, \ldots, \omega_n) = [0, 1)$.
(3) For any $m \geq 1$ and any $\omega' = (\omega_1', \ldots, \omega_m') \in \Sigma_\beta^m$, the concatenation $\omega * \omega' = (\omega_1, \ldots, \omega_n, \omega_1', \ldots, \omega_m')$ is admissible.

The full intervals also have the following properties which is given by Fan and Wang [35]. For any $\ell \geq 0$, let 0^ℓ be the empty word when $\ell = 0$, and otherwise, let $0^\ell = \underbrace{0, \ldots, 0}_{\ell}$. The concatenation $\omega * \emptyset$ is defined by ω.

Theorem 3.2.8 (Fan and Wang [35]). *Let $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_\beta^n$ with $n \geq 1$.*
(1) If $(\omega_1, \dots, \omega_{n-1}, \omega_n')$ with $\omega_n' > 0$ is admissible, then $I_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ is full for any $0 \leq \omega_n < \omega_n'$.
(2) If $I_N(\omega_1, \dots, \omega_n)$ is full, then for any $(\omega_1', \dots, \omega_m') \in \Sigma_\beta^m$, we have

$$|In + m(\omega_1, \dots, \omega_n, \omega_1', \dots, \omega_m')| = |I_n(\omega_1, \dots, \omega_n)| \cdot |I_m(\omega_1', \dots, \omega_m')|.$$

(3) The basic intervals $I_{n+\Gamma_n+1}(\omega_1, \dots, \omega_n, 0^{\Gamma_n+1})$ and $I_{n+t_n+1}(\varepsilon_1^*, \dots, \varepsilon_n^*, 0^{t_n+1})$ are full.

The following inequalities on the estimation of the lengths of basic intervals will be used which follows from Theorem 3.2.8(3) (see also [49]). For all word $(\omega_1, ..., \omega_n) \in \Sigma_\beta^n$, we have

$$\beta^{-(n+\Gamma_n+1)} \leq |I(\omega_1, \dots, \omega_n)| \leq \beta^{-n},$$
$$\beta^{-(n+t_n+1)} \leq |I(\varepsilon_1^*, \dots, \varepsilon_n^*)| \leq \beta^{-(n+t_n)}. \tag{3.2.2}$$

The following theorem in Fang and Wang [35] gives a way to evaluate the length of an arbitrary basic interval $I_n(\omega_1, \dots, \omega_n)$ by comparing the suffixes of $(\omega_1, \dots, \omega_n)$ with the prefixes of β-expansion of the unit 1. For all $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_\beta^n$. Define

$$k_n^*(\omega) = \inf\{0 \leq k < n : (\omega_{k+1}, \dots, \omega_n) = (\varepsilon_1^*, \dots, \varepsilon_{n-k}^*)\}.$$

If such k does not exist, let $k_n^*(\omega) = n$. For all $x \in [0, 1)$, let

$$k_n^*(x) = k_n^*(\varepsilon_1(x, \beta), \dots, \varepsilon_n(x, \beta)). \tag{3.2.3}$$

Theorem 3.2.9 (Fan and Wang [35]). *For each $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_\beta^n$, the length of $I_n(\omega)$ satisfies*

$$|I_n(\omega)| = \beta^{-k_n^*(\omega)} \cdot |I_{n-k_n^*(\omega)}(\varepsilon_1^*, \dots, \varepsilon_{n-k_n^*(\omega)}^*)|.$$

Moreover, we will apply the following theorem due to Bugeaud and Wang [20] which is a useful tool to estimate the number of full basic intervals.

Theorem 3.2.10 (Bugeaud and Wang [20]). *There is at least one full basic interval for all $n + 1$ consecutive basic intervals of order n.*

3.2.4 Classic and uniform Diophantine Approximation on β-dynamical system

Diophantine approximation research the quantitative properties of the distribution of the orbits in a dynamical system. It devotes to studying the size of dynamically defined limsup sets in the sense of measure and dimension. In this section, we will provide some results on classic and uniform Diophantine approximation on the β-dynamical system. For more information of the classical Diophantine Approximation, the readers can refer to [17, 22, 27, 50] and the references therein. The readers can also refer to [18, 19, 48] for more properties of the uniform Diophantine approximation.

The following result due to Philipp [67] will give a useful result related to the Diophantine approximation.

Theorem 3.2.11 (Philipp [67]). *Let $\{B_n\}_{n=1}^{\infty}$ be an arbitrary sequence of intervals contained in $[0,1)$. For any positive integer N and $x \in [0,1)$, denote by $A(N,x)$ the number of positive integer $n \leq N$ such that $T_\beta^n x \in B_n$. Put*

$$\phi(N) = \sum_{n \leq N} \nu_\beta(B_n).$$

Then

$$A(N,x) = \phi(N) + O\left(\phi^{\frac{1}{2}}(N) \log^{\frac{3}{2}+\epsilon} \phi(N)\right), \ \epsilon > 0$$

for almost all $x \in [0,1)$ where the constant implied by O is an absolute constant.

Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a function with $\varphi(n) \to 0$ as $n \to +\infty$. Let $B_n = B(x, \varphi(n))$ which is an interval whose center is x with radius $\varphi(n)$. By the equivalence between ν_β and $\mathcal{L}$ (see (3.2.1)), the following corollary is straightforward.

Corollary 3.2.12. *The set*

$$\{x \in [0,1) : |T_\beta^n x - x| \leq \varphi(n) \text{ for infinitely many } n \in \mathbb{N}\}$$

is of full Lebesgue measure if $\sum\limits_{n=1}^{\infty} \varphi(n) = +\infty$. Otherwise, it is of null Lebesgue measure.

Recall the definition of the exponents $v_\beta(x)$ and $\hat{v}_\beta(x)$ related to the classic and uniform Diophantine Approximation respectively (see Section 2.4). Shen and Wang gave the following dimensional result on the sets of points with classic Diophantine Approximation.

Theorem 3.2.13 (Shen and Wang [72]). *Let $\beta > 1$. Let $0 \leq v \leq +\infty$. Then*

$$\dim_{\mathrm{H}}\{x \in [0,1) : v_\beta(x) \geq v\} = \frac{1}{1+v}.$$

Bugeaud and Liao [18] studied the set of points with uniform Diophantine properties and established the theorem as follows.

Theorem 3.2.14 (Bugeaud and Liao [18]). *Let $\beta > 1$. Let $0 < \hat{v} < 1$ and $v > 0$. If $v < \frac{\hat{v}}{1-\hat{v}}$, then the set*

$$U_\beta(\hat{v}, v) := \{x \in [0,1) : \hat{v}_\beta(x) = \hat{v},\ v_\beta(x) = v\}$$

is empty. Otherwise, we have

$$\dim_{\mathrm{H}} U_\beta(\hat{v}, v) = \frac{v - (1+v)\hat{v}}{(1+v)(v-\hat{v})}.$$

Moreover,

$$\dim_{\mathrm{H}}\{x \in [0,1) : \hat{v}_\beta(x) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2.$$

3.2.5 Approximation of β

Let $\beta > 1$. Recall that the infinite β-expansion of 1 is $\varepsilon^*(1,\beta) = (\varepsilon_1^*, \varepsilon_2^*, \ldots)$. We will apply the approximation of β to construct the Cantor subset as follows. For all N with $\varepsilon_N^* > 0$, let $\beta_N > 1$ be the unique solution of the equation :

$$1 = \frac{\varepsilon_1^*}{x} + \cdots + \frac{\varepsilon_N^*}{x^N}.$$

Then

$$\varepsilon^*(\beta_N) = (\varepsilon_1^*, \ldots, \varepsilon_N^* - 1)^\infty.$$

Hence $1 < \beta_N < \beta$ and β_N is increasing to β as N goes to infinity. The number β_N is called *an approximation of β*. Moreover, by Theorem 3.2.4(3), $\Sigma_{\beta_N}^n \subseteq \Sigma_\beta^n$ for all $n \geq 1$ and $\Sigma_{\beta_N} \subseteq \Sigma_\beta$. We therefore have the following facts.

Proposition 3.2.15 (Shen and Wang [72]). *For every $\omega \in \Sigma_{\beta_N}^n$, when regarding ω as an element of Σ_β^n, we have*

$$\beta^{-(n+N)} \leq |I_n(\omega, \beta)| \leq \beta^{-n}. \tag{3.2.4}$$

Moreover, every $\omega \in \Sigma_{\beta_N}^n (n \geq N)$ end with 0^N is full when regarding ω as an element of Σ_β^n.

3.2.6 Results of β-expansion in the parameter space

In this section, we will recall some important results of β-expansion in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$. The readers can refer to [21, 43, 50, 65, 70] for more information.

Definition 3.2.16. We call a word $\omega = (\omega_1, \ldots, \omega_n)$ self-admissible if for all $1 \le i < n$,

$$(\omega_{i+1}, \ldots, \omega_n) \le_{\text{lex}} (\omega_1, \ldots, \omega_{n-i}).$$

An infinite sequence $\omega = (\omega_1, \omega_2, \ldots)$ is called self-admissible if $\sigma^i \omega <_{\text{lex}} \omega$ for all $i \ge 1$.

Denote by Λ_n the set of all self-admissible words with length n, i.e.,

$$\Lambda_n = \{\omega = (\omega_1, \omega_2, \ldots, \omega_n) : \text{ for every } 1 \le i < n,\ \sigma^i \omega \le_{\text{lex}} (\omega_1, \ldots, \omega_{n-i})\}.$$

For convenience, for all $1 < \beta_1 < \beta_2$, let

$$\Lambda_n(\beta_1, \beta_2) = \{\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n : \exists\, \beta \in (\beta_1, \beta_2] : \text{ s.t. } \varepsilon_1(\beta) = \omega_1, \ldots, \varepsilon_n(\beta) = \omega_n\}. \tag{3.2.5}$$

The definition of self-admissible word immediately gives the following result. The proof is evident and will be omitted.

Proposition 3.2.17. *For any $m \ge n \ge 1$, let $\omega \in \Lambda_n$. Let $\beta > 1$ whose infinite β-expansion of 1 satisfy $(\varepsilon_1^*, \ldots, \varepsilon_n^*) <_{\text{lex}} \omega$. Then for all $v_1, v_2, \ldots, v_i \in \Sigma_\beta^m$ $(i \ge 1)$, the concatenation $\omega * v_1 * \cdots * v_j$ is still self-admissible for all $1 \le j \le i$.*

The characterization of the the β-expansion of 1 was given by Parry [65].

Theorem 3.2.18 (Parry [65])**.** *An infinite sequence $(\omega_1, \omega_2, \ldots)$ is the β-expansion of 1 for some $\beta > 1$ if and only if it is self-admissible.*

Now we consider the *cylinders in the parameter space* $\{\beta \in \mathbb{R} : \beta > 1\}$.

Definition 3.2.19. For any $\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n$. The cylinder $I_n^P(\omega)$ associated to ω in the parameter space is the set of $\beta \in (1, +\infty)$ whose β-expansion of 1 has the prefix $(\omega_1, \ldots, \omega_n)$, i.e.

$$I_n^P(\omega) := \{\beta \in (1, +\infty) : \varepsilon_1(1, \beta) = \omega_1, \ldots, \varepsilon_n(1, \beta) = \omega_n\}.$$

The cylinders in the parameter space are intervals (see [70, Lemma 4.1]). The length of the cylinders of $\omega \in \Lambda_n$ in the parameter space is denoted by $|I_n^P(\omega)|$. For simplicity, the left endpoint and right endpoint of $I_n^P(\omega)$ are written as $\underline{\beta}(\omega)$ and $\overline{\beta}(\omega)$ respectively.

To estimate the length of cylinders in the parameter space, we need the notion of *recurrence time* $\tau(\omega)$ (see [50]) of the self-admissible word $\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n$. Define

$$\tau(\omega) := \inf\{1 \le k < n : \sigma^k(\omega_1, \ldots, \omega_n) = (\omega_1, \ldots, \omega_{n-k})\}.$$

If we cannot find such an integer k, we set $\tau(\omega) = n$. In this case, the self-admissible word ω is said to be *non-recurrent*.

The above definition of recurrence time immediately provides the following properties. For all integer $k \geq 0$, let $\omega^k = (\underbrace{\omega, \dots, \omega}_{k})$ where $\omega^0 = \emptyset$.

Remark 3.2.20. (1) Write

$$t(\omega) := n - \left\lfloor \frac{n}{\tau(\omega)} \right\rfloor \tau(\omega).$$

Then we have

$$(\omega_1, \dots, \omega_n) = \left((\omega_1, \dots, \omega_{\tau(\omega)})^{\lfloor \frac{n}{\tau(\omega)} \rfloor}, \omega_1, \dots, \omega_{t(\omega)} \right).$$

(2) If $\omega = (\omega_1, \dots, \omega_n)$ is non-recurrent, then the word $(\omega_1, \dots, \omega_n, 0^\ell)$ is still non-recurrent for all $\ell \geq 1$.

The following result gives the upper and lower bounds of the length of the cylinder $I_n^P(\omega)$.

Lemma 3.2.21 (Schemling [70], Li, Persson, Wang and Wu [50]). *Let $\omega = (\omega_1, \dots, \omega_n) \in \Lambda_n$. We have the following inequalities :*

(1) $|I_n^P(\omega)| \leq \overline{\beta}(\omega)^{-n+1}$;

(2)

$$|I_n^P(\omega)| \geq \begin{cases} C(\omega)\overline{\beta}(\omega)^{-n}, & \text{when } t(\omega) = 0; \\ C(\omega)\overline{\beta}(\omega)^{-n} \left(\dfrac{\omega_{t(\omega)+1}}{\overline{\beta}(\omega)} + \cdots + \dfrac{\omega_{\tau(\omega)} + 1}{\overline{\beta}(\omega)^{\tau(\omega)-t(\omega)}} \right), & \text{otherwise}, \end{cases}$$

where

$$C(\omega) := \frac{(\underline{\beta}(\omega) - 1)^2}{\underline{\beta}(\omega)}. \tag{3.2.6}$$

The study of the parameter space usually concerns on the set of parameters with respect to which the approximation properties of the orbit of 1 are prescribed. Persson and Schmeling [66] proved the following result.

Theorem 3.2.22 (Persson and Schmeling [66]). *Let $v \geq 0$. Then*

$$\dim_{\mathrm{H}}\{\beta \in (1,2) : v_\beta(1) \geq v\} = \frac{1}{1+v}.$$

Analogous to Theorem 3.2.14, Bugeaud and Liao [18] obtained the following theorem in the parameter space.

Theorem 3.2.23 (Bugeaud and Liao [18]). *Let $0 < \hat{v} < 1$ and $v > 0$. If $v < \frac{\hat{v}}{1-\hat{v}}$, then the set*

$$U(\hat{v}, v) := \{\beta \in (1,2) : \hat{v}_\beta(1) = \hat{v},\ v_\beta(1) = v\}$$

is empty. Otherwise, we have

$$\dim_{\mathrm{H}} U(\hat{v}, v) = \frac{v - (1+v)\hat{v}}{(1+v)(v-\hat{v})}.$$

Moreover,

$$\dim_{\mathrm{H}}\{\beta \in (1,2) : \hat{v}_\beta(1) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2.$$

Chapitre 4 Irregular sets on the lengths of basic intervals

In topology, the notion of residual set is usually used to describe a set being large. In a metric space X, a set R is said to be *residual* if its complement is of the first category. Moreover, in a complete metric space a set is residual if it contains a dense G_δ set, see [63]. Hence, in order to prove Theorem 2.1.2, it suffices to construct a set $U \subseteq [0,1]$ verifying the following three conditions :

(1) $U \subseteq E$;

(2) U is dense in $[0,1]$;

(3) U is a G_δ set.

As is discussed in Section 1.2, We are interested in the size of the irregular sets on the lengths of basic intervals in the topological sense. We will prove that the extremely irregular set is residual in this section.

4.1 Accumulation points related to the irregular sets

To state our results, we begin by introducing some notation. For all $x \in [0,1)$, the set of accumulation points of $\frac{-\log_\beta |I_n(x)|}{n}$ as $n \nearrow \infty$ is denoted by $A(D(x))$, that is,

$$A(D(x)) = \left\{ y \in [1, 1+\lambda(\beta)] : \lim_{k\to\infty} \frac{-\log_\beta |I_{n_k}(x)|}{n_k} = y \text{ for some } \{n_k\}_{k\geq 1} \nearrow \infty \right\}.$$

For an integer $n \geq 1$, denote by $k_n^*(x)$ the largest length of the suffixes of $(\varepsilon_1(x,\beta), \dots, \varepsilon_n(x,\beta))$ agreeing with the prefix of the digit sequence of the unit 1. In other words,

$$k_n^*(x) = k_n^*(x,\beta) := \inf\{k \geq 0 : (\varepsilon_{k+1}(x,\beta), \dots, \varepsilon_n(x,\beta)) = (\varepsilon_1^*, \dots, \varepsilon_{n-k}^*)\}. \quad (4.1.1)$$

Define

$$\rho(x) = \rho(x,\beta) := \limsup_{n\to\infty} \frac{t_{n-k_n^*(x)}}{n}.$$

where t_n is defined as (2.1.2). Since $n - k_n^*(x) \leq n$ and by the definition of Γ_n (see 2.1.3), we know that $t_{n-k_n^*(x)} \leq \Gamma_n$ which implies $\rho(x) \leq \lambda(\beta)$ for any $x \in [0,1)$. Actually, we will show the upper density $\overline{D}(x)$ is exactly equal to $1 + \rho(x)$ as the following theorem which is key to check which points the set $A(D(x))$ contains.

Lemma 4.1.1. *For all $x \in [0,1)$, we have $\overline{D}(x) = 1 + \rho(x)$.*

Proof. On the one hand, by the definition of $k_n^*(x)$ (3.2.3) and Theorem 3.2.9, we have

$$|I_n(x)| = \beta^{-k_n^*(x)} \left| I_{n-k_n^*(x)} \left(\varepsilon_1^*, \ldots, \varepsilon_{n-k_n^*(x)}^*\right) \right|. \tag{4.1.2}$$

It immediately follows from Theorem 3.2.8(3) that

$$\begin{aligned} \left| I_{n-k_n^*(x)} \left(\varepsilon_1^*, \ldots, \varepsilon_{n-k_n^*(x)}^*\right) \right| &\geq \left| I_{n-k_n^*(x)+t_{n-k_n^*(x)}+1} \left(\varepsilon_1^*, \ldots, \varepsilon_{n-k_n^*(x)}^*, 0^{t_{n-k_n^*(x)}+1}\right) \right| \\ &= \beta^{-\left(n-k_n^*(x)+t_{n-k_n^*(x)}+1\right)}. \end{aligned} \tag{4.1.3}$$

Combining (4.1.2) and (4.1.3), we get

$$\overline{D}(x) \leq \limsup_{n\to\infty} \frac{n + t_{n-k_n^*(x)} + 1}{n} = 1 + \rho(x). \tag{4.1.4}$$

One the other hand, we need to find a sequence $\{n_i\}_{i\geq 1}$ satisfying $\overline{D}(x) = 1+\tau(x)$. In fact, by the definition of $\rho(x)$, we can find a sequence $\{n_i\}_{i\geq 1}$ such that $\rho(x) = \lim\limits_{i\to\infty} \frac{t_{n_i-k_{n_i}^*(x)}}{n_i}$. In addition, by (3.2.2),

$$\beta^{-\left(n_i-k_{n_i}^*(x)+t_{n_i-k_{n_i}^*(x)}+1\right)} \leq |I(\varepsilon_1^*, \ldots \varepsilon_{n-k_n^*(x)}^*)| \leq \beta^{-\left(n_i-k_{n_i}^*(x)+t_{n_i-k_{n_i}^*(x)}\right)}. \tag{4.1.5}$$

Consequently, applying (4.1.2) and (4.1.5), we deduce that

$$\frac{n_i + t_{n_i-k_{n_i}^*(x)}}{n_i} \leq \frac{-\log_\beta |I_{n_i}(x)|}{n_i} \leq \frac{n_i + t_{n_i-k_{n_i}^*(x)} + 1}{n_i},$$

that is,

$$\lim_{i\to\infty} \frac{-\log_\beta |I_{n_i}(x)|}{n_i} = 1 + \rho(x). \tag{4.1.6}$$

Combination of (4.1.4) and (4.1.6) gives the desired result. □

Now we devote to proving that the set $A(D(x))$ is always a closed interval as the following result.

Lemma 4.1.2. *For any $x \in [0,1)$, we have $A(D(x)) = [1, 1+\rho(x)]$.*

Proof. We divide the proof into two cases by showing that $A(D(x)) = \{1\}$ when $\rho(x) = 0$ and $A(D(x)) = [1, 1+\rho(x)]$ when $\rho(x) > 0$.

Case I : $\rho(x) = 0$. Note that $\underline{D}(x) = 1$ by (2.1.4) and $\overline{D}(x) = 1+\rho(x)$ by Lemma 4.1.1, then $\underline{D}(x) = \overline{D}(x) = 1$. So $A(D(x)) = \{1\}$.

Case II : $\rho(x) > 0$. It follows from (2.1.4) and Lemma 4.1.1 that $\underline{D}(x) = 1$ and $\overline{D}(x) = 1+\rho(x)$, for any $1 < a < 1+\rho(x)$, we can choose an increasing sequence $\{n_k\}_{k\geq 1}$ tending to ∞ as $k \to \infty$ such that

$$\frac{-\log_\beta |I_{n_k+1}(x)|}{n_k+1} \leq a \leq \frac{-\log_\beta |I_{n_k}(x)|}{n_k}.$$

Note that $|I_{n_k}(x)| \geq |I_{n_k+1}(x)|$. We have

$$\frac{-\log_\beta |I_{n_k}(x)|}{n_k} \leq \frac{-\log_\beta |I_{n_k+1}(x)|}{n_k} = \frac{-\log_\beta |I_{n_k+1}(x)|}{n_k+1} \cdot \frac{n_k+1}{n_k}.$$

Therefore,

$$\frac{-\log_\beta |I_{n_k+1}(x)|}{n_k+1} \leq a \leq \frac{-\log_\beta |I_{n_k+1}(x)|}{n_k+1} \cdot \frac{n_k+1}{n_k},$$

which implies $\lim\limits_{k\to\infty} \frac{-\log_\beta |I_{n_k+1}(x)|}{n_k+1} = a$. Thus $[1, 1+\rho(x)] \subseteq A(D(x))$.

Moreover, $\underline{D}(x) = 1$ and $\overline{D}(x) = 1+\rho(x)$ indicate that $A(D(x)) \subseteq [1, 1+\rho(x)]$.

Therefore, $A(D(x)) = [1, 1+\rho(x)]$. $\square$

The above theorem indicates that the set D_δ can be written as $\{x \in [0,1) : A(D(x)) = [1,\delta]\}$. An extreme case is that $\delta = 1+\lambda(\beta)$ which means that the accumulation points of $\frac{-\log_\beta |I_n(x)|}{n}$ can contain any possible value in $[1, 1+\lambda(\beta)]$. So the points in D is said to be extremely irregular.

4.2 Proof of Theorem 2.1.2

For $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathcal{A}^n$ and a positive integer m with $m \leq n$, or for $(\omega_1, \omega_2, \ldots, \omega_n, \ldots) \in \mathcal{A}^{\mathbb{N}}$ and a positive integer m, let $\omega|_m = (\omega_1, \omega_2, ..., \omega_m)$.

Recall that $\varepsilon^*(\beta) = (\varepsilon_1^*, \ldots, \varepsilon_n^*, \ldots)$ is the infinite β-expansion of 1 and t_n is defined as (2.1.2), we give a property of the full intervals as the following lemma.

Lemma 4.2.1. *Let $k \geq 1$ be an integer. If $I_n(\omega_1, \ldots, \omega_n)$ is full, we have*

$$\overline{I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1})} \subseteq \operatorname{int}(I_n(\omega_1, \ldots, \omega_n))$$

where $\operatorname{int}(I_n(\omega))$ *denotes the interior of* $I_n(\omega) \subseteq [0,1]$.

***Proof*.** If $I_n(\omega_1, \ldots, \omega_n)$ is full, we get that $(\omega_1, \ldots, \omega_n)$ can concatenate any admissible word by Theorem 3.2.7. By the definition of t_n, we have

$$(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1}) \in \Sigma_\beta^*.$$

Now we only need to show that the left and right endpoints of

$$I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1})$$

lie in $\operatorname{int}(I_n(\omega_1, \ldots, \omega_n))$ respectively.

Case I : Since $\varepsilon_1^*(\beta) = \lceil \beta \rceil - 1 \geq 1$, we have

$$\frac{\omega_1}{\beta} + \cdots + \frac{\omega_n}{\beta^n} + \frac{\varepsilon_1^*}{\beta^{n+1}} + \cdots + \frac{\varepsilon_k^*}{\beta^{n+k}} > \frac{\omega_1}{\beta} + \cdots + \frac{\omega_n}{\beta^n}.$$

This inequality and $I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1}) \subseteq I_n(\omega_1, \ldots, \omega_n)$ imply that the left endpoint of $I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1})$ belongs to $\mathrm{int}(I_n(\omega_1, \ldots, \omega_n))$.

Case II : By the definition of t_k, we have $(\varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k}, 1)$ is admissible. Besides, the fullness of $I(\omega_1, \ldots, \omega_n)$ ensures that $(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k}, 1) \in \Sigma_\beta^*$ by Theorem 3.2.7. The same argument as Case I gives that the left endpoint of $I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k}, 1)$ belongs to $\mathrm{int}(I(\omega_1, \ldots, \omega_n))$, that is,

$$x_0 := \frac{\omega_1}{\beta} + \cdots + \frac{\omega_n}{\beta^n} + \frac{\varepsilon_1^*}{\beta^{n+1}} + \cdots + \frac{\varepsilon_k^*}{\beta^{n+k}} + \frac{1}{\beta^{n+k+t_k+1}} \in \mathrm{int}(I(\omega_1, \ldots, \omega_n)).$$

Now we prove that x_0 is the right endpoint of $I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1})$. As a matter of fact, for every $x \in I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1})$, we easily get that

$$\begin{aligned} x &= \tfrac{\omega_1}{\beta} + \cdots + \tfrac{\omega_n}{\beta^n} + \tfrac{\varepsilon_1^*}{\beta^{n+1}} + \cdots + \tfrac{\varepsilon_k^*}{\beta^{n+k}} + \tfrac{\varepsilon_{n+k+t_k+2}^*}{\beta^{n+k+t_k+2}} + \cdots \\ &\le \tfrac{\omega_1}{\beta} + \cdots + \tfrac{\omega_n}{\beta^n} + \tfrac{\varepsilon_1^*}{\beta^{n+1}} + \cdots + \tfrac{\varepsilon_k^*}{\beta^{n+k}} + \tfrac{1}{\beta^{n+k+t_k+1}} = x_0, \end{aligned}$$

Now let

$$\begin{aligned} x_i &= \tfrac{\omega_1}{\beta} + \cdots + \tfrac{\omega_n}{\beta^n} + \tfrac{\varepsilon_1^*}{\beta^{n+1}} + \cdots + \tfrac{\varepsilon_k^*}{\beta^{n+k}} + \tfrac{\varepsilon_1^*}{\beta^{n+k+t_k+2}} + \cdots + \tfrac{\varepsilon_i^*}{\beta^{n+k+t_k+i+1}} \\ &\in I(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1}). \end{aligned}$$

Note that

$$1 = \frac{\varepsilon_1^*}{\beta} + \frac{\varepsilon_2^*}{\beta^2} + \cdots .$$

We deduce that

$$\lim_{i \to \infty} x_i = x_0.$$

So x_0 is the right endpoint of $I_{n+k+t_k+1}(\omega_1, \ldots, \omega_n, \varepsilon_1^*, \ldots, \varepsilon_k^*, 0^{t_k+1})$. □

Now we devote to constructing a set U with the desired properties. From Fan and Wang [35], we know that $\lambda(\beta)$ can also be written as

$$\lambda(\beta) = \limsup_{n \to \infty} \frac{t_n}{n},$$

where t_n is defined as (2.1.2). For each $k \geq 1$, recall that Σ_β^k is the set of all admissible words of length k and Γ_k is defined as (2.1.3). Define

$$U := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\varepsilon_1, \ldots, \varepsilon_k) \in \Sigma_\beta^k} \mathrm{int}\left(I_{k+\Gamma_k+\sum\limits_{i=1}^{k}(m_i+t_{m_i}+1)}(\varepsilon_1, \ldots, \varepsilon_k, 0^{\Gamma_k+1}, v_1, \ldots, v_k) \right),$$

where

$$v_k = (\varepsilon_1^*, \ldots, \varepsilon_{m_k}^*, 0^{t_{m_k}+1}), \tag{4.2.1}$$

and m_k is chosen to be a fast increasing sequence such that

$$\lambda(\beta) = \lim_{k\to\infty} \frac{t_{m_k}}{m_k}, \quad k + \Gamma_k + 1 + \sum_{j=1}^{k-1}(m_j + t_{m_j} + 1) \ll m_k. \tag{4.2.2}$$

We can see that U is well defined. This is because, for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \Sigma_\beta^k$, the interval $I_{k+\Gamma_k+1}(\varepsilon_1, \ldots, \varepsilon_k, 0^{\Gamma_k+1})$ is full by Theorem 3.2.8(3) and it follows from Theorem 3.2.7 that $(\varepsilon_1, \ldots, \varepsilon_k, 0^{\Gamma_k+1})$ can concatenate any β-admissible word. Analogously, $I_{m+t_{m_k}+1}(v_k)$ is full by Theorem 3.2.8(3) and v_k can concatenate any admissible word by Theorem 3.2.7 for each $k \geq 1$.

Clearly, U is a G_δ set since $\operatorname{int}(I_n(\omega))$ is open for all $\omega \in \Sigma_\beta^n$. For convenience, we rewrite the basic interval $I_n(\omega)$ as I_ω for each $\omega \in \Sigma_\beta^n$. Next we will show that U is a subset of E and is dense in $[0, 1]$.

Lemma 4.2.2. *U is dense in* $[0, 1]$.

Proof. Given $x \in [0, 1)$ and $r > 0$, we only need to find $y \in U$ such that $|x - y| \leq r$. Let the β-expansion of x be $\varepsilon(x, \beta) = (\varepsilon_1(x), \varepsilon_2(x), ...)$, specially, if $x = 0$, let $\varepsilon(x, \beta) = (0, 0, \ldots, 0, \ldots)$. Clearly there exists $\ell \in \mathbb{N}$ such that $\beta^{-\ell} \leq r$.

Let ω_k be defined as (4.2.1). Take

$$\begin{aligned}
u_1 &= (\varepsilon_1(x), ..., \varepsilon_\ell(x), 0^{\Gamma_\ell+1}, v_1, ..., v_\ell) \in \Sigma_\beta^{\ell_1}, \text{where } \ell_1 = \ell + \Gamma_\ell + 1 + \sum_{j=1}^{\ell}(m_j + t_{m_j} + 1), \\
u_2 &= (u_1, 0^{\Gamma_{\ell_1}+1}, v_1, ..., v_{\ell_1}) \in \Sigma_\beta^{\ell_2}, \text{where } \ell_2 = \ell_1 + \Gamma_{\ell_1} + 1 + \sum_{j=1}^{\ell_1}(m_j + t_{m_j} + 1), \\
&\vdots \\
u_k &= (u_{k-1}, 0^{\Gamma_{\ell_{k-1}}+1}, v_1, ..., v_{\ell_{k-1}}) \in \Sigma_\beta^{\ell_k}, \text{where } \ell_k = \ell_{k-1} + \Gamma_{\ell_{k-1}} + 1 + \sum_{j=1}^{\ell_{k-1}}(m_j + t_{m_j} + 1), \\
&\vdots
\end{aligned}$$

Let

$$S := \bigcap_{k=1}^{\infty} \operatorname{int}(I(u_k)).$$

For each $k \geq 1$, we have

$$\operatorname{int}(I(u_{k+1})) \subseteq \overline{I(u_{k+1})} \subseteq \operatorname{int}\left(I(u_k, 0^{\Gamma_{\ell_k}+1}, v_1, \ldots, v_{\ell_k-1})\right) \subseteq I(u_k) \subseteq \overline{I(u_k)},$$

where the second inclusion relation follows from Lemma 4.2.1. Thus, it follows that

$$\bigcap_{k=1}^{\infty} \operatorname{int}(I(u_k)) = \bigcap_{k=1}^{\infty} \overline{I(u_k)}.$$

Note that $\overline{I(u_{k+1})} \subseteq \overline{I(u_k)}$, it is obvious that $\bigcap_{k=1}^{\infty} \overline{I(u_k)}$ is nonempty. The intersection $\bigcap_{k=1}^{\infty} \operatorname{int}(I(u_k))$ is therefore nonempty which gives the fact that $S \neq \emptyset$. For every $y \in S$, we get $y \in U$ by the construction of U and S. Moreover, we have

$$|x-y| \leq \beta^{-\ell} \leq r$$

since both the β-expansions of x and y begin with $\varepsilon_1(x), ..., \varepsilon_\ell(x)$. Therefore, U is dense in $[0,1]$. □

Lemma 4.2.3. *$U \subseteq D$.*

Proof. For every $x \in U$, we only need to prove that there exists a sequence $\{n_k\}_{k\geq 1}$ such that

$$\lim_{k\to\infty} \frac{-\log_\beta |I_{n_k}|}{n_k} = 1 + \lambda(\beta).$$

Then it follows that $\overline{D}(x) = 1+\lambda(\beta)$ since $\overline{D}(x) \leq 1+\lambda(\beta)$. As a consequence, $U \subseteq E$.

In fact, for all $x \in U$, by the construction of U, there exist infinitely many k, such that the β-expansion of x starts with $\varepsilon_1, \ldots, \varepsilon_k, 0^{\Gamma_k+1}, \omega_1, \ldots, \omega_k$, where $\varepsilon_1, \ldots, \varepsilon_k \in \Sigma_\beta^k$ and ω_i is defined as (4.2.1) for all $1 \leq i \leq k$. Let $n_k = k+\Gamma_k+1+\sum_{j=1}^{k-1}(m_j+t_{m_j}+1)+m_k$, for convenience, we denote n_k as $n_k = h_k+m_k$ where $h_k = k+\Gamma_k+1+\sum_{j=1}^{k-1}(m_j+t_{m_j}+1) \ll m_k$ by (4.2.2). A simple observation on $I_n(x)$: if $n = n_k + t_{m_k} + 1$, by the construction of U, we have

$$I_n(x) = I(\varepsilon_1, \ldots, \varepsilon_{h_k}, \varepsilon_1^*, \ldots, \varepsilon_{m_k}^*, 0^{t_{m_k}+1}), \tag{4.2.3}$$

and $I_n(x)$ is full by Theorem 3.2.8(3).

Now we estimate the length of $I_n(x)$ when $n = n_k$. By the definition of t_n and the criterion of admissibility, it follows that

$$I(\varepsilon_1^*, \ldots, \varepsilon_{m_k}^*) = I(\varepsilon_1^*, \ldots, \varepsilon_{m_k}^*, 0^{t_{m_k}}),$$

so $I_{n_k}(x) = I(\varepsilon_1, \ldots, \varepsilon_{h_k}, \varepsilon_1^*, \ldots, \varepsilon_{m_k}^*, 0^{t_{m_k}})$ by Remark 2.1.1.

Hence, we have $I_{n_k}(x) = I_{n_k+t_{m_k}}(x)$. Since $I_{n_k+t_{m_k}+1}(x)$ is full, we have

$$\beta^{-(n_k+t_{m_k}+1)} \leq |I_{n_k}(x)| = |I_{n_k+t_{m_k}}(x)| \leq \beta^{-(n_k+t_{m_k})}.$$

By (4.2.2), it immediately follows that

$$\lim_{k\to\infty} \frac{-\log_\beta |I_{n_k}|}{n_k} = \lim_{k\to\infty} \frac{n_k + t_{m_k}}{n_k} = \lim_{k\to\infty} \frac{h_k + m_k + t_{m_k}}{h_k + m_k} = 1 + \lambda(\beta).$$

□

Proof of Theorem 2.1.2 Since U is dense in $[0,1]$ and it is a G_δ set, we easily get that U is residual in $[0,1]$ by Baire Category Theorem. Moreover, Lemma 4.2.3 ensures that D is residual in $[0,1]$. □

4.3 Further results

Noting that $D \cap D_\delta = \emptyset$ for all $1 < \delta < 1 + \lambda(\beta)$, we have the following corollary.

Corollary 4.3.1. *Let $\beta > 1$ with $\lambda(\beta) > 0$, then D_δ is of the first category for every $1 < \delta < 1 + \lambda(\beta)$.*

Furthermore, Theorem 2.1.2 implies $\overline{D} = [0,1]$ where $\overline{D}$ denote the closure of D, so $\dim_{\mathrm{B}} D = \dim_{\mathrm{B}} \overline{D} = 1$ by Proposition 3.1.3(5). Hence, we easily get the following corollary which implies $0 = \dim_{\mathrm{H}} D < \dim_{\mathrm{B}} D = 1$.

Corollary 4.3.2. *If $\lambda(\beta) > 0$, then* $\dim_{\mathrm{B}} D = 1$.

Let F be the *irregular set* which contains the points of $x \in (0,1]$ whose limit of $\frac{-\log_\beta |I_n(x)|}{n}$ does not exist, i.e.,

$$F = \{x \in (0,1] : \underline{D}(x) < \overline{D}(x)\}.$$

The set of irregular points is negligible from the measure-theoretical point of view [76]. For every $1 < \delta \leq 1 + \lambda(\beta)$, we have $D_\delta \subseteq F$, so by (2.1.5), it follows that

$$\dim_{\mathrm{H}} F \geq \dim_{\mathrm{H}} D_\delta = \frac{\lambda(\beta) + 1 - \delta}{\delta \cdot \lambda(\beta)} \to 1$$

as $\delta \to 1$ which implies $\dim_{\mathrm{H}} F = 1$. It follows from Proposition 3.1.5 that

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = 1.$$

Thus, we obtain that the set F has full dimension if $\lambda(\beta) > 0$, i.e., the set D can be large from the viewpoint of dimension theory. The next result shows that F is large from a topological point of view as well which follows immediately from $D \subseteq F$.

Corollary 4.3.3. *Let $\beta > 1$ with $\lambda(\beta) > 0$, then F is residual, therefore F is of second category.*

Chapitre 5 Extremely Exceptional Sets of Run-length function

Before proving Theorem 2.2.1, we introduce our method of getting the Hausdorff dimension of the set $E^{\varphi}_{\max}$. By Remark 2.2.2, we only need to consider the case that $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} = +\infty$, we suppose this condition is true in the remainder of this paper without otherwise specified. For any sufficiently large integer p, we can always construct a set $E_p \subseteq E^{\varphi}_{\max}$ with Hausdorff dimension being larger than $\frac{p-1}{p}$. Then by letting $p \to +\infty$, the relationship between E_p and $E^{\varphi}_{\max}$ gives that $E^{\varphi}_{\max}$ is of full dimension.

5.1 Proof of Theorem 2.2.1

5.1.1 Construction of Cantor subset E_p of $E^{\varphi}_{\max}$

Let $p \in \mathbb{N},\ p > 1$. Now we are going to construct the desired set $E_p \subseteq E^{\varphi}_{\max}$ whose Hausdorff dimension is larger than $\frac{p-1}{p}$. Our construction of the set E_p is divided into two steps.

Step I Fixed $\beta > 1$. Let

$$h = \min\{k \geq 2 : (1, 0^{k-2}, 1) \text{ is } \beta-\text{admissible}\}. \tag{5.1.1}$$

Note that t_1 defined by (2.1.2) is finite. By Theorem 3.2.4, we have $h \leq t_1 + 1$ is also an finite integer. Since the β-expansion of $x \in (0, 1]$ is infinite, we get that h is a finite integer. Furthermore, Theorem 3.2.8(1) implies that $I_h(1, 0^{h-1})$ is full. The facts that

$$\limsup_{n\to\infty} \frac{n}{\varphi(n)} = +\infty \text{ and } \lim_{n\to\infty} \varphi(n) = +\infty$$

give that there exists a subsequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ satisfying

$$\lim_{k\to\infty} \frac{n_k}{\varphi(n_k)} = +\infty \tag{5.1.2}$$

with $n_1 \geq e^{h+1}$ and

$$n_k \geq \varphi(n_k) \geq k n_{k-1} \tag{5.1.3}$$

for all $k \geq 2,\ k \in \mathbb{N}$.

Let $\mathcal{G}_1 = \{(10^{n_1-1})\}$ be a singleton. Then the basic interval $I_{n_1}(10^{n_1-1})$ is full since $n_1 \geq e^{h+1}$. For every $k \geq 1$, let $d_k = \lfloor \log n_k \rfloor$. Write

$$\mathcal{M}_{d_k} = \{(\varepsilon_1, \ldots, \varepsilon_{d_k}) \in \Sigma_\beta^{d_k} :\ \varepsilon_1 = 1 \text{ and } I_{d_k}(\varepsilon_1, \ldots, \varepsilon_{d_k}) \text{ is full}\}. \tag{5.1.4}$$

Then the choice of $n_1 \geq e^{h+1}$ ensures that $d_k > h$ for every $k \geq 2$. For any $j \in \mathbb{Z}^+$, let

$$t_{2j} = \left\lfloor \frac{n_{2j} - n_{2j-1}}{d_{2j-1}} \right\rfloor,\ t_{2j+1} = \left\lfloor \frac{\frac{p-1}{p} n_{2j+1} - n_{2j}}{d_{2j}} \right\rfloor.$$

As a result from the choice of n_k in (5.1.3), we get that $t_k \geq 1$ for each $k \geq 1$. Next, for all $k \geq 1$, define

$$\mathcal{G}_k = \{u_k = (u_{k-1}^{(1)}, \ldots, u_{k-1}^{(t_k)}, 0^{n_k - n_{k-1} - d_{k-1} t_k}) :\ u_{k-1}^{(i)} \in \mathcal{M}_{d_{k-1}} \text{ for all } 1 \leq i \leq t_k\}.$$

Then, we have for each $u_k \in \mathcal{G}_k$, the length of u_k satisfies that $|u_k| = n_k - n_{k-1}$. By Theorems 3.2.7 and 3.2.8(1), we obtain that every u_k in $\mathcal{G}_k$ $(k \geq 1)$ is admissible and it is full which demonstrates that every u_k in $\mathcal{G}_k$ $(k \geq 1)$ can be concatenated by any β-admissible word. So $\mathcal{D}_k$ can be well defined as follows :

$$\mathcal{D}_k = \{(u_1, \ldots, u_k) : u_i \in \mathcal{G}_i, \text{ for all } 1 \leq i \leq k\}. \tag{5.1.5}$$

Step II For each $u = (u_1, \ldots, u_k) \in \mathcal{D}_k$, note that the length of u, denoted by $|u|$, satisfying that

$$|u| = |u_1| + |u_2| + \cdots + |u_k| = n_1 + (n_2 - n_1) + \cdots + (n_k - n_{k-1}) = n_k.$$

Set

$$E_p = \bigcap_{k \geq 1} \bigcup_{u \in \mathcal{D}_k} I_{n_k}(u).$$

The following lemma provides a detailed exposition of showing $E_p \subseteq E_{\max}^{\varphi}$ for every $p \in \mathbb{N},\ p > 1$.

Lemma 5.1.1. *Assume that* $\limsup\limits_{n\to\infty} \frac{n}{\varphi(n)} = +\infty$. *For every* $p \in \mathbb{N},\ p > 1$, *we have* $E_p \subseteq E_{\max}^{\varphi}$.

Proof. On the one hand, the construction of E_p yields that the word $u \in \mathcal{D}_k$ verifies the following properties :

(1) The character of $u_{k-1}^{(i)} \in \mathcal{M}_{d_{k-1}} (1 \leq i \leq t_k)$ beginning with 1 ensures that the maximal length of zeros in every word $u \in \mathcal{D}_k$ only appears at the tail of $u_k \in \mathcal{G}_k$ for all $k \geq 1$;

(2) The maximal length of zeros in every word $u_{2j+1} \in \mathcal{G}_{2j+1}$ for all $j \geq 0$ is increasing with respect to j. Thus, for large enough $j \geq 1$, noticing that

$$n_{2j} - n_{2j-1} - d_{2j-1}t_{2j} + d_{2j-1} = n_{2j} - n_{2j-1} - d_{2j-1}\left\lfloor \frac{n_{2j} - n_{2j-1}}{d_{2j-1}} \right\rfloor + d_{2j-1} \leq 2d_{2j-1},$$

we have

$$r_{n_{2j}}(x, \beta) \leq \max\{n_{2j-1} - n_{2j-2} - d_{2j-2}t_{2j-1} + d_{2j-2}, n_{2j} - n_{2j-1} - d_{2j-1}t_{2j} + d_{2j-1}\} < 2n_{2j-1}.$$

Therefore,

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} \leq \liminf_{j\to\infty} \frac{r_{n_{2j}}(x,\beta)}{\varphi(n_{2j})} \leq \liminf_{j\to\infty} \frac{2n_{2j-1}}{\varphi(n_{2j})} \leq \lim_{j\to\infty} \frac{2n_{2j-1}}{2jn_{2j-1}} = 0,$$

where the last inequality follows from (5.1.3).

On the other hand, we note that there are at least $n_k - n_{k-1} - d_{k-1}t_k$ zeros in every word $u \in \mathcal{D}_k$. So it holds that

$$r_{n_{2j+1}}(x, \beta) \geq n_{2j+1} - n_{2j} - d_{2j}t_{2j+1} \geq n_{2j+1} - n_{2j} - d_{2j}\left(\frac{\frac{p-1}{p}n_{2j+1} - n_{2j}}{d_{2j}} + 1\right) > \frac{1}{p}n_{2j+1}.$$

Thus,

$$\limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} \geq \limsup_{j\to\infty} \frac{r_{n_{2j+1}}(x,\beta)}{\varphi(n_{2j+1})} \geq \lim_{j\to\infty} \frac{\frac{1}{p}n_{2j+1}}{\varphi(n_{2j+1})} = +\infty,$$

by (5.1.2). □

5.1.2 Hausdorff dimension of $E^{\varphi}_{\max}$

We first give the lower bound of $\dim_{\mathrm{H}} E_p$, we technically show that given $\beta > 1$,

$$\dim_{\mathrm{H}} E_p \geq \frac{\log \overline{\beta}}{\log \beta} \frac{p-1}{p}$$

for all $1 < \overline{\beta} < \beta$. For all $k \geq 2$, recall that $\mathcal{M}_{d_k}$ and $\mathcal{D}_k$ are defined as (5.1.4) and (5.1.5) respectively. Let $a_k := \sharp\mathcal{M}_{d_k}$ and $b_k := \sharp\mathcal{D}_k$.

Lemma 5.1.2. *Fixed $\beta > 1$, for each $1 < \overline{\beta} < \beta$, there exist integers $k(\overline{\beta})$, $c(\overline{\beta})$ relying on $\overline{\beta}$ such that for every $k > k(\overline{\beta})$, such that $a_k \geq \overline{\beta}^{d_k}$ and $b_k \geq c(\overline{\beta})\overline{\beta}^{p_k}$ where*

$$p_k = \begin{cases} n_{2j} - \dfrac{1}{p}\displaystyle\sum_{i=1}^{j} n_{2i-1} - \sum_{i=1}^{2j-1} d_i, & \text{when } \ k = 2j, \text{ for some } j \in \mathbb{N}; \\ \dfrac{p-1}{p}n_{2j+1} - \dfrac{1}{p}\displaystyle\sum_{i=1}^{j} n_{2i-1} - \sum_{i=1}^{2j} d_i, & \text{when } \ k = 2j+1, \text{ for some } j \in \mathbb{N}. \end{cases}$$

Proof. We first give the lower bound of a_k. Recall h defined as (5.1.1), for any $k \geq 1$, let

$$\mathcal{M}'_{d_k} = \{(\varepsilon_1, \ldots, \varepsilon_{d_k}) \in \Sigma_\beta^{d_k} : \ (\varepsilon_1, \ldots, \varepsilon_h) = (1, 0^{h-1}) \text{ and } I_{d_k-h}(\varepsilon_{h+1}, \ldots, \varepsilon_{d_k}) \text{ is full}\}.$$

Then, from the comparison of the definition of $\mathcal{M}_{d_k}$ and $\mathcal{M}'_{d_k}$, it holds that $\mathcal{M}'_{d_k} \subseteq \mathcal{M}_{d_k}$ which implies $\sharp\mathcal{M}'_{d_k} \leq \sharp\mathcal{M}_{d_k}$. Theorem 3.2.1 indicates that $\sharp\Sigma_\beta^{d_k-h} \geq \beta^{d_k-h}$. Furthermore, note that $\sharp\mathcal{M}'_{d_k}$ is just the number of the full words in $\Sigma_\beta^{d_k-h}$. Hence, by Theorem 3.2.10, we obtain

$$a_k \geq \sharp\mathcal{M}'_{d_k} \geq \left\lfloor \frac{\beta^{d_k-h}}{d_k - h} \right\rfloor.$$

It follows that there exists an integer $k(\overline{\beta})$ depending on $\overline{\beta}$ such that for every $k > k(\overline{\beta})$, we have

$$\left\lfloor \frac{\beta^{d_k-h}}{d_k - h} \right\rfloor \geq \overline{\beta}^{d_k}.$$

Thus, $a_k = \sharp\mathcal{M}_{d_k} \geq \sharp\mathcal{M}'_{d_k} \geq \overline{\beta}^{d_k}$ for every $k \geq k(\overline{\beta})$.

Now we estimate b_k. For all $j \geq \lfloor \frac{k(\overline{\beta})}{2} \rfloor + 1 \triangleq k'(\overline{\beta})$, by the construction of $\mathcal{G}_{2j}$ and $\mathcal{G}_{2j+1}$, we have

$$\sharp\mathcal{G}_{2j} = (\sharp\mathcal{M}_{d_{2j-1}})^{t_{2j}} \geq \overline{\beta}^{d_{2j-1}t_{2j}} \geq \overline{\beta}^{n_{2j}-n_{2j-1}-d_{2j-1}},$$

and

$$\sharp\mathcal{G}_{2j+1} = (\sharp\mathcal{M}_{d_{2j}})^{t_{2j+1}} \geq \overline{\beta}^{d_{2j}t_{2j+1}} \geq \overline{\beta}^{\frac{p-1}{p}n_{2j+1}-n_{2j}-d_{2j}}.$$

Then it follows from the relationship between D_k and G_k that for each $j \geq \lfloor \frac{k(\overline{\beta})}{2} \rfloor + 1$,

$$\begin{aligned} b_{2j} = \sharp\mathcal{D}_{2j} = \prod_{i=1}^{2j} \sharp\mathcal{G}_i \geq \prod_{i=k'(\overline{\beta})}^{2j} \sharp\mathcal{G}_i &\geq \overline{\beta}^{\sum\limits_{i=k'(\overline{\beta})}^{j}(n_{2i}-n_{2i-1}-d_{2i-1})} \overline{\beta}^{\sum\limits_{i=k'(\overline{\beta})}^{j}(\frac{p-1}{p}n_{2i-1}-n_{2i-2}-d_{2i-2})} \\ &\geq c(\overline{\beta})\overline{\beta}^{\sum\limits_{i=1}^{j}(n_{2i}-n_{2i-1}-d_{2i-1})} \overline{\beta}^{\sum\limits_{i=1}^{j}(\frac{p-1}{p}n_{2i-1}-n_{2i-2}-d_{2i-2})} \\ &= c(\overline{\beta})\overline{\beta}^{n_{2j}-\frac{1}{p}\sum\limits_{i=1}^{j}n_{2i-1}-\sum\limits_{i=1}^{2j-1}d_i}, \end{aligned} \tag{5.1.6}$$

where

$$c(\overline{\beta}) = \overline{\beta}^{-\left(\sum\limits_{i=1}^{k'(\overline{\beta})}(n_{2i}-n_{2i-1}-d_{2i-1}) + \sum\limits_{i=1}^{k'(\overline{\beta})}(\frac{p-1}{p}n_{2i-1}-n_{2i-2}-d_{2i-2})\right)}.$$

Here $c(\overline{\beta})$ is a constant depending on $\overline{\beta}$. The same way as (5.1.6) shows that,

$$b_{2j+1} = \sharp\mathcal{D}_{2j+1} = \prod_{i=1}^{2j+1} \sharp\mathcal{G}_i \geq c(\overline{\beta})\overline{\beta}^{\frac{p-1}{p}n_{2j+1}-\frac{1}{p}\sum\limits_{i=1}^{j}n_{2i-1}-\sum\limits_{i=1}^{2j}d_i}. \tag{5.1.7}$$

Combining (5.1.6) and (5.1.7), we finish our proof. $\square$

Now we give the lower bound of $\dim_{\mathrm{H}} E_p$ as the following result.

Lemma 5.1.3. *For each $p \in \mathbb{N},\ p > 1$. The Hausdorff dimension of E_p satisfies that*

$$\dim_{\mathrm{H}} E_p \geq \frac{p-1}{p}.$$

Proof. It suffices to show that

$$\dim_{\mathrm{H}} E_p \geq \frac{p-1}{p}\frac{\log \overline{\beta}}{\log \beta}$$

for all $1 < \overline{\beta} < \beta$. To complete our proof, it falls naturally into three parts.

(1) Distribute a probability measure μ supported on E_p. Let

$$\mu([0,1)) = 1,\ \text{and}\ \mu(I_{n_1}(u)) = 1,\ \text{for}\ u \in \mathcal{D}_1.$$

For all $k \geq 1$, and $u = (u_1, \ldots, u_{k+1}) \in \mathcal{D}_{k+1}$, we set

$$\mu(I_{n_{k+1}}(u)) = \frac{\mu(I_{n_k}(u_1, \ldots, u_k))}{\sharp\ \mathcal{G}_k}. \tag{5.1.8}$$

The Kolmogorov's consistency theorem guarantees that μ we defined above can be uniquely extended to a Borel measure supported on E_p.

(2) Estimate the exponent of $\mu(I_n)$ for any basic intervals I_n of order n with $I_n \cap E_p \neq \emptyset$ in [0,1). By (5.1.8) and Lemma 5.1.2, we get that

$$\mu(I_{n_i}) = \frac{1}{b_i} \leq \frac{1}{c(\overline{\beta})\overline{\beta}^{p_i}}, \tag{5.1.9}$$

for every $i > k(\overline{\beta})$, where $k(\overline{\beta})$ is an integer depending on $\overline{\beta}$ given in Lemma 5.1.2. For $n \geq 1$, there exists $k \geq 0$ such that $n_k < n \leq n_{k+1}$. Then, we distinguish four cases to get the lower bound of $\mu(I_n)$.

Case 1. $k = 2j$ and $n_{2j} + \ell d_{2j-1} \leq n < n_{2j} + (\ell+1)d_{2j-1}$ for some $0 \leq \ell \leq t_{2j} - 1$. Notice that the number of I_n containing $I_{n_{2j+1}}(u)$ $(u \in \mathcal{D}_{2j+1})$ is larger than a_{2j-1}^{ℓ}. Then

$$\mu(I_n) \leq \mu(I_{n_{2j}+\ell d_{2j-1}}) \leq \mu(I_{n_{2j}})a_{2j-1}^{-\ell} \leq \frac{1}{c(\overline{\beta})\overline{\beta}^{p_{2j}}\overline{\beta}^{d_{2j-1}\ell}},$$

where the last inequality follows from (5.1.9) and Lemma 5.1.2. Moreover, Theorem 3.2.8(2) implies

$$|I_n| \geq |I_{n_{2j}+(\ell+1)d_{2j}}| = \frac{1}{\beta^{n_{2j}+(\ell+1)d_{2j}}}.$$

Consequently,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\log \overline{\beta}^{p_{2j}+d_{2j-1}\ell} + \log c(\overline{\beta})}{\log \beta^{n_{2j}+(\ell+1)d_{2j}}}.$$

Case 2. $k = 2j$ and $n_{2j} + t_{2j}d_{2j-1} \le n < n_{2j+1}$. Then

$$\mu(I_n) = \mu(I_{n_{2j+1}}) \le \frac{1}{c(\overline{\beta})\overline{\beta}^{p_{2j+1}}},$$

by (5.1.9) and Lemma 5.1.2. Moreover, Theorem 3.2.8(2) forces that

$$|I_n| \ge |I_{n_{2j+1}}| = \frac{1}{\beta^{n_{2j+1}}}.$$

Hence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \ge \frac{\log \overline{\beta}^{p_{2j+1}} + \log c(\overline{\beta})}{\log \beta^{n_{2j+1}}}.$$

Case 3. $k = 2j+1$ and $n_{2j+1}+\ell d_{2j} \le n < n_{2j+1}+(\ell+1)d_{2j}$ for some $0 \le \ell \le t_{2j}-1$. Similarly, it follows that

$$\frac{\log \mu(I_n)}{\log |I_n|} \ge \frac{\log \overline{\beta}^{p_{2j+1}+d_{2j}\ell} + \log c(\overline{\beta})}{\log \beta^{n_{2j+1}+(\ell+1)d_{2j+1}}}.$$

Case 4. $k = 2j+1$ and $n_{2j+1} + t_{2j+1}d_{2j} \le n < n_{2j+2}$. Analogously,

$$\frac{\log \mu(I_n)}{\log |I_n|} \ge \frac{\log \overline{\beta}^{p_{2j+2}} + \log c(\overline{\beta})}{\log \beta^{n_{2j+2}}}.$$

From the above discussion, by letting $n \to \infty$, we get that $\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \ge \frac{p-1}{p}\frac{\log \overline{\beta}}{\log \beta}$ for all $I_n \cap E_p \ne \emptyset$. We immediately get that, for every $\eta > 0$, there exits an integer n_0 such that for every $n \ge n_0$ and basic interval I_n with order n,

$$\mu(I_n) \le |I_n|^{\frac{p-1}{p}\frac{\log \overline{\beta}}{\log \beta} - \eta}.$$

(3) Use the modified mass distribution principle to get the lower bound of $\dim_{\mathrm{H}} E_p$. By (2), it follows from Theorem 3.1.8 that $\dim E_p \ge \frac{p-1}{p}\frac{\log \overline{\beta}}{\log \beta} - \eta$. The arbitrariness of $\eta > 0$ and $1 < \overline{\beta} < \beta$ demonstrates that $\dim E_p \ge \frac{p-1}{p}$. □

Proof of Theorem 2.2.1 Applying Lemma 5.1.1, it holds that $E_p \subseteq E^{\varphi}_{\max}$ for every $p \in \mathbb{N},\ p > 1$. By setting $p \to \infty$, we get that

$$\dim_{\mathrm{H}} E^{\varphi}_{\max} \ge \lim_{p\to\infty} \dim_{\mathrm{H}} E_p \ge \lim_{p\to\infty} \frac{p-1}{p} = 1$$

where the second inequality follows from Lemma 5.1.3. It is obvious that $\dim_{\mathrm{H}} E^{\varphi}_{\max} \le 1$. Thus,

$$\dim_{\mathrm{H}} E^{\varphi}_{\max} = 1.$$

□

5.2 Proof of Theorem 2.2.3

For every integer $n \geq 1$, let Γ_n be defined as (2.1.3) and h be given as (5.1.1). Since $\limsup_{n\to\infty} \frac{n}{\varphi(n)} = +\infty$ and $\varphi(n) \to +\infty$ as $n \to +\infty$, we can choose an increasing subsequence $\{n_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ satisfying

$$\lim_{i\to\infty} \frac{n_i}{\varphi(n_i)} = +\infty \tag{5.2.1}$$

with $n_i - n_{i-1} > \max\{2h, i + \mathcal{G}_i\}$ and $\varphi(n_i) \geq (i-1)n_{i-1}$. Fix $(\varepsilon_1, \ldots, \varepsilon_k) \in \Sigma_\beta^k$, let

$$\omega_i^{(k)} = \begin{cases} (1, 0^{n_{k+i}-n_{k+i-1}-1}), & \text{when } \ i \text{ is odd;} \\ \left((1, 0^{h-1})^{\lfloor \frac{n_{k+i}-n_{k+i-1}}{h} \rfloor h}, 0^{n_{k+i}-n_{k+i-1}-\lfloor \frac{n_{k+i}-n_{k+i-1}}{h} \rfloor h}\right), & \text{when } \ i \text{ is even,} \end{cases} \tag{5.2.2}$$

for every $1 \leq i \leq 2k$. Now we define

$$U := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Sigma_\beta^k} \mathrm{int}\left(I_{n_{3k}}(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k}, \omega_1^{(k)}, \ldots, \omega_{2k}^{(k)}\right),$$

where $\mathrm{int}(I_{|\varepsilon|}(\varepsilon))$ denotes the interior of $I_{|\varepsilon|}(\varepsilon)$ for every $\varepsilon \in \Sigma_\beta^*$.

Remark 5.2.1. U is well defined. This is because, for all $(\varepsilon_1, \ldots, \varepsilon_k) \in \Sigma_\beta^k$, it follows from Theorem 3.2.8(3) that the interval $I_{k+\Gamma_k+1}(\varepsilon_1, \ldots, \varepsilon_k, 0^{\Gamma_k+1})$ is full. Since $n_k > k + \Gamma_k$ by the choice of n_k, we have the basic interval $I_{n_k}(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k})$ is full by Theorem 3.2.8(3). So the word $(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k})$ can concatenate any β-admissible word by Theorem 3.2.7. Similarly, $\omega_i^{(k)}$ $(1 \leq i \leq 2k)$ can concatenate any admissible word by the choice of n_k satisfying $n_k - n_{k-1} > 2h$ for all $k \geq 2$ and it is full for each $k \geq 1$ by Theorem 3.2.8(1).

It is obvious that U is a G_δ set since $\mathrm{int}(I_{|\omega|}(\omega))$ is open for all $\omega \in \Sigma_\beta^*$. So it remains to show that U is a subset of $E_{\max}^{\varphi}$ and is dense in $[0, 1]$.

Lemma 5.2.2. $U \subseteq E_{\max}^{\varphi}$.

Proof. For every $x \in U$, by the construction of U, there exist infinitely many k, such that the β-expansion of x starts with $(\varepsilon_1, ..., \varepsilon_k, 0^{n_k-k}, \omega_1^{(k)}, \ldots, \omega_{2k}^{(k)})$, where $(\varepsilon_1, ..., \varepsilon_k) \in \Sigma_\beta^k$ and $\omega_i^{(k)}$ is defined as (5.2.2) for all $1 \leq i \leq 2k$. Now we are concentrating on finding out the upper limit and lower limit of $\frac{r_n(x,\beta)}{\varphi(n)}$.

When $n = n_{3k}$, the construction of U gives that the maximal length of zeros can only appear in the tail of $\omega_i^{(k)}(1 \leq i \leq 2k)$ defined as (5.2.2). Moreover, n_k is increasing as k increases. Consequently, it comes to the conclusion that, for large enough k,

$$r_{n_{3k}}(x, \beta) \leq \max\{k + \Gamma_k, 2h, n_{3k-1} - n_{3k-2}\} \leq n_{3k-1}.$$

Thus, we have

$$\liminf_{n\to\infty}\frac{r_n(x,\beta)}{\varphi(n)}\le\liminf_{k\to\infty}\frac{r_{n_{3k}}(x,\beta)}{\varphi(n_{3k})}\le\liminf_{k\to\infty}\frac{n_{3k-1}}{\varphi(n_{3k})}\le\lim_{k\to\infty}\frac{n_{3k-1}}{(3k-1)n_{3k-1}}=0.$$

When $n=n_{3k-1}$, by the observation on U, there are at least $n_{3k-1}-n_{3k-2}$ zeros in $\omega_{2k-1}^{(k)}$ which is defined as (5.2.2). We therefore obtain $r_{n_{3k-1}}(x,\beta)\ge n_{3k-1}-n_{3k-2}$. As a result, we get

$$\limsup_{n\to\infty}\frac{r_n(x,\beta)}{\varphi(n)}\ge\limsup_{k\to\infty}\frac{r_{n_{3k-1}}(x,\beta)}{\varphi(n_{2k-1})}\ge\lim_{k\to\infty}\frac{n_{3k-1}-n_{3k-2}}{\varphi(n_{3k-1})}=+\infty.$$

In conclusion, it immediately holds that $x\in E_{\max}^{\varphi}$, so $U\subseteq E_{\max}^{\varphi}$. □

Proof of Theorem 2.2.3 Now we check that the set

$$\bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Sigma_\beta^k}\operatorname{int}\left(I_{n_{3k}}(\varepsilon_1,\ldots,\varepsilon_k,0^{n_k-k},\omega_1^{(k)},\ldots,\omega_{2k}^{(k)}\right)$$

is dense in [0,1]. That is, for all real number $x\in[0,1]$ and $r>0$, we need to find out a real number $y\in U$ satisfying $|x-y|\le r$. Assume that the β-expansion of x is $\varepsilon(x,\beta)=(\varepsilon_1(x),\varepsilon_2(x),\ldots)$. Let ℓ be an integer such that $\beta^{-\ell}\le r$. We get that $(\varepsilon_1(x),\ldots,\varepsilon_\ell(x))\in\Sigma_\beta^\ell$. Then let

$$y\in I_{3n_\ell}(\varepsilon_1(x),\ldots,\varepsilon_\ell(x),0^{n_\ell-\ell},\omega_1^{(\ell)},...,\omega_{2\ell}^{(\ell)})$$

where $\omega_i^{(\ell)}$ is defined as (5.2.2) for all $1\le i\le 2\ell$. Thus $|x-y|\le\beta^{-\ell}\le r$ since both the β-expansions of x and y begin with $(\varepsilon_1(x),...,\varepsilon_\ell(x))$. Hence, the set

$$\bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Sigma_\beta^k}\operatorname{int}\left(I_{n_{3k}}(\varepsilon_1,\ldots,\varepsilon_k,0^{n_k-k},\omega_1^{(k)},\ldots,\omega_{2k}^{(k)}\right)$$

is dense in $[0,1]$.

By the Baire category theorem, we consequently have U is residual in $[0,1]$. To sum up, $E_{\max}^{\varphi}$ is residual in $[0,1]$ by Lemma 5.2.2.

□

Chapitre 6 Diophantine approximation and run-length function

In this section, we devote to showing the size of sets considered in Section 1.4. We are first show the relationship between the exponents of the classic and uniform Diophantine Approximation and run-length function. And give the size of the set $E_{a,b}$ (in the sense of Lesbegue measure, Hausdorff dimension, topological property).

6.1 Proof of Theorem 2.3.1

6.1.1 Run-length function and Diophantine approximation

Lemma 6.1.1. *Let $\beta > 1$. For all $x \in [0,1)$, for any $0 < a < 1$, we have $\liminf\limits_{n\to\infty} \frac{r_n(x,\beta)}{n} = a$ if and only if $\hat{v}_\beta(x) = \frac{a}{1-a}$.*

Proof. It suffices to show the following two cases. The details are left to the readers.

On the one hand, suppose that $\hat{v}_\beta(x) < \frac{a}{1-a}$, then we have $v_0 = \frac{a}{2(1-a)} + \frac{\hat{v}_\beta(x)}{2} > \hat{v}_\beta(x)$. By the definition of $\hat{v}_\beta(x)$, there is a sequence $\{n_k\}_{k=1}^{\infty}$ such that, for all $1 \leq n \leq n_k$,

$$T_\beta^n x > \beta^{-v_0 n} \geq \beta^{-(\lfloor v_0 n \rfloor + 1)}.$$

So it holds that

$$r_{n_k + \lfloor v_0 n_k \rfloor}(x,\beta) < \lfloor v_0 n_k \rfloor + 1.$$

This implies

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} \leq \lim_{k\to\infty} \frac{r_{n_k+\lfloor v_0 n_k \rfloor}(x,\beta)}{n_k + \lfloor v_0 n_k \rfloor} \leq \lim_{k\to\infty} \frac{\lfloor v_0 n_k \rfloor + 1}{n_k + \lfloor v_0 n_k \rfloor}.$$

Note that

$$\lim_{k\to\infty} \frac{\lfloor v_0 n_k \rfloor + 1}{n_k + \lfloor v_0 n_k \rfloor} = \frac{v_0}{1+v_0} = \frac{2a + \hat{v}_\beta(x)(1-a) - a}{2 + \hat{v}_\beta(x)(1-a) - a} < a,$$

where the last inequality follows from

$$\frac{a-x}{b-x} < \frac{a}{b}, \text{ for all } 0 \leq a < b, \ x > 0.$$

We deduce that

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} < a.$$

A contradiction. Therefore, $\hat{v}_\beta(x) \geq \frac{a}{1-a}$,

On the other hand, suppose that $\hat{v}_\beta(x) > \frac{a}{1-a}$, then $v_0 = \frac{a}{2(1-a)} + \frac{\hat{v}_\beta(x)}{2} < \hat{v}_\beta(x)$. The definition of $\hat{v}_\beta(x)$ implies that for all $N \gg 1$, there exists $1 \leq n \leq N$, such that

$$T_\beta^n x \leq \beta^{-v_0 N}.$$

Then for all $k = N + \lfloor v_0 N \rfloor + 1 \gg 1$, we have

$$r_k(x, \beta) \geq \lfloor v_0 N \rfloor.$$

This implies

$$\liminf_{k\to\infty} \frac{r_k(x,\beta)}{k} \geq \lim_{N\to\infty} \frac{\lfloor v_0 N \rfloor}{N + \lfloor v_0 N \rfloor + 1} = \frac{v_0}{1+v_0} = \frac{2a + \hat{v}_\beta(x)(1-a) - a}{2 + \hat{v}_\beta(x)(1-a) - a} > a,$$

where the last inequality follows from

$$\frac{a+x}{b+x} > \frac{a}{b} \text{ for all } 0 \leq a < b,\ x > 0. \tag{6.1.1}$$

This contradicts with $\liminf\limits_{n\to\infty} \frac{r_n(x,\beta)}{n} = a$. Consequently, $\hat{v}_\beta(x) \leq \frac{a}{1-a}$.

Thus, we conclude that $\liminf\limits_{n\to\infty} \frac{r_n(x,\beta)}{n} = a$. □

Lemma 6.1.2. *Let $\beta > 1$. For all $x \in [0,1)$, for each $0 < b < 1$, we have $\limsup\limits_{n\to\infty} \frac{r_n(x,\beta)}{n} = b$ if and only if $v_\beta(x) = \frac{b}{1-b}$.*

Proof. It can be deduced by the same arguments as the proof of Lemma 6.1.1. □

Now we can give part of the proof of Theorem 2.3.1.

We will first show when $a > \frac{b}{1+b},\ 0 < b \leq 1$, $E_{a,b} = \emptyset$. In fact, if $\limsup\limits_{n\to\infty} \frac{r_n(x,\beta)}{n} = b$, then for all $\delta > 0$, there exits a sequence $\{n_k\}_{k=1}^\infty$ such that $r_{n_k}(x,\beta) \leq \lfloor (b+\delta)n_k \rfloor$ and $\varepsilon_{n_k}(x,\beta) > 0$. Thus, when we consider the prefix at the position $n_k + \lfloor bn_k \rfloor$, there are at most $\lfloor (b+\delta)n_k \rfloor$ consecutive 0's. Immediately, $r_{n_k + \lfloor bn_k \rfloor}(x,\beta) \leq \lfloor (b+\delta)n_k \rfloor$. Hence,

$$a = \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} \leq \lim_{k\to\infty} \frac{r_{n_k+\lfloor bn_k \rfloor}(x,\beta)}{n_k + \lfloor bn_k \rfloor} \leq \lim_{k\to\infty} \frac{(b+\delta)n_k}{n_k + \lfloor bn_k \rfloor} = \frac{b+\delta}{1+b}.$$

Letting $\delta \to 0$, we have

$$a \leq \frac{b}{1+b}. \tag{6.1.2}$$

Therefore, $E_{a,b}$ is empty when $a > \frac{b}{1+b},\ 0 < b \leq 1$.

When $0 < a \leq \frac{b}{1+b},\ 0 < b < 1$, Lemmas 6.1.1 and 6.1.2 give the fact that the sets we consider here are essentially the same as the sets studied in Bugeaud and Liao [18], that is

$$E_{a,b} = \left\{ x \in [0,1) : \hat{v}_\beta(x) = \frac{a}{1-a},\ v_\beta(x) = \frac{b}{1-b} \right\} = U_\beta\left(\frac{a}{1-a}, \frac{b}{1-b} \right).$$

Consequently, we can apply Theorem 3.2.14 to obtain

$$\dim_{\mathrm{H}} E_{a,b} = \dim_{\mathrm{H}} U_\beta \left(\frac{a}{1-a}, \frac{b}{1-b} \right) = 1 - \frac{b^2(1-a)}{b-a}.$$

However, Theorem 3.2.14 cannot be applied for the cases $a = 0,\ 0 < b < 1$ and $0 < a \le \frac{1}{2},\ b = 1$. Remark that $E_{a,1} \subseteq F_1$ where F_1 is defined by (2.3.2) and $\dim_{\mathrm{H}} F_1 = 0$ by [53, Theorem 1.1]. So $\dim_{\mathrm{H}} E_{a,1} = 0$ $(0 < a \le \frac{1}{2})$ and there is nothing to prove. For the other case, we have

$$E_{0,b} \subseteq \left\{ x \in [0,1) : v_\beta(x) \ge \frac{b}{1-b} \right\}.$$

Then we can use Theorem 3.2.13 to obtain the upper bound of $\dim_{\mathrm{H}} E_{0,b}$ which is $1-b$ for all $0 < b < 1$. Hence it remains to give the lower bound of $\dim_{\mathrm{H}} E_{0,b}$ for all $0 < b < 1$.

6.1.2 Lower bound of $\dim_{\mathrm{H}} E_{0,b}$ $(0 < b < 1)$

Now we give the lower bound of $\dim_{\mathrm{H}} E_{0,b}$ for the case $0 < b < 1$. In fact, we can also include the proof of the lower bound of $\dim_{\mathrm{H}} E_{a,b}$ for the case $0 < a \le \frac{b}{1+b},\ 0 < b < 1$, though the later case has already been given in the end of Section 5.1.1.

For all $k \ge 1$ and $N > 1$ with $\varepsilon_N^*(\beta) > 0$, choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ which satisfy $n_k < m_k < n_{k+1}$ with $n_1 > 2N$, and $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$. Moreover, $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ can be chosen to satisfy

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = a \tag{6.1.3}$$

and

$$\lim_{k\to\infty} \frac{m_k - n_k}{m_k} = b. \tag{6.1.4}$$

In fact, such sequences do exist by the following arguments.

(1) If $0 < a \le \frac{b}{1+b},\ 0 < b < 1$, let

$$n'_k = \left\lfloor \left(\frac{b(1-a)}{a(1-b)} \right)^k \right\rfloor \quad \text{and} \quad m'_k = \left\lfloor \frac{1}{1-b} \left(\frac{b(1-a)}{a(1-b)} \right)^k \right\rfloor.$$

Note that $a < b$, so $\frac{b(1-a)}{a(1-b)} > 1$. Then both sequences $\{n'_k\}_{k=1}^\infty$ and $\{m'_k\}_{k=1}^\infty$ are increasing to infinity as k tends to infinity. A small adjustment can attain the required sequences.

(2) If $a = 0,\ 0 < b < 1$, let

$$n'_k = k^k \quad \text{and} \quad m'_k = \left\lfloor \frac{1}{1-b} k^k \right\rfloor.$$

We can adjust these sequences to make sure that $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$.

Now let us construct a Cantor subset of $E_{a,b}$.

For all $d > 2N$, let

$$\mathcal{M}_d = \{\omega = (1, 0^{N-1}, \omega_1, \ldots, \omega_{d-2N}, 0^N) : (\omega_1, \ldots, \omega_{d-N}) \in \Sigma_{\beta_N}^{d-2N}\}. \tag{6.1.5}$$

Remark that $(1, 0^{N-1}, \omega_1, \ldots, \omega_{d-N}, 0^N) \in \Sigma_{\beta_N}^d$ ends with 0^N. Thus, by Proposition 3.2.15, every word belonging to $\mathcal{M}_d$ is full when regarding it as an element of Σ_β^d. Now let $\mathcal{G}_1 = \{\omega : \omega \in \mathcal{M}_{n_1}\}$. Next, for all $k \geq 1$, let $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ where $0 \leq p_k < m_k - n_k$. Define

$$\mathcal{G}_{k+1} = \{u_{k+1} = (1, 0^{m_k - n_k - 1}, u_k^{(1)}, \ldots, u_k^{(t_k)}, u_k^{(t_k+1)}) : \ u_k^{(i)} \in \mathcal{M}_{m_k - n_k} \text{ for all } 1 \leq i \leq t_k\}$$

where

$$u_k^{(t_k+1)} = \begin{cases} 0^{p_k}, & \text{when } \ p_k \leq 2N; \\ \omega \in \mathcal{M}_{p_k}, & \text{when } \ p_k > 2N. \end{cases}$$

It follows from Theorem 3.2.8 (1) and (3) that every $u_k \in \mathcal{G}_k$ is full which means that every word in $\mathcal{G}_k$ can be concatenated by any β-admissible word. Hence, we can define the set $\mathcal{D}_k$ as :

$$\mathcal{D}_k = \{(u_1, \ldots, u_k) : u_i \in \mathcal{G}_i, \text{ for all } 1 \leq i \leq k\}. \tag{6.1.6}$$

Notice that the length of $u_k \in \mathcal{G}_k$ satisfies $|u_k| = n_k - n_{k-1}$. For each $u = (u_1, \ldots, u_k) \in D_k$, we have

$$|u| = |u_1| + |u_2| + \cdots + |u_k| = n_1 + (n_2 - n_1) + \cdots + (n_k - n_{k-1}) = n_k.$$

Define

$$E_N = \bigcap_{k=1}^{\infty} \bigcup_{u \in \mathcal{D}_k} I_{n_k}(u).$$

The following lemma shows that E_N is a subset of $E_{a,b}$.

Lemma 6.1.3. *We have $E_N \subseteq E_{a,b}$ for every $0 \leq a \leq \frac{b}{1+b}$ and $0 < b < 1$.*

Proof. For every integer $n \geq 1$, there exists a $k \geq 1$ such that $n_k < n \leq n_{k+1}$. We distinguish three cases.

(1) If $n_k < n \leq n_k + m_{k-1} - n_{k-1} + 2N$, we have $m_{k-1} - n_{k-1} - 1 \leq r_n(x, \beta) \leq m_{k-1} - n_{k-1} + 2N$ by the construction of E_N. It follows that

$$\frac{m_{k-1} - n_{k-1} - 1}{n_k + m_{k-1} - n_{k-1} + 2N} \leq \frac{r_n(x, \beta)}{n} \leq \frac{m_{k-1} - n_{k-1} + 2N}{n_k}.$$

(2) If $n_k + m_{k-1} - n_{k-1} + 2N < n \leq m_k$, the construction of E_N gives $r_n(x,\beta) = n - n_k$. By (6.1.1), we have

$$\frac{m_{k-1} - n_{k-1} + 2N}{n_k + m_{k-1} - n_{k-1} + 2N} \leq \frac{r_n(x,\beta)}{n} \leq \frac{m_k - n_k}{m_k}.$$

(3) If $m_k \leq n \leq n_{k+1}$, we deduce from the construction of E_N that $m_k - n_k - 1 \leq r_n(x,\beta) \leq m_k - n_k + 2N$. Consequently,

$$\frac{m_k - n_k - 1}{n_{k+1}} \leq \frac{r_n(x,\beta)}{n} \leq \frac{m_k - n_k + 2N}{m_k}.$$

Combining the above three cases, by (6.1.3) and (6.1.4), we have

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} \geq a \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} \leq b.$$

Now we complete our proof by finding the subsequences such that the limit inferior and limit superior are reached. In fact, by (6.1.3), we get

$$\lim_{k\to\infty} \frac{r_{n_k+m_{k-1}-n_{k-1}}}{n_k + m_{k-1} - n_{k-1}} \leq \lim_{k\to\infty} \frac{m_{k-1} - n_{k-1} + 2N}{n_k + m_{k-1} - n_{k-1}} = a.$$

It follows from (6.1.4) that

$$\lim_{k\to\infty} \frac{r_{m_k}}{m_k} = \lim_{k\to\infty} \frac{m_k - n_k - 1}{m_k} = b.$$

□

Now we estimate the cardinality of the set $\mathcal{D}_k$ defined by (6.1.6). Write $q_k := \sharp\mathcal{D}_k$.

Lemma 6.1.4. *Let $\beta > 1$. Let β_N be an approximation of β. For every $\overline{\beta} < \beta_N$, there exist an integer $k(\overline{\beta}, \beta_N)$ and real numbers $c(\overline{\beta}, \beta_N)$, $c'(\overline{\beta}, \beta_N)$ such that, for all $k \geq k(\overline{\beta}, \beta_N)$, we have*

$$q_k \geq c'(\overline{\beta}, \beta_N) c(\overline{\beta}, \beta_N)^k \overline{\beta}^{\sum_{i=1}^{k-1}(n_{i+1}-m_i)}. \tag{6.1.7}$$

Proof. Recall the definition of $\mathcal{M}_d$ as (6.1.5). Theorem 3.2.1 implies

$$\sharp\mathcal{M}_d \geq \beta_N^{d-2N}$$

for all $d \geq N$. Since $\overline{\beta} < \beta_N$, there exists an integer d' which depends on $\overline{\beta}$ and β_N such that, for every $d \geq d'$, we have

$$\beta_N^{d-2N} \geq \overline{\beta}^d. \tag{6.1.8}$$

Moreover, the fact that $m_k - n_k$ is increasing and tends to $+\infty$ as $k \to +\infty$ ensures that we can find a large enough integer $k(\overline{\beta}, \beta_N)$ satisfying that, for all $k \geq k(\overline{\beta}, \beta_N)$,

$$\sharp\mathcal{M}_{m_k-n_k} \geq \beta_N^{m_k-n_k-2N} \geq \overline{\beta}^{m_k-n_k}. \tag{6.1.9}$$

Then, when $p_k \leq 2N$, we have

$$\sharp\mathcal{G}_{k+1} \geq (\sharp\mathcal{M}_{m_k-n_k})^{t_k} \geq \overline{\beta}^{(m_k-n_k)t_k} \geq \frac{1}{\overline{\beta}^{2N}}\overline{\beta}^{n_{k+1}-m_k}.$$

When $p_k > 2N$, we deduce that

$$\sharp G_{k+1} \geq (\sharp\mathcal{M}_{m_k-n_k})^{t_k} \cdot \sharp\mathcal{M}_{p_k} \geq \overline{\beta}^{(m_k-n_k)t_k} \cdot \beta_N^{p_k-2N} = \frac{1}{{\beta_N}^{d'}}\overline{\beta}^{(m_k-n_k)t_k}{\beta_N}^{p_k-2N+d'}.$$

Note that $p_k - 2N + d' > d'$. By (6.1.8), we have

$$\sharp\mathcal{G}_{k+1} \geq \frac{1}{{\beta_N}^{d'}}\overline{\beta}^{(m_k-n_k)t_k}\overline{\beta}^{p_k+d'} = \frac{\overline{\beta}^{d'}}{{\beta_N}^{d'}}\overline{\beta}^{n_{k+1}-m_k}.$$

Let $c(\overline{\beta}, \beta_N) := \min\{\frac{1}{\overline{\beta}^{2N}}, \frac{\overline{\beta}^{d'}}{{\beta_N}^{d'}}\}$. It follows that for all $k \geq k(\overline{\beta}, \beta_N)$,

$$\sharp\mathcal{G}_{k+1} \geq c(\overline{\beta}, \beta_N)\overline{\beta}^{n_{k+1}-m_k}.$$

Immediately, by the relationship between $\mathcal{D}_k$ and $\mathcal{G}_k$, for any $k \geq k(\overline{\beta}, \beta_N)$, it comes to the conclusion that

$$\begin{aligned} q_k = \sharp\mathcal{D}_k = \prod_{i=1}^{k} \sharp\mathcal{G}_k \geq \prod_{i=k(\overline{\beta},\beta_N)}^{k} \sharp\mathcal{G}_i &\geq c(\overline{\beta}, \beta_N)^{k-k(\overline{\beta},\beta_N)}\overline{\beta}^{\sum_{i=k(\overline{\beta},\beta_N)}^{k-1}(n_{i+1}-m_i)} \\ &\geq c'(\overline{\beta}, \beta_N)c(\overline{\beta}, \beta_N)^{k}\overline{\beta}^{\sum_{i=1}^{k-1}(n_{i+1}-m_i)}, \end{aligned}$$

where

$$c'(\overline{\beta}, \beta_N) = \overline{\beta}^{-\sum_{i=1}^{k(\overline{\beta},\beta_N)-1}(n_{i+1}-m_i)}.$$

□

Now we divide into three parts to complete our proof of the lower bound of $\dim_{\mathrm{H}} E_{a,b}$ by using the modified mass distribution principle (Theorem 3.1.8).

(1) Define a probability measure μ supported on E_N. Set

$$\mu([0,1)) = 1 \quad \text{and} \quad \mu(I_{n_1}(u)) = \frac{1}{\sharp\mathcal{G}_1}, \text{ for } u \in \mathcal{D}_1.$$

For each $k \geq 1$, and $u = (u_1, \dots, u_{k+1}) \in \mathcal{D}_{k+1}$, let

$$\mu(I_{n_{k+1}}(u)) = \frac{\mu(I_{n_k}(u_1, \dots, u_k))}{\sharp\mathcal{G}_{k+1}}.$$

For any $u \notin \mathcal{D}_k$ $(k \geq 1)$, let $\mu(I_{n_k}(u)) = 0$. It is routine to check that μ is well defined on E_N and it can be extended to a probability measure on $[0,1)$.

(2) Calculate the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|}$ for any $x \in E_N$. For convenience, we denote $I_n(x)$ by I_n without ambiguity. Then we have

$$\mu(I_{n_i}) = \frac{1}{q_i} \leq \frac{1}{c'(\beta_N, \overline{\beta})c(\beta_N, \overline{\beta})^i \overline{\beta}^{\sum_{j=1}^{i-1}(n_{j+1}-m_j)}} \tag{6.1.10}$$

for every $i > k(\beta_N, \overline{\beta})$, where $k(\beta_N, \overline{\beta})$ is an integer given in Lemma 6.1.4. For all $n \geq 1$, there is an integer $k \geq 1$ such that $n_k < n \leq n_{k+1}$. By the construction of E_N and the definition of μ, it is natural to estimate the lower bound of $\frac{\log \mu(I_n)}{\log |I_n|}$ by dividing into the following three cases.

Case 1. $n_k < n \leq m_k$. It follows from (6.1.10) that

$$\mu(I_n) = \mu(I_{n_k}) \leq c'(\beta_N, \overline{\beta})^{-1} c(\beta_N, \overline{\beta})^{-k} \overline{\beta}^{-\sum_{j=1}^{k-1}(n_{j+1}-m_j)}.$$

Furthermore, Theorem 3.2.8(2) implies

$$|I_n(x)| \geq |I_{m_k}(x)| = \frac{1}{\beta^{m_k}}.$$

As a consequence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\sum_{j=1}^{k-1}(n_{j+1}-m_j)\log \overline{\beta} + k \log c(\beta_N, \overline{\beta}) + \log c'(\beta_N, \overline{\beta})}{m_k \log \beta}.$$

Case 2. $n = m_k + i(m_k - n_k) + \ell$ for some $0 \leq i < t_k$ and $0 \leq \ell < m_k - n_k$. In this case, when $0 \leq \ell \leq N$, by (6.1.9) and (6.1.10), we have

$$\begin{aligned}\mu(I_n) = \mu(I_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}) \cdot \frac{1}{(\sharp \mathcal{M}_{m_k-n_k})^i} \\ &\leq c'(\beta_N, \overline{\beta})^{-1} c(\beta_N, \overline{\beta})^{-k} \overline{\beta}^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}.\end{aligned}$$

When $N < \ell < m_k - n_k$, we similarly see that

$$\begin{aligned}\mu(I_n) = \mu(I_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}) \cdot \frac{1}{(\sharp \mathcal{M}_{m_k-n_k})^i} \cdot \frac{1}{\Sigma_{\beta_N}^{\ell-2N}} \\ &\leq c'(\beta_N, \overline{\beta})^{-1} c(\beta_N, \overline{\beta})^{-k} \overline{\beta}^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)} {\beta_N}^{-\ell+2N}.\end{aligned}$$

Moreover, by (3.2.4), it holds that

$$|I_n| \geq |I_{m_k+i(m_k-n_k)+\ell}| \geq \frac{1}{\beta^{m_k+i(m_k-n_k)+\ell+2N}}.$$

Therefore,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)\log \overline{\beta}+(\ell-N)\log \beta_N+k\log c(\beta_N,\overline{\beta})+\log c'(\beta_N,\overline{\beta})}{(m_k+i(m_k-n_k)+\ell+N)\log \beta}.$$

Case 3. $n=m_k+t_k(m_k-n_k)+\ell$ where $0\leq \ell\leq p_k$. When $0\leq \ell \leq 2N$, we have

$$\begin{aligned}\mu(I_n)=\mu(I_{m_k+t_k(m_k-n_k)})&=\mu(I_{m_k}(x))\cdot\frac{1}{(\sharp\mathcal{M}_{m_k-n_k})^{t_k}}\\ &\leq c'(\beta_N,\overline{\beta})^{-1}c(\beta_N,\overline{\beta})^{-k}\overline{\beta}^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)}.\end{aligned}$$

When $2N<\ell\leq p_k$, we get

$$\begin{aligned}\mu(I_n)=\mu(I_{m_k+t_k(m_k-n_k)+\ell})&\leq \mu(I_{m_k}(x))\cdot\frac{1}{(\sharp\mathcal{M}_{m_k-n_k})^{t_k}}\cdot\frac{1}{\Sigma_{\beta_N}^{\ell-2N}}\\ &\leq c'(\beta_N,\overline{\beta})^{-1}c(\beta_N,\overline{\beta})^{-k}\overline{\beta}^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)}{\beta_N}^{-\ell+2N}.\end{aligned}$$

In addition, by (3.2.4),

$$|I_n|\geq |I_{m_k+t_k(m_k-n_k)+\ell}|\geq \frac{1}{\beta^{m_k+t_k(m_k-n_k)+\ell+N}}.$$

Hence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)\log \overline{\beta}+(\ell-2N)\log \beta_N+k\log c(\beta_N,\overline{\beta})+\log c'(\beta_N,\overline{\beta})}{(m_k+t_k(m_k-n_k)+\ell+N)\log \beta}.$$

In all three cases, using (6.1.1), we obtain

$$\liminf_{n\to\infty}\frac{\log \mu(I_n)}{\log |I_n|}\geq \lim_{k\to\infty}\frac{\sum_{j=1}^{k-1}(n_{j+1}-m_j)}{m_k}\frac{\log\overline{\beta}}{\log \beta}.$$

By (6.1.3) and (6.1.4), it immediately holds that

$$\lim_{k\to\infty}\frac{n_k}{m_k}=1-b,\quad \lim_{k\to\infty}\frac{n_{k+1}}{m_k}=\frac{b(1-a)}{a}\quad\text{and}\quad \lim_{k\to\infty}\frac{m_{k+1}}{m_k}=\frac{b(1-a)}{a(1-b)}.$$

By the Stolz-Cesàro Theorem, we have

$$\lim_{k\to\infty}\frac{\sum_{j=1}^{k-1}(n_{j+1}-m_j)}{m_k}=\lim_{k\to\infty}\frac{n_{k+1}-m_k}{m_{k+1}-m_k}=\lim_{k\to\infty}\frac{\frac{n_{k+1}}{m_k}-1}{\frac{m_{k+1}}{m_k}-1}=1-\frac{b^2(1-a)}{b-a}.$$

As a consequence,

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq \left(1 - \frac{b^2(1-a)}{b-a}\right) \frac{\log \overline{\beta}}{\log \beta}.$$

(3) Use the modified mass distribution principle (Theorem 3.1.8). We first let $\overline{\beta} \to \beta_N$, and then let $N \to \infty$. Applying Theorem 3.1.8, we finish our proof.

6.1.3 Proof of Corollary 2.3.2

Note that when $\frac{1}{2} < a \leq 1$, the inequality (6.1.2) implies $E_a = \emptyset$. We only need to consider the case $0 \leq a \leq \frac{1}{2}$. By Lemma 6.1.1, we have

$$E_a = \left\{ x \in [0,1) : \hat{v}_\beta(x) = \frac{a}{1-a} \right\}.$$

Thus, applying Theorem 3.2.14, we have, for all $0 < a \leq \frac{1}{2}$,

$$\dim_{\mathrm{H}} E_a = \dim_{\mathrm{H}} \left\{ x \in [0,1) : \hat{v}_\beta(x) = \frac{a}{1-a} \right\} = (1-2a)^2.$$

When $a = 0$, by noting that $E_{0,0} \subseteq E_0$, we deduce that E_0 has full Lebesgue measure and thus has Huasdorff dimension 1.

6.2 Proof of Theorem 2.3.3

Let $\beta > 1$. Define $M = \min\{i > 1 : \varepsilon_i^*(\beta) > 0\}$. For all $k \geq 1$, let Γ_k be defined by (2.1.3). We choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_k > 2k + \Gamma_k$ and $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$. In addition, $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ satisfy

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = 0,$$

and

$$\lim_{k\to\infty} \frac{m_k - n_k}{m_k} = 1.$$

In fact, let

$$n_k' = (2k + \Gamma_k)^{2k} \quad \text{and} \quad m_k' = (2k + 2 + \Gamma_{k+1})^{2k+1}.$$

Then by small adjustments, we can obtain the required sequences.

For all $k \geq 1$, write $n_{k+1} = (m_k - n_k)t_k + n_k + p_k$ where $0 \leq p_k < m_k - n_k$. Now we define

$$U := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Sigma_\beta^k} \mathrm{int}\left(I_{n_{k+1}}(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k})\right),$$

where $\mathrm{int}(I_{|\omega|}(\omega))$ stands for the interior of $I_{|\omega|}(\omega)$ for all $\omega \in \Sigma_\beta^*$.

Remark 6.2.1. For all $(\varepsilon_1, \ldots, \varepsilon_k) \in \Sigma_\beta^k$, it follows from Theorem 3.2.8(3) that $(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k})$ is full since $n_k > 2k + \Gamma_k$. Note that $m_k - n_k \geq M$. Then the word $(1, 0^{m_k-n_k-1})$ is full. By Theorem 3.2.8 (1), the word 0^{p_k} is full. Therefore, $(\varepsilon_1, \ldots, \varepsilon_k, n_k - k, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k})$ can be concatenated by any β-admissible word. Thus U is well defined.

The set $\mathrm{int}(I_{|\omega|}(\omega))$ is open for all $\omega \in \Sigma^*$ implies that U is a G_δ set. Consequently, it suffices to show that U is a subset of $E_{0,1}$ and is dense in $[0, 1]$.

Lemma 6.2.2. *The set U is a subset of $E_{0,1}$.*

Proof. For any $x \in U$, it follows from the construction of U that there exist infinitely many k such that $\varepsilon(x, \beta) = (\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k})$ for some $(\varepsilon_1, \ldots, \varepsilon_k) \in \Sigma_\beta^k$. Now we are going to give the upper limit and lower limit of $\frac{r_n(x,\beta)}{n}$.

Let $n = n_{k+1} + m_k - n_k - 1$. Since $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$, we obtain

$$r_{n_{k+1}+m_k-n_k-1}(x, \beta) = m_k - n_k - 1.$$

As a result,

$$\liminf_{n\to\infty} \frac{r_n(x, \beta)}{n} \leq \lim_{k\to\infty} \frac{r_{n_{k+1}+m_k-n_k-1}(x, \beta)}{n_{k+1} + m_k - n_k - 1} = \lim_{k\to\infty} \frac{m_k - n_k - 1}{n_{k+1} + m_k - n_k - 1} = 0.$$

Let $n = m_k$. Note that $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$. The definition of $r_n(x, \beta)$ shows that

$$r_{m_k}(x, \beta) = m_k - n_k - 1.$$

It therefore follows that

$$\limsup_{n\to\infty} \frac{r_n(x, \beta)}{n} \geq \lim_{k\to\infty} \frac{r_{m_k}(x, \beta)}{m_k} = \lim_{k\to\infty} \frac{m_k - n_k - 1}{m_k} = \lim_{k\to\infty} \frac{m_k - n_k - 1}{m_k} = 1.$$

By the above discussion, we conclude that

$$\liminf_{n\to\infty} \frac{r_n(x, \beta)}{n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_n(x, \beta)}{n} = 1.$$

Hence, $x \in E_{0,1}$ which gives $U \subseteq E_{0,1}$. □

Proof of Theorem 2.3.3 It remains to show that for all $n \geq 1$, the set

$$U_n = \bigcup_{k=n}^{\infty} \bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Sigma_\beta^k} \mathrm{int}\left(I_{n_{k+1}}\left(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k})^{t_k}, 0^{p_k}\right)\right)$$

is dense in [0,1]. Now we will concentrate on finding a real number $y \in U$ such that $|x-y| \leq r$ for every $x \in [0, 1)$ and $r > 0$. Assume $\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots)$. Let ℓ'

be an integer satisfying $\beta^{-\ell'} \le r$. Let $\ell = \max\{n, \ell'\}$. Since $(\varepsilon_1(x,\beta), \ldots, \varepsilon_\ell(x,\beta)) \in \Sigma_\beta^\ell$, we choose a point

$$y \in \operatorname{int}\left(I_{n_{\ell+1}}\left(\varepsilon_1, \ldots, \varepsilon_\ell, 0^{n_\ell - \ell}, (1, 0^{m_\ell - n_\ell})^{t_\ell}, 0^{p_\ell}\right)\right).$$

Then it holds that $|x-y| \le \beta^{-\ell} \le r$ and $y \in U_n$. To sum up, the set

$$\bigcup_{k=n}^{\infty} \bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Sigma_\beta^k} \operatorname{int}\left(I_{n_{k+1}}\left(\varepsilon_1, \ldots, \varepsilon_k, 0^{n_k - k}, (1, 0^{m_k - n_k})^{t_k}, 0^{p_k}\right)\right)$$

is dense in $[0,1]$.

Thus, we can conclude by the Baire Category Theorem that U is residual in $[0,1]$. Then, $E_{0,1}$ is residual in $[0,1]$ by Lemma 6.2.2.

□

6.3 Proof of Theorem 2.3.4

As the same discussion at the first part of Section 3, it holds that $\dim_{\mathrm{H}} E_{0,0}^P$ is of full Lebesgue measure by using the result of Cao and Chen [21] that the set

$$\left\{\beta \in (1,2) : \lim_{n\to\infty} \frac{r_n(\beta)}{\log_\beta n} = 1\right\}$$

is of full Lebesgue measure. By the same argument as the proof of Theorem 2.3.1 for the case $a > \frac{b}{1+b},\ 0 < b \le 1$ in the end of Section 3.1, we get that $E_{a,b}^P$ is empty when $a > \frac{b}{1+b},\ 0 < b \le 1$.

When $0 < a \le \frac{b}{1+b},\ 0 < b < 1$, Lemmas 6.1.1 and 6.1.2 imply that

$$E_{a,b}^P = \left\{\beta \in (1,2) : \hat{v}_\beta(1) = \frac{a}{1-a},\ v_\beta(1) = \frac{b}{1-b}\right\} = U\left(\frac{a}{1-a}, \frac{b}{1-b}\right).$$

Then by Theorem 3.2.23, it holds that

$$\dim_{\mathrm{H}} E_{a,b}^P = \dim_{\mathrm{H}} U\left(\frac{a}{1-a}, \frac{b}{1-b}\right) = 1 - \frac{b^2(1-a)}{b-a}.$$

But Theorem 3.2.23 is not applicable for the case of $a = 0,\ 0 < b < 1$ and $0 < a \le \frac{1}{2},\ b = 1$. Note that $E_{a,1}^P \subseteq F_1^P$ where F_1^P is defined by (2.3.2). So we first give the Hausdorff dimension of F_1^P. Since

$$F_1^P = \{\beta \in (1,2) : v_\beta(1) = +\infty\} \subseteq \{\beta \in (1,2) : v_\beta(1) \ge v\}$$

for all $v > 0$, we have

$$\dim_{\mathrm{H}} F_1^P \le \dim_{\mathrm{H}} \{\beta \in (1,2) : v_\beta(1) \ge v\} = \frac{1}{1+v}$$

where the last equality follows from Theorem 3.2.22. Letting $v \to +\infty$, we have $\dim_{\mathrm{H}} F_1^P \leq 0$. This implies $\dim_{\mathrm{H}} E_{a,1}^P \leq 0$. In conclusion, $\dim_{\mathrm{H}} E_{a,1}^P = 0$ for any $0 < a \leq \frac{1}{2}$. For the other case, we have

$$E_{0,b}^P \subseteq \left\{\beta \in (1,2) : v_\beta(1) \geq \frac{b}{1-b}\right\}.$$

By Theorem 3.2.22, we deduce that the upper bound of $\dim_{\mathrm{H}} E_{0,b}^P$ is $1-b$ for all $0 < b < 1$. Hence, we only need to give the lower bound of $\dim_{\mathrm{H}} E_{0,b}$ for all $0 < b < 1$. We also include our proof of the case $0 < a \leq \frac{b}{1+b},\ 0 < b < 1$.

For every $1 < \beta_1 < \beta_2 < 2$, instead of dealing with the Hausdorff dimension of the set $E_{a,b}^P$ directly, we will technically investigate the Hausdorff dimension of the following set. For all $0 \leq a \leq \frac{b}{1+b},\ 0 < b \leq 1$, let

$$E_{a,b}^P(\beta_1,\beta_2) = \left\{\beta \in [\beta_1,\beta_2) : \liminf_{n\to\infty} \frac{r_n(\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(\beta)}{n} = b\right\}. \tag{6.3.1}$$

Let $\widetilde{\beta}_N$ be an approximation of β_2 and N is large enough such that $\widetilde{\beta}_N \geq \beta_1$.

For every $k \geq 1$, similar to what we did in Section 3.2, we take two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_1 > 2N$ and $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$. In addition,

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = a \quad \text{and} \quad \lim_{k\to\infty} \frac{m_k - n_k}{m_k} = b.$$

We can choose such two sequences by the same way in Section 3.2.

Now let us construct a Cantor set contained in $E_{a,b}^P(\beta_1,\beta_2)$ as follows.

For any integer $d > 2N$, we set

$$\mathcal{M}'_d = \{\omega = (\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2)-1,\omega_1,\ldots,\omega_{d-2N},0^N) : (\omega_1,\ldots,\omega_{d-2N}) \in \Sigma_{\widetilde{\beta}_N}^{d-2N}\}. \tag{6.3.2}$$

Let

$$\mathcal{G}'_1 = \{(\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2),\omega_1,\ldots,\omega_{d-2N},0^N) : (\omega_1,\ldots,\omega_{d-2N}) \in \Sigma_{\widetilde{\beta}_N}^{d-2N}\}.$$

Note that $\left(\varepsilon_1^*(\widetilde{\beta}_N),\ldots,\varepsilon_N^*(\widetilde{\beta}_N)\right) <_{\mathrm{lex}} (\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2))$. Now we give some observations on the elements in G'_1 as follows.

Remark 6.3.1. (1) For all $\omega \in \mathcal{G}'_1$, since $\left(\omega_1,\ldots,\omega_{d-2N},0^N\right) \in \Sigma_{\widetilde{\beta}_N}^{d-N}$ $(d > 2N)$, by Proposition 3.2.17, ω is self-admissible.

(2) For every $u \in \mathcal{M}'_d$ $(d > 2N)$, we have $u \in \Sigma_{\widetilde{\beta}_N}^d$. By Lemma 3.2.17, the word $\omega * u$ is still self-admissible for every all $\omega \in \mathcal{G}'_1$.

For every $k \geq 1$, write $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ with $0 \leq p_k < m_k - n_k$, then define $\mathcal{G}'_{k+1} =$

$$\{u_{k+1} = \Big(\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2) - 1, 0^{m_k - n_k - N}, u_k^{(1)}, \ldots, u_k^{(t_k)}, u_k^{(t_k+1)}\Big) : \ u_k^{(i)} \in \mathcal{M}'_{m_k - n_k}, \ 1 \leq i \leq t_k\},$$

where

$$u_k^{(t_k+1)} = \begin{cases} 0^{p_k}, & \text{when } \ p_k \leq 2N; \\ \omega \in \mathcal{M}'_{p_k}, & \text{when } \ p_k > 2N. \end{cases}$$

Let

$$\mathcal{D}'_k = \{(u_1, \ldots, u_k) : u_i \in \mathcal{G}'_i, \ 1 \leq i \leq k\}. \tag{6.3.3}$$

Notice that every $u_k \in \mathcal{G}'_k$ ends with 0^N. This guarantees that $(u_1, \ldots, u_k)$ can concatenate with any u_{k+1} to be a new self-admissible word. As a result, the set $\mathcal{D}'_k$ is well-defined.

As the classical technique of constructing a Cantor set, let

$$E(\beta_1, \beta_2) = \bigcap_{k=1}^{\infty} \bigcup_{u \in D'_k} I^P_{n_k}(u).$$

Similar to the process of Section 3, we now give the following result which means that $E(\beta_1, \beta_2)$ is a subset of $E^P_{a,b}(\beta_1, \beta_2)$.

Lemma 6.3.2. *For every $1 < \beta_1 < \beta_2 < 2$, $E(\beta_1, \beta_2) \subseteq E^P_{a,b}(\beta_1, \beta_2)$ for all $0 \leq a \leq \frac{b}{1+b}$ and $0 < b < 1$.*

Proof. The proof is just as the same as the proof of Lemma 6.1.3 by dividing into three cases. We omit it here. □

Analogously, we now focus on the estimation of the cardinality of the set $\mathcal{D}'_k$. Let $q'_k := \sharp \mathcal{D}'_k$. We obtain the following lemma.

Lemma 6.3.3. *For every $1 < \beta_1 < \beta_2 < 2$, let $\widetilde{\beta}_N$ be the real number defined in this section. Then there exist an integer $k(\beta_1, \widetilde{\beta}_N)$ and real numbers $c(\beta_1, \widetilde{\beta}_N), c'(\beta_1, \widetilde{\beta}_N)$ such that, for every $k \geq k(\beta_1, \widetilde{\beta}_N)$, we have*

$$q'_k \geq c'(\beta_1, \widetilde{\beta}_N) c(\beta_1, \widetilde{\beta}_N)^k \beta_1^{\sum_{i=1}^{k-1}(n_{i+1} - m_i)}. \tag{6.3.4}$$

Proof. We use the similar method as Lemma 6.1.4, the details are left to the readers. □

Let

$$C(\beta_1) = \frac{(\beta_1 - 1)^2}{\beta_1}.$$

Notice that $\underline{\beta}(u) \geq \beta_1 > 1$ for any $u = (u_1, \ldots, u_n) \in \Lambda_n(\beta_1, \beta_2)$ where $\Lambda_n(\beta_1, \beta_2)$ is defined by (3.2.5). Then

$$C(\underline{\beta}(u)) = \frac{(\underline{\beta}(u) - 1)^2}{\underline{\beta}(u)} \geq C(\beta_1).$$

The following lemma gives the estimation of the length of the cylinders with non-empty intersection with the Cantor set $E(\beta_1, \beta_2)$ which will be useful to estimate the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu(B(\beta,r))}{\log |r|}$ for any $r > 0$ and $\beta \in E(\beta_1, \beta_2)$.

Lemma 6.3.4. *For any $\beta \in E(\beta_1, \beta_2)$, suppose that $\varepsilon(1, \beta) = (u_1, u_2, \ldots)$. Then we have*

$$|I_n^P(u_1, \ldots, u_n)| \geq C(\beta_1)\beta_2^{-(n+N)}$$

for any $n \geq 1$.

Proof. For any $n \geq 1$, we are going to take the word $(u_1, \ldots, u_n, 0^N)$ into account. We claim that the word $(u_1, \ldots, u_n, 0^N)$ is non-recurrent.

In fact, by the construction of $E(\beta_1, \beta_2)$, for any $1 \leq i < n$, we have

$$\sigma^i(u_1, \ldots, u_n, 0^N) \in \Sigma_{\widetilde{\beta}_N}^{n-i+N}.$$

Notice that $\omega \leq_{\text{lex}} (\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2) - 1) <_{\text{lex}} (\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2))$ for any $\omega \in \Sigma_{\widetilde{\beta}_N}^n$ with $n \geq N$. It comes to the conclusion that $\sigma^i(u_1, \ldots, u_n, 0^N) <_{\text{lex}} (\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2))$ for any $1 \leq i < n + N$ which implies that $(u_1, \ldots, u_n, 0^N)$ is non-recurrent. Thus, by Lemma 3.2.21(2), we have

$$|I_n^P(u_1, \ldots, u_n)| \geq |I_{n+N}^P(u_1, \ldots, u_n, 0^N)| \geq C(u_1, \ldots, u_n, 0^N)\overline{\beta}(u_1, \ldots, u_n, 0^N)^{-(n+N)}.$$

It follows from the fact $\overline{\beta}(u_1, \ldots, u_n, 0^N) \leq \beta_2$ that

$$|I_n^P(u_1, \ldots, u_n)| \geq C(\beta_1)\beta_2^{-(n+N)}.$$

□

Let us now focus on giving the lower bound of $\dim_{\mathrm{H}} E(\beta_1, \beta_2)$. As the conventional process, we define a measure supported on $E(\beta_1, \beta_2)$ which is similar to Section 5.1.2 by distributing the mass uniformly. We will give the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu\left(I_n^P(u)\right)}{\log |I_n^P(u)|}$ for any cylinder $I_n^P(u)$ which has non-empty intersection with $E(\beta_1, \beta_2)$. Without any confusion, here and subsequently, I_n^P stands for the cylinder $I_n^P(u)$ for all $u \in \Lambda_n$.

(1) Define a probability measure supported on $E(\beta_1, \beta_2)$. Let

$$\mu([\beta_1, \beta_2)) = 1 \quad \text{and} \quad \mu(I^P_{n_1}(u)) = \frac{1}{\sharp G'_1}, \text{ for } u \in D'_1.$$

For all $k \geq 1$, and $u = (u_1, \ldots, u_{k+1}) \in \mathcal{D}'_{k+1}$, define

$$\mu(I^P_{n_{k+1}}(u)) = \frac{\mu(I^P_{n_k}(u_1, \ldots, u_k))}{\sharp \mathcal{G}'_{k+1}}.$$

(2) Estimate the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu(I^P_n)}{\log |I^P_n|}$ where $I^P_n \cap E(\beta_1, \beta_2) \neq \emptyset$. It follows from the definition of the measure that

$$\mu(I^P_{n_i}) = \frac{1}{q'_i} \leq \frac{1}{c'(\beta_1, \widetilde{\beta}_N) c(\beta_1, \widetilde{\beta}_N)^i \beta_1^{\sum_{j=1}^{i-1}(n_{j+1}-m_j)}}, \tag{6.3.5}$$

for every $i > k(\beta_1, \widetilde{\beta}_N)$. For any $\beta \in E(\beta_1, \beta_2)$, suppose that $\varepsilon(1, \beta) = (u_1, u_2, \ldots)$. For each $n \geq 1$, there exists an integer $k \geq 1$ such that $n_k < n \leq n_{k+1}$. It falls naturally into three cases.

Case 1. $n_k < n \leq m_k$. It follows from (6.3.5) that

$$\mu(I^P_n) = \mu(I^P_{n_k}) \leq c'(\beta_1, \widetilde{\beta}_N)^{-1} c(\beta_1, \widetilde{\beta}_N)^{-k} \beta_1^{-\sum_{j=1}^{k-1}(n_{j+1}-m_j)}.$$

Furthermore, by the construction of $E(\beta_1, \beta_2)$, the word $(u_1, \ldots, u_{m_k})$ is non-recurrent. Thus, by Lemma 3.2.21, we have

$$|I^P_n| \geq |I^P_{m_k}| \geq C(u_1, \ldots, u_{m_k}) \overline{\beta}(u_1, \ldots, u_{m_k})^{-m_k} \geq c(\beta_1) \beta_2^{-m_k}.$$

As a consequence,

$$\frac{\log \mu(I^P_n)}{\log |I^P_n|} \geq \frac{\log c'(\beta_1, \widetilde{\beta}_N) + k \log c(\beta_1, \widetilde{\beta}_N) + \sum\limits_{j=1}^{k-1}(n_{j+1} - m_j) \log \beta_1}{\log c(\beta_1) + m_k \log \beta_1}.$$

Case 2. $n = m_k + i(m_k - n_k) + \ell$ for some $0 \leq i < t_k$ and $0 \leq \ell < m_k - n_k$. On the one hand, when $0 \leq \ell \leq 2N$, we have

$$\begin{aligned}\mu(I^P_n) = \mu(I^P_{m_k + i(m_k-n_k)+\ell}) &\leq \mu(I^P_{m_k}) \cdot \frac{1}{(\sharp M_{m_k-n_k})^i} \\ &\leq c'(\beta_1, \widetilde{\beta}_N)^{-1} c(\beta_1, \widetilde{\beta}_N)^{-k} \beta_1^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}.\end{aligned}$$

On the other hand, when $2N < \ell < m_k - n_k$, we have

$$\begin{aligned}\mu(I^P_n) = \mu(I^P_{m_k + i(m_k-n_k)+\ell}) &\leq \mu(I^P_{m_k}) \cdot \frac{1}{(\sharp M_{m_k-n_k})^i} \cdot \frac{1}{\Sigma^{\ell-2N}_{\widetilde{\beta}_N}} \\ &\leq c'(\beta_1, \widetilde{\beta}_N)^{-1} c(\beta_1, \widetilde{\beta}_N)^{-k} \beta_1^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)} \widetilde{\beta}_N^{-\ell+2N}.\end{aligned}$$

Moreover, by Lemma 6.3.4,

$$|I_n^P| = |I^P_{m_k+i(m_k-n_k)+\ell}| \geq C(\beta_1){\beta_2}^{-(m_k+i(m_k-n_k)+\ell+N)}.$$

Hence,

$$\frac{\log\mu(I_n^P)}{\log|I_n^P|} \geq \frac{\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)\log\beta_1+(\ell-2N)\log\widetilde{\beta}_N+k\log c(\beta_1,\widetilde{\beta}_N)+\log c'(\beta_1,\widetilde{\beta}_N)}{\log C(\beta_1)+(m_k+i(m_k-n_k)+\ell+N)\log\beta_2}.$$

Case 3. $n=m_k+t_k(m_k-n_k)+\ell$ for some $0\leq\ell\leq p_k$. Similarly, when $0\leq\ell\leq 2N$, we have

$$\begin{aligned}\mu(I_n^P) = \mu(I^P_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I^P_{m_k})\cdot\frac{1}{(\sharp M_{m_k-n_k})^{t_k}}\\ &\leq c'(\beta_1,\widetilde{\beta}_N)^{-1}c(\beta_1,\widetilde{\beta}_N)^{-k}\beta_1^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)}.\end{aligned}$$

When $2N<\ell\leq p_k$, it follows that

$$\begin{aligned}\mu(I_n^P) = \mu(I^P_{m_k+t_k(m_k-n_k)+\ell}) &\leq \mu(I^P_{m_k})\cdot\frac{1}{(\sharp M_{m_k-n_k})^{i}}\cdot\frac{1}{\Sigma^{\ell-2N}_{\widetilde{\beta}_N}}\\ &\leq c'(\beta_1,\widetilde{\beta}_N)^{-1}c(\beta_1,\widetilde{\beta}_N)^{-k}\beta_1^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}\widetilde{\beta}_N^{-\ell+2N}.\end{aligned}$$

Furthermore, we conclude from Lemma 6.3.4 that

$$|I_n^P| = |I^P_{m_k+t_k(m_k-n_k)+\ell}| \geq C(\beta_1){\beta_2}^{-(m_k+t_k(m_k-n_k)+\ell+N)}.$$

Therefore, we have

$$\frac{\log\mu(I_n^P)}{\log|I_n^P|} \geq \frac{\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)\log\beta_1+(\ell-2N)\log\tilde{\beta}_N+k\log c(\beta_1,\widetilde{\beta}_N)+\log c'(\beta_1,\widetilde{\beta}_N)}{\log C(\beta_1)+(m_k+t_k(m_k-n_k)+\ell+N)\log\beta_2}.$$

Just proceeding as the same analysis in Section 3.2, for all the above three cases, we obtain

$$\liminf_{n\to\infty}\frac{\log\mu(I_n^P)}{\log|I_n^P|} \geq \left(1-\frac{b^2(1-a)}{b-a}\right)\frac{\log\beta_1}{\log\beta_2}.$$

(3) Use the mass distribution principle (see [27, Page 60]). Now we take any $B(\beta,r)$ with center $\beta\in E(\beta_1,\beta_2)$ and sufficiently small enough r verifying

$$|I^P_{n+1}| \leq r < |I_n^P| \leq \beta_1^{-n+1}, \tag{6.3.6}$$

where the last inequalities is guaranteed by the fact that $\overline{\beta}(\omega) \geq \beta_1$ for any $\omega \in \Lambda_n(\beta_1, \beta_2)$. By Lemma 6.3.4, we have

$$|I_n^P| \geq C(\beta_1)\beta_2^{-(n+N)}.$$

As a result, the ball $B(\beta, r)$ intersects no more than $2\left\lfloor C(\beta_1)^{-1}\beta_2^N \left(\frac{\beta_2}{\beta_1}\right)^{n-1}\right\rfloor + 2$ cylinders of order n. Moreover, it follows from Lemma 6.3.4 that

$$r \geq |I_{n+1}^P| \geq C(\beta_1)\beta_2^{-(n+1+N)}. \tag{6.3.7}$$

Immediately, the combination of (6.3.6) and (6.3.7) gives

$$\begin{aligned}
&\liminf_{r\to 0} \frac{\log \mu\left(B(\beta, r)\right)}{\log r} \\
&\geq \liminf_{n\to\infty} \frac{\log\left(2\left\lfloor C(\beta_1)^{-1}\beta_2^N \left(\frac{\beta_2}{\beta_1}\right)^{n-1}\right\rfloor + 2\right) + \log \mu\left(I_n^P\right)}{\log |I_n^P|} \cdot \frac{\log |I_n^P|}{-\log C(\beta_1) + (n+1+N)\log \beta_2} \\
&\geq \liminf_{n\to\infty} \left(\frac{(n-1)(\log\beta_2 - \log\beta_1)}{-\log C(\beta_1) + (n+N)\log\beta_2} + \frac{\log \mu\left(I_n^P\right)}{\log |I_n^P|}\right) \cdot \frac{(n-1)\log\beta_1}{-\log C(\beta_1) + (n+1+N)\log\beta_2} \\
&\geq \left(\frac{\log\beta_2 - \log\beta_1}{\log\beta_2} + \left(1 - \frac{b^2(1-a)}{b-a}\right)\frac{\log\beta_1}{\log\beta_2}\right)\frac{\log\beta_1}{\log\beta_2}.
\end{aligned}$$

Therefore, by the mass distribution principle and letting $\beta_1 \to \beta_2$, we get our desired result.

6.4 Proof of Theorem 2.3.6

Akin to Section 5, we need to find a subset of $E_{0,1}^P$ which is a dense G_δ set in the interval $[1, 2]$. Since the process of our proof is almost the same as Section 5. We only provide the construction of the required set V' in this section.

For all $k \geq 1$, we first choose the sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k\}$ and $n_k < m_k < n_{k+1}$. In addition, the sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ is chosen to satisfy

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = 0,$$

and

$$\lim_{k\to\infty} \frac{m_k - n_k}{m_k} = 1.$$

Actually, let

$$n_k' = k^{2k} \quad \text{and} \quad m_k' = (k+1)^{2k+1}.$$

We can obtain the required sequences with some adjustments.

For all $k \geq 1$, denote $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ where $0 \leq p_k < m_k - n_k$. We now define

$$V' = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\varepsilon_1,\ldots,\varepsilon_k)\in\Lambda_k(1,2)} \mathrm{int}\left(I_{n_{k+1}}^{P}\left(\varepsilon_1,\ldots,\varepsilon_k,0^{n_k-k},(1,0^{m_k-n_k})^{t_k},0^{p_k}\right)\right)$$

where $\Lambda_k(1,2)$ is defined by (3.2.5). Since $m_k - n_k > n_k - k$, we get $(1, 0^{m_k-n_k}) <_{\mathrm{lex}} (\varepsilon_1, \cdots, \varepsilon_k, 0^{n_k-k})$ for all $k \geq 1$. By Lemma 3.2.17, the set V' is well defined.

Chapitre 7 Uniform recurrence properties of beta-expansions

7.1 Proof of Theorems 2.4.3

Before our proof, we will give some useful lemmas.

Lemma 7.1.1. *For any $x \in [0,1)$ whose β-expansion is not periodic and $r_\beta(x) > 0$, there exist two sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ such that*

$$r_\beta(x) = \limsup_{k\to\infty} \frac{m_k - n_k}{n_k} \tag{7.1.1}$$

and

$$\hat{r}_\beta(x) = \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}}. \tag{7.1.2}$$

Proof. Assume that $\varepsilon(x,\beta) = (\varepsilon_1, \varepsilon_2, \ldots)$. Let

$$n_1 = \min\{n \geq 1 : \varepsilon_{n+1} = \varepsilon_1\}, \quad m_1 = \max\{n \geq n_1 : |T_\beta^{n_1} x - x| < \beta^{-(n-n_1)}\}.$$

Suppose that for all $k \geq 1$, n_k and m_k have been defined. Set

$$n_{k+1} = \min\{n \geq n_k : \varepsilon_{n+1} = \varepsilon_1\},\ m_{k+1} = \max\{n \geq n_{k+1} : |T_\beta^{n_k} x - x| < \beta^{-(n-n_k)}\}.$$

Note that $r_\beta(x) > 0$. We always can find the position n_k such that ε_{n_k+1} returns to ε_1 which means n_k is well defined. Since β^{-n} is decreasing to 0 as n goes to infinity and $\varepsilon(x,\beta)$ is not periodic, m_k is well defined. By the definitions of n_k and m_k, for all $k \geq 1$, we have

$$\beta^{-(m_k - n_k)-1} \leq |T_\beta^{n_k} x - x| < \beta^{-(m_k - n_k)}.$$

Now we choose two subsequences $\{n_{i_k}\}_{k=1}^{\infty}$ and $\{m_{i_k}\}_{k=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ such that $\{m_{i_k} - n_{i_k}\}_{k=1}^{\infty}$ is not decreasing. Let $i_1 = 1$. Assume that i_k has been defined. Let

$$i_{k+1} = \min\{i > i_k : m_i - n_i > m_{i_k} - n_{i_k}\}.$$

Since $r_\beta(x) > 0$, it follows that $m_k - n_k$ goes to infinity as $k \to +\infty$. So i_{k+1} is well defined. Then we have the sequence $\{m_{i_k} - n_{i_k}\}_{k=1}^{\infty}$ is not decreasing. Without causing any confusion, we still use the same symbols $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ to substitute the subsequences $\{n_{i_k}\}_{k=1}^\infty$ and $\{m_{i_k}\}_{k=1}^\infty$. We claim that

$$r_\beta(x) = \limsup_{k\to\infty} \frac{m_k - n_k}{n_k} \quad \text{and} \quad \hat{r}_\beta(x) = \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}}.$$

In fact, assume that

$$\limsup_{k\to\infty} \frac{m_k - n_k}{n_k} = c.$$

On the one hand, there is a subsequence $\{j_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \frac{m_{j_k} - n_{j_k}}{n_{j_k}} = c.$$

This implies that for all $\delta > 0$, there is an integer k_0, for each $k \geq k_0$, $m_{j_k} - n_{j_k} \geq (c-\delta)n_{j_k}$. So

$$|T_\beta^{n_{j_k}} x - x| < \beta^{n_{j_k} - m_{j_k}} \leq \beta^{-(c-\delta)n_{j_k}}.$$

Consequently, $r_\beta(x) \geq c - \delta$ for all $\delta \geq 0$. On the other hand, there is an integer k_0, for any $k \geq k_0$, we have $m_k - n_k \leq (c+\delta)n_k$. So for all $n \geq n_{k_0}$, there is an integer k such that $n_k \leq n < n_{k+1}$. This means

$$|T_\beta^n x - x| \geq \beta^{-(m_k - n_k)-1} \geq \beta^{-(c+\delta)n_k}.$$

Hence, $r_\beta(x) < c + \delta$ for any $\delta > 0$. Immediately, $r_\beta(x) = c$. The same argument can deduce the equality (7.1.2), we leave it to the readers. □

Lemma 7.1.2. *For all $x \in [0,1)$ whose β-expansion is not periodic and $r_\beta(x) > 0$, let $\{m_k\}_{k=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ be defined in Lemma 7.1.1. Then there is a sequence $\{t_k\}_{k=1}^\infty$ such that when $t_k \geq m_k$, we have*

$$(\varepsilon_1, \ldots, \varepsilon_{m_k}) = (\varepsilon_1, \ldots, \varepsilon_{n_k}, \varepsilon_1, \ldots, \varepsilon_{m_k - n_k}). \tag{7.1.3}$$

When $t_k < m_k$, we have

$$(\varepsilon_1, \ldots, \varepsilon_{m_k}) = \left(\varepsilon_1, \ldots, \varepsilon_{n_k}, \varepsilon_1, \ldots, \varepsilon_{t_k - n_k}, \varepsilon_{t_k - n_k + 1} - 1, \varepsilon_1^*, \ldots, \varepsilon_{m_k - t_k - 1}^*\right). \tag{7.1.4}$$

or

$$(\varepsilon_1, \ldots, \varepsilon_{m_k}) = (\varepsilon_1, \ldots, \varepsilon_{n_k}, \varepsilon_1, \ldots, \varepsilon_{t_k}, \varepsilon_{t_k - n_k + 1} + 1, 0^{m_k - n_k - t_k - 1}). \tag{7.1.5}$$

Proof. Let

$$t_k = \max\{n > n_k : (\varepsilon_{n_k+1} \ldots, \varepsilon_n) = (\varepsilon_1, \ldots, \varepsilon_{n-n_k})\}. \tag{7.1.6}$$

That is, the position t_k is chosen to make sure that $(\varepsilon_{n_k+1}, \ldots, \varepsilon_{t_k})$ is the maximal block after position n_k which returns to $(\varepsilon_1, \ldots, \varepsilon_{t_k - n_k})$. Since $\varepsilon(x,\beta)$ is not periodic, t_k is well defined. By the definition of t_k, it holds that

$$(\varepsilon_1, \ldots, \varepsilon_{t_k}) = (\varepsilon_1, \ldots, \varepsilon_{n_k}, \varepsilon_1, \ldots, \varepsilon_{t_k - n_k}), \ \varepsilon_{t_k+1} \neq \varepsilon_{t_k - n_k + 1}. \tag{7.1.7}$$

Then $T_\beta^{n_k}x$ and x belong to the interval $I_{t_k-n_k}(\varepsilon_1,\dots,\varepsilon_{t_k-n_k})$ which implies

$$\beta^{-(m_k-n_k)-1}\le |T_\beta^{n_k}x-x|\le |I_{t_k-n_k}(\varepsilon_1,\dots,\varepsilon_{t_k-n_k})|\le \beta^{-(t_k-n_k)}.$$

Thus, $t_k\le m_k+1$.

When $t_k\ge m_k$, it follows from (7.1.7) that

$$(\varepsilon_1,\dots,\varepsilon_{m_k})=(\varepsilon_1,\dots,\varepsilon_{n_k},\varepsilon_1,\dots,\varepsilon_{m_k-n_k}).$$

When $t_k<m_k$ and $|\varepsilon_{t_k+1}-\varepsilon_{t_k-n_k+1}|\ge 2$, by (7.1.7), there is a full basic interval of order t_k-n_k+1 between x and $T_\beta^{n_k}$. As a consequence,

$$|T_\beta^{n_k}x-x|\ge \beta^{-(t_k-n_k+1)}\ge \beta^{-(m_k-n_k)},$$

which contradicts with the definition of m_k. Hence, $|\varepsilon_{t_k+1}-\varepsilon_{t_k-n_k+1}|=1$.

When $\varepsilon_{t_k-n_k+1}=\varepsilon_{t_k+1}+1$. By the definition of t_k, we have

$$x=\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k-n_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k-n_k+1}}{\beta^{t_k-n_k+1}}+\cdots\ge\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k-n_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k+1}+1}{\beta^{t_k-n_k+1}}.$$

Noting that $|T_\beta^{n_k}x-x|=x-T_\beta^{n_k}x<\beta^{-(m_k-n_k)}$, we have

$$T_\beta^{n_k}x>x-\beta^{-(m_k-n_k)}\ge\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k+1}+1}{\beta^{t_k-n_k+1}}-\frac{1}{\beta^{m_k-n_k}}$$

Note that

$$\frac{1}{\beta^{t_k-n_k+1}}-\frac{1}{\beta^{m_k-n_k}}=\frac{\varepsilon_{t_k+1}}{\beta^{t_k-n_k+1}}+\frac{\varepsilon_1^*}{\beta^{t_k-n_k+2}}+\cdots+\frac{\varepsilon^*_{m_k-n_k-t_k-1}-1}{\beta^{m_k-n_k}}+\cdots.$$

Thus,

$$T_\beta^{n_k}x>\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k-n_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k+1}}{\beta^{t_k-n_k+1}}+\frac{\varepsilon_1^*}{\beta^{t_k-n_k+2}}+\cdots+\frac{\varepsilon^*_{m_k-n_k-t_k-1}-1}{\beta^{m_k-n_k}}+\cdots.$$

Then,

$$(\varepsilon_1,\dots,\varepsilon_{m_k})=\left(\varepsilon_1,\dots,\varepsilon_{n_k},\varepsilon_1,\dots,\varepsilon_{t_k-n_k},\varepsilon_{t_k-n_k+1}-1,\varepsilon_1^*,\dots,\varepsilon^*_{m_k-t_k-1}\right).$$

When $\varepsilon_{t_k+1}=\varepsilon_{t_k-n_k+1}+1$, we have

$$x=\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k-n_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k-n_k+1}}{\beta^{t_k-n_k+1}}+\cdots\le\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k-n_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k+1}}{\beta^{t_k-n_k+1}}.$$

Noticing that $|T_\beta^{n_k}x-x|=T_\beta^{n_k}x-x<\beta^{-(m_k-n_k)}$, we have

$$T_\beta^{n_k}<x+\beta^{-(m_k-n_k)}\le\frac{\varepsilon_1}{\beta}+\cdots+\frac{\varepsilon_{t_k-n_k}}{\beta^{t_k-n_k}}+\frac{\varepsilon_{t_k+1}}{\beta^{t_k-n_k+1}}+\frac{1}{\beta^{m_k-n_k}}.$$

This implies

$$(\varepsilon_1,\dots,\varepsilon_{m_k})=(\varepsilon_1,\dots,\varepsilon_{n_k},\varepsilon_1,\dots,\varepsilon_{t_k},\varepsilon_{t_k-n_k+1}+1,0^{m_k-n_k-t_k-1}).$$

$\square$

Remark 7.1.3. The two sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ chosen here are exactly the same sequences in Bugeaud and Liao [18]. However, the property that $n_{k+1} > m_k = t_k > n_k$ which always holds in [18] fails in our case. Then uncertainty of the relationship between m_k, t_k and n_{k+1} makes the construction of $\varepsilon(x,\beta)$ of all $x \in R_\beta(r,\hat{r})$ be much more complicated.

Now we will investigate the relationship between m_k and n_k for the case $0 \le \frac{r}{1+r} < \hat{r} \le +\infty$.

Lemma 7.1.4. *For each $x \in R_\beta(\hat{r},r)$ with $0 \le \frac{r}{1+r} < \hat{r} \le +\infty$, let m_k and n_k be defined in Lemma 7.1.1. Then there is an integer k' such that $n_{k+1} < m_k$ for all $k \ge k'$.*

Proof. Since $\hat{r} > \frac{r}{1+r}$, by (7.1.1) and (7.1.2), there is a positive number $\varepsilon > 0$ and an integer k'', such that for any $k \ge k''$, we have

$$\frac{m_k - n_k}{n_{k+1}} \ge \left(\frac{r}{1+r} + \varepsilon\right)\frac{1}{1-\varepsilon}.$$

By (7.1.1), we obtain

$$\frac{r}{1+r} = 1 - \frac{1}{1+r} = 1 - \frac{1}{1 + \limsup\limits_{k\to\infty} \frac{m_k - n_k}{n_k}} = \limsup_{k\to\infty} \frac{m_k - n_k}{m_k}.$$

Consequently, there is an integer $k' \ge k''$, such that for all $k \ge k'$,

$$\frac{m_k - n_k}{n_{k+1}} \ge \left(\frac{r}{1+r} + \varepsilon\right)\frac{1}{1-\varepsilon} \ge \frac{1}{1-\varepsilon}\cdot\frac{m_k - n_k}{m_k},$$

which implies $n_{k+1} \le (1-\varepsilon)m_k$. $\square$

The following lemma gives the sequences we will use to construct the covering of $R_\beta(r,\hat{r})$ when proving the upper bound of $\dim_{\mathrm{H}} R_\beta(r,\hat{r})$ for the case $0 \le \hat{r} \le \frac{r}{1+r}$, $0 < r \le +\infty$.

Lemma 7.1.5. *For all $x \in R_\beta(\hat{r},r)$ with $0 \le \hat{r} \le 1$, $\frac{\hat{r}}{1-\hat{r}} \le r < +\infty$, there are two sequences $\{n'_k\}_{k=1}^{\infty}$ and $\{m'_k\}_{k=1}^{\infty}$ such that for any large enough k, there is a positive real number C satisfying*

$$n'_{k+1} \ge (1-\delta)m'_k, \quad n'_{k+1} \ge \frac{2+\hat{r}_\beta(x)}{\hat{r}_\beta(x)}n'_k \quad \text{and} \quad k \le C\log n'_k.$$

Proof. For all $x \in R_\beta(\hat{r},r)$ $\big(0 \le \hat{r} \le 1,\ \frac{\hat{r}}{1-\hat{r}} \le r < +\infty\big)$, the β-expansion of x is not periodic and $r_\beta(x) > 0$. Let $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be the sequences defined in Lemma 7.1.1.

By (7.1.2), for any $0<\delta_1<\frac{\hat{r}_\beta(x)}{2}$, for sufficiently large k, we have

$$m_k-n_k\geq(\hat{r}_\beta(x)-\delta_1)n_{k+1},\ \text{that is,}\ m_k\geq n_{k+1}+(\hat{r}_\beta(x)-\delta_1)n_k\geq(1+\hat{r}_\beta(x)-\delta_1)n_k. \tag{7.1.8}$$

Note that $\hat{r}_\beta(x)\leq\frac{r_\beta(x)}{1+r_\beta(x)}$. It follows from (7.1.1) and (7.1.2) that

$$\liminf_{k\to\infty}\frac{m_k-n_k}{n_{k+1}}\leq\limsup_{k\to\infty}\frac{m_k-n_k}{m_k}.$$

Consequently,

$$\liminf_{k\to\infty}\frac{m_k-n_k}{n_{k+1}}-\limsup_{k\to\infty}\frac{m_k-n_k}{m_k}=\liminf_{k\to\infty}\left(\frac{m_k-n_k}{n_{k+1}}-\frac{m_k-n_k}{m_k}\right)\leq 0.$$

We claim that, for all $\delta>0$, there are infinitely many k, such that

$$n_{k+1}\geq(1-\delta)m_k\quad\text{and}\quad n_{k+1}\geq\left(1+\frac{\hat{r}_\beta(x)}{2}\right)n_k=\frac{2+\hat{r}_\beta(x)}{2}n_k. \tag{7.1.9}$$

In fact, notice that $m_k-n_k>0,\ n_{k+1}m_k>0$ for all $k\geq 1$. When

$$\liminf_{k\to\infty}\left(\frac{m_k-n_k}{n_{k+1}}-\frac{m_k-n_k}{m_k}\right)=\liminf_{k\to\infty}\frac{(m_k-n_k)(m_k-n_{k+1})}{n_{k+1}m_k}<0,$$

by (7.1.8), we have

$$n_{k+1}>m_k\geq(1+\hat{r}_\beta(x)-\delta_1)n_k\geq\left(1+\frac{\hat{r}_\beta(x)}{2}\right)n_k.$$

When

$$\liminf_{k\to\infty}\left(\frac{m_k-n_k}{n_{k+1}}-\frac{m_k-n_k}{m_k}\right)=\liminf_{k\to\infty}\frac{(m_k-n_k)(m_k-n_{k+1})}{n_{k+1}m_k}=0,$$

then for all $0<\delta_2\leq\frac{(\hat{r}_\beta(x)-\delta_1)(\hat{r}_\beta(x)-2\delta_1)}{2(1+\hat{r}_\beta(x)-\delta_1)}$, there are infinitely many $k\geq k_0$ such that

$$m_k-n_{k+1}\leq\delta_2\cdot\frac{n_{k+1}m_k}{m_k-n_k}\leq\delta_2\cdot\frac{n_{k+1}m_k}{(\hat{r}_\beta(x)-\delta_1)n_{k+1}}=\frac{\delta_2}{\hat{r}_\beta(x)-\delta_1}\cdot m_k,$$

where the last inequality follows from the first part of (7.1.8).

Therefore, by (7.1.8) and the choice of δ_2,

$$n_{k+1}\geq\left(1-\frac{\delta_2}{\hat{r}_\beta(x)-\delta_1}\right)m_k\geq\left(1-\frac{\delta_2}{\hat{r}_\beta(x)-\delta_1}\right)(1+\hat{r}_\beta(x)-\delta_1)n_k\geq\left(1+\frac{\hat{r}_\beta(x)}{2}\right)n_k.$$

Letting $\delta=\frac{\delta_2}{\hat{r}_\beta(x)-\delta_1}$, we obtain the claim.

Now we choose the subsequence $\{n_{j_k}\}_{k=1}^\infty$ and $\{m_{j_k}\}_{k=1}^\infty$ of $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ satisfying (7.1.9). For simplicity, let $n'_k=n_{j_k}$ and $m'_k=m_{j_k}$. Then $\{n'_k\}_{k=1}^\infty$ increases at least exponentially since

$$n'_{k+1}\geq\frac{2+\hat{r}_\beta(x)}{\hat{r}_\beta(x)}n'_k.$$

In conclusion, the new sequences satisfy that, there is a large enough k_0 and a positive real number C such that for all $k\geq k_0$, we have $k\leq C\log n'_k$. □

We now give an estimation of the numbers of the sum of all the lengths of the blocks which are "fixed" in the prefix of length m_k (m_k is defined in Lemma 7.1.5) of the infinite sequence $\varepsilon(x,\beta)$ where $x \in R_\beta(\hat{r}, r)$ $\left(0 \le \hat{r} \le 1,\ \frac{\hat{r}}{1-\hat{r}} \le r < +\infty\right)$ for all sufficiently large k.

Lemma 7.1.6. *For all $x \in R_\beta(\hat{r}, r)$ with $0 \le \hat{r} \le 1$, $\frac{\hat{r}}{1-\hat{r}} \le r < +\infty$, let $\{m'_k\}_{k=1}^{\infty}$ and $\{n'_k\}_{k=1}^{\infty}$ be the sequences defined in Lemma 7.1.5. Then for all large enough integer k, we have*

$$\sum_{i=1}^{k}(m'_i - n'_i) \ge n'_{k+1}\left(\frac{\hat{r}_\beta(x) r_\beta(x)}{r_\beta(x) - \hat{r}_\beta(x)} - \epsilon'\right) \tag{7.1.10}$$

Proof. By Lemma 7.1.5, the sequences $\{n'_k\}_{k=1}^{\infty}$ and $\{m'_k\}_{k=1}^{\infty}$ are the subsequences of$\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$, the equalities (7.1.1) and (7.1.2) become :

$$r_\beta(x) \ge \limsup_{k\to\infty} \frac{m'_k - n'_k}{n'_k} \tag{7.1.11}$$

and

$$\hat{r}_\beta(x) \le \liminf_{k\to\infty} \frac{m'_k - n'_k}{n'_{k+1}}. \tag{7.1.12}$$

By (7.1.11) and (7.1.12), for any real number $0 < \epsilon < \frac{\hat{r}_\beta(x)}{2}$, there exits a large enough k_1, such that for all $k \ge k_1$,

$$m'_k - n'_k \le (r_\beta(x) + \epsilon) n'_k \tag{7.1.13}$$

and

$$m'_k - n'_k \ge (\hat{r}_\beta(x) - \epsilon) n'_{k+1}. \tag{7.1.14}$$

By (7.1.13) and (7.1.14), we have

$$n'_k \ge \frac{m'_k - n'_k}{r_\beta(x) + \epsilon} \ge \frac{\hat{r}_\beta(x) - \epsilon}{r_\beta(x) + \epsilon} n'_{k+1}.$$

Hence,

$$\sum_{i=1}^{k}(m'_i - n'_i) \ge \sum_{i=k_1}^{k}(\hat{r}_\beta(x) - \epsilon) n'_{i+1} \ge (\hat{r}_\beta(x) - \epsilon) n'_{k+1} \sum_{i=0}^{k-k_1}\left(\frac{\hat{r}_\beta(x) - \epsilon}{r_\beta(x) + \epsilon}\right)^i. \tag{7.1.15}$$

Let

$$n' = (\hat{r}_\beta(x) - \epsilon) n'_{k+1} \sum_{i=k-k_1+1}^{\infty}\left(\frac{\hat{r}_\beta(x) - \epsilon}{r_\beta(x) + \epsilon}\right)^i.$$

Note that $\frac{\hat{r}_\beta(x)-\epsilon}{r_\beta(x)+\epsilon} < 1$. There exists $k' \ge k$ such that for all $\ell \ge k'$, we have

$$n'_{k+1}\left(\frac{\hat{r}_\beta(x) - \epsilon}{r_\beta(x) + \epsilon}\right)^\ell < \frac{1}{\ell^2}.$$

Thus,

$$n'_{k+1}\sum_{i=k'+1}^{\infty}\left(\frac{\hat{r}_\beta(x)-\epsilon}{r_\beta(x)+\epsilon}\right)^i<\sum_{i=k'+1}^{\infty}\frac{1}{i^2}<+\infty.$$

This implies

$$n'=(\hat{r}_\beta(x)-\epsilon)n'_{k+1}\sum_{i=k-k_1+1}^{k'}\left(\frac{\hat{r}_\beta(x)-\epsilon}{r_\beta(x)+\epsilon}\right)^i+(\hat{r}_\beta(x)-\epsilon)n'_{k+1}\sum_{i=k'+1}^{\infty}\left(\frac{\hat{r}_\beta(x)-\epsilon}{r_\beta(x)+\epsilon}\right)^i<+\infty.$$

By (7.1.15), for any small enough real number $\epsilon'>0$, there is a sufficiently large integer $k\geq\{k',k_1\}$ such that

$$\begin{aligned}\sum_{i=1}^{k}(m'_i-n'_i)&\geq(\hat{r}_\beta(x)-\epsilon)n'_{k+1}\sum_{i=0}^{k-1}\left(\frac{\hat{r}_\beta(x)-\epsilon}{r_\beta(x)+\epsilon}\right)^i\\&=(\hat{r}_\beta(x)-\epsilon)n'_{k+1}\sum_{i=0}^{\infty}\left(\frac{\hat{r}_\beta(x)-\epsilon}{r_\beta(x)+\epsilon}\right)^i-n'\\&\geq n'_{k+1}\left(\frac{\hat{r}_\beta(x)r_\beta(x)}{r_\beta(x)-\hat{r}_\beta(x)}-\epsilon'\right).\end{aligned}$$

□

We will prove Theorem 2.4.3 by dividing into three cases : $r=\hat{r}=0$, $0\leq\frac{r}{1+r}<\hat{r}\leq+\infty$ and $0\leq\hat{r}\leq\frac{r}{1+r},\ 0<r\leq+\infty$.

7.1.1 Case for $r=\hat{r}=0$

Note that $\hat{r}_\beta(x)\leq r_\beta(x)$ for all $x\in[0,1)$. Then

$$\{x\in[0,1):r_\beta(x)=0\}\subseteq\{x\in[0,1):\hat{r}_\beta(x)=0\}.$$

So if we prove that $r_\beta(x)=0$ for $\mathcal{L}$-almost all $x\in[0,1)$, we have $\hat{r}_\beta(x)=0$ for $\mathcal{L}$-almost all $x\in[0,1)$.

Hence, we only need to prove

$$\mathcal{L}\{x\in[0,1):r_\beta(x)>0\}=0.$$

Since $\sum\limits_{n=1}^{\infty}\beta^{-\frac{1}{k}n}<\infty$ for all $k\geq1$, it follows from Corollary 3.2.12 that

$$\mathcal{L}\{x\in[0,1):|T_\beta^n x-x|\leq\beta^{-\frac{1}{k}n}\text{ for infinitely many }n\in\mathbb{N}\}=0.$$

Moreover, we have

$$\{x\in[0,1):r_\beta(x)>0\}=\bigcup_{k=1}^{\infty}\left\{x\in[0,1):r_\beta(x)>\frac{1}{k}\right\}$$

and

$$\left\{x \in [0,1) : r_\beta(x) > \frac{1}{k}\right\} \subseteq \{x \in [0,1) : |T_\beta^n x - x| \le \beta^{-\frac{1}{k}n} \text{ for infinitely many } n \in \mathbb{N}\}.$$

Thus,

$$\mathcal{L}\{x \in [0,1) : r_\beta(x) > 0\} = 0.$$

7.1.2 Case for $0 \le \frac{r}{1+r} < \hat{r} \le +\infty$

When $\varepsilon(x,\beta)$ is periodic, we have $\hat{r} = r = +\infty$ and the set with such $\varepsilon(x,\beta)$ is countable.

When $\varepsilon(x,\beta)$ is not periodic, let m_k and n_k be defined in Lemma 7.1.1. We will construct a countable set D such that $\varepsilon(x,\beta) \in D$ for all $x \in R_\beta(\hat{r}, r)$ with $0 \le \frac{r}{1+r} < \hat{r} \le +\infty$. By Lemma 7.1.4, there is an integer k', such that for all $k \ge k'$, we have $n_{k+1} < m_k$. Let t_k be defined by (7.1.6). Suppose that $t_k = an_k + p_k$ where $1 \le a \le \lfloor r \rfloor + 1$ and $1 \le p_k < n_k$. Then, by (7.1.7),

$$(\varepsilon_1, \dots, \varepsilon_{t_k}) = \left((\varepsilon_1, \dots, \varepsilon_{n_k})^a, \varepsilon_1, \dots, \varepsilon_{p_k}\right), \quad \varepsilon_{t_k+1} \ne \varepsilon_{p_k+1}. \tag{7.1.16}$$

where $\omega^k = (\underbrace{\omega, \dots, \omega}_{k})$ for any $\omega \in \Sigma_\beta^*$, $k \in \mathbb{N}$. Therefore, when $n_{k+1} \le t_k \le m_k + 1$, by the definitions of $\{n_k\}_{k=1}^\infty$ and $\{t_k\}_{k=1}^\infty$, we have

$$(\varepsilon_1, \dots, \varepsilon_{t_k}) = \left(\varepsilon_1, \dots, \varepsilon_{n_k}, \varepsilon_1, \dots, \varepsilon_{n_{k+1}-n_k}, \varepsilon_1, \dots, \varepsilon_{t_k - n_{k+1}}\right), \ \varepsilon_{t_k+1} = \varepsilon_{t_k - n_{k+1}+1}. \tag{7.1.17}$$

By comparing the equalities (7.1.7), (7.1.16) and (7.1.17), we can check that $n_{k+1} \ne a'n_k$ for all integer $1 \le a' \le a$. Suppose that there are two integers $1 \le a' \le a$ and $1 \le j_k < n_k$ such that $n_{k+1} = a'n_k + j_k$. Then

$$\left(\varepsilon_1, \dots, \varepsilon_{n_{k+1}}\right) = \left((\varepsilon_1, \dots, \varepsilon_{n_k})^{a'}, \varepsilon_1, \dots, \varepsilon_{j_k}\right).$$

When $t_k < n_{k+1} < m_k$, by (7.1.4), (7.1.5) and (7.1.16), it follows that

$$\left(\varepsilon_1, \dots, \varepsilon_{n_{k+1}}\right) = \left((\varepsilon_1, \dots, \varepsilon_{n_k})^a, \varepsilon_1, \dots, \varepsilon_{p_k}, \varepsilon_{p_k+1} - 1, \varepsilon_1^*, \dots, \varepsilon_{n_{k+1}-p_k-1}^*\right)$$

or

$$\left(\varepsilon_1, \dots, \varepsilon_{n_{k+1}}\right) = \left((\varepsilon_1, \dots, \varepsilon_{n_k})^a, \varepsilon_1, \dots, \varepsilon_{p_k}, \varepsilon_{p_k+1} + 1, 0^{n_{k+1}-p_k-1}\right).$$

Now we will construct the countable set D. For all $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_\beta^n$, define

$$s_1(\omega) = \min\left\{1 \le i < n : \omega_{i+1} = \omega_1\right\}.$$

If such i does not exist, let $s_1(\omega) = n$. If $s_1(\omega) < n$, let

$$t_1(\omega) = \max\left\{s_1(\omega) < i \le n : \left(\omega_{s_1(\omega)+1}, \ldots, \omega_i\right) = \left(\omega_1, \ldots, \omega_{i-s_1(\omega)}\right)\right\}.$$

Suppose that $s_k(\omega)$ and $t_k(\omega)$ have been defined, let

$$s_{k+1}(\omega) = \min\{s_k(\omega) \le i < n : \omega_{i+1} = \omega_1\}.$$

If this i does not exist, let $s_{k+1}(\omega) = n$. If $s_{k+1}(\omega) < n$, let

$$t_{k+1}(\omega) = \max\left\{s_k(\omega) < i \le n : \left(\omega_{s_k(\omega)+1}, \ldots, \omega_i\right) = \left(\omega_1, \ldots, \omega_{i-s_k(\omega)}\right)\right\}.$$

Let $k(\omega) = \max\{k : 1 \le s_k(\omega) < n\}$ and $k(\omega) = 0$ if such k does not exist.

For all $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$, when $k(\omega) > 0$, let

$$M'(\omega) = \bigcup_{a=1}^{\lfloor r\rfloor+1} \bigcup_{k=1}^{k(\omega)} \bigcup_{j=0}^{\lfloor rn\rfloor} \left\{\left(\omega^a, u, \omega_{t_k(\omega)+1} - 1, \varepsilon_1^*, \ldots, \varepsilon_j^*\right) : u = \left(\omega_1, \ldots, \omega_{t_k(\omega)}\right),\ \omega_{t_k(\omega)+1} > 0\right\}$$

where $\left(\omega^a, u, \omega_{t_k(\omega)+1} - 1, \varepsilon_1^*, \ldots, \varepsilon_j^*\right) = \left(\omega^a, u, \omega_{t_k(\omega)+1} - 1\right)$ when $j = 0$, and

$$M''(\omega) = \bigcup_{a=1}^{\lfloor r\rfloor+1} \bigcup_{k=1}^{k(\omega)} \bigcup_{j=0}^{\lfloor rn\rfloor} \left\{\left(\omega^a, u, \omega_{t_k(\omega)+1} + 1, 0^j\right) : u = \left(\omega_1, \ldots, \omega_{t_k(\omega)}\right),\ 0 \le \omega_{t_k(\omega)+1} < \lfloor\beta\rfloor\right\}$$

where $\left(\omega^a, u, \omega_{t_k(\omega)+1} + 1, 0^j\right) = \left(\omega^a, u, \omega_{t_k(\omega)+1} + 1\right)$ when $j = 0$. When $k(\omega) = 0$, let

$$M'(\omega) = \bigcup_{a=1}^{\lfloor r\rfloor+1} \bigcup_{j=0}^{\lfloor rn\rfloor} \left\{\left(\omega^a, \omega_1 - 1, \varepsilon_1^*, \ldots, \varepsilon_j^*\right), \omega_1 > 0\right\}$$

where $\left(\omega^a, \omega_1 - 1, \varepsilon_1^*, \ldots, \varepsilon_j^*\right) = (\omega^a, \omega_1 - 1)$ when $j = 0$, and

$$M''(\omega) = \bigcup_{a=1}^{\lfloor r\rfloor+1} \bigcup_{j=0}^{\lfloor rn\rfloor} \left\{\left(\omega^a, \omega_1 + 1, 0^j\right), 0 \le \omega_1 < \lfloor\beta\rfloor\right\}$$

where $(\omega^a, \omega_1 + 1, 0^j) = (\omega^a, \omega_1 + 1)$ when $j = 0$.

Now fix $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$ and let $M(\omega) = M'(\omega) \bigcup M''(\omega) \bigcup \{\omega\}$. Set $M_1(\omega) = M(\omega) \bigcup \{(\omega^2)\}$. Suppose that $M_k(\omega)$ has been defined. Let

$$M_{k+1}(\omega) = \bigcup_{v_k \in M_k(\omega)} M(v_k) \bigcup \left\{(\omega^k)\right\}.$$

Then we have $\sharp M_k(\omega) < \infty$ for all $k \ge 1$ and we also have $M_k(\omega) \subseteq \mathcal{A}^*$. Let

$$D = \bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \bigcup_{k=1}^{\infty} M_k(\omega).$$

By the former analysis in this section, for any $x \in R_\beta(\hat{r}, r)$ with $0 \le \frac{r}{1+r} < \hat{r} \le +\infty$, we have $\varepsilon(x, \beta) \in D$ and the set D is countable.

As a consequence, the set $R_\beta(\hat{r}, r)$ $\left(0 \le \frac{r}{1+r} < \hat{r} \le +\infty\right)$ is countable.

7.1.3 Case for $0 \le \hat{r} \le \frac{r}{1+r},\ 0 < r \le +\infty$

Note that $\{x \in [0,1) : r_\beta(x) = +\infty\} \subseteq \{x \in [0,1) : r_\beta(x) \ge r\}$ for all $0 \le r \le +\infty$. When $0 \le \hat{r} \le \frac{r}{1+r},\ r = +\infty$, by (2.4.2), it holds that

$$\dim_{\mathrm{H}}\{x \in [0,1) : r_\beta(x) \ge r\} = \frac{1}{1+r}.$$

Consequently, letting $r \to +\infty$, we conclude that

$$\dim_{\mathrm{H}}\{x \in [0,1) : r_\beta(x) = +\infty\} = 0.$$

By the fact that $R_\beta(r,\hat{r}) \subseteq \{x \in [0,1) : r_\beta(x) = +\infty\}$, we have $\dim R_\beta(r,\hat{r}) = 0$.

When $0 \le \hat{r} \le \frac{r}{1+r},\ 0 < r < +\infty$. Our proof is divided into two parts.

The upper bound of $\dim_{\mathrm{H}} R_\beta(\hat{r}, r)$

We now construct a covering of the set $R_\beta(\hat{r}, r)$ with $0 \le \hat{r} \le \frac{r}{1+r},\ 0 < r < +\infty$. Let $\{n'_k\}_{k=1}^\infty$ and $\{m'_k\}_{k=1}^\infty$ be the sequences such that

$$\lim_{k\to\infty} \frac{m'_k - n'_k}{n'_k} = r \quad \text{and} \quad \lim_{k\to\infty} \frac{m'_k - n'_k}{n'_{k+1}} \ge \hat{r}. \tag{7.1.18}$$

Given $k \ge 1$, we collect all of the points x with $\hat{r}_\beta(x) \ge \hat{r}$ and $r_\beta(x) = r$. We first calculate the possible choices of digits among the m'_k prefix of $(\omega_1, \omega_2, \ldots)$. For the "free" blocks in the prefix of length m'_k of the infinite sequence $\varepsilon(x,\beta)$. Write their lengths as $d_1, \cdots, d_k$. It follows immediately that $d_i = 0$ when $n'_i < m'_{i-1}$ and $d_i = n'_i - m'_{i-1}$ when $n'_i > m'_{i-1}$ for all $2 \le i \le k$. Moreover, $d_1 = n_1$. Let $m_0 = 0$. By (7.1.10) and the choice of m_k and n_k, it follows from Lemma 7.1.5 that, for any sufficiently large $k \ge \max\{k_0, k_1, k'\}$,

$$\sum_{i=1}^k d_i \le \sum_{i=1}^k (n'_i - m'_{i-1}) + \sum_{n'_i < m'_{i-1}, k_0 \le i \le k} (m'_{i-1} - n'_i). \tag{7.1.19}$$

Applying Lemma 7.1.5, we have

$$\sum_{n'_i < m'_{i-1,1 \le i \le k}} (m'_{i-1} - n'_i) \le \delta \sum_{n'_i < m'_{i-1}, k_0 \le i \le k} m'_{i-1} + n_0$$

where $n_0 = \delta \sum\limits_{n'_i < m'_{i-1}, k_0 \le i \le k} (m'_{i-1} - n'_i) < +\infty$. By (7.1.13) and Lemma 7.1.5, we have

$$\begin{aligned}\sum_{n'_i < m'_{i-1,1 \le i \le k}} (m'_{i-1} - n'_i) &\le \delta \sum_{n'_i < m'_{i-1}, k_0 \le i \le k} (1 + r + \epsilon) n_{i-1} + n_0 \\ &\le \delta \left(1 + \frac{\hat{r}}{2+\hat{r}} + \ldots + \left(\frac{\hat{r}}{2+\hat{r}}\right)^k\right)(1 + r + \epsilon) n_k + n_0.\end{aligned} \tag{7.1.20}$$

By (7.1.10), it holds that

$$\sum_{i=1}^{k}(n'_i - m'_{i-1}) = n'_k - \sum_{i=1}^{k-1}(m'_i - n'_i) \le n'_k \left(1 - \frac{\hat{r}r}{r-\hat{r}} + \epsilon'\right). \tag{7.1.21}$$

Combining (7.1.19), (7.1.20) and (7.1.21), we obtain

$$\sum_{i=1}^{k} d_i \le n'_k \left(1 - \frac{\hat{r}r}{r-\hat{r}} + \epsilon''\right),$$

where ϵ'' is a small enough real number. By Theorem 3.2.1, we deduce that, for every blocks with length d_i, there are no more than

$$\frac{\beta}{\beta-1}\beta^{d_i}$$

ways of the words can be chosen. Thus, there are at most

$$\left(\frac{\beta}{\beta-1}\right)^k \beta^{\sum_{i=1}^{k} d_i} \le \left(\frac{\beta}{\beta-1}\right)^k \beta^{n'_k\left(1-\frac{\hat{r}r}{r-\hat{r}}+\epsilon''\right)}$$

choices of the "free" blocks in total. Notice that there are at most n_k possible choices for the first index of the k blocks. This indicates that there are at most ${n'_k}^k$ possible choices for the position of the "free" blocks. For the "fixed" block, it follows from (7.1.3), (7.1.4), (7.1.5) that the block $(\varepsilon_1, \dots, \varepsilon_{m'_k - n'_k})$ has at most $2\lfloor\beta\rfloor(m'_k - n'_k + 1) \le 2\beta(m'_k - n'_k + 1)$ choices which means there are at most $(2\beta(m'_k - n'_k + 1))^k$ choices of the "fixed" blocks in total. By Lemma 7.1.5 and (7.1.18), for all sufficient large k, the set of all real number belonging to $R_\beta(\hat{r}, r)$ is contained in a union of no more than

$$\left(\frac{2(m'_k - n'_k)n'_k\beta^2}{\beta-1}\right)^k \beta^{n'_k\left(1-\frac{\hat{r}r}{r-\hat{r}}+\epsilon''\right)} \le \left(\frac{2(r+\epsilon''){n'_k}^2\beta^2}{\beta-1}\right)^{C\log n'_k} \beta^{n'_k\left(1-\frac{\hat{r}r}{r-\hat{r}}+\epsilon''\right)}$$

basic intervals of order m'_k whose length is at most

$$\beta^{-m'_k} \le \beta^{(1+r-\epsilon'')n'_k},$$

where the last inequalities follows from (7.1.18). Denote

$$s_0 = \frac{r - (1+r)\hat{r} + \epsilon''(r-\hat{r})}{(r-\hat{r})(r+1-\epsilon'')}.$$

Then for any $s > s_0$, we have

$$\mathcal{H}^s\left(R_\beta(\hat{r}, r)\right) \le \sum_{n=1}^{\infty} \beta \left(\frac{2(r+\epsilon'')n^2\beta^2}{\beta-1}\right)^{C\log n} \beta^{-(1+r-\epsilon'')ns + n\left(1-\frac{\hat{r}r}{r-\hat{r}}+\epsilon''\right)} < +\infty.$$

Letting $\epsilon'' \to 0$, we conclude that

$$\dim_{\mathrm{H}} R_\beta(\hat{r}, r) \le \frac{r-(1+r)\hat{r}}{(r-\hat{r})(r+1)}.$$

Construction of Cantor Set

We construct a Cantor subset of $R_\beta(\hat{r}, r)$ $\big(0 \le \hat{r} \le \frac{r}{1+r},\ 0 < r < +\infty\big)$ as follows. Fix $\delta > 0$. Let β_N be the approximation of β which is defined in Section 2. Notice that $\beta_N \to \beta$ as $N \to \infty$, we can choose sufficiently large integer N with $\varepsilon_N^*(\beta) > 0$ and M large enough such that

$$\frac{\sharp\Sigma_{\beta_N}^M}{M} - M - 1 \ge \frac{\beta_N^M}{M} - M - 1 \ge \beta^{M(1-\delta)}. \tag{7.1.22}$$

Now we choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_1 > 2M$. The sequence $\{m_k - n_k\}_{k=1}^\infty$ is non-decreasing. In addition,

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = \hat{r} \tag{7.1.23}$$

and

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_k} = r. \tag{7.1.24}$$

Actually, we can choose the following sequences.

(1) When $\hat{r} = 0$, $0 < r < +\infty$, let

$$n_k' = k^k \quad \text{and} \quad m_k' = \left\lfloor (r+1)\, k^k \right\rfloor.$$

By a small adjustment, we can obtain the required sequences.

(2) When $0 < \hat{r} \le \frac{r}{1+r}$, $0 < r < +\infty$, let

$$n_k' = \left\lfloor \left(\frac{r}{\hat{r}}\right)^k \right\rfloor \text{ and } m_k' = \left\lfloor (r+1)\left(\frac{r}{\hat{r}}\right)^k \right\rfloor.$$

Note that $r < \hat{r}$. Both of the sequences $\{n_k'\}_{k=1}^\infty$ and $\{m_k'\}_{k=1}^\infty$ increase to infinity as k increases. We can adjust these sequences to make sure that they satisfy the required properties. Without any ambiguity, the statement that $\omega \in \Sigma_{\beta_N}^*$ is full means that ω is full when regarding it as an element of Σ_β^*.

For all $k \in \mathbb{N}$, let $m_k = \ell_k n_k + p_k$ with $0 \le p_k < n_k$ and $n_{k+1} - m_k = t_k M + q_k$ with $0 \le q_k < M$. For any integer $n > N$ and $\omega \in \Sigma_{\beta_N}^n$, define the function a_{p_k} of ω by

$$a_{p_k}(\omega) = \begin{cases} 0^{p_k}, & \text{when } \ p_k \le N, \\ \omega|_{p_k - N}, 0^N, & \text{when } \ p_k > N, \end{cases}$$

where $\omega|_i = (\omega_1, \ldots, \omega_i)$. Then $a_{p_k}(\omega)$ is full when regarding it as an element of Σ_β^*. For convenience, denote

$$\omega^i = (\omega_{n-i+1}, \ldots, \omega_n, \omega_{i+1}, \ldots, \omega_{n-i})$$

for all $1 \leq i < n$ and $\omega^n = \omega$. Let

$$\mathcal{D}_1 = \left\{ v_1 = \left((u)^{\lfloor \frac{n_1}{M} \rfloor}, 0^{n_1 - \lfloor \frac{n_1}{M} \rfloor M} \right) : u \in \Sigma_{\beta_N}^M \text{ is full and } u \neq 0^M \right\}.$$

Set

$$\mathcal{G}_1 = \left\{ u_1 = \left((v_1)^{\ell_1}, a_{p_1}(v_1) \right) : v_1 \in \mathcal{D}_1 \right\}.$$

Fix $v_1 = \left((u)^{\lfloor \frac{n_1}{M} \rfloor}, 0^{n - \lfloor \frac{n_1}{M} \rfloor M} \right) \in \mathcal{D}_1$. Define

$$\mathcal{M} = \left\{ \omega \in \Sigma_{\beta_N}^M : \omega \text{ is full and } \omega \neq u^i, \ \forall\ 1 \leq i \leq n \right\}.$$

Suppose that $\mathcal{D}_{k-1}$ and $\mathcal{G}_{k-1}$ are well defined. Fix $u_{k-1} \in \mathcal{G}_{k-1}$. Let

$$\mathcal{D}_k = \left\{ v_k = \left(u_{k-1}^0, \ldots, u_{k-1}^{t_{k-1}-1}, 0^{q_{k-1}} \right) : u_{k-1}^i \in \mathcal{M}, \forall\ 0 \leq i \leq t_{k-1} - 1 \right\}.$$

Define

$$\mathcal{G}_k = \left\{ u_k = \left((u_{k-1}, v_k)^{\ell_k}, a_{p_k}(u_{k-1}, v_k) \right) : u_{k-1} \in \mathcal{G}_{k-1}, v_k \in \mathcal{D}_k \right\}.$$

We can see that all of the words in $\mathcal{M}$, $\mathcal{G}_k$ and $\mathcal{D}_k$ are full when regarding them as an element of Σ_β^*. Hence, by Theorem 3.2.7, every word in $\mathcal{M}$, $\mathcal{G}_k$ and $\mathcal{D}_k$ can concatenated all β-admissible word.

Now define

$$E_N = \bigcap_{k=1}^{\infty} \bigcup_{u_k \in \mathcal{G}_k} I_{m_k}(u_k).$$

We claim that E_N is a subset $R_\beta(\hat{r}, r)$.

In fact, for any $x \in E_N$, suppose that $\varepsilon(x, \beta) = (\varepsilon_1, \varepsilon_2, \ldots)$. We first prove that for every n with $m_k \leq n < n_{k+1}$, we have

$$(\varepsilon_{n+1}, \ldots, \varepsilon_{n+2M}) \neq (\varepsilon_1, \ldots, \varepsilon_{2M}) = (\varepsilon_1, \ldots, \varepsilon_M, \varepsilon_1, \ldots, \varepsilon_M),$$

where the last equality follows from the fact that $n_1 > 2M$ and the construction of E_N. In fact, if it is not true, then there is $m_k \leq n < n_{k+1} - 2M$ such that

$$(\varepsilon_{n+1}, \ldots, \varepsilon_{n+2M}) = (\varepsilon_1, \ldots, \varepsilon_{2M}).$$

Assume that $n = m_k + tM + q$ $(0 \leq t \leq t_k - 1)$. Then we have

$$(\varepsilon_{n+1}, \ldots, \varepsilon_{n+2M}) = \left(\varepsilon_{m_k+tM+q+1}, \ldots, \varepsilon_{m_k+(t+1)M+1}, \ldots, \varepsilon_{m_k+(t+2)M}, \ldots, \varepsilon_{m_k+(t+2)M+q} \right).$$

This implies

$$\left(\varepsilon_{m_k+(t+1)M+1}, \ldots, \varepsilon_{m_k+(t+2)M} \right) = (\varepsilon_{M-q+1}, \ldots, \varepsilon_M, \varepsilon_1, \ldots, \varepsilon_q) \notin \mathcal{M},$$

which is a contradiction. Note that $n \geq 2M$. Assume that

$$(\varepsilon_{n+1}, \ldots, \varepsilon_{n+j}) = (\varepsilon_1, \ldots, \varepsilon_j), \ \varepsilon_{n+j+1} \neq \varepsilon_{j+1}.$$

Without loss of generality, suppose that $\varepsilon_{n+j+1} > \varepsilon_j+1$. Notice that both $(\varepsilon_{n+1}, \ldots, \varepsilon_{n+j+1})$ and $(\varepsilon_1, \ldots, \varepsilon_{j+1})$ belong to $\Sigma_{\beta_N}^M$. Then

$$\begin{aligned}
&|T_\beta^n x - x| = T_\beta^n x - x \\
&\geq \frac{\varepsilon_{n+1}}{\beta} + \cdots + \frac{\varepsilon_{n+j}}{\beta^j} + \frac{\varepsilon_{n+j+1}}{\beta^{j+1}} - \left(\frac{\varepsilon_{n+1}}{\beta} + \cdots + \frac{\varepsilon_{n+j}}{\beta^j} + \frac{\varepsilon_j}{\beta^{j+1}} + \frac{\varepsilon_1^*}{\beta^{j+2}} + \cdots + \frac{\varepsilon_N^* - 1}{\beta^{j+N+1}} \right) \\
&\geq \beta^{-j+N+1}.
\end{aligned} \tag{7.1.25}$$

Hence, for any $m_k \leq n < n_{k+1}$, we have

$$|T_\beta^n x - x| \geq \beta^{-(2M+N+1)}.$$

We now show that $r_\beta(x) = r$.

On the one hand, for any $\delta > 0$, notice that

$$\lim_{k \to \infty} \frac{m_k - n_k + N}{n_k} = r.$$

Then there exits k_0 large enough such that for all $k \geq k_0$, we have $m_k - n_k + N < (r+\delta)n_k$. For all $n \geq n_{k_0}$, there is $k \geq k_0$, such that $n_k \leq n < n_{k+1}$. Note for every n with $m_k \leq n < n_{k+1}$, we have $(\varepsilon_{n+1}, \ldots, \varepsilon_{n+2M}) \neq (\varepsilon_1, \ldots, \varepsilon_{2M}$. By the construction of E_N, for any $n_k \leq i \leq m_k$, we have

$$(\varepsilon_{i+1}, \ldots, \varepsilon_{m_k+2M}) \neq (\varepsilon_1, \ldots, \varepsilon_{m_k-i+2M}).$$

The same argument as (7.1.25) gives

$$|T_\beta^n x - x| \geq \beta^{-(m_k-n_k+2M+N+1)} > \beta^{-(r+\delta)n_k} \geq \beta^{-(r+\delta)n}.$$

By the definition of $r_\beta(x)$, it holds that $r_\beta(x) < r+\delta$ for all $\delta > 0$. So $r_\beta(x) \leq r$.

On the other hand, for all $\delta > 0$, there is k_0 such that for any $k \geq k_0$. Thus, we have $m_k - n_k \geq (r-\delta)n_k$. When $n = n_k$, we obtain

$$|T_\beta^{n_k} x - x| < \beta^{-(m_k-n_k)} \leq \beta^{-(r-\delta)n_k}.$$

As a consequence, $r_\beta(x) \geq r - \delta$ for any $\delta > 0$, which implies $r_\beta(x) \geq r$.

The proof of $\hat{r}_\beta(x) = \hat{r}$ is similar to the argument of $r_\beta(x) = r$. We leave the details to the readers.

Our Cantor set is therefore constructed.

The lower bound of $\dim_{\rm H} R_\beta(\hat{r}, r)$

The rest of this section is devoted to estimating the lower bound of Hausdorff dimension of E_N by the modified mass distribution principal.

As the classical method of giving the lower bound of $\dim_{\rm H} E_N$, we first define a Borel probability measure μ on E_N. Set

$$\mu([0,1)) = 1, \text{ and } \mu\left(I_{m_1}(u_1)\right) = \frac{1}{\sharp\mathcal{G}_1} = \frac{1}{\sharp\mathcal{D}_1}, \text{ for } u_1 \in \mathcal{G}_1.$$

For each $k \geq 1$, if $u_{k+1} = \big((u_k, v_{k+1})^{\ell_{k+1}}, a_{p_{k+1}}(u_k, v_{k+1})\big) \in \mathcal{G}_{k+1}$, let

$$\mu\left(I_{m_{k+1}}(u_{k+1})\right) = \frac{\mu\left(I_{m_k}(u_k)\right)}{\sharp\mathcal{D}_{k+1}}. \tag{7.1.26}$$

If $u_{k+1} \notin \mathcal{G}_{k+1}$ $(k \geq 1)$, let $\mu\left(I_{m_{k+1}}(u_{k+1})\right) = 0$. For each $n \in \mathbb{N}$ and each basic interval I_n with $I_n \cap E_N \neq \emptyset$, let $k \geq 1$ be the integer such that $m_k < n \leq m_{k+1}$, and set

$$\mu(I_n) = \sum_{I_{m_{k+1}} \subseteq I_n} \mu\left(I_{m_{k+1}}\right),$$

where the sum is taken over all the basic intervals associate to $u_{k+1} \in \mathcal{G}_{k+1}$ contained in I_n. We can see that μ satisfies the consistency property which ensures that it can be uniquely extended to a Borel probability measure on E_N.

Now we will estimate the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|}$ for all basic cylinder I_n with $I_n \cap E_N \neq \emptyset$. We claim that, there exists k_0, for all $k \geq k_0$, we have

$$\sharp\mathcal{G}_k \geq c^k \beta^{(1-\delta)\sum\limits_{i=1}^{k-1}(n_{i+1}-m_i)} \tag{7.1.27}$$

for all $k \geq k_0$. Combining Theorems 3.2.1, 3.2.10 and the definition of $\mathcal{M}$, by (7.1.22), we have

$$\sharp\mathcal{M} \geq \frac{\sharp\Sigma_\beta^M}{M} - M \geq \beta^{M(1-\delta)}.$$

Immediately, for any $k \geq 1$, the relationship between $\mathcal{M}$ and $\mathcal{D}_k$ gives

$$\sharp\mathcal{D}_k = (\sharp\mathcal{M})^{t_{k-1}} \geq \beta^{Mt_{k-1}(1-\delta)} = \beta^{-M(1-\delta)} \cdot \beta^{M(1-\delta)(t_{k-1}+1)} \geq c\beta^{(1-\delta)(n_k - m_{k-1})},$$

where $c = \beta^{-M(1-\delta)}$. As a consequence,

$$\sharp\mathcal{G}_k = \prod_{i=1}^k \sharp\mathcal{D}_i \geq c^k \beta^{(1-\delta)\sum\limits_{i=1}^{k-1}(n_{i+1}-m_i)}.$$

Therefore, for all $i \geq 2$,

$$\mu(I_{m_i}) = \frac{1}{\sharp\mathcal{G}_i} \leq \frac{1}{c^k \beta^{(1-\delta)\sum\limits_{j=1}^{i-1}(n_{j+1}-m_j)}}. \tag{7.1.28}$$

In gerneral $n \geq 1$, there is an integer $k \geq 1$ such that $m_k < n \leq m_{k+1}$. We distinguish two cases to estimate $\mu(I_n)$.

Case 1. When $m_k < n \leq n_{k+1}$, write $n = m_k + tM + q$ with $0 \leq t \leq t_k$ and $0 \leq q < M$. then

$$\mu(I_n) \leq \mu(I_{m_k}) \cdot \frac{1}{(\sharp\mathcal{M})^t} \leq c^{-k} \beta^{-(1-\delta)\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)} \cdot \frac{1}{\beta^{M(1-\delta)t}}.$$

Case 2. When $n_{k+1} < n \leq m_{k+1}$, by the construction of E_N, it follows that

$$\mu(I_n) \leq \mu(I_{n_{k+1}}) = \mu(I_{m_{k+1}}) \leq c^{-(k+1)} \beta^{-(1-\delta)\sum\limits_{j=1}^{k}(n_{j+1}-m_j)}.$$

By (3.2.4), in both two cases,

$$|I_n| \geq \frac{1}{\beta^{n+N}}.$$

Since

$$\frac{a+x}{b+x} \geq \frac{a}{b} \text{ for all } 0 < a \leq b,\ x \geq 0,$$

it holds that

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq (1-\delta) \lim_{k\to\infty} \frac{\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)}{m_k}.$$

By (7.1.23) and (7.1.24),

$$\lim_{k\to\infty} \frac{n_k}{m_k} = \frac{1}{1+r},\ \lim_{k\to\infty} \frac{n_k}{m_{k-1}} = \frac{r}{\hat{r}(1+r)} \text{ and } \lim_{k\to\infty} \frac{m_k}{m_{k-1}} = \frac{r}{\hat{r}}.$$

By the Stolz-Cesàro theorem, we deduce that

$$\lim_{k\to\infty} \frac{\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)}{m_k} = \lim_{k\to\infty} \frac{n_k - m_{k-1}}{m_k - m_{k-1}} = \lim_{k\to\infty} \frac{\frac{n_k}{m_{k-1}} - 1}{\frac{m_k}{m_{k-1}} - 1} = \frac{r-(1+r)\hat{r}}{(1+r)(r-\hat{r})}.$$

Hence,

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq (1-\delta) \cdot \frac{r-(1+r)\hat{r}}{(1+r)(r-\hat{r})}.$$

Let $\delta \to 0$. The modified mass distribution principle (Theorem 3.1.8) gives

$$\dim_{\mathrm{H}} E_N \geq \frac{r-(1+r)\hat{r}}{(1+r)(r-\hat{r})}.$$

for all sufficient large enough N.

7.2 Proof of Theorem 2.4.2

When $\hat{r} = 0$, note that

$$R_\beta(0,0) \subseteq \{x \in [0,1) : \hat{r}_\beta(x) = 0\}.$$

Theorem 2.4.3 yields that the set $\{x \in [0,1) : \hat{r}_\beta(x) = 0\}$ is of full Lebesgue measure. When $\hat{r} > 1$, we have

$$\hat{r} > \frac{r}{1+r} \geq 0$$

for all $0 \leq r \leq +\infty$. It follows from Theorem 2.4.3 that the set $\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r}\}$ is countable. When $0 < \hat{r} \leq 1$, applying the same process as Section 3.3.1, we deduce that, for all $\theta > 0$,

$$\begin{aligned} \dim_{\mathrm{H}}\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r},\ r \leq r_\beta(x) \leq r+\theta\} &\leq \frac{r+\theta-(1+r+\theta)\hat{r}}{(1+r)(r+\theta-\hat{r})} \\ &\leq \frac{r-(1+r)\hat{r}}{(1+r)(r-\hat{r})} + \frac{\theta(1-\hat{r})}{(1+r)(r-\hat{r})} \\ &\leq \frac{r-(1+r)\hat{r}}{(1+r)(r-\hat{r})} + \frac{\theta(1-\hat{r})^2}{\hat{r}^2}. \end{aligned} \tag{7.2.1}$$

Now considering the final part of (7.2.1) as a function of r, we can find that

$$\frac{r-(1+r)\hat{r}}{(1+r)(r-\hat{r})} + \frac{\theta(1-\hat{r})^2}{\hat{r}^2} \leq \frac{(1-\hat{r})^2}{(1+\hat{r})^2} + \frac{\theta(1-\hat{r})^2}{\hat{r}^2}$$

for all $r \geq 0$. Note that

$$\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r}\} \subseteq \bigcup_{i=0}^{+\infty}\bigcup_{j=1}^{n}\left\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r},\ i+\frac{j-1}{n} \leq r_\beta(x) \leq i+\frac{j}{n}\right\}.$$

By the σ-stability of Hausdorff dimension (see [27]), we have

$$\dim_{\mathrm{H}}\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r}\} \leq \left(\frac{1-\hat{r}}{1+\hat{r}}\right)^2 + \frac{1}{n}\frac{(1-\hat{r})^2}{\hat{r}^2}.$$

Letting $n \to +\infty$, we therefore conclude that

$$\dim_{\mathrm{H}}\{x \in [0,1) : \hat{r}_\beta(x) \geq \hat{r}\} \leq \left(\frac{1-\hat{r}}{1+\hat{r}}\right)^2.$$

Finally, we use the maximization method of Bugeaud and Liao [18] for the estimation of the lower bound of $\dim_{\mathrm{H}}\{x \in [0,1) : \hat{r}_\beta(x) = \hat{r}\}$. In fact, fix $0 < \hat{r} \leq 1$, the function $r \mapsto \dim_{\mathrm{H}} R_\beta(\hat{r}, r)$ is continuous and reaches its maximum at the unique point $r = \frac{2\hat{r}}{1-\hat{r}}$. By calculation, we conclude that the maximum is exactly equal to $\left(\frac{1-\hat{r}}{1+\hat{r}}\right)^2$. Therefore,

$$\dim_{\mathrm{H}}\{x \in [0,1) : \hat{r}_\beta(x) = \hat{r}\} \geq \left(\frac{1-\hat{r}}{1+\hat{r}}\right)^2.$$

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Publications

No.	Authors	Title	Magazine	Issue, Date	Part
1.	Zheng, Lixuan Mu, Min Li, bing	The topological property of the irregular sets on the lengths in beta-expansions of basic intervals	Journal of Mathematical Analysis and Applications	2017, 449, no. 1, 127 – 137.	Chapter 4
2.	Zheng, Lixuan Wu, Min Li, bing	The exceptional sets on the run-length function of of beta-expansions	Fractals-Complex Geometry Patterns and Scaling in Nature and Society	Vol. 25, No. 06, 1750060 (2017)	Chapter 5
3.	Zheng, Lixuan	Diophantine approximation and run-length function on beta-expansions	Journal of Number Theory	2019, 202, no.1, 60-90	Chapter 6
4.	Zheng, Lixuan	Uniform recurrence properties for beta-dynamical system	Submitted		Chapter 7