

Particles, strings, fields and application to cosmology

Osmin Lacombe

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"Rien ne sert de penser, il faut réfléchir avant."

Pierre Dac

"Une civilisation sans la Science, ce serait aussi absurde qu'un poisson sans bicyclette."

Pierre Desproges

Abstract

This thesis aims for a better understanding of supersymmetry breaking mechanisms in string theory, their low-energy effective descriptions, and their role in cosmological applications. We first review indispensable supersymmetry tools, before addressing the issue of partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ global supersymmetry breaking. In this context, we study the most general deformation of a vector multiplet and show that it corresponds to the magnetic dual of a triplet of Fayet-lliopoulos parameters. The study of non-linear realizations of partial breaking leads to the analysis of the details of supersymmetric Dirac-Born-Infeld actions and their uniqueness, in particular through explicit computations of their fermionic contributions. In a second part, we review string theory elements with a particular accent on type IIB compactification ingredients. We then study the inflationary possibilities of a type IIB string theory moduli stabilization scenario with D7-branes using logarithmic quantum corrections to the Kähler potential. We show that a new type of inflationary scenario can successfully be implemented near a metastable de Sitter vacuum. The value of the vacuum energy in this scenario is nevertheless imposed by the inflationary stage. We discuss the possibility of fitting hybrid inflation with this moduli stabilization inflationary scenario. We show how to construct waterfall fields at the D7-brane intersections, study their effective theory description and explain how they can drive the end of inflation and lower the global minimum of the scalar potential towards the almost vanishing value describing our Universe.

Résumé

Cette thèse a pour but d'avancer dans la compréhension des mécanismes de brisure de supersymétrie en théorie des cordes, de leur descriptions effectives à basse énergie et de leurs applications cosmologiques. Dans un premier temps, nous rappelons certaines notions indispensables à l'étude des théories supersymétriques et en particulier de la brisure partielle $\mathcal{N} = 2 \to \mathcal{N} = 1$ en supersymétrie globale. Dans ce contexte, nous étudions la déformation la plus générale d'un multiplet vectoriel et montrons qu'elle correspond au dual magnétique d'un triplet de paramètres de Fayet-Iliopoulos. L'étude des réalisations non-linéaires de la supersymétrie brisée nous conduit à l'analyse détaillée des actions Dirac-Born-Infeld supersymétriques et de leur unicité, en particulier à travers le calcul détaillé des contributions fermioniques. Dans une seconde partie, nous introduisons des éléments de théorie de cordes, en mettant l'accent sur les compactifications en théorie de type IIB. Nous étudions alors les possibilités d'inflation d'un modèle de stabilisation des modules en type IIB, utilisant des D7-branes et des corrections quantiques logarithmiques au potentiel de Kähler. Nous exposons un scénario d'inflation d'un genre nouveau, réalisé au voisinage d'un vide de de Sitter métastable. Dans ce modèle, la valeur de l'énergie du vide est fixée par la phase inflationniste. Cela motive la mise en oeuvre d'inflation hybride dans notre modèle : nous montrons comment construire des champs de cascade aux intersections des D7-branes et étudions leur théorie effective. Nous expliquons que ces champs provoquent la fin de l'inflation et permettent d'abaisser l'énergie du vide jusqu'à la valeur qui décrit notre Univers.

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Contents

Abstract **v**

Acknowledgements **vii**

1 Introduction **1**

2 Supersymmetry breaking and non-linear realizations **5**
2.1 Supersymmetry basics . . . 5
2.1.1 SUSY algebra, representations and Lagrangians . . . 5
2.1.2 Spontaneous supersymmetry breaking . . . 10
2.2 Non-linear realizations of supersymmetry . . . 12
2.2.1 Non-linear realizations and the Volkov-Akulov action . . . 12
2.2.2 Non-linear realizations in superspace and constrained superfields . . . 15
2.3 Partial $\mathcal{N} = 2$ global supersymmetry breaking and induced FI terms . . . 18
2.3.1 $\mathcal{N} = 2$ supersymmetry, general deformations and SUSY breaking . . . 19
2.3.2 $\mathcal{N} = 2$ electromagnetic duality . . . 23
2.3.3 The APT model of partial SUSY breaking and beyond . . . 28
2.4 Supersymmetric Dirac-Born-Infeld actions and Fayet-Iliopoulos terms . . . 31
2.4.1 Deformed Dirac-Born-Infeld action . . . 33
2.4.2 On-shell Lagrangians and the use of field redefinitions . . . 42
2.4.3 Redefining the SDBI+FI Lagrangian . . . 43
2.4.4 SDBI action with induced FI term from γ deformation . . . 50
2.5 Summary . . . 58

3 String theory vacua and magnetic field SUSY breaking **61**
3.1 String theory elements . . . 61
3.1.1 Bosonic string . . . 61
3.1.2 Superstrings and type IIB spectrum . . . 66
3.1.3 Massless spectra . . . 71
3.1.4 Vacuum amplitudes . . . 73
3.1.5 Dp branes . . . 78
3.2 Type IIB compactifications and de Sitter vacua . . . 82
3.2.1 Compactification . . . 82
3.2.2 Moduli stabilization in type IIB . . . 87
3.2.3 Kähler potential constant and logarithmic quantum corrections . . . 90
3.3 Orbifold compactifications with fluxes . . . 92
3.3.1 Orbifold compactifications . . . 92
3.3.2 Magnetic fluxes supersymmetry breaking . . . 96
3.3.3 Toroidal model of matter waterfall field . . . 97
3.4 Summary . . . 104

4 String phenomenology and cosmology **107**
4.1 Early Universe Cosmology and String theory . . . 107
4.1.1 Scalar field inflation . . . 108
4.1.2 String inflation scenarios . . . 110
4.2 Inflation from *D*7-branes moduli stabilization . . . 111
4.2.1 Type IIB model of intersecting D7-branes and moduli stabilization . 112
4.2.2 Local de Sitter minimum . . . 114
4.2.3 Inflation possibilities from the model . . . 116
4.2.4 Physical observables and theoretical parameters . . . 121
4.2.5 Stability of the minimum . . . 123
4.3 New physics near the minimum: waterfall fields and hybrid inflation . . . 127
4.4 Effective theory with waterfall fields . . . 129
4.4.1 *D*-term from magnetic fields . . . 130
4.4.2 *F*-term from brane separation . . . 131
4.4.3 New vacuum . . . 134
4.5 Lowering the global minimum . . . 139
4.5.1 Field theoretical description . . . 140
4.5.2 Simple case studied previously . . . 140
4.5.3 Additional tachyons from other D7-brane stacks . . . 142
4.5.4 Adding a fourth magnetized stack . . . 144
4.6 Summary . . . 147

A Various conventions **151**
A.1 Spinor conventions and useful identities . . . 151
A.2 Field strength identities . . . 151
A.3 Theta functions and $so(2)$ characters . . . 152
A.4 Momenta and windings sums . . . 153

B Computational details **155**
B.1 Derivation of (2.4.63) . . . 155
B.2 Superfield expansions . . . 155

C Tachyons from magnetic fields on the entire D7-branes worldvolumes **159**

List of Figures

4.1 Scalar potential $V(\phi)$ for different values of x giving an AdS, Minkowski or dS vacuum. 116

4.2 Ratio of the values of the scalar potential V at the two extrema (*left panel*) and value of the slow-roll parameter η_V at the two extrema (*right panel*), as functions of the parameter x. 117

4.3 Hubble parameter $H(\phi)$ (*top left*), number of e-folds $N(\phi)$ (*top right*) and slow-roll parameters ϵ, η (*bottom*) for $q = 0, C = 1 \times 10^8$ and $x = 2.753 \times 10^{-7}$. 118

4.4 Hubble parameter $H(\phi)$ (*top left*), number of e-folds $N(\phi)$ (*top right*) and slow-roll parameters ϵ, η (*bottom*), for $q = 0$, $C = 1 \times 10^8$ and $x = 3.3 \times 10^{-4}$. The dashed horizontal line shows the value $\eta = -0.02$. 119

4.5 Hubble parameter solution for $x = 3.3 \times 10^{-4}$, $q = 0$, $C = 1 \times 10^8$ with initial speed $\dot{\phi} = -2H'(\phi) = -2 \times 10^{-5}\kappa^{-1}$ (*red dashed line*), and with vanishing initial speed (*plain orange line*). 120

4.6 Point-like particle rolling down the inverted potential $-V(\varphi)$ from an initial position φ_0. An overshooting case is shown (*left panel*) as well as the CdL instanton solution (*right panel*) for which the field stops exactly at the maximum $-V(\varphi_-)$. 125

4.7 Scalar potential $V(\phi)$ for $x = 1.0 \times 10^{-2}$ (*top left*) and the CdL bounce solution functions $\phi(\rho)$ (*top right*) and $\chi(\rho)$ (*bottom*). The initial field value is $\varphi_0 = 4.9224\kappa^{-1}$ and the final Euclidean time is $\rho_f = 4.1222$. 126

4.8 Ratio of the critical Hubble parameter H_c^2 and the Hubble parameter at the minimum $H_-^2 = \kappa^2 V(\varphi_-)/3$. The value $H_- = H_c$ (horizontal dashed line) determines the existence and domination of the CdL instanton over the HM instanton. 127

4.9 Value $V(\mathcal{V}, v_2)$ of the global minimum of the scalar potential as a function of the internal volume, for the parameters of (4.4.56). 138

4.10 Value of the global minimum of the effective scalar potential as a function of $\mathcal{V}$, without (*blue*), with one (*orange*), two (*green*) or three (*red*) tachyons. The parameters are such that $\mathcal{V}_{c2} = \mathcal{V}_-$, $\mathcal{V}_{c3} = 0.99\mathcal{V}_-$ and $\mathcal{V}_{c1} = 0.98\mathcal{V}_-$. . . . 144

4.11 Value $V(\mathcal{V}, v_i)$ of the global minimum of the scalar potential as a function of the internal volume, for the parameters (4.5.31). We show *(left panel)* the almost vanishing value of the global minimum, and focus *(right panel)* on the waterfall zone near $\mathcal{V}_- \approx 201.9$. 147

List of Abbreviations

AdS	**A**nti **d**e **S**itter
APT	**A**ntoniadis **P**artouche **T**aylor
b. c.	**b**oundary **c**ondition
CdL	**C**oleman **d**e **L**uccia
CMB	**C**osmological **M**icrowave **B**ackground
CS	**C**hern **S**imons
CY	**C**alabi **Y**au
D	**D**irichlet
DBI	**D**irac **B**orn **I**nfeld
dS	**d**e **S**itter
EH	**E**instein **H**ilbert
EM	**E**lectro**M**agnetic
FI	**F**ayet-**I**liopoulos
FLRW	**F**riedmann **L**emaître **R**obertson **W**alker
HM	**H**awking **M**oss
IR	**I**nfra **R**ed
KK	**K**aluza **K**lein
KKLT	**K**achru **K**allosh **L**inde **T**rivedi
LVS	**L**arge **V**olume **S**cenarios
M	**M**inkowski
N	**N**eumann
NG	**N**ambu **G**oto
NS	**N**eveu **S**chwarz
R	**R**amond
SDBI	**S**upersymmetric **D**irac **B**orn **I**nfeld
SM	**S**tandard **M**odel
SUGRA	**SU**per**GRA**vity
SUSY	**SU**per**SY**mmetry
UV	**U**ltra **V**iolet
VA	**V**olkov **A**kulov
VEV	**V**acuum **E**xpectation **V**alue

Chapter 1

Introduction

Modern theoretical physics is based on two major discoveries of the beginning of the twentieth century: quantum mechanics and general relativity. Their simultaneous development led to great advances in our understanding of the laws of Nature, from subatomic to extragalactic scales. Quantum field theory developed as a compelling framework to describe relativistic quantum phenomena and showed the immeasurable power of the use of symmetries in theoretical physics. The use of quantum field theory and symmetries led in the '70s to the formulation of what is now known as the Standard Model of particle physics, which describes almost perfectly all the particles we detect in collider experiments. In some cases, predictions of the existence of a particle came even before it was observed, almost fifty years before in the case of the Higgs boson. In parallel, general relativity paved the ground for huge progress in our understanding of physics of the large scales of our Universe. Discoveries made thanks to general relativity go from cosmic expansion to astrophysical objects such as black holes, by way of gravitational waves. These general relativity predictions have been successfully tested experimentally. In the last few years, international collaborations were even able to take the first black hole picture. The study of cosmic expansion led to the Standard Model of Cosmology, called ΛCDM. When associated with the inflation scenario, it explains very precisely observations such as the cosmic microwave background or the distribution of galaxy clusters.

Unfortunately, both these Standard Models suffer from theoretical and experimental inconsistencies. Some of their problems are intimately related. For instance, the Standard Model of particle physics is unable to describe the dark matter and dark energy content of the Standard Model of cosmology. Supersymmetry emerged as a strikingly powerful theory to tackle some of these problems, such as the hierarchy of high energy scales, the gauge coupling unification, or yet the existence of dark matter, through the prediction of new weakly interacting elementary particles. On the theoretical side, supersymmetry is a key ingredient for the consistency of string theory, that we introduce hereafter. Even if supersymmetry holds at high energies, it must nevertheless break below a certain energy scale, higher than our accessible energy scales, as we have observed no experimental evidence of its existence yet. In particular, unbroken supersymmetry predicts superpartners with the same masses as the Standard Model particles, that we should be able to detect. Supersymmetry breaking is a central topic of this thesis, from its low-energy description to some of its string theory realizations and their applications.

String theory is one of the best quantum gravity theory candidates. It naturally describes gravity and can produce chiral gauge theories in the same time. Nevertheless, string theory predictive power is often criticized. Constructing predictive models for the physics accessible at our energy scales is the main goal of string phenomenology. One of the major issues is to describe how to go from properties of a ten-dimensional space-time, essential for the consistency of supersymmetric strings theories, to predictions for our

four-dimensional space-time. This question is resolved, through compactification mechanisms, with the idea that six of the initial dimensions are described by a compact manifold, parametrized by moduli fields. For small compact manifold sizes, observers living in the non-compact dimensions, at low energies, only have access to integrated compact degrees of freedom, namely to the lowest level of the Kaluza-Klein tower of states, so that they only feel the four non-compact dimensions. Phenomenologically interesting compact manifolds must preserve some of the supersymmetries of the initial string theory spectrum. This is the case of Calabi-Yau manifolds or toroidal orbifolds, which provide a vast choice of possible internal spaces associated with a vast number of possible low-energy predictions and vacua, called the string theory landscape. Effective theory descriptions of the low-energy regimes of string theory are mandatory to link with cosmological observables or scattering experiments observables, by keeping only the relevant degrees of freedom among the infinity of the full string theory ones. Tentatives to reduce the number of low-energy theories belonging to the string landscape have been developing these last years under the name of the swampland program.

The work of this thesis is motivated by the desire of getting a better understanding of supersymmetry breaking mechanisms in string theory and their early Universe consequences. At its early stages, our Universe was hot and dense, hence ruled by the laws of physics at very high energies. It is thus natural to try and build cosmological models, especially inflationary models, in the string theory framework. There are two rather different parts in this thesis.

The first part of this thesis tackles $\mathcal{N} = 2 \to \mathcal{N} = 1$ partial supersymmetry breaking from the effective point of view of global supersymmetry. Partial supersymmetry studies are motivated by the fact that type II string theories, convenient for string phenomenology constructions, exhibit $\mathcal{N} = 2$ supersymmetry in the bulk, whereas $\mathcal{N} = 1$ supersymmetry is preferred for phenomenological applications. In string theory this partial supersymmetry breaking occurs naturally on D3 branes, non-perturbative objects spanning our four space-time dimensions and localized in the internal space. Nevertheless, partial supersymmetry breaking realizations in global supersymmetry are not simple and require some subtle elements such as deformations or electric and magnetic Fayet-Iliopoulos parameters. Interestingly, the study of non-linear realizations of supersymmetry in partially breaking models naturally lead to the study of supersymmetric Dirac-Born-Infeld actions, which precisely describe the string theory D3 branes. The main aspiration of the first part of this thesis is to discover new effective theory partial supersymmetry breaking mechanisms and unveil their string theory origin, following a "bottom-up" approach.

The second part of this thesis follows a rather opposite way, namely a "top-down" approach, through string cosmology model building. If string theory is indeed the good candidate for quantum gravity theory, its low-energy regime should be able to describe the physics at play at the early stages of our Universe, namely during inflation. Supersymmetry breaking at the string level plays an essential role in constructions of cosmologically relevant de Sitter vacua, and can be intimately related to the problem of the stabilization of internal space moduli. This approach allows us to tackle the question of the hierarchy between the diverse scales at play in our Universe, namely the relation between the Planck, inflation, supersymmetry breaking or dark energy scales, and the string or compact manifold scales. To use the slow-rolling scalar field inflation paradigm, one should look for effective scalar potentials presenting a flat region, as well as a de Sitter minimum. Both these two features are difficult to obtain in a controlled way in string theory, and they impose careful checks of the validity of approximations, which often require large internal volumes and small string coupling.

This thesis is organized as follows. In Chapter 2, we first describe basic supersymmetry ingredients necessary to study spontaneous supersymmetry breaking and non-linear realizations, that we address in a second time. We make full use of these concepts to review partial supersymmetry breaking from global $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in the framework of deformed vector multiplets. We analyze supersymmetry breaking through the introduction of the most generic deformation parameters and study the dual description of the latter in terms of a triplet of Fayet-Ilopoulos parameters, as well as their consequences when supersymmetry is non-linearly realized. In this context, we are naturally led to study supersymmetric Dirac-Born-Infeld actions and the question of their uniqueness, in particular through detailed computation of their fermionic contributions. In Chapter 3 we propel ourselves at higher energies and describe key elements of string theory, starting from textbook material necessary for the discussions that come next. In particular, the beginning of this chapter aims to discuss vacuum amplitudes, type IIB compactifications on Calabi-Yau or toroidal orbifolds, and Dp branes. We particularly emphasize the importance of moduli stabilization scenarios leading to phenomenologically interesting de Sitter spaces. At the end of the chapter, we show, on a particular model of toroidal compactification with magnetic fluxes background, how to construct open string states sharing the properties of waterfall fields. Such waterfall fields are key elements of hybrid inflation scenarios, and we postpone their low-energy description until the next chapter. Chapter 4 is devoted to the application of string theory model building to cosmology. After a short introduction to the inflation paradigm, together with a discussion on its past implementations in string theory, we go on with a detailed study of the inflationary possibilities in a type IIB moduli stabilization framework with D7 branes using logarithmic quantum corrections to the Kähler potential. We expose a new inflationary mechanism near a de Sitter minimum and explain how to tune the vacuum energy using the hybrid inflation scenario. This scenario makes use of the waterfall field constructed before, which, living at D7 branes intersections, fits perfectly the studied moduli stabilization framework.

The research presented in this thesis is based on the papers listed below:

[1] I. Antoniadis, H. Jiang and O. Lacombe, *$\mathcal{N} = 2$ supersymmetry deformations, electromagnetic duality and Dirac-Born-Infeld actions*, *JHEP* **07** (2019) 147, [1904.06339]

→ corresponds to sections 2.3 and 2.4

[2] I. Antoniadis, H. Jiang and O. Lacombe, *Note on supersymmetric Dirac-Born-Infeld action with Fayet-Iliopoulos term*, *JHEP* **05** (2020) 111, [1912.12627]

→ corresponds to section 2.4

[3] I. Antoniadis, O. Lacombe and G. K. Leontaris, *Inflation near a metastable de Sitter vacuum from moduli stabilisation*, *Eur. Phys. J. C* **80** (2020) 1014, [2007.10362]

→ corresponds to section 4.1.1 and section 4.2

[4] I. Antoniadis, O. Lacombe and G. K. Leontaris, *Hybrid inflation and waterfall field in string theory from D7-branes*, 2109.03243

→ corresponds to sections 3.3, 4.3 and 4.4

Chapter 2

Supersymmetry breaking and non-linear realizations

2.1 Supersymmetry basics

Supersymmetry (SUSY) was first introduced to address unanswered aspects of the Standard Model (SM) of particle physics, such as the unification of gauge couplings at high energies. It then proved useful in broader theoretical contexts, from string theory to dark-matter models. SUSY assumes the existence of $\mathcal{N}$ symmetries relating fermionic and bosonic fields and when $\mathcal{N} > 1$, it is referred to as "extended" SUSY. Such a strong symmetry has an important number of theoretical consequences and assures a better control of the theory, through non-renormalization theorems for instance. On the phenomenological side, supersymmetry predicts superpartners for each particle of the Standard Model. As these superpartners are not observed at our energy scales, even if SUSY exists at high energies, it must necessarily " break " at a certain scale: this phenomenon is called spontaneous symmetry breaking. In analogy to Nambu-Goldstone bosons [5, 6] taking part in the famous Brout-Englert-Higgs [7, 8] mechanism for electroweak symmetry breaking, SUSY breaking models predict the existence of a massless fermion, called Goldstino, for each broken SUSY generator.

This section is devoted to a short introduction to supersymmetric theories and Lagrangians. This is mostly textbook material, but we need to introduce key elements that we will use repeatedly in the rest of this thesis. In section 2.1.1 we recall basic elements of $\mathcal{N} = 1$ SUSY theories constructions, before tackling in section 2.1.2 descriptions of the aforementioned SUSY breaking phenomenon.

2.1.1 SUSY algebra, representations and Lagrangians

Algebra In relativistic quantum field theories, symmetries play an essential role and are separated between Poincaré symmetries and internal symmetries, all implemented through the introduction of bosonic charges. Supersymmetry can be defined through the introduction of a new symmetry that exchanges fermionic and bosonic degrees of freedom, hence related to a fermionic charge. Together with the bosonic charges mentioned above, they form a graded Lie algebra, which has been proven to be the only one compatible with an S-matrix description of quantum field theories [9]. A supersymmetric theory is thus a theory containing fields in specific representations of the supersymmetry algebra

$$\begin{aligned} &\{Q_\alpha^A, \bar{Q}_{B\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta_B^A, \quad \{Q_\alpha^A, Q_\beta^B\} = 0, \\ &[P_\mu, Q_\alpha^A] = [P_\mu, Q_\beta^B] = 0, \quad [M_{\mu\nu}, Q_\alpha^A] = i\sigma^{\mu\nu}{}_\alpha^\beta Q_\beta^A, \end{aligned} \tag{2.1.1}$$

where $P_\mu, M_{\mu\nu}$ are respectively the translation and Lorentz generators of the Poincaré algebra. The A, B indices play a role for extended supersymmetry, when $\mathcal{N} \geq 2$ supersymmetric charges are included. Note that we considered vanishing central charges $Z^{AB} = 0$ in the above algebra, hence leading to an $U(\mathcal{N})$ R-symmetry group in extended SUSY. In the following, we use Weyl spinors and conventions of [10], summarized in Appendix A.1.

Multiplets If SUSY is a symmetry of the theory, fields must lie in representations of the SUSY algebra (2.1.1). As P^2 is a Casimir operator and as P_μ commutes with the SUSY generators, fields lying in the same representation of the superalgebra have the same mass. We construct massless irreducible representations, called multiplets, in the standard way. We first boost to a reference frame with convenient momentum operator, and then act with creation operators, related to the supercharges, on a lowest-spin Clifford vacuum. Because of the commutation relations (2.1.1) between Lorentz and supersymmetry charges, creation operators increase the spin by $1/2$. Supersymmetry multiplets thus contain states of different spins. Because of the fermionic nature of the SUSY charges, creation operators cannot be applied twice on the vacuum, and only a finite number of different spins are present in each representation. For $\mathcal{N} = 1$ SUSY, the massless multiplets of interest for the rest of this thesis are chiral multiplets $\Phi = (\phi, \psi, \mathsf{F})$, containing a complex scalar, its fermionic partner, and a complex auxiliary field, together with vector multiplets $V = (\lambda, A_\mu, \mathsf{D})$ containing a gauge vector, its fermionic partner called gaugino and a real auxiliary field. The F and D auxiliary fields are necessary to assure that the superalgebra closes on the entire multiplets, without the use of the equation of motions. They are not physical and must be eliminated through their respective equation of motion. By construction, the different fields of the multiplets are related through supersymmetry, and their transformations can be expressed using supercharge representations. Namely, the chiral multiplet components infinitesimal supersymmetry transformations read

$$\begin{aligned}
\delta_\epsilon \phi &\equiv (\epsilon Q + \bar{\epsilon}\bar{Q})\phi = \sqrt{2}\epsilon\psi, \\
\delta_\epsilon \psi_\alpha &\equiv (\epsilon Q + \bar{\epsilon}\bar{Q})\psi = i\sqrt{2}(\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu \phi + \sqrt{2}\epsilon_\alpha \mathsf{F}, \\
\delta_\epsilon \mathsf{F} &\equiv (\epsilon Q + \bar{\epsilon}\bar{Q})\mathsf{F} = -i\sqrt{2}\partial_\mu \psi \sigma^\mu \bar{\epsilon},
\end{aligned} \tag{2.1.2}$$

while those of the vector multiplet are

$$\begin{aligned}
\delta_\epsilon \lambda_\alpha &= i\mathsf{D}\epsilon_\alpha + (\sigma^{\mu\nu}\epsilon)_\alpha F_{\mu\nu}, \\
\delta_\epsilon F_{\mu\nu} &= 2i\left(\epsilon\sigma_{[\nu}\partial_{\mu]}\bar{\lambda} - \partial_{[\mu}\lambda\sigma_{\nu]}\bar{\epsilon}\right), \\
\delta_\epsilon \mathsf{D} &= \epsilon\not{\partial}\bar{\lambda} - \partial_\mu \lambda \sigma^\mu \bar{\epsilon}.
\end{aligned} \tag{2.1.3}$$

In the second line, we used the convention that $a_{[\mu}b_{\nu]} = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu)$. One can check that the r.h.s. of the last line is indeed real, as is required for a real D, by making use of identities of Appendix A.1.

Superfields It is convenient to describe supersymmetry multiplets and Lagrangians through the use of superfields and superspace. We do not enter the details of the construction of superspace but only recall the main ingredients that we will use in the following. Superspace is implemented by the addition of Grassmann coordinates $\theta, \bar{\theta}$ to the space-time coordinates x^μ. Because of the fermionic nature of Grassmann variables, any

function of $(x^\mu, \theta, \bar{\theta})$ can be power expanded exactly as

$$\begin{aligned} S(x,\theta,\bar{\theta}) = s(x) + \theta\lambda(x) + \bar{\theta}\bar{\psi}(x) + \theta\theta f(x) + \bar{\theta}\bar{\theta}g(x) + \theta\sigma^\mu\bar{\theta}A_\mu(x) \\ + \theta\theta\bar{\theta}\bar{\chi}(x) + \bar{\theta}\bar{\theta}\theta\eta(x) + \theta\theta\bar{\theta}\bar{\theta}h(x). \end{aligned} \tag{2.1.4}$$

The SUSY generators are expressed in terms of the Grassmann coordinates as

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu, \qquad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu. \tag{2.1.5}$$

One can then find the infinitesimal transformations of the superfield components by acting with the above operators on $S(x,\theta,\bar{\theta})$, through the definition

$$\begin{aligned} \delta_\epsilon S(x,\theta,\bar{\theta}) &\equiv (\epsilon Q + \bar{\epsilon}\bar{Q})S(x,\theta,\bar{\theta}) \\ &= \delta_\epsilon s(x) + \theta\delta_\epsilon\lambda(x) + \bar{\theta}\delta_\epsilon\bar{\psi}(x) + \theta\theta\delta_\epsilon f(x) + \bar{\theta}\bar{\theta}\delta_\epsilon g(x) + \cdots \end{aligned} \tag{2.1.6}$$

As the Grassmann derivatives $\partial/\partial\theta^\alpha$ are not covariant, they do not even generate superfields when acting on superfields. It is thus useful to introduce covariant derivatives $D_\alpha, \bar{D}_{\dot{\alpha}}$ satisfying $\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = 0$, called superderivatives, which will be used to construct new superfields and Lagrangians. These derivatives can be expressed as

$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu\,, \qquad \bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i(\theta\bar{\sigma}^\mu)_{\dot{\alpha}}\partial_\mu\,, \tag{2.1.7}$$

and satisfy the following anti-commutation relations

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu, \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0. \tag{2.1.8}$$

Chiral multiplets, introduced above, are described by chiral superfields Φ constructed from the constraint

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{2.1.9}$$

This constraint eliminates some of the components of a generic superfield. The chiral superfield Φ can be expressed as

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta\mathsf{F}(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) \\ &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta\mathsf{F}(y). \end{aligned} \tag{2.1.10}$$

In the last line, we introduced the chiral coordinate

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \tag{2.1.11}$$

with the help of which we can rewrite the superderivatives as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^\mu}, \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \tag{2.1.12}$$

We can easily check that acting with $\epsilon Q + \bar{\epsilon}\bar{Q}$ on $\Phi(x,\theta,\bar{\theta})$ and using relation (2.1.6), we find back the transformation laws (2.1.2). An important property is that the product of chiral superfields is a chiral superfield. Of course, antichiral superfields $\bar{\Phi}$ can be constructed the exact same way, with D_α instead of $\bar{D}_{\dot{\alpha}}$ in the constraint (2.1.9).

Vector multiplets are described through real superfields V satisfying the constraint

$$V = V^\dagger. \tag{2.1.13}$$

This constraint relates the diverse components of the real superfield. By applying a supergauge transformation $V \to V + \Phi + \Phi^\dagger$ it can be brought to the Wess-Zumino gauge [11], in which it reads

$$V_{WZ}(x,\theta,\bar\theta) = -\theta\sigma^\mu\bar\theta A_\mu(x) + i\theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) + \frac{1}{2}\theta\theta\bar\theta\bar\theta \mathsf{D}(x), \tag{2.1.14}$$

and where only λ, A_μ and D, introduced previously, are non-vanishing. The λ and D components are supergauge invariant while A_μ transforms following the standard gauge transformation $A_\mu \to A_\mu - i\partial_\mu(\phi - \phi^*)$. As can be seen by acting with $\epsilon Q + \bar\epsilon\bar Q$, the Wess-Zumino gauge is not SUSY invariant. We can nevertheless construct the $\mathcal{N} = 1$ field strength chiral superfield

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V = -\frac{1}{4}\bar D_{\dot\alpha}\bar D^{\dot\alpha} D_\alpha V, \tag{2.1.15}$$

which is invariant under supergauge transformations. When expressed in terms of the y variable introduced in (2.1.11), it reads

$$W_\alpha(y,\theta,\bar\theta) = -i\lambda_\alpha(y) + \theta_\alpha \mathsf{D}(y) - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta(\sigma^\mu\partial_\mu\bar\lambda(y))_\alpha\,, \tag{2.1.16}$$

and only contains the gauge invariant fields. What's more, one can check that the field strength superfield satisfies the condition

$$DW = D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha} = \bar D\bar W, \tag{2.1.17}$$

which is related to the Bianchi identities of $F_{\mu\nu}$.

Lagrangians Supersymmetric Lagrangians can be constructed out of the above superfields by using components that are invariant under supersymmetry, up to a total derivative. This is the case of the top components, proportional to $\theta\theta\bar\theta\bar\theta$, of generic superfields or the $\theta\theta$ components of chiral superfields. The general $\mathcal{N} = 1$ SUSY invariant Lagrangians can be put in the following form

$$\mathcal{L} = \int d^2\theta d^2\bar\theta\, V + \left(\int d^2\theta\, \Phi\Big|_{\bar\theta=0} + c.c.\right), \tag{2.1.18}$$

where $V = V^\dagger$ is a real superfield and the complex conjugate of the chiral superfield Φ integral is added because Lagrangians must be hermitian. We recall that the action is obtained after space-time integration

$$S = \int d^4x\mathcal{L}, \tag{2.1.19}$$

hence it is indeed sufficient for Lagrangians to be invariant up to a total derivative, justifying the use of the top components of the real and chiral superfields. When chiral superfields are expressed in terms of x^μ, the $\bar\theta = 0$ condition of the second part of the Lagrangian is necessary to select their $\theta\theta$ component. When expressed in terms of y^μ, introduced in (2.1.11), chiral superfields only contain θ components and this condition is

not necessary. Hence, we will not use it in the rest of this thesis. This remark also explains why the $\theta\theta$ component of chiral superfields is sometimes called top component. The first part of the Lagrangian (2.1.18) is called the D-part while the second is the F-part, because they correspond to the D and F auxiliary fields of the superfields, as can be seen in eqs. (2.1.10) and (2.1.14). We stress the fact that the V and Φ superfields in (2.1.18) can be made of out of other superfields, resulting in composite superfields.

As already mentioned, a product of chiral superfields stays chiral, hence interaction terms can be constructed by multiplying chiral superfields, and a general renormalizable Lagrangian for chiral matter can be written as

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\, K^i_j \Phi_i \bar{\Phi}^j + \left(\int d^2\theta\, c^i \Phi_i + \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{3} \lambda^{ijk} \Phi_i \Phi_j \Phi_k + c.c. \right) \\ &= -K^i_j \left(i\psi_i \not{\partial} \bar{\psi}^j - \partial_\mu \phi_i \partial^\mu \phi^{*j} + \mathsf{F}_i \bar{\mathsf{F}}^j \right) \\ &\quad + c^i \mathsf{F}_i + m^{ij} (\phi_i \mathsf{F}_j - \frac{1}{2} \psi_i \psi_j) + \lambda^{ijk} (\phi_i \phi_j \mathsf{F}_k - \psi_i \psi_j \phi_k) + c.c.\,. \end{aligned} \tag{2.1.20}$$

The first part, containing kinetic terms, is the expansion of the Kähler potential $K(\Phi_i, \bar{\Phi}^i)$ while the second one, containing interaction terms, is the expansion of the superpotential $\mathsf{W}(\Phi_i)$. The superpotential is at most cubic in the fields in a renormalizable theory

$$\mathsf{W}(\Phi_i) = c^i \Phi_i + \frac{1}{2} m^{ij} \Phi_i \Phi_j + \frac{1}{3} \lambda^{ijk} \Phi_i \Phi_j \Phi_k. \tag{2.1.21}$$

In the second line of (2.1.20) we expressed the Lagrangian explicitly in components, by expanding the superfields and integrating. For diagonal K, this Lagrangian corresponds to the Wess-Zumino model [11]. Note that the last line can be written as

$$\mathcal{L}_{\mathsf{W}(\phi_i)} = \frac{\partial \mathsf{W}}{\partial \phi_i}(\phi_i) F_i + \frac{\partial^2 \mathsf{W}}{\partial \phi_i \partial \phi_j}(\phi_i) \psi_i \psi_j + c.c.\,. \tag{2.1.22}$$

Lagrangians for vector superfields can be constructed in the same way, by making use of the supergauge invariant field strength introduced in eq. (2.1.15). We first form the composite chiral scalar superfield $W^2 = W^\alpha W_\alpha$, then extract its top component

$$\begin{aligned}\mathcal{L} &= \frac{1}{4} \int d^2\theta W^2 + c.c. \\ &= -i\lambda \not{\partial} \bar{\lambda} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathsf{D}^2. \end{aligned} \tag{2.1.23}$$

The $1/4$ factor is introduced to have canonically normalized kinetic terms for the field strength and fermions. The second line is obtained through partial integration and the equality holds up to a total derivative. The Lagrangian (2.1.23) is the supersymmetric version of the Lagrangian of a free vector field, including the vector, the gaugino and auxiliary field D. As we mentioned already, the top component of the vector field is also invariant under supersymmetry, hence nothing forbids us to add the so-called Fayet-lliopoulos (FI) [12] Lagrangian to the theory

$$\mathcal{L}_{\mathrm{FI}} = 2\xi \int d^4\theta V = \xi \mathsf{D}, \tag{2.1.24}$$

which will play a crucial role in supersymmetry breaking and in the rest of this thesis.

Gauge invariant Lagrangians, coupling chiral matter and gauge vector fields, can be formed in a very simple way. We do not enter the details and directly write down the

general Lagrangian for a chiral matter field with canonical Kähler potential, coupled to a $U(1)$ gauge vector as

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\, \Phi_i e^{q_i V}\bar{\Phi}^i + \left(\int d^2\theta\, \frac{1}{4}W^2 + c^i\Phi_i + \frac{1}{2}m^{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda^{ijk}\Phi_i\Phi_j\Phi_k + c.c.\right)\\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mathcal{D}_\mu\phi_i\mathcal{D}^\mu\phi^{*i} + \frac{1}{2}\mathsf{D}^2 - i\lambda\not{\mathcal{D}}\bar{\lambda} - i\psi_i\not{\mathcal{D}}\bar{\psi}^i + \mathsf{F}_i\bar{\mathsf{F}}^i\\ &\quad + i\sqrt{2}(q_i\phi^{*i}\psi_i\lambda - q_i\phi_i\bar{\psi}^{*i}\bar{\lambda}) + q_i\mathsf{D}\phi_i\phi^{*i} + \mathcal{L}_{\mathsf{W}(\phi_i)},\end{aligned} \tag{2.1.25}$$

where here $\mathcal{D}$ are gauge invariant derivatives, and the various fields transform under supergauge transformations as

$$\Phi_i \to e^{-iq_i\Lambda}\Phi_i, \quad \bar{\Phi}^i \to e^{+iq_i\bar{\Lambda}}\bar{\Phi}^i, \quad V \to V + i(\Lambda - \bar{\Lambda}). \tag{2.1.26}$$

In (2.1.25), the $\mathcal{L}_{\mathsf{W}(\phi_i)}$ part contains the chiral matter interactions introduced in (2.1.22). The linear term of the superpotential c^i can only be present for matter of vanishing charge, since otherwise it is not gauge invariant. The other terms of the superpotential must also be gauge invariant, hence they couple fields whose charges sum to zero. There exists a straightforward non-abelian generalisation of the Lagrangian (2.1.25), making use of the non-abelian field strength $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V$.

2.1.2 Spontaneous supersymmetry breaking

Supersymmetry predicts equal masses for superpartners, *i.e.* physical fields lying in the same representation of the superalgebra. As superpartners for SM particles have not been observed in experiments, supersymmetry cannot be a symmetry of the theory at low energies. One thus has to invoke mechanisms breaking supersymmetry spontaneously.

Global $\mathcal{N} = 1$ SUSY theories have positive semi-definite hamiltonians. This is shown by expressing P^0 as a function of the supercharges, with the help of the superalgebra (2.1.1) and the spinor identities of Appendix A.1

$$H = P^0 = \frac{1}{4}(\bar{\sigma}^0)^{\dot{\alpha}\alpha}\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = \frac{1}{4}(Q_1\bar{Q}_{\dot{1}} + Q_2\bar{Q}_{\dot{2}} + \bar{Q}_{\dot{1}}Q_1 + \bar{Q}_{\dot{2}}Q_2). \tag{2.1.27}$$

Supersymmetric states are annihilated by all supercharges and thus have vanishing energy, while non-supersymmetric states have positive energies. This fact can also be understood by looking at the effective scalar potential of the theory. Consider for instance a $U(1)$ gauge theory of chiral matter coupled to a FI term, described by the sum of the Lagrangians of eqs. (2.1.24) and (2.1.25). The scalar potential is simply

$$V = \frac{1}{2}\mathsf{D}^2 + \mathsf{F}_i\bar{\mathsf{F}}^i, \tag{2.1.28}$$

where the auxiliary fields are expressed in terms of the scalar fields ϕ_i, through their equations of motion

$$\mathsf{D} = -q_i\phi_i\phi^{*i} - \xi, \tag{2.1.29}$$

$$\bar{\mathsf{F}}^i = -\frac{\partial \mathsf{W}}{\partial\Phi_i}(\phi_i) = -c^i - m^{ij}\phi_j - \lambda^{jki}\phi_j\phi_k, \quad \mathsf{F}_i = -\frac{\partial\bar{\mathsf{W}}}{\partial\bar{\Phi}^i}(\phi^{*i})\,. \tag{2.1.30}$$

As explained above, supersymmetry is broken if the vacuum energy is positive, hence if the scalar potential is non-vanishing, which implies that at least one of the auxiliary fields has a non-vanishing expectation value, *i.e.* $\langle\mathsf{D}\rangle \neq 0$ *or* $\langle\mathsf{F}_i\rangle \neq 0$. From the expressions of

eqs. (2.1.29) and (2.1.30) for the auxiliary fields, we see that this can be achieved in several ways, that we describe below.

F-term SUSY breaking The first possibility is to choose a chiral field content and a superpotential such that $\langle \mathsf{F}_i \rangle = 0$ cannot be solved simultaneously for all ϕ_i. The simplest case is the O'Raifeartaigh model [13] with three chiral superfields and superpotential

$$\mathsf{W}(\Phi_i) = c_1\Phi_1 + m^{23}\Phi_2\Phi_3 + \lambda^{122}\Phi_1\Phi_2^2. \tag{2.1.31}$$

The auxiliary fields

$$\bar{\mathsf{F}}^1 = -c^1 - \lambda^{122}\phi_2^2, \qquad \bar{\mathsf{F}}^2 = -m^{23}\phi_3 - 2\lambda^{122}\phi_1\phi_2, \qquad \bar{\mathsf{F}}^3 = -m^{23}\phi_2, \tag{2.1.32}$$

cannot vanish identically and SUSY is hence spontaneously broken. For $-c^1 < (m^{23})^2/2\lambda^{122}$ the global minimum is at $\langle\phi_2\rangle = \langle\phi_3\rangle = 0$ and has a flat direction for ϕ_1. The auxiliary field $\bar{\mathsf{F}}^1$ gets a non-vanishing VEV responsible for SUSY breaking, and the scalar potential is simply $V = (c^1)^2$.

D-term SUSY breaking Another possibility is to have a non-vanishing FI parameter ξ, in the absence of matter (*i.e.* only a SUSY vector field theory) or together with matter fields arranged so as to forbid $\langle \mathsf{D} \rangle = 0$. The simplest method can be achieved for a vanishing superpotential, hence vanishing F_i auxiliary fields. In this case, a non-zero ξ is mandatory in order to have $\langle \mathsf{D} \rangle \neq 0$, otherwise $\langle\phi_i\rangle = 0$ always lead to a supersymmetric minimum. For instance, one can consider a theory of a single chiral field Φ of charge q such that $q\xi > 0$. The vacuum is at $\langle\phi\rangle = 0$, which leads to $\langle \mathsf{D} \rangle = -\xi$ and a non-vanishing scalar potential $V = \frac{1}{2}\xi^2$.

An anomaly free theory requires nevertheless that $\sum_i q_i^3 = 0$, demanding at least two chiral fields of opposite charges. This is the famous Fayet-Iliopoulos model [12] containing two chiral superfields Φ_+, Φ_- of charges $\pm q$ with superpotential $\mathsf{W} = m\Phi_+\Phi_-$. The auxiliary fields equations (2.1.29) and (2.1.30) simply reduce to

$$\mathsf{D} = -\xi - q|\phi_+|^2 + q|\phi_-|^2, \qquad \bar{F}^+ = -m\phi_-, \qquad \bar{F}^- = -m\phi_+, \tag{2.1.33}$$

and lead to the scalar potential

$$V = \frac{1}{2}\xi^2 + (m^2 + q\xi)|\phi_+|^2 + (m^2 - q\xi)|\phi_-|^2 + \frac{1}{2}q^2(|\phi_+|^2 - |\phi_-|^2)^2. \tag{2.1.34}$$

Hence, when $m^2 > q\xi$, the minimum is obtained for vanishing matter field VEVs, in which case only SUSY is broken, with a scalar potential $V = \frac{1}{2}\xi^2$. When $m^2 < q\xi$, the minimum is at $\langle\phi_+\rangle = 0$ and $\langle\phi_-\rangle = v$, with $|v|^2 \equiv -(m^2 - q\xi)/q^2 > 0$, and the scalar potential constant term is $V = m^2/2q^2(q\xi - m^2)$. In this second case, the gauge symmetry is broken as well and one can study the new vacuum by expanding ϕ_- around its VEV. The Goldstone boson of the broken gauge symmetry is the imaginary part of ϕ_- and is eaten by the vector field which becomes massive.

The goldstino As we explained at the beginning of the current section, supersymmetry is spontaneously broken each time an auxiliary field gets a non-vanishing VEV. This was motivated by the form of the scalar potential (2.1.28), but it can also be seen by the following reasoning. If SUSY is broken in the vacuum, some fields should have expectation values the SUSY transformations of which do not vanish. From the field transformations eqs. (2.1.2) and (2.1.3) we see that the only Lorentz invariant possibilities are those for

non-vanishing auxiliary fields VEVs. For both the chiral or the vector multiplets they appear in the transformation laws of the fermion of the multiplet, namely

$$\delta_\epsilon \psi_\alpha = \sqrt{2}\epsilon_\alpha \langle \mathsf{F} \rangle + \cdots , \quad \delta_\epsilon \lambda_\alpha = i\epsilon_\alpha \langle \mathsf{D} \rangle + \cdots . \tag{2.1.35}$$

These non-linear transformations, or shifts of the fermions, are typical for Goldstone fields of a broken symmetry. As supersymmetry is characterized by fermionic charges, one should not be surprised by the fact that these fields are Goldstone fermions, called goldstini. The linear shift of the goldstino is proportional to the scale of SUSY breaking, parametrized by the VEV of the auxiliary field responsible for SUSY breaking, namely $\langle \bar{\mathsf{F}}^1 \rangle = c^1$ or $\langle \mathsf{D} \rangle = \xi$ in the examples of F-term and D-term SUSY breaking described above. In these cases, the goldstino is the sfermion or the gaugino but in more complicated models, the goldstino can be a combination of the diverse fermions of the theory.

2.2 Non-linear realizations of supersymmetry

Spontaneous breaking of supersymmetry implies the appearance of a massless Goldstone mode, the goldstino, which we mentioned just above. It also generates masses proportional to the SUSY breaking scale, hence removing the mass degeneracy between superpartners of some of the multiplets. We now explain how the low energy dynamics of the goldstino multiplet can often be studied by neglecting the other massive degrees of freedom. As for the case of bosonic symmetry breaking, these dynamics can be described through non-linear realizations, that we introduce hereafter.

2.2.1 Non-linear realizations and the Volkov-Akulov action

Non-linear realizations When a bosonic symmetry is spontaneously broken, the gauge group $\mathcal{G}$, leaving the whole action invariant, is reduced to a subgroup $\mathcal{H}$ leaving any minimum invariant. One can go from one minimum to the other by acting with elements of the coset $\mathcal{G}/\mathcal{H}$. Starting from a specific vacuum configuration, *i.e.* a field VEV ϕ_{0i}, any different vacuum configuration ϕ_j can thus be written through a group transformation as

$$\phi_j = R(\theta)_j^{\ i} \phi_{0i} = \exp\left(\sum_a i\theta^a (t_a)_j^{\ i} \right) \phi_{0i}, \tag{2.2.1}$$

where $R(\theta)$ is in a linear representation R of $\mathcal{G}$, and $\theta = \theta^a t_a$ is expressed in terms of the generators t_a of the complementary of the stabilizer $\mathcal{H}$. From the dynamics of the initial fields ϕ_j, one can extract the dynamics of the parameters θ^a, which are the Goldstone modes, by rewriting the action for ϕ_i in terms of $R(\theta)_j^{\ i}\phi_{0i}$ and expanding it in θ^a. This action is highly non-linear, as the transformations of the θ^a coordinates. This can be seen by rewriting $\delta_\xi \phi_i = \xi^b (T_b)_i^{\ j} \phi_j$, where T^b are generators of the whole group $\mathcal{G}$, as

$$\delta_\xi \phi_i = \delta_\xi \theta^a \frac{\partial R_i^{\ k}}{\partial \theta^a} \phi_{0k} = \xi^b (T_b)_i^{\ j} R(\theta)_j^{\ l} \phi_{0l}, \tag{2.2.2}$$

which can be generically rewritten as

$$\delta_\xi \theta^a = \xi^b f(\theta)_b^{\ a}. \tag{2.2.3}$$

The $f(\theta)$ function generically depends non-linearly on θ, hence the name of non-linear realization of the symmetry. The simplest example is θ-independent function, called shift,

which already appeared in the goldstino transformations shown in eq. (2.1.35). Schematically, we understand that the symmetry goes from a linear to a non-linear realization in the following way: while the orignal fields were freely traveling in Minkowski space, after symmetry breaking the new fields of interest are restricted to the curved manifold describing the different vacuum configurations.

Case of supersymmetry The above discussion was intended to give an effective theory description of the spontaneous breaking of bosonic symmetries, and generally applies to internal symmetries. When the superalgebra is broken, because we consider Lorentz-invariant vacua, the unbroken symmetries must lie in the Lorentz group, identified to the stabilizer $\mathcal{H}$. The "coset" generators of interest are thus those of space-time translations and supersymmetry transformations. Of course, the treatment is a bit more complicated than in the previous case, because the physical fields of the theory depend on the space-time coordinates x^μ, while the superfields also depend on the superspace $\theta, \bar{\theta}$ coordinates. Keeping in mind that we are interested in the dynamics of the fields parametrizing the coset space, we thus inspect the SUSY transformations of the aforementioned coordinates. They can be derived from the multiplication of supergroup elements

$$\begin{aligned} G(0,\epsilon,\bar{\epsilon})G(x^\mu,\theta,\bar{\theta}) &= \exp\left(i(\epsilon Q + \bar{\epsilon}\bar{Q})\right)\exp\left(i(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})\right) \\ &= G(x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \theta+\epsilon, \bar{\theta}+\bar{\epsilon}). \end{aligned} \tag{2.2.4}$$

The second line has been obtained using Hausdorff's formula and the supersymmetry algebra (2.1.1). It results in a finite expansion due to the fact that high order commutators vanish. The group multiplication rule (2.2.4) leads to the following ϵ-parametrized SUSY transformations

$$x^\mu \to x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \quad \theta \to \theta + \epsilon, \quad \bar{\theta} \to \bar{\theta} + \bar{\epsilon}. \tag{2.2.5}$$

We see that θ undergoes a shift under the action of supersymmetry, typical of goldstino fields, as already mentioned. Hence, following the strategy of Akulov and Volkov [14, 10], one can generalize the transformations of θ for a fermionic field $\kappa\lambda_\alpha(x)$ depending on the space-time coordinates and proportional to the κ scale. This gives

$$\lambda_\alpha(x) \to \lambda'_\alpha(x') = \lambda_\alpha(x) + \frac{1}{\kappa}\epsilon_\alpha, \quad x^\mu \to x'^\mu = x^\mu + i\kappa\left(\lambda\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\lambda}\right), \tag{2.2.6}$$

from which we deduce the local infinitesimal SUSY transformation

$$\delta_\epsilon\lambda_\alpha(x) = \lambda'_\alpha(x) - \lambda_\alpha(x) = \frac{1}{\kappa}\epsilon_\alpha - i\kappa(\lambda\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\lambda})\partial_\mu\lambda_\alpha. \tag{2.2.7}$$

Comparing eqs. (2.1.35) and (2.2.7) we see that κ^{-1} characterizes the SUSY breaking scale, related to the $\langle \mathsf{D} \rangle$ or $\langle \mathsf{F} \rangle$ auxiliary fields VEVs in the SUSY breaking models of section 2.1.2. The λ_α transformation was transposed from the θ one, which has been derived from the group multiplication (2.2.4) and the SUSY algebra. It is thus not a surprise that this transformation (non-linearly) realizes the SUSY algebra as well. We check it by computing the commutator of two SUSY transformations of λ_α. It reads

$$(\delta_\epsilon\delta_\xi - \delta_\xi\delta_\epsilon)\lambda_\alpha = -2i(\epsilon\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\epsilon})\partial_\mu\lambda_\alpha. \tag{2.2.8}$$

The commutator of two SUSY transformations is thus indeed a translation, with a parameter given by the right hand side of eq. (2.2.8).

The Volkov-Akulov Lagrangian In their original paper, Akulov and Volkov followed the same strategy as the one described above for non-linear realization of SUSY. They applied it to derive an invariant Lagrangian. Invariant Lagrangians on Riemann surfaces can be written with the help of volume forms, expressed as the product of invariant differential forms. From eq. (2.2.5), we see that the canonical 1-forms transform as

$$dx^\mu \to dx^\mu + id\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu d\bar{\theta}, \quad d\theta^\alpha \to d\theta, \quad d\bar{\theta}_{\dot{\alpha}} \to d\bar{\theta}_{\dot{\alpha}}, \tag{2.2.9}$$

such that the following 1-form

$$f^a = dx^a - id\theta\sigma^a\bar{\theta} + i\theta\sigma^a d\bar{\theta}, \tag{2.2.10}$$

is indeed invariant under non-linear transformations. As before, one can replace the θ coordinate with the fermionic field $\kappa\lambda_\alpha$ and use $d\theta = \kappa dx^\mu \partial\lambda/\partial x^\mu$ to obtain the associated 1-form

$$\omega^a \equiv \boldsymbol{A}^a_\mu dx^\mu, \qquad \boldsymbol{A}^a_\mu \equiv \delta^a_\mu + i\kappa^2\lambda\sigma^a\partial_\mu\bar{\lambda} - i\kappa^2\partial_\mu\lambda\sigma^a\bar{\lambda}. \tag{2.2.11}$$

The $\boldsymbol{A}$ matrix is the vielbein associated with the non-linear realization of supersymmetry. By construction, ω^a is invariant under the non-linear SUSY transformation (2.2.7). It can thus be used to form an invariant volume form, the integral of which is a natural candidate for an invariant action

$$\begin{aligned}\mathcal{S} &= -\frac{1}{2\kappa^2}\int \omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3 = -\frac{1}{2\kappa^2}\cdot\frac{1}{4!}\int \epsilon_{abcd}\,\omega^a\wedge\omega^b\wedge\omega^c\wedge\omega^d \\ &= -\frac{1}{2\kappa^2}\cdot\frac{1}{4!}\int \epsilon_{\mu\nu\rho\sigma}\,\det\boldsymbol{A}\,dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma = -\frac{1}{2\kappa^2}\int d^4x\,\det\boldsymbol{A}.\end{aligned} \tag{2.2.12}$$

To obtain the second line we used the definition (2.2.11) of the vielbein form and the relation

$$\epsilon_{abcd}\boldsymbol{A}^a_\mu\boldsymbol{A}^b_\nu\boldsymbol{A}^c_\rho\boldsymbol{A}^d_\sigma = \epsilon_{\mu\nu\rho\sigma}\det\boldsymbol{A}. \tag{2.2.13}$$

This action is the space-time integral of the famous goldstino Volkov-Akulov Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{VA}} &= -\frac{1}{2\kappa^2}\det\boldsymbol{A} \\ &= -\frac{1}{2\kappa^2}\left(1 + \kappa^2 i\lambda\partial\!\!\!/\bar{\lambda} + \kappa^2 i\bar{\lambda}\bar{\partial}\!\!\!/\lambda - 2\kappa^4\left((\lambda\partial\!\!\!/\bar{\lambda})(\bar{\lambda}\bar{\partial}\!\!\!/\lambda) + \lambda\sigma^\mu\partial_\nu\bar{\lambda}\partial_\mu\lambda\sigma^\nu\bar{\lambda}\right)\right) + O(\lambda^6) \\ &= -\frac{1}{2\kappa^2} - \frac{i}{2}\lambda\partial\!\!\!/\bar{\lambda} - \frac{i}{2}\bar{\lambda}\bar{\partial}\!\!\!/\lambda - \frac{\kappa^2}{2}\Big[-(\lambda\partial\!\!\!/\bar{\lambda})(\bar{\lambda}\bar{\partial}\!\!\!/\lambda) - 2(\lambda\partial\!\!\!/\bar{\lambda})^2 - 2(\bar{\lambda}\bar{\partial}\!\!\!/\lambda)^2 \\ &\quad + \partial_\mu(\lambda^2)\partial^\mu(\bar{\lambda}^2) + \bar{\lambda}^2\lambda\Box\lambda + \lambda^2\bar{\lambda}\Box\bar{\lambda}\Big] + O(\lambda^6) + \text{total derivatives}\,.\end{aligned} \tag{2.2.14}$$

Here we made use of the slash notation conventions shown in eq. (A.1.9). We see that the overall constant in eqs. (2.2.12) and (2.2.14) is necessary to have canonically normalized goldstino kinetic terms. The first term of the Lagrangian is the positive constant scalar potential $V = \frac{1}{2\kappa^2}$ typical of SUSY breaking as explained in section 2.1.2 and observed in the D-term and F-term breaking models with the identification $1/\kappa \to \langle \mathsf{D}\rangle, \langle F\rangle/\sqrt{2}$.

Standard realizations Finally we comment on the fact that the so-called standard non-linear realizations of supersymmetry can be implemented on other fields s (either bosonic of fermionic) using the transformation law

$$\delta_\epsilon s = -i\kappa(\lambda\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\lambda})\partial_\mu s, \tag{2.2.15}$$

similar to the goldstino transformation (2.2.7), except for the absence of shift. Space-time derivatives must then be covariantized using the inverse vielbein $\boldsymbol{A}^{-1}$. It is indeed easy to show that $(\boldsymbol{A}^{-1})^{\mu}_{a}\partial_{\mu}s$ transforms according to the standard realization law (2.2.15).

2.2.2 Non-linear realizations in superspace and constrained superfields

Superspace description of non-linear realizations In the previous section, the goldstino dynamics, described by non-linear realizations of supersymmetry, were stated in component form. To apply the techniques described in section 2.1 to construct invariant Lagrangians, it is useful to lift this description to superspace. The goldstino superfield can be constructed from its lower component as any other superfield

$$\Lambda_{\alpha} \equiv \exp(\theta Q + \bar{\theta}\bar{Q})\lambda_{\alpha} = \lambda_{\alpha} + \frac{1}{\kappa}\theta_{\alpha} + \cdots . \tag{2.2.16}$$

Similarly, any field s in a non-linear realization can be lifted to a superfield

$$S(x,\theta,\bar{\theta}) \equiv \exp(\theta Q + \bar{\theta}\bar{Q})s(x) = s(x) - i\kappa(\lambda\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\lambda})\partial_{\mu}s(x) + \cdots , \tag{2.2.17}$$

where we used the transformation law (2.2.15). In fact, composite fields in standard realizations can be constructed out of any arbitrary superfield $S(x,\theta,\bar{\theta})$ by identifying the θ_{α} coordinate with the goldstino $\kappa\lambda_{\alpha}$. The obtained composite field $\mathscr{S}(x)$ can then be lifted back in superspace following the usual way. Hence, a superfield with lowest component transforming in a standard realization is written as

$$\mathbb{S} = \exp(\theta Q + \bar{\theta}\bar{Q})\mathscr{S}(x) = \exp(\theta Q + \bar{\theta}\bar{Q})\left[\exp(\epsilon Q + \bar{\epsilon}\bar{Q})s(x)\right]_{\epsilon=-\kappa\lambda} . \tag{2.2.18}$$

Equipped with the above tools, one can construct Lagrangians invariant under non-linear realizations of SUSY through the use of superfields. Namely, following the methods explained around eq. (2.1.18), one has to construct a scalar quantity corresponding either to a real or chiral superfield, and can then extract the Lagrangian from its top component. The first interesting Lagrangian is the one for the goldstino superfield Λ_{α}. We first construct the chiral superfield Λ^2. As its top component is a space-time derivative, it is not a good Lagrangian candidate. The only natural possibility is then

$$\mathcal{L}_{\text{VA}} = -\frac{\kappa^2}{2}\int d^4\theta\Lambda^2\bar{\Lambda}^2, \tag{2.2.19}$$

which indeed gives back the Volkov-Akulov Lagrangian (2.2.14) once expressed in components.

Starting from a linear SUSY Lagrangian $\mathcal{L}_0$ in component form, we construct a superspace expression of a non-linear invariant Lagrangian containing matter to goldstino interactions, in the following way. We first dress the fields of $\mathcal{L}_0$ with the goldstino field, through (2.2.18), to obtain a superspace Lagrangian $\mathbb{L}_0$, and then write the final Lagrangian as

$$\mathcal{L}' = \kappa^4\int d^4\theta\Lambda^2\bar{\Lambda}^2\mathbb{L}_0 = \det\boldsymbol{A}\,\mathscr{L}_0. \tag{2.2.20}$$

We can see from (2.2.16) that $\kappa^4\Lambda^2\bar{\Lambda}^2 = \theta^2\bar{\theta}^2 + \cdots$, so that the first term in the component expansion of $\mathcal{L}'$ is indeed the initial Lagrangian $\mathcal{L}_0$, while other terms in the expansion contain the goldstino interactions. The low-energy effective Lagrangian can be obtained by expanding the total Lagrangian of the theory, which is the sum of the two Lagrangians

shown in eqs. (2.2.19) and (2.2.20). It reads

$$\mathcal{L} = \mathcal{L}_{\rm VA} + \mathcal{L}' = -\frac{1}{2\kappa^2} - i\lambda\partial\!\!\!/\bar{\lambda} + \frac{i}{\kappa^2}\lambda\sigma^\mu\bar{\lambda}\partial_\mu\mathcal{L}_0 + \frac{i}{\kappa^2}(\lambda\sigma^\mu\partial^\nu\bar{\lambda} - \partial^\nu\lambda\sigma^\mu\bar{\lambda})T_{\mu\nu} + \cdots, \quad (2.2.21)$$

and shows the universality of the goldstino coupling to the energy-momentum tensor $T_{\mu\nu}$ of the initial theory.

Constrained superfields We have seen that non-linear realizations of SUSY describe a particle, the goldstino, decoupling from other fields which became massive after SUSY breaking. When SUSY is preserved, particles are arranged in complete multiplets described by superfields. After SUSY breaking, a convenient way to eliminate the massive particles of the theory makes use of constrained superfields. Applying constraints eliminate certain components of the superfields, which are then expressed with respect to the remaining ones. This can indeed be explained in the case for the goldstino superfield constructed above, which satisfies the constraints

$$D_\alpha\Lambda_\beta = \frac{1}{\kappa}\epsilon_{\beta\alpha} + i\kappa(\sigma^\mu\bar{\Lambda})_\alpha\partial_\mu\Lambda_\beta, \quad \bar{D}_{\dot{\alpha}}\Lambda_\beta = -i\kappa(\Lambda\sigma^\mu)_{\dot{\alpha}}\partial_\mu\Lambda_\beta. \quad (2.2.22)$$

Starting from a fermionic field ψ_α and creating a superfield $\Psi_\alpha \equiv \exp(\theta Q + \bar{\theta}\bar{Q})\psi_\alpha$ by use of the linear SUSY transformations of (2.1.2), one would have obtained a superfield containing the fields of an entire chiral multiplet. The constraints (2.2.22) ensure that only the goldstino remains in the final superfield, which is thus not an entire multiplet anymore.

One may wonder if constraints different from (2.2.22) would allow to eliminate components and create similar goldstino superfields. In fact, there exists indeed a whole collection of constraints generating goldstino superfields, and one can go from one to the other through simple manipulations [15, 16, 17, 18]. The simplest constraint is the nilpotent constraint [16] on a chiral superfield Φ, which reads

$$\Phi^2 = 0. \quad (2.2.23)$$

Expressing the chiral superfield $\Phi(y)$ through its component expansion (2.1.10) gives the set of constraints

$$\phi^2 = 0, \qquad 2\theta\theta\phi F = 2(\theta\psi)^2, \qquad 2\sqrt{2}\phi\theta\psi = 0, \quad (2.2.24)$$

the second equality of which is solved, using (A.1.5), by

$$\phi = \frac{\psi^2}{2F}, \quad (2.2.25)$$

which automatically satisfies the other constraints due to the fermionic nature of ψ_α. The generic Lagrangian (2.1.20) for a single chiral superfield Φ subject to the nilpotent constraint only allows for a linear superpotential and reads

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\,\Phi\bar{\Phi} + \left(\frac{1}{\sqrt{2}\kappa}\int d^2\theta\,\Phi + c.c.\right) \\ &= -i\psi\partial\!\!\!/\bar{\psi} + F\bar{F} + \frac{\bar{\psi}^2}{2\bar{F}}\partial^2\left(\frac{\psi^2}{2F}\right) + \left(\frac{F}{\sqrt{2}\kappa} + c.c.\right). \end{aligned} \quad (2.2.26)$$

The F and $\bar{F}$ auxiliary fields' equations of motion read

$$F = -\frac{1}{\sqrt{2}\kappa} + f(\psi, \bar{\psi}), \qquad \bar{F} = -\frac{1}{\sqrt{2}\kappa} + \bar{f}(\psi, \bar{\psi}), \tag{2.2.27}$$

where the $f(\psi, \bar{\psi})$ term contains goldstino interactions. They lead to the Lagrangian

$$\mathcal{L}_{KS} = -\frac{1}{2\kappa^2} - i\psi\partial\!\!\!/\bar{\psi} + \frac{\kappa^2}{2}\bar{\psi}^2\partial^2\psi^2 - \frac{\kappa^4}{4}\psi^2\bar{\psi}^2\partial^2\psi\partial^2\bar{\psi}^2, \tag{2.2.28}$$

which can be related to the Volkov-Akulov Lagrangian $\mathcal{L}_{VA}$. Indeed, using a field redefinition [19] of the form

$$\psi_\alpha = \lambda_\alpha + i\frac{\kappa^2}{2}(\sigma^\mu\bar{\lambda})_\alpha\partial_\mu\lambda^2\left(1 + i\kappa^2\bar{\lambda}\bar{\partial}\!\!\!/\lambda\right) + \cdots, \tag{2.2.29}$$

one can show the equivalence

$$\mathcal{L}_{KS}\left(\psi_\alpha(\lambda_\alpha)\right) = \mathcal{L}_{VA}. \tag{2.2.30}$$

This expresses the fact that the simple Lagrangian for the nilpotent field Φ is on-shell equivalent to the one for the Volkov-Akulov goldstino.

The nilpotent constraint (2.2.23) on the chiral multiplet Φ only eliminates the scalar field. To eliminate the auxiliary field, instead of using its equations of motion, one can add a second constraint [15] of the form

$$\Phi\bar{D}^2\bar{\Phi} = \frac{2\sqrt{2}}{\kappa}\Phi. \tag{2.2.31}$$

Note that with this additional constraint, the Lagrangian (2.2.26) then reads

$$\begin{aligned}\mathcal{L} = \int d^4\theta\,\Phi\bar{\Phi} + \left(\frac{1}{\sqrt{2}\kappa}\int d^2\theta\,\Phi + c.c.\right) &= \int d^4\theta\,\Phi\bar{\Phi} + \left(\frac{1}{4}\int d^2\theta\,\Phi\bar{D}^2\bar{\Phi} + c.c.\right) \\ &= -\int d^4\theta\,\Phi\bar{\Phi}.\end{aligned} \tag{2.2.32}$$

As nicely shown in [16] the two constraints eqs. (2.2.23) and (2.2.31) can be derived by adding a term

$$\mathcal{L}_\gamma = \frac{1}{2}\gamma(x)\Phi^2 \tag{2.2.33}$$

to the Lagrangian (2.2.26) for an unconstrained chiral superfield Φ. The equations of motion for the Lagrange multiplier $\gamma(x)$ and for Φ are then

$$\Phi^2 = 0, \qquad -\frac{1}{4}\bar{D}^2\bar{\Phi}^2 = -\frac{\partial W}{\partial\Phi}(\Phi) = -\gamma\Phi - \frac{1}{\sqrt{2}\kappa}. \tag{2.2.34}$$

The second equality is just a superfield rewriting of the auxiliary field e.o.m. of (2.1.30). Multiplying it by Φ gives back the constraint (2.2.31).

As mentioned in [18], a superfield $\tilde{\Phi}$ can also be constructed from the goldstino superfield Λ_α through

$$\tilde{\Phi} = -\frac{\kappa^3}{4\sqrt{2}}\bar{D}^2\left(\Lambda^2\bar{\Lambda}^2\right) = \frac{\kappa}{\sqrt{2}}\lambda^2 + \cdots, \tag{2.2.35}$$

and satisfies both the two constraints of eqs. (2.2.23) and (2.2.31), together with the nice property

$$\tilde{\Phi}\bar{\tilde{\Phi}} = \frac{\kappa^2}{2}\Lambda^2\bar{\Lambda}^2. \tag{2.2.36}$$

The superspace Volkov-Akulov Lagrangian (2.2.19) then reads

$$\mathcal{L}_{\rm VA} = -\frac{\kappa^2}{2}\int d^4\theta \Lambda^2\bar{\Lambda}^2 = -\int d^4\theta \tilde{\Phi}\bar{\tilde{\Phi}}, \tag{2.2.37}$$

and gives back the Lagrangian (2.2.32) obtained for the nilpotent Φ after applying the second constraint. Replacing the F auxiliary field by its solution (2.2.27) in the lowest component (2.2.25) of the nilpotent superfield Φ, one indeed obtains the lowest component of $\tilde{\Phi}$ given in (2.2.35).

We conclude this section by mentioning that following [20], we can eliminate the lowest component of any superfield S, *i.e.* we can express it as a combination of the other components and of the goldstino field, through the constraint

$$\bar{\Phi}\Phi S = 0, \tag{2.2.38}$$

where Φ is the nilpotent superfield. For instance, we eliminate the scalar, fermion or auxiliary field of a given chiral superfield X by taking respectively $S = X$, $S_\alpha = D_\alpha X$ or $S = D^2X$. Indeed, in each case, the lowest component of S is the field of interest. When applied to the goldstino auxiliary superfield itself $S = \bar{D}^2\bar{\Phi}$, the constraint (2.2.38) can be derived from the second constraint (2.2.31) after left multiplication by $\bar{\Phi}$.

2.3 Partial $\mathcal{N} = 2$ global supersymmetry breaking and induced FI terms

In this section, we tackle the problem of partial supersymmetry breaking in the $\mathcal{N} = 2$ extended supersymmetry theory of a vector multiplet. As we explain below, this multiplet can be written in terms of $\mathcal{N} = 1$ multiplets, and will thus repeatedly use the elements introduced in the two previous sections.

Partial breaking of $\mathcal{N} = 2$ global supersymmetry to $\mathcal{N} = 1$ requires a deformation of supersymmetry transformations, implemented by the introduction of deformation constants [21, 22, 23]. They modify the transformations of fermions but leave intact the supersymmetry algebra of infinitesimal transformations. Some of these constants can be absorbed by shifting the auxiliary fields and do not correspond to genuine deformations. One therefore expects that a general deformation contains the same number of parameters as the number of real auxiliary fields in the SUSY multiplet. The general deformation of $\mathcal{N} = 2$ vector multiplets thus forms a triplet under its $SU(2)_R$ symmetry and consists technically in constant imaginary part for the auxiliary fields triplet. The latter is formed by the complex F and real D auxiliary components of the $\mathcal{N} = 1$ chiral and vector multiplet composing the $\mathcal{N} = 2$ double chiral vector $\mathcal{W} = (X, W)$. The deformation associated with F is known to give rise to a magnetic Fayet-Iliopoulos term proportional to the special coordinate $f_X \equiv \partial_X f$, where $f(X)$ is the holomorphic $\mathcal{N} = 2$ prepotential [21]. We extend this result to the D-auxiliary field, whose deformation modifies the Bianchi identity of W and we show that this modification is dual to the ordinary FI parameter under electromagnetic duality, as in the case of the F auxiliary field.

Partial SUSY breaking implies a special relation among the deformation parameters guaranteeing the existence of a linear combination of the two supersymmetries under which all fermions of the multiplet transform linearly (without constants). Studying the

general two-derivative effective action of a deformed $\mathcal{N}=2$ double chiral multiplet we show that it exhibits a partial $\mathcal{N}=2\to\mathcal{N}=1$ breaking at the minimum of the scalar potential for generic values of the parameter space. Special values may leave $\mathcal{N}=2$ unbroken or a runaway potential but one can never realize complete breaking of both supersymmetries, unless trivially in a free theory. This result was expected since one could obtain it by using a $SU(2)_R$ rotation from the cases studied in the literature [21, 24]. This analysis is however useful for unveiling the main properties of the D-deformation γ, that are relevant in the context of Dirac-Born-Infeld (DBI) actions, that we study in section 2.4.

2.3.1 $\mathcal{N}=2$ supersymmetry, general deformations and SUSY breaking

We first review the properties of an $\mathcal{N}=2$ vector multiplet and consider its most general deformation, which can be parameterized by three real constants. We then expose the criteria for SUSY breaking in presence of these deformations. These notions will be useful to tackle partial SUSY breaking, studied in the next subsections.

$\mathcal{N}=2$ vector multiplet: structure, transformations and symmetries Off-shell $\mathcal{N}=2$ SUSY can be described in superspace in some specific cases, by introducing a second pair of coordinates $\tilde\theta,\bar{\tilde\theta}$ related to the second supersymmetry. One can create an $\mathcal{N}=2$ vector multiplet starting with the following superfield

$$\mathcal{W}(y,\theta,\tilde\theta)=X(y,\theta)+\sqrt{2}i\tilde\theta W(y,\theta)-\tilde\theta\tilde\theta G(y,\theta),\qquad y^\mu=x^\mu+i\theta\sigma^\mu\bar\theta+i\tilde\theta\sigma^\mu\bar{\tilde\theta}\,, \tag{2.3.1}$$

which is an extension of the chiral superfield Φ of (2.1.10) and is chiral with respect to both supersymmetries

$$\bar D\mathcal{W}=\bar{\tilde D}\mathcal{W}=0\,, \tag{2.3.2}$$

where the $\tilde D_\alpha,\bar{\tilde D}_\alpha$ supercovariant derivatives are defined as in (2.1.7) with $\theta,\bar\theta$ replaced by $\tilde\theta,\bar{\tilde\theta}$. The transformations under the second supersymmetry, denoted $\tilde\delta$, are similar to those of eq. (2.1.2), namely

$$\begin{aligned}\tilde\delta X&=\sqrt{2}i\tilde\epsilon W\,,\\ \tilde\delta W_\alpha&=\sqrt{2}(\sigma^\mu\bar{\tilde\epsilon})_\alpha\partial_\mu X+\sqrt{2}iG\tilde\epsilon_\alpha,\\ \tilde\delta G&=-\sqrt{2}\partial_\mu W\sigma^\mu\bar{\tilde\epsilon}\,.\end{aligned} \tag{2.3.3}$$

The superfield (2.3.1) is reducible and describes the degrees of freedom of a $\mathcal{N}=2$ vector and tensor multiplet. To reduce them to those of a vector, one requires W to be the field-strength superfield of a $\mathcal{N}=1$ vector multiplet, satisfying $DW=\bar D\bar W$ according to (2.1.17). Furthermore, one can verify explicitly that $\frac14\bar D^2\bar X$ transforms in the same way as G in (2.3.3). Therefore we can set

$$G=\frac14\bar D^2\bar X, \tag{2.3.4}$$

without violating the $\mathcal{N}=2$ supersymmetry.

Since $\mathcal{W}$ is chiral with respect to both supersymmetries, we can apply the same method as in the $\mathcal{N}=1$ case of eq. (2.1.23), and construct a Lagrangian from its top component

$$\mathcal{L}_{\mathcal{N}=2\,\mathcal{W}^2+c.c}=\frac14\int d^2\theta d^2\tilde\theta\mathcal{W}^2+c.c.=\frac14\int d^2\theta\left(W^2-\frac12X\bar D^2\bar X\right)+c.c.\,. \tag{2.3.5}$$

We used expression (2.3.4) for G when expanding $\mathcal{W}$. On the other hand, the $\mathcal{N}=2$ Maxwell theory, in terms of $\mathcal{N}=1$ language, is described by a chiral multiplet X and a field strength multiplet W with action given by

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=2\,\text{Maxwell}} &= \int d^2\theta d^2\bar{\theta}\bar{X}X + \frac{1}{4}\int d^2\theta W^2 + \frac{1}{4}\int d^2\theta \bar{W}^2 \\ &= \frac{1}{4}\int d^2\theta \left(W^2 - \frac{1}{2}X\bar{D}^2\bar{X}\right) + c.c. \end{aligned} \tag{2.3.6}$$

We see that the above two actions are equivalent, implying that the extra constraint imposed on $\mathcal{W}$ is correct.

The $\mathcal{N}=2$ vector multiplet can thus be described with an $\mathcal{N}=2$ superfield of the form

$$\mathcal{W}(y,\theta,\tilde{\theta}) = X(y,\theta) + \sqrt{2}i\tilde{\theta}W(y,\theta) - \frac{1}{4}\tilde{\theta}\tilde{\theta}\bar{D}^2\bar{X}(y,\theta)\,, \tag{2.3.7}$$

where X, W are $\mathcal{N}=1$ chiral and vector multiplets, respectively. Their component forms are similar to those already given in eqs. (2.1.10) and (2.1.16) and read

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha \mathsf{D} - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha\,, \tag{2.3.8}$$

$$X = x + \sqrt{2}\theta\chi - \theta\theta\mathsf{F}\,, \tag{2.3.9}$$

$$\frac{1}{4}\bar{D}^2\bar{X} = \bar{\mathsf{F}} - \sqrt{2}i\theta\sigma^\mu\partial_\mu\bar{\chi} - \theta\theta\eta^{\mu\nu}\partial_\mu\partial_\nu\bar{x}\,. \tag{2.3.10}$$

Alternatively, the $\mathcal{N}=2$ vector multiplet (2.3.7) can be obtained from (2.3.1) by imposing the following irreducibility conditions:

$$\mathcal{D}_i\mathcal{D}_j\mathcal{W} = \epsilon_i{}^k\epsilon_j{}^l\bar{\mathcal{D}}_k\bar{\mathcal{D}}_l\bar{\mathcal{W}}\,, \qquad i,j,k,l = 1,2\,, \tag{2.3.11}$$

where $\mathcal{D}_1 = D$, $\mathcal{D}_2 = \tilde{D}$ correspond to the supercovariant derivatives of the first and second supersymmetry and the antisymmetric symbol is defined through $\epsilon_1{}^1 = \epsilon_2{}^2 = 0$, $\epsilon_1{}^2 = -\epsilon_2{}^1 = 1$.

We are especially interested in the auxiliary field part of the SUSY transformations of fermions, that can be read off eqs. (2.1.2), (2.1.3) and (2.3.3), with G replaced as in (2.3.4). The fermions transform under the $\mathcal{N}=2$ supersymmetries as

$$\begin{aligned} \delta\chi_\alpha &= -\sqrt{2}\mathsf{F}\epsilon_\alpha + \cdots\,, & \tilde{\delta}\chi_\alpha &= i\mathsf{D}\tilde{\epsilon}_\alpha + \cdots\,, \\ \delta\lambda_\alpha &= i\mathsf{D}\epsilon_\alpha + \cdots\,, & \tilde{\delta}\lambda_\alpha &= -\sqrt{2}\bar{\mathsf{F}}\tilde{\epsilon}_\alpha + \cdots\,, \end{aligned} \tag{2.3.12}$$

so that the full SUSY transformation of the fermions can be written as

$$\delta_{SUSY}\begin{pmatrix}\chi_\alpha \\ \lambda_\alpha\end{pmatrix} = \begin{pmatrix}-\sqrt{2}\mathsf{F} & i\mathsf{D} \\ i\mathsf{D} & -\sqrt{2}\bar{\mathsf{F}}\end{pmatrix}\begin{pmatrix}\epsilon_\alpha \\ \tilde{\epsilon}_\alpha\end{pmatrix} + \cdots\,. \tag{2.3.13}$$

Of course, the $\mathcal{N}=2$ vector multiplet $\mathcal{W}$ has $SU(2)_R$ invariance. A nice way to express this symmetry in superspace is by defining the following $SU(2)_R$ doublets

$$\vartheta^1 = \theta\,, \qquad \vartheta^2 = \tilde{\theta}\,, \qquad \eta_1 = \chi\,, \qquad \eta_2 = \lambda\,, \tag{2.3.14}$$

and expanding the vector superfield in components as

$$\begin{aligned}\mathcal{W}(y,\theta,\tilde{\theta}) &= x + \sqrt{2}(\theta\chi + \tilde{\theta}\lambda) - \theta\theta\mathsf{F} - \tilde{\theta}\tilde{\theta}\bar{\mathsf{F}} + i\sqrt{2}\theta\tilde{\theta}\mathsf{D} + \cdots \\ &= x + \sqrt{2}\vartheta^i\eta_i - \vartheta^i\vartheta^j Y_{ij} + \cdots\,, \end{aligned} \tag{2.3.15}$$

where we defined the auxiliary field matrix

$$Y_{ij} = Y_{ji} = \Big(\boldsymbol{Y}\cdot\boldsymbol{\sigma}\sigma_2\Big)_{ij}\,, \quad \boldsymbol{Y} = \Big(\operatorname{Im}\mathsf{F}, \operatorname{Re}\mathsf{F}, \frac{\mathsf{D}}{\sqrt{2}}\Big),$$
$$Y_{11} = \mathsf{F}\,, \quad Y_{22} = \bar{\mathsf{F}}\,, \quad Y_{12} = -\frac{i}{\sqrt{2}}\mathsf{D}, \tag{2.3.16}$$

with the help of the standard Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. We also construct the following triplet of fermionic coordinates transforming in the adjoint representation of $SU(2)_R$

$$\boldsymbol{\Theta} = \begin{pmatrix}\theta & \tilde{\theta}\end{pmatrix}\boldsymbol{\sigma}\sigma_2\begin{pmatrix}\theta\\ \tilde{\theta}\end{pmatrix} = \Big(i(\theta\theta-\tilde{\theta}\tilde{\theta}), (\theta\theta+\tilde{\theta}\tilde{\theta}), -2i\theta\tilde{\theta}\Big)\,. \tag{2.3.17}$$

in order to express the last term of (2.3.15) as an $SU(2)_R$ scalar

$$\boldsymbol{\Theta}\cdot\boldsymbol{Y} = \theta\theta\mathsf{F} + \tilde{\theta}\tilde{\theta}\bar{\mathsf{F}} - \sqrt{2}i\theta\tilde{\theta}\mathsf{D} = \vartheta^i\vartheta^j Y_{ij}\,. \tag{2.3.18}$$

Note that the $SU(2)_R$ symmetry can also be seen from the $SU(2)_R$ invariant reality conditions

$$Y^*_{ij} = \epsilon_i{}^k\epsilon_j{}^l Y_{kl}\,. \tag{2.3.19}$$

General deformation In order to study deformations of SUSY, we modify $\boldsymbol{Y}$ by adding a constant deformation $\boldsymbol{Y}_{\text{def}}$. The real part of $\boldsymbol{Y}_{\text{def}}$ can be absorbed by a trivial shift of the auxiliary fields in $\boldsymbol{Y}$. Hence we only need to focus on a pure imaginary $\boldsymbol{Y}_{\text{def}}$ [24]. Using the $SU(2)_R$ symmetry, we can rotate the vector $\boldsymbol{Y}_{\text{def}}$ to any specific direction. As we will see, this just indicates that the model always has $\mathcal{N}=1$ residual supersymmetry after deformation. However, the direction of the residual supersymmetry, which is important for the purpose of total supersymmetry breaking, depends on the deformation parameters. We therefore do not rotate the deformation vector $\boldsymbol{Y}_{\text{def}}$ and consider the following natural parametrization

$$\boldsymbol{Y}_{\text{def}} = \Big(\frac{i}{4\kappa}\cos\phi, \frac{i}{4\kappa}\sin\phi, \frac{i\gamma}{\sqrt{2}}\Big)\,, \qquad \gamma,\phi,\kappa\in\mathbb{R}\,. \tag{2.3.20}$$

It contains three deformation parameters, one for each auxiliary field, and as the real part of the deformation vector has no physical effects, we can equivalently choose

$$\boldsymbol{Y}_{\text{def}} = \Big(\frac{ie^{i\phi}}{4\kappa}, \frac{e^{i\phi}}{4\kappa}, \frac{i\gamma}{\sqrt{2}}\Big)\,. \tag{2.3.21}$$

In the remainder, we will use the second form (2.3.21) of the general deformation. The deformation $\boldsymbol{Y}_{\text{def}}$ induces a deformation $\mathcal{W}_{\text{def}}$ of the superfield $\mathcal{W}$, which reads

$$\mathcal{W}_{\text{def}} = -\boldsymbol{\Theta}\cdot\boldsymbol{Y}_{\text{def}} = -\frac{1}{2\kappa}e^{i\phi}\tilde{\theta}\tilde{\theta} - \sqrt{2}\gamma\theta\tilde{\theta}\,, \tag{2.3.22}$$

and modifies the irreducibility condition (2.3.11) to

$$\mathcal{D}_i\mathcal{D}_j\mathcal{W} - \epsilon_i{}^k\epsilon_j{}^l\bar{\mathcal{D}}_k\bar{\mathcal{D}}_l\bar{\mathcal{W}} = i\gamma_{ij}\,, \qquad \gamma_{ij} = 8\Big(\operatorname{Im}(\boldsymbol{Y})\cdot\boldsymbol{\sigma}\sigma_2\Big)_{ij}\in\mathbb{R}\,. \tag{2.3.23}$$

In particular, this modifies the standard Bianchi identity of $\mathcal{N}=1$ vector multiplets given in (2.1.17), and leads to the following equation

$$DW = \bar{D}\bar{W} - 4i\gamma. \tag{2.3.24}$$

Modifications similar to the one of (2.3.23) were obtained in [25] through eletromagnetic duality transformations, which we discuss in the next section. The Bianchi identity modification (2.3.24) appeared first in [26].

The deformation $\mathcal{W}_{\text{def}}$ in (2.3.22) of the $\mathcal{N}=2$ vector multiplet leads to a deformation of the $\mathcal{N}=1$ field strength auxiliary field, the components of which are expressed in (2.3.8). The D auxiliary field is thus deformed to

$$\mathsf{D} = d + i\gamma, \qquad d, \gamma \in \mathbb{R}\,. \tag{2.3.25}$$

Hence, D is now a complex field which includes γ, the deformation constant, together with d, the auxiliary field that should be eliminated.

Deformed SUSY transformations and SUSY breaking In order to discuss supersymmetry transformations and supersymmetry breaking, one should take into account both the deformations and the dynamical parts sourced by the auxiliary fields. It is convenient to redefine the following quantities

$$\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_{\text{def}} + \boldsymbol{\Upsilon}_{\text{dynamic}} = \left(\operatorname{Im}\mathsf{F} + \frac{i}{4\kappa}e^{i\phi}, \operatorname{Re}\mathsf{F} + \frac{1}{4\kappa}e^{i\phi}, \frac{d+i\gamma}{\sqrt{2}}\right), \tag{2.3.26}$$

$$\mathcal{W}_{\text{auxiliary}} \equiv -\boldsymbol{\Theta}\cdot\boldsymbol{\Upsilon} \equiv \mathcal{W}_{\text{def}} + \mathcal{W}_{\text{dynamic}}\,, \tag{2.3.27}$$

where $\boldsymbol{\Upsilon}_{\text{dynamic}}$ refers to the auxiliary fields vacuum expectation values in (2.3.16).

The deformed transformations of the second supersymmetry are given by

$$\tilde{\delta}X = \sqrt{2}i\tilde{\epsilon}^{\alpha}\Big(W_{\alpha} + i\gamma\theta_{\alpha}\Big), \tag{2.3.28}$$

$$\tilde{\delta}W_{\alpha} = \sqrt{2}i\Big(\frac{1}{2\kappa}e^{i\phi}\tilde{\epsilon}_{\alpha} + \frac{1}{4}\tilde{\epsilon}_{\alpha}\bar{D}^2\bar{X} - i(\sigma^{\mu}\bar{\tilde{\epsilon}})_{\alpha}\partial_{\mu}X\Big)\,. \tag{2.3.29}$$

One can check that the $\mathcal{N}=2$ SUSY algebra is not affected by these constant deformations. In the presence of deformations, the fermion transformation rules (2.3.13) get modified as

$$\delta_{SUSY}\begin{pmatrix}\chi_{\alpha}\\ \lambda_{\alpha}\end{pmatrix} = \begin{pmatrix}-\sqrt{2}\mathsf{F} & i(d+i\gamma)\\ i(d+i\gamma) & -\sqrt{2}\left(\bar{\mathsf{F}} + \frac{1}{2\kappa}e^{i\phi}\right)\end{pmatrix}\begin{pmatrix}\epsilon_{\alpha}\\ \tilde{\epsilon}_{\alpha}\end{pmatrix} = -\sqrt{2}\begin{pmatrix}\Upsilon_2 + i\Upsilon_1 & -i\Upsilon_3\\ -i\Upsilon_3 & \Upsilon_2 - i\Upsilon_1\end{pmatrix}\begin{pmatrix}\epsilon_{\alpha}\\ \tilde{\epsilon}_{\alpha}\end{pmatrix}, \tag{2.3.30}$$

with $\boldsymbol{\Upsilon} = (\Upsilon_1, \Upsilon_2, \Upsilon_3)$ given in (2.3.26). One can also introduce the following parametrization of $\boldsymbol{\Upsilon}$ [24]

$$\boldsymbol{\Upsilon} \equiv \left(\frac{i}{2}\left(A^2 - B^2\right), -\frac{1}{2}\left(A^2 + B^2\right), -i\Gamma\right), \tag{2.3.31}$$

so that

$$\mathcal{W}_{\text{auxiliary}} = -\boldsymbol{\Theta}\cdot\boldsymbol{\Upsilon} = A^2\theta\theta + B^2\tilde{\theta}\tilde{\theta} + 2\Gamma\theta\tilde{\theta}\,. \tag{2.3.32}$$

If $\Gamma = \pm AB$, or equivalently $\boldsymbol{\Upsilon}\cdot\boldsymbol{\Upsilon} = 0$, $\mathcal{W}_{\text{auxiliary}}$ can be diagonalized and becomes a complete square

$$\mathcal{W}_{\text{auxiliary}} = (A\theta \pm B\tilde{\theta})^2\,. \tag{2.3.33}$$

This means that there is only partial supersymmetry breaking. Indeed, supersymmetry is preserved, at least partially, if there exists a linear combination of the fermions which is invariant under the SUSY transformation

$$\delta_{SUSY}(c_1\chi_\alpha + c_2\lambda_\alpha) = 0. \tag{2.3.34}$$

This is possible if the transformation matrix (2.3.30) is not invertible, namely

$$\det\begin{pmatrix} -\sqrt{2}\mathsf{F} & i(d+i\gamma) \\ i(d+i\gamma) & -\sqrt{2}\left(\bar{\mathsf{F}}+\frac{1}{2\kappa}e^{i\phi}\right)\end{pmatrix} = 2\mathsf{F}\left(\bar{\mathsf{F}}+\frac{1}{2\kappa}e^{i\phi}\right)+(d+i\gamma)^2 = 0\,. \tag{2.3.35}$$

It is easy to see that this is indeed equivalent to

$$\boldsymbol{Y}\cdot\boldsymbol{Y} = 0, \tag{2.3.36}$$

with $\boldsymbol{Y}$ given by (2.3.26). This is therefore the criteria for the existence of residual $\mathcal{N}=1$ SUSY, hence realizing partial supersymmetry breaking $\mathcal{N}=2\to\mathcal{N}=1$.

The residual supersymmetry can be found solving the coefficients of (2.3.34) through

$$r \equiv \frac{c_2}{c_1} = \frac{iY_3}{Y_2 - iY_1} = \frac{Y_2+iY_1}{iY_3}\,. \tag{2.3.37}$$

The unbroken supercharge can then be defined as the linear combination

$$S = c_1 Q + c_2\tilde{Q}\,, \tag{2.3.38}$$

which according to the supersymmetry algebra (2.1.1), with $A=1,2$ for $Q,\tilde{Q}$, indeed satisfies the $\mathcal{N}=1$ algebra

$$\{S_\alpha, \bar{S}_{\dot\alpha}\} = 2i\sigma^m_{\alpha\dot\alpha}\partial_m\,, \tag{2.3.39}$$

provided that $|c_1|^2+|c_2|^2=1$. This condition can always be realized by a trivial rescaling of c_1, c_2. One can also explicitly verify that

$$\delta^S_\epsilon\lambda = (\epsilon S+\bar\epsilon\bar S)\lambda = \Big(c_1(Y_2+iY_1) - ic_2Y_3\Big)\epsilon + \Big(\bar c_1(\bar Y_2 - i\bar Y_1) + i\bar c_2\bar Y_3\Big)\bar\epsilon = 0\,, \tag{2.3.40}$$

and similarly $\delta^S_\epsilon\chi = 0$.

2.3.2 $\mathcal{N}=2$ electromagnetic duality

In this section, we derive electromagnetic (EM) duality fully at $\mathcal{N}=2$ level. In the next subsection, we will show the essential role of EM duality in the understanding of the mechanism of partial SUSY breaking. EM duality shows that the general deformations introduced above are dual to the triplet of FI parameters for $(\mathrm{Re}\,\mathsf{F}, \mathrm{Im}\,\mathsf{F}, \mathsf{D}/\sqrt{2})$, and can thus be regarded as magnetic FI terms. Some points of this section were already made using a different language in [25]. We use a strategy making full use of various "long","short", chiral, antichiral superfields introduced in [24].

"Long" and "short" multiplets We start by constructing the following $\mathcal{N}=2$ "long" chiral-chiral superfield [24]

$$\hat{\mathcal{Z}} = Y + \sqrt{2}\tilde\theta\chi - \tilde\theta\tilde\theta\Big(\frac{1}{4}\bar D^2\bar Y + \frac{i}{2}\Phi\Big)\,, \tag{2.3.41}$$

where Y, χ_α, Φ are $\mathcal{N} = 1$ chiral superfields. We can define related $\mathcal{N} = 2$ "short" antichiral-chiral superfield as

$$\mathcal{Z} = -\frac{i}{2}\left(\bar{D}^2\hat{\mathcal{Z}} - \bar{D}^2\bar{\hat{\mathcal{Z}}}\right), \tag{2.3.42}$$

which can be expanded in components as

$$\mathcal{Z} = \Phi - \sqrt{2}i\bar{\theta}\bar{D}L - \frac{1}{4}\bar{\theta}^2\bar{D}^2\bar{\Phi}, \qquad L \equiv D\ell + \bar{D}\bar{\ell}. \tag{2.3.43}$$

L is a real linear superfield, *i.e.* a multiplet satisfying the reality condition (2.1.13) together with the additional constraint $D^2L = 0$. Of course, the "short" multiplet contains less degrees of freedom than the "long" one, hence the terminology.

Similarly, one could define the $\mathcal{N} = 2$ "long" chiral-antichiral superfield

$$\hat{\mathcal{W}} = X + \sqrt{2}\bar{\theta}\bar{\Omega} - \bar{\theta}^2\left(\frac{1}{4}\bar{D}^2\bar{U} + \frac{i}{2}X\right), \tag{2.3.44}$$

where $U, \bar{\Omega}_{\dot{\alpha}}, X$ are chiral, *i.e.* annihilated by $\bar{D}_{\dot{\beta}}$. In particular, $\bar{\Omega}$ can be written as $\bar{\Omega}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}\mathbb{L}$ with $\mathbb{L}$ a complex linear superfield, only satisfying $\bar{D}^2\mathbb{L} = 0$. As for $\mathcal{Z}$, we reduce it to the $\mathcal{N} = 2$ "short" chiral-chiral superfield

$$\mathcal{W} = -\frac{i}{2}\left(\bar{D}^2\hat{\mathcal{W}} - \bar{D}^2\bar{\hat{\mathcal{W}}}\right), \tag{2.3.45}$$

the component form of which reads

$$\mathcal{W} = X + \sqrt{2}i\tilde{\theta}W - \frac{1}{4}\tilde{\theta}^2\bar{D}^2\bar{X}, \quad W_\alpha = \bar{D}_{\dot{\alpha}}\left(\frac{1}{2}\bar{D}^{\dot{\alpha}}\Omega_\alpha - D_\alpha\bar{\Omega}^{\dot{\alpha}}\right) = \frac{1}{2}\bar{D}^2D_\alpha\left(\mathbb{L} + \bar{\mathbb{L}}\right). \tag{2.3.46}$$

It especially implies that W_α satisfies the standard SUSY Bianchi identity (2.1.17). This enables us to define the potential associated with W_α, a real superfield $V = -2(\mathbb{L} + \bar{\mathbb{L}})$ satisfying $W_\alpha = -\frac{1}{4}\bar{D}^2D_\alpha V$, as demanded by (2.1.15) for the field strength superfield.

Since both $\hat{\mathcal{W}}$ and $\mathcal{Z}$ are chiral-antichiral, so is their product. As for the cases of eqs. (2.1.23) and (2.3.5) we can thus construct a SUSY invariant Lagrangian extracting their top component

$$\int d^2\theta d^2\bar{\theta}\ \mathcal{Z}\hat{\mathcal{W}}. \tag{2.3.47}$$

Similarly we can construct a Lagrangian from the two chiral-chiral superfields $\hat{\mathcal{Z}}, \mathcal{W}$

$$\int d^2\theta d^2\tilde{\theta}\ \mathcal{W}\hat{\mathcal{Z}}. \tag{2.3.48}$$

In fact, these two Lagrangians are equal when considered with imaginary couplings. Indeed we have

$$\begin{aligned} i\int d^2\theta d^2\bar{\theta}\hat{\mathcal{W}}\mathcal{Z} + c.c. &= \frac{1}{2}\int d^2\theta d^2\bar{\theta}\hat{\mathcal{W}}\left(\bar{D}^2\hat{\mathcal{Z}} - \bar{D}^2\bar{\hat{\mathcal{Z}}}\right) + c.c. \\ &= -\frac{1}{2}\cdot\frac{1}{4}\int d^2\theta d^2\tilde{\theta}d^2\bar{\theta}\hat{\mathcal{W}}\hat{\mathcal{Z}} + \frac{1}{2}\cdot\frac{1}{4}\int d^2\theta d^2\bar{\theta}d^2\tilde{\theta}\bar{\hat{\mathcal{W}}}\hat{\mathcal{Z}} + c.c. \\ &= \frac{1}{2}\int d^2\theta d^2\tilde{\theta}\left(\bar{D}^2\hat{\mathcal{W}} - \bar{D}^2\bar{\hat{\mathcal{W}}}\right)\hat{\mathcal{Z}} + c.c. = i\int d^2\theta d^2\tilde{\theta}\hat{\mathcal{Z}}\mathcal{W} + c.c.\,. \end{aligned} \tag{2.3.49}$$

EM duality without deformation To establish the electromagnetic duality, we first consider the case without deformation, described by the following action

$$\mathcal{L}=\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}})+i\int d^2\theta d^2\bar{\tilde{\theta}}\ \mathcal{Z}\hat{\mathcal{W}}+c.c.\,, \tag{2.3.50}$$

where $\mathcal{F}$ is the holomorphic prepotential. The $\mathcal{N}=2$ EM duality can be shown by eliminating different set of variables.

Electric side We first consider the electric side of the theory by integrating out $\hat{\mathcal{W}}$. The equation of motion of $\hat{\mathcal{W}}$ leads to

$$\mathcal{Z}=0 \qquad \rightarrow \qquad \Phi=0,\quad L=\text{const.} \tag{2.3.51}$$

Actually one can further show that $L=0$. Indeed from (2.3.49) we deduce the equality

$$\int d^2\theta\ell W+c.c.=\int d^2\theta\ell^{\alpha}(-\frac{1}{4}\bar{D}^2D_{\alpha}V)+c.c.\,. \tag{2.3.52}$$

The r.h.s. can be written as $-\int d^2\theta d^2\bar{\theta}V(D\ell+\bar{D}\bar{\ell})=-\int d^2\theta d^2\bar{\theta}\,VL$, which leads to $L=0$ from the equation of motion of V. We thus redefine the field $\ell=iZ$ such that

$$DZ-\bar{D}\bar{Z}=-iL=0\,. \tag{2.3.53}$$

The chirality of Z and the fact that it satisfies the above standard supersymmetric Bianchi identity, show that Z is the field-strength superfield of a standard vector multiplet. $\hat{\mathcal{Z}}$ becomes then the standard (short) $\mathcal{N}=2$ chiral-chiral superfield describing a vector multiplet.

The original Lagrangian (2.3.50), which we call electric Lagrangian after integrating out $\hat{\mathcal{W}}$, becomes

$$\begin{aligned}\mathcal{L}_e &= \int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}})+c.c.=\int d^2\theta\Big(-\frac{1}{4}\bar{D}^2\bar{Y}\,\mathcal{F}'-\frac{1}{2}\ell^2\mathcal{F}''\Big)+c.c.\\ &= \int d^4\theta\ \bar{Y}\mathcal{F}_Y+\frac{1}{2}\int d^2\theta\ \mathcal{F}''Z^2+c.c.\,,\end{aligned} \tag{2.3.54}$$

where $\mathcal{F}_Y\equiv\mathcal{F}'(Y)$. It is then the standard $\mathcal{N}=2$ action of a vector multiplet with prepotential $\mathcal{F}$, written in terms of its $\mathcal{N}=1$ components.

Magnetic side We use eq. (2.3.49) to rewrite the original Lagrangian (2.3.50) as

$$\mathcal{L}=\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}})+i\int d^2\theta d^2\tilde{\theta}\ \mathcal{W}\hat{\mathcal{Z}}+c.c.\,. \tag{2.3.55}$$

We now integrate out $\hat{\mathcal{Z}}$, which has the following equation of motion

$$\mathcal{W}=i\mathcal{F}'(\hat{\mathcal{Z}}). \tag{2.3.56}$$

The initial Lagrangian takes the new form

$$\mathcal{L}=\int d^2\theta d^2\tilde{\theta}\Big(\mathcal{F}(\hat{\mathcal{Z}})-\hat{\mathcal{Z}}\mathcal{F}'\Big), \tag{2.3.57}$$

the integrand of which is nothing but the Legendre transformation of $\mathcal{F}$. From (2.3.56), one looks for an inverse function

$$\hat{\mathcal{Z}} = -i\mathcal{H}'(\mathcal{W}) \,, \tag{2.3.58}$$

such that

$$\mathcal{F}(\hat{\mathcal{Z}}) - \hat{\mathcal{Z}}\mathcal{F}' = \mathcal{H}(\mathcal{W}) \,. \tag{2.3.59}$$

The construction is reminiscent of the relation between the Lagrangian and the Hamiltonian formulations in classical mechanics. The analogy is made through the identifications $-i\mathcal{W} \leftrightarrow p$, $\hat{\mathcal{Z}} \leftrightarrow \dot{x}$, $\mathcal{F} \leftrightarrow L$, $-\mathcal{H} \leftrightarrow H$. The dual magnetic theory is thus described by the "Hamiltonian"

$$\int d^2\theta d^2\tilde{\theta}\mathcal{H}(\mathcal{W}) + c.c. \,. \tag{2.3.60}$$

For clarity, we will write the magnetic theory in components. We expand the Lagrangian (2.3.55) in terms of $\mathcal{N} = 1$ superfields

$$\mathcal{L} = \int d^2\theta \Big(\left(\mathcal{F}' + iX\right) \left(-\frac{1}{4}\bar{D}^2\bar{Y} - \frac{i}{2}\Phi\right) - \frac{i}{4}Y\bar{D}^2\bar{X} - \frac{1}{2}\mathcal{F}''\ell^2 + \ell W \Big) + c.c. \,. \tag{2.3.61}$$

We integrate out Φ and ℓ, the equations of motion of which read

$$X = i\mathcal{F}'(Y) \,, \qquad \ell_\alpha = \frac{W_\alpha}{\mathcal{F}''(Y)} \,. \tag{2.3.62}$$

Substituting them back into eq. (2.3.61), we obtain the magnetic Lagrangian

$$\mathcal{L}_m = \int d^4\theta\, \bar{Y}\mathcal{F}_Y + \frac{1}{2}\int d^2\theta \frac{W^2}{\mathcal{F}''} + c.c. \,. \tag{2.3.63}$$

Defining a new function $\mathcal{H}$ such that

$$X = i\mathcal{F}'(Y) \,, \quad \mathcal{H}'(X) = iY \,, \quad \mathcal{F}''\mathcal{H}'' = \frac{d\mathcal{F}'}{dY}\frac{d\mathcal{H}'}{dX} = \frac{idX}{dY}\frac{-idY}{dX} = 1 \,, \tag{2.3.64}$$

enables us to rewrite the magnetic Lagrangian (or "Hamiltonian") as

$$\mathcal{L}_m = \int d^2\theta d^2\tilde{\theta}\mathcal{H}(\mathcal{W}) + c.c. = \int d^2\theta d^2\bar{\theta}\; \bar{X}\mathcal{H}_X + \frac{1}{2}\int d^2\theta \mathcal{H}''W^2 + c.c. \,. \tag{2.3.65}$$

The form $\mathcal{L}_m$ matches exactly with the form of the electric theory $\mathcal{L}_e$. Thus, the electric theory with chiral scalar Y and prepotential derivative $i\mathcal{F}_Y(Y)$ is equivalent, or dual, to the magnetic theory with chiral scalar $Y^D = X = i\mathcal{F}_Y(Y)$ and prepotential derivative $i\mathcal{F}^D_{Y^D}(Y^D) = i\mathcal{H}_X(X) = -Y$. This establishes the electromagnetic duality at fully $\mathcal{N} = 2$ level.

EM duality with deformation We now turn to the case with SUSY deformations and therefore modify the Lagrangian as follows

$$\mathcal{L} = \int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}} - \sqrt{2}\theta\tilde{\theta}\gamma) + i\int d^2\theta d^2\bar{\tilde{\theta}}\; \left(\mathcal{Z} + \frac{i}{\kappa}e^{i\phi}\right)\hat{\mathcal{W}} + c.c. \,. \tag{2.3.66}$$

In order to find the magnetic duals of the deformation parameters, we study EM duality in a similar fashion as above.

Electric side We first integrate out the $\hat{\mathcal{W}}$ superfield

$$\mathcal{Z}+\frac{i}{\kappa}e^{i\phi}=0 \qquad \rightarrow \qquad \Phi=-\frac{i}{\kappa}e^{i\phi}, \quad L=\text{const}\in\mathbb{R}\,, \tag{2.3.67}$$

and using the same argument as around eq. (2.3.52) we further find that $L=0$. Defining now $\ell_\alpha=i(Z_\alpha-i\theta_\alpha\gamma)$, we expand

$$\hat{\mathcal{Z}}-\sqrt{2}\theta\tilde{\theta}\gamma=Y+\sqrt{2}i\tilde{\theta}Z-\tilde{\theta}\tilde{\theta}\Big(\frac{1}{4}\bar{D}^2\bar{Y}+\frac{i}{2}\Phi\Big)\,, \tag{2.3.68}$$

where Z satisfies the constraint

$$DZ-\bar{D}\bar{Z}=-4i\gamma\,. \tag{2.3.69}$$

This is the modified Bianchi identity of Z. Note that $\mathcal{Z}$ is not affected by γ. One can now obtain the electric action as

$$\begin{aligned}\mathcal{L}_e&=\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}}-\sqrt{2}\theta\tilde{\theta}\gamma)+c.c.=\int d^2\theta\left(\Big(-\frac{1}{4}\bar{D}^2\bar{Y}-\frac{i}{2}\Phi\Big)\mathcal{F}'+\frac{1}{2}\mathcal{F}''Z^2\right)+c.c.\\&=\int d^2\theta d^2\bar{\theta}\;\bar{Y}\mathcal{F}_Y+\frac{1}{2}\int d^2\theta\;\mathcal{F}''Z^2-\frac{1}{2\kappa}e^{i\phi}\int d^2\theta\mathcal{F}_Y+c.c.\,,\end{aligned} \tag{2.3.70}$$

where Z must satisfy the generalized Bianchi identity (2.3.69).

Magnetic side Using identity (2.3.49), the deformed Lagrangian (2.3.66) can be written as

$$\begin{aligned}\mathcal{L}&=\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}}-\sqrt{2}\theta\tilde{\theta}\gamma)+i\int d^2\theta d^2\tilde{\theta}\;\mathcal{W}\hat{\mathcal{Z}}-\frac{1}{\kappa}e^{i\phi}\int d^2\theta d^2\bar{\tilde{\theta}}\;\hat{\mathcal{W}}+c.c.\\&=\int d^2\theta d^2\tilde{\theta}\mathcal{F}(\hat{\mathcal{Z}}')+i\int d^2\theta d^2\tilde{\theta}\;\mathcal{W}\hat{\mathcal{Z}}'\\&\qquad+i\int d^2\theta d^2\tilde{\theta}\;\sqrt{2}\theta\tilde{\theta}\gamma\mathcal{W}-\frac{1}{\kappa}e^{i\phi}\int d^2\theta d^2\bar{\tilde{\theta}}\;\hat{\mathcal{W}}+c.c.\,,\end{aligned} \tag{2.3.71}$$

where we have trivially shifted the argument of $\mathcal{F}$. The first two terms can be treated as before and we arrive at the magnetic theory

$$\begin{aligned}\mathcal{L}_m&=\int d^4\theta\,\bar{Y}\mathcal{F}_Y+\frac{1}{2}\int d^2\theta\frac{W^2}{\mathcal{F}''}+\gamma\int d^2\theta d^2\tilde{\theta}\;\theta^2\tilde{\theta}^2\mathsf{D}+\frac{i}{2\kappa}e^{i\phi}\int d^2\theta X+c.c.\\&=\int d^4\theta\,\bar{X}\mathcal{H}_X+\frac{1}{2}\int d^2\theta\mathcal{H}''W^2+2\gamma\int d^2\theta d^2\bar{\theta}\;V+\frac{i}{2\kappa}e^{i\phi}\int d^2\theta X+c.c.\,.\end{aligned} \tag{2.3.72}$$

We see that the magnetic theory therefore contains a triplet of FI terms

$$\begin{aligned}\mathcal{L}_{\text{FI}}&=2\gamma\int d^2\theta d^2\bar{\theta}\;V+\frac{i}{2\kappa}e^{i\phi}\int d^2\theta X+c.c.\\&=2\gamma\mathsf{D}+\frac{1}{\kappa}\sin\phi\,\text{Re}\,\mathsf{F}+\frac{1}{\kappa}\cos\phi\,\text{Im}\,\mathsf{F}=-4i\boldsymbol{\Upsilon}\cdot\boldsymbol{\Upsilon}_{\text{def}}\,.\end{aligned} \tag{2.3.73}$$

Comparing the two actions (2.3.70) and (2.3.72), we clearly see the duality between deformations and triplet of FI couplings: $X\leftrightarrow\mathcal{F}_Y$ and [modification of Bianchi identity $DZ-\bar{D}\bar{Z}=-4i\gamma$] $\leftrightarrow$ [FI D-term $\gamma\mathsf{D}$]. Additional elements on this deformed vector multiplet can be found in [27].

2.3.3 The APT model of partial SUSY breaking and beyond

In this part, we first recall the Antoniadis-Partouche-Taylor (APT) model [21] realizing $\mathcal{N}=2\to\mathcal{N}=1$ partial SUSY breaking and then generalise it with the deformations introduced above. To do so, we analyze the general $\mathcal{N}=2$ action based on an arbitrary deformed vector superfield, we compute the scalar potential and we show that the only non-trivial minima break SUSY partially from $\mathcal{N}=2\to\mathcal{N}=1$.

APT model Our construction of the APT model starts with the $\mathcal{N}=2$ chiral-chiral superfield introduced in eq. (2.3.7), together with the deformation (2.3.22). The new vector superfield reads

$$\begin{aligned}&\mathcal{W}_{\rm new}=\mathcal{W}-\frac{1}{2\kappa}e^{i\phi}\tilde\theta\tilde\theta-\sqrt{2}\gamma\theta\tilde\theta=X+\sqrt{2}i\tilde\theta W-\frac{1}{4}\tilde\theta\tilde\theta\Big(\bar D^2\bar X+4m\Big),\\&X=x+\sqrt{2}\theta\chi-\theta\theta\mathsf{F}\,,\quad m\equiv\frac{1}{2\kappa}e^{i\phi}\,.\end{aligned}\tag{2.3.74}$$

The APT model corresponds to the case where the only non-vanishing deformation parameter is κ, hence we set $\gamma=\phi=0$. The APT Lagrangian, realizing the partial breaking, is thus

$$\begin{aligned}\mathcal{L}=&-i\left(\int d^2\theta d^2\tilde\theta\;\mathcal{F}(\mathcal{W}_{\rm new})-e\int d^2\theta X\right)-\sqrt{2}\xi\int d^4\theta V+c.c.\\=&-i\left(\int d^2\theta\Big(-\frac{1}{4}\mathcal{F}'(X)(\bar D^2\bar X+4m)+\frac{1}{2}\mathcal{F}''(X)W^2\Big)-e\int d^2\theta X\right)-\sqrt{2}\xi\int d^4\theta V+c.c.\\=&-i\left(\int d^4\theta\;\bar X\mathcal{F}'(X)-\int d^2\theta\Big(eX+m\mathcal{F}'(X)-\frac{1}{2}\mathcal{F}''(X)W^2\Big)\right)-\sqrt{2}\xi\int d^4\theta V+c.c.\,,\end{aligned}\tag{2.3.75}$$

where the holomorphic function $\mathcal{F}$ is the prepotential and $m,e,\xi\in\mathbb{R}$. As we explained previously in section 2.3.2, eX and $m\mathcal{F}'$ are dual to each other. Adding them simultaneously into the Lagrangian is crucial for partial supersymmetry breaking. The action can be further rewritten in a compact form as

$$\mathcal{L}=\int d^4\theta K(X,\bar X)+\int d^2\theta\,\mathsf{W}(X)+\int d^2\bar\theta\,\bar{\mathsf{W}}(\bar X)\tag{2.3.76}$$

$$+\left(i\int d^2\theta\frac{\mathcal{F}''(X)}{2}W^2+c.c.\right)-2\sqrt{2}\xi\int d^4\theta V\,,\tag{2.3.77}$$

where the $\mathcal{N}=1$ Kähler potential $K(X,\bar X)$ and superpotential $\mathsf{W}(X)$ where introduced in eq. (2.1.20) and are related to the $\mathcal{N}=2$ prepotential $\mathcal{F}$ through

$$K(X,\bar X)=-i\bar X\mathcal{F}'(X)+iX\bar{\mathcal{F}}'(\bar X),\qquad \mathsf{W}(X)=ieX+im\mathcal{F}'(X)\,.\tag{2.3.78}$$

We now study the scalar potential in order to find the vacuum of the theory. We first define

$$\tau(x)=\tau_1(x)+i\tau_2(x)\equiv\mathcal{F}''(x)\in\mathbb{C},\quad \tau_1,\tau_2\in\mathbb{R},\tag{2.3.79}$$

and derive the solutions of the $\mathcal{N}=1$ auxiliary fields equations of motion similar to the simple case of eqs. (2.1.29) and (2.1.30). There are small differences with respect to the simple case: $K_{\bar XX}=\partial_X\partial_{\bar X}K$ is non-canonical, the gauge coupling is non-unitary $g^2\neq1$, and the FI parameter ξ takes a factor $-\sqrt{2}$ with respect to the FI Lagrangian (2.1.24). The Kähler metrics is derived from (2.3.78) and the gauge coupling is be obtained by comparing the term in front of the field strength kinetic terms W^2 in eqs. (2.1.23) and (2.3.77).

They read

$$K_{\bar{X}X} = -i\left(\mathcal{F}''(x) - \bar{\mathcal{F}}''(x^*)\right) = 2\tau_2(x), \qquad \frac{1}{g^2} = \mathrm{Re}(-2i\mathcal{F}''(x)) = 2\tau_2(x). \tag{2.3.80}$$

so that the equations of motion including the mentioned modifications are solved by

$$\mathsf{D} = -g^2(-\sqrt{2}\xi) = \frac{\xi}{\sqrt{2}\tau_2}\,, \tag{2.3.81}$$

$$\mathsf{F} = \mathsf{F}^X = -K^{X\bar{X}}\frac{\partial\bar{\mathrm{W}}}{\partial\bar{X}} = \frac{-i(m\bar{\tau}+e)}{2\tau_2}\,, \quad \bar{\mathsf{F}} = \bar{\mathsf{F}}^{\bar{X}} = -K^{\bar{X}X}\frac{\partial\mathrm{W}}{\partial X} = \frac{i(m\tau+e)}{2\tau_2}\,. \tag{2.3.82}$$

The scalar potential is expressed the same way as in (2.1.28), with modifications due to the small differences mentioned above, and reads

$$V(x) = \frac{1}{2g^2}\mathsf{D}^2 + K_{\bar{X}X}\bar{\mathsf{F}}^{\bar{X}}\mathsf{F}^X = \frac{1}{2g^2}\mathsf{D}^2 + 2K_{\bar{X}X}\bar{\mathsf{F}}\mathsf{F} = \frac{|m\tau(x)+e|^2+\xi^2}{2\tau_2}\,. \tag{2.3.83}$$

We highlighted the fact that the scalar potential V depends on the scalar field x through $\tau(x)$. To find the vacuum, namely the minimum of the scalar potential, we thus need to extremize V with respect to the x. Equivalently, assuming $\partial\tau(x)/\partial x \neq 0$, we can extremize with respect to τ_1, τ_2 and get the following solutions

$$\langle\tau_1\rangle = -\frac{e}{m}\,, \qquad \langle\tau_2\rangle = \left|\frac{\xi}{m}\right|. \tag{2.3.84}$$

A negative τ_2 also extremizes the scalar potential but is discarded by positivity of the kinetic term, hence the absolute value. This solution can be used in eqs. (2.3.81) and (2.3.82) to find the auxiliary field VEVs

$$\langle\mathsf{F}\rangle = \langle\bar{\mathsf{F}}\rangle = -\frac{m}{2}, \qquad \langle\mathsf{D}\rangle = \frac{m\,\mathrm{sgn}(m\xi)}{\sqrt{2}}\,, \tag{2.3.85}$$

which in turn gives the vacuum potential energy

$$V = |m\xi|\,. \tag{2.3.86}$$

From eq. (2.3.30) we deduce the fermions transformation laws

$$\delta\lambda = i\mathsf{D}\epsilon = i\frac{m\,\mathrm{sgn}(m\xi)}{\sqrt{2}}\epsilon, \qquad \tilde{\delta}\lambda = -\sqrt{2}\tilde{\epsilon}(\bar{\mathsf{F}}+m) = -\frac{1}{\sqrt{2}}m\tilde{\epsilon}, \tag{2.3.87}$$

$$\delta\chi = -\sqrt{2}F\epsilon = \frac{m}{\sqrt{2}}\epsilon, \qquad \tilde{\delta}\chi = i\tilde{\epsilon}\mathsf{D} = i\frac{m\,\mathrm{sgn}(m\xi)}{\sqrt{2}}\tilde{\epsilon} \tag{2.3.88}$$

which show that

$$\delta_{SUSY}(\lambda + i\,\mathrm{sgn}(m\xi)\chi) = 0\,, \tag{2.3.89}$$

so that a linear combination of two supersymmetries is preserved. According to the discussion of the previous subsection this shows that the $\mathcal{N}=2$ supersymmetry is only partially broken.

Generalization of APT model As we emphasized, the crucial point in APT model is the simultaneous turning on of electric coupling eX and magnetic coupling $m\mathcal{F}'(X)$. Since in the previous sections we found three deformation parameters, it is natural to generalize

the APT model by adding electric and magnetic couplings corresponding to the three deformations.

The Lagrangian is thus almost the same as before

$$\begin{aligned}\mathcal{L} &= -i\left(\int d^2\theta d^2\tilde{\theta}\ \mathcal{F}(\mathcal{W}_{\text{new}}) - e\int d^2\theta X\right) - \sqrt{2}\xi\int d^4\theta V + c.c. \qquad (2.3.90)\\ &= -i\left(\int d^2\theta d^2\bar{\theta}\ \bar{X}\mathcal{F}'(X) - \int d^2\theta\Big(eX + m\mathcal{F}'(X) - \frac{1}{2}\mathcal{F}''(X)W^2\Big)\right) - \sqrt{2}\xi\int d^4\theta V + c.c.\,,\end{aligned}$$

with the difference that we now allow complex parameters $m = m_R + im_I$, $\mathsf{D} = d + i\gamma$ with $m_R, m_I, \gamma, \xi, e \in \mathbb{R}$. Note that e is taken to be real since its phase can be absorbed by a rescaling of X. The auxiliary fields are solved as in eqs. (2.3.81) and (2.3.82) and read

$$\mathsf{F} = \frac{-i(\bar{m}\bar{\tau} + e)}{2\tau_2}, \qquad \bar{\mathsf{F}} = \frac{i(m\tau + e)}{2\tau_2}, \qquad d = \frac{\xi - \sqrt{2}\gamma\tau_1}{\sqrt{2}\tau_2}\,. \qquad (2.3.91)$$

As in (2.3.83) these solutions lead to the scalar potential

$$V = \frac{|m\tau + e|^2 + \xi^2 - 2\sqrt{2}\xi\gamma\tau_1 + 2\gamma^2(\tau_1^2 + \tau_2^2)}{2\tau_2}\,. \qquad (2.3.92)$$

Extremizing the scalar potential, one finds that the vacuum sits at

$$\langle\tau_1\rangle = \frac{-em_R + \sqrt{2}\gamma\xi}{|m|^2 + 2\gamma^2}, \qquad \langle\tau_2\rangle = \frac{\sqrt{(\sqrt{2}e\gamma + m_R\xi)^2 + m_I^2(e^2 + \xi^2)}}{|m|^2 + 2\gamma^2}\,, \qquad (2.3.93)$$

and that the auxiliary field VEVs are

$$\langle\bar{\mathsf{F}}\rangle = \frac{2ie\gamma^2 + i\sqrt{2}m\gamma\xi + emm_I - m\sqrt{(\sqrt{2}e\gamma + m_R\xi)^2 + m_I^2(e^2 + \xi^2)}}{2\sqrt{(\sqrt{2}e\gamma + m_R\xi)^2 + m_I^2(e^2 + \xi^2)}}\,, \qquad (2.3.94)$$

$$\langle\mathsf{F}\rangle = \bar{\mathsf{F}}^*\,, \qquad (2.3.95)$$

$$\langle d\rangle = \frac{2em_R\gamma + \sqrt{2}|m|^2\xi}{2\sqrt{(\sqrt{2}e\gamma + m_R\xi)^2 + m_I^2(e^2 + \xi^2)}}\,, \qquad (2.3.96)$$

We can verify that with these values the following equality always holds

$$\boldsymbol{Y}\cdot\boldsymbol{Y} = 0\,, \qquad (2.3.97)$$

where we recall that $\boldsymbol{Y}$ is defined through eq. (2.3.26). As explained in subsection 2.3.1, the above condition implies that there is always a residual $\mathcal{N} = 1$ supersymmetry. We finally emphasize that in what was just done, we excluded the singular points $\tau_2 = 0, +\infty$, of runaway behavior, and the trivial case of a free theory with quadratic prepotential.

More $U(1)$s towards the complete breaking of supersymmetry As we have just seen, a theory with only one $U(1)$ always has an $\mathcal{N} = 1$ supersymmetric vacuum, independent of the FI parameters and deformations as soon as $\tau_2 \neq 0, +\infty$, in an interacting theory. Hence it seems impossible to completely break supersymmetry. However, note that although $\mathcal{N} = 1$ is always preserved, the residual supersymmetry, as a linear combination of the two original supersymmetries in $\mathcal{N} = 2$, depends on the deformations and

FI parameters. Therefore if the theory contains two or more $U(1)$s with different residual supersymmetries, the full system breaks supersymmetry completely. Of course, the different sectors should communicate through matter (not necessarily charged) or gravitational interactions.

More specifically, consider the Lagrangian with two decoupled $U(1)$s

$$\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} \,. \tag{2.3.98}$$

The previous analysis applies individually to these two subsectors, described as in (2.3.26) by their respective auxiliary fields vectors

$$\boldsymbol{Y}^{(1)} = \boldsymbol{Y}^{(1)}_{\text{def}} + \boldsymbol{Y}^{(1)}_{\text{dynamic}} \,, \qquad \boldsymbol{Y}^{(2)} = \boldsymbol{Y}^{(2)}_{\text{def}} + \boldsymbol{Y}^{(2)}_{\text{dynamic}} \,. \tag{2.3.99}$$

The full system is characterized by

$$\boldsymbol{Y} = \boldsymbol{Y}^{(1)} + \boldsymbol{Y}^{(2)} \,, \tag{2.3.100}$$

for which, as we have seen in the last subsection, we always have

$$\boldsymbol{Y}^{(1)} \cdot \boldsymbol{Y}^{(1)} = \boldsymbol{Y}^{(2)} \cdot \boldsymbol{Y}^{(2)} = 0 \,. \tag{2.3.101}$$

Hence, as long as the two vectors are not aligned, *i.e.* $\boldsymbol{Y}^{(2)} \neq c\boldsymbol{Y}^{(1)}$, which is true generically in the parameter space, we immediately have

$$\boldsymbol{Y} \cdot \boldsymbol{Y} \neq 0 \,, \tag{2.3.102}$$

meaning that $\mathcal{N} = 2$ supersymmetry is broken completely.

2.4 Supersymmetric Dirac-Born-Infeld actions and Fayet-Iliopoulos terms

In the current section we study the fate of generic deformations of $\mathcal{N} = 2$ vector multiplet in presence of non-linear realization of one of the two initial supersymmetries, describing the low-energy theory after (partial) SUSY breaking as explained in section 2.2. This non-linear realization will be imposed through a nilpotent constraint on the deformed vector multiplet, resulting in a generalized supersymmetric Dirac-Born-Infeld (SDBI) action [22, 28, 23, 24]. In the bosonic sector, we find essentially the same result as in the previous unconstrained case of a general prepotential. This time there is no scalar potential but the parameters of the FI term and the general deformation can be absorbed into a redefinition of the DBI couplings, namely the non-linear supersymmetry breaking scale (related to the brane tension), the $U(1)$ gauge coupling and the theta-angle. We notice that, as in the unconstrained case, the D-deformation induces a FI D-term through the theta-angle. The FI term is constant and the theory is not free. In principle, one would expect that the presence of this term would break both supersymmetries but this is not the case. Instead, one $\mathcal{N} = 1$ linear supersymmetry remains but it changes direction. In the string theory context, it corresponds to rotate the brane in the bulk. As in the previous case, the complete breaking of supersymmetry can arise only in a system of at least two DBI actions preserving different linear supersymmetries, corresponding to two branes at angles. In the fermionic sector, things are more involved and

The fermionic sector of SDBI actions is richer, as it includes goldstino interactions. To motivate its analysis, we recall that in the case of $\mathcal{N} = 1$ non-linear supersymmetry, the

Volkov-Akulov action [14] was shown to be unique up to field redefinitions that do not change the theory on-shell, see e.g. [19]. We hence would like to tackle an equivalent problem in $\mathcal{N} = 2$ non-linear supersymmetry, namely the uniqueness of SDBI actions. The latter describe the effective field theory of a D3 brane in an $\mathcal{N} = 2$ supersymmetric bulk, obtained for instance upon compactification of the ten-dimensional type II superstrings on a Calabi-Yau threefold. Hence a natural arena to study their uniqueness is within string theory, where a comparison of SDBI with the corresponding amplitudes was done in [29, 30] at the level of mass dimension-8 interactions. Before going to string theory setups, it is nevertheless important to have a better understanding of the uniqueness of SDBI actions at the effective theory level. To study them, one can consider deformations that do not change the form of the action bosonic part. As mentioned in the above paragraph and shown in the current section, this can be achieved by adding FI terms and their generalisations. The generation mechanisms and the effect of such FI terms constitute an interesting open problem related to supersymmetry breaking, that becomes more restrictive in supergravity and even more in extended supersymmetric theories. New FI terms, that do not require gauging the R-symmetry when coupled to supergravity, were written recently within $\mathcal{N} = 1$ [31, 32, 33, 34] and $\mathcal{N} = 2$ [35] supersymmetry; their bosonic part is identical to an ordinary constant FI term, while their fermionic dependence is highly non-trivial.

The above discussion should convince us that, even before making use of complicated FI terms, it is interesting to study carefully the fermionic contributions of standard or induced FI terms. Indeed, the goldstino of the $\mathcal{N} = 2 \to \mathcal{N} = 1$ partial breaking belongs to a vector multiplet of the linear supersymmetry, that has non-trivial self-interactions due to the non-linear supersymmetry. After solving the nilpotent constraint, the SDBI action can be written as an integral over the $\mathcal{N} = 1$ superspace, where non-linear supersymmetry is not manifest [23]. In terms of components, upon superspace integration, the bosonic part of the action can be written in a closed form, while it is not the case for the part involving the $U(1)$ gauginos. On the other hand, an explicit form of the whole action was given within the formalism of non-linear supersymmetry, using variables where manifest linear supersymmetry is lost [36, 18]. In section 2.4.3 we show the first computation of non-trivial higher dimensional physical vertices corresponding to fermionic contributions in the SDBI action, in the presence of a standard or induced FI terms. Such terms being linear in the $\mathcal{N} = 1$ vector multiplet, they generate interactions containing an odd number of gauge fields. As these terms are not present in the standard SDBI action, we expect that they show the difference between the SDBI action and SDBI coupled to FI terms in appropriate on-shell physical operators. Some terms are not physical and can be eliminated by means of field redefinitions, others are constrained by the low-energy theorems of supersymmetry breaking. These considerations motivate to compute dimension-10 physical mixed operators involving gauginos and gauge fields – either four gauginos and two gauge bosons, or two gauginos and three gauge fields – in both cases of the SDBI action with a standard or induced FI term, referred in the following to as SDBI+FI or SDBI+γ, θ, respectively. In the first case of SDBI+FI, we use both the non-linear formalism and the standard constrained superfield one with manifest linear supersymmetry, while in the second case of SDBI+γ, θ we use only the constrained superfield method. Actually, in the SDBI+FI case, using the formalism of non-linear supersymmetry, we also compute a class of dimension-12 mixed operators with two gauginos and four bosons, of the form $(\lambda\partial\bar{\lambda})F^4$.

2.4.1 Deformed Dirac-Born-Infeld action

In order to implement the non-linear realization of one of the two initial supersymmetries, we will impose a nilpotent constraint on the deformed $\mathcal{N} = 2$ vector multiplet, and obtain a generalized supersymmetric Dirac-Born-Infeld action. We will first study the bosonic part of the action and find that it is almost identical to the standard bosonic DBI up to some renormalization of coupling constants. This is quite similar to the case of DBI+FI model where the FI parameter only renormalizes the coupling of the bosonic DBI [23].

We will also study SUSY breaking in our model and find that, as in section 2.3.3, there is always a residual $\mathcal{N} = 1$ supersymmetry independently of the deformation parameters. However, this unbroken $\mathcal{N} = 1$ supercharge, as a linear combination of $\mathcal{N} = 2$ supercharges, depends on the deformation parameters.

In order to differentiate the deformed SDBI from the SDBI+FI model, the bosonic parts of which are similar, we will then look at their fermionic part using the non-linear SUSY formalism of [18].

Nilpotent constraint in $\mathcal{N} = 2$ The SDBI action arises from the partial supersymmetry breaking of $\mathcal{N} = 2 \to \mathcal{N} = 1$. It was first constructed through the coset method by Bagger and Galperin [22]. In [28], Rocek and Tseytlin found the same action through a nilpotent constraint on the $\mathcal{N} = 2$ superfield, similar to the $\mathcal{N} = 1$ constraint (2.2.23) for the goldstino superfield. We now explain this elegant nilpotent construction and discuss the deformed SDBI.

Without phase deformation We effectively break the $\mathcal{N} = 2$ supersymmetry by assuming the presence of a Lorentz invariant condensate $\langle \mathcal{W} \rangle = \mathcal{W}_{\text{def}} \neq 0$, so that the vector multiplet is described as in eq. (2.3.74) by

$$\mathcal{W}_{\text{new}} = \mathcal{W} + \langle \mathcal{W} \rangle = \mathcal{W} + \mathcal{W}_{\text{def}} = X + \sqrt{2} i\tilde{\theta} W - \frac{1}{4}\tilde{\theta}\tilde{\theta}\Big(\bar{D}^2 \bar{X} + \frac{2}{\kappa}\Big) , \tag{2.4.1}$$

where the deformation γ is implicit in W, and we took a vanishing phase deformation $\phi = 0$. To obtain a superfield with one non-linearized supersymmetry, we can apply a nilpotent constraint [28] identical to eq. (2.2.23), namely

$$\mathcal{W}_{\text{new}}{}^2 = 0. \tag{2.4.2}$$

When expressed in terms of the $\mathcal{N} = 1$ superfields, this constraint reads

$$\frac{1}{\kappa} X = WW - \frac{1}{2} X \bar{D}^2 \bar{X} , \tag{2.4.3}$$

and can be solved to eliminate X in terms of W [22], as was done for the goldstino superfield in section 2.2.2. The solution reads

$$X = \kappa W^2 - \kappa^3 \bar{D}^2 \Big[\frac{W^2 \bar{W}^2}{1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} - \mathcal{B}^2}} \Big] , \tag{2.4.4}$$

where we have introduced

$$\mathcal{A} = \frac{\kappa^2}{2}(D^2 W^2 + \bar{D}^2 \bar{W}^2) = \bar{\mathcal{A}}, \qquad \mathcal{B} = i\frac{\kappa^2}{2}(D^2 W^2 - \bar{D}^2 \bar{W}^2) = \bar{\mathcal{B}} . \tag{2.4.5}$$

Before imposing the constraint (2.4.2), the most general $\mathcal{N}=2$ supersymmetric two-derivative action is given in (2.3.75) . It depends on a prepotential and contains two (electric) FI terms linear in the $\mathcal{N}=1$ superfields X and V. After imposing the nilpotent constraint, only the linear term of the prepotential expressed as a function of $\mathcal{W}_{\rm new}$ remains, and it gives a vanishing contribution upon integration over the chiral superspace. One is thus left only with the two FI terms leading to the SDBI action coupled to the standard FI Lagrangian. The SDBI Lagrangian arises from the term linear in X

$$\mathcal{L}_{\rm SDBI}=\frac{1}{4\kappa g^2}\Big(\int d^2\theta X+\int d^2\bar\theta\bar X\Big)\,, \tag{2.4.6}$$

which can more generally be considered with a complex coupling constant

$$\mathcal{L}_{\rm SDBI}=\frac{1}{8\pi\kappa}\,{\rm Im}\Big(\tau\int d^2\theta X\Big)\,,\qquad \tau\equiv\frac{4\pi i}{g^2}+\frac{\theta}{2\pi}\,. \tag{2.4.7}$$

In the absence of θ-angle and γ deformation, the above Lagrangian gives rise to the standard SDBI Lagrangian.

With phase deformation In the presence of a phase deformation $\phi\neq 0$, $\mathcal{W}_{\rm new}$ has the exact form of (2.3.74) and the component form nilpotent constraint (2.4.4) is modified to

$$\frac{1}{\kappa}e^{i\phi}X=WW-\frac{1}{2}X\bar D^2\bar X\,. \tag{2.4.8}$$

We absorb the phase into X by defining a new superfield $\tilde X=e^{i\phi}X$ so that the constraint reads

$$\frac{1}{\kappa}\tilde X=\frac{1}{\kappa}(e^{i\phi}X)=WW-\frac{1}{2}(e^{i\phi}X)\bar D^2(e^{-i\phi}\bar X)=WW-\frac{1}{2}\tilde X\bar D^2\bar{\tilde X}\,, \tag{2.4.9}$$

so that its solution is the same as (2.4.4), except that it is now expressed for the new chiral superfield

$$\tilde X=\kappa W^2-\kappa^3\bar D^2\Big[\frac{W^2\bar W^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\Big]\,. \tag{2.4.10}$$

The SDBI Lagrangian (2.4.7) can hence be written as

$$\mathcal{L}_{\rm SDBI}=\frac{1}{8\pi\kappa}\,{\rm Im}\Big(\tau\int d^2\theta X\Big)=\frac{1}{8\pi\kappa}\,{\rm Im}\Big(\tilde\tau\int d^2\theta\tilde X\Big)\,, \tag{2.4.11}$$

where $\tilde\tau=e^{-i\phi}\tau$ is the new coupling constant. Therefore the only effect of a non-vanshing phase deformation $\phi\neq 0$ is to rotate the phase of the complex coupling constant. In the following we will consider a general complex coupling constant having already incorporated the phase ϕ.

Bosonic part We study the bosonic part of the deformed SDBI action, *i.e.* the deformed DBI action. It turns out that in spite of the general deformations, the resulting bosonic action takes the well-known form of the DBI action. To evaluate the SDBI Lagrangian (2.4.7), we replace X by its solution (2.4.4) of the nilpotent constraint. To do so, we use the component expression (2.3.8) for W, the $\mathcal{N}=1$ field strength superfield included in the

$\mathcal{N}=2$ deformed vector multiplet, and compute the quantity

$$W^2 = C + \psi\theta + \theta\theta E\,, \tag{2.4.12}$$
$$C = -\lambda^2\,, \quad \psi_\beta = -2i\mathsf{D}\lambda_\beta + 2F_{\mu\nu}\sigma^{\mu\nu}{}_\beta{}^\alpha\lambda_\alpha\,, \quad E = \mathsf{D}^2 - \frac{1}{2}(F^2 + iF\tilde{F}) - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda}\,,$$

where

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}, \qquad F\tilde{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \qquad \mathsf{D} = d + i\gamma. \tag{2.4.13}$$

In the above equation we defined the dual field strengh $\tilde{F}^{\mu\nu}$ and recalled the deformed expression (2.3.25) for D, which contains the auxiliary field d together with the deformation γ. In the pure bosonic case, *i.e.* for $\lambda = \bar{\lambda} = 0$, we have

$$W^2\big|_{\lambda=\bar{\lambda}=0} = \theta\theta E = \theta\theta\Big[\mathsf{D}^2 - \frac{1}{2}(F^2 + iF\tilde{F})\Big], \quad \bar{W}^2\big|_{\lambda=\bar{\lambda}=0} = \bar{\theta}\bar{\theta}\bar{E} = \bar{\theta}\bar{\theta}\Big[\bar{\mathsf{D}}^2 - \frac{1}{2}(F^2 - iF\tilde{F})\Big]\,. \tag{2.4.14}$$

Hence in this bosonic case $W^2, \bar{W}^2$ only have non-vanishing $\theta\theta$ component, so that the $\mathcal{A}, \mathcal{B}$ fields of (2.4.5) can only contribute through their lowest components

$$A \equiv \mathcal{A}|_{\theta=0} = \mathcal{A}|_{\lambda=\bar{\lambda}=0} = -2\kappa^2(E + \bar{E}) = 2\kappa^2\Big(F^2 - 2(d^2 - \gamma^2)\Big)\,, \tag{2.4.15}$$
$$B \equiv \mathcal{B}|_{\theta=0} = \mathcal{B}|_{\lambda=\bar{\lambda}=0} = -2i\kappa^2(E - \bar{E}) = -2\kappa^2\Big(F\tilde{F} - 4d\gamma\Big)\,. \tag{2.4.16}$$

With these ingredients we compute the bosonic part of the integral of X as

$$\begin{aligned}\int d^2\theta X\Big|_{\lambda=\bar{\lambda}=0} &= \int d^2\theta\left(\kappa W^2 - \kappa^3\bar{D}^2\Big[\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\Big]\right)\Big|_{\lambda=\bar{\lambda}=0}\\ &= \kappa\int d^2\theta W^2\Big|_{\lambda=\bar{\lambda}=0} + 4\kappa^3\int d^2\theta d^2\bar{\theta}\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\Big|_{\lambda=\bar{\lambda}=0}\\ &= \kappa E + 4\kappa^3\frac{E\bar{E}}{1+A+\sqrt{1+2A-B^2}}\,.\end{aligned} \tag{2.4.17}$$

that can be decomposed into real and imaginary parts as

$$2\,\mathrm{Re}\int d^2\theta X\Big|_{\lambda=\bar{\lambda}=0} = \kappa(E+\bar{E}) + \frac{8\kappa^3\,E\bar{E}}{1+A+\sqrt{1+2A-B^2}} = \frac{1}{2\kappa}\left(1-\sqrt{1+2A-B^2}\right), \tag{2.4.18}$$

$$2\,\mathrm{Im}\int d^2\theta X\Big|_{\lambda=\bar{\lambda}=0} = -i\kappa(E-\bar{E}) = \frac{B}{2\kappa}. \tag{2.4.19}$$

The bosonic part of the SDBI Lagrangian (2.4.7) thus reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{2g^2\kappa}\,\mathrm{Re}\int d^2\theta X\Big|_{\lambda=\bar{\lambda}=0} + \frac{\theta}{16\pi^2\kappa}\,\mathrm{Im}\int d^2\theta X\Big|_{\lambda=\bar{\lambda}=0}\\ &= \frac{1}{8g^2\kappa^2}\left(1-\sqrt{1+2A-B^2}\right) + \frac{\theta}{64\pi^2\kappa^2}B\\ &= \frac{1}{8g^2\kappa^2}\left(1-\sqrt{1+4\kappa^2\Big(F^2-2(d^2-\gamma^2)\Big)-4\kappa^4\Big(F\tilde{F}-4d\gamma\Big)^2}\right) - \frac{\theta}{32\pi^2}\Big(F\tilde{F}-4d\gamma\Big).\end{aligned} \tag{2.4.20}$$

Note the term $\theta\gamma d$ which is reminiscent of the standard FI term ξd and might provide

an alternative realization of supersymmetry breaking when we are in presence of a γ deformation together with a non-vanishing θ-angle. In the following we often refer to it as the γ-induced FI term.

Solving the d auxiliary field gives

$$d = \frac{2\gamma F\tilde{F}\kappa^2}{8\gamma^2\kappa^2+1} - \frac{\gamma g^2\theta\sqrt{1+4\tilde{\kappa}^2F^2-4\tilde{\kappa}^4(F\tilde{F})^2}}{2\sqrt{2}\sqrt{\gamma^2\kappa^2\left(g^4\theta^2+64\pi^4\right)+8\pi^4}}\,, \qquad \tilde{\kappa}^2 \equiv \frac{\kappa^2}{1+8\gamma^2\kappa^2}, \tag{2.4.21}$$

where we introduced the renormalized coupling $\tilde{\kappa}$, and replacing d by the above solution one obtains the bosonic Lagrangian

$$\mathcal{L} = \frac{1}{8g^2\kappa^2} - \frac{\theta F\tilde{F}}{32\pi^2\left(8\gamma^2\kappa^2+1\right)} - \frac{1}{8g^2\kappa\tilde{\kappa}}\sqrt{1+\frac{\theta^2g^4\gamma^2\tilde{\kappa}^2}{8\pi^4}}\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\tilde{\kappa}F_{\mu\nu}\right)}. \tag{2.4.22}$$

This Lagrangian takes the form of a standard bosonic DBI action, except that the couplings are renormalized by the deformations. For a vanishing θ angle, it simply reads

$$\mathcal{L} = \frac{1}{8g^2\kappa^2} - \frac{1}{8g^2\kappa\tilde{\kappa}}\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\tilde{\kappa}F_{\mu\nu}\right)}\,. \tag{2.4.23}$$

and if we furthermore set $\gamma = 0$, it of course reduces to the conventional DBI Lagrangian

$$\mathcal{L} = \frac{1}{8\kappa^2g^2}\left[1-\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\kappa F_{\mu\nu}\right)}\right] = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+\cdots\,. \tag{2.4.24}$$

It is worth reminding that, as described around (3.1.106), in string theory this Lagrangian describes a D3 brane of tension T_3 and gauge coupling g^2_{D3}. These Dp brane constants are related through eq. (3.1.108), which for $p=3$ reads

$$g_{D3} = \frac{1}{2\pi\alpha'\sqrt{T_3}}, \tag{2.4.25}$$

so that looking at the overall factor in the Lagrangan (2.4.24), κ can be related to α' by

$$\kappa = \frac{1}{2g_{D3}\sqrt{2T_3}} = \frac{\pi\alpha'}{\sqrt{2}}\,. \tag{2.4.26}$$

SUSY breaking We investigate supersymmetry breaking in SDBI actions coupled to FI terms, standard or induced by the deformations, using the formalism developed previously in section 2.3.1. We recall that SUSY breaking is studied through the determinations of the VEVs of the different auxiliary fields. It is thus important to keep track of their contributions, before solving them.

SUSY breaking in standard DBI+FI We start by considering the standard DBI+FI model, restricting ourselves to the bosonic part. We look at the bosonic contribution to the Lagrangian of (2.4.6) coupled to the FI Lagrangian (2.1.24). It reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{4\kappa g^2}\Big(\int d^2\theta X + \int d^2\bar{\theta}\bar{X}\Big) + \frac{\xi}{\sqrt{2}}\int d^2\theta d^2\bar{\theta}V \\ &= \frac{1}{8g^2\kappa^2}\left[1-\sqrt{1+4\kappa^2\Big(F^2-2\mathrm{D}^2\Big)-4\kappa^4\Big(F\tilde{F}\Big)^2}\right]+\frac{\xi}{\sqrt{8}}\mathrm{D}+\cdots\,.\end{aligned} \tag{2.4.27}$$

The auxiliary field is solved by

$$\mathsf{D} = -\frac{g^2\xi\sqrt{1+4\kappa^2F^2-4\kappa^4\left(F\tilde{F}\right)^2}}{\sqrt{8}\sqrt{1+g^4\kappa^2\xi^2}}, \qquad \langle\mathsf{D}\rangle = -\frac{g^2\xi}{\sqrt{8}\sqrt{1+g^4\kappa^2\xi^2}}, \tag{2.4.28}$$

which leads to the following bosonic Lagrangian

$$\mathcal{L}_{\text{DBI+FI}} = \frac{1}{8g^2\kappa^2} - \frac{\sqrt{1+g^4\kappa^2\xi^2}}{8g^2\kappa^2}\sqrt{1+4\kappa^2F^2-4\kappa^4\left(F\tilde{F}\right)^2}\,. \tag{2.4.29}$$

We see that, just like the deformations, the FI parameter ξ renormalizes the couplings. As explained in section 2.3.1, to study SUSY breaking we look at the fermion transformations (2.3.30) dependence in the auxiliary fields. In the present case it gives

$$\delta_\epsilon\lambda = \sqrt{2}iY_3\epsilon + \cdots = i\mathsf{D}\epsilon + \cdots\,, \tag{2.4.30}$$

$$\tilde{\delta}_{\tilde{\epsilon}}\lambda = -\sqrt{2}(Y_2 - iY_1)\tilde{\epsilon} + \cdots = -\sqrt{2}(\bar{\mathsf{F}} + \frac{1}{2\kappa})\tilde{\epsilon} + \cdots\,. \tag{2.4.31}$$

In our case the auxiliary field $\bar{\mathsf{F}}$ is not independent and must be expressed in terms of $\langle\mathsf{D}\rangle$ using the constraint (2.4.4). We recall that, according to (2.3.38), the left-over supersymmetry is a linear combination of the $\mathcal{N}=2$ supersymmetries, whose charge is thus written as

$$S = c_1Q + c_2\tilde{Q}\,, \tag{2.4.32}$$

with eq. (2.3.37) giving the ratio r of the coefficients as

$$r = \frac{c_2}{c_1} = \frac{iY_3}{Y_2 - iY_1} = -\frac{ig^2\kappa\xi}{1+\sqrt{1+g^4\kappa^2\xi^2}}\,. \tag{2.4.33}$$

Again, one can check that indeed the supersymmetry transformation associated with S leaves the fermion invariant. We therefore see that the FI term does not break the supersymmetry in the DBI action. Instead, it rotates the supercharges in the $\mathcal{N}=2$ space by an angle φ such that

$$\tan\varphi = |r| = \frac{g^2\kappa\xi}{1+\sqrt{1+g^4\kappa^2\xi^2}}, \qquad S = \cos\varphi\, Q - i\sin\varphi\,\tilde{Q}. \tag{2.4.34}$$

SUSY breaking in deformed DBI We turn to the study of SUSY breaking in the case of the DBI action with general deformation. As in the above case, from (2.4.4) we can solve the auxiliary field in X in terms of the auxiliary field in W

$$-\mathsf{F}e^{-i\phi} = \kappa\mathsf{D}^2 + 4\kappa^3\frac{\mathsf{D}^2\bar{\mathsf{D}}^2}{1+a+\sqrt{1+2a-b^2}}\,, \tag{2.4.35}$$

where

$$\mathsf{D} = d + i\gamma\,, \quad \bar{\mathsf{D}} = d - i\gamma\,, \qquad a = -4\kappa^2(d^2-\gamma^2)\,, \quad b = 8\kappa^2 d\gamma\,. \tag{2.4.36}$$

More explicitly, the F, $\bar{\mathsf{F}}$ and d solutions can be deduced from eq. (2.4.21) and their VEVs are

$$\mathsf{F} = -e^{-i\phi}\frac{1+8i\kappa^2 d\gamma - \sqrt{(1-8d^2\kappa^2)(1+8\kappa^2\gamma^2)}}{4\kappa}\,, \qquad \bar{\mathsf{F}} = F^* \tag{2.4.37}$$

$$d = -\frac{\gamma g^2\theta}{2\sqrt{2}\sqrt{\gamma^2\kappa^2\,(g^4\theta^2+64\pi^4)+8\pi^4}}\,. \tag{2.4.38}$$

This enables us to construct the $\boldsymbol{Y}$ vector defined in eq. (2.3.26) which reads

$$\begin{aligned}\boldsymbol{Y} &= \Big(\frac{\mathsf{F}-\bar{\mathsf{F}}}{2i} - \frac{1}{4i\kappa}e^{i\phi}, \frac{\mathsf{F}+\bar{\mathsf{F}}}{2} + \frac{1}{4\kappa}e^{i\phi}, \frac{d+i\gamma}{\sqrt{2}}\Big)\\ &= \Big(\frac{-\sqrt{(1-8d^2\kappa^2)(1+8\kappa^2\gamma^2)}\sin\phi + (i-8d\kappa^2\gamma)\cos\phi}{4\kappa},\\ &\qquad \frac{\sqrt{(1-8d^2\kappa^2)(1+8\kappa^2\gamma^2)}\cos\phi + (i-8d\kappa^2\gamma)\sin\phi}{4\kappa}, \frac{d+i\gamma}{\sqrt{2}}\Big)\,.\end{aligned} \tag{2.4.39}$$

and we can check that it verifies

$$\boldsymbol{Y}\cdot\boldsymbol{Y} = 0\,. \tag{2.4.40}$$

As explained in subsection 2.3.1, this implies that there is always a residual $\mathcal{N}=1$ supersymmetry. However, one sees that the following $SU(2)_R$ invariant quantity is not zero

$$\boldsymbol{Y}\cdot\boldsymbol{Y}^* = \frac{1}{8\kappa^2} + \gamma^2 = \frac{1}{8\kappa\tilde{\kappa}}\,, \tag{2.4.41}$$

and that it defines the partial SUSY breaking scale of the theory. The unbroken supersymmetry can be worked out as before

$$S = \cos\varphi\, Q + \sin\varphi\,\tilde{Q}\,, \quad \tan\varphi = |r| = \frac{\sqrt{1+8\gamma^2\kappa^2}-\sqrt{1-8d^2\kappa^2}}{\sqrt{1+8\gamma^2\kappa^2}+\sqrt{1-8d^2\kappa^2}}\,. \tag{2.4.42}$$

Note that all the possible phase factors have already been absorbed into the definition of supercharges.

Thus, we see that we can only partially break the supersymmetry in $\mathcal{N}=2$. In order to break the supersymmetry completely, we need to consider multiple DBIs corresponding to several $U(1)$s, just like what we discussed in the generalized APT model. In fact, the situation is similar to D-branes in string theory whose low energy effective action (for a single D-brane) is the supersymmetric DBI, where half of the bulk supersymmetries broken by the D-brane are realized non-linearly on the world-volume. When the bulk has $\mathcal{N}=2$, for instance in type II superstring compactified on a Calabi-Yau threefold, the worldvolume theory has one linear and one non-linear supersymmetry, as in our case of study. A constant magnetic field along the internal directions induces an FI term that one would naively expect to break the linear supersymmetry. However, in the absence of other branes or orientifolds, the magnetic field just rotates the direction of linear supersymmetry or equivalently, upon T-duality, it rotates the brane. In order to realize complete supersymmetry breaking, one has to consider a system of at least two magnetized branes, or equivalently branes at angles in the T-dual version [37, 38, 39].

Fermionic part As we have seen above, the bosonic part of the deformed DBI action takes the standard form of the DBI action after eliminating the auxiliary field. The only role of the deformations is to renormalize the coupling constants. This is quite similar to

the standard DBI+FI model. So purely from the bosonic sector viewpoint, it seems that our deformed DBI is the same as the standard DBI+FI model. In order to find a possible difference, we should also analyze the fermionic part of the action.

The most straightforward way to consider the fermionic contributions is to directly expand the superfields from the (2.4.7) action [2]. This will be shown in section 2.4.3. Instead, we try here to make use of the non-linear supersymmetry formalism described in section 2.2.2. We will particularly use it in the form presented in [18], where the authors found that in the standard DBI+FI model, the FI parameter generates an extra term besides renormalizing the coupling constants. It is exactly this extra term that is responsible for the gauging of R-symmetry when coupled to supergravity [40]. We will use this non-linear supersymmetry formalism to obtain the fermionic part of the deformed DBI action. A first analysis indicates that the extra term arising from the FI parameter does not appear and all deformations can be absorbed in the parameters of the standard DBI, exactly as for the bosonic part. This suggests that if we couple the deformed DBI action to supergravity, it may not be necessary to gauge the R-symmetry.

Let us first review the non-linear supersymmetry formalism elaborated in [18], related to the description given in section 2.2.2. Consider a Lagrangian of the type

$$\mathcal{L} = F^X + \bar{F}^{\bar{X}}, \tag{2.4.43}$$

transforming non-linearly as

$$\delta\mathcal{L} = \delta F^X + \delta\bar{F}^{\bar{X}} = -2i\partial_a(\chi\sigma^a\bar{\epsilon}F^X) - 2i\partial_a(\epsilon\sigma^a\bar{\chi}\bar{F}^{\bar{X}}). \tag{2.4.44}$$

Here χ_α is the goldstino in the chiral basis, transforming in the following way

$$\delta\chi_\alpha = \epsilon_\alpha - 2i\chi\sigma^\mu\bar{\epsilon}\partial_\mu\chi_\alpha\ . \tag{2.4.45}$$

This "chiral" goldstino χ_α is related via a field redefinition [41] to the Volkov-Akulov (VA) goldstino, that we denote ψ_α, contrary to section 2.2.2, in order to avoid confusion with the gaugino λ_α. The result of [18] is that up to boundary terms we can rewrite (2.4.43) as

$$\mathcal{L} = \det A(\mathcal{B} + \bar{\mathcal{B}}), \qquad \mathcal{B} = e^{\delta_\epsilon} F^X\Big|_{\epsilon=-\psi}. \tag{2.4.46}$$

We recall that $\boldsymbol{A}$ was defined in eq. (2.2.11) and that $\det \boldsymbol{A}$ is proportional to the Volkov-Akulov Lagrangian. We now apply this formalism to obtain the fermionic parts of the SDBI Lagrangian coupled to standard or γ-induced FI terms.

Standard SDBI+FI The standard DBI Lagrangian was constructed in eq. (2.4.6) from the constrained X field. Its auxiliary field $F^X = -(\frac{1}{2\kappa} + \frac{1}{4}D^2X|)$ indeed transforms in the proper way (2.4.44), as shown in [42, 18] and we can thus apply the above formalism. Hence the Lagrangian eq. (2.4.6) can be rewritten as

$$\mathcal{L} = \frac{1}{4\kappa g^2}\left(\frac{1}{\kappa} + F^X + \bar{F}^X\right) = \frac{1}{4\kappa g^2}\left(\frac{1}{\kappa} + \det \boldsymbol{A}(\mathcal{B} + \bar{\mathcal{B}})\right), \tag{2.4.47}$$

where

$$\begin{aligned}\mathcal{B}+\bar{\mathcal{B}} &= e^{\delta_\epsilon^*}(F^X+\bar{F}^X)\Big|_{\epsilon=-\psi} = e^{\delta_\epsilon^*}\Big[(F^X+\bar{F}^X)_{\text{bosonic}}\Big]\Big|_{\epsilon=-\psi}\\ &= e^{\delta_\epsilon^*}\frac{1}{2\kappa}\Big[-2+1-\sqrt{1+4\kappa^2\Big(F^2-2\mathrm{D}^2\Big)-4\kappa^4\Big(F\tilde{F}\Big)^2}\Big]\Big|_{\epsilon=-\psi}\\ &= \frac{1}{2\kappa}\Big[-1-\sqrt{1+4\kappa^2\Big(\mathcal{F}^2-2\mathcal{D}^2\Big)-4\kappa^4\Big(\mathcal{F}\tilde{\mathcal{F}}\Big)^2}\Big]\,. \end{aligned} \tag{2.4.48}$$

Note that in the second equality, we used the property that the gaugino λ is related to the VA goldstino through a field redefinition of the form $\lambda_\alpha = \psi_\alpha + \cdots$ and thus satisfies $e^{\delta_\epsilon^*}\lambda|_{\epsilon=-\psi} = 0$. The rules to implement the operation $e^{\delta_\epsilon^*}$ can be found in [18]. We also introduced the following quantities

$$\begin{aligned}&\mathcal{D} = e^{\delta_\epsilon^*}\mathrm{D}|_{\epsilon=-\psi}, \qquad \mathcal{F}_{ab} = (\boldsymbol{A}^{-1})^\mu_a(\boldsymbol{A}^{-1})^\nu_b(\partial_\mu u_\nu - \partial_\nu u_\mu),\\ &u_\mu = \boldsymbol{A}^a_\mu e^{\delta_\epsilon^*}v_a|_{\epsilon=-\psi} = v_\mu + O(\lambda^2),\end{aligned} \tag{2.4.49}$$

where v_a is the $U(1)$ gauge field. Here $\mathcal{D}$ should be regarded as the new auxiliary field although it is composite. Plugging eq. (2.4.48) in the general expression (2.4.46) we deduce the standard supersymmetric DBI Lagrangian written in the non-linear supersymmetry formalism

$$\mathcal{L}_{\text{SDBI}} = \frac{1}{8g^2\kappa^2}\left(2-\det\boldsymbol{A}\left[1+\sqrt{1+4\kappa^2\Big(\mathcal{F}^2-2\mathcal{D}^2\Big)-4\kappa^4\Big(\mathcal{F}\tilde{\mathcal{F}}\Big)^2}\right]\right). \tag{2.4.50}$$

Note the constant term proportional to $\det\boldsymbol{A}$, in agreement with [36] and an updated version of [18]. In the next section we show eq. (2.4.50) by a direct computation of the DBI action expanded up to terms of dimension eight [2].

To obtain the coupling of the SDBI Lagrangian to an FI term, we proceed as usual and add the FI Lagrangian (2.1.24) that we parametrize here as in (2.4.27). The non-linear supersymmetry formalism methods used to rewrite the FI Lagrangian are more involved than for the SDBI part, because the Lagrangian is the auxiliary field of a chiral superfield. Nevertheless, upon decomposing the real superfield V into several constrained chiral superfields and making use of their properties, it was shown in [18] that the FI term can be rewritten as

$$\begin{aligned}\mathcal{L}_{\text{FI}} &= \frac{1}{2\sqrt{2}}\xi\det\boldsymbol{A}\,\mathcal{D}\\ &\quad-\frac{i}{\sqrt{2}}\xi\det\boldsymbol{A}\,\epsilon^{abcd}[(\boldsymbol{A}^{-1})^\nu_a\partial_\nu\psi]\sigma_b[(\boldsymbol{A}^{-1})^\eta_c\partial_\eta\bar{\psi}](\boldsymbol{A}^{-1})^\mu_d u_\mu\,.\end{aligned} \tag{2.4.51}$$

After eliminating the auxiliary field $\mathcal{D}$ the total SDBI+FI Lagrangian then reads

$$\begin{aligned}\mathcal{L}_{\text{SDBI+FI}} &= \frac{1}{8g^2\kappa^2}\left[2-\det\boldsymbol{A}\left(1+\sqrt{1+g^4\kappa^2\xi^2}\sqrt{1+4\kappa^2\mathcal{F}^2-4\kappa^4\big(\mathcal{F}\tilde{\mathcal{F}}\big)^2}\right)\right]\\ &\quad-\frac{i}{\sqrt{2}}\xi\det\boldsymbol{A}\cdot\epsilon^{abcd}[(\boldsymbol{A}^{-1})_a{}^n\partial_n\psi]\sigma_b[(\boldsymbol{A}^{-1})_c{}^k\partial_k\bar{\psi}](\boldsymbol{A}^{-1})_d{}^m u_m\,.\end{aligned} \tag{2.4.52}$$

The second line is responsible for R-symmetry gauging when coupled to supergravity [18]. Indeed in the supergravity framework $(\boldsymbol{A}^{-1})^\mu_a\partial_\mu\psi^\alpha$ is replaced by $\hat{D}_a\psi^\alpha =$

$e_a^\mu D_\mu \psi^\alpha - \frac{1}{2M_P}\Psi_a^\alpha + \dots$ where Ψ denotes the gravitino. The second line of (2.4.52) generates the coupling $-\frac{i}{4\sqrt{2}}\frac{\xi}{M_P^2}\epsilon^{klmn}\Psi_k\sigma_l\bar{\Psi}_m v_n$ which charges the gravitino under the $U(1)$ gauge and indicates the gauging of R-symmetry.

Deformed SDBI We now turn to the case of the γ-deformed SDBI case. We would like to use the same non-linear formalism as in the previous case to rewrite the deformed action (2.4.6)

$$\mathcal{L}_{\text{SDBI}+\gamma,\theta} = \frac{1}{8\pi\kappa}\,\text{Im}\left(\tau\int d^2\theta X\right) = \frac{\tau}{16\pi\kappa i}(-\frac{1}{4}D^2X|) + c.c.\,, \tag{2.4.53}$$

in terms of the new variables. Although, naively, it seems that he non-linear supersymmetry formalism applies in the presence of the γ deformation, a more careful analysis is the scope of the next part. There, explicit calculations are made to check this assumption and clarify the difference between the deformed DBI and the DBI+FI actions [2]. Assuming for the moment that the non-linear supersymmetry formalism indeed applies also in the presence of the γ-deformation and observing that

$$F^X = -\frac{\tau}{16\pi\kappa i}\Big(\frac{1}{2\kappa} + \frac{1}{4}D^2X|\Big) \tag{2.4.54}$$

transforms in the way like (2.4.44), we rewrite the Lagrangian as

$$\mathcal{L}_{\text{SDBI}+\gamma,\theta} = \frac{1}{4\kappa^2 g^2} + F^X + \bar{F}^X = \frac{1}{4\kappa^2 g^2} + \det A(\mathcal{B}+\bar{\mathcal{B}})\,. \tag{2.4.55}$$

We must thus compute

$$\begin{aligned}
\mathcal{B}+\bar{\mathcal{B}} &= e^{\delta_\epsilon^*}(F^X+\bar{F}^X)\Big|_{\epsilon=-\psi} = e^{\delta_\epsilon^*}\Big[(F^X+\bar{F}^X)_{\text{bosonic}}\Big]\Big|_{\epsilon=-\psi} \\
&= e^{\delta_\epsilon^*}\Bigg[\frac{1}{8g^2\kappa^2}\left(1-\sqrt{1+4\kappa^2\Big(F^2-2(d^2-\gamma^2)\Big)-4\kappa^4\Big(F\tilde{F}-4d\gamma\Big)^2}\right) \\
&\qquad -\frac{1}{4\kappa^2 g^2} - \frac{\theta}{32\pi^2}\Big(F\tilde{F}-4d\gamma\Big)\Bigg]\Bigg|_{\epsilon=-\psi} \\
&= \frac{1}{8g^2\kappa^2}\left(-1-\sqrt{1+4\kappa^2\Big(\mathcal{F}^2-2(\boldsymbol{d}^2-\gamma^2)\Big)-4\kappa^4\Big(\mathcal{F}\tilde{\mathcal{F}}-4\boldsymbol{d}\gamma\Big)^2}\right) \\
&\qquad -\frac{\theta}{32\pi^2}\Big(\mathcal{F}\tilde{\mathcal{F}}-4\boldsymbol{d}\gamma\Big)\,.
\end{aligned} \tag{2.4.56}$$

The deformed D auxiliary field was dressed through

$$\boldsymbol{d} = e^{\delta_\epsilon^*}d|_{\epsilon=-\psi}, \qquad \gamma = e^{\delta_\epsilon^*}\gamma|_{\epsilon=-\psi}, \tag{2.4.57}$$

where $\boldsymbol{d}$ is the new composite auxiliary field and γ being a constant, it is not modified. The complete result thus takes the form

$$\begin{aligned}\mathcal{L} &= \frac{1}{4\kappa^2 g^2} + \det \boldsymbol{A}(\mathcal{B}+\bar{\mathcal{B}}) \\ &= \frac{1}{4\kappa^2 g^2} + \frac{\det \boldsymbol{A}}{8g^2\kappa^2}\left(-1-\sqrt{1+4\kappa^2\Big(\mathcal{F}^2-2(\boldsymbol{d}^2-\gamma^2)\Big)-4\kappa^4\Big(\mathcal{F}\tilde{\mathcal{F}}-4\boldsymbol{d}\gamma\Big)^2}\right) \\ &\quad + \det \boldsymbol{A}\,\frac{\theta \boldsymbol{d}\gamma}{8\pi^2} - \frac{\theta}{32\pi^2}\det \boldsymbol{A}\,\mathcal{F}\tilde{\mathcal{F}}\,. \end{aligned} \tag{2.4.58}$$

Using the definition $\mathcal{F}_{ab} = (\boldsymbol{A}^{-1})^m_a(\boldsymbol{A}^{-1})^n_b f_{mn}$, with $f_{mn} = \partial_m u_n - \partial_n u_m$ the standard field strength of u_n, together with relation (2.2.13) for $\boldsymbol{A}^{-1}$, the last term can be written as

$$\begin{aligned}\det \boldsymbol{A}\,\mathcal{F}\tilde{\mathcal{F}} &= \det \boldsymbol{A}\,\frac{1}{2}\epsilon^{abcd}\mathcal{F}_{ab}\mathcal{F}_{cd} \\ &= \frac{1}{2}\det \boldsymbol{A}\,\epsilon^{abcd}(\boldsymbol{A}^{-1})^m_a(\boldsymbol{A}^{-1})^n_b(\boldsymbol{A}^{-1})^k_c(\boldsymbol{A}^{-1})^l_d f_{mn}f_{kl} \\ &= \frac{1}{2}\epsilon^{mnkl}f_{mn}f_{kl} = f\tilde{f}\,. \end{aligned} \tag{2.4.59}$$

As this term is a total derivative it can be dropped from the Lagrangian, which after elimination of the auxiliary field $\boldsymbol{d}$ becomes

$$\mathcal{L}_{\text{SDBI}+\gamma,\theta} = \frac{1}{8\kappa^2 g^2}\left[2-\det \boldsymbol{A}\left(1+\frac{\kappa}{\tilde{\kappa}}\sqrt{1+\frac{\theta^2 g^4\gamma^2\tilde{\kappa}^2}{8\pi^4}}\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\tilde{\kappa}\mathcal{F}_{\mu\nu}\right)}\right)\right]. \tag{2.4.60}$$

We see that the second term in (2.4.52) does not appear here, suggesting that there is no need to gauge the R-symmetry in order to couple to supergravity. Thus this case with deformation seems different from the SDBI+FI model.

2.4.2 On-shell Lagrangians and the use of field redefinitions

The goal of the next two parts is to understand if the physical SDBI fermionic contributions are indeed different in the case with standard FI term, denoted SDBI+FI, and in the case with γ-induced FI term, denoted SDBI+γ,θ.

The main strategy is to compute the components of the Lagrangians described in the previous parts. We then use field redefinitions to eliminate various on-shell vanishing unphysical terms and obtain the physical on-shell higher dimensional operators. The S-matrix is invariant under field redefinitions. Thus, to eliminate an unphysical term of certain dimension, we use a specific field redefinition and act it on terms of lower dimension. However, the field redefinition also acts on other terms in the Lagrangian, giving rise to many extra higher dimensional terms. Repeating this procedure allows us to eliminate all unphysical terms and get the on-shell Lagrangian. In general, this process is complicated and tedious.

Before performing the computations, it is worth pointing out a big simplification. At any step, we will only be interested in the physical Lagrangian up to some dimension, say dimension ℓ, and thus will always neglect terms with dimension higher than ℓ. The simplification occurs if the term $\mathcal{O}$ under consideration is proportional to an equation of motion of the free theory. In such a case, we must be able to eliminate $\mathcal{O}$ through certain field redefinition acting on the free kinetic terms. If the dimension of $\mathcal{O}$ is close to ℓ,

acting the field redefinition on other terms of the Lagrangian may only generate terms with dimension strictly higher than ℓ. If this is indeed the case, we do not need to work out the field redefinition explicitly and can simply discard the term $\mathcal{O}$. This circumstance brings us a big simplification.

2.4.3 Redefining the SDBI+FI Lagrangian

SDBI+FI from non-linear supersymmetry formalism The non-linear supersymmetry formalism was applied to the SDBI+FI model in the previous parts to obtain the SDBI+FI Lagrangian (2.4.52). For convenience we rewrite it here dropping the constant term

$$\begin{aligned}\mathcal{L}_{\text{SDBI+FI}} = &-\frac{1}{8\kappa^2 g^2}\det \boldsymbol{A}\left(1+\sqrt{1+16g^4\kappa^2\xi^2}\sqrt{1+4\kappa^2\mathcal{F}^2+4\kappa^4(\mathcal{F}\tilde{\mathcal{F}})^2}\right)\\ &+2\sqrt{2}i\,\kappa^2\xi\,\det \boldsymbol{A}\epsilon^{abcd}[(\boldsymbol{A}^{-1})^{\nu}_{a}\partial_\nu\lambda]\sigma_b[(\boldsymbol{A}^{-1})^{\rho}_{c}\partial_\rho\bar{\lambda}](\boldsymbol{A}^{-1})^{\mu}_{d}u_\mu\,. \end{aligned} \tag{2.4.61}$$

Here λ is the goldstino in the non-linear supersymmetry formalism, which is also the gaugino in the linear one, through field redefinition. We use the same symbol λ to denote both the goldstino in this section and the gaugino in section 2.4.4 for simplicity of notation. As explained in the previous part, u_μ is the gauge boson, and the composite gauge field strength $\mathcal{F}_{ab}$ is defined by

$$\mathcal{F}_{ab} = (\boldsymbol{A}^{-1})^{\mu}_{a}(\boldsymbol{A}^{-1})^{\nu}_{b}F_{\mu\nu},\quad F_{\mu\nu}=\partial_\mu u_\nu-\partial_\nu u_\mu,\quad \mathcal{F}^2\equiv\mathcal{F}_{ab}\mathcal{F}^{ab},\quad \mathcal{F}\tilde{\mathcal{F}}\equiv\mathcal{F}_{ab}\tilde{\mathcal{F}}^{ab}, \tag{2.4.62}$$

where again $\boldsymbol{A}$ is the vielbein matrix defined in (2.2.11).

We mention here that the two terms in the parenthesis of (2.4.61) have the same sign, in agreement with [36] but in contrast to the opposite sign in [18]. The coupling constants are related to those of [18] as follows: $\kappa = 1/(2\sqrt{2}m)$ and $g^2 = m/\beta$. The dual tensor $\tilde{\mathcal{F}}$ is defined as in (A.2.1) with a factor of i.

As shown in appendix B.1, the Lagrangian (2.4.61) can be further rewritten in a manifestly gauge invariant way as

$$\begin{aligned}\mathcal{L}_{\text{SDBI+FI}} = &-\frac{1}{8\kappa^2 g^2}\det \boldsymbol{A}\left(1+\sqrt{1+16g^4\kappa^2\xi^2}\sqrt{1+4\kappa^2\mathcal{F}^2+4\kappa^4(\mathcal{F}\tilde{\mathcal{F}})^2}\right)\\ &-2\sqrt{2}\,\kappa^2\xi\,\lambda\sigma_\gamma\partial_\rho\bar{\lambda}\,\tilde{F}^{\gamma\rho}\,. \end{aligned} \tag{2.4.63}$$

We will expand in components each part of the Lagrangian (2.4.63), up to terms of dimension 12 – except for dimension-12 terms of the form $(\lambda\partial\bar{\lambda})^2F^2$ which are more involved. In the following, all equalities are written up to total derivatives, or terms with mass dimension higher than 12, or $(\lambda\partial\bar{\lambda})^2F^2$ terms. In order to facilitate the mass-dimension counting of the various operators, we recall here the mass dimensions of various fields and couplings

$$[u]=1\,,\qquad [F]=2\,,\qquad [\lambda]=\frac{3}{2}\,,\qquad [\kappa]=-2\,,\qquad [g]=0\,,\qquad [\xi]=2\,. \tag{2.4.64}$$

The $\det \boldsymbol{A}$ part of (2.4.61) is related to the Volkov-Akulov action [14] describing goldstino dynamics, already presented in (2.2.14). When expanding $\mathcal{F}^2$ in (2.4.61), one finds the

gauge boson kinetic term and higher order interactions

$$\begin{aligned}\mathcal{F}^2 &= \eta^{ac}\eta^{bd}\mathcal{F}_{ab}\mathcal{F}_{cd} = \eta^{ac}\eta^{bd}(\boldsymbol{A}^{-1})^{\mu}_{a}(\boldsymbol{A}^{-1})^{\nu}_{b}(\boldsymbol{A}^{-1})^{\rho}_{c}(\boldsymbol{A}^{-1})^{\sigma}_{d}F_{\mu\nu}F_{\rho\sigma}\\ &= \eta^{ac}\eta^{bd}F_{\mu\nu}F_{\rho\sigma}\delta^{\mu}_{a}\delta^{\nu}_{b}\delta^{\rho}_{c}\delta^{\sigma}_{d} + 4\kappa^2\eta^{ac}\eta^{bd}F_{\mu\nu}F_{\rho\sigma}(-i\lambda\sigma^{\mu}\partial_a\bar{\lambda} + i\partial_a\lambda\sigma^{\mu}\bar{\lambda})\delta^{\nu}_{b}\delta^{\rho}_{c}\delta^{\sigma}_{d} + \cdots\\ &= F^2 + 4\kappa^2(i\lambda\sigma^{\mu}\partial_{\rho}\bar{\lambda} - i\partial_{\rho}\lambda\sigma^{\mu}\bar{\lambda})F_{\mu\nu}F^{\nu\rho} + \cdots\,. \end{aligned} \tag{2.4.65}$$

The same computation for $(\mathcal{F}\tilde{\mathcal{F}})^2$ gives

$$\begin{aligned}(\mathcal{F}\tilde{\mathcal{F}})^2 &= \left(F\tilde{F} + 4\kappa^2(i\lambda\sigma^{\mu}\partial_{\rho}\bar{\lambda} - i\partial_{\rho}\lambda\sigma^{\mu}\bar{\lambda})F_{\mu\nu}\tilde{F}^{\nu\rho} + O((\lambda\partial\bar{\lambda})^2F^2)\right)^2\\ &= \left(F\tilde{F} - \kappa^2(i\lambda\partial\!\!\!/\bar{\lambda} + i\bar{\lambda}\bar{\partial}\!\!\!/\lambda)F\tilde{F} + O((\lambda\partial\bar{\lambda})^2F^2)\right)^2\\ &= (F\tilde{F})^2 - 2\kappa^2(i\lambda\partial\!\!\!/\bar{\lambda} + i\bar{\lambda}\bar{\partial}\!\!\!/\lambda)(F\tilde{F})^2 + O((\lambda\partial\bar{\lambda})^2F^2)\,. \end{aligned} \tag{2.4.66}$$

This expansion is also obtained directly by noticing that $\mathcal{F}\tilde{\mathcal{F}} = \det \boldsymbol{A}^{-1}F\tilde{F}$. Collecting all the above terms, the SDBI+FI Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\text{SDBI+FI}} = A\Big[&1 + \kappa^2 i\lambda\partial\!\!\!/\bar{\lambda} + \kappa^2 i\bar{\lambda}\bar{\partial}\!\!\!/\lambda - \kappa^4(\lambda\partial\!\!\!/\bar{\lambda})(\bar{\lambda}\bar{\partial}\!\!\!/\lambda) - 2\kappa^4(\lambda\partial\!\!\!/\bar{\lambda})^2 - 2\kappa^4(\bar{\lambda}\bar{\partial}\!\!\!/\lambda)^2\\ &+\kappa^4\bar{\lambda}^2\lambda\Box\lambda + \kappa^4\lambda^2\bar{\lambda}\Box\bar{\lambda} + \kappa^4\partial_{\mu}(\lambda^2)\partial^{\mu}(\bar{\lambda}^2)\Big] + 2B\kappa^4(i\lambda\partial\!\!\!/\bar{\lambda} + i\bar{\lambda}\bar{\partial}\!\!\!/\lambda)F^2\\ &+ 8B\kappa^4\left(i\lambda\sigma^{\mu}\partial_{\rho}\bar{\lambda} - i\partial_{\rho}\lambda\sigma^{\mu}\bar{\lambda}\right)F_{\mu\nu}F^{\nu\rho} - 16B\kappa^6\left(i\lambda\sigma^{\mu}\partial_{\rho}\bar{\lambda} - i\partial_{\rho}\lambda\sigma^{\mu}\bar{\lambda}\right)F_{\mu\nu}F^{\nu\rho}F^2\\ &+ 2B\kappa^2\left(F^2 + \kappa^2(F\tilde{F})^2 - \kappa^2F^4\right) + C\kappa^2\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}\,\tilde{F}_{\mu\nu} + \cdots\,. \end{aligned} \tag{2.4.67}$$

The constants A, B and C are defined as

$$\begin{aligned}A &\equiv -\frac{1}{8\kappa^2g^2}\left(1 + \sqrt{1 + 16g^4\kappa^2\xi^2}\right),\\ B &\equiv -\frac{1}{8\kappa^2g^2}\sqrt{1 + 16g^4\kappa^2\xi^2},\\ C &\equiv -2\sqrt{2}\xi\,, \end{aligned} \tag{2.4.68}$$

and have mass dimensions $[A] = [B] = 4$ and $[C] = 2$.

Physical action with standard FI term As explained in section 2.4.2, we will proceed as follows to obtain the physical SDBI+FI action: we first eliminate the lowest dimensional non-physical operators, namely the dimension-6 ones, by means of field redefinitions acting on kinetic terms. We then compute the higher dimensional contributions coming from the field redefinitions acting on the other terms in the Lagrangian. We repeat this procedure for operators with higher and higher dimensions.

In the computations, we will make full use of the identities given in appendix A.1. In all equalities thereafter, ellipses " $\cdots$ " should be understood as total derivatives or higher dimensional terms which we are not interested in.

Field redefinition ① To eliminate the dimension-6 term contained in the last line of (2.4.67) we apply the following field redefinition

$$① \quad \lambda_{\alpha} \to \lambda_{\alpha} + ia(\sigma^{\mu\nu}\lambda)_{\alpha}F_{\mu\nu}, \qquad a = \frac{C}{4A} \in \mathbb{R}, \quad [a] = -2. \tag{2.4.69}$$

Note that due to the equality $\sigma^{\rho\gamma} = \frac{i}{2}\epsilon^{\rho\gamma\mu\nu}\sigma_{\mu\nu}$, this field transformation is equivalent to the one with $F_{\mu\nu}$ replaced by $\tilde{F}_{\mu\nu}$. The fermion kinetic terms transform as

$$\begin{aligned} i\lambda\not{\partial}\bar{\lambda} + \text{c.c.} \underset{(2.4.69)}{\longrightarrow} \; & i\lambda\not{\partial}\bar{\lambda} + \text{c.c.} - 4a\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}\tilde{F}_{\mu\nu} - 2a\lambda\sigma^{\mu}\bar{\lambda}\partial^{\nu}F_{\nu\mu} + 2a^2 i\lambda\sigma^{\mu}\bar{\lambda}\tilde{F}_{\nu\mu}\partial_{\rho}F^{\rho\nu} \\ & + \left(2a^2 i\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}F_{\mu\rho}F^{\rho}_{\ \nu} + \frac{a^2}{2}i\lambda\not{\partial}\bar{\lambda}F^2 + \text{c.c.}\right) + \text{total derivatives}\,, \end{aligned} \tag{2.4.70}$$

and they indeed cancel the dimension-6 terms of (2.4.67) with the chosen parameter a. The dimension-6 term itself transforms as

$$\begin{aligned} \lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}\tilde{F}_{\mu\nu} \underset{(2.4.69)}{\longrightarrow} \; & \frac{1}{2}\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}\tilde{F}_{\mu\nu} - ia\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}F_{\mu\rho}F^{\rho}_{\ \nu} - \frac{a}{2}i\lambda\not{\partial}\bar{\lambda}F^2 - \frac{a}{4}i\lambda\not{\partial}\bar{\lambda}F\tilde{F} \\ & + \frac{a^2}{4}\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}\left(\tilde{F}_{\mu\nu}F^2 - F_{\mu\nu}F\tilde{F} + \tilde{F}_{\mu\nu}F\tilde{F} - F_{\mu\nu}F^2\right) + \text{c.c.} + \cdots\,. \end{aligned} \tag{2.4.71}$$

Other terms in the Lagrangian transform as

$$\begin{aligned} i\lambda\sigma^{\mu}\partial_{\nu}\bar{\lambda}F_{\mu\rho}F^{\rho\nu} + \text{c.c.} \underset{(2.4.69)}{\longrightarrow} \; & i\lambda\sigma^{\mu}\partial_{\nu}\bar{\lambda}F_{\mu\rho}F^{\rho\nu} - a\lambda\sigma^{\mu}\bar{\lambda}F_{\alpha\nu}F^{\nu\rho}\partial_{\rho}F^{\alpha}_{\ \mu} + \frac{a}{2}\lambda\sigma^{\mu}\partial^{\nu}\bar{\lambda}F_{\mu\nu}F\tilde{F} \\ & - i\frac{a^2}{4}\lambda\sigma^{\mu}\bar{\lambda}F^2\partial_{\mu}(F\tilde{F}) - \frac{a^2}{2}i\lambda\sigma^{\mu}\partial_{\nu}\bar{\lambda}F_{\mu\rho}F^{\rho\nu}F^2 + \text{c.c.} + \cdots\,, \end{aligned} \tag{2.4.72}$$

$$\bar{\lambda}^2\lambda\Box\lambda + \text{c.c.} \underset{(2.4.69)}{\longrightarrow} \bar{\lambda}^2\lambda\Box\lambda + 2ia\bar{\lambda}^2\lambda\sigma^{\mu\nu}\partial_{\rho}\lambda\,\partial^{\rho}F_{\mu\nu} + \text{c.c.} + \cdots\,. \tag{2.4.73}$$

Field redefinition ② Although the field redefinition (2.4.67) eliminates the original dimension-6 term in (2.4.67), it introduces another dimension-6 operator in (2.4.70). Hence, we must combine the field redefinition (2.4.67) with another field redefinition on the gauge boson

$$② \quad u_{\mu} \to u_{\mu} + b\lambda\sigma_{\mu}\bar{\lambda}\,, \qquad b = -\frac{C}{16B} \in \mathbb{R}, \qquad [b] = -2\,. \tag{2.4.74}$$

This is equivalent to the following field-strength redefinition

$$F_{\mu\nu} \to F_{\mu\nu} + b\partial_{\mu}(\lambda\sigma_{\nu}\bar{\lambda}) - b\partial_{\nu}(\lambda\sigma_{\mu}\bar{\lambda}) \equiv F_{\mu\nu} + 2b\partial_{[\mu}(\lambda\sigma_{\nu]}\bar{\lambda})\,. \tag{2.4.75}$$

The gauge boson kinetic term transforms as

$$\begin{aligned} F^2 \underset{(2.4.74)}{\longrightarrow} \; & F^2 - 4b\lambda\sigma^{\nu}\bar{\lambda}\partial^{\mu}F_{\mu\nu} + 4b^2\Big[\bar{\lambda}^2\lambda\Box\lambda + \lambda^2\bar{\lambda}\Box\bar{\lambda} \\ & + \frac{1}{2}\partial_{\mu}(\lambda^2)\partial^{\mu}(\bar{\lambda}^2) + (\lambda\not{\partial}\bar{\lambda})(\bar{\lambda}\bar{\not{\partial}}\lambda) - \frac{1}{2}\left((\lambda\not{\partial}\bar{\lambda})^2 + (\bar{\lambda}\bar{\not{\partial}}\lambda)^2\right)\Big] + \text{total derivatives}, \end{aligned} \tag{2.4.76}$$

and cancels the dimension-6 operator coming from (2.4.70). This field redefinition acts on other terms in the Lagrangian as follows

$$F^4 \underset{(2.4.74)}{\longrightarrow} F^4 + 2b\partial^\nu(\lambda\sigma^\mu\bar\lambda)F_{\nu\mu}F^2 + \cdots\,, \tag{2.4.77}$$

$$(F\tilde F)^2 \underset{(2.4.74)}{\longrightarrow} (F\tilde F)^2 + 2b\partial^\nu(\lambda\sigma^\mu\bar\lambda)\tilde F_{\nu\mu}F\tilde F + \cdots\,, \tag{2.4.78}$$

$$\lambda\sigma^\mu\bar\lambda\partial^\nu F_{\nu\mu} \underset{(2.4.74)}{\longrightarrow} \lambda\sigma^\mu\bar\lambda\partial^\nu F_{\nu\mu} - b\partial_\rho(\lambda^2)\partial^\rho(\bar\lambda^2) - b\partial_\mu(\lambda\sigma^\mu\bar\lambda)\partial_\rho(\lambda\sigma^\rho\bar\lambda) + \cdots\,, \tag{2.4.79}$$

$$i\lambda\sigma^\mu\bar\lambda\tilde F_{\nu\mu}\partial_\rho F^{\rho\nu} \underset{(2.4.74)}{\longrightarrow} i\lambda\sigma^\mu\bar\lambda\tilde F_{\nu\mu}\partial_\rho F^{\rho\nu} + \Big(bi\bar\lambda^2\lambda\sigma^{ab}\partial_\rho\lambda\,\partial^\rho F_{ab} + \text{c.c.}\Big) + \cdots\,, \tag{2.4.80}$$

$$i\lambda\sigma^\mu\partial_\rho\bar\lambda F_{\mu\nu}F^{\nu\rho} + \text{c.c.} \underset{(2.4.74)}{\longrightarrow} i\lambda\sigma^\mu\partial_\rho\bar\lambda F_{\mu\nu}F^{\nu\rho} + ib\bar\lambda^2\,\lambda\sigma^{ab}\partial^\mu\lambda\,\partial_\mu F_{ab} + \text{c.c.} + \cdots\,. \tag{2.4.81}$$

After applying the field redefinitions ① and ②, the dimension-6 terms are eliminated completely and the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_{\text{SDBI+FI}} \to\; & A\left(1 + \kappa^2 i\lambda\partial\!\!\!/\bar\lambda + \kappa^2 i\bar\lambda\bar\partial\!\!\!/\lambda\right) + \left(A\kappa^4 + 4Bb^2\kappa^2 + \frac{C}{2}b\kappa^2\right)\partial^\mu(\lambda^2)\partial_\mu(\bar\lambda^2) \\
& + \#\kappa^2(\lambda\partial\!\!\!/\bar\lambda)(\bar\lambda\bar\partial\!\!\!/\lambda) - \#\kappa^2(\lambda\partial\!\!\!/\bar\lambda)^2 - \#\kappa^2(\bar\lambda\bar\partial\!\!\!/\lambda)^2 + \#\kappa^2\bar\lambda^2\lambda\Box\lambda + \#\kappa^2\lambda^2\bar\lambda\Box\bar\lambda \\
& + 2Aa^2\kappa^2\, i\lambda\sigma^\mu\bar\lambda\tilde F_{\nu\mu}\partial_\rho F^{\rho\nu} + 2B\kappa^2\left(F^2 + \kappa^2(F\tilde F)^2 - \kappa^2F^4\right) \\
& + \left(2B\kappa^2 + \frac{Aa^2}{2} - \frac{Ca}{2}\right)\kappa^2 i\lambda\partial\!\!\!/\bar\lambda F^2 - \frac{Ca}{4}\kappa^2 i\lambda\partial\!\!\!/\bar\lambda F\tilde F + \text{c.c.} \\
& + \left(8B\kappa^2 - \frac{C^2}{8A}\right)\kappa^2 i\lambda\sigma^\mu\partial_\rho\bar\lambda F_{\mu\nu}F^{\nu\rho} + \text{c.c.} \\
& + 8Bb\kappa^4\lambda\sigma^\mu\partial^\nu\bar\lambda\;\left(\tilde F_{\mu\nu}F^2 + F_{\mu\nu}F^2 - \tilde F_{\mu\nu}F\tilde F - F_{\mu\nu}F\tilde F\right) + \text{c.c.}\,. \\
& - 16B\kappa^6(i\lambda\sigma^\mu\partial_\rho\bar\lambda - i\partial_\rho\lambda\sigma^\mu\bar\lambda)F_{\mu\nu}F^{\nu\rho}F^2 + 4Ba^2\kappa^4 i\lambda\sigma^\mu\bar\lambda F^2\partial_\mu(F\tilde F) + \cdots\,.
\end{aligned} \tag{2.4.82}$$

Note that the four-fermion/one-gauge-boson dimension-10 terms coming from (2.4.73), (2.4.80) and (2.4.81) cancel each other. They are thus absent in Lagrangian (2.4.82).

Field redefinition ③ The second line of (2.4.82) contains goldstino self interactions whose coefficients are not shown explicitly. They can actually be removed completely by applying the field redefinition

$$③ \quad \lambda_\alpha \to \lambda_\alpha + m\lambda_\alpha(\lambda\partial\!\!\!/\bar\lambda) - n\lambda_\alpha(\bar\lambda\bar\partial\!\!\!/\lambda) + p\lambda\sigma^\mu\bar\lambda\partial_\mu\lambda_\alpha\,, \quad [m] = [n] = [p] = 4\,, \tag{2.4.83}$$

Field redefinitions of this form were used [19] to demonstrate in components the on-shell equivalence of the goldstino Lagrangians mentioned in eq. (2.2.30). Under the above redefinition the kinetic terms transform as

$$\begin{aligned}
i\lambda\partial\!\!\!/\bar\lambda + i\bar\lambda\bar\partial\!\!\!/\lambda \underset{(2.4.83)}{\longrightarrow}\; & i\lambda\partial\!\!\!/\bar\lambda + i\bar\lambda\bar\partial\!\!\!/\lambda + 2i(m-p)(\lambda\partial\!\!\!/\bar\lambda)^2 - 2i(\bar m - \bar p)(\bar\lambda\bar\partial\!\!\!/\lambda)^2 + ip\lambda^2\bar\lambda\Box\bar\lambda \\
& - i\bar p\bar\lambda^2\lambda\Box\lambda - 2i(n - \bar n - p + \bar p)(\lambda\partial\!\!\!/\bar\lambda)(\bar\lambda\bar\partial\!\!\!/\lambda) \\
& + \text{total derivatives} + O(\lambda^6).
\end{aligned} \tag{2.4.84}$$

We see that there are enough parameters in (2.4.84) to cancel any four-fermion terms except for $\partial_\mu(\bar\lambda^2)\partial^\mu(\lambda^2)$ which is thus the only physical dimension-8 contribution to the

Volkov-Akulov Lagrangian. Under field redefinition ③, other terms in the Lagrangian generate dimension-12 terms of the form $\partial^2\lambda^4F^2$, or terms with dimension higher than 12.

Field redefinition ④ The first term in the third line of (2.4.82) is proportional to the equation of motion of a free gauge boson and thus can be eliminated. This is realized by using the field redefinition of the gauge boson

$$④ \quad u_\mu \to u_\mu + fi\lambda\sigma^\rho\bar{\lambda}\tilde{F}_{\mu\rho}, \qquad f = \frac{Aa^2}{4B}, \quad [f] = -4, \tag{2.4.85}$$

or equivalently the field redefinition of the gauge field strength

$$F_{\mu\nu} \to F_{\mu\nu} + 2f\partial_{[\mu}\left(i\lambda\sigma^\rho\bar{\lambda}\tilde{F}_{\nu]\rho}\right). \tag{2.4.86}$$

Under this redefinition the gauge boson kinetic term becomes

$$F^2 \underset{(2.4.86)}{\longrightarrow} F^2 - 4fi\lambda\sigma^\rho\bar{\lambda}\tilde{F}_{\nu\rho}\partial_\mu F^{\mu\nu} + \cdots, \tag{2.4.87}$$

and thus the second term cancels with the first term in the third line of (2.4.82). The field redefinition (2.4.85) also acts on other terms

$$F^4 \underset{(2.4.86)}{\longrightarrow} F^4 + 2fi\lambda\sigma^\mu\bar{\lambda}F^2\partial_\mu(F\tilde{F}) + \cdots, \tag{2.4.88}$$

$$(F\tilde{F})^2 \underset{(2.4.86)}{\longrightarrow} (F\tilde{F})^2 + 2fi\lambda\sigma^\mu\bar{\lambda}F^2\partial_\mu(F\tilde{F}) - 8fi\partial_\mu(\lambda\sigma^\nu\bar{\lambda})F^{\mu\rho}F_{\rho\nu}F\tilde{F} + \cdots. \tag{2.4.89}$$

Hence after applying the field redefinitions ③ and ④, the Lagrangian further reduces to

$$\begin{aligned}
\mathcal{L}_{\text{SDBI+FI}} \to\ & A\left(1 + \kappa^2 i\lambda\partial\!\!\!/\bar{\lambda} + \kappa^2 i\bar{\lambda}\bar{\partial}\!\!\!/\lambda\right) + \left(A\kappa^4 + 4Bb^2\kappa^2 + \frac{C}{2}b\kappa^2\right)\partial^\mu(\lambda^2)\partial_\mu(\bar{\lambda}^2) \\
& + 2B\kappa^2\left(F^2 + \kappa^2(F\tilde{F})^2 - \kappa^2F^4\right) + \left(8B\kappa^2 - \frac{C^2}{8A}\right)\kappa^2 i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho} + \text{c.c.} \\
& + \left(2B\kappa^2 + \frac{Aa^2}{2} - \frac{Ca}{2}\right)\kappa^2 i\lambda\partial\!\!\!/\bar{\lambda}F^2 + \text{c.c.} - \frac{Ca}{4}\kappa^2 i\lambda\partial\!\!\!/\bar{\lambda}F\tilde{F} + \text{c.c.} \\
& - 16B\kappa^6 i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho}F^2 + \text{c.c.} \\
& + 8Bb\,\kappa^4\lambda\sigma^\mu\partial^\nu\bar{\lambda}\ \left(\tilde{F}_{\mu\nu}F^2 + F_{\mu\nu}F^2 - \tilde{F}_{\mu\nu}F\tilde{F} - F_{\mu\nu}F\tilde{F}\right) + \text{c.c.} \\
& + \left(4Ba^2 + 8Bf\right)\kappa^4 i\lambda\sigma^\mu\bar{\lambda}F^2\partial_\mu(F\tilde{F}) - 16fB\kappa^4 i\partial_\nu(\lambda\sigma^\mu\bar{\lambda})F^{\nu\rho}F_{\rho\mu}F\tilde{F} + \cdots.
\end{aligned} \tag{2.4.90}$$

Field redefinition ⑤ The dimension-10 terms in the fifth line of (2.4.90) arrange in such a way that they are eliminated through the field redefinition

$$⑤ \quad \lambda_\alpha \to \lambda_\alpha + (\sigma^{\mu\nu}\lambda)_\alpha h\left(F_{\mu\nu}F^2 - F_{\mu\nu}F\tilde{F}\right), \quad h = -i\frac{4Bb}{A}\kappa^2 \in i\mathbb{R}, \quad [h] = -6. \tag{2.4.91}$$

Indeed, just like (2.4.69), one can replace $F_{\mu\nu}$ with $\tilde{F}_{\mu\nu}$ in (2.4.91), due to the identity $\sigma^{\rho\gamma} = \frac{i}{2}\epsilon^{\rho\gamma\mu\nu}\sigma_{\mu\nu}$. The goldstino kinetic terms transform under (2.4.91) as

$$\begin{aligned}
i\lambda\partial\!\!\!/\bar{\lambda} + i\bar{\lambda}\bar{\partial}\!\!\!/\lambda \underset{(2.4.91)}{\longrightarrow}\ & i\lambda\partial\!\!\!/\bar{\lambda} + i\bar{\lambda}\bar{\partial}\!\!\!/\lambda - 2\left(ih\lambda\sigma^\mu\partial^\nu\bar{\lambda} - i\bar{h}\partial^\nu\lambda\sigma^\mu\bar{\lambda}\right)\left(F_{\mu\nu}F^2 - \tilde{F}_{\mu\nu}F\tilde{F}\right) \\
& - 2\left(ih\lambda\sigma^\mu\partial^\nu\bar{\lambda} + i\bar{h}\partial^\nu\lambda\sigma^\mu\bar{\lambda}\right)\left(\tilde{F}_{\mu\nu}F^2 - F_{\mu\nu}F\tilde{F}\right),
\end{aligned} \tag{2.4.92}$$

and cancel exactly with the dimension-10 terms of (2.4.90). Acting (2.4.91) on other terms in the Lagrangian, we only get dimension-14 or dimension-16 terms.

Therefore, no dimension-10 operator survives in the physical on-shell Lagrangian.

Field redefinition ⑥ We are still left with the dimension-8 terms of the form $\lambda\not{\partial}\bar{\lambda}F\tilde{F}$ and $\lambda\not{\partial}\bar{\lambda}F^2$ in the third line of (2.4.90). The first can be eliminated through the field redefinition

$$⑥ \quad \lambda_\alpha \;\to\; \lambda_\alpha + c\lambda_\alpha F\tilde{F}, \qquad c = \frac{Ca}{8A} \in \mathbb{R}\,, \qquad [c] = -4\,, \tag{2.4.93}$$

which acts on the fermion kinetic terms as

$$\begin{aligned} i\lambda\not{\partial}\bar{\lambda} + \text{c.c.} \underset{(2.4.93)}{\longrightarrow} \; & i\lambda\not{\partial}\bar{\lambda} - ci\lambda\sigma^\mu\bar{\lambda}\partial_\mu(F\tilde{F}) - ic^2\lambda\sigma^\mu\partial_\mu(\lambda F\tilde{F})F\tilde{F} + \text{c.c.} \\ & = i\lambda\not{\partial}\bar{\lambda} + 2ci\lambda\not{\partial}\bar{\lambda}F\tilde{F} + \text{c.c.} + \cdots\,, \end{aligned} \tag{2.4.94}$$

and thus eliminates the dimension-8 terms containing $\lambda\not{\partial}\bar{\lambda}F\tilde{F}$. The field redefinition (2.4.93) also acts on other terms as

$$\begin{aligned} i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho} + \text{c.c.} \underset{(2.4.93)}{\longrightarrow} \; & i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho} - ic\lambda\sigma^\mu\bar{\lambda}\partial_\nu(F\tilde{F})F_{\mu\rho}F^{\rho\nu} + \text{c.c.} + \cdots \\ & = i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho} + ic\partial_\nu(\lambda\sigma^\mu\bar{\lambda})F\tilde{F}F_{\mu\rho}F^{\rho\nu} + \frac{ic}{4}\lambda\sigma^\mu\bar{\lambda}F^2\partial_\mu(F\tilde{F}) \\ & \quad + \text{c.c.} + \cdots\,, \end{aligned} \tag{2.4.95}$$

$$i\lambda\not{\partial}\bar{\lambda}F^2 + \text{c.c.} \underset{(2.4.93)}{\longrightarrow} i\lambda\not{\partial}\bar{\lambda}F^2 - ic\lambda\sigma^\mu\bar{\lambda}F^2\partial_\mu(F\tilde{F}) + \text{c.c.} + \cdots\,, \tag{2.4.96}$$

$$i\lambda\not{\partial}\bar{\lambda}F\tilde{F} + \text{c.c.} \underset{(2.4.93)}{\longrightarrow} i\lambda\not{\partial}\bar{\lambda}F\tilde{F} + \text{c.c.} + \cdots\,. \tag{2.4.97}$$

To get to the second line of (2.4.95) we integrated by part, used Bianchi identities of F, as well as its antisymmetry.

Field redefinition ⑦ The other dimension-8 operator $\lambda\not{\partial}\bar{\lambda}F^2$ can be eliminated by the following field redefinition

$$⑦ \quad \lambda_\alpha \;\to\; \lambda_\alpha + e\lambda_\alpha F^2, \quad e = -\frac{B}{A}\kappa^2 - \frac{a^2}{4} + \frac{Ca}{4A} \in \mathbb{R}\,, \quad [e] = -4\,. \tag{2.4.98}$$

Indeed for $e \in \mathbb{R}$, the fermion kinetic terms transform as

$$\begin{aligned} i\lambda\not{\partial}\bar{\lambda} + i\bar{\lambda}\bar{\not{\partial}}\lambda \underset{(2.4.98)}{\longrightarrow} \; & i\lambda\not{\partial}\bar{\lambda} + i\bar{\lambda}\bar{\not{\partial}}\lambda + 2e(i\lambda\not{\partial}\bar{\lambda} + i\bar{\lambda}\bar{\not{\partial}}\lambda)F^2 + \left[ie^2\lambda\sigma^\mu\partial_\mu(\lambda F^2)F^2 + \text{c.c.}\right] \\ & = i\lambda\not{\partial}\bar{\lambda} + i\bar{\lambda}\bar{\not{\partial}}\lambda + 2e(i\lambda\not{\partial}\bar{\lambda} + i\bar{\lambda}\bar{\not{\partial}}\lambda)F^2 + \cdots\,, \end{aligned} \tag{2.4.99}$$

and implements the desired cancellation. The dimension-8 term itself transforms under (2.4.98) as

$$i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho} + \text{c.c.} \underset{(2.4.98)}{\longrightarrow} i\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho} + 2ei\lambda\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\nu}F^{\nu\rho}F^2 + \text{c.c.} + \cdot \tag{2.4.100}$$

Therefore, under combined field redefinitions ⑤, ⑥ and ⑦ in the Lagrangian (2.4.90), we arrive at

$$\begin{aligned}\mathcal{L}_{\text{SDBI+FI}} \to\ & A + A\kappa^2(i\lambda\partial\!\!\!/\bar\lambda + i\bar\lambda\bar\partial\!\!\!/\lambda) + \left(A\kappa^4 + 4Bb^2\kappa^2 + \frac{C}{2}b\kappa^2\right)\partial_\mu(\lambda^2)\partial^\mu(\bar\lambda^2)\\ & + 2B\kappa^2\left(F^2 + \kappa^2(F\tilde F)^2 - \kappa^2F^4\right) + \left(8B\kappa^2 - \frac{C^2}{8A}\right)\kappa^2\left(i\lambda\sigma^\mu\partial_\rho\bar\lambda - i\partial_\rho\lambda\sigma^\mu\bar\lambda\right)F_{\mu\nu}F^{\nu\rho}\\ & - \left(16B\kappa^4 + \left(8B\kappa^2 - \frac{C^2}{8A}\right)\left(\frac{2B}{A}\kappa^2 + \frac{a^2}{2} - \frac{Ca}{2A}\right)\right)\kappa^2\left(i\lambda\sigma^\mu\partial_\rho\bar\lambda - i\partial_\rho\lambda\sigma^\mu\bar\lambda\right)F_{\mu\nu}F^{\nu\rho}F^2\\ & + O(\lambda^2F^6) + O(\lambda^4F^2)\,. \end{aligned} \tag{2.4.101}$$

Rescaling and final on-shell Lagrangian We see that most of the dimension-12 terms cancelled in the Lagrangian (2.4.101). Kinetic terms can be brought to standard normalizations through the following rescaling

$$\lambda\bar\lambda \ \to\ -\frac{\lambda\bar\lambda}{2A\kappa^2g^2}\,, \tag{2.4.102}$$

$$F^2 \ \to\ -\frac{F^2}{8B\kappa^2g^2}\,. \tag{2.4.103}$$

Using expressions (2.4.69), (2.4.74) for a, b and (2.4.68) for A, B and C and defining the new constant $\bar\kappa$

$$\bar\kappa^2 \equiv \frac{\kappa^2}{\sqrt{1+16g^4\kappa^2\xi^2}}\,, \tag{2.4.104}$$

we can rewrite the Lagrangian (2.4.101) in a much simpler way

$$\begin{aligned}\mathcal{L}_{\text{SDBI+FI}} =& -\frac{1}{8\kappa^2g^2}\left(1+\frac{\kappa^2}{\bar\kappa^2}\right) - \frac{1}{2g^2}(i\lambda\partial\!\!\!/\bar\lambda + i\bar\lambda\bar\partial\!\!\!/\lambda) - \frac{\bar\kappa^2}{g^2}\,\partial_\mu(\lambda^2)\partial^\mu(\bar\lambda^2)\\ & - \frac{F^2}{4g^2} - \frac{\bar\kappa^2}{4g^2}\left((F\tilde F)^2 - F^4\right) - \frac{2\bar\kappa^2}{g^2}\left(i\lambda\sigma^\mu\partial_\rho\bar\lambda - i\partial_\rho\lambda\sigma^\mu\bar\lambda\right)F_{\mu\nu}F^{\nu\rho}\\ & + \frac{6\bar\kappa^4}{g^2}\left(i\lambda\sigma^\mu\partial_\rho\bar\lambda - i\partial_\rho\lambda\sigma^\mu\bar\lambda\right)F_{\mu\nu}F^{\nu\rho}F^2 + O\left((\lambda\partial\bar\lambda)^2F^2\right) + O(\text{dim }14). \end{aligned} \tag{2.4.105}$$

We come back to our comment below eq. (2.4.62) on the sign of the first term "1" in (2.4.61), different from the one of [18]. Changing it amounts to change the definition of A in (2.4.68) and one gets the same on-shell action up to the order we consider, appart from the cosmological constant term, which has no role in global supersymmetry. However, with this sign the limit $\xi = 0$ makes the redefinitions (2.4.91) and (2.4.98) singular, because A vanishes. This last fact justifies our choice of sign.

Below are a few comments on the dimension-8 operators present in (2.4.105). The four-fermion term in the first line corresponds to the expansion of the Volkov-Akulov action (2.2.14) with the redefined decay constant $\bar\kappa$. The F^4 in the second line corresponds to the expansion of the bosonic DBI action with the same redefined tension. The two-fermion two-boson term in the second line is a consequence of the low energy theorem (2.2.21) for the goldstino coupling to matter. Here, the stress-energy tensor of the bosonic DBI action is $T^{\mu\nu} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^2 + \cdots$. The trace part $\eta^{\mu\nu}F^2$ vanishes on-shell, hence to leading order we are left with the dimension-8 operator at the end of the second line in (2.4.105). The dimension-12 term in the third line can also be explained in a similar way.

Nevertheless, the relative coefficient between the bosonic DBI action and the fermionic terms, as well as the value of $\bar{\kappa}$ cannot be obtained from the low energy theorem.

To summarize, by applying the following series of field redefinitions on (2.4.61),

$$\lambda_\alpha \quad \xrightarrow[①③⑤⑥⑦]{} \quad \sqrt{-\frac{1}{2Ag^2}}\left(\lambda_\alpha + ia(\sigma^{\mu\nu}\lambda)_\alpha F_{\mu\nu} + m\lambda_\alpha(\lambda\partial\!\!\!/\bar{\lambda}) - n\lambda_\alpha(\bar{\lambda}\bar{\partial\!\!\!/}\lambda) + p\lambda\sigma^\mu\bar{\lambda}\partial_\mu\lambda_\alpha \right.$$
$$\left. + c\lambda_\alpha F\tilde{F} + e\lambda_\alpha F^2 + h(\sigma^{\mu\nu}\lambda)_\alpha\left(F_{\mu\nu}F^2 - F_{\mu\nu}F\tilde{F}\right)\right), \qquad (2.4.106)$$

$$u_\mu \quad \xrightarrow[②④]{} \quad -\frac{1}{8\kappa^2 g^2 B}\left(u_\mu + b\lambda\sigma_\mu\bar{\lambda} + f\lambda\sigma^\rho\bar{\lambda}\tilde{F}_{\mu\rho}\right), \qquad (2.4.107)$$

we arrive at the low energy on-shell Lagragian (2.4.105).

The on-shell Lagrangian (2.4.105) has the same functional form whenever the FI parameter ξ is zero or not, except for the trivial constant piece. It follows that the FI parameter ξ enters the on-shell Lagrangian only through the renormalization of the coupling constant κ. This suggests that (2.4.61) is on-shell equivalent to

$$\begin{aligned}\mathcal{L}'_{\text{SDBI+FI}} &= -\frac{1}{8\kappa^2 g^2}\left(1-\frac{\kappa^2}{\bar{\kappa}^2}\right) - \frac{1}{8g^2\bar{\kappa}^2}\det\boldsymbol{A}\left(1+\sqrt{1+4\bar{\kappa}^2\mathcal{F}^2+4\bar{\kappa}^4(\mathcal{F}\tilde{\mathcal{F}})^2}\right)\\ &= -\frac{1}{8\kappa^2 g^2}\left(1-\frac{\kappa^2}{\bar{\kappa}^2}\right) - \frac{1}{8g^2\bar{\kappa}^2}\det\boldsymbol{A}\left(1+\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\bar{\kappa}\mathcal{F}_{\mu\nu}\right)}\right). \qquad (2.4.108)\end{aligned}$$

It is easy to verify that by setting $\lambda = 0$ and thus $\det\boldsymbol{A} = 1$, (2.4.108) agrees with the bosonic truncation of the SDBI+FI model (2.4.61). In the purely fermionic case $F = 0$, (2.4.108) is reduced to the VA action. This is also consistent with the fact that the VA action provides the low energy description of the supersymmetry breaking. Together with our explicit computations, the above results provide strong evidence that (2.4.108) is equivalent to (2.4.61) on-shell. So the standard FI term plays a trivial role in the SDBI action by just redefining the coupling constant.

2.4.4 SDBI action with induced FI term from γ deformation

In the previous section, we started with the non-linear supersymmetry representation of SDBI+FI model derived in [18], considered its low energy expansion and obtained the on-shell physical Lagrangian with both bosons and fermions up to order of dimension 12 (the latter operators involving two gauginos). The non-linear supersymmetry formalism makes the non-linear supersymmetry of SDBI action explicit. However, the linear supersymmetry is obscure and rather invisible.

In this section, we start with the linear supersymmetry representation of the generalized SDBI+γ,θ action, with manifest $\mathcal{N} = 1$ supersymmetry, and then compute the on-shell physical Lagrangian by means of field redefinitions. To do so we first recall briefly the construction of SDBI+γ,θ action and discuss how to recover the SDBI+FI as a particular limit. We then expand the action up to operators of dimension 10 (included) and compute the on-shell physical Lagrangians of SDBI+γ,θ and SDBI+FI through field redefinitions. The final result of our computations confirms what we obtained for SDBI+FI in the previous section based on the non-linear supersymmetry formalism of [18].

SDBI $+\gamma,\theta$ action from a non-linear constraint The deformed SDBI Lagrangian was written in (2.4.7) as the integral the X superfield, solution of the $\mathcal{N}=2$ nilpotent constraint, which was given by (2.4.4). For simplicity we recall it here

$$\mathcal{L}_{\rm SDBI}=\frac{1}{8\pi\kappa}\,{\rm Im}\left(\tau\int d^2\theta X\right)=\frac{1}{4g^2\kappa}\left(\int d^2\theta X+{\rm c.c.}\right)-\frac{i\theta}{32\pi^2\kappa}\left(\int d^2\theta X-{\rm c.c.}\right), \tag{2.4.109}$$

where we used τ defined in (2.4.7) as $\tau=4\pi i/g^2+\theta/2\pi$. Using the $\mathcal{A}$, $\mathcal{B}$ superfields introduced in eq. (2.4.5) the chiral half-superspace reads

$$\begin{aligned}\frac{1}{\kappa}\int d^2\theta X&=\int d^2\theta\left(W^2-\kappa^2\overline{D}^2\Big[\frac{W^2\overline{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\Big]\right)\\&=\int d^2\theta W^2+4\kappa^2\int d^2\theta d^2\bar\theta\frac{W^2\overline{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\\&=\int d^2\theta W^2+\frac{4}{\kappa^2}\int d^2\theta d^2\bar\theta\,\frac{W^2\overline{W}^2}{D^2W^2\overline{D}^2\overline{W}^2}\left(1+\mathcal{A}-\sqrt{1+2\mathcal{A}-\mathcal{B}^2}\right).\end{aligned} \tag{2.4.110}$$

For our computational convenience, we introduce the following chiral superfield Φ and real superfield $\mathcal{M}$

$$\Phi\equiv\frac{W^2}{\overline{D}^2\overline{W}^2},\quad \mathcal{M}\equiv 1+\mathcal{A}-\sqrt{1+2\mathcal{A}-\mathcal{B}^2},\qquad \overline{D}_{\dot\alpha}\Phi=0,\quad \overline{\mathcal{M}}=\mathcal{M}, \tag{2.4.111}$$

and we recall that the W^2 expansion was given in (2.4.12).The SDBI Lagrangian (2.4.109) can then be written as

$$\mathcal{L}_{\rm SDBI}\;=\;\frac{1}{4g^2}\int d^2\theta\,W^2+\frac{1}{4g^2}\int d^2\bar\theta\,\overline{W}^2+\frac{2}{g^2\kappa^2}\int d^4\theta\,\Phi\overline{\Phi}\mathcal{M}\,. \tag{2.4.112}$$

We recall that its pure bosonic part (2.4.22), is written as

$$\mathcal{L}_{\rm boson}=\frac{1}{8g^2\kappa^2}+\frac{i\theta F\tilde F}{32\pi^2\left(8\gamma^2\kappa^2+1\right)}-\frac{1}{8g^2\kappa\tilde\kappa}\sqrt{1+\frac{\theta^2g^4\gamma^2\tilde\kappa^2}{8\pi^4}}\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\tilde\kappa F_{\mu\nu}\right)}. \tag{2.4.113}$$

where $\tilde\kappa$ is the renormalized constant

$$\tilde\kappa^2=\frac{\kappa^2}{1+8\gamma^2\kappa^2}\,. \tag{2.4.114}$$

To prepare for the next subsection, we write down the free Maxwell piece of (2.4.112) explicitly

$$\mathcal{L}_4=\frac{1}{4g^2}(\mathsf{D}^2+\bar{\mathsf{D}}^2)-\frac{1}{2g^2}(i\lambda\partial\!\!\!/\bar\lambda+i\bar\lambda\bar\partial\!\!\!/\lambda)-\frac{F^2}{4g^2}+\frac{\theta}{32\pi^2}\left(iF\tilde F+i(\bar{\mathsf{D}}^2-\mathsf{D}^2)\right), \tag{2.4.115}$$

where $\mathsf{D}=d+i\gamma$ is the complex deformed field (2.3.25).

As already mentioned, the model obtained above with three deformation parameters will be referred to as SDBI+γ,θ model. The SDBI+FI model we discussed in the previous section arises by setting the deformation parameter $\gamma=0$, and adding the standard FI term $\xi\int d^4\theta V\propto\xi d$ to (2.4.7) and (2.4.112). Actually SDBI+FI can be obtained from

SDBI+γ,θ since the last term in (2.4.115) contains $i\theta(\bar{\mathsf{D}}^2-\mathsf{D}^2)\sim\gamma\theta d$ which is the standard FI term ξd with

$$\xi\equiv-\frac{\theta\gamma}{8\sqrt{2}\pi^2}\,. \tag{2.4.116}$$

Moreover, in the limit $\gamma\to 0$, the non-linear third term in (2.4.112) reduces to the one in the standard SDBI. Hence we conclude that SDBI+FI Lagrangian can be obtained from the SDBI+γ,θ one by taking the double scaling limit

$$\mathcal{L}_{\text{SDBI}+\gamma,\theta}\longrightarrow\mathcal{L}_{\text{SDBI}+\text{FI}}+\text{total derivative},\quad\text{when}\quad\begin{cases}\gamma\to 0\\ \gamma\theta=-8\sqrt{2}\pi^2\xi\end{cases}\quad\text{fixed.} \tag{2.4.117}$$

Of course, this limit is ill-defined at the non-perturbative level, since θ goes to infinity.

Component expansion We would like to find the physical on-shell action of (2.4.112) including both bosonic and fermionic contributions, by performing a low energy perturbative expansion in mass dimension. The non-linear interacting piece in (2.4.112) is

$$\int d^4\theta\,\Phi\bar{\Phi}\mathcal{M}\;=\;\Phi\bar{\Phi}\big|_0\,\mathcal{M}|_{\theta\theta\bar{\theta}\bar{\theta}}+\Phi\bar{\Phi}\big|_\theta\,\mathcal{M}|_{\theta\bar{\theta}\bar{\theta}}+\cdots+\Phi\bar{\Phi}\big|_{\theta\theta\bar{\theta}\bar{\theta}}\,\mathcal{M}|_0\,. \tag{2.4.118}$$

The relevant superfield expansions are shown in appendix B.2. In the following equations, we expand explicitly the various contributions of the superfield multiplication shown in (2.4.118) and keep terms up to dimension 10. As we will in eq. (2.4.128), the D auxiliary field can be expanded as $\mathsf{D}=\mathsf{D}_0+\mathsf{D}_4+\cdots$, where D_0 is constant, D_4 has dimension 4, etc. Hence we mention that the various terms in $\partial_\mu\mathsf{D}$ have dimensions at least 5, a useful fact to remember when keeping track of the mass dimension of the various terms in the Lagrangian.

The terms important for the Lagrangian component expansion are shown below, where the $\sim$ symbol indicates that equalities hold up to dimension-10 terms included or total derivatives

$$\Phi\bar{\Phi}\big|_0\,\mathcal{M}|_{\theta\theta\bar{\theta}\bar{\theta}}\sim 0\,, \tag{2.4.119}$$

$$\Phi\bar{\Phi}\big|_{\bar{\theta}}\,\mathcal{M}|_{\bar{\theta}\theta\theta}\sim\kappa^2\lambda^2\bar{\lambda}\Box\bar{\lambda}\frac{1}{8\mathsf{D}^2}\left(1-\frac{1+8i\kappa^2 d\gamma}{\sqrt{Z}}\right)-\frac{i\kappa^2\bar{\mathsf{D}}}{4\mathsf{D}^2\bar{\mathsf{D}}^2}\lambda^2\bar{\lambda}\sigma^{ab}\partial_\mu\bar{\lambda}\,\partial^\mu F_{ab}, \tag{2.4.120}$$

$$\Phi\bar{\Phi}\big|_{\bar{\theta}\bar{\theta}}\,\mathcal{M}|_{\theta\theta}\sim\frac{-\kappa^2\lambda^2\Box(\bar{\lambda}^2)}{8\mathsf{D}^2}\left(1-\frac{1+8i\kappa^2 d\gamma}{\sqrt{Z}}\right)+\#\kappa^4(\lambda\not{\partial}\bar{\lambda})^2, \tag{2.4.121}$$

$$\begin{aligned}\Phi\bar{\Phi}\big|_{\theta\bar{\theta}}\,\mathcal{M}|_{\theta\bar{\theta}}\sim\kappa^2\frac{i\psi\sigma^\mu\bar{\psi}}{8\mathsf{D}^2\bar{\mathsf{D}}^2}&\left[\partial_\mu\left(\mathcal{E}_-+\mathsf{D}^2-\bar{\mathsf{D}}^2\right)\left(1-\frac{1}{\sqrt{Z}}\right)+\partial_\mu\left(\mathcal{E}_++\mathsf{D}^2+\bar{\mathsf{D}}^2\right)\frac{8i\kappa^2 d\gamma}{\sqrt{Z}}\right]\\&+\#\kappa^4(\lambda\not{\partial}\bar{\lambda})(\bar{\lambda}\bar{\not{\partial}}\lambda),\end{aligned} \tag{2.4.122}$$

$$\begin{aligned}\Phi\bar{\Phi}\big|_{\bar{\theta}\bar{\theta}\theta}\,\mathcal{M}|_\theta\sim\kappa^2\frac{i\psi\not{\partial}\bar{\psi}}{4\mathsf{D}^2}&\left(1+\frac{F^{+^2}}{\mathsf{D}^2}\right)\left[1-\frac{1+8i\kappa^2 d\gamma}{\sqrt{Z}}-2\kappa^2\mathcal{E}_-\left(\frac{1}{\sqrt{Z}}+\frac{8i\kappa^2 d\gamma-64\kappa^4 d^2\gamma^2}{Z\sqrt{Z}}\right)\right.\\&\left.-2\kappa^2\mathcal{E}_+\frac{1+8i\kappa^2 d\gamma}{Z\sqrt{Z}}\right]-\frac{\kappa^2}{8\mathsf{D}^2}\lambda^2\bar{\lambda}\Box\bar{\lambda}\left(1-\frac{1+8i\kappa^2 d\gamma}{\sqrt{Z}}\right)+\frac{i\kappa^2\bar{\mathsf{D}}}{4\mathsf{D}^2\bar{\mathsf{D}}^2}\lambda^2\bar{\lambda}\bar{\sigma}^{ab}\partial_\mu\bar{\lambda}\partial^\mu F_{ab}\\&+\#\kappa^4(\lambda\not{\partial}\bar{\lambda})(\bar{\lambda}\bar{\not{\partial}}\lambda)+\#\kappa^4(\lambda\not{\partial}\bar{\lambda})^2,\end{aligned} \tag{2.4.123}$$

$$\begin{aligned}\Phi\overline{\Phi}\big|_{\theta\theta\bar\theta\bar\theta}\,\mathcal{M}|_0 \sim{}& \left(1-4\kappa^2(d^2-\gamma^2)-\sqrt{Z}\right)\left[\frac{1}{32}+\frac{i\psi\partial\!\!\!/\bar\psi}{16\mathsf{D}^2\bar{\mathsf{D}}^2}\left(1+\frac{2F^{+2}}{\mathsf{D}^2}\right)+\frac{i\psi\sigma^\mu\bar\psi}{8\mathsf{D}^2\bar{\mathsf{D}}^2}\frac{\partial_\mu\mathsf{D}}{\mathsf{D}}\right.\\ &\left.+\frac{1}{32\mathsf{D}^2\bar{\mathsf{D}}^2}\partial_\mu(\lambda^2)\partial^\mu(\bar\lambda^2)\right]-\kappa^2\frac{i\psi\partial\!\!\!/\bar\psi}{8\mathsf{D}^2\bar{\mathsf{D}}^2}\left(\mathcal{E}_+\left(1-\frac{1}{\sqrt{Z}}\right)+\mathcal{E}_-\frac{8i\kappa^2 d\gamma}{\sqrt{Z}}\right)\\ &-\frac{\kappa^4}{16}\left(\mathcal{E}_-^2\left(\frac{1}{\sqrt{Z}}+\frac{64\kappa^4d^2\gamma^2}{Z\sqrt{Z}}\right)-\frac{\mathcal{E}_+^2}{Z\sqrt{Z}}+2\mathcal{E}_+\mathcal{E}_-\frac{8i\kappa^2d\gamma}{Z\sqrt{Z}}\right),\\ &+\#\kappa^4(\lambda\partial\!\!\!/\bar\lambda)(\bar\lambda\bar\partial\!\!\!/\lambda)+\#\kappa^4(\lambda\partial\!\!\!/\bar\lambda)^2+\text{c.c.}\,.\end{aligned}\tag{2.4.124}$$

The (anti-)self-dual tensors $F^\pm$ are defined in (A.2.1) while ψ, $\mathcal{E}_\pm$, and Z are introduced in (B.2.3), (B.2.11) and (B.2.15). In particular, Z is given by

$$Z=(1+8\kappa^2\gamma^2)(1-8\kappa^2d^2)=1-8\kappa^2\left(d^2-\gamma^2\right)-(8\kappa^2d\gamma)^2.\tag{2.4.125}$$

In (2.4.124) the final c.c. symbol refers to complex conjugation of the whole right-hand side, even if some terms are real by themselves. We show (2.4.124) in this form to stress the fact that this term is real. In equations (2.4.119) to (2.4.124), we put # in front of four-gaugino terms to indicate that the corresponding coefficients can be calculated but their specific values are not important. As we explain later, these terms can be eliminated in the end by a field redefinition.

Collecting all the above terms, the SDBI Lagrangian (2.4.112) can be expanded up to dimension 10 as

$$\begin{aligned}\mathcal{L}={}&\frac{1}{8\kappa^2g^2}\left(1-\sqrt{Z}\right)+\frac{\theta}{32\pi^2}(iF\tilde F+4\gamma d)-\frac{F^2}{4g^2\sqrt{Z}}+iF\tilde F\frac{2\kappa^2d\gamma}{g^2\sqrt{Z}}+\frac{\kappa^2F^4}{4g^2Z\sqrt{Z}}\\ &-i\kappa^2F\tilde FF^2\frac{4\kappa^2d\gamma}{g^2Z\sqrt{Z}}-\frac{\kappa^2(F\tilde F)^2}{4g^2}\left(\frac{1}{\sqrt{Z}}+\frac{64\kappa^4d^2\gamma^2}{Z\sqrt{Z}}\right)-\left(i\lambda\partial\!\!\!/\bar\lambda+\text{c.c.}\right)\frac{1}{2g^2\sqrt{Z}}\\ &+\frac{i\psi\partial\!\!\!/\bar\psi}{2g^2\mathsf{D}^2}\left(1-\frac{1+8i\kappa^2d\gamma}{\sqrt{Z}}+\frac{1}{4\kappa^2\bar{\mathsf{D}}^2}\left(1-4\kappa^2(d^2-\gamma^2)-\sqrt{Z}\right)\right)+\text{c.c.}\\ &+\#(\lambda\partial\!\!\!/\bar\lambda)(\bar\lambda\bar\partial\!\!\!/\lambda)-\#(\lambda\partial\!\!\!/\bar\lambda)^2-\#(\bar\lambda\bar\partial\!\!\!/\lambda)^2\\ &+\frac{\partial_\mu(\lambda^2)\partial^\mu(\bar\lambda^2)}{2g^2}\left[\frac{1}{2\mathsf{D}^2}\left(1-\frac{1+8i\kappa^2d\gamma}{\sqrt{Z}}\right)+\text{c.c.}+\frac{1}{4\kappa^2\mathsf{D}^2\bar{\mathsf{D}}^2}\left(1-4\kappa^2(d^2-\gamma^2)-\sqrt{Z}\right)\right]\\ &+\frac{i\psi\partial\!\!\!/\bar\psi}{2g^2\mathsf{D}^2}\left[\left(1-\frac{1+8i\kappa^2d\gamma}{\sqrt{Z}}\right)\frac{F^{+2}}{\mathsf{D}^2}+2\kappa^2F\tilde F\left(\frac{1}{\sqrt{Z}}+\frac{64\kappa^4d^2\gamma^2-8i\kappa^2d\gamma}{Z\sqrt{Z}}\right)+2\kappa^2F^2\frac{1+8i\kappa^2d\gamma}{Z\sqrt{Z}}\right.\\ &\left.+\frac{1}{2\bar{\mathsf{D}}^2}\left(F\tilde F+F^2\right)\left(1-\frac{1-8i\kappa^2d\gamma}{\sqrt{Z}}+\frac{1}{2\kappa^2\mathsf{D}^2}\left(1-4\kappa^2(d^2-\gamma^2)-\sqrt{Z}\right)\right)\right]+\text{c.c.}\\ &-\kappa^2\left(i\lambda\partial\!\!\!/\bar\lambda F\tilde F\left(\frac{1}{\sqrt{Z}}+\frac{64\kappa^4d^2\gamma^2}{Z\sqrt{Z}}\right)-i\lambda\partial\!\!\!/\bar\lambda\frac{F^2}{Z\sqrt{Z}}+i\lambda\partial\!\!\!/\bar\lambda(F\tilde F+F^2)\frac{8i\kappa^2d\gamma}{Z\sqrt{Z}}\right)+\text{c.c.}\\ &+\kappa^2\frac{i\psi\sigma^\mu\bar\psi}{4g^2\bar{\mathsf{D}}^2\mathsf{D}^2}\left(\frac{1}{\mathsf{D}}-\frac{1}{\bar{\mathsf{D}}}\right)\partial_\mu d\left(1-4\kappa^2(d^2-\gamma^2)-\sqrt{Z}\right)+\text{c.c.}-\frac{\psi\sigma^\mu\bar\psi}{g^2\bar{\mathsf{D}}^2\mathsf{D}^2}\gamma\partial_\mu d\left(1-\frac{1-8\kappa^2d^2}{\sqrt{Z}}\right)\\ &+\text{total derivatives}+O(\dim 12)\,,\end{aligned}\tag{2.4.126}$$

where each c.c. symbol indicates now complex conjugation of the first preceding term. Again, # coefficients in front of the four-fermion dimension-8 terms are not important since these terms can be removed through field redefinitions, as explained later. Note that the dimension-8 and dimension-10 contributions of the form $\lambda^2\bar\lambda\Box\bar\lambda$, $\lambda^2\bar\lambda\bar\sigma^{ab}\partial^\mu\bar\lambda\partial_\mu F_{ab}$ present in (2.4.120) and (2.4.123) cancel each other in the Lagrangian (2.4.126), and so do

their complex conjugates.

Physical action with γ deformation and θ angle We will now compute the physical (on-shell) SDBI+γ,θ Lagrangian out of the off-shell one in (2.4.126), making use of redefinitions as explained in section 2.4.2. It is clear from (B.2.16) that when $\langle \mathsf{D} \rangle \neq 0$ the SDBI Lagrangian (2.4.126) contains dimension-6 terms. As explained in section 2.4.3, these terms are unphysical and can be eliminated by means of a field redefinition. This field redefinition generates also extra higher dimensional operators when acting on other terms in the Lagrangian.

Below, we proceed as follows. We first solve the equation of motion of D and use it to obtain the D-solved Lagrangian. Since the Lagrangian is too complicated, we only show explicitly the dimension-6 terms and the gaugino kinetic terms. Then we demonstrate how to eliminate the dimension-6 terms through field redefinition and write down the D-solved Lagrangian without dimension-6 terms. After discussing the elimination of some other unphysical terms, we finally obtain the physical on-shell action of SDBI+γ,θ Lagrangian up to dimension 10 (included).

i) Solving the D auxiliary field. We recall that, in the presence of γ deformation, the D auxiliary field splits into its dynamic part d and deformed part γ [1], as shown in (2.3.25). It is not real anymore and we get

$$\mathsf{D} = d + i\gamma, \qquad \bar{\mathsf{D}} = d - i\gamma. \tag{2.4.127}$$

We expand D in terms of increasing dimensions and solve for the first two terms through Euler-Lagrange equations applied in (2.4.126). The solution reads

$$\begin{aligned}
&\mathsf{D} \equiv \mathsf{D}_0 + \mathsf{D}_4 + \cdots, \qquad \mathsf{D}_0 = d_0 + i\gamma,\\
&d_0 = -\frac{\gamma g^2 \theta}{2\sqrt{2}\sqrt{8\pi^4 + \gamma^2\kappa^2(g^4\theta^2 + 64\pi^4)}},\\
&\mathsf{D}_4 = -\frac{2i\kappa^2\gamma F\tilde{F}}{1+8\kappa^2\gamma^2} + d_0\frac{2\kappa^2 F^2}{1+8\kappa^2\gamma^2} + 8\kappa^2(d_0 - i\gamma)\frac{1+8\kappa^2\gamma^2+2\sqrt{Z_0}}{(1+8\kappa^2\gamma^2+\sqrt{Z_0})^2}(i\lambda\partial\!\!\!/\bar{\lambda} + i\bar{\lambda}\bar{\partial}\!\!\!/\lambda),\\
&Z_0 = (1-8\kappa^2 d_0^2)(1+8\kappa^2\gamma^2) = \frac{8\pi^4(1+8\kappa^2\gamma^2)^2}{\gamma^2\kappa^2 g^4\theta^2 + 8\pi^4(1+8\kappa^2\gamma^2)},
\end{aligned} \tag{2.4.128}$$

where Z_0 is the lowest term in the expansion of Z defined in (2.4.125).

We then plug the above solutions for D_0 and D_4 back into (2.4.126). Especially, the $\partial_\mu d$ factor in (2.4.126) can be replaced with $\partial_\mu \mathsf{D}_4$ at dimension-10 order. One can further integrate by parts to transfer the derivative in $\partial_\mu \mathsf{D}_4$ to other factors. The resulting terms with bare D_4 can be combined with other terms in the Lagrangian. We do not show the whole D-solved Lagrangian but rather present it in schematic form

$$\begin{aligned}
\mathcal{L}_{\mathsf{D-solved}} = \; & \mathcal{L}_{\text{boson}} + \mathcal{L}_4\left(\lambda\partial\!\!\!/\bar{\lambda}, \bar{\lambda}\bar{\partial}\!\!\!/\lambda\right) + \mathcal{L}_6\left(\lambda\sigma^\mu\partial^\nu F_{\mu\nu}, \lambda\sigma^\mu\partial^\nu\tilde{F}_{\mu\nu}\right)\\
& + \mathcal{L}_8\left(\lambda\partial\!\!\!/\bar{\lambda}F^2, \partial_\mu\lambda\sigma^\mu\bar{\lambda}F^2, \lambda\partial\!\!\!/\bar{\lambda}F\tilde{F}, \partial_\mu\lambda\sigma^\mu\bar{\lambda}F\tilde{F}, \lambda\sigma^\mu\partial^\nu F_{\mu\rho}F^\rho_\nu\right)\\
& + \mathcal{L}_{10}\left(\lambda\sigma^\mu\partial^\nu F_{\mu\nu}F^2, \lambda\sigma^\mu\partial^\nu F_{\mu\nu}F\tilde{F}, \lambda\sigma^\mu\partial^\nu\tilde{F}_{\mu\nu}F^2, \lambda\sigma^\mu\partial^\nu\tilde{F}_{\mu\nu}F\tilde{F}\right) + \cdots.
\end{aligned} \tag{2.4.129}$$

In the above schematic Lagrangian we indicated the dimension of each term by a subscript, and showed each field dependence (in linearly independent operators up to total derivatives).

The dimension-4 term $\mathcal{L}_4$ contains the gaugino kinetic terms

$$\begin{aligned}\mathcal{L}_4 &= -\frac{i\lambda\not{\partial}\bar{\lambda}}{2g^2\sqrt{Z_0}} + \frac{i\lambda\not{\partial}\bar{\lambda}\mathsf{D}_0\bar{\mathsf{D}}_0}{2g^2\mathsf{D}_0^2}\left(1 - \frac{1+8i\kappa^2 d_0\gamma}{\sqrt{Z}} + \frac{1}{4\kappa^2\bar{\mathsf{D}}_0^2}\left(1-4\kappa^2(d_0^2-\gamma^2)-\sqrt{Z_0}\right)\right) + \text{c.c.} \\ &= -\frac{1+8\kappa^2\gamma^2}{g^2\sqrt{Z_0}(1+8\kappa^2+\sqrt{Z_0})}\, i\lambda\not{\partial}\bar{\lambda} + \text{c.c.} + \text{total derivatives}\,. \end{aligned} \tag{2.4.130}$$

while $\mathcal{L}_6$ can be obtained by inserting the expression (B.2.16) into the third line of (2.4.126), replacing $\mathsf{D} \to \mathsf{D}_0$ and keeping only the dimension-6 operators

$$\begin{aligned}\mathcal{L}_6 &= \frac{i\,\psi\not{\partial}\bar{\psi}|_6}{2g^2\bar{\mathsf{D}}_0^2\mathsf{D}_0^2}\left(\bar{\mathsf{D}}_0^2\left(1-\frac{1+8i\kappa^2 d\gamma}{\sqrt{Z_0}}\right) + \frac{1}{4\kappa^2}\left(1-4\kappa^2(d_0^2-\gamma^2)-\sqrt{Z_0}\right)\right) + \text{c.c.} \\ &= \frac{-4\kappa^2(1+8\kappa^2\gamma^2)}{g^2\sqrt{Z_0}(1+8\kappa^2\gamma^2+\sqrt{Z_0})^2}\left(2\bar{\mathsf{D}}_0\lambda\sigma^\mu\partial_\nu\bar{\lambda}F^{+\nu}{}_\mu - 2\mathsf{D}_0\lambda\sigma^\mu\partial_\nu\bar{\lambda}F^{-\nu}{}_\mu - \mathsf{D}_0\lambda\sigma^\mu\bar{\lambda}\partial^\nu F_{\nu\mu}\right) \\ &\quad + \text{c.c.} + \cdots\,. \end{aligned} \tag{2.4.131}$$

ii) Eliminating dimension-6 terms. The dimension-6 part $\mathcal{L}_6$ shown in (2.4.131) can be completely eliminated through the field redefinition

$$\lambda \to \lambda + a\sigma^{\mu\nu}\lambda F_{\mu\nu}\,, \quad a = -i\frac{4\kappa^2}{1+8\kappa^2\gamma^2+\sqrt{Z_0}}\bar{\mathsf{D}}_0. \tag{2.4.132}$$

Under (2.4.132) the gaugino kinetic term transforms as

$$\begin{aligned} i\lambda\not{\partial}\bar{\lambda} + \text{c.c.} \underset{(2.4.132)}{\longrightarrow}\; & i\lambda\not{\partial}\bar{\lambda} - 2ia\lambda\sigma^\nu\partial^\mu\bar{\lambda}F^+_{\mu\nu} - 2i\bar{a}\lambda\sigma^\nu\partial^\mu\bar{\lambda}F^-_{\mu\nu} - i\bar{a}\lambda\sigma^\mu\bar{\lambda}\partial^\nu F_{\nu\mu} \\ & + 2ia\bar{a}\lambda\sigma^\mu\partial_\nu\bar{\lambda}F_\mu{}^\rho F_\rho{}^\nu + \frac{a\bar{a}}{2}i\lambda\not{\partial}\bar{\lambda}F^2 + \text{c.c.} + \cdots\,, \end{aligned} \tag{2.4.133}$$

and the last three terms in the first line indeed cancel the dimension-6 operator $\mathcal{L}_6$ of (2.4.131). The field redefinition (2.4.132) also acts on operators present in $\mathcal{L}_6$ and $\mathcal{L}_8$ as follows

$$\begin{aligned} i\lambda\sigma^\mu\partial_\nu\bar{\lambda}F_{\mu\rho}F^{\rho\nu} \underset{(2.4.132)}{\longrightarrow}\; & i\lambda\sigma^\mu\partial_\nu\bar{\lambda}F_{\mu\rho}F^{\rho\nu} + \left(i\frac{a}{4}\lambda\sigma^\mu\partial^\nu\bar{\lambda}(F_{\nu\mu}-\tilde{F}_{\nu\mu})F\tilde{F} + \text{c.c.}\right) \\ & - i\frac{\bar{a}}{4}\partial_\nu(\lambda\sigma^\mu\bar{\lambda})F^{*\nu}{}_\mu F^2 + i\left(\left(\frac{\bar{a}}{4}+\frac{a}{2}\right)\lambda\sigma^\mu\partial^\nu\bar{\lambda} - \frac{\bar{a}}{4}\partial^\nu\lambda\sigma^\mu\bar{\lambda}\right)F_{\mu\nu}F^2 + \cdots\,, \end{aligned} \tag{2.4.134}$$

$$i\lambda\not{\partial}\bar{\lambda}F^2 \underset{(2.4.132)}{\longrightarrow} i\lambda\not{\partial}\bar{\lambda}F^2 - i(a+\bar{a})\lambda\sigma^\mu\partial^\nu\bar{\lambda}F_{\nu\mu}F^2 - i(a-\bar{a})\lambda\sigma^\mu\partial^\nu\bar{\lambda}\tilde{F}_{\nu\mu}F^2 + \cdots\,, \tag{2.4.135}$$

$$i\lambda\not{\partial}\bar{\lambda}F\tilde{F} \underset{(2.4.132)}{\longrightarrow} i\lambda\not{\partial}\bar{\lambda}F\tilde{F} - i(a+\bar{a})\lambda\sigma^\mu\partial^\nu\bar{\lambda}F_{\nu\mu}F\tilde{F} - i(a-\bar{a})\lambda\sigma^\mu\partial^\nu\bar{\lambda}\tilde{F}_{\nu\mu}F\tilde{F} + \cdots\,, \tag{2.4.136}$$

$$\begin{aligned} \lambda\sigma^\mu\partial^\nu\bar{\lambda}\tilde{F}_{\nu\mu} \underset{(2.4.132)}{\longrightarrow}\; & \lambda\sigma^\mu\partial^\nu\bar{\lambda}\tilde{F}_{\nu\mu} + \frac{a}{4}\lambda\not{\partial}\bar{\lambda}F\tilde{F} + \frac{a}{2}\lambda\not{\partial}\bar{\lambda}F^2 + \left(a\lambda\sigma^\mu\partial_\nu\bar{\lambda}F_{\mu\rho}F^{\rho\nu} + \text{c.c.}\right) \\ & + \left(\frac{a\bar{a}}{4}\lambda\sigma^\mu\partial^\nu\bar{\lambda}(\tilde{F}_{\nu\mu}F^2 - F_{\nu\mu}F\tilde{F} + \tilde{F}_{\nu\mu}F\tilde{F} - F_{\nu\mu}F^2) + \text{c.c.}\right) + \cdots\,, \end{aligned} \tag{2.4.137}$$

$$\begin{aligned} \lambda\sigma^\mu\partial^\nu\bar{\lambda}F_{\nu\mu} \underset{(2.4.132)}{\longrightarrow}\; & \lambda\sigma^\mu\partial^\nu\bar{\lambda}F_{\nu\mu} + \frac{a+\bar{a}}{4}\lambda\not{\partial}\bar{\lambda}F\tilde{F} - \left(a\lambda\sigma^\mu\partial_\nu\bar{\lambda}F_{\mu\rho}F^{\rho\nu} - \text{c.c.}\right) \\ & + \left(\frac{a\bar{a}}{4}\lambda\sigma^\mu\partial^\nu\bar{\lambda}(\tilde{F}_{\nu\mu}F^2 - F_{\nu\mu}F\tilde{F} + \tilde{F}_{\nu\mu}F\tilde{F} - F_{\nu\mu}F^2) - \text{c.c.}\right) + \cdots\,, \end{aligned} \tag{2.4.138}$$

where various identities in appendix appendix A.1 are used and "$\cdots$" indicate either total

derivatives, terms with dimension higher than 10, or dimension-10 terms proportional to the free equations of motion. As we explained in section 2.4.2, these last ones can be eliminated by means of field redefinition without introducing extra terms at this order.

iii) Eliminating terms containing four fermions. We also remark that under the field redefinition (2.4.132) the four-gaugino dimension-8 terms indicated with # coefficients in (B.2.16) transform as

$$(\lambda\not{\partial}\bar{\lambda})^2 \underset{(2.4.132)}{\longrightarrow} (\lambda\not{\partial}\bar{\lambda})^2 + \#\lambda\not{\partial}\bar{\lambda}\text{“}\lambda\partial\bar{\lambda}F\text{”} + \cdots\,, \tag{2.4.139}$$

$$(\bar{\lambda}\bar{\not{\partial}}\lambda)^2 \underset{(2.4.132)}{\longrightarrow} (\bar{\lambda}\bar{\not{\partial}}\lambda)^2 + \#\bar{\lambda}\bar{\not{\partial}}\lambda\text{“}\overline{\lambda\partial\bar{\lambda}F}\text{”} + \cdots\,, \tag{2.4.140}$$

$$(\lambda\not{\partial}\bar{\lambda})(\bar{\lambda}\bar{\not{\partial}}\lambda) \underset{(2.4.132)}{\longrightarrow} (\lambda\not{\partial}\bar{\lambda})(\bar{\lambda}\bar{\not{\partial}}\lambda) + \#\lambda\not{\partial}\bar{\lambda}\text{“}\lambda\partial\bar{\lambda}F\text{”} + \#\bar{\lambda}\bar{\not{\partial}}\lambda\text{“}\overline{\lambda\partial\bar{\lambda}F}\text{”} + \cdots\,, \tag{2.4.141}$$

where “$\lambda\partial\bar{\lambda}F$” schematically denotes a sum of various contractions containing one field strength, two fermions and one derivative. Their specific form is not important but the crucial point is that all these terms are proportional to the equation of a free fermion and thus can be eliminated.

As we have shown in section 2.4.3, the pure four-fermion terms of dimension 8 in (2.4.139), (2.4.140), (2.4.141) can be eliminated through the field redefinition (2.4.83)

$$\lambda_\alpha \to \lambda_\alpha + m\lambda_\alpha(\lambda\not{\partial}\bar{\lambda}) + n\lambda_\alpha(\partial_\mu\lambda\sigma^\mu\bar{\lambda}) + p\lambda\sigma^\mu\bar{\lambda}\partial_\mu\lambda_\alpha\,, \tag{2.4.142}$$

with constant coefficients m, n, p, under which the free fermion kinetic terms transform as (2.4.84). It is easy to check that the above four-fermion terms can be eliminated completely with appropriate constants m, n, p. Since the dimension-6 terms are eliminated through (2.4.132), acting (2.4.142) on the remaining terms in the Lagrangian can only generate terms with dimension strictly higher than 10 which we do not consider.

The field redefinition (2.4.132) also generates four fermion and one gauge boson mixed terms with dimension 10 in (2.4.139), (2.4.140), (2.4.141). Since they are proportional to the equation of free fermion, they can be eliminated through the following schematic field redefinition

$$\lambda_\alpha \to \lambda_\alpha + j\lambda_\alpha\text{“}\lambda\partial\bar{\lambda}F\text{”}\,. \tag{2.4.143}$$

Acting on the free fermion kinetic term, we have

$$\lambda\not{\partial}\bar{\lambda} \underset{(2.4.143)}{\longrightarrow} \lambda\not{\partial}\bar{\lambda} + j\lambda\not{\partial}\bar{\lambda}\text{“}\lambda\partial\bar{\lambda}F\text{”} + \bar{j}\bar{\lambda}\bar{\not{\partial}}\lambda\text{“}\overline{\lambda\partial\bar{\lambda}F}\text{”} + \cdots\,, \tag{2.4.144}$$

which indeed allows us to remove the dimension-10 terms in (2.4.139), (2.4.140), (2.4.141) with appropriate tensor structures and coefficients. In particular, no other dimension-10 terms would be generated due to this field redefinition (2.4.143).

To conclude, the four fermion terms in (2.4.126) with unspecified coefficients can be completely eliminated at this order without introducing extra terms. The only leftover four fermion operator of dimension 8 is the one written in the fifth line that corresponds to the expansion of the VA action.

iv) D-solved Lagrangian after eliminating dimension-6 operators and non Voklkov-Akulov 4-fermion terms. The field redefinition (2.4.132) used to eliminate $\mathcal{L}_6$ also acts on other terms

as we see from (2.4.133) to (2.4.138). Collecting all these terms, we arrive at

$$\begin{aligned}
\mathcal{L}_{\text{SDBI}+\gamma,\theta} = {} & \frac{1}{8g^2\kappa^2}\left(1-\frac{1+8\gamma^2\kappa^2}{\sqrt{Z_0}}\right)+\frac{i\theta F\tilde{F}}{32\pi^2(1+8\kappa^2\gamma^2)}-\frac{F^2}{4g^2\sqrt{Z_0}}+\frac{F^4-\left(F\tilde{F}\right)^2}{4g^2(1+8\kappa^2\gamma^2)\sqrt{Z_0}} \\
& -i\lambda\not{\partial}\bar{\lambda}\frac{1+8\kappa^2\gamma^2}{g^2\sqrt{Z_0}(1+8\gamma^2+\sqrt{Z_0})}-\partial_\mu(\lambda^2)\partial^\mu(\bar{\lambda}^2)\frac{2\kappa^2(1+8\kappa^2\gamma^2)}{g^2\sqrt{Z_0}(1+8\kappa^2\gamma^2+\sqrt{Z_0})^2}+\text{c.c.} \\
& -(i\lambda\sigma^\rho\partial_\nu\bar{\lambda}-i\partial_\nu\lambda\sigma^\rho\bar{\lambda})F^\nu{}_\mu F^\mu_\rho\frac{4\kappa^2}{g^2\sqrt{Z_0}\left(1+8\kappa^2\gamma^2+\sqrt{Z_0}\right)} \\
& +i\lambda\not{\partial}\bar{\lambda}F^2\frac{\kappa^2}{g^2\sqrt{Z_0}\left(1+8\kappa^2\gamma^2+\sqrt{Z_0}\right)}-i\lambda\not{\partial}\bar{\lambda}F\tilde{F}\frac{2\kappa^2}{g^2\sqrt{Z_0}\left(1+8\kappa^2\gamma^2+\sqrt{Z_0}\right)}+\text{c.c.} \\
& -\lambda\not{\partial}\bar{\lambda}F^2\frac{16\kappa^4 d_0\gamma(1+8\kappa^2\gamma^2)}{g^2\sqrt{Z_0}\left(1+8\kappa^2\gamma^2+\sqrt{Z_0}\right)^2}-\lambda\not{\partial}\bar{\lambda}F\tilde{F}\frac{16 d_0\gamma\kappa^4(1+8\kappa^2\gamma^2)^2}{g^2\sqrt{Z_0}\left(1+8\kappa^2\gamma^2+\sqrt{Z_0}\right)^3}+\text{c.c.} \\
& +\frac{8\kappa^4 d_0^2\quad\lambda\sigma^\mu\partial^\nu\bar{\lambda}}{g^2\mathrm{D}_0\sqrt{Z_0}(1+8\kappa^2\gamma^2+\sqrt{Z_0})^2}\left(\tilde{F}_{\mu\nu}F\tilde{F}+F_{\mu\nu}F\tilde{F}-\tilde{F}_{\mu\nu}F^2-F_{\mu\nu}F^2\right)+\text{c.c.}+\ldots,
\end{aligned} \tag{2.4.145}$$

where here c.c. indicate complex conjugation of entire lines. Several terms in the Lagrangian (2.4.145) still remain to be eliminated.

v) Eliminating dimension-8 and dimension-10 terms. The dimension-10 operators in the last line of (2.4.145) can be eliminated through the field redefinition

$$\lambda_\alpha\to\lambda_\alpha+h(\sigma^{\mu\nu}\lambda)_\alpha\left(F_{\mu\nu}F^2-F_{\mu\nu}F\tilde{F}\right),\qquad h=-i\frac{4\kappa^4 d_0^2}{\mathrm{D}_0(1+8\kappa^2\gamma^2+\sqrt{Z_0})}, \tag{2.4.146}$$

which is the analog to (2.4.91) used in section 2.4.3. The fermion kinetic terms transform as in (2.4.92) and cancel the dimension-10 operators.

Dimension-8 terms in the fourth and fifth line of (2.4.145) can also be eliminated by field redefinitions

$$\lambda_\alpha\to\lambda_\alpha+b\lambda_\alpha F^2,\qquad \lambda_\alpha\to\lambda_\alpha+c\lambda_\alpha F\tilde{F}, \tag{2.4.147}$$

with appropriate b,c coefficients. This is again analog to the field redefinitions (2.4.93) and (2.4.98) in section 2.4.3. Acting (2.4.147) on fermion kinetic terms eliminates the above dimension-8 operators. Additional terms generated by the field redefinitions (2.4.146) and (2.4.147) have dimension at least 12. The leftover dimension-8 operators containing two gauginos and two gauge bosons are those of the third line and correspond to the standard goldstino coupling to the energy momentum tensor, foretold by the low energy theorems.

Therefore, we can discard the last three lines of (2.4.145) by using (2.4.146) and (2.4.147), and the physical on-shell Lagrangian contains only the first three lines of (2.4.145).

vi) Rescaling and final result. Finally, to obtain canonical kinetic terms, we rescale

$$\lambda\bar{\lambda}\quad\to\quad\frac{\sqrt{Z_0}\left(1+8\kappa^2\gamma^2+\sqrt{Z_0}\right)}{2(1+8\kappa^2\gamma^2)}\lambda\bar{\lambda}\,, \tag{2.4.148}$$

$$F_{\mu\nu}\quad\to\quad Z_0^{1/4}F_{\mu\nu}\,. \tag{2.4.149}$$

As mentioned just above, the first three lines of (2.4.145) form the only remaining physical part of the Lagrangian, and after applying the rescaling one gets the following on-shell Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{SDBI}+\gamma,\theta} &= \frac{1}{8g^2\kappa^2}\left(1-\frac{\kappa^2}{\bar\kappa^2}\right)+\frac{i\theta F\tilde F\bar\kappa^2}{32\pi^2\kappa^2}-\frac{F^2}{4g^2}+\frac{\bar\kappa^2}{4g^2}F^4-\frac{\bar\kappa^2}{4g^2}\left(F\tilde F\right)^2\\ &\quad-\frac{1}{2g^2}\left(i\lambda\not\partial\bar\lambda+\text{c.c.}\right)-\frac{\bar\kappa^2}{g^2}\partial_\mu(\lambda^2)\partial^\mu(\bar\lambda^2)+\frac{2\bar\kappa^2}{g^2}(i\lambda\sigma^\rho\partial_\nu\bar\lambda-i\partial_\nu\lambda\sigma^\rho\bar\lambda)F^\nu_{\ \mu}F^\mu_{\ \rho}\\ &\quad+O(\dim 12),\end{aligned}\tag{2.4.150}$$

where we defined

$$\bar\kappa^2=\kappa^2\frac{\sqrt{Z_0}}{1+8\kappa^2\gamma^2}=\frac{2\sqrt{2}\pi^2\kappa^2}{\sqrt{8\pi^4(1+8\kappa^2\gamma^2)+g^4\gamma^2\kappa^2\theta^2}}\,.\tag{2.4.151}$$

This perturbative low energy expansion agrees with (2.4.105), up to an additive constant which plays no role in global supersymmetry.

After dropping the total derivative term $\theta F\tilde F$, it is easy to see that this expansion agrees on-shell with the low energy expansion of the action

$$\mathcal{L}'_{\text{SDBI}+\gamma,\theta}=\frac{1}{8g^2\kappa^2}\left(1+\frac{\kappa^2}{\bar\kappa^2}\right)-\frac{1}{8g^2\bar\kappa^2}\det\boldsymbol{A}\left(1+\sqrt{-\det(\eta_{\mu\nu}+2\sqrt{2}\bar\kappa\mathcal{F}_{\mu\nu})}\right).\tag{2.4.152}$$

One can also compare this action with the bosonic truncation given in (2.4.113) which can be rewritten as follows

$$\mathcal{L}_{\text{boson}}=\frac{1}{8g^2\kappa^2}-\frac{1}{8g^2\bar\kappa^2}\sqrt{-\det\left(\eta_{\mu\nu}+2\sqrt{2}\bar\kappa F_{\mu\nu}\right)}\,.\tag{2.4.153}$$

where we have rescaled F as $F_{\mu\nu}\to Z_0^{1/4}F_{\mu\nu}$ and dropped the total derivative term $\theta F\tilde F$. It is obvious that (2.4.153) indeed agrees with the bosonic truncation of (2.4.152) by setting $\lambda=0$ and thus $\det\boldsymbol{A}=1$. Instead, in the pure fermionic case $F=0$, (2.4.152) becomes the VA action, showing again that the VA action is the low energy description of spontaneous supersymmetry breaking. Considering our explicit low energy expansion up to dimension 10 as well as the above limits, we conclude that (2.4.152) is indeed on-shell equivalent to the original SDBI+γ,θ action.

To study the SDBI+FI model, we can consider the double scaling limit $\gamma\to 0$ with $\gamma\theta=-8\sqrt{2}\pi^2\xi$ fixed, as explained in (2.4.117). In this limit, the value of $\bar\kappa$ in (2.4.151) gives the value in (2.4.104). Hence the result (2.4.152) also agrees with the explicit computation (2.4.108) in the last section based on the non-linear formalism. Therefore, this also provides a non-trivial test of the non-linear supersymmetric formalism of [18].

2.5 Summary

In this chapter, we first introduced the basic concepts of $\mathcal{N}=1$ supersymmetry, highlighting the superspace construction of SUSY Lagrangians and the study of spontaneous SUSY breaking. We explained why non-linear supersymmetry realizations are very efficient ways to describe the low-energy limit of theories with spontaneously broken symmetries. We then explained how this formalism applies to $\mathcal{N}=1$ supersymmetry, first reviewing the standard Volkov-Akulov Lagrangian construction and then discussing more recent progress and tools such as constrained superfields. Motivated by the study of more

generic SUSY breaking scenarios we then switched to the presentation of aspects of partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ SUSY breaking. Theories with global $\mathcal{N} = 2$ supersymmetries are low-energy limits of $\mathcal{N} = 2$ supergravity theories, which in turn naturally appear as effective actions of compactified superstring theories.

We considered the general deformations of $\mathcal{N} = 2$ supersymmetry transformations for a vector multiplet and have shown that they are dual to the triplet of FI parameters under EM duality. We have then studied the effect of the deformations to the general $\mathcal{N} = 2$ two-derivative action with generic prepotential, as well as to the DBI action realizing one of the supersymmetries non-linearly. We computed the scalar potential and showed that for generic FI terms and deformation parameters, the vacuum is always $\mathcal{N} = 1$ supersymmetric. The complete breaking of supersymmetry requires the presence of at least two $U(1)$'s in analogy with the situation of branes at angles in string theory.

We also showed that the D-deformation induces an FI term proportional to the theta-angle. However, within the bosonic sector of the theory, all deformations can be absorbed to a redefinition of the DBI parameters (brane tension and coupling constants) after the elimination of the auxiliary field. This is also the case of the standard DBI + FI action, implying that the FI parameter and deformation are unobservable within the bosonic sector. This property is reminiscent of a brane rotation in string theory. An important difference, however, seemed at first to appear in the fermionic sector, where it was observed that the FI term leads to an extra contribution to the action written explicitly in the formalism of non-linear supersymmetry [18]. Applying this formalism in our case, where the FI term is generated by the deformation via the theta-angle, we did not find any extra contribution.

This seemingly observed difference in the fermionic actions for the two FI terms is important in the study of the uniqueness of SDBI actions. We thus studied the on-shell SDBI action implemented with either a standard FI term or a γ-deformation induced FI term. We have computed its low-energy expansion up to mass dimension-12 terms. The first non-trivial computation started for operators of dimension 10 and we have shown that these operators vanish on-shell and can be eliminated by field redefinitions. This result suggests that in either case, the deformation or the FI parameter does not break the linear supersymmetry but just modifies the goldstino decay constant by rotating the remaining residual supersymmetry. Considering the nature of the SDBI action realizing partial supersymmetry breaking with both linear and non-linear supersymmetry, it is not surprising to see the trivial role of the deformation or the FI parameter on-shell. On the other hand, the rotation modifies the field transformations of the linear supersymmetry in a non-linear way, although without constant in the gaugino transformation, which makes the result non-obvious even if the final result has a simple interpretation within string theory. Obviously, the rotation argument breaks down in the presence of another referent SDBI action and supersymmetry breaking should occur in this system.

Interesting questions remain, such as the existence of a deformation of the SDBI action that breaks spontaneously the linear supersymmetry, the coupling to supergravity, as well as its possible realization in string theory. Indeed, such terms are expected to modify on-shell fermionic dependences of higher than dimension-8 operators, motivating corresponding computations in string theory.

Chapter 3

String theory vacua and magnetic field SUSY breaking

3.1 String theory elements

String theory has been developing for more than five decades after pioneering works in the late '60s [43, 44, 45, 46, 47], which were at first not related to quantum gravity. Nevertheless, its natural UV completion or the presence of the graviton in its low energy spectrum, rapidly promoted string theory as one of the most reasonable candidates to describe quantum gravity. A complete review of advances in string theory is not the subject of this thesis and would in practice be impossible considering the enormous amount of discoveries developed through the past years. In this section, we will nevertheless make reminders of basics string theory elements, used in the rest of the thesis, based on various references [48, 49, 50, 51, 52].

We start with a quick review of the bosonic string, before switching to string theories including supersymmetry, superstring theories, and several useful ingredients such as vacuum amplitudes and D branes. The bosonic string is not consistent on its own and supersymmetry is necessary to construct consistent string theories. It leads to different descriptions: type I, type IIA, type IIB or yet heterotic superstring theories. The second string revolution showed that these theories are related through a web of dualities, reviewed in [53, 54]. Hence, we will not enter the details of each superstring theory construction but will rather recall some important points about type IIB string theory, which will be the framework for the constructions in the rest of this thesis.

3.1.1 Bosonic string

String action, equations of motion, mode expansions We first introduce basic concepts of string theory, through the description of bosonic string theory. Even if, as we will explain in this section, this theory is not consistent, it is a natural and instructive theory to start with. Fundamental degrees of freedom are those of open or closed strings of length l_s related to the Regge slope $\alpha' = l_s^2/2$. When evolving in time, strings span a $2d$ surface called worldsheet. The open string worldsheet possesses a boundary whereas the closed string one does not. The worldsheet is embedded in the ambient space-time by worldsheet coordinates $X^\mu(\tau, \sigma)$, where $\mu = 0, \dots, D-1$ are the D-dimensional space-time indices and τ, σ are the worldsheet parameters, with $\sigma = 0, \pi$ corresponding to the two extremities of the strings. In order to keep track of the nature of the extremities of the

strings, one has to impose the following boundary conditions (b. c.)

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma), \qquad \text{closed strings,} \tag{3.1.1}$$

$$X^\mu(\tau,0) = X_0^\mu,\ X^\mu(\tau,\pi) = X_1^\mu, \quad \text{open strings, Dirichlet boundary conditions,} \tag{3.1.2}$$

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,0) = \frac{\partial X^\mu}{\partial\sigma}(\tau,\pi) = 0, \quad \text{open strings, Neumann boundary conditions.} \tag{3.1.3}$$

The induced worldsheet metric can be constructed from the space-time metric

$$G_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \quad \alpha,\beta = \tau,\sigma, \tag{3.1.4}$$

and a natural worldsheet action is then the integral of its determinant, similar to the line element integral in the point-particle case. This leads to the Nambu-Goto (NG) [55, 56] worldsheet action

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det G_{\alpha\beta}}. \tag{3.1.5}$$

The square root being difficult to quantize, one can introduce an auxiliary metric $h_{\alpha\beta}$, the elimination of which gives back the NG action. The action with this auxiliary metric is the Polyakov [57, 58, 59] action

$$S_{\text{Polyakov}} = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{-\det h}\, h^{\alpha\beta} g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{3.1.6}$$

which as explained above, has to be considered together with the constraint

$$\frac{\delta}{\delta h^{\alpha\beta}} S_{\text{Polyakov}} = 0, \tag{3.1.7}$$

and the boundary conditions of eqs. (3.1.1) to (3.1.3). The equations of motion for $\partial_\alpha X^\mu$ and the constraint on $h_{\alpha\beta}$ read

$$\nabla^\alpha\nabla_\alpha X^\mu = \nabla^2 X^\mu = 0, \quad T_{\alpha\beta} = \partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\rho\lambda}\partial_\rho X\cdot\partial_\lambda X = 0, \tag{3.1.8}$$

where ∇ is the worldsheet covariant derivative, and the central dots denote space-time contractions $X\cdot X = X^\mu X_\mu = g_{\mu\nu}X^\mu X^\nu$. The Polyakov action is invariant under local worldsheet reparametrizations

$$\sigma,\tau \to \sigma' = f_\sigma(\sigma,\tau), \tau' = f_\tau(\sigma,\tau), \quad X^\mu(\sigma,\tau) \to X'^\mu,\ X'^\mu(\sigma',\tau') = X^\mu(\sigma,\tau), \tag{3.1.9}$$

local Weyl transformations

$$h_{\alpha\beta} \to e^{w(\sigma,\tau)}h_{\alpha\beta}, \quad X^\mu \to X^\mu, \tag{3.1.10}$$

and global Poincaré transformations $x^\mu \to a^\mu_\nu x^\nu + b^\mu$ leaving $h_{\alpha\beta}$ invariant. These symmetries can be used to gauge fix the Polyakov action through $h_{\alpha\beta} = \eta_{\alpha\beta}$. It then reads

$$S_{\text{g. f. Polyakov}} = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\left(\dot{X}^2 - X'^2\right), \qquad \dot{X} \equiv \frac{\partial X}{\partial\tau},\ X' \equiv \frac{\partial X}{\partial\sigma}, \tag{3.1.11}$$

and the equations of motion (3.1.8) become

$$\partial^2 X^\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2} \right) X^\mu = 0, \tag{3.1.12}$$

$$T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0, \quad T_{\tau\sigma} = T_{\sigma\tau} = \dot{X}\cdot X' = 0. \tag{3.1.13}$$

The first equation is a simple wave equation, solved as usual by introducing the left-moving and right-moving coordinates $\sigma^\pm$, given below in eq. (3.1.15). The wave equation factorizes and is solved by a sum of independent left-moving and right-moving solutions. Namely eqs. (3.1.12) and (3.1.13) transform to

$$\partial_+\partial_- X^\mu = 0, \quad T_{++} = \partial_+ X\cdot\partial_+ X = 0, \quad T_{--} = \partial_- X\cdot\partial_- X = 0, \quad T_{+-} = 0, \tag{3.1.14}$$

$$\sigma^\pm = \tau \pm \sigma, \qquad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma), \tag{3.1.15}$$

so that a general solution is written as

$$X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-). \tag{3.1.16}$$

The left-moving and right-moving solutions can be expanded according to their boundary conditions. For instance, for closed strings the expansion reads

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2in\sigma^+}, \tag{3.1.17}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2in\sigma^-}. \tag{3.1.18}$$

For open strings with *e.g.* Neumann boundary conditions (3.1.3) at both ends, NN, which mix X_L and X_R, the expansion is

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + 2i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}\cos(n\sigma), \tag{3.1.19}$$

and corresponds to standing waves solutions, the left-moving and right-moving sectors of which are not independent. In the above expansions, the x^μ and p^μ quantities correspond to the center-of-mass position and the momentum. For convenience we also define

$$\alpha^\mu_0 \equiv \sqrt{\frac{\alpha'}{2}}p^\mu, \tag{3.1.20}$$

so that derivatives of mode expansions similar to those of eqs. (3.1.17) to (3.1.19) can be written with sums including $n = 0$ terms.

Light-cone quantization There is still a residual symmetry remaining after gauge fixing of the worldsheet metric. As was explained in [60], there is a very convenient way to fix it by using the so-called light-cone gauge, which makes use of the space-time light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}), \qquad X^i, \quad i = 1,\ldots,D-2, \tag{3.1.21}$$

$$\eta_{+-} = \eta_{-+} = -1, \quad \eta_{++} = \eta_{--} = 0, \quad \eta_{ij} = \delta_{ij}. \tag{3.1.22}$$

Namely, this procedure uses the left-over reparametrization symmetry to redefine the worldsheet parameters through

$$\sigma^+ \to \hat{\sigma}^+(\sigma^+) = \frac{X_L^+(\sigma^+)}{\alpha' p^+}, \quad \sigma^- \to \hat{\sigma}^-(\sigma^-) = \frac{X_R^+(\sigma^-)}{\alpha' p^+}, \tag{3.1.23}$$

so that the worldsheet time goes to

$$\tau = \frac{1}{2}\left(\sigma^+ + \sigma^-\right) \to \hat{\tau} = \frac{X^+(\tau,\sigma)}{2\alpha' p^+} = \frac{\hat{X}^+(\hat{\tau},\hat{\sigma})}{\alpha' p^+}. \tag{3.1.24}$$

This means that the ambiant and worldsheet times are identified, namely $\hat{X}^+$ reads

$$\hat{X}^+(\hat{\sigma},\hat{\tau}) = 2\alpha' p^+ \hat{\tau}. \tag{3.1.25}$$

It does not have oscillators anymore and only carries a non-vanishing momentum p^+. Of course, once this parametrization choice is realized, one can drop the hat symbols. The energy-momentum constraints (3.1.13) can be combined and lead to

$$\begin{aligned} 0 = T_{\sigma\sigma} \pm T_{\tau\sigma} = \frac{1}{2}\left(\dot{X}^2 \pm X'^2\right)^2 &= -(\dot{X}^+ \pm X^{+'})(\dot{X}^- \pm X^{-'}) + \frac{1}{2}\sum_i (\dot{X}^i \pm X^{i'})(\dot{X}^i \pm X^{i'}) \\ &= -2\alpha' p^+ (\dot{X}^- \pm X^{-'}) + \frac{1}{2}\sum_i (\dot{X}^i \pm X^{i'})(\dot{X}^i \pm X^{i'}). \end{aligned} \tag{3.1.26}$$

The last line is obtained using the X^+ expansion (3.1.25) and can be used to express the α^-, $\tilde{\alpha}^-$ oscillators with respect to the α^i, $\tilde{\alpha}^i$ ones, so that only an integration constant, corresponding to the center-of-mass degree of freedom $x^-(\tau,\sigma)$, remains undetermined. The conclusion is that, in the light-cone gauge, the only independent oscillators are α^i, $\tilde{\alpha}^i$, so that the quantization can be made in $D-2$ dimension. In particular one finds

$$\frac{\alpha' p^-}{2} = \sqrt{\frac{\alpha'}{2}}\alpha_0^- = \frac{1}{2p^+}\sum_i \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i\right), \tag{3.1.27}$$

and a similar expression in terms of $\tilde{\alpha}_n^i$, so that the on-shell mass formula becomes

$$M^2 = -p^\mu p_\mu = 2p^- p^+ - \sum_i p^i p^i = \frac{1}{\alpha'}\sum_{n\neq 0}\sum_i (\alpha_n^i \alpha_{-n}^i + \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i). \tag{3.1.28}$$

We can now follow the standard methods of the first quantization, implemented in light-cone gauge. The center-of-mass position, momentum and oscillators can be promoted to quantum operators satisfying the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^i, \alpha_n^j] = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\eta^{ij}\delta_{m+n,0}, \quad [\alpha_m^i, \tilde{\alpha}_n^j] = 0. \tag{3.1.29}$$

Annihilation and creation operators satisfying the standard commutation relations can be defined as $a_n^i = \alpha_n^i/\sqrt{n}$, $a^{+i}_{\ n} = \alpha_{-n}^i/\sqrt{n}$, and similarly for right-moving oscillators. The vacuum is annihilated by all annihilation operators and is labelled by its momentum p. One generates the Hilbert space of states $|\phi\rangle$ by acting on the vacuum with creation operators. For instance, for closed strings, with vacuum state denoted $|0,\tilde{0},p\rangle$, these states

can be written as

$$|\phi\rangle = \prod_{a=1}^{\infty} (\alpha_{-a}^{i_a})^{m_a} \prod_{b=1}^{\infty} (\tilde{\alpha}_{-b}^{i_b})^{\tilde{m}_b} |0, \tilde{0}, p\rangle \,, \qquad \forall n > 1 \;\; \alpha_n^i |0, \tilde{0}, p\rangle = \tilde{\alpha}_n^i |0, \tilde{0}, p\rangle = 0, \quad (3.1.30)$$

where only a finite set of the oscillator "multiplicities" m_a, $\tilde{m}_b$, is non vanishing, so that the products over a and b are finite.

Hamiltonian and energy-momentum expansion The Hamiltonian of the theory can be computed from the gauge fixed action (3.1.11) and expanded using the wave equation solutions as in eqs. (3.1.17) to (3.1.19). For closed strings, it gives in light-cone gauge

$$H = \frac{1}{2} \int_0^{\pi} d\sigma (2\pi\alpha' P^2 + \frac{1}{2\pi\alpha'} X'^2) = \sum_i \frac{p^i p^i}{2p^+} + \frac{1}{\alpha' p^+} \left(\sum_i \sum_{n>0} (\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i) + E_0 + \tilde{E}_0 \right), \quad (3.1.31)$$

where E_0, $\tilde{E}_0$ are the zero point energies, on which we come back around eq. (3.1.37), and the dot product is applied in the $D-2$ space dimensions with indices $i = 1, \ldots, D-2$. The P^i are the momenta conjugate to X^i defined as usual from the derivatives of the Lagrangian with respect to $\dot{X}^i$.

The energy-momentum tensor can also be expanded, and for closed strings it reads

$$T_{++} = \partial_+ X \cdot \partial_+ X = 2\alpha' \sum_{n,m} \alpha_n \cdot \alpha_m e^{-2i(n+m)\sigma^+} \equiv 4\alpha' \sum_p L_p e^{-2ip\sigma^+}, \quad (3.1.32)$$

$$T_{--} = 4\alpha' \sum_p \tilde{L}_p e^{-2ip\sigma^-}, \quad (3.1.33)$$

$$L_p \equiv \frac{1}{2} \sum_n \alpha_{p-n} \cdot \alpha_n, \quad \tilde{L}_p \equiv \frac{1}{2} \sum_n \tilde{\alpha}_{p-n} \cdot \tilde{\alpha}_n. \quad (3.1.34)$$

The Virasoro generators L_p, $\tilde{L}_p$ thus correspond to the modes of the energy-momentum tensor and it is easy to show from eq. (3.1.29) that they satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}, \quad (3.1.35)$$

with central charge $c = (D-2)/12$. The Virasoro generators are intended to be normal ordered, a fact which only affects the L_0 and $\tilde{L}_0$ operators. The normal ordering reads

$$\begin{aligned} L_0 = \frac{1}{2} \sum_n \alpha_{-n} \cdot \alpha_n &= \frac{1}{2} \alpha_0^2 + \frac{1}{2} \sum_{n>1} \alpha_{-n} \cdot \alpha_n + \frac{1}{2} \sum_{n>1} \alpha_n \cdot \alpha_{-n} \\ &= \frac{\alpha' p^2}{4} + \sum_{n>1} \alpha_{-n} \cdot \alpha_n + E_0, \end{aligned} \quad (3.1.36)$$

and similarly for $\tilde{L}_0$. To obtain the last line we used the commutation relations (3.1.29) and defined

$$E_0 = \tilde{E}_0 = \frac{1}{2} \sum_{i=1}^{D-2} \sum_{n>1} n \simeq -\frac{D-2}{24}. \quad (3.1.37)$$

The last evaluation can be obtained in several ways; a common way is to use the zeta function regularization. A usual convention is to define an already normal ordered L_0 generator, thus differing from (3.1.36) by the E_0 constant. We will apply this convention

and redefine L_0 as

$$L_0 \equiv \frac{\alpha' p^2}{4} + N, \qquad N \equiv \sum_{n>1} \alpha_{-n} \cdot \alpha_n = \sum_{n>1} \alpha^i_{-n} \alpha^i_n, \tag{3.1.38}$$

where we also introduced the number operator N. The same holds for $\tilde{L}_0$, $\tilde{N}$.

Using the energy-momentum tensor mode expansion, the constraint (3.1.14) can be expressed from the L_p, $\tilde{L}_p$ operators. The physical states $|\phi\rangle$ are thus determined by imposing $L_p |\phi\rangle = \tilde{L}_p |\phi\rangle = 0$ for $p > 0$ and $(L_0 + E_0) |\phi\rangle = (\tilde{L}_0 + E_0) = 0$ with our newly defined L_0, $\tilde{L}_0$. Note that due to the algebra (3.1.35) and the presence of a non-vanishing central charge $c \neq 0$, the constraint $L_{-p} |\phi\rangle = 0$ cannot be imposed together with $L_p |\phi\rangle = 0$ for $p > 0$.

The constraints $(L_0 + E_0) |\phi\rangle = (\tilde{L}_0 + E_0) |\phi\rangle = 0$ give the closed string level matching condition $N = \tilde{N}$. Finally, the closed string mass formula (3.1.28) can be normal ordered as in eq. (3.1.36) and reads

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + \tilde{N} + E_0 + \tilde{E}_0), \qquad N = \tilde{N}, \tag{3.1.39}$$

which shows that the string spectrum contains a tower of states, the mass of which are separated by $2/\sqrt{\alpha'}$. For the first level, with $N = \tilde{N} = 0$, we find a negative square mass $M^2 = 4E_0/\alpha' < 0$, which shows that the vacuum is tachyonic.

Critical dimension We conclude this part on bosonic string theory by stating that in order for the theory to be self-consistent, a necessary condition is that the space-time dimension D takes a specific value, called critical dimension, $D = 26$. This critical value can be obtained in several ways; the most obvious one is to require the cancellation of the conformal anomaly, related to the UV behavior of string theory amplitudes. As any anomaly, it would characterize the breaking of a symmetry at the quantum level, namely the worldsheet conformal symmetry. Choosing an appropriate field content in the theory is necessary to cancel this anomaly. Worldsheet fields are space-time coordinates, the number of which is just de space-time dimension, so that the cancellation of conformal anomaly requires the specific dimension $D = 26$. This value for the critical dimension can be obtained in other related ways, *e.g.* through the computations of beta functions in the background field methods, or by requiring Lorentz invariance in light-cone gauge quantization.

3.1.2 Superstrings and type IIB spectrum

We just saw that the bosonic string theory contains a tachyon in the vacuum. What's more, it does not contain any fermion in its low energy spectrum, which is in direct contradiction with our Standard Model of particle physics. A way to fix these two issues is to consider supersymmetric string theories, which can be implemented in several ways, with worldsheet [61, 62] or space-time [63, 64, 65, 66] supersymmetry. We will focus on the worldsheet implementation of supersymmetry, in the way it was introduced by Ramond, Neveu and Schwarz (RNS). In the following construction we will often keep the space-time dimension D implicit, nevertheless we mention now that the superstring critical dimension, introduced for bosonic strings in the last paragraph of section 3.1.1, is $D = 10$.

Supersymmetric worldsheet action, mode expansions The first objective is to write a supersymmetric equivalent of the bosonic worldsheet action. We thus introduce supersymmetric partners to the worldsheet bosonic coordinates X^μ, corresponding to two Weyl spinors ψ_+^μ, ψ_-^μ, implementing the $\mathcal{N}=2$ worldsheet supersymmetry. The natural supersymmetric version of the gauge fixed action (3.1.11) can be written as

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \not{\partial}\psi_\mu\right), \tag{3.1.40}$$

where

$$\psi^\mu = \begin{pmatrix}\psi_+^\mu \\ \psi_-^\mu\end{pmatrix}, \quad \psi_+^\mu = (\psi_+^\mu)^*, \ \psi_-^\mu = (\psi_-^\mu)^*, \quad \bar{\psi}^\mu = i(\psi^\mu)^\dagger\rho^0 = i(\psi_+^\mu, -\psi_-^\mu), \tag{3.1.41}$$

$$\not{\partial} = \rho^\alpha\partial_\alpha, \quad \rho^\tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \rho^\sigma = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{3.1.42}$$

and the ρ^α matrices form a real representation of the Clifford algebra, implying that ψ_+, ψ_- are real Majorana spinors. The worldsheet action is indeed invariant under the following SUSY transformations

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu, \quad \delta_\epsilon\psi^\mu = \not{\partial} X^\mu \epsilon. \tag{3.1.43}$$

Redefining the worldsheet derivatives through the convenient coordinates of eq. (3.1.15), one can rewrite this action as

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma \left(\partial_- X\cdot\partial_+ X - i\psi_-\cdot\partial_+\psi_- - i\psi_+\cdot\partial_-\psi+\right), \tag{3.1.44}$$

leading to the worldsheet fermions equations of motion

$$\partial_+\psi_-^\mu = 0, \qquad \partial_-\psi_+^\mu = 0, \tag{3.1.45}$$

the solution of which are left and right-moving fermions $\psi_-(\sigma^-)$, $\psi_+(\sigma^+)$. As in the bosonic case, the energy-momentum tensor can be derived from the action, together with the supercurrent $J_{\alpha\beta}$ related to the SUSY transformations of the Lagrangian. The supercurrent is fermionic and thus carries worldsheet indices $\alpha = \tau, \sigma$, or yet $\alpha = +, -$ in the coordinates (3.1.15), together with spinor indices $\beta = +, -$. Nevertheless, it has only two independent components, as does the energy-momentum tensor, which read

$$J_+ \equiv J_{++} = \psi_+\cdot\partial_+ X = 0, \qquad J_- \equiv J_{--} = \psi_-\cdot\partial_- X = 0, \tag{3.1.46}$$

$$T_{++} = \partial_+ X\cdot\partial_+ X + \frac{i}{2}\psi_+\cdot\partial_+\psi_+ = 0, \quad T_{--} = \partial_- X\cdot\partial_- X + \frac{i}{2}\psi_-\cdot\partial_-\psi_- = 0. \tag{3.1.47}$$

In order the action to be invariant under field variations, worldsheet fermions have to satisfy the equation of motions (3.1.45) together with boundary conditions coming from the σ boundary variation of the action

$$\delta S = \cdots + \int d\tau \left(\psi_+\cdot\delta\psi_+ - \psi_-\cdot\delta\psi_-\right)|_{\sigma=\pi} - \left(\psi_+\cdot\delta\psi_+ - \psi_-\cdot\delta\psi_-\right)|_{\sigma=0} = 0. \tag{3.1.48}$$

We see that this relation imposes the boundary constraints

$$\psi_+(\tau,0) = \pm\psi_-(\tau,0), \quad \psi_+(\tau,\pi) = \pm\psi_-(\tau,\pi), \qquad \text{open strings}, \tag{3.1.49}$$

$$\psi_\pm(\tau,\sigma+\pi) = \pm\psi_\pm(\tau,\sigma), \qquad \text{closed strings}. \tag{3.1.50}$$

In the open string case, these conditions are imposed separately at each string endpoint. As the overall sign between the two spinors has no meaning, we can freely impose the sign at one of the two ends, for instance $\psi_+(\tau,0) = \psi_-(\tau,0)$ and are left with two choices at the opposite end, corresponding to periodic or anti-periodic boundary conditions

$$\psi_+(\tau,\pi) = +\psi_-(\tau,\pi), \quad \text{open strings Ramond (R) b. c.,} \tag{3.1.51}$$
$$\psi_+(\tau,\pi) = -\psi_-(\tau,\pi), \quad \text{open strings Neveu-Schwarz (NS) b. c.,} \tag{3.1.52}$$

which lead to the following expansions for the wave equations solutions

$$\psi_-^\mu(\sigma^-) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^-}, \quad \psi_+^\mu(\sigma^+) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^+}, \qquad \text{R sector,} \tag{3.1.53}$$
$$\psi_-^\mu(\sigma^-) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^-}, \quad \psi_+^\mu(\sigma^+) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^+}, \qquad \text{NS sector.} \tag{3.1.54}$$

These expansions correspond to two different sectors of the theory, namely the Ramond (R) and Neveu-Schwarz (NS) sector. The Majorana condition (3.1.41) on the worldsheet spinors implies that $(d_n^\mu)^\dagger = d_{-n}^\mu$, so that the $d_{\pm n}^\mu$ can be used as annihilation and creation operators, and similarly for the $b_{\pm r}^\mu$. As in the bosonic case, we can write the worldsheet fermions in light-cone gauge and use the residual worldsheet symmetries to set $\psi_\alpha^+(\tau,\sigma) = 0$ in the NS sector, while we must still keep its zero modes in the R sector. Moreover, as in the bosonic case, writing the constraints of eqs. (3.1.46) and (3.1.47) in light-cone gauge allows to express the $\psi_\alpha^-(\tau,\sigma)$ oscillators with respect to the ψ_α^i ones. Hence the first quantization can be done in light-cone gauge by lifting the transverse (labelled by i) modes to quantum operators following the (anti-)commutation relations analog to (3.1.29). These relations read

$$\{b_r^i, b_s^j\} = \eta^{ij}\delta_{r+s,0}, \quad \{d_n^i, d_m^j\} = \eta^{ij}\delta_{m+n,0}, \tag{3.1.55}$$

such that the original spinors indeed satisfy $\{\psi_\alpha^i(\tau,\sigma), \psi_\beta^j(\tau,\sigma')\} = i\pi\eta^{ij}\delta_{\alpha\beta}\delta(\sigma-\sigma')$.

In the closed string case, the boundary conditions

$$\psi_\alpha(\tau,\sigma+\pi) = +\psi_\alpha(\tau,\sigma) \quad \text{closed strings R b. c.,} \tag{3.1.56}$$
$$\psi_\alpha(\tau,\sigma+\pi) = -\psi_\alpha(\tau,\sigma) \quad \text{closed strings NS b. c.,} \tag{3.1.57}$$

correspond to the expansions

$$\psi_-^i(\sigma^-) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^i e^{-in\sigma^-}, \quad \psi_+^i(\sigma^+) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \tilde{d}_n^i e^{-in\sigma^+}, \qquad \text{R sector,} \tag{3.1.58}$$
$$\psi_-^i(\sigma^-) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^i e^{-ir\sigma^-}, \quad \psi_+^i(\sigma^+) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_r^i e^{-ir\sigma^+}, \qquad \text{NS sector.} \tag{3.1.59}$$

The boundary conditions can be chosen differently for the left-moving $\psi_-(\sigma^-)$ and right-moving $\psi_+(\sigma^+)$ solutions, so that we are in presence of four different sectors: the NSNS, NSR, RNS, and RR sectors. The $\tilde{d}_{\pm n}^i$, $\tilde{b}_{\pm r}^i$ are promoted to quantum operators satisfying anticommutation relations similar to those of eq. (3.1.55) while anti-commuting with the left-moving operators $\tilde{d}_{\pm n}^i$, $\tilde{b}_{\pm r}^i$

In both the closed and open strings cases, string states are obtained by acting on the

vacuum with the bosonic and fermionic creation operators of the sector under consideration. Namely, the states of each sector are constructed as

$$|\phi\rangle = \prod_{a=1}^{\infty} (\alpha_{-a}^{i_a})^{m_a} \prod_{b=1}^{\infty} (d_{-b}^{i_b})^{m_b} |0,p\rangle_{\mathrm{R}}\,, \qquad \text{Ramond (R) sector,} \tag{3.1.60}$$

$$|\phi\rangle = \prod_{a=1}^{\infty} (\alpha_{-a}^{i_a})^{m_a} \prod_{r=\frac{1}{2}}^{\infty} (b_{-r}^{i_r})^{m_r} |0,p\rangle_{\mathrm{NS}}\,, \qquad \text{Neveu-Scharz (NS) sector.} \tag{3.1.61}$$

The bosonic oscillators are of course identical in any sector. As their left and right-moving periodicities can be chosen separately, close strings presents four different types of states, corresponding to the sectors mentioned above

$$|\phi\rangle_{\mathrm{RR}} = \prod_{a>1} (\alpha_{-a}^{i_a})^{m_a} \prod_{b>1} (\tilde{\alpha}_{-b}^{j_b})^{\tilde{m}_b} \prod_{c>1} (d_{-c}^{j_c})^{m_c} \prod_{e>1} (\tilde{d}_{-e}^{k_e})^{\tilde{m}_e} \left|0,\tilde{0},p\right\rangle_{\mathrm{RR}}\,, \tag{3.1.62}$$

$$|\phi\rangle_{\mathrm{RNS}} = \prod_{a>1} (\alpha_{-a}^{i_a})^{m_a} \prod_{b>1} (\tilde{\alpha}_{-b}^{j_b})^{\tilde{m}_b} \prod_{c>1} (d_{-c}^{j_c})^{m_c} \prod_{s>\frac{1}{2}} (\tilde{b}_{-s}^{i_s})^{\tilde{m}_s} \left|0,\tilde{0},p\right\rangle_{\mathrm{RNS}}\,, \tag{3.1.63}$$

$$|\phi\rangle_{\mathrm{NSR}} = \prod_{a>1} (\alpha_{-a}^{i_a})^{m_a} \prod_{b>1} (\tilde{\alpha}_{-b}^{j_b})^{\tilde{m}_b} \prod_{r>\frac{1}{2}} (b_{-r}^{i_r})^{m_r} \prod_{e>0} (\tilde{d}_{-e}^{i_e})^{\tilde{m}_e} \left|0,\tilde{0},p\right\rangle_{\mathrm{NSR}}\,, \tag{3.1.64}$$

$$|\phi\rangle_{\mathrm{NSNS}} = \prod_{a>1} (\alpha_{-a}^{i_a})^{m_a} \prod_{b>1} (\tilde{\alpha}_{-b}^{j_b})^{\tilde{m}_b} \prod_{r>\frac{1}{2}} (b_{-r}^{j_r})^{m_r} \prod_{s>\frac{1}{2}} (\tilde{b}_{-s}^{i_s})^{\tilde{m}_s} \left|0,\tilde{0},p\right\rangle_{\mathrm{NSNS}}\,. \tag{3.1.65}$$

NS and R vacua By definition, the vacua $|0\rangle_{\mathrm{R}}$, $|0\rangle_{\mathrm{NS}}$ of each sector are annihilated by all annihilation operators. This leads to a standard space-time scalar vacuum in the NS sector, whereas it is different in the R sector. The difference comes from the fact that $|0\rangle_{\mathrm{R}}$ is in particular annihilated by the d_0^μ. During the discussion on light-cone gauge quantization, made before eq. (3.1.55), we mentioned that one must keep the ψ_α^+ zero modes in the Ramond sector, hence indeed keeping the μ index on the d_0^μ and not only a transverse index i. The d_0^μ also satisfy anti-commutators as in eq. (3.1.55), namely $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$. This algebra is similar, up to a factor of 2, to the space-time Clifford algebra of $(\Gamma^\mu)_b^a$ matrices, showing that $|0\rangle_{\mathrm{R}}$ can be described by a Majorana space-time spinor ξ_a with $2^{D/2}$ real components. We recall that spinor representations can be constructed by defining $(D-2)/2$ creation and $(D-2)/2$ annihilation matrices $\Gamma^{i\pm}$ satisfying standard anticommutation relations, and by act with creation matrices on a Clifford vacuum $|\xi\rangle$. As usual, this vacuum state is annihilated by all the annihilation matrices. A spinor ξ_a is thus constructed as

$$\Gamma^{0\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1), \quad \Gamma^{i\pm} = \frac{1}{2}(\Gamma^{2i} \pm i\Gamma^{2i+1}) \text{ for i} \neq 0, \tag{3.1.66}$$

$$\xi_a = (\Gamma^{\frac{D-2}{2}+})^{s_{\frac{D-2}{2}}+\frac{1}{2}} \cdots (\Gamma^{0+})^{s_0+\frac{1}{2}}\xi, \qquad \forall\ i = 0,\ldots,\tfrac{D-2}{2} \quad \Gamma^{i-}\xi = 0, \tag{3.1.67}$$

$$a = (s_0, s_1, \ldots, s_{\frac{D-2}{2}}) = (\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2}). \tag{3.1.68}$$

The R vacuum $\xi_a = |a\rangle_{\mathrm{R}}$ must thus be labelled by a spinor index a, defined as in eq. (3.1.68). Note that in the case where $D = 2$, which is the string worldsheet case, the spinor index a consists of only one spin s_0 and can thus be written as $a = \pm$. This justifies the (3.1.41) definitions of the worldsheet spinors.

As all creation operators are space-time vectors, the spin-statistic of any state is identical to the spin-statistic of the vacuum it is constructed from. For instance, the closed strings RR and NSNS states are space-time bosons while their RNS and NSR states are space-time fermions.

Supercurrents and energy-momentum tensor expansions The supercurrents and energy-momentum tensors of eqs. (3.1.46) and (3.1.47) can be expanded as in the bosonic string case, giving expressions similar to eqs. (3.1.32) to (3.1.34), with additional contributions to the energy-momentum modes L_p coming from the worldsheet fermionic action and new supercurrent mode expansions

$$F_m = \sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot d_{n+m}: \quad \text{(R)}, \qquad G_r = \sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot b_{r+n}: \quad \text{(NS)}, \tag{3.1.69}$$

where : : indicate normal ordering. For concreteness we also explicitly write the normal ordered L_0 (see the bosonic case discussion under eq. (3.1.36)) as

$$L_0 = \frac{1}{2}\alpha_0^2 + N, \quad N = \sum_{n>0}\alpha_{-n}\cdot\alpha_n + \sum_{n>0} n d_{-n}\cdot d_n, \qquad \text{R sector}, \tag{3.1.70}$$

$$L_0 = \frac{1}{2}\alpha_0^2 + N, \quad N = \sum_{n>0}\alpha_{-n}\cdot\alpha_n + \sum_{r>\frac{1}{2}} r b_{-r}\cdot b_r \qquad \text{NS sector}. \tag{3.1.71}$$

These operators satisfy the Super Virasoro algebra, and can be used to implement the constraints and to obtain physical states $|\phi\rangle$ which must satisfy

$$L_p|\phi\rangle = 0, \forall p>0, \qquad L_0|\phi\rangle_{\mathrm{R}} = E_0 + E_0^{f\,\mathrm{R}}|\phi\rangle_{\mathrm{R}}, \quad L_0|\phi\rangle_{\mathrm{NS}} = E_0 + E_0^{f\,\mathrm{NS}}|\phi\rangle_{\mathrm{NS}}, \tag{3.1.72}$$

$$F_m|\phi\rangle_{\mathrm{R}} = 0, \forall m\geq 0, \qquad G_r|\phi\rangle_{\mathrm{NS}} = 0, \forall r>0, \tag{3.1.73}$$

where $E_0^{f\,\mathrm{NS}}$ and $E_0^{f\,\mathrm{R}}$ are the fermionic contributions to the zero-point energy in the NS and R sector, computed as in eqs. (3.1.36) and (3.1.37) by normal ordering the original L_0 operator. As shown in eqs. (3.1.70) and (3.1.71) the original non-ordered operator contains either a $nd_{-n}\cdot d_n$ sum in the R sector, which after regularization cancels exactly the bosonic contribution E_0, or a $rb_{-r}\cdot b_r$ sum in the NS sector, which simply adds to E_0. This results in

$$E_0^{NS} = E_0 + E_0^{f\,\mathrm{NS}} = -(D-2)/16, \qquad E_0^{\mathrm{R}} = E_0 + E_0^{f\,\mathrm{R}} = 0. \tag{3.1.74}$$

The mass formula (3.1.39) still holds in superstrings theory, using the new L_0 and zero-point energy E_0^{S} depending on the choice of sector S=R,NS. In particular, this formula shows that the R vacuum is not tachyonic, whereas the NS vacuum is.

As usual, one finds the low-energy spectrum acting on the vacuum with creation generators. Doing so, it is easy to see that at integer levels, *i.e.* levels $N\in\mathbb{N}$, the R sector contains twice the number of degrees of freedom as the NS sector at the equivalent (having the same mass) half-integer levels. Moreover, the R sector does not contain any state at half-integer levels which could have paired with the existing NS sector states at integer levels (for instance, the NS tachyonic vacuum). As mentioned earlier, the NS sector (in one copy) generates space-time bosons whereas the R sector generates space-time fermions. Hence, the aforementioned discrepancy in the number of degrees of freedom at each level explicitly breaks the potential space-time supersymmetry.

GSO projection A way to solve this problem, together with the problem of having a tachyonic vacuum, is to reduce the spectrum by projecting out the unwanted physical states. A consistent way to do so was introduced by Gliozzi, Scherk and Olive [67] and is known as the GSO projection. It uses the fermionic number operator

$$F = \begin{cases} \sum_{r=\frac{1}{2}}^{\infty} b_{-r}\cdot b_r, & \text{NS sector} \\ \sum_{n=1}^{\infty} d_{-n}\cdot d_n & \text{R sector} \end{cases}, \tag{3.1.75}$$

to define the G_{parity} operator

$$G_{\text{parity}} = \begin{cases} (-1)^{F+1}, & \text{NS sector} \\ \pm\Gamma_*(-1)^F, & \text{R sector} \end{cases}, \tag{3.1.76}$$

with $\Gamma_* = i^{(D-2)/2}\Gamma^0\Gamma^1\cdots\Gamma^{D-1}$ the chirality matrix made of out of the space-time Γ^i matrices satisfying the Clifford algebra in the representation under consideration. The GSO projection consists in keeping only states with $G_{\text{parity}} = 1$. In the R sector there is an arbitrary in the definition of G_{parity}, as can be seen from the $\pm$ sign in front of the Γ_* matrix, corresponding to the choice of states with positive or negative chirality. This chirality arbitrary is fixed after applying the GSO projection on the vacuum, *i.e.* after choosing a definite chirality for the vacuum state. According to the discussion under eq. (3.1.65), the vacuum is *a priori* described by a Majorana spinor having $2^{D/2}$ real components. The GSO projection amounts to make a chirality choice, similar to a Weyl condition, for this vacuum spinor, which then reduces to a Majorana-Weyl spinor with $2^{D/2-1}$ degrees of freedom. To conclude, we see that the GSO projection eliminates half of the space-time fermions and all the integer level space-time bosons. In particular, it eliminates the NS tachyonic vacuum.

3.1.3 Massless spectra

Open strings For open strings, only one copy of the S=R,NS sector has to be considered and the massless spectrum corresponds to the $e_i\, b^i_{-1/2}\,|0,p\rangle_{\text{NS}}$ state coming from the NS sector, and the R vacuum $e_a\,|a,p\rangle_{\text{R}}$, where e_i and e_a are polarizaton vectors and k indicates that the vacuum has definite momentum given by the $p^i \propto \alpha^i_0$ eigenvalue. The physical constraints eqs. (3.1.72) and (3.1.73) can be applied with L_0 and $G_{1/2}$ for the NS state and with L_0 and F_0 for the R state. On the NS state, the constraints lead to $p^i p_i = 0$, $e_i p^i = 0$. These two equations show that this state is invariant under $e_i \to e_i + p_i$, which suggests that it is a gauge field state. It has $D-2$ degrees of freedom coming from the i index. On the R state the constraints lead $p_i p^i = 0$, $p_i(\Gamma^i)^a_b e_a = 0$. This last condition is nothing but the Dirac equation for a massless spinor, which again reduces the degrees of freedom of the Majorana-Weyl vacuum to $2^{D/2-2}$. For $D = 10$ we see that the numbers $D-2 = 8$ and $2^{D/2-2} = 8$ are identical, showing that there is the same number of degrees of freedom for the R state and NS state. They correspond to a gauge field, in the 8_V representation of $SO(D-2) = SO(8)$ (the little group for massless particles), and its gaugino supersymmetric partner, in the 8_S representation. This massless spectrum suggests that worldsheet supersymmetry induces space-time supersymmetry. We summarize this massless spectrum in the following table.

NS	R
$e_i\, b^i_{-1/2}\,\|0,p\rangle_{\text{NS}}$	$e_a\,\|a,p\rangle_{\text{R}}$
8_V	8_S
$A^i \leftrightarrow A^i + \partial^i\lambda$	λ_a
gauge boson	gaugino

TABLE 3.1: Open superstrings massless spectrum

Type II closed string In closed superstrings, the left-moving and right-moving degrees of freedom decouple, hence one also has to impose a $\tilde{G}_{\text{parity}} = 1$ condition on the right-moving solution. As mentioned above eq. (3.1.76), there is an arbitrary in the choice of the chirality of GSO projection, namely in the $\pm$ appearing in the R sector. This sign can be chosen equal or opposite for left and right-movers, corresponding to a non-chiral theory named type IIB or a chiral theory named type IIA. We focus on type IIB and expose its massless spectrum content. The massless states are tensor products of left and right-movers' massless states corresponding to the open string massless states. There exist hence four possibilities coming from the NSNS, RR, NSR, RNS sectors which are respectively written as $b^i_{-1/2} \otimes \tilde{b}^j_{-1/2} \left|0, \tilde{0}, p\right\rangle_{\text{NSNS}}$, $\left|a, p\right\rangle_R \otimes \left|\tilde{b}, p\right\rangle_R$, $b^i_{-1/2} \left|0, p\right\rangle_{\text{NS}} \otimes \left|\tilde{a}, p\right\rangle_{\text{R}}$, $\left|a, p\right\rangle_{\text{R}} \otimes \tilde{b}^i_{-1/2} \left|\tilde{0}, p\right\rangle_{\text{NS}}$. They respectively sit in the $8_V \otimes 8_V$, $8_S \otimes 8_S$, $8_V \otimes 8_S$, $8_S \otimes 8_V$ representations of the massless particles little group. These representations can be decomposed into irreducible representations to extract the fundamental particles corresponding to the above states.

We do not address the general product representations decomposition into irreducible representations in detail. We only focus on the $8_S \otimes 8_S$ decomposition. Expressing spinors as in eq. (3.1.67) the product $\left|a, p\right\rangle_R \otimes \left|\tilde{b}, p\right\rangle_R$ can be written as

$$\left|a, p\right\rangle_R \otimes \left|\tilde{b}, p\right\rangle_R = \xi_a \otimes \tilde{\xi}_b = \prod_{i=0}^{\frac{D-2}{2}} (\Gamma^{i+})^{s_i + \frac{1}{2}} \xi \otimes \prod_{j=0}^{\frac{D-2}{2}} (\Gamma^{j+})^{\tilde{s}_j + \frac{1}{2}} \tilde{\xi} \tag{3.1.77}$$

$$\simeq \sum_{i=0}^{D} \alpha_i \bar{\tilde{\xi}} \Gamma^{\mu_1 \mu_2 \cdots \mu_i} \xi, \tag{3.1.78}$$

where $\bar{\tilde{\xi}} = \tilde{\xi}^t C$ is built from the conjugation matrix C, and $\Gamma^{\mu_1 \cdots \mu_m} = \Gamma^{[\mu_1} \cdots \Gamma^{\mu_m]}$ with $m \leq D$ form a basis of the Clifford algebra. The commutator in the exponents is justified by the fact that, due to the anticommutation relations of gamma matrices, symmetric products reduce to products containing smaller numbers of matrices. As for the expansion in the second line of eq. (3.1.78), it is justified by the fact that a product of two $2^{D/2}$ dimensional spinor representations is expected to transform as a 2^D tensor, which can be decomposed into lower-dimensional tensorial representations. The $\Gamma^{\mu_1 \cdots \mu_m}$ transforming as anti-symmetric tensors, they are identified to p-forms denoted $[p]$, and the fact that the ξ_a, $\tilde{\xi}_b$ spinors are of the same chirality implies vanishing $\alpha_{2k+1} = 0$ coefficients in the second line expansion. Finally, Hodge duality allows to rewrite $(D-p)$-forms as p-forms, such that we can write : $8_S \otimes 8_S \simeq [0] + [2] + [4]$.

This massless spectrum forms the gravity multiplet of $\mathcal{N} = 2$ SUSY in $D = 10$ dimension. We observe that, as in the open string case, worldsheet SUSY induces space-time SUSY, here with respect to the 32 supercharges carried by two Q_L, Q_R Majorana-Weyl supercharge spinors. We expose the type IIB spectrum in table 3.2.

NS NS	RR	NS R	R NS
$b^i_{-1/2} \otimes \tilde{b}^j_{-1/2} \left\|0, \tilde{0}, p\right\rangle_{\text{NSNS}}$	$\left\|a, p\right\rangle_R \otimes \left\|\tilde{b}, p\right\rangle_R$	$b^i_{-1/2} \left\|0, p\right\rangle_{\text{NS}} \otimes \left\|\tilde{a}, p\right\rangle_{\text{R}}$	$\left\|a, p\right\rangle_{\text{R}} \otimes \tilde{b}^i_{-1/2} \left\|\tilde{0}, p\right\rangle_{\text{NS}}$
$8_V \otimes 8_V$	$8_S \otimes 8_S$	$8_V \otimes 8_S$	$8_S \otimes 8_V$
$35_{\text{sym}} + 28_{\text{antisym}} + 1$	$[0] + [2] + [4]$	$8_C + 56_S$	$8_C + 56_S$
g_{ij} B_{ij} ϕ	C_0 C_2 C_4	$\lambda^{1\dot{a}}$ ψ^1_{ai}	$\lambda^{2\dot{a}}$ ψ^2_{ai}
graviton B form dilaton	axion 2, 4-forms	dilatino gravitino	dilatino gravitino

TABLE 3.2: Closed superstrings Type IIB massless spectrum

One can show, for instance by using the background fields methods to express the superstring action, that the string coupling constant is related to the expectation value of the dilaton

$$g_s = e^{\langle\phi\rangle}, \tag{3.1.79}$$

a fact that we use several times in the following.

3.1.4 Vacuum amplitudes

We can construct string theory vacuum amplitudes in analogy with the partition functions in field theories, by considering a string propagating freely and closing onto itself. Depending on the type of string and the specificity of the operation of gluing the string onto itself, the worldsheet can have different topologies. For closed strings, they correspond to a torus or a Klein bottle, when the string is glued back onto the original state respectively without or with the action of the Ω worldsheet parity operator, *i.e.* without our with orientation reversal. For open strings, these two possibilities correspond respectively to the topologies of an annulus or a Möbius strip.

Bosonic strings vacuum amplitudes The amplitude describing the propagation of a single closed string state $|\phi\rangle$ during a certain time, parametrized by τ_2, and closing onto itself up to a translation of the σ worldsheet coordinate, parametrized by τ_1, can be written as

$$Z(\phi,\tau) = \langle\phi|\, e^{-(2\pi\alpha' p^+\tau_2)H} e^{i(2\pi\alpha' p^+\tau_1)P_\sigma}\, |\phi\rangle = \langle\phi|\, e^{-\tau_2\pi\alpha'\sum_i p^i p^i} q^{N+E_0}\bar{q}^{\tilde{N}+\tilde{E}_0}\, |\phi\rangle\,, \tag{3.1.80}$$

with

$$q \equiv e^{2\pi i\tau},\ \bar{q} = e^{-2\pi i\bar{\tau}} \qquad \tau \equiv \tau_1 + i\tau_2,\ \tau_1,\tau_2 \in \mathbb{R}. \tag{3.1.81}$$

The Hamiltonian, of expansion shown in (3.1.31), is the time evolution operator. The σ-translation operator $P_\sigma = \frac{1}{2}\int_0^\pi P_i X'^i$ is defined from the conjugate momenta P^i and simply reads

$$P_\sigma = 1/\alpha' p^+ (N - \tilde{N}), \tag{3.1.82}$$

for closed strings in light-cone gauge. The $2\pi\alpha' p^+$ factors in the exponential of the first equality of eq. (3.1.80) are introduced for convenience, in order to obtain the simple τ_1, τ_2 dependence in the last equality.

To obtain the total partition function, one then has to trace over the Hilbert space $\mathcal{H}$ of physical states $|\phi\rangle$ and integrate over the τ_1, τ_2 parameters. In fact, the τ_1 integral automatically selects physical states, satisfying the level matching condition, as can be seen from the expression (3.1.82) of P_σ. The $\tau = \tau_1 + i\tau_2$ parameter can be identified to the torus Teichmüller parameter and in order to obtain distinct tori when integrating over τ, one should restrict to the fundamental domain, defined for instance through the relation $\mathcal{F} = \{-\frac{1}{2} < \tau_1 \leq \frac{1}{2},\ |\tau| \geq 1\}$. As can be seen from eqs. (3.1.62) to (3.1.65), any state $|\phi\rangle = \left|p^i, \{o_n^i\}\right\rangle$ is characterized by its momenta p^i and oscillators, denoted generically as $\{o_n^i\}$, so that the Hilbert space factorizes. Hence $Z(\phi,\tau)$ of eq. (3.1.80) can be rewritten as a product $Z_p(p^i,\tau)\, Z_{\text{osc.}}(\{o_n^i\},\tau)$ and the integral over all the possible momenta simply

gives

$$\begin{aligned}\prod_{i=1}^{D-2}\int\frac{dp^i}{2\pi}Z_p(p^i,\tau)Z_{\text{osc.}}(\{o_n^i\},\tau) &= \prod_{i=1}^{D-2}\int\frac{dp^i}{2\pi}\left\langle p^i\right|e^{-\tau_2\pi\alpha'\sum_i p^i p^i}\left|p^i\right\rangle Z_{\text{osc.}}(\{o_n^i\},\tau)\\ &= \left(\int\frac{dp}{2\pi}\left\langle p\right|e^{-\tau_2\pi\alpha' p^2}\left|p\right\rangle\right)^{D-2}\prod_{i=1}^{D-2}Z_{\text{osc.}}(\{o_n^i\},\tau)\\ &= \frac{1}{(4\pi^2\alpha'\tau_2)^{(D-2)/2}}\prod_{i=1}^{D-2}Z_{\text{osc.}}(\{o_n^i\},\tau). \end{aligned}\tag{3.1.83}$$

In the partition function, the sum over elements of the oscillators Hilbert space amounts to trace $Z_{\text{osc.}}(\tau)$ and the integral over the τ parameter space is made using the $SL(2,\mathbb{Z})$ invariant measure $d^2\tau/\tau_2^2$, so that the final torus amplitude reads

$$\mathsf{T} = \int_{\mathcal{F}}\frac{d\tau^2}{\tau_2^2}\frac{1}{\tau_2^{(D-2)/2}}\text{tr}_{\mathcal{H}_{\text{osc.}}}Z(\tau) = \int_{\mathcal{F}}\frac{d\tau^2}{\tau_2^2}\frac{1}{\tau_2^{(D-2)/2}}\text{tr}_{\mathcal{H}_{\text{osc.}}}\left(q^{N+E_0}\bar{q}^{\tilde{N}+\tilde{E}_0}\right).\tag{3.1.84}$$

The $\mathcal{H}_{\text{osc.}}$ trace can be written explicitely by expanding the state $|\phi\rangle = \left|\{o_n^i\}\right\rangle$ in terms of its oscillators. When considering the bosonic string, this expansion is similar to eq. (3.1.30) for both the left and right oscillation modes $\{o_n^i\} = \{\{\alpha_n^i\},\{\tilde{\alpha}_n^i\}\}$, so that the trace reads

$$\begin{aligned}\text{tr}_{\mathcal{H}_{\text{osc.}}}Z_{\text{osc.}}(\tau) &= \text{tr}_{\mathcal{H}_{\text{osc.}}}\left(q^{N+E_0}\bar{q}^{\tilde{N}+\tilde{E}_0}\right) = |q|^{2E_0}\prod_{i=1}^{D-2}\sum_{\{o_n^i\}\in\mathcal{H}_{\text{osc.}}}\left\langle\{o_n^i\}\right|q^N\bar{q}^{\tilde{N}}\left|\{o_n^i\}\right\rangle\\ &= |q|^{2E_0}\prod_{i=1}^{D-2}\left(\sum_{\{\alpha_n^i\}\in\mathcal{H}_{\text{osc.}}}\left\langle\{\alpha_n^i\}\right|q^N\left|\{\alpha_n^i\}\right\rangle\right)\left(\sum_{\{\tilde{\alpha}_n^i\}\in\mathcal{H}_{\text{osc.}}}\left\langle\{\tilde{\alpha}_n^i\}\right|\bar{q}^{\tilde{N}}\left|\{\tilde{\alpha}_n^i\}\right\rangle\right)\\ &= |q|^{-\frac{D-2}{12}}\left|\sum_{\{m\}}\langle 0|\prod_{a>1}(\alpha_a^1)^{m_a}q^N\prod_{b>1}(\alpha_{-b}^1)^{m_b}|0\rangle\right|^{2(D-2)} = |q|^{-\frac{D-2}{12}}\left|\sum_{\{m\}}\prod_{a>1}q^{m_a a}\right|^{2(D-2)}\\ &= |q|^{-\frac{D-2}{12}}\left|\prod_{a>1}\sum_{m=0}^{\infty}q^{ma}\right|^{2(D-2)} = |q|^{-\frac{D-2}{12}}\left|\prod_{a>1}\frac{1}{1-q^a}\right|^{2(D-2)}\\ &= \left|q^{\frac{1}{24}}\prod_{a>1}(1-q^a)\right|^{-2(D-2)} = |\eta(\tau)|^{-2(D-2)}. \end{aligned}\tag{3.1.85}$$

In the last line, we used the definition of the Dedekind η function given in eq. (A.3.4). To go from the first to the last line, we used the definition of N given in eq. (3.1.38), the value (3.1.37) of E_0 and the oscillators commutation relations. We emphasize that the oscillators contribution from one transverse space coordinate to the total amplitude thus simply reads

$$\begin{aligned}\frac{1}{\eta(\tau)} &: \quad \text{left-moving bosonic coordinate,}\\ \frac{1}{\bar{\eta}(\bar{\tau})} &: \quad \text{right-moving bosonic coordinate .}\end{aligned}\tag{3.1.86}$$

As mentioned at the beginning of this section, one can also consider the vacuum amplitude of a closed string propagating freely and glued back onto itself after orientation reversal. Orientation reversal is described by the action of the Ω worldsheet parity operator, which thus transforms the σ coordinate as $\sigma \to \pi - \sigma$. Reversing the orientation

of the string before gluing it back thus amounts to the insertion of Ω in the single state amplitude (3.1.80). The worldsheet parity operator exchanges left and right sectors, *i.e.* left and right oscillator modes, so that

$$\begin{aligned}\Omega\left|\phi\right\rangle = \Omega\left|\phi_L,\phi_R\right\rangle = \Omega\prod_{a=1}^{\infty}(\alpha_{-a}^{i_a})^{m_a}\prod_{b=1}^{\infty}(\tilde{\alpha}_{-b}^{i_b})^{\tilde{m}_b}\left|0,\tilde{0},p\right\rangle = \prod_{b=1}^{\infty}(\alpha_{-b}^{i_b})^{m_b}\prod_{a=1}^{\infty}(\tilde{\alpha}_{-a}^{i_a})^{\tilde{m}_a}\left|0,\tilde{0},p\right\rangle \\ = \left|\phi_R,\phi_L\right\rangle. \qquad (3.1.87)\end{aligned}$$

We deduce that the commutation relations of the oscillators (their orthogonality) impose that only left-right symmetric (or antisymmetric) states stay in the amplitude (3.1.80). Indeed, these symmetric states satisfy $\langle\phi|\,\Omega\,|\phi\rangle = \langle\phi_L,\phi_L|\,\Omega\,|\phi_L,\phi_L\rangle = \langle\phi_L,\phi_L|\phi_L,\phi_L\rangle = \langle\phi|\phi\rangle$ (for antisymmetric ones, there is a - sign), while for non left-right symmetric ones we have $\langle\phi|\,\Omega\,|\phi\rangle = 0$. The total amplitude after insertion of the Ω operator can be written in a way similar to the torus one (3.1.84) and, due to the above fact, it simplifies as follows

$$\mathsf{K} = \frac{1}{2}\int_{\mathcal{F}'}\frac{d\tau^2}{\tau_2^2}\frac{1}{\tau_2^{(D-2)/2}}\mathrm{tr}_{\mathcal{H}_{\text{osc.}}}\left(q^{N+E_0}\bar{q}^{\tilde{N}+\tilde{E}_0}\Omega\right) = \frac{1}{2}\int_{\mathcal{F}'}\frac{d\tau^2}{\tau_2^2}\frac{1}{\tau_2^{(D-2)/2}}\mathrm{tr}_{\mathcal{H}_{\text{left osc.}}}(q\bar{q})^{N+E_0}. \quad (3.1.88)$$

We see that the Klein bottle amplitude only depends on τ_2, appearing through $q\bar{q}$, which can be understood from the fact that $2i\tau_2$ is the Teichmüller parameter of its covering torus. The fundamental domain $\mathcal{F}'$ can be chosen to be $\mathcal{F}' = \tau_2 > 0$.

In order to have a unoriented theory, *i.e.* a worldsheet parity invariant theory, one should only keep invariant states. This can be achieved at the level of vacuum amplitudes by introducing the worldsheet parity projector $\frac{1}{2}(1+\Omega)$, so that the total unoriented amplitude is $\frac{1}{2}\mathsf{T}+\mathsf{K}$.

Open strings vacuum amplitudes can be worked out in a similar way, propagating the states using the Hamiltonian operator of open strings. However when gluing an open string back onto itself, there is no possibility of σ coordinate translation, because the string endpoints are references for this coordinate. Hence the amplitudes only depend on the τ_2 parameter, and the annulus amplitude reads

$$\mathsf{A} = \frac{N^2}{2}\int_{\tau_2>0}\frac{d\tau_2}{\tau_2^2}\frac{1}{\tau_2^{(D-2)/2}}\mathrm{tr}_{\mathcal{H}_{\text{osc.}}}q^{\frac{1}{2}(N+E_0)}, \qquad (3.1.89)$$

where the N^2 factor comes from the trace over the ab endpoints $U(N)$ Chan-Paton indices, on which we come back in the next section, below eq. (3.1.103). Note the $1/2$ factor in the exponential, coming from the expression of the Hamiltonian in the open string case.

Unoriented open string theories can be constructed in the same way as unoriented closed ones, acting with the Ω operator. This leads to the Möbius strip amplitude

$$\mathsf{M} = \frac{\epsilon N}{2}\int_{\tau_2>0}\frac{d\tau_2}{\tau_2^2}\frac{1}{\tau_2^{(D-2)/2}}\mathrm{tr}_{\mathcal{H}_{\text{osc.}}}\left(q^{\frac{1}{2}(N+E_0)}\Omega\right), \qquad (3.1.90)$$

The ϵN comes from the trace over the Chan-Paton indices after the insertion of Ω, which leads to an orthogonal, respectively symplectic (for even N), gauge group for $\epsilon = 1$, respectively $\epsilon = -1$. These groups are of order $N(N\mp 1)/2$ and correspond to the O^- and O^+ projections.

The appearance of these groups can be understood in the following way. As Ω exchanges the open string endpoints, it also exchanges the Chan-Paton indices: an ab string is transformed to a ba string. On a generic state, described via the Chan-Paton matrix λ_{ab}, the Ω action can be represented by a unitary matrix M such that $\Omega : \lambda \to M\lambda^t M^{-1}$. The

transposition of λ expresses the endpoints exchange. As $\Omega^2 = 1$, the M matrix should satisfy the constraint

$$\lambda = M(M\lambda^t M^{-1})^t M^{-1} = M(M^t)^{-1}\lambda M^t M^{-1} = \lambda, \quad i.e. \quad M = \pm M^t = \epsilon M^t. \tag{3.1.91}$$

For simplicity we follow our discussion supposing that the two possibilities for M are

$$\epsilon = +1 \ : \quad M = \mathbb{1}_N, \qquad \epsilon = -1 \ : \quad M = i \begin{pmatrix} 0 & \mathbb{1}_{N/2} \\ -\mathbb{1}_{N/2} & 0 \end{pmatrix}. \tag{3.1.92}$$

In the open string case under consideration, Ω acts on oscillators as $\Omega : \alpha_n^i \to (-1)^n \alpha_n^i$. This can be seen from the wave equation solution (3.1.19), the fact that $\Omega : \sigma \to \pi - \sigma$ and the relation $\cos(n\pi - n\sigma)) = (-1)^n \cos(n\sigma)$. Hence, we see that an open string state is invariant under worldsheet parity if the combined action of Ω on both the Chan-Paton matrix and the oscillators leaves the state invariant. Consider the $\alpha_{-1}^i |0, p\rangle$ state, which corresponds to a vector, as the $b_{-1/2}^i |0, p\rangle$ open superstring vector of (3.1). Under Ω, its oscillator changes sign, and for $\epsilon = 1$ the Chan-Paton matrix λ goes to $M\lambda^t M^{-1} = \lambda^t$ according to (3.1.92), thus we should impose $\lambda = -\lambda^t$ and only $N(N-1)/2$ vectors are kept by the projection, which corresponds to an orthogonal gauge group. For $\epsilon = -1$ the similar condition would keep the $N(N+1)/2$ vectors of a symplectic gauge group.

Superstrings vacuum amplitudes Superstrings vacuum amplitudes are constructed as in the bosonic case, tracing over the factorized Hilbert space. The factorized Hilbert space consists of the product of the Hilbert space of momenta, bosonic oscillators and fermionic oscillators, in the R or NS sector. Furthermore, one implements the GSO projection in superstring amplitudes by introducing the GSO projector $\frac{1}{2}(1 + G_{\text{parity}})$, where G_{parity} was defined in eq. (3.1.76). In the NS sector, this projector is just $\frac{1}{2}(1 - (-1)^F)$ while in the R sector it can be written as $\frac{1}{2}(1 \mp (-1)^F)$ depending on the choice of chirality of the R vacuum, see the discussion below eq. (3.1.76).

The general formulae for the superstrings amplitudes are thus similar to the bosonic expressions of eqs. (3.1.84) and (3.1.88) to (3.1.90), with the addition of the GSO projector and with oscillator Hilbert spaces now comprising fermionic oscillators. Of course the superstrings expressions of eqs. (3.1.70), (3.1.71) and (3.1.74) for N, E_0, $E_0^{f\,S}$ (S=R,NS) must be used in the superstring amplitudes and depend on the sector under consideration.

We do not expose the computation of the oscillators traces but explain shortly the differences between the fermionic coordinates and the bosonic coordinates oscillators contributions. The derivation for the fermionic coordinates is similar to the one for bosonic coordinates (3.1.85), except that due to the fermionic nature of the d_{-n}^i, b_{-r}^i oscillators, each oscillator "multiplicity" can only take the values $m_n, m_r = 0, 1$. Thus, in the second-to-last line of (3.1.85), the term $\prod_{a>1} \sum_{m=0}^{\infty} q^{ma}$ must be replaced by $\prod_{a>1}(1 + q^{a-\nu_S})$ where $\nu_R = 0$ and $\nu_{NS} = \frac{1}{2}$ depend on the sector S=R,NS. Using the fermionic dependences in N and E_0^S, the computation of the oscillators traces then leads to the following fermionic

coordinate contributions

$$\frac{\vartheta_3^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}(\tau) \;:\quad \text{fermionic coordinate, no } G_{\text{parity}} \text{ insertion, NS sector,} \tag{3.1.93}$$

$$-\frac{\vartheta_4^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}(\tau) \;:\quad \text{fermionic coordinate, } G_{\text{parity}} \text{ insertion, NS sector,} \tag{3.1.94}$$

$$-\frac{\vartheta_2^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}(\tau) \;:\quad \text{fermionic coordinate, no } G_{\text{parity}} \text{ insertion, R sector,} \tag{3.1.95}$$

$$\mp\frac{\vartheta_1^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}(\tau) \;:\quad \text{fermionic coordinate, } G_{\text{parity}} \text{ insertion, R sector,} \tag{3.1.96}$$

where the infinite products have been rewritten in terms of the modular ϑ_i functions defined in eqs. (A.3.1) and (A.3.2). The total superstring amplitude is thus obtain by multiplying the momenta contribution, bosonic contribution and fermionic contribution, summing over the R and NS sectors with or without G_{parity} operator. Putting everything together, and considering $D-2$ transverse coordinates, we see that the total contribution can be written in the R, NS sectors as

$$\frac{V_{D-2}}{\tau_2^{\frac{D-2}{2}}\eta^{D-2}} = \frac{\vartheta_3^{\frac{D-2}{2}} - \vartheta_4^{\frac{D-2}{2}}}{2\eta^{\frac{D-2}{2}}\tau_2^{\frac{D-2}{2}}\eta^{D-2}} \;:\quad \text{total GSO projected contribution, NS sector,} \tag{3.1.97}$$

$$-\frac{(S/C)_{D-2}}{\tau_2^{\frac{D-2}{2}}\eta^{D-2}} = \frac{-\vartheta_2^{\frac{D-2}{2}} \mp \vartheta_4^{\frac{D-2}{2}}}{2\eta^{\frac{D-2}{2}}\tau_2^{\frac{D-2}{2}}\eta^{D-2}} \;:\quad \text{total GSO projected contribution, R sector,} \tag{3.1.98}$$

where V, S and C are level-one $so(2p)$ characters [48] defined in eq. (A.3.3). The choice of the S or C character in the R sector corresponds to the arbitrary choice of chiraltiy for the vacuum already mentioned several times. In Type IIB string theory, which has the same chirality choice for the left and right R vacua, the torus amplitude thus reads

$$\mathsf{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{|V_8 - S_8|^2}{\tau_2^4 |\eta|^{16}}. \tag{3.1.99}$$

Similar expressions can be obtained for the unoriented closed string amplitudes K and the oriented and unoriented open strings ones A, M.

We conclude this short description of vacuum amplitudes by mentioning that by definition, amplitudes are traces over the Hilbert space of physical states, with insertion of the exponential of the Hamiltonian. By expanding the vacuum amplitudes in powers of q, *i.e.* by power expanding $\eta(\tau)$, $\vartheta_i(\tau)$, one can extract the spectrum of the theory. Namely the multiplicities of the states are expansion coefficients while the masses are related (up to a factor) to the powers of q. For instance, the expansion of eq. (3.1.99) in terms of powers of $q\bar{q}$ starts as

$$\mathsf{T} \propto \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}(1-1)\left(64\,(q\bar{q})^0 + 2048\,(q\bar{q})^{\frac{1}{2}} + \cdots\right), \tag{3.1.100}$$

and we recover the 64 degrees of freedom of the massless spectrum of table 3.2 coming from each sector, NSNS, RNS, NSR, RR.

Tadpole cancellation and necessity of closed and open strings We conclude this part with a short but essential comment on the consistency of unoriented superstrings theories. By inspecting the ultraviolet behavior of the above amplitudes for small time, one is able to see that the Klein bottle, annulus and Möbius amplitudes present UV divergences. The torus amplitude is protected by modular invariance. These UV divergences correspond to transverse channel IR divergences originating from the exchange of tachyonic and massless modes, the latter being impossible to regularize through the introduction of a cutoff scale.

Cancellation of these divergences can be achieved by combining them with each other, which leads to the so-called tadpole cancellation conditions. As open string amplitudes include Chan-Paton factors, tadpole cancellation conditions are satisfied only for specific gauge groups. In the bosonic case, the gauge group is $SO(8192)$ [68, 69, 70]. In superstring theories and their orientifold compactifications, see section 3.2, tadpole cancellation conditions depend on the specificities of the theory but can be related in a very nice way to the presence of Dp branes, which we introduce in section 3.1.5, and Op planes implementing the orientifolding. These non-perturbative objects can be seen as the worldsheet boundaries spanned by the open strings' endpoints and carry charges under the massless RR Cp forms (they also source NSNS fields). Tadpole cancellation conditions are thus global RR charges neutrality conditions and ask for the simultaneous presence of both Dp branes and Op planes. They thus impose specific relations between the numbers of such non-perturbative objects, which are of great importance in superstring compactification models. We highlight that the necessary presence of Dp branes to cancel Op branes implies the need of open strings in any superstring theory containing unoriented closed strings.

Finally, one striking fact is that tadpoles conditions are related to cancellations of the effective action anomalies. The cancellation mechanism was discovered for type I superstrings by Green and Schwarz [71] and similar mechanisms have been unveiled in more generic superstring orientifold compactifications setups [72, 73, 74, 75, 76].

3.1.5 Dp branes

Construction Dp branes are extended objects spanning p spatial directions, called the worldvolume of the brane. Their existence concretized in 1995 [77, 78, 79] and played a major role in the "second revolution" of strings. Reviews on Dp branes can be found in [80, 81, 82, 83]. As Dp branes do not span the entire space-time (except for D$(D-1)$-branes) they explicitly break the D dimensional Poincaré symmetry of the theory. Dp branes are dynamical non-perturbative objects, the excitations of which are described by open strings, with different boundary conditions on the directions spanned by the Dp brane or the transverse directions. Namely, open string endpoints are attached to Dp branes, *i.e.* constrained through Dirichlet boundary conditions (3.1.2), DD, in the transverse directions, while they are free to move on the Dp brane, hence satisfying Neumann boundary conditions (3.1.3), NN, on the worldwolume directions. Concretely, we describe fluctuations of a Dp brane, taken for simplicity parallel to the first $p+1$ space-time direction, $\mu = 0, \dots, p$, and thus localized in the transverse directions $j = p+1, \dots, D-1$, by open strings satisfying the following boundary conditions

$$X^j(\tau, \sigma = 0) = X^j(\tau, \sigma = \pi) = X^j_{\mathrm{D}p}(\tau), \ j = p+1, \dots, D-1, \qquad \text{DD}, \tag{3.1.101}$$

$$\partial_\sigma X^\mu(\tau, \sigma = 0) = \partial_\sigma X^\mu(\tau, \sigma = \pi) = 0, \ \mu = 0, \dots, p \qquad \text{NN}, \tag{3.1.102}$$

where X^j_{Dp} are the localizations of the Dp brane in the transverse directions. Similar boundary conditions are applied to the fermionic partners of the bosonic coordinates. One can solve the equation of motions from the superstring action and find the bosonic and fermionic coordinates expansions. The NN bosonic coordinate expansion was for instance shown in eq. (3.1.19). There is no momentum in the DD expansion because the string's center of mass is localized on the Dp brane. The oscillators structures are in fact similar in both the sectors, and the massless states correspond to those described in table 3.1. It thus consists in the R vacuum $|a,p\rangle$, which is a space-time fermion, and the NS states $b^\mu_{-1/2}|0,p\rangle$, $b^j_{-1/2}|0,p\rangle$, which from the worldvolume point of view correspond to a gauge boson A^μ and $D-1-p$ scalars ϕ^j. This is the field content of a $U(1)$ vector multiplet with 16 supercharges, in $p+1$ dimensions. In fact, Dp branes preserve half of the bulk supersymmetries described in type IIB at the end of section 3.1.3. Indeed the open strings boundary conditions impose a relation between degrees of freedom of the left Q_L and right Q_R SUSY charges of the bulk. Only a linear combination $Q = Q_L + \Gamma^{(p)}_* Q_R$, is preserved and this reduces the initial 32 degrees of freedom to only 16. Here $\Gamma^{(p)}_*$ is the Dp-brane chirality matrix.

D-brane stacks and Chan-Paton indices One can also consider stacks of N parallel Dp branes, *i.e.* Dp branes spanning the same worldvolume directions but localized at different or equal points in the transverse directions. When branes are localized at the same points, they are said to be coincident, and we expect the symmetry of this configuration to reflected in the spectrum of the theory. Interactions between parallel branes vanish exactly, an important fact is that allows to consider stable configurations with parallel stack of branes, labeled by an index a. As explained before, open strings are attached to Dp branes and in the case of parallel branes, their endpoints can sit on different branes. For an open string stretching between two branes Dp_a, Dp_b, located at $X^j_a(\tau)$, $X^j_b(\tau)$, the DD boundary conditions of (3.1.101) transform to

$$X^j(\tau,\sigma=0) = X^j_a, \quad X^j(\tau,\sigma=\pi) = X^j_b, \qquad j = p+1,\dots,D-1. \tag{3.1.103}$$

Open strings should thus carry two a, b indices labeling the brane at which each end is attached. These indices are called Chan-Paton indices and add additional degrees of freedom to open string states. For N parallel branes, these indices take the values $a,b = 1,\dots,N$ and there are N^2 possibilities for the ab couple of indices. A generic open strings state can be described by an hermitian Chan-Paton matrix λ_{ab} and written as $\lambda_{ab}|\phi,ab\rangle$. The matrix λ_{ab} is a representation of the $U(N)$ gauge group formed by N coincident branes. The $U(1)^N = U(1)_1 \times U(1)_2 \times \cdots \times U(1)_N$ Cartan subalgebra is generated by strings with endpoints attached at the same brane, *i.e.* aa strings, each of these states indeed generating a $U(1)$ factor. Strings interactions rules show that open strings carry $+1$ charge at one end and -1 charge at the other end, under the $U(1)_a$, $U(1)_b$ groups associated to the Dp_a, Dp_b branes they are attached to. In fact, writing $U(N) = SU(N) \times U(1)$, they transform in the $\square_{+1} \times \overline{\square}_{-1}$ adjoint representation of $U(N)$. In the unoriented case described in some detail under eq. (3.1.90), as the string endpoints are identified, the states are in the $\square_{\pm 1} \times \square_{\pm 1} = \begin{smallmatrix}\square\\\square\end{smallmatrix} + \square\square + 1$ where the two first representations correspond to the O^- and the O^+ projections.

The expansion (3.1.103) leads to a mass formula similar to eq. (3.1.39), but for open strings with separated endpoints, which reads

$$M_{ab}^2 = \sum_{j=p+1}^{D-1} \left(\frac{X_a^j - X_b^j}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'}(N + E_0^S), \tag{3.1.104}$$

where as usual, S=R,NS is the sector under consideration. For coincident branes, *i.e.* equal X_a^j for all a, we see that there exist massless states for each Chan-Paton states ab. Nevertheless if substacks of $N_1, \dots, N_m$ branes are separated in transverse space, the lowest-lying ab states, with a and b in different stacks, acquire positive masses, while the lowest-lying aa states remain massless. This is the signature of the $U(N) \to U(N_1) \times \cdots \times U(N_p)$ symmetry breaking. Of course, when each of the N initial branes is at a different localization, $U(N)$ is maximally broken to $U(1)^N$.

Effective action As for the fundamental string, one can consider a natural Dp brane worldvolume action by simply integrating the square root of its worldvolume metric determinant. One could express the transverse coordinates with respect to the ones spanned by the Dp branes, but a more convenient way is to use worldvolume coordinates σ^α, $\alpha = 0, \dots, p$ and to embed the Dp brane into the total space-time through $X^\mu(\sigma^\alpha)$. The natural bosonic action would then be

$$S = -T_p \int d^{p+1}\sigma \sqrt{-\det G_{\alpha\beta}}, \qquad G_{\alpha\beta} \equiv \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} g_{\mu\nu}, \tag{3.1.105}$$

which is similar to the Nambu-Goto action (3.1.5), and where T_p is the brane tension to determine. We see that in a gauge where the σ^α coordnates are identified to the first $\mu = 0, \dots, p$ space-time coordinates X^μ, this action will naturally describe the dynamics of the X^j transverse coordinates, identified to the $D - 1 - p$ scalars ϕ^i described below eq. (3.1.102). One would thus expect that the worldvolume action also includes coupling to the wolrd-volume gauge boson A^μ and to the other fields of the theory under consideration, namely the C_p forms, dilaton ϕ and B_2 in type IIB. A natural extension, to include the ϕ, B_2 and A^μ can be found using the background fields methods and lead to the Dirac-Born-Infeld action

$$S_{\text{DBI}} = -\mu_P \int d^{p+1}\sigma\, e^{-\phi} \sqrt{-\det(G_{\alpha\beta} + 2\pi\alpha' \mathcal{F}_{\alpha\beta})}, \qquad \mu_p = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}}}, \tag{3.1.106}$$

$$B_{\alpha\beta} \equiv \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} B_{\mu\nu}, \quad F_{\alpha\beta} = \partial_{[\alpha} A_{\beta]}, \quad \mathcal{F}_{\alpha\beta} \equiv F_{\alpha\beta} + \frac{1}{2\pi\alpha'} B_{\alpha\beta}. \tag{3.1.107}$$

We can easily extract the expression of the string tension from eq. (3.1.106). We can also obtain the Yang-Mills action through the first terms of the $F^{\alpha\beta}$ expansion. From this Yang-Mills action we deduce the expression for the gauge coupling constant. These two constants are

$$T_p = \mu_p e^{-\langle\phi\rangle} = \frac{\mu_p}{g_s} = \frac{1}{g_s (2\pi)^p \alpha'^{\frac{p+1}{2}}}, \qquad g_{Dp}^2 = g_s (2\pi)^{p-2} \alpha'^{\frac{p-3}{2}} = \frac{1}{(2\pi\alpha')^2 T_p}. \tag{3.1.108}$$

We used the expression (3.1.79) relating the string coupling g_s to the dilaton VEV. A second natural action is the integral of the C_{p+1} RR form over the $p+1$ dimensional Dp brane worldvolume. Indeed, the match in the dimension of this differential form with the one

of the brane worldvolume allows to write

$$S = \mu_p \int d^{p+1}\sigma\, C_{p+1}. \tag{3.1.109}$$

Moreover, one can also construct $p+1$ dimensional differential forms by combining lower dimensional RR forms C_{p-1}, C_{p-3}, ... with the $\mathcal{F}$ 2-form. This leads to the Chern-Simons (CS) action (in flat space-time)

$$S_{\text{CS}} = \mu_p \int d^{p+1}\sigma \left(C \wedge e^{2\pi\alpha'\mathcal{F}}\right)_{p+1}, \qquad C \equiv \sum_{i=0}^{p+1} C_i, \tag{3.1.110}$$

where $()_{p+1}$ indicates that only the $p+1$ forms are extracted when expanding the wedge product of C with the exponential term. Hence, the CS action contains a sum over terms such as $C_{p-1} \wedge \mathcal{F}$, $C_{p-1} \wedge \mathcal{F}$, $C_{p-3} \wedge \mathcal{F}^2$, ..., which show that non-trivial $\mathcal{F}$ backgrounds can indue lower dimensional $D(p-1)$, $D(p-3)$, ..., charges. The coupling of the RR charges to the Dp branes is one of various arguments signaling that in type IIB theories, only Dp branes with odd p are interesting, as only the C_p odd forms exist.

The full SUSY worldvolume action is the supersymmetric extension of $S = S_{\text{DBI}} + S_{\text{CS}}$. We do not show explicitely its construction here. Of course, the case of a D3 brane was studied in some detail in the previous sections, where partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ SUSY breaking was studied.

T-duality and Wilson lines We conclude shortly by signaling the effect of T-duality on Dp branes. We introduce compactifcation elements in section 3.2.1 but mention that, in the case of circle compactification, T-duality along a compact direction X^n changes the compact radius from R to α'/R, or similarly changes the sign of the X^n_R right coordinate, *i.e.* $X^n_R \to -X^n_R$. This last fact amounts to a change of the boundary condition, from Dirichlet to Neumann, as can be seen from the solutions of the wave equation (which are sine or cosine functions), and hence T-duality turns a Dp brane into a $D(p-1)$ brane. It also brings type IIB theory to type IIA. T-duality is a very convenient way to describe similar phenomena with different objects.

As an example, we explain how T-duality allows a natural interpretation of Wilson lines as the T-dual of brane separations introduced previously. Open strings stretching between two branes separated in the compact direction X^n can wind several times, but as explained before, cannot carry momentum because their endpoints are fixed to the branes. The mass formula (3.1.104) gets a contribution from the winding number similar to the closed string one (3.2.3). The new mass formula reads

$$M^{(n)^2}_{ab} = \left(\frac{Rw}{\alpha'} + \frac{X^n_a - X^n_b}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N + E^S_0), \tag{3.1.111}$$

and we see that brane separations amount to shifts in winding number w, by $(X^n_a - X^n_b)/2\pi R$. We now consider a T-duality over the compact direction X^n , hence changing the radius R to $R' = \alpha'/R$. Replacing R by R' in the above mass formula, and writing the brane positions X^n_a in terms of the radius R as $X^n_a = \theta^n_a R$, it can be written as

$$M^{(n)^2}_{ab} = \left(\frac{w}{R'} + \frac{\theta^n_a - \theta^n_b}{2\pi R'}\right)^2 + \frac{1}{\alpha'}(N + E^S_0). \tag{3.1.112}$$

This mass formula takes the exact same form as for open strings with NN boundary conditions, as we expected from T-duality. These boundary conditions allow for momentum

number $k' \equiv w$, as in the closed string case (3.2.3), but no winding. This is understandable because with NN boundary conditions, the string endpoints move freely in the compact direction and one can unwind strings at no cost.

We see from eq. (3.1.112) that the T-dual of the brane separation implies a shift in the momentum $k' = w$ by $(\theta_a^n - \theta_b^n)/2\pi$. This kind of shift has a natural interpretation in terms of Wilson lines, which are (periodic) constant backgrounds for A_n^a, the gauge field of the $U(1)_a$ factor of the Dp_a brane. The effect of such gauge fields can be described in the background field method by adding a term of the form

$$\delta S = \frac{1}{4\pi\alpha'}\left[q\int d\tau A_\mu \partial_\tau X^\mu\right]_{\sigma=\pi}^{\sigma=0} = \frac{1}{4\pi\alpha'}\int d\tau (A_n^a - A_n^b)\partial_\tau X^n, \tag{3.1.113}$$

to the worldsheet action (3.1.11). The last equality simply takes into account that the string endpoints are charge ± 1 under $U(1)_a$ and $U(1)_b$, as mentioned earlier. This additional term simply amounts to momentum shift $p^n = k/R + (A_n^a - A_n^b)$, as can be seen from the equation of motions or the definition of the canonical momentum. Writing the periodic Wilson line as $A_n^a \equiv \theta_n^a/2\pi R'$, and using the open string mass formula gives exactly (3.1.112). This shows that the T-dual of the brane separation is indeed a Wilson line on the dual circle. As brane separations shift windings and Wilson lines shit momenta, this duality could have been guessed from the fact that T-duality exchanges these numbers. Of course, this holds for higher dimensional tori.

3.2 Type IIB compactifications and de Sitter vacua

The use of type II theories in string phenomenology is motivated by the fact that they allow for a rather good understanding of moduli stabilization mechanisms, and applications to cosmology, on which we come back in some detail section 4.1. Total moduli stabilization can be realized in type IIA superstrings at the classical level, by the inclusion of fluxes. Nevertheless de Sitter (dS) solutions seem very difficult to find in this context [84, 85, 86] and cosmological consequences are thus difficult to implement [87, 88]. Moreover the back reaction of the fluxes is difficult to evaluate in a controlled way and can go against some of the hypotheses, such as the use of Calabi-Yau internal spaces [89]. On the other hand, type IIB moduli stabilization makes use of quantum corrections, due to the no-scale structure of its classical scalar potential, and this can in principle be done in a controlled way through the large volume and small string coupling approximations. Before introducing these quantum corrections in section 3.2.3, we first review the general ideas behind string compactifications in section 3.2.1, as well as standard ways to stabilize all closed strings moduli while obtaining de Sitter vacua, in section 3.2.2. Cosmological motivations for dS vacua will be explained again in section 4.1.

3.2.1 Compactification

When considering phenomenological consequences of superstrings theory there is a crucial need to go from the initial ten-dimensional space-time to the four-dimensional Minkowski space-time M_4 that describes our accessible world at low energies. The usual way is to consider a six-dimensional compact space X_6, with a compactification scale small enough to generate high energy scales, so that these compact directions are invisible at our scales. The X_6 compact space is referred to as "internal" space whereas the M_4 remaining dimensions are referred as "our" space-time. The total space-time is considered to factorize as $M_4 \times X_6$.

Toroidal compactification *Circle compactification* Considering first only the X^n coordinate compactified on a circle of radius R, *i.e.* taking the identification $X^n \sim X^n + 2\pi R$, one should allow for new string boundary conditions. For instance, closed strings can wind the compact direction several times, namely

$$X^n(\tau, \sigma + \pi) = X^n(\tau, \sigma) + 2\pi R w \sim X^n(\tau, \sigma), \qquad w \in \mathbb{Z}. \tag{3.2.1}$$

Moreover, the momentum in the compact direction is now quantized,

$$p^n = \frac{k}{R}, \quad k \in \mathbb{Z}. \tag{3.2.2}$$

These two combined new features lead to a modification the mass formula (3.1.39), which takes the new form

$$M^2 = \frac{k^2}{R^2} + \frac{R^2 w^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} + E_0 + \tilde{E}_0), \tag{3.2.3}$$

where we see that a Kaluza-Klein (KK) tower of states is generated. The mass gap between each KK level being inversely proportional to the compactification radius, only the zero modes should be observable and thus for small internal radius. The above mass formula can be understood from the field theory point of view by Fourier expanding the massless particles wave functions and looking at the resulting action. Take for instance a scalar field such that $\phi(x^\mu, x^n + 2\pi R) = \phi(x^\mu, x^n + 2\pi R)$, with $\mu \neq n$, and Fourier expand it according to

$$\phi(x^\mu, x^n) = \sum_{k \in \mathbb{Z}} \phi_k(x^\mu) e^{ikx^n/R}, \qquad \mu \neq n. \tag{3.2.4}$$

The kinetic terms $\int d^D x (\partial\phi)^2$ generate kinetic terms for the zero modes ϕ_0, as well as a sum $\sum_k \int d^{D-1}x(\partial^\mu \phi_k \partial \phi_{-k} + \frac{k^2}{R}\phi_k\phi_{-k})$, which gives kinetic terms and mass terms for the higher Fourier modes. This gives exactly the KK mass of (3.2.3). On the other hand, the winding number mass contribution is purely stringy. It is obtained in the same way as the mass formula (3.1.39). The difference is that the mode expansion includes winding contribution $2Rw\sigma$, so that (3.2.1) is satisfied.

Toroidal compactification To go from ten to four dimensions, compactification should be realized on more than one compact direction. A direct extension of circle compactification is toroidal compactification, where the internal space is a d dimensional torus T^d

$$T^d \equiv \mathbb{R}^d / 2\pi\Lambda_d, \quad \Lambda_d = \left\{ \sum_{i=1}^{d} q^i \mathbf{R}_i \ ; \ q^i \in \mathbb{Z} \right\}. \tag{3.2.5}$$

The $\mathbf{R}_i$ are linearly independent vectors of $\mathbb{R}^d$, of norm R_i, generating the Λ_d lattice. The dual lattice Λ_d^* is generated by the dual vectors $\mathbf{R}_i^*$ satisfying

$$\mathbf{R}_k \cdot \mathbf{R}^{*l} = \mathbf{R}_k^M \mathbf{R}_M^{*l} = \delta_k^l, \tag{3.2.6}$$

where here the scalar product is the euclidean $\mathbb{R}^d$ scalar product. The torus metrics are defined through

$$g_{ij} = \mathbf{R}_i \cdot \mathbf{R}_j, \quad g^*_{ij} \equiv g^{ij} = \mathbf{R}^{*i} \cdot \mathbf{R}^{*j} = (g_{ij})^{-1}, \mathbf{R}^{*i} = g^{ij}\mathbf{R}_j, \tag{3.2.7}$$

where the last equalities are obtained through eq. (3.2.6). The T^d torus windings and momenta are lattice and dual lattice vectors

$$\mathbf{L} = n^l \mathbf{R}_l \in \Lambda_d, \quad n^l \in \mathbb{Z}, \qquad \mathbf{p} = m_k \mathbf{R}^{*k} \in \Lambda_d^*, \quad m_k \in \mathbb{Z}, \tag{3.2.8}$$

so that the closed string boundary condition, similar to the compact circle case (3.2.1), is

$$\mathbf{X}(\tau, \sigma + \pi) = \mathbf{X}(\tau, \sigma) + 2\pi \mathbf{L} \sim \mathbf{X}(\tau, \sigma), \tag{3.2.9}$$

where here $\mathbf{X}$ denotes the torus coordinate vectors. Introducing left and right momenta

$$\mathbf{p}_{L,R} = \mathbf{p} \pm \mathbf{L} = \left(m_k \pm g_{kl} n^l \right) \mathbf{R}^{*k}, \tag{3.2.10}$$

the generalization of the mass formula (3.2.3), in absence of B_2 background, reads

$$M^2 = \mathbf{p}_L^2 + \mathbf{p}_R^2 + \frac{2}{\alpha'}(N + \tilde{N} + E_0 + \tilde{E}_0), \tag{3.2.11}$$

and the new level matching condition is $\tilde{N} = N + \mathbf{p} \cdot \mathbf{L}$. We also mention that due to the momentum and winding discrete numbers, the momentum integral (3.1.83) of the torus amplitude is replaced by a lattice sum Λ, so that the contribution for the d coordinates is replaced as

$$\frac{1}{\tau_2^{\frac{d}{2}} |\eta(\tau)|^{2d}} \longrightarrow \Lambda = \sum_{m,n} \frac{q^{\frac{\alpha'}{4} \mathbf{p}_L \cdot \mathbf{p}_L} \bar{q}^{\frac{\alpha'}{4} \mathbf{p}_R \cdot \mathbf{p}_R}}{|\eta(\tau)|^{2d}} \,. \tag{3.2.12}$$

One can apply the above formalism to compact spaces $X_6 = T^6$, $T^2 \times T^2 \times T^2, \ldots$. These kinds of compactifications are called toroidal compactifications and will be used in the following to construct toroidal orbifolds, which reduce the number of supersymmetries of the compact theories.

Calabi-Yau compactification One should also consider more general internal spaces than simple tori. Except in the last line of the above paragraph, we have not mentioned supersymmetry in the two previous simple examples. Nevertheless, one usually would like to keep at least $\mathcal{N} = 1$ SUSY in the compactified low energy theory. This is justified by phenomenological implications of SUSY, but also because SUSY theories are easier to study and have many already mentioned interesting features such as the absence of vacuum tachyons or nice stability properties. In order for the $4d$ low-energy theory to conserve some supersymmetry, there must exist globally well-defined supercharges on the X_6 compact space. A globally well-defined supercharge is a local supercharge which is not rotated upon parallel transport on X_6. The rotations a local spinor undergoes under parallel transport around closed loops on X_6 form the holonomy group of X_6. Hence, for a $6d$ space, it is generally the $SO(6)$ group, the spinor representation of which contains no singlet. In order to preserve SUSY, the internal space X_6 must thus be a manifold with special holonomy, *i.e.* a manifold with holonomy group being a subgroup of $SO(6)$ with spinorial representation containing at least one singlet, associated to a conserved supercharge. This is the case of manifolds with $SU(3)$ holonomy. The spinorial $SO(6)$ representation $\mathbf{4}$ decomposes as $\mathbf{4} = \mathbf{3} + \mathbf{1}$ so that the initial $\mathbf{16}$ spinorial representation of $SO(10)$ on $X_6 \times M_4$ decomposes as $(\mathbf{3}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{2}') + (\mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}')$ under $SU(3) \times SO(3,1)$. The last two representations contain the preserved $\mathcal{N} = 1$ supercharges of the 4d theory.

CY spaces There are no generic ways to derive the metrics of compact manifolds with $SU(3)$ holonomy group, but the so-called Calabi-Yau condition is sufficient to ensure the existence of metrics for manifolds with $SU(N)$ holonomy. This condition asks the manifold to be Kähler with vanishing first Chern class. We recall that a $2N$-dimensional Kähler manifold is a hermitian manifold with closed Kähler form J. A globally well-defined complex structure J_i^j on a complex manifold allows to define holomorphic coordinates z^i, $\bar{z}^i$ for which $J = \text{diag}(i, \dots, i, -i, \dots, -i)$. For an hermitian manifold, the metric $g_{i\bar{j}}$ only contains mixed terms. The Kähler condition then reads

$$dJ = 0, \qquad J \equiv J_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = J_i^k g_{k\bar{j}}\, dz^i \wedge d\bar{z}^j = i g_{k\bar{j}}\, dz^i \wedge d\bar{z}^j, \tag{3.2.13}$$

where J is the Kähler 2-form. The Kähler condition ensures that parallel transport does not mix the z^i and $\bar{z}^j$ coordinates, hence reducing the generic $SO(2N)$ holonomy group to the $U(N)$ group. The possible $U(1)$ factor holonomy can be eliminated through the vanishing of the first Chern class condition. *6d* manifolds X_6 satisfying the Calabi-Yau condition, called Calabi-Yau (CY) threefolds, hence have special $SU(3)$ holonomy. Calabi-Yau manifolds can be classified through their topologically invariant Hodge numbers $h_{p,q}(X_6)$. Hodge numbers are the dimensions of the p, q Hodge cohomology classes entering the Hodge decomposition, which for X_6 Kähler manifolds reads

$$H^3(X_6) = H^{3,0}(X_6) + H^{2,1}(X_6) + H^{1,1}(X_6) + H^{1,2}(X_6) + H^{0,3}(X_6), \tag{3.2.14}$$

where the p, q exponents denote respectively the number of holomorphic and antiholomorphic indices. For instance $dz^1 \wedge dz^2 \wedge d\bar{z}^{\bar{1}}$ is a $2,1$ form. For CY threefolds, $h_{2,1} = h_{1,2}$ and $h_{3,0}(X_6) = \dim H^{3,0}(X_6) = 1$ and thus there exists only a unique holomorphic 3,0 form Ω. This form can be used to define the complex structure moduli in the following way.

Consider a variation $\delta g_{i\bar{j}}$ of the diagonal metric terms. In order the new metric to remain a Kähler manifold metric, it should be hermitian, and new coordinates cancelling the above metric variation should be found. As this cannot be done by holomorphic coordinate transformations, it should correspond to a change in the complex structure. By using the holomorphic 3,0-form Ω, we can expand $\delta g_{i\bar{j}}$ on a base α_K, $K = 1, \dots, h_{2,1}$ of $2,1$-forms

$$I_{ij\bar{k}} = \Omega_{ijl} g^{l\bar{m}} \delta g_{\bar{m}\bar{k}} = \sum_{K=1}^{h_{2,1}} u_K \alpha_{K\, ij\bar{k}}. \tag{3.2.15}$$

The u_K complex coefficients are called complex structures moduli, because as explained just above, they parametrize the possible changes of complex structures.

In a simpler way, the Kähler form J can be expanded on a base ω_α, $\alpha = 1, \dots, h_{1,1}$ of (real) 1,1-forms as

$$J = \sum_{\alpha=1}^{h_{1,1}} t_\alpha \omega_\alpha, \tag{3.2.16}$$

where the t_α real coefficients are called Kähler moduli. The integral of $J \wedge J \wedge J$ over X_6 measures the internal volume, ans the integral J^p measures the size of $2p$-cycles. Kähler moduli are thus naturally related to the sizes of the even dimensional internal cycles, similarly to the compact circle radius or the area of the internal tori described earlier.

In fact, once the CY three-folds Hodge numbers $h_{1,1}$ and $h_{2,1}$ are specified, fixing the Kähler and complex structure moduli totally determines the geometry of X_6

Massless modes In the case of circle compactifications, we mentioned how we could Fourier expand wave functions in the compact direction. The same idea can be used in

more complex internal geometries such as tori or CY manifolds, by first considering a factorization ansatz and looking for the zero modes of Laplace or Dirac operators in the internal manifold, which are related to the massless wave equations. We expose briefly the effect of compactifiation on massless particles through two examples: the spinors and the metric examples.

We first consider a $10d$ spinor wave function $\Psi(x^M)$, depending on the $10d$ coordinates x^M. We already mentioned that the spinor representation of $SO(10)$ splits under $SO(6) \times SO(1,3)$ as $\mathbf{16} = (\mathbf{4},\mathbf{2}) + (\bar{\mathbf{4}},\mathbf{2}')$ so that the spinor compactification ansatz can be written as

$$\Psi(x^M) = \psi(x^\mu)\chi(x^n), \qquad \mu = 0,\dots,4, \quad n = 5,\dots,9, \tag{3.2.17}$$

where ψ and χ are respectively $4d$ and $6d$ spinors. The initial $10d$ Dirac operator $\not{D}$ can be decomposed into $4d$ and $6d$ ones when acting on the $(\mathbf{4},\mathbf{2})$ or $(\bar{\mathbf{4}},\mathbf{2}')$ representations. Hence, according to the factorization ansatz (3.2.17), the Dirac equation $\not{D}\Psi(x^M) = 0$ splits and the number of solutions to the $6d$ Dirac equations $\not{D}\chi(x^n) = 0$, $\not{D}^\dagger\chi(x^n) = 0$, *i.e.* the dimension of the kernel of the Dirac operator, gives the number of left or right-handed $4d$ spinors coming from each initial $10d$ spinor. In fact, the difference between the dimensions of the left and right kernels, called the Dirac index, gives the number of chiral fermions. This difference is non-zero when left and right Dirac operators are different, which is for instance the case when magnetic fields are present in the internal directions. The topologically invariant Dirac index is determined through index theorems depending on the situation under study.

We now evoke the metric case. Each of the initial $10d$ metric $g_{MN}(x^M)$ indices can be taken either in the internal space X_6 or in our $4d$ space-time M_4. When both indices are taken in M_4, the compactification ansatz can be written as $g_{\mu\nu}(x^M) = g_{\mu\nu}(x^\mu)\phi(x^n)$, and the internal scalar $\phi(x^n)$ satisfies a Laplace equation, which has a unique solution in compact space. Hence, the $10d$ metric generates a unique $4d$ metric $g_{\mu\nu}(x^\mu)$. The potential $4d$ vector $g_{\mu n}$, which would be a gauge boson for X_6 continuous isometries, is absent in CY compactifications because such isometries do not exist. Finally, the g_{mn} $4d$ scalars can be decomposed using the complex coordinates of the CY threefold introduced before (3.2.13) and they lead to $h_{1,1}$ Kähler moduli parametrizing the $g_{i\bar{j}}$ degrees of freedom as well as $h_{2,1}$ complex structure moduli corresponding to the g_{ij} degrees of freedom.

We have shown in table 3.2 the type IIB spectrum. We have just described the $10d$ massless spinors and metric dimensional reduction and do not enter the details for the other massless particles. Nevertheless we expose the important result that the $4d$ massless spectrum arranges in a $\mathcal{N} = 2$ supergravity multiplet containing the metric $g_{\mu\nu}$ and the $C_{ijk\mu}$ vector, $2h_{2,1}$ vector multiplets containing the complex structure moduli and vectors $C_{ij\bar{k}\mu}$, $h_{1,1}$ hypermultiplets containing the Kähler moduli, $B_{i\bar{j}}$, $C_{i\bar{j}}$, $C_{i\bar{j}\mu\nu}$ and the universal hypermultiplet containing the dilaton ϕ, C_0 and $B_{\mu\nu}$, $C_{\mu\nu}$. Of course the supersymmetric partners are present in all the multiplets.

CY orientifolds For phenomenological reasons, one usually wants to reduce supersymmetry of the CY compactified theory from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. A common method is to introduce orientifold projections that reduce the spectrum and keep only $\mathcal{N} = 1$ SUSY. Orientifold projections where introduced in section 3.1.4 when evoking the unoriented strings amplitudes. Similarly to what was discussed there, the orientifolding operation consists in quotienting the theory by a combination of the worldsheet parity operator and a $\mathbb{Z}_2$ symmetry of X_6. This symmetry cannot be chosen arbitrarily and must preserve $\mathcal{N} = 1$ SUSY. The details of CY orientifold procedures are beyond the scope of this section. We nevertheless mention that they introduce Op orientifold planes located at the

fixed point of the $\mathbb{Z}_2$ action, the RR charge of which must be cancelled by Dp branes. As explained in section 3.1.5, excitations of these non-perturbative states are described by open string states, and the CY orientifold theories are thus a good playground to implement phenomenological models with open string matter fields. Another key point is that CY orientifolds allow for chiral spectra.

3.2.2 Moduli stabilization in type IIB

In the previous part, we introduced the notion of closed string moduli, which correspond to massless scalar fields of the low-energy theory. Such massless scalars do not seem to be present in our current world. Through their interactions with Standard Model particles, they should change our observable physics drastically and they would even be responsible for long-range interactions, fifth forces, which we do not observe. Hence, phenomenologically interesting string theory models should come along with scenarios that give a mass to these particles, *i.e.* stabilize them at a certain VEV. As massless particles do not have any associated energy scale, low-energy effective theories are natural playgrounds to study moduli stabilization. We will thus be interested in ways leading to the generation of a non-vanishing scalar potential for the closed string moduli fields.

Effective action The effective action of the low-energy theory, *i.e.* for the massless fields described in the previous section, can be obtained from the $10d$ type IIB effective action after reduction over a CY internal space X_6. As we have seen, the diverse massless fields arrange in supermultiplets and the $4d$ effective theory should be described within the framework of local supersymmetry, supergravity [90]. We refer to [91] for a recent review of supergravity constructions. Supergravity theories are described by a Kähler potential $\mathcal{K}$ and superpotential $\mathcal{W}$, similar to those introduced around eq. (2.1.21) for the case of global supersymmetry. The Kähler potential for CY compactifications can be written at leading order in α' as

$$\kappa^2\mathcal{K} = -\ln(S+\bar{S}) - \ln(i\int \Omega\wedge\bar{\Omega}) - 2\ln(e^{-\frac{3}{2}\phi}\int J\wedge J\wedge J), \tag{3.2.18}$$

where J and Ω are the Kähler form and 3,0-homomorphic form introduced in the previous section. As we have explained, they depend respectively on the Kähler and complex structure moduli. In fact, the supergravity variables also include the RR p-forms and the B_2 2-form. For instance, the usual axio-dilaton field was defined as $S = e^{-\phi} + iC_0$. The Kähler potential (3.2.18) is the classical Kähler potential and computable leading corrections should be added. We expose some of them in section 3.2.3. The term in the last logarithm of (3.2.18) is proportional to the internal volume because, as discussed under eq. (3.2.16), $J\wedge J\wedge J$ is proportional to the CY volume form. We also mentioned there that the cycle volumes can be related to real Kähler moduli. For instance, when the square of the internal volume is a product of three 4-cycles $\mathcal{V}_i$, with real Kähler moduli τ_i for $i = 1,2,3$, the last term of eq. (3.2.18) can be rewritten as

$$\kappa^2\mathcal{K}_{\tau_i} = -2\ln(\sqrt{\tau_1\tau_2\tau_3}). \tag{3.2.19}$$

In toroidal compactifications, the Ω and J forms can be expanded explicitely in terms of the tori data and the classical Kähler potential (3.2.18) takes a simple expression. For instance for a factorized $T^6 = T_1^2\times T_2^2\times T_3^2$ it reads

$$\kappa^2\mathcal{K} = -\ln(S+\bar{S}) - \sum_{i=1}^{3}\ln(\mathcal{U}_i+\bar{\mathcal{U}}_i) - \sum_{i=1}^{3}\ln(\mathcal{T}_i+\bar{\mathcal{T}}_i), \tag{3.2.20}$$

where $\mathcal{T}_i = e^{-\phi}/\alpha'^2 \mathcal{A}_j \mathcal{A}_k + i f(C_p)$, $i \neq j,k$. Here $\mathcal{A}_i$ are the areas of the tori and $\mathcal{U}_i$ their complex structures, related to the angle between they lattice vectors, see eq. (3.3.13). We do not show explicitly the RR dependent functions f. The structure of the Kähler potential of eqs. (3.2.18) and (3.2.19) is called no-scale structure, for reasons explained below.

When RR and NSNS 3-form field strength $F_3 = dC_2$ and $H_3 = dB_2$ fluxes are present in the $10d$ theory, their backreaction modifies the space-time geometry. Indeed, these fluxes must obey quantization conditions, expressed by integrations over 3-cycles, and when their $10d$ kinetic terms are integrated over X_6, they affect the $4d$ metric. We do not enter the details of the supergravity construction in presence of flux background but recall that $4d$ Poincaré invariance imposes the imaginary self-dual (ISD) condition, which reads

$$\star\, G_3 = iG_3, \qquad G_3 = F_3 - iSH_3, \tag{3.2.21}$$

where $\star$ denotes the $6d$ Hodge dual operator. The fluxes being quantized, this ISD condition is in fact a condition on the moduli, which are thus fixed at specific values. In the effective theory, this amounts to the generation of masses. These masses scale as the flux densities, hence in $1/R^3$ where R^6 is the scale of the compact X_6. This scale can be compared to the KK scale, which goes as $1/R$, according to (3.2.3). Hence, in the large volume limit, *i.e.* large R limit, we can indeed study the moduli stabilization consistently. In the same limit, the fluxes are very diluted and the total metric stays close to the initial one on $M_4 \times X_6$ (without fluxes), so that we can keep using the supergravity low-energy limit mentioned earlier. Of course, we need new terms taking into account the flux background, the dominant contributions of which have been shown [92, 93] to correspond to a superpotential $\mathcal{W}$ of the form

$$\mathcal{W} = \int_{X_6} G_3 \wedge \Omega. \tag{3.2.22}$$

As expected, the superpotential depends on the axio-dilaton S through G_3 defined in (3.2.21) and the complex structure moduli through Ω.

Complex structure moduli stabilization To study the stabilization of moduli, one can look at the scalar potential of the $4d$ supergravity theory. The F-term scalar potential expression in supergravity is a generalization of the global supersymmetry one, shown in eqs. (2.1.28) and (2.1.30). In supergravity, the F^i auxiliary fields depend on the Kähler potential, and the F-term scalar potential gets an additional negative contribution, proportional to the square modulus of the superpotential. It reads

$$V_F = e^{\kappa^2 \mathcal{K}} \left(\sum_{i,\bar{j}} \mathcal{K}^{i\bar{j}} D_i \mathcal{W} D_{\bar{j}} \bar{\mathcal{W}} - 3\kappa^2 |\mathcal{W}|^2 \right), \quad D_i = \partial_i + \mathcal{K}_i, \tag{3.2.23}$$

where the sum over $i, \bar{j}$ is over all the moduli. As the superpotential (3.2.22) does not depend on the Kähler moduli, and due to the no-scale structure of eqs. (3.2.18) and (3.2.19), the sum over the Kähler moduli in eq. (3.2.23) exactly cancels the $-3\kappa^2|\mathcal{W}|^2$ part, and the remaining contribution is only

$$V_F = e^{\kappa^2 \mathcal{K}} \left(\sum_{a,\bar{b}} \mathcal{K}^{a\bar{b}} D_a \mathcal{W} D_{\bar{b}} \bar{\mathcal{W}} \right). \tag{3.2.24}$$

It depends only on the complex structure moduli and axio-dilaton S. Minimizing this positive definite potential leads to the conditions $D_a \mathcal{W} = 0$, which can be computed directly and shown to be exactly equivalent to the ISD condition (3.2.21). This confirms that the

introduction of a 3-form field strength flux through the superpotential is a good effective theory description and indeed leads to the complex moduli and axio-dilaton stabilization. After stabilization of these moduli, the superpotential $\mathcal{W}$ is fixed to a constant value $\mathcal{W}_0$.

Kähler moduli stabilization: KKLT and LVS From the above discussion, we deduce that the Kähler moduli are not stabilized by the fluxes. Usual ways to stabilize them require additional ingredients, namely non-perturbative contributions. In the next section we expose a recently discovered way to stabilize these moduli in a fully perturbative framework.

KKLT The main ingredient used to stabilize the Kähler moduli in the way introduced by Kachru, Kallosh, Linde and Trivedi (KKLT) [**?**], is the presence of non-perturbative contributions to the superpotential, coming from either D3-branes instantons (wrapping internal 4-cycles) or gluino condensation on D7-brane stacks (also wrapping internal 4-cycles). These non-perturbative contributions take a common form

$$\mathcal{W}_{\text{NP}} = Ae^{-2\pi c\tau}, \tag{3.2.25}$$

where here τ is the Kähler modulus related to the size of the wrapped 4-cycle, and A depends on the complex structure moduli, and is thus considered constant when these last ones have been fixed by the fluxes. The constant c depends on the origin of the non-perturbative term, for instance it is proportional to $1/N_c$ for gluino condensation.The total superpotential after complex structures stabilization is thus $\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_{\text{NP}}$. As it depends on the Kähler moduli, the no-scale structure is broken and the scalar potential is minimized for $D_\tau \mathcal{W} = 0$. This condition, which can be computed using the Kähler potential (3.2.19) moduli dependence with $\tau_i = \tau$, in the simple case where we neglect the τ_i RR forms imaginary part, reads

$$\mathcal{W}_0 = -Ae^{-2\pi c\langle\tau\rangle}\left(1 + \frac{4\pi c\langle\tau\rangle}{3}\right), \qquad \tau = \text{Re}(\tau). \tag{3.2.26}$$

We can solve explicitly $\langle\tau\rangle$ in terms of the real parameters $\mathcal{W}_0$, c, and A using the Lambert function W_0. Here we implicitly considered $\mathcal{W}_0$ to be negative. As mentioned under eq. (3.2.21), the large volume limit is necessary to keep the supergravity effective theory description, and in our context it is simply $\tau \gg 1$. We see from the solution (3.2.26) that this requires an exponentially small superpotential $\mathcal{W}_0$, which is not quite generic, as it demands almost exact cancellations between all the fluxes contributions.

The resulting scalar potential at the minimum reads

$$V_{\text{eff}} = -3e^{\mathcal{K}}|\mathcal{W}_0 + \mathcal{W}_{\text{NP}}|^2 = -\frac{2\pi^2 c^2 A^2 e^{-4\pi c\langle\tau\rangle}}{3\langle\tau\rangle}, \tag{3.2.27}$$

and its negative value leads to an AdS vacuum. This AdS vacuum can in theory be uplifted to a dS vacuum by addition of new ingredients such as $\overline{D3}$ branes or magnetic fluxes for D7-brane gauge fields [94], both breaking supersymmetry and generating positive contributions to the scalar potential. There are several issues with the $\overline{D3}$ configurations concerning their stablity, the need for a precise tuning of the fluxes, the quantum corrections to the value of the positive contribution and hence the tuning of the global dS minimum.

LVS One way to get around the tuning of $\mathcal{W}_0$ to an exponentially small value is to consider the first α' correction to the Kähler potential. This correction has good reasons to

be included because, in concrete models, it is of the same magnitude as the $\mathcal{W}_{\rm NP}$ contributions. Models [95, 96] using both the superpotential non-perturbative corrections $\mathcal{W}_{\rm NP}$ and the leading α' correction ξ to the Kähler potential are called Large Volume Scenario (LVS) models, as the stabilization is realized at exponentially large internal volume. The first example considered two Kähler moduli, of real parts τ_b and τ_s, parametrizing the sizes of one "big" and one "small" 4-cycles, this last one being wrapped by a D3 instanton. A concrete example of a CY with such Kähler moduli was found before the LVS scenario in [97, 98, 99]. The Kähler potential and superpotential in such a context take the form

$$\kappa^2\mathcal{K} = -2\ln\left(\frac{1}{9\sqrt{2}}(\tau_b^{\frac{3}{2}} - \tau_s^{\frac{3}{2}}) + \xi\right), \tag{3.2.28}$$

$$\mathcal{W} = \mathcal{W}_0 + A_s e^{-2\pi c_s \tau_s}. \tag{3.2.29}$$

We come back to the correction ξ in the next section. In the large volume limit, *i.e.* for $\tau_b^{3/2} - \tau_s^{3/2} \sim \tau_b^{3/2} \gg 1$, the scalar potential is minimized for τ_b, τ_s at

$$\langle\tau_s\rangle \sim \xi^{\frac{2}{3}}, \qquad \langle\tau_b\rangle \sim e^{\frac{4\pi c_s\langle\tau_s\rangle}{3}}, \tag{3.2.30}$$

and we see that for c_s not too small, we indeed have $\langle\tau_s\rangle \gg \langle\tau_s\rangle$. The scalar potential has a negative value at this minimum, as can be seen by looking at its asymptotic behavior at large and small volumes, and hence leads to an AdS vacuum. As in the KKLT case, additional ingredients can in principle be added to uplift the vacuum to a dS minimum.

3.2.3 Kähler potential constant and logarithmic quantum corrections

Quantum corrections We have not shown explicitly the 10d supergravity effective action of type IIB superstring theory. It contains kinetic terms for the massless spectrum fields of table 3.2 and local terms related to the sources of the various fluxes, as $D3$ branes. In addition to the metric kinetic terms, described by the standard Einstein-Hilbert (EH) term linear in the scalar curvature $\mathcal{R}$, one can also include the leading order gravitational term, depending on the fourth power of the Riemann tensor. Such $\mathcal{R}^4$ terms are induced from graviton scattering and do not receive any other perturbative corrections beyond one-loop [100, 101].

As described in section 3.2.1, the low energy limit of type IIB theory is described by its effective action obtained upon compactification to four dimension of the $10d$ space $M_4 \times X_6$. Under this dimensional reduction of the $10d$ action, the $\mathcal{R}^4$ couplings induce a novel EH term localized in the bulk, denoted as $\mathcal{R}_{(4)}$ in the following. The resulting 4d effective action relevant to our discussion takes the form [102, 103]

$$S_{\rm grav} = \frac{1}{(2\pi)^7\alpha'^4}\int\limits_{M_4\times X_6} e^{-2\phi}\mathcal{R} + \frac{\chi}{(2\pi)^4\alpha'}\int\limits_{M_4}\left(2\zeta(3)e^{-2\phi} + 4\zeta(2)\right)\mathcal{R}_{(4)}\,, \tag{3.2.31}$$

where we used the Riemann ζ function, satisfying $\zeta(2) = \pi^2/6$. The proportionality factor of the $\mathcal{R}_{(4)}$ term depends on the Euler characteristic $\chi(X_6)$ of the internal manifold given by

$$\chi(X_6) = \frac{3}{4\pi^3}\int\limits_{X_6}\mathcal{R}\wedge\mathcal{R}\wedge\mathcal{R}\,\cdot \tag{3.2.32}$$

From (3.2.31) and (3.2.32) it is readily inferred that the $\mathcal{R}_{(4)}$ term only exists in four dimensions and that in the large volume limit, it is localized at points in the internal space

where the Euler number is concentrated. This localized terms induce corrections to the Kähler moduli part of the supergravity Kähler potential, proportional to the Euler characteristic, that we show explicitly in the following. These corrections modify the dilaton fixing procedure and the Kähler moduli normalisation [104], but in the end, they can be written as constant shifts ξ in the Kähler potential. We come back to it below.

A second phenomenon takes place at the points of $4d$ localized gravity kinetic terms. In the bulk, $10d$ gravitons, represented by closed strings, can be emitted from these points towards distinct Dp brane and orientifold sources, leading to local tadpoles [105]. In a geometric configuration with $D7$-brane stacks, as well as $O7$-orientifold planes, spanning four out of the six internal dimensions, a novel type of radiative corrections emerge. More concretely, by momentum conservation, gravitons emitted from the localized $\mathcal{R}_{(4)}$ vertices and ending on $D7$ brane sources propagate effectively in the two dimensions transverse to the $D7$ branes, acquiring a logarithmic propagator as a function of the distance. As a result, for a generic distribution with three $D7$-brane stacks at the boundaries of the compactified space, "far" away from the localized EH term, they give rise to corrections depending logarithmically on the size of the bulk [106, 103]

$$\delta S_{\rm grav} \sim -\frac{4\zeta(2)}{(2\pi)^3}\chi(X_6)\int_{M_4}\left(\sum_{k=1}^{3} e^{2\phi}T_k \ln(R_\perp^k/\mathtt{w})\right)\mathcal{R}_{(4)}\,. \tag{3.2.33}$$

Here, T_k is the effective tension of the k-th $D7$-brane stack, $R_\perp^k$ stands for the size of its $2d$ transverse space, and $\mathtt{w}$ is the width of the $\mathcal{R}_{(4)}$ localisation, playing the role of an effective ultraviolet cutoff for the graviton propagator in the bulk [103]. The localisation width $\mathtt{w}$ can be evaluated in the large Euler characteristic limit and the above logarithmic correction written in terms of the standard ξ correction [106]. Incorporating both these corrections into the Kähler potential (3.2.19) , we obtain the following Kähler moduli dependance

$$\kappa^2\mathcal{K}(\tau_k) = -2\ln\left(\sqrt{\tau_1\tau_2\tau_3}+\xi+\sum_k\gamma_k\ln\tau_k\right) = -2\ln\left(\mathcal{V}+\xi+\gamma\ln\mathcal{V}\right), \tag{3.2.34}$$

where in the last equality we assumed for simplicity the same tension $T_k \equiv T = e^{-\phi}T_0$ for all the brane stacks, which amounts to identical $\gamma_k \equiv \gamma/2$. The parameters ξ and γ are then given by [103, 106]

$$\gamma \equiv -\frac{1}{2}g_s T_0 \xi\,, \qquad \text{with } \xi = -\frac{\chi}{4}\times\begin{cases}\frac{\pi^2}{3}g_s^2 & \text{for orbifolds}\\ \zeta(3) & \text{for smooth CY}\end{cases}. \tag{3.2.35}$$

In the above formulae we have also shown the case of toroidal orbifolds, where the logarithmic correction is generated at one string loop order. The orbifold Euler characteristic $\chi_{\rm orb}$ is defined in analogy with the CY one, as the difference between the numbers of closed string $\mathcal{N}=2$ hyper and vector multiplets, $\chi_{\rm orb} = 4(n_H - n_V)$. A precise formula is given in (4.4.1). From the supergravity formula (3.2.23) we check that these corrections break the no-scale structure and induce a non-zero F-term effective potential V_F. The inclusion of ξ was already involved in the LVS scenario described in the last paragraph of section 3.2.2.

Fully perturbative moduli stabilization scenario Considering both the corrections described above leads to a new fully perturbative moduli stabilization scenario [107] that we describe shortly here. We describe it more precisely in section 4.2, where we also study its

cosmological consequences. In addition to the above quantum corrected Kähler potential, one can consider D-terms associated with magnetized $U(1)$ factors of the $D7$-brane stacks [94]. The D-term effective potential V_D can be minimised to fix the ratios τ_i/τ_j. The sum of F-term and D-term contributions constitutes the effective scalar potential V_{eff} which, after minimising the ratios in the large volume limit, can be cast in the form

$$\begin{aligned} V_{\text{eff}}(\mathcal{V}) = V_F + V_D &\simeq \frac{3\mathcal{W}_0^2}{2\kappa^4\mathcal{V}^3}\left(2\gamma(\ln\mathcal{V}-4)+\xi\right)+\frac{d}{\kappa^4\mathcal{V}^2} \\ &\equiv \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{3\sigma}{2\mathcal{V}^2}\right). \end{aligned} \tag{3.2.36}$$

The constant d parametrizes the D-terms, $\kappa = \sqrt{8\pi G_N}$ is the reduced Planck length, and we have defined

$$q \equiv \frac{\xi}{2\gamma} = -\frac{1}{g_s T_0}, \quad \sigma \equiv \frac{2d}{9{\mathcal{W}_0}^2\gamma}, \quad C \equiv -3{\mathcal{W}_0}^2\gamma\,. \tag{3.2.37}$$

Here $\mathcal{W}_0$ is the constant superpotential remaining after complex moduli stabilization obtained in the way described in section 3.2.2. As will be explained in section 4.2, within the above procedure, positive square masses are provided to all the Kähler moduli fields and at the same time a local de Sitter vacuum is obtained at weak coupling and large volume for γ and q negative, implying positive tension and negative Euler number, $T_0 > 0$ and $\chi < 0$.

3.3 Orbifold compactifications with fluxes

We introduced the main ingredients used in string compactifications in section 3.2.1. We explained the basics of Calabi-Yau compactifications and evoked quickly CY orientifolds constructions used to reduce the number of supersymmetries. These constructions are not easy and not generically solvable. We also mentioned the simpler case of toroidal compactifications, and we now introduce new ingredients leading to solvable toroidal orbifold compactifications. As they can be seen as singular limits of Calabi-Yau orientifolds, they are very helpful to unveil some of the properties of the latter. We highlight again that the main goal of these construction is to reduce supersymmetry and obtain phenomenologically interesting models.

3.3.1 Orbifold compactifications

General ideas Toroidal orbifolds are obtained as quotients of tori by discrete symmetry groups G such as $\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_3$ ... Namely this consists in identifying points through

$$\mathbf{X} \sim g\mathbf{X} + 2\pi\mathbf{L}, \qquad g \in G. \tag{3.3.1}$$

This differs from the simple torus identification (3.2.9) by the action of the discrete symmetry group element $g \in G$. We recall that here, $\mathbf{L} \in \Lambda_d$ is a vector of the T^d torus lattice. Hence, for the above identification to be self-consistent, the θ action must preserve the torus lattice, *i.e.* must be a symmetry of the torus lattice with $g\mathbf{L} = \mathbf{L}' \in \Lambda_d$. The action of the symmetry group on the worldsheet fermions is dictated by worldsheet supersymmetry.

The orbifold identification (3.3.1) allows for new closed string boundary conditions

$$\mathbf{X}(\tau,\sigma+\pi) = g\mathbf{X}(\tau,\sigma) + 2\pi\mathbf{L}. \tag{3.3.2}$$

When $g = \mathbb{1}$, this boundary condition is already allowed in the simple torus case and leads to "untwisted" sector. The other elements of the symmetry group introduce new sectors in the theory, called "twisted" sectors. They correspond to strings with center of mass located at the orbifold fixed points

$$\mathbf{X}_{\text{f.p.}} = g\mathbf{X}_{\text{f.p.}}(\tau,\sigma) + 2\pi\mathbf{L}, \qquad g \in G, \tag{3.3.3}$$

which are singularities of the quotient space. States with these new boundary conditions describe strings which close on themselves thanks to a θ action. The orbifold partition function is then obtained by introducing a projector $1/|G|\sum_{g\in G} g$, similar to the world-sheet parity projector $(1+\Omega)/2$. This projector truncates the spectrum to orbifold invariant states, and summing over the Hilbert spaces $\mathcal{H}_h$ of all the sectors of the theory, the total partition function can be written as

$$Z = \frac{1}{|G|}\sum_{h,g\in G} Z\begin{bmatrix}h\\g\end{bmatrix} = \frac{1}{|G|}\sum_{h,g\in G} \text{tr}_{\mathcal{H}_h}\left(g e^{-\tau_2\pi\alpha'\sum_i p^i p^i} q^{N^h+E_0}\bar{q}^{\tilde{N}^h+\tilde{E}_0}\right), \tag{3.3.4}$$

where in the last equality we used the amplitude expression of eq. (3.1.80), and we highlighted that the number operators N^h are sector dependent.

To be concrete, we consider the bosonic $\mathbb{Z}_2$ action on the single compact coordinate X^9, *i.e.* for circle compactification of the X^9 coordinate. The symmetry group is then $G = \{\mathbb{1}, g = -\mathbb{1}\}$ and the action of g on a bosonic coordinate is simply $X^9 \to -X^9$. We deduce from the X^9 mode expansion that the action of g on a untwisted state (3.1.30), *i.e.* a state satisfying untwisted b. c., with momentum and winding numbers k, w, is

$$g\prod_a(\alpha^9_{-a})^{m_a}\prod_b(\tilde{\alpha}^9_{-b})^{m_b}\left|0,\tilde{0},k,w\right\rangle = (-1)^{N+\tilde{N}}\prod_a(\alpha^9_{-a})^{m_a}\prod_b(\tilde{\alpha}^9_{-b})^{m_b}\left|0,\tilde{0},-k,-w\right\rangle. \tag{3.3.5}$$

Hence, when g is inserted, only the $k = w = 0$ states contribute in the trace over $\mathcal{H}_{\mathbb{1}}$, so that no lattice sum appear in the amplitude for the corresponding coordinate. The computation of the trace is similar to the one of eq. (3.1.85), with the difference that there is a $(-1)^N$ factor for left and right-moving sectors, changing the $1/(1-q^a)$ term to $1/(1+q^a)$. This results in a total contribution from the left and right moving closed strings sectors of the form

$$\frac{1}{2}\left|\frac{2\eta}{\vartheta_2}\right| \quad : \quad \text{untwisted bosonic coordinate , } g \text{ insertion.} \tag{3.3.6}$$

The twisted sectors contributions can be found either through modular invariance or by mode expanding the wave equation solutions with twisted boundary conditions. The boundary condition $X^9(\tau,\sigma+\pi) = -X^9(\tau,\sigma)$ does not allow for winding nor momentum, and it forces the string center of mass to sit at one of the two fixed points of the circular orbifold, at $x^9 = 0, \pi$. The antiperiodic boundary condition induces half integer modes, and the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_\pi$ are constructed the usual way, by acting with creation operators on the two vacuum states located at the two orbifold fixed points. The Hilbert spaces traces computations can again be done as in eq. (3.1.85) and give for each orbifold fixed point a combined left and right contribution of

$$\frac{1}{2}\left|\frac{\eta}{\vartheta_4}\right| \quad : \quad \text{orbifold fixed point twisted bosonic coordinate , no } g \text{ insertion,} \tag{3.3.7}$$

$$\frac{1}{2}\left|\frac{\eta}{\vartheta_3}\right| \quad : \quad \text{orbifold fixed point twisted bosonic coordinate , } g \text{ insertion.} \tag{3.3.8}$$

We recall that the ϑ_i functions are given in eq. (A.3.1). Hence, by applying formula (3.3.4) in our simple example, we express the circle orbifold bosonic torus amplitude as

$$\mathsf{T} \sim \frac{1}{2}\sum_{m,n}\frac{q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2}}{|\eta(\tau)|^2}+\frac{1}{2}\left|\frac{2\eta}{\vartheta_2}\right|+\frac{2}{2}\left|\frac{\eta}{\vartheta_4}\right|+\frac{2}{2}\left|\frac{\eta}{\vartheta_3}\right|, \tag{3.3.9}$$

where in the last two terms we took into account the sum over the two orbifold fixed points Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_\pi$. This gives the factor 2 in front of the single orbifold fixed point contributions (3.3.7) and (3.3.8). The lattice sum of the first term is similar to the one introduced in (3.2.12) for a d dimentional torus T^d, except that it simplifies in the circle case.

Of course, the above torus amplitude is multiplied by the non-compact coordinate contributions, and must include the fermionic contributions as well. We do not enter the details of the generic methods but rather expose, in the next paragraph, a particular example that we use in the following.

A particular model: the $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold We describe below the construction of a *D7* brane configuration dual to a $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold and write its vacuum amplitudes. We consider for simplicity a factorised 6-torus $T^6=T_1^2\times T_2^2\times T_3^2$ with $i=1,2,3$ indices denoting the (45), (67) and (89) internal directions respectively. We use the general torus T^d definitions of (3.3.10) and below, that we recall here in our particular case. The i-th torus T_i^2 is defined as

$$T_i^2\equiv\mathbb{R}^2/2\pi\Lambda_i,\quad \Lambda_i=\left\{q\,\mathbf{R}_{ix}+r\,\mathbf{R}_{iy};q,r\in\mathbb{Z}\right\}, \tag{3.3.10}$$

with $\mathbf{R}_{ix}$, $\mathbf{R}_{iy}$ two linearly independent vectors of norm R_{ix}, R_{iy} and relative angle α_i. The dual lattice Λ_i^* is generated by the dual vectors $\mathbf{R}_i^{*x},\mathbf{R}_i^{*y}$ satisfying $\mathbf{R}_{ik}\cdot\mathbf{R}_i^{*l}=\delta_k^l$. The torus metric reads

$$g_{kl}^{(i)}=\mathbf{R}_{ik}\cdot\mathbf{R}_{il}=\frac{\mathcal{A}_i}{\mathrm{Re}(U_i)}\begin{pmatrix}1 & \mathrm{Im}(U_i)\\ \mathrm{Im}(U_i) & |U_i|^2\end{pmatrix}, \tag{3.3.11}$$

and its inverse can be used to raise the indices and express the dual vectors $\mathbf{R}_i^{*k}=g^{(i)kl}\mathbf{R}_{il}$. In the above metric we have defined by $\mathcal{A}_i$ the unit cell area of the torus T_i^2

$$\mathcal{A}_i\equiv\sqrt{\det g^{(i)}}=\frac{\mathrm{vol}(2\pi\Lambda_i)}{(2\pi)^2}=R_{ix}R_{iy}\sin\alpha_i,\quad\text{with}\quad\mathbf{R}_{ix}\cdot\mathbf{R}_{iy}=R_{ix}R_{iy}\cos\alpha_i, \tag{3.3.12}$$

and by U_i, the torus complex structure modulus

$$U_i\equiv i\frac{R_{iy}}{R_{ix}}e^{-i\alpha_i}=\frac{1}{{R_{ix}}^2}(\mathcal{A}_i+i\mathbf{R}_{ix}\cdot\mathbf{R}_{iy}). \tag{3.3.13}$$

The area $\mathcal{A}_i$ of the torus, which is a 2-cycle, is nothing but its Kähler modulus, real in absence of B_2 background.

The $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold is constructed by quotienting the factorized torus by a $\mathbb{Z}_2\times\mathbb{Z}_2$ action. The $\mathbb{Z}_2\times\mathbb{Z}_2$ group contains four elements $G=(\mathbb{1},g,f,h)$ which act on the complexified torus coordinates $Z^i=X^{2+2i}+iX^{3+2i}$, as

$$\begin{aligned}&\mathbb{1}:(Z^1,Z^2,Z^3)\to(Z^1,Z^2,Z^3), && \alpha:(Z^1,Z^2,Z^3)\to(Z^1,-Z^2,-Z^3)\\ &\beta:(Z^1,Z^2,Z^3)\to(-Z^1,Z^2,-Z^3), && \gamma=\alpha\circ\beta:(Z^1,Z^2,Z^3)\to(-Z^1,-Z^2,Z^3)\end{aligned} \tag{3.3.14}$$

Each of the α, β, γ symmetry group elements leaves a torus invariant and has $4 \times 4 = 16$ fixed points in the remaining two. $O5$ planes, spanning the torus left invariant by the symmetry group element α, β or γ, sit at these fixed points. Their RR charge is cancelled by the presence of $D5$ branes. Moreover, orientifolding through the action of the Ω operator requires the presence of D9 branes cancelling the RR charges of the O9 planes.

This $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold containing $D9$ and $D5$ branes was described in [108, 109]. For our future purpose we rather consider using T-duality, mentioned quickly at the end of section 3.1.5, to describe the equivalent model containing $D7$ branes.

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\times$	$\times$
$D7_2$	$\times$	$\times$	$\cdot$
$D7_3$	$\times$	$\cdot$	$\times$

$\longleftrightarrow$ T-duality along (45) $\longrightarrow$

	(45)	(67)	(89)
$D9_1$	$\times$	$\times$	$\times$
$D5_2$	$\cdot$	$\times$	$\cdot$
$D5_3$	$\cdot$	$\cdot$	$\times$

In the above tables, a cross $\times$ represents the $D7$ worldvolume spanning the corresponding torus, while a dot $\cdot$ indicates the transverse directions where the $D7$ brane is localized. In the following we will introduce magnetic fields and circled crosses $\otimes$ will represent directions of a magnetic flux for the worldvolume $U(1)$ gauge fields.

The torus, Klein bottle, annulus and Möbius amplitudes are computed using standard methods [110, 111, 48] generalizing to the superstring case what was described in the previous paragraph for bosonic coordinates. The specific amplitudes for the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ model can be found in *e.g.* [108]. The torus amplitude (without discrete torsion) reads

$$\begin{aligned}
4\mathsf{T} = {} & |T_{oo}|^2 \Lambda_1\Lambda_2\Lambda_3 + 16\left(|T_{og}|^2 \Lambda_1 + |T_{of}|^2 \Lambda_2 + |T_{oh}|^2 \Lambda_3\right)\left|\frac{\eta^2}{\vartheta_2^2}\right|^2 \\
& + 16\left(|T_{go}|^2 \Lambda_1 + |T_{fo}|^2 \Lambda_2 + |T_{ho}|^2 \Lambda_3\right)\left|\frac{\eta^2}{\vartheta_4^2}\right|^2 \\
& + 16\left(|T_{gg}|^2 \Lambda_1 + |T_{ff}|^2 \Lambda_2 + |T_{hh}|^2 \Lambda_3\right)\left|\frac{\eta^2}{\vartheta_3^2}\right|^2 \\
& + 64\left(|T_{gh}|^2 + |T_{gf}|^2 + |T_{fg}|^2 + |T_{fh}|^2 + |T_{hg}|^2 + |T_{hf}|^2\right)\left|\frac{\eta^3}{\vartheta_2\vartheta_3\vartheta_4}\right|^2,
\end{aligned} \tag{3.3.15}$$

where ϑ_i's are the Riemann theta-functions and η the Dedekind function, depending on the world-sheet torus modular parameter τ, given in appendix A.3. The tori lattice sums Λ_i are given in appendix A.4. There is a clash on notations with the defining torus lattice of (3.3.10), but this should not cause any problem. The characters T_{kj} are expressed in terms of the 16 $\mathbb{Z}_2 \times \mathbb{Z}_2$ characters τ_{kl} constructed from quadruple products of the four level-one $SO(2)$ characters, see Appendix A.3. The T_{kj} characters used for the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ model are defined in [48, 108]

$$\begin{aligned}
T_{ko} &= \tau_{ko} + \tau_{kg} + \tau_{kh} + \tau_{kf}, \quad & T_{kg} &= \tau_{ko} + \tau_{kg} - \tau_{kh} - \tau_{kf}, \\
T_{kh} &= \tau_{ko} - \tau_{kg} + \tau_{kh} - \tau_{kf}, \quad & T_{kf} &= \tau_{ko} - \tau_{kg} - \tau_{kh} + \tau_{kf},
\end{aligned} \tag{3.3.16}$$

for $k = o, f, h, g$. In the torus amplitude (3.3.15) we recognize the bosonic contributions eqs. (3.3.6) to (3.3.8) of the orbifold amplitudes.

In the current $D7$-branes setup, the world-sheet involution Ω projection is implemented by adding the Klein-bottle amplitude K to the half torus $\frac{1}{2}\mathsf{T}$ of equation (3.3.15),

following the conventions of [48]. The Klein-bottle amplitude reads

$$8\mathsf{K} = \Big(W_1P_2P_3 + P_1W_2P_3 + P_1P_2W_3\Big)\,T_{oo} + 2\times 16\left[P_1T_{go} + P_2T_{fo} + P_3T_{ho}\right]\left(\frac{\eta}{\vartheta_4}\right)^2, \quad (3.3.17)$$

where the Klein-bottle lattice sums P_i, W_i are given in Appendix A.4. The open string spectrum can be obtained through the annulus and Möbius amplitudes; we describe them in the following subsections, in the presence of magnetic fields. We then use (x,y,z) arguments for the T_{kj} open-string characters referring to the internal oscillator shifts. For instance, the first character of (A.3.5) reads

$$\begin{aligned}\tau_{oo}(x,y,z) = {}& V_2(0)O_2(x)O_2(y)O_2(z) + O_2(0)V_2(x)V_2(y)V_2(z)\\ & -S_2(0)S_2(x)S_2(y)S_2(z) - C_2(0)C_2(x)C_2(y)C_2(z),\end{aligned} \quad (3.3.18)$$

and the $T_{kj}(x,y,z)$ follow the same logic. Of course in the torus amplitude (3.3.15), T_{kj} stand for $T_{kj}(0,0,0)$.

3.3.2 Magnetic fluxes supersymmetry breaking

We now give a few elements of toroidal compactifications in the presence of worldvolume magnetic fields, that we use in the following subsections. We generically denote by $H_a^{(i)}$ a magnetic field introduced on the $D7_a$ stack, in the i-th internal plane, with $i = 1,2$ and 3 for (45), (67) and (89) respectively.

Magnetic fields modify the world-sheet action by introducing boundary terms [112, 113]. The solution of the wave equations depends on the charge of the open string. Neutral strings have standard oscillators while charged ones see their modes shifted by the magnetic field through the theta function argument

$$\zeta_a^{(i)} = \frac{1}{\pi}\mathrm{Arctan}(2\pi\alpha' q_a H_a^{(i)}). \quad (3.3.19)$$

In the following we choose a normalisation for the $U(1)$ charges at the endpoints of an open string $q = \pm 1, 0$.

For NN boundary conditions, this argument appears through a factor $\eta/\vartheta_1(\zeta_a^{(i)}\tau)$, replacing the standard P/η^2 bosonic oscillators contribution of a (compact) complex coordinate, which is the square of the single bosonic coordinate (3.1.86) and contains a lattice sum P. The argument in ϑ_1 contains in particular the field-theory Landau levels, replacing the lattice momenta sums. For ND or DN boundary conditions, it gives an argument to the ϑ_4 function appearing in the $\eta/\vartheta_4(\zeta_a^{(i)}\tau)$ factors. The dipole strings (with ends of opposite charges, *i.e.* attached to the same D brane) have special quantised zero-modes inducing "boosted" string momenta [112]

$$\tilde{m}_a^{(i)} = \frac{m_a^{(i)}}{\sqrt{1+\left(2\pi\alpha' H_a^{(i)}\right)^2}}. \quad (3.3.20)$$

The magnetic fields $H_a^{(i)}$ are quantized through the standard Dirac quantization on fluxes

$$m\int_{T^2} H = 2\pi n, \quad (3.3.21)$$

where m is the wrapping number and n the flux quantum. This leads to the magnetic field quantization

$$2\pi H_a^{(i)} \mathcal{A}_i = k_a^{(i)}, \qquad k_a^{(i)} = \frac{n_a^{(i)}}{m_a^{(i)}} \in \mathbb{Q}\,, \tag{3.3.22}$$

with $k_a^{(i)}$ the ratio of wrapping number $m_a^{(i)}$ and flux number $n_a^{(i)}$ of the D7$_a$ brane on the i-th torus T_i^2. We recall that the T_i^2 area is $4\pi^2\mathcal{A}_i$, see (3.3.12). Note that due to the $\mathbb{Z}_2$ quotient, $n_a^{(i)}$ can take half-integer values. This does not change the allowed values for $k_a^{(i)}$.

In the next sections we will extract the open string mass spectrum from the annulus amplitude. The masses can also be extracted by looking at the different spins of the internal components of the massless states (without magnetic fields), through the mass shift formula given in [113]

$$\Delta m^2 = \frac{1}{2\alpha'} \sum_i \left[(2n_i + 1) \left| \zeta_L^{(i)} + \zeta_R^{(i)} \right| + 2\Sigma_i \left(\zeta_L^{(i)} + \zeta_R^{(i)} \right) \right]. \tag{3.3.23}$$

The L, R subscripts indicate the string endpoints and have to be replaced by the corresponding brane in the oscillator shift defined in (3.3.19). The first term with the sum corresponds to the Landau levels, while the second one corresponds to the magnetic moments for the internal Σ_i helicities. Landau levels appear only for NN boundary conditions. This formula, which can be derived from the annulus amplitude, can be understood using the field theoretical description of the magnetized brane. We see from the above mass formula that the spectrum is not supersymmetric anymore as it depends on the spin of the considered state. Hence introducing magnetic fields on the internal tori is indeed a way to break supersymmetry [113]. Models with magnetic fields can be also made in CY compactifications, using *e.g.* $D7$ branes gauge fields, but we do not describe them here.

3.3.3 Toroidal model of matter waterfall field

In this section, we use all the above toroidal orbifold ingredients to implement a model of matter waterfall field, which will be used in section 4.4 when constructing a type IIB superstring inflationary model. We thus postpone the explanation of our motivations to section 4.4 and we only expose the goal of the current section. Our objective is to construct a matter field, at an intersection of the $D7$-branes stacks , whose mass-squared depends non-trivially on the total internal volume and becomes negative below a certain volume. To do so, we will construct states with positive and negative mass contributions, coming respectively from brane separations and magnetic fields.

One magnetized stack We first consider a toy model with only one magnetic field. We turn this magnetic field on the $D7_2$ stack and align it with the common $U(1)$. Hence, the whole stack is magnetized and there is no neutral $D7_2$ brane. We choose the magnetization to be on the third torus T_3^2 (*i.e.* in the (89) direction). Then, according to the notation introduced in equation (3.3.22), we denote the magnetic field by $H_2^{(3)}$, and the associated oscillator shift by $\zeta_2^{(3)}$. The configuration is summarized in the following table:

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\times$	$\times$
$D7_2$	$\times$	$\cdot$	$\otimes$
$D7_3$	$\times$	$\times$	$\cdot$

The annulus amplitude is computed using the techniques and conventions of [114, 48, 108]. For our $D7$-branes model, the different contributions to the annulus amplitude read

$$\begin{aligned}8\mathsf{A}_0 =& \left({N_1}^2 W_1 P_2 P_3 + {N_3}^2 P_1 P_2 W_3 + 2N_2 \bar{N}_2 P_1 W_2 \tilde{P}_3\right) T_{oo}(0,0,0) \\ &+ 2N_1 N_3 P_2\, T_{fo}(0,0,0) \left(\frac{\eta}{\vartheta_4(0)}\right)^2 ,\end{aligned} \tag{3.3.24}$$

$$\begin{aligned}8\mathsf{A}_1 =& \left(-2iN_1 N_2 T_{ho}(0,0,\zeta_2^{(3)}\tau)\frac{k_2^{(3)}\eta}{\vartheta_1(\zeta_2^{(3)}\tau)} + 2iN_1 \bar{N}_2 T_{ho}(0,0,-\zeta_2^{(3)}\tau)\frac{k_2^{(3)}\eta}{\vartheta_1(-\zeta_2^{(3)}\tau)}\right)\left(\frac{\eta}{\vartheta_4(0)}\right)^2 \\ &+ \left(2N_3 N_2 P_1 T_{go}(0,0,\zeta_2^{(3)}\tau) + 2N_3 \bar{N}_2 P_1 T_{go}(0,0,-\zeta_2^{(3)}\tau)\right)\frac{\eta}{\vartheta_4(0)}\frac{\eta}{\vartheta_4(\zeta_2^{(3)}\tau)} ,\end{aligned} \tag{3.3.25}$$

$$8\mathsf{A}_2 = -iN_2{}^2 P_1 W_2 T_{oo}(0,0,2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\eta}{\vartheta_1(2\zeta_2^{(3)}\tau)} + i\bar{N}_2{}^2 P_1 W_2 T_{oo}(0,0,-2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\eta}{\vartheta_1(-2\zeta_2^{(3)}\tau)} , \tag{3.3.26}$$

where $\mathsf{A}_0, \mathsf{A}_1, \mathsf{A}_2$ correspond respectively to the neutral, charged ± 1 and charged ± 2 strings with respect to the magnetized $U(1)$. In the above expressions, P_i, W_i are the standard momentum and winding sums defined in Appendix A.4, while $\tilde{P}_3$ is the sum over boosted momenta $\tilde{m}_3 = m_3/\sqrt{1+(2\pi\alpha' H_2^{(3)})^2}$ coming along with dipole strings, as explained above eq. (3.3.20). Note also that for notational simplicity, the parameter τ is used instead of the direct channel annulus parameter $\frac{i}{2}\,\mathrm{Im}\tau$.

The T_{ko} characters were introduced in eq. (3.3.16) and Appendix A.3. Their dependence in the magnetic fields is explained around eq. (3.3.18). As explained shortly in section 3.3.2, it is easy to trace back the different state contributions to the amplitude: each of ND or DN mixed boundary conditions contributes by a η/ϑ_4 factor (instead of $1/\eta^2$ for standard bosonic coordinates), with oscillator shift $\zeta_i\tau$ when a magnetic field is present on the N boundary, and each NN boundary condition with magnetic fields introduces Landau levels through a η/ϑ_1 factor with oscillator shifts.

We present hereafter the Möbius contributions M_0 and M_2, corresponding to neutral and doubly charged strings. The Möbius amplitude identifies the string endpoints, hence it does not contain simply charged contribution and its non-vanishing contributions read

$$\begin{aligned}8\mathsf{M}_0 =& -\left(N_1 W_1 P_2 P_3 + N_3 P_1 P_2 W_3\right)\hat{T}_{oo}(0,0,0) + \left(N_1 W_1 + N_3 P_1\right)\hat{T}_{og}(0,0,0)\left(\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\right)^2 \\ &+ \left(N_1 P_2 + N_3 P_2\right)\hat{T}_{of}(0,0,0)\left(\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\right)^2 + \left(N_1 P_3 + N_3 W_3\right)\hat{T}_{oh}(0,0,0)\left(\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\right)^2 ,\end{aligned} \tag{3.3.27}$$

$$
\begin{aligned}
8\mathsf{M}_2 = {} & iN_2P_1W_2\hat{T}_{oo}(0,0,2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_1(2\zeta_2^{(3)}\tau)} - i\bar{N}_2P_1W_2\hat{T}_{oo}(0,0,-2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_1(-2\zeta_2^{(3)}\tau)}\,, \\
& - N_2P_1\,\hat{T}_{og}(0,0,2\zeta_2^{(3)}\tau)\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_2(2\zeta_2^{(3)}\tau)} - \bar{N}_2P_1\,\hat{T}_{og}(0,0,-2\zeta_2^{(3)}\tau)\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_2(-2\zeta_2^{(3)}\tau)} \\
& - N_2W_2\,\hat{T}_{of}(0,0,2\zeta_2^{(3)}\tau)\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_2(2\zeta_2^{(3)}\tau)} - \bar{N}_2W_2\,\hat{T}_{of}(0,0,-2\zeta_2^{(3)}\tau)\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_2(-2\zeta_2^{(3)}\tau)} \\
& + \left(iN_2\,\hat{T}_{oh}(0,0,2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_1(2\zeta_2^{(3)}\tau)} - i\bar{N}_2\,\hat{T}_{oh}(0,0,-2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\hat{\eta}}{\hat{\vartheta}_1(-2\zeta_2^{(3)}\tau)}\right)\left(\frac{2\hat{\eta}}{\hat{\vartheta}_2(0)}\right)^2.
\end{aligned}
\tag{3.3.28}
$$

The Möbius amplitude modifies the unitary groups of the unmagnetized branes to orthogonal groups (branes on top of orientifolds). On the other hand, it acts on the magnetized branes by forming states in the antisymmetric representation. The hatted $\hat{T}_{ij}$ characters and $\hat{\vartheta}$ functions are related as usual to the choice of a real basis of characters [48].

The different Chan-Paton multiplicities, the significance of which was explained in section 3.1.5, are as follows: N_1, N_3 for the string endpoints aligned with the $D7_1, D7_3$ branes and $N_2, \bar{N}_2$ for the $D7_2$ string endpoints aligned with the $U(1)$ magnetic field, with charge ± 1. N_1 and N_3 are real because they index orthogonal groups. These Chan-Paton multiplicities include the wrapping numbers factors so that they are in fact written as

$$
N_a = N'_a m_a^{(j)} m_a^{(k)}, \qquad a \neq j \neq k \neq a, \tag{3.3.29}
$$

where N'_a is the true number of branes in the a-th stack. Replacing eq. (3.3.29) in the amplitudes of eqs. (3.3.24) to (3.3.28), one can read the chiral fermions multiplicities through the "intersection number" defined for each magnetized torus T_j^2 as

$$
I_{ab}^{(j)} = n_a^{(j)} m_b^{(j)} - m_a^{(j)} n_b^{(j)}. \tag{3.3.30}
$$

This intersection number $I_{ab} = \prod_j I_{ab}^{(j)}$ is the index of the Dirac operator of the charged fermions. Taking a specific example, the total multiplicity of the doubly charged state between the $D7_2$ brane and its orientifold image, described by the amplitude A_2 given in eq. (3.3.26), can be written as

$$
{N_2}^2\, k_2^{(3)} - \bar{N}_2^2\, k_2^{(3)} = 2{N'_2}^2 (m_2^{(1)} m_2^{(3)})^2 k_2^{(3)} = 2{N'_2}^2 {m_2^{(1)}}^2 m_2^{(3)} n_2^{(3)} = {N'_2}^2 {m_2^{(1)}}^2 I_{22'}. \tag{3.3.31}
$$

The ${N'_2}^2 {m_2^{(1)}}^2$ is just the Chan-Paton multiplicity for the unmagnetized torus while $I_{22'}$ is related the chiral fermions multiplicity (which is then modified by the orientifold projection). This multiplicity can be understood from the field theoretical point of view as the degeneracy of each Landau level [113].

As usual, the various multiplicities are subject to tadpole cancellation conditions, modified in general by the presence of 3-form fluxes needed for complex structure moduli stabilization.

The massless states of the original orbifold model are modified by the magnetic field. The charged states receive different contributions (according to the internal spins) resulting to the mass shift (3.3.23). We show in the following table the smallest mass shifts for each state (*i.e.* the new lowest lying states after magnetic deformation). In the table, the lines and columns entries represent the two possible string endpoints of each state.

	$D7_1$	$D7_2$	$D7_3$
$D7_1$	$\alpha' m^2 = 0$	$\alpha' m^2 = -2\lvert\zeta_2^{(3)}\rvert$	$\alpha' m^2 = -\lvert\zeta_2^{(3)}\rvert$
$D7_2$		$\alpha' m^2 = -2\lvert\zeta_2^{(3)}\rvert$	$\alpha' m^2 = -\lvert\zeta_2^{(3)}\rvert$
$D7_3$			$\alpha' m^2 = 0$

We see that tachyonic states can appear in the spectrum [113, 114]. In order to eliminate them, we introduce appropriate brane separations and/or Wilson lines. The brane separation and Wilson lines contributions to the masses were described in section 3.1.5. We recall that in the annulus amplitudes of equations (3.3.24) to (3.3.26), brane separations introduce shifts in W_i the winding numbers and Wilson lines introduce shifts of the momentum numbers in P_i according to the endpoint charges of the strings. We then introduce Wilson lines and separations as follows:

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\times$	$\times$
$D7_2$	$\times$	$\cdot$	$\otimes$
$D7_3$	$\times$	$\times$	$\cdot$

$\longrightarrow$

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\times$	$\times$
$D7_2$	$\times$	$\cdot_{\pm x_2}$	$\otimes$
$D7_3$	$\times_{A_3}$	$\times$	$\cdot$

The index A_3 indicates a $U(1)$ Wilson line gauge field, that we take again along the diagonal abelian factor of the $D7_3$ stack and turned on within the torus T_1^2 in the (45) internal plan. The x_2 index represents the brane positions of the $D7_2$ brane stack (and $-x_2$ for its orientifold image).

If the Wilson line modulus is projected out by the orbifolding procedure, the model would generally only allow for discrete Wilson lines that can be expressed in the dual lattice as

$$A_3 = a_{3x}\mathbf{R}_1^{*x} + a_{3y}\mathbf{R}_1^{*y}, \quad \text{with} \quad a_{3x}, a_{3y} \in \mathbb{Q}\,. \tag{3.3.32}$$

For $\mathbb{Z}_2$ orbifolds we typically get $a_{3x/y} = \frac{1}{2}$ (if non-vanishing). This Wilson line gives a mass for the charged fields of the which is a generalization of the circle contribution (3.1.112) to the T^2 torus case

$$\alpha' m^2 = \alpha' A_3 \cdot A_3 = \alpha' a_{3k} a_{3l} g^{(1)kl} = \frac{\alpha'}{\mathcal{A}_1 \mathrm{Re}(U_1)} \left| a_{3y} + iU_1 a_{3x} \right|^2 \equiv \frac{\alpha' a_3^2(U_1)}{\mathcal{A}_1}\,, \tag{3.3.33}$$

where the dimensionful area $\mathcal{A}_1$, the dimensionless complex structure U_1, and the torus metric $g^{(1)}$ were defined in (3.3.12), (3.3.13) and (3.3.11). In the last equality we separated the complex structure and Kähler modulus ($\mathcal{A}_1$) dependences.

Similarly, the $D7_2$ brane position x_2 can be expressed as

$$x_2 \equiv x_2^x\, \mathbf{R}_{2x} + x_2^y\, \mathbf{R}_{2y} \quad \text{with} \quad x_2^x, x_2^y \in \mathbb{Q}, \tag{3.3.34}$$

where we assumed again discretisation of the positions at symmetric points of the fundamental cell. We recall that $\mathbf{R}_2^x$ and $\mathbf{R}_2^y$ are the torus lattice vectors defined in (3.3.10). As in the circle compactification case (3.1.111), the displacement x_2 of the $D7_2$ stack from the origin generates a mass for the strings stretched between the brane stack and its image with respect to the orientifold plane located at the origin

$$\alpha' m^2 = \frac{4\, x_2 \cdot x_2}{\alpha'} = \frac{4\, x_2^k x_2^l g^{(2)}_{kl}}{\alpha'} = \frac{4\mathcal{A}_2}{\alpha' \mathrm{Re}(U_2)} \left| x_2^x - iU_2 x_2^y \right|^2 \equiv \frac{y(U_2)\, \mathcal{A}_2}{\alpha'}. \tag{3.3.35}$$

In the last equality we isolated again the complex structure modulus dependence from the Kähler modulus one. For more general toroidal orbifolds, the point group symmetry

has to be compatible with the stabilised complex structure moduli, so that the Wilson line and brane separation quantization already incorporates the U_i dependence.

The new lowest lying mass states are shown in the table below.

	$D7_1$	$D7_2$	$D7_3$
$D7_1$	$\alpha' m^2 = 0$	$\alpha' m^2 = 0$	$\alpha' m^2 = 0$
$D7_2$		$\alpha' m^2 = -2\lvert\zeta_2^{(3)}\rvert + \frac{y\mathcal{A}_2}{\alpha'}$	$\alpha' m^2 = -\lvert\zeta_2^{(3)}\rvert + \frac{\alpha' a_3^2}{\mathcal{A}_1}$
$D7_3$			$\alpha' m^2 = 0$

In the small field approximation (induced by the large volume limit), the oscillator shift reads

$$\zeta_2^{(3)} = \frac{1}{\pi}\text{Arctan}(2\pi\alpha' q H_2^{(3)}) = \frac{1}{\pi}\text{Arctan}\left(\frac{\alpha' k_2^{(3)}}{\mathcal{A}_3}\right) \approx \frac{\alpha' k_2^{(3)}}{\pi\mathcal{A}_3}. \tag{3.3.36}$$

The Kähler moduli stabilization model we will use in section 4.2 fixes the $\mathcal{A}_i$ ratios and the tori areas are power fractions of the total volume: $\mathcal{A}_i \equiv \alpha' r_i \mathcal{V}^{1/3}$, with $r_1 r_2 r_3 = 1$. Hence, the masses of the lowest-lying states read

$$\alpha' m_{23}^2 = -\lvert\zeta_2^{(3)}\rvert + \frac{\alpha' a_3^2}{\mathcal{A}_1} \approx -\frac{\lvert k_2^{(3)}\rvert}{\pi r_3 \mathcal{V}^{1/3}} + \frac{a_3^2}{r_1 \mathcal{V}^{1/3}}, \tag{3.3.37}$$

$$\alpha' m_{22}^2 = -2\lvert\zeta_2^{(3)}\rvert + \frac{y\mathcal{A}_2}{\alpha'} \approx -\frac{2\lvert k_2^{(3)}\rvert}{\pi r_3 \mathcal{V}^{1/3}} + y r_2 \mathcal{V}^{1/3}. \tag{3.3.38}$$

Thus, when $\pi r_3 a_3^2 > r_1 \lvert k_2^{(3)}\rvert$ the m_{23}^2 mass is positive for any value of the volume. For instance, considering $a_{3x} = a_{3y} = \frac{1}{2}$ and taking $r_1 = r_3$, as will be the case in the following, the condition to eliminate the tachyon in the intersection of $D7_2$ and $D7_3$ branes is

$$4\text{Re}(U_1)\lvert k_2^{(3)}\rvert < \pi\lvert 1 + iU_1\rvert^2. \tag{3.3.39}$$

For instance, in square torus this condition is reduced to $\lvert k_2^{(3)}\rvert < \pi$ corresponding from (3.3.22) to $n_2^{(3)} < \pi m_2^{(3)}$, *i.e* to a flux number smaller than the wrapping number. Concerning the second lowest lying massive state on the $D7_2$ branes, we observe that

$$m_{22}^2 \xrightarrow[\ln \mathcal{V} \to \pm\infty]{} \pm\infty, \tag{3.3.40}$$

hence, depending on the flux $\lvert k_2^{(3)}\rvert$ and separation x, m_{22} turns negative when the volume falls below a specific value, *e.g.* $\mathcal{V}_-$, as required for our waterfall field candidate.

Magnetic fields on each stack We now consider the following configuration with magnetic fields on each stack, again denoted by a circled cross $\otimes$.

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\otimes$	$\times$
$D7_2$	$\times$	$\cdot$	$\otimes$
$D7_3$	$\otimes$	$\times$	$\cdot$

The different contributions to the annulus amplitude A_0, A_1 and A_2 corresponding to the neutral, single and doubly charged strings, read

$$8\mathsf{A}_0 = \left(N_1\bar{N}_1 W_1 \tilde{P}_2 P_3 + N_2\bar{N}_2 P_1 W_2 \tilde{P}_3 + N_3\bar{N}_3 \tilde{P}_1 P_2 W_3\right) T_{oo}(0,0,0), \tag{3.3.41}$$

$$
\begin{aligned}
4\mathsf{A}_1 =& -i\Big(N_1N_2T_{fo}(0,\zeta_1^{(2)}\tau,\zeta_2^{(3)}\tau)+\bar N_1N_2T_{fo}(0,-\zeta_1^{(2)}\tau,\zeta_2^{(3)}\tau)\Big)\frac{k_2^{(3)}\eta^3}{\vartheta_4(0)\vartheta_4(\zeta_1^{(2)}\tau)\vartheta_1(\zeta_2^{(3)}\tau)}\\
&+i\Big(N_1\bar N_2T_{fo}(0,\zeta_1^{(2)}\tau,-\zeta_2^{(3)}\tau)+\bar N_1\bar N_2T_{fo}(0,-\zeta_1^{(2)}\tau,-\zeta_2^{(3)}\tau)\Big)\frac{k_2^{(3)}\eta^3}{\vartheta_4(0)\vartheta_4(\zeta_1^{(2)}\tau)\vartheta_1(-\zeta_2^{(3)}\tau)}\\
&-i\Big(N_1N_3T_{fo}(\zeta_3^{(1)}\tau,\zeta_1^{(2)}\tau,0)+N_1\bar N_3T_{fo}(-\zeta_3^{(1)}\tau,\zeta_1^{(2)}\tau,0)\Big)\frac{k_1^{(2)}\eta^3}{\vartheta_4(\zeta_3^{(1)}\tau)\vartheta_1(\zeta_1^{(2)}\tau)\vartheta_4(0)}\\
&+i\Big(\bar N_1N_3T_{fo}(\zeta_3^{(1)}\tau,-\zeta_1^{(2)}\tau,0)+\bar N_1\bar N_3T_{fo}(-\zeta_3^{(1)}\tau,-\zeta_1^{(2)}\tau,0)\Big)\frac{k_1^{(2)}\eta^3}{\vartheta_4(\zeta_3^{(1)}\tau)\vartheta_1(-\zeta_1^{(2)}\tau)\vartheta_4(0)}\\
&-i\Big(N_2N_3T_{go}(\zeta_3^{(1)}\tau,0,\zeta_2^{(3)}\tau)+N_2\bar N_3T_{go}(-\zeta_3^{(1)}\tau,0,\zeta_2^{(3)}\tau)\Big)\frac{k_3^{(1)}\eta^3}{\vartheta_4(\zeta_3^{(1)}\tau)\vartheta_4(0)\vartheta_1(\zeta_2^{(3)}\tau)}\\
&+i\Big(\bar N_2\bar N_3T_{go}(-\zeta_3^{(1)}\tau,0,-\zeta_2^{(3)}\tau)+\bar N_2N_3T_{go}(\zeta_3^{(1)}\tau,0,-\zeta_2^{(3)}\tau)\Big)\frac{k_3^{(1)}\eta^3}{\vartheta_4(\zeta_3^{(1)}\tau)\vartheta_4(0)\vartheta_1(-\zeta_2^{(3)}\tau)},
\end{aligned}
\tag{3.3.42}
$$

$$
\begin{aligned}
8\mathsf{A}_2 =& -iN_1^2W_1P_3T_{oo}(0,2\zeta_1^{(2)}\tau,0)\frac{2k_1^{(2)}\eta}{\vartheta_1(2\zeta_1^{(2)}\tau)}+i\bar N_1^2W_1P_3T_{oo}(0,-2\zeta_1^{(2)}\tau,0)\frac{2k_1^{(2)}\eta}{\vartheta_1(-2\zeta_1^{(2)}\tau)}\\
&-iN_2^2P_1W_2T_{oo}(0,0,2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\eta}{\vartheta_1(2\zeta_2^{(3)}\tau)}+i\bar N_2^2P_1W_2T_{oo}(0,0,-2\zeta_2^{(3)}\tau)\frac{2k_2^{(3)}\eta}{\vartheta_1(-2\zeta_2^{(3)}\tau)}\\
&-iN_3^2P_2W_3T_{oo}(2\zeta_3^{(1)}\tau,0,0)\frac{2k_3^{(1)}\eta}{\vartheta_1(2\zeta_3^{(1)}\tau)}+i\bar N_3^2P_2P_3T_{oo}(-2\zeta_3^{(1)}\tau,0,0)\frac{2k_3^{(1)}\eta}{\vartheta_1(-2\zeta_3^{(1)}\tau)}.
\end{aligned}
\tag{3.3.43}
$$

Exactly the same comments as those under eqs. (3.3.24) to (3.3.26) apply here. The Möbius contributions have similar forms as those in eqs. (3.3.27) and (3.3.28), and are omitted here since they play no role in our arguments. They act as for the single magnetized brane of the previous paragraph, generating states in antisymmetric representations of the gauge groups. They also modify the chiral fermion multiplicity described around equation (3.3.30).

The masses of the lowest-lying states of the spectrum are shown in the following table

	$D7_1$	$D7_2$	$D7_3$
$D7_1$	$\alpha' m^2 = -2\lvert\zeta_1^{(2)}\rvert$	$\alpha' m^2 = \lvert\zeta_2^{(3)}\rvert - \lvert\zeta_1^{(2)}\rvert$	$\alpha' m^2 = \lvert\zeta_1^{(2)}\rvert - \lvert\zeta_3^{(1)}\rvert$
$D7_2$		$\alpha' m^2 = -2\lvert\zeta_2^{(3)}\rvert$	$\alpha' m^2 = \lvert\zeta_3^{(1)}\rvert - \lvert\zeta_2^{(3)}\rvert$
$D7_3$			$\alpha' m^2 = -2\lvert\zeta_3^{(1)}\rvert$

We see that two different kinds of states appear: the $D7_a$–$D7_a$ (doubly charged) states, and the mixed $D7_a$–$D7_b$ ones, with $a \neq b$. The mass of the former can be uplifted as in the previous subsection and will be explained below. We can use neither Wilson lines nor brane separations to increase the mixed states masses, since these can be introduced only in directions without magnetic fields, *i.e.* along both worldvolumes (for Wilson lines), or transverse to both stacks (for separations). In the directions along the magnetic field, zero modes of gauge potentials are gauge artifacts and thus unphysical. We must then specify the fields $H_a^{(i)}$ in order to eliminate the tachyons, at least at large volumes. By a simple inspection of the table above, it follows that the only way to eliminate all three potential

tachyons in the $D7_a$–$D7_b$ brane intersections is to choose

$$|\zeta_1^{(2)}| = |\zeta_2^{(3)}| = |\zeta_3^{(1)}|. \tag{3.3.44}$$

The corresponding lowest-lying states then become massless.

As for a the single magnetized brane, to uplift the tachyons on the $D7_a$–$D7_a$ sectors, we can introduce distance separations between branes and their images (in the direction orthogonal to their worldvolume), or Wilson lines *i.e.* constant background gauge fields (on the unmagnetised worldvolume torus). We show below a configuration keeping only one potential tachyonic state that can play the role of the waterfall field:

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\otimes$	$\times$
$D7_2$	$\times$	$\cdot$	$\otimes$
$D7_3$	$\otimes$	$\times$	$\cdot$

$\longrightarrow$

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\otimes$	$\times_{A_1}$
$D7_2$	$\times$	$\cdot_{\pm x_2}$	$\otimes$
$D7_3$	$\otimes$	$\times_{A_3}$	$\cdot$

Using the notation of the previous subsection, we introduce (discrete) Wilson lines along the third torus T_3^2 for the $D7_1$ stack and along the second torus T_2^2 for the $D7_3$ stack, while we separate the $D7_2$ stack from its orientifold image in its transverse directions. The masses for the double charge states in the three brane stacks now become:

$$\alpha' m_{11}^2 = -2|\zeta_1^{(2)}| + \frac{\alpha' a_1^2}{\mathcal{A}_3} \approx -\frac{2\alpha'|k_1^{(2)}|}{\pi\mathcal{A}_2} + \frac{\alpha' a_1^2}{\mathcal{A}_3} \approx -\frac{2|k_1^{(2)}|}{\pi r_2 \mathcal{V}^{1/3}} + \frac{a_1^2}{r_3 \mathcal{V}^{1/3}}, \tag{3.3.45}$$

$$\alpha' m_{22}^2 = -2|\zeta_2^{(3)}| + \frac{y\mathcal{A}_2}{\alpha'} \approx -\frac{2\alpha'|k_2^{(3)}|}{\pi\mathcal{A}_3} + \frac{y\mathcal{A}_2}{\alpha'} = -\frac{2|k_2^{(3)}|}{\pi r_3 \mathcal{V}^{1/3}} + y r_2 \mathcal{V}^{1/3}, \tag{3.3.46}$$

$$\alpha' m_{33}^2 = -2|\zeta_3^{(1)}| + \frac{\alpha' a_3^2}{\mathcal{A}_2} \approx -\frac{2\alpha'|k_3^{(1)}|}{\pi\mathcal{A}_1} + \frac{\alpha' a_3^2}{\mathcal{A}_2} \approx -\frac{2|k_3^{(1)}|}{\pi r_1 \mathcal{V}^{1/3}} + \frac{a_3^2}{r_2 \mathcal{V}^{1/3}}. \tag{3.3.47}$$

To obtain the second equality of each equation we used large volume expansions for $\zeta_a^{(j)}$ as in eq. (3.3.36). The Wilson lines and brane position parameters are defined as in eqs. (3.3.33) and (3.3.34). Similarly to the single magnetic field case, by choosing appropriately a_1, a_3, and the values of the magnetic fluxes $|k_1^{(2)}|$ and $|k_3^{(1)}|$, one can eliminate the $D7_1$–$D7_1$ and $D7_3$–$D7_3$ tachyons. For instance, as explained after eq. (3.3.38), for $a_i = 1/2$ typical for $\mathbb{Z}_2$ orbifolds, this requires flux numbers smaller than wrapping numbers. On the other hand, the $D7_2$–$D7_2$ state becomes tachyonic at and below a critical value of the volume that can be chosen to be around $\mathcal{V}_-$, defined in (4.2.41)), as required for the waterfall field.

Magnetic fields on entire worldvolumes In the previous case we saw that in order to eliminate the mixed-state tachyons from brane intersections we had to impose condition (3.3.44). We now relax this condition by introducing magnetic fields in all worldvolume tori as shown below:

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\otimes$	$\otimes$
$D7_2$	$\otimes$	$\cdot$	$\otimes$
$D7_3$	$\otimes$	$\otimes$	$\cdot$

The masses of the potential tachyonic states can be extracted by computing the annulus amplitude as done before and they are shown in the following table:

	$D7_1$	$D7_2$	$D7_3$
$D7_1$	$\alpha' m^2 = -2\left\|\zeta_1^{(2)} + \zeta_1^{(3)}\right\|$	$\alpha' m^2 = \left\|\zeta_1^{(3)} \pm \zeta_2^{(3)}\right\| - \left\|\zeta_1^{(2)} \pm \zeta_2^{(1)}\right\|$	$\alpha' m^2 = \left\|\zeta_1^{(2)} \pm \zeta_3^{(2)}\right\| - \left\|\zeta_1^{(3)} \pm \zeta_3^{(1)}\right\|$
$D7_2$		$\alpha' m^2 = -2\left\|\zeta_2^{(1)} + \zeta_2^{(3)}\right\|$	$\alpha' m^2 = \left\|\zeta_2^{(1)} \pm \zeta_3^{(1)}\right\| - \left\|\zeta_2^{(3)} \pm \zeta_3^{(2)}\right\|$
$D7_3$			$\alpha' m^2 = -2\left\|\zeta_3^{(1)} + \zeta_3^{(2)}\right\|$

where the $\pm$ signs in the same equality have to be identical (*e.g.* if the first $\pm$ is a $+$, the second is a $+$ as well.)

The mixed states $D7_a$–$D7_b$, $a \neq b$ can be eliminated by choosing an appropriate field configuration, satisfying a system of inequalities defined by the positivity of the corresponding mass expressions in the table:

$$\begin{cases} \left|\zeta_1^{(3)} \pm \zeta_2^{(3)}\right| - \left|\zeta_1^{(2)} \pm \zeta_2^{(1)}\right| \geq 0 \\ \left|\zeta_1^{(2)} \pm \zeta_3^{(2)}\right| - \left|\zeta_1^{(3)} \pm \zeta_3^{(1)}\right| \geq 0 \\ \left|\zeta_2^{(1)} \pm \zeta_3^{(1)}\right| - \left|\zeta_2^{(3)} \pm \zeta_3^{(2)}\right| \geq 0 \end{cases} \tag{3.3.48}$$

This system is solved by the following configurations

$$\begin{aligned} (A{-}1) \quad & \zeta_1^{(3)} = \zeta_2^{(1)} = \zeta_3^{(2)}, \quad \zeta_1^{(2)} = \zeta_2^{(3)} = \zeta_3^{(1)}; \\ 2) \quad & \zeta_1^{(3)} = \zeta_2^{(1)} = -\zeta_3^{(2)}, \quad \zeta_1^{(2)} = \zeta_2^{(3)} = -\zeta_3^{(1)}; \\ 3) \quad & \zeta_1^{(3)} = -\zeta_2^{(1)} = \zeta_3^{(2)}, \quad \zeta_1^{(2)} = -\zeta_2^{(3)} = \zeta_3^{(1)}; \\ 4) \quad & \zeta_1^{(3)} = -\zeta_2^{(1)} = -\zeta_3^{(2)}, \quad \zeta_1^{(2)} = -\zeta_2^{(3)} = -\zeta_3^{(1)}; \\ (B{-}1) \quad & \zeta_1^{(2)} = \zeta_1^{(3)}, \quad \zeta_2^{(1)} = \zeta_2^{(3)}, \quad \zeta_3^{(1)} = \zeta_3^{(2)}; \\ 2) \quad & \zeta_1^{(2)} = -\zeta_1^{(3)}, \quad \zeta_2^{(1)} = -\zeta_2^{(3)}, \quad \zeta_3^{(1)} = -\zeta_3^{(2)}; \end{aligned} \tag{3.3.49}$$

for which all inequalities are saturated and the lowest lying mixed states become massless.

For the solutions $(A-i)$, all the doubly charged states $D7_a$–$D7_a$ have identical tachyonic masses equal to $\alpha' m^2 = -2\left|\zeta_1^{(2)} + \zeta_1^{(3)}\right|$, while for solution $(B-1)$ they can have different masses. Solution $(B-2)$ is the supersymmetry preserving one, with all lowest lying states remaining massless. In both $(A-i)$ and $(B-1)$ cases, the study of tachyonic states and their elimination through Wilson lines and brane separations is identical to what we did before. Nevertheless, we see that we are allowed to have more complex configurations than with only one magnetic field on each brane.

3.4 Summary

In this chapter we reviewed crucial elements of string theory used in string phenomenology. We started from a quick review of bosonic and superstring worldsheet constructions, their light-cone gauge quantizations and the resulting mass formulae. We described the massless spectrum of open strings and of the closed string sector of type IIB superstrings. We then introduced the torus, Klein bottle, annulus and Möbius vacuum amplitudes and explained their role in constructions with unoriented strings, the tadpole cancellation conditions they generate and their interpretations in terms of non-perturbative objects, namely Dp branes and Op planes.

After this general introduction we focused on compactification mechanisms. They are at the heart of string phenomenology models, as easily understood from the necessity

to go from the ten dimensions of superstrings to our observed four-dimensional spacetime. We motivated that most of our discussions were done in the type IIB framework, which is most convenient for string phenomenology applications, and explained the basic ingredients of circle, toroidal and Calabi-Yau compactifications, as well as the use of orientifolding procedures to reduce supersymmetry. This led us to the subject of closed string moduli and their stabilization, which we tackled in some detail. We recalled the importance of a recently discovered quantum logarithmic correction to the Kähler potential and motivated its use in moduli stabilization models involving $D7$ branes. We postponed the cosmological implications of the low effective theory emerging from the quantum corrected Kähler potential until chapter 4.

Finally, we introduced toroidal orbifold compactifications, which lead to deeper comprehension of the complicated Calabi-Yau compactifications. Toroidal orbifolds allow for exact results and can be seen as singular limits (in moduli space) of CY spaces. A famous example is the $T^4/\mathbb{Z}_2$ orbifold, which is the singular limit of $K3$ with moduli at the boundary of the moduli space. General properties of toroidal orbifolds were reviewed before exposing the explicit example of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and showing its vacuum amplitudes. We then explained how internal $U(1)$ magnetic fields can be used to break supersymmetry through their interactions with internal spins, and lead to potential tachyons in the spectrum.

In this context, we investigated the possibility of generating a "waterfall field", which we use for cosmological applications in chapter 4. This kind of field is characterized by an internal-volume dependent mass, positive at large volumes and negative below a certain volume. The existence of such a field comes from the combination of two types of contributions to charged open string scalar masses: a positive supersymmetric contribution corresponding to Wilson lines or brane separations, and a possibly negative non-supersymmetric contribution coming from the presence of magnetic fields. The first contribution can be described in the low-energy theory by a superpotential and its associated F-term, while the second one is described by an appropriate FI term entering the D-auxiliary component of the $U(1)$. We come back on the effective theory description in section 4.4.

Chapter 4

String phenomenology and cosmology

4.1 Early Universe Cosmology and String theory

The inflation paradigm was developed in the is 80's [115, 116, 117] to answer several issues of standard cosmology, *i.e*. cosmology with radiation and matter dominations. Among others, the two most important ones are the horizon and the flatness problems. The first one is related to the difficulty to explain why our Universe is so homogeneous and isotropic, when standard cosmology tells us that we can observe points far away from each other, in opposite directions, that have never been causally connected. Another way to state this problem is by defining the horizon distance at some time as the distance traveled by light since the Big Bang singularity. Regions of our Universe that, according to standard cosmology, causally connect only today, or equivalently scales that "enter" the horizon today, would then have no reason to share similar properties. The second problem is related to the small amount of curvature density observed today, which in standard cosmology would necessarily be the result of an extreme fine-tuning of the curvature density in the past, hence the name of flatness problem.

A way to resolve these problems is called inflation and postulates a long period of accelerated expansion of the Universe before the epoch of standard cosmology. During this epoch, distances are blown up exponentially while the horizon is almost fixed, hence scales exit horizon very quickly. They can then reenter it during the standard cosmology eras. If inflation has lasted enough, scales that come back into the horizon today would have already been inside the horizon during inflation, and this resolves the horizon problem. The flatness problem is also resolved by the fact that, during the inflationary stage, the curvature density drops very fast. Hence, even if the curvature density started from a large value, if inflation lasted enough, it would have reached the seemingly tuned value needed for today's observations. One can evaluate the minimum amount of inflation, computed in terms of e-folds, necessary to solve these two problems.

It is easy to see, from the Friedmann equations, that inflation can be described by an approximately constant energy density, leading to an exponential expansion in a quasi de Sitter space. The pressure related to the source of energy must be negative in order to have acceleration of the expansion rate. The most famous example is a scalar field, called inflaton, rolling down a scalar potential at a sufficiently small speed. Models using a slowly rolling scalar field are called slow-roll inflation models.

When inflation happens below the Planck scale, it can in principle be described by an effective theory. Nevertheless, if inflation lasted for a certain amount of time, scales that reentered the horizon today would have been sub-Planckian in the past and should thus be described by a quantum gravity theory. The question of the description of inflation in a UV complete quantum gravity theory, such as string theory, thus naturally arises. As we

have explained in section 3.2.2, moduli scalar fields arise from string theory compactifications. We recall that moduli have vanishing flat scalar potential and one need to specify moduli stabilization mechanisms in order to generate mass terms and scalar potential for these fields. It is thus natural to study inflation in the context of moduli stabilization and to search for an inflaton candidate between these moduli.

We will first introduce definitions and elementary facts used in scalar field inflation models in section 4.1.1. We then review shortly in section 4.1.2 some of the problems one has to face in string cosmology and then evoke various models studied in the past.

4.1.1 Scalar field inflation

In this section we expose general facts about scalar field inflation models. Rather than entering the general concepts of inflation, we give a list of definitions and introduce all the notions used in the following sections.

In first approximation, our expanding Universe is homogeneous and isotropic and can be described by the standard $4d$ Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + d\Omega^2\right), \tag{4.1.1}$$

parametrized by the scale factor $a(t)$ and curvature parameter $k = -1, 0, 1$. As usual, r is a radial coordinate and here $d\Omega^2$ is the metric on a sphere.

Inflation is characterized by an accelerated expansion of the Universe and occurs when $\ddot{a} > 0$. The dots denote derivatives with respect to the FLRW time t. We first give the definition of the Hubble parameter

$$H(t) = \frac{\dot{a}}{a}. \tag{4.1.2}$$

We then recall the Friedmann equations, which are the Einstein equations associated to the FLRW metric, for an expanding Universe filled with a single scalar field ϕ in scalar potential $V(\phi)$. They read

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + \kappa^2 V(\phi), \tag{4.1.3}$$

$$2\dot{H} = -\dot{\phi}^2. \tag{4.1.4}$$

The Klein-Gordon equation for the scalar field in FLRW background is

$$\ddot{\phi} + 3H\dot{\phi} + \kappa^2 V'(\phi) = 0\,. \tag{4.1.5}$$

We recall that our ϕ is dimensionless. Following the Hamilton-Jacobi method [118], we make a change of variable to take the inflaton ϕ as the time variable. This change of variable is given by rewriting $\dot{H} = \frac{dH}{d\phi}\dot{\phi}$ in equation (4.1.4), leading to

$$\frac{dH}{d\phi} = H'(\phi) = -\frac{1}{2}\dot{\phi}\,. \tag{4.1.6}$$

Using (4.1.3) and expressing $\dot{\phi}$ as a function of H and V, we obtain

$$H'(\phi) = \mp\frac{1}{\sqrt{2}}\sqrt{3H^2(\phi) - \kappa^2 V(\phi)}\,. \tag{4.1.7}$$

We also use the exact slow-roll parameters defined as [119]

$$\eta(\phi) = 2\,\frac{H''(\phi)}{H(\phi)}\,, \qquad \epsilon(\phi) = -\frac{\dot{H}}{H^2} = 2\left(\frac{H'(\phi)}{H(\phi)}\right)^2\,. \tag{4.1.8}$$

From the first expression of ϵ in (4.1.8), we obtain

$$\frac{\ddot{a}}{aH^2} = 1 - \epsilon, \tag{4.1.9}$$

so that $\epsilon < 1$ is the natural criterion characterising inflation. The parameters defined in (4.1.8) are different from their slow-roll approximations expressed in terms of the potential only

$$\eta_V(\phi) = \frac{V''(\phi)}{V(\phi)}, \quad \epsilon_V(\phi) = \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \tag{4.1.10}$$

with slow-roll limit $\epsilon \underset{\epsilon\ll 1}{\to} \epsilon_V$, $\eta \underset{\eta\ll 1}{\to} \eta_V - \epsilon_V$.

When these two slow-roll parameters are small, $\dot{\phi}$ can be neglected in (4.1.3) (and $\ddot{\phi}$ in (4.1.5)) and we get the following Hubble constant slow-roll solution

$$H_{sr}(\phi) = \kappa\sqrt{\frac{V(\phi)}{3}}. \tag{4.1.11}$$

In general (slow-roll or not), the number of e-folds N before the end of inflation is defined by

$$N = \ln\left(\frac{a_{end}}{a}\right) = \int_t^{t_{end}} H dt = \int_\phi^{\phi_{end}} H\frac{d\phi}{\dot{\phi}} = -\frac{1}{2}\int_\phi^{\phi_{end}} \frac{H}{H'}d\phi = \frac{1}{\sqrt{2}}\int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{\epsilon}}. \tag{4.1.12}$$

From equation (4.1.12) we observe that in order to obtain the correct amount of e-folds near the minimum, *i.e.* when $V' = 0$, we would rather use definition (4.1.8) of ϵ rather than ϵ_V of (4.1.10). Nevertheless, in our study, the definition (4.1.10) of η_V is sufficient and is the one we will use (dropping the subscript V). Indeed, we only use this slow-roll parameter to determine when the modes exit the horizon. As we will see in the next sections, at this point $\epsilon_* \ll \eta_* \ll 1$ and thus, according to (4.1.10), the slow-roll expression η_V gives the correct estimate.

We recall now that any model of inflation is constrained by the observed inhomogeneities in the energy density power spectrum, related to the temperature anisotropies of the cosmological microwave background (CMB) radiation, or to the number density of galaxy clusters. The three observational constraints are the following:

- In order to solve the horizon problem, inflation must last for at least 60 e-folds after horizon exit of the interesting modes.This is at least true for high inflation scale. Instead for lower energies, *e.g.* for TeV scale inflation, the required numbered of e-folds decreases up to about 50. Considering high inflation scale we must thus obtain

$$N_* = \int_{\phi_{end}}^{\phi_*} \frac{d\phi}{\sqrt{\epsilon}} \gtrsim 60. \tag{4.1.13}$$

  The star in N_* or ϕ_* denotes the values of variables taken at horizon exit.

- The observed spectral index n_S measuring the deviation from a scale-invariant power spectrum, is related to the slow-roll parameters at horizon exit by

$$n_S - 1 = 2\eta_* - 6\epsilon_* \simeq -0.04\,. \tag{4.1.14}$$

- The spectral amplitude $\mathcal{A}_S$ induced by observations is

$$\mathcal{A}_S = \frac{\kappa^4 V_*}{24\pi^2\epsilon_*} \simeq 2.2 \times 10^{-9}. \tag{4.1.15}$$

4.1.2 String inflation scenarios

As we mentioned in the introduction of this section, scalar fields similar to the one needed for slow-roll inflation appear naturally in string theory compactifications, they are moduli of the theory. These massless scalar fields must be given a mass through moduli stabilization mechanisms. If inflation is to be supported by a string modulus, the latter must present an almost flat potential in order to satisfy slow-roll conditions. In the meantime, other moduli must have great masses to ensure that the effective theory describing a single inflaton scalar field is correct. The question of generating a potential with a single flat valley, for the inflaton, with steep slopes in other moduli direction is *a prioiri* not so simple because we expect that a generic stabilization mechanism would treat the moduli in approximatively the same way. Moreover, when quantum corrections are used to create huge masses for almost all the moduli, it is not obvious how to keep the flatness of the inflaton potential.

Approximations Even when string theory is used to implement inflation models, one is forced to consider it low energy limit because full non-perturbative control is impossible. The scalar potential is usually derived in the supergravity limit obtained after a long list of approximations, some of which we expose now. First of all, one often consider only the lowest KK modes, which is a correct approximation as long as the different levels do not mix. Another approximation, necessary in order to keep only the first terms in string loop expansion, is that the string coupling g_s is small. As we explained in section 3.2.3, loop corrections are nevertheless often used to generate a potential for the moduli and it is important to evaluate if the neglected corrections are indeed small compared to the considered one. A similar approximation takes place with the α' expansion of the background fields formulations of the worldsheet theory. The background field variations must remain small compared to the string scale. Finally, when non-perturbative objects are employed in the string setup, such as Dp branes in Type IIB compactifications, one often consider them static and neglect their back reaction on the internal manifold geometry.

Hence one major issue in string phenomenology, and string cosmology, is to deal with these approximations and inspect how much they influence the low energy predictions, in particular, the inflationary phase. In order all these approximations to be valid and for inflation to be described by a low energy supergravity approximation of string theory, one need at least the following hierarchy of scales

$$M_{SUSY} < H_{\text{inflation}} < M_{KK} < \frac{1}{\sqrt{\alpha'}} < M_{\text{Pl}}. \tag{4.1.16}$$

The last inequality is obtained by writing the $4d$ Planck mass with respect to the internal volume and the string length, as in (4.4.5), once the small g_s regime is imposed. A discussion on the pros and cons as well as the approximations of type IIB superstring theory to construct dS spaces can be found in the recent review of [120].

de Sitter spaces Another issue important in string cosmology is the value of the cosmological constant, which can be negative, vanishing or positive, leading to an Anti-de Sitter (AdS), Minkowski (M) or de Sitter (dS) space. Cosmological observations today tend towards a tiny positive value of the cosmological constant, or dark energy, hence suggesting that we live in a dS Universe. In generic supergravity setups dS vacua are not obvious to obtain because of the form of the scalar potential (3.2.23) and its negative contribution quadratic in the superpotential magnitude. If one imposes supersymmetry, only this second contribution is non-vanishing at the minimum, and as the D-part of the potential also vanishes, we get a AdS space related to the constant superpotential $\mathcal{W}_0$. Thus, dS spaces can only be obtained through supersymmetry breaking, as it was already clear from the global SUSY discussion of Chapter 2. SUSY breaking must be realized in a controlled way, in the string theory sense. We already evoked methods to obtain dS vacua in section 3.2. These methods are nevertheless subject to controverse, questioning in particular the validity of the approximations used to uplift the AdS vacuum. The use of $\overline{D3}$ branes and their backreaction on the space-time geometry is one of the strongest criticism [121, 122]. The fact that dS space is difficult to obtain in effective theories describing the low energy regime of string theory even led recently to a set of conjectures [123, 124, 125] stating that effective theories with stable dS minimum belong to the string theory "swampland", *i.e.* they are not compatible with a string theory UV completion. The only way to obtain a positive dark energy would then be through quintessence models. Again, the review [120] tackles these aspects in details.

String cosmology models We conclude this section by listing shortly models and ingredients that have been developed to address cosmology through string theory. Reviews on string cosmology can be found for instance in [126, 127, 128, 129]. The string gas cosmology, explaining our number of spatial dimensioins through winding modes annihilations, was developed in [130]. Braneworld cosmology scenarios [131, 132, 133, 134] were rapidly related to string theory setups with branes and antibranes configurations generating a scalar potential for the position moduli [135, 136, 137]. Alternatives to inflation were studied through the ekpyrotic and cyclic Universe scenarios [138, 139, 140]. Application of the KKLT moduli stabilization scenario to cosmology was initiated in [141] and study of the specificities of Kähler moduli inflation in [142].

4.2 Inflation from $D7$-branes moduli stabilization

This section is devoted to the study of cosmological inflation, in a class of effective models emerging in the framework of type IIB moduli stabilization with the logarithmic corrections introduced in section 3.2.3. Our first goal is to investigate the possibility of realistic inflation without imposing the present tiny value of the vacuum energy at the minimum of the scalar potential. We will then address this issue in sections 4.3 to 4.5, together with the question of the end of the inflationary phase, by introducing a "waterfall" field in the context of hybrid inflation.

In section 4.2.1 we introduce the ingredients used to ensure moduli stabilization with the aforementioned logarithmic quantum corrections, and derive the expression of the effective scalar potential. In the simplest case, the scalar potential of the effective four

dimensional theory can be expressed in terms of the volume modulus, which we will identify to the inflaton, and two other orthogonal combinations of Kähler moduli fields. As we show in section 4.2.2, essentially only one free parameter, denoted x, controls the shape of the potential and in particular delimits its two extrema. More precisely, the requirement of a dS minimum confines x in a very small region where the potential stays almost flat, and the two extrema of the potential are very close to each other.

In this restrictive context, we investigate in section 4.2.3 the possible implementations of inflationary scenarios. We start by examining various existing inflationary scenarios including in particular hilltop inflation. Varying x, we adjust the value of the slow-roll parameter η so that inflation starts near the maximum with the correct value of the spectral index. However in this case the slow-roll parameters remain small all the way to the minimum and inflation doesn't stop, producing much more than the required 60 e-folds. Alternatively, imposing the correct number of e-folds, the resulting spectral index does not reproduce the observable value. Hilltop inflation is thus ruled out. We then proceed with a novel proposal where the horizon exit occurs near the inflection point of the potential and inflation takes place essentially near its minimum, with the required number of e-folds being produced. This is a reasonable and completely justifiable assumption, in analogy with the concept of hilltop inflation. Both are characterised with the property that the slow-roll parameter ϵ is negligible at both extrema and consequently, a short interval of the inflaton trajectory is enough to accumulate the required number of e-folds. We find that an inflationary phase is feasible near the minimum and the desired number of e-folds can readily be achieved. This model also predicts a ratio of tensor-to-scalar primordial fluctuations $r \simeq 4 \times 10^{-4}$. Moreover, because of the proximity of the two extrema of the potential, the inflaton is restricted to a short range of values ensuring small field inflation, compatible with the validity of the effective field theory.

On the other hand, since the minimum is generated from quantum corrections, it is metastable and is expected to decay to the true minimum in the runaway direction of large volume. We perform in section 4.2.5 an estimate of its lifetime due to either tunnelling by the Coleman-de Lucia instanton [143, 144], or passing over the barrier by the Hawking-Moss instanton [145]. Our analysis shows that, in the x-region where inflation is viable, the false vacuum decay is due to the latter, leading to an extremely long lifetime.

4.2.1 Type IIB model of intersecting D7-branes and moduli stabilization

We come back on a moduli stabilization model developed in [107] within the type IIB string framework in presence of logarithmic quantum corrections. Complex structure moduli and the dilaton are supposed to be stabilised in the standard supersymmetric way by turning on 3-form fluxes, as described in section 3.2.2. The model takes into account the quantum corrections in a three intersecting $D7$ branes configuration [106]. As explained in section 3.2.3 these corrections break the no-scale structure of the effective theory and give a non-zero contribution to the F-part of the supergravity scalar potential. If one also considers the $U(1)$ anomalous symmetries of the $D7$ branes, Fayet-Iliopoulos D-terms must be introduced in the scalar potential [94] and can be used to uplift the scalar potential to a de Sitter minimum, with all Kähler moduli stabilised.

We denote by τ_i for $i = 1,2,3$ the real parts of the $D7$-branes worldvolume Kähler moduli, and for simplicity we drop their imaginary parts which are absorbed by the anomalous $U(1)$'s to become massive, and play no role in the minimization procedure. The Kähler potential of the model [107] was given in (3.2.34) and we recall it here

$$\mathcal{K} = -\frac{2}{\kappa^2}\ln\left((\tau_1\tau_2\tau_3)^{\frac{1}{2}} + \xi + \sum_{k=1}^{3}\gamma_k \ln(\tau_k)\right) = -\frac{2}{\kappa^2}\ln\left(\mathcal{V} + \xi + \gamma\ln\mathcal{V}\right). \tag{4.2.1}$$

The compactification volume $\mathcal{V}$ is expressed from the τ_i simply as

$$\mathcal{V} = (\tau_1\tau_2\tau_3)^{\frac{1}{2}}. \tag{4.2.2}$$

In fact, some dilaton dependence is hidden in the τ_i moduli, but do not play any role after moduli stabilization from fluxes. We recall that ξ is an α'^3 correction and that γ_i are model dependant parameters for the logarithmic quantum corrections associated with the $D7$ branes [106]. Both these corrections were introduced in section 3.2.3. As in eq. (3.2.34), in the last equality of eq. (4.2.1), we took identical γ_k for simplicity, given by

$$\gamma_1 = \gamma_2 = \gamma_3 \equiv \frac{\gamma}{2} = -\frac{1}{2} g_s T_0 \, \xi, \tag{4.2.3}$$

where T_0/g_s is the effective $D7$-brane tension. Note the minus sign in the last equality of (4.2.3).

One can extract the F-part of the scalar potential using the supergravity formula (3.2.23) with Kähler potential (4.2.1) and a constant superpotential contribution $\mathcal{W}_0$ left over from the 3-form fluxes upon the complex structure moduli stabilization, described in section 3.2.2. The F-part depends only on the volume $\mathcal{V}$ and after defining $\mu = \exp(\xi/2\gamma)$, its exact expression is

$$V_F = \frac{3\gamma\mathcal{W}_0^2}{\kappa^4} \frac{2(\gamma+2\mathcal{V}) + (4\gamma-\mathcal{V})\ln(\mu\mathcal{V})}{(\mathcal{V}+2\gamma\ln(\mu\mathcal{V}))^2 \, (6\gamma^2+\mathcal{V}^2+8\gamma\mathcal{V}+\gamma(4\gamma-\mathcal{V})\ln(\mu\mathcal{V}))}, \tag{4.2.4}$$

In the large volume limit, V_F takes the much simpler form

$$V_F = \frac{3\mathcal{W}_0^2}{2\kappa^4\mathcal{V}^3}\left(2\gamma(\ln\mathcal{V}-4)+\xi\right) + \cdots \tag{4.2.5}$$

The D-part of the scalar potential coming from the $D7$ branes can also be expressed very simply in the large worldvolume limit

$$V_D = \frac{d_1}{\kappa^4\tau_1^3} + \frac{d_2}{\kappa^4\tau_2^3} + \frac{d_3}{\kappa^4\tau_3^3} + \cdots \tag{4.2.6}$$

where the d_i for $i = 1,2,3$ are model dependent constants related to the $U(1)$ anomalies. Contrary to the F-part, this D-part depends on the three τ_i fields. Instead of these Kähler moduli we will rather work with the canonically normalized fields

$$t_i = \frac{1}{\sqrt{2}}\ln(\tau_i), \tag{4.2.7}$$

from which we obtain the following base, after isolating the volume from the two other perpendicular directions

$$\phi = \frac{1}{\sqrt{3}}(t_1+t_2+t_3) = \frac{\sqrt{6}}{3}\ln(\mathcal{V}), \tag{4.2.8}$$

$$u = \frac{1}{\sqrt{2}}(t_1-t_2), \tag{4.2.9}$$

$$v = \frac{1}{\sqrt{6}}(t_1+t_2-2t_3). \tag{4.2.10}$$

In terms of these fields, the D-part of the potential (4.2.6) reads

$$V_D = \frac{e^{-\sqrt{6}\phi}}{\kappa^4}\left(d_1 e^{-\sqrt{3}v-3u} + d_2 e^{-\sqrt{3}v+3u} + d_3 e^{2\sqrt{3}v}\right) + \cdots \tag{4.2.11}$$

so that the total scalar potential is

$$\begin{aligned} V_F + V_D =& \frac{3\mathcal{W}_0^2}{2\kappa^4} e^{-3\sqrt{\frac{3}{2}}\phi}\left(\gamma\left(\sqrt{6}\phi - 4\right) + \xi\right) \\ &+ \frac{e^{-\sqrt{6}\phi}}{\kappa^4}\left(d_1 e^{-\sqrt{3}v-3u} + d_2 e^{-\sqrt{3}v+3u} + d_3 e^{2\sqrt{3}v}\right) + \cdots \end{aligned} \tag{4.2.12}$$

4.2.2 Local de Sitter minimum

We study now the minimum of the scalar potential (4.2.12). The field ϕ will be associated with the inflaton and its evolution will determine the inflation era. We must then stabilise the two other canonically normalized fields u, v at their values u_0 and v_0 dictated by the minimization of V_D in (4.2.6). Their values at the minimum read

$$u_0 = \frac{1}{6}\ln\left(\frac{d_1}{d_2}\right), \quad v_0 = \frac{1}{6\sqrt{3}}\ln\left(\frac{d_1 d_2}{d_3^2}\right), \tag{4.2.13}$$

for which the potential V_D becomes

$$V_D(\phi, u_0, v_0) = \frac{3(d_1 d_2 d_3)^{\frac{1}{3}}}{\kappa^4 \mathcal{V}^2} = \frac{d}{\kappa^4 \mathcal{V}^2} = \frac{d}{\kappa^4} e^{-\sqrt{6}\phi}, \tag{4.2.14}$$

with $d \equiv 3(d_1 d_2 d_3)^{\frac{1}{3}}$. Hence after stabilization of the two transverse moduli, the total scalar potential reduces in the large volume limit to

$$V(\mathcal{V}) = V_F + V_D \simeq \frac{3\mathcal{W}_0^2}{2\kappa^4 \mathcal{V}^3}\left(2\gamma(\ln \mathcal{V} - 4) + \xi\right) + \frac{d}{\kappa^4 \mathcal{V}^2} \equiv \frac{C}{\kappa^4}\left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right), \tag{4.2.15}$$

where we defined

$$q \equiv \frac{\xi}{2\gamma}, \quad \sigma \equiv \frac{2d}{9{\mathcal{W}_0}^2\gamma}, \quad C \equiv -3{\mathcal{W}_0}^2\gamma > 0. \tag{4.2.16}$$

The last inequality is obtained for $\gamma < 0$, which is a condition for a dS vacuum to exist at large volume, as we explain below. The parameter q essentially shifts the local extrema towards large volumes. It is essential in the string context but does not play a role for inflation. Thus, for simplicity, we will take it zero in the numerical study of section 4.2.3, before coming back to its significance in section 4.2.4. C is an overall constant which plays no role in the minimization but will be related to the observed spectral amplitude, when the model is considered as a candidate for an inflationary scenario. Thus, σ is the only effective parameter of the model.

In section 4.2.3 we will study the inflationary possibilities from the above model. The inflaton will be identified with the canonically normalized ϕ/κ, thus we express here the potential (4.2.15) in terms of the (dimensionless) inflaton ϕ

$$V(\phi) \simeq -\frac{C}{\kappa^4} e^{-3\sqrt{\frac{3}{2}}\phi}\left(\sqrt{\frac{3}{2}}\phi - 4 + q + \frac{3}{2}\sigma e^{\sqrt{\frac{3}{2}}\phi}\right). \tag{4.2.17}$$

We emphasize again that ϕ is dimensionless. In order to minimize and study the slow-roll parameters we compute the first two derivatives of V. From (4.2.17) we get

$$V'(\phi) = 3\sqrt{\frac{3}{2}}\frac{C}{\kappa^4}e^{-3\sqrt{\frac{3}{2}}\phi}\left(\sqrt{\frac{3}{2}}\phi + q - \frac{13}{3} + \sigma e^{\sqrt{\frac{3}{2}}\phi}\right), \tag{4.2.18}$$

$$V''(\phi) = -\frac{27}{2}\frac{C}{\kappa^4}e^{-3\sqrt{\frac{3}{2}}\phi}\left(\sqrt{\frac{3}{2}}\phi + q - \frac{14}{3} + \frac{2}{3}\sigma e^{\sqrt{\frac{3}{2}}\phi}\right). \tag{4.2.19}$$

Solving $V'(\phi) = 0$ leads to the two solutions

$$\phi_- = -\sqrt{\frac{2}{3}}\left(q - \frac{13}{3} + W_0\left(-e^{-x-1}\right)\right), \tag{4.2.20}$$

$$\phi_+ = -\sqrt{\frac{2}{3}}\left(q - \frac{13}{3} + W_{-1}\left(-e^{-x-1}\right)\right), \tag{4.2.21}$$

with ϕ_- the local minimum and ϕ_+ the local maximum, with $\phi_- < \phi_+$. $W_{0/-1}$ are the two branches of the Lambert function (or product logarithm) and x is defined through the relation

$$x \equiv q - \frac{16}{3} - \ln(-\sigma) \quad \leftrightarrow \quad \sigma = -e^{q-\frac{16}{3}-x}. \tag{4.2.22}$$

As mentioned above, it is clear from (4.2.20) and (4.2.21) that when x is kept constant, varying q shifts the local extrema. The critical value $x_c \simeq 0.072132$ gives a Minkowski minimum, *i.e.* with $V(\phi_-) = 0$. The region $0 < x < x_c$ gives a dS minimum and $x > x_c$ gives an AdS one. The region $x < 0$ corresponds to the case where the two branches of the Lambert function join and the potential loses its local extrema. The shape of the potential in the three regimes is shown in Figure 4.1.

The values of the potential and its derivatives at the extrema can be derived from (4.2.20) and (4.2.21):

$$V(\phi_{-/+}) = -\frac{C}{6\kappa^4}e^{-13+3q+3W_{0/-1}(-e^{-x-1})}\left(2 + 3W_{0/-1}(-e^{-x-1})\right), \tag{4.2.23}$$

$$V'(\phi_{-/+}) = 0, \tag{4.2.24}$$

$$V''(\phi_{-/+}) = \frac{9C}{2\kappa^4}e^{-13+3q+3W_{0/-1}(-e^{-x-1})}\left(1 + W_{0/-1}(-e^{-x-1})\right). \tag{4.2.25}$$

Firstly, from (4.2.23) we see that only the parameter x determines the ratio between the values of the potential at the extrema. Indeed we get

$$\frac{V(\phi_+)}{V(\phi_-)} = \frac{\left(W_0(-e^{-x-1})\right)^3\left(2 + 3W_{-1}(-e^{-x-1})\right)}{\left(W_{-1}(-e^{-x-1})\right)^3\left(2 + 3W_0(-e^{-x-1})\right)}. \tag{4.2.26}$$

This ratio is plotted in the left panel of Figure 4.2. Secondly, we see that at the two extrema, the slow-roll parameter $\eta_V = \frac{V''}{V}$ also depends on x only. It reads

$$\eta(\phi_{-/+}) = \frac{V''(\phi_{-/+})}{V(\phi_{-/+})} = -9\frac{1 + W_{0/-1}(-e^{-x-1})}{\frac{2}{3} + W_{0/-1}(-e^{-x-1})}. \tag{4.2.27}$$

Again, from (4.2.27) and (4.2.26) we see that x is the only important parameter of the model for the shape of the potential. From Figure 4.2 we see that as soon as $x \lesssim 0.05$ there is no scale separation anymore between the values of the potential at the two local extrema.

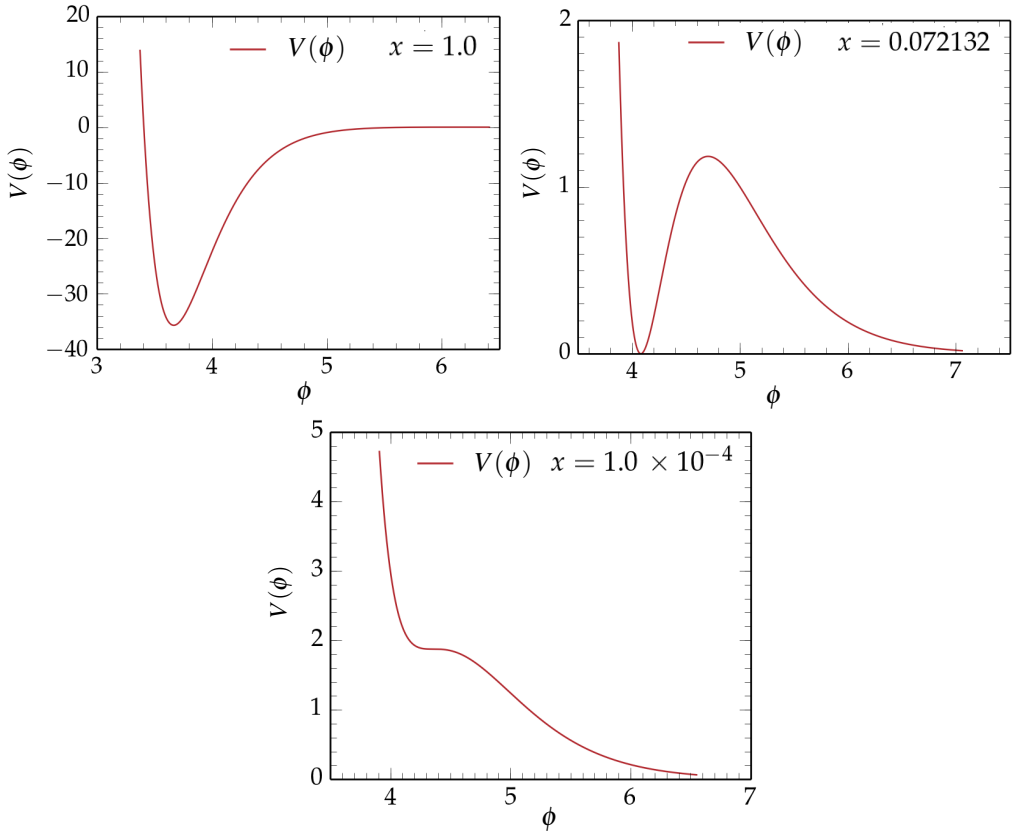


FIGURE 4.1: Scalar potential $V(\phi)$ for different values of x giving an AdS, Minkowski or dS vacuum.

4.2.3 Inflation possibilities from the model

Hilltop inflation The hilltop inflation scenario [116, 146] emerged more than thirty years ago. The idea is the following: the inflaton starts rolling from a local maximum down to the minimum of the potential. In the vicinity of the local maximum, the slow-roll parameter ϵ is negligible while η is determined by the observed spectral index. The horizon exit occurs near the maximum, and the 60 remaining e-folds are obtained from there to a point before the minimum, where $\epsilon = 1$ and inflation stops. As $\epsilon \to 0$ at the maximum, one can generate an infinite number of e-folds. In reality, the number of e-folds is dictated by the initial condition. The closest to the maximum the inflaton starts rolling down, the largest the number of e-folds is.

The fact that the inflaton starts rolling from the maximum of the potential may be motivated if one considers that this maximum was related to a symmetry restoration point. At higher temperatures this point could have been a symmetric minimum of the potential, which became a maximum after spontaneous symmetry breaking occurring when temperature cooled down. Hence if the inflaton sits at a symmetric point at higher temperatures, it is natural to take initial conditions near the maximum.

In our model, according to (4.2.27) the values of the η slow-roll parameter at the maximum only depends on x. Solving (4.2.27) in order to have $\eta(\phi_+) \simeq -0.02$ we find $x \simeq 2.753 \times 10^{-7}$. This value can also be obtained graphically from the right panel of Figure 4.2.

In order to study the possibility of hilltop inflation we then take $x \simeq 2.753 \times 10^{-7}$,

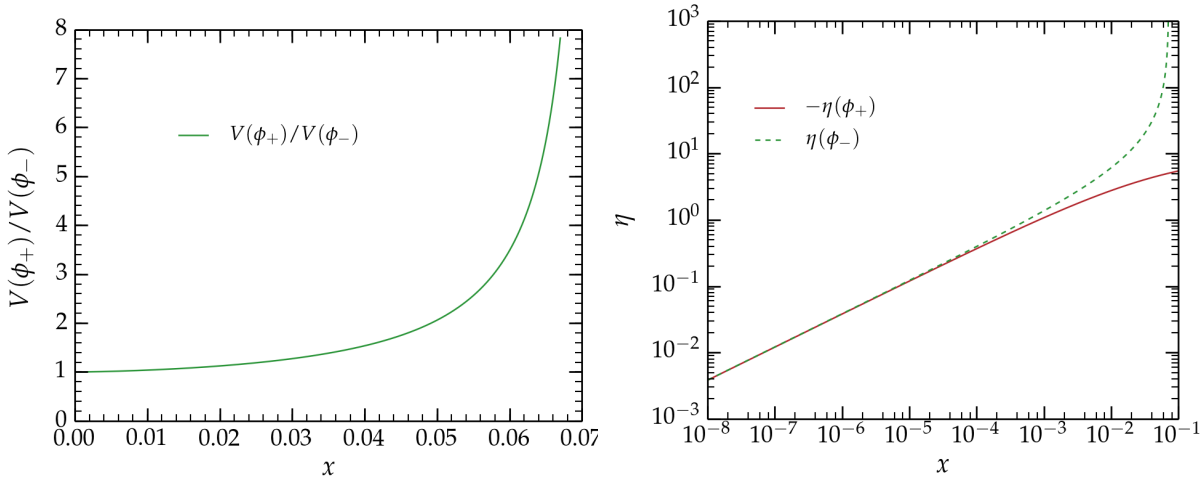


FIGURE 4.2: Ratio of the values of the scalar potential V at the two extrema (*left panel*) and value of the slow-roll parameter η_V at the two extrema (*right panel*), as functions of the parameter x.

$q = 0$ and $C = 1.0 \times 10^8$ and solve (4.1.7) numerically.[1] This allows to find the Hubble parameter solution all along the inflation trajectory. Due to the shape of the potential (containing exponentials and linear terms), we had to use very high precision data types. This was achieved through a `C++` code using `cpp_dec_float` data types provided by the `multiprecision` package of the `boost` library.

Plots of all the interesting parameters are shown in Figure 4.3. The horizontal axes show the values of the inflaton ϕ. As the field starts from the maximum and goes towards the minimum, the arrow of time is from right to left (*i.e.* for decreasing ϕ.)

We see from the bottom panel of Figure 4.3 that the slow-roll parameters ϵ and η stay small, hence the slow-roll regime holds all the way from the maximum down the minimum. Since $\epsilon \ll 1$, inflation continues until the minimum and there is no natural criterion marking the end of inflation. Of course, the value of the potential at the minimum being of order of the inflation scale, some new physics should be added to lower the potential near the actual cosmological constant. Nevertheless we see from Figure 4.3 that there is a huge number of e-folds all along the inflationary trajectory, *i.e.* between the would be horizon exit at $\eta_* = -0.02$ (near the maximum) and the minimum. Hence, this model cannot accommodate hilltop-inflation scenario because the constraints $N_* \gtrsim 60$ and $\eta_* = -0.02$ cannot be satisfied together by adjusting x, the only relevant parameter here.

Inflation around the minimum from the inflection point *General idea* We now consider the case where the e-folds are obtained only near the minimum. This allows to alleviate the constraint $\eta(\phi_+) \simeq -0.02$. We start with initial conditions near the maximum with no initial speed. We come back to this point at the end of the section. The inflationary phase corresponds to the inflaton rolling down its potential. As it goes from the maximum to the minimum, the second derivative $V''(\phi)$ changes sign and if $\eta(\phi_+) < -0.02$, it will pass through the value $\eta(\phi_*) = -0.02$ before the inflection point. The x parameter of the model can then be chosen so that at least 60 e-folds are obtained from this point to the end of inflation. From the above argument we see that in order this scenario to correctly

[1] The C coefficient being an overall scaling, its value is not important in the numerical studies. Hence we choose it, such that the potential is of order 1. The true value of C is determined by the amplitude (4.1.15) in the end.

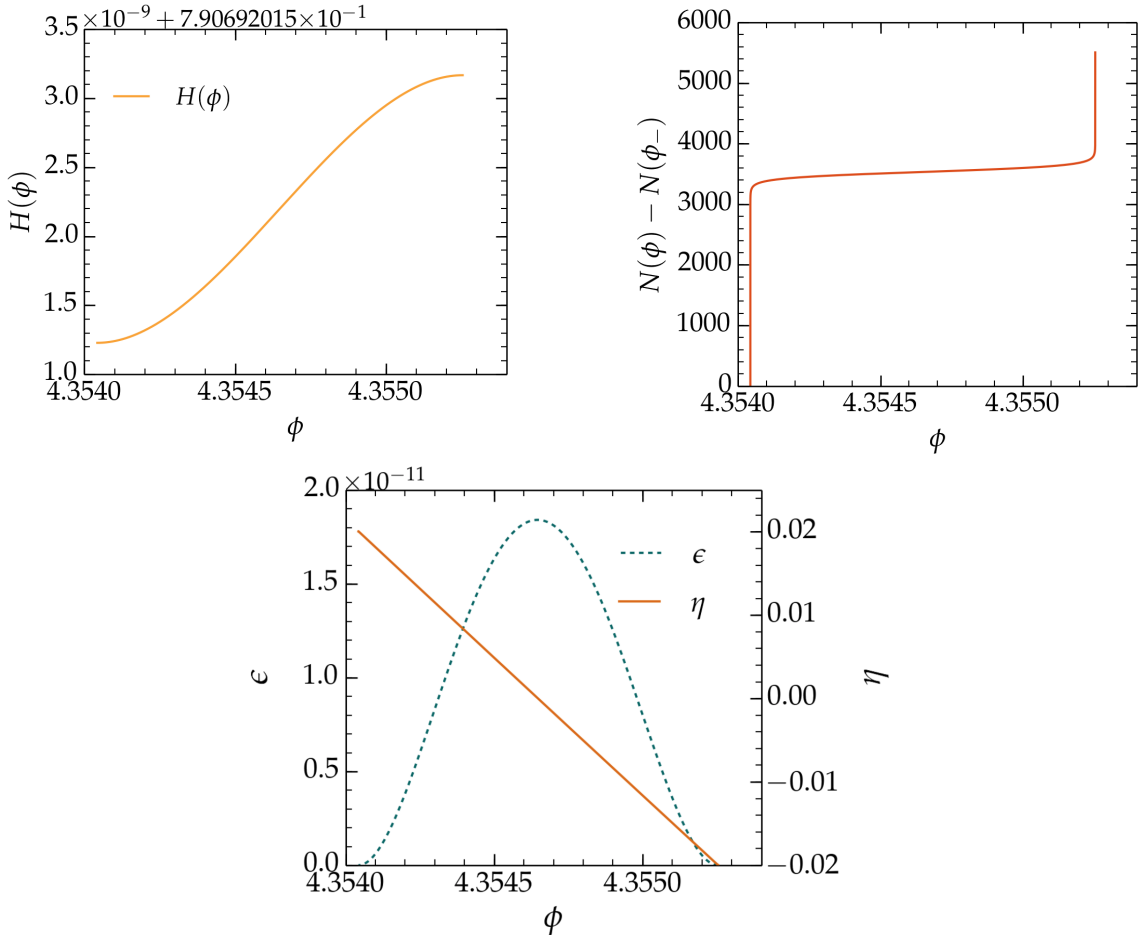


FIGURE 4.3: Hubble parameter $H(\phi)$ (*top left*), number of e-folds $N(\phi)$ (*top right*) and slow-roll parameters ϵ, η (*bottom*) for $q = 0$, $C = 1 \times 10^8$ and $x = 2.753 \times 10^{-7}$.

match the observational data, the initial position of the inflaton has to be higher than the inflection point, where η is negative, so that $\eta = -0.02$ is taken at the horizon exit.

As in the hilltop case of section 4.2.3, we solved the evolution equation (4.1.7) numerically starting near the maximum with vanishing inflaton initial speed. We got $N_* \simeq 60$ for $x \simeq 3.3\ 10^{-4}$. In that case, $\phi_- = 4.334$ and $\phi_+ = 4.376$. The e-folds are computed from the horizon exit $\phi_* \simeq 4.354$ at which $\eta(\phi_*) = -0.02$, to the minimum ϕ_-. Is should be observed that the corresponding inflaton field displacement

$$\Delta\phi \simeq 0.02\,, \tag{4.2.28}$$

is much less than one in Planck units, corresponding to small field inflation, compatible with the validity of the effective field theory. We show the numerical solution in Figure 4.4. Again, the horizontal axes correspond to the inflaton ϕ. As the field starts from the maximum and goes towards the minimum, the arrow of time is from right to left (*i.e.* for decreasing ϕ.)

In Figure 4.4 we see that at the minimum neither ϵ nor η is bigger than one (η is however close to 1). It is easy to understand from the plot of the ϵ parameter and formula (4.1.12) that almost all e-folds are obtained near the minimum because ϵ is very tiny there. The vertical line shows the value of ϕ_* for which $\eta(\phi_*) = -0.02$. It is very close to the inflection point, hence the modes exit the horizon near (a bit before) the inflection point.

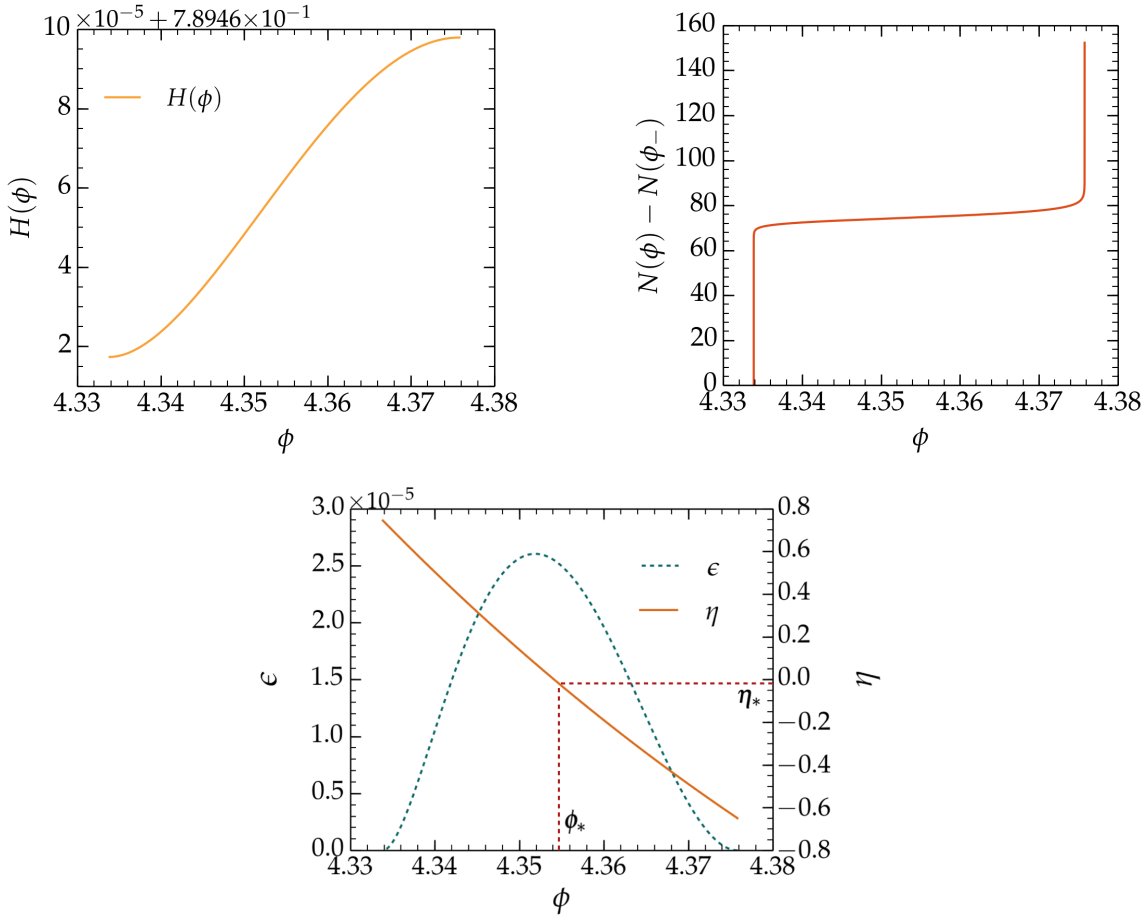


FIGURE 4.4: Hubble parameter $H(\phi)$ (*top left*), number of e-folds $N(\phi)$ (*top right*) and slow-roll parameters ϵ, η (*bottom*), for $q = 0$, $C = 1 \times 10^8$ and $x = 3.3 \times 10^{-4}$. The dashed horizontal line shows the value $\eta = -0.02$.

Study near the minimum We wish to describe carefully what happens close to the minimum. Indeed, one has to check that the field goes on the other side of the minimum ($\phi < \phi_-$). We then expect that the field stops at a value ϕ_{stop} and goes back towards the minimum, starting its oscillation phase, usually related to the reheating period and the inflaton decay [147, 148, 149]. Nevertheless, one usually assumes tha the inflaton potential (almost) vanishes at the minimum. The inflaton decays into other particles and the cosmological constant stays then neglible in front of the radiation and matter densities for a sufficiently long period of time (assuming a solution to the cosmological constant problem).

In our model, for the parameter $x \simeq 3.3 \times 10^{-4}$ chosen here, there is no scale separation between the inflation scale H_* (at horizon exit) and the scale at the minimum, see Figure 4.2. Therefore the standard reheating scenario cannot occur, because the potential energy of the inflaton (or equivalently the cosmological constant) remains important at the minimum. Hence, if nothing is modified in the model the energy density of the created particles stays small compared to the cosmological constant.

In the `C++` program used to solve numerically (4.1.7), the field values are stored in `cpp_dec_float` types of variable, available in the `multiprecision` package of the `boost` library. These variables allow to store numbers of the form $a \times 10^n$ with a 100 digits precision on the coefficient a. When we reach the minimum, the field evolves very slowly and if $\phi_{stop} - \phi_-$ is small, so is $H(\phi_{stop}) - H(\phi_-)$ and the 100 digits precision is not enough to

determine when the inflaton stops (*i.e.* when $\partial H/\partial\phi = 0$). In order to bypass this difficulty, we expand the Hubble parameter H around the slow-roll solution H_{sr} by defining a new variable δH through

$$H = H_{sr} + \delta H = \sqrt{\frac{V}{3}} + \delta H \,. \tag{4.2.29}$$

Replacing the first derivative of δH,

$$\delta H' = H' - H'_{sr} = H' - \frac{1}{2\sqrt{3}}\frac{V'}{\sqrt{V}} \,, \tag{4.2.30}$$

in (4.1.7), one finds the new form of equation (4.1.7)

$$\delta H' = \pm\frac{1}{\sqrt{2}}\sqrt{6H_{sr}\delta H} - \frac{1}{2\sqrt{3}}\frac{V'}{\sqrt{V}} \,, \tag{4.2.31}$$

where we neglected the δH^2 term. The advantage of this new formulation is that now, even if δH is small compared to H_{sr}, their values are not stored using the same coeffiicient and we do not have to store precisely $H = H_{sr} + \delta H$. The numerical solution of the evolution equation (4.2.31) confirms that the inflaton indeed reaches $\phi_{stop} < \phi_-$, but stays very close. Due to the small values of the slow-roll parameters (see Figure 4.4) the inflaton is still in a slow-roll regime near the minimum and the oscillations are very slow. In fact we checked numerically that the number of e-folds during the first oscillations is greater than the one from ϕ_* to ϕ_-. As mentioned earlier, this is easily understandable considering the fact that due to the large value of $V_- = V(\phi_-)$ (same scale as the inflation scale) the kinetic energy of the inflaton is not big enough to produce particles which would change significantly the equation of state of the Universe.

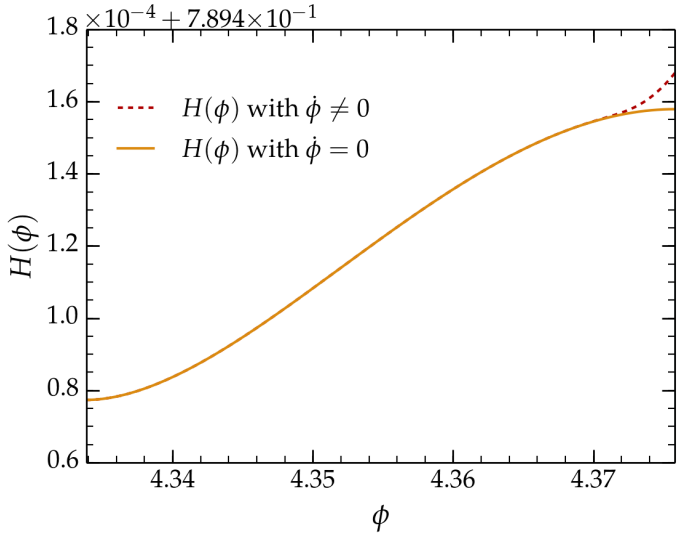


FIGURE 4.5: Hubble parameter solution for $x = 3.3 \times 10^{-4}$, $q = 0$, $C = 1 \times 10^8$ with initial speed $\dot\phi = -2H'(\phi) = -2 \times 10^{-5}\kappa^{-1}$ (*red dashed line*), and with vanishing initial speed (*plain orange line*).

Initial conditions In the above study we started with the inflaton near the maximum with vanishing speed. In fact, one can change these initial conditions without altering the conclusions of the study as long as the inflaton starts between the maximum and the inflection point with a relatively small speed. Indeed, the constraint that the inflaton starts

higher that the inflection point comes from the fact that it has to cross the $\eta = -0.02$ point when rolling towards the minimum. Nevertheless, an argument of symmetry restoration similar to the one explained in section 4.2.3 for the hilltop scenario motivates that the field starts near the maximum. In that case, if the initial speed stays relatively small, the inflaton is damped sufficiently near the maximum, such that the study does not change with respect with the case with vanishing initial speed. A solution for a non-zero initial speed is shown in Figure 4.5. If the initial speed is too large, equation (4.1.3) shows that the major contribution to the Hubble parameter comes from the inflaton kinetic energy. As V does not vary much from the maximum to the minimum, the inflaton only sees a flat potential until it reaches the wall at small ϕ. In that case the previous study does not hold, and slow-roll inflation is not obtained.

4.2.4 Physical observables and theoretical parameters

In this section we study the implications of the inflationary scenario described in section 4.2.3 to physical observables and we discuss the relation of the parameters of the model to those of the fundamental string theory.

Inflation scale We see from Figure 4.4 that when the modes exit the horizon, the value of the slow-roll parameter related to the amplitude of primordial fluctuations is $\epsilon_* \simeq 2.5 \times 10^{-5}$, implying a value for the ratio r of tensor to scalar perturbations

$$r = 16\epsilon_* \simeq 4 \times 10^{-4}\,. \tag{4.2.32}$$

From (4.1.15) we deduce that

$$\kappa^4 V_* = 24\pi^2 \epsilon_* A_S \simeq 1.48 \times 10^{-11}. \tag{4.2.33}$$

This constraint fixes the overall amplitude of the scalar potential. Indeed for the x value of interest $V_* \simeq V(\phi_-)$ and we see from (4.2.23) that the value of the potential at the minimum reads

$$\kappa^4 V(\phi_-) = -\frac{C}{6} e^{-13+3q+3W_0(-e^{-x-1})} \left(2 + 3W_0(-e^{-x-1})\right) \equiv C e^{3q} \times w(x), \tag{4.2.34}$$

with

$$w(x) = -\frac{1}{6} e^{-13+3W_0(-e^{-x-1})} \left(2 + 3W_0(-e^{-x-1})\right). \tag{4.2.35}$$

For the value of $x \simeq 3.3 \times 10^{-4}$ realising the inflationary scenario described in section 4.2.3, we obtain $w(x) \simeq 1.87 \times 10^{-8}$. Together with the constraint (4.2.33), equation (4.2.34) fixes the overall constant to

$$C e^{3q} = \frac{\kappa^4 V(\phi_-)}{w(x)} \simeq \frac{\kappa^4 V_*}{w(x)} \simeq \frac{1.46 \times 10^{-11}}{1.87 \times 10^{-8}} \simeq 7.81 \times 10^{-4}. \tag{4.2.36}$$

Note that for $q = 0$ the value of C is different from the one used in the plots of Figure 4.4, but as we explained in section 4.2.2, the overall constant just scales the potential and has no implication in the study of the inflation phase dynamics (in particular, it does not appear in the slow-roll parameters computation).

From (4.2.33) we deduce that the inflation scale is

$$H_* \simeq \kappa \sqrt{\frac{V_*}{3}} \simeq 2.2 \times 10^{-6} \kappa^{-1} \simeq 5.28 \times 10^9 \ \text{TeV}\,. \tag{4.2.37}$$

In the last equality we used the value of the Planck scale that we recall here: $\kappa^{-1} \simeq 2.4 \times 10^{18}$ GeV. As mentioned earlier, we observe from Figure 4.4 that the value of the potential at the horizon exit and at the minimum are almost identical. Hence the positive value of the potential at the de Sitter minimum is given by $\kappa^4 V_{dS} \simeq \kappa^4 V_*$ and is way above the observed cosmological constant today.

String parameters We now relate the parameters of the model to those of the underlying string theory and examine the constraints implied by the inflationary scenario described above. The string parameters are: ξ, γ, related to the quantum corrections, d associated with the anomalous $U(1)$ charges of the $D7$ branes, and $\mathcal{W}_0$ the constant superpotential remaining after complex structure moduli and axion-dilaton stabilization. For the sake of clarity, we write again the expressions of the quantum corrections parameters

$$\xi = -\frac{\zeta(3)}{4}\chi_{CY}, \qquad \gamma = -\frac{1}{2}g_s T_0 \xi\,, \tag{4.2.38}$$

and of the parameters entering the scalar potential, already introduced in (4.2.16)

$$q \equiv \frac{\xi}{2\gamma} = -\frac{1}{g_s T_0}, \quad \sigma \equiv \frac{2d}{9{\mathcal{W}_0}^2\gamma} < 0, \quad C \equiv -3{\mathcal{W}_0}^2\gamma > 0, \tag{4.2.39}$$

As already mentioned, note that in order to have $C > 0$ we need $\gamma < 0$ and hence a negative Euler number χ_{CY}. We have also defined the x parameter by

$$x \equiv q - \frac{16}{3} - \ln(-\sigma)\,. \tag{4.2.40}$$

From (4.2.20) we deduce that the volume at the minimum is a function of q and x only:

$$\mathcal{V}_- = \exp\left(\frac{3}{\sqrt{6}}\phi_-\right) = e^{-q} \times \exp\left(\frac{13}{3} - W_0\left(-e^{-x-1}\right)\right). \tag{4.2.41}$$

Thus, for a given value of x, one obtains large volume for large (negative) q. In fact from (4.2.39), q is indeed negative for positive T_0, implying a surplus (locally) of D7-branes relative to orientifold O7-planes [106]. Then large values of q are reached as long as g_s is small. Hence the weak coupling and large volume limits are related in a simple way.

We now turn back to the string parameters $\mathcal{W}_0$ and d, which are partially fixed by the observational constraint through (4.2.36). From the expressions (4.2.39), the superpotential reads

$$\mathcal{W}_0^2 = -\frac{C}{3\gamma} \simeq -\frac{2.64 \times 10^{-4}}{\gamma} e^{-3q}, \tag{4.2.42}$$

whereas the d parameter from the $U(1)$ D-terms of $D7$ branes is

$$d = \frac{9\mathcal{W}_0^2\gamma}{2}\sigma = \frac{3C}{2}e^{q-\frac{16}{3}-x} \simeq \frac{3}{2} \times 7.81 \times 10^{-4} e^{-3q} \times e^{q-\frac{16}{3}-x} \simeq 5.65 \times 10^{-6} e^{-2q}\,. \tag{4.2.43}$$

We see from (4.2.42) that for values of $-\gamma$ around 10^{-2}, the value $\mathcal{W}_0 \sim 1$ is reached as soon as $-q \gtrsim 5$. On the other hand, from (4.2.43) we see that $d \sim 1$ is reached for $-q \gtrsim 7$.

We conclude that for $-q = 1/(g_s T_0)$ not much greater than a few units, our inflationary model can be accommodated in the weak string coupling and large volume limits. This justifies that the large volume limit could safely be taken in the expressions (4.2.5) – (4.2.6) of the the scalar potential contributions V_F and V_D. Moreover, the superpotential

$\mathcal{W}_0$ and D-term coefficient d take values of order one. In fact, $\mathcal{W}_0$ around unity can be naturally obtained from combinations of integer fluxes.

4.2.5 Stability of the minimum

For the value of x considered in section 4.2.3 in order to get an inflationary period, the values of the potential at the minimum and maximum are very close. Hence it is important to know if the inflaton can escape from the local minimum, or the false vacuum, and tunnel through the barrier of the potential before evolving classically towards the true minimum in the runaway direction at large field values. We recall that the shape of the potential for the value of the parameter x giving an inflationary epoch is similar to the one shown in the right panel of Figure 4.1.

To evaluate the false vacuum stability we use the methods developed by Coleman et al. [143, 144]. In order to keep their conventions, we will use the dimension-full inflaton

$$\varphi = \frac{\phi}{\kappa}. \tag{4.2.44}$$

This definition of the inflaton field φ will be considered only in the current section. We are interested in the probability that the inflaton tunnels from the false vacuum at $\varphi_- = \kappa^{-1}\phi_-$ to the true vacuum located at the runaway direction $\varphi = +\infty$. Following the Coleman-de Luccia (CdL) argument [144], this probability is expressed as a decay rate per unit volume and time by

$$\Gamma = Ae^{-B}, \qquad \text{with} \qquad B = S_E(\varphi) - S_E(\varphi_-), \tag{4.2.45}$$

where S_E is the Euclidean tunnelling action to minimize. Here φ denotes the instanton solution of the scalar field action. (For a recent review on vacuum stability, see [150]). For a scalar field coupled to gravity the Euclidean action reads

$$S_E = \int d^4x\sqrt{g}\left(-\frac{1}{2\kappa^2}R + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right). \tag{4.2.46}$$

Following the strategy of CdL, one looks for a solution with an $O(4)$ symmetry. Such a Euclidean space-time can be described by the following metric

$$ds^2 = d\rho^2 + \chi(\rho)^2(d\Omega_3)^2, \tag{4.2.47}$$

were $(d\Omega_3)^2$ is the metric of the unit 3-sphere. The Euclidean scale factor $\chi(\rho)$ gives the curvature of the 3-sphere at given ρ. The Euclidean field equations for the scalar field φ and the scale factor χ are

$$\varphi'' + 3\frac{\chi'}{\chi}\varphi' = \frac{\partial V}{\partial\varphi}, \tag{4.2.48}$$

$$\chi'' = -\frac{\kappa^2}{3}\chi\left(\varphi'^2 + V(\varphi)\right) \qquad \left(\text{or} \quad \chi'^2 = 1 + \frac{\kappa^2}{3}\chi^2\left(\frac{1}{2}\varphi'^2 - V(\varphi)\right)\right) \tag{4.2.49}$$

The prime denotes, in this section, derivative with respect to ρ which plays the role of the time variable. Using the $O(4)$ symmetry of (4.2.47) and the field equations (4.2.49), one can rewrite the Euclidean action (4.2.46) as

$$S_E = 2\pi^2\int d\rho\left(\chi^3\left(\frac{1}{2}\varphi'^2 + V\right) + \frac{3}{\kappa^2}\left(\chi^2\chi'' + \chi\chi'^2 - \chi\right)\right) = -2\pi^2\int d\rho\chi^3(\rho)V\left(\varphi(\rho)\right). \tag{4.2.50}$$

The two simplest solutions to (4.2.48) – (4.2.49) are the ones where the field φ sits at an extremum of the potential V. These solutions read

$$\varphi(\rho) = \varphi_\pm, \quad \chi(\rho) = \frac{1}{H_\pm}\sin(H_\pm \rho), \quad \text{with} \quad H_\pm = \kappa\sqrt{\frac{V(\varphi_\pm)}{3}}, \tag{4.2.51}$$

and are defined for $\rho \in [0, H_\pm^{-1}\,\pi]$. Once plugged back in (4.2.47), the solution for $\chi(\rho)$ in (4.2.51) gives simply the four sphere metric, which is the Euclidean extension of de Sitter space-time with $H = H_\pm$, obtained by analytic continuation of the real time to the Euclidean time ρ [151]. The solution with the inflaton sitting at the top of the barrier, $\varphi(\rho) = \varphi_+$ is related to the Hawking-Moss (HM) instanton [145]. We come back to this solution later. We compute the action of solution (4.2.51) through equation (4.2.50). It reads

$$S_E(\varphi_\pm) = -2\pi^2 \int d\rho \frac{1}{H_\pm^3}\sin^3(H_\pm\rho)V(\varphi_\pm) = -\frac{24\pi^2}{\kappa^4 V(\varphi_\pm)}. \tag{4.2.52}$$

More complex solutions to (4.2.48) – (4.2.49), referred as CdL instantons, can be found by imposing the following boundary conditions

$$\varphi'(0) = 0, \quad \chi(0) = 0, \quad \chi'(0) = 1, \quad \varphi(\rho_f) = \varphi_-\,, \tag{4.2.53}$$

where the final time ρ_f is determined through $\chi(\rho = \rho_f) = 0$. At that time one should get $\varphi'(\rho_f) = 0$. As one can see, the value $\varphi_0 = \varphi(0)$ is not specified for the CdL instanton. Similarly to what was described originally in the case without gravity [143], the boundary conditions (4.2.53) associated with the field equations (4.2.48) – (4.2.49) correspond to the classical problem of a field evolving in the reverse potential $-V$ and subject to a friction force proportional to $3\chi'/\chi$. Starting from an *a priori* unknown position φ_0, it rolls down to the local minimum $-V(\varphi_+)$, passes it, climbs the hill and stops exactly at the local maximum $-V(\varphi_-)$. One has to find the value of φ_0 for which the field stops exactly at φ_- with no speed. Figure 4.6 sketches the situation of a point-like particle rolling down the inverted potential $-V(\varphi)$. Going back to the real time picture, φ_0 can be interpreted as the position reached by the scalar field after it tunneled the barrier from the valse vacuum φ_-. Then, from this point, the field evolves classically until the true (runaway) minimum at infinity.

Exact analytical solutions for the CdL instantons are difficult to find in general. They have been studied in the case without gravity for simple triangular or squared potentials [152][153]. Analytical solutions were also presented in the original papers [143][144] in the famous "thin-wall" approximation. This limit nevertheless demands that the false and true vacua, $V(\varphi_-)$ and $V(\infty) = 0$, are very close with respect to the height of the barrier $\Delta V = V(\varphi_+) - V(\varphi_-)$. However, this approximation does not hold for the case we are studying, as can be seen for instance from the right panel of Figure 4.1.

CdL instantons can also be searched numerically following the undershooting, overshooting method proposed initially in the original paper without gravity [143]. The idea is very simple: we start with any value φ_0 and solve numerically (4.2.48) – (4.2.49) with initial conditions (4.2.53). If the solution overshoots, *i.e.* if the field continues rolling after having reached the local maximum $-V(\varphi_-)$ of the reversed potential, we start again with a new initial position φ_0 closer to the minimum $-V(\varphi_+)$. If the solution undershoots, *i.e.* if the field does not reach the local maximum $-V(\varphi_-)$, we start with a new φ_0 a bit further from the minimum $-V(\varphi_+)$. We repeat this operation until we find the initial value φ_0 for which the field stops exactly at the maximum of the reversed potential, which is the false vacuum. We sketch the overshooting situation on the left panel of Figure 4.6 and the CdL instanton solution on the right panel.

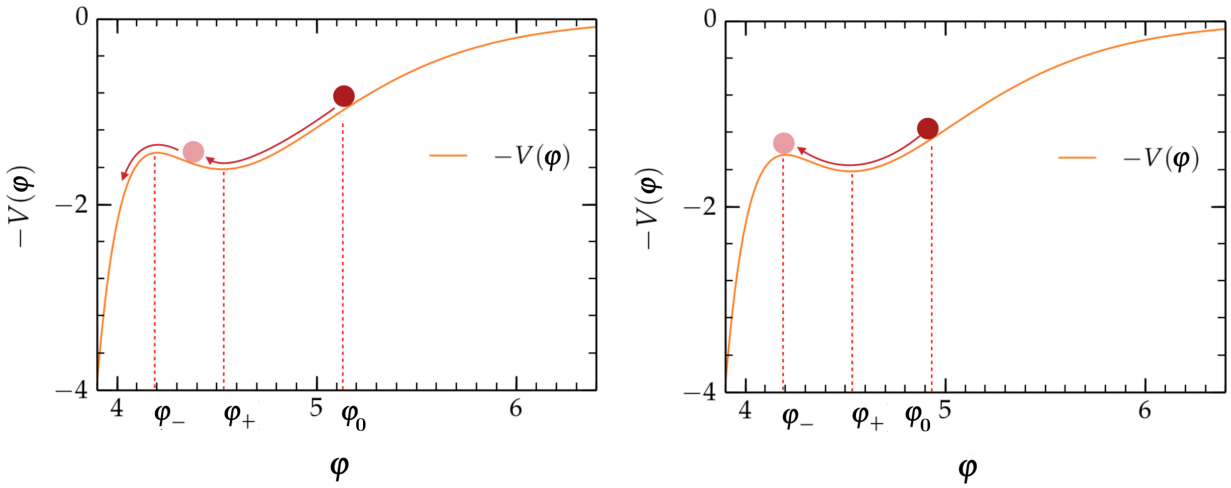


FIGURE 4.6: Point-like particle rolling down the inverted potential $-V(\varphi)$ from an initial position φ_0. An overshooting case is shown (*left panel*) as well as the CdL instanton solution (*right panel*) for which the field stops exactly at the maximum $-V(\varphi_-)$.

In Figure 4.7, we show the CdL instanton solution obtained with this method for the value of the parameter $x = 1.0 \times 10^{-2}$. In this case, the field values at the two extrema are $\varphi_- = \phi_- \kappa^{-1} = 4.2446\kappa^{-1}$ and $\varphi_+ = \phi_+ \kappa^{-1} = 4.4756\kappa^{-1}$, while the initial value is $\varphi_0 = \kappa^{-1}\phi_0 = 4.9224\kappa^{-1}$.

We tried to obtain a solution similar to the one of Figure 4.7 for $x = 3.3 \times 10^{-4}$, which satisfies the observational constraints of the inflationary epoch, as explained in section 4.2.3 . Nevertheless we were not able to find a numerical solution satisfying the boundary conditions. In fact, if the potential barrier is too flat, the existence of CdL instanton is not guaranteed anymore. A criterion for the existence of CdL instanton has been established in the past following various arguments [153] [154]. It demands that the scale factor at the minimum H_- stays below a critical value H_c defined by

$$H_c^2 = -\frac{V_{\varphi\varphi}(\varphi_+)}{4} - \frac{\kappa^2 \Delta V}{3}, \tag{4.2.54}$$

where $V_{\varphi\varphi} = \partial^2 V / \partial \varphi^2$ and as above, $\Delta V = V(\varphi_+) - V(\varphi_-)$ is the height of the barrier. When H_- approches H_c, the potential is flat near the maximum hence the ΔV contribution can be neglected in front of that of $V_{\varphi\varphi}(\varphi_+)$ in (4.2.54). In this case, we can use relations (4.2.23) and (4.2.25) to express the ratio

$$\frac{H_c^2}{H_-^2} \simeq -\frac{3}{4}\frac{V_{\varphi\varphi}(\varphi_+)}{\kappa^2 V(\varphi_-)} = \frac{81 W_0(-e^{-x-1})^3 \left(1 + W_{-1}(-e^{-x-1})\right)}{4 W_{-1}(-e^{-x-1})^3 \left(2 + 3W_0(-e^{-x-1})\right)}. \tag{4.2.55}$$

For $H_- > H_c$ the existence of the CdL instantons is not guaranteed anymore. Figure 4.8 shows the ratio of the Hubble scales in (4.2.55). In fact, the Hubble scale critical value H_c also marks the dominance of the Hawking-Moss instanton [145] solution. As explained before, this solution is the one for which the inflaton stays at the top of the barrier $\varphi(\rho) = \varphi_+$. This solution can be seen as describing the inflaton going above the potential barrier instead of properly tunnelling. As mentioned after (4.2.51) such solutions always exist, but when $H_- < H_c$ their action is higher than the CdL solutions and give thus negligible contribution to the tunnelling rate (4.2.45).

In Figure 4.8, we see that $H_- > H_c$ holds for $x = 3.3 \times 10^{-4}$, which explains that we

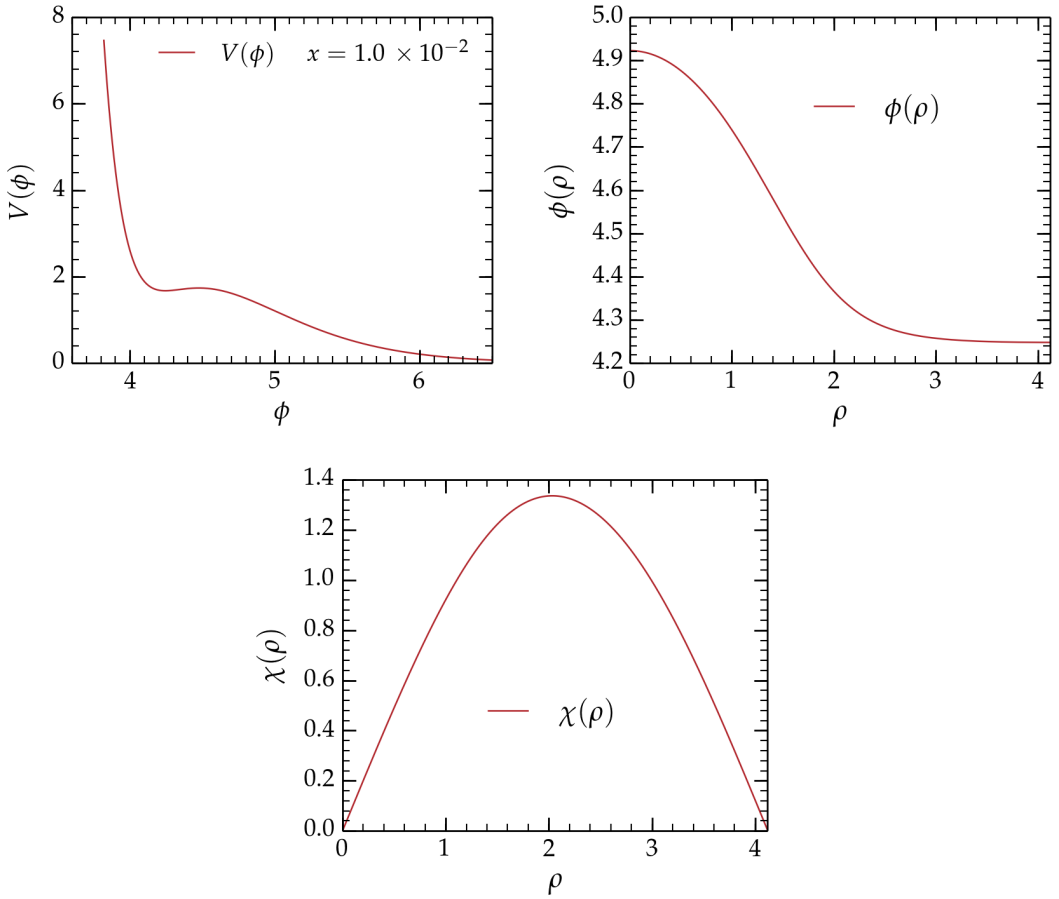


FIGURE 4.7: Scalar potential $V(\phi)$ for $x = 1.0 \times 10^{-2}$ (*top left*) and the CdL bounce solution functions $\phi(\rho)$ (*top right*) and $\chi(\rho)$ (*bottom*). The initial field value is $\varphi_0 = 4.9224\kappa^{-1}$ and the final Euclidean time is $\rho_f = 4.1222$.

are not able to find the standard CdL instanton. Hence in this case only the HM instanton contribute to the tunnelling rate. The tunnelling coefficient B introduced in (4.2.45) is then computed from (4.2.52) and reads

$$B = S_E(\varphi_+) - S_E(\varphi_-) = -\frac{24\pi^2}{\kappa^4 V(\varphi_+)} + \frac{24\pi^2}{\kappa^4 V(\varphi_-)} \simeq \frac{8\pi^2 \Delta V}{3H_*^4}. \tag{4.2.56}$$

In the last equality, ΔV is the height of the barrier and H_* is the inflation scale, *i.e.* the Hubble parameter when the modes exit the horizon. We recall that since in our model the potential is almost flat along the inflationary trajectory, we have $H_- \simeq H_+ \simeq H_*$. For $x = 3.3 \times 10^{-4}$ we find $\Delta V \simeq 2.0 \times 10^{-4} V_*$ and from (4.2.33) – (4.2.37) we deduce

$$B \simeq 3.3 \times 10^9, \qquad i.e. \qquad \Gamma = A e^{-3.3 \times 10^9}. \tag{4.2.57}$$

It follows that the decay rate of the local minimum is extremely small and the vacuum is practically stable.

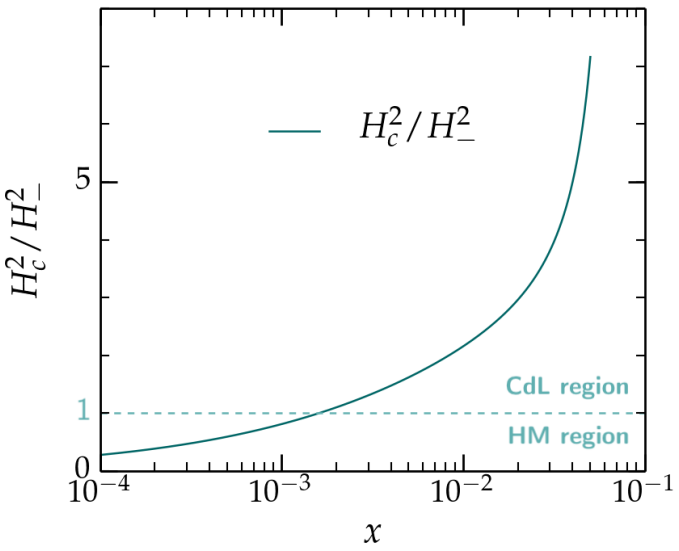


FIGURE 4.8: Ratio of the critical Hubble parameter H_c^2 and the Hubble parameter at the minimum $H_-^2 = \kappa^2 V(\varphi_-)/3$. The value $H_- = H_c$ (horizontal dashed line) determines the existence and domination of the CdL instanton over the HM instanton.

4.3 New physics near the minimum: waterfall fields and hybrid inflation

We now summarize the specificities of the above moduli stabilization model and its inflationary possibility, explained in section 4.2.3, some of the possible theoretical issues, and suggest new ingredients to solve them.

Slow roll inflation compatible with observations can be realized for $x \simeq 3.3\ 10^{-4}$, where the x parameter was defined in eq. (4.2.22). Then, the field separation between the two extrema of the effective potential is given by $\phi_+ - \phi_- = 0.042$. The inflaton starts rolling near the maximum with no initial speed, these initial conditions being motivated if one considers that this maximum is related to a symmetry restoration point. The inflationary phase corresponds to the inflaton rolling down its potential. An analysis of the slow roll parameters ϵ, η, defined in (4.1.10) have shown that $\epsilon \ll |\eta|$ holds in the whole region of the field space $[\phi_-, \phi_+]$. Hence the spectral index of primordial density fluctuation $n_s \simeq 1 + 2\eta$ is fixed by η which has to be around -0.02 at the horizon exit $\phi \equiv \phi^*$ to agree with the data.

As the inflaton ϕ goes down from the maximum to the minimum, the second derivative $V''(\phi)$ changes sign and as the slow roll parameter $\eta(\phi_+) < -0.02$, it passes through the value $\eta(\phi_*) = -0.02$ before the inflection point. The x parameter of the model is chosen so that at least 60 e-folds are obtained from this point to the end of inflation. The required number of $N_* \simeq 60$ e-folds is computed from the horizon exit $\phi_* \simeq \phi_- + 0.02$ at which $\eta(\phi_*) = -0.02$, to the minimum ϕ_-. The modes exit the horizon just before the inflection point is reached and most of the e-folds are obtained around the minimum. Furthermore, it should be emphasised that the corresponding inflaton field displacement is $\Delta\phi \simeq 0.02$, which is much less than one in Planck units, corresponding to small field inflation compatible with the validity of the effective field theory.

We stressed in section 4.2.4 that in this model, the dS vacuum energy is constrained by the choice of the value of the parameter x. For the value of interest for inflation, the

potential at the minimum $V(\phi_-) \simeq V(\phi^*)$ is practically of the same order that the inflation scale. This amount of vacuum energy is way much greater than the observed value today, hence it could not be the true vacuum of the theory. Indeed, with such a big value, the Universe would continue expanding and never reach the standard cosmology with radiation and matter domination eras.

As suggested in [3], the introduction of new physics near the minimum of the potential brings in a natural scenario for the end of the inflation epoch. This relates the model to the hybrid inflation proposal [155], where a second field Y is added to the model. This "waterfall" field Y adds another direction to the scalar potential. If falling towards this direction becomes favorable at a certain point of the inflaton trajectory, this immediately ends the inflation era and the theory reaches another minimum at a different energy scale, which should coincide with the true vacuum today dominated by the observed dark energy.

The main features of the hybrid scenario adapted to our model are described by the following potential

$$V_Y(\phi, Y) = V(\phi) + \frac{1}{2} m_Y(\phi)^2 Y^2 + \frac{\lambda}{4} Y^4 \,, \tag{4.3.1}$$

where $V(\phi)$ is the inflaton potential (4.2.17) and the extra terms contain the dependence in Y together with its coupling to the inflaton ϕ. Depending on the sign of its effective squared mass ${m_Y}^2(\phi)$, the waterfall field Y stays in two separate phases. When ${m_Y}^2 > 0$, the minimum in the Y-field direction is at the origin

$$\langle Y \rangle = 0, \quad \text{when} \quad {m_Y}^2(\phi) > 0 \,, \tag{4.3.2}$$

and the extra contribution to the scalar potential vanishes

$$V_Y(\phi, 0) = V(\phi) \,. \tag{4.3.3}$$

When the mass of Y becomes tachyonic, a phase transition occurs and the new vacuum is obtained for Y at a non-vanishing VEV

$$\langle Y \rangle = \pm \frac{|m_Y|}{\sqrt{\lambda}} \equiv \pm v, \quad \text{when} \quad {m_Y}^2(\phi) < 0. \tag{4.3.4}$$

The value of the potential V_Y at the minimum of this broken phase is

$$V_Y(\phi, v) = V(\phi) - \frac{{m_Y}^4(\phi)}{4\lambda} \,. \tag{4.3.5}$$

For suitable $m_Y(\phi)$ during the inflationary phase, when the field ϕ rolls down the potential, the system is in the symmetric phase and the Y field is stabilised with a vanishing VEV and a large mass. The inflationary phase is then equivalent to the one field inflation model. Subsequently, if ${m_Y}^2$ turns negative near the inflaton minimum ϕ_-, a phase transition occurs and the Y field attains its value given in (4.3.4) at the new minimum. This amounts to a change of the potential $V(\phi)$ near the minimum, by a negative constant $V_{down} = -m_Y^4/(4\lambda) < 0$. The effect of such a downlift is double: it decreases the value of the cosmological constant and if the waterfall direction is steep enough, it gives a natural criterion to stop inflation ($\epsilon > 1$). In the next sections we propose a possible implementation of hybrid inflation in our string theory framework by showing how the waterfall field can be identified with an open string state on $D7$-branes stacks. There, it will be more convenient to think of the inflaton as the internal volume $\mathcal{V}$, related to the inflaton ϕ through (4.2.8). This way we also avoid any confusion with the dilaton.

4.4 Effective theory with waterfall fields

In order to implement hybrid inflation, motivated in section 4.3, we now explain how to combine the Kähler moduli stabilization inflationary model described in section 4.2 with the tachyonic waterfall field toy model developed in section 3.3.3. We recall that the latter is based on a $T^2 \times T^2 \times T^2/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and that the matter fields live on magnetized $D7$ branes.

Before going further, we recall that a large internal space Euler characteristic is required in our moduli stabilization setup. Indeed, as mentioned in section 3.2.3, this condition is necessary to control the approximations in the computation [103, 106] of the localisation width of the induced $4d$ graviton kinetic terms, *i.e.* of the logarithmic Kähler quantum corrections γ_k and γ of eqs. (3.2.34) and (4.2.1). For toroidal orbifolds, the Euler characteristic is defined as

$$\chi_{\text{orb}} = \frac{1}{|G|} \sum_{g,h \in G} \chi(g,h), \tag{4.4.1}$$

where G is the symmetry group of the orbifold and $\chi(g,h)$ the number of fixed points under both twists g and h, taken zero when there is a common fixed torus. In the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example, $G = (\mathbb{1}, \alpha, \beta, \gamma)$ acts as shown in eq. (3.3.14), and a non-trivial (g,h) pair is either (α,β), (α,γ) or (β,γ). They have $\chi(h,g) = \chi(g,h) = 4^3 = 64$. Hence the Euler characteristic is $\chi = 1/4 \times 2 \times 64 \times 3 = 96$, with the factor of 2 coming from the interchange of g and h in the sum. As $96 \gg 1$, the orbifold model is a valid and simple candidate for a specific model.

To study the novelty introduced by the waterfall direction, we will first compute the effective field theory scalar potential for the Kähler moduli and the newly introduced matter fields, and then describe the new vacuum of the theory. The moduli scalar potential depends on the total internal volume $\mathcal{V} = \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3/\alpha'^3 = \sqrt{\tau_1\tau_2\tau_3}$ through the F-part described in eqs. (4.2.4) and (4.2.5), and on the Kähler moduli τ_a through the D-part. In the toroidal orbifold model, the Kähler moduli τ_a are simply related to the 2-tori areas $\mathcal{A}_a$. Moreover, the scalar potential gets a new F-part depending on the matter fields. As we are interested in the waterfall direction, we only keep track of possible tachyonic matter field contributions to the scalar potential, and put the other (massive) matter fields to zero. The canonically normalized tachyonic field, coming from the $D7_2 - D7_2$ state of section 3.3.3, is denoted φ_- (and its charge conjugate φ_+) in the following. They must not be confused with the notations introduced in section 4.2.5 when studying the moduli scalar potential vacuum stability.

For simplicity, we recall the brane configuration of section 3.3.3 in the following table.

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\otimes$	$\times_{A_1}$
$D7_2$	$\times$	$\cdot_{\pm x_2}$	$\otimes$
$D7_3$	$\otimes$	$\times_{A_3}$	$\cdot$

(4.4.2)

For simplicity we will consider wrapping numbers $m_2^{(1)} = m_2^{(3)} = 1$ and $N_2' = 1$ such that the $D7_2$ gauge group is restricted to $U(1)_2$. See eq. (3.3.29) for the definition of N_2'. The number of chiral fermions after orientifold projection is denoted by given by $n_{22'}^{(\Omega)}$. The tachyonic state will hence also have multiplicity $n_{22'}^{(\Omega)}$, corresponding to the different Landau states and related to the intersection numbers. In the following we will often refer to "the tachyon" while describing all the degenerate tachyonic scalars together, because once the tachyon gets a non-vanishing VEV, a specific direction is fixed for all the Landau states, producing a massive field and $n_{22'}^{(\Omega)} - 1$ Goldstone modes.

4.4.1 *D*-term from magnetic fields

The magnetic fields can be described in the effective theory through a *D*-term scalar potential

$$\begin{aligned} V_D &= \sum_a \frac{g^2_{U(1)_a}}{2}\left(\xi_a + \sum_n q_a^n |\varphi_a^n|^2\right)^2 + \cdots \\ &= \sum_{a=1,3} \frac{g^2_{U(1)_a}}{2}\xi_a^2 + \frac{g^2_{U(1)_2}}{2}\left(\xi_2 + 2|\varphi_+|^2 - 2|\varphi_-|^2 + \cdots\right)^2 + \cdots . \end{aligned} \tag{4.4.3}$$

In the first line, the sum runs over the n charged scalar fields. As explained above, in the second line of (4.4.3) we have only kept the tachyonic field (and its charge conjugate) contributions, with charges $q_a = \pm 2$.

The Fayet-Iliopoulos parameters ξ_a and gauge couplings $g^2_{U(1)_a}$ used in the *D*-term scalar potential depend on the Kähler moduli. Indeed, from the *D*-term (4.4.3) and from the string frame expressions (3.3.23), we can write the magnetic field contribution to the mass of the matter fields in the configuration of table (4.4.2) as

$$m^2_{H_2} \equiv 2g^2_{U(1)_2}\xi_2 = \frac{2|\zeta_2^{(3)}|}{\alpha'} \approx \frac{2|k_2^{(3)}|}{\pi\alpha'}\frac{\alpha'}{\mathcal{A}_3} \approx \frac{2|k_2^{(3)}|}{\pi}\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\alpha'}{\mathcal{A}_3}. \tag{4.4.4}$$

We recall that $\zeta_2^{(3)}$ is given in equation (3.3.19) and hence the third equality holds in the small magnetic field (large volume) limit. In order to go to the supergravity frame, we used the four dimensional Planck constant expression

$$\frac{1}{\kappa^2} \equiv \frac{1}{\kappa_4^2} = \frac{\tilde{\mathcal{V}}}{\kappa_{10}^2} = \frac{\tilde{\mathcal{V}}}{\alpha' g_s^2 (4\pi^2\alpha')^3} = \frac{\mathcal{V}}{\alpha' g_s^2}, \tag{4.4.5}$$

where we restored the string units in the total volume $\tilde{\mathcal{V}} = (4\pi^2)^3\alpha'^3\mathcal{V} = (4\pi^2)^3\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3$.

The gauge couplings are expressed in terms of the magnetized *D*7-brane worldvolumes as

$$\frac{1}{g^2_{U(1)_a}} = \frac{|m_a^{(j)} m_a^{(k)}|}{g_s\alpha'^2}\left|\mathcal{A}_j + i\alpha' k_a^{(j)}\right|\left|\mathcal{A}_k + i\alpha' k_a^{(k)}\right|, \qquad \text{with} \quad a \neq j \neq k \neq a. \tag{4.4.6}$$

In the small magnetic fields (large areas) limit, the couplings (4.4.6) reduce to

$$\frac{1}{g^2_{U(1)_a}} \approx |m_a^{(j)} m_a^{(k)}|\frac{\mathcal{A}_j\mathcal{A}_k}{g_s\alpha'^2} = |m_a^{(j)} m_a^{(k)}|\frac{\mathcal{V}}{g_s}\frac{\alpha'}{\mathcal{A}_a}, \qquad \text{with} \quad a \neq j \neq k \neq a. \tag{4.4.7}$$

Combining equations (4.4.4) to (4.4.6), we deduce the expressions for the moduli dependent Fayet-Iliopoulos term

$$\xi_2 = \frac{m^2_{H_2}}{2g^2_{U(1)_2}} \approx |m_2^{(1)} m_2^{(3)}|\frac{g_s|k_2^{(3)}|}{\pi\kappa^2\mathcal{V}}\frac{\mathcal{A}_1}{\alpha'}. \tag{4.4.8}$$

We obtain similar expressions for ξ_1 and ξ_3 for the configuraition of (4.4.2), so that the D-term part of the scalar potential (4.4.3) reads

$$V_D \approx \frac{1}{\kappa^4 \mathcal{V}^2}\left(d_1\frac{\mathcal{A}_3}{\mathcal{A}_2}+d_2\frac{\mathcal{A}_1}{\mathcal{A}_3}+d_3\frac{\mathcal{A}_2}{\mathcal{A}_1}\right)+m_{H_2}^2\left(|\varphi_+|^2-|\varphi_-|^2\right)+2g_{U(1)_2}^2\left(|\varphi_+|^2-|\varphi_-|^2\right)^2, \tag{4.4.9}$$

where we defined the Kähler moduli D-term parameters

$$d_a \equiv \frac{g_{U(1)_a}^2}{2}\xi_a^2 = \frac{1}{2}g_s^3|m_a^{(j)}m_a^{(k)}|\left(\frac{k_a^{(j)}}{\pi}\right)^2, \quad (a,j,k)=\sigma(1,2,3) \text{ with } \sigma \text{ a 3-cycle.} \tag{4.4.10}$$

Note again that the above d_a correspond to the specific flux configuration of (4.4.2).

4.4.2 F-term from brane separation

Appart from the D-term potential, the effective field theory contains a positive mass contribution for the tachyonic scalars of the model described in section 3.3.3. These scalars come from strings stretching between the $D7_2$ brane stack and its image, and the positive contribution to their mass is due to the distance separation between the brane and its orientifold image. It is generated by the VEV of an adjoint scalar coming from strings with both ends on the $D7_2$ stack and preserves supersymmetry, in contrast to the tachyonic contribution from the magnetic field discussed above.

More precisely, this contribution is described by a trilinear superpotential obtained by an appropriate $\mathcal{N}=1$ truncation of an $\mathcal{N}=4$ supersymmetric theory within the untwisted orbifold sector:

$$\mathcal{W}_{C_i^{7_a}} \ni \mathrm{Tr}\left(C_1^{7_a}\left[C_2^{7_a},C_3^{7_a}\right]\right). \tag{4.4.11}$$

The $C_j^{7_a}$ for $j=1,2,3$ are the three $\mathcal{N}=1$ chiral multiplets that are part of an $\mathcal{N}=4$ vector multiplet living on the $D7_a$ brane stack. $C_a^{7_a}$ parametrise the brane position in the transverse plane while $C_j^{7_a}$ with $j\neq a$ are the internal components of the 8d gauge fields along the two planes of the worldvolume of the $D7_a$ brane [156, 157]. As explained above, the couplings of interest are given by equation (4.4.11), with $a=2$. We can then identify the relevant superpotential in our case from[2]:

$$\mathcal{W}_{C_i^{7_2}} = w_{ijk}C_i^{7_2}C_j^{7_2}C_k^{7_2} \ni c\,\Phi_2\Phi_+\Phi_-. \tag{4.4.12}$$

Here Φ_i are the un-normalized fields: Φ_2 is the modulus associated with the $D7_2$ brane position x_2 of section 3.3.3, hence $C_2^{7_2}$, while Φ_- (and Φ_+) is the tachyonic matter field of interest (and its charge conjugate) assimilated to $C_1^{7_2}$ and $C_3^{7_2}$. When Φ_2 acquires a non vanishing VEV $\langle\Phi_2\rangle \sim x_2$, the superpotential (4.4.12) generates a (supersymmetric) mass for the matter fields Φ_+ and Φ_-.

The physical mass for the canonically normalized fields φ_i can be computed from the physical Yukawa couplings derived from the supergravity action [158, 159, 50] and expressed as

$$\mathcal{W}_{tach} = Y_{ijk}\,\varphi_i\varphi_j\varphi_k, \qquad \text{with} \quad Y_{ijk} = w_{ijk}\left(\mathcal{K}_{i\bar{i}}\mathcal{K}_{j\bar{j}}\mathcal{K}_{k\bar{k}}\right)^{-\frac{1}{2}}e^{\frac{\kappa^2}{2}\mathcal{K}}. \tag{4.4.13}$$

$\mathcal{K}_{i\bar{i}}$ are the Kähler metrics of the matter fields of interest (assuming no kinetic mixing), and w_{ijk} is the trilinear coupling of the holomorphic superpotential, which in our case is

[2] In our conventions the superpotential and all un-normalized fields are dimensionless.

simply related to c defined in (4.4.12). In the type IIB string framework, and for the untwisted fields appearing in (4.4.11), the Kähler metrics of the matter fields on magnetized tori read [50, 160, 161, 156, 162]

$$\kappa^2 \mathcal{K}_{C_1^{7_2}\bar{C}_1^{7_2}} = \frac{\pi e^{\phi_4}}{(U_1+\bar{U}_1)}\sqrt{\frac{\alpha'\mathcal{A}_1}{\mathcal{A}_2\mathcal{A}_3}}\left|\frac{m_2^{(3)}}{m_2^{(1)}}\right|\left|\frac{\mathcal{A}_3+i\alpha' k_2^{(3)}}{\mathcal{A}_1+i\alpha' k_2^{(1)}}\right|, \tag{4.4.14}$$

$$\kappa^2 \mathcal{K}_{C_3^{7_2}\bar{C}_3^{7_2}} = \frac{\pi e^{\phi_4}}{(U_3+\bar{U}_3)}\sqrt{\frac{\alpha'\mathcal{A}_3}{\mathcal{A}_1\mathcal{A}_2}}\left|\frac{m_2^{(1)}}{m_2^{(3)}}\right|\left|\frac{\mathcal{A}_1+i\alpha' k_2^{(1)}}{\mathcal{A}_3+i\alpha' k_2^{(3)}}\right|, \tag{4.4.15}$$

$$\kappa^2 \mathcal{K}_{C_2^{7_2}\bar{C}_2^{7_2}} = \frac{\pi e^{\phi_4}}{\alpha'^2(U_2+\bar{U}_2)}\sqrt{\frac{\alpha'\mathcal{A}_2}{\mathcal{A}_1\mathcal{A}_3}}\left|m_2^{(1)}m_2^{(3)}\right|\left|\mathcal{A}_1+i\alpha' k_2^{(1)}\right|\left|\mathcal{A}_3+i\alpha' k_2^{(3)}\right|. \tag{4.4.16}$$

The $m_a^{(i)}$, $n_a^{(i)}$ integers are related to the quantised magnetic field $H_a^{(i)}$ with $k_a^{(i)}$ given in equation (3.3.22). We recall that in the present example we take $m_2^{(1)} = m_2^{(3)} = 1$, as mentioned under eq. (4.4.2). The four dimensional dilaton ϕ_4 is related to the ten dimensional one through the total volume

$$e^{\phi_4} = e^{\phi}\mathcal{V}^{-\frac{1}{2}} = \frac{e^{\phi}\alpha'^{3/2}}{\sqrt{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}}. \tag{4.4.17}$$

The 10d dilaton is part of the axio-dilaton multiplet defined as

$$S = e^{-\phi} + iC_0, \qquad \text{with} \quad g_s = \langle e^{\phi}\rangle. \tag{4.4.18}$$

In the configuration of (4.4.2), $H_2^{(3)}$ is turned on and $H_2^{(1)}$ vanishes. In the large volume limit, *i.e.* when $\alpha' k_2^{(3)} \ll \mathcal{A}_3$, the magnetic flux is diluted and the Kähler metrics approach the unmagnetized ones. We will check later that the magnetic fields are indeed small for our purposes. In that case the Kähler metrics read

$$\kappa^2 \mathcal{K}_{C_1^{7_2}\bar{C}_1^{7_2}} = \frac{\pi e^{\phi_4}}{(U_1+\bar{U}_1)}\sqrt{\frac{\alpha'\mathcal{A}_3}{\mathcal{A}_1\mathcal{A}_2}}\left|\frac{m_2^{(3)}}{m_2^{(1)}}\right| = \frac{1}{(U_1+\bar{U}_1)(\mathcal{T}_3+\bar{\mathcal{T}}_3)}, \tag{4.4.19}$$

$$\kappa^2 \mathcal{K}_{C_3^{7_2}\bar{C}_3^{7_2}} = \frac{\pi e^{\phi_4}}{(U_3+\bar{U}_3)}\sqrt{\frac{\alpha'\mathcal{A}_1}{\mathcal{A}_2\mathcal{A}_3}}\left|\frac{m_2^{(1)}}{m_2^{(3)}}\right| = \frac{1}{(U_3+\bar{U}_3)(\mathcal{T}_1+\bar{\mathcal{T}}_1)}, \tag{4.4.20}$$

$$\kappa^2 \mathcal{K}_{C_2^{7_2}\bar{C}_2^{7_2}} = \frac{\pi e^{\phi_4}}{(U_2+\bar{U}_2)}\sqrt{\frac{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}{\alpha'^3}}\left|m_2^{(1)}m_2^{(3)}\right| = \frac{1}{(S+\bar{S})(U_2+\bar{U}_2)}, \tag{4.4.21}$$

where we used that in the toroidal case the $\mathcal{T}_i$ moduli are expressed in terms of the tori areas through

$$\mathcal{T}_i = \frac{e^{-\phi}\mathcal{A}_j\mathcal{A}_k}{\alpha'^2} + ia_i\,, \quad i\neq j\neq k\neq i. \tag{4.4.22}$$

In the last equalities of eqs. (4.4.19) to (4.4.21) we also explicitly took $m_2^{(1)} = m_2^{(3)} = 1$. These Kähler metrics follow from a Kähler potential of the usual form

$$\begin{aligned}\kappa^2\mathcal{K} = &-\ln\left[(S+\bar{S})(U_2+\bar{U}_2) - |C_2^{7_2}|^2\right]\\ &-\ln\left[(\mathcal{T}_2+\bar{\mathcal{T}}_2)\prod_{i,j=1,3}\left(|\epsilon_{i2j}|(\mathcal{T}_i+\bar{\mathcal{T}}_i)(U_j+\bar{U}_j) - |\epsilon_{i2j}||C_j^{7_2}|^2+\cdots\right)\right].\end{aligned} \tag{4.4.23}$$

In the last line, there is an implicit summation on the j index, and ϵ_{i2j} is the standard fully antisymmetric symbol. In the above Kähler potential we did not include the quantum

corrections of equation (3.2.34).

From equations (4.4.19) to (4.4.21) we see that the physical Yukawa couplings (4.4.13) read

$$Y_{ijk} = \kappa^3 w_{ijk} \left(\frac{1}{(S+\bar{S})(\mathcal{T}_1+\bar{\mathcal{T}}_1)(\mathcal{T}_3+\bar{\mathcal{T}}_3)\prod_l (U_l+\bar{U}_l)} \right)^{-\frac{1}{2}} \left((S+\bar{S}) \prod_l (\mathcal{T}_l+\bar{\mathcal{T}}_l)(U_l+\bar{U}_l) \right)^{-\frac{1}{2}}$$
$$= \kappa^3 w_{ijk} \frac{1}{\sqrt{\mathcal{T}_2+\bar{\mathcal{T}}_2}} = \kappa^3 w_{ijk}\, g_s^{1/2} \sqrt{\frac{\mathcal{A}_2}{\alpha' \mathcal{V}}}\,. \tag{4.4.24}$$

We have made use of the definitions (4.4.18), (4.4.17) and (4.4.22) to express the various moduli in terms of the physical quantities. From (4.4.24) we can extract the internal volume dependence of the canonically normalized tachyonic fields superpotential (4.4.12)

$$\mathcal{W}_{tach} = g_s^{1/2} \kappa^3 \sqrt{\frac{\mathcal{A}_2}{\alpha' \mathcal{V}}} \varphi_2 \varphi_+ \varphi_-, \tag{4.4.25}$$

which generates a F-term scalar potential.

Mass term When φ_2 gets a non-vanishing VEV $\langle \varphi_2 \rangle \neq 0$, the F-term gives a mass to the tachyonic fields

$$V_F \ni \kappa^{-4} \sum_i \left| \frac{\partial \mathcal{W}_{tach}}{\kappa \partial \varphi_i} \right|^2 = \frac{g_s}{\left| m_2^{(1)} m_2^{(3)} \right|} |\langle \varphi_2 \rangle|^2 \frac{\mathcal{A}_2}{\alpha' \mathcal{V}} \left(|\varphi_+|^2 + |\varphi_-|^2 \right) \equiv m_{x_2}^2 \left(|\varphi_+|^2 + |\varphi_-|^2 \right). \tag{4.4.26}$$

In the above equation, we defined m_{x_2} as the physical mass coming from the brane position x_2.

From equation (4.4.21) we read the Φ_2 Kähler metric and deduce the expression for the canonically normalized field

$$\varphi_2 = \frac{\kappa^{-1}}{\sqrt{(U_2+\bar{U}_2)(S+\bar{S})}} \Phi_2 = \frac{\kappa^{-1} g_s^{1/2}}{\sqrt{U_2+\bar{U}_2}} \Phi_2. \tag{4.4.27}$$

We recall that Φ_2 is the dimensionless complexified scalar modulus related to the brane position on T_2^2, given by

$$\Phi_2 = x_2^x - i U_2 x_2^y. \tag{4.4.28}$$

Hence from equations (4.4.26) to (4.4.28) we deduce that

$$m_{x_2}^2 = |\langle \varphi_2 \rangle|^2 g_s \frac{\mathcal{A}_2}{\alpha' \mathcal{V}} = \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\mathcal{A}_2}{\alpha' |U_2+\bar{U}_2|} \left| x_2^x - i U_2 x_2^y \right|^2 \equiv y(U_2) \frac{g_s^2}{\kappa^2 \mathcal{V}} \frac{\mathcal{A}_2}{\alpha'}. \tag{4.4.29}$$

Replacing $\kappa^2 \mathcal{V}/g_s^2$ by α' through (4.4.5), we find back the string mass formula (3.3.35) derived in section 3.3.3, except for irrelevant powers of 2 which come from the fact that in the current part we derived the mass term without explicitly applying the orientifold and orbifold projections. In the following we use the last form of (4.4.29).

Quartic term In order to analyse the phase transition of the waterfall field, we need to keep track of the quartic terms in addition to the mass terms. For the D-term scalar potential the quartic contributions were already included in the expansion (4.4.3). The full F-term scalar potential can be computed through the supergravity formula using the total superpotential $\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_{C_i^{72}}$ containing the flux-dependent constant and the

$C_i^{7_2}$ dependent part of eqs. (4.4.11) and (4.4.12), together with the total Kähler potential including the $C_i^{7_2}$ dependence of (4.4.23) and the quantum corrections of eq. (3.2.34). From this F-term we can extract the quartic contribution of the waterfall field.

Nevertheless, the leading corrections in g_s are easily obtained by expanding the Kähler potential (4.4.23) with respect to the tachyonic field φ_- (or rather its non-canonically normalized "parent" $C_1^{7_2}$ or $C_3^{7_2}$), thus neglecting the $C_2^{7_2}$ dependence in the logarithm of the first line together with the one-loop quantum corrections. The leading quartic contribution for the tachyonic scalar field potential then simply reads

$$V_F \ni \frac{g_s^2}{(U_2+\bar{U}_2)}\frac{\mathcal{A}_2}{\alpha'\mathcal{V}}|\langle\Phi_2\rangle|^2|\varphi_-|^4 = y(U_2)\frac{g_s^2}{\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'}|\varphi_-|^4 = \kappa^2 m_{x_2}^2|\varphi_-|^4. \tag{4.4.30}$$

Thus, it turns out that the leading quartic contribution comes entirely from the expansion comes entirely from the expansion of the $e^{\kappa^2\mathcal{K}}$ factor in the supergravity formula. The dependence on the moduli of this term is identical to the one of the mass term as it comes from the $\mathcal{K}^{i\bar{j}}\mathcal{D}_i\mathcal{W}\mathcal{D}_{\bar{j}}\overline{\mathcal{W}}$ part of the F-term scalar potential, with the derivative taken with respect to the φ_+ field.

4.4.3 New vacuum

Summing the D-term and F-term contributions (4.4.3), (4.4.26) and (4.4.30) for the matter fields with the F-term scalar potential for the volume modulus, we obtain the effective scalar potential to minimize in order to obtain the physical vacuum. It reads

$$\begin{aligned} V(\mathcal{A}_i,\varphi_\pm) &= V_F(\mathcal{V}) + V_F(\mathcal{A}_i,\varphi_\pm) + V_D(\mathcal{A}_i,\varphi_\pm) + \cdots \\ &= V_F(\mathcal{V}) + m_{x_2}^2\left(|\varphi_+|^2+|\varphi_-|^2\right) + \kappa^2 m_{x_2}^2|\varphi_-|^4 + \cdots \\ &\quad + \sum_{b=1,3}\frac{g_{U(1)_b}^2}{2}\xi_b^2 + \frac{g_{U(1)_2}^2}{2}\left(\xi_2+2|\varphi_+|^2-2|\varphi_-|^2\right)^2 + \cdots \end{aligned} \tag{4.4.31}$$

Kähler moduli minimization We first minimize the scalar potential with respect to the internal areas moduli $\mathcal{A}_i$, letting free the total volume $\mathcal{V}$, neglecting for the moment the matter fields. This is similar to what was done in [3] and shown in sections 4.2.1 and 4.2.2, with nevertheless a slightly different expression for the D-term before the minimization. This is why we perform again the minimization in our precise model. Defining the ratios

$$u \equiv \frac{\mathcal{A}_3}{\mathcal{A}_2}, \qquad v \equiv \frac{\mathcal{A}_1}{\mathcal{A}_3}, \qquad \frac{1}{uv} = \frac{\mathcal{A}_2}{\mathcal{A}_1}, \tag{4.4.32}$$

the D-term part of the scalar potential (4.4.9) reads

$$V_D(\mathcal{A}_i) = V_D(\mathcal{V},u,v) = \frac{1}{\kappa^4\mathcal{V}^2}\left(d_1u + d_2v + \frac{d_3}{uv}\right), \tag{4.4.33}$$

where the d_i parameters are defined in equation (4.4.10). V_D is minimized by

$$u_0 = \left(\frac{d_2d_3}{d_1^2}\right)^{\frac{1}{3}}, \qquad v_0 = \left(\frac{d_1d_3}{d_2^2}\right)^{\frac{1}{3}}, \tag{4.4.34}$$

which gives the following tori moduli

$$\mathcal{A}_1 = \alpha' \left(\frac{d_3}{d_2}\right)^{\frac{1}{3}} \mathcal{V}^{\frac{1}{3}}, \quad \mathcal{A}_2 = \alpha' \left(\frac{d_1}{d_3}\right)^{\frac{1}{3}} \mathcal{V}^{\frac{1}{3}}, \quad \mathcal{A}_3 = \alpha' \left(\frac{d_2}{d_1}\right)^{\frac{1}{3}} \mathcal{V}^{\frac{1}{3}}, \tag{4.4.35}$$

while its expression at the minimum becomes:

$$V_D(\mathcal{V}) = V_D(\mathcal{V}, u_0, v_0) = \frac{3(d_1 d_2 d_3)^{\frac{1}{3}}}{\kappa^4 \mathcal{V}^2} \equiv \frac{d}{\kappa^4 \mathcal{V}^2}. \tag{4.4.36}$$

In the last equality we defined

$$d \equiv 3(d_1 d_2 d_3)^{\frac{1}{3}} = \frac{3}{2} g_s^3 \left|\frac{m_1^{(3)} m_2^{(1)} m_3^{(2)}}{m_1^{(2)} m_2^{(3)} m_3^{(1)}}\right|^{\frac{1}{3}} \left(\frac{n_1^{(2)} n_2^{(3)} n_3^{(1)}}{\pi^3}\right)^{\frac{2}{3}}, \tag{4.4.37}$$

giving back the D-term contribution shown in equation (4.2.15), but with a specific value of d related to the parameters of our model.

From (4.4.4), (4.4.6), (4.4.10) and (4.4.29), one finds that the masses and couplings for the matter fields take the form

$$\begin{aligned} m_{H_2}^2 &= 2\sqrt{2} \left|m_2^{(1)} m_2^{(3)}\right|^{-\frac{1}{2}} \frac{\sqrt{g_s}}{\kappa^2} \frac{\left(d_1^2 d_2\right)^{\frac{1}{6}}}{\mathcal{V}^{\frac{4}{3}}}, \qquad g_{U(1)_2}^2 = \frac{g_s}{\mathcal{V}^{\frac{2}{3}}} \left(\frac{d_1}{d_3}\right)^{\frac{1}{3}} \left|m_2^{(1)} m_2^{(3)}\right|^{-1}, \\ m_{x_2}^2 &= \frac{g_s^2}{\kappa^2} y(U_2) \left(\frac{d_1}{d_3}\right)^{\frac{1}{3}} \frac{1}{\mathcal{V}^{\frac{2}{3}}}. \end{aligned} \tag{4.4.38}$$

For the configuration of section 3.3.3, the condition $|\zeta_1^{(2)}| = |\zeta_2^{(3)}| = |\zeta_3^{(1)}|$ in (3.3.44) is necessary to eliminate the tachyons from different brane intersections. Together with the moduli stabilization condition (4.4.35), this gives the following relations that the fluxes must satisfy

$$n_2^{(3)} = \left|\frac{m_3^{(1)} m_3^{(2)}}{m_2^{(1)} m_1^{(2)}}\right| n_1^{(2)}, \qquad n_3^{(1)} = \left|\frac{m_3^{(1)} m_1^{(3)}}{m_2^{(1)} m_2^{(3)}}\right| n_1^{(2)}, \tag{4.4.39}$$

leading to the following expression for the D-term parameter introduced in eq. (4.4.37):

$$d = \frac{3}{2} g_s^3 \left|\frac{m_3^{(1)} m_3^{(2)} m_1^{(3)}}{m_2^{(1)} m_2^{(3)} m_1^{(2)}}\right| \left(\frac{n_1^{(2)}}{\pi}\right)^2 = \frac{3}{2} g_s^3 \left(\frac{k}{\pi}\right)^2, \quad \text{with} \quad k^2 \equiv (n_1^{(2)})^2 \left|\frac{m_3^{(1)} m_3^{(2)} m_1^{(3)}}{m_2^{(1)} m_2^{(3)} m_1^{(2)}}\right|. \tag{4.4.40}$$

We recall that the wrapping numbers $m_a^{(j)}$ are also subject, together with the brane multiplicities N_a (or N_a' introduced in eq. (3.3.29)), to tadpole cancellation conditions. For instance, in the absence of 3-form fluxes the tadpole cancellation conditions for the $m_a^{(j)}$ simply read $N_a' m_a^{(j)} m_a^{(k)} - 16 = 0$, $a \neq j \neq k \neq a$, for each $D7_a$ brane stack. We also recall that we have chosen $m_2^{(1)} = m_2^{(3)} = 1$, even if we kept generality in eqs. (4.4.37) to (4.4.40). Note that the model of section 3.3.3 with more general flux configurations leads to less constrained flux parameters $n_a^{(j)}$ than those of eq. (4.4.39).

Global minimum and waterfall direction After stabilization of the transverse moduli ratios (u and v), the left-over parameters of the total scalar potential (4.4.31) can be replaced using equations (4.4.38). In the simple case under consideration, with fluxes as in

(4.4.39), the mass and coupling of eq. (4.4.38) read

$$m_{H_2}^2 = 2\frac{g_s^2}{\kappa^2\mathcal{V}^{\frac{4}{3}}}\frac{k}{\pi}\left|\frac{1}{\prod_{a\neq j} m_a^{(j)}}\right|^{\frac{1}{6}}, \qquad g_{U(1)_2}^2 = \frac{g_s}{\mathcal{V}^{\frac{2}{3}}}\left|\frac{1}{\prod_{a\neq j} m_a^{(j)}}\right|^{\frac{1}{3}},$$
$$m_{x_2}^2 = \frac{g_s^2}{\kappa^2\mathcal{V}^{\frac{2}{3}}}\, y(U_2)\left|\frac{m_2^{(1)2} m_2^{(3)2}}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{\frac{1}{3}}. \tag{4.4.41}$$

Neglecting the massive φ_+ field and expressing the volume modulus dependent contribution $V_F(\mathcal{V}) + V_D(\mathcal{V})$ through (4.2.15), the scalar potential (4.4.31) is written as

$$V(\mathcal{V}, \varphi_-) = \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) + \frac{1}{2}m_Y^2(\mathcal{V})|\varphi_-|^2 + \frac{\lambda(\mathcal{V})}{4}|\varphi_-|^4, \tag{4.4.42}$$

taking the same form as eq. (4.3.1), with φ_- playing the role of the waterfall field Y. Its mass and coupling read

$$\frac{1}{2}m_Y^2(\mathcal{V}) = (m_{x_2}^2 - m_{H_2}^2) = \frac{g_s^2\, y(U_2)}{\kappa^2\mathcal{V}^{\frac{2}{3}}}\left|\frac{1}{m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}}\right|^{\frac{1}{3}}\left(1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{\frac{2}{3}}\right), \tag{4.4.43}$$

$$\lambda(\mathcal{V}) = 4\left(2g_{U(1)_2}^2 + \kappa^2 m_{x_2}^2\right) = \frac{4g_s}{\mathcal{V}^{\frac{2}{3}}}\left|\frac{1}{\prod_{a\neq j} m_a^{(j)}}\right|^{\frac{1}{3}}\left(2 + g_s\, y(U_2)\right). \tag{4.4.44}$$

In eq. (4.4.43) we explicitly took $m_2^{(1)} = m_2^{(3)} = 1$ and defined the critical volume $\mathcal{V}_{c2}$ at which φ_- becomes tachyonic, *i.e.* for $\mathcal{V} < \mathcal{V}_{c2}$,

$$\mathcal{V}_{c2} \equiv \left(\frac{2k}{\pi y(U_2)}\right)^{\frac{3}{2}}\left|m_1^{(2)} m_1^{(3)} m_3^{(1)} m_3^{(2)}\right|^{\frac{1}{4}}. \tag{4.4.45}$$

As expected, $\mathcal{V}_{c2}$ depends on the fluxes through k defined in eq. (4.4.40) and on the $D7_2$ brane position through $y(U_2)$ defined through (4.4.29). We also remark from eq. (4.4.44) that the main contribution to the quartic coupling λ comes from the D-term part of the potential, since the F-term contribution is suppressed by a power of g_s.

When the mass term m_Y^2 of eq. (4.4.43) becomes negative, the waterfall field Y (our tachyonic field φ_-) rolls down its potential to the new vacuum at $\langle\varphi_-\rangle = \langle Y\rangle = \pm v_2$. From eqs. (4.3.5) and (4.4.42) we see that the value of the potential at this new vacuum is

$$V(\mathcal{V}, v_2) = V_F(\mathcal{V}) + V_D(\mathcal{V}) - \frac{m_Y^4}{4\lambda}(\mathcal{V})$$
$$= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) - \frac{C_2}{\kappa^4\mathcal{V}^{\frac{2}{3}}}\left(1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2. \tag{4.4.46}$$

We recall here the expressions of various parameters introduced before

$$q = \frac{\xi}{2\gamma}, \quad \sigma = \frac{2d}{9{\mathcal{W}_0}^2\gamma}, \quad C = -3{\mathcal{W}_0}^2\gamma > 0, \quad y(U_2) = \frac{\left|x_2^x - iU_2 x_2^y\right|^2}{|U_2 + \bar{U}_2|}, \quad d = \frac{3}{2}g_s^3\left(\frac{k}{\pi}\right)^2, \tag{4.4.47}$$

and define C_2, the coefficient of the tachyonic contribution to the vacuum energy through

$$C_2 \equiv -\frac{g_s^3 y^2(U_2)}{4\left(2+g_s\, y(U_2)\right)}\left|\frac{1}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right|^{\frac{1}{3}}. \tag{4.4.48}$$

We observe that C_2 and $\mathcal{V}_{c2}$ are not independent, their are related by

$$C_2 = \beta_2 \frac{d}{3\mathcal{V}_{c2}^{\frac{4}{3}}}, \qquad \beta_2 \equiv \frac{2}{2+g_s y(U_2)} \in [0,1], \tag{4.4.49}$$

where the parameter β_2 expresses the relative contributions to the quartic coupling from the F-term versus the D-term. From (4.4.29) we see that as $y(U_2) > 0$, β_2 lies between 0 and 1. For $\beta_2 = 1$ the D-term dominates whereas for $\beta_2 = 0$ the F-term dominates. It is clear from (4.4.49) that the tachyonic contribution becomes maximal (in absolute value) for $\beta_2 = 1$, when the quartic coupling is dominated by the D-term one.

Let us discuss now the physics of the waterfall direction. As explained in section 4.3, the waterfall field can generate the desired scenario for the end of inflation. It has to become tachyonic when the volume modulus $\mathcal{V}$ (identified to the inflaton ϕ through eq. (4.2.8)) reaches the bottom of its potential $V_F(\mathcal{V}) + V_D(\mathcal{V})$. This situation corresponds to $\mathcal{V}_{c2} \approx \mathcal{V}_-$, where $\mathcal{V}_-$ is the value of the volume at its minimum, expressed by (4.2.41). From eq. (4.4.45) we see that the value $\mathcal{V}_{c2}$ depends on $y(U_2)$, k and a ratio of flux and wrapping numbers, hence it is easy to choose $\mathcal{V}_{c2}$ near $\mathcal{V}_-$.

We see from the scalar potential expression (4.4.46) that once C and d are determined by the inflationary phase, and $\mathcal{V}_{c2}$ fixed to $\mathcal{V}_-$, the coefficient C_2 is the only parameter to tune the minimum. From the relation of eq. (4.4.49) we see that in fact, only β_2 can be used for fixed C and d. As β_2 depends only on the product $g_s y(U_2)$, we express $\mathcal{V}_{c2}$ in terms of d, g_s and $g_s y(U_2)$ using eqs. (4.4.40) and (4.4.45). It reads:

$$\mathcal{V}_{c2} = \frac{4\sqrt{2}}{3}\frac{\sqrt{\pi}}{(y(U_2)g_s)^{\frac{3}{2}}} g_s^{-\frac{3}{2}} d \left|\frac{m_1^{(2)}}{n_1^{(2)}}\right|^{\frac{1}{2}} \tag{4.4.50}$$

Hence, in principle we can first fix the product $g_s y(U_2)$ to have the desired β_2 and tune the minimum, then choose the values of g_s and of the ratio of the flux/wrapping numbers on the second torus to tune the critical volume.

Example of numerical values We give now an explicit example of parameters supporting the above discussion. As explained in section 4.2.4, in the inflationary scenario discussed above the values of x and C are fixed by observational constraints to

$$x \approx 3.3 \times 10^{-4}, \qquad C = e^{-3q} \times 7.81 \times 10^{-4} \equiv e^{-3q} C_0. \tag{4.4.51}$$

From eqs. (4.2.16), (4.2.22) and (4.2.41) we extract for $q = 0$ the values of $\mathcal{V}_-$, the minimum of the modulus part of the potential, and the d magnetic flux parameter

$$\mathcal{V}_- \approx 201.9, \qquad d \approx 5.65 \times 10^{-6}. \tag{4.4.52}$$

We compute numerically the global minimum of the potential given in (4.4.46) and see that in order to have an almost vanishing value at the minimum, we need to tune the tachyonic coefficient to $C_2 \approx 5.136 \times 10^{-9}$, which through equation (4.4.49) would impose

the value

$$\beta^{\Lambda=0} = 5.136 \times 10^{-9} \frac{3\mathcal{V}_-^{\frac{4}{3}}}{d} \approx 3.228. \tag{4.4.53}$$

Nevertheless, as $\beta_2 \in [0,1]$, we see that we cannot tune the vacuum energy to zero in the simple model of the current section. We come back to this point in detail in the next section. From eq. (4.4.49) we see that the largest value $\beta_2 \approx 1$ is obtained for small $g_s y(U_2)$ and taking for instance $g_s y(U_2) \approx 10^{-2}$ in equation (4.4.50) we obtain

$$\mathcal{V}_{c2} \approx \mathcal{V}_- \approx 201.9 = 1.89 \times 10^{-2} g_s^{-\frac{3}{2}} \left| \frac{m_1^{(2)}}{n_1^{(2)}} \right|^{\frac{1}{2}}, \tag{4.4.54}$$

which has to be satisfied together with the relations on d and β_2 given by eqs. (4.4.40) and (4.4.49)

$$\beta_2 = \frac{2}{2 + g_s y(U_2)} \approx 1, \qquad d = \frac{3}{2} g_s^3 \left| \frac{m_3^{(1)} m_3^{(2)} m_1^{(3)}}{m_1^{(2)}} \right| \left(\frac{n_1^{(2)}}{\pi} \right)^2 \approx 5.65 \times 10^{-6}. \tag{4.4.55}$$

We recall that we consider the case $m_2^{(1)} = m_2^{(3)} = 1$. The following parameters

$$\begin{aligned} &g_s = 2.596 \times 10^{-3}, \qquad n_1^{(2)} = 1, \qquad m_1^{(2)} = 2, \\ &m_1^{(3)} = 10, \quad m_3^{(1)} = 17, \qquad m_3^{(2)} = 25, \qquad y(U_2) = 3.85, \end{aligned} \tag{4.4.56}$$

give the desired values for d, $\mathcal{V}_{c2}$ and $\beta_2 \approx 1$. Of course, there is an infinite set of other choices of parameters giving the same values. We show in the left panel of Figure 4.9 the value of the potential at the global minimum (including the waterfall), located at $\mathcal{V}_0 \approx 160$. However, as explained above, this value is not vanishing. The next section tackles this point in details.

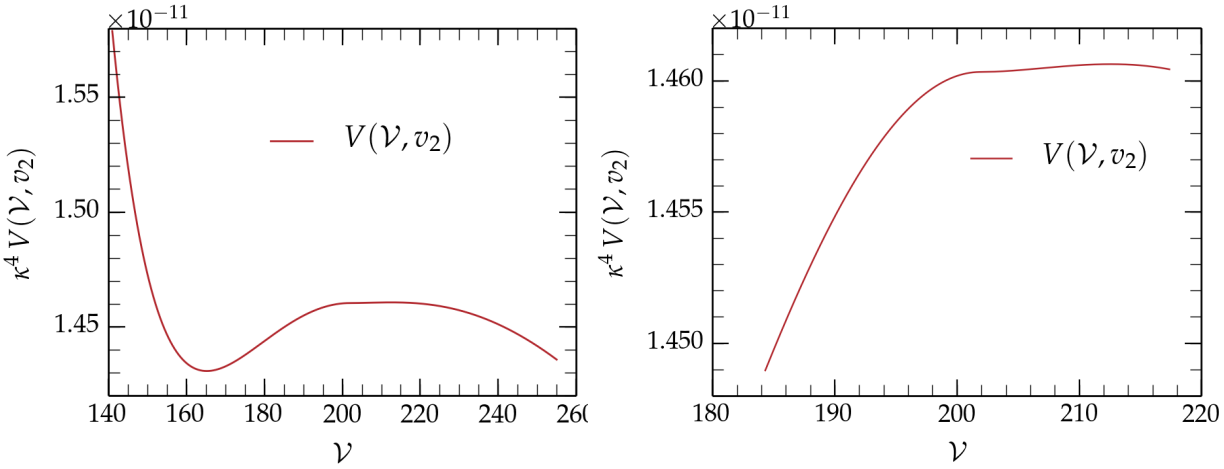


FIGURE 4.9: Value $V(\mathcal{V}, v_2)$ of the global minimum of the scalar potential as a function of the internal volume, for the parameters of (4.4.56).

As one can see from the right panel of Figure 4.9, the tachyonic field gives indeed a "warterfall" direction. Falling in this direction leads to an increase of the slow-roll parameters, which marks the end of the inflationary phase. A precise computation of the slow-roll parameters along the inflaton trajectory is necessary to extract the extra number of e-folds until the end of inflation and compare it to the case without waterfall field [3]. This number depends on $\mathcal{V}_- - \mathcal{V}_{c2}$ and is model dependent.

Validity of our approximations Before the end of this section, we stress the fact that with the parameters (4.4.56), the volume modulus is large and the flux numbers $k_a^{(i)}$ small, so that the large volume approximations of *e.g.* eqs. (4.4.4) and (4.4.7) hold. We also want to check that the vacuum expectation value v_2 of the waterfall field stays small (in κ units), so that the quartic expansion of (4.4.3) holds. From eqs. (4.3.4) and (4.4.43) this VEV is expressed as

$$\langle \varphi_- \rangle = \langle Y \rangle = \pm v_2 = \pm \frac{|m_Y|}{\sqrt{\lambda}} = \pm \frac{1}{\sqrt{2}\kappa} \sqrt{\frac{g_s y(U_2)}{2 + g_s y(U_2)}} \left| 1 - \left(\frac{\mathcal{V}_{c2}}{\mathcal{V}} \right)^{\frac{2}{3}} \right|^{\frac{1}{2}} . \tag{4.4.57}$$

It follows that $\langle \varphi_- \rangle$ is entirely determined by $g_s y(U_2)$ and $\mathcal{V}_{c2}$. With the parameters of (4.4.56) and the volume modulus $\mathcal{V}_{c2} \geqslant \mathcal{V}_0 \gtrsim 160$, which is the range of Figure 4.9, one finds a VEV v_2 satisfying $0 \leq \kappa |v_2| \lesssim 0.4\sqrt{g} \approx 0.02$. The quartic expansion (4.4.3) is thus indeed sufficient.

4.5 Lowering the global minimum

We have seen in section 4.4.3 that in our type IIB framework with three orthogonal $D7$ branes, a waterfall field can be implemented through a doubly charged state stretching between *e.g.* the $D7_2$ brane and its orientifold image. The mass of such a state depends on the internal volume (our inflaton) and we showed that under a certain critical volume this state becomes tachyonic, generating a waterfall direction typical of hybrid inflation models described in section 4.3.

The first motivation for the introduction of such a waterfall direction was that it is responsible for the end of inflation. The second motivation was that the waterfall field, through its negative contribution to the scalar potential, lowers the value of the global minimum and can in principle tune the cosmological constant to the almost vanishing value observed today. Nevertheless, as we explained near the end of section 4.4.3, due to the relation (4.4.49) we are not able to choose independently the position $\mathcal{V}_{c2}$ and the depth of the waterfall related to C_2. The remaining freedom in the choice of the waterfall depth lies in the β_2 coefficient, whose value $\beta^{\Lambda=0} \approx 3.228$, needed to tune the vacuum energy to zero, cannot be reached in our example where $\beta_2 \leq 1$. In this section we investigate how to modify the model in order to bypass the constraint imposed by eq. (4.4.49) and lower the global minimum.

We first verify in section 4.5.1 that in the field theoretical description, where we can choose freely the mass and coupling parameters while keeping their volume dependence, the tuning of the global minimum is indeed possible. In section 4.5.2 we come back to the simple case studied in the previous section. We show that the natural tentative to tune the vacuum through the use of the parameter q, not constrained by the inflationary phase, does not work. We also study if the contributions of the γ, ξ quantum corrections to the tachyonic mass and coupling, gives extra freedom and helps to evade relation (4.4.49). We show that, as these quantum corrections stay small, they do not play an important role. We hence examine in section 4.5.3 if adding more tachyons, coming from the two other $D7$-brane stacks, allows to tune the vacuum energy to zero. We find that even if these additional tachyons lower indeed the global minimum, their contribution still determined by d, constrained by the inflationary phase, is not sufficient to tune the vacuum energy to zero. Nevertheless in section 4.5.4, we show that adding a forth magnetised stack, parallel to an already present one, adds additional tachyonic contributions to the scalar potential, allowing to tune the vacuum energy.

4.5.1 Field theoretical description

We first look at the possibility to tune the vacuum energy of the model with arbitrary parameters, *i.e.* in the field theoretical description. We thus take arbitrary values for the mass and quartic parameters of the F-terms and D-terms, but keep the volume dependences as in the string theory setup of the previous sections. The scalar potential is written as

$$\begin{aligned} V(\mathcal{V},\varphi_-) &= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{3\sigma}{2\mathcal{V}^2}\right)+\frac{1}{2}\left(m_F^2(\mathcal{V})-m_D^2(\mathcal{V})\right)\varphi_-^2 \\ &\quad+\frac{1}{4}\left(\lambda_F(\mathcal{V})+\lambda_D(\mathcal{V})\right)\varphi_-^4 \\ &= \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{3\sigma}{2\mathcal{V}^2}\right)+\frac{1}{2}\left(\frac{\mu_F^0}{\mathcal{V}^{\frac{2}{3}}}-\frac{\mu_D^0}{\mathcal{V}^{\frac{4}{3}}}\right)\varphi_-^2+\frac{1}{4}\frac{(\lambda_F^0+\lambda_D^0)}{\mathcal{V}^{\frac{2}{3}}}\varphi_-^4\,. \end{aligned} \tag{4.5.1}$$

As described in the previous sections on a particular example, for $\mathcal{V}<\mathcal{V}_{c2}$ the matter field becomes tachyonic and gets a non-vanishing VEV $v\neq 0$. The scalar potential gets a contribution $-m^4/4\lambda(\mathcal{V})$ when φ_- sits at its VEV and the dependence of the global minimum in the volume then reads

$$V(\mathcal{V},v)=\frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V}-4+q}{\mathcal{V}^3}-\frac{3\sigma}{2\mathcal{V}^2}\right)-\frac{{\mu_F^0}^2}{4(\lambda_F^0+\lambda_D^0)}\frac{1}{\mathcal{V}^{\frac{2}{3}}}\left(1-\left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2, \tag{4.5.2}$$

with $\mathcal{V}_{c2}^{2/3}=\mu_D^0/\mu_F^0$. The D-term parameter μ_D^0 is related to the flux parameter d and thus to x, relevant during the inflationary phase. We see from eq. (4.5.2) that in the field theoretical description, one can tune μ_D^0, $\mathcal{V}_c=(\mu_D^0/\mu_F^0)^{3/2}$ and the coefficient $C_2={\mu_F^0}^2/4(\lambda_F^0+\lambda_D^0)$ independently. This was not the case in the simple configuration described in section 4.4.3 due to relation (4.4.49) between C_2, d and $\mathcal{V}_{c2}$, which translates the fact that in our string theory setup, the $\mu_D^0,\mu_F^0,\lambda_D^0$ and λ_F^0 parameters cannot be chosen independently.

In the next subsections we will investigate if more complex configurations can allow the tuning of the scalar potential at the global minimum within our string theory setup.

4.5.2 Simple case studied previously

Dependence on the q parameter We now come back on the discussion on the tuning of the global minimum in the configuration discussed since the beginning of section 4.4, *i.e.* with the flux configuration (4.4.2). The value of the scalar potential at the global minimum was expressed in (4.4.46). We first examine if the use of the parameter q could liberate the constraint on the waterfall depth C_2 (which is related to $\mathcal{V}_{c2}\approx\mathcal{V}_-$), by shifting $\mathcal{V}_-$ arbitrarily. From eqs. (4.2.16), (4.2.22) and (4.2.41) we express the following parameters dependence in q and x:

$$\mathcal{V}_-=e^{-q}\exp\left(\frac{13}{3}-W_0\left(-e^{-x-1}\right)\right),\quad \sigma=-e^{q-\frac{16}{3}-x},\quad d=-\frac{3}{2}C\sigma,\quad C\equiv-3{W_0}^2\gamma. \tag{4.5.3}$$

We recall again that the parameters x and C determine the inflationary phase [3] and are fixed from the observations to the values of eq. (4.4.51). From the definition (4.5.3) and constraints (4.4.51), it seems that the q parameter could indeed help to tune the vacuum energy, by shifting the value of $\mathcal{V}_-$, and thus the tachyonic contribution's coefficient C_2

defined in eq. (4.4.49). Applying the constraint on the waterfall position $\mathcal{V}_{c2} \approx \mathcal{V}_-$, the C_2 dependence on q reads

$$C_2 = \beta_2 \frac{d}{3\mathcal{V}_c^{\frac{4}{3}}} \approx \beta_2 \frac{d}{3\mathcal{V}_-^{\frac{4}{3}}} = \frac{-\beta_2 C\sigma}{2\mathcal{V}_-^{\frac{4}{3}}} = \beta_2 \frac{C_0}{2} e^{-\frac{2}{3}q} \times \frac{e^{-\frac{16}{3}-x}}{\left(\exp\left(\frac{13}{3} - W_0\left(-e^{-x-1}\right)\right)\right)^{\frac{4}{3}}}. \tag{4.5.4}$$

Replacing C_2 through (4.5.4), we see the scalar potential of (4.4.46) is nevertheless scale invariant with respect to q. It can indeed be expressed as

$$V(\mathbb{V}, v) = \frac{C_0}{\kappa^4} \frac{4 - \ln \mathbb{V}}{\mathbb{V}^3} + \frac{d_0}{\kappa^4 \mathbb{V}^2} \left(1 - \frac{\beta_2}{3}\left(1 - \left(\frac{\mathbb{V}}{\mathbb{V}_-}\right)^{\frac{2}{3}}\right)^2\right), \tag{4.5.5}$$

in terms of the q-dependent variables

$$\mathbb{V} = e^q \mathcal{V}, \quad \mathbb{V}_- = e^q \mathcal{V}_- = \exp\left(\frac{13}{3} - W_0\left(-e^{-x-1}\right)\right), \quad d_0 \equiv \frac{3}{2} C_0 e^{-\frac{16}{3}-x} = e^{2q} d\,, \tag{4.5.6}$$

which absorb the explicit q-dependence of V. Hence, $V(\mathbb{V}, v)$ only depends on x, C_0, both fixed by the inflationary phase, and β_2. As mentioned already, it is clear that the greater β_2 is, the lower the global minimum is. Hence the value $\beta_2 = 1$ gives the lowest minimum, which is then totally fixed by x and C_0.

We conclude that in the simple case studied in the previous section, the value of the vacuum at the global minimum is totally fixed by the constraints on the inflationary phase and the waterfall scenario implementation, and that neither the q nor β_2 parameters can help to lower it.

Influence of γ corrections to the squared mass and quartic term In the previous sections we neglected the contributions to the F-term squared mass and quartic terms for φ_- coming from γ and ξ factors. We now examine if these corrections could add supplementary freedom allowing us to choose independently the mass and quartic coupling of the tachyonic field. As explained in the field theory description of section 4.5.1, in this way one could tune the vacuum energy.

The aforementioned corrections can be read from the F-term supergravity formula through the expansion in the φ_- (or $C_2^{7_2}$) variable of the Kähler potential. The first corrections (in the g_s and γ expansion) to the mass and quartic contributions read

$$\frac{1}{2} m_\gamma^2 = \kappa^{-2} \frac{6 g_s \mathcal{W}_0^2 \gamma}{(U_1 + \bar{U}_1)^2 (U_2 + \bar{U}_2)(U_3 + \bar{U}_3)^2} \frac{\ln \mathcal{V} - 2 + q}{\mathcal{V}^3} = -2\kappa^{-2} C_0 \frac{g_s}{f(U_i)} \frac{\ln \mathbb{V} - 2}{\mathbb{V}^3}, \tag{4.5.7}$$

$$\frac{1}{4} \lambda_\gamma = \frac{2 g_s \mathcal{W}_0^2 \gamma}{(U_1 + \bar{U}_1)^2 (U_2 + \bar{U}_2)(U_3 + \bar{U}_3)^2} \frac{5 \ln \mathcal{V} - 9 + 5q}{\mathcal{V}^3} = -\frac{2}{3} C_0 \frac{g_s}{f(U_i)} \frac{5 \ln \mathbb{V} - 9}{\mathbb{V}^3}, \tag{4.5.8}$$

with $f(U_i) = (U_1 + \bar{U}_1)^2 (U_2 + \bar{U}_2)(U_3 + \bar{U}_3)^2$ and $\mathbb{V}$ introduced in eq. (4.5.6). Recall that $-C > 0$, so that these parameters are indeed positive. The mass and quartic terms of

eqs. (4.4.43) and (4.4.44) associated with these additional contributions now read:

$$\begin{aligned}\frac{1}{2}{m'_Y}^2(\mathcal{V}) &= (m_{x_2}^2 - m_{H_2}^2 + \frac{1}{2}m_\gamma^2)\\ &= \frac{g_s^2\, x_2(U_2)}{\kappa^2\mathcal{V}^{\frac{2}{3}}}\left(\frac{1}{m_1^{(2)}m_1^{(3)}m_3^{(1)}m_3^{(2)}}\right)^{\frac{1}{3}}\left(1-\left(\frac{\mathcal{V}_{c2}}{\mathcal{V}}\right)^{\frac{2}{3}}\right) - \frac{2Cg_s}{f(U_i)}\frac{\ln\mathcal{V}-2+q}{\kappa^2\mathcal{V}^3}, \qquad (4.5.9)\\ \frac{1}{4}\lambda'(\mathcal{V}) &= \left(2g_{U(1)_2}^2 + \kappa^2 m_{x_2}^2 + \frac{1}{4}\lambda_\gamma^2\right)\\ &= \frac{g_s}{\mathcal{V}^{\frac{2}{3}}}\left(\frac{1}{\prod_{a\neq j} m_a^{(j)}}\right)^{\frac{1}{3}}\left(2+g_s\, y(U_2)\right) - \frac{2}{3}C\frac{g_s}{f(U_i)}\frac{5\ln\mathcal{V}-9+5q}{\mathcal{V}^3}. \qquad (4.5.10)\end{aligned}$$

The new critical volume cannot be computed analytically now. Nevertheless the γ correction is suppressed by a factor $\mathcal{V}^{7/3}$ and stays small for the values considered previously at large volume. Indeed, from eqs. (4.5.9) and (4.5.10) we see that the coefficients in front of the previous contributions and γ corrections scale as $g_s^2 y(U_2)$ against $g_s C$ for the mass, and g_s against Cg_s for the quartic coupling λ, so that it is not possible to balance the huge volume suppression $\mathcal{V}^{7/3}$ of the γ corrections.

We also remark that the corrections of eqs. (4.5.7) and (4.5.8) are effectively independent of the q parameter since they only depend on $\mathbb{V}$, as the other contributions.

4.5.3 Additional tachyons from other D7-brane stacks

We now study the possibility of having several tachyons similar to the one described previously. We start with the addition of a second tachyon, generating a second waterfall direction. As the position of the second waterfall is only constrained to be at volumes $\mathcal{V} < \mathcal{V}_{c2} \approx \mathcal{V}_-$, we expect to have more freedom on the height of this second waterfall scalar potential contribution. We consider the following configuration:

	(45)	(67)	(89)
$D7_1$	$\cdot$	$\otimes$	$\times_{A_1}$
$D7_2$	$\times$	$\cdot_{\pm x_2}$	$\otimes$
$D7_3$	$\otimes$	$\times$	$\cdot_{\pm x_3}$

(4.5.11)

The $D7_3$ brane tachyon is not eliminated by a Wilson line anymore. We introduce a position x_3 for the brane on the third torus T_3^2, eliminating the tachyon at large volumes, exactly as the one from the $D7_2$ brane. The mass of the string state is indeed of the form

$$\alpha' m_{33} = -\frac{2|k_3^{(1)}|\alpha'}{\pi\mathcal{A}_1} + \frac{z(U_3)A_3}{\alpha'}, \qquad (4.5.12)$$

where the function $z(U_3)$ plays a role similar to $y(U_2)$ in the previous sections and is directly related to the brane position x_3. As for the tachyon studied previously, we describe the new effective theory of the second tachyon ψ_- through its masses m_{x_3} and m_{H_3}, generated respectively an F-term and a D-term, and the corresponding quartic couplings. Their expressions are similar to those of eqs. (4.4.43) and (4.4.44) for the $D7_2$–$D7_2$ tachyon φ_-, replacing the fluxes and tori areas by the respective ones for the $D7_3$–$D7_3$ state. The corresponding parameters for this $D7_3$–$D7_3$ state are denoted with a 3 subscript. For instance, $\mathcal{V}_{c3}$ is the critical volume of this second tachyon, corresponding to the position of the second waterfall.

For $\mathcal{V}_{c3} < \mathcal{V}_{c2}$, the study of the first phase transition does not change with respect to the single tachyon configuration. Indeed for $\mathcal{V}_{c3} < \mathcal{V} \leq \mathcal{V}_{c2}$, the second tachyon sits at its vanishing VEV $\langle \psi_- \rangle = 0$ and does not contribute to the potential. Then, when $\mathcal{V} \leq \mathcal{V}_{c3}$ the second tachyonic field gets a non-vanishing VEV $v_3 \neq 0$ and its contribution to the scalar potential reads

$$V(\mathcal{V}, v_3) = -\frac{m_Z^4}{\lambda_Z}(\mathcal{V}) = -\frac{C_3}{\kappa^4 \mathcal{V}^{\frac{2}{3}}} \left(1 - \left(\frac{\mathcal{V}_{c3}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2, \tag{4.5.13}$$

with

$$\mathcal{V}_{c3} \equiv \left(\frac{2k}{\pi z(U_3)}\right)^{\frac{3}{2}} \left|\frac{m_1^{(2)} m_1^{(3)} m_2^{(1)} m_2^{(3)}}{m_3^{(1)^5} m_3^{(2)^5}}\right|^{\frac{1}{4}}, \quad C_3 = \beta_3 \frac{d}{3\mathcal{V}_{c3}^{\frac{4}{3}}}, \quad \beta_3 = \frac{2}{2 + g_s z(U_3) |m_3^{(1)} m_3^{(2)}|} \in [0,1]. \tag{4.5.14}$$

For $\mathcal{V} < \mathcal{V}_{c3} \leq \mathcal{V}_{c2}$, the dependence of the global minimum of the scalar potential hence reads

$$V(\mathcal{V}, v_2, v_3) = \frac{C}{\kappa^4} \left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) - \sum_{a=2,3} \frac{C_a}{\kappa^4 \mathcal{V}^{\frac{2}{3}}} \left(1 - \left(\frac{\mathcal{V}_{ca}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2. \tag{4.5.15}$$

A short comment is in order on the way the global minimum is determined when several tachyons appear. The mass and coupling of φ_- expressed in eqs. (4.4.43) and (4.4.44) and the similar ones for ψ_- are the "bare" ones. As the "first" tachyon gets a VEV before the second one, contributions should appear due to interaction terms. These interaction terms come from the supergravity formula for the scalar potential through the expansion of the Kähler potential. Namely, corrections to the F-term mass and quartic coupling of ψ_- due to the VEV of the φ_- field are of the form:

$$m^2_{x_3, corrections} \sim \kappa^2 (m^2_{x_3} + m^2_{x_2}) \langle \varphi_- \rangle^2 + \kappa^4 m^2_{x_3} \langle \varphi_- \rangle^4 + \cdots \tag{4.5.16}$$

$$\lambda_{corrections} \sim \kappa^2 m^2_{x_3, corrections} = \kappa^4 (m^2_{x_3} + m^2_{x_2}) \langle \varphi_- \rangle^2 + \kappa^4 m^2_{x_3} \langle \varphi_- \rangle^4 + \cdots \tag{4.5.17}$$

As long as $\langle \varphi_- \rangle$ stays small (compared to κ) these corrections are negligible in front of the "bare" parameters and only shift the values of C_3 or $\mathcal{V}_{c3}$ by a small amount. Conversely, once ψ_- gets a non-vanishing VEV, corrections to the first tachyon parameters also appear but are negligible and only shift lightly the values of C_2 or $\mathcal{V}_{c2}$.

We now turn back to the study of the global minimum. We see through (4.5.14) that the amplitude C_3 of the tachyonic contribution and its critical volume $\mathcal{V}_{c3}$ are directly related. To get a large tachyonic contribution, we need to increase C_3, implying a smaller critical volume $\mathcal{V}_{c3}$. Nevertheless, at small volumes the moduli part (the first contribution) of the scalar potential (4.5.15) dominates because it increases as $1/\mathcal{V}^3$, against $1/\mathcal{V}^2$ for the tachyonic contributions. Hence if $\mathcal{V}_{c3}$ is small, the tachyonic contribution only appears at small volumes and cannot compensate the moduli part. In fact, it turns out that the largest contribution to the scalar potential from the second tachyon is for $\mathcal{V}_{c3} \approx \mathcal{V}_{c2}$ and hence $C_3 \approx C_2$. We see from Figure 4.10 that the second tachyon (*green curves*) contribution indeed lowers the value of the global minimum but is not sufficient to tune the vacuum energy to zero.

We are thus naturally led to consider adding a third tachyon on the last brane $D7_1$. The treatment is identical to the one for the first two and its contribution is described by a critical volume $\mathcal{V}_{c1}$ related to the corresponding coefficient C_1. When $\mathcal{V} < \mathcal{V}_{c1} \leq \mathcal{V}_{c3} \leq \mathcal{V}_{c2}$, all three tachyons sit at their respective non-vanishing VEV. The value of the global

minimum of the scalar potential is then as in eq. (4.5.15) but with a sum over the three tachyons:

$$V(\mathcal{V}, v_1, v_2, v_3) = \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) - \sum_{a=1,2,3} \frac{C_a}{\kappa^4 \mathcal{V}^{\frac{2}{3}}}\left(1 - \left(\frac{\mathcal{V}_{ca}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2 . \quad (4.5.18)$$

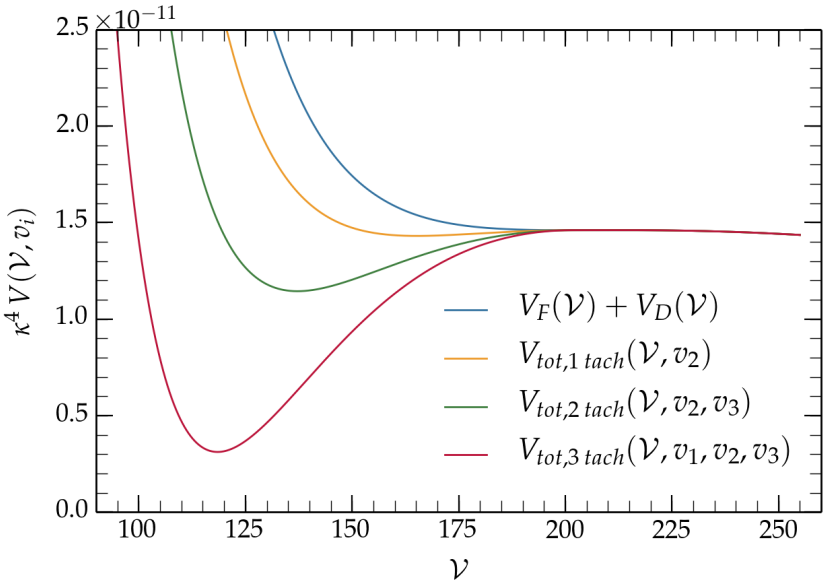


FIGURE 4.10: Value of the global minimum of the effective scalar potential as a function of $\mathcal{V}$, without (*blue*), with one (*orange*), two (*green*) or three (*red*) tachyons. The parameters are such that $\mathcal{V}_{c2} = \mathcal{V}_-$, $\mathcal{V}_{c3} = 0.99\mathcal{V}_-$ and $\mathcal{V}_{c1} = 0.98\mathcal{V}_-$.

From Figure 4.10 we see that the third tachyon is not sufficient yet to lower the global minimum to zero. In fact, this is understandable by the fact that when $\mathcal{V}_{c1} \approx \mathcal{V}_{c2} \approx \mathcal{V}_{c3}$, the value of the global minimum expressed as in (4.5.15) (but with the sum on $a = 1, 2, 3$ tachyons) is almost similar to the one with only one tachyon, but with an effective tachyonic contribution coefficient equal to $C_1 + C_2 + C_3$ instead of C_2. As the C_i are also related to the $\mathcal{V}_{ci}$ the only parameter to tune is $\beta_1 + \beta_2 + \beta_3 \leq 3$, which is always smaller than the desired value $\beta_2^{\Lambda=0} \approx 3.228$ introduced in (4.4.53). One can also wonder if having magnetic fluxes on the entire worldvolumes would allow to relax the relation between the $\mathcal{V}_{ci}$ and the C_i in order to go above this bound, but we show in Appendix C that a configuration as in section 3.3.3 does not help.

It is now clear, as can be understood from the above discussion, that the addition of a fourth tachyon would allow for an effective $\beta = \sum_{i\ tachyons} \beta_i$ that could be higher that the value $\beta^{\Lambda=0} \approx 3.228$, allowing to tune the vacuum energy to zero. In the next subsection we implement this idea in an example with a fourth $D7$-brane stack, parallel to one of the stacks previously studied.

4.5.4 Adding a fourth magnetized stack

As explained in the previous subsection, a fourth tachyon seems necessary to tune the vacuum energy of the minimum. One way to achieve this is by adding a fourth $D7$ brane stack, parallel to one of the one already present, say $D7_2$. We thus consider the following

configuration

	(45)	(67)	(89)
$D7_1$	$\cdot_{\pm x_1}$	$\otimes$	$\times$
$D7_{2a}$	$\times$	$\cdot_{\pm x_{2a}}$	$\otimes$
$D7_{2b}$	$\times$	$\cdot_{\pm x_{2b}}$	$\otimes$
$D7_3$	$\otimes$	$\times$	$\cdot_{\pm x_3}$

(4.5.19)

The two $D7_{2i}$ branes can be studied exactly as before. The $D7_{2i}$–$D7_1$, $D7_{2i}$–$D7_3$, $D7_{2i}$–$D7_{2i}$ states are hence identical to the ones studied in section 3.3.3. The necessary condition to eliminate the mixed-state tachyons is similar to (3.3.44):

$$|\zeta_1^{(2)}| = |\zeta_{2a}^{(3)}| = \zeta_{2b}^{(3)}| = |\zeta_3^{(1)}|. \tag{4.5.20}$$

The new ingredient comes from the $D7_{2a}$–$D7_{2b}$ states. The magnetic fields produce the following mass for the lowest lying states

$$\alpha' m^2 = -|\zeta_{2a}^{(3)}| - |\zeta_{2a}^{(3)}| = -2|\zeta_{2a}^{(3)}|, \tag{4.5.21}$$

where in the last equality we used equation (4.5.20). The $D7_{2a}$–$D7_{2b}$ states also receive contributions from their relative distance, *i.e.* from the separation in the second torus T_2^2 due to the different brane localisations x_{2a} and x_{2b}. We recall that

$$x_{2i} \equiv x_{2i}^x \, \mathbf{R}_{2x} + x_{2i}^y \, \mathbf{R}_{2y} \quad \text{with} \quad x_{2i}^x, x_{2i}^y \in \mathbb{Q}, \quad i = a, b. \tag{4.5.22}$$

The mass contribution is then similar to the one of (3.3.35), with x_2 replaced by $x_{2ab} = x_{2a} - x_{2b}$. It reads

$$\alpha' m^2 = \frac{x_{2ab} \cdot x_{2ab}}{\alpha'} = \frac{x_{2ab}^k x_{2ab}^l g_{kl}^{(2)}}{\alpha'} = \frac{4\mathcal{A}_2}{\alpha' \mathrm{Re}(U_2)} \left| x_{2ab}^x - i U_2 x_{2ab}^y \right|^2 \equiv \frac{y_{ab}(U_2)\, \mathcal{A}_2}{\alpha'}. \tag{4.5.23}$$

The total $D7_{2a}$–$D7_{2b}$ lowest lying state mass then reads

$$\alpha' m_{22}^2 = -2|\zeta_{2a}^{(3)}| + \frac{y_{ab}\mathcal{A}_2}{\alpha'} \approx -\frac{2\alpha'|k_{2a}^{(3)}|}{\pi \mathcal{A}_3} + \frac{y_{ab}\mathcal{A}_2}{\alpha'}. \tag{4.5.24}$$

In the effective theory, the new mass contributions come from a D-term and F-term as in the previous cases. The second brane orthogonal to the T_2^2 torus give additional contributions to the D-term scalar potential obtained from the previous formula (4.4.3), where we recall that the sum runs over the different $U(1)$ factors:

$$\begin{aligned} V_D &= \sum_a \frac{g_{U(1)_a}^2}{2} \left(\xi_a + \sum_n q_a^n |\varphi_a^n|^2 \right)^2 + \cdots \\ &= \frac{g_{U(1)_{2a}}^2}{2} \left(\xi_{2a} - 2|\varphi_{2a\,-}|^2 - |\varphi_{2ab\,-}|^2 + \cdots \right)^2 + \frac{g_{U(1)_{2b}}^2}{2} \left(\xi_{2b} - 2|\varphi_{2b\,-}|^2 - |\varphi_{2ab\,-}|^2 + \cdots \right)^2 \\ &\quad + \sum_{a=1,3} \frac{g_{U(1)_a}^2}{2} \left(\xi_a - 2|\varphi_{a\,-}|^2 + \cdots \right)^2 + \cdots, \end{aligned} \tag{4.5.25}$$

with the FI terms ξ_i expressed from the fluxes as in eq. (4.4.8). The additional $D7_{2b}$ brane adds a contribution to the d_2 term, defined in (4.4.10), which now reads

$$d_2 = \frac{g_{U(1)_{2a}}^2}{2} \xi_{2a}^2 + \frac{g_{U(1)_{2b}}^2}{2} \xi_{2b}^2 = g_{U(1)_{2a}}^2 \xi_{2a}^2. \tag{4.5.26}$$

In the last equality we used the flux condition (4.5.20) and the fact that for unit wrapping numbers $g^2_{U(1)_{2a}} = g^2_{U(1)_{2b}}$ since the two stacks are parallel, as can be seen from equation (4.4.7). The D-term contributions to the masses and quartic couplings of the $\varphi_{2a,-}$, $\varphi_{2b,-}$ and $\varphi_{2ab,-}$ fields can be expressed by expanding the scalar potential (4.5.25). The masses have the same expressions while there is a factor of 2 difference between the quartic couplings of the doubly charged states and the bi-charged $D7_{2a}$–$D7_{2b}$ state.

The F-term contributions to the mass and quartic couplings can be derived as in the previous subsections, see eq. (4.4.29) and around, and read

$$\begin{aligned} &m^2_{x_{2i}} = y_i(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'}, \quad i=a,b, \qquad m^2_{x_{2ab}} = y_{ab}(U_2)\frac{g_s^2}{\kappa^2\mathcal{V}}\frac{\mathcal{A}_2}{\alpha'}, \\ &\lambda_{x_{2i}} = 4\kappa^2 m^2_{x_{2i}}, \qquad \lambda_{x_{2ab}} \sim 4\kappa^2 m^2_{x_{2ab}}, \end{aligned} \tag{4.5.27}$$

where we recall that y_{ab} was defined in eq. (4.5.23) using $x_{2ab} = x_{2a} - x_{2b}$ and the $y_i(U_2)$ are of course defined with respect to the respective brane positions x_{2i}. There are some subtleties for the low-energy derivation of the mass and quartic couplings for the $D7_{2a}$ – $D7_{2b}$ tachyon, because it does not appear in the same way as the $D7_{2i}$ – $D7_{2i}$ tachyons in the Kähler potential and has a different superpotential expression. Nevertheless, as expected from the string mass formula, we obtain the dependences as in (4.5.27).

The minimization procedure follows as in the case with the three tachyons of section 4.5.3. In the present case, there are four tachyons coming from the doubly charged states between each stack and its image, and a fifth one from the $D7_{2a}$–$D7_{2b}$ sector. The value of the scalar potential at the minimum hence reads

$$V(\mathcal{V}, v_i) = \frac{C}{\kappa^4}\left(-\frac{\ln\mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) - \sum_{i=1}^{5}\frac{C_i}{\kappa^4\mathcal{V}^{\frac{2}{3}}}\left(1 - \left(\frac{\mathcal{V}_{ci}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2, \tag{4.5.28}$$

where the sum runs over the five tachyons mentioned above, hence $i = 1, 2a, 2b, 2ab, 3$. The critical volumes and tachyonic contribution coefficients can be computed as before and read

$$\begin{aligned} &\mathcal{V}_{c1}{}^{\frac{2}{3}} \equiv \frac{2|k_1^{(2)}|}{w(U_1)\pi}\left(\frac{d_2}{d_1}\right)^{\frac{1}{3}}, \quad \mathcal{V}_{c2i}{}^{\frac{2}{3}} \equiv \frac{2|k_{2a}^{(3)}|}{y_i(U_2)\pi}\left(\frac{d_3}{d_2}\right)^{\frac{1}{3}}, \; i=a,b,ab, \quad \mathcal{V}_{c3}{}^{\frac{2}{3}} \equiv \frac{2|k_3^{(1)}|}{z(U_3)\pi}\left(\frac{d_1}{d_3}\right)^{\frac{1}{3}}, \\ &C_i = \beta_i\frac{d}{3\mathcal{V}_{ci}^{\frac{4}{3}}}, \quad i=1,2ab,3, \qquad C_i = \frac{1}{2}\beta_i\frac{d}{3\mathcal{V}_{ci}^{\frac{4}{3}}}, \quad i=2a,2b, \\ &\beta_i = \frac{2}{2+g_s f_i}, \quad i=1,2a,2b,3 \quad \text{with} \quad (f_1,f_{2a},f_{2b},f_3) = (w,y_a,y_b,z), \quad \beta_{2ab} = \frac{1}{1+g_s y_{ab}}. \end{aligned} \tag{4.5.29}$$

There is a small subtlety coming from the addition of a second parallel brane $D7_{2b}$, which modifies the d_2 parameter as in eq. (4.5.26), and is responsible for the factor $\frac{1}{2}$ in C_{2a}, C_{2b}. This factor is not present in C_{2ab}, because of the factor 2 between the D-term quartic couplings mentioned under eq. (4.5.26). As in the case with three tachyons discussed under (4.5.18), we look at the maximum value of the tachyonic amplitude, reached for almost equal $\mathcal{V}_{c,i} \approx \mathcal{V}_-$ and saturated values $\beta_i = 1$:

$$C_1 + C_{2a} + C_{2b} + C_{2ab} + C_3 \approx 4\frac{d}{3\mathcal{V}_-^{\frac{4}{3}}} > \beta^{\Lambda=0}\frac{d}{3\mathcal{V}_-^{\frac{4}{3}}}. \tag{4.5.30}$$

Hence the value $\beta^{\Lambda=0}$ introduced in eq. (4.4.53) can be reached with the current configuration, *i.e.* through the addition of the fourth brane $D7_{2b}$, and the value of the global minimum of the potential can be tuned to almost zero. With the saturated bound of equation (4.5.30), the sum of the tachyonic contributions to the global minimum is greater than the moduli contribution and an AdS vacuum is obtained. There are several options to tune the global minimum: one can either lower the β_i parameters or choose smaller tachyonic critical volumes (except for the first waterfall field responsible for the end of inflation).

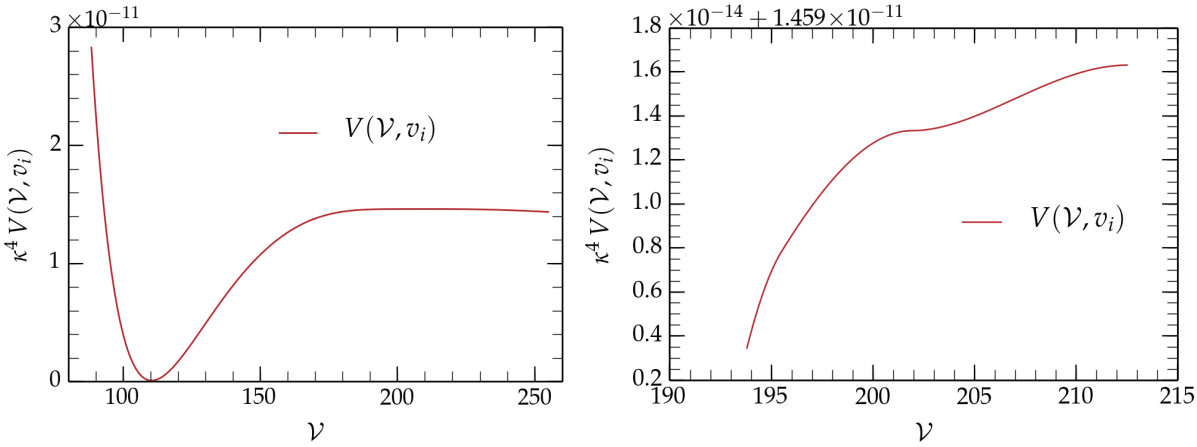


FIGURE 4.11: Value $V(\mathcal{V}, v_i)$ of the global minimum of the scalar potential as a function of the internal volume, for the parameters (4.5.31). We show *(left panel)* the almost vanishing value of the global minimum, and focus *(right panel)* on the waterfall zone near $\mathcal{V}_- \approx 201.9$.

Taking for simplicity only unit wrapping numbers, the choice of parameters

$$\begin{aligned} &g_s = 8.025 \times 10^{-3} \qquad n_1^{(2)} = 12, \qquad m_i^{(j)} = 1, \qquad x(U_1) = 0.185, \\ &z(U_3) = 0.189, \quad y_a(U_2) = 0.0881, \quad y_b(U_2) = 0.098, \quad y_{ab}(U_2) = 0.09, \end{aligned} \tag{4.5.31}$$

gives the following values for d, the critical volumes and the β_i coefficients:

$$\begin{aligned} &d = 5.65 \times 10^{-6}, \quad \beta_i \approx 1, \ i = 1, 2a, 2b, 2ab, 3, \quad \mathcal{V}_{c2a} = 201.9 \approx \mathcal{V}_-, \\ &\mathcal{V}_{c1} = 187.6, \quad \mathcal{V}_{c2b} = 172.1, \quad \mathcal{V}_{c2ab} = 195.5, \quad \mathcal{V}_{c3} = 181.7. \end{aligned} \tag{4.5.32}$$

Figure 4.11 shows the value of the global minimum of the scalar potential as a function of the internal volume for the parameters of (4.5.31). We see that with this choice the cosmological constant can indeed be tuned to an almost vanishing positive value.

4.6 Summary

Reconciling moduli stabilization and de Sitter vacua is essential in order to obtain an effective potential appropriate for cosmological inflation. In a type IIB string theory framework with intersecting D7 branes, we have shown that a non-vanishing potential is generated with the internal volume modulus playing the role of the inflaton ϕ, triggering exponential growth of the Universe. The essential ingredients in the stabilization mechanism are space-filling $D7$ branes. They generate logarithmic radiative corrections induced when effectively massless closed strings traverse their codimension-two bulk towards localized gravity sources. Moreover, the dS vacuum is obtained due to the positive D-term contributions coming from their $U(1)$ gauge factors.

We have shown that in the large volume limit, the induced effective potential for Kähler moduli receives a minimalist structure with a shape, and in particular, the volume separation of its two local extrema, that can be parametrized in terms of a single non-negative parameter, x. The largest separation, albeit rather small, occurs at a critical value $x_c > 0$. Beyond this point only AdS solutions are admissible. As x attains smaller values, the distance between the two extrema diminishes and at the final admissible point $x = 0$, it collapses to zero. The upshot of the above picture is that there exists a non-zero value $x < x_c$ at which a new inflationary small-field scenario is successfully implemented.

This scenario is distinct from other well-known solutions and its benchmarks are:

- Most of the required number of e-folds (~ 60) are collected in the vicinity of the minimum of the potential, while the horizon exit arises near (from above) the inflection point.
- The variation of the inflaton field is small compared to the Planck scale as in small field inflation models, consistently with the validity of the effective field theory and swampland distance conjecture [163].
- The prediction for the tensor-to-scalar ratio of primordial density fluctuations in the early universe is $r \approx 4 \times 10^{-4}$.
- The potential, induced by radiative corrections, presents a false vacuum that is expected to decay to the true one towards the direction of large ϕ values. We implemented well-established methods evaluating the possibility of tunneling [144] or passing over the potential barrier [145] and computed the vacuum decay rate. We found that the false vacuum has an extremely long lifetime.

While inflation is successfully described close to a sufficiently long-lived minimum of the potential, the cosmological constant in this scenario acquires a rather large value compared to the one observed today. We have thus described how this problem can be evaded within the context of hybrid inflation, realized when a second field creates a "waterfall" direction in the potential that stops inflation quickly, as soon as the slow-roll parameter ϵ exceeds unity.

We naturally implemented this idea within the same framework by considering the case where the waterfall field is realized by appropriate open string excitations located on three $D7$-brane stacks. For concreteness we used a toroidal orbifold example and studied the charged states on the branes and their intersections. Generically, they receive tachyonic contributions due to the coupling of the magnetic field with the internal spin and a positive supersymmetric contribution when Wilson lines along the branes worldvolumes are turned on, or branes separation in the transverse directions. We have shown that for appropriate magnetic fluxes and brane separations, most of the would-be tachyons can be eliminated. The remaining ones appear for an internal volume close to the minimum of the Kähler moduli potential, so that they play the role of waterfall fields. Hence, the magnetic fluxes generate the appropriate coupling with the waterfall fields, necessary to realize the transition to the true vacuum after inflation. At first, we implemented the hybrid inflation scenario considering only one tachyonic state. We observed that the value of the scalar potential at the global minimum was still too important to account for today's dark energy density. Adding more tachyons in the same compactification framework helped to lower the global minimum and we showed that with four tachyons we could successfully tune the vacuum energy to its almost vanishing, albeit positive, value.

In conclusion, it is worth emphasizing that the successful implementation of cosmological inflation in the above analysis is based only on a few simple characteristics occurring in generic type IIB string vacua. The few coefficients involved depend on well

defined topological properties such as the Euler characteristic of the compactification manifold, and the coefficients of the D-terms determined by the $D7$ brane configuration. Consequently, this analysis can in principle apply to an ample class of vacua in the string landscape. Moreover, the main features of the proposed framework describing waterfall fields tally with the general principles of hybrid inflation, establishing a firm ground for the implementation of this scenario in string theory.

Appendix A

Various conventions

A.1 Spinor conventions and useful identities

We work with Weyl spinors and use conventions of [10]. We recall conventions and useful identities below:

$$\psi\chi = \psi^\alpha\chi_\alpha = -\chi_\alpha\psi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \tag{A.1.1}$$

$$\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\bar{\sigma}^\mu\chi, \quad (\chi\sigma^\mu\bar{\psi})^* = \psi\sigma^\mu\bar{\chi}, \quad \chi\sigma^{\mu\nu}\psi = -\psi\sigma^{\mu\nu}\chi, \quad (\chi\sigma^{\mu\nu}\psi)^* = \bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\psi}, \tag{A.1.2}$$

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \quad \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \quad \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}, \quad \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta}\,, \tag{A.1.3}$$

$$\chi_\alpha \equiv (\sigma^{\mu\nu}\lambda)_\alpha \implies \chi^\alpha = -(\lambda\sigma^{\mu\nu})^\alpha, \quad \bar{\chi}^{\dot{\alpha}} = (\bar{\sigma}^{\mu\nu}\bar{\lambda})^{\dot{\alpha}}, \quad \bar{\chi}_{\dot{\alpha}} = -(\bar{\lambda}\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}\,, \tag{A.1.4}$$

$$\theta\psi\,\theta\phi = -\frac{1}{2}\theta\theta\,\psi\phi, \qquad \psi\chi\,\bar{\varphi}\bar{\eta} = -\frac{1}{2}\,\psi\sigma^\mu\bar{\varphi}\,\chi\sigma_\mu\bar{\eta} \quad \text{(Fierz identity)}\,. \tag{A.1.5}$$

We also list some properties of Pauli σ-matrices:

$$\sigma^{\mu\nu} \equiv \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \quad \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = -2\eta^{\mu\nu} \quad \to \quad \sigma^\mu\bar{\sigma}^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu}, \tag{A.1.6}$$

$$\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\gamma}) = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\gamma} - \eta^{\mu\gamma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\gamma}, \tag{A.1.7}$$

$$\sigma^{\mu\nu}\sigma^\rho = \frac{1}{2}\left(\eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\gamma}\sigma_\gamma\right), \quad \bar{\sigma}^\mu\sigma^{\nu\rho} = -\frac{1}{2}\left(\eta^{\mu\nu}\bar{\sigma}^\rho - \eta^{\mu\rho}\bar{\sigma}^\nu + i\epsilon^{\mu\nu\rho\gamma}\bar{\sigma}_\gamma\right). \tag{A.1.8}$$

Our slash notation conventions are as follows

$$\lambda\partial\!\!\!/\bar{\chi} \equiv \lambda\sigma^\mu\partial_\mu\bar{\chi}, \qquad \bar{\lambda}\bar{\sigma}^\mu\partial_\mu\chi \equiv \bar{\lambda}\bar{\partial}\!\!\!/\chi, \qquad (\lambda\partial\!\!\!/\bar{\chi})^* = -\bar{\lambda}\bar{\partial}\!\!\!/\chi = \partial_\mu\chi\sigma^\mu\bar{\lambda}\,. \tag{A.1.9}$$

A.2 Field strength identities

We define the dual $\tilde{F}^{\gamma\rho}$ of the antisymmetric field-strength tensor $F_{\mu\nu}$ and the associated self-dual or anti-self dual tensors as follows

$$\tilde{F}^{\gamma\rho} = \frac{i}{2}\epsilon^{\gamma\rho\mu\nu}F_{\mu\nu}, \qquad F^+_{\mu\nu} = \frac{F_{\mu\nu} + \tilde{F}_{\mu\nu}}{2}, \qquad F^-_{\mu\nu} = \frac{F_{\mu\nu} - \tilde{F}_{\mu\nu}}{2}. \tag{A.2.1}$$

The above tensors satisfy the following properties

$$\tilde{F}^{\mu}_{\ \rho}\tilde{F}^{\rho\nu} = -\frac{1}{2}\eta^{\mu\nu}F^2 - F^{\mu}_{\ \rho}F^{\rho\nu}\,, \qquad F^2 \equiv F^{\mu\nu}F_{\mu\nu} = \tilde{F}^2\,, \tag{A.2.2}$$

$$F^{\mu\rho}F_{\rho\alpha}F^{\alpha\nu} = \frac{1}{4}\tilde{F}^{\mu\nu}F\tilde{F} - \frac{1}{2}F^{\mu\nu}F^2, \qquad F_{\mu\alpha}\tilde{F}^{\alpha}_{\ \nu} = \frac{1}{4}\eta_{\mu\nu}F\tilde{F}\,, \tag{A.2.3}$$

$$F^{+\mu}_{\ \ \rho}F^{-\rho\nu} = F^{+\nu}_{\ \ \rho}F^{-\rho\mu} = \frac{1}{4}\left(F^{\mu}_{\ \rho}F^{\rho\nu} - \tilde{F}^{\mu}_{\ \rho}\tilde{F}^{\rho\nu}\right) = \frac{1}{8}\eta_{\mu\nu}F^2 + \frac{1}{2}F^{\mu}_{\ \rho}F^{\rho\nu}. \tag{A.2.4}$$

From (A.1.8) and (A.2.1) we derive the useful identities

$$\sigma^{\mu}\bar{\sigma}^{\nu\rho}F_{\nu\rho} = -2F^{-\mu}_{\ \ \nu}\sigma^{\nu}, \qquad F_{\nu\rho}\sigma^{\nu\rho}\sigma^{\mu} = 2F^{+\mu}_{\ \ \nu}\sigma^{\nu}. \tag{A.2.5}$$

A.3 Theta functions and $so(2)$ characters

The Jacobi theta functions are introduced as

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau) &= \sum_n q^{\frac{1}{2}(n+\alpha)^2}e^{2\pi i(n+\alpha)(z+\beta)} \\ &= e^{2\pi i\alpha(z+\beta)}q^{\frac{\alpha^2}{2}}\prod_{n=1}^{\infty}(1-q^n)(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i(z+\beta)})(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i(z+\beta)}).\end{aligned} \tag{A.3.1}$$

From them we define the four theta functions

$$\begin{aligned}\vartheta_1(z|\tau) &\equiv \vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(z|\tau), & \vartheta_2(z|\tau) &\equiv \vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}(z|\tau), \\ \vartheta_3(z|\tau) &\equiv \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(z|\tau), & \vartheta_4(z|\tau) &\equiv \vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}(z|\tau),\end{aligned} \tag{A.3.2}$$

used in the superstring amplitudes we consider in this document. When the ϑ_i functions are written without z dependence, as in $\vartheta_1(\tau)$ it means that we took $z = 0$.

The four level-one $so(2p)$ characters read

$$\begin{aligned}O_{2p}(z) &= \frac{\vartheta_3^p(z|\tau) + \vartheta_4^p(z|\tau)}{2\eta^p}, & V_{2p}(z) &= \frac{\vartheta_3^p(z|\tau) - \vartheta_4^p(z|\tau)}{2\eta^p}, \\ S_{2p}(z) &= \frac{\vartheta_2^p(z|\tau) + i^{-p}\vartheta_1^p(z|\tau)}{2\eta^p}, & C_{2p}(z) &= \frac{\vartheta_2^p(z|\tau) - i^{-p}\vartheta_1^p(z|\tau)}{2\eta^p},\end{aligned} \tag{A.3.3}$$

with the Dedekind function defined by

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{A.3.4}$$

The space-time characters used in $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifolds are constructed from the $so(2)$ characters (A.3.3) and read :

$$\begin{aligned}
\tau_{oo} &= V_2O_2O_2O_2 + O_2V_2V_2V_2 - S_2S_2S_2S_2 - C_2C_2C_2C_2\\
\tau_{og} &= O_2V_2O_2O_2 + V_2O_2V_2V_2 - C_2C_2S_2S_2 - S_2S_2C_2C_2\\
\tau_{oh} &= O_2O_2O_2V_2 + V_2V_2V_2O_2 - C_2S_2S_2C_2 - S_2C_2C_2S_2\\
\tau_{of} &= O_2O_2V_2O_2 + V_2V_2O_2V_2 - C_2S_2C_2S_2 - S_2C_2S_2C_2\\
\\
\tau_{go} &= V_2O_2S_2C_2 + O_2V_2C_2S_2 - S_2S_2V_2O_2 - C_2C_2O_2V_2\\
\tau_{gg} &= O_2V_2S_2C_2 + V_2O_2C_2S_2 - S_2S_2O_2V_2 - C_2C_2V_2O_2\\
\tau_{gh} &= O_2O_2S_2S_2 + V_2V_2C_2C_2 - C_2S_2V_2V_2 - S_2C_2O_2O_2\\
\tau_{gf} &= O_2O_2C_2C_2 + V_2V_2S_2S_2 - S_2C_2V_2V_2 - C_2S_2O_2O_2\\
\\
\tau_{ho} &= V_2S_2C_2O_2 + O_2C_2S_2V_2 - C_2O_2V_2C_2 - S_2V_2O_2S_2\\
\tau_{hg} &= O_2C_2C_2O_2 + V_2S_2S_2V_2 - C_2O_2O_2S_2 - S_2V_2V_2C_2\\
\tau_{hh} &= O_2S_2C_2V_2 + V_2C_2S_2O_2 - S_2O_2V_2S_2 - C_2V_2O_2C_2\\
\tau_{hf} &= O_2S_2S_2O_2 + V_2C_2C_2V_2 - C_2V_2V_2S_2 - S_2O_2O_2C_2\\
\\
\tau_{fo} &= V_2S_2O_2C_2 + O_2C_2V_2S_2 - S_2V_2S_2O_2 - C_2O_2C_2V_2\\
\tau_{fg} &= O_2C_2O_2C_2 + V_2S_2V_2S_2 - C_2O_2S_2O_2 - S_2V_2C_2V_2\\
\tau_{fh} &= O_2S_2O_2S_2 + V_2C_2V_2C_2 - C_2V_2S_2V_2 - S_2O_2C_2O_2\\
\tau_{ff} &= O_2S_2V_2C_2 + V_2C_2O_2S_2 - C_2V_2C_2O_2 - S_2O_2S_2V_2
\end{aligned} \tag{A.3.5}$$

The T_{kj} characters used in the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ model of section 3.3.3 are [48, 108]

$$\begin{aligned}
T_{ko} &= \tau_{ko} + \tau_{kg} + \tau_{kh} + \tau_{kf}, \quad T_{kg} = \tau_{ko} + \tau_{kg} - \tau_{kh} - \tau_{kf},\\
T_{kh} &= \tau_{ko} - \tau_{kg} + \tau_{kh} - \tau_{kf}, \quad T_{kf} = \tau_{ko} - \tau_{kg} - \tau_{kh} + \tau_{kf},
\end{aligned} \tag{A.3.6}$$

for $k = o, f, h, g$.

A.4 Momenta and windings sums

In absence of B-field background, the T_i^2 torus momenta, lying on the dual lattice Λ_i^* defined under equation (3.3.10), read

$$\mathbf{p}_i = m_k \mathbf{R}_i^{*k}, \quad m_k \in \mathbb{Z}. \tag{A.4.1}$$

Defining the T_i^2 torus windings, lying on the lattice Λ_i, through

$$\mathbf{L}_i = n^l \mathbf{R}_{il} \quad n^l \in \mathbb{Z}, \tag{A.4.2}$$

we introduce left and right momenta

$$\mathbf{p}_{iL,R} = \left(m_k \pm g_{kl}^{(i)} n^l\right) \mathbf{R}_i^{*k}. \tag{A.4.3}$$

The T_i^2 torus partition function is then defined by

$$\Lambda_i = \sum_{m,n} \frac{q^{\frac{\alpha'}{4}\mathbf{p}_{iL}\cdot\mathbf{p}_{iL}}\,\bar{q}^{\frac{\alpha'}{4}\mathbf{p}_{iR}\cdot\mathbf{p}_{iR}}}{|\eta(\tau)|^4}\,. \tag{A.4.4}$$

The Klein-bottle windings and momenta sums read

$$W_i = \sum_n \frac{q^{\frac{1}{2\alpha'}\mathbf{L}_i\cdot\mathbf{L}_i}}{\eta(2i\mathrm{Im}\tau)^2}\,, \qquad P_i = \sum_n \frac{e^{-2\pi\ell\alpha'\mathbf{p}_i\cdot\mathbf{p}_i}}{\eta(i\ell)^2}\,, \tag{A.4.5}$$

and the annulus ones are

$$W_i = \sum_n \frac{e^{-2\pi\ell\frac{1}{4\alpha'}\mathbf{L}_i\cdot\mathbf{L}_i}}{\eta(i\ell)^2}\,, \qquad P_i = \sum_m \frac{q^{\frac{\alpha'}{2}\mathbf{p}_i\cdot\mathbf{p}_i}}{\eta(i\mathrm{Im}\tau/2)^2}\,, \tag{A.4.6}$$

with ℓ being the modulus of the double cover of either the Klein bottle or the annulus [48].

Appendix B

Computational details

B.1 Derivation of (2.4.63)

The extra term in the second line of (2.4.61), containing the gauge potential u_μ, arises from the FI term. To make gauge invariance manifest, we rewrite it in terms of the field-strength $F_{\mu\nu}$ as we describe below. Using the property (2.2.13) of determinant for $\boldsymbol{A}^{-1}$ the second line of (2.4.61) can be written as

$$\begin{aligned} i\det\boldsymbol{A}\epsilon^{abcd}[(\boldsymbol{A}^{-1})_a{}^\nu\partial_\nu\lambda]\sigma_b[(\boldsymbol{A}^{-1})_c{}^\rho\partial_\rho\bar\lambda](\boldsymbol{A}^{-1})_d{}^\mu u_\mu &= i\det\boldsymbol{A}\,\epsilon^{\nu\gamma\rho\mu}\ \det\boldsymbol{A}^{-1}\boldsymbol{A}^e_\gamma\ \partial_\nu\lambda\sigma_e\partial_\rho\bar\lambda u_\mu \\ &= i\epsilon^{\nu\gamma\rho\mu}\boldsymbol{A}^e_\gamma\ \partial_\nu\lambda\sigma_e\partial_\rho\bar\lambda u_\mu\ . \end{aligned} \tag{B.1.1}$$

This can be further simplified by using several integrations by parts and Fierz identities (A.1.5)

$$\begin{aligned} i\epsilon^{\nu\gamma\rho\mu}\boldsymbol{A}^e_\gamma\ \partial_\nu\lambda\sigma_e\partial_\rho\bar\lambda u_\mu &= i\epsilon^{\nu\gamma\rho\mu}(\delta^e_\gamma + i\kappa^2\lambda\sigma^e\partial_\gamma\bar\lambda - i\kappa^2\partial_\gamma\lambda\sigma^e\bar\lambda)\ \partial_\nu\lambda\sigma_e\partial_\rho\bar\lambda u_\mu \\ &= -i\epsilon^{\nu\gamma\rho\mu}\delta^e_\gamma\ \lambda\sigma_e\partial_\rho\bar\lambda\ \partial_\nu u_\mu - i\kappa^2\epsilon^{\nu\gamma\rho\mu}(i\lambda\sigma^e\partial_\gamma\bar\lambda - i\partial_\gamma\lambda\sigma^e\bar\lambda)\ \lambda\sigma_e\partial_\rho\bar\lambda\ \partial_\nu u_\mu \\ &\quad - i\kappa^2\epsilon^{\nu\gamma\rho\mu}\partial_\nu(i\lambda\sigma^e\partial_\rho\bar\lambda - i\partial_\rho\lambda\sigma^e\bar\lambda)\ \lambda\sigma_e\partial_\rho\bar\lambda\ u_\mu + \text{total derivative} \\ &= -\frac{i}{2}\epsilon^{\gamma\rho\nu\mu}\ \lambda\sigma_\gamma\partial_\rho\bar\lambda\ F_{\nu\mu} - \frac{i}{2}\kappa^2\epsilon^{\nu\gamma\rho\mu}(-2)(i\lambda^2\ \partial_\gamma\bar\lambda\partial_\rho\bar\lambda - i\lambda\partial_\gamma\lambda\ \bar\lambda\partial_\rho\bar\lambda)F_{\nu\mu} \\ &\quad - i\kappa^2\epsilon^{\nu\gamma\rho\mu}(-2)(i\lambda\partial_\nu\lambda\partial_\gamma\bar\lambda\partial_\rho\bar\lambda - i\lambda\partial_\gamma\lambda\partial_\nu\bar\lambda\partial_\rho\bar\lambda)\ u_\mu + \text{total derivative} \\ &= -\lambda\sigma_\gamma\partial_\rho\bar\lambda\ \tilde F^{\gamma\rho} + 0 + \frac{i}{4}\kappa^2\epsilon^{\nu\gamma\rho\mu}\partial_\gamma(\lambda^2)\partial_\rho(\bar\lambda^2)F_{\nu\mu} + 0 + \text{total derivative} \\ &= -\lambda\sigma_\gamma\partial_\rho\bar\lambda\ \tilde F^{\gamma\rho} - \frac{i}{4}\kappa^2\epsilon^{\nu\gamma\rho\mu}\lambda^2\partial_\rho(\bar\lambda^2)\partial_\gamma F_{\nu\mu} + \text{total derivative} \\ &= -\lambda\sigma_\gamma\partial_\rho\bar\lambda\ \tilde F^{\gamma\rho} + \text{total derivative.} \end{aligned} \tag{B.1.2}$$

We also repeatedly used relations $\epsilon^{\nu\gamma\rho\mu}\partial_\mu\partial_\nu = 0$ and $\epsilon^{\nu\gamma\rho\mu}\partial_\mu\lambda\partial_\nu\lambda = 0$. Once rewritten as (B.1.2) it is obvious that the second line of (2.4.61) is gauge invariant.

B.2 Superfield expansions

We first recall the component expansions of the superfields W_α and W^2

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha\mathsf{D} - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\bar\lambda)_\alpha\ , \tag{B.2.1}$$

$$W^2 = C + 2\psi\theta + \theta\theta E\ , \tag{B.2.2}$$

$$C = -\lambda^2,\quad \psi_\beta = F_{\mu\nu}(\sigma^{\mu\nu}\lambda)_\beta - i\mathsf{D}\lambda_\beta \equiv \Upsilon_\beta - i\mathsf{D}\lambda_\beta,\quad E = \mathsf{D}^2 - \frac{1}{2}(F^2 + F\tilde F) - 2i\lambda\not\partial\bar\lambda\ . \tag{B.2.3}$$

We also use the following chiral and anti-chiral superfield expansions

$$-\frac{1}{4}\overline{D}^2\overline{W}^2(y) = \bar{E} + 2i\theta\partial\!\!\!/\bar{\psi} + \theta^2\Box\bar{C}\,, \tag{B.2.4}$$

$$-\frac{1}{4}D^2W^2(\bar{y}) = E + 2i\bar{\theta}\bar{\partial}\!\!\!/\psi + \theta^2\Box C\,. \tag{B.2.5}$$

Then, the chiral superfield Φ defined in (2.4.111) has the following filed component expansion, depending of the chiral coordinates $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$,

$$\Phi(y) = \frac{W^2}{\overline{D}^2\overline{W}^2}(y) = \phi(y) + \chi(y)\theta + \theta^2\mathcal{G}(y), \tag{B.2.6}$$

$$\phi = -\frac{C}{4\bar{E}}, \quad \chi_\alpha = -\frac{\psi_\alpha}{2\bar{E}} + i\frac{C(\partial\!\!\!/\bar{\psi})_\alpha}{2\bar{E}^2}, \quad \mathcal{G} = -\frac{E}{4\bar{E}} - \frac{i\psi\partial\!\!\!/\bar{\psi}}{2\bar{E}^2} + \frac{C\Box\bar{C}}{4\bar{E}^2} - \frac{C(\partial\!\!\!/\bar{\psi})^2}{2\bar{E}^3}\,. \tag{B.2.7}$$

We can now compute the component expansion of the real superfield $\Phi\bar{\Phi}$

$$\begin{aligned}\Phi\bar{\Phi}(x) \;=\;& \phi\bar{\phi} + (\bar{\chi}\bar{\theta}\phi + \text{c.c.}) + (\bar{\theta}^2\phi\bar{\mathcal{G}} + \text{c.c.}) + \chi\theta\bar{\chi}\bar{\theta} + i\theta\sigma^\mu\bar{\theta}\left(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}\right)\\ &+ \left(\bar{\theta}^2\theta\left[\bar{\mathcal{G}}\chi - \frac{i}{2}\sigma^\mu\left(\bar{\chi}\partial_\mu\phi - \partial_\mu\bar{\chi}\phi\right)\right] + \text{c.c.}\right)\\ &+ \theta^2\bar{\theta}^2\left[\mathcal{G}\bar{\mathcal{G}} - \partial^\mu\phi\partial_\mu\bar{\phi} - \frac{i}{4}\chi\partial\!\!\!/\bar{\chi} + \frac{i}{4}\partial_\mu\chi\sigma^\mu\bar{\chi} + \frac{1}{4}\Box(\phi\bar{\phi})\right].\end{aligned} \tag{B.2.8}$$

Finally, for the real superfields $\mathcal{A}$ and $\mathcal{B}$ defined in (2.4.5), we have the following component expansions

$$\begin{aligned}\mathcal{A}(x) =& \frac{\kappa^2}{2}(D^2W^2 + \overline{D}^2\overline{W}^2)\\ =& -2\kappa^2\Bigg[\left(\mathsf{D}^2 + \bar{\mathsf{D}}^2\right) + \mathcal{E}_+ + 2i\left(\bar{\theta}\bar{\sigma}^\mu\partial_\mu(\Psi - i\mathsf{D}\lambda) + \theta\sigma^\mu\partial_\mu(\bar{\Psi} + i\bar{\mathsf{D}}\bar{\lambda})\right)\\ &+ \bar{\theta}\bar{\theta}\Box C + \theta\theta\Box\bar{C} - i\theta\sigma^\mu\bar{\theta}\partial_\mu(\mathcal{E}_- + \mathsf{D}^2 - \bar{\mathsf{D}}^2)\\ &+ \bar{\theta}\bar{\theta}\theta\Box(\Psi - i\mathsf{D}\lambda) + \theta\theta\bar{\theta}\Box(\bar{\Psi} + i\bar{\mathsf{D}}\bar{\lambda}) + \theta^2\bar{\theta}^2\frac{1}{4}\Box\left(\mathcal{E}_+ + \mathsf{D}^2 + \bar{\mathsf{D}}^2\right)\Bigg],\end{aligned} \tag{B.2.9}$$

$$\begin{aligned}\mathcal{B}(x) =& i\frac{\kappa^2}{2}(D^2W^2 - \overline{D}^2\overline{W}^2)\\ =& -2i\kappa^2\Bigg[\left(\mathsf{D}^2 - \bar{\mathsf{D}}^2\right) + \mathcal{E}_- + 2i\left(\bar{\theta}\bar{\sigma}^\mu\partial_\mu(\Psi - i\mathsf{D}\lambda) - \theta\sigma^\mu\partial_\mu(\bar{\Psi} + i\bar{\mathsf{D}}\bar{\lambda})\right)\\ &+ \bar{\theta}\bar{\theta}\Box C - \theta\theta\Box\bar{C} - i\theta\sigma^\mu\bar{\theta}\partial_\mu(\mathcal{E}_+ + \mathsf{D}^2 + \bar{\mathsf{D}}^2)\\ &+ \bar{\theta}\bar{\theta}\theta\Box(\Psi - i\mathsf{D}\lambda) - \theta\theta\bar{\theta}\Box(\bar{\Psi} + i\bar{\mathsf{D}}\bar{\lambda}) + \theta^2\bar{\theta}^2\frac{1}{4}\Box\left(\mathcal{E}_- + \mathsf{D}^2 - \bar{\mathsf{D}}^2\right)\Bigg],\end{aligned} \tag{B.2.10}$$

where

$$\mathcal{E}_\pm \equiv E \pm \bar{E} - \left(\mathsf{D}^2 \pm \bar{\mathsf{D}}^2\right). \tag{B.2.11}$$

Since the auxiliary fields D and $\bar{\mathsf{D}}$ are not dynamical and should be eliminated at the end, we can isolate their contribution in the above two real superfields

$$\mathcal{A} \;\equiv \mathcal{A}_{scalar} + \mathcal{A}' = -2\kappa^2\left(\mathsf{D}^2 + \bar{\mathsf{D}}^2\right) + \mathcal{A}' = -4\kappa^2\left(d^2 - \gamma^2\right) + \mathcal{A}', \tag{B.2.12}$$

$$\mathcal{B} \;\equiv \mathcal{B}_{scalar} + \mathcal{B}' = -2i\kappa^2\left(\mathsf{D}^2 - \bar{\mathsf{D}}^2\right) + \mathcal{B}' = 8\kappa^2 d\gamma + \mathcal{B}', \tag{B.2.13}$$

Then, the superfield $\mathcal{M}$ defined in (2.4.111) can be expanded up to mass dimension 10 (included)

$$\begin{aligned}
\mathcal{M} &= 1+\mathcal{A}-\sqrt{1+2\mathcal{A}-\mathcal{B}^2} \\
&= 1-4\kappa^2\left(d^2-\gamma^2\right)+\mathcal{A}'-\sqrt{1-8\kappa^2\left(d^2-\gamma^2\right)-64\kappa^4 d^2\gamma^2+2\mathcal{A}'-16\kappa^2 d\gamma\mathcal{B}'-\mathcal{B}'^2} \\
&= 1-4\kappa^2\left(d^2-\gamma^2\right)+\mathcal{A}'-\sqrt{Z}\left[1+\frac{\mathcal{A}'}{Z}-\frac{8\kappa^2 d\gamma\mathcal{B}'}{Z}-\frac{\mathcal{B}'^2}{2Z}\right. \\
&\quad \left.-\frac{1}{8Z^2}\left(2\mathcal{A}'-\mathcal{B}'^2-16\kappa^2 d\gamma\mathcal{B}'\right)^2+\frac{1}{16Z^3}\left(2\mathcal{A}'-16\kappa^2 d\gamma\mathcal{B}'\right)^3\right]+O(\kappa^6). \qquad \text{(B.2.14)}
\end{aligned}$$

where we introduced

$$Z=(1+8\kappa^2\gamma^2)(1-8\kappa^2 d^2)=1-8\kappa^2\left(d^2-\gamma^2\right)-(8\kappa^2 d\gamma)^2\,. \qquad \text{(B.2.15)}$$

From (B.2.3) we can compute explicitly the useful expansion

$$\begin{aligned}
i\psi\partial\!\!\!/\bar{\psi} &= i\Psi\partial\!\!\!/\bar{\Psi}-\bar{\mathrm{D}}\Psi\partial\!\!\!/\bar{\lambda}-\partial_\mu(\bar{\mathrm{D}})\Psi\sigma^\mu\bar{\lambda}+\mathrm{D}\lambda\partial\!\!\!/\bar{\Psi}+i\mathrm{D}\bar{\mathrm{D}}\lambda\partial\!\!\!/\bar{\lambda} \\
&= 2i\lambda\sigma^\mu\partial_\nu\bar{\lambda}F_\mu^{\ \rho}F_\rho^{\ \nu}+\frac{i}{2}\lambda\partial\!\!\!/\bar{\lambda}F^2+\bar{\mathrm{D}}\lambda\sigma^\mu\partial_\nu\bar{\lambda}(F^\nu_{\ \mu}+F^{*\nu}_{\ \ \mu})+\partial^\nu(\bar{\mathrm{D}})\lambda\sigma^\mu\bar{\lambda}(F_{\nu\mu}+\tilde{F}_{\nu\mu}) \\
&\quad -\mathrm{D}\lambda\sigma^\mu\partial^\nu\bar{\lambda}(F_{\nu\mu}-\tilde{F}_{\nu\mu})-\mathrm{D}\lambda\sigma^\mu\bar{\lambda}\partial^\nu F_{\nu\mu}+i\mathrm{D}\bar{\mathrm{D}}\lambda\partial\!\!\!/\bar{\lambda}+i\mathrm{D}\partial_\mu\bar{\mathrm{D}}\lambda\sigma^\mu\bar{\lambda}\,. \qquad \text{(B.2.16)}
\end{aligned}$$

Appendix C

Tachyons from magnetic fields on the entire D7-branes worldvolumes

In this appendix we study the tachyons generated by a configuration with three $D7$-brane stacks with magnetic fields on the entire worldvolumes. This is motivated because we saw in section 4.5.3 that eqs. (4.4.49) and (4.5.14) fix the relation between the critical volumes and the amplitudes through the flux parameter d. This parameter plays a crucial role in the inflationary phase and is fixed by observations. In the simple flux configuration of section 4.5.3, all fluxes were taken equal, hence $d_1 = d_2 = d_3$. One may wonder if allowing for different d_a would relax relations between the tachyonic contribution scalings and the critical volumes, by introducing d_a in the relations similar to eqs. (4.4.49) and (4.5.14).

According to the study of section 3.3.3, it is possible to have different doubly charged states masses (and hence different d_a) by putting magnetic fields on the entire brane worldvolumes, as shown in the following table.

	(45)	(67)	(89)
$D7_1$	$\cdot_{\pm x_1}$	$\otimes$	$\otimes$
$D7_2$	$\otimes$	$\cdot_{\pm x_2}$	$\otimes$
$D7_3$	$\otimes$	$\otimes$	$\cdot_{\pm x_3}$

We recall that the magnetic fields are subject to conditions $(A-i)$ or $(B-i)$ of equation (3.3.49) to eliminate the mixed states tachyons. In order to have the possibility for different (non-vanishing) d_a, we choose the configuration of fluxes in condition $(B-1)$ of equation (3.3.49) that we recall here for simplicity

$$(B-1) \qquad \zeta_1^{(2)} = \zeta_1^{(3)}, \qquad \zeta_2^{(1)} = \zeta_2^{(3)}, \qquad \zeta_3^{(1)} = \zeta_3^{(2)}. \tag{C.0.1}$$

An important point is that when magnetic fields are plugged on the entire worldvolumes, one cannot use Wilson lines A_i any more to eliminate the tachyons from the doubly charges states. The only way is to use brane separations x_i, which indeed eliminate tachyons at large volumes but lead to tachyons under a certain critical volume. This was phenomenon was described in details in the previous subsections. In the present case, we thus have to consider one tachyon for each doubly charged $D7_i$–$D7_i$ state. As before, the tachyonic masses contributions generated by the magnetic fluxes and brane separation at

the string level read

$$\begin{aligned}
\alpha' m_{11}^2 &= -2\left|\zeta_1^{(2)} + \zeta_1^{(3)}\right| + \frac{w\mathcal{A}_1}{\alpha'} \underset{(B-1)}{=} -4\left|\zeta_1^{(2)}\right| + \frac{w\mathcal{A}_1}{\alpha'} \approx -\frac{4|k_1^{(2)}|\alpha'}{\pi\mathcal{A}_2} + \frac{w\mathcal{A}_1}{\alpha'}, \\
\alpha' m_{22}^2 &= -2\left|\zeta_2^{(1)} + \zeta_2^{(3)}\right| + \frac{y\mathcal{A}_2}{\alpha'} \underset{(B-1)}{=} -4\left|\zeta_2^{(1)}\right| + \frac{y\mathcal{A}_2}{\alpha'} \approx -\frac{4|k_2^{(1)}|\alpha'}{\pi\mathcal{A}_1} + \frac{y\mathcal{A}_2}{\alpha'}, \\
\alpha' m_{33}^2 &= -2\left|\zeta_3^{(1)} + \zeta_3^{(2)}\right| + \frac{z\mathcal{A}_3}{\alpha'} \underset{(B-1)}{=} -4\left|\zeta_3^{(1)}\right| + \frac{z\mathcal{A}_3}{\alpha'} \approx -\frac{4|k_3^{(1)}|\alpha'}{\pi\mathcal{A}_1} + \frac{z\mathcal{A}_3}{\alpha'}.
\end{aligned} \tag{C.0.2}$$

In the low energy effective theory this corresponds to d_a parameters of the form

$$d_a = \frac{1}{2} g_s^3 |m_a^{(j)} m_a^{(k)}| \left(\frac{2k_a^{(j)}}{\pi}\right)^2, \quad a \neq j \neq k \neq a. \tag{C.0.3}$$

Remember that the moduli stabilisation conditions depend on these d_a and are given by (4.4.35). Together with (C.0.1) these conditions allow to express *e.g.* $n_1^{(3)}$, $n_2^{(1)}$ and $n_3^{(2)}$ with respect to $n_1^{(2)}$, $n_2^{(3)}$, $n_3^{(1)}$ and the $m_a^{(j)}$, hence leaving only three independent flux numbers together with the wrapping numbers.

After some straightforward manipulations we check that when the volume is inferior to all the critical volumes, *i.e.* when for any value of a, $\mathcal{V} < V_{c,a}$, the scalar potential reads

$$V(\mathcal{V}, v_1, v_2, v_3) = \frac{C}{\kappa^4}\left(-\frac{\ln \mathcal{V} - 4 + q}{\mathcal{V}^3} - \frac{3\sigma}{2\mathcal{V}^2}\right) - \sum_{a=1}^{3} \frac{C_a}{\kappa^4 \mathcal{V}^{\frac{2}{3}}}\left(1 - \left(\frac{\mathcal{V}_{ca}}{\mathcal{V}}\right)^{\frac{2}{3}}\right)^2 \tag{C.0.4}$$

with again

$$\begin{aligned}
&\mathcal{V}_{c1}{}^{\frac{2}{3}} \equiv \frac{4|k_1^{(2)}|}{w(U_1)\pi}\left(\frac{d_2}{d_1}\right)^{\frac{1}{3}}, \quad \mathcal{V}_{c2}{}^{\frac{2}{3}} \equiv \frac{4|k_2^{(1)}|}{y(U_2)\pi}\left(\frac{d_2}{d_1}\right)^{\frac{1}{3}}, \quad \mathcal{V}_{c3}{}^{\frac{2}{3}} \equiv \frac{4|k_3^{(1)}|}{z(U_3)\pi}\left(\frac{d_1}{d_3}\right)^{\frac{1}{3}}, \\
&C_a = \beta_a \frac{d}{3\mathcal{V}_{ca}}, \qquad \beta_a = \frac{2}{2 + g_s f_a(U_a)|m_a^{(j)} m_a^{(k)}|} \in [0,1], \quad (f_1, f_2, f_3) = (w, y, z).
\end{aligned} \tag{C.0.5}$$

Hence we see that even with different d_a as in the current configuration, the relations (C.0.5) between the critical volumes and the amplitudes of the tachyonic contributions only imply $d = 3(d_1 d_2 d_3)^{\frac{1}{3}}$, as in the simpler case with only one magnetic field per brane. The potential is thus identical to the one with three tachyons (*red curve*) of Figure 4.10.

Bibliography

[1] I. Antoniadis, H. Jiang and O. Lacombe, $\mathcal{N} = 2$ *supersymmetry deformations, electromagnetic duality and Dirac-Born-Infeld actions*, *JHEP* **07** (2019) 147, [1904.06339].

[2] I. Antoniadis, H. Jiang and O. Lacombe, *Note on supersymmetric Dirac-Born-Infeld action with Fayet-Iliopoulos term*, *JHEP* **05** (2020) 111, [1912.12627].

[3] I. Antoniadis, O. Lacombe and G. K. Leontaris, *Inflation near a metastable de Sitter vacuum from moduli stabilisation*, *Eur. Phys. J. C* **80** (2020) 1014, [2007.10362].

[4] I. Antoniadis, O. Lacombe and G. K. Leontaris, *Hybrid inflation and waterfall field in string theory from D7-branes*, 2109.03243.

[5] Y. Nambu, *Quasiparticles and Gauge Invariance in the Theory of Superconductivity*, *Phys. Rev.* **117** (1960) 648–663.

[6] J. Goldstone, *Field Theories with Superconductor Solutions*, *Nuovo Cim.* **19** (1961) 154–164.

[7] P. W. Higgs, *Broken Symmetries and the Masses of Gauge Bosons*, *Phys. Rev. Lett.* **13** (1964) 508–509.

[8] F. Englert and R. Brout, *Broken Symmetry and the Mass of Gauge Vector Mesons*, *Phys. Rev. Lett.* **13** (1964) 321–323.

[9] R. Haag, J. T. Lopuszanski and M. Sohnius, *All Possible Generators of Supersymmetries of the s Matrix*, *Nucl. Phys. B* **88** (1975) 257.

[10] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, Princeton, NJ, USA, 1992.

[11] J. Wess and B. Zumino, *Supergauge Transformations in Four-Dimensions*, *Nucl. Phys. B* **70** (1974) 39–50.

[12] P. Fayet and J. Iliopoulos, *Spontaneously Broken Supergauge Symmetries and Goldstone Spinors*, *Phys. Lett. B* **51** (1974) 461–464.

[13] L. O'Raifeartaigh, *Spontaneous Symmetry Breaking for Chiral Scalar Superfields*, *Nucl. Phys. B* **96** (1975) 331–352.

[14] D. V. Volkov and V. P. Akulov, *Possible universal neutrino interaction*, *JETP Lett.* **16** (1972) 438–440.

[15] M. Rocek, *Linearizing the Volkov-Akulov Model*, *Phys. Rev. Lett.* **41** (1978) 451–453.

[16] Z. Komargodski and N. Seiberg, *From Linear SUSY to Constrained Superfields*, *JHEP* **09** (2009) 066, [0907.2441].

[17] N. Cribiori, G. Dall'Agata and F. Farakos, *From Linear to Non-linear SUSY and Back Again*, *JHEP* **08** (2017) 117, [1704.07387].

[18] N. Cribiori, F. Farakos and M. Tournoy, *Supersymmetric Born-Infeld actions and new Fayet-Iliopoulos terms*, *JHEP* **03** (2019) 050, [1811.08424].

[19] S. M. Kuzenko and S. J. Tyler, *Relating the Komargodski-Seiberg and Akulov-Volkov actions: Exact nonlinear field redefinition*, *Phys. Lett.* **B698** (2011) 319–322, [1009.3298].

[20] G. Dall'Agata, E. Dudas and F. Farakos, *On the origin of constrained superfields*, *JHEP* **05** (2016) 041, [1603.03416].

[21] I. Antoniadis, H. Partouche and T. R. Taylor, *Spontaneous breaking of N=2 global supersymmetry*, *Phys. Lett. B* **372** (1996) 83–87, [hep-th/9512006].

[22] J. Bagger and A. Galperin, *A New Goldstone multiplet for partially broken supersymmetry*, *Phys. Rev.* **D55** (1997) 1091–1098, [hep-th/9608177].

[23] I. Antoniadis, J. P. Derendinger and T. Maillard, *Nonlinear N=2 Supersymmetry, Effective Actions and Moduli Stabilization*, *Nucl. Phys. B* **808** (2009) 53–79, [0804.1738].

[24] I. Antoniadis, J.-P. Derendinger and C. Markou, *Nonlinear $\mathcal{N} = 2$ global supersymmetry*, *JHEP* **06** (2017) 052, [1703.08806].

[25] E. A. Ivanov and B. M. Zupnik, *Modified N=2 supersymmetry and Fayet-Iliopoulos terms*, *Phys. Atom. Nucl.* **62** (1999) 1043–1055, [hep-th/9710236].

[26] S. M. Kuzenko, *The Fayet-Iliopoulos term and nonlinear self-duality*, *Phys. Rev.* **D81** (2010) 085036, [0911.5190].

[27] I. Antoniadis, J.-P. Derendinger, H. Jiang and G. Tartaglino-Mazzucchelli, *Magnetic deformation of super-Maxwell theory in supergravity*, *JHEP* **08** (2020) 079, [2005.11374].

[28] M. Rocek and A. A. Tseytlin, *Partial breaking of global D = 4 supersymmetry, constrained superfields, and three-brane actions*, *Phys. Rev.* **D59** (1999) 106001, [hep-th/9811232].

[29] A. A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, *Shifman, M.A. (ed.): The many faces of the superworld* (1999) 417–452, [hep-th/9908105].

[30] E. A. Bergshoeff, A. Bilal, M. de Roo and A. Sevrin, *Supersymmetric nonAbelian Born-Infeld revisited*, *JHEP* **07** (2001) 029, [hep-th/0105274].

[31] N. Cribiori, F. Farakos, M. Tournoy and A. van Proeyen, *Fayet-Iliopoulos terms in supergravity without gauged R-symmetry*, *JHEP* **04** (2018) 032, [1712.08601].

[32] I. Antoniadis, A. Chatrabhuti, H. Isono and R. Knoops, *Fayet–iliopoulos terms in supergravity and d-term inflation*, *Eur. Phys. J. C.* **78** (2018) .

[33] I. Antoniadis, A. Chatrabhuti, H. Isono and R. Knoops, *The cosmological constant in Supergravity*, *Eur. Phys. J. C* **78** (2018) 718, [1805.00852].

[34] I. Antoniadis and F. Rondeau, *New Kähler invariant Fayet–Iliopoulos terms in supergravity and cosmological applications*, *Eur. Phys. J. C* **80** (2020) 346, [1912.08117].

[35] I. Antoniadis, J.-P. Derendinger, F. Farakos and G. Tartaglino-Mazzucchelli, *New Fayet-Iliopoulos terms in $\mathcal{N} = 2$ supergravity*, *JHEP* **07** (2019) 061.

[36] S. Bellucci, N. Kozyrev, S. Krivonos and A. Sutulin, *Space-filling D3-brane within coset approach*, *JHEP* **08** (2015) 094, [1505.07386].

[37] I. Antoniadis and T. Maillard, *Moduli stabilization from magnetic fluxes in type I string theory*, *Nucl. Phys. B* **716** (2005) 3–32, [hep-th/0412008].

[38] M. Bianchi and E. Trevigne, *The Open story of the magnetic fluxes*, *JHEP* **08** (2005) 034, [hep-th/0502147].

[39] I. Antoniadis, A. Kumar and T. Maillard, *Magnetic fluxes and moduli stabilization*, *Nucl. Phys. B* **767** (2007) 139–162, [hep-th/0610246].

[40] D. Z. Freedman, *Supergravity with Axial Gauge Invariance*, *Phys. Rev. D* **15** (1977) 1173.

[41] S. Samuel and J. Wess, *A Superfield Formulation of the Nonlinear Realization of Supersymmetry and Its Coupling to Supergravity*, *Nucl. Phys.* **B221** (1983) 153–177.

[42] M. Klein, *Couplings in pseudosupersymmetry*, *Phys. Rev. D* **66** (2002) 055009, [hep-th/0205300].

[43] G. Veneziano, *Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories*, *Nuovo Cim. A* **57** (1968) 190–197.

[44] J. A. Shapiro, *Narrow-resonance model with regge behavior for pi pi scattering*, *Phys. Rev.* **179** (1969) 1345–1353.

[45] M. A. Virasoro, *Alternative constructions of crossing-symmetric amplitudes with regge behavior*, *Phys. Rev.* **177** (1969) 2309–2311.

[46] J. Scherk and J. H. Schwarz, *Dual Models for Nonhadrons*, *Nucl. Phys. B* **81** (1974) 118–144.

[47] T. Yoneya, *Connection of Dual Models to Electrodynamics and Gravidynamics*, *Prog. Theor. Phys.* **51** (1974) 1907–1920.

[48] C. Angelantonj and A. Sagnotti, *Open strings*, *Phys. Rept.* **371** (2002) 1–150, [hep-th/0204089].

[49] K. Becker, M. Becker and J. H. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 12, 2006.

[50] L. E. Ibanez and A. M. Uranga, *String theory and particle physics: An introduction to string phenomenology*. Cambridge University Press, 2, 2012.

[51] R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013, 10.1007/978-3-642-29497-6.

[52] E. Kiritsis, *String theory in a nutshell*. Princeton University Press, USA, 2019.

[53] J. H. Schwarz, *Lectures on superstring and M theory dualities: Given at ICTP Spring School and at TASI Summer School*, *Nucl. Phys. B Proc. Suppl.* **55** (1997) 1–32, [hep-th/9607201].

[54] A. Sen, *An Introduction to nonperturbative string theory*, in *A Newton Institute Euroconference on Duality and Supersymmetric Theories*, pp. 297–413, 2, 1998. hep-th/9802051.

[55] Y. Nambu, *Duality and hydrodynamics*, *Lectures at the Copenhagen Conference* (1970) .

[56] T. Goto, *Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model*, *Prog. Theor. Phys.* **46** (1971) 1560–1569.

[57] S. Deser and B. Zumino, *A Complete Action for the Spinning String*, *Phys. Lett. B* **65** (1976) 369–373.

[58] L. Brink, P. Di Vecchia and P. S. Howe, *A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String*, *Phys. Lett. B* **65** (1976) 471–474.

[59] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys. Lett. B* **103** (1981) 207–210.

[60] P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, *Quantum dynamics of a massless relativistic string*, *Nucl. Phys. B* **56** (1973) 109–135.

[61] A. Neveu and J. H. Schwarz, *Factorizable dual model of pions*, *Nucl. Phys. B* **31** (1971) 86–112.

[62] P. Ramond, *Dual Theory for Free Fermions*, *Phys. Rev. D* **3** (1971) 2415–2418.

[63] M. B. Green and J. H. Schwarz, *Supersymmetrical Dual String Theory*, *Nucl. Phys. B* **181** (1981) 502–530.

[64] M. B. Green and J. H. Schwarz, *Supersymmetrical Dual String Theory. 2. Vertices and Trees*, *Nucl. Phys. B* **198** (1982) 252–268.

[65] M. B. Green and J. H. Schwarz, *Supersymmetrical Dual String Theory. 3. Loops and Renormalization*, *Nucl. Phys. B* **198** (1982) 441–460.

[66] N. Berkovits, *Super Poincare covariant quantization of the superstring*, *JHEP* **04** (2000) 018, [hep-th/0001035].

[67] F. Gliozzi, J. Scherk and D. I. Olive, *Supersymmetry, Supergravity Theories and the Dual Spinor Model*, *Nucl. Phys. B* **122** (1977) 253–290.

[68] M. R. Douglas and B. Grinstein, *Dilaton Tadpole for the Open Bosonic String*, *Phys. Lett. B* **183** (1987) 52.

[69] S. Weinberg, *Cancellation of One Loop Divergences in SO(8192) String Theory*, *Phys. Lett. B* **187** (1987) 278–282.

[70] M. Bianchi and A. Sagnotti, *The Partition Function of the SO(8192) Bosonic String*, *Phys. Lett. B* **211** (1988) 407–416.

[71] M. B. Green and J. H. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, *Phys. Lett. B* **149** (1984) 117–122.

[72] G. Pradisi and A. Sagnotti, *Open String Orbifolds*, *Phys. Lett. B* **216** (1989) 59–67.

[73] A. Sagnotti, *A Note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett. B* **294** (1992) 196–203, [hep-th/9210127].

[74] E. G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d manifolds*, *Phys. Rev. D* **54** (1996) 1667–1676, [hep-th/9601038].

[75] G. Aldazabal, D. Badagnani, L. E. Ibanez and A. M. Uranga, *Tadpole versus anomaly cancellation in D = 4, D = 6 compact IIB orientifolds*, *JHEP* **06** (1999) 031, [hep-th/9904071].

[76] M. Bianchi and J. F. Morales, *Anomalies \& tadpoles*, *JHEP* **03** (2000) 030, [hep-th/0002149].

[77] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys. B* **451** (1995) 96–108, [hep-th/9504090].

[78] A. Strominger, *Open p-branes*, *Phys. Lett. B* **383** (1996) 44–47, [hep-th/9512059].

[79] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[80] J. Polchinski, S. Chaudhuri and C. V. Johnson, *Notes on D-branes*, hep-th/9602052.

[81] J. Polchinski, *Tasi lectures on D-branes*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 293–356, 11, 1996. hep-th/9611050.

[82] C. P. Bachas, *Lectures on D-branes*, in *A Newton Institute Euroconference on Duality and Supersymmetric Theories*, pp. 414–473, 6, 1998. hep-th/9806199.

[83] C. V. Johnson, *D-branes*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2005, 10.1017/CBO9780511606540.

[84] D. Andriot, *New constraints on classical de Sitter: flirting with the swampland*, *Fortsch. Phys.* **67** (2019) 1800103, [1807.09698].

[85] D. Andriot, *On the de Sitter swampland criterion*, *Phys. Lett. B* **785** (2018) 570–573, [1806.10999].

[86] D. Andriot, *Open problems on classical de Sitter solutions*, *Fortsch. Phys.* **67** (2019) 1900026, [1902.10093].

[87] C. Caviezel, P. Koerber, S. Kors, D. Lust, T. Wrase and M. Zagermann, *On the Cosmology of Type IIA Compactifications on SU(3)-structure Manifolds*, *JHEP* **04** (2009) 010, [0812.3551].

[88] R. Flauger, S. Paban, D. Robbins and T. Wrase, *Searching for slow-roll moduli inflation in massive type IIA supergravity with metric fluxes*, *Phys. Rev. D* **79** (2009) 086011, [0812.3886].

[89] B. S. Acharya, F. Benini and R. Valandro, *Fixing moduli in exact type IIA flux vacua*, *JHEP* **02** (2007) 018, [hep-th/0607223].

[90] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, *Yang-Mills Theories with Local Supersymmetry: Lagrangian, Transformation Laws and SuperHiggs Effect*, *Nucl. Phys. B* **212** (1983) 413.

[91] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 5, 2012.

[92] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four folds*, *Nucl. Phys. B* **584** (2000) 69–108, [hep-th/9906070].

[93] T. R. Taylor and C. Vafa, *R R flux on Calabi-Yau and partial supersymmetry breaking*, *Phys. Lett. B* **474** (2000) 130–137, [hep-th/9912152].

[94] S. Kachru, R. Kallosh, A. D. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, *Phys. Rev. D* **68** (2003) 046005, [hep-th/0301240].

[95] C. P. Burgess, R. Kallosh and F. Quevedo, *De Sitter string vacua from supersymmetric D terms*, *JHEP* **10** (2003) 056, [hep-th/0309187].

[96] V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, *Systematics of moduli stabilisation in Calabi-Yau flux compactifications*, *JHEP* **03** (2005) 007, [hep-th/0502058].

[97] J. P. Conlon, F. Quevedo and K. Suruliz, *Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking*, *JHEP* **08** (2005) 007, [hep-th/0505076].

[98] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys. B* **359** (1991) 21–74.

[99] P. Candelas, X. De La Ossa, A. Font, S. H. Katz and D. R. Morrison, *Mirror symmetry for two parameter models. 1.*, *Nucl. Phys. B* **416** (1994) 481–538, [hep-th/9308083].

[100] P. Candelas, A. Font, S. H. Katz and D. R. Morrison, *Mirror symmetry for two parameter models. 2.*, *Nucl. Phys. B* **429** (1994) 626–674, [hep-th/9403187].

[101] M. T. Grisaru, A. E. M. van de Ven and D. Zanon, *Four Loop Divergences for the N=1 Supersymmetric Nonlinear Sigma Model in Two-Dimensions*, *Nucl. Phys. B* **277** (1986) 409–428.

[102] I. Antoniadis, S. Ferrara, R. Minasian and K. S. Narain, *R**4 couplings in M and type II theories on Calabi-Yau spaces*, *Nucl. Phys. B* **507** (1997) 571–588, [hep-th/9707013].

[103] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, *String loop corrections to the universal hypermultiplet*, *Class. Quant. Grav.* **20** (2003) 5079–5102, [hep-th/0307268].

[104] I. Antoniadis, R. Minasian and P. Vanhove, *Noncompact Calabi-Yau manifolds and localized gravity*, *Nucl. Phys. B* **648** (2003) 69–93, [hep-th/0209030].

[105] K. Becker, M. Becker, M. Haack and J. Louis, *Supersymmetry breaking and alpha-prime corrections to flux induced potentials*, *JHEP* **06** (2002) 060, [hep-th/0204254].

[106] I. Antoniadis and C. Bachas, *Branes and the gauge hierarchy*, *Phys. Lett. B* **450** (1999) 83–91, [hep-th/9812093].

[107] I. Antoniadis, Y. Chen and G. K. Leontaris, *Logarithmic loop corrections, moduli stabilisation and de Sitter vacua in string theory*, *JHEP* **01** (2020) 149, [1909.10525].

[108] I. Antoniadis, Y. Chen and G. K. Leontaris, *Perturbative moduli stabilisation in type IIB/F-theory framework*, *Eur. Phys. J. C* **78** (2018) 766, [1803.08941].

[109] M. Larosa and G. Pradisi, *Magnetized four-dimensional Z(2) x Z(2) orientifolds*, *Nucl. Phys. B* **667** (2003) 261–309, [hep-th/0305224].

[110] G. Aldazabal, A. Font, L. E. Ibanez and G. Violero, *D = 4, N=1, type IIB orientifolds*, *Nucl. Phys. B* **536** (1998) 29–68, [hep-th/9804026].

[111] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys. B* **376** (1992) 365–386.

[112] M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, *Phys. Lett. B* **247** (1990) 517–524.

[113] A. Abouelsaood, C. G. Callan, Jr., C. R. Nappi and S. A. Yost, *Open Strings in Background Gauge Fields*, *Nucl. Phys. B* **280** (1987) 599–624.

[114] C. Bachas, *A Way to break supersymmetry*, hep-th/9503030.

[115] C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, *Type I strings on magnetized orbifolds and brane transmutation*, *Phys. Lett. B* **489** (2000) 223–232, [hep-th/0007090].

[116] A. H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, *Phys. Rev. D* **23** (1981) 347–356.

[117] A. D. Linde, *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems*, *Phys. Lett. B* **108** (1982) 389–393.

[118] A. Albrecht and P. J. Steinhardt, *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, *Phys. Rev. Lett.* **48** (1982) 1220–1223.

[119] D. Salopek and J. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, *Phys. Rev. D* **42** (1990) 3936–3962.

[120] A. R. Liddle and D. Lyth, *Cosmological inflation and large scale structure*. Cambridge University Press, 9, 2000.

[121] M. Cicoli, S. De Alwis, A. Maharana, F. Muia and F. Quevedo, *De Sitter vs Quintessence in String Theory*, *Fortsch. Phys.* **67** (2019) 1800079, [1808.08967].

[122] I. Bena, M. Grana and N. Halmagyi, *On the Existence of Meta-stable Vacua in Klebanov-Strassler*, *JHEP* **09** (2010) 087, [0912.3519].

[123] I. Bena, G. Giecold, M. Grana, N. Halmagyi and S. Massai, *The backreaction of anti-D3 branes on the Klebanov-Strassler geometry*, *JHEP* **06** (2013) 060, [1106.6165].

[124] G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, *De Sitter Space and the Swampland*, 1806.08362.

[125] H. Ooguri, E. Palti, G. Shiu and C. Vafa, *Distance and de Sitter Conjectures on the Swampland*, *Phys. Lett. B* **788** (2019) 180–184, [1810.05506].

[126] P. Agrawal, G. Obied, P. J. Steinhardt and C. Vafa, *On the Cosmological Implications of the String Swampland*, *Phys. Lett. B* **784** (2018) 271–276, [1806.09718].

[127] F. Quevedo, *Lectures on string/brane cosmology*, *Class. Quant. Grav.* **19** (2002) 5721–5779, [hep-th/0210292].

[128] L. McAllister and E. Silverstein, *String Cosmology: A Review*, *Gen. Rel. Grav.* **40** (2008) 565–605, [0710.2951].

[129] D. Baumann and L. McAllister, *Inflation and String Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 5, 2015, 10.1017/CBO9781316105733.

[130] H. Năstase, *Cosmology and String Theory*, vol. 197 of *Fundamental Theories of Physics*. Springer, 2019, 10.1007/978-3-030-15077-8.

[131] R. H. Brandenberger and C. Vafa, *Superstrings in the Early Universe*, *Nucl. Phys. B* **316** (1989) 391–410.

[132] P. Binetruy, C. Deffayet and D. Langlois, *Nonconventional cosmology from a brane universe*, *Nucl. Phys. B* **565** (2000) 269–287, [hep-th/9905012].

[133] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, *Brane cosmological evolution in a bulk with cosmological constant*, *Phys. Lett. B* **477** (2000) 285–291, [hep-th/9910219].

[134] J. M. Cline, C. Grojean and G. Servant, *Cosmological expansion in the presence of extra dimensions*, *Phys. Rev. Lett.* **83** (1999) 4245, [hep-ph/9906523].

[135] C. Csaki, M. Graesser, L. Randall and J. Terning, *Cosmology of brane models with radion stabilization*, *Phys. Rev. D* **62** (2000) 045015, [hep-ph/9911406].

[136] C. P. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh and R.-J. Zhang, *The Inflationary brane anti-brane universe*, *JHEP* **07** (2001) 047, [hep-th/0105204].

[137] J. Garcia-Bellido, R. Rabadan and F. Zamora, *Inflationary scenarios from branes at angles*, *JHEP* **01** (2002) 036, [hep-th/0112147].

[138] K. Dasgupta, C. Herdeiro, S. Hirano and R. Kallosh, *D3 / D7 inflationary model and M theory*, *Phys. Rev. D* **65** (2002) 126002, [hep-th/0203019].

[139] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, *Phys. Rev. D* **64** (2001) 123522, [hep-th/0103239].

[140] P. J. Steinhardt, N. Turok and N. Turok, *A Cyclic model of the universe*, *Science* **296** (2002) 1436–1439, [hep-th/0111030].

[141] E. I. Buchbinder, J. Khoury and B. A. Ovrut, *New Ekpyrotic cosmology*, *Phys. Rev. D* **76** (2007) 123503, [hep-th/0702154].

[142] S. Kachru, R. Kallosh, A. D. Linde, J. M. Maldacena, L. P. McAllister and S. P. Trivedi, *Towards inflation in string theory*, *JCAP* **10** (2003) 013, [hep-th/0308055].

[143] J. P. Conlon and F. Quevedo, *Kahler moduli inflation*, *JHEP* **01** (2006) 146, [hep-th/0509012].

[144] S. R. Coleman, *The Fate of the False Vacuum. 1. Semiclassical Theory*, *Phys. Rev. D* **15** (1977) 2929–2936.

[145] S. R. Coleman and F. De Luccia, *Gravitational Effects on and of Vacuum Decay*, *Phys. Rev. D* **21** (1980) 3305.

[146] S. Hawking and I. Moss, *Supercooled Phase Transitions in the Very Early Universe*, *Adv. Ser. Astrophys. Cosmol.* **3** (1987) 154–157.

[147] L. Boubekeur and D. H. Lyth, *Hilltop inflation*, *JCAP* **07** (2005) 010, [hep-ph/0502047].

[148] L. Abbott, E. Farhi and M. B. Wise, *Particle Production in the New Inflationary Cosmology*, *Phys. Lett. B* **117** (1982) 29.

[149] A. Albrecht, P. J. Steinhardt, M. S. Turner and F. Wilczek, *Reheating an Inflationary Universe*, *Phys. Rev. Lett.* **48** (1982) 1437.

[150] M. S. Turner, *Coherent Scalar Field Oscillations in an Expanding Universe*, *Phys. Rev. D* **28** (1983) 1243.

[151] T. Markkanen, A. Rajantie and S. Stopyra, *Cosmological Aspects of Higgs Vacuum Metastability*, *Front. Astron. Space Sci.* **5** (2018) 40, [1809.06923].

[152] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2, 2011, 10.1017/CBO9780511524646.

[153] K.-M. Lee and E. J. Weinberg, *Tunneling without barriers*, *Nucl. Phys. B* **267** (1986) 181–202.

[154] L. G. Jensen and P. J. Steinhardt, *Bubble Nucleation and the Coleman-Weinberg Model*, *Nucl. Phys. B* **237** (1984) 176–188.

[155] V. Balek and M. Demetrian, *A Criterion for bubble formation in de Sitter universe*, *Phys. Rev. D* **69** (2004) 063518, [gr-qc/0311040].

[156] A. D. Linde, *Hybrid inflation*, *Phys. Rev. D* **49** (1994) 748–754, [astro-ph/9307002].

[157] D. Lüst, S. Reffert and S. Stieberger, *Flux-induced soft supersymmetry breaking in chiral type IIB orientifolds with D3 / D7-branes*, *Nucl. Phys. B* **706** (2005) 3–52, [hep-th/0406092].

[158] P. G. Camara, L. E. Ibanez and A. M. Uranga, *Flux-induced SUSY-breaking soft terms on D7-D3 brane systems*, *Nucl. Phys. B* **708** (2005) 268–316, [hep-th/0408036].

[159] V. S. Kaplunovsky and J. Louis, *Model independent analysis of soft terms in effective supergravity and in string theory*, *Phys. Lett. B* **306** (1993) 269–275, [hep-th/9303040].

[160] A. Brignole, L. E. Ibàñez and C. Munoz, *Soft supersymmetry breaking terms from supergravity and superstring models*, *Adv. Ser. Direct. High Energy Phys.* **18** (1998) 125–148, [hep-ph/9707209].

[161] L. E. Ibàñez, C. Munoz and S. Rigolin, *Aspect of type I string phenomenology*, *Nucl. Phys. B* **553** (1999) 43–80, [hep-ph/9812397].

[162] D. Lüst, P. Mayr, R. Richter and S. Stieberger, *Scattering of gauge, matter, and moduli fields from intersecting branes*, *Nucl. Phys. B* **696** (2004) 205–250, [hep-th/0404134].

[163] A. Font and L. E. Ibanez, *SUSY-breaking soft terms in a MSSM magnetized D7-brane model*, *JHEP* **03** (2005) 040, [hep-th/0412150].

[164] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys. B* **766** (2007) 21–33, [hep-th/0605264].