The growth rate of the number of periodic orbits for vector fields and uniform Diophantine approximation of beta-transformations

Wanlou Wu

École Doctorale MSTIC

Laboratoire d'Analyse et de Mathématiques Appliquées

Thèse

Présentée pour l'obtention du grade de Docteur

DE L'UNIVERSITÉ PARIS-EST

par

Wanlou WU

Taux de croissance du nombre d'orbites périodiques pour les champs de vecteurs et approximation diophantienne uniforme des bêta-transformations

---

Spécialité : Mathématiques

Soutenue le 31 octobre 2019 devant le jury composé de :

M. Jérôme BUZZI	Université Paris-Sud	Rapporteur
M. Yongluo CAO	Université Soochow	Président
M. Lingmin LIAO	Université Paris-Est Créteil	Directeur de thèse
M. Tomas PERSSON	Université de Lund	Rapporteur
M. Michał RAMS	Instytut Matematyczny PAN	Examinateur
Mme. Yun WANG	Université Soochow	Examinateur
M. Dawei YANG	Université Soochow	Directeur de thèse

Laboratoire d'Analyse et de Mathématiques Appliquées
Paris-Est Créteil
61, avenue du Général de Gaulle
94460 Créteil, FRANCE

School of Mathematical Sciences
Soochow University
1, Shizi Street
215006 Suzhou City, Jiangsu Province, CHINA

Acknowledgments

This thesis is completed under the direction of Professors Lingmin Liao and Dawei Yang, for whom I want to express my deepest gratitude. Apart from their guidance in mathematics, correction of my manuscripts and financial support for my traveling to conferences, I also want to thank their patience, humility and encouragement. During the doctoral period, Mr. Liao and Mr. Yang made a strict request to me. They also patiently urged me to learn, helped me to improve my learning methods and teaching methods. At the same time, they provided me with a lot of opportunities for academic exchange and learning, encouraged me to communicate with other scholars, which improve my ability to study and research independently. In life, they provided me with many very valuable suggestions.

I also would like to express my deep gratitude to the referees, Professor Jérôme Buzzi and Professor Tomas Persson, for their patience of my poor writings. Moreover, I want to thank the following strong group of mathematicians: Jérôme Buzzi, Yongluo Cao, Lingmin Liao, Tomas Persson, Michał Rams, Yun Wang, and Dawei Yang, for their interests upon joining the committee of my oral defense.

In my early graduate years in Suzhou, I would like to thank Professor Yongluo Cao and Professor Yong Zhang for their guidance on how to give lectures and do study, which has a great influence on me. I learned the basic analysis tools from both of them, which are fundamental for this thesis. For my life in China, there are also many other professors and friends, who gave me concern and help in my study and life. I would like to take this opportunity to express my acknowledgments: Huyi Hu, Gang Liao, Jiansong Liu, Zeya Mi, Wenxin Qin, Congcong Qu, Enhui Shi, Juan Wang, Yun Zhao, Rui Zou, Yuntao Zang.

During the preparation of this thesis, I benefit a lot from "Laboratoire d'Analyse et de Mathématiques Appliquées"(LAMA). Especially, I want to thank our director Stéphane Sabourau and our secretary Sonia Boufala, for their effective works. Besides, I also want to thank Sylvie Cash, the secretary of our graduate school, for her kind help. I also want to take this opportunity to thank my colleagues in LAMA for their sharing different cultures and mathematics with me: Mohd Ali Khameini Ahmad, Timothee Crin-Barat, Zikang Dong, Edouard Daviaud, Alaa Marachli, Guillaume Saës, Kunkun Song, Jin Tan, Lixuan Zheng.

Lastly, I want to thank my parents and my wife, Dongsheng Tang. While completing her work, my wife is responsible for all the affairs of the family. Without her help, understanding, tolerance and support, I believe that the life of doctoral student in these years will be very different.

List of publications

1. Wu Wanlou, Yang Dawei, Zhang Yong. On the growth rate of periodic orbits for vector fields. Advances in Mathematics, 346:170-193, 2019, https://doi.org/10.1016/j.aim.2019.01.048.

2. Wu Wanlou, Li Bo. On the F-expanding of Homoclinic class. Journal of Applied Analysis and Computation, 9(3):1083-1101, 2019, DOI:10.11948/2156-907X.20180287.

3. Wu Wanlou, Liu Jiangsong. Approximation property on entropies for surface diffeomorphisms. Proceedings of the American Mathematical Society,148(1):223-233, 2020, https://doi.org/10.1090/proc/14670.

4. Wu Wanlou, Zheng Lixuan, Dimension Theory of Diophantine approximation related to Beta-transformations, preprint arXiv:1907.13031, 2019.

摘要

不变量、轨道的渐近形态是动力系统的两个重要研究课题. 不变量主要包括拓扑不变量和渐近不变量. 轨道的渐近形态包含两层含义: 一是轨道的极限集的构成和大小 (Lebesgue 测度、Hausdorff 维数), 二是轨道趋近于其极限集的方式. 本文对动力系统中的这两个主题进行了研究: 第一, 我们研究了连续时间的动力系统的周期轨道条数增长率这一渐近不变量与系统的拓扑熵这一拓扑不变量之间的关系; 第二, 在某一类给定的离散动力系统 (β-动力系统)中, 我们刻画了点的轨道按照某种给定速率逼近给定点的集合的大小.

我们第一个方面的研究建立了 C^1-通有的向量场的周期轨道条数的增长率与其拓扑熵之间的关系：周期轨道条数的增长率大于或等于拓扑熵. 这个结论将 Katok 的关于 $C^{1+\alpha}(\alpha > 0)$ 曲面微分同胚的周期轨道条数的增长率与其拓扑熵之间的关系推广到了任意维的 C^1-通有向量场. 相比离散时间的动力系统, 我们需要处理奇点和修剪流所带来的困难. 通过估计周期轨道的周期与回复轨道的回复时间的差别, 我们对廖山涛追踪引理这一基本工具, 给出了一点改进.

在第二个方面的研究中, 我们考虑经典的定义在 $[0,1]$ 上的 β-变换 $(\beta > 1)$ T_β:

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor,$$

其中 $\lfloor\cdot\rfloor$ 表示取整函数. 对于定义在自然数集上的两个实的正函数 ψ_1 和 ψ_2, 用 $\mathcal{L}(\psi_1)$ 表示区间 $[0,1]$ 中的所有满足性质: 存在无穷多个正整数 $n \in \mathbb{N}$, 使得 $T_\beta^n x < \psi_1(n)$ 成立的点 x 组成的集合. 用 $\mathcal{U}(\psi_2)$ 表示区间 $[0,1]$ 中的所有满足性质: 对于任意充分大的 N, 不等式 $T_\beta^n x < \psi_2(N)$ 有一个小于或等于 N 的解 n 的点 x 组成的集合. 假设 $\underline{v}_1$ (相应的, $\overline{v}_1$) 是以 β 为底的 $\psi_1(n)$ 的对数的相反数除以 n 的下极限 (相应的, 上极限). 符号 $\underline{v}_2$ 和 $\overline{v}_2$ 表示用函数 ψ_2 代替函数 ψ_1 所得到的下极限与上极限. 从 Philipp 的结果来看, 如果级数 $\sum \psi_1(n)$ (相应的, 级数 $\sum \psi_2(n)$) 收敛, 那么集合 $\mathcal{L}(\psi_1)$ (相应的, $\mathcal{U}(\psi_2)$) 的 Lebesgue 测度为零. 我们计算这些集合的 Hausdorff 维数. 集合 $\mathcal{L}(\psi_1)$ 的 Hausdorff 维数已经被找到并且维度公式完全由 $\underline{v}_1$ 决定. 我们研究了集合 $\mathcal{L}(\psi_1)$ 和 $\mathcal{U}(\psi_2)$ 的交集的 Lebesgue 测度和 Hausdorff 维数, 并使用 $\underline{v}_1$, $\overline{v}_1$, $\underline{v}_2$ 和 $\overline{v}_2$ 给出了刻画. 作为推论, 我们得到了集合 $\mathcal{U}(\psi_2)$ 的 Lebesgue 测度和 Hausdorff 维数的严格估计, 并使用 $\underline{v}_2$ 和 $\overline{v}_2$ 给出了刻画. Bugeaud 和 Liao 的结果只考虑了 ψ_1 和 ψ_2 都是指数函数的特殊情况, 我们的结果将 ψ_1 和 ψ_2 推广到一般的正函数.

关键词: 向量场; 周期轨道; 增长率; 丢番图逼近; beta-展式; Hausdorff 维数.

作　　者: 吴 万 楼

指导老师: 廖 灵 敏(教 授)

杨 大 伟(教 授)

Abstract

In this thesis, we study two problems in the dynamical systems. One concerns the relationship between the number of periodic orbits and the topological entropy of the vector fields. The other is on the size of the set of points whose orbits approximate a given point by some given speed in beta-dynamical systems.

For the first problem, we prove that for C^1-generic vector fields : the growth rate of the number of periodic orbits is greater than or equal to the topological entropy. This extends a classical result of Katok for $C^{1+\alpha}(\alpha > 0)$ surface diffeomorphisms to C^1-generic vector fields of any dimension. Comparing with time-discrete dynamical systems, the main difficulty comes from the existence of singularities and the shear of the flow. By estimating the difference between the period of the periodic orbit and the recurrent time of the recurrent orbit, we improve Liao's shadowing lemma which is a basic tool to deal with the difficulty caused by singularities.

For the second one, we consider the classical beta-transformation T_β over the unit interval $[0,1]$. For two positive functions ψ_1 and ψ_2 defined on the set of natural numbers, let $\mathcal{L}(\psi_1)$ be the set of points x in the interval $[0,1]$ such that $T_\beta^n x < \psi_1(n)$ for infinite many n, and $\mathcal{U}(\psi_2)$ be the set of points x in the interval $[0,1]$ such that for any N large enough, the inequality $T_\beta^n x < \psi_2(N)$ has a solution n less than or equal to N. Let $\underline{v}_1$ (respectively, $\overline{v}_1$) be the lower limit (respectively, upper limit) of the inverse of the logarithm in base beta of $\psi_1(n)$ divided by n. The notations $\underline{v}_2$ and $\overline{v}_2$ are defined by replacing ψ_1 with ψ_2. From the result of Philipp, if the series $\sum \psi_1(n)$ (respectively, the series $\sum \psi_2(n)$) converges, then the Lebesgue measure of $\mathcal{L}(\psi_1)$ (respectively, $\mathcal{U}(\psi_2)$) is zero. We calculate the Hausdorff dimension of these sets. The Hausdorff dimension of $\mathcal{L}(\psi_1)$ has already been found and the dimensional formula is totally determined by $\underline{v}_1$. We characterize the Lebesgue measure and the Hausdorff dimension of the intersection of $\mathcal{L}(\psi_1)$ and $\mathcal{U}(\psi_2)$, using $\underline{v}_1$, $\overline{v}_1$, $\underline{v}_2$ and $\overline{v}_2$. As a corollary, sharp estimations of the Hausdorff dimension of the set $\mathcal{U}(\psi_2)$ are obtained by using $\underline{v}_2$ and $\overline{v}_2$. Our results generalize those of Bugeaud and Liao where only quite special ψ_1 and ψ_2 have been considered.

Key Words: vector fields; periodic orbits; growth rate; Diophantine approximation; β-transformations; Hausdorff dimension.

Résumé

Dans cette thèse, nous étudions deux problèmes en systèmes dynamiques. L'un concerne la relation entre le nombre d'orbites périodiques et l'entropie topologique des champs de vecteurs. L'autre concerne la taille de l'ensemble des points dont les orbites se rapprochent d'un point donné avec une vitesse donnée dans les bêta-systèmes dynamiques.

Pour le premier problème, nous démontrons que pour les champs de vecteurs C^1 génériques : le taux de croissance du nombre d'orbites périodiques est supérieur ou égal à l'entropie topologique. Ceci étend un résultat classique de Katok pour des $C^{1+\alpha}(\alpha > 0)$ difféomorphismes de surface à des champs de vecteurs C^1 génériques de toute dimension. En comparant avec la systèmes dynamiques discrets dans le temps, la difficulté principale provient de l'existence de singularités et du cisaillement du flux. En estimant la différence entre la période de l'orbite périodique et le temps récurrent de l'orbite récurrente, nous améliorons le lemme d'ombrage de Liao, qui est un outil fondamental pour traiter la difficulté causée par les singularités.

Pour le second, nous considérons la bêta-transformation T_β sur l'intervalle $[0,1]$. Pour deux fonctions positives réelles ψ_1 et ψ_2 définies sur l'ensemble des entiers natures, notons $\mathcal{L}(\psi_1)$ l'ensemble des points x dans l'intervalle $[0,1]$ tels que $T_\beta^n x < \psi_1(n)$ pour une infinité de n, et $\mathcal{U}(\psi_2)$ l'ensemble des points x dans l'intervalle $[0,1]$ tels que pour N suffisamment grand, l'inégalité $T_\beta^n x < \psi_2(N)$ admet une solution n inférieure ou égale à N. Soit $\underline{v}_1$ (respectivement, $\overline{v}_1$) la limite inférieure (respectivement, limite supérieure) de l'inverse du logarithme en base β de $\psi_1(n)$ divisé par n. Les notations $\underline{v}_2$ et $\overline{v}_2$ sont définies en remplaçant ψ_1 par ψ_2. D'après le résultat de Philipp, si la série de $\sum \psi_1(n)$ (respectivement, $\sum \psi_2(n)$) converge, la mesure de Lebesgue de $\mathcal{L}(\psi_1)$ (respectivement, $\mathcal{U}(\psi_2)$) est zéro. Nous calculons la dimension de Hausdorff de ces ensembles. La dimension de Hausdorff de $\mathcal{L}(\psi_1)$ a été bien trouvée et la formule dimensionnelle est uniquement déterminée par $\underline{v}_1$. Nous étudions la mesure de Lebesgue et la dimension de Hausdorff de l'ensemble de l'intersection de $\mathcal{L}(\psi_1)$ et $\mathcal{U}(\psi_2)$, et en donnons des caractérisations en utilisant $\underline{v}_1$, $\overline{v}_1$, $\underline{v}_2$ et $\overline{v}_2$. En corollaire, des estimations précises de la dimension de Hausdorff de l'ensemble $\mathcal{U}(\psi_2)$ sont obtenues à l'aide de $\underline{v}_2$ et de $\overline{v}_2$. Nos résultats généralisent ceux de Bugeaud et de Liao où seuls les exemple de ψ_1 et ψ_2 très spéciaux ont été pris en compte.

Mots clés: champs de vecteurs; orbites périodiques; taux de croissance; approximation Diophantienne; β-transformations; dimension de Hausdorff.

Contents

1 Aperçu des travaux

Cette thèse porte sur deux sujets principaux en systèmes dynamiques. D'une part, l'étude détaillée des orbites périodiques est une partie importante de la théorie qualitative des systèmes dynamiques et des difféomorphismes. Beaucoup de travaux sont consacrés à démontrer, dans des cas spécifiques tels que le problème des trois corps, qu'il existe une infinité d'orbites périodiques [1]. Puisque les données périodiques et l'entropie topologique sont les deux invariants les plus importants des systèmes dynamiques et le taux de croissance du nombre d'orbites périodiques peut reflecter la complexité de ce système dynamique [2–4], nous nous intéresserons à la relation entre le taux de croissance du nombre d'orbites périodiques et l'entropie topologique d'un système dynamique à temps continu. D'autre part, nous nous intéressons à la dimension de Hausdorff d'un ensemble de mesure de Lebesgue zéro. Considérons les β-transformations T_β sur l'intervalle $[0,1)$. Étant donnée deux fonctions positives $\psi_1,\ \psi_2 : \mathbb{N} \to \mathbb{R}^+$, définissons

$$\mathcal{L}(\psi_1) := \left\{x \in [0,1) : T_\beta^n x < \psi_1(n),\ \text{pour une infinité d'entiers } n \in \mathbb{N}\right\},$$

$$\mathcal{U}(\psi_2) := \left\{x \in [0,1) : \forall\ N \gg 1,\ \exists\ n \in [0,N],\ t.\ q.\ T_\beta^n x < \psi_2(N)\right\},$$

où $\gg$ signifie assez grand. Nous estimons la dimension de Hausdorff de $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$, et de $\mathcal{U}(\psi_2)$.

1.1 Taux de croissance du nombre d'orbites périodiques

Dans l'étude des systèmes dynamiques, il est très attrayant d'identifier les phénomènes spécifiques importants associés au comportement asymptotique de systèmes dynamiques lisses. Les invariants topologiques asymptotiques et invariants topologiques ont été étudiés par de nombreux mathématiciens célèbres tels que Kolmogorov, Sinai. Il existe trois grandes classes d'invariants asymptotiques :

- types de récurrence,
- croissance du nombre d'orbites de divers types et de la complexité des familles d'orbites,
- distribution asymptotique et comportement statistique d'orbites.

Les deux premières classes sont de nature purement topologique et la dernière est naturellement liée à la théorie ergodique.

Soit M une variété riemannienne lisse, compacte et sans bord. Soit $f : M \to M$ une transformation préservant une mesure sur $(M, \mathcal{A})$. L'ensemble de toutes les mesures invariantes et l'ensemble de toutes les mesures ergodiques sont respectivement notés par $\mathcal{M}(f)$ et $\mathcal{E}(f)$. Étant donné deux partitions finies mesurables de M,

$$\xi = \{A_1, A_2, \cdots, A_k\}, \quad \eta = \{B_1, B_2, \cdots, B_l\}.$$

On note $\xi \vee \eta$ le *joint* de ξ et de η, c'est-à-dire la partition mesurable formée des intersections

$$\xi \vee \eta := \{A_i \cap B_j : 1 \leq i \leq k,\ 1 \leq j \leq l\}.$$

Pour un entier $n \geq 1$, définissons le join $\bigvee_{i=0}^{n-1} f^{-i}(\xi)$ par

$$\bigvee_{i=0}^{n-1} f^{-i}(\xi) := \xi \vee f^{-1}(\xi) \vee \cdots \vee f^{-n+1}(\xi).$$

Soit $\mu \in \mathcal{M}(f)$ une mesure invariante et $\xi = \{A_1, A_2, \cdots, A_k\}$ une partition mesurable finie. L'entropie métrique de la partition mesurable ξ est définie par

$$H_\mu(\xi) := -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

La limite

$$\lim_{n\to\infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} f^{-i}(\xi) \right)$$

existe. On l'appelle l'entropie de la transformation f pour la partition ξ, et on la note $h_\mu(f, \xi)$. Le nombre

$$h_\mu(f) := \sup \{h_\mu(f, \xi) : \xi \text{ est une partition finie mesurable de } M\}$$

est appelé l'entropie de la transformation f par rapport à mesure μ.

Soit ξ un recouvrement ouvert de M. Soit $N\left(\bigvee_{i=0}^{n-1} f^{-i}(\xi)\right)$ le cardinal minimal d'un sous-recouvrement de $\bigvee_{i=0}^{n-1} f^{-i}(\xi)$. L'entropie de la transformation f par rapport à ξ est définie comme suit:

$$h(f, \xi) := \lim_{n\to\infty} \frac{1}{n} \log N \left(\bigvee_{i=0}^{n-1} f^{-i}(\xi) \right).$$

L'entropie topologique de la transformation f est

$$h_{top}(f) := \sup\{h(f, \xi) : \xi \text{ est un recouvrement ouvert de } M\}.$$

Les orbites périodiques représentent la classe d'orbites la plus distinctive. Nous définissons l'ensemble

$$P_n(f) := \{x \in M : x = f^n(x)\},$$

alors $\sharp P_n(f)$ donne le nombre total de points pour lesquels l'entier positif n est une période, pas nécessairement la plus petite période possible. Ces nombres sont des invariants topologiques. Si n est un nombre premier, alors

$$\sharp P_n(f) - \sharp P_1(f)$$

donne exactement le nombre de points périodiques avec la plus petite période n. Ainsi,

$$\frac{\sharp P_n(f) - \sharp P_1(f)}{n}$$

est le nombre d'orbites périodiques de période n. En général, la relation entre le nombre de points périodiques de période n et le nombre d'orbites périodiques de période n est compliquée. Si on note $\sharp \hat{P}_n(f)$ le nombre de points périodiques de plus petite période n, alors

$$\frac{\sharp \hat{P}_n(f)}{n}$$

est le nombre de orbites périodiques de période n. Les nombres $\sharp \hat{P}_n(f)$ sont également des invariants topologiques et peuvent être exprimés par $\sharp P_n(f)$ et inversement via une fonction de la théorie des nombres. Cependant, il est plus pratique de travailler avec $\sharp P_n(f)$ qu'avec $\sharp \hat{P}_n(f)$.

Pour décrire la croissance asymptotique du nombre de points périodiques on définit le taux de croissance exponentiel $p(f)$ de la suite $\sharp P_n(f)$:

$$p(f) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \max\{\sharp P_n(f),\ 1\}.$$

Remarquons que nous écrivons $\max\{\sharp P_n(f),\ 1\}$ au lieu de $\sharp P_n(f)$ afin d'éviter d'écrire $\log 0$. Si $p(f) = 0$, il est parfois utile de considérer le taux de croissance polynomial des points périodiques donné par

$$\overline{\lim_{n\to\infty}} \frac{1}{\log n} \log \max\{\sharp P_n(f),\ 1\}.$$

Si $p(f) < \infty$, le taux de croissance du nombre de points périodiques est au plus exponentiel. En 1965, Artin et Mazur [5, Théorème] ont prouvé qu'il existe un ensemble dense $\mathcal{D} \subset C^r(M)$, dans l'espace des difféomorphismes de classe C^r d'une variété compacte M vers elle-même avec la topologie C^r uniforme, telle que pour toute fonction $f \in \mathcal{D}$, le nombre $\sharp P_n(f)$ croît de façon exponentielle avec n. Ils ont également introduit la fonction zeta ζ_f dynamique de f définie sur le plan $\mathbb{C}$ par

$$\zeta_f(z) := \exp\left(\sum_{n=1}^{\infty} P_n(f) \frac{z^n}{n}\right).$$

La fonction ζ_f encode toutes les informations sur le nombre de points périodiques de f.

Pour beaucoup de cas, par exemple, si $f : [0,1] \to [0,1]$ est la fonction $f(x) = 2x \bmod 1$, on a

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) = h_{top}(f).$$

En 1978, Bowen [6] posait la question suivante : la propriété

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) = h_{top}(f)$$

est-elle générique en C^r-topologie? Bowen [7] a prouvé que pour un système f satisfaisant l'axiome A de toute dimension, la limite supérieure du taux de croissance du nombre de points périodiques est égale à son entropie topologique, à savoir:

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) = h_{top}(f).$$

Mais il s'agit d'une situation quelque peu particulière. Il est bien connu que les difféomorphismes satisfaisant l'axiome A ne sont pas denses dans $C^1(M)$. Katok [8, Théorème 4.3] a montré que pour un $C^{1+\alpha}(\alpha > 0)$ difféomorphisme f sur une variété compacte et pour toute mesure de probabilité borélienne f-invariante et d'exposant de Lyapunov non nul, la limite supérieure du taux de croissance du nombre de points périodiques de f est supérieure ou égale à son entropie métrique, c'est-à-dire

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) \geq h_\mu(f), \quad \text{où } \mu \text{ est une mesure hyperbolique.}$$

En particulier, si f est un $C^{1+\alpha}$ difféomorphisme d'une surface, on a

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) \geq h_{top}(f).$$

En 1999, Kaloshin [9, Théorème 6] a montré qu'en général $\sharp P_n(f)$ peut croître beaucoup plus rapidement que l'entropie topologique de f.

En comparant le systèmes dynamiques en temps discret avec les systèmes dynamiques en temps continu, nous souhaitons traiter les systèmes dynamiques générées par les champs de vecteurs comme un système dynamique à temps continu. Soit $\mathcal{X}^1(M)$ l'espace de tous les champs des vecteurs C^1 sur M avec la norme C^1. Un champ de vecteurs $X \in \mathcal{X}^1(M)$ génère un flot

$$\varphi_t = \varphi_t^X.$$

Pour le flot, il faut compter les orbites périodiques au lieu des points périodiques. On peut le faire de deux manières différentes. Il serait plus proche du cas à temps discret de compter les orbites périodiques pondérées par leur longueurs, ce qui revient à compter les points périodiques dans le cas du temps discret. D'autre part, on peut compter le nombre d'orbites périodiques sans pondération par leur longueurs. Si le nombre d'orbites périodiques augmente de manière exponentielle, les deux manières sont équivalentes. Étant donné que la plupart des orbites d'une longueur donnée par T auront une longueur proche de T, les taux de croissance du nombre d'orbites périodique obtenu par ces deux méthodes sont identiques.

L'une des difficultés principales est la présence de singularités. Les flots avec des singularités ont une dynamique riche et compliquée comme les *attracteur de Lorenz* dans [10], [11, pp.368–381]. Aux singularités, on ne peut pas définir le *flot de Poincaré linéaire* (voir Définition 3.1). Par conséquent, nous perdons certaines propriétés de compacité. Même s'il n'y a pas de singularités, nous ne pouvons pas utiliser la théorie habituelle de Pesin comme dans Lian et Young [12] puisque le champ de vecteurs est seulement C^1. De plus, on peut avoir un “cisaillement”des flot. Ceci est crucial dans ce travail car nous auront essentiellement besoin de contrôler les périodes. Soit

$$[x] := \{y \in M : y \in \mathrm{Orb}(x)\},$$

où $\mathrm{Orb}(x) := \{\varphi_t(x) : t \in \mathbb{R}\}$. Définissons l'ensemble

$$P_T(X) := \{[x] \subset M : 0 < \pi(x) \leq T\},$$

où $\pi(x)$ est la période minimale de x. Soit

$$\sharp P_T(X) := \sum_{[x] \in P_T(X)} \pi(x).$$

Nous obtenons le résultat suivant.

Théorème A *Il y a un ensemble G_δ dense $\mathcal{R} \subset \mathcal{X}^1(M)$ tel que pour tout $X \in \mathcal{R}$, on a*

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) \geq h_{top}(X).$$

1.2 Approximation diophantienne

L'histoire de l'approximation diophantienne est assez ancienne: elle inclut les premières estimations de π, les calculs relatifs aux études astronomiques et la théorie du developpement en fraction continue. L'un des objectifs principaux de l'approximation diophantienne est l'étude de l'approximation des nombres réels par des nombres rationnels. Le problème de savoir à quelle vitesse un nombre irrationnel donné peut-il être approché par des nombres rationnels p/q de dénominateur q inférieur à un entier positif fixé q_0 a été largement traité par les mathématiciens. En 1842, Dirichlet [13] prouva un théorème cèlèbre en approximation diophantienne.

Théorème de Dirichlet *Soit θ, Q deux nombres réels avec $Q \geq 1$, il existe un entier n avec $1 \leq n \leq Q$ tel que*

$$\|n\theta\| < Q^{-1},$$

où $\|\xi\|$ est la distance entre ξ et le nombre entier le plus proche.

Le théorème de Dirichlet est appelé un *théorème d'approximation uniforme* dans [14, p.p.2]. Une forme faible du théorème de Dirichlet, appelée *théorème d'approximation asymptotique* dans [14, p.p.2], souvent citée comme un corollaire du théorème de Dirichlet dans la littérature, existe dans le livre de Legendre [15, 1808, p.p.18-19] (en utilisant les propriétés des fractaions continues): pour tout nombre réel θ, il existe une infinité d'entiers n tels que

$$\|n\theta\| < n^{-1}.$$

Le premier exige que les occurrences de telles approximations ne soient pas trop lacunaire, alors que ce dernier ne demande qu'une infinité de solutions à une certiane inégalité. Pour le cas général, Khintchine en 1924 [16] montra que, pour une fonction positive, $\psi : \mathbb{N} \to \mathbb{R}^+$, si $x \mapsto x\psi(x)$ est décroissante, alors

$$\mathcal{L}_\psi := \{\theta \in \mathbb{R} : \|n\theta\| < \psi(n) \text{ pour une infinité d'entiers } n \in \mathbb{N}\}$$

est de mesure de Lebesgue égale à zéro si la série $\sum \psi(n)$ converge et est de mesure de Lebesgue pleine sinon. En enlevant la condition de décroissance sur ψ, des résultats similaires sont attendus (ceci constitue la conjecture de Duffin-Shaeffer [17, 18]). On renvoie [19, 20] pour les derniers progrès. Dans le cas où $\mathcal{L}_\psi$ est de mesure de Lebesgue nulle, il est tout de même naturel de tenter de calculer sa dimension de Hausdorff. Le premier résultat sur la dimension de Hausdorff de $\mathcal{L}_\psi$ remonte à Jarník Bosicovitch [21, 22]. Il a été montré que l'ensemble

$$\left\{\theta \in \mathbb{R} : \|n\theta\| < \frac{1}{n^\tau} \text{ pour une infinité d'entiers } n \in \mathbb{N}\right\}$$

a pour dimension de Hausdorff $\dfrac{2}{1+\tau}$ pour tout $\tau > 1$.

Par analogie avec la théorie de l'approximation diophantienne classique, Hill et Velani [23] ont étudié les propriétés d'approximation des orbites de systèmes dynamiques et présenté *le problème des cibles rétrécissantes*. Plus précisément, pour un système dynamique $(X, \mathcal{B}, \mu, T)$ muni d'une métrique d, pour un $x_0 \in X$ et une fonction positive $\psi : \mathbb{N} \to \mathbb{R}^+$, considérons la propriété métrique de l'ensemble des points bien *ψ-approximable par le point* x_0, c'est-à-dire, l'ensemble

$$\mathcal{L}(T, \psi, x_0) := \{x \in X : d(T^n x, x_0) < \psi(n) \text{ pour une infinité d'entiers } n \in \mathbb{N}\}.$$

Si ψ est une fonction constante égale à c et $x_0 \in supp(\mu)$, le théorème de récurrence de Poincaré implique que $\mathcal{L}(T, \psi, x_0)$ a la même μ-mesure que la boule $B(x_0, c)$. Hill et Velani [23] ont posé la question sur la taille de $\mathcal{L}(T, \psi, x_0)$ si ψ diminue avec le temps. Ils ont étudié le cas où T est une fonction rationnelle sur la sphère de Riemann $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Après, de nombreux études apparaissent par exemple, le lemme de Borel-Cantelli dynamique [24], le problème des cibles rétrécissantes [25, 26], le temps d'entrée [27], le temps de retour [28–31], le temps d'attente [32] etc.

Pour une rotation d'angle irrationnel $([0, 1], R_\alpha)$, Kim a montré [33, Proposition 2.6] que pour Lebesgue presque tout point $x \in [0, 1]$ et $\alpha \in [0, 1]$,

$$|R_\alpha^n x - x_0| < \psi(n) \quad \text{pour une infinité d'entiers } n \in \mathbb{N},$$

si et seulement si $\sum \psi(n) = \infty$. Bugeaud [34, Théorème 1], Troubetzkoy et Schmeling [35, Théorème 3.2] ont prouvé que la taille de $\mathcal{L}(R_\alpha, \psi, x_0)$ est liée au degré polynomial $\lim\limits_{n\to\infty} \dfrac{-\log \psi(n)}{\log n}$ de ψ. Plus précisément pour tout $\tau > 1$,

$$\dim_H \left(\left\{x \in [0, 1] : |R_\alpha^n x - x_0| < \frac{1}{n^\tau} \text{ pour une infinité d'entiers } n \in \mathbb{N}\right\}\right) = \frac{1}{\tau},$$

où $\dim_H$ désigne la dimension de Hausdorff d'un ensemble.

Pour le cas où T est une fonction rationnelle expansive de degré plus grand que 2 et X est l'ensemble de Julia correspondant ([23, Théorème 1, Théorème 4], [36, Théorème 1]) ainsi que le cas où T est linéaire donnée par une matrice à coéfficients entiers et X est le tore de dimension n, Hill et Velani [37, Théorème 1, Théorème 2] ont démontré que la taille de $\mathcal{L}(T, \psi, x_0)$ est relié à l'exposant $\varliminf_{n\to\infty} \dfrac{-\log_\beta \psi(n)}{n}$ de ψ.

Etant donné $\beta > 1$, la β-transformation T_β sur $[0, 1)$ est définie par

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor,$$

où $\lfloor \xi \rfloor$ est le plus grand entier inférieur ou égal à ξ. Pour le système $([0, 1), T_\beta)$, Philipp [38, Théorème 2A, B, C] a démontré que l'ensemble $\mathcal{L}(T_\beta, \psi, x_0)$ a mesure de Lebesgue nulle si et seulement si $\sum \psi(n) < \infty$. Shen et Wang [39, Théorème 1.1] ont établi le résultat suivant sur la dimension de Hausdorff de $\mathcal{L}(T_\beta, \psi, x_0)$.

Théorème SW *Pour tout $\beta > 1$ et tout $x_0 \in [0, 1]$, on a*

$$\dim_H \mathcal{L}(T_\beta, \psi, x_0) = \frac{1}{1+v}, \quad \textit{où} \quad v := \varliminf_{n\to\infty} \frac{-\log_\beta \psi(n)}{n}.$$

Parallèlement au théorème d'approximation asymptotique, il convient également d'étudier les propriétés d'approximation uniforme comme dans le théorème de Dirichlet. L'approximation diophantienne uniforme a été étudiée par Bugeaud et Liao [40] pour les β-transformations. Étant donné $x \in [0, 1)$ et $\beta > 1$, soit

$$\nu_\beta(x) := \sup \left\{ v \geq 0 : T_\beta^n x < (\beta^n)^{-v} \text{ pour une infinité d'entiers } n \in \mathbb{N} \right\},$$

$$\hat{\nu}_\beta(x) := \sup \left\{ v \geq 0 : \forall\ N \gg 1,\ T_\beta^n x < (\beta^N)^{-v} \text{ a une solution } n \in [0, N] \right\}.$$

Bugeaud et Liao [40] ont prouvé le théorème suivant.

Théorème BL *([40, Théorème 1.4]) Pour tout $v \in (0, \infty)$ et tout $\hat{v} \in (0, 1)$, si $v < \hat{v}/(1-\hat{v})$, alors l'ensemble*

$$\{x \in [0, 1] : \nu_\beta(x) = v\} \cap \{x \in [0, 1] : \hat{\nu}_\beta(x) \geq \hat{v}\}$$

est vide. Sinon on a

$$\dim_H \left(\{x \in [0, 1] : \nu_\beta(x) = v\} \cap \{x \in [0, 1] : \hat{\nu}_\beta(x) = \hat{v}\} \right) = \frac{v - \hat{v} - v\hat{v}}{(1+v)(v-\hat{v})}.$$

Les exposants ν_β et $\hat{\nu}_\beta$ ont été introduits dans [41] (voir aussi [42, Ch.7]) et sont également fortement liés à la fonction run-length (voir [43–46]). Le but de cette thèse est d'étudier les ensembles introduits dans [40] quand la fonction d'approximation $n \mapsto \beta^{-nv}$ est remplacée par une fonction positive générale. Plus précisément, étant donné deux fonctions positives $\psi_1,\ \psi_2 : \mathbb{N} \to \mathbb{R}^+$, on pose :

$$\mathcal{L}(\psi_1) := \left\{ x \in [0, 1] : T_\beta^n x < \psi_1(n) \text{ pour une infinité d'entiers } n \in \mathbb{N} \right\},$$

$$\mathcal{U}(\psi_2) := \left\{ x \in [0, 1] : \forall\ N \gg 1,\ T_\beta^n x < \psi_2(N) \text{ a une solution } n \in [0, N] \right\}.$$

On souhaite déterminer les dimensions de Hausdorff de $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$, et de $\mathcal{U}(\psi_2)$. Soit

$$\underline{v}_1 = \varliminf_{n\to\infty} \frac{-\log_\beta \psi_1(n)}{n}, \quad \overline{v}_1 = \varlimsup_{n\to\infty} \frac{-\log_\beta \psi_1(n)}{n};$$
$$\underline{v}_2 = \varliminf_{n\to\infty} \frac{-\log_\beta \psi_2(n)}{n}, \quad \overline{v}_2 = \varlimsup_{n\to\infty} \frac{-\log_\beta \psi_2(n)}{n}.$$

Si $\overline{v}_1 < 0$, par la définition de $\underline{v}_1$, il existe une suite $\{n_j\}$ de sorte que

$$\lim_{j\to\infty} \frac{-\log_\beta \psi_1(n_j)}{n_j} = \underline{v}_1.$$

Ensuite, pour $\varepsilon > 0$ assez petit, il existe un entier $j_0 \in \mathbb{N}$ tel que

$$1 < \beta^{-n_j(\underline{v}_1+\varepsilon)} \leq \psi_1(n_j), \text{ pour tout } j \geq j_0.$$

Remarquons

$$T_\beta^n x < 1 < \psi_1(n_j), \text{ pour tout } x \in [0,1) \text{ et tout } n \in \mathbb{N},$$

pour tout $x \in [0,1)$, nous avons

$$T_\beta^{n_j} x \leq 1 < \psi_1(n_j).$$

Alors

$$[0,1) \subseteq \mathcal{L}(\psi_1).$$

D'autre part, notons $\{n_i\}$ la suite telle que

$$\psi_2(n_i) > 1, \text{ pour } i = 1,2,3\cdots.$$

Alors pour tout $x \in [0,1)$ et tout entier $n \in [1, n_i]$, nous avons

$$T_\beta^n x < 1 < \psi_2(n_i).$$

Ainsi, nous pouvons remplacer $\psi_2(n)$ par la fonction

$$\widetilde{\psi}_2(n) = \begin{cases} \psi_2(n), & \text{if } n \neq n_j \\ 1, & \text{if } n = n_j \end{cases}, \ j = 1,2,\cdots$$

La taille (mesure de Lebesgue ou dimension de Hausdorff) de l'ensemble de $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ (respectivement, $\mathcal{U}(\psi_2)$) est identique à celle de l'ensemble de $\mathcal{L}(\psi_1) \cap \mathcal{U}(\widetilde{\psi}_2)$ (respectivement, $\mathcal{U}(\widetilde{\psi}_2)$). Par conséquent, dans cette thèse, nous supposons toujours que $\underline{v}_1 \geq 0$ et $\underline{v}_2 \geq 0$. Nous établissons les théorèmes suivants sur la dimension de Hausdorff des ensembles $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ et $\mathcal{U}(\psi_2)$. Théorème B donne la dimension de Hausdorff de $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ pour le cas particulier que $\underline{v}_1$, $\overline{v}_1$, $\underline{v}_2$ et $\overline{v}_2$ est 0 ou ∞.

Théorème B:

(1) *Si* $\underline{v}_1 = \overline{v}_1 = \underline{v}_2 = \overline{v}_2 = 0$, *alors*

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 1;$$

(2) *Si* $\underline{v}_2 = \infty$ *et* $0 \leq \underline{v}_1 \leq \overline{v}_1 \leq \infty$, *alors* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *est dénombrable;*

(3) *Si* $\underline{v}_1 = \infty$ *et* $0 \leq \underline{v}_2 \leq \overline{v}_2 \leq \infty$, *alors* $\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 0$.

Remrques: *Pour l'énoncé (1), l'ensemble* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *n'est pas nécessaire pour mesure de Lebesgue pleine. En fait, si la série* $\sum \psi_1(n)$ *converge, par [38, Théorème 2A, B, C],*

$$m (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 0,$$

où $m(A)$ *désigne la mesure de Lebesgue de* A*. L'ensemble* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *peut également avoir mesure de Lebesgue pleine. Par exemple, si*

$$\psi_1(n) = \psi_2(n) = \frac{\log^2 n}{\sqrt{n}},$$

selon Dmitry, Konstantoulas et Florian [47, Théorème 1.1],

$$m (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 1.$$

Pour l'énoncé (3), si $\underline{v}_2 = \infty$, $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *est dénombrable. Si* $1 < \underline{v}_2 < \infty$, *alors* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *est vide (c.f. Lemma 6.4). Si* $0 < \underline{v}_2 \leq 1$, *alors* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *n'est pas dénombrable (c.f. Proposition 6.5).*

Théorème C *Si* $\underline{v}_2 > 1$, *alors* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *est dénombrable. Si* $\overline{v}_1/(2+\overline{v}_1) \leq \underline{v}_2 \leq 1 < \overline{v}_2$, *alors on a*

$$0 \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

Si $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2 \leq \overline{v}_2 \leq 1$ *et* $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2$, *alors*

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

Si $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$ *et* $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2 \leq 1$, *alors*

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

Si $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2 \leq \overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, *alors*

$$\frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)} \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

Si $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$ *et* $\overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, *alors*

$$\frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)} \leq \dim_H\left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

Nous remarquons que les Théorèmes B et C incluent tous les cas possibles. Pour l'ensemble $\mathcal{U}(\psi_2)$, nous estimerons sa dimension de Hausdorff.

Théorème D *Si* $\underline{v}_2 > 1$, *alors* $\mathcal{U}(\psi_2)$ *est dénombrable. Si* $\underline{v}_2 \leq 1 < \overline{v}_2$, *alors*

$$0 \leq \dim_H\left(\mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

Si $\overline{v}_2 \leq 1$, *alors*

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H\left(\mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

Nous montrerons dans les exemples 6.9, 6.10, 6.11, 6.12, 6.13, 6.14 et 6.15 que les bornes supérieure et inférieure dans les Théorèmes C et D peuvent toutes être atteintes. Lorsque $\underline{v}_1 = \overline{v}_1 = 0$, nous avons le résultat sous la forme du Théorème E.

Théorème E *Supposons* $\underline{v}_1 = \overline{v}_1 = 0$. *Si* $\underline{v}_2 > 0$, *alors* $\mathcal{U}(\psi_2) \subseteq \mathcal{L}(\psi_1)$. *On a donc dans ce cas,*

$$\dim_H\left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) = \dim_H\left(\mathcal{U}(\psi_2)\right).$$

2 Introduction

In this section, we introduce the origin of the problems, the background of the research and the latest research results. The result on the growth rate of periodic orbits for vector fields is Theorem A, which is the main result of my cooperation with Dawei Yang and Yong Zhang published in *Advances in Mathematics*. While Theorems B, C, D, and E study the Diophantine approximation of the orbit of a point x under the transformation T_β. This is a joint work with Lixuan Zheng.

This thesis focuses on two main subjects. On the one hand, the detailed study of periodic orbits is an important part of the qualitative theory of dynamical systems and of diffeomorphisms. Much work is devoted to demonstrating in specific cases such as the three-body problem that infinitely many periodic orbits exist [1]. Since the periodic data and topological entropy are the most important two invariants of dynamical systems and for diffeomorphisms, the growth rate of the number of periodic points can reflect the complexity of this dynamical system [2–4], we focus on the relationship between the growth rate of periodic data and topological entropy of a continuous-time dynamical system. On the other hand, we are interested in the Hausdorff dimension of a set with zero-Lebesgue measure. Roughly speaking, for the β-dynamical system $([0,1), T_\beta)$, where $\beta > 1$, given two positive functions $\psi_1,\ \psi_2 : \mathbb{N} \to \mathbb{R}^+$, define

$$\mathcal{L}(\psi_1) := \left\{x \in [0,1) : T_\beta^n x < \psi_1(n),\ \text{for infinitely many } n \in \mathbb{N}\right\},$$

$$\mathcal{U}(\psi_2) := \left\{x \in [0,1) : \forall\ N \gg 1,\ \exists\ n \in [0,N],\ s.t.\ T_\beta^n x < \psi_2(N)\right\},$$

where $\gg$ means large enough. We estimate the Hausdorff dimension of the two sets $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ and $\mathcal{U}(\psi_2)$ completely for any $\beta > 1$ and any two positive functions $\psi_1,\ \psi_2$.

2.1 Growth rate of the number of periodic orbits

In the study of dynamical systems, to identify important specific phenomena associated with the asymptotic behavior of smooth dynamical systems is a very attracting task. Asymptotic topological invariants and topological invariants attract the attention of many famous mathematicians such as Kolmogorov, Sinai. There are three broad classes of asymptotic invariants :

- types of recurrence,
- growth of the numbers of orbits of various kinds and of the complexity of orbit families,
- asymptotic distribution and statistical behavior of orbits.

The first two classes are of a purely topological nature and the last class is naturally related to ergodic theory.

Let M be a boundary-less compact smooth Riemannian manifold, $f : M \to M$ a measure-preserving transformation on $(M, \mathcal{A})$. The set of all invariant measures and the set of all ergodic measures are denoted by $\mathcal{M}(f)$ and $\mathcal{E}(f)$, respectively. Given two finite measurable partitions of M,

$$\xi = \{A_1, A_2, \cdots, A_k\}, \quad \eta = \{B_1, B_2, \cdots, B_l\},$$

the union $\xi \vee \eta$ is defined as

$$\xi \bigvee \eta := \{A_i \cap B_j : 1 \leq i \leq k,\ 1 \leq j \leq l\}.$$

For an integer $n \geq 1$, the union $\bigvee\limits_{i=0}^{n-1} f^{-i}(\xi)$ is defined as

$$\bigvee_{i=0}^{n-1} f^{-i}(\xi) := \xi \vee f^{-1}(\xi) \vee \cdots \vee f^{-n+1}(\xi).$$

Given an invariant measure $\mu \in \mathcal{M}(f)$ and a finite measurable partition

$$\xi = \{A_1, A_2, \cdots, A_k\},$$

the metric entropy of the measurable partition ξ is defined as

$$H_\mu(\xi) := -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i).$$

The metric entropy of the transformation f with respect to ξ is defined as

$$h_\mu(f, \xi) := \lim_{n\to\infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} f^{-i}(\xi) \right).$$

The metric entropy of the transformation f with respect to μ is defined as

$$h_\mu(f) := \sup \left\{ h_\mu(f, \xi) : \xi \text{ is a finite measurable partition of } M \right\}.$$

Let ξ be an open cover of M, $N\left(\bigvee\limits_{i=0}^{n-1} f^{-i}(\xi)\right)$ be the minimal cardinality of a subcover of $\bigvee\limits_{i=0}^{n-1} f^{-i}(\xi)$. The topological entropy of the transformation f with respect to ξ is defined as

$$h(f, \xi) := \lim_{n\to\infty} \frac{1}{n} \log N \left(\bigvee_{i=0}^{n-1} f^{-i}(\xi) \right).$$

The topological entropy of the transformation f is defined as

$$h_{top}(f) := \sup\{h(f, \xi) : \xi \text{ is an open cover of } M\}.$$

Metric entropy gives a quantitative measure of the complexity of a dynamical system as seen via an invariant measure. Topological entropy was found by extracting from the same concept an invariant of topological dynamics only. The metric entropy of the union of two invariant sets is the sum of the entropies of the invariant sets, weighted by their measures, whereas for topological entropy the entropy of a union is the maximum of the entropies of the two components. Thus, the topological entropy measures the maximal dynamical complexity versus an average complexity reflected by metric entropy. Therefore, metric entropy is not greater than topological entropy and measures assigning most weight to regions of high complexity should have metric entropy close to the topological entropy. It means that the topological entropy is the supremum of the metric entropies. This is the famous variational principle.

Variational Principle ([48, p.p. 187]) If f is a continuous map on a compact metric space M, then

$$h_{top}(f) = \sup \{h_\mu(f) : \mu \in \mathcal{M}(f)\} .$$

Periodic orbits represent the most distinctive class of orbits. We define the set

$$P_n(f) := \{x \in M : x = f^n(x)\},$$

let $\sharp P_n(f)$ be the cardinal number of the set $P_n(f)$. Then $\sharp P_n(f)$ gives the total number of points for which the positive integer n is a period, not necessarily the smallest possible period. Those numbers are topological invariants. If n is a prime number, then

$$\sharp P_n(f) - \sharp P_1(f)$$

gives exactly the number of periodic points with smallest period n. Thus,

$$\frac{\sharp P_n(f) - \sharp P_1(f)}{n}$$

is the number of periodic orbits with period n. In general, the connection between the number of periodic points with period n and the number of periodic orbits with period n is complicated. If denote by $\sharp \hat{P}_n(f)$ the number of periodic points with smallest period n, then

$$\frac{\sharp \hat{P}_n(f)}{n}$$

is the number of periodic orbits with period n. The numbers $\sharp \hat{P}_n(f)$ are also topological invariants and can be expressed through $\sharp P_n(f)$ and vice versa via some number-theoretic function. However, it is more convenient to work with $\sharp P_n(f)$ than with $\sharp \hat{P}_n(f)$.

The most natural measure of asymptotic growth of the number of periodic points is the exponential growth rate $p(f)$ for the sequence $\sharp P_n(f)$:

$$p(f) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \max\{\sharp P_n(f),\ 1\}.$$

We remark that we write $\max\{\sharp P_n(f),\ 1\}$ instead of $\sharp P_n(f)$ in order to avoid taking $\log 0$. If

$$p(f) = 0,$$

it is sometimes useful to consider the polynomial growth rate of the number of periodic points given by

$$\overline{\lim_{n\to\infty}} \frac{1}{\log n} \log \max\{\sharp P_n(f),\ 1\}.$$

If

$$p(f) < \infty,$$

then the growth rate of the number of periodic points is at most exponential. In 1965, Artin and Mazur [5, Theorem] proved that there is a dense set $\mathcal{D} \subset C^r(M)$, the space of all C^r diffeomorphisms of a compact manifold M into itself with the uniform C^r-topology, such that for any diffeomorphism $f \in \mathcal{D}$, the number $\sharp P_n(f)$ grows at most exponentially with n. They also introduced the dynamical ζ_f-function of f defined on domaine complexe $\mathbb{C}$ by

$$\zeta_f(z) := \exp\left(\sum_{n=1}^{\infty} \sharp P_n(f) \frac{z^n}{n}\right).$$

One can incorporate all the information about the numbers of periodic points into the ζ_f-function of f.

Especially, for example, if $f : [0,1] \to [0,1]$ is the map $f(x) = 2x \bmod 1$, one has

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) = h_{top}(f).$$

In 1978, Bowen [6] asked the following question: is the property that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) = h_{top}(f)$$

generic with respect to the C^r-topology? For an Axiom A system f in any dimension, Bowen [7] proved that the upper limit of the growth rate of the number of periodic points is equal to its topological entropy, namely,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) = h_{top}(f).$$

But this is a somewhat special situation. It is well known that Axiom A diffeomorphisms are not dense in $C^1(M)$. Katok [8, Theorem 4.3] showed that for a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism f on a compact manifold and any f-invariant Borel probability measure with non-zero Lyapunov exponents, the upper limit of the growth rate of the number of periodic points for f is larger than or equal to its metric entropy, i.e.,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) \geq h_\mu(f), \text{ where } \mu \text{ is a hyperbolic measure.}$$

In particular, if f is a $C^{1+\alpha}$ surface diffeomorphism, then one has

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) \geq h_{top}(f).$$

For the C^1 setting, Chen [49, Main Theorem] proved that there is a dense G_δ set $\mathcal{G} \subset C^1(M)$ such that for any $f \in \mathcal{G}$, one has

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \sharp P_n(f) \geq h_{top}(f).$$

In 1999, Kaloshin [9, Theorem 6] showed that in general $\sharp P_n(f)$ can grow much faster than the topological entropy of f.

Comparing with discrete-time dynamical systems, we want to know how to measure the growth rate for continuous-time dynamical systems. The continuous-time dynamical systems usually can be generated by vector fields. For any point $x \in M$, give a vector $X(x)$ in the tangent space T_xM. Thus, the map

$$x \mapsto X(x)$$

forms a section of the tangent bundle

$$TM := \bigcup_{x\in M} T_xM$$

which is called a *vector field* on M. Let $\mathcal{X}^1(M)$ be the space of all C^1 vector fields on M with the C^1 norm. A vector field $X \in \mathcal{X}^1(M)$ generates a flow

$$\varphi_t = \varphi_t^X.$$

For flow, since the periodic points are uncountable, one should count periodic orbits instead of periodic points. One can do in two different ways. It would be closest to the discrete-time case to count periodic orbits weighted by their length, which is what counting of periodic points amount to in the discrete-time case. On the other hand, one can count just the number of periodic orbits without weighting by their lengths. If the number of periodic orbits grows exponentially, then the distinction is immaterial. Because most orbits of length up to T will have length close to T, the growth rate of the number of the periodic orbit obtained by these two methods is the same. In my thesis, we count periodic orbits weighted by their length. This is because when consider the probability measure which is supported on the periodic orbit, we should count the periodic orbit's length.

For vector fields, one of the main difficulties is the presence of singularities. Flows with singularities have rich and complicated dynamics such as the *Lorenz attractor* in [10], [11, p.p. 368–381]. At singularities, one can not define the *linear Poincaré flow* (see Definition 3.1). Hence we lose some compact properties. Even there is no singularities, we are not able to use the usual Pesin theory as in Lian and Young [12] since the vector field is only C^1. Additionally, one may have "shear" for flows. For flows, the shadowing time of periodic orbit in shadowing lemma is the reparameterization of the time of quasi-hyperbolic orbits. We must estimate the difference between those times since we have to control the periods by the nature of this work. Define

$$[x] := \{y \in M : y \in \mathrm{Orb}(x)\},$$

where

$$\mathrm{Orb}(x) := \{\varphi_t(x) : t \in \mathbb{R}\}.$$

Let

$$P_T(X) := \{[x] \subset M : 0 < \pi(x) \leq T\},$$

where

$$\pi(x) = \min\{t > 0 : \varphi_t(x) = x\}.$$

Then, add those periods of the periodic orbits whose periods is not larger than T. Denoted by

$$\sharp P_T(X) := \sum_{[x]\in P_T(X)} \pi(x).$$

We get the following result.

Theorem A. *(WU, D. Yang and Y. Zhang) There is a dense G_δ set $\mathcal{R} \subset \mathcal{X}^1(M)$ such that for any $X \in \mathcal{R}$, one has*

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) \geq h_{top}(X).$$

2.2 Diophantine approximation

The history of Diophantine approximation is quite old: it includes early estimates for π, computations related to astronomical studies and the theory of continued fraction expansion. One main goal of Diophantine approximation is the study of the approximation of real numbers by rational numbers. The problem, how closely can a given irrational number be approximated by the rational numbers p/q with denominator q no larger than a fixed positive integer q_0, has been widely concerned by mathematicians. In 1842, Dirichlet [13] proved an illustrious theorem in Diophantine approximation.
Dirichlet Theorem Given two real numbers $\theta,\ Q$ with $Q \geq 1$, there is an integer n with $1 \leq n \leq Q$ such that

$$\|n\theta\| < Q^{-1},$$

where $\|\xi\|$ denotes the distance from ξ to the nearest integer.

Dirichlet Theorem is called a *uniform approximation theorem* in [14, p.p.2]. A weak form of Dirichlet Theorem, called an *asymptotic approximation theorem* in [14, p.p. 2], which was often refered to as a corollary of Dirichlet Theorem in the litterature, has already existed in the book of Legendre [15, 1808, p.p. 18-19] (using a continued fraction fact): for any real number θ, there are infinitely many $n \in \mathbb{N}$ such that

$$\|n\theta\| < n^{-1}.$$

The former requires that occurrences of such approximations be not too lacunary, while the latter asks to only for infinitely many solutions to some inequality. For the general case, Khintchine in 1924 [16] showed that for a positive function $\psi : \mathbb{N} \to \mathbb{R}^+$, if $x \mapsto x\psi(x)$ is non-increasing, then

$$\mathcal{L}_\psi := \{\theta \in \mathbb{R} : \|n\theta\| < \psi(n),\ \text{for infinitely many } n \in \mathbb{N}\}$$

has Lebesgue measure zero if the series $\sum \psi(n)$ converges and has full Lebesgue measure otherwise. Deleting the non-increasing condition on ψ, the famous conjecture of Duffin-Schaeffer [17, 18] is the expected similar result. For the latest progresses on this conjecture, one can see [19, 20]. In the case where the set has Lebesgue measure zero, it is natural to calculate the Hausdorff dimension of $\mathcal{L}_\psi$. The first result on the Hausdorff dimension of $\mathcal{L}_\psi$ dates back to Jarník-Bosicovitch Theorem [21, 22]. It was shown that for any $\tau > 1$, one has

$$\dim_H\left(\left\{\theta \in \mathbb{R} : \|n\theta\| < \frac{1}{n^\tau}, \text{ for infinitely many } n \in \mathbb{N}\right\}\right) = \frac{2}{1+\tau},$$

where $\dim_H(\cdot)$ denotes the Hausdorff dimension of a set.

In analogy with the classical Diophantine approximation, Hill and Velani [23] studied the approximation properties of the orbits of a dynamical system and introduced the so called *shrinking target problems*: for a measurable preserving dynamical system $(X, \mathcal{B}, \mu, T)$ with a metric d, fix $x_0 \in X$ and a positive function $\psi : \mathbb{N} \to \mathbb{R}^+$, define the set of the *ψ-well asymptotically approximable points* by x_0 as

$$\mathcal{L}(T, \psi, x_0) := \{x \in X : d(T^n x, x_0) < \psi(n), \text{ for infinitely many } n \in \mathbb{N}\},$$

what is the size (Lebesgue measure, Hausdorff dimension) of the set $\mathcal{L}(T, \psi, x_0)$? If ψ is a constant function and $x_0 \in supp(\mu)$, Poincaré Recurrence Theorem implies that the set $\mathcal{L}(T, \psi, x_0)$ has the same μ-measure as that of the ball $B(x_0, \psi)$. Hill and Velani [23] asked the question what will happen if ψ decreases with the time. It is that if the radius $r(n)$ of $B(n) = B(x_0; r(n))$ tends to 0 as n tending to ∞, what is the size (Lebesgue measure, Hausdorff dimension) of the subset $\mathcal{L}(T, \psi, x_0)$? They studied the case where T is an expanding rational map of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This arises a rich subsequent work on the so called quantitative recurrent properties [50] such as dynamical Borel-Cantelli lemma [24], shrinking target problems [25, 26], hitting time [27], recurrence time [28–31] and waiting time [32], etc.

For the irrational rotation $([0, 1], R_\alpha)$, Kim showed [33, Proposition 2.6] that for Lebesgue almost all $x \in [0, 1]$ and $\alpha \in [0, 1]$, one has

$$|R_\alpha^n x - x_0| < \psi(n), \quad \text{for infinitely many } n \in \mathbb{N},$$

if and only if the series $\sum \psi(n) = \infty$. Bugeaud [34, Theorem 1], Troubetzkoy and Schmeling [35, Theorem 3.2] proved that the size of $\mathcal{L}(R_\alpha, \psi, x_0)$ is related to the polynomial degree $\lim\limits_{n\to\infty} \dfrac{-\log \psi(n)}{\log n}$ of ψ and that for any $\tau > 1$,

$$\dim_H\left(\left\{x \in [0, 1] : |R_\alpha^n x - x_0| < \frac{1}{n^\tau}, \text{ for infinitely many } n \in \mathbb{N}\right\}\right) = \frac{1}{\tau}.$$

If (X, T) is the dynamical system with T an expanding rational map with degree larger than 2 and X the corresponding Julia set ([23, Theorem 1, Theorem 4], [36, Theorem 1]) as well as the case where T is a linear map given by a matrix with integer coefficients and X is an n-dimensional torus, Hill and Velani [37, Theorem 1, Theorem

2] proved that the size of $\mathcal{L}(T, \psi, x_0)$ is related to the exponent index $\varliminf_{n\to\infty} \frac{-\log_\beta \psi(n)}{n}$ of ψ.

If T is the β-transformation T_β $(\beta > 1)$ on $[0, 1)$ defined by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ denotes the largest integer less than or equal to ξ. For the β-dynamical system $([0, 1), T_\beta)$, Philipp [38, Theorem 2A, B, C] proved that the set $\mathcal{L}(T_\beta, \psi, x_0)$ has Lebesgue measure zero if and only if the series $\sum \psi(n) < \infty$ and has full Lebesgue measure if and only if the series $\sum \psi(n) = \infty$. Shen and Wang [39, Theorem 1.1] established the following result on the Hausdorff dimension of $\mathcal{L}(T_\beta, \psi, x_0)$.

Theorem SW *For any real number $\beta > 1$ and $x_0 \in [0, 1]$, one has*

$$\dim_H \mathcal{L}(T_\beta, \psi, x_0) = \frac{1}{1 + v}, \qquad \text{where } v := \varliminf_{n\to\infty} \frac{-\log_\beta \psi(n)}{n}.$$

Parallel to the asymptotic approximation theorem, it is also worth of studying the uniform approximation properties as in Dirichlet Theorem. The uniform Diophantine approximation was studied by Bugeaud and Liao [40] related to β-transformations T_β. Given $x \in [0, 1)$ and $\beta > 1$, let

$$\nu_\beta(x) := \sup \left\{ v \geq 0 : T_\beta^n x < (\beta^n)^{-v}, \text{ for infinitely many } n \in \mathbb{N} \right\},$$

$$\hat{\nu}_\beta(x) := \sup \left\{ v \geq 0 : \forall\ N \gg 1,\ T_\beta^n x < (\beta^N)^{-v} \text{ has a solution } n \in [0, N] \right\}.$$

Bugeaud and Liao [40] proved the following theorem.

Theorem BL(*[40, Theorem 1.4]) For any $v \in (0, \infty)$ and any $\hat{v} \in (0, 1)$, if the real number $v < \hat{v}/(1 - \hat{v})$, then the set*

$$\{x \in [0, 1] : \nu_\beta(x) = v\} \cap \{x \in [0, 1] : \hat{\nu}_\beta(x) \geq \hat{v}\}$$

is empty. Otherwise, one has

$$\dim_H \left(\{x \in [0, 1] : \nu_\beta(x) = v\} \cap \{x \in [0, 1] : \hat{\nu}_\beta(x) = \hat{v}\}\right) = \frac{v - \hat{v} - v\hat{v}}{(1 + v)(v - \hat{v})}.$$

The exponents ν_β and $\hat{\nu}_\beta$ were introduced in [41](see also [42, Ch.7]). The exponent $\nu_\beta(x)$ measures the speed with which x can be approximated in a nonuniform way by points with finite orbit, while the exponent $\hat{\nu}_\beta(x)$ measures the speed of the corresponding uniform approximation. The exponents ν_β and $\hat{\nu}_\beta$ are also strongly related to the run-length function of β-expansions (see [43–46]). The aim of this thesis is to study the Diophantine approximation sets in [40] when the approximation speed function $n \mapsto \beta^{-nv}$ is replaced by a general positive function. More precisely, fix two positive functions $\psi_1,\ \psi_2 : \mathbb{N} \to \mathbb{R}^+$, define

$$\mathcal{L}(\psi_1) := \left\{ x \in [0, 1] : T_\beta^n x < \psi_1(n), \text{ for infinitely many } n \in \mathbb{N} \right\},$$

$$\mathcal{U}(\psi_2) := \left\{x \in [0,1] : \forall\ N \gg 1,\ T_\beta^n x < \psi_2(N) \text{ has a solution } n \in [0,N]\right\}.$$

We would like to calculate the Hausdorff dimension of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ and the set $\mathcal{U}(\psi_2)$. Let

$$\underline{v}_1 := \varliminf_{n\to\infty} \frac{-\log_\beta \psi_1(n)}{n}, \quad \overline{v}_1 := \varlimsup_{n\to\infty} \frac{-\log_\beta \psi_1(n)}{n};$$

$$\underline{v}_2 := \varliminf_{n\to\infty} \frac{-\log_\beta \psi_2(n)}{n}, \quad \overline{v}_2 := \varlimsup_{n\to\infty} \frac{-\log_\beta \psi_2(n)}{n}.$$

If $\underline{v}_1 < 0$, by the definition of $\underline{v}_1$, then there is a sequence of $\{n_j\}$ such that

$$\lim_{j\to\infty} \frac{-\log_\beta \psi_1(n_j)}{n_j} = \underline{v}_1.$$

Then, for $\varepsilon > 0$ small enough, there exists an integer j_0 such that

$$1 < \beta^{-n_j(\underline{v}_1+\varepsilon)} \le \psi_1(n_j), \text{ for any } j \ge j_0.$$

By the fact

$$T_\beta^n x < 1, \text{ for any } x \in [0,1) \text{ and any } n \in \mathbb{N},$$

we have

$$T_\beta^{n_j} x < 1 \le \psi_1(n_j), \quad \text{for any } x \in [0,1).$$

This implies

$$[0,1) \subseteq \mathcal{L}(\psi_1).$$

On the other hand, if we take all the integer n_i with

$$\psi_2(n_i) > 1, \text{ for } i = 1,2,3\cdots,$$

then for any $x \in [0,1)$ and any integer $n \in [1, n_i]$, we have

$$T_\beta^n x < 1 < \psi_2(n_i).$$

Thus, we can replace $\psi_2(n)$ by the function

$$\widetilde{\psi}_2(n) = \begin{cases} \psi_2(n), & \text{if } n \neq n_j \\ 1, & \text{if } n = n_j \end{cases}, \ j = 1,2,\cdots.$$

The size (Lebesgue measure or Hausdorff dimension) of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ (respectively, $\mathcal{U}(\psi_2)$) is the same as that of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\widetilde{\psi}_2)$ (respectively, $\mathcal{U}(\widetilde{\psi}_2)$). Therefore, in this thesis, we always assume $\underline{v}_1 \ge 0$ and $\underline{v}_2 \ge 0$. We establish the following theorems to estimate the Hausdorff dimension of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ and the set $\mathcal{U}(\psi_2)$. Theorem B gives the Hausdorff dimension of $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ for the special case where one or more of $\underline{v}_1$, $\overline{v}_1$, $\underline{v}_2$ and $\overline{v}_2$ are 0 or ∞.

Theorem B. *(WU, L. Zheng)*

(1) If $\underline{v}_1 = \overline{v}_1 = \underline{v}_2 = \overline{v}_2 = 0$, *then*

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 1.$$

(2) If $\underline{v}_2 = \infty$ *and* $0 \leq \underline{v}_1 \leq \overline{v}_1 \leq \infty$, *then* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *is countable.*

(3) If $\underline{v}_1 = \infty$ *and* $0 \leq \underline{v}_2 \leq \overline{v}_2 \leq \infty$, *then* $\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 0$.

Remark. *For Item* (1), *the set* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *is not necessary of full Lebesgue measure. In fact, if the series* $\sum \psi_1(n)$ *converges, by [38, Theorem 2A, B, C],*

$$m (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 0,$$

where $m(A)$ *denotes the Lebesgue measure of* A. *The set* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *can also be of full Lebesgue measure. For example, if*

$$\psi_1(n) = \psi_2(n) = \frac{\log^2 n}{\sqrt{n}},$$

according to Dmitry, Konstantoulas, and Florian [47, Theorem 1.1],

$$m (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = 1.$$

For Item (3), *if* $\underline{v}_2 = \infty$, *then* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *is countable. If* $1 < \underline{v}_2 < \infty$, *then the set* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *is empty (see Lemma 6.4). If* $0 < \underline{v}_2 \leq 1$, *then the set* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *is uncountable (see Proposition 6.5).*

Theorem C. *(WU, L. Zheng) If* $\underline{v}_2 > 1$, *then* $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ *is countable. If* $\overline{v}_1/(2+\overline{v}_1) \leq \underline{v}_2 \leq 1 < \overline{v}_2$, *then*

$$0 \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2 \leq \overline{v}_2 \leq 1$ *and* $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2$, *then*

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$ *and* $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2 \leq 1$, *then*

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

If $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2 \leq \overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, *then*

$$\frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)} \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$ *and* $\overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, *then*

$$\frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)} \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

We remark that Theorems B and C include all possible cases. We also estimate the Hausdorff dimension of $\mathcal{U}(\psi_2)$.

Theorem D. *(WU, L. Zheng) If $\underline{v}_2 > 1$, then $\mathcal{U}(\psi_2)$ is countable. If $\underline{v}_2 \leq 1 < \overline{v}_2$, then*

$$0 \leq \dim_H(\mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\overline{v}_2 \leq 1$, then

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H(\mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

We will show in Examples 6.9, 6.10, 6.11, 6.12, 6.13, 6.14 and 6.15, that the upper and lower bounds of Theorems C and D can be all reached. When $\underline{v}_1 = \overline{v}_1 = 0$, we have the result as Theorem E.

Theorem E. *(WU, L. Zheng) Assume $\underline{v}_1 = \overline{v}_1 = 0$. If $\underline{v}_2 > 0$, then $\mathcal{U}(\psi_2) \subseteq \mathcal{L}(\psi_1)$. Therefore,*

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \dim_H(\mathcal{U}(\psi_2)).$$

3 Flows

In this section, we introduce the basic knowledge about flow. On the whole, we review the existence of a global flow on the compact manifolds. First, we will look more closely at the Poincaré flow, the linear Poincaré flow, the extended linear Poincaré flow, the scaled linear Poincaré flow. For study the dynamics of the flow, we will also be concerned with the sectional Poincaré map and the rescaled sectional Poincaré map. Secondly, we introduce the measure and flow for vector fields. What is more, the transgression of a measure is a joint work with D. Yang and Y. Zhang.

A continuous-time dynamical system is usually given infinitesimally (for example, by means of differential equations) and the reconstruction of the dynamics from this infinitesimal description involves some kind of integration process. Given a differentiable manifold M of dimension n, the time evolution is given by a smooth function

$$F(x,t)=\varphi^t(x),\ x\in M,\ t\in\mathbb{R},$$

which satisfies the group property

$$\varphi^t\circ\varphi^s=\varphi^{t+s}.$$

But, this smooth function may not be defined for all x and t.

We first consider the local aspect of the situation, fix $x\in M$ and vary t, one obtains a parameterized smooth curve on M. Denoted by $\xi(x)$ the tangent vector to the curve at $t=0$ (the point x). The vector $\xi(x)$ belongs to the tangent space T_xM which is a n-dimensional linear space "attached"to M at the point x. Thus, the map

$$x\mapsto\xi(x)$$

forms a section of the tangent bundle

$$TM:=\bigcup_{x\in M}T_xM$$

or a *vector field* on M. Therefore, a vector field on M is an assignment of a tangent vector to each point in M. Let $U\subset M$ be a coordinate neighborhood with coordinates $(s_1,\cdots,s_n)$, the tangent bundle TU is a direct product $U\times\mathbb{R}^n$ and a vector field is determined by a map from U to $\mathbb{R}^n$, n real-valued functions $v_1,\cdots,v_n$. Denote by $\dfrac{\partial}{\partial s_i}$ the basic vector fields which associate to every point the i-th vector of the standard basis in $\mathbb{R}^n$, every vector field can be represented locally as

$$\sum_{i=1}^n v_i(s_1,\cdots,s_n)\frac{\partial}{\partial s_i}.$$

When the initial point x is represented by coordinates $s_1^0,\cdots,s_n^0$, the evolution of this point is obtained by solving the system of first-order ordinary differential equations

$$\frac{ds_i}{dt}=v_i(s_1,\cdots,s_n)$$

with initial conditions

$$s_i(0) = s_i^0, \quad \text{for } i = 1, 2, \cdots, n.$$

If the functions v_i are continuously differentiable, the solution for sufficiently small time exists, is unique and depends smoothly on the initial condition. Thus, for small values of t, the transformation φ^t can be recovered from the vector field. For larger t, one should take compositions of maps defined in local coordinates. The vecter field is called *complete*, if the solutions exist for all real values of t. If the manifold M is compact and has no boundary, then it can be covered by a finite number of coordinate charts. Inside any charts, the solutions exist for a fixed length of time. Since every point $x \in M$ belongs to a coordinate neighborhood which is not very small, it implies that any C^1 vector field on a clsoed compact manifold without boundary is complete and defines a *smooth flow*, a one-parameter group of diffeomorphisms of M.

3.1 Poincaré flow

Denoted by $\mathcal{X}^1(M)$ the space of all C^1 vector fields on differentiable manifold M with the C^1 norm. Given a vector field $X \in \mathcal{X}^1(M)$, let

$$\varphi_t := \varphi_t^X$$

be the flow generated by $X \in \mathcal{X}^1(M)$. A point $\sigma \in M$ is a *singularity* if the vector $X(\sigma) = 0$. Denote by $\mathrm{Sing}(X)$ the set of all singularities of the vector field X. A point x is *regular* if the vector $X(x) \neq 0$. A regular point p is *periodic*, if $\varphi_{t_0}^X(p) = p$ for some $t_0 > 0$. A critical point is either a singularity or a periodic point. Denote the *normal bundle* of X by

$$\mathcal{N}^X := \bigcup_{x \in M \setminus \mathrm{Sing}(X)} \mathcal{N}_x, \qquad \text{where } \mathcal{N}_x := \{v \in T_xM : \ v \perp X(x)\}.$$

For the flow φ_t^X, its derivative with respect to the space variable is called the *tangent flow* and is denoted by $\mathrm{d}\varphi_t^X$.

Definition 3.1. *Given $x \in M \setminus \mathrm{Sing}(X)$, $v \in \mathcal{N}_x$ and $t \in \mathbb{R}$, the linear Poincaré flow $\psi_t^X(v)$ is the orthogonal projection of $d\varphi_t^X(v)$ on $\mathcal{N}_{\varphi_t^X(x)}$ along the flow direction $X(\varphi_t^X(x))$,*

$$\psi_t^X(v) := \mathrm{d}\varphi_t^X(v) - \frac{\langle \mathrm{d}\varphi_t^X(v), X(\varphi_t^X(x))\rangle}{\| X(\varphi_t^X(x)) \|^2} X(\varphi_t^X(x)),$$

where $< \cdot, \cdot >$ is the inner product on T_xM given by the Riemannian metric.

This flow could also be defined in a more general way by Liao [51]. In Liao's work [51], he introduced one-parameter transformation group. And the flow is a special case. The linear Poincaré flow ψ_t loses the compactness due to the existence of singularities. This difficulty can be overcome by extending the linear Poincaré flow (see [52] used the terminology of "extended linear Poincaré flow"). For understanding the accumulation directions on singularity, one can define the *transgression* of the tangent flow $\mathrm{d}\varphi_t^X$: for the sphere bundle

$$SM := \{v \in TM : \ \|v\| = 1\},$$

the map

$$(t,v)\to \frac{\mathrm{d}\varphi_t^X(v)}{\|\mathrm{d}\varphi_t^X(v)\|},\quad t\in\mathbb{R},\quad v\in SM,$$

defines a flow on SM. At any point $x\in M$, one has

$$S_xM := \{v\in T_xM:\ \|v\|=1\}.$$

For any point $v\in S_xM$, one can define a fiber

$$\mathcal{N}_v := \{u\in T_xM:\ u\perp v\}$$

and then define a bundle $\mathcal{N}SM$. Then one can consider another bundle on M:

$$NSM := \{(v_1,v_2):\ v_1\in S_xM,\ v_2\in T_xM,\ v_1\perp v_2\},$$

and define the following flow on NSM after Liao: for any $v_1\in S_xM,\ v_2\in T_xM$ and any $t\in\mathbb{R}$, define

$$\begin{aligned}\Theta_t(v_1,v_2) &= \left(\frac{\mathrm{d}\varphi_t^X(v_1)}{\|\ \mathrm{d}\varphi_t^X(v_1)\ \|},\mathrm{d}\varphi_t^X(v_2)-\frac{\langle \mathrm{d}\varphi_t^X(v_1),\mathrm{d}\varphi_t^X(v_2)\rangle}{\|\ \mathrm{d}\varphi_t^X(v_1)\ \|^2}\mathrm{d}\varphi_t^X(v_1)\right)\\ &= \left(\mathrm{Proj}_S\left(\Theta_t\right),\mathrm{Proj}_N(\Theta_t)\right).\end{aligned}$$

Then, the linear Poincaré flow ψ_t can be "embedded" in the flow Θ_t. In fact, if one considers the subsets

$$\left\{\left(\frac{X(x)}{\|X(x)\|},\ v\right)\right\}\subset NSM,$$

then for any regular point $x\in M$ and any $v\in\mathcal{N}_x$, one has

$$\psi_t^X(v) := \mathrm{Proj}_N\Theta_t\left(\frac{X(x)}{\|X(x)\|},\ v\right).$$

For a compact invariant set Λ, its *transgression* $\widetilde{\Lambda}$ is defined by

$$\widetilde{\Lambda} := \mathrm{Closure}\left\{\frac{X(x)}{\|X(x)\|}\ :\ x\in\Lambda\setminus\mathrm{Sing}(X)\right\}$$

in SM. Therefore, $\mathrm{Proj}_N\Theta_t$ is a continuous flow on $\widetilde{\Lambda}$. The *extended linear Poincaré flow* $\widetilde{\psi}$ on $\mathcal{N}_{\widetilde{\Lambda}}SM$ is defined as the compactification of the fibered flow $\mathrm{Proj}_N\Theta_t$ on

$$\bigcup_{x\in\Lambda\setminus\mathrm{Sing}(X)}\left\{\frac{X(x)}{\|X(x)\|}\right\}$$

over the base flow $\mathrm{Proj}_S\Theta_t$. Thus, we have the compactness. The extended linear Poincaré flow is continuous with respect to the vector fields, the time and the vector.

Lemma 3.2. *([52, Lemma 3.1]) The extended linear Poincaré flow $\widetilde{\psi}_t^X(v)$ varies continuously with respect to the vector field X, the time t and the vector v.*

In Liao's work [51, 53, 54], one can find the *rescaled linear Poincaré flow* ψ_t^* which is defined by

$$\psi_t^*(v) := \frac{\| X(x) \|}{\| X(\varphi_t^X(x)) \|}\psi_t^X(v) = \frac{\psi_t^X(v)}{\| \mathrm{d}\varphi_t^X|_{<X(x)>} \|},$$

for any $x \in M \setminus \mathrm{Sing}(X)$, $v \in \mathcal{N}_x$ and $t \in \mathbb{R}$. This rescaled linear Poincaré flow will help us to overcome some difficulties produced by singularities since it gives uniform estimations on some non-compact sets.

Lemma 3.3. *([55, Lemma 2.1]) For any $\tau > 0$, there is a constant $C_\tau > 0$ such that for any $t \in [-\tau, \tau]$, one has*

$$\|\psi_t^*\| \le C_\tau,$$

where $\|\psi_t^*\| := \sup\{|\psi_t^*(v)| : v \in \mathcal{N} \text{ and } |v| = 1\}$.

Fix $\alpha > 0$, the normal manifold $N_x(\alpha)$ of x is defined as:

$$N_x(\alpha) := \exp_x(\mathcal{N}_x(\alpha)),$$

where

$$\mathcal{N}_x(\alpha) := \{v \in \mathcal{N}_x : \quad |v| \le \alpha\}.$$

Take $\alpha' > 0$ small enough such that for any $x \in M$, $\exp_x$ is a diffeomorphism from $\mathcal{N}_x(\alpha)$ to its image $N_x(\alpha)$. The sectional return map of a cross section of a periodic point is defined by Poincaré to study the dynamics in a small neighborhood of a periodic orbit of a vector field. By generalizing this idea to every regular point, Gan and Yang [55] first use the notations $\mathcal{P}$ and $\mathcal{P}^*$ to define the sectional Poincaré map and the rescaled sectional Poincaré map (also see [56]) between any two cross sections at any two points in the same regular orbit and to study the local sectional dynamics to understand Liao's powerful tools "canonical equations"in a more geometrical way.

Given $t > 0$ and $x \in M\backslash\mathrm{Sing}(X)$, the flow φ_t defines a local holonomy map $P_{x,\ \varphi_t(x)}$ from $N_x(\alpha')$ to $N_{\varphi_t(x)}(\alpha')$ in a small neighborhood of x. The sectional Poincaré map

$$\mathcal{P}_t : \mathcal{N}_x(\alpha') \to \mathcal{N}_{\varphi_t(x)}(\alpha')$$

is the lift of the holonomy map $P_{x,\ \varphi_t(x)}$ and can be expressed by

$$\mathcal{P}_t := \exp_{\varphi_t(x)}^{-1} \circ P_{x,\varphi_t(x)} \circ \exp_x .$$

The rescaled sectional Poincaré map

$$\mathcal{P}_t^* : \mathcal{N}_x(\alpha') \to \mathcal{N}_{\varphi_t(x)}(\alpha')$$

is defined by

$$\mathcal{P}_t^*(v) := \frac{\mathcal{P}_t(v)}{|X(\varphi_t(x))|}, \qquad \text{for any } v \in \mathcal{N}_x(\alpha').$$

3.2 Measures, entropy and dominated splitting of flows

A measure μ is called φ_t-invariant, if μ is φ_{t_0}-invariant for any fixed $t_0 \in \mathbb{R}$. An invariant measure μ is said to be φ_t-ergodic, if μ is φ_{t_0}-ergodic for any fixed $t_0 \in \mathbb{R}$. Let $\mathcal{M}(\varphi)$ denote the space of all φ_t-invariant measures and $\mathcal{E}(\varphi)$ denote those are ergodic. Assume that μ is a φ_t-invariant measure which is not concentrated on $\mathrm{Sing}(X)$, for the *linear Poincaré flow*

$$\psi_t : \mathcal{N} \to \mathcal{N},$$

by Oseledec Theorem [3, Theorem S.2.9], for μ-almost every point x, there is a measurable splitting

$$\mathcal{N}_x = \bigoplus_{i=1}^{k(x)} \mathcal{E}_i(x)$$

and numbers

$$\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{k(x)}(x),$$

such that

$$\lim_{t\to\infty} \frac{1}{t} \log \parallel \psi_t(v) \parallel = \lambda_i(x),\ \forall\ v \in \mathcal{E}_i(x) \setminus \{0\},\ i = 1, 2, \ldots, k(x),$$

where $1 \le k(x) \le d-1$. These quantities are called the *Lyapunov exponents* at point x of ψ_t and the sub-bundle $\mathcal{E}_i(x)$ is called the Oseledec subspace of $\lambda_i(x)$.

Definition 3.4. *An ergodic measure μ of the flow φ_t is regular, if it is not supported on a singularity. A regular ergodic measure is hyperbolic, if the Lyapunov exponents of the linear Poincaré flow ψ_t are all non-zero.*

Remark. *We can also define the hyperbolicity of an ergodic measure by using the tangent flow $\mathrm{d}\varphi_t$ as usual. However, for ergodic measures that are not supported on singularities, there will be one zero Lyapunov exponent for the tangent flow along the flow direction.*

For a regular hyperbolic ergodic measure μ, one has the splitting

$$\mathcal{N} = \bigoplus_{i=1}^{k} \mathcal{E}_i,\ (1 \le k \le d-1).$$

We rewrite as

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u,$$

where all the Lyapunov exponents along $\mathcal{E}^s$ are negative and all the Lyapunov exponents along $\mathcal{F}^u$ are positive. The splitting

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u$$

is called the *hyperbolic Oseledec splitting* with respect to the hyperbolic ergodic invariant measure μ.

Lemma 3.5. *[Definition of the transgression of a measure] If μ is a φ_t-invariant ergodic measure on M with $\mu(\mathrm{Sing}(X)) = 0$, then there is an ergodic $\mathrm{Proj}_S\Theta_t$-invariant measure $\widetilde{\mu}$ on $\widetilde{\Lambda}$, the transgression of $\Lambda = Supp(\mu)$, such that the Lyapunov exponents of flow $\widetilde{\psi}_t$ with respect to the measure $\widetilde{\mu}$ are the same as the Lyapunov exponents of the linear Poincaré flow ψ_t with respect to the measure μ. The measure $\widetilde{\mu}$ is called the* transgression *of μ.*

Proof. Let

$$\mathbb{P} : TM \to M$$

be the projection which is a continuous surjection defined as

$$\mathbb{P}(v) = x, \quad \text{for any vector } v \in T_xM.$$

Thus, $\mathbb{P}$ induces a bijective map from

$$\widehat{\Lambda} := \bigcup_{x \in \Lambda \setminus \mathrm{Sing}(X)} \left\{ \frac{X(x)}{\|X(x)\|} \right\}$$

to $\Lambda \setminus \mathrm{Sing}(X)$. The fact $\mu(\mathrm{Sing}(X)) = 0$ implies

$$\mu(\Lambda \setminus \mathrm{Sing}(X)) = 1.$$

Define the measure

$$\widetilde{\mu} := (\mathbb{P}|_{\widehat{\Lambda}})_* \mu$$

on $\widehat{\Lambda}$. Since

$$\widetilde{\mu}(\widehat{\Lambda}) = \mu(\mathbb{P}(\widehat{\Lambda})) = \mu(\Lambda \setminus \mathrm{Sing}(X)) = 1,$$

$\widetilde{\mu}$ is a probability measure on $\widehat{\Lambda}$, hence on $\widetilde{\Lambda} = \mathrm{Closure}(\widehat{\Lambda})$. For any Borel set $A \subset \widetilde{\Lambda}$ with $\widetilde{\mu}(\psi_{-t}(A)) = \widetilde{\mu}(A)$ for every $t \in \mathbb{R}$, one has

$$\mu(\mathbb{P}(\mathrm{Proj}_S\Theta_{-t}(A))) = \mu(\mathbb{P}A), \quad \text{for every } t \in \mathbb{R}.$$

Since μ is an ergodic φ_t-invariant measure,

$$\widetilde{\mu}(A) = \mu(\mathbb{P}A) = 0 \text{ or } 1.$$

It means that $\widetilde{\mu}$ is an ergodic $\mathrm{Proj}_S\Theta_t$-invariant measure.

Applying Oseledec Theorem to the linear Poincaré flow

$$\psi_t : \mathcal{N}_\Lambda \to \mathcal{N}_\Lambda,$$

for μ-almost every point x, there is splitting

$$\mathcal{N}_x = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_m$$

and numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m$$

such that

$$\lim_{t\to\infty}\frac{1}{t}\log\parallel\psi_t(v)\parallel=\lambda_i,\ \forall\ v\in\mathcal{E}_i\setminus\{0\},\ i=1,2,\ldots,m,$$

where $1\le m\le\dim(M)-1$. Since $\widetilde{\psi}_t$ is defined on $\widetilde{\Lambda}=\mathrm{Closure}(\widehat{\Lambda})$, for any point $X(x)/\|X(x)\|\in\widehat{\Lambda}$, any $v\in\mathcal{E}_{X(x)/\|X(x)\|}:=\mathcal{E}_x$ and any t, one has

$$\widetilde{\psi}_t(v):=\mathrm{Proj}_N\Theta_t\left(\frac{X(x)}{\|X(x)\|},\ v\right).$$

Since μ-almost every point x is contained in $\Lambda\setminus\mathrm{Sing}(X)$ and $\widetilde{\mu}$-almost every point is contained in $\widehat{\Lambda}$, by the definition of the extended Poincaré flow, for the splitting

$$\mathcal{N}_x=\bigoplus_{i=1}^{m}\mathcal{E}_i,\quad\forall\ v\in\mathcal{E}_i\setminus\{0\},$$

one has

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log\parallel\widetilde{\psi}_t(v)\parallel &= \lim_{t\to\infty}\frac{1}{t}\log\parallel\mathrm{Proj}_N\Theta_t(X(x)/\|X(x)\|,v)\parallel\\ &= \lim_{t\to\infty}\frac{1}{t}\log\parallel\psi_t(v)\parallel=\lambda_i.\end{aligned}$$

Therefore, the Lyapunov exponents of $\widetilde{\psi}_t$ with respect to the measure $\widetilde{\mu}$ are the same as the Lyapunov exponents of the linear Poincaré flow ψ_t with respect to the measure μ. □

Let $f:M\to M$ be a continuous map of a compact metric space M with distance function d. The set of all invariant measures and the set of all ergodic measures are denoted by $\mathcal{M}(f)$ and $\mathcal{E}(f)$, respectively. The metric d_n^f is defined as

$$d_n^f(x,y):=\max_{0\le i\le n-1}d(f^ix,f^iy).$$

Denote by $B_f(x,n,\varepsilon)$ the open ball

$$\{y\in M:\ d_n^f(x,y)<\varepsilon\}.$$

A subset $E\subset M$ is said to be (n,ε)-*spanning*, if

$$M\subset\bigcup_{x\in E}B_f(x,n,\varepsilon).$$

Let $N_f(n,\varepsilon)$ be the minimal cardinality of an (n,ε)-spanning set. Define

$$h(f,\varepsilon):=\overline{\lim_{n\to\infty}}\frac{1}{n}\log N_f(n,\varepsilon).$$

Since $h(f,\varepsilon)$ does not decrease with ε, the topological entropy $h_{top}(f)$ is defined as

$$h_{top}(f):=\lim_{\varepsilon\to0}h(f,\varepsilon).$$

A priori, this quantity might depend on the metric d. Actually, it does not. One can see [3, p.p. 113]. A subset $K \subset M$ is said to be (n,ε)-*separated*, if any two different points $x,\ y \in K$ implies

$$d_n^f(x,y) > \varepsilon.$$

Let $S_f(n,\varepsilon)$ be the largest cardinality of an (n,ε)-separated set. One can also define the topological entropy $h_{top}(f)$ via the number $S_f(n,\varepsilon)$ as similar as $N_f(n,\varepsilon)$ (see [48, p.p. 170]). Katok [8, Theorem 1.1] gave a new formula for the metric entropy $h_\mu(f)$ of f-invariant ergodic measure μ as

$$h_\mu(f) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{\log N_f(n,\varepsilon,\delta)}{n},$$

where $N_f(n,\varepsilon,\delta)$ is the minimal cardinality of an (n,ε)-spanning set of measure larger than or equal to $1-\delta$.

The definition of topological entropy $h_{top}(X)$ and metric entropy $h_\mu(X)$ for vector field $X \in \mathcal{X}^1(M)$ (or flow φ_t) is completely parallel to that for the discrete-time case. The counterpart of

$$d_n^f(x,y) := \max_{0\le i\le n-1} d(f^i x, f^i y)$$

is the following nondecreasing family of metrics

$$d_T^X(x,y) := \max_{0\le t\le T} d(\varphi_t(x),\varphi_t(y)).$$

Then, the only property worth special notice is that

$$h_{top}(X) := h_{top}(\varphi_1).$$

Usually, we consider the time-one map φ_1. Define,

$$h_{top}(X) := h_{top}(\varphi_1), \qquad h_\mu(X) := h_\mu(\varphi_1).$$

The topological entropy for a flow is obviously invariant under flow equivalence. It changes under time change and hence under orbit equivalence in a rather complicated way. One can show that for a flow without fixed points any time change preserves vanishing of the topological entropy. If the topological entropy for a map or a flow vanishes, the subexponential asymptotic of any of the quantities involved in its definition may provide a useful insight into the complexity of the orbit structure.

Definition 3.6. *Let $\Lambda \subset M \setminus \mathrm{Sing}(X)$ be an invariant (not necessarily compact) set. An invariant splitting*

$$\mathcal{N}_\Lambda = \mathcal{E} \oplus \mathcal{F}$$

with respect to the linear Poincaré flow ψ_t is dominated, *if there are $C>0$ and $\eta>0$ such that for any $x\in\Lambda$ and any fixed $t>0$, one has*

$$\| \psi_t|\mathcal{E}_x \| \cdot \| \psi_{-t}|\mathcal{F}_{\varphi_t(x)} \| \le C\mathrm{e}^{-\eta t}.$$

Lemma 3.7. *(WU, D. Yang and Y. Zhang) If the invariant splitting*

$$\mathcal{N}_{\Lambda\setminus \mathrm{Sing}(X)} = \mathcal{E} \oplus \mathcal{F}$$

is a dominated splitting with respect to the linear Poincaré flow ψ_t on an invariant set Λ, then the extended linear Poincaré flow $\widetilde{\psi}_t$ has dominated splitting

$$\mathcal{N}_{\widetilde{\Lambda}} SM = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$$

and the bundles $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$ are continuous on $\widetilde{\Lambda}$.

Proof. By the definition of dominated splitting, there are $C > 0$ and $\eta > 0$ such that for any $x \in \Lambda \setminus \mathrm{Sing}(X)$ and any fixed $t > 0$, one has

$$\| \psi_t|\mathcal{E}_x \| \cdot \| \psi_{-t}|\mathcal{F}_{\varphi_t(x)} \| \leq C\mathrm{e}^{-\eta t}. \tag{1}$$

Thus, on the set

$$\Gamma := \left\{ \frac{X(x)}{\|X(x)\|} : \ x \in \Lambda \setminus \mathrm{Sing}(X) \right\} \subset SM,$$

the lifts of $\mathcal{E}$ and $\mathcal{F}$ in $\mathcal{N}_\Gamma SM$ still satisfy the inequality (1). Thus these two bundles can be extended on the closure of Γ, which is $\widetilde{\Lambda}$, in a unique and continuous way. $\square$

4 Hausdorff dimension and β-expansions

In this section, we introduce the basic knowledge of Hausdorff dimension and β-expansions. In the first part, we describe the origin, definition and basic properties of the Hausdorff dimension. In the second one, we introduce the β-expansion, especially the properties of "full cylinders".

4.1 Hausdorff dimension

Last few decades, mathematics has been concerned largely with irregular sets and non-smooth functions which provide a much better representation of many natural phenomena than do the figures of classical geometry. Fractal geometry provides a general framework for the study of such irregular sets. The notion of dimension is central to fractal geometry, because dimension indicates how much space a set occupies near to each of its points. Among the wide variety of fractal dimension, the definition of the Hausdorff dimension, based on a construction of Carathéodory, is the oldest and probably the most important. Hausdorff dimension has the advantage of being defined for any set and is mathematically convenient, as it is based on measures, which are relatively easy to manipulate. Even though it is hard to calculate or to estimate by computational methods in many cases, Hausdorff measure and dimension are essential for understanding of the mathematics of fractals.

For a non-empty subset $F \subset \mathbb{R}^n$, n-dimensional Euclidean space, the *diameter* $|F|$ of F is defined as the largest distance apart of any pair of points in F, i.e. ,

$$|F| := \sup\{|x-y| : x,\ y \in F\}.$$

Given a number $\delta > 0$, a countable (or finite) collection of sets $\{F_i\}$ is called a δ-*cover* of F, if

$$0 \le |F_i| \le \delta,\ \text{for } i = 1, 2, \cdots$$

and

$$F \subset \bigcup_{i=1}^{\infty} F_i.$$

Fix a non-negative number s, for any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(F) := \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : F \subset \bigcup_{i=1}^{\infty} U_i \text{ with } |U_i| \le \delta\right\}.$$

Since δ decreases, the class of permissible covers of F is reduced. Thus the infimum $\mathcal{H}^s_\delta$ increases. Therefore, we write

$$\mathcal{H}^s(F) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(F).$$

This limit exists for any subset $F \subset \mathbb{R}^n$, though the limiting value can be 0 or ∞. $\mathcal{H}^s(F)$ is called the *s-dimensional Hausdorff measure* of F.

For any subset $F \subset \mathbb{R}^n$ and any $\delta < 1$, by the definition of $\mathcal{H}^s_\delta(F)$, $\mathcal{H}^s_\delta(F)$ is non-increasing with s. Therefore, $\mathcal{H}^s(F)$ is also non-increasing. In fact, for $t > s$, one has

$$\sum_i |U_i|^t \leq \sum_i |U_i|^{t-s}|U_i|^t \leq \delta^{t-s}\sum_i |U_i|^s.$$

Therefore,

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F).$$

Letting $\delta \to 0$, if $\mathcal{H}^s(F) < \infty$, then

$$\mathcal{H}^t(F) = 0, \quad \text{for any } t > s.$$

Therefore, a graph of $\mathcal{H}^s(F)$ with respect to s shows that there is a critical value of s at which $\mathcal{H}^s(F)$ "jumps" from ∞ to 0. The critical value is said to be the *Hausdorff dimension* of F. Some authors refer to Hausdorff dimension as *Hausdorff-Besicovitch dimension* (see [57–59]). As the Figure 1 shows, the *Hausdorff dimension* of a set F

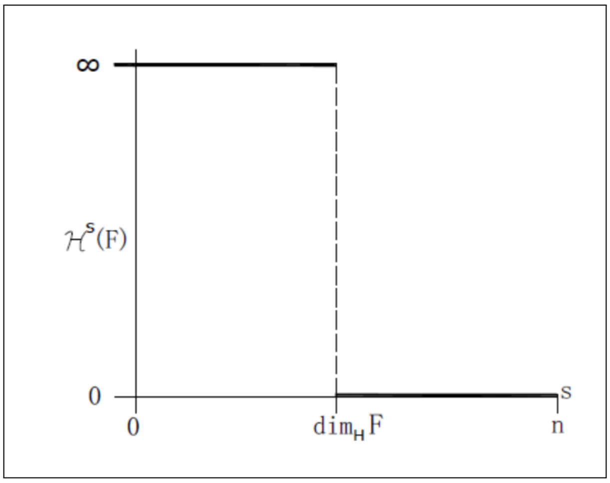


Figure 1 – Graph of $\mathcal{H}^s(F)$ with respect to s for a set F.

can be expressed as

$$\dim_{\mathrm{H}} F := \inf\{s \geq 0 : \ \mathcal{H}^s(F) = 0\} := \sup\{s \geq 0 : \ \mathcal{H}^s(F) = \infty\}.$$

For a set F, the s-dimensional Hausdorff measure and Hausdorff dimension of F have the relation:

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leq s < \dim_{\mathrm{H}} F \\ 0 & \text{if } s > \dim_{\mathrm{H}} F. \end{cases}$$

If $s = \dim_{\mathrm{H}} F$, then $\mathcal{H}^s(F)$ may be 0 or ∞ or may satisfy

$$0 < \mathcal{H}^s(F) < \infty.$$

Since Hausdorff measures generalize the familiar ideas of length, area, volume and so on, Hausdorff dimension also has some "good"properties as that.

Proposition 4.1. *Hausdorff dimension satisfies the following properties:*

(1) Monotonicity: For subset $E,\ F$, if $E \subset F$, then

$$\dim_H E \leq \dim_H F.$$

(2) Countable stability: If $F_1,\ F_2, \cdots$ is a countable sequence of sets, then

$$\dim_H \left(\bigcup_{i=1}^{\infty} F_i \right) = \sup_i \{\dim_H F_i\}.$$

(3) Countable sets: If F is countable, then

$$\dim_H F = 0.$$

(4) Open sets: If the subset $F \subset \mathbb{R}^n$ is open, then

$$\dim_H F = n.$$

(5) Smooth sets: If F is a smooth m-dimensional submanifold of $\mathbb{R}^n$, then

$$\dim_H F = m.$$

The Borel-Cantelli Lemma [60, Lemmas 1 and 2] has been found to be extremely useful for the derivations of many theorems of measure. A classical form of Borel-Cantelli Lemma can be stated as follows.

Lemma 4.2. *(Borel-Cantelli Lemma [60, Lemmas 1 and 2]) Let μ be a measure on a set X with σ-algebra F and $\{A_n\}$ be a sequence in F. If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then*

$$\mu \left(\limsup_{n \to \infty} A_n \right) = 0.$$

The Mass distribution principle [61, p.p. 60] is a classical tool to estimate the lower bound of the Hausdorff dimension of a set.

Lemma 4.3. *([61, Mass distribution principle, p.p. 60]) Assume that E is a Borel measurable set and μ is a Borel measure with $\mu(E) > 0$. If there exist numbers $c > 0$ and $\delta > 0$ such that*

$$\mu(U) \leq c|U|^s, \quad \textit{for any } U \textit{ with } |U| \leq \delta,$$

then

$$\dim_{\mathrm{H}} E \geq s.$$

4.2 β-expansions

The notion of β-expansion was introduced by Rényi [62] in 1957. For any $\beta > 1$, the β-transformation T_β on $[0, 1)$ is defined by

$$T_\beta x = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ denotes the largest integer less than or equal to ξ. Let

$$\lceil \beta \rfloor = \begin{cases} \beta - 1, & \text{if } \beta \text{ is a positive integer,} \\ \lfloor \beta \rfloor, & \text{otherwise.} \end{cases}$$

Definition 4.4. *The expansion of a number $x \in [0, 1)$ with respect to the base β, also called the β-expansion of x, is the sequence of integers $\{\varepsilon_n\}_{n\geq 1} := \{\varepsilon_n(x, \beta)\}_{n\geq 1}$ from $\{0, 1, \cdots, \lceil \beta \rfloor\}$ such that*

$$x = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots + \frac{\varepsilon_n}{\beta^n} + \cdots , \tag{2}$$

defined by the following equivalent properties:

$$\sum_{i>n} \frac{\varepsilon_i}{\beta^i} < \frac{1}{\beta^n}, \quad \textit{for all } n \geq 0;$$

$$\varepsilon_1 = \lfloor \beta x \rfloor, \; \varepsilon_n = \lfloor \beta T_\beta^{n-1} x \rfloor, \quad \textit{for all } n \geq 2.$$

We also write

$$d_\beta(x) = (\varepsilon_1, \cdots, \varepsilon_n, \cdots).$$

We can extend the definition of the β-transformation to the point 1 as

$$T_\beta 1 = \beta - \lfloor \beta \rfloor,$$

one can obtain

$$1 = \frac{\varepsilon_1(1,\beta)}{\beta} + \frac{\varepsilon_2(1,\beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(1,\beta)}{\beta^n} + \cdots ,$$

where

$$\varepsilon_1(1, \beta) = \lfloor \beta \rfloor, \; \varepsilon_n = \lfloor \beta T_\beta^{n-1} 1 \rfloor, \text{ for all } n \geq 2.$$

We also write

$$d_\beta(1) = (\varepsilon_1(1, \beta), \cdots, \varepsilon_n(1, \beta), \cdots).$$

If $d_\beta(1)$ is finite, i.e. , there is an integer $m > 0$ such that $\varepsilon_m(1, \beta) \neq 0$ and

$$\varepsilon_i(1, \beta) = 0, \text{ for all } i > m,$$

then β is called a *simple Parry number*. In this case, the infinite β-expansion of 1 is defined as:

$$(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \cdots, \varepsilon_n^*(\beta), \cdots) := (\varepsilon_1(1, \beta), \varepsilon_2(1, \beta), \cdots, \varepsilon_m(1, \beta) - 1)^\infty,$$

where $(\omega)^\infty$ denotes the periodic sequences $(\omega, \omega, \cdots)$. If $d_\beta(1)$ is infinite, then define the infinite β-expansion of 1 as

$$(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \cdots, \varepsilon_n^*(\beta), \cdots) := (\varepsilon_1(1,\beta), \varepsilon_2(1,\beta), \cdots, \varepsilon_n(1,\beta), \cdots).$$

Endow the set $\{0, 1, \cdots, \lceil \beta \rfloor\}^{\mathbb{N}}$ with the product topology and define the one-sided shift operator σ as:

$$\sigma\left((\omega_n)_{n\geq 1}\right) := (\omega_{n+1})_{n\geq 1},$$

for any infinite sequence $(\omega_n)_{n\geq 1}$ in $\{0, 1, \cdots, \lceil \beta \rfloor\}^{\mathbb{N}}$. The lexicographical order $<_{lex}$ on $\{0, 1, \cdots, \lceil \beta \rfloor\}^{\mathbb{N}}$ is defined as:

$$\omega = (\omega_1, \omega_2, \cdots) <_{lex} \omega' = (\omega_1', \omega_2', \cdots),$$

if $\omega_1 < \omega_1'$ or there is an integer $k \geq 2$ such that

$$\omega_i = \omega_i', \text{ for all } 1 \leq i < k$$

but $\omega_k < \omega_k'$. Denote by $\omega \leq_{lex} \omega'$ if $\omega <_{lex} \omega'$ or $\omega = \omega'$.

Definition 4.5. *A finite word $(\omega_1, \omega_2, \cdots, \omega_n)$ is called β-admissible, if there is $x \in [0,1]$ such that the β-expansion of x begins with $(\omega_1, \omega_2, \cdots, \omega_n)$. An infinite sequence $(\omega_1, \omega_2, \cdots, \omega_n, \cdots)$ is called β-admissible, if there is $x \in [0,1]$ such that the β-expansion of x is $(\omega_1, \omega_2, \cdots, \omega_n, \cdots)$.*

Denote by Σ_β the set of all infinite β-admissible sequences and Σ_β^n the set of all infinite β-admissible sequences with length n. The β-admissible sequences are characterized by Parry [63] and Rényi [62].

Theorem 4.6. *Given a real number $\beta > 1$,*

(1) ([63, Lemma 1]) A word $\omega = (\omega_n)_{n\geq 1} \in \Sigma_\beta$ if and only if

$$\sigma^k(\omega) \leq_{lex} (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \cdots, \varepsilon_n^*(\beta), \cdots), \text{ for all } k \geq 0.$$

(2) ([63, Lemma 3]) For any $x_1,\ x_2 \in [0,1]$, $x_1 < x_2$ if and only if

$$d_\beta(x_1) <_{lex} d_\beta(x_2).$$

(3) ([63, Lemma 4]) For any $\beta_1 < \beta_2$, one has

$$\Sigma_{\beta_1}^n \subseteq \Sigma_{\beta_2}^n, \quad \Sigma_{\beta_1} \subseteq \Sigma_{\beta_2}.$$

Theorem 4.7. *([62, Theorem 2]) For any $\beta > 1$, one has*

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1},$$

where $\sharp$ denotes the cardinality of a finite set.

For every $(\omega_1, \cdots, \omega_n) \in \Sigma_\beta^n$, we call

$$I_n(\omega_1, \cdots, \omega_n) := \{x \in [0,1] : d_\beta(x) \text{ starts with } (\omega_1, \cdots, \omega_n)\}$$

an *n-th order basic interval* with respect to β. Denote by $I_n(x)$ the n-th order basic interval containing x. The basic intervals are also called *cylinders* by some authors. It is crucial to estimate the lengths of the basic intervals. We will use the key notion of "full basic interval"as follows (see [64,65]).

Definition 4.8. *For any $(\omega_1, \cdots, \omega_n) \in \Sigma_\beta^n$, a basic interval*

$$I_n(\omega_1, \cdots, \omega_n)$$

is said to be full if its length is β^{-n}.

Proposition 4.9. *([65, Lemma 3.1] and [39, Lemma 2.5]) For any $(\omega_1, \cdots, \omega_n) \in \Sigma_\beta^n$, the following statements are equivalent:*

(1) $I_n(\omega_1, \cdots, \omega_n)$ is a full basic interval.

(2) $T_\beta^n I_n(\omega_1, \cdots, \omega_n) = [0, 1)$.

(3) For any $\omega' = (\omega'_1, \cdots, \omega'_m) \in \Sigma_\beta^m$, the concatenation

$$(\omega_1, \cdots, \omega_n, \omega'_1, \cdots, \omega'_m) \in \Sigma_\beta^{n+m}, \ \textit{i.e. , is } \beta\textit{-admissible.}$$

Proposition 4.10. *([39, Corollary 2.6])*

(1) If $(\omega_1, \cdots, \omega_{n+1})$ is a β-admissible sequence with $\omega_{n+1} \neq 0$, then

$$I_{n+1}(\omega_1, \cdots, \omega'_{n+1})$$

is full for any $0 \leq \omega'_{n+1} < \omega_{n+1}$.

(2) For every $\omega \in \Sigma_\beta^n$, if $I_n(\omega)$ is full, then for any $\omega' \in \Sigma_\beta^m$, one has

$$|I_{n+m}(\omega, \omega')| = |I_n(\omega)| \cdot |I_m(\omega')| = \frac{|I_m(\omega')|}{\beta^n}.$$

(3) For any $\omega \in \Sigma_\beta^n$, if $I_{n+m}(\omega, \omega')$ is a full basic interval contained in $I_n(\omega)$ with the smallest order, then

$$|I_{n+m}(\omega, \omega')| \geq \frac{|I_n(\omega)|}{\beta}.$$

Next, we define a sequence of numbers β_N approximating β. Let $\{\varepsilon_k^*(\beta) : k \geq 1\}$ be the infinite β-expansion of 1. For any $\varepsilon_N^*(\beta) > 0$, let β_N be the unique real solution of the equation

$$1 = \frac{\varepsilon_1^*(\beta)}{z} + \cdots + \frac{\varepsilon_N^*(\beta)}{z^N}. \tag{3}$$

Therefore, $\beta_N < \beta$ and the sequence $\{\beta_N : N \geq 1\}$ increases and converges to β when N tends to infinity.

Lemma 4.11. *([39, Lemma 2.7]) For every $\omega \in \Sigma_{\beta_N}^n$ viewed as an element of Σ_β^n, one has*

$$\frac{1}{\beta^{n+N}} \leq |I_n(\omega_1, \cdots, \omega_n)| \leq \frac{1}{\beta^n}.$$

5 Growth rate of periodic orbits for vector fields

In this section, we complete the proof of Theorem A in three subsections. Depending on whether the vector field is star or not, we mainly solve two problems: for the non-star vector field, the growth rate of the number of periodic orbits is infinite; for the star vector field, the growth rate of the number of periodic orbits is equal or larger than its topological entropy.

In the study of differentiable dynamical systems, one of the main subjects is to describe the dynamics of most dynamical systems. These theories were established in the last century. See [66] for instance. An important progress is due to Peixoto [67]:
Theorem (Peixoto) *For any closed surface M, A C^1 vector field on M is C^1 structurally stable vector field if and only if it is Morse-Smale and every vector field could be accumulated by a structurally stable one in the C^1 topology.*

Smale was interested in the generalization of Peixotos result and he asked whether Morse-Smale vector fields are dense in the space of vector fields. Soon, Levinson and Thom pointed out that Morse-Smale vector fields would not be dense (without a rigorous proof). See [66]. Smale noticed the point and he constructed his famous horseshoe (for two dimensional diffeomorphisms or three-dimensional vector fields) [1] which shows that the dynamics may be very complicated and Morse-Smale systems would not be dense in the space of diffeomorphisms or vector fields. There are two kinds of typical dynamical systems: Morse-Smale system or a system with a horseshoe. Their dynamical behavior is quite different:

1. The dynamics of Morse-Smale system is very simple: the chain recurrent set of a Morse-Smale system is a set containing finitely many hyperbolic periodic orbits or singularities. The topological entropy is robustly zero.

2. The dynamics of a system with a horseshoe is very complicated: its chain recurrent set contains a non-trivial basic set with dense periodic orbits. The topological entropy is robustly positive.

Is there other typical dynamics beyond the above two ones? Palis formulated the idea for diffeomorphisms, and he conjectured that
Conjecture (Palis [48, 68, 69]) *Every system can be approximated either by Morse-Smale systems or by systems exhibiting a horseshoe (non-trivial hyperbolic basic set).*

Important progress has been made for the conjecture of Palis for diffeomorphisms: in C^1 topology, Pujals-Sambarino [70] proved it for two-dimensional diffeomorphisms (as a corollary of a stronger result); Bonatti-Gan-Wen [71] gave a prove for three-dimensional diffeomorphisms; and finally Crovisier [72] proved the conjecture for any-dimensional diffeomorphisms. In this study, star systems was catched up.

Definition 5.1. *A vector field $X \in \mathcal{X}^1(M)$ is* star, *if there is a C^1 neighborhood $\mathcal{U}$ of X such that every critical element of any vector field $Y \in \mathcal{U}$ is hyperbolic.*

5.1 The Reduction of Theorem A

5.1.1 Proof of Theorem A

For a generic non-star vector field, the growth rate of the number of periodic orbits for this vector field is infinite. We postpone the proof of Theorem 5.2 in Section 5.1.2.

Theorem 5.2. *There is a dense G_δ set $\mathcal{R} \subset \mathcal{X}^1(M)$ such that if vector field $X \in \mathcal{R}$ is not star, then*

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) = \infty.$$

For star vector fields, we have two steps. First, based on that any regular ergodic measure of star vector field is hyperbolic, we show that the hyperbolic Oseledec splitting is a dominated splitting (Theorem 5.3). Secondly, we prove that if the hyperbolic Oseledec splitting with respect to a regular hyperbolic measure is a dominated splitting, then the growth rate of the number of periodic orbits is larger than or equal to the metric entropy (Theorem 5.4).

Theorem 5.3. *If μ is a regular ergodic invariant measure of a C^1 star vector field X with $h_\mu(X) > 0$, then μ is a hyperbolic measure and its hyperbolic Oseledec splitting*

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u$$

is a dominated splitting.

The proof of Theorem 5.3 is in Section 5.1.3.

Theorem 5.4. *Let μ be a regular ergodic invariant hyperbolic measure of a vector field $X \in \mathcal{X}^1(M)$. If the hyperbolic Oseledec splitting*

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u$$

is a dominated splitting, then

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) \geq h_\mu(X) := h_\mu(\varphi_1).$$

For Theorem 5.4, we have to deal with the re-parametrization problem. In Liao's shadowing lemma, the period of periodic point which shadows the recurrent point is re-parametrization of the recurrent time. For our goal, we have to estimate the difference between the recurrent time of the recurrent orbit and its re-parametrization (the time of the periodic orbit). In Section 5.2, we give the shadowing lemma with time control. Theorem 5.4 is proved in Section 5.3.

Proof of Theorem A. Take a dense G_δ set $\mathcal{R} \subset \mathcal{X}^1(M)$ as Theorem 5.2. For any $X \in \mathcal{R}$, if X is not a star vector field, by Theorem 5.2, one has

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) = \infty > h_{top}(X).$$

If X is a star vector field, then any ergodic invariant measure μ of the star vector field X is a hyperbolic measure by [73, Theorem E]. If $h_\mu(X)=0$, then

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\log\sharp P_T(X)\geq h_\mu(X).$$

By the variational principle,

$$h_{top}(\varphi_1)=\sup\{h_\mu(\varphi_1):\mu \text{ is an ergodic measure of X}\},$$

one has

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\log\sharp P_T(X)\geq h_{top}(X).$$

Now, we can suppose

$$h_\mu(X)>0.$$

According to Theorem 5.3, the hyperbolic Oseledec splitting

$$\mathcal{N}=\mathcal{E}^s\oplus\mathcal{F}^u$$

with respect to the ergodic invariant measure μ is a dominated splitting. By Theorem 5.4, one has

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\log\sharp P_T(X)\geq h_\mu(X).$$

By the variational principle,

$$h_{top}(\varphi_1)=\sup\{h_\mu(\varphi_1):\mu \text{ is an ergodic measure of X}\},$$

one has

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\log\sharp P_T(X)\geq h_{top}(X).$$

The proof of Theorem A is complete. $\square$

5.1.2 Non-star vector fields

Lemma 5.5. *There is a dense G_δ set $\mathcal{R}\subset\mathcal{X}^1(M)$ such that for given $T,\ k\in\mathbb{N}$, if for every C^1 neighborhood $\mathcal{U}$ of a vector field $X\in\mathcal{R}$, there is a vector field $Y\in\mathcal{U}$ having k periodic orbits whose periods belong to $(T,\frac{3T}{2})$, then the vector field X has k periodic orbits whose periods belong to $(T,\frac{3T}{2})$.*

Proof. Fix a countable topological base $\{O_1,\dots,O_i,\dots\}$ of M. Let

$$\{U_1,U_2,\dots,U_n,\dots\}$$

be the family of finite unions of $\{O_i\}$. We define

$$\mathcal{H}^k_{n,T}:=\left\{X\in\mathcal{X}^1(M):\begin{array}{l}X\text{ has } k \text{ hyperbolic periodic orbits with}\\ \text{period belonging to } (T,3T/2)\text{ in } U_n\end{array}\right\},$$

$$\mathcal{N}^k_{n,T} := \left\{ X : \begin{array}{l} \exists\ C^1 \text{ neighborhood } \mathcal{U} \text{ of } X, \text{ s.t. for any } Y \in \mathcal{U}, \text{ either } Y \\ \text{has less than } k \text{ periodic orbits with periods belonging to} \\ (T, 3T/2) \text{ or all } k \text{ periodic orbits with period belonging to} \\ (T, 3T/2) \text{ of } Y \text{ are not in } U_n \end{array} \right\}.$$

By the definition, the set $\mathcal{N}^k_{n,T}$ is open. By the stability of hyperbolicity, $\mathcal{H}^k_{n,T}$ is open.

Claim.

$$\overline{\mathcal{H}^k_{n,T} \cup \mathcal{N}^k_{n,T}} = \mathcal{X}^1(M).$$

Proof of Claim: For any vector field $X \in \mathcal{X}^1(M)$, if $X \notin \mathcal{N}^k_{n,T}$, then for any C^1 neighborhood $\mathcal{U}$ of the vector field X, there is a vector field $Y \in \mathcal{U}$ which has k periodic orbits whose periods belong to $(T, 3T/2)$ belonging to U_n. Thus, there is a sequence $\{X_m\}_{m\in\mathbb{N}} \subset \mathcal{H}^k_{n,T}$ such that

$$X_m \to X.$$

Therefore,

$$X \in \overline{\mathcal{H}^k_{n,T}}.$$

Consequently, the set

$$\mathcal{H}^k_{n,T} \bigcup \mathcal{N}^k_{n,T}$$

is open and dense. □

Let

$$\mathcal{R} := \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcap_{T=1}^{\infty} \left(\mathcal{H}^k_{n,T} \cup \mathcal{N}^k_{n,T}\right).$$

By the definition, $\mathcal{R}$ is a residual subset of $\mathcal{X}^1(M)$. For given $T > 0$ and any vector field $X \in \mathcal{R}$, if there exists a C^1 neighborhood $\mathcal{U}$ of $X \in \mathcal{R}$ such that any $Y \in \mathcal{U}$ has k periodic points with period belonging to $(T, 3T/2)$, then

$$X \notin \mathcal{N}^k_{n_0,T}, \qquad \text{for some } n_0.$$

Therefore,

$$X \in \mathcal{H}^k_{n_0,T}.$$

□

We prove Theorem 5.2 based on the generic property of vector fields.

Proof of Theorem 5.2. Take a dense G_δ set $\mathcal{R} \subset \mathcal{X}^1(M)$ as Lemma 5.5. For any vector field $X \in \mathcal{R}$, if the vector field is not a star vector field, then for any C^1 neighborhood $\mathcal{U}$ of the vector field X, there is a vector field $Y \in \mathcal{U}$ which has a non-hyperbolic periodic point x. Let T' be the period of x with respect to the vector field Y, then $\psi^Y_{T'}$ has a eigenvalue λ whose module is 1. For any positive $N \in \mathbb{N}$, we consider the following two cases.

λ is real. In this case, $\lambda = \pm 1$. We may assume that $\lambda = 1$ (The case $\lambda = -1$ can be proved similarly). After a C^1 perturbation, one can assume that Y is locally linear in a small neighborhood of the periodic orbit. Therefore, there is an infinite subset $B \subseteq M$ such that

$$\varphi^Y_{T'}|B = Id.$$

We can find at least e^N fixed points of $\varphi^Y_{T'}$ in a small cross section at x. By Lemma 5.5, X has e^N periodic orbits whose periods belong to $([T']-1, 3([T'-1])/2)$. Consequently, one has

$$\frac{1}{2T'}\log \sharp P_{2T'}(X) \geq \frac{N}{2}.$$

λ is not real. In this case, $D\varphi^Y_{T'}$ is a rotation map on the sub-eigenspace V with respect to the eigenvalue λ. One can also assume that Y is locally linear in a small neighborhood of the periodic orbit after a C^1 perturbation, one can also assume that $\varphi^Y_{T'}|V$ is a rational rotation by perturbation. Thus there is a positive $k \in \mathbb{N}$ such that

$$\varphi^Y_{kT'}|V = Id.$$

Therefore, one can find at least e^{NT} fixed points of φ^Y_T, where $T = kT'$. Consequently,

$$\frac{1}{T}\log \sharp P_T(Y) \geq N.$$

In any case, for any C^1 neighborhood $\mathcal{U}$ of the vector field X and every positive $n \in \mathbb{N}$, there are vector field $Y \in \mathcal{U}$ and time $T_0 = T_0(n)$ such that the vector field Y has at least e^{nT_0} periodic orbits whose periods is T_0. By Lemma 5.5, the vector field X has at least e^{nT_0} periodic orbits whose periods belong to $(T_0, 3T_0/2)$. By the arbitrariness of n, one has

$$\overline{\lim_{T\to\infty}}\frac{1}{T}\log \sharp P_T(X) = \infty.$$

□

5.1.3 Star vector fields

Liao has proved the following estimates for star flows in [53, Proposition 4.4].

Lemma 5.6. *For every star vector field $X \in \mathcal{X}^1(M)$, there are a C^1 neighborhood $\mathcal{U}$ of the vector field X and constants $\eta > 0$, $T_0 > 0$, such that for any periodic orbit $\mathcal{O}$ of every vector field $Y \in \mathcal{U}$ with $\pi(\mathcal{O}) \geq T_0$ and the natural hyperbolic splitting*

$$\mathcal{N}_{\mathcal{O}} = \mathcal{E} \oplus \mathcal{F}$$

with respect to ψ^Y_t, we have

(1) For any $x \in \mathcal{O}$ and every $t \geq T_0$, one has

$$\frac{\| \psi^Y_t|\mathcal{E}_x \|}{m(\psi^Y_t|\mathcal{F}_x)} \leq \mathrm{e}^{-2\eta t}.$$

(2) For any $x \in \mathcal{O}$,

$$\prod_{i=0}^{[\pi(\mathcal{O})/T_0]-1} \| \psi_{T_0}^Y|\mathcal{E}_{\varphi_{iT_0}^Y(x)} \| \leq \mathrm{e}^{-\eta\pi(\mathcal{O})}, \qquad \prod_{i=0}^{[\pi(\mathcal{O})/T_0]-1} m\left(\varphi_{T_0}^Y|\mathcal{F}_{\varphi_{iT_0}^Y(x)}\right) \geq \mathrm{e}^{\eta\pi(\mathcal{O})}.$$

Next, we introduce the ergodic closing lemma for flows and give the statement about the relationship between periodic orbits and metric entropy.

Definition 5.7. *A point $x \in M \setminus \mathrm{Sing}(X)$ is* strongly closable*, if for any C^1 neighborhood $\mathcal{U}$ of the vector field X, $L > 0$ and any $\delta > 0$, there are vector field $Y \in \mathcal{U}$, $y \in M$ and $\tau_0 > 0$ such that*

(1) $\varphi_{\tau_0}^Y(y) = y$.

(2) $d(\varphi_t^Y(y), \varphi_t^X(x)) < \delta, \quad \forall \ \ 0 \leq t \leq \tau_0$.

(3) $X = Y$ on $M \backslash B$, where $B = \bigcup_{t\in[-L,0]} B(\varphi_t^X(x), \delta)$.

Denote by $\Sigma(X)$ the set of all strongly closable points of the vector field X, Wen [74, Theorem 3.9] gave the following flow version of the ergodic closing lemma.

Theorem 5.8. *([74, Theorem 3.9]) For any C^1 vector field X and any φ_t^X-invariant Borel probability measure μ, one has*

$$\mu(\mathrm{Sing}(X) \cup \Sigma(X)) = 1.$$

Proof of Theorem 5.3. According to [73, Theorem E], μ is a hyperbolic measure. Let

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u$$

be the hyperbolic Oseledec splitting with respect to the hyperbolic measure μ. By Lemma 5.6, there are $\eta > 0$, $T_0 > 0$ and a C^1 neighborhood $\mathcal{U}$ of the vector field X, such that for every periodic orbit $\mathcal{O}$ of any vector field $Y \in \mathcal{U}$ with $\pi(\mathcal{O}) \geq T_0$ and the natural hyperbolic splitting

$$\mathcal{N}_{\mathcal{O}} = \mathcal{E} \oplus \mathcal{F}$$

with respect to ψ_t^Y, one has

$$\frac{\| \psi_t^Y|\mathcal{E}_x \|}{m(\psi_t^Y|\mathcal{F}_x)} \leq \mathrm{e}^{-2\eta t}, \qquad \forall \ t \geq T_0, \forall \ x \in \mathcal{O};$$

$$\prod_{i=0}^{[\pi(\mathcal{O})/T_0]-1} \| \psi_{T_0}^Y|\mathcal{E}_{\varphi_{iT_0}^Y(x)} \| \leq \mathrm{e}^{-\eta\pi(\mathcal{O})}, \qquad \prod_{i=0}^{[\pi(\mathcal{O})/T_0]-1} m\left(\psi_{T_0}^Y|\mathcal{F}_{\varphi_{iT_0}^Y(x)}\right) \geq \mathrm{e}^{\eta\pi(\mathcal{O})}, \quad \forall \ x \in \mathcal{O}.$$

Since $h_\mu(X) > 0$ and the metric entropy on any critical element is zero, we will assume that μ does not support on any critical element for the rest of proof. Define

$$B(\mu) = \left\{ x : \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(x))dt = \int f d\mu, \ \forall \text{ continuous function } f : M \to \mathbb{R} \right\}.$$

Since μ is ergodic, one has
$$\mu(B(\mu))=1.$$
Thus, by Theorem 5.8, one has
$$\mu(B(\mu)\cap Supp(\mu)\cap \Sigma(X))=1.$$
For μ almost every point x, there are vector fields $\{X_n\}_{n\in\mathbb{N}}\subset\mathcal{U}$, points $\{x_n\}_{n\in\mathbb{N}}\subset M$ and $\{\tau_n:\tau_n>0\}_{n\in\mathbb{N}}$ with $\varphi^{X_n}_{\tau_n}(x_n)=x_n$ satisfying:

- $d(\varphi_t^{X_n}(x_n),\varphi_t^X(x))<\frac{1}{n}$, for $\forall\ t\in[0,\tau_n]$.

- $\|X_n-X\|_{C^1}<\frac{1}{n}$.

Consider the ergodic measure μ_n which is supported on the orbit of x_n. Since the point x is strongly closable point, for any continuous function f, one has
$$\lim_{n\to\infty}\int f d\mu_n=\lim_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n}f(\varphi_t(x_n))dt=\lim_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n}f(\varphi_t(x))dt=\int fd\mu.$$
Thus,
$$\mu_n\to\mu,\quad \text{in the sense of weak}^* \text{ topology}.$$
As μ is not supported on any critical element, one has
$$\tau_n\to\infty \text{ as } n\to\infty.$$

Claim. *There are only finitely many sinks or sources among* $\{Orb(x_n)\}$.

Proof of Claim. If not, for fixed $x\in B(\mu)\cap Supp(\mu)\cap\Sigma(X)$, we may assume that $Orb(x_n)$ are sinks, then one only has
$$\prod_{i=0}^{[\tau_n/T_0]-1}\|\psi^{X_n}_{T_0}(\varphi^{X_n}_{iT_0}(x_n))\|\leq \mathrm{e}^{-\eta\tau_n}.$$
By the definition of the extended linear Poincaré flow, one has
$$\prod_{i=0}^{[\tau_n/T_0]-1}\|\widetilde{\psi}^{X_n}_{T_0}(\varphi^{X_n}_{iT_0}(x))\|\leq \mathrm{e}^{-\eta\tau_n}.$$
Since $\|X_n-X\|_{C^1}<\frac{1}{n}$ and
$$d(\varphi_t^{X_n}(x_n),\varphi_t^X(x))<\frac{1}{n},\ \text{for any } t\in[0,\tau_n],$$
one has
$$\prod_{i=0}^{[\tau_n/T_0]-1}\|\widetilde{\psi}^{X}_{T_0}(\varphi^{X}_{iT_0}(x))\|\leq \mathrm{e}^{-\eta\tau_n}.$$

By Lemma 3.5, the definition of the transgression of a measure, one has

$$\int \log \| \widetilde{\psi}_{T_0}^{X} \| \, d\widetilde{\mu}(x) \leq -\eta.$$

It means that

$$\int \log \| \psi_{T_0}^{X} \| \, d\mu(x) \leq -\eta.$$

Therefore, the Lyapunov exponents of the linear Poincaré flow ψ_t are negative. By the Ruelle inequality [75, Theorem 2], one can get

$$h_\mu(\varphi_{T_0}) = 0.$$

Since μ is an ergodic measure,

$$h_\mu(\varphi_{T_0}) = |T_0| h_\mu(\varphi_1) = |T_0| h_\mu(X) > 0.$$

This is a contradiction. The claim is thus proved. $\square$

By Lemma 5.6, for the non-trivial hyperbolic splitting

$$\mathcal{N}_{Orb(x_n)} = \mathcal{E} \oplus \mathcal{F}$$

with respect to $\psi_t^{X_n}$, we have

$$\prod_{i=0}^{[\tau_n/T_0]-1} \| \psi_{T_0}^{X_n} | \mathcal{E}_{\varphi_{iT_0}^{X_n}(y)} \| \leq \mathrm{e}^{-\eta\tau_n}, \qquad \prod_{i=0}^{[\tau_n/T_0]-1} m(\psi_{T_0}^{X_n} | \mathcal{F}_{\varphi_{iT_0}^{X_n}(y)}) \geq \mathrm{e}^{\eta\tau_n}, \ \forall \ y \in Orb(x_n). \tag{4}$$

We may assume that the indices of $Orb(x_n)$ are the same, then there is a dominated splitting on the limit point x as

$$\mathcal{N}_x = \mathcal{G}_x \oplus \mathcal{H}_x,$$

where $\mathcal{G}_x = \lim_{n\to\infty} \mathcal{E}_{x_n}$ and $\mathcal{H}_x = \lim_{n\to\infty} \mathcal{F}_{x_n}$. We only need to prove

$$\mathcal{G}_x = \mathcal{E}_x^s, \ \mathcal{H}_x = \mathcal{F}_x^u.$$

By Lemma 3.5, the inequalities (4) means

$$\int \log \| \widetilde{\psi}_{T_0}^{X_n} | \mathcal{E}_x \| \, d\widetilde{\mu}_n(x) \leq -\eta, \qquad \int \log m(\widetilde{\psi}_{T_0}^{X_n} | \mathcal{F}_x) d\widetilde{\mu}_n(x) \geq \eta.$$

By Lemma 3.2, since

$$\|X_n - X\|_{C^1} < \frac{1}{n}, \text{ for every positive } n \in \mathbb{N},$$

one has

$$\int \log \| \widetilde{\psi}_{T_0}^{X} | \mathcal{E}_x \| \, d\widetilde{\mu}_n(x) \leq -\eta, \qquad \int \log m(\widetilde{\psi}_{T_0}^{X} | \mathcal{F}_x) d\widetilde{\mu}_n(x) \geq \eta.$$

By Lemma 3.2 again, one has

$$\int \log \| \widetilde{\psi}_{T_0}^X|\mathcal{G}_x \| \, d\widetilde{\mu}(x) \le -\eta, \qquad \int \log m(\widetilde{\psi}_{T_0}^X|\mathcal{H}_x) d\widetilde{\mu}(x) \ge \eta.$$

According to the Birkhoff ergodic theorem and Lemma 3.5, one has

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \log \| \psi_{T_0}^X|\mathcal{G}_{\varphi_{iT_0}(x)} \| &= \lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \log \| \widetilde{\psi}_{T_0}^X|\mathcal{G}_{\varphi_{iT_0}(x)} \| \\
&= \int \log \| \widetilde{\psi}_{T_0}^X|\mathcal{G}_x \| \, d\widetilde{\mu}(x) \le -\eta,
\end{aligned}$$

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \log m(\psi_{T_0}^X|\mathcal{H}_{\varphi_{iT_0}(x)}) &= \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \log m(\widetilde{\psi}_{T_0}^X|\mathcal{H}_{\varphi_{iT_0}(x)}) \\
&= \int \log m(\widetilde{\psi}_{T_0}^X|\mathcal{H}_x) d\widetilde{\mu}(x) \ge \eta.
\end{aligned}$$

It means that

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} \log \| \psi_{T_0}^X|\mathcal{G}_{\varphi_{iT_0}(x)} \| \le -\eta < 0.$$

That implies that

$$\mathcal{G}_x \subset \mathcal{E}_x^s.$$

If $\mathcal{E}_x^s \not\subseteq \mathcal{G}_x$, then there is a non-zero vector v belong to $\mathcal{E}_x^s$ but not belong to $\mathcal{G}_x$. One has the dominated splitting

$$v = v_1 + v_2,$$

where $v_1 \in \mathcal{G}_x,\ 0 \ne v_2 \in \mathcal{H}_x$. Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \log\|\psi_t^X(v)\| = \lim_{t\to+\infty} \frac{1}{t} \log\|\psi_t^X(v_2)\| \ge \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \log m(\widetilde{\psi}_{T_0}^X|\mathcal{H}_{\varphi_{iT_0}(x)}) > 0.$$

This contradicts to the fact that the Lyapunov exponents along $\mathcal{E}_x^s$ are negative. Consequently, one has

$$\mathcal{E}_x^s \subset \mathcal{G}_x.$$

Therefore, one has

$$\mathcal{E}_x^s = \mathcal{G}_x.$$

Similarly, one obtains

$$\mathcal{F}_x^u = \mathcal{H}_x.$$

□

5.2 A shadowing lemma with time control

For the linear Poincaré flow, one has the shadowing lemma of Liao for some quasi-hyperbolic orbit segments.

Definition 5.9. *Assume that $\Lambda \subset M \setminus \mathrm{Sing}(X)$ is an invariant (not necessarily compact) set having a dominated splitting*

$$\mathcal{N}_\Lambda = \mathcal{E} \oplus \mathcal{F}$$

with respect to the linear Poincaré flow. Given $\eta > 0$ and $T_0 > 0$, an orbit arc $\varphi^X_{[0,T]}(x) \subset \Lambda$ with $T > T_0$ is (η, T_0)-quasi hyperbolic *(associated to Λ) if there is a time partition*

$$0 = t_0 < t_1 < t_2 < \cdots < t_l = T$$

with $t_{i+1} - t_i \leq T_0,\ i = 0, \cdots, l-1$ such that for $k = 0, 1, \cdots, l-1$, one has

$$\prod_{i=0}^{k-1} \| \psi^*_{t_{i+1}-t_i} | \mathcal{E}_{\varphi^X_{t_i}(x)} \| \leq e^{-\eta t_k}, \qquad \prod_{i=k}^{l-1} m\left(\psi^*_{t_{i+1}-t_i} | \mathcal{F}_{\varphi^X_{t_i}(x)}\right) \geq \mathrm{e}^{\eta(T-t_k)}.$$

For obtaining a periodic orbit, we have the Liao's shadowing Lemma (see [54, Theorem 5.5] and [76, Theorem I]) which means the recurrent quasi-hyperbolic orbits whose initial point and terminal point are far away from $\mathrm{Sing}(X)$ can be shadowed by periodic orbits.

Theorem 5.10. *(Liao's shadowing Lemma) Suppose $\Lambda \subset M \setminus \mathrm{Sing}(X)$ is an invariant set with a dominated splitting*

$$\mathcal{N}_\Lambda = \mathcal{E} \oplus \mathcal{F}.$$

Given $\varepsilon_0 > 0$ and two constants $\eta > 0,\ T_0 \geq 1$, for every $\varepsilon > 0$, there is $\delta > 0$ such that for any orbit segment $\varphi^X_{[0,T]}(x) \subset \Lambda$ with the following properties:

- *$d(x, \mathrm{Sing}(X)) \geq \varepsilon_0$ and $d(\varphi^X_T(x), \mathrm{Sing}(X)) \geq \varepsilon_0$.*
- *$\varphi^X_{[0,T]}(x)$ is (η, T_0)-quasi hyperbolic.*
- *$d(x, \varphi^X_T(x)) < \delta$.*

Then there exist a C^1 increasing homeomorphism

$$\theta : [0, T] \to \mathbb{R}$$

and a periodic point $p \in M$ whose period is $\theta(T)$ such that:

(1) $1 - \varepsilon < \theta'(t) < 1 + \varepsilon,\ \forall\ t \in [0, T]$.

(2) $d(\varphi^X_t(x), \varphi^X_{\theta(t)}(p)) \leq \varepsilon |X(\varphi^X_t(x))|,\ \forall\ t \in [0, T]$.

In Theorem 5.10, for any $t \in [0,T]$, one can have

$$\varphi^X_{\theta(t)}(p) \in \exp_{\varphi_t(x)}\left(\mathcal{N}_{\varphi_t(x)}(2\varepsilon|X(\varphi_t(x))|)\right),$$

where for any regular point y and any $\chi > 0$, $\mathcal{N}_y(\chi)$ is defined as

$$\mathcal{N}_y(\chi) := \{v \in \mathcal{N}_y : \ \|v\| \le \chi\}.$$

In fact, one can get more information on the periodic orbit.

Proposition 5.11. *Under the setting of Theorem 5.10, if the time $T = mT_0$ for some $m \in \mathbb{R}^+$, then there is $N = N(\eta, T_0)$ such that*

$$|\theta(\tau) - \tau| \le N \cdot d(x, \varphi^X_T(x)), \quad \forall\, \tau \in \mathbb{N} \cap [0,T].$$

For a normed vector space A and $r > 0$, define

$$A(r) = \{v \in A : \ \|v\| < r\}.$$

Recall that $N_x(\chi) = \exp_x \mathcal{N}_x(\chi)$.

Lemma 5.12. *For the flow φ^X_t generated by the vector field $X \in \mathcal{X}^1(M^n)$, there are two constants $C > 0$, $\delta > 0$ such that for any $y \in N_x(\delta|X(x)|)$, there is a unique $t = t(y) \in [0,2]$ such that $\varphi_t(y) \in N_{\varphi_1(x)}(\delta)$ and*

$$|t(y) - 1| < C \cdot d(x,y).$$

Proof. We take $\varepsilon_0 > 0$ such that the exponential map $\exp_x$ is a diffeomorphism on the ball $T_xM(\varepsilon_0)$. For any point $x \in M$ and any point y which is close to the point x, one can lift the local orbit of the point y to the tangent space T_xM in the following way: for any $v \in T_xM$, if $\|v\| < \varepsilon_0$, then one can define a local flow

$$\widetilde{\varphi}_t(v) = \exp_x^{-1} \circ \varphi_t \circ \exp_x(v).$$

Then, the flow $\widetilde{\varphi}_t$ is a local flow generated by a C^1 vector field $\widetilde{X}_x$ on T_xM, where

$$\widetilde{X}_x(v) = D(\exp_x^{-1}) \circ X(\exp_x(v)).$$

Thus, one has

$$K := \sup_{x\in M,\ v \in T_xM(\varepsilon_0)} \{|\widetilde{X}_x(v)|,\ \|D\widetilde{X}_x(v)\|\} < \infty.$$

Now for any regular point $x \in M$, one can identify T_xM to the Euclidean space $\mathbb{R}^n$ by some isometrical transformation satisfying

$$e_1 = \frac{X(x)}{|X(x)|}$$

for an orthonormal basis $e = \{e_1, \cdots, e_n\}$ of the Euclidean space $\mathbb{R}^n$. In this way, the flow $\widetilde{\varphi}_t$ can be regard as the solution of the differential equation:

$$\frac{dz}{dt} = \widetilde{X}_x(z).$$

Given $\varepsilon > 0$, by reducing ε_0 if necessary, one can assume that the map $D\exp_x(v)$ is ε-close to the identity map for any x and $v \in T_xM(\varepsilon_0)$.

Claim. *There is $\delta > 0$ such that for any regular point x, one has*

$$N_x(\delta|X(x)|) \cap \mathrm{Sing}(X) = \emptyset.$$

Proof of the Claim. It suffices to consider the flow $\widetilde{\varphi}_t$ and the vector field $\widetilde{X}_x$. Given $\delta < \varepsilon/K$, for any $v \in N_x(\delta|X(x)|)$, one has

$$|\widetilde{X}_x(v)| \geq |\widetilde{X}_x(0)| - \max_{\xi \in T_x M(\varepsilon_0)} \|D\widetilde{X}_x(\xi)\|.|v| \geq |X(x)| - K\delta|X(x)| \geq (1-\varepsilon)|X(x)| > 0.$$

Since the map $D\exp_x$ is ε-close to identity, one can conclude. $\square$

Claim. *By reducing δ if necessary, for any regular point x, any $y \in N_x(\delta|X(x)|/2)$ and any $t \in [\delta/3, 2\delta/3]$, there is a unique $s = s(t,y) \in [0,\delta]$ such that*

$$\varphi_s(y) \in N_{\varphi_t(x)}(\delta).$$

Proof of the Claim. One can work in the local chart introduced above. By reducing δ if necessary such that for any $|v| \leq \delta|X(x)|$, one has

$$\sup_{t\in(-\delta,\delta)} \frac{|\widetilde{X}_x(v)|}{|\widetilde{X}_x(\widetilde{\varphi}_t(v))|} < 1 + \frac{\varepsilon}{K}, \quad \sup_{t\in(-\delta,\delta)} \angle(\widetilde{X}_x(v), \widetilde{X}_x(\widetilde{\varphi}_t(v))) < \frac{\varepsilon}{K}.$$

For any $v \in N_x(\delta|X(x)|/3)$ and the time t_0 satisfying

$$|\widetilde{\varphi}_{t_0}(v)| = \delta\left|\widetilde{X}_x(\widetilde{\varphi}_{t_0}(0))\right|, \qquad |\widetilde{\varphi}_s(v)| \leq \delta\left|\widetilde{X}_x(\widetilde{\varphi}_s(0))\right|, \ \forall\ s \in [0,t_0),$$

one has

$$|\widetilde{\varphi}_{t_0}(v)| = \left|v + \int_0^{t_0} \widetilde{X}_x(\widetilde{\varphi}_t(v))dt\right| \leq |v| + (1+\varepsilon)t_0|\widetilde{X}_x(0)|.$$

Thus,

$$\delta\left|\widetilde{X}_x(0)\right| \leq \delta(1+\varepsilon/K)\left|\widetilde{X}_x(0)\right| \leq \delta/3\left|\widetilde{X}_x(0)\right| + (1+\varepsilon)t_0\left|\widetilde{X}_x(0)\right|.$$

Consequently, one has

$$t_0 \geq \frac{2\delta}{3(1+\varepsilon)} \geq \frac{\delta}{2}.$$

Similar estimate gives the fact that $t_0 \leq 2\delta$ by reversing the inequalities. $\square$

Recall the definitions of the *sectional Poincaré map* and the *rescaled sectional Poincaré map*, the above claim allows one to consider the non-linear dynamics along the flows: for any regular point x and t, the sectional Poincaré map

$$\mathcal{P}_t : \mathcal{N}_x(\delta|X(x)|) \to \mathcal{N}_{\varphi_t(x)}(\delta)$$

is the lift of the holonomy map induced by the local flow from $\exp_x(\mathcal{N}_x(\delta|X(x)|))$ to $\exp_{\varphi_t(x)}(\mathcal{N}_{\varphi_t(x)}(\delta))$. The rescaled sectional Poincaré map

$$\mathcal{P}^*_t : \mathcal{N}_x(\delta) \to \mathcal{N}_{\varphi_t(x)}(\delta)$$

is defined by

$$\mathcal{P}_t^*(v) = \frac{\mathcal{P}_t(|X(x)|v)}{|X(\varphi_t(x))|}, \quad \forall\, v \in \mathcal{N}_x(\delta).$$

In the local coordinate, by the choice of e_1, one can denote the space $N_x(r)$ by

$$N_x(r) = \sum_{i=2}^{n} y_i e_i,$$

where $r = \sqrt{\sum_{i=2}^n y_i^2}$ and $(y_2, \cdots, y_n) \in \mathbb{R}^{n-1}$. By abuse of the notions, denote $\sum y_i e_i$ by y. In this local coordinate, one can present $\mathcal{P}_t^*$. Define the map $\tau : \mathcal{N}_x(\delta/2) \to \mathbb{R}^1$ such that

$$\frac{\widetilde{\varphi}_{\tau(y)} \circ \widetilde{\varphi}_t(|\widetilde{X}_x(x)|y)}{|\widetilde{X}_x(\widetilde{\varphi}_t(x))|} \in \mathcal{N}_{\varphi_t(x)}(\delta),$$

for any $y \in \mathcal{N}_x(\delta/2)$. From the above facts, the map τ is an injective. In the local chart, the rescaled sectional Poincaré map $\mathcal{P}_t^*$ can be written by

$$\mathcal{P}_t^*(y) := \frac{\widetilde{\varphi}_{\tau(y)} \circ \widetilde{\varphi}_t(|\widetilde{X}_x(x)|y)}{|\widetilde{X}_x(\widetilde{\varphi}_t(x))|}.$$

We are going to estimate $\tau(y)$. For $t \in [\delta/3, 2\delta/3]$, consider the function

$$H(x,t,y,\tau) = \left\langle \frac{\widetilde{\varphi}_{\tau(y)} \circ \widetilde{\varphi}_t(|\widetilde{X}_x(x)|y)}{|\widetilde{X}_x(\widetilde{\varphi}_t(x))|}, \frac{\widetilde{X}_x(\widetilde{\varphi}_t(x))}{|\widetilde{X}_x(\widetilde{\varphi}_t(x))|} \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the local Euclidean coordinate. From the definition of map τ, one has

1. $H(x, t, y_0, \tau(y_0)) = 0$ from the definition of map τ.
2. $\partial H/\partial y$ and $\partial H/\partial \tau$ are equi-continuous with respect to x and t.
3. $\partial H/\partial \tau|_{y=0,\tau=0} = \left\langle \frac{\widetilde{X}_x(\widetilde{\varphi}_t(x))}{|\widetilde{X}_x(\widetilde{\varphi}_t(x))|}, \frac{\widetilde{X}_x(\widetilde{\varphi}_t(x))}{|\widetilde{X}_x(\widetilde{\varphi}_t(x))|} \right\rangle = 1.$

By the Implicit Function Theorem, one has

$$\frac{\partial \tau}{\partial y} = -\frac{\partial H/\partial y}{\partial H/\partial \tau}.$$

Since φ_t is C^1 and $\varphi_t(y) \in T_x M(\varepsilon_0)$, $\partial H/\partial \tau$ is uniformly bounded away from zero and $\partial H/\partial y$ is uniformly bounded. Therefore, $\partial \tau/\partial y$ is uniformly bounded with respect to y. This means

$$|s(t,y) - t| = |\tau(y)| \le C_0 \cdot d(x,y),$$

where C_0 is a constant decided by $\partial \tau/\partial y$. If we split 1 into different time t, where $t \in [\delta/3, 2\delta/3]$, then we can get at most

$$\left\lfloor \frac{3}{\delta} \right\rfloor + 1$$

different t. By adding the time consecutively, one can get the constant C. $\square$

Fix $\eta > 0$, $T_0 \geq 1$, the following Lemma 5.13 will show that the distance between (η, T_0)-quasi hyperbolic orbit and its shadowing periodic orbit can be controlled by the distance between the starting point and terminal point of this quasi hyperbolic orbit. The idea about proof of Lemma 5.13 can refer to [3, Page 269, Corollary 6.4.17].

Lemma 5.13. *Under the assumption of Proposition 5.11, taking $\alpha = e^{-\eta/2}$, there is a constant $C > 0$ such that for the (η, T_0)-quasi hyperbolic orbit $\varphi_{[0,mT_0]}(x)$ and the shadowing orbit $\varphi_{[0,\theta(mT_0)]}(p)$ in Theorem 5.10, one has*

$$d(\varphi_{iT_0}(x),\ \varphi_{\theta(iT_0)}(p)) \leq C \cdot \alpha^{min\{i,\ m-i\}} \cdot d(x,\ \varphi_{mT_0}(x)),\ \forall\ i \in [0, n].$$

Proof. By Theorem 5.10, one can assume that

$$\varphi^X_{\theta(t)}(p) \in \exp_{\varphi_t(x)}\left(\mathcal{N}_{\varphi_t(x)}(2\varepsilon|X(\varphi_t(x))|)\right),\ \text{for any } t \in [0, T].$$

Thus, one can lift the dynamics in the normal fibers and consider a sequence of rescaled sectional Poincaré maps

$$\left\{\mathcal{P}^*_{T_0,\varphi_{iT_0}(x)} : \mathcal{N}_{\varphi_{iT_0}(x)} \to \mathcal{N}_{\varphi_{(i+1)T_0}(x)}\right\}.$$

Since $\varphi_{[0,mT_0]}(x)$ is a (η, T_0)-quasi hyperbolic orbit, by Definition 5.9, for the dominated splitting

$$\mathcal{N}_\Lambda = \mathcal{E} \oplus \mathcal{F}$$

with respect to the linear Poincaré flow, one has

$$\prod_{i=0}^{k-1} \parallel \psi^*_{T_0}|\mathcal{E}_{\varphi_{(i-1)T_0}(x)} \parallel \leq e^{-k\eta}, \quad \prod_{i=k}^{m-1} m\left(\psi^*_{T_0}|\mathcal{F}_{\varphi_{iT_0}(x)}\right) \geq \mathrm{e}^{(m-k)\eta},\ \text{for } k = 0, \cdots, m.$$

By Liao's shadowing Lemma 5.10, one has

$$d(\varphi_t(x), \varphi_{\theta(t)}(p)) \leq \varepsilon|X(\varphi_t(x))|,\ \forall\ t \in [0, T].$$

For the maps $\left\{\mathcal{P}^*_{T_0,\varphi_{iT_0}(x)}\right\}$, the periodic orbit of p also admits a dominated splitting

$$\mathcal{N}_x = \mathcal{E}_p \oplus \mathcal{F}_p$$

with respect to $\left\{D\mathcal{P}^*_{T_0,\varphi_{iT_0}(x)}\right\}$. Thus, they have plaques in the normal fibers. The distance in the $\mathcal{E}$-plaques are denoted by $d_\mathcal{E}$ and the the distance in the $\mathcal{F}$-plaques are denoted by $d_\mathcal{F}$. One can split the distance into $\mathcal{E}$-distance and $\mathcal{F}$-distance. There is a constant $C \geq 1$ such that

$$d_\mathcal{F}(p, x) \leq Cd(p, x), \quad d_\mathcal{E}(p, x) \leq Cd(p, x), \quad d(p, x) \leq d_\mathcal{E}(p, x) + d_\mathcal{F}(p, x).$$

Therefore,

$$\begin{aligned}
d(\varphi_{iT_0}(x),\varphi_{\theta(iT_0)}(p)) &\le d_{\mathcal{E}}(\varphi_{iT_0}(x),\varphi_{\theta(iT_0)}(p))+d_{\mathcal{F}}(\varphi_{iT_0}(x),\varphi_{\theta(iT_0)}(p))\\
&\le \alpha^i d_{\mathcal{E}}(x,p)+\alpha^{m-i}d_{\mathcal{F}}(\varphi_{mT_0}(x),\varphi_{\theta(mT_0)}(p))\\
&\le \alpha^i d(x,p)+\alpha^{m-i}d(\varphi_{mT_0}(x),\varphi_{\theta(mT_0)}(p))\\
&\le \alpha^{\min\{i,m-i\}}(d(x,p)+d(\varphi_{mT_0}(x),\varphi_{\theta(mT_0)}(p)))
\end{aligned}$$

By adapting a generalized shadowing lemma of Gan [77, Theorem 1.1], by enlarging C if necessary, one has

$$d(x,\ p)\le Cd(x,\varphi_T(x)),\quad d(\varphi_T(x),\varphi_{\theta(T)}(p))\le Cd(x,\varphi_T(x)).$$

Therefore,

$$d(\varphi_{iT_0}(x),\varphi_{\theta(iT_0)}(p))\le C^2\cdot\alpha^{min\{i,\ m-i\}}\cdot d(x,\varphi_{mT_0}(x)),\ \forall\ i\in[0,m].$$

$\square$

Proof of Proposition 5.11. By Theorem 5.10, given $\varepsilon_0>0$, $\eta>0$ and $T_0\ge 1$, for every $\varepsilon>0$, there is $\tilde{\delta}>0$ such that for any (η,T_0)-quasi hyperbolic orbit segment $\varphi^X_{[0,T]}(x)\subset\Lambda$ satisfying

$$d(x,\mathrm{Sing}(X))\ge\varepsilon_0,\quad d(\varphi_T(x),\mathrm{Sing}(X))\ge\varepsilon_0,\quad d(x,\varphi_T(x))<\tilde{\delta},$$

there is a periodic point $p\in M$ and a C^1 strictly increasing function θ such that $\varphi_{\theta(T)}(p)=p$ and

$$d(\varphi_t(x),\varphi_\theta(t)(p))\le\varepsilon|X(\varphi_t(x))|,\ \text{for any}\ t\in[0,T].$$

Consider a time partition

$$0=t_0<t_1<t_2<t_3<\cdots<t_m=T$$

with $t_{i+1}-t_i=T_0$, taking $\alpha=\mathrm{e}^{-\eta/2}\in(0,1)$, by Lemma 5.13, there is a constant $C_1>0$ such that

$$d(\varphi_\theta(t_i)(p),\varphi_{t_i}(x))\le C_1\cdot\alpha^{\min\{i,m-i\}}\cdot d(x,\varphi_T(x)),\ i=0,1,\dots,m.$$

On the fact that

$$d(\varphi_t(x),\varphi_{\theta(t)}(p))\le,\varepsilon|X(\varphi_t(x))|,\ \text{for every}\ t\in[0,T],$$

by Lemma 5.12, there is a constant $C_2=C_2(T_0)>0$ such that for any

$$\varphi_\theta(t_i)(p)\in N_{\varphi_{t_i}(x)}(\delta),\ i=0,1,\dots,m,$$

one has

$$|\theta(t_{i+1})-\theta(t_i)-(t_{i+1}-t_i)|\le C_2\cdot d(\varphi_{t_i}(x),\varphi_{\theta(t_i)}(p)),\ \forall\ i=0,1,\dots,m.$$

Let $N := N(\eta, T_0) = \dfrac{2C_2 \cdot C_1}{1-\alpha}$, one has

$$\begin{aligned} |\theta(T) - T| &\le \sum_{i=0}^{m-1} |\theta(t_{i+1}) - \theta(t_i) - (t_{i+1} - t_i)| \le C_2 \cdot \sum_{i=0}^{m-1} d(\varphi_{t_i}(x), \varphi_{\theta(t_i)}(p)) \\ &\le C_2 \cdot C_1 \cdot d(\varphi_T(x), x) \sum_{i=0}^{m-1} \alpha^{\min\{i,\ m-i\}} \le N \cdot d(\varphi_T(x), x). \end{aligned}$$

From the above discussion, for any $\tau \in \mathbb{N} \cap [0, T]$, we have

$$|\theta(\tau) - \tau| \le \sum_{i=0}^{m-1} |\theta(t_{i+1}) - \theta(t_i) - (t_{i+1} - t_i)| \le N \cdot d(x, \varphi_T^X(x)).$$

□

5.3 Periodic orbits of vector fields by shadowing

5.3.1 Pesin block of vector fields

For a regular hyperbolic ergodic measure μ and its hyperbolic Oseledec splitting

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u,$$

by definition of ψ_t^*, for μ-almost every point x, one has

$$\lambda^-(\mu) = \lim_{t\to\infty} \frac{1}{t} \log \| \psi_t^* | \mathcal{E}_x^s \| < 0, \qquad \lambda^+(\mu) = \lim_{t\to\infty} \frac{1}{t} \log m(\psi_t^* | \mathcal{F}_x^u) > 0 \tag{5}$$

Lemma 5.14. *If the hyperbolic Oseledec splitting of a regular hyperbolic ergodic measure μ is a dominated splitting, then for any $\varepsilon > 0$, there exists $T(\varepsilon) \in \mathbb{R}$ such that for μ-almost every point $x \in M$ and every $T \ge T(\varepsilon)$, one has*

$$\lim_{k\to\infty} \frac{1}{k \cdot T} \sum_{i=0}^{k-1} \log \| \psi_T^* | \mathcal{E}^s_{\varphi_{iT}(x)} \| \text{ exists and is contained in } [\lambda^-(\mu), \lambda^-(\mu) + \varepsilon),$$

$$\lim_{k\to\infty} \frac{1}{k \cdot T} \sum_{i=0}^{k-1} \log \| \psi_{-T}^* | \mathcal{F}^u_{\varphi_{-iT}(x)} \| \text{ exists and is contained in } (-\lambda^+(\mu) - \varepsilon, -\lambda^+(\mu)].$$

Proof. Let R be the support of the regular hyperbolic ergodic measure μ and $\widetilde{R}$ be the transgression of R:

$$\widetilde{\Lambda} := \text{Closure} \left\{ \frac{X(x)}{\|X(x)\|} \ : \ x \in R \setminus \text{Sing}(X) \right\}.$$

By Lemma 3.7, $\widetilde{R}$ admits a dominated splitting

$$\mathcal{N}_{\widetilde{R}} SM = \mathcal{E}^s \oplus \mathcal{F}^u$$

with respect to the extended linear Poincaré flow. By Lemma 3.5, one has

$$\lambda^-(\mu) = \lambda^-(\widetilde{\mu}) = \lim_{t\to+\infty} \frac{1}{t} \int \log \parallel \widetilde{\psi}_t|\mathcal{E}^s \parallel \mathrm{d}\widetilde{\mu},$$

where $\widetilde{\mu}$ is the transgression of μ. Therefore, for any $\varepsilon > 0$, there is $T(\varepsilon) > 0$ large enough such that for every $T \geq T(\varepsilon)$, one has

$$\left| \frac{1}{T} \int \log \parallel \widetilde{\psi}_T|\mathcal{E}^s_x \parallel \mathrm{d}\widetilde{\mu} - \lambda^-(\mu) \right| < \varepsilon.$$

Since every vector

$$\frac{X(x)}{\parallel X(x) \parallel} \in \widetilde{R}$$

is a unit vector, by Lemma 3.2, for the fixed $T \geq T(\varepsilon)$, $\widetilde{\psi}_T(\cdot)$ is continuous on

$$NSM := \{(v_1, v_2) : v_1 \in S_xM,\ v_2 \in T_xM,\ v_1 \perp v_2\}.$$

Thus, by the Birkhoff Ergodic Theorem, one has

$$\lim_{k\to\infty} \frac{1}{k\cdot T} \sum_{i=0}^{k-1} \log \parallel \psi^*_T|\mathcal{E}^s_{\varphi_{iT}(x)} \parallel = \frac{1}{T} \int \log \parallel \widetilde{\psi}_T|\mathcal{E}^s_x \parallel d\widetilde{\mu} < \lambda^-(\mu) + \varepsilon.$$

Since the norms are sub-multiplicative, one has

$$\lim_{k\to\infty} \frac{1}{k\cdot T} \sum_{i=0}^{k-1} \log \parallel \psi^*_T|\mathcal{E}^s_{\varphi_{iT}(x)} \parallel \quad \geq \quad \lim_{t\to\infty} \frac{1}{t} \log \parallel \psi^*_t|\mathcal{E}^s_x \parallel = \lambda^-(\mu).$$

The conclusion for sub-bundle $\mathcal{F}^u$ can be obtained similarly. □

Definition 5.15. *Let μ be a regular hyperbolic ergodic measure of $X \in \mathcal{X}^1(M)$,*

$$\mathcal{N}_\Lambda = \mathcal{E}^s \oplus \mathcal{F}^u$$

be the hyperbolic Oseledec splitting, where Λ is a Borel set with μ-total measure. Given $\lambda \in (0,1)$, $L > 0$ and $k \in \mathbb{R}^+$, the Pesin block *$\Lambda^L_\lambda(k)$ is defined as:*

$$\Lambda^L_\lambda(k) := \left\{ x \in \Lambda : \begin{array}{l} \prod_{i=0}^{n-1} \parallel \psi^*_L|\mathcal{E}^s_{\varphi_{iL}(x)} \parallel \leq k\lambda^n,\ \forall\ n \geq 1, \\ \prod_{i=0}^{n-1} \parallel \psi^*_{-L}|\mathcal{F}^u_{\varphi_{-iL}(x)} \parallel \leq k\lambda^n,\ \forall\ n \geq 1,\ d(x, \mathrm{Sing}(X)\cap\Lambda) \geq \frac{1}{k} \end{array} \right\}.$$

Proposition 5.16. *If the hyperbolic Oseledec splitting of a regular hyperbolic ergodic measure μ is a dominated splitting, then the* Pesin block *$\Lambda^L_\lambda(k)$ is a compact set with the property that*

$$\mu(\Lambda^L_\lambda(k)) \to 1 \text{ as } k \to +\infty.$$

where $\lambda := \mathrm{e}^{-\eta}, 0 < \eta < \min\{-\lambda^-(\mu), \lambda^+(\mu)\}$, $L \geq T(\min\{-\lambda^-(\mu), \lambda^+(\mu)\} - \eta)$ as in Lemma 5.14.

Proof. By Definition 5.15, $\Lambda_\lambda^L(k)$ is a compact set. According to the hyperbolicity of the regular measure and the choice of λ, for μ-almost every point $x \in M$, by Lemma 5.14, there is a constant $C(x)$ satisfying

$$\prod_{i=0}^{n-1} \| \psi_L^* | \mathcal{E}^s_{\varphi_{iL}(x)} \| \leq C(x) \cdot \lambda^n, \qquad \prod_{i=0}^{n-1} \| \psi_{-L}^* | \mathcal{F}^u_{\varphi_{-iL}(x)} \| \leq C(x) \cdot \lambda^n, \ \forall\, n \geq 1.$$

Let

$$\Gamma_\lambda^L(k) := \left\{ x \in \Lambda : \ \prod_{i=0}^{n-1} \| \psi_L^* | \mathcal{E}^s_{\varphi_{iL}(x)} \| \leq k\lambda^n, \ \prod_{i=0}^{n-1} \| \psi_{-L}^* | \mathcal{F}^u_{\varphi_{-iL}(x)} \| \leq k\lambda^n, \ \forall\, n \geq 1 \right\}.$$

Therefore,

$$\mu\left(\bigcup_{k>0} \Gamma_\lambda^L(k) \right) = 1.$$

For two real numbers $0 < k_1 < k_2$, one has

$$\Gamma_\lambda^L(k_1) \subset \Gamma_\lambda^L(k_2).$$

Consequently,

$$\mu(\Gamma_\lambda^L(k)) \to 1 \quad \text{as} \quad k \to \infty.$$

According to the facts that

$$\Lambda_\lambda^L(k) \subset \Gamma_\lambda^L(k) \text{ and } \mu(\mathrm{Sing}(X)) = 0,$$

for any $\varepsilon > 0$, there is $K = K(\varepsilon) \in \mathbb{N}$ such that

$$\left| \mu(\Gamma_\lambda^L(k)) - \mu(\Lambda_\lambda^L(k)) \right| < \varepsilon, \quad \forall\, k \geq K.$$

Then

$$\mu(\Lambda_\lambda^L(k)) \to 1 \quad \text{as} \quad k \to \infty.$$

□

5.3.2 Constructing many periodic orbits: proof of Theorem 5.4

We have the following version of Poincaré Recurrence Theorem for flows. It can be deduced by the case of diffeomorphisms. Hence, the proof is omitted.

Proposition 5.17. *Let μ be an φ_t-invariant measure. For any fixed time t_0 and any set B with positive μ-measure, there is a set $R \subset B$ with $\mu(R) = \mu(B)$ and a sequence of integers $0 < n_1 < n_2 < n_3 < \cdots < n_i < \cdots$ such that for every $x \in R$, we have*

1. *$\varphi_{n_i \cdot t_0}(x) \in R$, for all $i \in \mathbb{N}$.*
2. *$d(x, \varphi_{n_i \cdot t_0}(x)) \to 0$, as $i \to \infty$.*

In fact, one can have the following stronger recurrent property.

Proposition 5.18. *Assume that f is a homeomorphism on a compact metric space M. Let μ be an ergodic invariant measure of f. If Λ is a set with positive measure of μ, then given $\delta > 0$ and $\varepsilon > 0$, one has*

$$\lim_{n\to\infty} \mu(\Lambda_n) = \mu(\Lambda),$$

where $\Lambda_n := \{x \in \Lambda : \ \exists\ m \in [n, (1+\varepsilon)n],\ s.t.\ f^m(x) \in \Lambda,\ d(f^m(x), x) < \delta\}$.

Proof. Given $\delta > 0$ and $\varepsilon > 0$, take a finite measurable partition $\mathcal{P} := \{P_i\}_{i=1}^{\ell}$ of M such that

$$\mathrm{diam}(P_i) \le \delta$$

and

$$P_i \subset \Lambda \quad \text{ or } \quad P_i \cap \Lambda = \emptyset, \text{ for } i = 1, 2, \cdots, \ell.$$

Consider the set

$$\Lambda_n(\mathcal{P}) = \{x \in \Lambda : \exists i \in [1, \ell], m \in [n, (1+\varepsilon)n], \text{s.t. } f^m(x) \in \Lambda \text{ and } x, f^m(x) \in P_i \in \mathcal{P}\}.$$

Fix $P_i \subset \Lambda$, define

$$\mathcal{P}^i_{n,\varepsilon} = \left\{ x \in P_i : \sum_{j=0}^{n-1} \chi_{P_i}(f^j(x)) \le n\mu(P_i)(1 + \frac{\varepsilon}{3}), \sum_{j=0}^{\lfloor n(1+\varepsilon) \rfloor} \chi_{P_i}(f^j(x)) \ge n\mu(P_i)(1 + \frac{2\varepsilon}{3}) \right\},$$

where χ_{P_i} is the characteristic function of set P_i. Therefore,

$$\mathcal{P}^i_{n,\varepsilon} \subset P_i \cap \Lambda_n(\mathcal{P}).$$

By The Birkhoff Ergodic Theorem,

$$\mu(P_i \setminus \mathcal{P}^i_{n,\varepsilon}) \to 0 \text{ as } n \to \infty.$$

This implies the proposition. $\square$

Now we are ready to prove Theorem 5.4.

Proof of Theorem 5.4. The proof will follow some steps.

Choose a Pesin block. Since the hyperbolic Oseledec splitting

$$\mathcal{N} = \mathcal{E}^s \oplus \mathcal{F}^u$$

with respect to the ergodic hyperbolic measure μ is a dominated splitting, one can take $\lambda \in (0, 1)$ which is larger than and close to $\mathrm{e}^{-\min\{-\lambda^-(\mu),\ \lambda^+(\mu)\}}$ and L as Lemma 5.14 to define Pesin block (Definition 5.15) $\Lambda^L_\lambda(k)$:

$$\Lambda^L_\lambda(k) := \left\{ x \in \Lambda : \begin{array}{l} \prod_{i=0}^{n-1} \parallel \psi^*_L|\mathcal{E}^s_{\varphi_{iL}(x)} \parallel \le k\lambda^n,\ \forall\ n \ge 1, \\ \prod_{i=0}^{n-1} \parallel \psi^*_{-L}|\mathcal{F}^u_{\varphi_{-iL}(x)} \parallel \le k\lambda^n, \forall\ n \ge 1, d(x, \mathrm{Sing}(X) \cap \Lambda) \ge \frac{1}{k} \end{array} \right\}.$$

By Proposition 5.16, taking $k > 0$ large enough such that

$$\mu(\Lambda_\lambda^L(k)) > 0.$$

One can fix $\lambda_0 \in (\lambda, 1)$, then there is a positive integer $j = j(k) \in \mathbb{N}$ such that for any $x \in \Lambda_\lambda^L(k)$, one has

$$\prod_{i=0}^{n-1} \| \psi^*_{j(k)L}|\mathcal{E}^s_{\varphi_{ij(k)L}(x)} \| \leq \lambda_0^n, \quad \prod_{i=0}^{n-1} \| \psi^*_{-j(k)L}|\mathcal{F}^u_{\varphi_{-ij(k)L}(x)} \| \leq \lambda_0^n, \forall\, n \geq 1.$$

Then, for the set $\Lambda_{\lambda_0}^{L_0}(k)$ defined by

$$\Lambda_{\lambda_0}^{L_0}(k) = \left\{ x \in \Lambda : \begin{array}{l} \prod_{i=0}^{n-1} \| \psi^*_{j(k)L}|\mathcal{E}^s_{\varphi_{ij(k)L}(x)} \| \leq \lambda_0^n, \ \forall\, n \geq 1, \\ \prod_{i=0}^{n-1} \| \psi^*_{-j(k)L}|\mathcal{F}^u_{\varphi_{-ij(k)L}(x)} \| \leq \lambda_0^n, \forall\, n \geq 1, d(x, \mathrm{Sing}(X) \cap \Lambda) \geq \frac{1}{k} \end{array} \right\},$$

where $L_0 = jL$. One has

$$\mu(\Lambda_{\lambda_0}^{L_0}(k)) \geq \mu(\Lambda_\lambda^L(k)) > 0.$$

Hereafter, fix this k. By Proposition 5.17, taking $B = \Lambda_{\lambda_0}^{L_0}(k)$, for μ-almost every point $x \in \Lambda_{\lambda_0}^{L_0}(k)$, the forward orbit of x will return to $\Lambda_{\lambda_0}^{L_0}(k)$ and will be arbitrarily close to x. Let $\eta_0 = -\log \lambda_0$. If

$$\varphi_{nL_0}(x) \in \Lambda_{\lambda_0}^{L_0}(k) \text{ for some } n \in \mathbb{N},$$

then $\varphi_{[0,nL_0]}(x)$ is (η_0, L_0)-quasi hyperbolic orbit arc.

The shadowing constants. Let

$$C := \max\left\{1, \max_{x \in M} |X(x)|\right\}.$$

Given $\varepsilon_0 = 1/k$, $\eta = \eta_0$, $T_0 = L_0$ and $\varepsilon > 0$, for $\varepsilon_1 = \varepsilon/3C$, by Theorem 5.10, there is a constant $\delta = \delta(\varepsilon)$ much smaller than ε such that for any x, $\varphi_{nL_0}(x) \in \Lambda_{\lambda_0}^{L_0}(k)$, if

$$d(x, \varphi_{nL_0}(x)) < \delta,$$

then there is a point $p \in M$ and C^1 strictly increasing function

$$\theta : [0, nL_0] \to \mathbb{R}$$

such that $p = \varphi_{\theta(nL_0)}(p)$ and

$$d(\varphi_t(x), \varphi_{\theta(t)}(p)) \leq \varepsilon_1 |X(\varphi_t(x))| < \varepsilon/3, \text{ for all } t \in [0, nL_0].$$

Moreover, by Proposition 5.11, one has

$$|\theta(t) - t| \leq N\delta, \quad \text{for any integer } t \in [0, nL_0],$$

where N is a constant which is independent of x and n. One can also assume that $N\delta$ is much smaller than ε.

A seperation set K_n. For $\varepsilon > 0$ and $n \in \mathbb{N}$, we claim that there is a finite set $K_n = K_n(k, \varepsilon) \subset \Lambda_{\lambda_0}^{L_0}(k)$ with the following properties:

1. For points $x, y \in K_n$, there is an integer $t \in [0, nL_0]$ such that
$$d(\varphi_t(x), \varphi_t(y)) > \varepsilon.$$
2. For any $x \in K_n$, there is an integer $m = m(n)$ with $n < m \leq (1+\varepsilon)n$ such that
$$\varphi_{mL_0}(x) \in \Lambda_{\lambda_0}^{L_0}(k) \text{ and } d(x, \varphi_{mL_0}(x)) < \delta(\varepsilon).$$
3. $\displaystyle \varliminf_{\varepsilon \to 0} \varliminf_{n \to \infty} \frac{1}{nL_0} \log \sharp K_n \geq \mathrm{h}_\mu(\varphi_1).$

The construction of K_n. Now, we give the precise construction of K_n. We consider the following sets:

$$\Lambda_{\lambda_0}^{L_0}(k, n) = \left\{x \in \Lambda_{\lambda_0}^{L_0}(k) : \exists\, m \in [n, (1+\varepsilon)n], s.t.\ \varphi_{mL_0}(x) \in \Lambda,\ d(x, \varphi_{mL_0}(x)) < \delta(\varepsilon)\right\}.$$

By the construction, one has
$$\mu(\Lambda_{\lambda_0}^{L_0}(k)) > 0.$$
Applying Proposition 5.18, by taking $f = \varphi_{L_0}$, one has
$$\lim_{n \to \infty} \mu(\Lambda_{\lambda_0}^{L_0}(k, n)) = \mu(\Lambda_{\lambda_0}^{L_0}(k)).$$

Take a maximal choice of $K_n = K_n(k, \varepsilon) \subset \Lambda_{\lambda_0}^{L_0}(k, n)$ such that Item 1 is satisfied. By the definition of $\Lambda_{\lambda_0}^{L_0}(k, n)$, Item 2 is satisfied. For Item 3, let us recall the new formula of the metric entropy by Katok [8, Theorem 1.1], see also section 5.1. By the maximality of K_n, one has

$$\sharp K_n \geq N_{\varphi_1}(nL_0, \varepsilon, 1 - \mu(\Lambda_{\lambda_0}^{L_0}(k, n))).$$

Thus,

$$\varliminf_{\varepsilon \to 0} \varliminf_{n \to \infty} \frac{1}{nL_0} \log \sharp K_n \geq \varliminf_{\varepsilon \to 0} \varliminf_{n \to \infty} \frac{1}{nL_0} \log N_{\varphi_1}(nL_0, \varepsilon, 1 - \mu(\Lambda_{\lambda_0}^{L_0}(k, n))) \geq \mathrm{h}_\mu(\varphi_1).$$

The construction of K_n is hence complete.

Estimate the growth rate of the periodic orbits. Now we can complete the poof of Theorem 5.4. For every $x \in K_n$, there is an integer m_x with $n \leq m_x \leq n(1+\varepsilon)$ such that
$$\varphi_{m_x L_0}(x) \in \Lambda_{\lambda_0}^{L_0}(k).$$
By Theorem 5.10, there is a strictly increasing map

$$\theta_x : [0, m_x L_0] \to \mathbb{R}$$

and a periodic point $p = p_x$ of period $\theta(m_x L_0)$ such that

$$d(\varphi_t(x), \varphi_{\theta_x(t)}(p)) < \varepsilon_1 |X(\varphi_t(x))| < \varepsilon/3, \quad \forall\, t \in [0, m_x L_0].$$

By Proposition 5.11, one has

$$|\theta_x(\tau) - \tau| \le N \cdot d(x, \varphi_{mL_0}(x)) \le N\delta, \quad \forall\, \tau \in \mathbb{N} \cap [0, mL_0].$$

For two different points $x,\ y \in K_n$, by the construction of K_n, there is an integer $j \in \mathbb{N} \cap [0, nL_0]$ such that

$$d(\varphi_j(x), \varphi_j(y)) > \varepsilon.$$

Thus,

$$\begin{aligned} d(\varphi_{\theta_x(j)}(p_x), \varphi_{\theta_y(j)}(p_y)) &\ge d(\varphi_j(x), \varphi_j(y)) - d(\varphi_j(x), \varphi_{\theta_x(j)}(p_x)) - d(\varphi_j(y), \varphi_{\theta_y(j)}(p_y)) \\ &> \varepsilon - \varepsilon/3 - \varepsilon/3 = \varepsilon/3. \end{aligned}$$

In fact, we have the following disjoint property:

Claim. *For $C := \sup_{z \in M}\{\|X(z)\|\} < \infty$, one has*

$$\varphi_{(-\varepsilon/32C,\ \varepsilon/32C)}(p_x) \cap \varphi_{(-\varepsilon/32C,\ \varepsilon/32C)}(p_y) = \emptyset.$$

Proof of the Claim. By Proposition 5.11, taking $\delta \in \left(0, \dfrac{\varepsilon}{64CN}\right)$, one has

$$|\theta_x(j) - j| \le N\delta, \qquad |\theta_y(j) - j| \le N\delta.$$

Therefore,

$$|\theta_x(j) - \theta_y(j)| \le |\theta_x(j) - j| + |\theta_y(j) - j| \le 2N\delta < \varepsilon/32C.$$

Thus, for any $t \in (-t_0, t_0)$, one has

$$d(\varphi_{\theta_x(j)+t}(p_x), \varphi_{\theta_x(j)}(p_x)) < \varepsilon/16, \quad d(\varphi_{\theta_y(j)+t}(p_y), \varphi_{\theta_y(j)}(p_y)) < \varepsilon/16.$$

Consequently, for any $t, s \in (-t_0, t_0)$,

$$\begin{aligned} d(\varphi_{\theta_x(j)+t}(p_x), \varphi_{\theta_y(j)+s}(p_y)) &\ge d(\varphi_{\theta_x(j)}(p_x), \varphi_{\theta_y(j)}(p_y)) - d(\varphi_{\theta_x(j)+t}(p_x), \varphi_{\theta(j)}(p_x)) \\ &- d(\varphi_{\theta_y(j)+s}(p_y), \varphi_{\theta_y(j)}(p_y)) > \varepsilon/3 - \varepsilon/8 > 0. \end{aligned}$$

This implies the claim. □

From the claim, in the periodic orbit $\mathrm{Orb}(p_x)$, any orbit segment $\varphi_{[0,1]}(z)$ contains at most $32C/\varepsilon$ points in the set $\{p_x\}_{x \in K_n}$. Consequently, one has

$$\sum_{[x] \in P_T(X),\ nL_0(1-\varepsilon) - N\delta \le \pi(x) \le nL_0(1+\varepsilon) + N\delta} \pi(x) \ge \varepsilon/32C \cdot \sharp K_n.$$

Thus,

$$\sharp P_{nL_0(1+\varepsilon)+N\delta}(X) \ge \varepsilon/32C \cdot \sharp K_n.$$

Therefore,

$$\begin{aligned}
\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) &\geq \overline{\lim_{n\to\infty}} \frac{1}{nL_0(1+\varepsilon)+N\delta} \log \sharp P_{nL_0(1+\varepsilon)+N\delta}(X) \\
&= \lim_{n\to\infty} \frac{nL_0}{nL_0(1+\varepsilon)+N\delta} \overline{\lim_{n\to\infty}} \frac{1}{nL_0} \log \# P_{nL_0(1+\varepsilon)+N\delta}(X) \\
&\geq \frac{1}{1+\varepsilon} \underline{\lim_{n\to\infty}} \frac{1}{nL_0} \log(\varepsilon/32C \cdot \sharp K_n) \\
&\geq \frac{1}{1+\varepsilon} \underline{\lim_{n\to\infty}} \frac{1}{nL_0} \log \sharp K_n
\end{aligned}$$

Thus, by letting $\varepsilon \to 0$, one has

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \log \sharp P_T(X) \;\geq\; h_\mu(X).$$

$\square$

Theorem 5.4 is now proved, hence the proof of Theorem A is complete.

6 Uniform Diophantine approximate

6.1 Proofs of Theorems B and C

First, we prove that if the upper limit $\overline{v}_2 = \infty$, then the Hausdorff dimension of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ and the set $\mathcal{U}(\psi_2)$ are zero.

Proposition 6.1. *If $\overline{v}_2 = \infty$, then*

$$\mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \nu_\beta(x) = \infty\}.$$

Thus,

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \dim_H(\mathcal{U}(\psi_2)) = 0.$$

Proof. For every point $x \in \mathcal{U}(\psi_2)$, we distinguish two cases:

Case 1: There is an integer n_0 such that

$$T_\beta^{n_0} x = 0.$$

Therefore, the β-expansion of x is finite. Thus,

$$T_\beta^n x = 0, \quad \text{for any } n \geq n_0.$$

By the definition of the exponent $\nu_\beta(x)$, we have

$$\nu_\beta(x) = \infty.$$

Case 2: For any $n \in \mathbb{N}$, we always have

$$T_\beta^n x > 0.$$

Since the point $x \in \mathcal{U}(\psi_2)$, there is $N_0 \geq 1$ such that for any $N \geq N_0$, there is an integer $n \in [0, N]$ such that

$$0 < T_\beta^n x < \psi_2(N).$$

Since

$$\limsup_{n\to\infty} \frac{-\log_\beta \psi_2(n)}{n} = \overline{v}_2 = \infty,$$

for any $L > 0$ large enough, there is a sequence $\{n_i\}$ such that

$$\psi_2(n_i) \leq \beta^{-n_i L}.$$

Let $m_1 := \min\{n_i : \ n_i \geq N_0\}$, there is an integer $j_1 \in [0, m_1]$ such that

$$0 < T_\beta^{j_1} x < \psi_2(m_1) \leq \beta^{-m_1 L} \leq \beta^{-j_1 L}.$$

Take $m_2 := \min\left\{n_i > m_1 : T_\beta^{j_1} x > \beta^{-n_i L}\right\}$, there is an integer $j_2 \in [0, m_2]$ such that

$$0 < T_\beta^{j_2} x < \psi_2(m_2) \leq \beta^{-m_2 L} \leq \beta^{-j_2 L}.$$

Since

$$T_\beta^{j_2}x < \psi_2(m_2) \le \beta^{-m_2L} < T_\beta^{j_1}x,$$

$$j_2 \neq j_1.$$

Repeat this process, one can get a sequence of pairwise disjoint integers $\{j_i : i \ge 1\}$ such that

$$0 < T_\beta^{j_i}x < \psi_2(m_i) \le \beta^{-m_iL} \le \beta^{-j_iL}.$$

Therefore,

$$x \in \left\{x \in [0,1] : T_\beta^n x < \beta^{-nL}, \text{ for infinitely many integers } n \in \mathbb{N}\right\}.$$

By the arbitrariness of L, we have

$$\nu_\beta(x) = \infty.$$

Hence, in all cases, we have

$$\mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \nu_\beta(x) = \infty\}.$$

By **Theorem SW**,

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \le \dim_H(\mathcal{U}(\psi_2)) \le \dim_H(\{x \in [0,1] : \nu_\beta(x) = \infty\}) = 0.$$

□

We discuss the relation between $\nu_\beta(x)$, $\hat{\nu}_\beta(x)$ and $\underline{v}_1$, $\overline{v}_1$, $\underline{v}_2$, $\overline{v}_2$, which are important to the proof of Theorem C.

Lemma 6.2. *For $0 \le \underline{v}_1 \le \overline{v}_1 < \infty$ and $0 \le \underline{v}_2 \le \overline{v}_2 < \infty$, one has*

(1) $\{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) > \overline{v}_2\} \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$;

(2) $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \nu_\beta(x) \ge \underline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \ge \underline{v}_2\}$.

Proof. (1) For any point $x \in \{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\}$ and any $\varepsilon > 0$ small enough, there is a sequence $\{n_i\}$ such that

$$T_\beta^{n_i}x < \beta^{-n_i(\overline{v}_1+\varepsilon)}.$$

By the definition of $\overline{v}_1$, for the above ε, there is an integer i_0 such that

$$\psi_1(n_i) > \beta^{-n_i(\overline{v}_1+\varepsilon)}, \quad \text{for any } i \ge i_0.$$

Then,

$$T_\beta^{n_i}x < \beta^{-n_i(\overline{v}_1+\varepsilon)} < \psi_1(n_i), \text{ for any } i \ge i_0.$$

Thus, $x \in \mathcal{L}(\psi_1)$. Therefore,

$$\{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\} \subseteq \mathcal{L}(\psi_1).$$

By similar discussion,
$$\{x \in [0,1] : \hat{\nu}_\beta(x) > \overline{v}_2\} \subseteq \mathcal{U}(\psi_2).$$
Thus,
$$\{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) > \overline{v}_2\} \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2).$$

(2) For any point $x \in \mathcal{L}(\psi_1)$, there is a sequence $\{n_i\}$ such that
$$T_\beta^{n_i} x < \psi_1(n_i).$$
By the definition of the exponent $\underline{v}_1$, for any $\varepsilon > 0$, there is an integer i_0 such that for any $i \geq i_0$, one has
$$T_\beta^{n_i} x < \psi_1(n_i) < \beta^{-n_i(\underline{v}_1 - \varepsilon)}.$$
Thus,
$$x \in \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1 - \varepsilon\}.$$
Therefore,
$$\mathcal{L}(\psi_1) \subseteq \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1 - \varepsilon\}.$$
By the arbitrariness of ε, one can obtain
$$\mathcal{L}(\psi_1) \subseteq \bigcap_{\varepsilon>0} \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1 - \varepsilon\} = \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1\}.$$
By similar discussion,
$$\mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$
Thus,
$$\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$

□

To prove Theorem B, we deal with the special case that $\nu_\beta(x) = \infty$ and $\hat{\nu}_\beta(x) = \infty$.

Lemma 6.3. *If the exponents $\nu_\beta(x) = \infty$ and $\hat{\nu}_\beta(x) = \infty$, then one has*

$$(1)\ \bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\} \subseteq \{x \in [0,1] : \nu_\beta(x) = \infty\}.$$

$$(2)\ \{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\} = \bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}.$$

Proof. (1) For any point $x \in \bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}$, there is an integer n_0 such that
$$d_\beta(x) = (\omega, 0^\infty), \quad \text{for some } \omega \in \Sigma_\beta^{n_0}.$$
Thus , for any $n \geq n_0$, one has
$$T_\beta^n x = 0.$$

By the definition of the exponent $\nu_\beta(x)$, we have

$$\nu_\beta(x) = \infty.$$

Therefore,

$$\bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\} \subseteq \{x \in [0,1] : \nu_\beta(x) = \infty\}.$$

(2) By the similar discussion of Item (1) and the definition of the exponent $\hat{\nu}_\beta(x)$, one has

$$\bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\} \subseteq \{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\}.$$

What is left is to show

$$\{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}.$$

By contrary, for any point $x \in \{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\}$, we suppose

$$x \notin \bigcup_{n=1}^{\infty} \bigcup_{\omega \in \Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}.$$

Then

$$T_\beta^n x > 0, \quad \text{for every } n \in \mathbb{N}.$$

Denote the β-expansion of x by

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \cdots ,$$

where $a_i \in \{0, \cdots, \lceil \beta \rfloor\}$, for all $i \geq 1$. We can take two increasing sequences $\{n_i' : i \geq 1\}$ and $\{m_i' : i \geq 1\}$ with the properties:

(1) For every $i \geq 1$, one has

$$a_{n_i'} > 0, \quad a_{n_i'+1} = \cdots = a_{m_i'-1} = 0, \quad a_{m_i'} > 0.$$

(2) For every $a_n = 0$, there is an integer i such that

$$n_i' < n < m_i'.$$

By the choice of the sequence $\{n_i' : i \geq 1\}$ and $\{m_i' : i \geq 1\}$, for every $i \geq 1$, one has

$$n_i' < m_i' < n_{i+1}'.$$

Since the exponent $\hat{\nu}_\beta(x) > 0$, one has

$$\limsup_{i\to\infty}(m'_i - n'_i) = \infty.$$

Take $n_1 = n'_1$ and $m_1 = m'_1$, suppose m_k, n_k have been defined. Let $i_1 = 1$ and $i_{k+1} := \min\{i > i_k : m'_i - n'_i > m_k - n_k\}$, for $k \geq 1$. Then, define

$$n_{k+1} := n'_{i_{k+1}}, \quad m_{k+1} := m'_{i_{k+1}}.$$

Therefore, the sequence $\{i_k : k \geq 1\}$ is well defined. By this way, we take the maximal subsequences $\{n_k : k \geq 1\}$ and $\{m_k : k \geq 1\}$ of $\{n'_i : i \geq 1\}$ and $\{m'_i : i \geq 1\}$, respectively, in such a way that the sequence $\{m_k - n_k : k \geq 1\}$ is non-decreasing. Notice

$$\beta^{n_k - m_k} < T_\beta^{n_k} x < \beta^{n_k - m_k + 1}.$$

As the similar discussion in [40], one has

$$\hat{\nu}_\beta(x) = \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} \leq 1.$$

This contradicts our assumption $\hat{\nu}_\beta(x) = \infty$. Thus, we proved

$$\{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{\omega\in\Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}.$$

Therefore,

$$\{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\} = \bigcup_{n=1}^{\infty} \bigcup_{\omega\in\Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}.$$

This means that the set $\{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\}$ is countable. $\square$

Lemma 6.4. *The set $\{x \in [0,1] : 1 < \hat{\nu}_\beta(x) < \infty\}$ is empty.*

Proof. This follows from the proof of Item (2) of Lemma 6.3. $\square$

Proposition 6.5. *The sets*

$$\{x \in [0,1] : \nu_\beta(x) = \infty\}, \quad \{x \in [0,1] : \hat{\nu}_\beta(x) = 1\}$$

are uncountable.

Proof. For any real number $a > 1$, we give a correspondence:

$$\Psi(a) \mapsto 10^{\lfloor 2^a \rfloor} 10^{\left\lfloor 2^{a^{2^2}} \right\rfloor} 10^{\left\lfloor 2^{a^{3^2}} \right\rfloor} 1 \cdots 10^{\left\lfloor 2^{a^{k^2}} \right\rfloor} 1 \cdots .$$

The infinite string is a β-expansion of some $x \in [0,1)$. Denote this x by x_a. Then, we can obtain a correspondence:

$$\Phi(a) \mapsto x_a$$

from $\{a : a > 1\}$ to $\{x_a : a > 1\}$. One can check $\nu_\beta(x_a) = \infty$ and $\hat{\nu}_\beta(x_a) = 1$. Therefore,

$$\{x_a : a > 1\} \subseteq \{x \in [0,1] : \nu_\beta(x) = \infty\}$$

and

$$\{x_a : a > 1\} \subseteq \{x \in [0,1] : \hat{\nu}_\beta(x) = 1\}.$$

For different $a_1 > 1$ and $a_2 > 1$, there is an positive integer $k_0 \in \mathbb{N}$ such that

$$\left| 2^{a_1^{k^2}} - 2^{a_2^{k^2}} \right| > 1, \text{ for any } k \geq k_0.$$

Thus,

$$\Psi(a_1) \neq \Psi(a_2).$$

Then,

$$\Phi(a_1) \neq \Phi(a_2).$$

Hence, the cardinality of the set $\{a : a > 1\}$ is less than or equal to that of $\{x_a : a > 1\}$. Similarly, the cardinality of the set $\{x_a : a > 1\}$ is less than or equal to that of

$$\{x \in [0,1] : \nu_\beta(x) = \infty\} \quad (\{x \in [0,1] : \hat{\nu}_\beta(x) = 1\}).$$

Since the set $\{a : a > 1\}$ is uncountable,

$$\{x \in [0,1] : \nu_\beta(x) = \infty\} \, (\{x \in [0,1] : \hat{\nu}_\beta(x) = 1\})$$

is uncountable. □

Proof of Theorem B. (1) If the numbers $\underline{v}_1 = \overline{v}_1 = \underline{v}_2 = \overline{v}_2 = 0$, then

$$\lim_{n\to\infty} \frac{-\log_\beta \psi_1(n)}{n} = 0, \qquad \lim_{n\to\infty} \frac{-\log_\beta \psi_2(n)}{n} = 0.$$

Thus, for any positive integer m large enough, there is an integer $n_0 > 0$ such that

$$\beta^{-n/m} \leq \psi_1(n), \quad \beta^{-n/m} \leq \psi_2(n), \quad \text{for any } n \geq n_0.$$

Therefore, for any positive integer m, one has

$$\{x \in [0,1] : \nu_\beta(x) \geq 1/m\} \subseteq \mathcal{L}(\psi_1), \quad \{x \in [0,1] : \hat{\nu}_\beta(x) = 1/m\} \subseteq \mathcal{U}(\psi_2).$$

By the fact

$$\{x \in [0,1] : \hat{\nu}_\beta(x) = 1/m\} \subseteq \{x \in [0,1] : \nu_\beta(x) \geq 1/m\},$$

one has

$$\{x \in [0,1] : \hat{\nu}_\beta(x) = 1/m\} \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2).$$

According to [40, Theorem 1.5], we have

$$1 = \sup_{m\in\mathbb{N}^+} \dim_H (\{x \in [0,1] : \hat{\nu}_\beta(x) = 1/m\}) \leq \dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)).$$

Thus,
$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2))=1.$$

(2) If $\underline{v}_2=\infty$, then
$$\lim_{n\to\infty}\frac{-\log_\beta\psi_2(n)}{n}=\infty.$$
For any $L>0$ large enough, there is an integer n_0 such that
$$\psi_2(n)\le\beta^{-nL},\ \text{for any } n\ge n_0.$$
Therefore, for any point $x\in\mathcal{U}(\psi_2)$, one has
$$\hat{\nu}_\beta(x)>L.$$
By the arbitrariness of L,
$$\hat{\nu}_\beta(x)=\infty.$$
Thus,
$$\mathcal{U}(\psi_2)\subseteq\{x\in[0,1]:\hat{\nu}_\beta(x)=\infty\}.$$
On the other hand, if the point $x\in\{x\in[0,1]:\hat{\nu}_\beta(x)=\infty\}$, then, by Lemma 6.3,
$$x\in\bigcup_{n=1}^{\infty}\bigcup_{\omega\in\Sigma_\beta^n}\{x\in[0,1]:d_\beta(x)=(\omega,0^\infty)\}.$$
Then, there is an integer n_0 such that
$$T_\beta^n x=0,\quad \text{for any } n\ge n_0.$$
Therefore, $x\in\mathcal{U}(\psi_2)$. Thus,
$$\{x\in[0,1]:\hat{\nu}_\beta(x)=\infty\}\subseteq\mathcal{U}(\psi_2).$$
Consequently,
$$\{x\in[0,1]:\hat{\nu}_\beta(x)=\infty\}=\mathcal{U}(\psi_2).$$
By the fact that $\{x\in[0,1]:\hat{\nu}_\beta(x)=\infty\}$ is countable, $\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)$ is countable.

(3) If $\underline{v}_1=\infty$, then, by **Theorem SW**,
$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2))\le\dim_H(\mathcal{L}(\psi_1))=0.$$

□

Now, we prove Theorem C. We divide the proof into three propositions as Propositions 6.6, 6.7, 6.8.

Proposition 6.6. *For any $0\le\underline{v}_2\le\overline{v}_2<\infty$, one has*
$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2))\le\frac{1}{1+\overline{v}_2}.$$

Proof. By definition of the exponent $\overline{v}_2$, we can take a subsequence $\{n_k : k \geq 1\}$ such that
$$\lim_{k\to\infty} \frac{-\log_\beta \psi_2(n_k)}{n_k} = \overline{v}_2.$$
Then, for any $\varepsilon > 0$, there is an integer k_0 such that
$$\beta^{-n_k(\overline{v}_2+\varepsilon)} \leq \psi_2(n_k) \leq \beta^{-n_k(\overline{v}_2-\varepsilon)}, \quad \text{for any } k \geq k_0.$$
For every point $x \in \mathcal{U}(\psi_2)$, we distinguish two cases:

Case 1: If there is an integer n_0 such that
$$T_\beta^{n_0} x = 0,$$
then the β-expansion of x is finite. Thus,
$$T_\beta^n x = 0, \quad \text{for any } n \geq n_0.$$
Hence,
$$x \in \left\{x \in [0,1] : T_\beta^n x < \beta^{-n(\overline{v}_2-\varepsilon)}, \text{ for infinitely many integers } n\right\}.$$

Case 2: For any $n \in \mathbb{N}$, we always have
$$T_\beta^n x > 0.$$
Since $x \in \mathcal{U}(\psi_2)$, there is an integer $N_0 \geq 1$ such that for any $N \geq N_0$, there is an integer $n \in [0, N]$ such that
$$0 < T_\beta^n x < \psi_2(N).$$
For the above $\varepsilon > 0$, let
$$K := \max\{N_0,\ n_{k_0}\}.$$
For any $n_k \geq K$, there is an integer $j_k \in [0, n_k]$ such that
$$0 < T_\beta^{j_k} x < \psi_2(n_k) \leq \beta^{-n_k(\overline{v}_2-\varepsilon)}.$$
In fact, let $k_1 := \min\{k : n_k \geq K\}$, there is an integer $j_{k_1} \in [0, n_{k_1}]$ such that
$$0 < T_\beta^{j_{k_1}} x < \psi_2(n_{k_1}) \leq \beta^{-n_{k_1}(\overline{v}_2-\varepsilon)} \leq \beta^{-j_{k_1}(\overline{v}_2-\varepsilon)}.$$
Take $k_2 := \min\left\{k > k_1 : T_\beta^{j_{k_1}} x > \beta^{-n_k(\overline{v}_2-\varepsilon)}\right\}$. There is an integer $j_{k_2} \in [0, n_{k_2}]$ such that
$$0 < T_\beta^{j_{k_2}} x < \psi_2(n_{k_2}) \leq \beta^{-n_{k_2}(\overline{v}_2-\varepsilon)} \leq \beta^{-j_{k_2}(\overline{v}_2-\varepsilon)}.$$
Since
$$T_\beta^{j_{k_2}} x < \psi_2(n_{k_2}) \leq \beta^{-m_2 L} < T_\beta^{j_{k_1}} x,$$
$$j_{k_2} \neq j_{k_1}.$$

Repeat this process, one can get a sequence of pairwise disjoint integers $\{k_i : i \geq 1\}$ such that

$$0 < T_\beta^{j_{k_i}} x < \psi_2(n_{k_i}) \leq \beta^{-n_{k_i}(\overline{v}_2-\varepsilon)} \leq \beta^{-j_{k_i}(\overline{v}_2-\varepsilon)}.$$

Therefore,

$$x \in \left\{x \in [0,1] : T_\beta^n x < \beta^{-n(\overline{v}_2-\varepsilon)}, \text{ for infinitely many integers } n \in \mathbb{N}\right\}.$$

Hence, in all cases, we have

$$\mathcal{U}(\psi_2) \subseteq \left\{x \in [0,1] : T_\beta^n x < \beta^{-n(\overline{v}_2-\varepsilon)}, \text{ for infinitely many integers } n \in \mathbb{N}\right\}.$$

By **Theorem SW** [39, Theorem 1.1],

$$\dim_H \mathcal{U}(\psi_2) \leq \frac{1}{1+\overline{v}_2}.$$

Then,

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \dim_H(\mathcal{U}(\psi_2)) \leq \frac{1}{1+\overline{v}_2}.$$

□

Proposition 6.7. *If $\underline{v}_2 > 1$, then $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ is countable. If $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2 \leq 1$, then*

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2.$$

If $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$, then

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

Proof. By the Item (2) of Lemma 6.2, one has

$$\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$

The argument on the upper bound of the Hausdorff dimension of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ can be obtained by a natural covering of the set

$$\{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$

According to the Item (2) and (3) of Theorem B, we only need to consider the case $\nu_\beta(x) \in [\underline{v}_1, \infty)$ and $\hat{\nu}_\beta(x) \in [\underline{v}_2, \infty)$. For any point $x \in \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$, there is a number $v_\beta \in [\underline{v}_1, \infty)$ such that

$$x \in \mathbb{B} := \{x \in [0,1] : \nu_\beta(x) = v_\beta\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$

Denote its β-expansion by

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \cdots,$$

where $a_i \in \{0, \cdots, \lceil \beta \rfloor\}$, for all $i \geq 1$. By the same way as Lemma 6.3, we take the maximal subsequences $\{n_k : k \geq 1\}$ and $\{m_k : k \geq 1\}$ of $\{n'_i : i \geq 1\}$ and $\{m'_i : i \geq 1\}$, respectively. Notice

$$\beta^{n_k - m_k} < T_\beta^{n_k} x < \beta^{n_k - m_k + 1}.$$

We have the following claim.

Claim. $v_\beta = \limsup_{k\to\infty} \frac{m_k - n_k}{n_k}, \quad \underline{v}_2 \le \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}}.$

Proof of Claim. Without loss of generality, we assume

$$\limsup_{k\to\infty} \frac{m_k - n_k}{n_k} = c_1, \quad \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = c_2.$$

First, we show $v_\beta = v_\beta(x) = c_1$. For any $\varepsilon > 0$, there is an integer $k_0 > 0$ such that

$$m_k - n_k \le n_k(c_1 + \varepsilon), \quad \text{for any } k \ge k_0.$$

Since

$$T_\beta^{n_k} x > \beta^{n_k - m_k},$$

we have

$$T_\beta^{n_k} x > \beta^{n_k - m_k} \ge \beta^{-n_k(c_1+\varepsilon)}.$$

In general, for any $n \ge n_{k_0}$, there is an integer $k \ge k_0$ such that $n_k \le n < n_{k+1}$. By the choice of the sequence $\{n_k\}$, we have

$$T_\beta^{n} x > T_\beta^{n_k} x > \beta^{-n_k(c_1+\varepsilon)} > \beta^{-n(c_1+\varepsilon)}.$$

It means

$$v_\beta = v_\beta(x) < c_1 + \varepsilon.$$

On the other hand, by the definition of c_1, taking a subsequence $\{n_{k_i}\}$ and $\{m_{k_i}\}$ such that

$$\frac{m_{k_i} - n_{k_i}}{n_{k_i}} \ge c_1 - \varepsilon,$$

one has

$$T_\beta^{n_{k_i}} x < \beta^{n_{k_i} - m_{k_i} + 1} \le \beta^{-n_{k_i}(c_1-\varepsilon)+1}.$$

Thus,

$$v_\beta = \nu_\beta(x) \ge c_1 - \varepsilon.$$

By the arbitrariness of ε, we have

$$v_\beta = c_1.$$

Next, we will prove $\underline{v}_2 \le c_2$. By the definition of the exponent $\underline{v}_2$, for any $\varepsilon > 0$, there is an integer $n_0 = n_0(\varepsilon) > 0$ such that

$$\psi_2(n) \le \beta^{-n(\underline{v}_2 - \varepsilon)}, \quad \text{for any } n \ge n_0.$$

By the definition of c_2, one can take a subsequence $\{k_i : i \ge 1\}$ such that

$$\lim_{i\to\infty} \frac{m_{k_i} - n_{k_i}}{n_{k_{i+1}}} = c_2.$$

For the above $\varepsilon > 0$, there is an integer $i_0 = i_0(\varepsilon) > 0$ such that

$$m_{k_i} - n_{k_i} \le n_{k_{i+1}}(c_2 + \varepsilon), \quad \text{for any } i \ge i_0.$$

By contrary,, we suppose $c_2 < \underline{v}_2$. Then, for any $\varepsilon \in \left(0, \frac{\underline{v}_2 - c_2}{4}\right)$ and any integer $J \geq K$, where $K := \max\{n_0(\varepsilon),\ n_{i_0(\varepsilon)}\}$, there is an integer $n_{k_{i+1}} > J$ such that for any integer $n \in [1, n_{k_{i+1}}]$, one has

$$T_\beta^n x > T_\beta^{n_{k_i}} x > \beta^{n_{k_i} - m_{k_i}} \geq \beta^{-n_{k_{i+1}}(c_2+\varepsilon)} > \beta^{-n_{k_{i+1}}(\underline{v}_2-\varepsilon)} \geq \psi_2(n_{k_{i+1}}).$$

This contracts the fact $x \in \mathcal{U}(\psi_2)$. Therefore,

$$\underline{v}_2 \leq c_2 = \liminf_{k\to\infty} \frac{m_k - n_k}{n_{k+1}}.$$

□

Now, we consider

$$v_\beta = \limsup_{n\to\infty} \frac{m_k - n_k}{n_k} = \limsup_{n\to\infty} \frac{m_k}{n_k} - 1, \tag{6}$$

$$\underline{v}_2 \leq \liminf_{n\to\infty} \frac{m_k - n_k}{n_{k+1}} \leq \liminf_{n\to\infty} \frac{m_k - n_k}{m_k} = 1 - \limsup_{n\to\infty} \frac{n_k}{m_k}. \tag{7}$$

Since

$$\left(\limsup \frac{n_k}{m_k}\right) \cdot \left(\limsup \frac{m_k}{n_k}\right) \geq 1,$$

we derive from (6) and (7) that

$$v_\beta \geq \frac{\underline{v}_2}{1 - \underline{v}_2}, \qquad \underline{v}_2 \leq \frac{v_\beta}{1 + v_\beta}. \tag{8}$$

If $\underline{v}_2 > 1$, then $\underline{v}_2 > v_\beta/(1+v_\beta)$, for any $v_\beta \geq v_1$. This contradicts (8). It means that $\mathbb{B}$ is empty. By Lemma 6.3,

$$\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2) = \bigcup_{\omega\in\Sigma_\beta^n} \{x \in [0,1] : d_\beta(x) = (\omega, 0^\infty)\}.$$

Hence, $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ is countable.

If $\underline{v}_2 \leq 1$, by the inequality (8), for any $v_\beta < \underline{v}_2/(1-\underline{v}_2)$, the set $\mathbb{B}$ is empty. Therefore, we only consider the condition $v_\beta \geq \underline{v}_2/(1-\underline{v}_2)$. According to the way as Lemma 6.3, take the sequences $\{n_k : k \geq 1\}$ and $\{m_k : k \geq 1\}$ such that

$$v_\beta = \lim_{n\to\infty} \frac{m_k - n_k}{n_k}, \qquad \text{and} \qquad \underline{v}_2 \leq \liminf_{n\to\infty} \frac{m_k - n_k}{n_{k+1}}.$$

Given $0 < \varepsilon < \underline{v}_2/2$, for k large enough, one has

$$(v_\beta - \varepsilon)n_k \leq m_k - n_k \leq (v_\beta + \varepsilon)n_k, \tag{9}$$

$$m_k - n_k \geq (\underline{v}_2 - \varepsilon)n_{k+1}. \tag{10}$$

By inequality (9), one has

$$(1+v_\beta-\varepsilon)m_{k-1} \le (1+v_\beta-\varepsilon)n_k \le m_k.$$

Therefore, the sequence $\{m_k : k \ge 1\}$ increases at least exponentially. Since

$$n_k \ge m_{k-1}, \text{ for every } k \ge 2,$$

the sequence $\{n_k : k \ge 1\}$ also increases at least exponentially. Thus, there is a positive constant C such that

$$k \le C \log_\beta n_k.$$

Combining (9) and (10), one obtains

$$(\underline{v}_2-\varepsilon)n_{k+1} \le (v_\beta+\varepsilon)n_k.$$

Thus, for k large enough, there is an integer n_0 and a postive real number ε_1 small enough such that the sum of all lengths of the blocks of 0 in the prefix of length n_k of the infinite sequence $a_1a_2\cdots$ is at least equal to

$$\begin{aligned}\sum_{i=1}^{k}(\underline{v}_2-\varepsilon)n_i - n_0 &= n_k(\underline{v}_2-\varepsilon)\left[\frac{1-\left(\dfrac{\underline{v}_2-\varepsilon}{v_\beta+\varepsilon}\right)^k}{1-\left(\dfrac{\underline{v}_2-\varepsilon}{v_\beta+\varepsilon}\right)}\right] - n_0 \\ &\ge n_k\left(\frac{v_\beta\cdot\underline{v}_2}{v_\beta-\underline{v}_2}-\varepsilon_1\right).\end{aligned}$$

Among the digits $a_1\cdots a_{m_k}$, there are k blocks of digits which are 'free'. Denote their lengths by $l_1,\cdots,l_k$. Let $\varepsilon_2 = \dfrac{(v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2)\varepsilon_1}{v_\beta-\underline{v}_2}$. One has

$$\sum_{i=1}^{k} l_i \le n_k - n_k\left(\frac{v_\beta\cdot\underline{v}_2}{v_\beta-\underline{v}_2}-\varepsilon_1\right) = n_k(1+\varepsilon_2)\frac{v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2}{v_\beta-\underline{v}_2}.$$

By Theorem 4.7, there are at most $\dfrac{\beta\cdot\beta^{l_i}}{\beta-1}$ ways to choose the block with length l_i. Thus, one has in total at most

$$\left(\frac{\beta}{\beta-1}\right)^k\cdot\beta^{\sum_{i=1}^k l_i} \le \left(\frac{\beta}{\beta-1}\right)^k\cdot\beta^{n_k(1+\varepsilon_2)(v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2)/(v_\beta-\underline{v}_2)}$$

possible choices of the digits $a_1\cdots a_{m_k}$. On the other hand, there are at most k, where $k \le C\log_\beta n_k$, blocks of 0 in the prefix of length n_k of the infinite sequence $a_1a_2\cdots$. Since there are at most n_k possible choices for their first index, one has in total at most $(n_k)^{C\log_\beta n_k}$ possible choices. Consequently, the set of those $x \in \mathbb{B}$ is covered by

$$\left(\frac{\beta n_k}{\beta-1}\right)^{C\log_\beta n_k}\cdot\beta^{n_k(1+\varepsilon_2)(v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2)/(v_\beta-\underline{v}_2)}$$

basic intervals of length at most β^{-m_k}. Moreover, by (9) and by letting $\varepsilon_3 = \dfrac{\varepsilon}{1+v_\beta}$, we have

$$\beta^{-m_k} \leq \beta^{-(1+v_\beta)(1-\varepsilon_3)n_k}.$$

Take $\varepsilon' = \max\{\varepsilon_2,\ \varepsilon_3\}$. The set of those $x \in \mathbb{B}$ is covered by

$$\left(\frac{\beta n_k}{\beta-1}\right)^{C\log_\beta n_k} \cdot \beta^{n_k(1+\varepsilon')(v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2)/(v_\beta-\underline{v}_2)}$$

basic intervals of length at most $\beta^{-(1+v_\beta)(1-\varepsilon')n_k}$. We consider the series

$$\sum_{N\geq 1}(N)^{C\log_\beta N}\beta^{N(1+\varepsilon')(v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2)/(v_\beta-\underline{v}_2)}\beta^{-(1+v_\beta)(1-\varepsilon')Ns}.$$

The critical exponent s_0 such that the series converges if $s > s_0$ and diverges if $s < s_0$ is given by

$$s_0 = \frac{1+\varepsilon'}{1-\varepsilon'} \cdot \frac{v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2}{(1+v_\beta)(v_\beta-\underline{v}_2)}.$$

By a standard covering argument and the arbitrariness of ε', the Hausdorff dimension of the set $\mathbb{B}' := \{x \in [0,1] : \nu_\beta(x) = \nu_\beta\} \cap \mathcal{U}(\psi_2)$ is at most equal to

$$\dim_H(\mathbb{B}') \leq \frac{v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2}{(1+v_\beta)(v_\beta-\underline{v}_2)}. \tag{11}$$

For $v_\beta \geq \underline{v}_2/(1-\underline{v}_2)$, fix a positive integer L large enough. We consider the set

$$\mathbb{D} := \{x \in [0,1] : v_\beta \leq \nu_\beta(x) < v_\beta + 1/L\} \cap \mathcal{U}(\psi_2).$$

Repeat the above discussion, if $\underline{v}_2 < 1$, then

$$\dim_H(\mathbb{D}) \leq \frac{v_\beta-\underline{v}_2-v_\beta\cdot v_2}{(1+v_\beta)(v_\beta-\underline{v}_2)} + \frac{\underline{v}_2^2/L}{1-\underline{v}_2}.$$

If $\underline{v}_2 = 1$, then $v_\beta = \infty$. By **Theorem SW**,

$$\dim_H(\mathbb{D}) = 0.$$

If $\underline{v}_2 < 1$, since the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ is a subset of

$$\bigcup_{N=0}^{+\infty}\bigcup_{i=1}^{L}\{x \in [0,1] : v_1 + N + (i-1)/L \leq \nu_\beta(x) < v_1 + N + i/L\} \cap \mathcal{U}(\psi_2),$$

let L tend to infinite, one has

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \sup_{v_\beta \geq \underline{v}_2/(1-\underline{v}_2)}\left\{\frac{v_\beta-\underline{v}_2-v_\beta\cdot\underline{v}_2}{(1+v_\beta)(v_\beta-\underline{v}_2)}\right\}.$$

Regard the right side as a function of v_β with $v_\beta \geq \underline{v}_2/(1-\underline{v}_2)$, if $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2$, then the maximum is attained for $v_\beta = 2\underline{v}_2/(1-\underline{v}_2)$. Therefore,

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2.$$

If $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$, then the maximum is attained for $v_\beta = \underline{v}_1$. Thus,

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

□

To get the lower bound of the Hausdorff dimension of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$, we consider the set

$$\{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) > \overline{v}_1\}.$$

Proposition 6.8. *If $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2 \leq 1$, then*

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2.$$

If $\overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, then

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

Proof. By Item (1) of Lemma 6.2,

$$\{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) > \overline{v}_2\} \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2).$$

If $\overline{v}_2 = 1$, then

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq 0$$

always holds. If $\overline{v}_2 < 1$, then we fix $\delta > 0$ with $\overline{v}_2 + \delta < 1$, we consider the lower bound of the Hausdorff dimension of

$$\mathbb{F} := \{x \in [0,1] : \nu_\beta(x) = v_\beta + \delta\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \overline{v}_2 + \delta\},$$

where $v_\beta \geq \overline{v}_1$ is a real number. By **Theorem BL**, if

$$\frac{\overline{v}_1 + \delta}{\overline{v}_2 + \delta} < \frac{1}{1-(\overline{v}_2+\delta)},$$

then $\mathbb{F}$ is empty. Therefore, we consider the case $\dfrac{\overline{v}_1 + \delta}{\overline{v}_2 + \delta} \geq \dfrac{1}{1-(\overline{v}_2+\delta)}$. In fact, if $\overline{v}_2 > 0$, then there is a $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$, one has

$$\frac{v_\beta + \delta}{\overline{v}_2 + \delta} \geq \frac{\overline{v}_1 + \delta}{\overline{v}_2 + \delta} \geq \frac{1}{1-(\overline{v}_2+\delta)} > 1.$$

For any $\delta \in (0, \delta_0]$, we will construct a Cantor subset E_δ of $\mathbb{F}$. Let

$$n'_k = \left\lfloor \left(\frac{v_\beta + \delta}{\overline{v}_2 + \delta} \right)^k \right\rfloor, \quad m'_k = \lfloor (1 + v_\beta + \delta) n'_k \rfloor, \quad k = 1, 2, \cdots .$$

If $\overline{v}_2 = 0$, let

$$n'_k = k^k, \quad m'_k = \lfloor (1 + v_\beta + \delta) n'_k \rfloor, \quad k = 1, 2, \cdots .$$

Making an adjustment, we can choose two subsequences $\{n_k\}$ and $\{m_k\}$ with

$$n_k < m_k < n_{k+1}, \quad \text{for every } k \geq 1$$

such that $\{m_k - n_k\}$ is a non-decreasing sequence and

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_k} = v_\beta + \delta, \qquad \lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = \overline{v}_2 + \delta. \tag{12}$$

Consider the set of real numbers $x \in [0, 1)$ whose β-expansion

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \cdots ,$$

satisfies that for all $k \geq 1$,

$$a_{n_k} = 1, \; a_{n_k+1} = \cdots = a_{m_k-1} = 0, \; a_{m_k} = 1,$$

$$a_{m_k+(m_k-n_k)} = a_{m_k+2(m_k-n_k)} = \cdots = a_{m_k+t_k(m_k-n_k)} = 1,$$

where t_k is the largest integer such that $m_k + t_k(m_k - n_k) < n_{k+1}$. Then,

$$t_k \leq \frac{n_{k+1} - m_k}{m_k - n_k} \leq \frac{2}{\overline{v}_2 + \delta},$$

for k large enough. Therefore, the sequence $\{t_k : k \geq 1\}$ is bounded. Fix N, let β_N be the real number defined by the infinite β-expansion of 1 as equality (3). We replace the digit 1 for a_{n_k}, a_{m_k} and $a_{m_k+i(m_k-n_k)}$ for any $1 \leq i \leq t_k$ by the block $0^N 1 0^N$. Fill other places by blocks belonging to Σ_{β_N}. Thus, we have constructed the Cantor type subset E_δ. Since $\{t_k\}$ is bound, one has

$$\lim_{k\to\infty} \frac{m_k - n_k - 1 + 2N}{n_k + (4k-2)N + \sum_{i=1}^{k-1} 2Nt_i} = \lim_{k\to\infty} \frac{m_k - n_k}{n_k} = v_\beta + \delta,$$

$$\lim_{k\to\infty} \frac{m_k - n_k - 1 + 2N}{n_{k+1} + (4k+2)N + \sum_{i=1}^{k} 2Nt_i} = \lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1}} = \overline{v}_2 + \delta.$$

According to the construction, the sequence $d_\beta(x)$ is in Σ_{β_N}.

Claim. $E_\delta \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$.

Proof of Claim. Given $\varepsilon > 0$, by (12), there exists an integer k_0 such that

$$m_k - n_k \le (v_\beta + \delta + \varepsilon) n_k, \ m_k - n_k \le (\overline{v}_2 + \delta + \varepsilon) n_{k+1}, \text{ for any } k \ge k_0.$$

By the definitions of $\overline{v}_1$ and $\overline{v}_2$, there is an integer n_0 such that

$$\beta^{-n(\overline{v}_1+\delta+\varepsilon)} \le \psi_1(n), \quad \beta^{-n(\overline{v}_2+\delta+\varepsilon)} \le \psi_2(n), \text{ for any } n \ge n_0.$$

Let $N_0 = \max\{n_{k_0}, \ n_0\}$, for any point $x \in E_\delta$ and any $n_k \ge N_0$, one has

$$T_\beta^{n_k} x < \beta^{n_k - m_k + 1} \le \beta^{-n_k(v_\beta + \delta + \varepsilon - 1/n_k)} \le \beta^{-n_k(\overline{v}_1 + \delta + \varepsilon - 1/n_k)} \le \psi_1(n_k).$$

It means $x \in \mathcal{L}(\psi_1)$. On the other hand, for $N \ge N_0$, there is an integer i such that

$$n_{k+i} \le N < n_{k+i+1}.$$

Therefore,

$$T_\beta^{n_{k+i}} x < \beta^{n_{k+i} - m_{k+i} + 1} \le \beta^{-N(\overline{v}_2 + \delta + \varepsilon - 1/n_{k+i+1})} \le \psi_2(N).$$

It means $x \in \mathcal{U}(\psi_2)$. Then, $x \in \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$. Therefore,

$$E_\delta \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2).$$

$\square$

For a positive integer n, denote by $I_n(a_1, \cdots, a_n)$ the basic interval composed of $x \in [0, 1)$ whose β-expansion starts with $a_1, \cdots, a_n$. For abbreviation, we denote $I_n(a_1, \cdots, a_n)$ briefly by I_n. We distribute the mass uniformly when meet a block in Σ_{β_N} and keep the mass when go through the positions where the digits are determined by construction of E_δ. The Bernoulli measure μ on E_δ is defined as follows.

If $n < n_1$, then define

$$\mu(I_n) = \frac{1}{\sharp \Sigma_{\beta_N}^n}.$$

If $n_1 \le n \le m_1 + 4N$, then define

$$\mu(I_n) = \frac{1}{\sharp \Sigma_{\beta_N}^{n_1 - 1}}.$$

If there is an integer t with $0 \le t \le t_1 - 1$ such that

$$m_1 + 4N + (t+1)(m_1 - n_1) + 2Nt < n \le m_1 + 4N + (t+1)(m_1 - n_1) + 2N(t+1),$$

then define

$$\mu(I_n) = \frac{1}{\sharp \Sigma_{\beta_N}^{n_1 - 1}} \cdot \frac{1}{\left(\sharp \Sigma_{\beta_N}^{m_1 - n_1 - 1}\right)^{t+1}}.$$

If there is an integer t with $0 \le t \le t_1$ such that

$$m_1 + 4N + t(m_1 - n_1) + 2Nt < n \le c,$$

where $c := \min\{n_2 + 4N + 2Nt_1, m_1 + 4N + (t+1)(m_1 - n_1) + 2Nt\}$, then define

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^{n_1-1}} \cdot \frac{1}{\left(\sharp\Sigma_{\beta_N}^{m_1-n_1-1}\right)^t} \cdot \frac{1}{\sharp\Sigma_{\beta_N}^{n-(m_1+4N+t(m_1-n_1)+2Nt)}}.$$

For $k \geq 2$, let

$$l_k := n_k + 4(k-1)N + \sum_{i=1}^{k-1} 2Nt_i, \quad h_k := m_k + 4kN + \sum_{i=1}^{k-1} 2Nt_i,$$

$$p_k := m_k - n_k - 1, \quad q_k := h_k + t_k(m_k - n_k) + 2Nt_k.$$

If $l_k \leq n \leq h_k$, then define

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^{n_1-1}} \cdot \frac{1}{\prod_{i=1}^{k-1}\left(\sharp\Sigma_{\beta_N}^{p_i}\right)^{t_i} \cdot \left(\sharp\Sigma_{\beta_N}^{l_{i+1}-q_i-1}\right)} = \mu(I_{l_k}) = \mu(I_{h_k}).$$

If there is an integer t with $0 \leq t \leq t_k - 1$ such that

$$h_k + (t+1)(m_k - n_k) + 2Nt < n \leq h_k + (t+1)(m_k - n_k) + 2N(t+1),$$

then define

$$\mu(I_n) = \mu(I_{h_k}) \cdot \frac{1}{\left(\sharp\Sigma_{\beta_N}^{p_k}\right)^{t+1}}.$$

If there is an integer t with $0 \leq t \leq t_k$ such that

$$h_k + t(m_k - n_k) + 2Nt < n \leq \min\{l_{k+1}, h_k + (t+1)(m_k - n_k) + 2Nt\},$$

then define

$$\mu(I_n) = \mu(I_{h_k}) \cdot \frac{1}{\left(\sharp\Sigma_{\beta_N}^{p_k}\right)^t} \cdot \frac{1}{\sharp\Sigma_{\beta_N}^{n-(h_k+t(m_k-n_k)+2Nt)}}.$$

By the construction and Proposition 4.9, I_{h_k} is full. For calculating the local dimension of the measure μ, we discuss different cases as follows.
Case A: If $n = h_k$, then

$$\begin{aligned}
\liminf_{k\to\infty} \frac{\log_\beta \mu(I_{h_k})}{\log_\beta |I_{h_k}|} &= \liminf_{k\to\infty} \frac{n_1 - 1 + \sum_{i=1}^{k-1}(t_i p_i + l_{i+1} - q_i - 1)}{h_k} \cdot \log_\beta \beta_N \\
&= \liminf_{k\to\infty} \frac{n_1 - 1 + \sum_{i=1}^{k-1}(l_{i+1} - h_i - 2Nt_i - 1)}{h_k} \cdot \log_\beta \beta_N.
\end{aligned}$$

Recall that $\{t_k : k \geq 1\}$ is bounded and $\{m_k : k \geq 1\}$ grows exponentially fast in terms of k, therefore,

$$\liminf_{k\to\infty} \frac{\log_\beta \mu(I_{h_k})}{\log_\beta |I_{h_k}|} = \liminf_{k\to\infty} \frac{\sum_{i=1}^{k-1}(n_{i+1} - m_i)}{m_k} \log_\beta \beta_N.$$

By equalities (12), one has

$$\lim_{k\to\infty}\frac{m_k}{n_k}=1+v_\beta+\delta,\quad \lim_{k\to\infty}\frac{m_{k+1}}{m_k}=\frac{v_\beta+\delta}{\overline{v}_2+\delta},\quad \lim_{k\to\infty}\frac{n_{k+1}}{m_k}=\frac{v_\beta+\delta}{(\overline{v}_2+\delta)(1+v_\beta+\delta)}.$$

According to Stolz-Cesàro Theorem,

$$\begin{aligned}\liminf_{k\to\infty}\frac{\sum_{i=1}^{k-1}(n_{i+1}-m_i)}{m_k} &= \liminf_{k\to\infty}\frac{n_{k+1}-m_k}{m_{k+1}-m_k}\\ &= \liminf_{k\to\infty}\frac{\frac{n_{k+1}}{m_k}-1}{\frac{m_{k+1}}{m_k}-1}=\frac{v_\beta-\overline{v}_2-(v_\beta+\delta)(\overline{v}_2+\delta)}{(1+v_\beta+\delta)(v_\beta-\overline{v}_2)}.\end{aligned}$$

Thus,

$$\liminf_{k\to\infty}\frac{\log_\beta\mu(I_{h_k})}{\log_\beta|I_{h_k}|}=\frac{v_\beta-\overline{v}_2-(v_\beta+\delta)(\overline{v}_2+\delta)}{(1+v_\beta+\delta)(v_\beta-\overline{v}_2)}\cdot\log_\beta\beta_N.$$

Case B**:** For an integer n large enough, if there is $k\geq 2$ such that $l_k\leq n\leq h_k$, then

$$\liminf_{k\to\infty}\frac{\log_\beta\mu(I_n)}{\log_\beta|I_n|}\geq\liminf_{k\to\infty}\frac{\log_\beta\mu(I_n)}{\log_\beta|I_{h_k}|}=\liminf_{k\to\infty}\frac{\log_\beta\mu(I_{h_k})}{\log_\beta|I_{h_k}|}.$$

Case C**:** For an integer n, if there is an integer t with $0\leq t\leq t_k-1$ such that

$$h_k+(t+1)(m_k-n_k)+2Nt<n\leq h_k+(t+1)(m_k-n_k)+2N(t+1),$$

then one has

$$\mu(I_n)\leq\mu(I_{h_k})\cdot\beta_N^{-(t+1)p_k}.$$

Since I_{h_k} is full, by Proposition 4.10,

$$|I_n|=|I_{h_k}|\cdot|I_{n-h_k}(\omega')|,$$

where ω' is an admissible block in $\Sigma_{\beta_N}^{n-h_k}$. By Lemma 4.11,

$$|I_n|\geq|I_{h_k}|\cdot\beta^{-(n-h_k+N)}.$$

Hence,

$$\frac{-\log_\beta\mu(I_n)}{-\log_\beta|I_n|}\geq\frac{-\log_\beta\mu(I_{h_k})+(t+1)p_k\log_\beta\beta_N}{-\log_\beta|I_{h_k}|+((t+1)p_k+N(2t+1))}\geq\frac{-\log_\beta\mu(I_{h_k})}{-\log_\beta|I_{h_k}|}\cdot\varphi(N),$$

where $\varphi(N)<1$ and $\varphi(N)$ tends to 1 as N tends to infinity. If there is an integer t with $0\leq t\leq t_k$ such that

$$h_k+t(m_k-n_k)+2Nt<n\leq\min\{l_{k+1},h_k+(t+1)(m_k-n_k)+2Nt\},$$

then letting $l:=n-(h_k+t(m_k-n_k)+2Nt)$, one has

$$\mu(I_n)\leq\mu(I_{h_k})\cdot\beta_N^{-tp_k-l}.$$

Since I_{h_k} is full, by Proposition 4.10,

$$|I_n| = |I_{h_k}| \cdot |I_{n-h_k}(\omega')|,$$

where ω' is an admissible block in $\Sigma_{\beta_N}^{n-h_k}$. By Lemma 4.11,

$$|I_{n-h_k}(\omega')| \geq \beta^{-(n-h_k+N)}.$$

Therefore,

$$|I_n| \geq |I_{h_k}| \cdot \beta^{-(n-h_k+N)}.$$

Hence,

$$\frac{-\log_\beta \mu(I_n)}{-\log_\beta |I_n|} \geq \frac{-\log_\beta \mu(I_{h_k}) + (tp_k + l)\log_\beta \beta_N}{-\log_\beta |I_{h_k}| + (tp_k + l + t + N(2t+1))} \geq \frac{-\log_\beta \mu(I_{h_k})}{-\log_\beta |I_{h_k}|} \cdot \varphi(N).$$

Therefore, in all cases,

$$\liminf_{k\to\infty} \frac{\log_\beta \mu(I_n)}{\log_\beta |I_n|} \geq \frac{v_\beta - \overline{v}_2 - (v_\beta + \delta)(\overline{v}_2 + \delta)}{(1 + v_\beta + \delta)(v_\beta - \overline{v}_2)} \cdot \log_\beta \beta_N \cdot \varphi(N).$$

Given a point $x \in E_\delta$, let r be a number with $|I_{n+1}(x)| \leq r < |I_n(x)|$. We consider the ball $B(x, r)$. By Lemma 4.11, every n-th order basic interval I_n satisfies

$$|I_n| \geq \beta^{-(n+N)}.$$

Hence, the ball $B(x, r)$ interests at most $\lfloor 2\beta^N \rfloor + 2$ basic intervals of order n. On the other hand,

$$r \geq |I_{n+1}(x)| \geq \beta^{-(n+1+N)} = \beta^{-(1+N)} \cdot \beta^{-n} \geq \beta^{-(1+N)} \cdot |I_n(x)|.$$

Therefore,

$$\liminf_{r\to 0} \frac{\log_\beta \mu(B(x,r))}{\log_\beta r} = \liminf_{n\to\infty} \frac{\log_\beta \mu(I_n(x))}{\log_\beta |I_n(x)|}.$$

By the arbitrariness of the number $\delta \in (0, \delta_0]$, one has

$$\liminf_{k\to\infty} \frac{\log_\beta \mu(I_n)}{\log_\beta |I_n|} \geq \frac{v_\beta - \overline{v}_2 - v_\beta \cdot \overline{v}_2}{(1 + v_\beta)(v_\beta - \overline{v}_2)} \cdot \log_\beta \beta_N \cdot \varphi(N).$$

Let N tend to infinity, by Mass Distribution Principle [61, p.p. 60], one has

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \frac{v_\beta - \overline{v}_2 - v_1\overline{v}_2}{(1 + v_\beta)(v_\beta - \overline{v}_2)}. \tag{13}$$

Regard the right side as a function of v_β with $v_\beta \geq \overline{v}_2/(1 - \overline{v}_2)$, if $\overline{v}_1/(2 + \overline{v}_1) < \overline{v}_2$, then the maximum is attained for $v_\beta = 2\overline{v}_2/(1 - \overline{v}_2)$. Therefore,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \left(\frac{1 - \overline{v}_2}{1 + \overline{v}_2}\right)^2.$$

If $\overline{v}_2 \leq \overline{v}_1/(2 + \overline{v}_1)$, then the maximum is attained for $v_\beta = \overline{v}_1$. Thus,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1 + \overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

$\square$

Proof of Theorem C. If $\underline{v}_2 > 1$, according to Proposition 6.7, the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ is countable. By the definition of the Hausdorff dimension, one has

$$\dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \geq 0.$$

If $\underline{v}_1/(2+\underline{v}_1) \leq \underline{v}_2 \leq 1 < \overline{v}_2$, by Proposition 6.6, then

$$\dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \frac{1}{1+\overline{v}_2}.$$

Combining with Proposition 6.7, we have

$$0 \leq \dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\overline{v}_1/(2+\overline{v}_1) < \underline{v}_2 \leq \overline{v}_2 \leq 1$ and $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2$, by Propositions 6.6 and 6.7, we also have

$$\dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

Since $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2$, according to Propositions 6.8,

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right).$$

Then,

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$ and $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2 \leq 1$, by Propositions 6.6, 6.7 and 6.8, then

$$\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2 \leq \dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

If $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2 \leq \overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, combining Proposition 6.6 with Proposition 6.7, then one has

$$\dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

According to Proposition 6.8, we have

$$\dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \geq \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

Therefore,

$$\frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)} \leq \dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \min\left\{\frac{1}{1+\overline{v}_2},\ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$ and $\overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, by Propositions 6.6, 6.7 and 6.8, then

$$\frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1 \cdot \overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)} \leq \dim_H \left(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)\right) \leq \frac{\underline{v}_1 - \underline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1 - \underline{v}_2)}.$$

$\square$

6.2 Proofs of Theorems D and E

We will give the proofs of Theorems D and E. First, we prove Theorem D.

Proof of Theorem D. By Lemma 6.2, one has

$$\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2) \subseteq \{x \in [0,1] : \nu_\beta(x) \geq \underline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$

Replace the role of $\underline{v}_1$ by v_β, for every sufficiently large positive integer L, we consider the set

$$\mathbb{D} := \{x \in [0,1] : v_\beta \leq \nu_\beta(x) < v_\beta + 1/L\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) \geq \underline{v}_2\}.$$

In fact, by the similar discussions in Proposition 6.7, if $\underline{v}_2 > 1$, then

$$\underline{v}_2 \geq (v_\beta + 1/L)/(1 + v_\beta + 1/L), \quad \text{for any } v_\beta.$$

Therefore, the set $\mathbb{D}$ is empty. Thus,

$$\mathcal{U}(\psi_2) = \{x \in [0,1] : \hat{\nu}_\beta(x) = \infty\}.$$

According to the discussion in Proposition 6.7 and Lemma 6.3, $\mathcal{U}(\psi_2)$ is countable.

If $\underline{v}_2 = 1$, then $v_\beta = \infty$. By **Theorem SW**,

$$\dim_H(\mathbb{D}) = 0.$$

If $\underline{v}_2 < 1$, by the similar discussions in Proposition 6.7, then for $v_\beta \geq \underline{v}_2/(1-\underline{v}_2)$, one has

$$\dim_H(\mathbb{D}) \leq \frac{v_\beta - \underline{v}_2 - v_\beta \cdot v_2}{(1+v_\beta)(v_\beta - \underline{v}_2)} + \frac{\underline{v}_2^2/L}{1-\underline{v}_2}.$$

Since the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$ is a subset of

$$\bigcup_{N=0}^{\infty} \bigcup_{i=1}^{L} \{x \in [0,1] : \underline{v}_1 + N + (i-1)/L \leq \nu_\beta(x) < v_1 + N + i/L\} \bigcap \mathcal{U}(\psi_2),$$

let L tend to infinite, one has

$$\dim_H(\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \sup_{v_\beta \geq \underline{v}_2/(1-\underline{v}_2)} \left\{ \frac{v_\beta - \underline{v}_2 - v_\beta \cdot \underline{v}_2}{(1+v_\beta)(v_\beta - \underline{v}_2)} \right\}.$$

Regard the right side as a function of v_β with $v_\beta \geq \underline{v}_2/(1-\underline{v}_2)$, the maximum is attained for $v_\beta = 2\underline{v}_2/(1-\underline{v}_2)$. Therefore,

$$\dim_H(\mathcal{U}(\psi_2)) \leq \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2.$$

Combining with Proposition 6.6, one has

$$0 \leq \dim_H(\mathcal{U}(\psi_2)) \leq \min\left\{\frac{1}{1+\overline{v}_2}, \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2\right\}.$$

If $\overline{v}_2 \leq 1$, then $\underline{v}_2 \leq \overline{v}_2 \leq 1$, one also has

$$\dim_H (\mathcal{U}(\psi_2)) \leq \min\left\{ \frac{1}{1+\overline{v}_2}, \ \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2 \right\}.$$

To obtain the lower bound of the Hausdorff dimension of the set $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$, we need to construct a Cantor type subset E of $\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)$. By Lemma 6.2, one has

$$\{x \in [0,1] : \nu_\beta(x) > \overline{v}_1\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) > \overline{v}_2\} \subseteq \mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2).$$

We replace the role of $\overline{v}_1$ by v_β with $v_\beta \geq \overline{v}_2/(1-\overline{v}_2)$. Fix $\delta > 0$, consider

$$\{x \in [0,1] : \nu_\beta(x) = v_\beta + \delta\} \cap \{x \in [0,1] : \hat{\nu}_\beta(x) = \overline{v}_2 + \delta\}.$$

If $\overline{v}_2 = 0$, then let

$$n'_k = k^k, \quad m'_k = \lfloor (1+v_\beta+\delta)n'_k \rfloor, \quad \text{for } k = 1, 2, \cdots.$$

If $\overline{v}_2 > 0$, then, let

$$n'_k = \left\lfloor \left(\frac{v_\beta+\delta}{\overline{v}_2+\delta}\right)^k \right\rfloor, \quad m'_k = \lfloor (1+v_\beta+\delta)n'_k \rfloor, \quad k = 1, 2, \cdots.$$

Arguing as in the proof of Proposition 6.8, one has

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \frac{v_\beta - \overline{v}_2 - v_\beta \overline{v}_2}{(1+v_\beta)(v_\beta - \overline{v}_2)}.$$

Regarding the right hand side of the inequality as a function of v_β and taking $v_\beta \geq \overline{v}_2/(1-\overline{v}_2)$ into account, the maximum is attained at $v_\beta = 2\overline{v}_2/(1-\overline{v}_2)$. Then,

$$\dim_H (\mathcal{U}(\psi_2)) \geq \left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2.$$

□

Proof of Theorem E. If $\underline{v}_1 = \overline{v}_1 = 0$, then

$$\lim_{n\to\infty} \frac{-\log_\beta \psi_1(n)}{n} = 0.$$

If $\underline{v}_2 > 0$, by the definition of $\underline{v}_2$ and $\underline{v}_1 = \overline{v}_1 = 0$, then for any $\varepsilon \in (0, \underline{v}_2/2)$, there is an integer n_0 such that for any $n \geq n_0$, one has

$$\psi_2(n) \leq \beta^{-n(\underline{v}_2-\varepsilon)} < \beta^{-n\varepsilon} \leq \psi_1(n).$$

For any point $x \in \mathcal{U}(\psi_2)$, we distinguish two cases:
Case 1: There is an integer n_0 such that

$$T_\beta^{n_0} x = 0.$$

Therefore, the β-expansion of x is finite. Thus,

$$T_\beta^n x = 0, \quad \text{for any } n \geq n_0.$$

Hence,

$$x \in \mathcal{L}(\psi_1).$$

Case 2: For any $n \in \mathbb{N}$, we always have

$$T_\beta^n x > 0.$$

Thus, there is an integer $N_0 \geq 1$ such that for any $N \geq N_0$, there is an integer $n \in [0, N]$ such that

$$0 < T_\beta^n x < \psi_2(N).$$

For the above $\varepsilon > 0$, let

$$K := \max\{N_0,\ n_0\}.$$

For any $n \geq K$, there is an integer $j \in [0, n]$ such that

$$0 < T_\beta^j x < \psi_2(n) \leq \beta^{-n(\underline{v}_2 - \varepsilon)} \leq \beta^{-n\varepsilon} \leq \psi_1(n).$$

In fact, let $n_1 := \min\{n : n \geq K\}$, there is an integer $j_1 \in [0, n_1]$ such that

$$0 < T_\beta^{j_1} x < \psi_2(n_1) \leq \beta^{-n_1(\underline{v}_2 - \varepsilon)} \leq \beta^{-n_1\varepsilon} \leq \beta^{-j_1\varepsilon}.$$

Take $n_2 := \min\left\{n > n_1 : T_\beta^{j_1} x > \beta^{-n_k(\underline{v}_2 - \varepsilon)}\right\}$. There is an integer $j_2 \in [0, n_2]$ such that

$$0 < T_\beta^{j_2} x < \psi_2(n_2) \leq \beta^{-n_2(\underline{v}_2 - \varepsilon)} \leq \beta^{-n_2\varepsilon} \leq \beta^{-j_2\varepsilon}.$$

Since

$$T_\beta^{j_2} x < \psi_2(n_2) \leq \beta^{-n_2(\underline{v}_2 - \varepsilon)} < T_\beta^{j_1} x,$$
$$j_2 \neq j_1.$$

Repeat this process, one can get a sequence of pairwise disjoint integers $\{j_i : i \geq 1\}$ such that

$$0 < T_\beta^{j_i} x < \psi_2(n_i) \leq \beta^{-n_i(\underline{v}_2 - \varepsilon)} \leq \beta^{-n_i\varepsilon} \leq \beta^{-j_i\varepsilon}.$$

Since the cardinality of $\{j_i : i \geq 1\}$ is infinite, there is an integer i_0 such that for any $i \geq i_0$, one has

$$0 < T_\beta^{j_i} x < \psi_2(n_i) \leq \beta^{-n_i(\underline{v}_2 - \varepsilon)} \leq \beta^{-n_i\varepsilon} \leq \beta^{-j_i\varepsilon} \leq \psi_1(j_i).$$

Therefore,

$$x \in \mathcal{L}(\psi_1).$$

Hence, in all cases, we have

$$\mathcal{U}(\psi_2) \subseteq \mathcal{L}(\psi_1).$$

Therefore,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \dim_H (\mathcal{U}(\psi_2)).$$

□

6.3 Examples

In this section, we will show that the upper and lower bounds of the Hausdorff dimension in Theorems C and D can be all reached. Examples 6.9, 6.10 and 6.11 explain that the upper bound estimation $\frac{1}{1+\overline{v}_2}$, $\left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2$ and $\frac{\underline{v}_1-\underline{v}_2-\underline{v}_1\underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1-\underline{v}_2)}$ are reachable, respectively.

Example 6.9. *Let $\psi_1(n) = 1$, for $n = 1, 2, \cdots$ and*

$$\psi_2(n) = \begin{cases} \beta^{-3n}, & \text{if } n = k^k \\ 1, & \text{if } n \neq k^k \end{cases}, \quad k = 1, 2, \cdots$$

Then, $\underline{v}_1 = \overline{v}_1 = 0$, $\underline{v}_2 = 0$, $\overline{v}_2 = 3$ and

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{1}{4} = \frac{1}{1+\overline{v}_2}.$$

Proof. By Proposition 6.6, we only need to show that

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq 1/4.$$

Now, we construct a Cantor subset E of the set $\mathcal{L}(\psi_1) \bigcap \mathcal{U}(\psi_2)$ such that

$$\dim_H(E) \geq 1/4.$$

For $k = 1, 2, \cdots$, let

$$n_k := k^k, \qquad m_k := 4k^k,$$

we construct a Cantor subset E by the same way as Proposition 6.8, then

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq 1/4 = \frac{1}{1+\overline{v}_2}.$$

$\square$

Example 6.10. *Let $\psi_1(n) = 1$, for $n = 1, 2, \cdots$ and*

$$\psi_2(n) = \begin{cases} \beta^{-2n}, & \text{if } n = 4^k \\ \beta^{-n/2}, & \text{if } n \neq 4^k \end{cases}, \quad k = 1, 2, \cdots$$

Then, $\underline{v}_1 = \overline{v} = 0$, $\underline{v}_2 = 1/2$, $\overline{v}_2 = 2$ and

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{1}{9} = \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2.$$

Proof. Since $\underline{v}_1/(2+\underline{v}_1) < \underline{v}_2$, by Proposition 6.7,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2 = \frac{1}{9}.$$

The proof is completed by showing

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) \geq \frac{1}{9} = \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2.$$

For $k = 1, 2, \cdots$, let

$$n_k := 4^k, \qquad m_k := 3\cdot 4^k,$$

we construct a Cantor subset E by the same way as Proposition 6.8, then

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) \geq \frac{1}{9} = \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2.$$

□

Example 6.11. *For $k = 1, 2, \cdots$, let*

$$\psi_1(n) = \begin{cases} \beta^{-n/2}, & \text{if } n = 3^k \\ \beta^{-n}, & \text{if } n \neq 3^k \end{cases}, \quad \psi_2(n) = \begin{cases} \beta^{-n/2}, & \text{if } n = 3^k \\ \beta^{-n/6,} & \text{if } n \neq 3^k. \end{cases}$$

Then, $\underline{v}_1 = 1/2$, $\overline{v}_1 = 1$ $\underline{v}_2 = 1/6$, $\overline{v}_2 = 1/2$ and

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) = \frac{1}{2} = \frac{\underline{v}_1-\underline{v}_2-\underline{v}_1\underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1-\underline{v}_2)}.$$

Proof. Since $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$, by Proposition 6.7, one has

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) \leq \frac{\underline{v}_1-\underline{v}_2-v_1\cdot\underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1-\underline{v}_2)} = \frac{1}{2}.$$

It suffices to show

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) \geq 1/2.$$

For $k = 1, 2, \cdots$, let

$$n_k := 4^k, \qquad m_k := 3\cdot 4^k,$$

we construct a Cantor subset E by the same way as Proposition 6.8, then

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) \geq \frac{1}{2} = \frac{\underline{v}_1-\underline{v}_2-\underline{v}_1\underline{v}_2}{(1+\underline{v}_1)(\underline{v}_1-\underline{v}_2)}.$$

□

Examples 6.12 and 6.13 explain that the lower bound estimation 0 and $\left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2$ are reachable, respectively.

Example 6.12. *Let $\psi_1(n) = 1$ for $n = 1, 2, \cdots$ and*

$$\psi_2(n) = \begin{cases} \beta^{-3n}, & \text{if } n = 4^k \\ \beta^{-n}, & \text{if } n \neq 4^k \end{cases}, \quad k = 1, 2, \cdots$$

Then, $\underline{v}_1 = \overline{v}_1 = 0$, $\underline{v}_2 = 1$, $\overline{v}_2 = 3$ and

$$\dim_H(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)) = 0.$$

Proof. It remains to prove

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq 0.$$

In fact, by Proposition 6.7, one has

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \left(\frac{1-\underline{v}_2}{1+\underline{v}_2}\right)^2 = \left(\frac{1-1}{1+1}\right)^2 = 0.$$

□

Example 6.13. *For $k = 1, 2, \cdots$, let*

$$\psi_1(n) = \begin{cases} \beta^{-3n}, & \text{if } n = 2k+1 \\ \beta^{-10n/3}, & \text{if } n = 2k \end{cases}, \quad \psi_2(n) = \begin{cases} \beta^{-21n/32}, & \text{if } n = 2k+1 \\ \beta^{-2n/3}, & \text{if } n = 2k. \end{cases}$$

Then, $\underline{v}_1 = 3$, $\overline{v}_1 = 10/3$, $\underline{v}_2 = 21/32$, $\overline{v}_2 = 2/3$ and

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{1}{25} = \left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2.$$

Proof. Since $\overline{v}_1/(2+\overline{v}_1) < \overline{v}_2$, by Proposition 6.8, one has

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \geq \left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2.$$

The proof is completed by showing

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{1}{25} = \left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2.$$

In fact, since $\underline{v}_2 \leq \underline{v}_1/(2+\underline{v}_1)$, by Proposition 6.7, we have

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{1}{25}.$$

Therefore,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{1}{25} = \left(\frac{1-\overline{v}_2}{1+\overline{v}_2}\right)^2.$$

□

In the following two examples, we consider the cases that $\underline{v}_1 = \overline{v}_1$ and $\underline{v}_2 = \overline{v}_2$, respectively, which explain that the lower bound estimation $\dfrac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}$ is reachable.

Example 6.14. *Let $\psi_1(n) = \beta^{-n}$ for $n = 1, 2, \cdots$ and*

$$\psi_2(n) = \begin{cases} 1, & \text{if } n = 2k+1 \\ \beta^{-n/4}, & \text{if } n = 2k. \end{cases}, k = 1, 2, \cdots$$

Then, $\underline{v}_1 = \overline{v}_1 = 1$, $\underline{v}_2 = 0$, $\overline{v}_2 = 1/4$ and

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{1}{3} = \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

Proof. Since $\overline{v}_2 \le \overline{v}_1/(2+\overline{v}_1)$, by Proposition 6.8, what is left is to show

$$\dim_H\left(\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)\right) \le \frac{1}{3} = \frac{\overline{v}_1-\overline{v}_2-\overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1-\overline{v}_2)}.$$

For any point $x\in[0,1]$, denote its β-expansion by

$$x = \frac{a_1}{\beta}+\frac{a_2}{\beta^2}+\cdots+\frac{a_n}{\beta^n}+\cdots,$$

where $a_i\in\{0,\cdots,\lceil\beta\rfloor\}$, for all $i\ge 1$. Let

$$a_{n'_i}>0,\quad a_{n'_i+1}=\cdots=a_{m'_i-1}=0,\quad a_{m'_i}>0.$$

If $x\in\mathcal{L}(\psi_1)\cap\mathcal{U}(\psi_2)$, since $\underline{v}_1>0$, then one has

$$\limsup_{i\to\infty}(m'_i-n'_i)=\infty.$$

Arguing as in the proof of Proposotion 6.7, we take the maximal subsequences $\{n_k : k\ge 1\}$ and $\{m_k : k\ge 1\}$ of $\{n'_i : i\ge 1\}$ and $\{m'_i : i\ge 1\}$, respectively, such that the sequence $\{m_k-n_k : k\ge 1\}$ is non-decreasing. Notice

$$\beta^{n_k-m_k}<T_\beta^{n_k}x<\beta^{n_k-m_k+1},$$

one has

$$\limsup_{k\to\infty}\frac{m_k-n_k}{n_k}=1.$$

Since $x\in\mathcal{U}(\psi_2)$, there is an integer k_0 such that for any $k\ge k_0$, one has

$$m_k-n_k\ge\lfloor n_{k+1}/4\rfloor.$$

If not, for any $j\ge1$, there is an integer k_j such that

$$m_{k_j}-n_{k_j}<n_{k_{j+1}}/4.$$

Since one of $n_{k_{j+1}}$ and $n_{k_{j+1}}+1$ is even, denote it by $l_{k_{j+1}}$, for any integer $n\in[1,l_{k_{j+1}}]$, one has

$$T_\beta^n x>\beta^{n_{k_j}-m_{k_j}}>\beta^{-l_{k_{j+1}}/4}=\psi_2(l_{k_{j+1}}).$$

It contradicts the fact that $x\in\mathcal{U}(\psi_2)$.

Choose the subsequence $\{n_{k_i} : i\ge1\}$ and $\{m_{k_i} : i\ge1\}$ of $\{n_k : k\ge1\}$ and $\{m_k : k\ge1\}$, respectively, such that

$$\lim_{i\to\infty}\frac{m_{k_i}-n_{k_i}}{n_{k_i}}=1.$$

For simplicity, let $\{n_k : k\ge1\}$ and $\{m_k : k\ge1\}$ stand for $\{n_{k_i} : i\ge1\}$ and $\{m_{k_i} : i\ge1\}$, respectively. For any $\varepsilon>0$, there is an integer k' such that for any $k\ge k'$, one has

$$(1-\varepsilon)n_k\le m_k-n_k\le(1+\varepsilon)n_k,\quad m_k-n_k\ge\frac{n_{k+1}}{4}-2.$$

Arguing as in the proof of Proposition 6.7, one has

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{1}{3} = \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

□

Example 6.15. *Let*

$$\psi_1(n) = \begin{cases} \beta^{-n/3}, & \text{if } n = 2k+1 \\ \beta^{-2n/3}, & \text{if } n = 2k \end{cases}, \quad k = 1, 2, \cdots$$

and $\psi_2(n) = \beta^{-2n/11}$ *for* $n = 1, 2, \cdots$. *Then,* $\underline{v}_1 = 1/3$, $\overline{v}_1 = 2/3$, $\underline{v}_2 = \overline{v}_2 = 2/11$ *and*

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{9}{20} = \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

Proof. Since $\overline{v}_2 \leq \overline{v}_1/(2+\overline{v}_1)$, by Proposition 6.8, what is left is to show

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{9}{20} = \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

In fact, since $\underline{v}_2 = \dfrac{2}{11} \leq \dfrac{1}{7} = \underline{v}_1/(2+\underline{v}_1)$, by Proposition 6.7,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) \leq \frac{9}{20}.$$

Then,

$$\dim_H (\mathcal{L}(\psi_1) \cap \mathcal{U}(\psi_2)) = \frac{\overline{v}_1 - \overline{v}_2 - \overline{v}_1\overline{v}_2}{(1+\overline{v}_1)(\overline{v}_1 - \overline{v}_2)}.$$

□

7 Future Studies

7.1 Lyapunov exponents approximation

Given a φ_t-invariant measure μ which is not concentrated on Sing(X), from the Oseledec Theorem [3, Theorem S.2.9], for the *linear Poincaré flow* $\psi_t : \mathcal{N} \to \mathcal{N}$ and μ-almost every point x, there is a measurable splitting

$$\mathcal{N}_x = \bigoplus_{i=1}^{k(x)} \mathcal{E}_i(x)$$

and a sequence of numbers

$$\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{k(x)}(x),$$

such that

$$\lim_{t\to\infty} \frac{1}{t} \log \parallel \psi_t(v) \parallel = \lambda_i(x),\ \forall\ v \in \mathcal{E}_i(x) \setminus \{0\},\ i = 1, 2, \ldots, k(x),$$

where $1 \leq k(x) \leq d-1$. Therefore, we get Lyapunov exponents $\lambda_i(x)$ at point x with respect to ψ_t and the corresponding Oseledec subspace $\mathcal{E}_i(x)$. Lyapunov exponents for a differential equation are a natural generalization of the eigenvalues of the matrix in the linear part of the equation. Lyapunov exponents describe the asymptotic evolution of a tangent map: positive or negative exponents correspond to exponential growth or decay of the norm, respectively, whereas vanishing exponents mean lack of exponential behavior. When the measure μ is an ergodic invariant measure of φ_t, Lyapunov exponents $\lambda_i(x)$ is a constant, denoted by λ_i. An ergodic invariant measure is called *hyperbolic* if its Lyapunov exponents are different from zero. For a periodic orbit of flow, we have an atomic measure (invariant measure) on periodic orbit. Then, there also are Lyapunov exponents and Oseledec subspace on periodic orbit. What is the relationship between Lyapunov exponents of hyperbolic ergodic invariant measure and that of an atomic measure (invariant measure) on periodic orbit?

For the discrete-time dynamics (diffeomorphism), the closing lemma of Katok [8] in Pesin Theory affirms that the closure of hyperbolic periodic points contains the support of a given hyperbolic measure. Katok and Mendoza showed [3, Theorem S.5.5] (see also [78, Theorem 15.4.7]) that the smallest absolute value in all of the Lyapunov exponents of a hyperbolic measure can be approximated from the upper side by the smallest absolute value in all of the Lyapunov exponents of a hyperbolic periodic orbit. In uniform hyperbolic systems, a classical result of Sigmund [79] illustrates that periodic measures are dense in the set of invariant measures. From the nonuniform hyperbolic case, Hirayama [80] showed that periodic measures are dense in the set of invariant measures supported by a total measure set with respect to a hyperbolic mixing measure. Wang and Sun [4, Theorem 1.1] proved that Lyapunov exponents of a hyperbolic ergodic measure of a $C^{1+\alpha}(\alpha > 0)$ diffeomorphism are approximated by Lyapunov exponents of hyperbolic atomic measures on periodic orbits.

For the continuous-time dynamics (flow), we can consider the following questions:

(1) Does the closure of hyperbolic periodic points contains the support of a given hyperbolic measure of flow?

(2) Can Lyapunov exponents of a hyperbolic ergodic measure of a flow be approximated by Lyapunov exponents of hyperbolic atomic measures on periodic orbits?

7.2 Uniform Diophantine approximation of dynamical systems

Given a measurable dynamical system $(M, \mathcal{B}, \mu, T)$ with a metric d, fixed $x_0 \in M$ and a positive function $\psi : \mathbb{N} \to \mathbb{R}^+$, we have the set of the *ψ-well asymptotically approximable points* by x_0 as

$$\mathcal{L}(M, \psi, x_0) := \{x \in M : d(T^n x, x_0) < \psi(n), \text{ for infinitely many } n \in \mathbb{N}\}.$$

For any real number $\beta > 1$ and $x_0 \in [0, 1]$, **Theorem SW** shows

$$\dim_H \mathcal{L}(T_\beta, \psi, x_0) = \frac{1}{1+v}, \qquad \text{where } \ v := \liminf_{n\to\infty} \frac{-\log_\beta \psi(n)}{n}.$$

In fact, in my thesis, we only consider the case $x_0 = 0$ and define the sets

$$\mathcal{L}(\psi_1) := \left\{x \in [0, 1] : T_\beta^n x < \psi_1(n), \text{ for infinitely many integers } n \in \mathbb{N}\right\},$$

$$\mathcal{U}(\psi_2) := \left\{x \in [0, 1] : \forall\ N \gg 1, T_\beta^n x < \psi_2(N) \text{ has a solution } n \in [0, N]\right\}.$$

Naturally, if $x_0 \neq 0$, then define the sets

$$\mathcal{L}(\psi_1,\ x_0) := \left\{x \in [0, 1] : d(T_\beta^n x,\ x_0) < \psi_1(n), \text{ for infinitely many integers } n \in \mathbb{N}\right\},$$

$$\mathcal{U}(\psi_2,\ x_0) := \left\{x \in [0, 1] : \forall\ N \gg 1, d(T_\beta^n x,\ x_0) < \psi_2(N) \text{ has a solution } n \in [0, N]\right\},$$

what is the size (Lebesgue measure, Hausdorff dimension) of the set $\mathcal{U}(\psi_2, x_0)$ and the set $\mathcal{L}(\psi_1,\ x_0) \cap \mathcal{U}(\psi_2, x_0)$?

On the other hand, in dynamical systems, people often consider diffeomorphisms on compact Riemannian manifolds such as uniform hyperbolic maps, nonuniform hyperbolic maps and average conformal maps. Define the sets

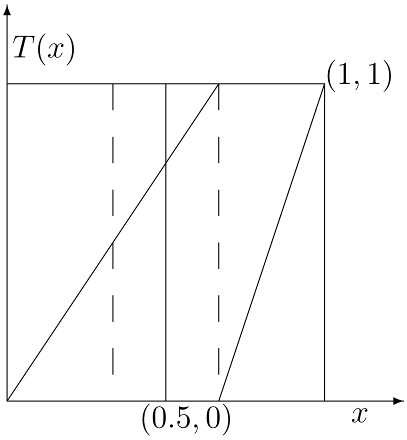
$$\mathcal{L}(T, \psi_1, x_0) := \{x \in [0, 1] : d(T^n x,\ x_0) < \psi_1(n), \text{ for infinitely many integers } n \in \mathbb{N}\},$$

$$\mathcal{U}(T, \psi_2, x_0) := \{x \in [0, 1] : \forall\ N \gg 1, d(T^n x,\ x_0) < \psi_2(N) \text{ has a solution } n \in [0, N]\},$$

what is the size (Lebesgue measure, Hausdorff dimension) of the set $\mathcal{U}(T, \psi_2, x_0)$ and the set $\mathcal{L}(T, \psi_1, x_0) \cap \mathcal{U}(T, \psi_2, x_0)$? For example, we can consider the size (Lebesgue measure, Hausdorff dimension) of the set $\mathcal{U}(T, \psi_2, x_0)$ and the set $\mathcal{L}(T, \psi_1, x_0) \cap \mathcal{U}(T, \psi_2, x_0)$ for different maps T as followings.

Example 7.1. *For real number $a \in (1/2, 1)$, let $T : [0, 1] \to [0, 1]$ be the map defined by*

$$T(x) = \begin{cases} \dfrac{x}{a} \ mod\ 1 & if\ x \in [0, a) \\ \\ \dfrac{x-a}{1-a} \ mod\ 1 & if\ x \in [a, 1]. \end{cases}$$

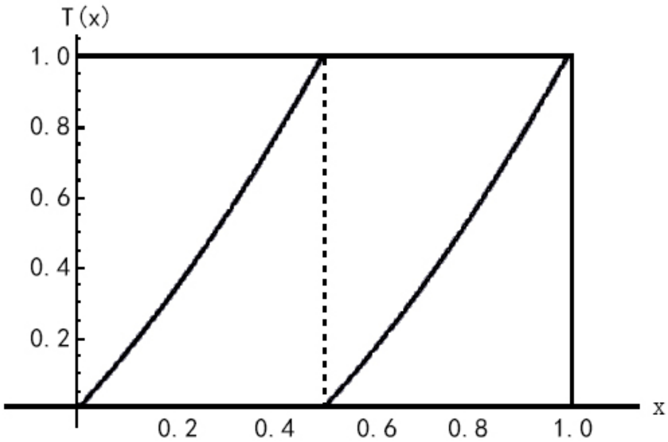


what is the size (Lebesgue measure, Hausdorff dimension) of the set $\mathcal{U}(T,\psi_2,x_0)$ and the set $\mathcal{L}(T,\psi_1,x_0)\cap\mathcal{U}(T,\psi_2,x_0)$?

If the image of T has two non-linear branches as the following 7.2, what is the size (Lebesgue measure, Hausdorff dimension) of the set $\mathcal{U}(T,\psi_2,x_0)$ and the set $\mathcal{L}(T,\psi_1,x_0)\cap\mathcal{U}(T,\psi_2,x_0)$?

Example 7.2. *Let* $T:[0,1]\to[0,1]$ *be the map defined by:*

$$T(x)=\begin{cases} x^2+\dfrac{3x}{2}, & \text{if } x\in[0,1/2)\\ x^2+\dfrac{x}{2}-\dfrac{1}{2}, & \text{if } x\in[1/2,1]. \end{cases}$$



References

[1] S. Smale. Diffeornorphisms with many periodic points. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 63–80. Princeton University Press, Princeton, 1965. 1, 11, 37

[2] A. Livšic. Certain properties of the homology of Y-systems. *Mat. Zametki*, 10:555–564, 1971. 1, 11

[3] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza. 1, 11, 26, 29, 50, 88

[4] Z. Wang and W. Sun. Lyapunov exponents of hyperbolic measures and hyperbolic periodic orbits. *Trans. Amer. Math. Soc.*, 362(8):4267–4282, 2010. 1, 11, 88

[5] M. Artin and B. Mazur. On periodic points. *Ann. of Math. (2)*, 81:82–99, 1965. 3, 14

[6] R. Bowen. Entropy and the fundamental group. In *The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977)*, volume 668 of *Lecture Notes in Math.*, pages 21–29. Springer, Berlin, 1978. 3, 14

[7] R. Bowen. Topological entropy and axiom A. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 23–41. Amer. Math. Soc., Providence, R.I., 1970. 4, 14

[8] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Inst. Hautes Études Sci. Publ. Math.*, (51):137–173, 1980. 4, 14, 29, 57, 88

[9] V. Kaloshin. An extension of the Artin-Mazur theorem. *Ann. of Math. (2)*, 150(2):729–741, 1999. 4, 15

[10] E. Lorenz. Deterministic nonperiodic flow. *Journal of the atmospheric sciences*, 20(2):130–141, 1963. 4, 15

[11] J. Marsden and M. McCracken. *The Hopf bifurcation and its applications.* Springer-Verlag, New York, 1976. With contributions by P. Chernoff, G. Childs, S. Chow, J. R. Dorroh, J. Guckenheimer, L. Howard, N. Kopell, O. Lanford, J. Mallet-Paret, G. Oster, O. Ruiz, S. Schecter, D. Schmidt and S. Smale, Applied Mathematical Sciences, Vol. 19. 4, 15

[12] Z. Lian and L. Young. Lyapunov exponents, periodic orbits, and horseshoes for semiflows on Hilbert spaces. *J. Amer. Math. Soc.*, 25(3):637–665, 2012. 4, 15

[13] L. Dirichlet. Verallgemeinerung eines satzes aus der lehre von den kettenbrüchen nebst einigen anwendungen auf die theorie der zahlen. *SB Preuss. Akad. Wiss*, 1842:93–95, 1842. 5, 16

[14] M. Waldschmidt. Recent advances in Diophantine approximation. In *Number theory, analysis and geometry*, pages 659–704. Springer, New York, 2012. 5, 16

[15] A. Legendre. *Essai sur la théorie des nombres*. Cambridge Library Collection. Cambridge University Press, Cambridge, 2009. Reprint of the second (1808) edition. 5, 16

[16] A. Khintchine. Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.*, 92(1-2):115–125, 1924. 6, 16

[17] R. Duffin and A. Schaeffer. Khintchine's problem in metric Diophantine approximation. *Duke Math. J.*, 8:243–255, 1941. 6, 17

[18] Dimitris K. and James M. On the duffin-schaeffer conjecture. *arXiv:1907.04593*, 2019. 6, 17

[19] V. Beresnevich, G. Harman, A. Haynes, and S. Velani. The Duffin-Schaeffer conjecture with extra divergence II. *Math. Z.*, 275(1-2):127–133, 2013. 6, 17

[20] A. Haynes, A. Pollington, and S. Velani. The Duffin-Schaeffer conjecture with extra divergence. *Math. Ann.*, 353(2):259–273, 2012. 6, 17

[21] V. Jarník. Diophantische approximationen und hausdorffsches mass. *Rec. Math. Moscou*, 36:371–382, 1929. 6, 17

[22] A. Besicovitch. Sets of Fractional Dimensions (IV): On Rational Approximation to Real Numbers. *J. London Math. Soc.*, 9(2):126–131, 1934. 6, 17

[23] R. Hill and S. Velani. The ergodic theory of shrinking targets. *Invent. Math.*, 119(1):175–198, 1995. 6, 7, 17

[24] N. Chernov and D. Kleinbock. Dynamical Borel-Cantelli lemmas for Gibbs measures. *Israel J. Math.*, 122:1–27, 2001. 6, 17

[25] B. Fayad. Mixing in the absence of the shrinking target property. *Bull. London Math. Soc.*, 38(5):829–838, 2006. 6, 17

[26] J. Tseng. On circle rotations and the shrinking target properties. *Discrete Contin. Dyn. Syst.*, 20(4):1111–1122, 2008. 6, 17

[27] J. L. Fernández, M. V. Melián, and D. Pestana. Expanding maps, shrinking targets and hitting times. *Nonlinearity*, 25(9):2443–2471, 2012. 6, 17

[28] L. Barreira and B. Saussol. Hausdorff dimension of measures via Poincaré recurrence. *Comm. Math. Phys.*, 219(2):443–463, 2001. 6, 17

[29] J.L. Fernández, M.V. Melián, and D. Pestana. Quantitative recurrence properties of expanding maps. *arXiv:math/0703222*, 2007. 6, 17

[30] B. Tan and B. Wang. Quantitative recurrence properties for beta-dynamical system. *Adv. Math.*, 228(4):2071–2097, 2011. 6, 17

[31] L. Peng, B. Tan, and B. Wang. Quantitative Poincaré recurrence in continued fraction dynamical system. *Sci. China Math.*, 55(1):131–140, 2012. 6, 17

[32] S. Galatolo. Dimension via waiting time and recurrence. *Math. Res. Lett.*, 12(2-3):377–386, 2005. 6, 17

[33] D. Kim. The shrinking target property of irrational rotations. *Nonlinearity*, 20(7):1637–1643, 2007. 6, 17

[34] Y. Bugeaud. A note on inhomogeneous Diophantine approximation. *Glasg. Math. J.*, 45(1):105–110, 2003. 6, 17

[35] S. Trubetskoĭ and Ĭ. Shmeling. Inhomogeneous Diophantine approximations and angular recurrence for billiards in polygons. *Mat. Sb.*, 194(2):129–144, 2003. 6, 17

[36] R. Hill and S. Velani. Metric Diophantine approximation in Julia sets of expanding rational maps. *Inst. Hautes Études Sci. Publ. Math.*, (85):193–216, 1997. 7, 17

[37] R. Hill and S. Velani. The shrinking target problem for matrix transformations of tori. *J. London Math. Soc. (2)*, 60(2):381–398, 1999. 7, 17

[38] W. Philipp. Some metrical theorems in number theory. *Pacific J. Math.*, 20:109–127, 1967. 7, 9, 18, 20

[39] L. Shen and B. Wang. Shrinking target problems for beta-dynamical system. *Sci. China Math.*, 56(1):91–104, 2013. 7, 18, 36, 68

[40] Y. Bugeaud and L. Liao. Uniform Diophantine approximation related to b-ary and β-expansions. *Ergodic Theory Dynam. Systems*, 36(1):1–22, 2016. 7, 18, 64, 65

[41] M. Amou and Y. Bugeaud. Exponents of Diophantine approximation and expansions in integer bases. *J. Lond. Math. Soc. (2)*, 81(2):297–316, 2010. 7, 18

[42] Y. Bugeaud. *Distribution modulo one and Diophantine approximation*, volume 193 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012. 7, 18

[43] P. Erdős and A. Rényi. On a new law of large numbers. *J. Analyse Math.*, 23:103–111, 1970. 7, 18

[44] J. Liu and M. Lü. Hausdorff dimension of some sets arising by the run-length function of β-expansions. *J. Math. Anal. Appl.*, 455(1):832–841, 2017. 7, 18

[45] C. Cao and Y. Chen. The run-length function of the β-expansion of the unit. *J. Number Theory*, 177:248–262, 2017. 7, 18

[46] L. Zheng. Diophantine approximation and run-length function on β-expansions. *arXiv preprint arXiv:1805.04744*, 2018. 7, 18

[47] K. Dmitry, K. Ioannis, and K. Florian. Diophantine approximation and run-length function on β-expansions. *arXiv:1904.08584*, 2019. 9, 20

[48] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. 13, 29, 37

[49] R. Chen. Entropy and periodic orbits for generic diffeomorphisms. *Bull. Korean Math. Soc.*, 50(3):727–730, 2013. 15

[50] M. Boshernitzan. Quantitative recurrence results. *Invent. Math.*, 113(3):617–631, 1993. 17

[51] S. Liao. On (η, d)-contractible orbits of vector fields. *Systems Sci. Math. Sci.*, 2(3):193–227, 1989. 23, 25

[52] M. Li, S. Gan, and L. Wen. Robustly transitive singular sets via approach of an extended linear Poincaré flow. *Discrete Contin. Dyn. Syst.*, 13(2):239–269, 2005. 23, 24

[53] S. Liao. A basic property of a certain class of differential systems. *Acta Math. Sinica*, 22(3):316–343, 1979. 25, 41

[54] S. Liao. Some uniformity properties of ordinary differential systems and a generalization of an existence theorem for periodic orbits. *Beijing Daxue Xuebao*, (2):1–19, 1985. 25, 46

[55] S. Gan and D. Yang. Morse-Smale systems and horseshoes for three dimensional singular flows. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(1):39–112, 2018. 25

[56] S. Crovisier and D. Yang. Homoclinic tangencies and singular hyperbolicity for three-dimensional vector fields. *arXiv preprint arXiv:1702.05994*, 2017. 25

[57] S. Albeverio, M. Pratsiovytyi, and G. Torbin. Fractal probability distributions and transformations preserving the Hausdorff-Besicovitch dimension. *Ergodic Theory Dynam. Systems*, 24(1):1–16, 2004. 32

[58] M. Manstavičius. Hausdorff-Besicovitch dimension of graphs and p-variation of some Lévy processes. *Bernoulli*, 13(1):40–53, 2007. 32

[59] S. Gouravaraju and R. Ganguli. Estimating the hausdorff–besicovitch dimension of boundary of basin of attraction in helicopter trim. *Applied Mathematics and Computation*, 218(21):10435–10442, 2012. 32

[60] O. Barndorff-Nielsen. On the rate of growth of the partial maxima of a sequence of independent identically distributed random variables. *Math. Scand.*, 9:383–394, 1961. 33

[61] K. Falconer. *Fractal geometry*. John Wiley & Sons, Ltd., Chichester, 1990. Mathematical foundations and applications. 33, 78

[62] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar*, 8:477–493, 1957. 34, 35

[63] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, 11:401–416, 1960. 35

[64] B. Li and J. Wu. Beta-expansion and continued fraction expansion over formal Laurent series. *Finite Fields Appl.*, 14(3):635–647, 2008. 36

[65] A. Fan and B. Wang. On the lengths of basic intervals in beta expansions. *Nonlinearity*, 25(5):1329–1343, 2012. 36

[66] Anosov D. *Dynamical Systems in the 1960s: The Hyperbolic Revolution. In: Bolibruch .A.A. et al. (eds)*. Springer, Berlin, Heidelberg, 2006. Mathematical Events of the Twentieth Century. 37

[67] M. Peixoto. Structural stability on two-dimensional manifolds. *Topology*, 1:101–120, 1962. 37

[68] J. Palis. A global view of dynamics and a conjecture on the denseness of finitude of attractors. Number 261, pages xiii–xiv, 335–347. 2000. Géométrie complexe et systèmes dynamiques (Orsay, 1995). 37

[69] J. Palis. Open questions leading to a global perspective in dynamics. *Nonlinearity*, 21(4):T37–T43, 2008. 37

[70] E. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. of Math. (2)*, 151(3):961–1023, 2000. 37

[71] C. Bonatti, S. Gan, and L. Wen. On the existence of non-trivial homoclinic classes. *Ergodic Theory Dynam. Systems*, 27(5):1473–1508, 2007. 37

[72] S. Crovisier. Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems. *Ann. of Math. (2)*, 172(3):1641–1677, 2010. 37

[73] Y. Shi, S. Gan, and L. Wen. On the singular-hyperbolicity of star flows. *J. Mod. Dyn.*, 8(2):191–219, 2014. 39, 42

[74] L. Wen. On the C^1 stability conjecture for flows. *J. Differential Equations*, 129(2):334–357, 1996. 42

[75] D. Ruelle. An inequality for the entropy of differentiable maps. *Bol. Soc. Brasil. Mat.*, 9(1):83–87, 1978. 44

[76] S. Liao. An existence theorem for periodic orbits. *Beijing Daxue Xuebao*, (1):1–20, 1979. 46

[77] S. Gan. A generalized shadowing lemma. *Discrete Contin. Dyn. Syst.*, 8(3):627–632, 2002. 51

[78] L. Barreira and Y. Pesin. *Nonuniform hyperbolicity*, volume 115 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007. Dynamics of systems with nonzero Lyapunov exponents. 88

[79] K. Sigmund. Generic properties of invariant measures for Axiom A diffeomorphisms. *Invent. Math.*, 11:99–109, 1970. 88

[80] M. Hirayama. Periodic probability measures are dense in the set of invariant measures. *Discrete Contin. Dyn. Syst.*, 9(5):1185–1192, 2003. 88