Mathematical analysis and numerical approximations of magnetohydrodynamics equations with several types of boundary conditions

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Mathematical analysis and numerical approximations of magnetohydrodynamics equations with several types of boundary conditions

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Introduction générale

La magnétohydrodynamique (abrégé par la suite par "MHD") est le champ de la physique qui décrit le comportement d'un fluide électriquement conducteur (tel que les métaux liquides, les plasmas, l'eau salée etc ...) sous l'effet de champs magnétiques [37, 43]. Ce domaine de la physique a été découvert en 1942 par H. Alfvén dans [6] (travaux pour lesquels il obtiendra le prix Nobel en 1942), et à la suite des travaux de Hartmann sur les métaux liquides en 1937.

La MHD joue un rôle prépondérant dans le domaine de l'astrophysique. En effet, l'espace est majoritairement composé de plasmas, et ceux-ci sont particulièrement sensibles aux champs électromagnétiques. On peut mentionner par exemple les phénomènes de vents solaires, c'est à dire une éjection de plasmas du soleil [97].

Toujours dans le domaine des plasmas, et particulièrement inspirée des phénomènes astrophysiques, la MHD a été mise en avant ces dernières années avec les essais sur la fusion nucléaire, notamment avec les tokamaks (voir par exemple [100]). L'idée est de fusionner de petits noyaux atomiques, comme par exemple le tritium et le deutérium, en reproduisant les réactions chimiques apparaissant naturellement dans le Soleil. Cela implique donc des conditions extrêmes de températures (plus de 100 millions de degrés) pour obtenir la fusion, ce qui forme un plasma. Afin de contrôler cette fusion, il n'est pas envisageable, à de telles températures, d'utiliser un confinement physique. On s'oriente donc vers un confinement magnétique, et cela fait intervenir les équations de la MHD.

On retrouve également la MHD dans différents domaines de l'industrie et l'ingénierie, par exemple pour le refroidissement des réacteurs nucléaires par du métal liquide, le pompage des métaux au moyen de pompes électromagnétiques, la simulation de l'électrolyse lors de la production d'aluminium, la production d'énergie MHD (voir [63, 43, 73]).

La MHD se caractérise par un système d'équations aux dérivées partielles couplées: la mécanique des fluides est gouvernée par les équations de Navier-Stokes, et l'électromagnétisme par les équations de Maxwell. La combinaison de ces deux domaines, la mécanique des fluides et l'électromagnétisme, génère deux effets de couplage: le mouvement d'un matériau conducteur sous l'effet d'un champ magnétique induit un courant électrique, qui va ainsi modifier le champ électromagnétique en place. De plus, le courant et le champ magnétique génèrent la force de Lorentz, une force mécanique exercée sur les particules du fluide dans la direction normale à la fois au champ magnétique et au courant électrique.

Dans cette thèse, on va prendre le cas particulier de la MHD incompressible, c'est à dire que les fluides visqueux sont incompressibles, et on se place dans un domaine borné Ω de $\mathbb{R}^3$. On obtient ainsi le système d'équations aux dérivées partielles qui suit (voir par exemple [37]), que l'on appellera système (MHD) dans ce qui suit :

$$\text{(MHD)} \begin{cases} -\nu\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla P - \kappa(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b} = \boldsymbol{f} & \text{dans } \Omega, \\ \kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g} & \text{dans } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{dans } \Omega, \\ \operatorname{div}\boldsymbol{b} = 0 & \text{dans } \Omega. \end{cases}$$

Les inconnues de ces équations sont le champ de vitesses $\boldsymbol{u}$, le champ magnétique $\boldsymbol{b}$, et la pression P, tandis que $\boldsymbol{f}$ représente les forces extérieures et $\boldsymbol{g}$ est un terme source. Les paramètres ν, μ et κ sont respectivement la viscosité du fluide, la perméabilité magnétique et le terme de couplage. Le terme $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b}$ représente la force de Lorentz et le terme $\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b})$ le mouvement du matériau conducteur dans le champ magnétique précédemment mentionné.

L'étude d'équations aux dérivées partielles dans un domaine borné implique l'apparition de conditions aux limites. Les conditions au bord du domaine les plus fréquemment considérées sont les conditions de Dirichlet:

$$\boldsymbol{u} = \mathbf{0} \quad \text{sur } \Gamma$$

qui sont des conditions de non-glissement, signifiant que le fluide adhère à la paroi. Cette formulation, introduite par G. Stokes en 1845 [93], a cependant des limites: en effet, suivant les travaux de Serrin [88], ces conditions peuvent ne pas être réalistes, entraînant des phénomènes de couche limite. Afin de modéliser les différentes situations, on doit considérer plusieurs types de conditions au bord. Dans cette thèse, nous en considérerons deux: des conditions dites de type Navier et des conditions sur la pression.

Les conditions de Navier se présentent comme suit: on considère des parois solides, c'est à dire que le fluide ne peut pas les traverser. Ainsi, la vitesse du fluide dans la direction normale aux bords est nulle, ce qui se retranscrit dans l'égalité:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{sur } \Gamma.$$

Afin de prendre en compte la réaction de la paroi par rapport au mouvement du fluide, Navier établit une condition de glissement avec friction (voir [74]), qui permet de mesurer l'effet de frottement du fluide contre la paroi, en considérant la composante tangentielle du tenseur des contraintes proportionnelle à la composante tangentielle, notée τ, du champ de vitesses :

$$2[D(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{0} \quad \text{sur } \Gamma$$

où $\alpha \geq 0$ est le coefficient de friction et:

$$D(\boldsymbol{u})_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),\ 1 \leqslant i, j \leqslant 3$$

Ainsi, on caractérise les conditions de Navier par :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{et} \quad 2[D(\boldsymbol{u})\boldsymbol{n}]_\tau + \alpha \boldsymbol{u}_\tau = \boldsymbol{0} \quad \text{sur } \Gamma$$

Notons que, lorsque α tend vers l'infini, on retrouve formellement les conditions de Dirichlet. Dans le cas d'un bord plat et d'un coefficient de friction α nul, cette condition de Navier est équivalente aux conditions de type Navier:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{et} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{sur } \Gamma \tag{0.0.1}$$

C'est cette dernière condition que nous allons en particulier considérer dans la thèse.

Les conditions impliquant une pression au bord apparaissent dans plusieurs applications physiques, comme, par exemple, dans les réseaux hydrauliques, les pipelines ou la circulation du sang. On peut les retrouver également dans la modélisation du réseau de veines dans le cerveau, à partir d'une angiographie en trois dimensions obtenue par résonance magnétique. On doit en outre coupler la pression au bord avec une condition sur la vitesse tangentielle:

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{et} \quad P = P_0 \quad \text{sur } \Gamma. \tag{0.0.2}$$

On se concentrera dans cette thèse sur deux axes essentiels. Le premier concerne l'analyse mathématique du système (MHD) avec deux types de conditions aux limites. La principale originalité tient au choix de conditions aux limites non classiques pour le champ de vitesse. Le premier objectif est d'étudier le système stationnaire (MHD) dans le cas plus général où le domaine Ω est un ouvert borné avec une frontière Γ pouvant être non connexe. Nous considérons un premier type de conditions aux limites pour $\boldsymbol{u}$, $\boldsymbol{b}$ et P :

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{sur } \Gamma \quad \text{et} \quad P = P_0 + \alpha_i \quad \text{sur } \Gamma_i \quad 1 \leq i \leq I, \tag{0.0.3}$$

où Γ_i sont les composantes connexes du bord et les coefficients α_i sont des constantes. Cette forme de condition aux limites sur la pression a été introduite pour des systèmes de Stokes et de Navier-Stokes d'abord dans le cas hilbertien dans [33, 34], où les auteurs ont étudié les différences $\alpha_i - \alpha_0$, $i = 1 \cdots I$ qui représentent les sauts de pression entre les entrées et sorties des sections Γ_i dans les réseaux de tuyaux. Ensuite une extension de ces résultats en théorie L^p est établie dans [14].

Nous démontrons l'existence, l'unicité et la régularité des solutions en théorie hilbertienne dans une première étape, puis plus généralement en théorie L^p, $1 < p < \infty$. Un deuxième objectif est d'étudier le système (MHD) avec des conditions aux limites de type Navier (0.0.1) imposées sur le champ de vitesse et le champ magnétique. Le cadre fonctionnel que nous avons choisi est celui des espaces de Banach basés sur les espaces L^p, $1 < p < \infty$. En particulier, on se place dans des domaines simplement connexes, avec des frontières connexes, pour établir des résultats d'existence de solutions faibles et fortes.

On trouve de nombreux travaux sur l'analyse mathématique du système (MHD) dans la littérature, mais essentiellement dans le cas de conditions au bord de Dirichlet pour le champ de vitesses. Pour le cas stationnaire, on peut citer les travaux [9, 103] pour l'existence et la régularité de solutions, ou les travaux de [5, 4] pour le cas des conditions mixtes pour le champ magnétique. Gunzburger, Meir et Peterson [52] ont étudié le système MHD avec des conditions aux bord de Dirichlet non homogènes en dimension 3. En supposant que Ω est simplement connexe et de classe $\mathcal{C}^{1,1}$ ou convexe et que les données au bord sont petites, ils montrent l'existence et l'unicité des solutions faibles $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$. On peut citer aussi les travaux [5, 3, 4, 24]. Ces résultats ont été récemment généralisés par Amrouche et Boukassa [9] pour le cas d'un ouvert Ω multiplement connexe et un bord Γ non connexe. Ils ont démontré des résultats d'existence de solutions faibles $\boldsymbol{W}^{1,p}(\Omega)$ pour $p \geq 2$ et fortes $\boldsymbol{W}^{2,p}(\Omega)$ pour $p \geq 6/5$. Pour le cas non stationnaire, on peut à titre d'exemple citer les travaux [84, 101, 50, 53]. Comme pour les équations de Navier-Stokes en dimension 3, des résultats d'existence de solutions faibles globale en temps ainsi que des résultats d'existence et d'unicité des solutions fortes locales en temps ont été démontrés.

Le deuxième axe de cette thèse concerne l'approximation numérique du système (MHD) avec les conditions aux limites de type Navier (0.0.1) à la fois pour le champ de vitesse et pour le champ magnétique. Le premier objectif est d'analyser un nouveau schéma numérique de Galerkin discontinu (DG). Il existe une littérature importante qui s'intéresse à l'étude des schémas numériques pour la résolution des équations (MHD) avec des conditions aux limites de type Dirichlet pour le champ de vitesse. Plusieurs discrétisations basées sur des éléments finis conformes ont été proposées pour le cas linéaire et non linéaire du système (MHD) [17, 41, 52]. Dans le cas d'un domaine Ω polyédrique non convexe, une méthode d'éléments finis conforme avec régularisation est analysée dans [54]. Cette méthode permet de prendre en compte la contrainte $\operatorname{div} \boldsymbol{b} = 0$ dans la formulation discrète. Une autre façon pour contourner les difficultés liées à l'approximation du champ magnétique a été proposée dans [85] [86] en utilisant les éléments finis de Nédélec et en introduisant un multiplicateur de Lagrange. Le système (MHD) a aussi été approché par la méthode Galerkin discontinue (DG). On trouve dans [61] le premier résultat concernant l'approximation du système linéarisé (MHD), avec des conditions aux limites de type Dirichlet pour le champ de vitesse, par la méthode DG. Une extension de cette étude au cas non linéaire a été récemment établie dans [81].

Le deuxième objectif concerne l'approximation numérique du système de Navier-Stokes non stationnaire, avec des conditions aux limites de type Navier. Nous étudions ce système par deux méthodes qui consistent à remplacer l'équation de continuité $\operatorname{div} \boldsymbol{u} = 0$ dans Ω par une équation de divergence perturbée. Ces deux méthodes sont:

– La méthode de compressibilité artificielle qui consiste à remplacer l'équation $\operatorname{div} \boldsymbol{u} = 0$ dans Ω par

$$\operatorname{div} \boldsymbol{u}^\epsilon + \epsilon p_t^\epsilon = 0 \quad \text{dans } \ \Omega \times]0, T[, \tag{0.0.4}$$

où $\epsilon > 0$ est le paramètre de perturbation.

– La méthode de pseudo-compressibilité qui consiste à insérer un terme de stabilisation de la pression dans l'équation de conservation de masse:

$$\operatorname{div} \boldsymbol{u}^\epsilon - \epsilon \Delta p_t^\epsilon = 0 \quad \text{dans } \ \Omega \times]0, T[\qquad \text{et} \qquad \frac{\partial p_t^\epsilon}{\partial \boldsymbol{n}} = 0 \quad \text{sur } \ \Gamma. \tag{0.0.5}$$

La plupart des travaux portant sur l'approximation des équations de Navier-Stokes avec les deux méthodes mentionnées ci-dessus considèrent des conditions classiques de non-glissement. Nous citons [25, 95, 80, 91] pour l'approche par la méthode de compressibilité artificielle et [91, 80] pour l'approche via la méthode de pseudo-compressibilité. Nous citons aussi [23] pour une extension au cas du système de Brinkman-Forcheimer avec les conditions aux limites de type Navier. D'autres méthodes sont proposées pour déterminer numériquement une solution approchée pour le système de Navier-Stokes : la méthode de stabilisation de la pression (c.f. [69, 82, 89]) qui consiste à approcher l'équation d'incompressibilité par $\operatorname{div} \boldsymbol{u}^\epsilon - \epsilon \Delta p = 0$, $\varepsilon > 0$ et la méthode de pénalisation (c.f. [90, 62]) qui prend la forme suivante: $\operatorname{div} \boldsymbol{u}^\epsilon - \epsilon p = 0$, $\varepsilon > 0$. Cette dernière technique a été appliquée pour l'analyse d'un schéma d'éléments finis pour le système staionnaire de la MHD avec des conditions aux limites de type Dirichlet pour le champ de vitesse [38, 94]. Comme le système (MHD) est un couplage des équations de Navier-Stokes et des équations de Maxwell, mettre en place une étude du système non stationnaire de Navier-Stokes serait donc intéressant pour aller vers l'étude dans le cas d'évolution du système (MHD).

Ce manuscrit est composé de quatre parties. Le travail présenté dans chaque partie a été accompli sous la supervision de Nour Seloula, Maître de conférences à Caen. Nous présentons ci-dessous le contenu de chaque chapitre, l'ensemble des motivations et des contributions.

Le **Chapitre 1** est dédié à l'étude du système suivant : trouver la vitesse $\boldsymbol{u}$, la pression P, le champ magnétique $\boldsymbol{b}$ et les constantes α_i solutions de

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{dans } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \quad \text{et} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{et} \quad \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma, \\ P = P_0 \quad \text{sur } \Gamma_0 \quad \text{et} \quad P = P_0 + \alpha_i \ \text{ sur } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{et} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ \forall 1 \leqslant i \leqslant I, \end{cases} \tag{0.0.6}$$

où $\boldsymbol{f}$, $\boldsymbol{g}$ et P_0 sont des données. Nous nous plaçons dans un ouvert Ω borné lipschitzien qu'on supposera de classe $\mathcal{C}^{1,1}$. La frontière $\Gamma = \partial\Omega$ se décompose en $\Gamma = \bigcup_{i=1}^{I} \Gamma_i$ avec Γ_i les composantes connexes de Γ. Nous verrons plus loin l'importance de prescrire les flux à travers chacune des composantes connexes du bord dans (0.0.6).
Soulignons ici, qu'en utilisant l'identité

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2} \nabla |\boldsymbol{u}|^2,$$

le terme de convection $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ du système (MHD) est remplacé par le terme non linéaire $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$. La pression $P = p + \frac{1}{2}\nabla|\boldsymbol{u}|^2$ est la pression de Bernoulli (ou dynamique) qui dépend de la pression p du fluide.

Dans un premier temps, on cherche, sous des hypothèses de régularité minimale pour $\boldsymbol{f}$, $\boldsymbol{g}$ et P_0, l'existence et l'unicité de solutions faibles $(\boldsymbol{u}, \boldsymbol{b}, P, \alpha_i)$ dans $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}$. L'idée consiste à écrire une formulation variationnelle équivalente puis à appliquer le théorème de point fixe de Schauder. On cherche alors une solution $(\boldsymbol{u}, \boldsymbol{b})$ dans l'espace $\boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ avec $\boldsymbol{V}_N(\Omega)$ naturellement donné par

$$\boldsymbol{V}_N(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\, \operatorname{div} \boldsymbol{v} = 0 \ \text{ dans } \Omega,\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \ \text{ sur } \Gamma,\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0,\ \forall 1 \leqslant i \leqslant I\}.$$

Remarquons que, contrairement au cas de conditions au bord de type Dirichlet, l'espace $\boldsymbol{H}^{-1}(\Omega)$ n'est pas adapté pour les termes sources $\boldsymbol{f}$ et $\boldsymbol{g}$. En effet, prenons $\boldsymbol{v} \in \boldsymbol{V}_N(\Omega)$.
Pour écrire une formulation variationnelle, nous pouvons considérer la dualité

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{[\boldsymbol{H}_0^{2,2}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{2,2}(\mathbf{curl},\Omega)}$$

avec

$$\boldsymbol{H}_0^{2,2}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma\}.$$

On doit alors supposer que $\boldsymbol{f}$ et $\boldsymbol{g}$ appartiennent à l'espace $[\boldsymbol{H}_0^{2,2}(\mathbf{curl}, \Omega)]'$. Cependant, comme $\boldsymbol{v}$ appartient à l'espace $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, alors l'hypothèse précédente sur $\boldsymbol{f}$ et $\boldsymbol{g}$ peut être affaiblie en considérant l'espace $[\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$, dual de $\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)$ défini par:

$$\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^6(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma\}.$$

$\boldsymbol{\mathcal{D}}(\Omega)$ étant dense dans cet espace, son dual est un sous-espace de $\boldsymbol{\mathcal{D}}'(\Omega)$ qui peut être caractérisé par [14, Lemma 2.5] :

$$[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]' = \{\boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\Psi}; \quad \boldsymbol{F} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega),\ \boldsymbol{\Psi} \in \boldsymbol{L}^2(\Omega)\}$$

Ainsi, comme $\boldsymbol{V}_N(\Omega) \hookrightarrow \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)$, la dualité précédente pour le cas de $\boldsymbol{f}$ par exemple peut s'écrire sous la forme suivante:

$$\langle \boldsymbol{f}, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)} = \int_\Omega \boldsymbol{F}\cdot \boldsymbol{v}\,d\boldsymbol{x} + \int_\Omega \boldsymbol{\Psi}\cdot \mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x}.$$

Nous considérons ce choix de l'espace $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ pour les données $\boldsymbol{f}, \boldsymbol{g}$ afin d'obtenir les solutions $\boldsymbol{u}$ et $\boldsymbol{b}$ dans $\boldsymbol{H}^1(\Omega)$.

L'espace $\boldsymbol{\mathcal{D}}_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\ \operatorname{div}\boldsymbol{v} = 0 \text{ dans } \Omega\}$ est inclus dans $\boldsymbol{V}_N(\Omega)$ mais n'est pas dense. Il n'est donc pas possible d'interpréter un problème variationnel avec l'espace $\boldsymbol{V}_N(\Omega)$ et en particulier d'appliquer le théorème de De Rham qui permet de retrouver la pression P et de revenir vers le système de départ (0.0.6).

Notons aussi que la donnée $\boldsymbol{g}$ doit vérifier deux conditions de compatibilité. La première est donnée par

$$\operatorname{div}\boldsymbol{g} = 0 \quad \text{dans } \Omega. \tag{0.0.7}$$

Cette condition est due au fait que la deuxième équation du système (MHD) est de forme rotationnelle : $\boldsymbol{g} = \mathbf{curl}\,\boldsymbol{\Psi}$ avec $\boldsymbol{\Psi} = \mathbf{curl}\,\boldsymbol{b} - (\boldsymbol{u}\times\boldsymbol{b})$. Pour la deuxième condition de compatibilité, nous avons besoin d'introduire l'espace suivant

$$\boldsymbol{K}_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div}\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{0} \quad \text{dans } \Omega,\ \ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}\ \text{ sur } \Gamma\}.$$

Rappelons que $\boldsymbol{K}_N(\Omega)$ est un espace vectoriel de dimension I (cf. [8, Proposition 3.18] et [15, Corollary 4.2]) et il est engendré par les vecteurs ∇q_i^N, $i = 1,\dots,I$ avec $q_i^N \in H^1(\Omega)$ solutions du problème :

$$\begin{cases} -\Delta q_i^N = 0 \qquad \text{in } \Omega, \quad q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant},\ \ 1 \le k \le I \\ \left\langle \partial_n q_i^N,\, 1\right\rangle_{\Gamma_k} = \delta_{i\,k},\ \ 1 \le k \le I, \quad \text{and} \quad \left\langle \partial_n q_i^N,\, 1\right\rangle_{\Gamma_0} = -1. \end{cases} \tag{0.0.8}$$

Il est clair maintenant que $\boldsymbol{g}$ vérifie :

$$\langle \boldsymbol{g},\, \nabla q_i^N\rangle_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)} = \langle \mathbf{curl}\,\boldsymbol{\Psi},\, \nabla q_i^N\rangle_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)} = 0, \quad 1 \le i \le I, \tag{0.0.9}$$

où l'on observe que l'ouvert Ω étant de classe $\mathcal{C}^{1,1}$, les fonctions q_i^N appartiennent à $H^2(\Omega)$ et donc les vecteurs ∇q_i^N sont bien des éléments de $\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)$.

Après avoir étudié le cas hilbertien, on s'intéresse dans un second temps à la régularité de ces solutions. On commence par montrer l'existence des solutions dans la classe d'espaces $\boldsymbol{W}^{1,p}(\Omega)$ pour $p > 2$. L'idée est de réécrire le système sous la forme suivante :

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{dans } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} = \boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \quad \text{et} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{et} \quad \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma, \\ P = P_0 \quad \text{sur } \Gamma_0 \quad \text{et} \quad P = P_0 + \alpha_i \ \text{ sur } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{et} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ \forall 1 \leqslant i \leqslant I. \end{cases}$$

On démontre la régularité $\boldsymbol{W}^{1,p}(\Omega)$ pour le champ de vitesse en appliquant les résultats de régularité pour le problème de Stokes $(\mathcal{S}_\mathcal{N})$ (cf. [15, Theorem 5.6, Theorem 5.7] pour le cas $h = 0$) :

$$(\mathcal{S}_\mathcal{N}) \begin{cases} -\Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{dans } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma, \\ P = P_0 \quad \text{sur } \Gamma_0 \quad \text{et} \quad P = P_0 + c_i \ \text{ sur } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I, \end{cases}$$

où h est donnée. Pour le champ magnétique $\boldsymbol{b}$, nous appliquons les résultats de régularité du problème elliptique $(\mathcal{E}_\mathcal{N})$ (cf. [15, Proposition 5.1, Corollary 5.4]) :

$$(\mathcal{E}_\mathcal{N}) \begin{cases} -\Delta \boldsymbol{b} = \boldsymbol{g} \quad \text{et} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma \\ \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I. \end{cases}$$

Un des points clefs pour étudier les problèmes $(\mathcal{S}_\mathcal{N})$ et $(\mathcal{E}_\mathcal{N})$ concerne une inégalité de type Poincaré, établie dans [15, Corollary 3.2], qui permet de contrôler la norme $\boldsymbol{W}^{1,p}(\Omega)$ $(1 < p < \infty)$ par la norme $\boldsymbol{L}^p$ de la divergence, du rotationnel et des flux à travers les composantes connexes du bord :

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\, \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} \big), \ \ \forall \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega), \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ sur } \Gamma.$$

Remarquons ici que si $\boldsymbol{v}$ appartient à l'espace $\boldsymbol{V}_N(\Omega)$, la norme $\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ est équivalente à la norme $\|\cdot\|_{\boldsymbol{H}^1(\Omega)}$, d'ou l'importance des conditions de flux nuls sur les composantes connexes du bord.

Dans ce travail, on généralise le résultat de régularité donné dans [15, Theorem 5.7] pour $(\mathcal{S}_\mathcal{N})$ au cas d'un champ de vitesse à divergence non nulle (cas $h \neq 0$). Pour le problème $(\mathcal{E}_\mathcal{N})$, nous étendons le résultat de régularité de [15, Proposition 5.1] à un cadre plus général avec des données dans des espaces fonctionnels plus larges, plus précisément dans $(\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega))'$ avec p' (respectivement r') le conjugué de p (respectivement de r) et

$$(\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega))' = \{\boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\Psi}; \ \ \boldsymbol{F} \in \boldsymbol{L}^r(\Omega), \ \boldsymbol{\Psi} \in \boldsymbol{L}^p(\Omega)\}, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{3}.$$

Par un argument de bootstrap, nous montrons la régularité $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ de la solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ avec des données $\boldsymbol{f}, \boldsymbol{g} \in (\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega))'$, $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ et $\boldsymbol{g}$ satisfaisant les conditions de compatibilité (0.0.9) et (0.0.7). Si on suppose que les données sont plus régulières et que Ω est de classe $\mathcal{C}^{2,1}$, on peut obtenir de la même façon la régularité $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ pour $p \geq \frac{6}{5}$.

Le résultat de régularité $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ pour les valeurs petites de p avec $\frac{3}{2} < p < 2$ est plus délicat à démontrer. Pour le faire nous utilisons un théorème de point fixe de Banach appliqué au système linéarisé suivant : trouver $(\boldsymbol{u}, \boldsymbol{b}, P, c_i)$ solution de

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{dans } \Omega, \\ \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g} \quad \text{et} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{et} \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{sur}\, \Gamma, \\ P = P_0 \quad \text{et} \quad P = P_0 + c_i \quad \text{sur}\, \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \quad \text{et} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ \forall\, 1 \leqslant i \leqslant I \end{cases} \tag{0.0.10}$$

où $\boldsymbol{w}$ et $\boldsymbol{d}$ sont deux vecteurs donnés. Pour commencer nous montrons l'existence et l'unicité d'une solution faible $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c_i}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}$ ainsi que des estimations pour le champ de vitesse, le champ magnétique et la pression. Pour cela nous supposons $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, $\boldsymbol{d} \in \boldsymbol{L}^3(\Omega)$ et $\operatorname{div} \boldsymbol{d} = 0$ dans Ω. Nous nous intéressons ensuite à la régularité $\boldsymbol{W}^{1,p}(\Omega)$ et $\boldsymbol{W}^{2,p}(\Omega)$. Une question naturelle est de savoir dans quel espace doivent vivre les données $\boldsymbol{w}$ et $\boldsymbol{d}$ pour montrer de telles régularités. Nous commençons par étudier la régularité $\boldsymbol{W}^{2,p}(\Omega)$ avec $p \geq \frac{6}{5}$. Nous montrons que les hypothèses précédentes sur $\boldsymbol{w}$ et $\boldsymbol{d}$ ne sont pas suffisantes. En effet, le but est d'utiliser les résultats de régularité pour les problèmes de Stokes $(\mathcal{S}_\mathcal{N})$ et le problème elliptique $(\mathcal{E}_\mathcal{N})$ qui nécessitent un second membre dans $\boldsymbol{L}^p(\Omega)$. Cela revient à s'assurer que les termes $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}$ et $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})$ sont dans l'espace $\boldsymbol{L}^p(\Omega)$ pour tout $p \geq \frac{6}{5}$. Nous procédons en deux étapes. La première étape de la preuve consiste à supposer les données $\boldsymbol{w}$ et $\boldsymbol{d}$ suffisamment régulières : $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ et $\boldsymbol{d} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$. La deuxième étape consiste à supprimer cette hypothèse de régularité en utilisant des estimations établies dans la première étape. La régularité demandée pour $p \geq \frac{6}{5}$ est alors prouvée avec les données $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ et $\boldsymbol{d} \in \boldsymbol{W}_\sigma^{1,3/2}(\Omega)$.

Pour montrer la régularité des solutions dans $\boldsymbol{W}^{1,p}(\Omega)$ pour $p > 2$, nous supposons $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ avec $r = \frac{3p}{3+p}$ et $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ et que $\boldsymbol{w}$ et $\boldsymbol{d}$ vérifient les conditions suivantes:

$$\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^s(\Omega), \ \ \boldsymbol{d} \in \boldsymbol{W}^{1,s}(\Omega), \quad \text{et} \quad \operatorname{div} \boldsymbol{d} = 0, \tag{0.0.11}$$

avec

$$s = \frac{3}{2} \quad \text{si } 2 < p < 3, \quad s > \frac{3}{2} \quad \text{si } p = 3 \quad \text{et} \quad s = r \quad \text{si } p > 3. \tag{0.0.12}$$

Ces conditions impliquent que les termes $(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}$ et $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})$ sont dans $\boldsymbol{L}^r(\Omega)$ avec $r \geq 6/5$. Nous pouvons alors déduire par le travail précédent l'existence et l'unicité des solutions $(\boldsymbol{u}, \boldsymbol{b}, P)$ dans $\boldsymbol{W}^{2,r}(\Omega) \times \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)$, ce qui implique, grâce à l'injection de Sobolev $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, la régularité $\boldsymbol{W}^{1,p}(\Omega)$ pour $\boldsymbol{u}$ et $\boldsymbol{b}$ lorsque $p > 2$.

Nous terminons l'étude du système linéarisé (0.0.10) par la preuve de la régularité $\boldsymbol{W}^{1,p}(\Omega)$ avec $p < 2$. Comme le système est linéaire, nous utilisons un argument de dualité. Une des difficultés que nous rencontrons dans cette approche consiste à donner un sens aux traces au bord et à obtenir, par le biais de lemmes de densité, les formules de Green adéquates. Pour y faire face, nous écrivons tout d'abord une formulation variationnelle équivalente au système linéarisé en vérifiant que tous les termes sont bien définis. Afin d'obtenir la régularité $\boldsymbol{W}^{1,p}(\Omega)$, par dualité, pour ces petites valeurs de p, on est amenés à étudier une variante du système linéaire (0.0.10) :

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{dans } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) + \nabla \chi = \boldsymbol{g} \quad \text{et} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{et} \quad \chi = 0 \quad \text{sur } \Gamma, \\ P = P_0 \text{ sur } \Gamma_0 \quad \text{et} \quad P = P_0 + c_i \quad \text{sur } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \quad \text{et} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \qquad 1 \leq i \leq I. \end{cases} \tag{0.0.13}$$

où le potentiel scalaire χ est une inconnue supplémentaire du système. Remarquons ici que la condition de compatibilité (0.0.7) n'est plus nécessaire pour résoudre le système (0.0.13). De plus, en passant à la divergence dans la seconde équation de (0.0.13), la solution χ peut être obtenue directement comme solution du problème de Dirichlet suivant:

$$\Delta \chi = \operatorname{div} \boldsymbol{g} \quad \text{dans } \Omega \qquad \text{et} \quad \chi = 0 \quad \text{sur } \Gamma.$$

Cela permet d'étendre les résultats de régularité $\boldsymbol{W}^{1,p}(\Omega)$ avec $p > 2$ et $\boldsymbol{W}^{2,p}(\Omega)$ avec $p \geq \frac{6}{5}$ obtenus précédemment pour le système (0.0.13).

Nous complétons cette étude du système linéarisé avec deux preuves différentes de la régularité $\boldsymbol{W}^{2,p}(\Omega)$ pour $1 < p < \frac{6}{5}$, et pour terminer nous améliorons la régularité de la pression en considérant une donnée P_0 moins régulière.

Après cette étude du système linéarisé (0.0.10), nous pouvons retourner au système non linéaire (0.0.6) pour montrer l'existence des solutions $(\boldsymbol{u}, \boldsymbol{b}, P)$ dans $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ pour $\frac{3}{2} < p < 2$. Pour cela, on applique un théorème du point fixe de Banach sur le problème linéarisé (0.0.10) en supposant que les données sont petites et plus précisement pour

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_1,$$

pour une certaine constante $\delta_1 > 0$. Pour montrer l'unicité, nous devons supposer que les données sont encore plus petites :

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_2,$$

avec $\delta_2 \in (0,\, \delta_1]$

Les résultats présentés dans ce chapitre ont fait l'objet d'une note publiée aux Comptes Rendus de l'Académie des Sciences de Paris [79]

Dans le **Chapitre 2**, nous considérons le système (MHD) avec des conditions aux limites de type Navier :

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla P - \kappa(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{dans } \Omega, \\ \kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g} \quad \text{et} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{et} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{sur } \Gamma, \\ \boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{et} \quad \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{sur } \Gamma. \end{cases} \tag{0.0.14}$$

Afin de ne pas compliquer l'analyse du système (0.0.14), nous avons choisit de supposer que Ω est un ouvert borné de $\mathbb{R}^3$, de classe $\mathcal{C}^{1,1}$ et simplement connexe. Cette dernière hypothèse est due aux conditions aux limites considérées dans (0.0.14). Comme pour le problème étudié au Chapitre 1, on s'intéressera aux solutions faibles et aux solutions fortes, aussi bien dans un cadre hilbertien que dans le cadre de la théorie L^p.

Nous commençons par le cas hilbertien. Pour chercher des solutions faibles $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$, plusieurs questions se posent : comment choisir $\boldsymbol{f}$ et $\boldsymbol{g}$ pour que les conditions au bord $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}$ et $\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0}$ sur Γ aient un sens d'une part, et d'autre part y a-t-il des conditions de compatibilité nécessaires pour la résolution? Contrairement au cas du système (MHD) avec une condition de Dirichlet pour le champ de vitesse, il n'est pas convenable ici de prendre une fonction $\boldsymbol{f}$ dans $\boldsymbol{H}^{-1}(\Omega)$. En effet, si on veut écrire une formulation variationnelle, on doit préciser la dualité $\langle \boldsymbol{f},\, \boldsymbol{v}\rangle$ tout en choisissant $\boldsymbol{f}$ dans un sous-espace de distributions pour pouvoir ensuite remonter au problème initial. Comme les fonctions test appartiennent à l'espace

$$\boldsymbol{H}_0^{2,2}(\operatorname{div},\Omega) := \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\ \operatorname{div}\boldsymbol{v}\in L^2(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{sur}\,\Gamma\},$$

il est naturel de prendre $\boldsymbol{f}$ dans l'espace dual $(\boldsymbol{H}_0^{2,2}(\operatorname{div},\Omega))'$ où l'on rappelle que $\boldsymbol{\mathcal{D}}(\Omega)$ est dense dans $\boldsymbol{H}_0^{2,2}(\operatorname{div},\Omega)$ (c.f. [87, Proposition 1.0.2] et [87, Proposition 1.0.4]) et que

$$[\boldsymbol{H}_0^{2,2}(\operatorname{div},\Omega)]' = \{\boldsymbol{F}+\nabla\chi;\quad \boldsymbol{F}\in\boldsymbol{L}^2(\Omega),\, \chi\in L^2(\Omega)\}.$$

Comme dans le cas du système (0.0.6), on peut prendre $\boldsymbol{f}$ dans un espace plus gros

$$[\boldsymbol{H}_0^{6,2}(\operatorname{div},\Omega)]' = \{\boldsymbol{F}+\nabla\chi;\quad \boldsymbol{F}\in\boldsymbol{L}^{\frac{6}{5}}(\Omega),\, \chi\in L^2(\Omega)\}. \tag{0.0.15}$$

Avec ce choix d'espace pour $\boldsymbol{f}$, la condition au bord $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}$ a un sens dans $\boldsymbol{H}^{-1/2}(\Gamma)$. En effet, avec la caractérisation (0.0.15), si on cherche $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$, $\boldsymbol{b}\in\boldsymbol{H}^1(\Omega)$ et $P\in L^2(\Omega)$, alors $\Delta\boldsymbol{u}=\boldsymbol{f}-\nabla P-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b}$ appartient à $[\boldsymbol{H}_0^{6,2}(\mathrm{div},\ \Omega)]'$ car $\boldsymbol{u}\cdot\nabla\boldsymbol{u}\in\boldsymbol{L}^{3/2}(\Omega)\hookrightarrow\boldsymbol{L}^{6/5}(\Omega)$ et $\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b}\in\boldsymbol{L}^{3/2}(\Omega)\hookrightarrow\boldsymbol{L}^{6/5}(\Omega)$. Donc $\boldsymbol{u}$ appartiendrait à l'espace

$$\boldsymbol{M}^{6,2}(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{H}^{1}(\Omega);\ \ \Delta\,\boldsymbol{v}\in[\boldsymbol{H}_0^{6,2}(\mathrm{div},\,\Omega)]'\big\}.$$

En utilisant un raisonnement similaire à [13, Lemma 4.1] et [8], on peut vérifier que $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ est dense dans $\boldsymbol{M}^{6,2}(\Omega)$ et par suite on a la formule de Green suivante : pour tout $\boldsymbol{v}\in\boldsymbol{M}^{6,2}(\Omega)$ et $\boldsymbol{\varphi}\in\boldsymbol{H}^{1}(\Omega)$ avec $\mathrm{div}\,\boldsymbol{\varphi}=0$ dans Ω et $\boldsymbol{\varphi}\cdot\boldsymbol{n}=0$ sur Γ :

$$-\langle\Delta\,\boldsymbol{v},\,\boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\,\Omega)]'\times\boldsymbol{H}_0^{6,2}(\mathrm{div},\,\Omega)}=\int_\Omega\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma,$$

où la dualité sur Γ est définie par $\langle\cdot,\cdot\rangle_\Gamma=\langle\cdot,\cdot\rangle_{\boldsymbol{H}^{-\frac12}(\Gamma)\times\boldsymbol{H}^{\frac12}(\Gamma)}$ ce qui donne bien un sens à la condition au bord $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}$.

Ce qui précède est aussi valable pour la condition au bord $\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n}=\mathbf{0}$ sur Γ mais avec une donnée plus régulière $\boldsymbol{g}\in\boldsymbol{L}^{6/5}(\Omega)\hookrightarrow[\boldsymbol{H}_0^{6,2}(\mathrm{div};\Omega)]'$. Contrairement au cas de la donnée $\boldsymbol{f}$, des conditions de compatibilité sur $\boldsymbol{g}$ sont nécessaires pour résoudre le système (0.0.14). La première condition est $\mathrm{div}\,\boldsymbol{g}=0$ dans Ω qui découle directement du fait que $\boldsymbol{g}=\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b})+\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b}$ dans Ω. De plus, si on pose $\boldsymbol{\Psi}=\mathbf{curl}\,\boldsymbol{b}-(\boldsymbol{u}\times\boldsymbol{b})$, alors $\boldsymbol{\Psi}\in\boldsymbol{L}^2(\Omega)$ et $\mathbf{curl}\,\boldsymbol{\Psi}=\boldsymbol{g}\in\boldsymbol{L}^{\frac65}(\Omega)$. Comme $\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n}=\mathbf{0}$ sur Γ et $(\boldsymbol{u}\times\boldsymbol{b})\times\boldsymbol{n}=\mathbf{0}$, on déduit que $\boldsymbol{\Psi}\times\boldsymbol{n}=\mathbf{0}$ sur Γ ce qui a un sens dans $\boldsymbol{H}^{-1/2}(\Gamma)$. Donc, on a d'une part : pour tout $\chi\in H^2(\Omega)$,

$$\int_\Omega\mathbf{curl}\,\boldsymbol{\Psi}\cdot\nabla\chi\,dx=-\langle\boldsymbol{\Psi}\times\boldsymbol{n},\nabla\chi\rangle_{\boldsymbol{H}^{-1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma)}=0.$$

D'autre part, comme $\boldsymbol{g}$ appartient à $\boldsymbol{L}^{6/5}(\Omega)$ et est à divergence nulle, la trace $\boldsymbol{g}\cdot\boldsymbol{n}$ appartient à $W^{-\frac56,\frac65}(\Gamma)$. Alors, pour tout $\chi\in H^2(\Omega)$,

$$\int_\Omega\mathbf{curl}\,\boldsymbol{\Psi}\cdot\nabla\chi\,dx=\langle\boldsymbol{g}\cdot\boldsymbol{n},\chi\rangle_{W^{-\frac56,\frac65}(\Gamma)\times W^{\frac56,6}(\Gamma)}.$$

Par conséquent, $\boldsymbol{g}\cdot\boldsymbol{n}=0$ sur Γ est une deuxième condition de compatibilité à considérer pour la donnée $\boldsymbol{g}$.

On est alors en mesure de montrer l'existence des solutions faibles. Nous utilisons pour cela un théorème de point fixe de Leray-Schauder. Ensuite, nous montrons la régularité $\boldsymbol{W}^{1,p}(\Omega)$ pour $p>2$ et la régularité $\boldsymbol{W}^{2,p}(\Omega)$ pour $p\geqslant\frac65$. Pour obtenir les résultats de régularité pour le champ de vitesse, nous utilisons la régularité du problème de Stokes $(\mathcal{S}_\mathcal{T})$ suivant (cf. [15])

$$(\mathcal{S}_\mathcal{T})\begin{cases}-\Delta\boldsymbol{u}+\nabla P=\boldsymbol{f}\quad\text{et}\quad & \mathrm{div}\,\boldsymbol{u}=0\quad\text{dans }\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n}=a\quad\text{et}\quad & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}\quad\text{sur }\Gamma.\end{cases}$$

Nous avons besoin de généraliser le résultat de [13, Theorem 4.4] portant sur l'existence des solutions dans $\boldsymbol{W}^{1,p}(\Omega)$, $p > 2$ avec des conditions au bord non homogènes et pour des données $\boldsymbol{f}$ dans des espaces plus larges $[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$ définis par

$$[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]' = \{\boldsymbol{F} + \nabla\psi \,|\, \boldsymbol{F} \in \boldsymbol{L}^r(\Omega),\ \psi \in L^p(\Omega)\}$$

Pour obtenir les résultats de régularité pour le champ magnétique, nous passons par la régularité du problème elliptique $(\mathcal{E}_{\mathcal{T}})$ suivant (cf. [15])

$$(\mathcal{E}_{\mathcal{T}}) \begin{cases} -\Delta \boldsymbol{b} = \boldsymbol{g} \quad \text{et} \quad & \mathrm{div}\, \boldsymbol{b} = 0 \quad \text{dans } \Omega, \\ \boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{et} \quad & \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma. \end{cases}$$

Dans le **Chapitre 3**, nous étudions l'approximation du système (MHD) avec les conditions aux limites de type Navier (0.0.14), à la fois pour le champ de vitesse et pour le champ magnétique, par des formulations de type Galerkin discontinues. Les points de départ de ce travail sont les travaux de P. Houston et al. [61] et les travaux de W. Qiu and K. Shi [81].

La méthode de Galerkin discontinue est une classe des méthodes d'éléments finis utilisant des fonctions de base complètement discontinues. Ces fonctions de base et les fonctions tests sont choisies dans le même espace sans aucune continuité imposée aux interfaces entre les éléments (un tétraèdre ou un hexaèdre en 3D, un triangle ou un quadrilatère en 2D). Cette spécificité permet à cette méthode une grande flexibilité au maillage. En effet, en l'absence de la sensibilité à la régularité des maillages, la méthode est adaptée pour représenter des géométries industrielles qui sont parfois complexes et qui nécessitent des maillages non structurés et non conformes. De plus, le degré p des polynômes d'approximation peut être augmenté dans les éléments du maillage où la solution est plus régulière, ce qui peut être combiné avec un raffinement du maillage h dans les parties de moindre régularité de la solution. L'ensemble forme alors un maillage hp. De plus, l'approximation avec cette méthode impose d'avoir recours à une formulation faible locale où la prise en compte des conditions aux limites est faite au sens faible, directement dans la formulation du problème sans les imposer dans la définition de l'espace d'approximation. Ceci permet une mise en œuvre simplifiée de la méthode. Une intégration par parties fait alors apparaître des termes de bord où les flux physiques sont approchés par des flux numériques aux interfaces à cause des discontinuités entre les mailles. Cette méthode possède d'autre propriétés importantes. Elle est localement conservative, elle est aisément parallélisable ce qui donne à la méthode un grand intérêt. Cependant, les méthodes DG sont plus coûteuses en temps et en mémoire que les méthodes d'éléments finis standards à cause des valeurs doubles de la solution aux interfaces. Cela augmente considérablement le nombre de degrés de liberté locaux sur lesquels les inconnues du problème reposent.

Introduite il y a exactement 48 ans par Reed et Hill (1973) [83] pour la résolution d'équations de transport de neutrons, la méthode DG a connu un certain nombre de développements au cours de ces dernières années et reste encore aujourd'hui une méthode très utilisée. Son champ d'application a par la suite été étendu aux problèmes avec diffusion, nous citons [19, 32, 48] pour les équations de Navier-Stokes, [57, 67, 36] pour l'électromagnétisme.

Ce chapitre comporte quatre sections. Après avoir introduit les notations nécessaires dans la première section, nous nous sommes intéressés au système linéarisé pour lequel nous avons fait une étude complète d'une méthode DG dans la deuxième section. La principale difficulté dans l'approximation des équations de Maxwell est l'incorporation de la contrainte $\operatorname{div} \boldsymbol{b} = 0$ dans le schéma DG. Cette contrainte peut être imposée par la technique dite de régularisation. Cette méthode consiste formellement à remplacer l'opérateur $\mathbf{curl}(\mathbf{curl})$ par l'opérateur laplacien vectoriel en rajoutant un terme $\mathbf{grad}(\operatorname{div})$ à l'équation $\kappa\mu\, \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \kappa\, \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g}$ du système (0.0.14). Ce terme est nul car toute solution $\boldsymbol{b}$ du problème (0.0.14) est à divergence nulle. Ainsi, en vertu de l'identité:

$$\mathbf{curl}(\mathbf{curl}) - \mathbf{grad}(\operatorname{div}) = -\Delta,$$

tout champ magnétique $\boldsymbol{b}$ solution de (0.0.14) vérifie aussi l'équation

$$-\kappa\mu\Delta\boldsymbol{b} - \kappa\, \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \quad \text{et} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{dans } \Omega.$$

Cette méthode permet alors d'avoir un problème régularisé naturellement posé dans l'espace

$$\boldsymbol{H}_0(\operatorname{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \quad \operatorname{div} \boldsymbol{v} \in L^2(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{sur } \Gamma\}.$$

Si le domaine Ω est polyédrique et convexe ou encore s'il a une frontière régulière (c.f. [46, théorème3.7]), l'espace $\boldsymbol{H}_0(\operatorname{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$ coïncide avec l'espace

$$\boldsymbol{H}^1_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{sur } \Gamma\}.$$

La technique de régularisation des équations de Maxwell a été initialement introduite par Werner [99] et Leis [64]. Ses applications sont variées, citons par exemple l'étude des équations de Maxwell en régime harmonique et à basse fréquence [77], l'étude des équations de Maxwell dans un domaine polyédrique [35] ou l'étude d'une méthode de décomposition de domaine pour les équations de Maxwell [7]. En revanche, lorsque le domaine possède des singularités géométriques, comme des coins ou arêtes rentrants, l'approche n'est pas adaptée et la solution DG du problème régularisé peut converger mais pas vers la solution du problème physique de départ (voir les résultats numériques dans [58]). La même situation a été observée dans le cas d'approximation des basses fréquences par les éléments finis conformes (voir [7, 35]). Cela s'explique par le manque de régularité de la solution qui n'est pas dans ce cas $\boldsymbol{H}^1(\Omega)$ mais seulement $\boldsymbol{H}^{1/2}(\Omega)$ [46]. Dans ce cas une approche alternative consiste à introduire un multiplicateur de Lagrange r scalaire comme une inconnue supplémentaire du problème [61, 81]. Ce multiplicateur de Lagrange r peut être vu comme une pseudo-pression associée à la contrainte de divergence nulle vérifiée par toute solution du problème. La taille du système discret devient alors plus grande et à la place de la deuxième équation du système (0.0.14), le couple $(\boldsymbol{b}, r)$ est solution de:

$$\kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} + \nabla r - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}) = \boldsymbol{g} \quad \text{et} \quad \operatorname{div}\boldsymbol{b} = 0 \qquad \text{dans } \Omega,$$

avec une condition au bord $r = 0$ sur Γ. Notons ici qu'en passant à la divergence dans l'équation précédente nous obtenons $\Delta r = \operatorname{div}\boldsymbol{g}$ dans Ω et dans le cas particulier où le terme source $\boldsymbol{g}$ est à divergence nulle, nous avons $r = 0$.

Cette méthode a été appliquée dans [61] pour l'approximation du système (MHD) linéarisé. Elle a été récemment étendue au cas non linéaire dans [81]. Soulignons cependant que la condition aux limites considérée dans ces travaux pour le champ de vitesse est la condition classique de type Dirichlet. Nous proposons dans la section 2 de ce chapitre une formulation régularisée, ou encore augmentée en imposant une condition aux limites de type Navier à la fois pour le champ de vitesse et le champ magnétique. Nous supposons que Ω présente une régularité $\mathcal{C}^{1,1}$ où Ω est à bord polyédral mais convexe. On introduit donc une formulation discontinue basée sur la méthode symétrique avec pénalisation interne («Interior Penalty») classique de [47]. La principale différence est le choix du terme de stabilisation considéré. Au lieu de la norme L^2 du saut de la vitesse à travers les éléments, nous considérons des termes de stabilisation basés sur le saut normal et sur le saut tangentiel des solutions approchées. Tout d'abord, on montre que le problème continu est bien posé. Ensuite, comme pour les méthodes d'éléments finis standards, nous montrons que le problème discret admet une unique solution. La preuve s'appuie sur la stabilité des forme linéaires associées au problème variationnel approché.

Une fois la stabilité de la méthode mise en place, on est amené à discuter la convergence. Dans la troisième section, on établit des estimations d'erreur *a priori* de la vitesse $\boldsymbol{u}$, du champ magnétique $\boldsymbol{b}$ et de la pression p. On considère d'abord l'erreur en norme énergie $|||(\boldsymbol{u}_h, \boldsymbol{b}_h)|||^2 = \|\boldsymbol{u}_h\|_{v,h}^2 + \|\boldsymbol{b}_h\|_{m,h}^2$, pour le champ de vitesse et le champ magnétique (voir la sous section 3.2.3 pour les notations), ensuite en norme L^2 pour la pression p.

La quatrième section est consacrée à l'extension de l'approximation par la méthode DG au système non linéaire (0.0.14). La principale difficulté est de montrer la stabilité des formes bilinéaires issues des termes de convection de Navier-Stokes O_h et du terme de couplage C_h sur les espaces discrets. Afin de porter une attention particulière à la stabilité de la forme de couplage C_h vitesse/champ magnétique, précisons ici sa définition :

$$C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) := \kappa \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{v}_h \times \boldsymbol{b}_h) \cdot \mathbf{curl}\, \boldsymbol{b}_h \, d\boldsymbol{x} - \kappa \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{v}_h \times \boldsymbol{b}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds. \quad (0.0.16)$$

Dans un travail récent [81], les auteurs proposent une analyse d'un schéma DG pour le système (MHD) dans $\mathbb{R}^3$ où les conditions aux limites sont de type Dirichlet pour le champ de vitesse et de type Navier (0.0.1) pour le champ magnétique. Ils considèrent le cas général d'un domaine Ω lipschitzien. Ils montrent la continuité de C_h en appliquant un argument L^6 - L^3 - L^2 pour le terme donné sur les mailles puis l'inégalité de trace discrète est utilisée pour le terme donné sur le bord. Cette approche est naturelle dans le cas d'un domaine peu régulier. En effet, sans hypothèse de régularité sur le domaine, et au vu des conditions aux limites considérées pour le champ magnétique $\boldsymbol{b}$, la solution $\boldsymbol{b}$ du système (0.0.14) est seulement $\boldsymbol{H}^{1/2}(\Omega)$ et donc $\boldsymbol{L}^3(\Omega)$ par les injections de Sobolev. Cependant, la solution $\boldsymbol{u}$ est bien $\boldsymbol{H}^1(\Omega)$ et donc $\boldsymbol{L}^6(\Omega)$ ce qui permet d'appliquer l'inégalité discrète dérivée dans [47, 49] (voir aussi [39]) qui affirme que la norme $\boldsymbol{L}^6$ est contrôlée par la norme DG notée par $|\!|\!|\cdot|\!|\!|_h$ (voir (3.4.18) avec $r = 6$) :

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^6(\Omega)} \le C |\!|\!|\boldsymbol{v}_h|\!|\!|_h := C\Big(\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{h_e} \|[\![\boldsymbol{v}_h]\!]\|_{0,e}^2 \Big)^{1/2}. \quad (0.0.17)$$

Etant donné que la norme DG pour $\boldsymbol{b}$ est différente (voir la définition de $|\boldsymbol{b}_h|_C$ dans (0.0.18)), une analyse fonctionnelle discrète est établie dans [81] afin de montrer que la norme $\boldsymbol{L}^3$ est contrôlée par cette norme DG. Plus précisément, l'inégalité discrète suivante est montrée pour l'espace polynomial par morceaux

$$\|\boldsymbol{b}_h\|_{\boldsymbol{L}^3(\Omega)} \le C|\boldsymbol{b}_h|_C^2 := \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{b}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{m_0}{h_e} \|[\![\boldsymbol{b}_h]\!]_T\|_{0,e}^2 \,, \quad (0.0.18)$$

Notons ici que sans hypothèse de régularité sur le domaine et avec des conditions aux limites de type Navier à la fois pour le champ de vitesse et pour le champ magnétique, les solutions $\boldsymbol{u}$ et $\boldsymbol{b}$ du système (0.0.14) sont toutes les deux $\boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Donc l'argument précédent ne peut pas être appliqué pour montrer la continuité de la forme C_h. Des hypothèses additionnelles de régularité sur la solution exacte sont donc indispensables. Dans ce travail, nous montrons une nouvelle inégalité discrète analogue à (0.0.17) sur les espaces discontinus sous réserve que la frontière du domaine soit suffisamment régulière. Dans notre cas, le champ magnétique, tout comme le champ de vitesses, ont bien la régularité $\boldsymbol{H}^1(\Omega)$. Plus précisément, pour $1 \le p \le 6$, nous montrons que la norme L^p est contrôlée par la norme DG suivante :

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)} \le C\, \|\boldsymbol{v}_h\|_{v,h}^2 := \sum_{T\in\mathcal{T}_h} \Big(\|\mathrm{div}\, \boldsymbol{v}_h\|_{0,T}^2 + \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \Big) + \sum_{e\in\mathcal{F}_h} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2 \,,$$

avec $C = C(\Omega)$ une constante indépendante de h. Cette inégalité est naturelle car la norme DG $\|\cdot\|_{v,h}$ est l'équivalent discret de la norme $\boldsymbol{H}_T^1(\Omega)$. Les techniques de démonstration ont été inspirées par les travaux de Girault et Rivière [47, Lemma 6.2] pour la preuve de (0.0.17).

Cependant, la preuve n'est pas triviale et repose sur un résultat de régularité de l'opérateur elliptique de second ordre $-\Delta = \mathbf{curl}(\mathbf{curl}) - \mathbf{grad}(\mathrm{div})$ avec la condition aux limites de type Navier dans un domaine borné $\Omega \in \mathbb{R}^3$ de classe $C^{2,1}$.

Cette nouvelle inégalité discrète permet, d'une part, d'appliquer un argument L^4-L^4-L^2 pour montrer la continuité de la forme C_h et de la forme O_h (voir (3.2.37) pour la définition de O_h et (0.0.16) pour la définitions de C_h) sur les espaces discrets. D'autre part, elle permet de prouver l'existence d'une solution au problème discret grâce au théorème de point fixe de Brouwer. Des estimations de l'erreur entre les solutions exactes et les solutions discrètes sont ensuite établies.

Le **Chapitre 4** concerne l'étude du système de Navier-Stokes non stationnaire avec des conditions aux limites de type Navier: trouver $\boldsymbol{u}$ et p tels que :

$$(\mathcal{NS}) \begin{cases} \boldsymbol{u}_t - \Delta\boldsymbol{u} + \nabla p + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \boldsymbol{f} \quad \text{et} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{dans } \Omega\times]0,T[, \\ \qquad\qquad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{et} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{sur } \Gamma\times]0,T[, \\ \qquad\qquad\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \text{dans } \Omega. \end{cases}$$

Nous supposons que Ω est un domaine borné simplement connexe, de bord Γ de classe $\mathcal{C}^{1,1}$. Il est connu que la condition d'incompressibilité $\operatorname{div}\boldsymbol{u} = 0$ dans Ω rend assez difficile la mise en oeuvre des approximations numériques du système de Navier-Stokes. Une des méthodes numériques pour traiter ce problème est l'approximation par un système perturbé à la manière introduite dans Temam [95] et [25]. Le problème perturbé dépend d'un paramètre ε destiné à tendre vers zéro. Dans ce chapitre, deux problèmes perturbés sont analysés. Nous nous intéressons dans un premier temps à la méthode de compressibilité artificielle (0.0.4). Nous commençons par une étude d'existence de solutions faibles pour le problème de Navier Stokes $(\mathcal{NS})$ en utilisant la méthode de Faedo Galerkin. Contrairement au cas du système de Navier-Stokes avec une condition de Dirichlet, où l'inégalité de Poincaré joue un rôle important, nous avons besoin ici des inégalités pour les champs de vecteurs faisant intervenir les opérateurs **curl** et div. Ensuite, nous montrons que le problème perturbé possède au moins une solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ qui converge vers $(\boldsymbol{u}, p)$ solution du système non perturbé $(\mathcal{NS})$ lorsque ε tend vers zéro. Nous terminons par une analyse d'erreur pour cette méthode.

Dans un deuxième temps, nous étudions une approximation du système $(\mathcal{NS})$ via la méthode de pseudo-compressibilité (0.0.5). Après avoir établi des estimations *a priori* pour la solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ du problème perturbé, nous montrons une estimation d'erreur entre la solution $(\boldsymbol{u}, p)$ du système $(\mathcal{NS})$ et $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$. La preuve est donnée en deux étapes : on commence par montrer une estimation pour la partie d'erreur introduite par la perturbation de l'opérateur linéaire. Ensuite nous donnons une estimation pour la partie d'erreur associée au terme non linéaire.

Pour terminer nous considérons un schéma d'ordre 2 en temps pour approcher le problème perturbé avec la méthode de pseudo-compressibilité. Nous montrons la stabilité du schéma ainsi qu'une estimation d'erreur d'ordre 2.

Chapter 1

MHD system with boundary condition involving the pressure

1.1 Introduction

Let Ω be an open bounded set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$. In this work, we consider the following incompressible stationary magnetohydrodynamics (MHD) system: find the velocity field $\boldsymbol{u}$, the pressure P, the magnetic field $\boldsymbol{b}$ and the constant vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$ such that for $1 \leq i \leq I$:

$$\begin{cases} -\nu\,\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla P - \kappa(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in } \Omega, \\ \kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + \alpha_i \ \text{ on } \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{b}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ \forall 1 \leqslant i \leqslant I \end{cases} \tag{1.1.1}$$

where Γ is the boundary of Ω which is not necessary connected. Here $\Gamma = \bigcup_{i=0}^{I}\Gamma_i$ where Γ_i are the connected components of Γ with Γ_0 the exterior boundary which contains Ω and all the other boundaries. We denote by $\boldsymbol{n}$ the unit vector normal to Γ. The constants ν, μ and κ are constant kinematic, magnetic viscosity and a coupling number respectively. We refer to [37, 42] for further discussion of typical values for these parameters. The vector $\boldsymbol{f}$, $\boldsymbol{g}$ and the scalar h and P_0 are given. In this work, we assume that $\nu = \mu = \kappa = 1$ for convenience. Using the identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2$, the classical nonlinear term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in the Navier-Stokes equations is replaced by $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$. The pressure $P = p + \frac{1}{2}|\boldsymbol{u}|^2$ is then the Bernoulli (or dynamic) pressure, where p is the kinematic pressure. The boundary conditions involving the pressure are used in various physical applications. For example, in hydraulic networks, as oil ducts, microfluidic channels or the blood circulatory system. Pressure driven flows occur also in the modeling of the cerebral venous network from three-dimensional angiographic images obtained by magnetic resonance. We note that the MHD system (1.1.1)

has been extensively studied by many authors. We note that most of the contributions are often given where Dirichlet type boundary conditions on the velocity field are imposed. At a continuous level, we can refer, for example to [103, 9] for the existence and the regularity of the solutions of (1.1.1), to [5] for the global solvability of (1.1.1) under mixed boundary conditions for the magnetic field. For the discretization approaches of (1.1.1), a few related contributions include mixed finite elements [56, 86, 51], discontinuous Galerkin finite elements [81] or iterative penalty finite element methods [38] and so on. The boundary condition under the form $P = P_0 + \alpha_i$ on Γ_i, $i = 1, \ldots, I$ was first introduced in [34, 33] for the Stokes and the Navier-Stokes systems in steady hilbertian case. The authors studied the differences $\alpha_i - \alpha_0$, $i = 1 \ldots I$ which represent the unknown pressure drop on inflow and outflow sections Γ_i in a network of pipes. This work is extended to L^p-theory for $1 < p < \infty$ in [14]. In our work, we study the MHD system (1.1.1) with pressure boundary condition, together with no tangential flow and no tangential magnetic field on the boundary. Up to our knowledge, with these type of booundary conditions, this work is the first one to give a complete L^p-theory for the MHD system (1.1.1) not only for large values of $p \geq 2$ but also for small values $3/2 < p < 2$ in $\Omega \subset \mathbb{R}^3$ domain with a boundary Γ not necessary connected.
These results were announced in a CRAS note in Comptes Rendus Mathématique [79].

1.2 Main results

In this section, we briefly discuss the main results of this chapter, for which the following notations are needed:
For $p \in [1, \infty)$, p' denotes the conjugate exponent of p, i.e. $\frac{1}{p'} = 1 - \frac{1}{p}$. We introduce the following space

$$\boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega);\, \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}\,, \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{3} \tag{1.2.1}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{\,r,p}(\mathbf{curl},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

The closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega)$ is denoted by $\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)$ with

$$\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega);\ \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}\ \text{ on } \Gamma\}\,.$$

The dual space of $\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)$ is denoted by $[\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)]'$ and its characterization is given in Proposition 1.3.1. We introduce also the kernel

$$\boldsymbol{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \text{div}\ \boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0},\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}\ \text{ on } \Gamma\},$$

which is spanned by the functions $\nabla q_i^N \in \boldsymbol{W}^{1,q}(\Omega)$ for any $1 < q < \infty$ [15, Corollary 4.2] and q_i^N is the unique solution of the problem

$$\begin{cases} -\Delta q_i^N = 0 \qquad \text{in } \Omega, \quad q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant}, \;\; 1 \le k \le I \\ \big\langle \partial_n q_i^N,\, 1 \big\rangle_{\Gamma_k} = \delta_{i\,k}, \;\; 1 \le k \le I, \quad \text{and} \quad \big\langle \partial_n q_i^N,\, 1 \big\rangle_{\Gamma_0} = -1. \end{cases} \tag{1.2.2}$$

We will use the symbol σ to represent a set of divergence free functions. For exemple the space $\boldsymbol{L}^p_\sigma(\Omega)$ is the space of functions in $\boldsymbol{L}^p(\Omega)$ with divergence free. We will denote by C an unspecified positive constant which may depend on Ω and the dependence on other parameters will be specified if necessary.

The first theorem is concerned with the existence of weak solutions in the case of Hilbert spaces for the following MHD problem:

$$\begin{cases} -\;\Delta \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \quad \text{and} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + \alpha_i \;\; \text{on } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \quad \text{and} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \;\; \forall 1 \leqslant i \leqslant I \end{cases} \tag{MHD}$$

The proof is given in Subsection 1.6.1 (see Theorem 1.6.1). We note that in the case when $\partial\Omega$ is not connected, to ensure the solvability of problem (*MHD*), we need to impose the conditions for $\boldsymbol{u}$ and $\boldsymbol{b}$ on the connected components Γ_i: $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0$ and $\langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0$ for $1 \le i \le I$. (See [15] and [14] for an equivalent form of these conditions). Of course, if $\partial\Omega$ is connected, the above conditions are no longer necessary.

Theorem 1.2.1. (Weak solutions of the (*MHD*) system in $\boldsymbol{H}^1(\Omega)$). *Let $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$, $h = 0$ and $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ with the compatibility conditions*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}^2_N(\Omega), \quad \langle \boldsymbol{g}, \boldsymbol{v} \rangle_{\Omega_{6,2}} = 0, \tag{1.2.3}$$

$$\operatorname{div} \boldsymbol{g} = 0 \quad \text{in } \Omega, \tag{1.2.4}$$

where $\langle \cdot, \cdot \rangle_{\Omega_{r,p}}$ denotes the duality product between $[\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)]'$ and $\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)$. Then the (MHD) problem has at least one weak solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{\alpha}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} + \|P\|_{L^2(\Omega)} \le M,$$

where $M = C(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-1/2}(\Gamma)})$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$ defined by

$$\alpha_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{6,2}} - \langle P_0, \nabla q_i^N \cdot \boldsymbol{n} \rangle_\Gamma + \int_\Omega (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \cdot \nabla q_i^N \, dx - \int_\Omega (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \cdot \nabla q_i^N \, dx, \tag{1.2.5}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

In addition, suppose that $\boldsymbol{f}$, $\boldsymbol{g}$ and P_0 are small in the sense that

$$C_1 C_2^2 M \leq \frac{2}{3C_{\mathcal{P}}^2}, \tag{1.2.6}$$

where $C_{\mathcal{P}}$ is the constant in (1.3.6) *and C_1, C_2 are the constants defined in* (1.6.15)*. Then the weak solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ of (MHD) is unique.*

The next two theorems are concerned with generalized solutions in $\boldsymbol{W}^{1,p}(\Omega)$ for $p > 2$ and strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$ for $p \geq 6/5$. The existence of weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ for $\frac{3}{2} < p < 2$ is not trivial. We will precise this case later.

Theorem 1.2.2. *(Weak solutions in $\boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$ for the (MHD) system). Let $p > 2$. Suppose that $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $h = 0$ and $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ with the compatibility condition* (1.2.4) *and*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\Omega_{r',p'}} = 0. \tag{1.2.7}$$

Then the weak solution for the (MHD) system given by Theorem 1.2.1 satisfies

$$(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega).$$

Moreover, we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega))'} + \|\boldsymbol{g}\|_{(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega))'} + \|P_0\|_{W^{1/r',r}(\Gamma)})$$

Theorem 1.2.3. *(Strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$ for the (MHD) system). Let us suppose that Ω is of class $\mathcal{C}^{2,1}$ and $p \geq \frac{6}{5}$. Let $\boldsymbol{f}$, $\boldsymbol{g}$ and P_0 with the compatibility conditions* (1.2.4) *and* (1.2.7) *and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \in \boldsymbol{L}^p(\Omega), \quad h = 0 \quad \text{and} \quad P_0 \in W^{1-\frac{1}{p},p}(\Gamma).$$

Then the weak solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ for the (MHD) system given by Theorem 1.2.1 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)})$$

We refer to Theorem 1.6.3 and Theorem 1.6.4 for the proof of the above result, where we use the estimates obtained in the Hilbert case and a bootstrap argument using regularity results of some Stokes and elliptic problems in [15] and [14].
To deal with the regularity of the solutions of the (*MHD*) system in $\boldsymbol{W}^{1,p}(\Omega)$ with $\frac{3}{2} < p < 2$, we need to study the following linearized MHD system: Find ($\boldsymbol{u}$, $\boldsymbol{b}$, P, $\boldsymbol{c}$) with $\boldsymbol{c} = (c_1, \ldots, c_I)$ such that for $1 \leq i \leq I$:

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} + \nabla \mathrm{P} - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g} \quad \text{and} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ \mathrm{P} = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \mathrm{P} = P_0 + c_i \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0. \end{cases} \tag{1.2.8}$$

The next theorem gives existence of weak and strong solutions for the linearized problem (1.2.8)

Theorem 1.2.4. *(Existence of weak and strong solutions of the linearized MHD problem). Suppose that*

$$\boldsymbol{f},\boldsymbol{g}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]',\quad P_0\in W^{1-\frac{1}{r},r}(\Gamma),\quad h\in W^{1,r}(\Omega)$$

with the compatibility conditions (1.2.4) *and* (1.2.7).
(1). *For any* $p\geq 2$, *if* $\mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^s(\Omega)$, $\boldsymbol{d}\in\boldsymbol{W}^{1,s}_\sigma(\Omega)$ *where* s *is given by*

$$s=\frac{3}{2}\quad\text{if } 2<p<3,\quad s>\frac{3}{2}\quad\text{if } p=3\quad\text{and}\quad s=r\quad\text{if } p>3,$$

then the linearized system (1.2.8) *has a unique solution* $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ *with* $\boldsymbol{c}=(c_1,\ldots,c_I)$. *Moreover, we have the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|P\|_{W^{1,r}(\Omega)}\leq C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)})\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}$$
$$+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}+\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]}+(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)})\,\|h\|_{W^{1,r}(\Omega)}\big)$$

(2). *Let* $\frac{3}{2}<p<2$. *If* $\mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^{3/2}(\Omega)$ *and* $\boldsymbol{d}\in\boldsymbol{W}^{1,3/2}_\sigma(\Omega)$, *then the linearized problem* (1.2.8) *has a unique solution* $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$. *Moreover, we have the following estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}$$
$$+\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]}+(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\,\|h\|_{W^{1,r}(\Omega)}\Big)$$

and

$$\|P\|_{W^{1,r}(\Omega)}\leq C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})^2\times\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}$$
$$+\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]}+(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\,\|h\|_{W^{1,r}(\Omega)}\Big).$$

(3). *Also for any* $p\in(1,\infty)$, *if* Ω *is of class* $\mathcal{C}^{2,1}$, $h=0$ *in* Ω *and*

$$\boldsymbol{f}\in\boldsymbol{L}^p(\Omega),\ \ \boldsymbol{g}\in\boldsymbol{L}^p(\Omega),\ \ \mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^{3/2}(\Omega),\ \ \boldsymbol{d}\in\boldsymbol{W}^{1,3/2}(\Omega),\ \text{ and }\ P_0\in W^{1-\frac{1}{p},p}(\Gamma)$$

with the compatibility conditions (1.2.4) *and* (1.2.7), *then* $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)\times\mathbb{R}^I$ *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|P\|_{W^{1,p}(\Omega)}\leq C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})$$
$$\times\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}+\|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\big)$$

where $C=C(\Omega,p)$ *if* $p\geq 6/5$ *and* $C=C(\Omega,p)(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})$ *if* $1<p<6/5$.

We refer to Theorem 1.5.4, Theorem 1.5.7, Theorem 1.5.1 and Theorem 1.5.11 for the proof of the above results. Note that in the above theorem, to prove the existence of weak solutions in $\boldsymbol{W}^{1,p}(\Omega)$ with $3/2<p<2$, we use a duality argument.

We also note that we proved more general existence results in Corollary 1.5.10 where the regularity of the pressure is improved by supposing a data P_0 less regular.

Finally, the next result shows the existence and uniqueness of weak solutions with $3/2 < p < 2$ for the nonlinear (*MHD*) problem (see Theorem 1.6.5). The proof is essentially based on the estimates obtained above for the linearized problem (1.2.8).

Theorem 1.2.5. *(Regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $\frac{3}{2} < p < 2$ for the (MHD) system). Assume that $\frac{3}{2} < p < 2$ and r with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Let us consider $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ and $h \in W^{1,r}(\Omega)$ with the compatibility conditions (1.2.4) and (1.2.7).*

(i) *There exists a constant δ_1 such that, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_1$$

Then, the (MHD) problem has at least one solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the following estimates:

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C_1\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \\ + \|h\|_{W^{1,r}(\Omega)} \Big)\end{aligned} \tag{1.2.9}$$

$$\begin{aligned}\|P\|_{W^{1,r}(\Omega)} \leqslant C_1(1 + C^*\eta)\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \\ + \|h\|_{W^{1,r}(\Omega)}\Big),\end{aligned} \tag{1.2.10}$$

where $\delta_1 = (2C^2C^*)^{-1}$, $C_1 = C(1 + C^*\eta)^2$ with $C > 0$, $C^* > 0$ are the constants given in (1.6.25) and η defined by (1.6.26). Furthermore, we have for all $1 \leqslant i \leqslant I$

$$\alpha_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_\Omega - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N \, d\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N \, d\boldsymbol{x} + \int_\Gamma (h - P_0) \nabla q_i^N \cdot \boldsymbol{n} \, d\sigma$$

(ii) *Moreover, if the data satisfy that*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_2,$$

for some $\delta_2 \in]0, \delta_1]$, then the weak solution of (MHD) problem is unique.

1.3 Notations and preliminary results

Before studying the MHD problem (*MHD*), we introduce some basic notations and specific functional framework to this chapter. If we do not state otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$, which is not necessary connected, of class at least $\mathcal{C}^{1,1}$ and sometimes of class $\mathcal{C}^{2,1}$. We denote by Γ_i, $0 \leq i \leq I$, the connected components of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$.

The vector fields and matrix fields as well as the corresponding spaces are denoted by bold font. We will use C to denote a generic positive constant which may depend on Ω and the dependence on other parameters will be specified if necessary. For $1 < p < \infty$, $\boldsymbol{L}^p(\Omega)$ denotes the usuel vector-valued $\boldsymbol{L}^p$-space over Ω. As usual, we denote by $\boldsymbol{W}^{m,p}(\Omega)$ the Sobolev space of functions in $\boldsymbol{L}^p(\Omega)$ whose weak derivatives of order less than or equal to m are also in $\boldsymbol{L}^p(\Omega)$. In the case $p = 2$, we shall write $\boldsymbol{H}^m(\Omega)$ instead to $\boldsymbol{W}^{m,2}(\Omega)$. If $p \in [1, \infty)$, p' denotes the conjugate exponent of p, i.e. $\frac{1}{p'} = 1 - \frac{1}{p}$. We define the spaces

$$\boldsymbol{X}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega),\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

which is equipped with the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}\,.$$

The subspaces $\boldsymbol{X}^p_N(\Omega)$ and $\boldsymbol{V}^p_N(\Omega)$ are defined by

$$\boldsymbol{X}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega),\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}\ \text{ on}\,\Gamma\},$$

$$\boldsymbol{V}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^p_N(\Omega);\ \mathrm{div}\ \boldsymbol{v} = 0\ \text{ in}\,\Omega\}.$$

When $p = 2$, we will use the notation $\boldsymbol{X}_N(\Omega)$ instead to $\boldsymbol{X}^2_N(\Omega)$. We denote by $\boldsymbol{\mathcal{D}}(\Omega)$ the set of smooth functions (infinitely differentiable) with compact support in Ω. For $p, r \in [1, \infty)$, we introduce the following space

$$\boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^r(\Omega);\, \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}\,, \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{3} \tag{1.3.1}$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^{\,r,p}(\mathbf{curl},\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^r(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

It can be shown that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega)$ (cf. [87, Proposition 1.0.2] for the case $r = p$). The closure of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega)$ is denoted by $\boldsymbol{H}^{\,r,p}_0(\mathbf{curl}, \Omega)$ with

$$\boldsymbol{H}^{\,r,p}_0(\mathbf{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^{\,r,p}(\mathbf{curl}, \Omega);\ \ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}\ \text{ on } \Gamma\}\,.$$

$\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}^{\,r,p}_0(\mathbf{curl}, \Omega)$ and its dual space denoted by $[\boldsymbol{H}^{\,r,p}_0(\mathbf{curl}, \Omega)]'$ can be characterized as follows (cf. [14, Lemma 2.4], [14, Lemma 2.5] and [87, Proposition 1.0.6] for the case $r = p$):

Proposition 1.3.1. *A distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}^{r,p}_0(\mathbf{curl}, \Omega)]'$ iff there exists $\boldsymbol{F} \in \boldsymbol{L}^{r'}(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi}$. Moreover, we have the estimate :*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}^{r,p}_0(\mathbf{curl},\Omega)]'} \leq \inf_{\boldsymbol{f}=\boldsymbol{F}+\mathbf{curl}\,\boldsymbol{\psi}} \max\{\|\boldsymbol{F}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)}\}.$$

Next we introduce the kernel

$$\boldsymbol{K}^p_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{0},\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}\ \text{ on } \Gamma\}.$$

Thanks to [15, Corollary 4.2], we know that this kernel is of finite dimension and spanned by the functions ∇q_i^N, $1 \leq i \leq I$, where q_i^N is the unique solution of the problem

$$\begin{cases} -\Delta q_i^N = 0 \quad \text{ in } \Omega, \quad q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^N|_{\Gamma_k} = \text{constant}, \;\; 1 \leq k \leq I \\ \big\langle \partial_n q_i^N, 1 \big\rangle_{\Gamma_k} = \delta_{i\,k}, \;\; 1 \leq k \leq I, \quad \text{and} \quad \big\langle \partial_n q_i^N, 1 \big\rangle_{\Gamma_0} = -1. \end{cases} \tag{1.3.2}$$

Moreover, the functions ∇q_i^N, $1 \leq i \leq I$, belong to $\boldsymbol{W}^{1,q}(\Omega)$ for any $1 < q < \infty$. We will use also the symbol σ to represent a set of divergence free functions. In other words, if $\boldsymbol{X}$ is a Banach space, then $\boldsymbol{X}_\sigma = \{\boldsymbol{v} \in \boldsymbol{X};\;\; \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$.
We recall some useful results that play an important role in the proof of the regularity of solutions in this chapter. We begin with the following result (see [15, Theorem 3.2.])

Theorem 1.3.1. *The space $\boldsymbol{X}_N^p(\Omega)$ is continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant C, such that for any $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|\big). \tag{1.3.3}$$

And more generally (see [15, Corollary 5.3])

Corollary 1.3.2. *Let $m \in \mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then the space*

$$\boldsymbol{X}^{m,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\, \operatorname{div} \boldsymbol{v} \in W^{m-1,p}(\Omega),\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{m-1,p}(\Omega),\;\; \boldsymbol{v}\cdot\boldsymbol{n} \in \boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)\}$$

is continuously embedded in $\boldsymbol{W}^{m,p}(\Omega)$ and we have the following estimate: for any function $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{W^{m-1,p}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)}\Big) \tag{1.3.4}$$

We also recall the following result (cf. [15, Corollary 3.2]) which gives a Poincaré inequality for every function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ.

Corollary 1.3.3. *On the space $\boldsymbol{X}_N^p(\Omega)$, the seminorm*

$$\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{\mathrm{L}^p(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{I}} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_\mathrm{i}}| \tag{1.3.5}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$ for any $1 < p < \infty$. In particular, we have the following Poincaré inequality for every function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_{\mathcal{P}}\big(\|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|\big), \tag{1.3.6}$$

where $C_{\mathcal{P}} = C_{\mathcal{P}}(\Omega) > 0$. Moreover, the norm (1.3.5) is equivalent to the full norm $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$ on $\boldsymbol{X}_N^p(\Omega)$.

Let us consider the following Stokes problem:

$$
(\mathcal{S}_{\mathcal{N}})\begin{cases} -\Delta \boldsymbol{u} + \nabla P = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \ \text{ on } \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I. \end{cases}
$$

Then, the following proposition is an extension of that in [15, Theorem 5.7] to the case of non-zero divergence condition ($h \neq 0$). It is concerned with the existence and uniqueness of the weak and strong solutions for the Stokes problem $(\mathcal{S}_{\mathcal{N}})$.

Proposition 1.3.2. *We assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$, h and P_0 such that*

$$
\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]', \quad h \in W^{1,r}(\Omega) \quad \text{and} \quad P_0 \in W^{1-1/r,r}(\Gamma),
$$

with $r \leq p$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}$. Then, the problem $(\mathcal{S}_{\mathcal{N}})$ has a unique solution $(\boldsymbol{u},\,P) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ and constants $c_1, \ldots, c_I$ satisfying the estimate:

$$
\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}\big), \tag{1.3.7}
$$

and $c_1, \ldots, c_I$ are given by

$$
c_i = \langle \boldsymbol{f},\, \nabla q_i^N \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)} + \int_\Gamma (h - P_0)\, \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma. \tag{1.3.8}
$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$ and $P_0 \in W^{1-\frac{1}{p},p}(\Gamma)$, then $(\boldsymbol{u},\,P)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimate

$$
\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leqslant C_S\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \sum_{i=1}^{I} |c_i|\Big), \tag{1.3.9}
$$

where $C_S = C_S(\Omega) > 0$.

Proof. To reduce the non vanishing divergence problem $(\mathcal{S}_{\mathcal{N}})$ to the case where $\operatorname{div} \boldsymbol{u} = 0$ in Ω, we consider the problem

$$
\Delta\,\theta = h \quad \text{in}\,\Omega \quad \text{and} \quad \theta = 0 \quad \text{on}\,\Gamma.
$$

Since $h \in W^{1,r}(\Omega)$, it has a unique solution $\theta \in W^{3,r}(\Omega) \hookrightarrow W^{2,p}(\Omega)$, with (cf. [46, Theorem 1.8])

$$
\|\theta\|_{W^{2,p}(\Omega)} \leq C\|h\|_{W^{1,r}(\Omega)}. \tag{1.3.10}
$$

Taking $\boldsymbol{w} = \nabla\,\theta$ and defining

$$
\widetilde{\boldsymbol{w}} = \boldsymbol{w} - \sum_{i=1}^{I} \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N, \tag{1.3.11}
$$

we see that $\widetilde{\boldsymbol{w}} \in \boldsymbol{W}^{1,p}(\Omega)$ with div $\widetilde{\boldsymbol{w}} = h$, $\mathbf{curl}\,\widetilde{\boldsymbol{w}} = \mathbf{0}$ in Ω, $\widetilde{\boldsymbol{w}} \times \boldsymbol{n} = \mathbf{0}$ on Γ and $\langle \widetilde{\boldsymbol{w}} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ for any $1 \leq i \leq I$. Finally, taking $\boldsymbol{z} = \boldsymbol{u} - \widetilde{\boldsymbol{w}}$, we see that the problem $(\mathcal{S}_{\mathcal{N}})$ can be reduced to the following problem for $\boldsymbol{z}$ and P:

$$\begin{cases} -\Delta \boldsymbol{z} + \nabla P = \boldsymbol{f} + \Delta \widetilde{\boldsymbol{w}} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma, \\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \ \text{ on } \Gamma_i, \\ \langle \boldsymbol{z} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ 1 \leq i \leq I. \end{cases} \tag{1.3.12}$$

Since $\boldsymbol{w} = \nabla\theta$ and $\Delta(\nabla q_i^N) = 0$, it follows from (1.3.11) that $\Delta\,\widetilde{\boldsymbol{w}} = \nabla(\Delta\theta) \in \boldsymbol{L}^r(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$ and $\int_\Omega \Delta\widetilde{\boldsymbol{w}} \cdot \nabla q_i^N \, d\boldsymbol{x} = 0$, we deduce from [15, Theorem 5.7], the existence of a unique solution $(\boldsymbol{z},\, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ of (1.3.12) with $\boldsymbol{c} = (c_1, \ldots, c_I)$ given by (1.3.8). Moreover, using (1.3.10), we have $\|\Delta\widetilde{\boldsymbol{w}}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \leq C\,\|h\|_{W^{1,r}(\Omega)}$ and then $(\boldsymbol{z}, P)$ satisfies the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leq C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'} + \|h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}\Big). \tag{1.3.13}$$

As a consequence, $(\boldsymbol{u}, P) = (\boldsymbol{z} + \widetilde{\boldsymbol{w}}, P) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ is the unique solution of $(\mathcal{S}_{\mathcal{N}})$ and the estimate (1.3.7) follows from (1.3.10) and (1.3.13).
Now, we suppose that $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $h \in W^{1,p}(\Omega)$ and $P_0 \in W^{1-1/p,p}(\Gamma)$. We know that $(\boldsymbol{u}, P)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$. We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$. Since $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on Γ, we have $\boldsymbol{z} \cdot \boldsymbol{n} = 0$ on Γ and then $\boldsymbol{z}$ belongs to $\boldsymbol{X}_N(\Omega)$. By Theorem 1.3.1, the function $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Then, $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \ \operatorname{div} \boldsymbol{u} = h \in W^{1,p}(\Omega), \ \ \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \ \text{ and } \ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma.$$

We deduce from Corollary 1.3.2 (with $m = 2$) that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. □

We need also some regularity results for the following elliptic problem

$$(\mathcal{E}_{\mathcal{N}}) \begin{cases} -\Delta \boldsymbol{b} = \boldsymbol{g} \quad \text{and} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma \\ \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases}$$

which can be seen as a Stokes problem without pressure. We note that $(\mathcal{E}_N)$ is well-posed. Indeed, observe that the condition $\operatorname{div} \boldsymbol{g} = 0$ in Ω is necessary to solve $(\mathcal{E}_N)$ and then we can verify that it is equivalent to the following problem:

$$\begin{cases} -\Delta\,\boldsymbol{b} = \boldsymbol{g} & \text{in } \Omega \\ \operatorname{div}\,\boldsymbol{b} = 0, \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} & \text{on}\,\Gamma, \\ \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases} \tag{1.3.14}$$

where we have replaced the condition div $\boldsymbol{b} = 0$ in Ω by div $\boldsymbol{b} = 0$ on Γ. Next, we know that for any $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$ such that div $\boldsymbol{b} \in W^{1,p}(\Omega)$, we have (cf. [10] or [55] for $\boldsymbol{b} \in \boldsymbol{W}^{2,p}(\Omega)$):

$$\operatorname{div} \boldsymbol{b} = \operatorname{div}_\Gamma \boldsymbol{b}_t + \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} - 2K\boldsymbol{b} \cdot \boldsymbol{n} \quad \text{on } \Gamma, \tag{1.3.15}$$

where $\boldsymbol{b}_t$ is the tangential component of $\boldsymbol{b}$, K denotes the mean curvature of Γ and $\operatorname{div}_\Gamma$ is the surface divergence.

Then, using (1.3.15), the problem (1.3.14) is equivalent to: find $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta\, \boldsymbol{b} = \boldsymbol{g} & \text{in}\,\Omega, \\ \boldsymbol{b} \times \boldsymbol{n} = 0 \quad \text{and} \quad \dfrac{\partial \boldsymbol{b}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} - 2K\boldsymbol{b} \cdot \boldsymbol{n} = 0 & \text{on}\,\Gamma, \\ \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases} \tag{1.3.16}$$

where the condition $\dfrac{\partial \boldsymbol{b}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} - 2K\boldsymbol{b} \cdot \boldsymbol{n} = 0$ on Γ is a Fourier-Robin type boundary condition.

We begin with the following regularity result for $(\mathcal{E}_\mathcal{N})$ which can be found in [15, Corollary 5.4.].

Theorem 1.3.3. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ satisfying the compatibility conditions*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \int_\Omega \boldsymbol{g} \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0, \tag{1.3.17}$$

$$\operatorname{div} \boldsymbol{g} = 0 \quad \text{in}\,\Omega. \tag{1.3.18}$$

Then the elliptic problem $(\mathcal{E}_\mathcal{N})$ has a unique solution $\boldsymbol{b} \in \boldsymbol{W}^{2,p}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leqslant C_E \, \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} . \tag{1.3.19}$$

We need also the following useful result for $(\mathcal{E}_\mathcal{N})$ which gives an improvement of that in [15, Proposition 5.1]. Indeed, we consider the dual space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ (c.f. (1.3.1) and Proposition (1.3.1)) for data in the right-hand side instead of $[\boldsymbol{H}_0^{p',p'}(\mathbf{curl}, \Omega)]'$.

Lemma 1.3.4. *Let Ω of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ satisfying the compatibility conditions*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)} = 0, \tag{1.3.20}$$

$$\operatorname{div} \boldsymbol{g} = 0 \quad \text{in}\,\Omega. \tag{1.3.21}$$

Then, the elliptic problem $(\mathcal{E}_\mathcal{N})$ has a unique solution $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C \, \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \tag{1.3.22}$$

Proof. Using the characterization of the dual space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$ given in Proposition 1.3.1, we can write $\boldsymbol{g}$ as:

$$\boldsymbol{g} = \boldsymbol{G} + \mathbf{curl}\,\Psi, \quad \text{where} \quad \boldsymbol{G} \in \boldsymbol{L}^r(\Omega) \quad \text{and} \quad \Psi \in \boldsymbol{L}^p(\Omega). \tag{1.3.23}$$

Note that, from (1.3.2), for any $1 \leqslant i \leqslant I$, $\langle \mathbf{curl}\,\Psi, \nabla q_i^N \rangle_\Omega = 0$, then it follows from (1.3.20) and (1.3.23) that $\boldsymbol{G}$ also satisfies the compatibility condition (1.3.20). Similarly, by (1.3.23), we have $\operatorname{div}\boldsymbol{G} = 0$. Thanks to Theorem 1.3.3, the following problem:

$$\begin{cases} -\Delta \boldsymbol{b}_1 = \boldsymbol{G} \quad \text{in}\,\Omega \quad \text{and} \quad \operatorname{div}\boldsymbol{b}_1 = 0 \quad \text{in}\,\Omega, \\ \boldsymbol{b}_1 \times \boldsymbol{n} = 0 \quad \text{on}\,\Gamma, \\ \langle \boldsymbol{b}_1 \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I \end{cases}$$

has a unique solution $\boldsymbol{b}_1 \in \boldsymbol{W}^{2,r}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{b}_1\|_{\boldsymbol{W}^{2,r}(\Omega)} \leqslant C\,\|\boldsymbol{G}\|_{\boldsymbol{L}^r(\Omega)}\,. \tag{1.3.24}$$

Next, since $\mathbf{curl}\,\Psi \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'$ and satisfies the compatibility conditions (1.3.20), by [15, Proposition 5.1.] the following problem

$$\begin{cases} -\Delta \boldsymbol{b}_2 = \mathbf{curl}\,\Psi \quad \text{in}\,\Omega \quad \text{and} \quad \operatorname{div}\boldsymbol{b}_2 = 0 \quad \text{in}\,\Omega, \\ \boldsymbol{b}_2 \times \boldsymbol{n} = 0 \quad \text{on}\,\Gamma, \\ \langle \boldsymbol{b}_2 \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I \end{cases}$$

has a unique solution $\boldsymbol{b}_2 \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{b}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C\,\|\mathbf{curl}\,\Psi\|_{\boldsymbol{L}^p(\Omega)}\,. \tag{1.3.25}$$

Since $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$, $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Then, $\boldsymbol{b} = \boldsymbol{b}_1 + \boldsymbol{b}_2$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and it is the unique solution of $(\mathcal{E}_\mathcal{N})$. The estimate (1.3.22) follows from (1.3.24) and (1.3.25). $\square$

Remark 1.3.4.

(1). We note that the regularity $\mathcal{C}^{2,1}$ in Lemma 1.3.4 can be reduced to $\mathcal{C}^{1,1}$. Indeed, we can verify that the Stokes problem $(\mathcal{E}_\mathcal{N})$ is equivalent to the following variational frmulation (c.f. [15, Proposition 5.1]): Find $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$ such that for any $\boldsymbol{a} \in \boldsymbol{V}_N^{p'}(\Omega)$:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{b} \cdot \mathbf{curl}\,\boldsymbol{a}\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{g},\, \boldsymbol{a} \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)}. \tag{1.3.26}$$

Thanks to [15, Lemma 5.1], if Ω is of class $\mathcal{C}^{1,1}$, the following infi-sup condition holds: there exists a constant $\beta > 0$, such that:

$$\inf_{\substack{\boldsymbol{a}\in \boldsymbol{V}_N^{p'}(\Omega) \\ \boldsymbol{a}\neq 0}} \sup_{\substack{\boldsymbol{b}\in \boldsymbol{V}_N^{p}(\Omega) \\ \boldsymbol{b}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{b} \cdot \mathbf{curl}\,\boldsymbol{a}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p'}(\Omega)}} \geq \beta. \tag{1.3.27}$$

So, problem (1.3.26) *has a unique solution* $\boldsymbol{u} \in \boldsymbol{V}_N^p(\Omega) \subset \boldsymbol{W}^{1,p}(\Omega)$ *since the right-hand sides defines an element of* $(\boldsymbol{V}_N^p(\Omega))'$.

(2). In the classical study of the Stokes and Navier-Stokes equations, the pressure P *is obtained thanks to a variant of De Rham's theorem (see [10, Theorem 2.8]). Indeed, let* $\Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain and* $\boldsymbol{f} \in \boldsymbol{W}^{-1,p}(\Omega)$, $1 < p < \infty$ *satisfying*

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{D}}'(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Then there exists $P \in L^p(\Omega)$ *such that* $\boldsymbol{f} = \nabla P$. *Unlike the case of Dirichlet boundary condition, the pressure in the* (*MHD*) *problem can be found independently of the velocity* $\boldsymbol{u}$ *and the magnetic field* $\boldsymbol{b}$. *Indeed, the pressure* P *is a solution of the problem*

$$\begin{cases} \Delta P = \operatorname{div} \boldsymbol{f} - \operatorname{div}((\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}) + \operatorname{div}((\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}) \quad \text{in } \Omega \\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + \alpha_i \quad \text{on } \Gamma_i. \end{cases}$$

So, when we talk about the regularity $\boldsymbol{W}^{1,p}(\Omega)$ *or* $\boldsymbol{W}^{2,p}(\Omega)$ *it concerns* $(\boldsymbol{u}, \boldsymbol{b})$ *and we mean that* $(\boldsymbol{u}, \boldsymbol{b}, P)$ *is the weak or strong solution of the* (*MHD*) *problem.*

1.4 The linearized MHD system: L^2-theory

In this section we take $\boldsymbol{w}$ and $\boldsymbol{d}$ such that:

$$\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega), \quad \boldsymbol{d} \in \boldsymbol{L}^3(\Omega), \qquad \operatorname{div} \boldsymbol{d} = 0 \ \text{ in } \Omega, \tag{1.4.1}$$

and we consider the following linearized MHD system: Find ($\boldsymbol{u}$, $\boldsymbol{b}$, P, $\boldsymbol{c}$) with $\boldsymbol{c} = (c_1, \ldots, c_I)$ such that for $1 \leq i \leq I$:

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} + \nabla \mathrm{P} - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g} \quad \text{and} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ \mathrm{P} = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \mathrm{P} = P_0 + c_i \ \text{ on } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0. \end{cases} \tag{1.4.2}$$

The aim of this section is to show, under minimal regularity assumptions on $\boldsymbol{f}$, $\boldsymbol{g}$, h and P_0, the existence and the uniqueness of weak solutions $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c})$ in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$. Classically, the idea is to write an equivalent variational formulation and use Lax Milgram if the bilinear form involved in the variational formulation is coercive. It is natural to look for a solution $(\boldsymbol{u}, \boldsymbol{b})$ in $\boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ with

$$\boldsymbol{V}_N(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega, \ \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \Gamma, \ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ \forall 1 \leqslant i \leqslant I\}.$$

Unlike the case of Dirichlet type boundary conditions, the space $\boldsymbol{H}^{-1}(\Omega)$ is not suitable for source terms in the right hand side to find solutions in $\boldsymbol{H}^1(\Omega)$. Let us analyse the case of $\boldsymbol{f}$, it holds true also for $\boldsymbol{g}$. Since $\boldsymbol{v} \in \boldsymbol{V}_N(\Omega)$, then we can firstly consider the duality pairing $\langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{2,2}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{2,2}(\mathbf{curl},\Omega)}$ in view to write an equivalent variational formulation. Then, we must suppose that $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{2,2}(\mathbf{curl},\Omega)]'$. But, we have $\boldsymbol{v}$ belongs to $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$. Then, the previous hypothesis on $\boldsymbol{f}$ can be weakened by considering the space $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ which is a subspace of $\boldsymbol{H}^{-1}(\Omega)$. Indeed, thanks to the characterization given in Proposition 1.3.1, we have for $r = 6$ and $p = 2$,

$$[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]' = \{\boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi};\;\; \boldsymbol{F} \in \boldsymbol{L}^{6/5}(\Omega),\; \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega)\}. \tag{1.4.3}$$

Then, since $\boldsymbol{V}_N(\Omega) \hookrightarrow \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)$, the previous duality is replaced by

$$\langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)} = \int_\Omega \boldsymbol{F}\cdot\boldsymbol{v}\,d\boldsymbol{x} + \int_\Omega \boldsymbol{\psi}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x}.$$

In the sequel, we will consider the space $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ for $\boldsymbol{f}$ and $\boldsymbol{g}$ to obtain solutions in $\boldsymbol{H}^1(\Omega)$.

Proposition 1.4.1. *Let us suppose $h = 0$. Let $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ and $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ with the compatibility conditions*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^2(\Omega), \quad \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\Omega_{6,2}} = 0, \tag{1.4.4}$$

$$\operatorname{div}\boldsymbol{g} = 0 \quad \text{on}\,\Omega, \tag{1.4.5}$$

where $\langle\cdot,\cdot\rangle_{\Omega_{6,2}}$ denotes the duality product between $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ and $\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)$.
Then the following two problems are equivalent:
(i) Find $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ solution of (1.4.2).
(ii) Find $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ and $\boldsymbol{c} \in \mathbb{R}^I$ such that: for all $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$

$$\begin{aligned}&\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx + \int_\Omega (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\boldsymbol{v}\,dx - \int_\Omega (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\boldsymbol{v}\,dx + \int_\Omega \mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{\Psi}\,dx\\ &+ \int_\Omega (\mathbf{curl}\,\boldsymbol{\Psi})\times\boldsymbol{d}\cdot\boldsymbol{u}\,dx = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\Omega_{6,2}} + \langle \boldsymbol{g}, \boldsymbol{\Psi}\rangle_{\Omega_{6,2}} - \langle P_0, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_0} - \sum_{i=1}^{I}\langle P_0 + c_i, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_i}\end{aligned} \tag{1.4.6}$$

and $\boldsymbol{c} = (c_1, \cdots, c_I)$ satisfying for $1 \le i \le I$:

$$c_i = \langle \boldsymbol{f}, \nabla q_i^N\rangle_{\Omega_{6,2}} - \langle P_0, \nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma - \int_\Omega (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\nabla q_i^N\,dx + \int_\Omega (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\nabla q_i^N\,dx, \tag{1.4.7}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Proof. Using the same arguments as in [15, Lemma 5.5], we can prove that $\mathcal{D}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in the space

$$\mathcal{E}(\Omega) = \{(\boldsymbol{u},\, P) \in \boldsymbol{H}_\sigma^1(\Omega) \times L^2(\Omega);\;\; \Delta\boldsymbol{u} + \nabla P \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'\}.$$

Moreover, we have the following Green formula: For any $(\boldsymbol{u},\, P) \in \mathcal{E}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\langle -\Delta\, \boldsymbol{u} + \nabla\, P,\, \boldsymbol{\varphi}\rangle_{\Omega_{6,2}} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \langle P,\, \boldsymbol{\varphi} \cdot \boldsymbol{n}\rangle_\Gamma. \tag{1.4.8}$$

Using Green formula (1.4.8), we deduce that any $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ satisfying (1.4.2) also solves (1.4.6). It remains to recover the relation (1.4.7). Let us take $\boldsymbol{v} \in \boldsymbol{H}^1_\sigma(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and set:

$$\boldsymbol{v}_0 = \boldsymbol{v} - \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N \tag{1.4.9}$$

Observe that $\boldsymbol{v}_0 \in \boldsymbol{H}^1_\sigma(\Omega)$, $\boldsymbol{v}_0 \times \boldsymbol{n} = \boldsymbol{0}$ on Γ and due to the properties of q_i^N, we have for all $1 \leq i \leq I$, $\langle \boldsymbol{v}_0 \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$. Then $\boldsymbol{v}_0$ belongs to $\boldsymbol{V}_N(\Omega)$. Multiplying the first equation on the left of the problem (1.4.2) with $\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N$, integrating by parts in Ω, we obtain

$$\begin{aligned}
&\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx + \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \boldsymbol{v}\, dx - \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \boldsymbol{v}\, dx - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega + \langle P_0, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\Gamma_0} \\
&+ \sum_{i=1}^{I} \langle P_0 + c_i, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\Gamma_i} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}_0\, dx + \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \boldsymbol{v}_0\, dx - \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \boldsymbol{v}_0\, dx \\
&- \langle \boldsymbol{f}, \boldsymbol{v}_0\rangle_\Omega + \langle P_0, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_{\Gamma_0} + \sum_{i=1}^{I} \langle P_0 + c_i, \boldsymbol{v}_0 \cdot \boldsymbol{n}\rangle_{\Gamma_i} + \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \Big[\int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx \Big] \\
&+ \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \Big[- \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N\, dx - \langle \boldsymbol{f}, \nabla q_i^N\rangle_\Omega + \langle P_0, \nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma \Big] + \sum_{i=1}^{I} c_i\, \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0
\end{aligned}$$

Comparing with the variational formulation (1.4.6) for the test function $(\boldsymbol{v}_0, \boldsymbol{0})$, we obtain for all $1 \leqslant i \leqslant I$:

$$\begin{aligned}
\sum_{i=1}^{I} c_i\, \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = \sum_{i=1}^{I} \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \Big[&- \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx + \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N\, dx \\
&+ \langle \boldsymbol{f}, \nabla q_i^N\rangle_\Omega - \langle P_0, \nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma \Big]
\end{aligned}$$

Finally, taking $\boldsymbol{v} = \nabla q_j^N$, due to the properties of q_i^N in (1.3.2), we obtain the relation (1.4.7) for all $1 \leqslant j \leqslant I$.

Conversely, let $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ be a solution of (1.4.6) and $\boldsymbol{c} = (c_1, \cdots, c_I)$ satisfying (1.4.7). We want to show it implies (i). We note that $\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ is not a subspace of $\boldsymbol{V}_N(\Omega)$, so it is not possible to prove directly that (1.4.6)-(1.4.7) implies (i). In particular, we can not apply the De Rham's lemma to recover the pressure. As a consequence, we need to extend (1.4.6) for all divergence free functions $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{X}_N(\Omega) \times \boldsymbol{X}_N(\Omega)$. For this purpose, let $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{X}_N(\Omega) \times \boldsymbol{X}_N(\Omega)$ and we consider the decomposition (1.4.9) for $\boldsymbol{v}$ to obtain $\boldsymbol{v}_0 \in \boldsymbol{V}_N(\Omega)$. Similarly, we set

$$\boldsymbol{\Psi}_0 = \boldsymbol{\Psi} - \sum_{i=1}^{I} \langle \boldsymbol{\Psi} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N, \tag{1.4.10}$$

which implies that $\boldsymbol{\Psi}_0$ is a function of $\boldsymbol{V}_N(\Omega)$. Replacing in (1.4.6), we obtain:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx+\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\boldsymbol{v}\,dx-\int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\boldsymbol{v}\,dx+\int_\Omega\mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{\Psi}\,dx$$
$$+\int_\Omega(\mathbf{curl}\,\boldsymbol{\Psi})\times\boldsymbol{d}\cdot\boldsymbol{u}\,dx-\langle\boldsymbol{f},\boldsymbol{v}\rangle_\Omega-\langle\boldsymbol{g},\boldsymbol{\Psi}\rangle_\Omega+\langle P_0,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_0}+\sum_{i=1}^{I}\langle P_0+c_i,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_i}$$
$$=\sum_{i=1}^{I}\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big[\underbrace{-\int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\nabla q_i^N\,dx+\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\nabla q_i^N\,dx-\langle\boldsymbol{f},\nabla q_i^N\rangle_\Omega+\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma}_{=-c_i}\Big]$$
$$+\sum_{i=1}^{I}\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}c_i+\sum_{i=1}^{I}\langle\boldsymbol{\Psi}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\langle\boldsymbol{g},\nabla q_i^N\rangle_\Omega,$$

where we have used the fact that for all $1\leqslant i\leqslant I$: $\sum_{j=1}^{I}c_j\,\langle\nabla q_i^N\cdot\boldsymbol{n},1\rangle_{\Gamma_j}=c_i$. Note that the compatibility condition (1.4.4) implies that $\langle\boldsymbol{g},\nabla q_i^N\rangle_\Omega=0$ for all $1\leqslant i\leqslant I$. Thus, the right hand side of the above relation is equal to zero and then for any $(\boldsymbol{v},\boldsymbol{\Psi})\in\boldsymbol{X}_N(\Omega)\times\boldsymbol{X}_N(\Omega)$, we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx+\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\boldsymbol{v}\,dx-\int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\boldsymbol{v}\,dx+\int_\Omega\mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{\Psi}\,dx$$
$$+\int_\Omega(\mathbf{curl}\,\boldsymbol{\Psi})\times\boldsymbol{d}\cdot\boldsymbol{u}\,dx=\langle\boldsymbol{f},\boldsymbol{v}\rangle_\Omega+\langle\boldsymbol{g},\boldsymbol{\Psi}\rangle_\Omega-\langle P_0,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_0}-\sum_{i=1}^{I}\langle P_0+c_i,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_i}.\tag{1.4.11}$$

That means that problem (1.4.11) and (1.4.6) are equivalent. So, in the sequel, we will prove that problem (1.4.11) implies (i).

Choosing $(\boldsymbol{v},\boldsymbol{0})$ with $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ as a test function in (1.4.11), we have

$$\langle-\Delta\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}-(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}-\boldsymbol{f},\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0$$

So by De Rham's theorem, there exists a distribution $P\in\mathcal{D}'(\Omega)$, defined uniquely up to an additive constant such that

$$-\Delta\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}-(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}-\boldsymbol{f}=-\nabla P\quad\text{in }\Omega.\tag{1.4.12}$$

Since $\boldsymbol{u}$ and $\boldsymbol{b}$ belong to $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$, the terms $(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ belong to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)\hookrightarrow\boldsymbol{H}^{-1}(\Omega)$. As $\boldsymbol{f}\in[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'\hookrightarrow\boldsymbol{H}^{-1}(\Omega)$, we deduce that $\nabla P\in H^{-1}(\Omega)$ and then $P\in L^2(\Omega)$ with a trace in $H^{-\frac{1}{2}}(\Gamma)$ (we refer to [11]). Next, choosing $(\boldsymbol{0},\boldsymbol{\Psi})$ with $\boldsymbol{\Psi}\in\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in (1.4.11), we have

$$\langle\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b}-\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})-\boldsymbol{g},\boldsymbol{\Psi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0$$

Then, applying [76, Lemma 2.2], we have $\chi\in L^2(\Omega)$ defined uniquely up to an additive constant such that

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b}-\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})-\boldsymbol{g}=\nabla\chi\quad\text{in }\Omega\qquad\text{and}\qquad\chi=0\quad\text{on }\Gamma$$

We note that the trace of χ is well defined and belongs to $H^{-1/2}(\Gamma)$. Taking the divergence of the above equation, the function χ is solution of the harmonic problem

$$\Delta\chi=0\quad\text{in }\Omega\qquad\text{and}\qquad\chi=0\quad\text{on }\Gamma.$$

So, we deduce that $\chi = 0$ in Ω which gives the second equation in (1.4.2). Moreover, by the fact that $\boldsymbol{u}$ and $\boldsymbol{b}$ belong to the space $\boldsymbol{V}_N(\Omega)$, we have $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{b} = 0$ in Ω and $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ.

It remains to show the boundary conditions on the pressure. Multiplying equation (1.4.12) by $\boldsymbol{v} \in \boldsymbol{X}_N(\Omega)$, using the decomposition (1.4.9) and integrating on Ω, we obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}_0\,dx + \int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\boldsymbol{v}_0\,dx - \int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\boldsymbol{v}_0\,dx - \langle\boldsymbol{f},\boldsymbol{v}_0\rangle_\Omega + \langle P,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_\Gamma$$
$$= \sum_{i=1}^I\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big[-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\nabla q_i^N\,dx + \int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\nabla q_i^N\,dx + \langle\boldsymbol{f},\nabla q_i^N\rangle_\Omega - \langle P,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma\Big]$$

Taking $(\boldsymbol{v},\boldsymbol{0})$ test function in (1.4.6), we have:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}_0\,dx + \int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\boldsymbol{v}_0\,dx - \int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\boldsymbol{v}_0\,dx - \langle\boldsymbol{f},\boldsymbol{v}_0\rangle_\Omega + \langle P_0,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_{\Gamma_0}$$
$$+\sum_{i=1}^I\langle P_0+c_i,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_{\Gamma_i} = \sum_{i=1}^I\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big[-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\nabla q_i^N\,dx + \int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\nabla q_i^N\,dx$$
$$+\langle\boldsymbol{f},\nabla q_i^N\rangle_\Omega - \langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_0} - \sum_{j=1}^I\langle P_0+c_j,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_j}\Big]$$

Substracting both equations and using again the decomposition (1.4.9), we obtain:

$$\underbrace{\langle P,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^I\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\langle P,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma}_{\langle P,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma}$$
$$= \underbrace{\langle P_0,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_{\Gamma_0} + \sum_{i=1}^I\langle P_0+c_i,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_{\Gamma_i}}_{\langle P_0,\boldsymbol{v}_0\cdot\boldsymbol{n}\rangle_\Gamma + c_i\sum_{i=1}^I\langle\boldsymbol{v}_0\cdot\boldsymbol{n},1\rangle_{\Gamma_i}} + \sum_{i=1}^I\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big[\underbrace{\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_0} + \sum_{j=1}^I\langle P_0+c_j,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_j}}_{\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma + c_i}\Big]$$

Since $\langle\boldsymbol{v}_0\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = 0$, $1\le i\le I$, we have

$$\langle P,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = \langle P_0,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma + \sum_{i=1}^I\langle c_i,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_i} = \langle P_0,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_0} + \sum_{i=1}^I\langle P_0+c_i,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_i}$$

Next, the argument to deduce that $P = P_0$ on Γ_0 and $P = P_0 + c_j$ on Γ_j is very similar to that of [14, Proposition 3.7], hence we omit it. □

Remark 1.4.1. *(i) Note that the compatibility condition* (1.4.4) *is necessary. Indeed, if we choose* $\boldsymbol{v} = \boldsymbol{0}$ *and* $\boldsymbol{\psi} = \nabla q_i^N$ *in* (1.4.6)*, we have* $\langle\boldsymbol{g},\nabla q_i^N\rangle_{\Omega_{6,2}} = 0$, $1\le i\le I$*. Observe that since* Ω *is of class* $\mathcal{C}^{1,1}$*, the functions* q_i^N *belong to* $H^2(\Omega)$ *and then the vectors* ∇q_i^N *belong to* $\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)$*. From the characterization* (1.4.3)*, this condition is actually written as* $\int_\Omega \boldsymbol{F}\cdot\nabla q_i^N\,d\boldsymbol{x} = 0$, $1\le i\le I$*. In the case where* Ω *is simply connected, the compatibility condition* (1.4.4) *is not necessary to solve* (1.4.2) *because the kernel* $\boldsymbol{K}_N^2(\Omega) = \{\boldsymbol{0}\}$*.*

(ii) If $\boldsymbol{g}$ *is the* **curl** *of an element* $\boldsymbol{\xi}\in\boldsymbol{L}^2(\Omega)$*, then* $\boldsymbol{g}$ *is still an element of* $[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$*. Moreover, since* $\operatorname{div}\boldsymbol{g} = 0$ *in* Ω*, it always satisfies the compatibility condition* (1.4.4)*.*

We now prove the solvability of the problem (1.4.6).

Theorem 1.4.2. *Let Ω be $\mathcal{C}^{1,1}$ and we suppose $h = 0$. Let*

$$\boldsymbol{f},\, \boldsymbol{g} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]' \quad \text{and} \quad P_0 \in H^{-\frac{1}{2}}(\Gamma)$$

with the compatibility conditions (1.4.4)-(1.4.5). *Then the problem* (1.4.2) *has a unique weak solution* $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ *which satisfies the estimates:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)}) \tag{1.4.13}$$

$$\|P\|_{L^2(\Omega)} \leqslant C(1 + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)})(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)}) \tag{1.4.14}$$

Moreover, if Ω is $\mathcal{C}^{2,1}$, $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{L}^{6/5}(\Omega)$ and $P_0 \in W^{1/6,6/5}(\Gamma)$, then $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)$ and we have the following estimate:

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,6/5}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,6/5}(\Omega)} + \|P\|_{W^{1,6/5}(\Omega)} &\leqslant C(1 + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}) \\ &\quad \times (\|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|P_0\|_{W^{1/6,6/5}(\Gamma)})\end{aligned} \tag{1.4.15}$$

Proof. We know, according to Proposition 1.4.1, that the linearized problem (1.4.2) is equivalent to (1.4.6)-(1.4.7). The existence and uniqueness of weak solution $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ follow from Lax-Milgram theorem. Let us define the bilinear continuous forms $a : \boldsymbol{Z}_N(\Omega) \times \boldsymbol{Z}_N(\Omega) \to \mathbb{R}$ and $a_{\boldsymbol{w},\boldsymbol{d}} : \boldsymbol{Z}_N(\Omega) \times \boldsymbol{Z}_N(\Omega) \to \mathbb{R}$ as follows:

$$\begin{aligned} a((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{\Psi})) &= \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx + \int_\Omega \mathbf{curl}\, \boldsymbol{b} \cdot \mathbf{curl}\, \boldsymbol{\Psi}\, dx \\ a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{\Psi})) &= \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \boldsymbol{v}\, dx + \int_\Omega (\mathbf{curl}\, \boldsymbol{\Psi}) \times \boldsymbol{d} \cdot \boldsymbol{u}\, dx - \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \boldsymbol{v}\, dx \end{aligned} \tag{1.4.16}$$

where $\boldsymbol{Z}_N(\Omega) = \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ equipped with the product norm

$$\|(\boldsymbol{v},\, \boldsymbol{\Psi})\|^2_{\boldsymbol{Z}_N(\Omega)} = \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{\Psi}\|^2_{\boldsymbol{H}^1(\Omega)}\,. \tag{1.4.17}$$

Next, we introduce the linear form $\mathcal{L} : \boldsymbol{Z}_N(\Omega) \to \mathbb{R}$ defined as follows

$$\mathcal{L}(\boldsymbol{v}, \boldsymbol{\Psi}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega_{6,2}} + \langle \boldsymbol{g}, \boldsymbol{\Psi} \rangle_{\Omega_{6,2}} - \langle P_0, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_0} - \sum_{i=1}^{I} \langle P_0 + c_i, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\Gamma_i}$$

So, the variational formulation (1.4.6) can be rewritten as: for any $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{Z}_N(\Omega)$

$$\mathcal{A}((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{\Psi})) = a((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{\Psi})) + a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{\Psi})) = \mathcal{L}(\boldsymbol{v}, \boldsymbol{\Psi}) \tag{1.4.18}$$

Since $((\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}) \cdot \boldsymbol{u} = \boldsymbol{0}$, then we have $a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{u}, \boldsymbol{b})) = 0$ for all $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}_N(\Omega)$. Since $\boldsymbol{v}$ and $\boldsymbol{\Psi}$ belong to $\boldsymbol{V}_N(\Omega)$, we have from [15, Corollary 3.2.] that the application

$\boldsymbol{v}\mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$ (respectively $\boldsymbol{\Psi}\mapsto \|\mathbf{curl}\,\boldsymbol{\Psi}\|_{\boldsymbol{L}^2(\Omega)}$) is a norm on $\boldsymbol{V}_N(\Omega)$ equivalent to the norm $\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}$ (respectively $\|\boldsymbol{\Psi}\|_{\boldsymbol{H}^1(\Omega)}$). As a consequence,

$$\begin{aligned}\mathcal{A}((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{\Psi})) = |a((\boldsymbol{v},\boldsymbol{\Psi}),(\boldsymbol{v},\boldsymbol{\Psi}))| &= \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\Psi}\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\geqslant \frac{2}{C_{\mathcal{P}}^2}\,\|(\boldsymbol{v},\boldsymbol{\Psi})\|^2_{\boldsymbol{Z}_N(\Omega)}\end{aligned} \tag{1.4.19}$$

where $C_{\mathcal{P}}$ is the constant given in (1.3.6). This shows that the bilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive on $\boldsymbol{Z}_N(\Omega)$. Moreover, applying Cauchy-Schwarz inequality, we have:

$$\begin{aligned}|a((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{\Psi}))| &\leqslant \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{\Psi}\|_{\boldsymbol{L}^2(\Omega)}\\ &\leqslant C\,\|(\boldsymbol{u},\boldsymbol{b})\|_{\boldsymbol{Z}_N(\Omega)}\|(\boldsymbol{v},\boldsymbol{\Psi})\|_{\boldsymbol{Z}_N(\Omega)}\end{aligned} \tag{1.4.20}$$

Now, using Hölder inequality, we have

$$\begin{aligned}|a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{\Psi}))| &\leqslant \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^6(\Omega)} + \|\mathbf{curl}\,\boldsymbol{\Psi}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\\ &+ \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^6(\Omega)}\end{aligned}$$

Now, using again the equivalence of norms $\|\cdot\|_{\boldsymbol{V}_N(\Omega)}$ and $\|\cdot\|_{\boldsymbol{H}^1(\Omega)}$, we obtain for $C_1>0$ the constant of the embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow \boldsymbol{L}^6(\Omega)$

$$|a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{\Psi}))| \leqslant \big(C_1^2\,\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + C_1\,\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\big)\,\|(\boldsymbol{u},\,\boldsymbol{b})\|_{\boldsymbol{Z}_N(\Omega)}\|(\boldsymbol{v},\,\boldsymbol{\Psi})\|_{\boldsymbol{Z}_N(\Omega)} \tag{1.4.21}$$

From (1.4.20) and (1.4.21), we can deduce that the form $\mathcal{A}(\cdot,\cdot)$ is continuous. Using similar arguments, we can verify that the right hand side of (1.4.18) defines an element in the dual space of $\boldsymbol{Z}_N(\Omega)$. Thus, by Lax-Milgram lemma, there exists a unique $(\boldsymbol{u},\boldsymbol{b})\in \boldsymbol{Z}_N(\Omega)$ satisfying (1.4.18). So, due to Theorem 1.3.1, we obtain the existence of a unique weak solution $(\boldsymbol{u},\boldsymbol{b})\in \boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)$. Using (1.4.19), the variational formulation and trace theorem, we obtain the estimate (1.4.13). The existence of the pressure follows from De Rham's theorem. Moreover for the pressure estimate, we can write

$$\|P\|_{L^2(\Omega)}\leq C\,\|\nabla P\|_{\boldsymbol{H}^{-1}(\Omega)}\leq C\big(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}+\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}+\|(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}+\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\|_{\boldsymbol{H}^{-1}(\Omega)}\big).$$

We know that $\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)}\leq C\,\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'}$ and $\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}\leq C\,\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$. For the two remaining terms, we proceed as follows

$$\|(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}\leq C\,\|(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\|_{\boldsymbol{L}^{6/5}(\Omega)}\leq C\,\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)},$$

so, we have

$$\|(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\|_{\boldsymbol{H}^{-1}(\Omega)}\leq CC_1\,\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.$$

Proceeding similarly, we get

$$\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\|_{\boldsymbol{H}^{-1}(\Omega)}\leq C\,\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$$

Hence, using the above estimates together with the estimate (1.4.13), we deduce the pressure estimate (1.4.14).

Now, if $\boldsymbol{f},\boldsymbol{g}\in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ and $P_0\in W^{1/6,6/5}(\Gamma)$, then we already know that $(\boldsymbol{u},\boldsymbol{b},P)\in \boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2(\Omega)$ is solution of (1.4.2). We deduce that $(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ belongs to $\boldsymbol{L}^{6/5}(\Omega)$. Similarly, we have $\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}) = (\boldsymbol{d}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{d}$ belongs to $\boldsymbol{L}^{6/5}(\Omega)$. Observe that $(\boldsymbol{u}, P, \boldsymbol{c})$ is solution of the following Stokes problem

$$\begin{cases} -\Delta\boldsymbol{u} + \nabla P = \boldsymbol{F} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma \quad \text{and} \quad P = P_0 \quad \text{on } \Gamma_0, \quad P = P_0 + c_i \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases}$$

with $\boldsymbol{F} = \boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ in $\boldsymbol{L}^{6/5}(\Omega)$. Thanks to the regularity of Stokes problem $(\mathcal{S}_N)$ (see Proposition 1.3.2), $(\boldsymbol{u}, P)$ belongs to $\boldsymbol{W}^{2,6/5}(\Omega)\times W^{1,6/5}(\Omega)$ with the corresponding estimate. Next, since $\boldsymbol{b}$ is a solution of the following elliptic problem

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} = \boldsymbol{G} \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{b}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma, \\ \langle \boldsymbol{b}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases}$$

with $\boldsymbol{G} = \boldsymbol{g} + \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})$ in $\boldsymbol{L}^{6/5}(\Omega)$ satisfying the compatibility conditions (1.4.4)-(1.4.5), we deduce from Theorem 1.3.3 that $\boldsymbol{b}$ belongs to $\boldsymbol{W}^{2,6/5}(\Omega)$. The estimate (1.4.15) then follows from the regularity estimates of the above Stokes problem on $(\boldsymbol{u}, P)$ and elliptic problem on $\boldsymbol{b}$. □

1.5 The linearized MHD system: L^p-theory

After the study of weak solutions in the case of Hilbert spaces, we are interested in the study of weak and strong solutions in L^p-theory for the linearized system (1.4.2). We begin by studying strong solutions. If $p \geq 6/5$, it follows that $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{6/5}(\Omega)$, $W^{1-1/p,p}(\Gamma) \hookrightarrow W^{1/6,6/5}(\Gamma)$. Then, due to Theorem 1.4.2, we have $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{W}^{2,6/5}(\Omega)\times\boldsymbol{W}^{2,6/5}(\Omega)$. In the next subsection we will prove that this solution belongs to $\boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)$ for any $p > 6/5$.

1.5.1 Strong solution in $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq 6/5$

The aim of this section is to give an answer to the question of the existence of a regular solution $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ for the linearized MHD problem (1.4.2). When $p < 3$, we have the embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Then, supposing $\boldsymbol{d} \in \boldsymbol{W}^{1,3/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ implies that the term $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ belongs to $\boldsymbol{L}^p(\Omega)$. If $p < 3/2$, $\boldsymbol{W}^{1,p^*}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$ with $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{3}$ and then supposing $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ implies that the term $(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\Omega)$. So, we can use the well-known regularity of the Stokes problem $(\mathcal{S}_N)$ in order to prove the regularity $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ with $p < 3/2$ for $(\boldsymbol{u}, P)$ since the right hand side $\boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ belongs to $\boldsymbol{L}^p(\Omega)$. Similarly, the term $\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}) = (\boldsymbol{d}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{d}$ belongs to $\boldsymbol{L}^p(\Omega)$ and we can use the regularity of the elliptic problem $(\mathcal{E}_N)$ to prove the regularity $\boldsymbol{W}^{2,p}(\Omega)$ with $p < 3/2$ for $\boldsymbol{b}$. Now, if $3/2 \leq p < 3$, the terms $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ and $(\boldsymbol{d}\cdot\nabla)\boldsymbol{u}$ still belong to $\boldsymbol{L}^p(\Omega)$ but the situation is different for the terms $(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}$ if $\mathbf{curl}\,\boldsymbol{w}$ and $\nabla\boldsymbol{d}$ belong only to $\boldsymbol{L}^{3/2}(\Omega)$.

Indeed, we must suppose $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^s(\Omega)$ and $\nabla \boldsymbol{d} \in \boldsymbol{L}^s(\Omega)$ with

$$s = \frac{3}{2} \quad \text{if } p < \frac{3}{2}, \quad s > \frac{3}{2} \quad \text{if } p = \frac{3}{2} \quad \text{and} \quad s = p \quad \text{if } p > \frac{3}{2}.$$

Now, if $p \geq 3$, the problem arises for the terms $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}$ and $(\boldsymbol{d} \cdot \nabla)\boldsymbol{u}$ if we suppose only $\boldsymbol{d}$ in $\boldsymbol{L}^3(\Omega)$. So, we must suppose that $\boldsymbol{d} \in \boldsymbol{L}^{s'}(\Omega)$ with

$$s' = 3 \quad \text{if } p < 3, \quad s' > 3 \quad \text{if } p = 3 \quad \text{and} \quad s' = p \quad \text{if } p > 3.$$

So, to conserve the assumptions $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{W}^{1,3/2}(\Omega)$ and prove strong solutions $(\boldsymbol{u}, \boldsymbol{b}, P)$ in $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$, we first assume that $\boldsymbol{w}$ and $\boldsymbol{d}$ are more regular and belong to $\boldsymbol{\mathcal{D}}(\overline{\Omega})$. We will then prove a priori estimates allowing to remove this latter regularity. We refer to [12, Theorem 2.4] for a similar proof for the Oseen problem. The details are given in the following regularity result in a solenoidal framework.

Theorem 1.5.1. *Let Ω be $\mathcal{C}^{2,1}$ and $p \geq 6/5$. Assume that $h = 0$, and let $\boldsymbol{f}, \boldsymbol{g}$, $\boldsymbol{w}$, $\boldsymbol{d}$ and P_0 satisfying* (1.4.5)*,*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \;\; \boldsymbol{g} \in \boldsymbol{L}^p(\Omega), \;\; \mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega), \;\; \boldsymbol{d} \in \boldsymbol{W}^{1,3/2}_{\sigma}(\Omega), \;\; \text{and} \;\; P_0 \in W^{1-\frac{1}{p},p}(\Gamma)$$

with the compatibility condition

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}^{p'}_N(\Omega), \quad \int_\Omega \boldsymbol{g} \cdot \boldsymbol{v}\, d\boldsymbol{x} = 0. \tag{1.5.1}$$

Then, the weak solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ of the problem (1.4.2) *given by Theorem 1.4.2 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ which also satisfies the estimate:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}) \\ &\times \big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} \big) \end{aligned} \tag{1.5.2}$$

Proof. We prove it in two steps:

First step: We consider the case of $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$ and $\boldsymbol{d} \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$. We know that for all $p \geq 6/5$ we have

$$\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^{6/5}(\Omega) \quad \text{and} \quad W^{1-1/p,p}(\Gamma) \hookrightarrow H^{1/2}(\Gamma).$$

Thanks to Theorem 1.4.2, there exists a unique solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega) \times \mathbb{R}^I$ verifying the estimates (1.4.13)-(1.4.14).
Since $\boldsymbol{u} \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$ and $\mathbf{curl}\,\boldsymbol{b} \in \boldsymbol{W}^{1,\frac{6}{5}}(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$, it follows that $(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} \in \boldsymbol{L}^6(\Omega)$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^2(\Omega)$. Note that $\boldsymbol{L}^2(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ if $p \leq 2$, then we have three cases:

Case $\frac{6}{5} < p \leqslant 2$: Since $\boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^p(\Omega)$, thanks to the existence of strong solutions for Stokes equations (see Theorem 1.3.2), we have that $(\boldsymbol{u}, P) \in \boldsymbol{W}^{2,p}(\Omega) \times$

$W^{1,p}(\Omega)$. Moreover, we have $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) \in \boldsymbol{L}^p(\Omega)$. Thanks to the regularity of elliptic problem (see Theorem 1.3.3), we have that $\boldsymbol{b} \in \boldsymbol{W}^{2,p}(\Omega)$.

Case $2 \leqslant p \leqslant 6$**:** From the previous case, $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$. Since

$$\boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega),$$

then $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and $(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^6(\Omega)$. Hence we hava that $\boldsymbol{f} - \mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^p(\Omega)$. Again, by Theorem 1.3.2, it follows that $(\boldsymbol{u}, P) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, we have that $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) \in \boldsymbol{L}^6(\Omega)$. Thanks to Theorem 1.3.3, we have that $\boldsymbol{b} \in \boldsymbol{W}^{2,p}(\Omega)$.

Case $p > 6$**:** We know that $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,6}(\Omega) \times \boldsymbol{W}^{2,6}(\Omega) \times W^{1,6}(\Omega)$. Since

$$\boldsymbol{W}^{2,6}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\infty}(\Omega)$$

then $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \in \boldsymbol{L}^q(\Omega)$, $(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^q(\Omega)$ and $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) \in \boldsymbol{L}^q(\Omega)$ for any $q \geq 1$. Again, according to the regularity of Stokes and elliptic problems, we have $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and we have the following estimate:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \\ &\leqslant C_{SE}\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} \\ &+ \sum_{i=1}^{I} |c_i| + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)} \Big), \end{aligned} \tag{1.5.3}$$

where $C_{SE} = \max(C_S, C_E)$ with C_S the constant given in (1.3.9) and C_E the constant given in (1.3.19).

To prove the estimate (1.5.2), we must bound the terms $\|(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$, $\|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$, $\|\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)}$ and $\sum_{i=1}^{I} |c_i|$ in the right hand side of (1.5.3).

For this, let $\epsilon > 0$ and $\rho_{\epsilon/2}$ the classical mollifier. We consider $\widetilde{\boldsymbol{y}} = \widetilde{\mathbf{curl}\, \boldsymbol{w}}$ and $\widetilde{\boldsymbol{d}}$ the extensions by $\mathbf{0}$ of $\boldsymbol{y}$ and $\boldsymbol{d}$ to $\mathbb{R}^3$, respectively. We decompose $\mathbf{curl}\, \boldsymbol{w}$ and $\boldsymbol{d}$:

$$\mathbf{curl}\, \boldsymbol{w} = \boldsymbol{y}_1^\epsilon + \boldsymbol{y}_2^\epsilon \quad \text{where} \quad \boldsymbol{y}_1^\epsilon = \widetilde{\mathbf{curl}\, \boldsymbol{w}} * \rho_{\epsilon/2} \quad \text{and} \quad \boldsymbol{y}_2^\epsilon = \mathbf{curl}\, \boldsymbol{w} - \boldsymbol{y}_1^\epsilon, \tag{1.5.4}$$

$$\boldsymbol{d} = \boldsymbol{d}_1^\epsilon + \boldsymbol{d}_2^\epsilon \quad \text{where} \quad \boldsymbol{d}_1^\epsilon = \widetilde{\boldsymbol{d}} * \rho_{\epsilon/2} \tag{1.5.5}$$

(i) Estimate of the term $\|(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. First, we look for the estimate depending on $\boldsymbol{y}_2^\varepsilon$. Observe that $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^m(\Omega)$ with

$$\frac{1}{m} = \frac{1}{p} - \frac{2}{3} \text{ if } p < 3/2, \quad m = \frac{3s}{2s-3} \in [1, \infty[\text{ if } p = 3/2 \quad \text{and} \quad m = \infty \text{ if } p > 3/2.$$

Using the Hölder inequality and Sobolev embedding, we have

$$\|\boldsymbol{y}_2^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^s(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)} \leq C \|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^s(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}$$

where $\frac{1}{p} = \frac{1}{m} + \frac{1}{s}$ and s the real number defined as:

$$s = \frac{3}{2} \quad \text{if } p < \frac{3}{2}, \quad s > \frac{3}{2} \quad \text{if } p = \frac{3}{2} \quad \text{and} \quad s = p \quad \text{if } p > \frac{3}{2}. \tag{1.5.6}$$

Moreover, we have

$$\|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^s(\Omega)} = \left\|\mathbf{curl}\, \boldsymbol{w} - \widetilde{\mathbf{curl}\, \boldsymbol{w}} * \rho_{\epsilon/2}\right\|_{\boldsymbol{L}^s(\Omega)} \leq \epsilon.$$

Then, it follows that

$$\|\boldsymbol{y}_2^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant C\epsilon \, \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} . \tag{1.5.7}$$

To get the estimate depending on $\boldsymbol{y}_1^\epsilon$, we consider two steps (similar to [14, Theorem 3.5]):

- **Case** $\frac{6}{5} \leq p \leqslant 6$**:** there exists $q \in [\frac{3}{2}, \infty]$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{6}$. By Hölder inequality, we have

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{y}_1^\epsilon\|_{\boldsymbol{L}^q(\Omega)} \, \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} .$$

Let $t \in [1,\, 3]$ such that $1 + \frac{1}{q} = \frac{2}{3} + \frac{1}{t}$, we obtain

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \, \|\rho_\epsilon\|_{\boldsymbol{L}^t(\Omega)} \, \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \leq C_\varepsilon \, \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \, \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} ,$$

where C_ε is the constant absorbing the norm of the mollifier. Since $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, it follows from (1.4.13) that

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_2 C_\epsilon \, \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \big(\, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)} \big),$$

where C_2 is the constant of the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$. Since, $p \geqslant \frac{6}{5}$, we deduce that

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_3 C_\epsilon \, \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} \Big), \tag{1.5.8}$$

where C_3 is a constant which depends on C_2, $\boldsymbol{L}^p(\Omega) \hookrightarrow [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$ and $W^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$.

- **Case** $p > 6$: we know that the embedding

$$\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,m}(\Omega),$$

is compact for any $m \in [1, p^*[$ if $p < 3$, for any $m \in [1,\, \infty[$ if $p = 3$ and for $m \in [1, \infty]$ if $p > 3$. We choose the exponent m such that $6 < m < +\infty$. So, we have:

$$\boldsymbol{W}^{2,p}(\Omega) \underset{\text{compact}}{\hookrightarrow} \boldsymbol{W}^{1,m}(\Omega) \underset{\text{continuous}}{\hookrightarrow} \boldsymbol{L}^6(\Omega).$$

Hence, for any $\varepsilon' > 0$, we know that there exists a constant $C_{\varepsilon'}$ such that the following interpolation inequality holds:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,m}(\Omega)} \leqslant \varepsilon' \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \tag{1.5.9}$$

For $t > 2$ such that $1 + \frac{1}{p} = \frac{2}{3} + \frac{1}{t}$, we have

$$\begin{aligned}\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} &\leqslant C \|\boldsymbol{y}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,m}(\Omega)} \\ &\leqslant C \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\rho_{\epsilon/2}\|_{\boldsymbol{L}^t(\mathbb{R}^3)} \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,m}(\Omega)}.\end{aligned}$$

Using (1.5.9), we obtain

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant CC_\epsilon \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} (\varepsilon' \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}) \tag{1.5.10}$$

Thus, choosing $\varepsilon' > 0$ small enough, we can deduce from (1.5.8) or (1.5.10) that

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant CC_\epsilon \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} (\varepsilon' \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}). \tag{1.5.11}$$

(ii) Estimate of the term $\|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$. Using the decomposition (1.5.5), as previously, we have for the part $\boldsymbol{d}_2^\varepsilon$:

$$\|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^{s'}(\Omega)} \leqslant \left\|\boldsymbol{d} - \widetilde{\boldsymbol{d}} * \rho_{\epsilon/2}\right\|_{\boldsymbol{L}^{s'}(\Omega)} \leq \epsilon. \tag{1.5.12}$$

Recall that $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,k}(\Omega)$ for $k = p^* = \dfrac{3p}{3-p}$ if $p < 3$, for any $k \in [1, \infty[$ if $p = 3$ and $k = \infty$ if $p > 3$. Using the Hölder inequality and (1.5.12), we have

$$\|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^k(\Omega)} \|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^{s'}(\Omega)} \leqslant C\epsilon \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)}, \tag{1.5.13}$$

where $\frac{1}{p} = \frac{1}{k} + \frac{1}{s'}$ for s' given by:

$$s' = 3 \quad \text{if } p < 3, \quad s' > 3 \quad \text{if } p = 3 \quad \text{and} \quad s' = p \quad \text{if } p > 3. \tag{1.5.14}$$

It remains to prove the estimate depending on $\boldsymbol{d}_1^\epsilon$. We have three cases:

- **Case** $p \leq 2$: Using the Hölder inequality, we have

$$\|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^k(\Omega)} \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)},$$

where $\frac{1}{p} = \frac{1}{k} + \frac{1}{2}$. Let $t \in [1, 3/2]$ such that $1 + \frac{1}{k} = \frac{1}{3} + \frac{1}{t}$, we obtain (since $r \geq 3$)

$$\begin{aligned}\|(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} &\leqslant \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\rho_{\epsilon/2}\|_{\boldsymbol{L}^t(\mathbb{R}^3)} \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}.\end{aligned} \tag{1.5.15}$$

where C_4 is a constant which depends on $\boldsymbol{L}^p(\Omega) \hookrightarrow [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$ and $W^{1-\frac{1}{p},p}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$.

- **Case** $2 < p < 3$: Assuming $2 < q < p^*$, from the relation

$$\boldsymbol{W}^{2,p}(\Omega) \underset{\text{compact}}{\hookrightarrow} \boldsymbol{W}^{1,q}(\Omega) \underset{\text{continuous}}{\hookrightarrow} \boldsymbol{H}^1(\Omega)$$

we have for any $\epsilon' > 0$, there exists a constant C_ϵ such that

$$\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,q}(\Omega)} \leqslant \varepsilon' \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \tag{1.5.16}$$

Let k be defined by $\frac{1}{p} = \frac{1}{q} + \frac{1}{k}$ and $t \geqslant 1$ defined by $1 + \frac{1}{k} = \frac{1}{3} + \frac{1}{t}$. Thus, since $k > 3$, the following estimate holds:

$$\|(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^k(\Omega)} \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^q(\Omega)} \leq \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^t(\mathbb{R}^3)} \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^q(\Omega)}.$$

Next using (1.5.16) yields,

$$\|(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leq C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} (\varepsilon' \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}). \tag{1.5.17}$$

- **Case** $p \geq 3$: For $\frac{1}{p} = \frac{1}{s'} + \frac{1}{p*}$ with s' defined in (1.5.14), we have

$$\|(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^{s'}(\Omega)} \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^{p*}(\Omega)}.$$

Let t be defined by $1 + \frac{1}{s'} = \frac{1}{3} + \frac{1}{t}$. Thus, using (1.5.16) with $q = p*$, we obtain:

$$\|(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leq C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} (\varepsilon' \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}). \tag{1.5.18}$$

Choosing $\epsilon' > 0$ small enough, we deduce from (1.5.15), (1.5.17) or (1.5.18) that

$$\|(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leq C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} (\varepsilon' \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}). \tag{1.5.19}$$

(iii) Estimate of the term $\|\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)}$. Note that, since div $\boldsymbol{u} = 0$ and div $\boldsymbol{d} = 0$:

$$\mathbf{curl}\,(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{d} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{d}.$$

- **The term** $\|\boldsymbol{d} \cdot \nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$. Using the decomposition (1.5.5) and exactly the same analysis as in **(ii)** for the term $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}$ with $\mathbf{curl}\,\boldsymbol{b}$ replaced by $\nabla \boldsymbol{u}$, we obtain the following estimates:

$$\|\boldsymbol{d}_2^\epsilon \cdot \nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^k(\Omega)} \|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^{s'}(\Omega)} \leqslant C\epsilon \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}, \tag{1.5.20}$$

where s' is defined in (1.5.14) and

$$\|\boldsymbol{d}_1^\epsilon \cdot \nabla \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} (\varepsilon' \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}). \tag{1.5.21}$$

- **The term** $\|\boldsymbol{u} \cdot \nabla \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$. The analysis is similar to the case **(i)**. We consider:

$$\nabla \boldsymbol{d} = \boldsymbol{z}_1^\epsilon + \boldsymbol{z}_2^\epsilon \qquad \text{where} \qquad \boldsymbol{z}_1^\epsilon = \widetilde{\nabla \boldsymbol{d}} * \rho_{\epsilon/2} \quad \text{and} \quad \boldsymbol{z}_2^\epsilon = \nabla \boldsymbol{d} - \widetilde{\nabla \boldsymbol{d}} * \rho_{\epsilon/2}, \tag{1.5.22}$$

$\widetilde{\nabla \boldsymbol{d}}$ is the extension by zero of $\nabla \boldsymbol{d}$ to $\mathbb{R}^3$. Observe that

$$\|\boldsymbol{z}_2^\epsilon\|_{\boldsymbol{L}^s(\Omega)} \leqslant \|\nabla \boldsymbol{d} - \widetilde{\nabla \boldsymbol{d}} * \rho_{\epsilon/2}\|_{\boldsymbol{L}^s(\Omega)} \leq \epsilon,$$

with s given in (1.5.6). Using the above estimates and the same arguments as in the case **(i)**, the influence of $\boldsymbol{z}_2^\epsilon$ in the bound of $\|\boldsymbol{u}\cdot\nabla\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$ is given by:

$$\|\boldsymbol{u}\cdot\boldsymbol{z}_2^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \le C\,\|\boldsymbol{z}_2^\epsilon\|_{\boldsymbol{L}^s(\Omega)}\,\|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)} \le C\epsilon\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}\,. \tag{1.5.23}$$

And for the bound depending on $\boldsymbol{z}_1^\epsilon$, proceeding in the same way as in the case **(i)**, we derive:

$$\|\boldsymbol{z}_1^\epsilon\cdot\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leqslant CC_\epsilon\,\|\nabla\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\,(\varepsilon'\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_{\varepsilon'}\,\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}) \tag{1.5.24}$$

(iv) Estimate of the constants $\sum_{i=1}^I |c_i|$. We note that

$$\begin{aligned}|c_i| &\leqslant |\int_\Omega \boldsymbol{f}\cdot\nabla q_i^N\,d\boldsymbol{x}| + |\int_\Omega(\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{d})\cdot\nabla q_i^N\,d\boldsymbol{x}| + |\int_\Omega(\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{u})\cdot\nabla q_i^N)\,d\boldsymbol{x}| + |\int_\Gamma P_0\nabla q_i^N\cdot\boldsymbol{n}\,d\sigma|\\
&\leqslant \|\boldsymbol{f}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\,\|\nabla q_i^N\|_{\boldsymbol{L}^6(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\,\|\nabla q_i^N\|_{\boldsymbol{L}^6(\Omega)}\\
&+ \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\,\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\,\|\nabla q_i^N\|_{\boldsymbol{L}^6(\Omega)} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)}\,\|\nabla q_i^N\cdot\boldsymbol{n}\|_{H^{\frac{1}{2}}(\Gamma)}\end{aligned}$$

Thanks to [87, Corollary 3.3.1], we know that the functions ∇q_i^N belong to $\boldsymbol{W}^{1,q}(\Omega)$ for any $q\ge 2$ where each q_i^N is the unique solution of the problem (1.3.2).
Now, the estimate (1.4.13) yields:

$$\sum_{i=1}^I |c_i| \leqslant C(1+\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)})(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}) \tag{1.5.25}$$

Using the embeddings

$$\boldsymbol{W}^{1,3/2}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega),\;\; \boldsymbol{L}^p(\Omega)\hookrightarrow[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]',\;\; W^{1-\frac{1}{p},p}(\Gamma)\hookrightarrow H^{-\frac{1}{2}}(\Gamma),$$

choosing ϵ and ϵ' such that

$$\epsilon' C_{SE}C_\epsilon(\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}) < \frac{1}{2},$$

we deduce from (1.5.7),(1.5.11), (1.5.15), (1.5.19)-(1.5.21), (1.5.23)-(1.5.25), the weak estimate (1.4.13) and the embedding $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$ that the estimate (1.5.2) holds in all cases.

Second step: The case of $\mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d}\in\boldsymbol{W}_\sigma^{1,3/2}(\Omega)$.
Let $\boldsymbol{w}_\lambda\in\boldsymbol{\mathcal{D}}(\Omega)$ and $\boldsymbol{d}_\lambda\in\boldsymbol{\mathcal{D}}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{w}_\lambda\to\mathbf{curl}\,\boldsymbol{w}$ in $\boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d}_\lambda\to\boldsymbol{d}$ in $\boldsymbol{W}^{1,3/2}(\Omega)$.
Consequently, the following problem:

$$\begin{cases} -\Delta\boldsymbol{u}_\lambda + (\mathbf{curl}\,\boldsymbol{w}_\lambda)\times\boldsymbol{u}_\lambda + \nabla\mathrm{P}_\lambda - (\mathbf{curl}\,\boldsymbol{b}_\lambda)\times\boldsymbol{d}_\lambda = \boldsymbol{f} \quad\text{and}\quad \operatorname{div}\boldsymbol{u}_\lambda = 0 \quad\text{in }\Omega,\\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b}_\lambda - \mathbf{curl}(\boldsymbol{u}_\lambda\times\boldsymbol{d}_\lambda) = \boldsymbol{g} \quad\text{and}\quad \operatorname{div}\boldsymbol{b}_\lambda = 0 \quad\text{in }\Omega,\\ \boldsymbol{u}_\lambda\times\boldsymbol{n} = 0 \quad\text{and}\quad \boldsymbol{b}_\lambda\times\boldsymbol{n} = 0 \quad\text{on }\Gamma,\\ \mathrm{P}_\lambda = P_0 \;\text{ on }\Gamma_0 \quad\text{and}\quad \mathrm{P}_\lambda = P_0 + c_i \;\text{ on }\Gamma_i,\\ \langle\boldsymbol{u}_\lambda\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = 0, \quad\text{and}\quad \langle\boldsymbol{b}_\lambda\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = 0, \quad 1\le i\le I.\end{cases}$$

has a unique solution $(\boldsymbol{u}_\lambda,\boldsymbol{b}_\lambda,P_\lambda,\boldsymbol{c}_\lambda)\in\boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)\times\mathbb{R}^I$ and satisfies:

$$\begin{aligned}&\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P_\lambda\|_{W^{1,p}(\Omega)} \le C(1+\|\mathbf{curl}\,\boldsymbol{w}_\lambda\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}_\lambda\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})\\ &\times\big(\,\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\big),\end{aligned} \tag{1.5.26}$$

where C is independent of λ. Finally, these uniform bounds enable us to pass to the limit $\lambda\to 0$. As a consequence, $(\boldsymbol{u}_\lambda,\boldsymbol{b}_\lambda,P_\lambda,\boldsymbol{c}_\lambda)$ converges to $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})$ the solution of the linearized MHD problem (1.4.2) and satisfies the estimate (1.5.2). □

1.5.2 Weak solution in $\boldsymbol{W}^{1,p}(\Omega)$ with $1 < p < +\infty$

In this subsection, we study the regularity $\boldsymbol{W}^{1,p}(\Omega)$ of the weak solution for the linearized MHD problem (1.4.2). We begin with the case $p > 2$. The next theorem will be improved in Corollary 1.5.10 where we consider a data P_0 less regular.

In the following, we denote by $\langle \cdot, \cdot \rangle_{\Omega_{r,p}}$ the duality product between $[\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)]'$ and $\boldsymbol{H}_0^{r,p}(\mathbf{curl}, \Omega)$.

Theorem 1.5.2. *(**Generalized solution in** $\boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$). Suppose that Ω is of class $\mathcal{C}^{1,1}$ and $p > 2$. Assume that $h = 0$, and let $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ and $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ with the compatibility condition*

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \quad \langle \boldsymbol{g}, \boldsymbol{v} \rangle_{\Omega_{r',p'}} = 0, \tag{1.5.27}$$

$$\operatorname{div} \boldsymbol{g} = 0 \quad \text{in}\, \Omega. \tag{1.5.28}$$

and

$$\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^s(\Omega), \;\; \boldsymbol{d} \in \boldsymbol{W}_\sigma^{1,s}(\Omega), \tag{1.5.29}$$

with

$$s = \frac{3}{2} \quad \text{if } 2 < p < 3, \quad s > \frac{3}{2} \quad \text{if } p = 3 \quad \text{and} \quad s = r \quad \text{if } p > 3. \tag{1.5.30}$$

$$r \geq 1 \qquad \text{such that} \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{3}. \tag{1.5.31}$$

Then the linearized MHD problem (1.4.2) has a unique solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the following estimate:

$$\begin{aligned}
&\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big) \\
&\times \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-1/r,r}(\Gamma)} \big).
\end{aligned} \tag{1.5.32}$$

Proof. **A) Existence:** Applying Proposition 1.3.2, there exists a unique solution $(\boldsymbol{u}_1, P_1, \boldsymbol{\alpha}^{(1)}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ solution of the following problem:

$$\begin{cases}
-\Delta \boldsymbol{u}_1 + \nabla P_1 = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_1 = 0 \quad \text{in } \Omega, \\
\boldsymbol{u}_1 \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\
P_1 = P_0 \quad \text{on } \Gamma_0, \;\; P_1 = P_0 + \alpha_i^{(1)} \quad \text{on } \Gamma_i, \\
\langle \boldsymbol{u}_1 \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \;\; \forall 1 \leqslant i \leqslant I
\end{cases}$$

where $\alpha_i^{(1)} = \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} - \int_\Gamma P_0 \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma$ and satisfying the estimate:

$$\|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P_1\|_{W^{1,r}(\Omega)} \leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \Big). \tag{1.5.33}$$

Next, since $\boldsymbol{g}$ satisfies the compatibility conditions (1.5.27)-(1.5.28), due to Lemma 1.3.4, the following problem:

$$\begin{cases} -\Delta \boldsymbol{b}_1 = \boldsymbol{g} \quad \text{and} \quad \operatorname{div} \boldsymbol{b}_1 = 0 \quad \text{in } \Omega, \\ \boldsymbol{b}_1 \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ \langle \boldsymbol{b}_1 \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \quad \forall 1 \leqslant i \leqslant I \end{cases}$$

has a unique solution $\boldsymbol{b}_1 \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{b}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \tag{1.5.34}$$

Then, since $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^s(\Omega)$ and $\boldsymbol{u}_1 \in \boldsymbol{W}^{1,p}(\Omega)$, we have $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_1 \in \boldsymbol{L}^r(\Omega)$. Indeed, if $p < 3$, then $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p*}(\Omega)$ with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$ and $\frac{1}{s} + \frac{1}{p*} = \frac{1}{r}$. If $p = 3$, then there exists $\varepsilon > 0$ such that $\dfrac{1}{\frac{3}{2}+\varepsilon} + \frac{1}{p*} = \frac{2}{3}$. Finally, if $p > 3$, then $p* = \infty$ and $r = s$. Next, since $s \geqslant \frac{3}{2}$, then $\boldsymbol{W}^{1,s}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ so $\boldsymbol{d} \in \boldsymbol{L}^3(\Omega)$, and by the definition of r in (1.5.31), we have $(\mathbf{curl}\, \boldsymbol{b}_1) \times \boldsymbol{d} \in \boldsymbol{L}^r(\Omega)$. Then $\boldsymbol{f}_1 = -(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_1 + (\mathbf{curl}\, \boldsymbol{b}_1) \times \boldsymbol{d}$ belongs to $\boldsymbol{L}^r(\Omega)$. Furthermore, we set $\boldsymbol{g}_1 = \mathbf{curl}(\boldsymbol{u}_1 \times \boldsymbol{d}) = (\boldsymbol{d} \cdot \nabla)\boldsymbol{u}_1 - (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{d}$. By the same way, using the definitions of s and r, we can check that $\boldsymbol{g}_1 \in \boldsymbol{L}^r(\Omega)$. Moreover, $\boldsymbol{g}_1$ satisfies the compatibility conditions (1.5.27)-(1.5.28). Observe that with the values of s given in (1.5.30) for $p > 2$, we have $r \in [\frac{6}{5}, 3)$ and satisfies

$$s = \frac{3}{2} \quad \text{if } \frac{6}{5} < r < \frac{3}{2}, \quad s > \frac{3}{2} \quad \text{if } r = \frac{3}{2} \quad \text{and} \quad s = r \quad \text{if } r > \frac{3}{2}. \tag{1.5.35}$$

So, $s \geq \frac{3}{2}$ and then $\mathbf{curl}\, \boldsymbol{w}$ is at least in $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{d}$ is at least in $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$. We deduce from Theorem 1.5.1 that the following problem:

$$\begin{cases} -\Delta \boldsymbol{u}_2 + (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_2 + \nabla \mathrm{P}_2 - (\mathbf{curl}\, \boldsymbol{b}_2) \times \boldsymbol{d} = \boldsymbol{f}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_2 = 0 \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b}_2 - \mathbf{curl}(\boldsymbol{u}_2 \times \boldsymbol{d}) = \boldsymbol{g}_1 \quad \text{and} \quad \operatorname{div} \boldsymbol{b}_2 = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}_2 \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b}_2 \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ \mathrm{P}_2 = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \mathrm{P}_2 = \alpha_i^{(2)} \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u}_2 \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{b}_2 \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \tag{1.5.36}$$

has a unique solution $(\boldsymbol{u}_2, \boldsymbol{b}_2, P_2, \boldsymbol{\alpha}^{(2)}) \in \boldsymbol{W}^{2,r}(\Omega) \times \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$\begin{aligned} &\|\boldsymbol{u}_2\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\boldsymbol{b}_2\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|P_2\|_{W^{1,r}(\Omega)} \\ &\leqslant C(1 + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})(\|\boldsymbol{f}_1\|_{\boldsymbol{L}^r(\Omega)} + \|\boldsymbol{g}_1\|_{\boldsymbol{L}^r(\Omega)}) \end{aligned} \tag{1.5.37}$$

with

$$\alpha_i^{(2)} = \langle (\mathbf{curl}(\boldsymbol{b}_1 + \boldsymbol{b}_2)) \times \boldsymbol{d}, \nabla q_i^N \rangle_{\Omega_{r',p'}} - \langle (\mathbf{curl}\, \boldsymbol{w}) \times (\boldsymbol{u}_1 + \boldsymbol{u}_2), \nabla q_i^N \rangle_{\Omega_{r',p'}}. \tag{1.5.38}$$

Finally, using the embedding $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$, the solution of the linearized MHD problem (1.4.2) is given by $(\boldsymbol{u}_1+\boldsymbol{u}_2, \boldsymbol{b}_1+\boldsymbol{b}_2, P_1+P_2, \boldsymbol{\alpha}^{(1)}+\boldsymbol{\alpha}^{(2)}) \in \boldsymbol{W}^{1,p}(\Omega)\times \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times \mathbb{R}^I$.

In particular, the constants $c_i = \alpha_i^{(1)} + \alpha_i^{(2)}$ are given by

$$c_i = \langle \boldsymbol{f}, \nabla q_i^N\rangle_{\Omega_{r',p'}} - \int_\Gamma P_0 \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma + \langle (\mathbf{curl}\, \boldsymbol{b})\times \boldsymbol{d}, \nabla q_i^N\rangle_{\Omega_{r',p'}} - \langle (\mathbf{curl}\, \boldsymbol{w})\times \boldsymbol{u}, \nabla q_i^N\rangle_{\Omega_{r',p'}}. \quad (1.5.39)$$

B) Estimates: The terms on $\boldsymbol{f}_1$ and $\boldsymbol{g}_1$ in (1.5.37) can be controlled as:

$$\|\boldsymbol{f}_1\|_{\boldsymbol{L}^r(\Omega)} \le C\big(\|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} \|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{b}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} \big). \quad (1.5.40)$$

$$\|\boldsymbol{g}_1\|_{\boldsymbol{L}^r(\Omega)} \le C\big(\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\nabla \boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)} \|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} \big). \quad (1.5.41)$$

Then, using the above estimates and the embeddings $\boldsymbol{W}^{1,s}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ for $s \ge 3/2$, the estimate (1.5.37) becomes

$$\begin{aligned} &\|\boldsymbol{u}_2\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|\boldsymbol{b}_2\|_{\boldsymbol{W}^{2,r}(\Omega)} + \|P_2\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big) \\ &\times \big(\|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big)(\|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}_1\|_{\boldsymbol{W}^{1,p}(\Omega)}). \end{aligned} \quad (1.5.42)$$

Thanks to (1.5.33), (1.5.34) and (1.5.42), the solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ satisfies

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big)^2 \\ &\times \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-1/r,r}(\Gamma)} \big). \end{aligned} \quad (1.5.43)$$

This estimate is not optimal and can be improved. For this, we will consider $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{1,p}(\Omega)\times \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)$ the solution of (1.4.2) obtained in the existence part.
Note that, due to the hypothesis on $\boldsymbol{d}$ and $\mathbf{curl}\, \boldsymbol{w}$, the terms $(\mathbf{curl}\, \boldsymbol{w})\times \boldsymbol{u}$, $(\mathbf{curl}\, \boldsymbol{b})\times \boldsymbol{d}$ and $\mathbf{curl}(\boldsymbol{u}\times \boldsymbol{d})$ belong to $\boldsymbol{L}^r(\Omega)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Thus, according to the regularity of the Stokes problem $(\mathcal{S}_{\mathcal{N}})$ (see Proposition 1.3.2) and the elliptic problem $(\mathcal{E}_N)$ (see Lemma 1.3.4), we have:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \\ &\leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|(\mathbf{curl}\, \boldsymbol{w})\times \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \\ &+ \|(\mathbf{curl}\, \boldsymbol{b})\times \boldsymbol{d}\|_{\boldsymbol{L}^r(\Omega)} + \|\mathbf{curl}(\boldsymbol{u}\times \boldsymbol{d})\|_{\boldsymbol{L}^r(\Omega)} \Big) \end{aligned} \quad (1.5.44)$$

We proceed in a way similar to the proof of the Theorem 1.5.1: we bound the three last terms of (1.5.44), using the decomposition of $\mathbf{curl}\, \boldsymbol{w}$, $\boldsymbol{d}$ and $\nabla \boldsymbol{d}$ given in (1.5.4)-(1.5.5) and (1.5.22) respectively.

(i) The term $\|(\mathbf{curl}\, \boldsymbol{w})\times \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}$: Using the decomposition (1.5.4) for $\boldsymbol{y} = \mathbf{curl}\, \boldsymbol{w}$, we obtain:

$$\|\boldsymbol{y}_2^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant \|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^s(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^{p^*}(\Omega)} \leqslant C\epsilon \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \quad (1.5.45)$$

where $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ is $p<3$, for $p^* = \frac{3s}{2s-3}$ if $p=3$ and $p^* = \infty$ if $p>3$. Next, for the term $\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}$, let us consider the first the case $p<3$. We have

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant \|\boldsymbol{y}_1^\epsilon\|_{\boldsymbol{L}^t(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)} \leqslant \|\boldsymbol{y}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\rho_{\epsilon/2}\|_{\boldsymbol{L}^k(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)}$$

with $\frac{1}{r}=\frac{1}{t}+\frac{1}{m}$ and $1+\frac{1}{t}=\frac{2}{3}+\frac{1}{k}$. Choosing $6<m<p^*$, the embedding $\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow \boldsymbol{L}^m(\Omega)$ is compact. Following this choice, we have $t\in]\frac{3}{2},\frac{2p}{2+p}[$ and $k\in]1,\frac{6p}{5p+6}[$. Then, for any $\varepsilon'>0$, there exists $C_{\varepsilon'}>0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)}\leq \varepsilon'\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+C_{\varepsilon'}\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}.$$

So, we deduce

$$\|\boldsymbol{y}_1^\epsilon\times\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}\leqslant \varepsilon' C_\varepsilon\|\boldsymbol{y}\|_{\boldsymbol{L}^{3/2}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+C_1C_{\varepsilon'}C_\epsilon\|\boldsymbol{y}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \tag{1.5.46}$$

where C_ϵ is the constant which absorbes the norm of the mollifier and C_1 is the constant of the Sobolev embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$. If $p\geq 3$, we have

$$\|\boldsymbol{y}_1^\epsilon\times\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}\leqslant \|\boldsymbol{y}_1^\epsilon\|_{\boldsymbol{L}^s(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)}\leqslant \|\boldsymbol{y}\|_{\boldsymbol{L}^s(\Omega)}\left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^1(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^m(\Omega)}, \tag{1.5.47}$$

where we choose $m=\infty$ if $p>3$ and $m\in(1,\infty)$ if $p=3$.

(ii) The term $\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\|_{\boldsymbol{L}^r(\Omega)}$: Using the decomposition (1.5.5) for $\boldsymbol{d}$, we have:

$$\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^r(\Omega)}\leqslant\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^3(\Omega)}\leqslant C\epsilon\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \tag{1.5.48}$$

Next, in order to bound the term $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}_1^\epsilon$, we have two cases:

• The case $2<p\leq 6$: we have

$$\begin{aligned}\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^r(\Omega)} &\leqslant \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^t(\Omega)}\leqslant\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^k(\mathbb{R}^3)}\\ &\leqslant C_\epsilon\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}\end{aligned} \tag{1.5.49}$$

with $\frac{1}{r}=\frac{1}{2}+\frac{1}{t}$ and $1+\frac{1}{t}=\frac{1}{3}+\frac{1}{k}$, so we have to take $t=\frac{6p}{6-p}$ and $k=\frac{2p}{2+p}$ which are well-defined.

• The case $p>6$. We have:

$$\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^r(\Omega)}\leqslant\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^q(\Omega)}\|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^t(\Omega)}\leqslant\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^q(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^k(\mathbb{R}^3)} \tag{1.5.50}$$

with $\frac{1}{r}=\frac{1}{q}+\frac{1}{t}$ and $1+\frac{1}{t}=\frac{1}{k}+\frac{1}{3}$. We choose $3<q<p$, and then we have that $t\in]3,p[$ and $k\in]1,\frac{3p}{2p+3}[$. The interpolation estimate of $\boldsymbol{W}^{1,q}(\Omega)$ between $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{W}^{1,p}(\Omega)$ gives (cf. [75]):

$$\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^q(\Omega)}\leqslant\|\boldsymbol{b}\|^\theta_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{b}\|^{1-\theta}_{\boldsymbol{H}^1(\Omega)},$$

with $\theta=\frac{p(q-2)}{q(p-2)}$. Applying the Young inequality, we obtain for small $\epsilon'>0$:

$$\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^q(\Omega)}\leqslant\epsilon'\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}+C_{\epsilon'}\|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}.$$

Replacing this estimate in (1.5.50), we obtain:

$$\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^r(\Omega)}\leqslant\epsilon' C_\epsilon\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}+C_{\epsilon'}C_\epsilon\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}. \tag{1.5.51}$$

(iii) The term $\|\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})\|_{\boldsymbol{L}^r(\Omega)}$: Since $\operatorname{div}\boldsymbol{u}=0$ and $\operatorname{div}\boldsymbol{d}=0$ in Ω, thus we rewrite $\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})=(\boldsymbol{d}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}$.

• **The term $\|(\boldsymbol{d}\cdot\nabla)\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}$:** following the same proof as for the term $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ by replacing $\mathbf{curl}\,\boldsymbol{b}$ with $\nabla\boldsymbol{u}$, we obtain

$$\|(\boldsymbol{d}_2^\epsilon\cdot\nabla)\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}\leqslant C\epsilon\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \tag{1.5.52}$$

and

$$\|(\boldsymbol{d}_1^\epsilon \cdot \nabla)\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \left(\varepsilon' \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + C_{\epsilon'} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\right) \tag{1.5.53}$$

• **The term $\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}\|_{\boldsymbol{L}^r(\Omega)}$:** In the same way, we remark that we can control this term as for $\|(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)}$ by replacing $\mathbf{curl}\,\boldsymbol{w}$ with $\nabla\boldsymbol{d}$. Applying the decomposition (1.5.22) for $\nabla\boldsymbol{d}$, we thus prove that:

$$\|\boldsymbol{z}_2^\epsilon \cdot \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant C\epsilon \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \tag{1.5.54}$$

and

$$\|\boldsymbol{z}_1^\epsilon \cdot \boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant CC_\epsilon \|\nabla\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)} \left(\varepsilon' \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + C_{\varepsilon'} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\right) \tag{1.5.55}$$

Finally, taking the estimates (1.5.45)-(1.5.55) together with the embedding $\boldsymbol{W}^{1,s}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ for $s \geq 3/2$ and (1.5.44), then choosing $\epsilon, \epsilon' > 0$ small enough and using the estimate (1.4.13), we thus obtain (1.5.32). □

Remark 1.5.1.
(i) *The case $p > 2$ can be analyzed in a similar way to the case $p = 2$ to prove that the space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ is optimal to obtain the regularity $\boldsymbol{W}^{1,p}(\Omega)$.*
(ii) *Why do we take $P_0 \in W^{1-1/r,r}(\Gamma)$ instead of $W^{-1/p,p}(\Gamma)$? If we take $P_0 \in W^{-1/p,p}(\Gamma)$, we obtain that $P \in L^p(\Omega)$ as in the classic case of Navier-Stokes equations with Dirichlet boundary conditions. But we are not able to solve the Stokes problem $(\mathcal{S}_\mathcal{N})$ because, in this case, $\boldsymbol{f} = \mathbf{curl}(\mathbf{curl}\,\boldsymbol{u}) + \nabla P \notin [\boldsymbol{H}^{r',p'}(\mathbf{curl},\Omega)]'$.*

We also need to study the case where the divergence is not free for the velocity field. The following problem appears as the dual problem associated to the linearized MHD problem (1.4.2) in the study of weak solutions for $p < 2$:

$$\begin{cases} -\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u} + \nabla \mathrm{P} - (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}) + \nabla\chi = \boldsymbol{g} \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma, \\ \mathrm{P} = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \mathrm{P} = P_0 + c_i \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{b}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0. \end{cases} \tag{1.5.56}$$

Observe that the second equation in (1.4.2) is replaced by $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}) + \nabla\chi = \boldsymbol{g}$ in Ω with $\chi = 0$ on Γ. The scalar χ represents the Lagrange multiplier associated with magnetic divergence constraint. Note that, taking the divergence in the above equation, χ is a solution of the following Dirichlet problem:

$$\Delta\chi = \operatorname{div}\boldsymbol{g} \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma. \tag{1.5.57}$$

In particular, if $\boldsymbol{g}$ is divergence-free, we have $\chi = 0$. Nevertheless, the introduction of χ will be useful to enforce zero divergence condition over the magnetic field. First, we give the following result for the case $h = 0$.

Corollary 1.5.3. *Suppose that $p \geqslant 2$ and $h = 0$. Let $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ with the compatibility condition* (1.5.27) *and $\boldsymbol{w}$, $\boldsymbol{d}$ defined with* (1.5.29)-(1.5.31). *Then the problem* (1.5.56) *has a unique solution $(\boldsymbol{u}, \boldsymbol{b}, P, \chi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ where $\boldsymbol{c} = (c_1, \ldots, c_I)$ is given by* (1.5.39). *Moreover, we have the following estimates:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big) \\ &\times \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}\big). \end{aligned} \tag{1.5.58}$$

$$\|\chi\|_{W^{1,r}(\Omega)} \leq C\,\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \tag{1.5.59}$$

Proof. As mentioned before, the scalar χ can be found directly as a solution of the Dirchlet problem (1.5.57). Since $\boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $\operatorname{div} \boldsymbol{g} \in W^{-1,r}(\Omega)$ and then χ belongs to $W^{1,r}(\Omega)$ and satisfies the estimate (1.5.59). We set $\boldsymbol{g}' = \boldsymbol{g} - \nabla\chi$. It is clear that $\boldsymbol{g}'$ is an element of the dual space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Moreover, it is clear that div $\boldsymbol{g}' = 0$ in Ω and $\boldsymbol{g}'$ satisfies the compatibility conditions (1.5.27). So, problem (1.5.56) becomes

$$\begin{cases} -\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u} + \nabla \mathrm{P} - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g}' \quad \text{and} \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ \mathrm{P} = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \mathrm{P} = P_0 + c_i \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{b}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0. \end{cases} \tag{1.5.60}$$

Thanks to Theorem 1.5.2, problem (1.5.60) has a unique solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c})$ in $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big) \\ &\times \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}'\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}\big). \end{aligned} \tag{1.5.61}$$

Using (1.5.59), the previous estimate still holds when $\boldsymbol{g}'$ is replaced by $\boldsymbol{g}$. □

The next theorem gives a generalization for the case $h \neq 0$.

Theorem 1.5.4. *Suppose that $p \geqslant 2$. Let $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ and $h \in W^{1,r}(\Omega)$ with the compatibility condition* (1.5.27) *and $\boldsymbol{w}$, $\boldsymbol{d}$ defined with* (1.5.29)-(1.5.31). *Then the problem* (1.5.56) *has a unique solution $(\boldsymbol{u}, \boldsymbol{b}, P, \chi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the following estimate for $(\boldsymbol{u}, \boldsymbol{b}, P)$:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big)^2 \\ &\times \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}\big). \end{aligned} \tag{1.5.62}$$

Proof. The idea is to lift the data h by using the Stokes problem:

$$\begin{cases} -\Delta\boldsymbol{u}_1 + \nabla P_1 = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_1 = h \quad \text{in } \Omega, \\ \boldsymbol{u}_1 \times \boldsymbol{n} = 0 \quad \text{on } \Gamma, \\ P_1 = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P_1 = P_0 + \alpha_i^{(1)} \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u}_1\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ \forall 1 \leqslant i \leqslant I \end{cases}$$

Thanks to Proposition 1.3.2, there exists a unique solution $(\boldsymbol{u}_1, P_1, \boldsymbol{\alpha}^{(1)}) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times \mathbb{R}^I$ satisfying the estimate:

$$\|\boldsymbol{u}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P_1\|_{W^{1,r}(\Omega)} \leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \Big) \tag{1.5.63}$$

where $\alpha_i^{(1)} = \langle \boldsymbol{f}, \nabla q_i^N\rangle_\Omega + \displaystyle\int_\Gamma (h - P_0)\, \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma.$

Next, since $\boldsymbol{g}$ satisfies the compatibility condition (1.5.27), due to [15, Theorem 5.2], the following problem:

$$\begin{cases} -\Delta \boldsymbol{b}_1 + \nabla\chi = \boldsymbol{g} \quad \text{and} \quad \operatorname{div} \boldsymbol{b}_1 = 0 \quad \text{in } \Omega, \\ \boldsymbol{b}_1 \times \boldsymbol{n} = 0 \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma, \\ \langle \boldsymbol{b}_1\cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0 \quad \forall 1 \leqslant i \leqslant I \end{cases}$$

has a unique solution $(\boldsymbol{b}_1, \chi) \in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{b}_1\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\chi\|_{W^{1,r}(\Omega)} \leqslant C\, \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \tag{1.5.64}$$

Finally, we consider $(\boldsymbol{u}_2,\, \boldsymbol{b}_2,\, P_2,\, \boldsymbol{\alpha}^{(2)}) \in \boldsymbol{W}^{2,r}(\Omega)\times \boldsymbol{W}^{2,r}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ the solution of (1.5.36) satisfying (1.5.38) and (1.5.42). Therefore, $(\boldsymbol{u}_1+\boldsymbol{u}_2,\, \boldsymbol{b}_1+\boldsymbol{b}_2, P_1+P_2, \chi, \boldsymbol{\alpha}^{(1)}+\boldsymbol{\alpha}^{(2)})$ is the solution of (1.5.56). Estimate (1.5.62) follows from (1.5.63),(1.5.64) and (1.5.42). □

Note that the estimate (1.5.62) is not optimal and will be improved in the next result.

Proposition 1.5.5. *Under the assumptions of Theorem 1.5.4, the problem* (1.5.56) *has a unique solution* $(\boldsymbol{u}, \boldsymbol{b}, P, \chi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega)\times \boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ *satisfying* (1.5.59) *and the following estimate:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \\ \leqslant\ & C\big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big)\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \\ +\ \ & \|P_0\|_{W^{1-1/r,r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)}\big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big)\Big). \end{aligned} \tag{1.5.65}$$

Proof. We can reduce the non vanishing divergence problem (1.5.56) for the velocity to the case where $\operatorname{div}\boldsymbol{u} = 0$ in Ω, by solving the following Dirichlet problem:

$$\Delta\theta = h \quad \text{in } \Omega \quad \text{and} \quad \theta = 0 \quad \text{on } \Gamma \tag{1.5.66}$$

For $h \in W^{1,r}(\Omega)$, problem (1.5.66) has a unique solution $\theta \in W^{3,r}(\Omega) \hookrightarrow W^{2,p}(\Omega)$ satisfying the following estimate:

$$\|\theta\|_{W^{2,p}(\Omega)} \leqslant C\, \|h\|_{W^{1,r}(\Omega)} \tag{1.5.67}$$

Setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, then (1.5.56) becomes: Find $(\boldsymbol{z}, \boldsymbol{b}, P, \chi, \boldsymbol{c})$ solution of problem:

$$\begin{cases} -\Delta \boldsymbol{z} + (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{z} + \nabla P - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} + \nabla h - (\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta \quad \text{in } \Omega \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{z} \times \boldsymbol{d}) + \nabla\chi = \boldsymbol{g} + \mathbf{curl}(\nabla\theta \times \boldsymbol{d}) \quad \text{in } \Omega \\ \operatorname{div} \boldsymbol{z} = 0, \quad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma \\ P = P_0 \quad \text{on } \Gamma_0, \quad P = P_0 + c_i \quad \text{on } \Gamma_i \\ \langle \boldsymbol{z} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases} \tag{1.5.68}$$

which is a problem treated in the proof of Corollary 1.5.3. Since $\nabla\theta \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p*}(\Omega)$, by using the definition of s in (1.5.6), we have $(\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta \in \boldsymbol{L}^r(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ if $p < 3$, $p^* = \frac{rs}{s-r}$ if $p = 3$ and $p^* = \infty$ if $p > 3$. So $\boldsymbol{f} + \nabla h - (\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Now, we consider the term $\mathbf{curl}(\nabla\theta \times \boldsymbol{d}) = \boldsymbol{d} \cdot \nabla\nabla\theta - \nabla\theta \cdot \nabla\boldsymbol{d}$. Since $\boldsymbol{d} \in \boldsymbol{W}^{1,s}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ $(s \geqslant 3/2)$, then $\boldsymbol{d} \cdot \nabla\nabla\theta$ belongs to $\boldsymbol{L}^r(\Omega)$. Moreover, since $\nabla\boldsymbol{d} \in \boldsymbol{L}^s(\Omega)$, using the same arguments for the term $(\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta$, we deduce that $\nabla\theta \cdot \nabla\boldsymbol{d}$ belongs to $\boldsymbol{L}^r(\Omega)$. So, $\boldsymbol{g} + \mathbf{curl}(\nabla\theta \times \boldsymbol{d}) \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ and satisfies (1.5.27). Thanks to Theorem 1.5.2, there exists a unique solution $(\boldsymbol{z}, \boldsymbol{b}, P, \chi, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying (1.5.59) and

$$\begin{aligned} &\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big) \\ &\times \big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\nabla h\|_{\boldsymbol{L}^r(\Omega)} + \|(\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta\|_{\boldsymbol{L}^r(\Omega)} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} \\ &+ \|\mathbf{curl}(\nabla\theta \times \boldsymbol{d})\|_{\boldsymbol{L}^r(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)} \Big). \end{aligned} \tag{1.5.69}$$

with

$$\begin{aligned} c_i = {} & \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} + \int_\Gamma (h - P_0) \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma - \langle (\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta, \nabla q_i^N \rangle_{\Omega_{r',p'}} \\ & - \langle (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{z}, \nabla q_i^N \rangle_{\Omega_{r',p'}} + \langle (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}, \nabla q_i^N \rangle_{\Omega_{r',p'}} \end{aligned}$$

To bound the terms $\|(\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta\|_{\boldsymbol{L}^r(\Omega)}$ and $\|\mathbf{curl}(\nabla\theta \times \boldsymbol{d})\|_{\boldsymbol{L}^r(\Omega)}$ in (1.5.69), we write by using (1.5.67)

$$\begin{aligned} \|(\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta\|_{\boldsymbol{L}^r(\Omega)} &\leq \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} \|\nabla\theta\|_{\boldsymbol{L}^{p^*}(\Omega)} \leq \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} \|\nabla\theta\|_{\boldsymbol{W}^{1,p}(\Omega)} \\ &\leq C \quad \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} \|h\|_{W^{1,r}(\Omega)} . \end{aligned} \tag{1.5.70}$$

In addition we have

$$\begin{aligned} \|\mathbf{curl}(\nabla\theta \times \boldsymbol{d})\|_{\boldsymbol{L}^r(\Omega)} &\leq \|\boldsymbol{d} \cdot \nabla\nabla\theta\|_{\boldsymbol{L}^r(\Omega)} + \|\nabla\boldsymbol{d} \cdot \nabla\theta\|_{\boldsymbol{L}^r(\Omega)} \\ &\leq \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\nabla\nabla\theta\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\boldsymbol{d}\|_{\boldsymbol{L}^s(\Omega)} \|\nabla\theta\|_{\boldsymbol{L}^{p^*}(\Omega)} \\ &\leq C \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)} \|h\|_{W^{1,r}(\Omega)} . \end{aligned} \tag{1.5.71}$$

Now plugging the estimates (1.5.70) and (1.5.71) in (1.5.69) gives

$$\begin{aligned} &\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leqslant C\big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big) \\ &\times \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-1/r,r}(\Gamma)} \\ &+ \|h\|_{W^{1,r}(\Omega)} \big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^s(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,s}(\Omega)}\big)\Big). \end{aligned} \tag{1.5.72}$$

Thus, summing the resulting estimate (1.5.72) along with estimate (1.5.67), we get the bound for $(\boldsymbol{u}, \boldsymbol{b}, P)$ in (1.5.65). $\square$

We are interested now on the existence of solution for the linearized problem (1.4.2) in $\boldsymbol{W}^{1,p}(\Omega)$ with $p < 2$. Since the problem is linear, we will use a duality argument developed by Lions-Magenes [66]. This way ensures the uniqueness of solutions. For this, we must derive that problem (1.4.2) has an equivalent variational formulation. We then need adequate density lemma and Green formulae, adapted to our functional framework, to define rigorously each term. We introduce the space

$$\boldsymbol{\mathcal{V}}(\Omega) := \Big\{(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau) \in \boldsymbol{W}^{1,p'}(\Omega) \times \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,(p^*)'}(\Omega) \times W_0^{1,(p^*)'}(\Omega);\ \operatorname{div} \boldsymbol{v} \in W_0^{1,(p^*)'}(\Omega),$$
$$\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma,\ \theta = 0, \text{ on } \Gamma_0 \text{ and } \theta = cste \text{ on } \Gamma_i,\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = \langle \boldsymbol{a} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0,\ \forall 1 \leqslant i \leqslant I\Big\}$$

and we recall that $\langle \cdot, \cdot \rangle_{\Omega_{p^*,p}}$ denotes the duality between $\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)$ and $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$.

Lemma 1.5.6. *We suppose Ω of class $\mathcal{C}^{1,1}$. Let $\frac{3}{2} < p < 2$. Assume that $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $h = 0$ and $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ satisfying the compatibility conditions* (1.5.27)-(1.5.28)*, together with $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{W}_\sigma^{1,3/2}(\Omega)$. Then, the following two problems are equivalent:*
(1). *$(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfies the linearized problem* (1.4.2)*.*
(2). *Find $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}_\sigma^{1,p}(\Omega) \times \boldsymbol{W}_\sigma^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ with $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ and $\langle \boldsymbol{b} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for any $1 \leqslant i \leqslant I$ such that:*
For any $(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau) \in \boldsymbol{\mathcal{V}}(\Omega)$,

$$\langle \boldsymbol{u}, -\Delta \boldsymbol{v} - (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{v} + (\mathbf{curl}\, \boldsymbol{a}) \times \boldsymbol{d} + \nabla\theta \rangle_{\Omega_{p^*,p}} - \int_\Omega P \operatorname{div} \boldsymbol{v}\, dx$$
$$+ \langle \boldsymbol{b}, \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{a} + \mathbf{curl}(\boldsymbol{v} \times \boldsymbol{d}) + \nabla\tau \rangle_{\Omega_{p^*,p}} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega_{r',p'}} + \langle \boldsymbol{g}, \boldsymbol{a} \rangle_{\Omega_{r',p'}} - \int_\Gamma P_0 \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma, \quad (1.5.73)$$

$$c_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} - \int_\Gamma P_0 \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma + \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N\, dx - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx \quad (1.5.74)$$

Proof. ***(1) $\Rightarrow$ (2)*** Let $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}_\sigma^{1,p}(\Omega) \times \boldsymbol{W}_\sigma^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ solution of the linearized problem (1.4.2). Let us take $(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau) \in \boldsymbol{\mathcal{V}}(\Omega)$. We want to multiply the system (1.4.2) by $(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau)$ and integrate by parts. Let us study these terms one by one.
Firstly, we note that the duality pairing $\langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\Omega_{r',p'}}$ is well defined. Indeed, since $-\Delta \boldsymbol{u} = \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u}$ and $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, it follows that $-\Delta \boldsymbol{u} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Besides, recall that $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega)$, so $\mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$, and since

$$\frac{1}{r'} = 1 - \frac{1}{r} = 1 - \frac{3+p}{3p} = \frac{1}{p'} - \frac{1}{3}, \quad (1.5.75)$$

then we have $\boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{r'}(\Omega)$. Hence $\boldsymbol{v} \in \boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)$. Now, by the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)$, we have

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{v} \rangle_{\Omega_{r',p'}} = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, dx = \langle \boldsymbol{u}, \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v} \rangle_{\Omega_{p^*,p}} \quad (1.5.76)$$

The last duality pairing is again well defined: the embedding $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$ implies $\boldsymbol{u} \in \boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)$, and since $\mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$ then $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v} \in [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$. Thus, due to the relation $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v} = -\Delta \boldsymbol{v} + \nabla \operatorname{div} \boldsymbol{v}$, we deduce that:

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{v}\rangle_{\Omega r',p'} = \langle \boldsymbol{u}, -\Delta \boldsymbol{v} + \nabla \operatorname{div} \boldsymbol{v}\rangle_{\Omega p^*,p}$$

Observe that since $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$, we have $\operatorname{div} \boldsymbol{v} \in \boldsymbol{W}_0^{1,(p^*)'}(\Omega)$, and it follows that $\nabla \operatorname{div} \boldsymbol{v} \in \boldsymbol{L}^{(p^*)'}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$. Therefore, since $\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v}$ belongs to $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$, we deduce that $\Delta \boldsymbol{v}$ also belongs to $[\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)]'$. This proves that the last duality makes sense. Next, since $\operatorname{div} \boldsymbol{v} = 0$ on Γ and $\operatorname{div} \boldsymbol{u} = 0$ in Ω, we have

$$\langle \boldsymbol{u}, \nabla \operatorname{div} \boldsymbol{v}\rangle_{\Omega p^*,p} = \int_\Omega \boldsymbol{u} \cdot \nabla \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} = -\int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} + \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{n} \operatorname{div} \boldsymbol{v}\, d\sigma = 0 \quad .$$

We conclude that

$$\langle -\Delta \boldsymbol{u}, \boldsymbol{v}\rangle_{\Omega r',p'} = \langle \boldsymbol{u}, -\Delta \boldsymbol{v}\rangle_{\Omega p^*,p}$$

We now treat the term $\langle (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}, \boldsymbol{v}\rangle_{\Omega r',p'}$. Since we have $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$, then, by definition of r, $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \in \boldsymbol{L}^r(\Omega)$. Besides, $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega) \hookrightarrow \boldsymbol{L}^{(p')^*}(\Omega)$. So

$$\langle (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}, \boldsymbol{v}\rangle_{\Omega r',p'} = \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \boldsymbol{v}\, dx = -\int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{v} \cdot \boldsymbol{u}\, dx$$

with the integral well defined thanks to (1.5.75) and

$$\frac{1}{r} + \frac{1}{(p')^*} = \frac{1}{r} + \frac{1}{p'} - \frac{1}{3} = \frac{1}{r} + \frac{1}{r'} = 1.$$

For the term $\langle -(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}, \boldsymbol{v}\rangle_{\Omega_{r',p'}}$, we proceed as for the previous pairing: we have $\mathbf{curl}\, \boldsymbol{b} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$, so $(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \in \boldsymbol{L}^r(\Omega)$. Therefore:

$$\langle -(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}, \boldsymbol{v}\rangle_{\Omega r',p'} = -\int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \boldsymbol{v}\, dx = -\int_\Omega \mathbf{curl}\, \boldsymbol{b} \cdot (\boldsymbol{d} \times \boldsymbol{v})\, dx$$

Using again the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}_0^{p^*,p}(\mathbf{curl}, \Omega)$, we obtain

$$-\int_\Omega \mathbf{curl}\, \boldsymbol{b} \cdot (\boldsymbol{d} \times \boldsymbol{v})\, dx = \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\boldsymbol{v} \times \boldsymbol{d})\, dx.$$

It remains us to treat the pressure term. Since $\nabla P \in \boldsymbol{L}^r(\Omega)$, we have as for the both previous terms the well defined of the integral:

$$\langle \nabla P, \boldsymbol{v}\rangle_{\Omega r',p'} = \int_\Omega \nabla P \cdot \boldsymbol{v}\, dx$$

Using the same arguments as in [14, Proposition 3.7], and taking into account the boundary conditions on the pressure P, we have

$$\begin{aligned}
\langle \nabla P, \boldsymbol{v}\rangle_{\Omega r',p'} &= \int_\Gamma P\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma - \int_\Omega P \operatorname{div} \boldsymbol{v}\, dx \\
&= \int_{\Gamma_0} P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma + \sum_{i=1}^{I} \int_{\Gamma_i} (P_0 + c_i)\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma - \int_\Omega P \operatorname{div} \boldsymbol{v}\, dx
\end{aligned}$$

However, since $\langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}=0$ for all $1\leqslant i\leqslant I$, we have $\displaystyle\sum_{i=1}^{I}\int_{\Gamma_i} c_i\,\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma=0$. Therefore, we obtain:

$$\langle \nabla P,\boldsymbol{v}\rangle_{\Omega r',p'}=\int_\Gamma P_0\,\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma-\int_\Omega P\ \mathrm{div}\,\boldsymbol{v}\,dx$$

Now, multiplying the equation $\mathrm{div}\,\boldsymbol{u}=0$ in Ω by θ, we obtain by the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{H}^{p^*,p}(\mathrm{div},\Omega)$

$$0=-\int_\Omega \theta\,\mathrm{div}\,\boldsymbol{u}\,dx=\int_\Omega \boldsymbol{u}\cdot\nabla\theta\,dx-\int_\Gamma \theta\,\boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma,$$

where we have used the fact that $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow\boldsymbol{L}^{p*}(\Omega)$ which implies that $\boldsymbol{u}\in\boldsymbol{H}^{p^*,p}(\mathrm{div},\Omega)$. Combining the boundary conditions of θ on Γ_i, $0\leqslant i\leqslant I$, with zero fluxs of the velocity $\langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}=0$ for $1\leqslant i\leqslant I$, we have: $\displaystyle\int_\Gamma \theta\,\boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma=0$. Thus, summing the above resulting terms, we obtain:

$$\begin{aligned}\langle \boldsymbol{f},\boldsymbol{v}\rangle_{\Omega r',p'}=\langle \boldsymbol{u},\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}\rangle_{\Omega p^*,p}&-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{v}\cdot\boldsymbol{u}\,dx-\int_\Omega \mathbf{curl}(\boldsymbol{d}\times\boldsymbol{v})\cdot\boldsymbol{b}\,dx\\&-\int_\Omega P\,\mathrm{div}\,\boldsymbol{v}\,dx+\int_\Gamma P_0\,\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma+\int_\Omega \boldsymbol{u}\cdot\nabla\theta\,dx.\end{aligned}\tag{1.5.77}$$

Now, we treat the terms of the second equation of (1.4.2). For the term $\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b},\boldsymbol{a}\rangle_{\Omega r',p'}$, the duality pairing is well defined and following the same reasoning than for (1.5.76), we have

$$\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b},\boldsymbol{a}\rangle_{\Omega_{r',p'}}=\int_\Omega \mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{a}\,dx=\langle \boldsymbol{b},\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{a}\rangle_{\Omega_{p^*,p}}$$

Next, for the term $\langle \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}),\boldsymbol{a}\rangle_{\Omega r',p'}$, the duality pairing is again well defined: since $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow\boldsymbol{L}^{p^*}(\Omega)$ and $\boldsymbol{d}\in\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$, then $\boldsymbol{u}\times\boldsymbol{d}\in\boldsymbol{L}^p(\Omega)$ so $\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$. Similarly, by the density of $\boldsymbol{\mathcal{D}}(\Omega)$ in $\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)$, we have

$$\langle \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d}),\boldsymbol{a}\rangle_{\Omega r',p'}=\int_\Omega \mathbf{curl}\,\boldsymbol{a}\cdot(\boldsymbol{u}\times\boldsymbol{d})\,dx=\int_\Omega \boldsymbol{u}\cdot(\boldsymbol{d}\times\mathbf{curl}\,\boldsymbol{a})\,dx$$

Here also the integrals are well defined. Indeed, for the first integral, $\mathbf{curl}\,\boldsymbol{a}\in\boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{u}\in\boldsymbol{L}^{p^*}(\Omega)$, $\boldsymbol{d}\in\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$ and $\frac{1}{p'}+\frac{1}{p^*}+\frac{1}{3}=1$.

It remains us to multiply the equation $\mathrm{div}\ \boldsymbol{b}=0$ in Ω by τ. By density of $\mathcal{D}(\Omega)$ in $W_0^{1,(p^*)'}(\Omega)$, we have the Green formula: for any $\tau\in W_0^{1,(p^*)'}(\Omega)$

$$-\int_\Omega(\mathrm{div}\,\boldsymbol{b})\,\tau\,dx=\int_\Omega \boldsymbol{b}\cdot\nabla\tau\,dx$$

In summary, these terms provided by the second equation of (1.4.2) give:

$$\langle \boldsymbol{g},\boldsymbol{a}\rangle_{\Omega r',p'}=\langle \boldsymbol{b},\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{a}\rangle_{\Omega p^*,p}-\int_\Omega \boldsymbol{u}\cdot(\boldsymbol{d}\times\mathbf{curl}\,\boldsymbol{a})\,dx+\int_\Omega \boldsymbol{b}\cdot\nabla\tau\,dx\tag{1.5.78}$$

Finally, adding (1.5.77) and (1.5.78), we obtain the variational formulation (1.5.73).

We now want to determine the constants c_i in (1.5.74). Let us take $\boldsymbol{v}\in\boldsymbol{W}_\sigma^{1,p}(\Omega)$ with $\boldsymbol{v}\times\boldsymbol{n}=0$ on Γ, and set:

$$\boldsymbol{v}_0=\boldsymbol{v}-\sum_{i=1}^{I}\Big(\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma\Big)\nabla q_i^N\tag{1.5.79}$$

So, $\boldsymbol{v}_0$ belongs to $\boldsymbol{W}^{1,p}_\sigma(\Omega)$ and satisfies $\boldsymbol{v}\times\boldsymbol{n}=0$ on Γ, $\langle \boldsymbol{v}_0\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}=0$, $1\le i\le I$. Multiplying the first equation of (1.4.2) by $\boldsymbol{v}$ and integrating by parts, we obtain

$$\begin{aligned}&\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{v}\cdot\boldsymbol{u}\,dx+\int_\Omega \mathbf{curl}(\boldsymbol{v}\times\boldsymbol{d})\cdot\boldsymbol{b}\,dx\\&+\int_{\Gamma_0}P_0\,\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma+\sum_{i=1}^I\int_{\Gamma_i}(P_0+c_i)\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,dx\end{aligned}\tag{1.5.80}$$

We now take a test function $(\boldsymbol{v}_0,0,0,0)$ in (1.5.73). Note that it is possible because of the definition of (1.5.79):

$$\begin{aligned}&\langle\boldsymbol{u},-\Delta\boldsymbol{v}_0\rangle_{\Omega p^*,p}-\int_\Omega\boldsymbol{u}\cdot(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{v}_0\,dx-\int_\Omega P\,\mathrm{div}\,\boldsymbol{v}_0\,dx+\int_\Omega\boldsymbol{b}\cdot\mathbf{curl}(\boldsymbol{v}_0\times\boldsymbol{d})\,dx\\&=\langle\boldsymbol{f},\boldsymbol{v}_0\rangle_{\Omega r',p'}-\int_\Gamma P_0\,\boldsymbol{v}_0\cdot\boldsymbol{n}\,d\sigma\end{aligned}$$

By definition of q_i^N, we have $\mathrm{div}\,\boldsymbol{v}_0=0$. Besides, $\mathbf{curl}\,\boldsymbol{v}_0=\mathbf{curl}\,\boldsymbol{v}$, so it follows from the same density argument used previously:

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{v}_0\cdot\boldsymbol{u}\,dx+\int_\Omega\boldsymbol{b}\cdot\mathbf{curl}(\boldsymbol{v}_0\times\boldsymbol{d})\,dx=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}_0\,dx-\int_\Gamma P_0\,\boldsymbol{v}_0\cdot\boldsymbol{n}\,d\sigma$$

Decomposing $\boldsymbol{v}_0$ with (1.5.79) in the previous equality, we have:

$$\begin{aligned}&\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,dx-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{v}\cdot\boldsymbol{u}+\int_\Omega\boldsymbol{b}\cdot\mathbf{curl}(\boldsymbol{v}\times\boldsymbol{d})\,dx-\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,dx+\int_\Gamma P_0\,\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma\\&=\sum_{i=1}^I\Big(\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma\Big)\Big[-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\nabla q_i^N\cdot\boldsymbol{u}\,dx+\int_\Omega\boldsymbol{b}\cdot\mathbf{curl}(\nabla q_i^N\times\boldsymbol{d})\,dx-\int_\Omega\boldsymbol{f}\cdot\nabla q_i^N\,dx+\int_\Gamma P_0\,\nabla q_i^N\cdot\boldsymbol{n}\,d\sigma\Big]\end{aligned}$$

Injecting (1.5.80) in this calculus, we thus obtain:

$$\begin{aligned}-\sum_{i=1}^I c_i\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma\quad&=\quad\sum_{i=1}^I\Big(\int_{\Gamma_i}\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma\Big)\Big[-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\nabla q_i^N\cdot\boldsymbol{u}\,dx+\int_\Omega\boldsymbol{b}\cdot\mathbf{curl}(\nabla q_i^N\times\boldsymbol{d})\,dx\\&\quad-\quad\int_\Omega\boldsymbol{f}\cdot\nabla q_i^N\,dx+\int_\Gamma P_0\,\nabla q_i^N\cdot\boldsymbol{n}\,d\sigma\Big]\end{aligned}$$

Therefore, taking $\boldsymbol{v}=\nabla q_i^N$ and since, for all $1\leqslant i,k\leqslant I$, $\langle\nabla q_i^N\cdot\boldsymbol{n},1\rangle_{\Gamma_k}=\delta_{i,k}$, we have:

$$c_i=-\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\cdot\nabla q_i^N\,dx-\underbrace{\int_\Omega\boldsymbol{b}\cdot\mathbf{curl}(\nabla q_i^N\times\boldsymbol{d})\,dx}_{\int_\Omega(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\cdot\nabla q_i^N\,dx}+\int_\Omega\boldsymbol{f}\cdot\nabla q_i^N\,dx-\int_\Gamma P_0\,\nabla q_i^N\cdot\boldsymbol{n}\,d\sigma$$

which gives the relation (1.5.74).

(2) ⇒ (1) Conversely, let $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})\in\boldsymbol{W}^{1,p}_\sigma(\Omega)\times\boldsymbol{W}^{1,p}_\sigma(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$ solution of (1.5.73)-(1.5.74) with $\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{b}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, and $\langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=\langle\boldsymbol{b}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$ for all $1\leqslant i\leqslant I$. We want to prove that $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})$ satisfies (1.4.2). Let us take $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega)$, $\boldsymbol{a}=\boldsymbol{0}$, $\theta=\tau=0$ as test functions in (1.5.73). We obtain

$$\langle-\Delta\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}-(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}-\boldsymbol{f},\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0,\quad\forall\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega)$$

So by De Rham's theorem, there exists $P \in \mathrm{L}^p(\Omega)$ such that

$$-\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} + \nabla P = \boldsymbol{f}$$

So, $(\boldsymbol{u}, \boldsymbol{b}, P)$ satisfies the first equation of (1.4.2). Let us now take $\boldsymbol{a} \in \boldsymbol{W}^{1,p'}_{\sigma}(\Omega)$ with $\boldsymbol{a} \times \boldsymbol{n} = 0$ on Γ, $\boldsymbol{v} = \mathbf{0}$, $\theta = \tau = 0$ as test functions in (1.5.73). We obtain

$$\langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) - \boldsymbol{g}, \boldsymbol{a} \rangle_{\Omega_{p^*,p}}$$

Applying a De Rham Lemma version for functionals acting on vector fields with vanishing tangential components (see [76, Lemma 2.2]), there exists $\chi \in L^2(\Omega)$ defined uniquely up to an additive constant such that:

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) + \nabla \chi = \boldsymbol{g} \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma. \tag{1.5.81}$$

But taking the divergence in (1.5.81), χ is solution of the following Dirichlet problem:

$$\Delta \chi = \operatorname{div} \boldsymbol{g} = 0 \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \text{ on } \Gamma$$

Since $\boldsymbol{g}$ satisfies the compatibility condition (1.5.28), then χ is equal to zero and we have:

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g}$$

So $(\boldsymbol{u}, \boldsymbol{b})$ satisfies the second equation in (1.4.2).
Next, if we choose $\boldsymbol{v} = \boldsymbol{a} = \mathbf{0}$, $\tau = 0$ and $\theta \in \mathcal{D}(\Omega)$, then we obtain $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Similarly, if we choose $\boldsymbol{v} = \boldsymbol{a} = \mathbf{0}$, $\theta = 0$ and $\tau \in \mathcal{D}(\Omega)$, we obtain $\operatorname{div} \boldsymbol{b} = 0$ in Ω.

It remains to prove the boundary condition given on the pressure P. To this end, we follow a method from [14, Proposition 3.7]. Let us take as test functions $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ and $\operatorname{div} \boldsymbol{v} \in W^{1,p^*}_0(\Omega)$, $\boldsymbol{a} = \mathbf{0}$, $\theta \in W^{1,(p^*)'}(\Omega)$ and $\tau = 0$ in the variational formulation (1.5.73). Thus, applying Green formulae as previously, we have:

$$\begin{aligned}\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega_{r',p'}} &= \langle \boldsymbol{u}, -\Delta \boldsymbol{v} \rangle_{\Omega_{r',p'}} - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{v} \cdot \boldsymbol{u} \, dx + \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\boldsymbol{v} \times \boldsymbol{d}) \, dx \\ &+ \int_\Omega \nabla\theta \cdot \boldsymbol{u} \, dx - \int_\Omega \nabla\theta \cdot \boldsymbol{u} \, dx - \int_\Omega P \operatorname{div} \boldsymbol{v} \, dx + \int_\Gamma P \, \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma\end{aligned}$$

We decompose $\boldsymbol{v}$ as in (1.4.9) and to simplify the presentation, we set $\boldsymbol{z} = \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \, d\sigma \Big) \nabla q_i^N$. So, $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{z}$. By definition of q_i^N for $1 \leqslant i \leqslant I$, we have $\Delta \boldsymbol{z} = 0$ and $\operatorname{div} \boldsymbol{z} = 0$ in Ω. Thus:

$$\begin{aligned}&\langle \boldsymbol{f}, \boldsymbol{v}_0 \rangle_{\Omega_{r',p'}} + \langle \boldsymbol{f}, \boldsymbol{z} \rangle_{\Omega_{r',p'}} = \langle \boldsymbol{u}, -\Delta \boldsymbol{v}_0 - (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{v}_0 + \nabla\theta \rangle_{\Omega_{p^*,p}} + \langle \boldsymbol{b}, \mathbf{curl}(\boldsymbol{v}_0 \times \boldsymbol{d}) \rangle_{\Omega_{p^*,p}} - \int_\Omega \nabla\theta \cdot \boldsymbol{u} \, dx \\ &- \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{z} \cdot \boldsymbol{u} \, dx + \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\boldsymbol{z} \times \boldsymbol{d}) \, dx - \int_\Omega P \operatorname{div} \boldsymbol{v}_0 \, dx + \int_\Gamma P \boldsymbol{v}_0 \cdot \boldsymbol{n} \, d\sigma + \int_\Omega P \, \boldsymbol{z} \cdot \boldsymbol{n} \, d\sigma\end{aligned}$$

Taking $(\boldsymbol{v}_0, \mathbf{0}, \theta, 0)$ as a test function in the variational formulation (1.5.73), we obtain

$$\int_\Gamma P \, \boldsymbol{v}_0 \cdot \boldsymbol{n} \, d\sigma - \int_\Gamma P_0 \, \boldsymbol{v}_0 \cdot \boldsymbol{n} \, d\sigma - \int_\Omega \nabla\theta \cdot \boldsymbol{u} \, dx = 0$$

Note that, since $\operatorname{div} \boldsymbol{u} = 0$ in Ω, $\theta = 0$ on Γ_0, $\theta = \beta_i$ on Γ_i and $\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$, it follows that $\displaystyle\int_\Omega \nabla\theta \cdot \boldsymbol{u}\, dx = -\int_\Omega \theta \operatorname{div} \boldsymbol{u}\, dx + \int_\Gamma \theta \boldsymbol{u} \cdot \boldsymbol{n}\, d\sigma = 0$. Therefore:

$$\int_\Gamma (P - P_0)\boldsymbol{v}_0 \cdot \boldsymbol{n}\, d\sigma = 0$$

Then, we deduce that

$$\langle \boldsymbol{f}, \boldsymbol{z}\rangle_{\Omega_{r',p'}} + \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{z} \cdot \boldsymbol{u}\, dx - \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\boldsymbol{z} \times \boldsymbol{d})\, dx - \int_\Omega P\, \boldsymbol{z} \cdot \boldsymbol{n}\, d\sigma = 0. \tag{1.5.84}$$

Now, using (1.5.84) and the fact that $\displaystyle\int_{\Gamma_i} \boldsymbol{v}_0 \cdot \boldsymbol{n}\, d\sigma = 0$ for all $1 \leqslant i \leqslant I$, we have

$$\begin{aligned}
\int_\Gamma P\boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma &= \int_\Gamma P\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\, d\sigma + \int_\Gamma P\, \boldsymbol{z} \cdot \boldsymbol{n}\, d\sigma \\
&= \int_\Gamma P_0\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\, d\sigma + \langle \boldsymbol{f}, \boldsymbol{z}\rangle_{\Omega_{r',p'}} + \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{z} \cdot \boldsymbol{u}\, dx - \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\boldsymbol{z} \times \boldsymbol{d})\, dx \\
&= \int_\Gamma P_0\, \boldsymbol{v}_0 \cdot \boldsymbol{n}\, d\sigma + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma \Big) \Big[\langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx \\
&\quad - \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\nabla q_i^N \times \boldsymbol{d})\, dx \Big].
\end{aligned} \tag{1.5.85}$$

However, we have from (1.5.73), for all $1 \leqslant i \leqslant I$,

$$\begin{aligned}
c_i &= \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} - \int_\Gamma P_0\, \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma + \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N\, dx - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx \\
&= \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{r',p'}} - \int_\Gamma P_0\, \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma - \int_\Omega \boldsymbol{b} \cdot \mathbf{curl}(\nabla q_i^N \times \boldsymbol{d})\, dx - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N\, dx.
\end{aligned}$$

Therefore, replacing in (1.5.85), we have:

$$\int_\Gamma P\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_\Gamma P_0\, \boldsymbol{v}_0 \cdot \boldsymbol{n} d\sigma + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma \Big) \Big[c_i + \int_\Gamma P_0\, \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma \Big]$$

Moreover, applying directly the decomposition (1.4.9), we have:

$$\int_\Gamma P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_\Gamma P_0\, \boldsymbol{v}_0 \cdot \boldsymbol{n} d\sigma + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma \Big) \int_\Gamma P_0\, \nabla q_i^N \cdot \boldsymbol{n}\, d\sigma$$

Thus, combining the last two equations, we obtain:

$$\int_\Gamma P\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \int_\Gamma P_0 \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma + \sum_{i=1}^{I} \Big(\int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma \Big) c_i = \int_\Gamma (P_0 + c)\boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma,$$

with $c = 0$ on Γ_0 and $c = c_i$ on Γ_i for all $1 \leqslant i \leqslant I$. We conclude as in [14, Theorem 3.2.] to prove that $P = P_0$ on Γ_0 and $P = P_0 + c_i$ on Γ_i . □

We are now in position to prove the following theorem

Theorem 1.5.7. *We suppose Ω of classe $\mathcal{C}^{1,1}$. Let $\frac{3}{2} < p < 2$. Assume that $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$, $h \in W^{1,r}(\Omega)$ with the compatibility conditions (1.5.27)-(1.5.28), together with*

$\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ *and* $\boldsymbol{d} \in \boldsymbol{W}^{1,3/2}_{\sigma}(\Omega)$. *Then the linearized problem* (1.4.2) *has a unique solution* $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. *Moreover, we have the following estimates:*

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} \\ &+ \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|\boldsymbol{g}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]} + (1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}) \|h\|_{W^{1,r}(\Omega)} \Big)\end{aligned} \tag{1.5.86}$$

$$\begin{aligned}&\|P\|_{W^{1,r}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})^2 \times \Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} \\ &+ \|\boldsymbol{g}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + (1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}) \|h\|_{W^{1,r}(\Omega)} \Big).\end{aligned} \tag{1.5.87}$$

Proof. Since $2 < p' < 3$, thanks to Theorem 1.5.4, we have for any $(\boldsymbol{F}, \boldsymbol{G}, \phi) \in [\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega)]' \times [\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega)]' \perp \boldsymbol{K}^p_N(\Omega) \times W^{1,(p^*)'}_0(\Omega)$ that the following problem

$$\begin{cases} -\Delta \boldsymbol{v} - (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{v} + \nabla\theta + (\mathbf{curl}\,\boldsymbol{a}) \times \boldsymbol{d} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = \phi \ \text{ in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{a} + \mathbf{curl}(\boldsymbol{v} \times \boldsymbol{d}) + \nabla\tau = \boldsymbol{G} \quad \text{and} \quad \operatorname{div} \boldsymbol{a} = 0 \ \text{ in } \Omega, \\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{a} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \tau = 0 \quad \text{on } \Gamma, \quad \theta = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta = \beta_i \ \text{ on } \Gamma_i, \\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \text{and} \quad \langle \boldsymbol{a} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \ \ 1 \leq i \leq I. \end{cases} \tag{1.5.88}$$

has a unique solution $(\boldsymbol{v}, \boldsymbol{a}, \theta, \tau, \boldsymbol{\beta}) \in \boldsymbol{W}^{1,p'}(\Omega) \times \boldsymbol{W}^{1,p'}(\Omega) \times W^{1,(p^*)'}(\Omega) \times W^{1,(p^*)'}_0(\Omega) \times \mathbb{R}^I$ with $\operatorname{div} \boldsymbol{v} \in W^{1,(p^*)'}_0(\Omega)$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_I)$ such that:

$$\beta_i = \langle \boldsymbol{F}, \nabla q^N_i \rangle_{\Omega_{p^*,p}} + \langle (\mathbf{curl}\,\boldsymbol{a}) \times \boldsymbol{d}, \nabla q^N_i \rangle_{\Omega_{p^*,p}} - \langle (\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{v}, \nabla q^N_i \rangle_{\Omega_{p^*,p}} + \int_\Gamma \phi \nabla q^N_i \cdot \boldsymbol{n}\, d\sigma.$$

Moreover, this solution also satisfies the estimates:

$$\begin{aligned}&\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\theta\|_{W^{1,(p^*)'}(\Omega)} \leqslant C\Big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big) \\ &\times \Big(\|\boldsymbol{F}\|_{[\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega)]'} + \|\boldsymbol{G}\|_{[\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega)]'} + \big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\big) \|\phi\|_{W^{1,(p^*)'}(\Omega)} \Big).\end{aligned} \tag{1.5.89}$$

We note that, from Theorem 1.5.4 for $2 < p' < 3$, the value of s is $3/2$. Using (1.5.89), we have

$$\begin{aligned}&|\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega r',p'} + \langle \boldsymbol{g}, \boldsymbol{a} \rangle_{\Omega r',p'} - \int_\Gamma P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma| \\ &\leqslant \|\boldsymbol{f}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} \|\boldsymbol{v}\|_{\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)} + \|\boldsymbol{g}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} \|\boldsymbol{a}\|_{\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)} \\ &\leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \Big)\Big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\boldsymbol{a}\|_{\boldsymbol{W}^{1,p'}(\Omega)} \Big) \\ &\leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}^{r',p'}_0(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} \Big)\Big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big) \\ &\times \Big(\|\boldsymbol{F}\|_{\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega)} + \|\boldsymbol{G}\|_{\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega)} + (1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}) \|\phi\|_{W^{1,(p^*)'}(\Omega)} \Big)\end{aligned} \tag{1.5.90}$$

We deduce that the linear mapping $(\boldsymbol{F}, \boldsymbol{G}, \phi) \to \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega_{r',p'}} + \langle \boldsymbol{g}, \boldsymbol{a} \rangle_{\Omega_{r',p'}} - \int_\Gamma P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma$ defines an element of the dual space of $\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega) \times \boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega) \times W^{-1,p^*}(\Omega)$. It follows from Riesz' representation theorem that there exists a solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ in $\boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega) \times \boldsymbol{H}^{p^*,p}_0(\mathbf{curl},\Omega) \times W^{-1,p^*}(\Omega)$ of the problem

$$\langle \boldsymbol{u}, \boldsymbol{F} \rangle_{\Omega p^*,p} + \langle \boldsymbol{b}, \boldsymbol{G} \rangle_{\Omega p^*,p} - \langle P, \phi \rangle_{W^{-1,p^*}(\Omega) \times W^{1,(p^*)'}_0(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{\Omega r',p'} + \langle \boldsymbol{g}, \boldsymbol{a} \rangle_{\Omega r',p'} - \int_\Gamma P_0\, \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma$$

which is the variational formulation (1.5.73). Moreover, it satisfies the estimate:

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)}+\big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\big)^{-1}\|P\|_{W^{-1,p^*}(\Omega)}\\&\leqslant C\big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)}\big)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\big).\end{aligned}\tag{1.5.91}$$

In order to recover the solution of (1.4.2) through the equivalence result given in Lemma 1.5.6, it remains us to prove that $\boldsymbol{u},\boldsymbol{b}\in\boldsymbol{W}^{1,p}(\Omega)$, $P\in W^{1,r}(\Omega)$, that $\langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$, $\langle\boldsymbol{b}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$ for all $1\leqslant i\leqslant I$ and to recover the relation of (1.5.74). We firstly want to show that $\displaystyle\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma=0$ and $\displaystyle\int_{\Gamma_i}\boldsymbol{b}\cdot\boldsymbol{n}\,d\sigma=0$. We choose $(\mathbf{0},\mathbf{0},\theta,0)$ with $\theta\in W^{1,(p^*)'}(\Omega)$ satisfying $\theta=0$ on Γ_0 and $\theta=\delta_{ij}$ on Γ_j for all $1\leqslant j\leqslant I$ and a fixed $1\leqslant i\leqslant I$. Then:

$$0=\langle\boldsymbol{u},\nabla\theta\rangle_{\Omega_{p^*,p}}=\int_\Omega\boldsymbol{u}\cdot\nabla\theta\,dx=\int_\Gamma\theta\boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma-\int_\Omega\operatorname{div}\boldsymbol{u}\,\theta\,dx=\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,d\sigma$$

For the condition $\displaystyle\int_{\Gamma_i}\boldsymbol{b}\cdot\boldsymbol{n}\,d\sigma=0$ for all $1\leqslant i\leqslant I$, we set $\tilde{\boldsymbol{b}}=\boldsymbol{b}-\displaystyle\sum_{i=1}^{I}\langle\boldsymbol{b}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\nabla q_i^N$. Observe that by the definition of q_i^N, $\tilde{\boldsymbol{b}}$ is also solution of (1.5.73) and satisfies the condition $\langle\tilde{\boldsymbol{b}}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$.

Next, taking test functions $(\mathbf{0},\mathbf{0},\theta,0)$ and $(\mathbf{0},\mathbf{0},0,\tau)$ with $\theta\in W^{1,(p^*)'}(\Omega)$ as above and $\tau\in\mathcal{D}(\Omega)$, we respectly recover $\operatorname{div}\boldsymbol{u}=0$ and $\operatorname{div}\boldsymbol{b}=0$ in Ω. Besides, since $\boldsymbol{u},\boldsymbol{b}\in\boldsymbol{H}_0^{p^*,p}(\mathbf{curl},\Omega)$, we have $\boldsymbol{u}$ and $\boldsymbol{b}$ belong to $\boldsymbol{X}_N^p(\Omega)$. From Theorem 1.3.1, we deduce that $\boldsymbol{u},\boldsymbol{b}\in\boldsymbol{W}^{1,p}(\Omega)$. Thus, the estimate (1.5.86) follows from (1.3.4) and (1.5.91). Finally, in order to prove that $P\in W^{1,r}(\Omega)$, we take the test functions $(\boldsymbol{v},\mathbf{0},0,0)$ with $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega)$, and we obtain as in the proof of Lemma 1.5.6 that:

$$\nabla P=\boldsymbol{f}+\Delta\boldsymbol{u}-(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\quad\text{in }\Omega.$$

Then taking the divergence, P is solution of the following problem

$$\begin{aligned}\Delta P=\operatorname{div}\boldsymbol{f}+\operatorname{div}((\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}-(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u})\quad\text{in }\Omega,\\ P=P_0\quad\text{on }\Gamma_0\quad\text{and}\quad P=P_0+c_i\quad\text{on }\Gamma_i.\end{aligned}\tag{1.5.92}$$

Since $\mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow\boldsymbol{L}^{p^*}(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\in\boldsymbol{L}^r(\Omega)$. Besides, $\mathbf{curl}\,\boldsymbol{b}\in\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{d}\in\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$. So $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\in\boldsymbol{L}^r(\Omega)$. Hence, we obtain that $\Delta P\in W^{-1,r}(\Omega)$. Since P_0 belongs to $W^{1-1/r,r}(\Gamma)$, we deduce that the solution P of (1.5.92) belongs to $\in W^{1,r}(\Omega)$. Moreover, it satisfies the estimate

$$\|P\|_{W^{1,r}(\Omega)}\leqslant\|\operatorname{div}\boldsymbol{f}\|_{W^{-1,r}(\Omega)}+\|\operatorname{div}((\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}-(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u})\|_{W^{-1,r}(\Omega)}+\|P_0\|_{W^{1-1/r,r}(\Gamma)}$$

Applying the characterization of $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$ given in Proposition 1.3.1, we write $\boldsymbol{f}=\boldsymbol{F}+\mathbf{curl}\,\boldsymbol{\Psi}$ with $\boldsymbol{F}\in\boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\Psi}\in\boldsymbol{L}^p(\Omega)$. So

$$\|\operatorname{div}\boldsymbol{f}\|_{W^{-1,r}(\Omega)}=\|\operatorname{div}\boldsymbol{F}\|_{W^{-1,r}(\Omega)}=\sup_{\theta\in W_0^{1,r'}(\Omega)}\frac{|\langle\operatorname{div}\boldsymbol{F},\theta\rangle|}{\|\theta\|_{W^{1,r'}(\Omega)}}=\sup_{\theta\in W_0^{1,r'}(\Omega)}\frac{|\langle\boldsymbol{F},\nabla\theta\rangle|}{\|\theta\|_{W^{1,r'}(\Omega)}}\leqslant\|\boldsymbol{F}\|_{\boldsymbol{L}^r(\Omega)},$$

which implies that

$$\|\operatorname{div}\boldsymbol{f}\|_{W^{-1,r'}(\Omega)}\leqslant\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}\tag{1.5.93}$$

In the same way, we have:

$$\begin{aligned} \|\mathrm{div}((\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d})\|_{W^{-1,r}(\Omega)} &\leqslant \|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\|_{\boldsymbol{L}^r(\Omega)} \leq \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq C_d\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}, \end{aligned} \tag{1.5.94}$$

where C_d is the constant related to the Sobolev embedding $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$. Next,

$$\begin{aligned} \|\mathrm{div}((\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u})\|_{W^{-1,r}(\Omega)} &\leqslant \|(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^{p^*}(\Omega)} \\ &\leq C\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}, \end{aligned} \tag{1.5.95}$$

where we have used the Sobolev embedding $\boldsymbol{W}^{1,p}(\Omega)\hookrightarrow\boldsymbol{L}^{p^*}(\Omega)$. Using estimates (1.5.93), (1.5.94), (1.5.95) combined with the estimate (1.5.86), we obtain the estimate (1.5.87) for the pressure. $\square$

The following result gives the regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $p<2$ when the divergence of the velocity field $\boldsymbol{u}$ does not vanish.

Corollary 1.5.8. *Let $\frac{3}{2}<p<2$. Assume that $\boldsymbol{f},\boldsymbol{g}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, $P_0\in W^{1-\frac{1}{r},r}(\Gamma)$, $h\in W^{1,r}(\Omega)$ with the compatibility conditions* (1.5.27)-(1.5.28)*, together with $\mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d}\in\boldsymbol{W}_\sigma^{1,3/2}(\Omega)$. Then the linearized problem* (1.4.2) *has a unique solution $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)\times\mathbb{R}^I$. Moreover, we have the following estimates:*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+ \\ &+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}+\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]}+(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\|h\|_{W^{1,r}(\Omega)}\Big) \end{aligned} \tag{1.5.96}$$

and

$$\begin{aligned} &\|P\|_{W^{1,r}(\Omega)}\leq C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})^2\times\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+ \\ &+\|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]}+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}+(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\|h\|_{W^{1,r}(\Omega)}\Big). \end{aligned} \tag{1.5.97}$$

Proof. We can reduce the non-vanishing divergence problem for the velocity to the case where $\mathrm{div}\,\boldsymbol{u}=0$, by solving the Dirichlet problem:

$$\Delta\theta=h \quad \text{in } \Omega \quad \text{and} \quad \theta=0 \text{ on } \Gamma.$$

For $h\in W^{1,r}(\Omega)$, the solution θ belongs to $W^{3,r}(\Omega)$ and satisfies the estimate

$$\|\theta\|_{W^{3,r}(\Omega)}\leq C\|h\|_{W^{1,r}(\Omega)}. \tag{1.5.98}$$

Setting $\boldsymbol{z}=\boldsymbol{u}-\nabla\theta$, we obtain that $(\boldsymbol{z},\boldsymbol{b},P,\boldsymbol{c})$ is the solution of the problem treated in the Theorem 1.5.7 with $\boldsymbol{f}$ and $\boldsymbol{g}$ replaced by $\tilde{\boldsymbol{f}}=\boldsymbol{f}+\nabla h-(\mathbf{curl}\,\boldsymbol{w})\times\nabla\theta$ and $\tilde{\boldsymbol{g}}=\boldsymbol{g}+\mathbf{curl}(\nabla\theta\times\boldsymbol{d})$ respectively. Indeed, we have $\nabla h\in\boldsymbol{L}^r(\Omega)\hookrightarrow[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, and since $\nabla\theta\in\boldsymbol{W}^{2,r}(\Omega)\hookrightarrow\boldsymbol{L}^{p^*}(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{w})\times\nabla\theta\in\boldsymbol{L}^r(\Omega)\hookrightarrow[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, and $\nabla\theta\times\boldsymbol{d}\in\boldsymbol{L}^p(\Omega)$ so $\mathbf{curl}(\nabla\theta\times\boldsymbol{d})\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$. Therefore, $\tilde{\boldsymbol{f}},\tilde{\boldsymbol{g}}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$. Besides, since we add a $\mathbf{curl}$, then $\tilde{\boldsymbol{g}}$ still satisfies the compatibility conditions (1.5.27)-(1.5.28). Thus, applying Theorem 1.5.7, we have the estimate:

$$\begin{aligned} \|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leqslant C\Big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big) \\ &\times\Big(\Big\|\tilde{\boldsymbol{f}}\Big\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+\|\tilde{\boldsymbol{g}}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}+\|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\Big) \end{aligned}$$

We want to control the terms on $\tilde{\boldsymbol{f}}$ and $\tilde{\boldsymbol{g}}$.

$$\begin{aligned}\left\|\tilde{\boldsymbol{f}}\right\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} &\leqslant \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\nabla h\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|(\mathbf{curl}\,\boldsymbol{w})\times\nabla\theta\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}\\ &\leqslant \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\nabla h\|_{\boldsymbol{L}^r(\Omega)} + \|(\mathbf{curl}\,\boldsymbol{w})\times\nabla\theta\|_{\boldsymbol{L}^r(\Omega)}\\ &\leqslant \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|h\|_{W^{1,r}(\Omega)} + C\,\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\nabla\theta\|_{\boldsymbol{W}^{1,p}(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\|\tilde{\boldsymbol{g}}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} &\leqslant \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\mathbf{curl}(\boldsymbol{d}\times\nabla\theta)\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'}\\ &\leqslant \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{d}\times\nabla\theta\|_{\boldsymbol{L}^p(\Omega)}\\ &\leqslant \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + C_d\,\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\|\nabla\theta\|_{\boldsymbol{L}^{p^*}(\Omega)}\,.\end{aligned}$$

Therefore, from these two last estimates combined with (1.5.98), we obtain the estimate (1.5.96).

Moreover, we also obtain from the Theorem 1.5.7:

$$\begin{aligned}\|P\|_{W^{1,r}(\Omega)} \leqslant C&\Big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)^2\\ &\times\Big(\left\|\tilde{\boldsymbol{f}}\right\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\tilde{\boldsymbol{g}}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}\Big).\end{aligned}$$

Using the same arguments as previously, we can obtain the estimate (1.5.97). □

In this subsection, we always take $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ to obtain $P \in W^{1,r}(\Omega)$. However, since the pressure is decoupled from the system, we can improve its regularity given in the previous results by choosing a convenient boundary condition. For this, we begin by the following regularity concerning the Stokes problem $(\mathcal{S}_\mathcal{N})$ which is an improvement of [23, Theorem 2.2.6]:

Theorem 1.5.9. *Let Ω be of class $\mathcal{C}^{1,1}$. Let us assume $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$ and $h \in W^{1,r}(\Omega)$ with $1 \leqslant r \leqslant p$ and $\frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3}$. Then*

1. *If $r < 3$ and $P_0 \in W^{-\frac{1}{r^*},r^*}(\Gamma)$, the Stokes problem $(\mathcal{S}_\mathcal{N})$ has a unique solution $(\boldsymbol{u},\,P) \in \boldsymbol{W}^{1,p}(\Omega)\times L^{r^*}(\Omega)$.*
2. *If $r \geqslant 3$ and $P_0 \in W^{-\frac{1}{q},q}(\Gamma)$ for any finite number $q > 1$, the Stokes problem $(\mathcal{S}_\mathcal{N})$ has a unique solution $(\boldsymbol{u},\,P) \in \boldsymbol{W}^{1,p}(\Omega)\times L^{q}(\Omega)$.*

Proof. Taking the divergence in the first equation of $(\mathcal{S}_\mathcal{N})$, we have:

$$\begin{cases}\Delta P = \operatorname{div}\boldsymbol{f} + \Delta h & \text{in } \Omega,\\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \text{ on } \Gamma_i.\end{cases}$$

We split this problem in two parts: find P_1 such that

$$(\mathcal{P}_1) \qquad \Delta P_1 = \operatorname{div}\boldsymbol{f} + \Delta h \quad \text{in } \Omega \qquad \text{and} \quad P_1 = 0 \text{ on } \Gamma,$$

and find P_2 such that

$$(\mathcal{P}_2) \qquad \Delta P_2 = 0 \quad \text{in } \Omega, \quad P_2 = P_0 \text{ on } \Gamma_0 \quad \text{and} \quad P_2 = P_0 + c_i \text{ on } \Gamma_i.$$

We note that the regularity of P_1 is only dependent of $\operatorname{div}\boldsymbol{f}$ and Δh, and then we choose P_0 in order to recover for P_2 the same regularity as than for P_1. Then, we obtain the regularity of P by adding P_1 and P_2. Let us analyze problem $(\mathcal{P}_1)$. Since $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, there exists $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\Psi} \in \boldsymbol{L}^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\Psi}$. So $\operatorname{div}\boldsymbol{f} = \operatorname{div}\boldsymbol{F} \in W^{-1,r}(\Omega)$. Moreover, we have $\Delta h \in W^{-1,r}(\Omega)$. Then, $\operatorname{div}\boldsymbol{f} + \Delta h$ belongs to $\in W^{-1,r}(\Omega)$ which implies that problem $(\mathcal{P}_1)$ has a unique solution $P_1 \in W^{1,r}(\Omega)$. Next, we determine the regularity of P_2 with respect to the data P_0. We note that P_0 must be chosen so that the solution P_2 of $(\mathcal{P}_2)$ could belong to a class of spaces containing spaces for P_1. We distinguish the following cases:

Case $r < 3$: If $P_0 \in W^{-\frac{1}{r^*},r^*}(\Gamma)$ with $r^* = \frac{3r}{3-r}$, the solution P_2 of problem $(\mathcal{P}_2)$ belongs to $L^{r^*}(\Omega)$. Since $P_1 \in W^{1,r}(\Omega) \hookrightarrow L^{r^*}(\Omega)$, we deduce that $P = P_1 + P_2$ belongs to $L^{r^*}(\Omega)$.

Case $r \geqslant 3$:

We have $P_1 \in W^{1,r}(\Omega) \hookrightarrow L^q(\Omega)$ for any finite number $q > 1$ if $r = 3$, for $q = \infty$ if $r > 3$. Thus, taking $P_0 \in W^{-\frac{1}{q},q}(\Gamma)$ for any $q > 1$ we have $P_2 \in L^q(\Omega)$ and then $P \in L^q(\Omega)$. □

Remark 1.5.2. *Observe that, using the above splitting, if $P_0 \in W^{1-1/r,r}(\Gamma)$, we have immediately $P \in W^{1,r}(\Omega)$.*

The regularity result given in Theorem 1.5.9 enables us to improve the pressure in the linearized MHD system (1.4.2). In particular, we have the following result

Corollary 1.5.10. *Let $p > \frac{3}{2}$, $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, $h \in W^{1,r}(\Omega)$, $P_0 \in W^{-\frac{1}{r^*},r^*}(\Gamma)$ satisfying the compatibility condition* (1.5.27)-(1.5.28)*. We suppose that*

- $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ *and* $\boldsymbol{d} \in \boldsymbol{W}_\sigma^{1,\frac{3}{2}}(\Omega)$ *if* $\frac{3}{2} < p < 2$.
- $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^s(\Omega)$ *and* $\boldsymbol{d} \in \boldsymbol{W}_\sigma^{1,s}(\Omega)$ *if $p \geqslant 2$, where s is defined in* (1.5.6)

Then, the solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ given in Proposition 1.5.5 and Theorem 1.5.7 of the linearized MHD problem (1.4.2) *belongs to $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times L^{r^*}(\Omega)$.*

Proof. We are going to take advantage of the regularity results for the Stokes problem $(\mathcal{S}_\mathcal{N})$ given in Theorem 1.5.9. Then, we can rewrite (1.4.2) in the following way: Find $(\boldsymbol{u}, P, \boldsymbol{c})$ such that

$$(\widetilde{\mathcal{S}_\mathcal{N}}) \begin{cases} -\Delta\boldsymbol{u} + \nabla P = \widetilde{\boldsymbol{f}} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in } \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \\ P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \ \text{ on } \Gamma_i, \\ \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ 1 \le i \le I, \end{cases}$$

with $\widetilde{\boldsymbol{f}} = \boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}$ and find $\boldsymbol{b}$ solution of the following elliptic problem

$$(\widetilde{\mathcal{E}_\mathcal{N}}) \begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} = \widetilde{\boldsymbol{g}} \quad \text{in } \Omega \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{b}\times\boldsymbol{n} = 0 \quad \text{on } \Gamma \\ \langle \boldsymbol{b}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \quad \forall 1 \leqslant i \leqslant I, \end{cases}$$

with $\widetilde{\boldsymbol{g}} = \boldsymbol{g} + \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})$. As in the previous proofs, we can easily verify that the assumptions on $\boldsymbol{f}$, $\mathbf{curl}\,\boldsymbol{w}$ and $\boldsymbol{d}$ imply that the term $\widetilde{\boldsymbol{f}}$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$ for both cases $p < 2$ and $p \ge 2$. Thanks to Theorem 1.5.9, there exists a unique solution $(\boldsymbol{u}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p}(\Omega) \times L^{r^*}(\Omega) \times \mathbb{R}^I$ for the problem $(\widetilde{\mathcal{S}_\mathcal{N}})$. Besides, the existence of $\boldsymbol{b}$ is independent of the pressure. Indeed, $\widetilde{\boldsymbol{g}}$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$ and satisfies the compatibility conditions (1.5.27)-(1.5.28). It follows from Lemma 1.3.4 that problem $(\widetilde{\mathcal{E}_\mathcal{N}})$ has a unique solution $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$. □

1.5.3 Strong solution in $\boldsymbol{W}^{2,p}(\Omega)$; $1 < p < 6/5$

The aim of this subsection is to complete the L^p-theory for the linearized MHD problem (1.4.2) by the proof of strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$ with $1 < p < 6/5$. One of the approach that we can use is to consider $\boldsymbol{w}$ and $\boldsymbol{d}$ more regular in a first step and then remove this regularity in a second step. Since the proof with this approach highly mimics that of Theorem 1.5.1, we put it in the Appendix at the end of this chapter (see Section 1.7). We are going to give a shorter and different proof where we take advantage of the regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $1 < p < 2$ for the linearized MHD problem (1.4.2).

Theorem 1.5.11 (Strong solution in $\boldsymbol{W}^{2,p}(\Omega)$ with $1 < p < \frac{6}{5}$)**.** *Suppose that Ω is of class $\mathcal{C}^{2,1}$ and $1 < p < \frac{6}{5}$. Assume $h = 0$, and let $\boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$, $P_0 \in W^{1-\frac{1}{p},p}(\Gamma)$, $\mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{W}^{1,3/2}_{\sigma}(\Omega)$ with the compatibility conditions* (1.4.5)-(1.5.1).

Then the linearized problem (1.4.2) *has a unique solution* $(\boldsymbol{u}, \boldsymbol{b}, P, c) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^I$ *satisfying the following estimate:*

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leqslant C\Big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)^2 \\ \times \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\Big)\end{aligned} \tag{1.5.99}$$

Proof. Observe that

$$\boldsymbol{L}^p(\Omega) \hookrightarrow [\boldsymbol{H}^{r',(p^*)'}(\mathbf{curl},\Omega)]', \quad P_0 \in W^{1-1/p,p}(\Gamma) \hookrightarrow W^{1-1/r,r}(\Gamma), \tag{1.5.100}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Since $1 < p < 6/5$, we have $\frac{3}{2} < p^* < 2$. Then applying the regularity $\boldsymbol{W}^{1,p}(\Omega)$ for the MHD system (1.4.2) for small values of p (see Theorem 1.5.7 with $h = 0$), we can deduce the existence of a solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) \in \boldsymbol{W}^{1,p^*}(\Omega) \times \boldsymbol{W}^{1,p^*}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p^*}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p^*}(\Omega)} \\ \leq\ & C(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}\Big),\end{aligned} \tag{1.5.101}$$

where we have used the embedding (1.5.100) and

$$\|P\|_{W^{1,r}(\Omega)} \leq C(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,3/2}(\Omega)})^2\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}\Big),$$

Since $\boldsymbol{W}^{1,p^*}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$ with $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{3}$, the terms $(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}$ belong to $\boldsymbol{L}^p(\Omega)$. $(\boldsymbol{u},\ P, \boldsymbol{c})$ is then a solution of the problem $(\widetilde{\mathcal{S}_\mathcal{N}})$ with $h = 0$ and a RHS $\widetilde{\boldsymbol{f}}$ in $\boldsymbol{L}^p(\Omega)$. We deduce from Proposition 1.3.2 that $(\boldsymbol{u}, P)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leqslant C_S\Big(\|\boldsymbol{f}\|_{[\boldsymbol{L}^p(\Omega)} + \|(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\Big). \tag{1.5.102}$$

Next, $\boldsymbol{b}$ is a solution of the elleptic problem $(\widetilde{\mathcal{E}_\mathcal{N}})$ with a RHS $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})$ which belongs to $\boldsymbol{L}^p(\Omega)$. Thanks to Theorem 1.3.3, the solution $\boldsymbol{b}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leqslant C_E\big(\|\boldsymbol{g}\|_{[\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)}\big). \tag{1.5.103}$$

Moreover, we have the following bounds

$$\|(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}\, \|\boldsymbol{u}\|_{\boldsymbol{L}^{p^{**}}(\Omega)} \leq C\, \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{3/2}(\Omega)}\, \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p^*}(\Omega)}, \tag{1.5.104}$$

$$\|(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}\leq\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^{p^*}(\Omega)}\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\leq C\,\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p^*}(\Omega)}, \tag{1.5.105}$$

and similarly,

$$\|\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)}\leq\|(\boldsymbol{d}\cdot\nabla)\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}\leq C\,\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p^*}(\Omega)}(\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}+\|\nabla\boldsymbol{d}\|_{\boldsymbol{L}^{3/2}(\Omega)}).$$

Collecting the above bounds together with (1.5.101) in (1.5.102)-(1.5.103) leads to the bound (1.5.99).

□

Proceeding as in the proof of Corollary 1.5.8, we can extend the previous result to the non-vanishing divergence case. The proof of the following result can be found in the Appendix (see Section 1.7).

Corollary 1.5.12. *Let Ω be of class $\mathcal{C}^{2,1}$ and $1<p<\frac{6}{5}$. Assume that $\boldsymbol{f},\boldsymbol{g}\in\boldsymbol{L}^p(\Omega)$, $P_0\in W^{1-\frac{1}{p},p}(\Gamma)$, $h\in W^{1,p}(\Omega)$, $\mathbf{curl}\,\boldsymbol{w}\in\boldsymbol{L}^{3/2}(\Omega)$ and $\boldsymbol{d}\in\boldsymbol{W}^{1,3/2}_\sigma(\Omega)$ with the compatibility conditions* (1.4.5)-(1.5.1). *Then the linearized problem* (1.4.2) *has a unique solution $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})\in\boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)\times\mathbb{R}^I$. Moreover, we have the following estimates:*

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|P\|_{W^{1,p}(\Omega)}\\&\leqslant C(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})^2(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}+\|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\\&+(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})\,\|h\|_{W^{1-\frac{1}{p},p}(\Gamma)})\end{aligned} \tag{1.5.106}$$

1.6 The nonlinear MHD system

In this section, we consider the nonlinear problem and we study the existence of generalized and strong solutions for (*MHD*).

1.6.1 Existence and uniqueness: L^2-theory

In this subsection, we establish the existence and uniqueness for the weak solution in the Hilbert case for the problem (*MHD*). The following result is one of the main results of this chapter. First, we recall that $\langle\cdot,\cdot\rangle_{\Omega_{r,p}}$ denotes the duality product between $[\boldsymbol{H}^{r,p}_0(\mathbf{curl},\Omega)]'$ and $\boldsymbol{H}^{r,p}_0(\mathbf{curl},\Omega)$ and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Theorem 1.6.1. (Weak solutions of (*MHD*) system in $\boldsymbol{H}^1(\Omega)$). *Let Ω be of classe $\mathcal{C}^{1,1}$ and let*

$$\boldsymbol{f},\ \boldsymbol{g}\in[\boldsymbol{H}^{6,2}_0(\mathbf{curl},\Omega)]',\quad h=0\quad\text{and}\quad P_0\in H^{-\frac{1}{2}}(\Gamma)$$

satisfying the compatibility conditions

$$\forall\,\boldsymbol{v}\in\boldsymbol{K}^2_N(\Omega),\quad\langle\boldsymbol{g},\boldsymbol{v}\rangle_{\Omega_{6,2}}=0, \tag{1.6.1}$$

$$\operatorname{div}\boldsymbol{g}=0\quad\text{in }\Omega. \tag{1.6.2}$$

Then the (*MHD*) *problem has at least one weak solution $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{\alpha})\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}^I$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}+\|P\|_{\boldsymbol{L}^2(\Omega)}\leq M, \tag{1.6.3}$$

where $M = C(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-1/2}(\Gamma)})$ *with*

$$\alpha_i = \langle \boldsymbol{f}, \nabla q_i^N, \rangle_{\Omega_{6,2}} - \langle P_0, \nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma + \int_\Omega (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \cdot \nabla q_i^N \, dx - \int_\Omega (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \cdot \nabla q_i^N \, dx. \quad (1.6.4)$$

In addition, suppose that $\boldsymbol{f}$, $\boldsymbol{g}$ *and* P_0 *are small in the sense that*

$$C_1 C_2^2 M \leq \frac{2}{3C_{\mathcal{P}}^2}, \quad (1.6.5)$$

where $C_{\mathcal{P}}$ *is the constant in* (1.3.6) *and* C_1, C_2 *are given in* (1.6.15). *Then the weak solution* $(\boldsymbol{u}, \boldsymbol{b}, P)$ *of* (*MHD*) *is unique.*

We first recall that the space $\boldsymbol{V}_N(\Omega)$ denotes

$$\boldsymbol{V}_N(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \Gamma,\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0,\ \forall 1 \leqslant i \leqslant I\}$$

and we give the following definition.

Definition 1.6.2. *Given* $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$ *and* $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ *with the compatibility conditions* (1.6.1)-(1.6.2), $(\boldsymbol{u}, \boldsymbol{b}, \alpha_i) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega) \times \mathbb{R}$ *is called a weak solution of* (*MHD*) *problem if it satisfies: for all* $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$,

$$\begin{aligned} &\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega (\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{v}\, d\boldsymbol{x} - \int_\Omega (\mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{b}) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega \mathbf{curl}\,\boldsymbol{b} \cdot \mathbf{curl}\,\boldsymbol{\Psi}\, d\boldsymbol{x} \\ &+ \int_\Omega (\mathbf{curl}\,\boldsymbol{\Psi} \times \boldsymbol{b}) \cdot \boldsymbol{u}\, d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\Omega_{6,2}} + \langle \boldsymbol{g}, \boldsymbol{\Psi}\rangle_{\Omega_{6,2}} - \langle P_0, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\Gamma_0} - \sum_{i=1}^{I} \langle P_0 + \alpha_i, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\Gamma_i} \end{aligned} \quad (1.6.6)$$

and

$$\alpha_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_{\Omega_{6,2}} - \langle P_0, \nabla q_i^N \cdot \boldsymbol{n}\rangle_\Gamma + \int_\Omega (\mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{b}) \cdot \nabla q_i^N \, d\boldsymbol{x} - \int_\Omega (\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{u}) \cdot \nabla q_i^N \, d\boldsymbol{x}. \quad (1.6.7)$$

To interpret (1.6.6)-(1.6.7), it is convenient to remove the constraint of fluxs of the test functions through Γ_i. In the following Lemma, we prove that (1.6.6) can be extended to any test function $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{X}_N^2(\Omega) \times \boldsymbol{X}_N^2(\Omega)$.

Lemma 1.6.1. *Let* $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{6,2}(\mathbf{curl}, \Omega)]'$ *and* $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ *with the compatibility conditions* (1.6.1)-(1.6.2). *Then, the following two statements are equivalent:*

(i) $(\boldsymbol{u}, \boldsymbol{b}, \alpha_i) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega) \times \mathbb{R}$ *satisfies* (1.6.6)-(1.6.7) *for any* $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$.

(ii) $(\boldsymbol{u}, \boldsymbol{b}, \alpha_i) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega) \times \mathbb{R}$ *satisfies* (1.6.6)-(1.6.7) *for any* $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{X}_N^2(\Omega) \times \boldsymbol{X}_N^2(\Omega)$.

Proof. Since $\boldsymbol{V}_N(\Omega) \subset \boldsymbol{X}_N^2(\Omega)$, then (ii) implies (i). Conversely, let $(\boldsymbol{u}, \boldsymbol{b}, \alpha_i) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega) \times \mathbb{R}$ satisfying (1.6.6)-(1.6.7) for any $(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ and we want to show it implies (ii). The proof is similar to than given for the linearized problem (see Proposition 1.4.1).
Let $(\widetilde{\boldsymbol{\Psi}}, \widetilde{\boldsymbol{v}}) \in \boldsymbol{X}_N^2(\Omega) \times \boldsymbol{X}_N^2(\Omega)$. We set

$$\boldsymbol{\Psi} = \widetilde{\boldsymbol{\Psi}} - \sum_{i=1}^{I} \langle \widetilde{\boldsymbol{\Psi}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N \quad \text{and} \quad \boldsymbol{v} = \widetilde{\boldsymbol{v}} - \sum_{i=1}^{I} \langle \widetilde{\boldsymbol{v}} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} \nabla q_i^N.$$

Then, $(\boldsymbol{\Psi}, \boldsymbol{v})$ belongs to $\boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$. Replacing in (1.6.6)-(1.6.7), we obtain

$$\int_\Omega \mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\Psi}}\,d\boldsymbol{x}+\int_\Omega(\mathbf{curl}\,\widetilde{\boldsymbol{\Psi}}\times\boldsymbol{b})\cdot\boldsymbol{u}\,d\boldsymbol{x}-\langle\boldsymbol{g},\widetilde{\boldsymbol{\Psi}}\rangle_{\Omega_{6,2}}+\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{v}}\,d\boldsymbol{x}$$
$$+\int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\widetilde{\boldsymbol{v}}\,d\boldsymbol{x}-\int_\Omega(\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b})\cdot\widetilde{\boldsymbol{v}}\,d\boldsymbol{x}-\langle\boldsymbol{f},\widetilde{\boldsymbol{v}}\rangle_{\Omega_{6,2}}+\langle P_0,\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\rangle_{\Gamma_0}+\sum_{i=1}^{I}\langle P_0+\alpha_i,\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\rangle_{\Gamma_i}$$
$$=\sum_{i=1}^{I}\langle\widetilde{\boldsymbol{\Psi}}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big(\int_\Omega\mathbf{curl}\,\boldsymbol{b}\cdot\underbrace{\mathbf{curl}(\nabla q_i^N)}_{=0}\,d\boldsymbol{x}+\int_\Omega\underbrace{(\mathbf{curl}\,\nabla q_i^N)}_{=0}\times\boldsymbol{b}\cdot\boldsymbol{u}\,d\boldsymbol{x}-\underbrace{\langle\boldsymbol{g},\nabla q_i^N\rangle_{\Omega_{6,2}}}_{=0\text{ due to (1.6.1)}}\Big)\tag{1.6.8}$$
$$+\sum_{i=1}^{I}\langle\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big(\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\underbrace{(\mathbf{curl}\,\nabla q_i^N)}_{=0}\,d\boldsymbol{x}-\int_\Omega(\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b})\cdot\nabla q_i^N\,d\boldsymbol{x}+\int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\nabla q_i^N\,d\boldsymbol{x}$$
$$-\langle\boldsymbol{f},\nabla q_i^N\rangle_{\Omega_{6,2}}+\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_0}+\sum_{j=1}^{I}\langle P_0+\alpha_j,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_j}\Big)$$

So, we have

$$\sum_{i=1}^{I}\langle\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}\Big(-\int_\Omega(\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b})\cdot\nabla q_i^N\,d\boldsymbol{x}+\int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\nabla q_i^N\,d\boldsymbol{x}-\langle\boldsymbol{f},\nabla q_i^N\rangle_{\Omega_{6,2}}$$
$$+\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_0}+\sum_{j=1}^{I}\langle P_0+\alpha_j,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_j}\Big)$$

Since q_i^N satisfies (1.3.2), we have in particular that $\sum_{j=1}^{I}\langle\alpha_j,\nabla q_i^n\cdot\boldsymbol{n}\rangle_{\Gamma_j}=\alpha_i\delta_{ij}$. Then, we obtain

$$\sum_{j=1}^{I}\langle P_0+\alpha_j,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_j}=\sum_{j=1}^{I}\langle P_0,\nabla q_i^N\cdot\boldsymbol{n}\rangle_{\Gamma_j}+\alpha_i\tag{1.6.9}$$

Replacing (1.6.7) in (1.6.9), we obtain from (1.6.8) that

$$\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{v}}\,d\boldsymbol{x}+\int_\Omega(\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\widetilde{\boldsymbol{v}}\,d\boldsymbol{x}-\int_\Omega(\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b})\cdot\widetilde{\boldsymbol{v}}\,d\boldsymbol{x}+\int_\Omega\mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\Psi}}\,d\boldsymbol{x}$$
$$+\int_\Omega(\mathbf{curl}\,\widetilde{\boldsymbol{\Psi}}\times\boldsymbol{b})\cdot\boldsymbol{u}\,d\boldsymbol{x}=\langle\boldsymbol{f},\widetilde{\boldsymbol{v}}\rangle_{\Omega_{6,2}}+\langle\boldsymbol{g},\widetilde{\boldsymbol{\Psi}}\rangle_{\Omega_{6,2}}-\langle P_0,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\Gamma_0}-\sum_{i=1}^{I}\langle P_0+\alpha_i,\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}\rangle_{\Gamma_i}\tag{1.6.10}$$

which is (1.6.6) with test functions $(\widetilde{\boldsymbol{\Psi}},\widetilde{\boldsymbol{v}})\in\boldsymbol{X}_N^2(\Omega)\times\boldsymbol{X}_N^2(\Omega)$. This completes the proof.

□

We can now prove the following result

Theorem 1.6.2. *Let* $\boldsymbol{f},\boldsymbol{g}\in[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ *and* $P_0\in H^{-\frac{1}{2}}(\Gamma)$ *with the compatibility conditions* (1.6.1)-(1.6.2)*. Then, the following two statements are equivalent:*
(i) $(\boldsymbol{u},\boldsymbol{b},P,\alpha_i)\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}$ *is a solution of* (*MHD*),
(ii) $(\boldsymbol{u},\boldsymbol{b},\alpha_i)\in\boldsymbol{V}_N(\Omega)\times\boldsymbol{V}_N(\Omega)\times\mathbb{R}$ *is a weak solution of* (*MHD*), *in the sense of Definition* (1.6.2).

Proof. The proof that **(i)** implies **(ii)** is very similar to that of Proposition 1.4.1, hence we omit it. Let $(\boldsymbol{u},\boldsymbol{b},\alpha_i)\in\boldsymbol{V}_N(\Omega)\times\boldsymbol{V}_N(\Omega)\times\mathbb{R}$ satisfying (1.6.6)-(1.6.7) for any $(\boldsymbol{v},\boldsymbol{\Psi})\in\boldsymbol{V}_N(\Omega)\times\boldsymbol{V}_N(\Omega)$. Due to Lemma 1.6.1, we have that $(\boldsymbol{u},\boldsymbol{b},\alpha_i)\in\boldsymbol{V}_N(\Omega)\times\boldsymbol{V}_N(\Omega)\times\mathbb{R}$ satisfies also (1.6.6)-(1.6.7) for any $(\boldsymbol{v},\boldsymbol{\Psi})\in\boldsymbol{X}_N^2(\Omega)\times\boldsymbol{X}_N^2(\Omega)$. Choosing $\boldsymbol{v}\in\boldsymbol{\mathcal{D}}_\sigma(\Omega)$ and $\boldsymbol{\Psi}=\boldsymbol{0}$ as test functions in (1.6.6), we have

$$\langle-\Delta\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u}-(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b}-\boldsymbol{f},\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega)\times\boldsymbol{\mathcal{D}}(\Omega)}=0.$$

So, by De Rham's theorem, there exists an unique $P \in \boldsymbol{L}^2(\Omega)$ such that

$$-\Delta \boldsymbol{u} + \nabla P + (\mathbf{curl}\, \boldsymbol{u}) \times \boldsymbol{u} - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{b} - \boldsymbol{f} = 0 \quad \text{in } \Omega. \tag{1.6.11}$$

Next, choosing $(\boldsymbol{0}, \boldsymbol{\Psi})$ with $\boldsymbol{\Psi} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$ in (1.6.6), we have

$$\langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{g}, \boldsymbol{\Psi} \rangle_{\boldsymbol{\mathcal{D}'}(\Omega) \times \boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Then, applying [76, Lemma 2.2], we have $\chi \in L^2(\Omega)$ defined uniquely up to an additive constant such that

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) - \boldsymbol{g} = \nabla \chi \quad \text{in } \Omega \qquad \text{and} \qquad \chi = 0 \quad \text{on } \Gamma$$

Since χ is solution of the following harmonic problem

$$\Delta \chi = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \chi = 0 \quad \text{on } \Gamma.$$

We deduce that $\chi = 0$ in Ω which gives the second equation in (*MHD*). Moreover, by the fact that $\boldsymbol{u}$ and $\boldsymbol{b}$ belong to the space $\boldsymbol{V}_N(\Omega)$, we have $\operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{b} = 0$ in Ω and $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. The proof of the boundary conditions on the pressure is fairly similar to that given in [14, Proposition 3.7]. □

Proof of Theorem 1.6.1:

We use the Schauder fixed point Theorem. We make use of the product space $\boldsymbol{Z}_N(\Omega) = \boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega)$ defined in (1.4.17). We define the mapping $G : \boldsymbol{Z}_N(\Omega) \to \boldsymbol{Z}_N(\Omega)$ such that $G(\boldsymbol{w}, \boldsymbol{d}) = (\boldsymbol{u}, \boldsymbol{b})$ with $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}_N(\Omega)$ a solution of the linearized problem (1.4.18). By Theorem 1.4.2, for each pair $(\boldsymbol{w}, \boldsymbol{d}) \in \boldsymbol{Z}_N(\Omega)$ the solution $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}_N(\Omega)$ of problem (1.4.18) exists, is unique and satisfies the following estimate:

$$\|(\boldsymbol{u}, \boldsymbol{b})\|^2_{\boldsymbol{Z}_N(\Omega)} = (\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)})^{1/2} \leqslant C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)}\big) := M \tag{1.6.12}$$

for some constant $C > 0$ independent of $\boldsymbol{w}$ and $\boldsymbol{d}$. We define the ball

$$\boldsymbol{B}_r = \{(\boldsymbol{v}, \boldsymbol{\Psi}) \in \boldsymbol{Z}_N(\Omega); \quad \|(\boldsymbol{v}, \boldsymbol{\Psi})\|_{\boldsymbol{Z}_N(\Omega)} \leqslant r\},$$

where $r = M = C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)}\big)$. By the definition of G and (1.6.12), it follows that $G(\boldsymbol{B}_r) \subset \boldsymbol{B}_r$ and then G is a mapping of the ball $\boldsymbol{B}_r$ into itself. Now, we want to prove that the operator G is compact on $\boldsymbol{B}_r$. For this, let $\{(\boldsymbol{w}_k, \boldsymbol{d}_k)\}_{k \geq 1}$ be an arbitrary sequence in $\boldsymbol{B}_r$. Since $\boldsymbol{H}^1(\Omega)$ is reflexive, there exists a subsequence still denoted $\{(\boldsymbol{w}_k, \boldsymbol{d}_k)\}_{k \geq 1}$ and a pair $(\boldsymbol{w}, \boldsymbol{d})$ in $\boldsymbol{B}_r$ such that $(\boldsymbol{w}_k, \boldsymbol{d}_k)$ converges weakly to $(\boldsymbol{w}, \boldsymbol{d})$ in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ as $k \to \infty$. We set $(\boldsymbol{u}_k, \boldsymbol{b}_k) = G(\boldsymbol{w}_k, \boldsymbol{d}_k)$ and $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}}) = G(\boldsymbol{w}, \boldsymbol{d})$. We have to prove that $(\boldsymbol{u}_k, \boldsymbol{b}_k)$ converges strongly to $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}})$ in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$ as $k \to \infty$. By the compactness of the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$, we have that $\boldsymbol{w}_k \to \boldsymbol{w}$ strongly in $\boldsymbol{L}^4(\Omega)$ and $\boldsymbol{d}_k \to \boldsymbol{d}$ strongly in $\boldsymbol{L}^4(\Omega)$. By definition $(\boldsymbol{u}_k, \boldsymbol{b}_k)$ and $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}})$ are solutions of

$$a((\boldsymbol{u}_k, \boldsymbol{b}_k), (\boldsymbol{v}, \boldsymbol{\Psi})) + a_{\boldsymbol{w}_k, \boldsymbol{d}_k}((\boldsymbol{u}_k, \boldsymbol{b}_k), (\boldsymbol{v}, \boldsymbol{\Psi})) = \mathcal{L}(\boldsymbol{v}, \boldsymbol{\Psi}),$$

and

$$a((\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}}), (\boldsymbol{v}, \boldsymbol{\Psi})) + a_{\boldsymbol{w}, \boldsymbol{d}}((\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}}), (\boldsymbol{v}, \boldsymbol{\Psi})) = \mathcal{L}(\boldsymbol{v}, \boldsymbol{\Psi}). \tag{1.6.13}$$

where the forms a and $a_{\boldsymbol{w},\boldsymbol{d}}$ are defined in (1.4.16). By subtracting the above problems, we obtain

$$\begin{aligned} & (\mathbf{curl}(\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}), \mathbf{curl}\,\boldsymbol{v}) + (\mathbf{curl}(\boldsymbol{b}_k - \widetilde{\boldsymbol{b}}), \mathbf{curl}\,\boldsymbol{\Psi}) + ((\mathbf{curl}\,\boldsymbol{w}_k) \times \boldsymbol{u}_k - (\mathbf{curl}\,\boldsymbol{w}) \times \widetilde{\boldsymbol{u}}, \boldsymbol{v}) \\ + \; & (\boldsymbol{u}_k, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}_k) - (\widetilde{\boldsymbol{u}}, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}) - ((\mathbf{curl}\,\boldsymbol{b}_k) \times \boldsymbol{d}_k, \boldsymbol{v}) + ((\mathbf{curl}\,\widetilde{\boldsymbol{b}}) \times \boldsymbol{d}, \boldsymbol{v}) = 0 \end{aligned} \tag{1.6.14}$$

Since

$$((\mathbf{curl}\,\boldsymbol{w}_k) \times \boldsymbol{u}_k - (\mathbf{curl}\,\boldsymbol{w}) \times \widetilde{\boldsymbol{u}}, \boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{w} \times (\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}), \boldsymbol{v}) + (\mathbf{curl}(\boldsymbol{w}_k - \boldsymbol{w}) \times \boldsymbol{u}_k, \boldsymbol{v})$$

and

$$\begin{aligned} & (\boldsymbol{u}_k, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}_k) - (\widetilde{\boldsymbol{u}}, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}) - ((\mathbf{curl}\,\boldsymbol{b}_k) \times \boldsymbol{d}_k, \boldsymbol{v}) + ((\mathbf{curl}\,\widetilde{\boldsymbol{b}}) \times \boldsymbol{d}, \boldsymbol{v}) \\ = \; & (\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}) - (\boldsymbol{u}_k, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}) - (\mathbf{curl}(\boldsymbol{b}_k - \widetilde{\boldsymbol{b}}) \times \boldsymbol{d}, \boldsymbol{v}) \\ + \; & ((\mathbf{curl}\,\boldsymbol{b}_k) \times \boldsymbol{d}, \boldsymbol{v}) + (\boldsymbol{u}_k, (\mathbf{curl}\,\boldsymbol{\Psi}) \times \boldsymbol{d}_k) - ((\mathbf{curl}\,\boldsymbol{b}_k) \times \boldsymbol{d}_k, \boldsymbol{v}). \end{aligned}$$

Then, replacing in (1.6.14), we obtain

$$\begin{aligned} & a((\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}}), (\boldsymbol{v}, \boldsymbol{\Psi})) + a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}}), (\boldsymbol{v}, \boldsymbol{\Psi})) \\ & = -(\mathbf{curl}(\boldsymbol{w}_k - \boldsymbol{w}) \times \boldsymbol{u}_k, \boldsymbol{v}) + (\boldsymbol{u}_k, \mathbf{curl}\,\boldsymbol{\Psi} \times (\boldsymbol{d} - \boldsymbol{d}_k)) - (\mathbf{curl}\,\boldsymbol{b}_k \times (\boldsymbol{d} - \boldsymbol{d}_k), \boldsymbol{v}) \end{aligned}$$

By Hölder inequality and the fact that $(\boldsymbol{u}_k, \boldsymbol{b}_k)$ belongs to $\boldsymbol{B}_r$, we have

$$\begin{aligned} |(\mathbf{curl}(\boldsymbol{w}_k - \boldsymbol{w}) \times \boldsymbol{u}_k, \boldsymbol{v})| & \leqslant C_2 \left\| \boldsymbol{w}_k - \boldsymbol{w} \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{u}_k \right\|_{\boldsymbol{H}^1(\Omega)} \left\| \boldsymbol{v} \right\|_{\boldsymbol{H}^1(\Omega)} \\ & \leqslant C_2 M \left\| \boldsymbol{w}_k - \boldsymbol{w} \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{v} \right\|_{\boldsymbol{H}^1(\Omega)}, \end{aligned}$$

$$\begin{aligned} |(\boldsymbol{u}_k, \mathbf{curl}\,\boldsymbol{\Psi} \times (\boldsymbol{d} - \boldsymbol{d}_k))| & \leqslant \left\| \boldsymbol{u}_k \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \mathbf{curl}\,\boldsymbol{\Psi} \right\|_{\boldsymbol{L}^2(\Omega)} \left\| \boldsymbol{d} - \boldsymbol{d}_k \right\|_{\boldsymbol{L}^4(\Omega)} \\ & \leqslant C_1 C_2 M \left\| \boldsymbol{\Psi} \right\|_{\boldsymbol{H}^1(\Omega)} \left\| \boldsymbol{d} - \boldsymbol{d}_k \right\|_{\boldsymbol{L}^4(\Omega)} \end{aligned}$$

and

$$\begin{aligned} |(\mathbf{curl}\,\boldsymbol{b}_k \times (\boldsymbol{d} - \boldsymbol{d}_k), \boldsymbol{v})| & \leqslant C \left\| \mathbf{curl}\,\boldsymbol{b}_k \right\|_{\boldsymbol{L}^2(\Omega)} \left\| \boldsymbol{d} - \boldsymbol{d}_k \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{v} \right\|_{\boldsymbol{L}^4(\Omega)} \\ & \leqslant C_1 C_2 M \left\| \boldsymbol{d} - \boldsymbol{d}_k \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{v} \right\|_{\boldsymbol{H}^1(\Omega)} \end{aligned}$$

where $C_1 > 0$ and $C_2 > 0$ are such that

$$\left\| \mathbf{curl}\,\boldsymbol{v} \right\|_{\boldsymbol{L}^2(\Omega)} \leq C_1 \left\| \boldsymbol{v} \right\|_{\boldsymbol{H}^1(\Omega)} \quad \text{and} \quad \left\| \boldsymbol{v} \right\|_{\boldsymbol{L}^4(\Omega)} \leq C_2 \left\| \boldsymbol{v} \right\|_{\boldsymbol{H}^1(\Omega)}. \tag{1.6.15}$$

Choosing $(\boldsymbol{v}, \boldsymbol{\Psi}) = (\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}})$, thanks to the coercivity of the form a in (1.4.19) and the fact that $a_{\boldsymbol{w},\boldsymbol{d}}((\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}}), (\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}})) = 0$, we obtain

$$\begin{aligned} & \frac{2}{C_{\mathcal{P}}^2} \big(\left\| \boldsymbol{u}_k - \widetilde{\boldsymbol{u}} \right\|_{\boldsymbol{H}^1(\Omega)}^2 + \| \boldsymbol{b}_k - \widetilde{\boldsymbol{b}} \|_{\boldsymbol{H}^1(\Omega)}^2 \big) \leqslant |\, a((\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}}), (\boldsymbol{u}_k - \widetilde{\boldsymbol{u}}, \boldsymbol{b}_k - \widetilde{\boldsymbol{b}})) \,| \\ & \leqslant C_2 M \left\| \boldsymbol{w}_k - \boldsymbol{w} \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{u}_k - \widetilde{\boldsymbol{u}} \right\|_{\boldsymbol{H}^1(\Omega)} + C_1 C_2 M \left\| \boldsymbol{d} - \boldsymbol{d}_k \right\|_{\boldsymbol{L}^4(\Omega)} (\left\| \boldsymbol{u}_k - \widetilde{\boldsymbol{u}} \right\|_{\boldsymbol{H}^1(\Omega)} + \| \boldsymbol{b}_k - \widetilde{\boldsymbol{b}} \|_{\boldsymbol{H}^1(\Omega)}) \\ & \leqslant \frac{1}{2} \left\| \boldsymbol{u}_k - \widetilde{\boldsymbol{u}} \right\|_{\boldsymbol{H}^1(\Omega)}^2 + \frac{1}{2} \| \boldsymbol{b}_k - \widetilde{\boldsymbol{b}} \|_{\boldsymbol{H}^1(\Omega)}^2 + C_2^2 M^2 \left\| \boldsymbol{w}_k - \boldsymbol{w} \right\|_{\boldsymbol{L}^4(\Omega)}^2 + \frac{3}{2} C_1^2 C_2^2 M^2 \left\| \boldsymbol{d}_k - \boldsymbol{d} \right\|_{\boldsymbol{L}^4(\Omega)}^2. \end{aligned}$$

So, we have

$$\left\| \boldsymbol{u}_k - \widetilde{\boldsymbol{u}} \right\|_{\boldsymbol{H}^1(\Omega)}^2 + \| \boldsymbol{b}_k - \widetilde{\boldsymbol{b}} \|_{\boldsymbol{H}^1(\Omega)}^2 \leqslant C\big(\left\| \boldsymbol{w}_k - \boldsymbol{w} \right\|_{\boldsymbol{L}^4(\Omega)}^2 + \left\| \boldsymbol{d}_k - \boldsymbol{d} \right\|_{\boldsymbol{L}^4(\Omega)}^2 \big) \longrightarrow 0 \quad \text{as } k \longrightarrow 0,$$

where $C = C_{\mathcal{P}}^2 \max(C_2^2 M^2, \, \frac{3}{2} C_1^2 C_2^2 M^2)$ is independent of k. Hence, this gives the compactness of G. From Schauder's theorem we then find that G has a fixed point $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}}) = G(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}}) \in \boldsymbol{B}_r$. This fixed point is solution of (1.6.13). Moreover, $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{b}})$ satisfies (1.6.12).

Now, we establish the uniqueness of the solution of (MHD). For this, let $(\boldsymbol{u}_1, \boldsymbol{b}_1, P_1)$ and $(\boldsymbol{u}_2, \boldsymbol{b}_2, P_2)$ in $\boldsymbol{V}_N(\Omega) \times \boldsymbol{V}_N(\Omega) \times L^2(\Omega)$ be two solutions of (MHD). We set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{b} = \boldsymbol{b}_1 - \boldsymbol{b}_2$ et $P = P_1 - P_2$ and we want to prove that $\boldsymbol{u} = \boldsymbol{b} = \boldsymbol{0}$ and $P = 0$. Choose $\boldsymbol{v} = \boldsymbol{u}$ and $\boldsymbol{\Psi} = \boldsymbol{b}$ in (1.6.6), then $(\boldsymbol{u}, \boldsymbol{b})$ satisfies

$$\begin{aligned} &\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}\|^2_{\boldsymbol{L}^2(\Omega)} + ((\mathbf{curl}\,\boldsymbol{u}_1) \times \boldsymbol{u}_1 - (\mathbf{curl}\,\boldsymbol{u}_2) \times \boldsymbol{u}_2, \boldsymbol{u}) \\ &- ((\mathbf{curl}\,\boldsymbol{b}_1) \times \boldsymbol{b}_1 - (\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}_2, \boldsymbol{u}) + ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}_1, \boldsymbol{u}_1) - ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}_2, \boldsymbol{u}_2) = 0 \end{aligned}$$

Observe that

$$((\mathbf{curl}\,\boldsymbol{u}_1) \times \boldsymbol{u}_1, \boldsymbol{u}) - ((\mathbf{curl}\,\boldsymbol{u}_2) \times \boldsymbol{u}_2, \boldsymbol{u}) = ((\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}_1, \boldsymbol{u}).$$

and

$$\begin{aligned} &- ((\mathbf{curl}\,\boldsymbol{b}_1) \times \boldsymbol{b}_1 - (\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}_2, \boldsymbol{u}) + ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}_1, \boldsymbol{u}_1) - ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}_2, \boldsymbol{u}_2) \\ &= ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}, \boldsymbol{u}_2) - ((\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}, \boldsymbol{u}) \end{aligned}$$

which gives

$$\begin{aligned} &\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}\|^2_{\boldsymbol{L}^2(\Omega)} + ((\mathbf{curl}\,\boldsymbol{u}_1) \times \boldsymbol{u}_1 - (\mathbf{curl}\,\boldsymbol{u}_2) \times \boldsymbol{u}_2, \boldsymbol{u}) \\ &- ((\mathbf{curl}\,\boldsymbol{b}_1) \times \boldsymbol{b}_1 - (\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}_2, \boldsymbol{u}) + ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}_1, \boldsymbol{u}_1) - ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}_2, \boldsymbol{u}_2) \\ &= \|\mathbf{curl}\,\boldsymbol{u}\|^2 + \|\mathbf{curl}\,\boldsymbol{b}\|^2 + ((\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}_1, \boldsymbol{u}) \\ &+ ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}, \boldsymbol{u}_2) - ((\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}, \boldsymbol{u}) \end{aligned}$$

Then,

$$\|\mathbf{curl}\,\boldsymbol{u}\|^2 + \|\mathbf{curl}\,\boldsymbol{b}\|^2 = ((\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}, \boldsymbol{u}) - ((\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}_1, \boldsymbol{u}) - ((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}, \boldsymbol{u}_2) \tag{1.6.16}$$

We want to bound the terms in the RHS of (1.6.16). We have

$$\begin{aligned} |((\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}_1, \boldsymbol{u})| &\leqslant \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_1\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \\ &\leqslant C_1 C_2 \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}_1\|_{\boldsymbol{L}^4(\Omega)} \\ &\leqslant C_1 C_2^2 \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}_1\|_{\boldsymbol{H}^1(\Omega)} \leqslant C_1 C_2^2 M \|\boldsymbol{u}\|^2_{\boldsymbol{H}_1(\Omega)}, \end{aligned}$$

$$\begin{aligned} |((\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}, \boldsymbol{u}_2)| &\leqslant \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{b}\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{L}^4(\Omega)} \\ &\leqslant C_1 C_2^2 \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{H}^1(\Omega)} \\ &\leqslant C_1 C_2^2 M \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} |((\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{b}, \boldsymbol{u})| &\leqslant \|\mathbf{curl}\,\boldsymbol{b}_2\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{b}\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \\ &\leqslant C_1 C_2^2 \|\boldsymbol{b}_2\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \\ &\leqslant \frac{1}{2} C_1 C_2^2 M (\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}). \end{aligned}$$

Using these estimates in (1.6.16) together with Poincaré's type inequality (1.3.6), we obtain

$$\frac{1}{C_{\mathcal{P}}^2}\big(\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}\,\big)\leqslant\|\mathbf{curl}\,\boldsymbol{u}\|^2+\|\mathbf{curl}\,\boldsymbol{b}\|^2\leqslant\frac{3}{2}C_1C_2^2M(\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}),$$

where $C_{\mathcal{P}}$ is the constant in (1.3.6). From this relation, we obtain

$$(\frac{1}{C_{\mathcal{P}}^2}-\frac{3}{2}C_1C_2^2M)\big(\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)}+\|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}\,\big)\leq 0.$$

This, together with condition (1.6.5), implies that $\boldsymbol{u}=\boldsymbol{b}=\boldsymbol{0}$. The construction of the pressure $P\in L^2(\Omega)$ follows from De Rham's Theorem (see 1.6.2).

1.6.2 Weak solution: L^p-theory, $\ p\geq 2$

In this subsection, we study the regularity of weak solution of system (*MHD*) in L^p-theory. We start with the case $p\geq 2$. The proof is done essentially using the existence of weak solution in the hilbertian case and a bootstrap argument. To take advantage of the regularity of the Stokes problem $(\mathcal{S}_{\mathcal{N}})$ and the elliptic problem $(\mathcal{E}_{\mathcal{N}})$, we can rewrite the (*MHD*) problem in the following way:

$$\begin{cases}-\Delta\boldsymbol{u}+\nabla P=\boldsymbol{f}-(\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u}+(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{d}\qquad\text{in }\Omega,\\ \operatorname{div}\boldsymbol{u}=h\quad\text{in }\Omega,\\ \boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}\quad\text{on }\Gamma,\\ P=P_0\quad\text{on }\Gamma_0\quad\text{and}\quad P=P_0+c_i\ \text{ on }\Gamma_i,\\ \langle\boldsymbol{u}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,\ 1\leq i\leq I,\end{cases}$$

and

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b}=\boldsymbol{g}+\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b})\quad\text{in }\Omega,\\ \operatorname{div}\boldsymbol{b}=0\quad\text{in }\Omega,\\ \boldsymbol{b}\times\boldsymbol{n}=\boldsymbol{0}\quad\text{on }\Gamma\\ \langle\boldsymbol{b}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,\quad\forall 1\leqslant i\leqslant I.\end{cases}$$

The following result can be improved in the same way as in Corollary 1.5.10 by considering a data P_0 less regular.

Theorem 1.6.3 (Regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $p>2$)**.** *Let $p>2$ and $r=\frac{3p}{3+p}$. Suppose that $\boldsymbol{f},\boldsymbol{g}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, $h=0$, $P_0\in W^{1-\frac{1}{r},r}(\Gamma)$ with the compatibility conditions* (1.5.27)-(1.5.28)*. Then the weak solution for the* (*MHD*) *system given by Theorem 1.6.1 satisfies*

$$(\boldsymbol{u},\boldsymbol{b},P)\in\boldsymbol{W}^{1,p}(\Omega)\times\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega).$$

Moreover, we have the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|P\|_{W^{1,r}(\Omega)}\leqslant C(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega))'}+\|\boldsymbol{g}\|_{(\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega))'}+\|P_0\|_{W^{1/r',r}(\Gamma)})\quad(1.6.17)$$

Proof. Since $p>2$, we have $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'\hookrightarrow[\boldsymbol{H}_0^{6,2}(\mathbf{curl},\Omega)]'$ and $W^{1-1/r,r}(\Gamma)\hookrightarrow H^{-1/2}(\Gamma)$. Thanks to Theorem 1.6.1, there exists $(\boldsymbol{u},\boldsymbol{b},P,\boldsymbol{c})\in\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)\times L^2(\Omega)\times\mathbb{R}^I$ solution of (*MHD*). By using the embedding $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$, it follows that $(\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b}$ belong to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$. To apply the regularity of Stokes problem and obtain weak solutions $(\boldsymbol{u},P)\in\boldsymbol{W}^{1,p}(\Omega)\times W^{1,r}(\Omega)$, we must justify that the RHS $\boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Similarly, to apply the regularity of the elliptic problem $(\mathcal{E}_\mathcal{N})$ and obtain a solution $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$, we also need to justify that the RHS $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. As a consequence, we distinguish according to the values of p the following cases:

(i) If $p \leq 3$, then $\boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^{3/2}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega) = \boldsymbol{L}^r(\Omega)$. Since $\boldsymbol{L}^r(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, thanks to the regularity of the Stokes problem $(\mathcal{S}_\mathcal{N})$, see Proposition 1.3.2, we have that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $P \in W^{1,r}(\Omega)$. Since $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{b}$, we have with the same argument above that $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. It follows that $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Moreover, $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ satisfies the compatibility conditions (1.5.27)-(1.5.28). Consequently, thanks to the regularity of the elliptic problem $(\mathcal{E}_\mathcal{N})$, see Lemma 1.3.4, we have that $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$.

(ii) If $p > 3$, from the previous case, we have that $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{1,3}(\Omega) \times \boldsymbol{W}^{1,3}(\Omega) \times W^{1,3/2}(\Omega)$. Therefore, $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{L}^q(\Omega) \times \boldsymbol{L}^q(\Omega)$, for any $1 < q < \infty$. Then, $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \in \boldsymbol{L}^m(\Omega)$ for all $1 \leq m < 3$. In particular, we take $\frac{1}{m} = \frac{1}{p} + \frac{1}{3}$ with $3/2 < m < 3$. So, we have that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Using the same arguments, we obtain that $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^{\frac{3p}{3+p}}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$. Then, the required regularity for $(\boldsymbol{u}, \boldsymbol{b}, P)$ follows by applying the regularity of the Stokes problem $(\mathcal{S}_\mathcal{N})$. Further, we have $\boldsymbol{u} \times \boldsymbol{b} \in \boldsymbol{L}^t(\Omega)$ for any $1 < t < \infty$. In particular, taking $t = p$ with $3 < t < \infty$, we have that $\boldsymbol{u} \times \boldsymbol{b} \in \boldsymbol{L}^p(\Omega)$. So, due to the characterization of the space $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, we have $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ belongs to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ and we finish the proof by applying the regularity of the elliptic problem $(\mathcal{E}_\mathcal{N})$.

□

1.6.3 Strong solution: L^p-theory, $p \geq 6/5$

In this subsection, we study the existence of strong solutions for more regular data. The following theorem gives the regularity $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq 6/5$.

Theorem 1.6.4 (Regularity $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$)**.** *Let us suppose that Ω is of class $\mathcal{C}^{2,1}$ and $p \geq \frac{6}{5}$. Let $\boldsymbol{f}$, $\boldsymbol{g}$ and P_0 satisfy* (1.5.27)-(1.5.28) *and*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \in \boldsymbol{L}^p(\Omega), \quad h = 0 \quad \text{and} \quad P_0 \in W^{1-\frac{1}{p},p}(\Gamma).$$

Then the weak solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ for the (MHD) system given by Theorem 1.6.1 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the following estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leqslant C(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}) \tag{1.6.18}$$

Proof. To start the proof, the idea is to use the regularity result for weak solutions in $\boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$ given in Theorem 1.6.3 instead of the weak solutions in the Hilbert case $\boldsymbol{H}^1(\Omega)$. Observe that if $6/5 \leq p \leq 3/2$, we have $2 < p^* < 3$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Now, denoting $r(p^*) = \frac{3p^*}{3+p^*}$, we obtain that $r(p^*) = p$. Then, we have from the hypothesis of this Theorem that $\boldsymbol{f} \in \boldsymbol{L}^{r(p^*)}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^{r(p^*)}(\Omega)$ and $P_0 \in W^{1-1/r(p^*),r(p^*)}(\Gamma)$. Since $\boldsymbol{L}^{r(p^*)}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{(r^{p^*})',(p^*)'}(\mathbf{curl}, \Omega)]'$ and $p^* > 2$, we deduce from the regularity result of the (*MHD*) problem (see Theorem 1.6.4) that $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{1,p^*}(\Omega) \times \boldsymbol{W}^{1,p^*}(\Omega) \times W^{1,r(p^*)}(\Omega)$. Then, we have the following three cases:

(i) Case $\frac{6}{5} \leq p < \frac{3}{2}$: We have $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \in \boldsymbol{L}^t(\Omega)$ with $\frac{1}{t} = \frac{2}{p} - 1$. Since $t > p$, it follows that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\Omega)$. The same argument gives that $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belongs to $\boldsymbol{L}^p(\Omega)$. Consequently, thanks to the existence of strong solutions for the Stokes problem $(\mathcal{S}_\mathcal{N})$ (see Proposition 1.3.2), we deduce that $(\boldsymbol{u}, P) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Since $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ belongs also to $\boldsymbol{L}^t(\Omega)$ with $t > p$, we deduce from the regularity result of the elliptic problem $(\mathcal{E}_\mathcal{N})$ (see Theorem 1.3.3) that $\boldsymbol{b} \in \boldsymbol{W}^{2,p}(\Omega)$.

(ii) Case $p = \frac{3}{2}$: We have in this case $p^* = 3$. From above, we know that $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{1,3}(\Omega) \times \boldsymbol{W}^{1,3}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$. Since $\boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^q(\Omega)$ for any $1 < q < \infty$, we deduce that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belong to $\boldsymbol{L}^t(\Omega)$ with $\frac{1}{t} = \frac{1}{3} + \frac{1}{q}$. Choosing $q > 3$ gives $t > \frac{3}{2}$. Thanks to the regularity of the Stokes problem $(\mathcal{S}_\mathcal{N})$, we have that $(\boldsymbol{u}, P) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$. Using the same arguments, we have $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{u}) \in \boldsymbol{L}^t(\Omega)$ with $t > \frac{3}{2}$. Then, we apply the regularity of the elliptic problem $(\mathcal{E}_\mathcal{N})$ to obtain $\boldsymbol{b} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$.

(iii) Case $p > \frac{3}{2}$: We know that $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. Then, $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{L}^q(\Omega) \times \boldsymbol{L}^q(\Omega)$ with $1 < q < \infty$. We deduce that the terms $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belong to $\boldsymbol{L}^t(\Omega)$ with $1 \leq t < 3$. So, we have the following two cases:

(a) If $\frac{3}{2} < p < 3$, we have $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$, $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ and $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ belong to $\boldsymbol{L}^p(\Omega)$. Thanks to the regularity of $(\mathcal{S}_\mathcal{N})$ and $(\mathcal{E}_\mathcal{N})$, we deduce that $(\boldsymbol{u}, \boldsymbol{b}, P)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$.

(b) If $p \geq 3$, from the above result, we have $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,3-\varepsilon}(\Omega) \times \boldsymbol{W}^{2,3-\varepsilon}(\Omega) \times W^{1,3-\varepsilon}(\Omega)$ with $0 < \varepsilon < \frac{3}{2}$. Observe that $(3-\varepsilon)^* = \frac{3(3-\varepsilon)}{\varepsilon} > 3$. This implies that $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{b} \in \boldsymbol{L}^\infty(\Omega)$. Since $\mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{W}^{1,3-\varepsilon}(\Omega) \hookrightarrow \boldsymbol{L}^{(3-\varepsilon)^*}(\Omega)$, it follows that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belong to $\boldsymbol{L}^{(3-\varepsilon)^*}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Again, according the regularity of Stokes problem, we have $(\boldsymbol{u}, P) \in \boldsymbol{W}^{2,3}(\Omega) \times W^{1,3}(\Omega)$. Similarly, $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \in \boldsymbol{L}^3(\Omega)$ and the regularity of the elliptic problem $(\mathcal{E}_\mathcal{N})$ implies that $\boldsymbol{b} \in \boldsymbol{W}^{1,3}(\Omega)$. Finally, using the embeddings $\boldsymbol{W}^{2,3}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^q(\Omega)$ for $1 < q < \infty$, all the terms $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$, $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ and $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ belong to $\boldsymbol{L}^q(\Omega)$. To conclude, we apply once again the regularity of Stokes problem $(\mathcal{S}_\mathcal{N})$ and elliptic problem $(\mathcal{E}_\mathcal{N})$.

$\square$

1.6.4 Existence result of the MHD system for $3/2 < p < 2$

The next theorem tells us that it is possible to extend the regularity $\boldsymbol{W}^{1,p}(\Omega)$ of the solution of the nonlinear (*MHD*) problem for $\frac{3}{2} < p < 2$. For this, we apply Banach's fixed-point theorem over the linearized problem (1.4.2).

Theorem 1.6.5 (Regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $\frac{3}{2} < p < 2$)**.** *Assume that $\frac{3}{2} < p < 2$ and let r be defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$. Let us consider $\boldsymbol{f}, \boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$, $P_0 \in W^{1-\frac{1}{r},r}(\Gamma)$ and $h \in W^{1,r}(\Omega)$ with the compatibility conditions* (1.5.27)-(1.5.28).

(i) *There exists a constant δ_1 such that, if*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_1$$

Then, the (MHD) problem has at least a solution $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{\alpha}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the following estimates:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C_1\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)}\big) \tag{1.6.19}$$

$$\|P\|_{W^{1,r}(\Omega)} \leqslant C_1(1 + C^*\eta)\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)}\big) \tag{1.6.20}$$

where $\delta_1 = (2C^2C^*)^{-1}$, $C_1 = C(1+C^*\eta)^2$ *with* $C>0$, $C^*>0$ *are the constants given in* (1.6.25) *and* η *defined by* (1.6.26). *Furthermore,* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_I)$ *satisfies*

$$\alpha_i = \langle \boldsymbol{f}, \nabla q_i^N \rangle_\Omega - \int_\Omega (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u} \cdot \nabla q_i^N \, d\boldsymbol{x} + \int_\Omega (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d} \cdot \nabla q_i^N \, d\boldsymbol{x} + \int_\Gamma (h - P_0) \nabla q_i^N \cdot \boldsymbol{n} \, d\sigma$$

(ii) *Moreover, if the data satisfy that*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_2,$$

for some $\delta_2 \in]0, \delta_1]$, *then the weak solution of* (*MHD*) *is unique.*

Proof.

(i) Existence: Let us define the space

$$\boldsymbol{Z}^p(\Omega) = \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}_\sigma^{1,p}(\Omega)$$

For given $(\boldsymbol{w}, \boldsymbol{d}) \in \boldsymbol{B}_\eta$, define the operator T by $T(\boldsymbol{w}, \boldsymbol{d}) = (\boldsymbol{u}, \boldsymbol{b})$ where $(\boldsymbol{u}, \boldsymbol{b})$ is the unique solution of the linearized problem (1.4.2) given by Theorem 1.5.7 and the neighbourhood $\boldsymbol{B}_\eta$ is defined by

$$\boldsymbol{B}_\eta = \{(\boldsymbol{w}, \boldsymbol{d}) \in \boldsymbol{Z}^p(\Omega),\ \|(\boldsymbol{w}, \boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)} \leq \eta\}, \quad \eta > 0.$$

Here $\boldsymbol{Z}^p(\Omega)$ is equipped with the norm

$$\|(\boldsymbol{w}, \boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)} = \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

We have to prove that T is a contraction from $\boldsymbol{B}_\eta$ to itself. Let $(\boldsymbol{w}_1, \boldsymbol{d}_1), (\boldsymbol{w}_2, \boldsymbol{d}_2) \in \boldsymbol{B}_\eta$. We show that there exists $\theta \in (0,1)$ such that:

$$\|T(\boldsymbol{w}_1, \boldsymbol{d}_1) - T(\boldsymbol{w}_2, \boldsymbol{d}_2)\|_{\boldsymbol{Z}^p(\Omega)} = \|(\boldsymbol{u}_1, \boldsymbol{b}_1) - (\boldsymbol{u}_2, \boldsymbol{b}_2)\|_{\boldsymbol{Z}^p(\Omega)} \leqslant \theta\, \|(\boldsymbol{w}_1, \boldsymbol{d}_1) - (\boldsymbol{w}_2, \boldsymbol{d}_2)\|_{\boldsymbol{Z}^p(\Omega)} \quad (1.6.21)$$

Thanks to Corollary 1.5.8, each $(\boldsymbol{u}_i, \boldsymbol{b}_i, P_i, \boldsymbol{c}^i)$, $i = 1, 2$, belongs to $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times W^{1,r}(\Omega)$ and verifies:

$$\begin{cases} -\Delta \boldsymbol{u}_i + (\mathbf{curl}\, \boldsymbol{w}_i) \times \boldsymbol{u}_i + \nabla P_i - (\mathbf{curl}\, \boldsymbol{b}_i) \times \boldsymbol{d}_i = \boldsymbol{f} \text{ and } \operatorname{div} \boldsymbol{u}_i = h \text{ in } \Omega \\ \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{b}_i - \mathbf{curl}(\boldsymbol{u}_i \times \boldsymbol{d}_i) = \boldsymbol{g} \text{ and } \operatorname{div} \boldsymbol{b}_i = 0 \text{ in } \Omega \\ \boldsymbol{u}_i \times \boldsymbol{n} = 0 \text{ and } \boldsymbol{b}_i \times \boldsymbol{n} = 0 \text{ on } \Gamma \\ P_i = P_0 \text{ on } \Gamma_0 \text{ and } P_i = P_0 + c_j^{(i)} \text{ on } \Gamma_j \\ \langle \boldsymbol{u}_i \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 \text{ and } \langle \boldsymbol{b}_i \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0, \ \forall 1 \leqslant j \leqslant I \end{cases} \quad (1.6.22)$$

together with following estimate for $i = 1, 2$:

$$\|(\boldsymbol{u}_i, \boldsymbol{b}_i)\|_{\boldsymbol{Z}^p(\Omega)} \leqslant C\Big(1 + \|\mathbf{curl}\, \boldsymbol{w}_i\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}_i\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)\Big(\gamma_1 + (1 + \|\mathbf{curl}\, \boldsymbol{w}_i\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}_i\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})\gamma_2\Big) \quad (1.6.23)$$

where $\gamma_1 = \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|P_0\|_{W^{1-\frac{1}{r},r}(\Gamma)}$ and $\gamma_2 = \|h\|_{W^{1,r}(\Omega)}$.

Next, the differences $(\boldsymbol{u}, \boldsymbol{b}, P, \boldsymbol{c}) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{b}_1 - \boldsymbol{b}_2, P_1 - P_2, \boldsymbol{c}^1 - \boldsymbol{c}^2)$ satisfy

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{w}_1) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{d}_1 = \boldsymbol{f}_2 \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \\ \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}_1) = \boldsymbol{g}_2 \text{ and } \operatorname{div} \boldsymbol{b} = 0 \text{ in } \Omega \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ and } \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma \\ P = 0 \text{ on } \Gamma_0 \quad \text{and} \quad P = c_j \quad \text{on } \Gamma_j \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 \text{ and } \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 \ \forall 1 \leqslant j \leqslant I \end{cases} \quad (1.6.24)$$

with $\boldsymbol{f}_2 = -(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}_2 + (\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{d}$ and $\boldsymbol{g}_2 = \mathbf{curl}(\boldsymbol{u}_2 \times \boldsymbol{d})$. Observe that $\boldsymbol{f}_2$ and $\boldsymbol{g}_2$ belong to $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$. Indeed, Since $\boldsymbol{u}_2, \boldsymbol{b}_2 \in \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$, $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{W}^{1,\frac{3}{2}}(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}_2$ and $(\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{d}$ belong to $\boldsymbol{L}^r(\Omega) \hookrightarrow [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$. Besides, $\boldsymbol{u}_2 \times \boldsymbol{d} \in \boldsymbol{L}^p(\Omega)$ so $\mathbf{curl}(\boldsymbol{u}_2 \times \boldsymbol{d}) \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$. Moreover, since $\boldsymbol{g}_2$ is a $\mathbf{curl}$, then it satisfies the conditions (1.5.27)-(1.5.28). Hence, we apply the Theorem 1.5.7 and we have the estimate:

$$\|(\boldsymbol{u},\boldsymbol{b})\|_{\boldsymbol{Z}^p(\Omega)} \leqslant C(1 + \|\mathbf{curl}\,\boldsymbol{w}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}_1\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})(\|\boldsymbol{f}_2\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} + \|\boldsymbol{g}_2\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'})$$

By the definition of the norm on $[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'$, the Hölder inequality and the embeddings

$$\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega) \text{ and } \boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega),$$

it follows:

$$\begin{aligned}
\|\boldsymbol{f}_2\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} &\leqslant \|(\mathbf{curl}\,\boldsymbol{w}) \times \boldsymbol{u}_2\|_{\boldsymbol{L}^r(\Omega)} + \|(\mathbf{curl}\,\boldsymbol{b}_2) \times \boldsymbol{d}\|_{\boldsymbol{L}^r(\Omega)} \\
&\leqslant \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{L}^{p^*}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}_2\|_{\boldsymbol{L}^p(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \\
&\leqslant C(C_w \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} + C_d \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{b}_2\|_{\boldsymbol{W}^{1,p}(\Omega)})
\end{aligned}$$

and

$$\|\boldsymbol{g}_2\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\Omega)]'} = \|\boldsymbol{u}_2 \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{u}_2\|_{\boldsymbol{L}^{p^*}(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \leqslant CC_d \|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}$$

where $C_w > 0$ and $C_d > 0$ are respectively defined by $\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \leqslant C_w \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)}$ and $\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \leqslant C_d \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}$. Therefore, recalling that $(\boldsymbol{w}_1, \boldsymbol{d}_1)$ belongs to $\boldsymbol{B}_\eta$, we have:

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leqslant C(1 + C^* \|(\boldsymbol{w}_1,\boldsymbol{d}_1)\|_{\boldsymbol{Z}^p(\Omega)})(C_w \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} + C_d \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,p}(\Omega)}) \|(\boldsymbol{u}_2,\boldsymbol{b}_2)\|_{\boldsymbol{Z}^p(\Omega)} \\
&\leqslant C(1 + C^*\eta)C^* \|(\boldsymbol{w},\boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)} \|(\boldsymbol{u}_2,\boldsymbol{b}_2)\|_{\boldsymbol{Z}^p(\Omega)}
\end{aligned}$$

with $C^* = C_w + C_d$. Combining with (1.6.23), we thus obtain:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C_1 C^* \|(\boldsymbol{w},\boldsymbol{d})\|_{\boldsymbol{Z}^p(\Omega)} (\gamma_1 + (1 + C^*\eta)\gamma_2) \tag{1.6.25}$$

where $C_1 = C(1 + C^*\eta)^2$. Hence, if we choose, for instance:

$$\eta = (C^*)((2CC^*\gamma)^{-\frac{1}{3}} - 1) \text{ and } \gamma = \gamma_1 + \gamma_2 < (2CC^*)^{-1} \tag{1.6.26}$$

then $C_1 C^*(\gamma_1 + (1 + C^*\eta)\gamma_2) < \frac{1}{2}$. Therefore T is a contraction and we obtain the unique fixed-point $(\boldsymbol{u}^*, \boldsymbol{b}^*) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}_\sigma(\Omega)$ satisfying

$$\|(\boldsymbol{u}^*,\boldsymbol{b}^*)\|_{\boldsymbol{Z}^p(\Omega)} \leqslant C\Big(1 + C^* \|(\boldsymbol{u}^*,\boldsymbol{b}^*)\|_{\boldsymbol{Z}^p(\Omega)}\Big)\Big(\gamma_1 + (1 + C^* \|(\boldsymbol{u}^*,\boldsymbol{b}^*)\|_{\boldsymbol{Z}^p(\Omega)})\gamma_2\Big)$$

Next, since $(\boldsymbol{u}^*, \boldsymbol{b}^*) \in \boldsymbol{B}_\eta$, we obtain

$$\|(\boldsymbol{u}^*,\boldsymbol{b}^*)\|_{\boldsymbol{Z}^p(\Omega)} \leqslant C(1 + C^*\eta)(\gamma_1 + (1 + C^*\eta)\gamma_2) \leqslant C_1\gamma \tag{1.6.27}$$

which implies the estimate (1.6.19):
Now, we want to prove the estimate for the associated pressure. Taking the divergence in the first equation of problem (MHD), we have that P^* is a solution of the following problem:

$$\begin{cases}
\Delta P^* = \operatorname{div} \boldsymbol{f} + \operatorname{div}((\mathbf{curl}\,\boldsymbol{b}^*) \times \boldsymbol{b}^* - (\mathbf{curl}\,\boldsymbol{u}^*) \times \boldsymbol{u}^*) + \Delta h \qquad \text{in } \Omega, \\
P^* = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P^* = P_0 + c_i \quad \text{on } \Gamma_i
\end{cases}$$

with

$$\|P^*\|_{W^{1,r}(\Omega)} \leqslant \|\operatorname{div} \boldsymbol{f}\|_{W^{-1,r}(\Omega)} + \|\operatorname{div}((\mathbf{curl}\, \boldsymbol{b}^*) \times \boldsymbol{b}^* - (\mathbf{curl}\, \boldsymbol{u}^*) \times \boldsymbol{u}^*)\|_{W^{-1,r}(\Omega)} + \|\Delta h\|_{W^{-1,r}(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}$$

Following the same calculus as in the proof of Theorem 1.5.7, we obtain

$$\|P^*\|_{W^{1,r}(\Omega)} \leqslant C(1 + C^*\eta)^2(\gamma_1 + (1 + C^*\eta)\gamma_2).$$

which implies (1.6.20) and the proof of existence is completed.

(ii) Uniqueness:

Let $(\boldsymbol{u}_1, \boldsymbol{b}_1, P_1)$ and $(\boldsymbol{u}_2, \boldsymbol{b}_2, P_2)$ two solutions of the problem (MHD). Then, we set $(\boldsymbol{u}, \boldsymbol{b}, P) = (\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{b}_1 - \boldsymbol{b}_2, P_1 - P_2)$ which satisfies the problem:

$$\begin{cases} -\Delta \boldsymbol{u} + (\mathbf{curl}\, \boldsymbol{u}_1) \times \boldsymbol{u} + \nabla P - (\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{b}_1 = \boldsymbol{f}_2 \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b} - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}_1) = \boldsymbol{g}_2 \text{ and } \operatorname{div} \boldsymbol{b} = 0 \text{ in } \Omega \\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma \\ P = 0 \text{ on } \Gamma_0 \quad \text{and} \quad P = \alpha_i^{(1)} - \alpha_i^{(2)} \quad \text{on } \Gamma_i, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \text{ and } \langle \boldsymbol{b} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \ \forall\, 1 \leqslant i \leqslant I \end{cases}$$

where $\boldsymbol{f}_2, \boldsymbol{g}_2 \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]'$ are already given in (1.6.24) and satisfy the hypothesis of Theorem 1.5.7. Applying this theorem, we have the estimate

$$\begin{aligned} \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}^p(\Omega)} &\leqslant C(1 + \|\mathbf{curl}\, \boldsymbol{u}_1\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{b}_1\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})(C_w \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}(\Omega)} + C_d \|\boldsymbol{b}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{b}_2\|_{\boldsymbol{W}^{1,p}(\Omega)}) \\ &\leqslant C(1 + C^* \|(\boldsymbol{u}_1, \boldsymbol{b}_1)\|_{\boldsymbol{Z}^p(\Omega)}) C^* \|(\boldsymbol{u}_2, \boldsymbol{b}_2)\|_{\boldsymbol{Z}^p(\Omega)} \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}^p(\Omega)}. \end{aligned}$$

From (1.6.27), we obtain that:

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}^p(\Omega)} \leqslant C(1 + C^* C_1 \gamma) C^* C_1 \gamma \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}^p(\Omega)}$$

Thus, for γ small enough such that

$$C(1 + C^* C_1 \gamma) C^* C_1 \gamma < 1$$

we deduce that $\boldsymbol{u} = \boldsymbol{b} = \mathbf{0}$ and then we obtain the uniqueness of the velocity and the magnetic field which implies the uniqueness of the pressure P. □

1.7 Appendix

We begin this section by giving another proof of Theorem 1.5.11.

A second proof of Theorem 1.5.11: Let $\lambda > 0$, and let us assume $\boldsymbol{f}_\lambda, \boldsymbol{g}_\lambda \in \boldsymbol{D}(\Omega)$ such that $\boldsymbol{f}_\lambda$ and $\boldsymbol{g}_\lambda$ respectively converge to $\boldsymbol{f}$ and $\boldsymbol{g}$ in $\boldsymbol{L}^p(\Omega)$, and $P_0^\lambda \in C^\infty(\Gamma)$ which converges to P_0 in $W^{1-\frac{1}{p},p}(\Gamma)$.

We consider the problem: find $(\boldsymbol{u}_\lambda, \boldsymbol{b}_\lambda, P_\lambda, c_i^\lambda)$ solution of:

$$\begin{cases} -\Delta \boldsymbol{u}_\lambda + (\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_\lambda + \nabla P_\lambda - (\mathbf{curl}\, \boldsymbol{b}_\lambda) \times \boldsymbol{d} = \boldsymbol{f}_\lambda \text{ and } \operatorname{div} \boldsymbol{u}_\lambda = 0 \text{ in } \Omega \\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{b}_\lambda - \mathbf{curl}(\boldsymbol{u}_\lambda \times \boldsymbol{d}) = \boldsymbol{g}_\lambda \text{ and } \operatorname{div} \boldsymbol{b}_\lambda = 0 \text{ in } \Omega \\ P_\lambda = P_0^\lambda, \quad \text{on } \Gamma_0,\ P_\lambda = P_0^\lambda + c_i^\lambda \quad \text{on } \Gamma_i \\ \boldsymbol{u}_\lambda \times \boldsymbol{n} = 0, \quad \boldsymbol{b}_\lambda \times \boldsymbol{n} = 0 \quad \text{on } \Gamma \\ \langle \boldsymbol{u}_\lambda \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0, \quad \langle \boldsymbol{b}_\lambda \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0,\ \forall\, 1 \leqslant i \leqslant I \end{cases} \tag{1.7.1}$$

Note that, since $\boldsymbol{f}_\lambda, \boldsymbol{g}_\lambda \in \boldsymbol{D}(\Omega)$, in particular they belong to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Thus, applying Theorem 1.5.1, we have $(\boldsymbol{u}_\lambda, \boldsymbol{b}_\lambda) \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \times \boldsymbol{W}^{2,\frac{6}{5}}(\Omega) \hookrightarrow \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega) \times \boldsymbol{L}^6(\Omega)$. Therefore, $(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_\lambda$, $(\mathbf{curl}\, \boldsymbol{b}_\lambda) \times \boldsymbol{d}$ and $\mathbf{curl}(\boldsymbol{u}_\lambda \times \boldsymbol{d})$ belong to $\boldsymbol{L}^{\frac{6}{5}}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$. Hence, using the regularity results of the Stokes and elliptic problems, we obtain that the problem (1.7.1) has a unique solution $(\boldsymbol{u}_\lambda, \boldsymbol{b}_\lambda, P_\lambda) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ which also satisfies the estimate:

$$\begin{aligned} &\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P_\lambda\|_{W^{1,p}(\Omega)} \\ &\leqslant C_{SE}\Big(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \left\|P_0^\lambda\right\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \sum_{i=1}^{I} |c_i^\lambda| + \|(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \\ &+ \|(\mathbf{curl}\, \boldsymbol{b}_\lambda) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}(\boldsymbol{u}_\lambda \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)} \Big) \end{aligned} \tag{1.7.2}$$

with $C_{SE} = \max(C_S, C_E)$ where C_S is the constant given in the Proposition 1.3.2 and C_E the constant given in the Theorem 1.3.3. We now want to bound the right hand side terms $\|(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}$, $\|(\mathbf{curl}\, \boldsymbol{b}_\lambda) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$, $\|\mathbf{curl}(\boldsymbol{u}_\lambda \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)}$ and $\sum_{i=1}^{I} |c_i^\lambda|$ to obtain the estimate (1.5.99). In this purpose, we decompose $\mathbf{curl}\, \boldsymbol{w}$ and $\boldsymbol{d}$ as in (1.5.4)-(1.5.5).

Let $\epsilon > 0$ and $\rho_{\epsilon/2}$ the classical mollifier. We consider $\tilde{\boldsymbol{y}} = \widetilde{\mathbf{curl}\, \boldsymbol{w}}$ and $\tilde{\boldsymbol{d}}$ the extensions by 0 of $\boldsymbol{y} = \mathbf{curl}\, \boldsymbol{w}$ and $\boldsymbol{d}$ in $\mathbb{R}^3$ respectively. We take:

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{w} = \boldsymbol{y}_1^\epsilon + \boldsymbol{y}_2^\epsilon \text{ where } \boldsymbol{y}_1^\epsilon = \tilde{\boldsymbol{y}} * \rho_{\epsilon/2} \text{ and } \boldsymbol{y}_2^\epsilon = \mathbf{curl}\, \boldsymbol{w} - \boldsymbol{y}_1^\epsilon \\ \boldsymbol{d} = \boldsymbol{d}_1^\epsilon + \boldsymbol{d}_2^\epsilon \text{ where } \boldsymbol{d}_1^\epsilon = \tilde{\boldsymbol{d}} * \rho_{\epsilon/2} \text{ and } \boldsymbol{d}_2^\epsilon = \boldsymbol{d} - \boldsymbol{d}_1^\epsilon \end{aligned} \tag{1.7.3}$$

For each term, we start by bounding the part depending on $\boldsymbol{d}_2^\epsilon$ (resp. $\boldsymbol{y}_2^\epsilon$), and then we look at $\boldsymbol{d}_1^\epsilon$ (resp. $\boldsymbol{y}_1^\epsilon$).

(i) Estimate of the term $\|(\mathbf{curl}\, \boldsymbol{w}) \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}$:

First, following the definition of the mollifier, we classically obtain:

$$\|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} = \left\|\boldsymbol{y} - \tilde{\boldsymbol{y}} * \rho_{\epsilon/2}\right\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \leqslant \epsilon \tag{1.7.4}$$

Then, since we have $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$ and the Hölder inequality, we obtain:

$$\|\boldsymbol{y}_2^\epsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^{**}}(\Omega)} \leqslant C_1 \|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} \tag{1.7.5}$$

where C_1 is the constant related to the previous Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$. Thus, injecting (1.7.4) in (1.7.5), it follows:

$$\|\boldsymbol{y}_2^\epsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_1 \epsilon \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}$$

Now, for the term in $\boldsymbol{y}_1^\epsilon$, we apply the Hölder inequality to obtain:

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\boldsymbol{y}_1^\epsilon\|_{\boldsymbol{L}^m(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^q(\Omega)} \leqslant \|\boldsymbol{y}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^t(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^q(\Omega)}$$

with $m, q \geqslant p$ such that $\frac{1}{p} = \frac{1}{m} + \frac{1}{q}$ and $t > 1$ defined by $1 + \frac{1}{m} = \frac{2}{3} + \frac{1}{t}$. Note that this definition imposes $m > \frac{3}{2}$, and we can take in particular $m = 3$, and hence $q = p^*$. Since, following the properties of the mollifier, there exists $C_\epsilon > 0$ such that, for all $t > 1$:

$$\left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^t(\Omega)} \leqslant C_\epsilon \tag{1.7.6}$$

so we have

$$\|\boldsymbol{y}_1^\epsilon \times \boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_\epsilon \|\boldsymbol{y}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}$$

(ii) Estimate of the term $\|(\mathbf{curl}\,\boldsymbol{b}_\lambda) \times \boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$: As previously, we have from the definition of the mollifier:

$$\|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} = \left\|\boldsymbol{d} - \tilde{\boldsymbol{d}} * \rho_{\epsilon/2}\right\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} \leqslant \epsilon \tag{1.7.7}$$

Then, combining the Hölder inequality and the Sobolev embedding $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$, we have:

$$\|(\mathbf{curl}\,\boldsymbol{b}_\lambda) \times \boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\mathbf{curl}\,\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)} \|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^3(\Omega)} \leqslant C_2\epsilon \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{1,p^*}(\Omega)}$$

where C_2 is the constant related to the Sobolev embedding $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. We finally recall the Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p^*}(\Omega)$ to obtain:

$$\|(\mathbf{curl}\,\boldsymbol{b}_\lambda) \times \boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_2 C_3 \epsilon \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} \tag{1.7.8}$$

with C_3 the constant related to the Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p^*}(\Omega)$. It remains to bound the term in $\boldsymbol{d}_1^\epsilon$. Applying the Hölder inequality, we have:

$$\|(\mathbf{curl}\,\boldsymbol{b}_\lambda) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant \|\mathbf{curl}\,\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^q(\Omega)} \|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^m(\Omega)} \leqslant \|\mathbf{curl}\,\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^q(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\rho_\epsilon\|_{\boldsymbol{L}^t(\Omega)} \tag{1.7.9}$$

with $m, q \geqslant p$ such that $\frac{1}{p} = \frac{1}{m} + \frac{1}{q}$ and $t > 1$ such that $1 + \frac{1}{m} = \frac{1}{3} + \frac{1}{t}$. Note that these relations require $m > 3$, then $\frac{1}{p} < \frac{1}{q} + \frac{1}{3}$ so $q < p^*$. Therefore, we have the Sobolev embeddings:

$$\boldsymbol{W}^{2,p}(\Omega) \underset{\text{compact}}{\hookrightarrow} \boldsymbol{W}^{1,q}(\Omega) \underset{\text{continuous}}{\hookrightarrow} \boldsymbol{L}^{p^*}(\Omega)$$

hence there exists $\eta > 0$ and $C_\eta > 0$ such that

$$\|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{1,q}(\Omega)} \leqslant \eta \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_\eta \|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)} \tag{1.7.10}$$

Injecting (1.7.10) in (1.7.9), combining with (1.7.6) and the Sobolev embedding $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$, we obtain:

$$\|(\mathbf{curl}\,\boldsymbol{b}_\lambda) \times \boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_\epsilon C_2 \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} (\eta \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_\eta \|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)})$$

(iii) Estimate of the term $\|\mathbf{curl}(\boldsymbol{u}_\lambda \times \boldsymbol{d})\|_{\boldsymbol{L}^p(\Omega)}$:

Since $\operatorname{div} \boldsymbol{u}_\lambda = 0$ and $\operatorname{div} \boldsymbol{d} = 0$ in Ω, then $\mathbf{curl}(\boldsymbol{u}_\lambda \times \boldsymbol{d}) = (\boldsymbol{d} \cdot \nabla)\boldsymbol{u}_\lambda - (\boldsymbol{u}_\lambda \cdot \nabla)\boldsymbol{d}$. We thus bound these two terms.

- **Estimate of the term $\|(\boldsymbol{d} \cdot \nabla)\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)}$:**

  The reasoning is exactly the same as for (ii) with $\mathbf{curl}\,\boldsymbol{b}_\lambda$ replacing by $\nabla \boldsymbol{u}_\lambda$. Then we have:

  $$\|(\boldsymbol{d}_2^\epsilon \cdot \nabla)\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_2 C_3 \epsilon \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}$$

  and

  $$\|(\boldsymbol{d}_1^\epsilon \cdot \nabla)\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_2 C_\epsilon \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} (\eta \|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_\eta \|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)})$$

- **Estimate of the term $\|(\boldsymbol{u}_\lambda \cdot \nabla)\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)}$:**

  Again, the reasoning is the same as for (i), with $\nabla\boldsymbol{d}$ instead of $\mathbf{curl}\,\boldsymbol{w}$. Then we have:

$$\|(\boldsymbol{u}_\lambda \cdot \nabla)\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_1\epsilon\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}$$

  and

$$\|(\boldsymbol{u}_\lambda \cdot \nabla)\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^p(\Omega)} \leqslant C_\epsilon\,\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}$$

(iv) Estimate of the term $\sum_{i=1}^{I}|c_i^\lambda|$:

With a triangle inequality, we have:

$$\begin{aligned}|c_i^\lambda| \leqslant |\langle \boldsymbol{f}_\lambda, \nabla q_i^N\rangle| + |\langle P_0^\lambda, \nabla q_i^N\cdot\boldsymbol{n}\rangle| + |\int_\Omega (\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}_\lambda\cdot\nabla q_i^N\,dx|\\ + |\int_\Omega (\mathbf{curl}\,\boldsymbol{b}_\lambda)\times\boldsymbol{d}\cdot\nabla q_i^N\,dx|\end{aligned}$$

We can't directly bound $|\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}_\lambda\cdot\nabla q_i^N\,dx|$ and $|\int_\Omega(\mathbf{curl}\,\boldsymbol{b}_\lambda)\times\boldsymbol{d}\cdot\nabla q_i^N\,dx|$ with an Hölder inequality: we must use again the decomposition of $\mathbf{curl}\,\boldsymbol{w}$ and $\boldsymbol{d}$ in (1.7.3).

- **Estimate of the term $|\int_\Omega(\mathbf{curl}\,\boldsymbol{w})\times\boldsymbol{u}_\lambda\cdot\nabla q_i^N\,dx|$:**

  From (1.7.4), the Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega)\hookrightarrow\boldsymbol{L}^{p^{**}}(\Omega)$ and the Hölder inequality, we obtain:

$$\begin{aligned}|\int_\Omega \boldsymbol{y}_2^\epsilon\times\boldsymbol{u}_\lambda\cdot\nabla q_i^N\,dx| &\leqslant \|\boldsymbol{y}_2^\epsilon\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^{**}}(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^{p'}(\Omega)}\\ &\leqslant \epsilon C_1\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^{p'}(\Omega)}\end{aligned}$$

  with C_1 defined in (1.7.5). Next, applying an Hölder inequality, we have:

$$\begin{aligned}|\int_\Omega \boldsymbol{y}_1^\epsilon\times\boldsymbol{u}_\lambda\cdot\nabla q_i^N\,dx| &\leqslant \|\boldsymbol{y}_1^\epsilon\|_{\boldsymbol{L}^q(\Omega)}\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^m(\Omega)}\\ &\leqslant \|\boldsymbol{y}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\,\left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^t(\Omega)}\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^m(\Omega)}\\ &\leqslant C_\epsilon\,\|\boldsymbol{y}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\,\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^m(\Omega)}\end{aligned}$$

  with $m, q \geqslant p$ such that $\frac{1}{q}+\frac{1}{p^*}+\frac{1}{m}=1$ and $t>1$ such that $1+\frac{1}{q}=\frac{2}{3}+\frac{1}{t}$, and C_ϵ already defined in (1.7.6).

- **Estimate of the term $\int_\Omega(\mathbf{curl}\,\boldsymbol{b}_\lambda)\times\boldsymbol{d}\cdot\nabla q_i^N\,dx$:**

  Again, combining (1.7.7), the Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega)\hookrightarrow\boldsymbol{W}^{1,p^*}(\Omega)$ and the Hölder inequality, we have:

$$\begin{aligned}|\int_\Omega (\mathbf{curl}\,\boldsymbol{b}_\lambda)\times\boldsymbol{d}_2^\epsilon\cdot\nabla q_i^N\,dx| &\leqslant \|\mathbf{curl}\,\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}\,\|\boldsymbol{d}_2^\epsilon\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^{p'}(\Omega)}\\ &\leqslant C_3\epsilon\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^{p'}(\Omega)}\,\|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}\end{aligned}$$

  where C_3 is defined in (1.7.8).

  Now, we look at the part with $\boldsymbol{d}_1^\epsilon$, with the Hölder inequality:

$$\begin{aligned}|\int_\Omega (\mathbf{curl}\,\boldsymbol{b}_\lambda)\times\boldsymbol{d}_1^\epsilon\cdot\nabla q_i^N\,dx| &\leqslant \|\mathbf{curl}\,\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^q(\Omega)}\,\|\boldsymbol{d}_1^\epsilon\|_{\boldsymbol{L}^m(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^\alpha(\Omega)}\\ &\leqslant \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{1,q}(\Omega)}\,\|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)}\,\left\|\rho_{\epsilon/2}\right\|_{\boldsymbol{L}^t(\Omega)}\,\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^\alpha(\Omega)}\end{aligned}$$

with $q, m, \alpha \geqslant p$ such that $\frac{1}{q}+\frac{1}{m}+\frac{1}{\alpha}=1$ and $t>1$ such that $1+\frac{1}{m}=\frac{1}{3}+\frac{1}{t}$. In order to recover (1.7.10), note that, taking α such that $1-\frac{1}{\alpha}=\frac{1}{p}$, we obtain the same assumptions on q and m. Therefore, since we can take the same q as in (1.7.10), we apply it with (1.7.6) to obtain:

$$|\int_\Omega (\mathbf{curl}\, \boldsymbol{b}_\lambda)\times \boldsymbol{d}_1^\epsilon\cdot\nabla q_i^N\,dx| \leqslant C_\epsilon C_2 \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} \left\|\nabla q_i^N\right\|_{\boldsymbol{L}^\alpha(\Omega)} (\eta \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_\eta \|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)})$$

Finally, noting that there exists $C_q>0$ such that, for all $m\geqslant 1$, $\left\|\nabla q_i^N\right\|_{\boldsymbol{L}^m(\Omega)}\leqslant C_q$, we obtain from the previous calculus that:

$$\begin{aligned}\sum_{i=1}^{I}|c_i^\lambda| \leqslant I C_q\Big(& \|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)} + \left\|P_0^\lambda\right\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \epsilon C_1\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + \epsilon C_3 \|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}\\ &+\eta C_\epsilon C_2\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)} + C_\epsilon\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}\\ &+C_\eta C_\epsilon C_2\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}\Big).\end{aligned}$$

Then, injecting $(i)-(iv)$ in (1.7.2), and taking ϵ small enough such that:

$$\epsilon\Big(C_2C_3+\max(2CC_1, IC_qC_3)\Big)<\frac{1}{4}$$

and η such that

$$\eta\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} CC_2C_\epsilon<\frac{1}{4}$$

where $C>0$ denotes the constant $C=(1+IC_q)$, we obtain

$$\begin{aligned}&\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|P_\lambda\|_{W^{1,p}(\Omega)}\\ &\leqslant C_{SE}\Big[C(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{g}_\lambda\|_{\boldsymbol{L}^p(\Omega)}+\left\|P_0^\lambda\right\|_{W^{1-\frac{1}{p},p}(\Gamma)})\\ &+\Big(C_\epsilon(1+2CC_2C_\eta)\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}+C_\epsilon C\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big)(\|\boldsymbol{u}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)}+\|\boldsymbol{b}_\lambda\|_{\boldsymbol{L}^{p^*}(\Omega)})\Big]\end{aligned}\tag{1.7.11}$$

Applying the estimate (1.5.86) in (1.7.11), we finally obtain the estimate:

$$\begin{aligned}&\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|\boldsymbol{b}_\lambda\|_{\boldsymbol{W}^{2,p}(\Omega)}+\|P_\lambda\|_{W^{1,p}(\Omega)}\\ &\leqslant C_{SE}\Big(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{g}_\lambda\|_{\boldsymbol{L}^p(\Omega)}+\left\|P_0^\lambda\right\|_{W^{1-\frac{1}{p},p}(\Gamma)}\Big)\Big[C+C_3C_f\Big(C_\epsilon(1+2CC_2C_\eta)\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\\ &+CC_\epsilon\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}\Big)\Big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)\Big]\\ &\leqslant C_{SE}\Big(\|\boldsymbol{f}_\lambda\|_{\boldsymbol{L}^p(\Omega)}+\|\boldsymbol{g}_\lambda\|_{\boldsymbol{L}^p(\Omega)}+\left\|P_0^\lambda\right\|_{W^{1-\frac{1}{p},p}(\Gamma)}\Big)\max\Big(C, C_3C_fC_\epsilon(1+2CC_2C_\epsilon), CC_\epsilon\Big)\\ &\times\Big(1+\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)}+\|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)^2\end{aligned}\tag{1.7.12}$$

where C_f is the constant of the estimate (1.5.86) and C_3 is defined in (1.7.8).

To conclude, from the estimate (1.7.12) we can extract subsequences of $\boldsymbol{u}_\lambda$, $\boldsymbol{b}_\lambda$ and P_λ, which are still denoted $\boldsymbol{u}_\lambda$, $\boldsymbol{b}_\lambda$ and P_λ, such that:

$$\boldsymbol{u}_\lambda \rightharpoonup \boldsymbol{u} \text{ and } \boldsymbol{b}_\lambda\rightharpoonup\boldsymbol{b} \text{ in } \boldsymbol{W}^{2,p}(\Omega),\;\; P_\lambda\rightharpoonup P \text{ in } W^{1,p}(\Omega)$$

where $(\boldsymbol{u},\boldsymbol{b},P)\in\boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ is solution of (1.4.2) and satisfies the estimate (1.5.99). $\square$

Next, we give here the proof of Corollary 1.5.12 which is the extension of the previous result to the case of non-zero divergence condition for $\boldsymbol{u}$.

Proof of Corollary 1.5.12: We proceed with the same reasoning as in the Corollary 1.5.8. We recover the solution of a linearized problem with a vanishing divergence by considering the Dirichlet problem:

$$\Delta\theta = h \text{ in } \Omega \text{ and } \theta = 0 \text{ on } \Gamma$$

and setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$, thus $(\boldsymbol{z}, \boldsymbol{b}, P, \boldsymbol{c})$ is the solution of (1.4.2) in the Theorem 1.5.11 with $\boldsymbol{f}$ and $\boldsymbol{g}$ replaced by $\tilde{\boldsymbol{f}} = \boldsymbol{f} + \nabla h - (\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta$ and $\tilde{\boldsymbol{g}} = \boldsymbol{g} + \mathbf{curl}(\nabla\theta \times \boldsymbol{d})$.

Indeed, the assumptions of the Theorem 1.5.11 are satisfied: since $\nabla\theta \in \boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$ and $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{w}) \times \nabla\theta \in \boldsymbol{L}^p(\Omega)$. Moreover, we have by hypothesis $\nabla h \in \boldsymbol{L}^p(\Omega)$ so $\tilde{\boldsymbol{f}} \in \boldsymbol{L}^p(\Omega)$. In the same way, we have $\tilde{\boldsymbol{g}} \in \boldsymbol{L}^p(\Omega)$ and since we only add a curl to $\boldsymbol{g}$, then $\tilde{\boldsymbol{g}}$ satisfies the conditions (1.5.27)-(1.5.28). Hence, we recover the estimate:

$$\begin{aligned} &\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \\ &\leqslant C_F\Big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)^2 \Big(\|\tilde{\boldsymbol{f}}\|_{\boldsymbol{L}^p(\Omega)} + \|\tilde{\boldsymbol{g}}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)}\Big) \end{aligned} \tag{1.7.13}$$

We must bound $\|\tilde{\boldsymbol{f}}\|_{\boldsymbol{L}^p(\Omega)}$ and $\|\tilde{\boldsymbol{g}}\|_{\boldsymbol{L}^p(\Omega)}$ term by term:

- Applying the Hölder inequality and the Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$, we have:

$$\begin{aligned} \|\tilde{\boldsymbol{f}}\|_{\boldsymbol{L}^p(\Omega)} &\leqslant \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla h\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|\nabla\theta\|_{\boldsymbol{L}^{p^{**}}(\Omega)} \\ &\leqslant \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} + C_1 \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \|h\|_{W^{1,p}(\Omega)} \end{aligned} \tag{1.7.14}$$

  where C_1 denotes the constant of the Sobolev embedding $\boldsymbol{W}^{2,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^{**}}(\Omega)$.

- Note that $\operatorname{div}\nabla\theta = \Delta\theta = h$, thus we rewrite $\mathbf{curl}(\nabla\theta \times \boldsymbol{d}) = -h\boldsymbol{d} + (\boldsymbol{d}\cdot\nabla)\nabla\theta - (\nabla\theta\cdot\nabla)\boldsymbol{d}$. Thus, we have:

$$\begin{aligned} \|\tilde{\boldsymbol{g}}\|_{\boldsymbol{L}^p(\Omega)} &\leqslant \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|h\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} + \|(\boldsymbol{d}\cdot\nabla)\nabla\theta\|_{\boldsymbol{L}^p(\Omega)} + \|(\nabla\theta\cdot\nabla)\boldsymbol{d}\|_{\boldsymbol{L}^p(\Omega)} \\ &\leqslant \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|h\|_{L^{p^{**}}(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^3(\Omega)} \|\theta\|_{W^{2,p^*}(\Omega)} + \|\nabla\theta\|_{\boldsymbol{L}^{p^{**}}(\Omega)} \|\nabla\boldsymbol{d}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} \\ &\leqslant \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + 2C_2C_3 \|h\|_{W^{1,p}(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} + C_1 \|h\|_{W^{1,p}(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)} \end{aligned} \tag{1.7.15}$$

  where C_2 and C_3 are the constants respectively related to the Sobolev embeddings $\boldsymbol{W}^{1,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ and $\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{p^*}(\Omega)$.

Hence, combining (1.7.14) and (1.7.15) with (1.7.13), it follows that:

$$\begin{aligned} &\|\boldsymbol{z}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \\ &\leqslant C_F\Big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)^2 \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} \\ &+ \|h\|_{W^{1,p}(\Omega)} \Big(1 + C_1 \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + (2C_2C_3 + C_1) \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big) \\ &\leqslant C_F\Big(1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)}\Big)^2 \Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)} + \|P_0\|_{W^{1-\frac{1}{p},p}(\Gamma)} \\ &+ \max\Big(1, 2C_2C_3 + C_1\Big) \|h\|_{W^{1,p}(\Omega)} (1 + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{\frac{3}{2}}(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{W}^{1,\frac{3}{2}}(\Omega)})\Big) \end{aligned}$$

Finally, we use triangle inequality and we get the estimate (1.5.106). $\square$

Chapter 2

MHD system with Navier-type boundary condition

2.1 Introduction

Let Ω be a bounded simply connected domain in $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ with a connected boundary Γ. We denote by $\boldsymbol{n}$ the unit vector normal to Γ and exterior to Ω. We consider the following magnetohydrodynamic system (MHD): find the velocity field $\boldsymbol{u}$, the dynamic pressure p, and the magnetic field $\boldsymbol{b}$ satisfying:

$$-\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla P - (\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b} = \boldsymbol{f} \qquad \text{in}\,\Omega, \tag{2.1.1}$$

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) = \boldsymbol{g} \qquad \text{in}\,\Omega, \tag{2.1.2}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in}\,\Omega, \tag{2.1.3}$$

$$\operatorname{div}\boldsymbol{b} = 0 \qquad \text{in}\,\Omega. \tag{2.1.4}$$

Here, $\boldsymbol{f}$ and $\boldsymbol{g}$ are given source terms. We complete the system with the following types of homogeneous boundary conditions for the velocity field $\boldsymbol{u}$ and the magnetic field $\boldsymbol{b}$:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma, \tag{2.1.5}$$

$$\boldsymbol{b}\cdot\boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma. \tag{2.1.6}$$

We state our main results. The first theorem gives the existence of weak solutions for the system

(2.1.1)-(2.1.6) in the Hilbert case. Suitable spaces for data $\boldsymbol{f}$ and $\boldsymbol{g}$ are specified. We need to introduce the space $\boldsymbol{H}_0^{r,p}(\operatorname{div},\Omega)$ defined by

$$\boldsymbol{H}_0^{r,p}(\operatorname{div},\Omega) := \{\boldsymbol{\phi}\in\boldsymbol{L}^r(\Omega);\,\operatorname{div}\boldsymbol{\phi}\in L^p(\Omega),\ \boldsymbol{\phi}\cdot\boldsymbol{n} = 0 \text{ on}\,\Gamma\}$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ which is a Banach space endowed with the norm:

$$\|\boldsymbol{\phi}\|_{\boldsymbol{H}_0^{r,p}(\operatorname{div},\Omega)} = \|\boldsymbol{\phi}\|_{\boldsymbol{L}^r(\Omega)} + \|\operatorname{div}\boldsymbol{\phi}\|_{L^p(\Omega)}$$

The space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)$ (c.f. [87, Proposition 1.0.2]). Its dual space denoted $[\boldsymbol{H}_0^{r',p'}(\operatorname{div},\Omega)]'$ can be characterized as follows (c.f [87, Proposition 1.0.4]):

Proposition 2.1.1. *A distribution* $\boldsymbol{f}$ *belongs to* $[\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)]'$ *iff there exists* $\boldsymbol{F} \in \boldsymbol{L}^{r'}(\Omega)$ *and* $\psi \in L^{p'}(\Omega)$ *such that* $\boldsymbol{f} = \boldsymbol{F} + \nabla\psi$. *Moreover, we have the estimate :*

$$\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r,p}(\mathrm{div},\Omega)]'} \leq \inf_{\boldsymbol{f}=\boldsymbol{F}+\nabla\psi} \max\{\|\boldsymbol{F}\|_{\boldsymbol{L}^{r'}(\Omega)}, \|\psi\|_{\boldsymbol{L}^{p'}(\Omega)}\}.$$

$$[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]' = \{\boldsymbol{F} + \nabla\psi \,|\, \boldsymbol{F} \in \boldsymbol{L}^r(\Omega),\ \psi \in L^p(\Omega)\} \tag{2.1.7}$$

Theorem 2.1.2. (Weak solutions in $\boldsymbol{H}^1(\Omega)$). *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'$ *and* $\boldsymbol{g} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ *satisfying the following compatibility conditions:*

$$\mathrm{div}\,\boldsymbol{g} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{g}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{2.1.8}$$

Then the problem (2.1.1)-(2.1.6) *has at least a weak solution*

$$(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}.$$

Further, the solution $(\boldsymbol{u}, \boldsymbol{b})$ *satisfies the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\Big(\, \|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\Big)$$

with $C > 0$ *which depends on the datas of the problem.*

Next, using the regularity of the Stokes problem and an elliptic problem combining with a bootstrap argument, we show the regularity $\boldsymbol{W}^{1,p}(\Omega)$ with $p > 2$ and the regularity $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq 6/5$. We obtain the following two theorems.

Theorem 2.1.3. (Strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$ with $p \geq 6/5$) *Let us suppose that* Ω *is of class* $\mathcal{C}^{2,1}$ *and* $p \geq 6/5$. *Let* $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ *satisfying the compatibility conditions* (2.1.8). *Then the solution* $(\boldsymbol{u}, \boldsymbol{b}, P)$ *of* (2.1.1)-(2.1.6) *given in Theorem 2.1.2 belongs to* $\boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$.

Theorem 2.1.4. (Generalized solutions in $\boldsymbol{W}^{1,p}(\Omega)$ with $p \geqslant 2$) *Let us suppose that* Ω *is of class* $\mathcal{C}^{1,1}$ *and* $p \geq 2$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$ *and* $\boldsymbol{g} \in \boldsymbol{L}^r(\Omega)$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ *satisfying the compatibility conditions* (2.1.8). *Then the weak solution* $(\boldsymbol{u}, \boldsymbol{b}, P)$ *of* (2.1.1)-(2.1.6) *given in Theorem 2.1.2 belongs to* $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

2.2 Notations and preliminaries

In this section, we review some notations and functional framework. The spaces mean vector or matrix valued spaces are denoted by bold letters, as well as its elements. We will use C to denote a generic positive constant which may depend on Ω and the dependence on other parameters will be specified if necessary. For $1 < p < \infty$, we define the spaces

$$\boldsymbol{X}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} = 0\ \text{ on}\,\Gamma\}.$$

$$\boldsymbol{V}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}_T^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} = 0\ \text{in}\,\Omega\}.$$

When $p = 2$, we will use the notation $\boldsymbol{X}_T(\Omega)$ (respectively $\boldsymbol{V}_T(\Omega)$) instead to $\boldsymbol{X}_T^2(\Omega)$ (respectively $\boldsymbol{V}_T^2(\Omega)$).

We note that the space $\boldsymbol{X}^p_T(\Omega)$ defined above is continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$ (c.f [15, Theorem 3.2]). Moreover, we have the following Poincaré inequality which can be found in [98, Theorem 2.1] (see also [15, Corollary 3.4] for the more general case where Ω is multiply connected): for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leqslant C\Big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} \Big) \tag{2.2.1}$$

The inequality (2.2.1) is a particular case of that in [15, Corollary 3.4] without the flux through the cuts because we suppose in this chapter that Ω is simply connected.

We will use somme regularity results concerning the following Stokes problem $(\mathcal{S}_\mathcal{T})$:

$$(\mathcal{S}_\mathcal{T})\begin{cases} -\Delta\boldsymbol{u} + \nabla P = \boldsymbol{f} \quad \text{and} \quad & \mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = a \quad \text{and} \quad & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \quad \text{on } \Gamma. \end{cases}$$

We first recall the following strong regularity result [13, Theorem 4.8.]:

Theorem 2.2.1. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $a \in W^{2-\frac{1}{p},p}(\Gamma)$, $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ with the following the compatibility condition:*

$$\int_\Gamma a\,ds = 0 \tag{2.2.2}$$

Then, the solution $(\boldsymbol{u}, P)$ of problem $(\mathcal{S}_\mathcal{T})$ given by Theorem 2.2.2 belongs to $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|a\|_{W^{2-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \big)$$

Using the characterization (2.1.7), we can extend the result in [13, Theorem 4.4] to prove the existence and uniqueness of solutions in $\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ of the Stokes problem $(\mathcal{S}_\mathcal{T})$, by supposing $\boldsymbol{f}$ in the dual space $[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$ instead of $[\boldsymbol{H}_0^{p',p'}(\mathrm{div},\Omega)]'$:

Theorem 2.2.2. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$, $a_u \in W^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{h}_u\times\boldsymbol{n} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ with $r \leqslant p$ and $\frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3}$ satisfying the following the compatibility condition (2.2.2). Then, the Stokes problem $(\mathcal{S}_\mathcal{T})$ has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)/\mathbb{R}} \leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'} + \|a_u\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h}_u\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)} \Big)$$

Proof. Next, with the characterization of the dual space $[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]'$ given in Proposition 2.1.1 and the linearity of the Stokes problem, we divide the problem into two parts, with $(\boldsymbol{u}_1,\pi_1)$ and $(\boldsymbol{u}_2,\pi_2)$ respectively solutions of:

$$\begin{cases} -\Delta\boldsymbol{u}_1 + \nabla\pi_1 = \boldsymbol{F}\\ \mathrm{div}\,\boldsymbol{u}_1 = 0 \quad \text{on } \Omega\\ \boldsymbol{u}_1\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{u}_1\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma \end{cases} \tag{2.2.3}$$

and

$$\begin{cases} -\Delta \boldsymbol{u}_2 + \nabla \pi_2 = \nabla \psi \\ \operatorname{div} \boldsymbol{u}_2 = 0 \quad \text{on } \Omega \\ \boldsymbol{u}_2 \cdot \boldsymbol{n} = a, \quad \mathbf{curl}\, \boldsymbol{u}_2 \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \quad \text{on } \Gamma \end{cases} \tag{2.2.4}$$

with $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\psi \in L^p(\Omega)$. Note that if $(\boldsymbol{u}_1, \pi_1)$ is solution of (2.2.3) and $(\boldsymbol{u}_2, \pi_2)$ solution of (2.2.4), then $(\boldsymbol{u}, \pi) = (\boldsymbol{u}_1 + \boldsymbol{u}_2, \pi_1 + \pi_2)$ is solution of $(\mathcal{S}_\mathcal{T})$.

Applying Theorem 2.2.1, the Stokes problem (2.2.3) has a unique solution $(\boldsymbol{u}_1, \pi_1) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R}$. However, since $\frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3}$, we have the following Sobolev embedding:

$$\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$$

so $(\boldsymbol{u}_1, \pi_1) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Next, with (2.1.7), we have that $\nabla\psi$ belongs to $[\boldsymbol{H}_0^{p'}(\operatorname{div}, \Omega)]'$, then with [13, Theorem 4.4], we obtain the unique solution $(\boldsymbol{u}_2, \pi_2) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Thus, we conclude by combining those results and we have $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. □

Remark 2.2.1. *We note that the regularity $\mathcal{C}^{2,1}$ in Theorem 2.2.2 can be reduced to $\mathcal{C}^{1,1}$. Indeed, we consider the homogeneous case $a = 0$. We can verify that the Stokes problem $(\mathcal{S}_\mathcal{T})$ is equivalent to the following variational formulation (c.f. [13, Theorem 4.4]): Find $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{V}_T^{p'}(\Omega)$:*

$$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h}_u \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}, \tag{2.2.5}$$

where $\langle \cdot,\, \cdot \rangle_\Omega$ denotes the duality product between $[\boldsymbol{H}_0^{r',p'}(\operatorname{div}, \Omega)]'$ and $[\boldsymbol{H}_0^{r',p'}(\operatorname{div}, \Omega)]$. Thanks to [13, Corollarty 3.5], if Ω is of class $\mathcal{C}^{1,1}$, the following infi-sup condition holds: there exists a constant $\beta > 0$, such that:

$$\inf_{\substack{\boldsymbol{v} \in \boldsymbol{V}_T^{p'}(\Omega) \\ \boldsymbol{v} \neq 0}} \sup_{\substack{\boldsymbol{u} \in \boldsymbol{V}_T^{p}(\Omega) \\ \boldsymbol{u} \neq 0}} \frac{\int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}(\Omega)}} \geq \beta. \tag{2.2.6}$$

So, problem (2.2.5) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}_T^p(\Omega) \subset \boldsymbol{W}^{1,p}(\Omega)$ since the right-hand sides defines an element of $(\boldsymbol{V}_T^p(\Omega))'$. The pressure can be recovered by De Rham argument. For the non homogeneous case, the boundary condition $\boldsymbol{u} \cdot \boldsymbol{n}$ can be lifted by solving the following Neumann problem:

$$\Delta\theta = \boldsymbol{0} \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial \theta}{\partial \boldsymbol{n}} = a \quad \text{on } \Gamma.$$

We need also regularity results for the following elliptic problem $(\mathcal{E}_\mathcal{T})$ (cf. [13])

$$(\mathcal{E}_\mathcal{T}) \begin{cases} -\Delta \boldsymbol{b} = \boldsymbol{g} \quad \text{and} \qquad \operatorname{div} \boldsymbol{b} = 0 \quad \text{in } \Omega, \\ \boldsymbol{b} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma \end{cases}$$

Theorem 2.2.3. *Let $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ satisfying the compatibility conditions*

$$\operatorname{div} \boldsymbol{g} = 0 \quad \text{in } \Omega \qquad \boldsymbol{g} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{2.2.7}$$

Then, problem $(\mathcal{E}_\mathcal{T})$ has a unique solution $\boldsymbol{b} \in \boldsymbol{W}^{2,p}(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{b}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leqslant C \, \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}$$

Next, we need to define the following space

$$\boldsymbol{M}^{r,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\;\; \Delta\,\boldsymbol{v} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div},\,\Omega)]'\},$$

which is a Banach space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{M}^{r,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\Delta\,\boldsymbol{v}\|_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\,\Omega)]'}.$$

The following Lemma gives a density result and a Green formula which allows to define the tangential traces of $\mathbf{curl}\,\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{b}$ on Γ. The proof is similar to that in [13, Lemma 4.1] and [8].

Lemma 2.2.4. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{M}^{r,p}(\Omega)$. Moreover, the linear mapping $\gamma\;:\;\boldsymbol{v}\to\mathbf{curl}\,\boldsymbol{v}|_\Gamma\times\boldsymbol{n}$ defined on $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended to a linear continuous map from $\gamma\;:\;\boldsymbol{M}^{r,p}(\Omega)$ to $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and we have the Green formula: for any $\boldsymbol{v}\in\boldsymbol{M}^{r,p}(\Omega)$ and any $\boldsymbol{\varphi}\in\boldsymbol{W}^{1,p'}(\Omega)\cap\boldsymbol{X}_T^{p'}(\Omega)$ with $\mathrm{div}\,\boldsymbol{\varphi}=0$ in Ω ,*

$$-\,\langle\Delta\,\boldsymbol{v},\,\boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\,\Omega)]'\times\boldsymbol{H}_0^{r',p'}(\mathrm{div},\,\Omega)} = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n},\,\boldsymbol{\varphi}\rangle_\Gamma, \tag{2.2.8}$$

where the duality on Γ is defined by $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}$.

2.3 Weak solutions in the Hilbert case

In this section, we are going to study the well-posedness of the MHD problem 2.1.1-(2.1.6) in the Hilbert space. We prove the existence of weak solutions in $\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)$ by applying a Leray-Schauder fixed point argument. A natural question is how to choose the data $\boldsymbol{f}$ and $\boldsymbol{g}$ in order to give a sens to the boundary conditions (2.1.5) – (2.1.6). It is natural to look for a solution $(\boldsymbol{u},\boldsymbol{b})$ in $\boldsymbol{V}_T(\Omega)\times\boldsymbol{V}_T(\Omega)$ with

$$\boldsymbol{V}_T(\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\;\;\mathrm{div}\,\boldsymbol{v}=0\;\;\text{in}\;\;\Omega,\;\;\boldsymbol{v}\cdot\boldsymbol{n}=0\quad\text{on}\;\Gamma\}$$

As discussed in Chapter 1, for boundary conditions of type (2.1.5)-(2.1.6), the space $\boldsymbol{H}^{-1}(\Omega)$ is not suitable for the external forces $\boldsymbol{f}$ and $\boldsymbol{g}$ to find solutions $(\boldsymbol{u},\boldsymbol{b})$ in $\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^1(\Omega)$. In view to write an equivalent variational formulation, we must precise the duality $\langle\boldsymbol{f},\,\boldsymbol{v}\rangle$ while choosing $\boldsymbol{f}$ in a subspace of distributions. Since the test functions belong to the space

$$\boldsymbol{H}_0^{2,2}(\mathrm{div};\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\;\;\mathrm{div}\,\boldsymbol{v}\in L^2(\Omega),\;\;\boldsymbol{v}\cdot\boldsymbol{n}=0\;\;\text{on}\;\Gamma\},$$

we must choose $\boldsymbol{f}$ in its dual space, so $\boldsymbol{f}\in[\boldsymbol{H}_0^{2,2}(\mathrm{div};\Omega)]'$. This hypothesis can be weakend by observing that $\boldsymbol{H}^1(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$ and then we can consider for $\boldsymbol{f}$ the space $[\boldsymbol{H}_0^{6,2}(\mathrm{div};\Omega)]'$ which contains $[\boldsymbol{H}_0^{2,2}(\mathrm{div};\Omega)]'$. Recall that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in this dual space and its characterization is given in (2.1.7). With this choice, the boundary condition $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ has a sens in $\boldsymbol{H}^{-1/2}(\Gamma)$. Indeed, in view of the characterization given in Proposition 2.1.1, if we have $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$, $\boldsymbol{b}\in\boldsymbol{H}^1(\Omega)$ and $P\in L^2(\Omega)$, then $\Delta\boldsymbol{u}=\boldsymbol{f}-\nabla P-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{b}$ belongs to $[\boldsymbol{H}_0^{6,2}(\mathrm{div},\;\Omega)]'$ and then $\boldsymbol{u}$ belongs to $\boldsymbol{M}^{6,2}(\Omega)$. We deduce thanks to Lemma 2.2.4 that the trace $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}$ has a sens in $\boldsymbol{H}^{-1/2}(\Gamma)$. This is also valid for the trace of $\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n}$ on Γ with a data more regular $\boldsymbol{g}\in\boldsymbol{L}^{6/5}(\Omega)\hookrightarrow[\boldsymbol{H}_0^{6,2}(\mathrm{div};\Omega)]'$. This additional regularity for $\boldsymbol{g}$ is needed in order to give a sens to $\boldsymbol{g}\cdot\boldsymbol{n}$ on the boundary (see (2.3.4) and (2.3.5)).

Theorem 2.3.1. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$ *and* $\boldsymbol{g} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$ *satisfying the following compatibility conditions:*

$$\mathrm{div}\, \boldsymbol{g} = 0 \quad \text{in } \Omega \qquad \boldsymbol{g} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{2.3.1}$$

Then the problem (2.1.1)-(2.1.6) *has at least a weak solution*

$$(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$$

Further, the solution $(\boldsymbol{u}, \boldsymbol{b})$ *satisfies the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} \Big) \tag{2.3.2}$$

with $C > 0$ *which depends on the data of the problem.*

Proof. **(i)** In a first step, we prove that the compatibility condition (2.3.1) is necessary. Let $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ a weak solution of (2.1.1)-(2.1.6). From (3.1.1), $\boldsymbol{g} = \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) + \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b}$, then we have immediately $\mathrm{div}\,\boldsymbol{g} = 0$ in Ω. Now, we want to prove the condition $\boldsymbol{g} \cdot \boldsymbol{n} = 0$ on Γ. We take $\boldsymbol{\Psi} = \mathbf{curl}\,\boldsymbol{b} - (\boldsymbol{u} \times \boldsymbol{b})$. Then, $\boldsymbol{\Psi} \in \boldsymbol{L}^2(\Omega)$ and $\mathbf{curl}\,\boldsymbol{\Psi} = \boldsymbol{g} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Moreover, since $\mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{n} = \mathbf{0}$ on Γ and

$$(\boldsymbol{u} \times \boldsymbol{b}) \times \boldsymbol{n} = \boldsymbol{b}(\underbrace{\boldsymbol{n} \cdot \boldsymbol{u}}_{=0}) - \boldsymbol{u}(\underbrace{\boldsymbol{n} \cdot \boldsymbol{b}}_{=0}) = 0 \quad \text{on } \Gamma, \tag{2.3.3}$$

we deduce that $\boldsymbol{\Psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ which has a sense in $\boldsymbol{H}^{-1/2}(\Gamma)$. Then, on the one hand, for all $\chi \in H^2(\Omega)$, we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\Psi} \cdot \nabla\chi \, dx = -\langle \boldsymbol{\Psi} \times \boldsymbol{n}, \nabla\chi \rangle_{\boldsymbol{H}^{-1/2}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)} = 0 \tag{2.3.4}$$

On the other hand, since $\boldsymbol{g} \in \boldsymbol{L}^{6/5}(\Omega)$ and is divergence-free, the trace $\boldsymbol{g} \cdot \boldsymbol{n}$ belongs to $W^{-\frac{5}{6},\frac{6}{5}}(\Gamma)$. Then, for all $\chi \in H^2(\Omega)$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\Psi} \cdot \nabla\chi \, dx = \langle \boldsymbol{g} \cdot \boldsymbol{n}, \chi \rangle_{W^{-\frac{5}{6},\frac{6}{5}}(\Gamma) \times W^{\frac{5}{6},6}(\Gamma)} \tag{2.3.5}$$

So $\boldsymbol{g} \cdot \boldsymbol{n} = 0$ in $W^{-\frac{5}{6},\frac{6}{5}}(\Gamma)$.

(ii) We now want to prove the existence of weak solutions. For this, we apply a Leray-Schauder fixed-point theorem. We define the Hilbert space $\boldsymbol{Z} := \boldsymbol{V}_T(\Omega) \times \boldsymbol{V}_T(\Omega)$ equipped with the norm

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}} = \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}.$$

We suppose that $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}$ is given and we set the following problem: Find $((\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}), \hat{p}) \in \boldsymbol{Z} \times L^2(\Omega)/\mathbb{R}$ such that

$$\begin{cases} -\Delta \hat{\boldsymbol{u}} + \nabla \hat{P} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} + \boldsymbol{f} & \text{and} \quad \mathrm{div}\,\hat{\boldsymbol{u}} = 0 \qquad \text{in } \Omega, \\ \mathbf{curl}\,\mathbf{curl}\,\hat{\boldsymbol{b}} = \boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) & \text{and} \quad \mathrm{div}\,\hat{\boldsymbol{b}} = 0 \qquad \text{in } \Omega, \\ \hat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 & \text{and} \quad \mathbf{curl}\,\hat{\boldsymbol{u}} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma, \\ \hat{\boldsymbol{b}} \cdot \boldsymbol{n} = 0 & \text{and} \quad \mathbf{curl}\,\hat{\boldsymbol{b}} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma. \end{cases} \tag{2.3.6}$$

Observe that problem (2.3.6) is decoupled. Indeed, $(\hat{\boldsymbol{u}}, \hat{p})$ is a solution of a Stokes problem of type $(\mathcal{S}_\mathcal{T})$ with homogeneous boundary conditions. $\hat{\boldsymbol{b}}$ is solution of an elliptic problem of type $(\mathcal{E}_\mathcal{T})$. So, to prove the existence and the uniqueness of solutions for (2.3.6), we can apply the regularity results for $(\mathcal{E}_\mathcal{T})$ and $(\mathcal{S}_\mathcal{T})$ given in Theorem 2.2.3 and 2.2.2. Let us first verify the conditions of Theorem 2.2.2. Since $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Similarly, $(\mathbf{curl}\, \boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Therefore:

$$-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} + \boldsymbol{f} \in \boldsymbol{L}^{\frac{6}{5}}(\Omega) \hookrightarrow [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$$

Applying Theorem 2.2.2, we deduce the existence and uniqueness of $(\hat{\boldsymbol{u}}, \hat{p}) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. Next to apply Theorem 2.2.3 and obtain $\hat{\boldsymbol{b}}$, we must verify that $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ belongs to $\boldsymbol{L}^{6/5}(\Omega)$, is divergence-free and has zero normal trace. First, since $\mathrm{div}\, \boldsymbol{g} = 0$ in Ω, it is obvious that $\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})$ is divergence-free. Next, since $\mathrm{div} \boldsymbol{u} = \mathrm{div}\, \boldsymbol{b} = 0$ in Ω, we have

$$\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} \tag{2.3.7}$$

So as above, using the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Then the RHS of the second equation (2.3.6) belongs to $\boldsymbol{L}^{\frac{6}{5}}(\Omega)$. Moreover, thanks to (2.3.3), we have $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \cdot \boldsymbol{n} = 0$ on Γ and then $(\boldsymbol{g} + \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b})) \cdot \boldsymbol{n} = 0$ on Γ. The existence of $\hat{\boldsymbol{b}}$ is guaranteed by Theorem 2.2.3. We conclude that there exists a unique solution $((\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}), \hat{p}) \in \boldsymbol{Z} \times L^2(\Omega)/\mathbb{R}$ solution of (2.3.6). Now, we define the following operator:

$$\begin{aligned} \mathcal{T} : \boldsymbol{Z} &\to \boldsymbol{Z} \times L^2(\Omega)/\mathbb{R} \to \boldsymbol{Z} \\ (\boldsymbol{u}, \boldsymbol{b}) &\mapsto (\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}, \hat{p}) \mapsto (\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}) \end{aligned}$$

where $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}})$ is the component of the solution $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}, \hat{P})$ of (2.3.6). We want to prove that $\mathcal{T}$ has a fixed point which is the solution of the MHD system (2.1.1)-(2.1.6). For this, we proceed in two steps to prove the following statements (see [44, Theorem 11.3]):

(1) The operator $\mathcal{T}$ is compact.

(2) There exists a constant $C > 0$ independent of $\boldsymbol{u}, \boldsymbol{b}$ and α such that $\|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{Z}} \leqslant C$ for all $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}$ and for all $\alpha \in [0, 1]$ such that $(\boldsymbol{u}, \boldsymbol{b}) = \alpha T(\boldsymbol{u}, \boldsymbol{b})$.

Proof of (1). Since $\boldsymbol{Z}$ is reflexive, it is sufficient to prove that T is continuous. Let $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}$, $(\boldsymbol{u}_k, \boldsymbol{b}_k) \in \boldsymbol{Z}$ such that $(\boldsymbol{u}_k, \boldsymbol{b}_k) \rightharpoonup (\boldsymbol{u}, \boldsymbol{b})$ weak in $\boldsymbol{Z}$. Let us define $(\hat{\boldsymbol{u}_k}, \hat{\boldsymbol{b}_k}) = \mathcal{T}(\boldsymbol{u}_k, \boldsymbol{b}_k)$, $\forall k \in \mathbb{N}$. We want to prove that $\mathcal{T}(\boldsymbol{u}_k, \boldsymbol{b}_k) = (\hat{\boldsymbol{u}_k}, \hat{\boldsymbol{b}_k}) \longrightarrow (\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}) = \mathcal{T}(\boldsymbol{u}, \boldsymbol{b})$ strongly in $\boldsymbol{Z}$ which amounts to prove that when $k \to +\infty$

$$\hat{\boldsymbol{u}_k} - \hat{\boldsymbol{u}} \longrightarrow 0 \quad \text{and} \quad \hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}} \longrightarrow 0 \quad \text{in} \quad \boldsymbol{H}^1(\Omega).$$

For all $k \in \mathbb{N}$, $(\hat{\boldsymbol{u}_k} - \hat{\boldsymbol{u}}, \hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}})$ satisfies the following problem:

$$\begin{cases} \Delta(\hat{\boldsymbol{u}_k} - \hat{\boldsymbol{u}}) + \nabla(\hat{P_k} - \hat{P}) = -(\boldsymbol{u}_k \cdot \nabla)\boldsymbol{u}_k + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{b}_k \times \boldsymbol{b}_k - \mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{b} & \text{in}\ \Omega, \\ \mathbf{curl}\,\mathbf{curl}(\hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}}) = \mathbf{curl}(\boldsymbol{u}_k \times \boldsymbol{b}_k) - \mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) & \text{in}\ \Omega, \\ \mathrm{div}(\hat{\boldsymbol{u}_k} - \hat{\boldsymbol{u}}) = 0 \quad \text{and} \quad \mathrm{div}(\hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}}) = 0 & \text{in}\ \Omega, \\ (\hat{\boldsymbol{u}_k} - \hat{\boldsymbol{u}}) \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}(\hat{\boldsymbol{u}_k} - \hat{\boldsymbol{u}}) \times \boldsymbol{n} = \boldsymbol{0}, & \text{on}\ \Gamma, \\ (\hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}}) \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}(\hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}}) \times \boldsymbol{n} = \boldsymbol{0}, & \text{on}\ \Gamma. \end{cases}$$

$(\hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}})$ satisfies a Stokes problem that we can split into two parts:

$$\begin{cases} -\Delta \boldsymbol{w}_k + \nabla q_k = -((\boldsymbol{u}_k - \boldsymbol{u}) \cdot \nabla)\boldsymbol{u} + \mathbf{curl}\, \boldsymbol{b} \times (\boldsymbol{b}_k - \boldsymbol{b}) & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w}_k = 0 & \text{in } \Omega, \\ \boldsymbol{w}_k \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{w}_k \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{2.3.8}$$

and

$$\begin{cases} -\nu \Delta \boldsymbol{y}_k + \nabla r_k = -(\boldsymbol{u}_k \cdot \nabla)(\boldsymbol{u}_k - \boldsymbol{u}) + \mathbf{curl}(\boldsymbol{b}_k - \boldsymbol{b}) \times \boldsymbol{b}_k & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{y}_k = 0 & \text{in } \Omega, \\ \boldsymbol{y}_k \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{y}_k \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{2.3.9}$$

where $\boldsymbol{w}_k + \boldsymbol{y}_k = \hat{\boldsymbol{u}}_k - \hat{\boldsymbol{u}}$ and $q_k + r_k = \hat{p_k} - \hat{p}$. Since $-((\boldsymbol{u}_k - \boldsymbol{u}) \cdot \nabla)\boldsymbol{u} + \mathbf{curl}\, \boldsymbol{b} \times (\boldsymbol{b}_k - \boldsymbol{b}) \in \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, thanks to Theorem 2.2.2, there exists $(\boldsymbol{w}_k, q_k) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ solution of (2.3.8) satisfying the estimate

$$\begin{aligned} & \|\boldsymbol{w}_k\|_{\boldsymbol{H}^1(\Omega)} + \|q_k\|_{L^2(\Omega)/\mathbb{R}} \\ & \leqslant C\Big(\|((\boldsymbol{u}_k - \boldsymbol{u}) \cdot \nabla)\boldsymbol{u}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{b} \times (\boldsymbol{b}_k - \boldsymbol{b})\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} \Big) \\ & \leqslant C\Big(\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)} + \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{b}_k - \boldsymbol{b}\|_{\boldsymbol{L}^3(\Omega)} \Big) \\ & \leqslant C\Big(\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_k - \boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)} + \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{b}_k - \boldsymbol{b}\|_{\boldsymbol{L}^3(\Omega)} \Big) \end{aligned}$$

Since $\hat{\boldsymbol{u}}_k \rightharpoonup \boldsymbol{u}$ and $\hat{\boldsymbol{b}}_k \rightharpoonup \boldsymbol{b}$ weakly in $\boldsymbol{H}^1(\Omega)$ and the embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$ is compact, we deduce that $\hat{\boldsymbol{u}}_k \longrightarrow \boldsymbol{u}$ and $\hat{\boldsymbol{b}}_k \longrightarrow \boldsymbol{b}$ strongly in $\boldsymbol{L}^3(\Omega)$. This implies that $\boldsymbol{w}_k \to 0$. Similarly, since $-(\boldsymbol{u}_k \cdot \nabla)(\boldsymbol{u}_k - \boldsymbol{u}) + \mathbf{curl}(\boldsymbol{b}_k - \boldsymbol{b}) \times \boldsymbol{b}_k \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, by Theorem 2.2.2 there exists $(\boldsymbol{y}_k, r_k) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$ solution of the Stokes problem (2.3.9). So, $(\boldsymbol{u}_k \cdot \nabla)(\boldsymbol{u}_k - \boldsymbol{u}) \rightharpoonup 0$ and $\mathbf{curl}(\boldsymbol{b}_k - \boldsymbol{b}) \times \boldsymbol{b}_k \rightharpoonup 0$ weakly in $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$ imply that $\boldsymbol{y}_k \longrightarrow 0$, weakly in $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ and $r_k \to 0$ weakly in $W^{1,\frac{3}{2}}(\Omega)$. But, the embedding $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{H}^1(\Omega)$ is compact, so we deduce that $\boldsymbol{y}_k \to 0$ strongly in $\boldsymbol{H}^1(\Omega)$. We conclude that $\hat{\boldsymbol{u}_k} \to \hat{\boldsymbol{u}}$ in $\boldsymbol{H}^1(\Omega)$.
Next, $\hat{\boldsymbol{b}}_k - \boldsymbol{b}$ is a solution of an elliptic problem with a right-hand side $\mathbf{curl}((\boldsymbol{u}_k - \boldsymbol{u}) \times \boldsymbol{b}_k) - \mathbf{curl}(\boldsymbol{u} \times (\boldsymbol{b} - \boldsymbol{b}_k))$ which is equal to $\mathbf{curl}((\boldsymbol{u}_k - \boldsymbol{u}) \times \boldsymbol{b}_k) - \mathbf{curl}(\boldsymbol{u} \times (\boldsymbol{b} - \boldsymbol{b}_k))$. Using the relation (2.3.7), we can verify that it is a function of $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$. Thanks to Theorem 2.2.3, there exists a solution $\hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}} \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. Since $\mathbf{curl}((\boldsymbol{u}_k - \boldsymbol{u}) \times \boldsymbol{b}_k) \rightharpoonup 0$ and $\mathbf{curl}(\boldsymbol{u} \times (\boldsymbol{b} - \boldsymbol{b}_k)) \rightharpoonup 0$ in $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$, we deduce that $\hat{\boldsymbol{b}_k} - \hat{\boldsymbol{b}} \rightharpoonup 0$ in $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. Finally, with the compact embedding $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$ in $\boldsymbol{H}^1(\Omega)$, we conclude that $\hat{\boldsymbol{b}_k} \to \hat{\boldsymbol{b}}$ in $\boldsymbol{H}^1(\Omega)$.

Proof of (2). Let $(\boldsymbol{u}, \boldsymbol{b}) = \alpha \mathcal{T}(\boldsymbol{u}, \boldsymbol{b})$ with $(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{Z}$ and $\alpha \in [0, 1]$. Since $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}) = \mathcal{T}(\boldsymbol{u}, \boldsymbol{b})$, then $(\boldsymbol{u}, \boldsymbol{b}) = \alpha(\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}) = (\alpha\hat{\boldsymbol{u}}, \alpha\hat{\boldsymbol{b}})$ and $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{b}}) = \mathcal{T}(\alpha\hat{\boldsymbol{u}}, \alpha\hat{\boldsymbol{b}})$ satisfies:

$$\begin{cases} -\Delta \hat{\boldsymbol{u}} + \nabla \hat{P} = \boldsymbol{f} - \alpha^2 (\hat{\boldsymbol{u}} \cdot \nabla)\hat{\boldsymbol{u}} + \alpha^2 (\mathbf{curl}\, \hat{\boldsymbol{b}}) \times \hat{\boldsymbol{b}} & \text{in } \Omega, \\ \mathbf{curl}\, \mathbf{curl}\, \hat{\boldsymbol{b}} = \boldsymbol{g} + \alpha^2\, \mathbf{curl}(\hat{\boldsymbol{u}} \times \hat{\boldsymbol{b}}) & \text{in } \Omega, \\ \operatorname{div} \hat{\boldsymbol{u}} = 0 \quad \text{and} \quad \operatorname{div} \hat{\boldsymbol{b}} = 0, & \text{in } \Omega, \\ \hat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \hat{\boldsymbol{u}} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \\ \hat{\boldsymbol{b}} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \hat{\boldsymbol{b}} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{2.3.10}$$

Multiplying the first equation (2.3.10) with $\hat{\boldsymbol{u}} \in \boldsymbol{V}_T(\Omega)$ and integrating by parts, we obtain

$$\|\mathbf{curl}\,\hat{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{f}, \hat{\boldsymbol{u}}\rangle_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)} + \alpha^2 \int_\Omega (\mathbf{curl}\,\hat{\boldsymbol{b}}) \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{u}}\, d\boldsymbol{x}$$

Similarly, multiplying the second equation (2.3.10) with $\hat{\boldsymbol{b}}$ and integrating by parts, we obtain:

$$\|\,\mathbf{curl}\,\hat{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(\Omega)} = \int_\Omega \boldsymbol{g}\cdot \hat{\boldsymbol{b}}\, d\boldsymbol{x} + \int_\Omega \mathbf{curl}\,\hat{\boldsymbol{b}} \cdot (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{b}})\, d\boldsymbol{x}$$

Since $\mathbf{curl}\,\hat{\boldsymbol{b}} \times (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{b}}) = -(\mathbf{curl}\,\hat{\boldsymbol{b}}) \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{u}}$, we obtain by adding the both relations:

$$\|\,\mathbf{curl}\,\hat{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\,\mathbf{curl}\,\hat{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{f}, \hat{\boldsymbol{u}}\rangle_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'\times \boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)} + \int_\Omega \boldsymbol{g}\cdot \hat{\boldsymbol{b}}\, d\boldsymbol{x}$$

So we have:

$$\begin{aligned}&\|\mathbf{curl}\,\hat{\boldsymbol{u}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\,\mathbf{curl}\,\hat{\boldsymbol{b}}\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\leqslant C\Big(\,\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'}\,\|\hat{\boldsymbol{u}}\|_{\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\,\|\hat{\boldsymbol{b}}\|_{\boldsymbol{L}^6(\Omega)}\Big)\end{aligned} \tag{2.3.11}$$

Combining (2.2.1) and Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, and then, applying Young inequality in (2.3.11), we obtain that:

$$\|\hat{\boldsymbol{u}}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\hat{\boldsymbol{b}}\|^2_{\boldsymbol{H}^1(\Omega)} \leqslant C\Big(\,\|\boldsymbol{f}\|^2_{[\boldsymbol{H}_0^{6,2}(\mathrm{div},\Omega)]'} + \|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)}\Big) \tag{2.3.12}$$

Thus we conclude that:

$$\|(\boldsymbol{u},\boldsymbol{b})\|_{\boldsymbol{Z}} = \alpha\|(\hat{\boldsymbol{u}},\hat{\boldsymbol{b}})\|_{\boldsymbol{Z}} \leqslant C_1$$

with C_1 which depends on datas of the problem but not of $\boldsymbol{u},\boldsymbol{b}$ and α. By using Leray-Schauder fixed point theorem, we deduce the existence of at least one solution in $((\boldsymbol{u},\boldsymbol{b}),P) \in \boldsymbol{Z}\times L^2(\Omega)/\mathbb{R}$ of problem (2.1.1)-(2.1.6). Since $\hat{\boldsymbol{u}} = \boldsymbol{u}$ and $\hat{\boldsymbol{b}} = \boldsymbol{b}$, the estimate (2.3.2) follows directly from (2.3.12) which completes the proof. □

2.4 Regularity of the weak solutions

In this section, we study the regularity of the weak solution for the MHD problem (2.1.1)-(2.1.6) in L^p-theory. We will use the regularity results for the Stokes problem $(\mathcal{S}_\mathcal{T})$ and $(\mathcal{E}_\mathcal{T})$. For this, we can rewrite the system (2.1.1)-(2.1.6) on the form

$$\begin{cases} -\Delta\boldsymbol{u} + \nabla p = \boldsymbol{F} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} = \boldsymbol{G} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{b} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma,\\ \boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma,\end{cases} \tag{2.4.1}$$

with $\boldsymbol{F} = \boldsymbol{f} - (\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u} + \kappa(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b}$ and $\boldsymbol{G} = \boldsymbol{g} + \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b})$. We begin with the following result concerning strong solutions in $\boldsymbol{W}^{2,p}(\Omega)$ when $p \geq 6/5$.

Theorem 2.4.1. *Let us suppose that Ω is of class $\mathcal{C}^{2,1}$ and $p \geq 6/5$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ satisfying the compatibility conditions* (2.3.1). *Then the solution* $(\boldsymbol{u},\boldsymbol{b},p)$ *of* (2.1.1)-(2.1.6) *given in Theorem 2.3.1 belongs to* $\boldsymbol{W}^{2,p}(\Omega)\times\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)/\mathbb{R}$.

Proof. Let $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ is the weak solution of (2.1.1)-(2.1.6) given by Theorem 2.3.1. Since $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, we have that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. So, we have three cases

(i) Case $\frac{6}{5} \leqslant p \leqslant \frac{3}{2}$. Since $\boldsymbol{F}$ belongs to $\boldsymbol{L}^{\frac{3}{2}}(\Omega)$, we obtain by Theorem 2.2.1 that $(\boldsymbol{u}, P) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R}$. Next, $\boldsymbol{G}$ belongs to $\boldsymbol{L}^{3/2}(\Omega)$, is divergence-free and satisfies the compatibility condition (2.3.1). Thanks to Theorem 2.2.3, the solution $\boldsymbol{b}$ belongs to $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega)$. So, we deduce that $(\boldsymbol{u}, \boldsymbol{b}, P)$ belongs to $\boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R}$. Since $p \leq 3/2$, we have $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ for any $p \leq 3/2$.

(ii) Case $\frac{3}{2} < p < 3$. By the previous case, we have $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times \boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)/\mathbb{R}$. In view of the embeddings

$$\boldsymbol{W}^{2,\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^t(\Omega), \quad \text{for } 1 \leqslant t < \infty, \tag{2.4.2}$$

we have $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \in \boldsymbol{L}^s(\Omega)$, $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^s(\Omega)$, with $1 \leqslant s < 3$. Thus, $\boldsymbol{F} \in \boldsymbol{L}^s(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{G} \in \boldsymbol{L}^s(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$. Applying again the regularity of the Stokes and elliptic problems $(\mathcal{S}_\mathcal{T})$ and $(\mathcal{E}_\mathcal{T})$, we deduce that $(\boldsymbol{u}, \boldsymbol{b}, P) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ for any $p < 3$.

(iii) Case $3 \leqslant p < +\infty$. From the above case, we have $(\boldsymbol{u}, \boldsymbol{b}, p) \in \boldsymbol{W}^{2,3-\delta}(\Omega) \times \boldsymbol{W}^{2,3-\delta}(\Omega) \times W^{1,3-\delta}(\Omega)/\mathbb{R}$ with $\delta \in (0, \frac{3}{2})$. It follows from the Sobolev embedding

$$\boldsymbol{W}^{2,3-\delta}(\Omega) \hookrightarrow \boldsymbol{W}^{1,q}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega), \quad \text{with } 3 < q < \infty \tag{2.4.3}$$

that $\boldsymbol{u} \in \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{b} \in \boldsymbol{L}^\infty(\Omega)$. Then, the terms $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belong to $\boldsymbol{L}^p(\Omega)$ and this implies that $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{G} \in \boldsymbol{L}^p(\Omega)$. The result follows similarly by applying the regularity of $(\mathcal{S}_\mathcal{T})$ and $(\mathcal{E}_\mathcal{T})$. □

Now, we study the regularity $\boldsymbol{W}^{1,p}(\Omega)$ for the solutions of the MHD system (2.1.1) -(2.1.6) for $p > 2$. The proof is based on the regularity of the Stokes and elliptic problems $(\mathcal{S}_\mathcal{T})$ and $(\mathcal{E}_\mathcal{T})$.

Theorem 2.4.2. *Let us suppose that* Ω *is of class* $\mathcal{C}^{2,1}$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)]'$ *and* $\boldsymbol{g} \in \boldsymbol{L}^r(\Omega)$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{3}$ *and* $p \geqslant 2$ *satisfying the compatibility condition* (2.3.1). *Then, the weak solution* $(\boldsymbol{u}, \boldsymbol{b}, P)$ *of* (2.1.1)-(2.1.6) *given in Theorem 2.2.2 belongs to* $\boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

Proof. Since $[\boldsymbol{H}_0^{r',p'}(\mathrm{div}, \Omega)]' \hookrightarrow [\boldsymbol{H}_0^{6,2}(\mathrm{div}, \Omega)]'$ and $\boldsymbol{L}^r(\Omega) \hookrightarrow \boldsymbol{L}^{\frac{6}{5}}(\Omega)$, it follows by applying Theorem 2.2.2 that the system (2.1.1)-(2.1.6) has a solution $(\boldsymbol{u}, \boldsymbol{b}, P)$ in $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. We proceed as in the proof of the previous Theorem, we have $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$ and $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \in \boldsymbol{L}^{\frac{3}{2}}(\Omega)$. So, we separate the proof in two cases: $p \in [2, 3]$ and $p > 3$.

(i) Case $r \leqslant \frac{3}{2}$**.** Then, $p \leq 3$ and with the characterization given in Proposition 2.1.1, we set $\boldsymbol{\phi} \in \boldsymbol{L}^r(\Omega)$ and $\psi \in L^p(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\phi} + \nabla\psi$. Then, we rewrite the first equation in the MHD system (2.1.1)-(2.1.6) as: $\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{F} \quad \text{in } \Omega$ with $\pi = P - \psi$ and $\boldsymbol{F} = (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} + \boldsymbol{\phi}$. Therefore, since $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} - (\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$, we have $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and then, applying Theorem 2.2.1, we obtain that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R}$. However, we have the Sobolev embeddings $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ and $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$, so $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$, and since $\psi \in L^p(\Omega)$, then $\pi \in L^p(\Omega)$ implies that $P \in L^p(\Omega)$.
Next, since $\boldsymbol{g} \in \boldsymbol{L}^r(\Omega)$ and $\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) \in \boldsymbol{L}^{\frac{3}{2}}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$, it directly follows from the regularity of the elliptic problem $(\mathcal{E}_\mathcal{T})$ that $\boldsymbol{b} \in \boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$.

(ii) Case $r \geq \frac{3}{2}$ and $p \leq 3$. We have $[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]' \hookrightarrow [\boldsymbol{H}_0^{(\frac{3}{2})',p'}(\mathrm{div},\Omega)]'$ and $\boldsymbol{L}^r \hookrightarrow \boldsymbol{L}^{\frac{3}{2}}(\Omega)$, so $(\boldsymbol{u},\boldsymbol{b},P) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Note that we have the following Sobolev embeddings:

$$\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^s(\Omega)$$

with $s = \frac{3p}{3+p}$ if $2 \leqslant p < 3$ and $s \in [1,\infty)$ if $p = 3$. Applying it, we obtain that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belong to $\boldsymbol{L}^{\frac{3p}{6-p}}(\Omega)$. Then, $\boldsymbol{F} \in \boldsymbol{L}^{\frac{3p}{6-p}}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$ and, applying Theorem 2.2.1, we obtain that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R}$ and with the same way than in the previous case, we obtain $(\boldsymbol{u},\boldsymbol{b},P) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

(iii) Case $p > 3$: Note that the conditions $\frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{3}$ and $r \leqslant p$ imply $\frac{3p}{3+p} \leqslant r \leqslant p$. We divide this case in two steps: $r \in [\frac{3p}{3+p}, 3)$ and $r \in [3,p]$.

• $r \in [\frac{3p}{3+p}, 3)$: Since the case $p = 3$ and $r \in [\frac{3p}{3+p}, 3)$ comes from the previous case, we have $(\boldsymbol{u},\boldsymbol{b}) \in \boldsymbol{W}^{1,3}(\Omega) \times \boldsymbol{W}^{1,3}(\Omega)$. We have the Sobolev embedding:

$$\boldsymbol{W}^{1,3}(\Omega) \hookrightarrow \boldsymbol{L}^s(\Omega), \quad \forall s \in [1,\infty)$$

then we have $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} \in \boldsymbol{L}^{3-\epsilon}(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$ with $\epsilon > 0$, and same thing for $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$. So we obtain $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$, and then from Theorem 2.2.1 and the previous reasoning on the pressure we obtain $(\boldsymbol{u},P) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R} \hookrightarrow \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. By the same way for the elliptic case, we have $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$.

• Case $r \in [3,p]$: For $0 < \delta < \frac{3}{2}$, we have $[\boldsymbol{H}_0^{r',p'}(\mathrm{div},\Omega)]' \hookrightarrow [\boldsymbol{H}_0^{3-\delta,p'}(\mathrm{div},\Omega)]'$. Then, applying the previous case, we obtain $(\boldsymbol{u},\boldsymbol{b}) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{1,p}(\Omega)$. Applying the Sobolev embedding:

$$\boldsymbol{W}^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^{\infty}(\Omega)$$

we have that $(\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u}$ and $(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b}$ belong to $\boldsymbol{L}^p(\Omega) \hookrightarrow \boldsymbol{L}^r(\Omega)$, so $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$. Then, from Theorem 2.2.1, we obtain $(\boldsymbol{u},P) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega)/\mathbb{R} \hookrightarrow \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. By the same way, we have $\boldsymbol{b} \in \boldsymbol{W}^{1,p}(\Omega)$. □

Remark 2.4.1. *We can also extend the existence of weak solution in* $\boldsymbol{W}^{1,p}(\Omega)$ *with* $p < 2$ *by using a duality argument in the same way as in Chapter 1.*

Chapter 3

Discontinuous Galerkin method for MHD system with Navier-type boundary condition

3.1 Introduction

Let Ω be a bounded simply connected domain in $\mathbb{R}^3$ with a connected boundary Γ. We assume that Ω is convex or has a boundary of class $\mathcal{C}^{1,1}$. We use these restrictions on the domain throughout this Chapter. Additional smoothness of the boundary will be precised whenever needed. We denote by $\boldsymbol{n}$ the unit vector normal to Γ and exterior to Ω. We consider the following Magnetohydrodynamic system (MHD): find the velocity field $\boldsymbol{u}$, the dynamic pressure p and the magnetic field $\boldsymbol{b}$ satisfying:

$$\begin{aligned} -\nu\,\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u})\times\boldsymbol{u} + \nabla p - \kappa(\mathbf{curl}\,\boldsymbol{b})\times\boldsymbol{b} &= \boldsymbol{f} && \text{in}\,\Omega\\ \kappa\mu\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u}\times\boldsymbol{b}) &= \boldsymbol{g} && \text{in}\,\Omega\\ \operatorname{div}\boldsymbol{u} &= 0, && \text{in}\,\Omega,\\ \operatorname{div}\boldsymbol{b} &= 0, && \text{in}\,\Omega. \end{aligned}$$

Here, ν^{-1} is the hydrodynamic Reynolds number, μ^{-1} is the magnetic Reynolds number and κ is the coupling number. The functions $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g}\in\boldsymbol{L}^2(\Omega)$ are given source terms. We complete the system with the following types of homogeneous boundary conditions for the velocity field $\boldsymbol{u}$ and the magnetic field $\boldsymbol{b}$:

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad &\text{and} \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma,\\ \boldsymbol{b}\cdot\boldsymbol{n} = 0 \qquad &\text{and} \qquad \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{n} = \mathbf{0} \quad \text{on}\,\Gamma. \end{aligned}$$

The scalar functions p will be required to have zero mean over Ω:

$$\int_\Omega p\,dx = 0, \tag{3.1.1}$$

and we require the data $\boldsymbol{g}$ to have the following compatibility condition (see in (2.3.1) Chapter 2)

$$\operatorname{div}\boldsymbol{g} = 0 \quad \text{in}\,\Omega \qquad \text{and} \quad \boldsymbol{g}\cdot\boldsymbol{n} = 0 \;\; \text{on}\,\Gamma. \tag{3.1.2}$$

The main difficulty for this model is to incorporate the divergence-free constraint for the magnetic field. The approach that we will use here consists in introducing a regularization term as in [77]. In order to impose the divergence-free constraint, an other approach consist in introducing a new unknown as a Lagrange multiplier r (see [61, 81]) and solving a system of the form: find the velocity field $\boldsymbol{u}$, the dynamic pressure p, the magnetic field $\boldsymbol{b}$, and the scalar function r satisfying:

$$-\nu\,\Delta\boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla p - \kappa(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \qquad \text{in}\,\Omega \tag{3.1.3}$$

$$\kappa\mu\;\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) + \nabla r = \boldsymbol{g} \qquad \text{in}\,\Omega \tag{3.1.4}$$

$$\operatorname{div}\boldsymbol{u} = 0, \qquad \text{in}\,\Omega, \tag{3.1.5}$$

$$\operatorname{div}\boldsymbol{b} = 0, \qquad \text{in}\,\Omega. \tag{3.1.6}$$

with $\displaystyle\int_\Omega r\,dx = 0$. Remark that in theory the scalar r should be null. Indeed, by taking formally the divergence of (3.1.4), we obtain $\Delta r = \operatorname{div}\boldsymbol{g}$ in Ω. In particular, if $\operatorname{div}\boldsymbol{g} = 0$, we have $r = 0$. Thus, in this case (or for $\boldsymbol{g} = \boldsymbol{0}$), the MHD system (3.1.3)-(3.1.6) is the same as (3.4.1)-(3.4.3) and that considered in Chapter 2. We note that the regularity of $\boldsymbol{u}$ and $\boldsymbol{b}$ on non-convex and non-smooth domains is lower than $\boldsymbol{H}^1(\Omega)$. It is well-known that on a general non-convex Lipschitz polyhedron, the space $\boldsymbol{H}(\operatorname{div},\Omega) \cap \boldsymbol{H}(\mathbf{curl},\Omega)$ is embedded in $\boldsymbol{H}^s(\Omega)$ for $\frac{1}{2} < s < 1$ only.

A full DG method is proposed in [61] for a linearized variant of the system (3.4.1)-(3.4.3) where Dirichlet type boundary conditions on the velocity field together with zero tangential trace for the magnetic field are imposed. In the recent work [81], a DG method is proposed and analyzed for the nonlinear MHD problem where with two types of boundary conditions for the magnetic field but still with Dirichlet type boundary conditions for the velocity field. In this chapter we consider a new variational setting for the formulation of MHD problem where the Navier-type boundary condition is imposed both for the velocity and for the magnetic field. To our knowledge, there is no work in the literature done concerning the DG method for the MHD problem with Navier-type boundary conditions for the velocity and the magnetic field.

3.2 A mixed discontinuous Galerkin method for linearized MHD system

We introduce the linearized problem: For given functions $\boldsymbol{w}$ such that $\mathbf{curl}\,\boldsymbol{w} \in \boldsymbol{L}^\infty(\Omega)$ and $\boldsymbol{d} \in \boldsymbol{L}^\infty(\Omega)$, find the velocity field $\boldsymbol{u}$, the dynamic pressure p and the magnetic field $\boldsymbol{b}$ satisfying:

$$-\nu\,\Delta\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{w} \times \boldsymbol{u} + \nabla p - \kappa(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{d} = \boldsymbol{f} \qquad \text{in}\,\Omega \tag{3.2.1}$$

$$\kappa\mu\;\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{d}) = \boldsymbol{g} \qquad \text{in}\,\Omega \tag{3.2.2}$$

$$\operatorname{div}\boldsymbol{u} = 0, \qquad \text{in}\,\Omega, \tag{3.2.3}$$

$$\operatorname{div}\boldsymbol{b} = 0, \qquad \text{in}\,\Omega, \tag{3.2.4}$$

with the boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on}\,\Gamma, \tag{3.2.5}$$

$$\boldsymbol{b}\cdot\boldsymbol{n} = 0 \qquad \text{and} \qquad \mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on}\,\Gamma. \tag{3.2.6}$$

and the scalar function p required to have zero mean over Ω:

$$\int_\Omega p\,dx = 0, \tag{3.2.7}$$

3.2.1 The continuous problem and its well-posedness

In this subsection, we first introduce a mixed variational formulation for the linearized problem (3.2.1)-(3.2.7) that incorporates the divergence constraint on $\boldsymbol{b}$. Next, we prove the existence and the uniqueness of the weak solution. To write (3.2.1)-(3.2.7) in a weak form, we review some notations and functional framework we will use in the study. We denote by $\boldsymbol{H}^1_T(\Omega)$ the space

$$\boldsymbol{H}^1_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega); \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on}\,\Gamma\} \tag{3.2.8}$$

With the assumption made on Ω, we have the following inequality for functions $\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega)$ (see V. Girault and P.A. Raviart [46, Theorem 3.9])

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \le C\big(\,\|\mathbf{curl}\ \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\boldsymbol{v}\|_{L^2(\Omega)}\big) \tag{3.2.9}$$

We recall the following Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$ for all real numbers $1 \le p \le 6$. So, we have from (3.2.9)

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega), \quad \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C\big(\,\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\boldsymbol{v}\|_{L^2(\Omega)}\big). \tag{3.2.10}$$

We employ the product space $\boldsymbol{W}(\Omega) = \boldsymbol{H}^1_T(\Omega) \times \boldsymbol{H}^1_T(\Omega)$ with the usual graph norm

$$\|(\boldsymbol{v},\boldsymbol{c})\|_{\boldsymbol{W}(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{c}\|^2_{\boldsymbol{H}^1(\Omega)})^{1/2} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega),\ \boldsymbol{c} \in \boldsymbol{H}^1_T(\Omega).$$

Furthermore, we set $\|\mathcal{L}\|_{\boldsymbol{L}^2(\Omega)} = \Big(\,\|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}\Big)^{1/2}$.

Based on the well-known vector identity

$$-\Delta\boldsymbol{b} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \nabla(\mathrm{div}\,\boldsymbol{b}),$$

and since $\mathrm{div}\,\boldsymbol{b} = 0$ in Ω, we can replace the curl-curl operator in the Maxwell equation by the vector Laplacian. The variational formulation of the linearized MHD system (3.2.1)-(3.2.7) aims to find $((\boldsymbol{u},\boldsymbol{b}),p) \in \boldsymbol{W}(\Omega) \times L^2_0(\Omega)$ such that

$$\begin{aligned}
\mathcal{A}((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c})) + \mathcal{B}((\boldsymbol{v},\boldsymbol{c}),p) &= \mathcal{L}((\boldsymbol{v},\boldsymbol{c})), \quad \forall(\boldsymbol{v},\boldsymbol{c}) \in \boldsymbol{W}(\Omega), && (3.2.11)\\
\mathcal{B}((\boldsymbol{u},\boldsymbol{b}),q) &= 0, \qquad\qquad \forall q \in L^2_0(\Omega), && (3.2.12)
\end{aligned}$$

where the forms $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ are defined as

$$\mathcal{A}((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c})) = A(\boldsymbol{u},\boldsymbol{v}) + M(\boldsymbol{b},\boldsymbol{c}) + O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) + C(\boldsymbol{d},\boldsymbol{v},\boldsymbol{b}) - C(\boldsymbol{d},\boldsymbol{u},\boldsymbol{c}), \tag{3.2.13}$$

with

$$\begin{aligned}
A(\boldsymbol{u},\boldsymbol{v}) &= \nu\Big(\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x} + \int_\Omega(\mathrm{div}\,\boldsymbol{u})\,(\mathrm{div}\,\boldsymbol{v})\,d\boldsymbol{x}\Big), && (3.2.14)\\
M(\boldsymbol{b},\boldsymbol{c}) &= \kappa\mu\Big(\int_\Omega \mathbf{curl}\,\boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{c}\,d\boldsymbol{x} + \int_\Omega(\mathrm{div}\,\boldsymbol{b})\,(\mathrm{div}\,\boldsymbol{c})\,d\boldsymbol{x}\Big), && (3.2.15)\\
O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega(\mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{v})\cdot\boldsymbol{u}\,d\boldsymbol{x}, && (3.2.16)\\
C(\boldsymbol{d},\boldsymbol{v},\boldsymbol{b}) &= \kappa\int_\Omega(\mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{d})\cdot\boldsymbol{v}\,d\boldsymbol{x}, && (3.2.17)
\end{aligned}$$

$$\mathcal{B}((\boldsymbol{v},\boldsymbol{c}),p) = B(\boldsymbol{v},q) = \int_\Omega (\operatorname{div}\boldsymbol{v})q\,d\boldsymbol{x}. \tag{3.2.18}$$

and

$$\mathcal{L}(\boldsymbol{v},\boldsymbol{c}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\boldsymbol{x} + \int_\Omega \boldsymbol{g}\cdot\boldsymbol{c}\,d\boldsymbol{x}. \tag{3.2.19}$$

The next proposition shows that the weak solution for the linearized MHD problem (3.2.1)-(3.2.7) is in fact a solution of the variational problem (3.2.11)-(3.2.12), and vice-versa. The proof is standard and similar to [52, Proposition 3.1].

Proposition 3.2.1. *We suppose that $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$, $\boldsymbol{g}\in\boldsymbol{L}^2(\Omega)$ satisfy the compatibility condition* (3.1.2). *The problem: Find $((\boldsymbol{u},\boldsymbol{b}),p)\in\boldsymbol{W}(\Omega)\times L^2_0(\Omega)$ satisfying* (3.2.1)-(3.2.7) *is equivalent to the variational formulation* (3.2.11)-(3.2.12).

We need to verify some properties of the forms defined above which will be useful to prove the existence and uniqueness of a solution to the problem (3.2.11)-(3.2.12). In the following Lemma, we prove the coercivity and the continuity of the form $\mathcal{A}$.

Lemma 3.2.2. *For any $(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{W}(\Omega)$, $(\boldsymbol{b},\boldsymbol{c})\in\boldsymbol{W}(\Omega)$, there exist constants $C>0$ such that*

$$\mathcal{A}((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{u},\boldsymbol{b})) \geq C\,\|(\boldsymbol{u},\boldsymbol{b})\|^2_{\boldsymbol{W}(\Omega)} \tag{3.2.20}$$

$$\mathcal{A}((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c})) \leq C\,\|(\boldsymbol{u},\boldsymbol{b})\|_{\boldsymbol{W}(\Omega)}\,\|(\boldsymbol{v},\boldsymbol{c})\|_{\boldsymbol{W}(\Omega)} \tag{3.2.21}$$

Proof. For each $(\boldsymbol{u},\boldsymbol{b})\in\boldsymbol{W}(\Omega)$, $(\boldsymbol{v},\boldsymbol{c})\in\boldsymbol{W}(\Omega)$, it is easy to see that the trilinear form $\mathcal{A}_0((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c}))$ defined by

$$\mathcal{A}_0((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c})) = O(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) + C(\boldsymbol{d},\boldsymbol{v},\boldsymbol{b}) - C(\boldsymbol{d},\boldsymbol{u},\boldsymbol{c}), \tag{3.2.22}$$

is antisymmetric with respect to its last two arguments and satisfies

$$\mathcal{A}_0((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{u},\boldsymbol{b})) = 0. \tag{3.2.23}$$

In view of (3.2.9),we have

$$\begin{aligned}
&\mathcal{A}((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{u},\boldsymbol{b}))\\
=\ &\nu\big(\,\|\mathbf{curl}\ \boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|^2_{L^2(\Omega)}\,\big) + \mu\kappa\big(\,\|\mathbf{curl}\ \boldsymbol{b}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{b}\|^2_{L^2(\Omega)}\,\big)\\
\geq\ & C\nu\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + C\mu\kappa\,\|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}\\
\geq\ & C\min(\nu,\mu\kappa)\,\|(\boldsymbol{u},\boldsymbol{b})\|^2_{\boldsymbol{W}(\Omega)},
\end{aligned}$$

which gives the estimate (3.2.20). For the proof of the continuity result in (3.2.21), we obtain by using the Cauchy-Schwarz inequality

$$\begin{aligned}
A(\boldsymbol{u},\boldsymbol{v}) + M(\boldsymbol{b},\boldsymbol{c}) \leq\ &\nu\big(\,\|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\,\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{u}\|_{L^2(\Omega)}\,\|\operatorname{div}\boldsymbol{v}\|_{L^2(\Omega)}\,\big)\\
&+\mu\kappa\big(\,\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)}\,\|\mathbf{curl}\,\boldsymbol{c}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div}\boldsymbol{b}\|_{L^2(\Omega)}\,\|\operatorname{div}\boldsymbol{c}\|_{L^2(\Omega)}\,\big)\\
\leq\ &C\max(\nu,\mu\kappa)\big(\,\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\,\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}\,\|\boldsymbol{c}\|_{\boldsymbol{H}^1(\Omega)}\,\big)\\
\leq\ &C\max(\nu,\mu\kappa)\Big(\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)}\,\Big)^{1/2}\Big(\,\|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{c}\|^2_{\boldsymbol{H}^1(\Omega)}\,\Big)^{1/2}.
\end{aligned} \tag{3.2.24}$$

Moreover, we have for all $\boldsymbol{u}, \boldsymbol{b} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{v}, \boldsymbol{c} \in \boldsymbol{H}^1(\Omega)$

$$\begin{aligned} O(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) &\leq \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^\infty(\Omega)} \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{W}(\Omega)} \|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}, \end{aligned} \tag{3.2.25}$$

$$\begin{aligned} C(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{b}) &\leq \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{W}(\Omega)} \|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}. \end{aligned} \tag{3.2.26}$$

and similarly

$$C(\boldsymbol{d}, \boldsymbol{u}, \boldsymbol{c}) \leq \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{W}(\Omega)} \|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}. \tag{3.2.27}$$

We deduce from (3.2.25), (3.2.26) and (3.2.27) that

$$\mathcal{A}_0((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{c})) \leq \left(\|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \right) \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{W}(\Omega)} \|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}. \tag{3.2.28}$$

Using (3.2.24) and (3.2.28), we obtain

$$\mathcal{A}((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{c})) \leq \left(\max(\nu, \mu\kappa) + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^\infty(\Omega)} + \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)}\right) \|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{W}(\Omega)} \|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)},$$

which finishes the proof. $\square$

Moreover, the bilinear form B is continuous in $\boldsymbol{H}^1_T(\Omega) \times L^2_0(\Omega)$ and satisfies the following inf-sup condition: there exists a constant $\beta > 0$ such that

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega)} \frac{B(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1_T(\Omega)}} \geqslant \beta \|q\|_{L^2(\Omega)}, \tag{3.2.29}$$

which is a direct consequence of its analogue with $\boldsymbol{H}^1_T(\Omega)$ replaced by $\boldsymbol{H}^1_0(\Omega)$ (see [22]).
Observe that

$$\sup_{(\boldsymbol{v},\boldsymbol{c}) \in \boldsymbol{W}(\Omega)} \frac{\mathcal{B}((\boldsymbol{v}, \boldsymbol{c}), q)}{\|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}} \geq \sup_{(\boldsymbol{v},\boldsymbol{c}) \in \boldsymbol{H}^1_T(\Omega) \times \{0\}} \frac{\mathcal{B}((\boldsymbol{v}, \boldsymbol{c}), q)}{\|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}}.$$

Hence, by the definition of $\mathcal{B}(\cdot, \cdot)$ and (3.2.29), we deduce the following inf-sup:

$$\sup_{(\boldsymbol{v},\boldsymbol{c}) \in \boldsymbol{W}(\Omega)} \frac{\mathcal{B}((\boldsymbol{v}, \boldsymbol{c}), q)}{\|(\boldsymbol{v}, \boldsymbol{c})\|_{\boldsymbol{W}(\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}. \tag{3.2.30}$$

The continuity of the linear forms $\mathcal{L}$ follows with the same arguments. We are now able to state the following existence and uniqueness result.

Theorem 3.2.3. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ satisfying* (3.1.2)*. The variational formulation* (3.2.11)-(3.2.12) *admits a unique solution $((\boldsymbol{u}, \boldsymbol{b}), p) \in \boldsymbol{W}(\Omega) \times L^2_0(\Omega)$ and there exists a constant $C > 0$ such that*

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{\boldsymbol{W}(\Omega)} \leq C \|\mathcal{L}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.2.31}$$

Proof. According to the inf-sup condition (3.2.30) and Lemma 3.2.2, the hypothesis of the Babuska-Brezzi theorem are satisfied. Therefore, the mixed variational formulation (3.2.11)-(3.2.12) is well-posed. The estimate (3.2.31) can be derived by choosing $(\boldsymbol{v}, \boldsymbol{c}) = (\boldsymbol{u}, \boldsymbol{b})$ and $q = p$ in (3.2.11)-(3.2.12) and using the coercivity property (3.2.20). $\square$

3.2.2 Mixed finite element discretization

In this section, we introduce a mixed DG approximation for (3.2.1)-(3.2.7). We provide the solvability and stability of the discrete scheme. To this end, let us introduce some notations. The domain Ω is discretized by a discrete family of conforming meshes $\mathcal{T}_h$ made of tetrahedra. The index h is indicative of the mesh size h which is defined as $h = \max\limits_{T\in\mathcal{T}_h} h_T$, where h_T is the diameter of T. The family is supposed to be regular in Ciarlet's sense [29], i.e. there exists $\varsigma > 0$ independent of h such that the ratio

$$\frac{h_T}{\rho_T} \leq \varsigma, \quad \forall T \in \mathcal{T}_h, \tag{3.2.32}$$

where ρ_T is the diameter of the inscribed circle in T. We shall use the assumption (3.2.32) throughout this work. Let us denote by $\mathcal{F}_h^I$ the set of internal faces and by $\mathcal{F}_h^\Gamma$ the set of external faces on Γ. We set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^\Gamma$. We denote by h_e the diameter of each face e. Let T^+ and T^- be two adjacent elements of $\mathcal{T}_h$ and let $\boldsymbol{n}^+$ (respectively $\boldsymbol{n}^-$) be the outward unit normal vector on ∂T^+ (respectively ∂T^-). For a vector field $\boldsymbol{u}$, we denote by $\boldsymbol{u}^\pm$ the trace of $\boldsymbol{u}$ from the interior of $T^\pm$. We define jumps

$$[\![\boldsymbol{v}]\!]_T := \boldsymbol{n}^+\times\boldsymbol{v}^+ + \boldsymbol{n}^-\times\boldsymbol{v}^-, \qquad [\![\boldsymbol{v}]\!]_N := \boldsymbol{v}^+\cdot\boldsymbol{n}^+ + \boldsymbol{v}^-\cdot\boldsymbol{n}^-, \quad [\![q]\!] := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-, \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}^+ - \boldsymbol{v}^-,$$

and averages

$$\{\!\{\boldsymbol{v}\}\!\} := \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-), \qquad \{\!\{q\}\!\} := \frac{1}{2}(q^+ + q^-),$$

and adopt the convention that for boundary faces $e \in \mathcal{F}_h^\Gamma$, we set $[\![\boldsymbol{v}]\!]_T = \boldsymbol{v}\times\boldsymbol{n}$, $[\![\boldsymbol{v}]\!]_N = \boldsymbol{v}\cdot\boldsymbol{n}$, $[\![q]\!] = q\boldsymbol{n}$, $\{\!\{\boldsymbol{v}\}\!\} = \boldsymbol{v}$ and $\{\!\{q\}\!\} = q$. Let $\mathcal{P}_k$ denotes the space of polynomials of total degree at most k on T with $k = 1, 2$ or 3. The corresponding vector-valued function space is denoted by $\boldsymbol{\mathcal{P}}_k$. Now, we introduce the following finite element spaces which respectively approximate $\boldsymbol{u}$, $\boldsymbol{b}$ and p:

$$\begin{aligned}
\boldsymbol{X}_h &:= \left\{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{v}_{h|T} \in \boldsymbol{\mathcal{P}}_k(T)\,,\ \forall T \in \mathcal{T}_h\right\}\\
\boldsymbol{C}_h &:= \left\{\boldsymbol{c}_h \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{c}_{h|T} \in \boldsymbol{\mathcal{P}}_k(T)\,,\ \forall T \in \mathcal{T}_h\right\}\\
Q_h &:= \left\{q_h \in L^2_0(\Omega);\ q_{h|T} \in \mathcal{P}_{k-1}(T)\,,\ \forall T \in \mathcal{T}_h\right\}
\end{aligned}$$

We denote by $\boldsymbol{W}_h$ the product space $\boldsymbol{X}_h \times \boldsymbol{C}_h$. The norm $\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)}$ is defined by

$$\|\cdot\|_{\boldsymbol{L}^2(\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} \|\cdot\|_{0,T}\,, \ \text{ for any } T \in \mathcal{T}_h,$$

with $\|\cdot\|_{0,T} = \|\cdot\|_{\boldsymbol{L}^2(T)}$. Similarly, we use the notation $\|\cdot\|_{0,e} = \|\cdot\|_{\boldsymbol{L}^2(e)}$ for any $e \in \mathcal{F}_h$.

The mixed DG scheme reads: Find $((\boldsymbol{u}_h, \boldsymbol{b}_h), p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$\begin{aligned}
\mathcal{A}_h((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{v}_h,\boldsymbol{c}_h)) + \mathcal{B}_h((\boldsymbol{v}_h,\boldsymbol{c}_h),p_h) &= \mathcal{L}((\boldsymbol{v}_h,\boldsymbol{c}_h)), &(3.2.33)\\
\mathcal{B}_h((\boldsymbol{u}_h,\boldsymbol{b}_h),q_h) &= 0, &(3.2.34)
\end{aligned}$$

for all $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$, where the discrete forms $\mathcal{A}_h$ and $\mathcal{B}_h$ are defined by

$$\begin{aligned}
\mathcal{A}_h((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{v}_h,\boldsymbol{c}_h)) &= A_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + M_h(\boldsymbol{b}_h,\boldsymbol{c}_h) + O_h(\boldsymbol{w},\boldsymbol{u}_h,\boldsymbol{v}_h)\\
&+ C_h(\boldsymbol{d},\boldsymbol{v}_h,\boldsymbol{b}_h) - C_h(\boldsymbol{d},\boldsymbol{u}_h,\boldsymbol{c}_h), &(3.2.35)\\
\mathcal{B}_h((\boldsymbol{u}_h,\boldsymbol{b}_h),q_h) &= B_h(\boldsymbol{u}_h,q_h), &(3.2.36)
\end{aligned}$$

with

$$\begin{aligned}
A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \quad := \quad & \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \boldsymbol{v}_h \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\ \boldsymbol{u}_h)(\mathrm{div}\ \boldsymbol{v}_h)\, d\boldsymbol{x} \\
- \quad & \nu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\ \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{v}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \Big) \\
- \quad & \nu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\mathrm{div}\ \boldsymbol{u}_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \int_e \{\!\{\mathrm{div}\, \boldsymbol{v}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \Big) \\
+ \quad & \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_T \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \, ds
\end{aligned}$$

with $\sigma_1, \sigma_2 > 0$ the stabilization parameters that will be chosen large enough to ensure the coercivity of the bilinear form A_h (see Lemma 3.2.5 below). The two last terms in the definition of A_h involving the tangential and normal jumps of the discrete vector fields across the edges are necessary to ensure the coercivity of the bilinear form A_h (see Lemma 3.2.5 below). We define the convective term with :

$$O_h(\boldsymbol{w}, \boldsymbol{u}_h,\, \boldsymbol{v}_h) := \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \tag{3.2.37}$$

Observe that due to the assumption on $\mathbf{curl}\, \boldsymbol{w}$, the form O_h can be written as

$$O_h(\boldsymbol{w}, \boldsymbol{u}_h,\, \boldsymbol{v}_h) := \int_\Omega (\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x},$$

where the integral is well defined. We also define the coupling form C_h as :

$$C_h(\boldsymbol{d}, \boldsymbol{v}_h, \boldsymbol{b}_h) := \kappa \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{v}_h \times \boldsymbol{d}) \cdot \mathbf{curl}\, \boldsymbol{b}_h \, d\boldsymbol{x} - \kappa \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{v}_h \times \boldsymbol{d}\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds. \tag{3.2.38}$$

The divergence constraint on the velocity is represented by B_h :

$$B_h(\boldsymbol{v}_h, p_h) := - \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{v}_h) p_h \, d\boldsymbol{x} + \sum_{e\in\mathcal{F}_h} \int_e \{\!\{p_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds. \tag{3.2.39}$$

For the curl-curl term in Maxwell equation, analogous to the continuous setting, we apply an augmentation technique where we replace the curl-curl operator by the vector Laplacian (c.f. [52, 77]). The form M_h is then defined by:

$$\begin{aligned}
M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) \quad := \quad & \kappa\mu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{b}_h \cdot \mathbf{curl}\, \boldsymbol{c}_h \, d\boldsymbol{x} + \kappa\mu \sum_{T\in\mathcal{T}_h} \int_T (\mathrm{div}\, \boldsymbol{b}_h)(\mathrm{div}\, \boldsymbol{c}_h)\, d\boldsymbol{x} \\
- \quad & \kappa\mu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\, \boldsymbol{b}_h\}\!\} \cdot [\![\boldsymbol{c}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{c}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds \Big) \\
- \quad & \kappa\mu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\mathrm{div}\, \boldsymbol{b}_h\}\!\} [\![\boldsymbol{c}_h]\!]_N \, ds + \{\!\{\mathrm{div}\, \boldsymbol{c}_h\}\!\} [\![\boldsymbol{b}_h]\!]_N \, ds \Big) \\
+ \quad & \sum_{e\in\mathcal{F}_h^I} \frac{\kappa\mu m_1}{h_e} \int_e [\![\boldsymbol{b}_h]\!]_T \cdot [\![\boldsymbol{c}_h]\!]_T \, ds + \sum_{e\in\mathcal{F}_h} \frac{\kappa\mu m_2}{h_e} \int_e [\![\boldsymbol{b}_h]\!]_N [\![\boldsymbol{c}_h]\!]_N \, ds
\end{aligned}$$

with $m_1, m_2 > 0$ are stabilization parameters that will be chosen large enough to ensure the coercivity of the bilinear form M_h (see Lemma 3.2.5 below).

Remark 3.2.1. *Another discontinuous Galerkin formulation is possible to solve the MHD problem (3.2.1)-(3.2.4). Indeed, the curl-curl operator can be discretized using a standard interior penalty approach (see [61, 57, 81]) and then the form M_h is defined as*

$$\begin{aligned} M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) \quad &:= \quad \kappa\mu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{b}_h \cdot \mathbf{curl}\, \boldsymbol{c}_h \, d\boldsymbol{x} - \kappa\mu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{b}_h\}\!\} \cdot [\![\boldsymbol{c}_h]\!]_T \, ds \\ &- \quad \kappa\mu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{c}_h\}\!\} \cdot [\![\boldsymbol{b}_h]\!]_T \, ds + \sum_{e\in\mathcal{F}_h^I} \frac{\kappa\mu m_0}{h_e} \int_e [\![\boldsymbol{b}_h]\!]_T \cdot [\![\boldsymbol{c}_h]\!]_T \, ds \end{aligned} \tag{3.2.40}$$

with $m_0 > 0$ a stabilization parameter sufficiently large. The divergence-free constraint for the magnetic field is imposed by introducing a Lagrange multiplier r_h. This approach is motivated by the lack of the regularity of $\boldsymbol{b}$ when Ω is not smooth. As mentionned above, if Ω is non-convex polyhedra, the magnetic field $\boldsymbol{b}$ belongs to $\boldsymbol{H}^{1/2}(\Omega)$ only. Due to our assumption on Ω, the magnetic field as well as the velocity field have the regularity $\boldsymbol{H}^1(\Omega)$.

3.2.3 Continuity and Ellipticity properties

To discuss the stability properties of the above forms, we need to introduce the following semi-norms

$$|\boldsymbol{u}_h|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathrm{div}\, \boldsymbol{u}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 , \qquad \forall \boldsymbol{u}_h \in \boldsymbol{X}_h + \boldsymbol{H}_T^1(\Omega).$$

and

$$\|\boldsymbol{u}_h\|_{v,h}^2 := |\boldsymbol{u}_h|_{1,h}^2 + \sum_{e\in\mathcal{F}_h} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 , \qquad \forall \boldsymbol{u}_h \in \boldsymbol{X}_h + \boldsymbol{H}_T^1(\Omega).$$

$$\|\boldsymbol{b}_h\|_{m,h}^2 := |\boldsymbol{b}_h|_{1,h}^2 + \sum_{e\in\mathcal{F}_h} \frac{m_1}{h_e} \|[\![\boldsymbol{b}_h]\!]_N\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{m_2}{h_e} \|[\![\boldsymbol{b}_h]\!]_T\|_{0,e}^2 , \qquad \forall \boldsymbol{b}_h \in \boldsymbol{C}_h + \boldsymbol{H}_T^1(\Omega).$$

$$\|q_h\|_{Q_h}^2 := \|q_h\|_{\mathrm{L}^2(\Omega)}^2 , \qquad \forall q_h \in Q_h.$$

We have the following result

Lemma 3.2.4. *The application $\boldsymbol{u} \mapsto \|\boldsymbol{u}\|_{v,h}$ (respectively $\boldsymbol{u} \mapsto \|\boldsymbol{u}\|_{m,h}$) is a norm on $\boldsymbol{X}_h$ (respectively on $\boldsymbol{C}_h$)*

Proof. Let $\boldsymbol{u} \in \boldsymbol{X}_h$ such that $\|\boldsymbol{u}\|_{v,h} = 0$ and we prove that $\boldsymbol{u} = \boldsymbol{0}$ in Ω. It follows that $\mathrm{div}\, \boldsymbol{u} = 0$ and $\mathbf{curl}\, \boldsymbol{u} = 0$ in T for all $T \in \mathcal{T}_h$. Moreover, $[\![\boldsymbol{u}]\!]_N = 0$ for all $e \in \mathcal{F}_h$. Thus $\boldsymbol{u}$ is continuous across every internal faces of $\mathcal{F}_h^I$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on every external faces of $\mathcal{F}_h^\Gamma$. So $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div}, \Omega)$. Furthermore, we have $[\![\boldsymbol{u}]\!]_T = \boldsymbol{0}$ for all $e \in \mathcal{F}_h^I$. Thus $\boldsymbol{u}$ is continuous across every internal faces of $\mathcal{F}_h^I$ and then $\boldsymbol{u}$ belongs to $\boldsymbol{H}(\mathbf{curl}, \Omega)$. We deduce that $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)$. Applying the Poincaré-type inequality (see [8]), we obtain that:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\, \boldsymbol{u}\|_{L^2(\Omega)}) = 0 \tag{3.2.41}$$

what achieves the proof for $\|\boldsymbol{u}\|_{v,h}$. The proof is exactly the same for the norm $\|\boldsymbol{u}\|_{m,h}$. □

Next, we recall the following discrete Poincaré-Friedrichs inequality for discontinuous finite element spaces (see [102, Lemma 3.1]) :

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \, \|\boldsymbol{v}_h\|_{v,h} , \tag{3.2.42}$$

where the constant $C > 0$ is independent of h. Since $\boldsymbol{X}_h = \boldsymbol{C}_h$, this inequality is still valid for any $\boldsymbol{b}_h \in \boldsymbol{C}_h$ and when the norm $\|\cdot\|_{v,h}$ is replaced by $\|\cdot\|_{m,h}$.

We have a first lemma which ensures the coercivity for the forms A_h and M_h.

Lemma 3.2.5. *For σ_1 and σ_2 large enough, there exists a positive constant C such that:*

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geqslant C\nu \|\boldsymbol{u}_h\|_{v,h}^2, \qquad \boldsymbol{u}_h \in \boldsymbol{X}_h. \tag{3.2.43}$$

For m_1 and m_2 large enough, there exists a positive constant C such that:

$$M_h(\boldsymbol{b}_h, \boldsymbol{b}_h) \geqslant C\kappa\mu \|\boldsymbol{b}_h\|_{m,h}^2, \qquad \boldsymbol{b}_h \in \boldsymbol{C}_h. \tag{3.2.44}$$

Besides, we have $O_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{u}_h) = 0$.

Proof. Trivially, we have:

$$O_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{u}_h) = \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u}_h) \cdot \boldsymbol{u}_h \, d\boldsymbol{x} = 0.$$

Now, we prove the ellipticity property for A_h. Let $\boldsymbol{u}_h \in \boldsymbol{X}_h$, we have:

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) &= \nu \sum_{T\in\mathcal{T}_h} (\|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 + \|\mathrm{div}\, \boldsymbol{u}_h\|_{0,T}^2) \\ &- 2\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds - 2\nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\mathrm{div}\, \boldsymbol{u}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \\ &+ \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 \end{aligned} \tag{3.2.45}$$

Applying the Cauchy-Schwarz inequality, we have:

$$2\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \leqslant 2\nu \Big(\sum_{e\in\mathcal{F}_h^I} h_e \|\{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\}\|_{0,e}^2 \Big)^{\frac{1}{2}} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}}$$

Now let $e \in \mathcal{F}_h^I$ such that $e = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Obviously,

$$\sqrt{h_e} \| \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\}\|_{0,e} \quad \leq \quad \frac{\sqrt{h_e}}{2} \sum_{i=1}^{2} \|(\mathbf{curl}\ \boldsymbol{u}_h)_{/T_i}\|_{0,e}.$$

Thanks to the equivalence of norms in finite dimensional spaces and to a classical scaling argument, we obtain

$$\sqrt{h_e} \| \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\}\|_{0,e} \quad \leq \quad \frac{c_1}{2} \Big(\| \,\mathbf{curl}\, \boldsymbol{u}_h\|_{0,\mathrm{T}_1} + \| \,\mathbf{curl}\, \boldsymbol{u}_h\|_{0,\mathrm{T}_2} \Big),$$

where c_1 is a constant independent of the discretization. A completely similar argument holds on a boundary face $e \in \mathcal{F}_h^\Gamma$. So by summing upon all faces it follows that

$$\begin{aligned} 2\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds &\leq 2c_1\nu \big(\sum_{T\in\mathcal{T}_h} \| \,\mathbf{curl}\ \boldsymbol{u}_h\|_{0,T}^2 \big)^{1/2} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}} \\ &\leq 2c_1\nu |\boldsymbol{u}_h|_{1,h} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Similarly, we can check that

$$2\nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\operatorname{div} \boldsymbol{u}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \leq 2c_2\nu |\boldsymbol{u}_h|_{1,h} \Big(\sum_{e\in\mathcal{F}_h} h_e^{-1} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2 \Big)^{\frac{1}{2}}.$$

As a consequence, we have

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \ &\geq \ \nu |\boldsymbol{u}_h|_{1,h}^2 + \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2 \\ &- \ \frac{2c_1\nu}{\sqrt{\sigma_1}} |\boldsymbol{u}_h|_{1,h} \Big(\sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 \Big)^{\frac{1}{2}} - \frac{2c_2\nu}{\sqrt{\sigma_2}} |\boldsymbol{u}_h|_{1,h} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2 \Big)^{\frac{1}{2}}, \end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned} A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \quad &\geq \quad \nu |\boldsymbol{u}_h|_{1,h}^2 + \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2 \\ &- \quad \frac{c\,\nu}{\sigma} \Big(|\boldsymbol{u}_h|_{1,h}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_T \right\|_{0,e}^2 + \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \left\| [\![\boldsymbol{u}_h]\!]_N \right\|_{0,e}^2 \Big), \end{aligned}$$

where $c = c_1 + c_2$ and $\sigma = \min(\sqrt{\sigma_1}, \sqrt{\sigma_2})$. So $A_h(\cdot,\cdot)$ is coercive for $\sigma > c$ and the coercivity constant C in (3.2.43) can be obtained from below independently of σ_1 and σ_2, for σ_1 and σ_2 sufficiently large. With the same manner, we prove the ellipticity property of M_h by using the definition of $\|\boldsymbol{b}_h\|_{m,h}$ and by supposing that the parameters m_1 and m_2 are sufficiently large. □

Remark 3.2.2. *Compared with the approach in [61, 57, 81], a semi-coercivity property for M_h, defined in* (3.2.40)*, is established with respect to the semi-norm*

$$|\boldsymbol{b}_h|_C^2 := \sum_{T\in\mathcal{T}_h} \left\| \mathbf{curl}\, \boldsymbol{b}_h \right\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h^I} \frac{m_0}{h_e} \left\| [\![\boldsymbol{b}_h]\!]_T \right\|_{0,e}^2 , \tag{3.2.46}$$

for a parameter m_0 sufficiently large.

In the sequel, the following inverse estimate will be useful (see [29])

Lemma 3.2.6. *For all $p \in \mathcal{P}_k(T)$, we have*

$$\|p\|_{0,\partial T} \leq C h_T^{-1/2} \|p\|_{0,T} , \tag{3.2.47}$$

where C is a positive constant depending only on k and the shape regularity parameter ς of the mesh.

Now, we give the following continuity results with respect of the norms $\|\cdot\|_{v,h}$ and $\|\cdot\|_{m,h}$. The proof is similar to that in [31, 16], but we write it here for the reader's convenience.

Lemma 3.2.7. *Let $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h$, $\boldsymbol{b}_h, \boldsymbol{c}_h \in \boldsymbol{C}_h$ and $p_h \in Q_h$. Then, we have:*

$$\begin{aligned} |A_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \quad &\leqslant \quad C\nu \left\| \boldsymbol{u}_h \right\|_{v,h} \left\| \boldsymbol{v}_h \right\|_{v,h} , \\ |B_h(\boldsymbol{v}_h, p_h)| \quad &\leqslant \quad C \left\| \boldsymbol{v}_h \right\|_{v,h} \left\| p_h \right\|_{0,\Omega} , \\ |M_h(\boldsymbol{b}_h, \boldsymbol{c}_h)| \quad &\leqslant \quad C\kappa\mu \left\| \boldsymbol{b}_h \right\|_{m,h} \left\| \boldsymbol{c}_h \right\|_{m,h} , \end{aligned}$$

with constants $C > 0$ that are independent of h, ν, μ and κ.

Proof. Let us write: $A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := I_1 + I_2 + I_3 + I_4$, with

$$\begin{aligned}
I_1 &:= \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\ \boldsymbol{v}_h \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h \, d\boldsymbol{x} \\
I_2 &:= -\nu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\ \boldsymbol{u}_h\}\!\} \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{v}_h\}\!\} \cdot [\![\boldsymbol{u}_h]\!]_T \, ds \Big) \\
I_3 &:= -\nu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\operatorname{div}\ \boldsymbol{u}_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \int_e \{\!\{\operatorname{div} \boldsymbol{v}_h\}\!\} [\![\boldsymbol{u}_h]\!]_N \, ds \Big) \\
I_4 &:= \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_T \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \, ds
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|I_1| &\leqslant \nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac12} + \nu \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{u}_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac12} \\
&\le \nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{u}_h\|_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{1/2} \\
&\le \nu \, \|\boldsymbol{u}_h\|_{v,h} \, \|\boldsymbol{v}_h\|_{v,h} .
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|I_4| &\leqslant \nu \Big(\sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \|[\![\boldsymbol{v}_h]\!]_T\|^2 \Big)^{\frac12} + \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{u}_h]\!]_N\|_{0,e}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|^2 \Big)^{\frac12} \\
&\le \nu \, \|\boldsymbol{u}_h\|_{v,h} \, \|\boldsymbol{v}_h\|_{v,h} .
\end{aligned}$$

Now, applying Cauchy-Schwarz inequality and the inverse inequality (3.2.47), we obtain:

$$\begin{aligned}
|I_2| &\leqslant \nu \Big(\sum_{e\in\mathcal{F}_h^I} h_e \, \|\{\!\{\mathbf{curl}\, \boldsymbol{u}_h\}\!\}\|_{0,e}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \, \|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2 \Big)^{\frac12} \\
&\quad + \nu \Big(\sum_{e\in\mathcal{F}_h^I} h_e \, \|\{\!\{\mathbf{curl}\, \boldsymbol{v}_h\}\!\}\|_{0,e}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \, \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac12} \\
&\leqslant \nu C \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{u}_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \, \|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2 \Big)^{\frac12} \\
&\quad + \nu C \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \, \|[\![\boldsymbol{u}_h]\!]_T\|_{0,e}^2 \Big)^{\frac12} .
\end{aligned}$$

So, we have $|I_2| \le C\nu \, \|\boldsymbol{u}_h\|_{v,h} \, \|\boldsymbol{v}_h\|_{v,h}$ and in a similar way, for T_3, we can verify that $|I_3| \le C\nu \, \|\boldsymbol{u}_h\|_{v,h} \, \|\boldsymbol{v}_h\|_{v,h}$. Adding these previous estimates, we deduce the continuity for A_h. To prove the continuity of the form M_h it suffices to use the same techniques as in the proof for A_h. Next, the form B_h is similar to the form used in [31, 59] for the Stokes system and we can prove similarly

$$\begin{aligned}
|B_h(\boldsymbol{v}_h, p_h)| &\leqslant \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{T\in\mathcal{T}_h} \|p_h\|_{0,T}^2 \Big)^{\frac12} + \Big(\sum_{e\in\mathcal{F}_h} \frac{h_e}{\sigma_2} \|\{\!\{p_h\}\!\}\|_{0,e}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 \Big)^{\frac12} \\
&\leqslant \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{T\in\mathcal{T}_h} \|p_h\|_{0,T}^2 \Big)^{\frac12} + C\sigma_2^{-\frac12} \Big(\sum_{T\in\mathcal{T}_h} \|p_h\|_{0,T}^2 \Big)^{\frac12} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \|[\![\boldsymbol{v}_h]\!]_N\|_{0,e}^2 \Big)^{\frac12} \\
&\leqslant C \, \|\boldsymbol{v}_h\|_{v,h} \, \|p_h\|_{0,\Omega} .
\end{aligned}$$

□

We shall need the following estimates for the trilinear forms O_h and C_h.

Lemma 3.2.8. *For all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h$ and $\boldsymbol{c}_h \in \boldsymbol{C}_h$, we have*

$$|O_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h)| \leqslant C \left\|\mathbf{curl}\, \boldsymbol{w}\right\|_{L^\infty(\Omega)} \left\|\boldsymbol{u}_h\right\|_{v,h} \left\|\boldsymbol{v}_h\right\|_{v,h}, \tag{3.2.48}$$

$$|C_h(\boldsymbol{d}; \boldsymbol{v}_h, \boldsymbol{c}_h)| \leqslant \kappa C \left\|\boldsymbol{d}\right\|_{\boldsymbol{L}^\infty(\Omega)} \left\|\boldsymbol{v}_h\right\|_{v,h} \left\|\boldsymbol{c}_h\right\|_{m,h}. \tag{3.2.49}$$

with constants $C > 0$ independent of h, ν, μ and κ.

Proof. Applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |O_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h)| &= \Big| \int_\Omega (\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \Big| \leq \left\|\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u}_h\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{L}^2(\Omega)} \\ &\leq 2 \left\|\mathbf{curl}\, \boldsymbol{w}\right\|_{\boldsymbol{L}^\infty(\Omega)} \left\|\boldsymbol{u}_h\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

By virtue of (3.2.42), the estimate (3.2.48) follows immediately. Next, by the definition of the form $\boldsymbol{C}_h$, we have

$$C_h(\boldsymbol{d}, \boldsymbol{v}_h, \boldsymbol{c}_h) = \kappa \sum_{T \in \mathcal{T}_h} \int_T (\boldsymbol{v}_h \times \boldsymbol{d}) \cdot \mathbf{curl}\, \boldsymbol{c}_h \, d\boldsymbol{x} - \kappa \sum_{e \in \mathcal{F}_h^I} \int_e \{\!\!\{\boldsymbol{v}_h \times \boldsymbol{d}\}\!\!\} \cdot [\![\boldsymbol{c}_h]\!]_T \, ds \tag{3.2.50}$$

Applying the Cauchy-Schwarz inequality and the inverse inequality (3.2.47), we obtain

$$\begin{aligned} |C_h(\boldsymbol{d}, \boldsymbol{v}_h, \boldsymbol{c}_h)| &\leq 2\kappa \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \Big(\sum_{T \in \mathcal{T}_h} \|\, \mathbf{curl}\, \boldsymbol{c}_h\|_{0,T}^2 \Big)^{1/2} \\ &+ C\kappa \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \Big(\sum_{e \in \mathcal{F}_h^I} h \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(e)}^2 \Big)^{1/2} \Big(\sum_{e \in \mathcal{F}_h^I} h^{-1} \|[\![\boldsymbol{c}_h]\!]_T\|_{\boldsymbol{L}^2(e)}^2 \Big)^{1/2} \\ &\leq C\kappa \left\|\boldsymbol{d}\right\|_{\boldsymbol{L}^\infty(\Omega)} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{c}_h\right\|_{m,h}. \end{aligned}$$

Thanks again to (3.2.42), we deduce the estimate (3.2.49). □

Now, by endowing the space $\boldsymbol{W}_h$ with the product norm $|||(\cdot,\cdot)|||$ defined by

$$|||(\boldsymbol{v}_h, \boldsymbol{c}_h)|||^2 = \left\|\boldsymbol{v}_h\right\|_{v,h}^2 + \left\|\boldsymbol{c}_h\right\|_{m,h}^2,$$

we deduce in view of Lemma 3.2.5, 3.2.7 and Lemma 3.2.8 the following coercivity and continuity properties for the form $\mathcal{A}_h(\cdot,\cdot,\cdot)$.

Lemma 3.2.9. *For any $(\boldsymbol{u}_h, \boldsymbol{v}_h) \in \boldsymbol{W}_h$, $(\boldsymbol{b}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$, there exist constants $C > 0$ such that*

$$\mathcal{A}_h((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h)) \geq C \min(\nu, \mu\kappa) |||(\boldsymbol{u}_h, \boldsymbol{b}_h)|||^2. \tag{3.2.51}$$

$$\mathcal{A}_h((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{v}_h, \boldsymbol{c}_h)) \leq C |||(\boldsymbol{u}_h, \boldsymbol{b}_h)|||\, |||(\boldsymbol{v}_h, \boldsymbol{c}_h)|||. \tag{3.2.52}$$

3.2.4 Inf-sup conditions

The well-posedness of the discrete linearized problem (3.2.33)-(3.2.34) requires an inf-sup condition which is an extension of the usual inf-sup condition for the Stokes problem to our Navier-type boundary conditions. Let us begin by introducing an approximation result for the space $\boldsymbol{X}_h$ (see [45]). For $k = 1, 2, 3$, there exists a continuous interpolation operator I_h defined from $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{X}_h$ such that, for all $T \in \mathcal{T}_h$ and $e \in \mathcal{F}_h$:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\ \ \forall q_h \in \mathcal{P}_{k-1}(T),\ \int_T q_h \operatorname{div}(I_h(\boldsymbol{v}) - \boldsymbol{v})\, dx = 0, \tag{3.2.53}$$

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega),\ \forall e \in \mathcal{F}_h,\ \ \forall q_h \in \mathcal{P}_{k-1}(e),\ \ \int_e q_h \, [\![I_h(\boldsymbol{v})]\!] \, ds = 0. \tag{3.2.54}$$

Moreover, for $s \in [1, k+1]$ the following interpolation estimate holds:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^s(\Omega), \quad \|I_h(\boldsymbol{v}) - \boldsymbol{v}\|_{1,T} \leqslant C h_T^{s-1} \|\boldsymbol{v}\|_{s,\Delta_T}\,, \tag{3.2.55}$$

where Δ_T is a suitable macro-element containing T.

We have the following lemma for the inf-sup condition of B_h:

Lemma 3.2.10. *There exists $C_B > 0$ only depending on Ω such that :*

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{B_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{v,h} \|q_h\|_{Q_h}} \geqslant C_B, \tag{3.2.56}$$

Proof. To prove the inf-sup condition (3.2.56), we show that for any $q_h \in Q_h$, we can associate $\boldsymbol{v}_h \in \boldsymbol{X}_h$ such that :

$$B_h(\boldsymbol{v}_h, q_h) = \|q_h\|^2_{L^2(\Omega)}\,, \tag{3.2.57}$$

$$\|\boldsymbol{v}_h\|_{v,h} \leqslant C \|q_h\|_{L^2(\Omega)}\,. \tag{3.2.58}$$

Applying the classical inf-sup condition for the Stokes problem with Dirchlet boundary condition, we know that for any $q \in L^2_0(\Omega)$, there exists $\boldsymbol{z} \in \boldsymbol{H}^1_0(\Omega)$ such that:

$$\begin{cases} \operatorname{div} \boldsymbol{z} = q, \\ \|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C \|q\|_{L^2(\Omega)} \end{cases}$$

In particular, for $q_h \in Q_h \subset L^2_0(\Omega)$, there exists $\boldsymbol{z} \in \boldsymbol{H}^1_0(\Omega)$ and hence in $\boldsymbol{H}^1_T(\Omega)$ such that $q_h = \operatorname{div} \boldsymbol{z}$. Now, let $\boldsymbol{v}_h = I_h(\boldsymbol{z})$ with I_h the interpolation operator from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{X}_h$ defined in (3.2.53)-(3.2.54). Then, we have

$$\forall T \in \mathcal{T}_h, \quad \int_T q_h \operatorname{div} \boldsymbol{v}_h \, d\boldsymbol{x} = \int_T q_h \operatorname{div} \boldsymbol{z} d\boldsymbol{x} = \|q_h\|^2_{0,T}\,.$$

Moreover, due to (3.2.54), we have $\displaystyle\int_e \{\!\{q_h\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds = 0$, $\forall e \in \mathcal{F}_h$. Thus $B_h(\boldsymbol{v}_h, q_h) = \|q_h\|^2_{L^2(\Omega)}$, which gives (3.2.57). Besides, by the continuity of the operator I_h, we deduce that

$$\|\boldsymbol{v}_h\|_{v,h} = \|I_h \boldsymbol{z}\|_{v,h} \leqslant C \|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|q\|_{L^2(\Omega)}\,. \tag{3.2.59}$$

This yields to

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{B_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{v,h}} \geqslant \frac{B_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{v,h}} \geqslant \frac{\|q_h\|^2_{L^2(\Omega)}}{C \|q_h\|_{L^2(\Omega)}} \geqslant C_B \|q_h\|_{L^2(\Omega)}\,. \tag{3.2.60}$$

□

The discrete inf-sup condition for B_h in (3.2.56) gives the following discrete analogue of (3.2.30).

$$\sup_{(\boldsymbol{v}_h,\boldsymbol{c}_h)\in \boldsymbol{W}_h} \frac{\mathcal{B}_h((\boldsymbol{v}_h,\boldsymbol{c}_h),q)}{\|(\boldsymbol{v}_h,\boldsymbol{c}_h)\|_{\boldsymbol{W}_h}} \geqslant C\,\|q_h\|_{L^2(\Omega)}\,. \tag{3.2.61}$$

Now we can come back to the discrete linearized problem (3.2.33)-(3.2.34) and show its well posedness by using the previous properties.

Theorem 3.2.11. *Assuming the stabilization parameters σ_1, σ_2, m_1 and m_2 sufficiently large, there exists a unique solution $((\boldsymbol{u}_h,\boldsymbol{b}_h),p_h)\in \boldsymbol{W}_h\times Q_h$ for the linear problem* (3.2.33)-(3.2.34).

Proof. Since the problem is linear and finite dimensional, we have only to show that if $\boldsymbol{f}=\boldsymbol{0}$ and $\boldsymbol{g}=\boldsymbol{0}$, then $((\boldsymbol{u}_h,\boldsymbol{b}_h),p_h)$ is the zero solution. Taking $\boldsymbol{v}_h=\boldsymbol{u}_h$, $q_h=p_h$ and $\boldsymbol{c}_h=\boldsymbol{b}_h$ as test functions in (3.2.33)-(3.2.34) and summing up the equations, we obtain:

$$\mathcal{A}_h((\boldsymbol{w},\boldsymbol{d}),(\boldsymbol{u}_h,\boldsymbol{u}_h),(\boldsymbol{b}_h,\boldsymbol{b}_h)) = A_h(\boldsymbol{u}_h,\boldsymbol{u}_h)+M_h(\boldsymbol{b}_h,\boldsymbol{b}_h)=0$$

Thus, with (3.2.43) and (3.2.44), we obtain that $\boldsymbol{u}_h=\boldsymbol{0}$, and $\boldsymbol{b}_h=\boldsymbol{0}$. Then, (3.2.34) gives:

$$\mathcal{B}_h((\boldsymbol{v}_h,\boldsymbol{c}_h),p_h)=B_h(\boldsymbol{v}_h,p_h)=0,\ \ \forall \boldsymbol{v}_h\in \boldsymbol{X}_h.$$

Applying (3.2.56), we can take $q_h=p_h$ and, for all $\boldsymbol{v}_h\in\boldsymbol{X}_h$ and $\boldsymbol{v}_h\neq\boldsymbol{0}$, we obtain:

$$\frac{B_h(\boldsymbol{v}_h,p_h)}{\|\boldsymbol{v}_h\|_{v,h}}\geqslant C_B\,\|p_h\|_{L^2(\Omega)}\,.$$

Thus we conclude that $p_h=0$. □

3.3 A priori error estimates for the linearized MHD system

In this section, we establish some a priori error estimates in terms of h in order to deduce the convergence of the DG scheme (3.2.33)-(3.2.34) for the linearized MHD model. Our analysis is based on an auxiliary formulation by using lifting operators. We shall suppose the following smoothness assumption on the analytical solution $((\boldsymbol{u},\boldsymbol{b}),p)$ of (3.2.1)-(3.2.7):

$$\boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega),\ \ \boldsymbol{b}\in\boldsymbol{H}^{k+1}(\Omega),\ \ \ p\in H^k(\Omega), \tag{3.3.1}$$

The following theorem gives our main convergence result.

Theorem 3.3.1. *Assume the analytical solution $((\boldsymbol{u},\boldsymbol{b}),p)$ of* (3.2.1)-(3.2.7) *satisfies the smoothness assumption* (3.3.1). *Let $((\boldsymbol{u}_h,\boldsymbol{b}_h),p_h)$ be the solution in $\boldsymbol{W}_h\times Q_h$ of the DG method* (3.2.33)-(3.2.34). *Then, we have the following estimates on the errors:*

$$|||(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{b}-\boldsymbol{b}_h)|||+\|p-p_h\|_{L^2(\Omega)}\leq Ch^k\big(\,\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}+\|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\big) \tag{3.3.2}$$

with C a positive constant independent of h but dependent on the datas of the problem.

3.3.1 Definition of an extended DG form

Following [18], we introduce some lifting operators in order to facilitate the error analysis by extending the forms A_h, B_h, M_h and D_h to the continuous level. To this end, we define the following spaces:

$$\boldsymbol{X}(h) = \boldsymbol{H}^1_T(\Omega) + \boldsymbol{X}_h, \quad \boldsymbol{C}(h) = \boldsymbol{H}^1_T(\Omega) + \boldsymbol{C}_h \quad \text{and} \quad Q(h) = L^2_0(\Omega) + Q_h.$$

We define the following lifting operators:

- $\mathcal{L}_1 : \boldsymbol{X}(h) \to \boldsymbol{X}_h$ such that:

$$\int_\Omega \mathcal{L}_1(\boldsymbol{v}) \cdot \boldsymbol{w}\, d\mathbf{x} = \sum_{e\in\mathcal{F}_h^I} \int_e [\![\boldsymbol{v}]\!]_T \cdot \{\!\{\boldsymbol{w}\}\!\}\, ds, \qquad \forall \boldsymbol{w} \in \boldsymbol{X}_h,$$

- $\mathcal{L}_2 : \boldsymbol{C}(h) \to \boldsymbol{C}_h$ such that:

$$\int_\Omega \mathcal{L}_2(\boldsymbol{c}) \cdot \boldsymbol{w}\, d\boldsymbol{x} = \sum_{e\in\mathcal{F}_h^I} \int_e [\![\boldsymbol{c}]\!]_T \cdot \{\!\{\boldsymbol{w}\}\!\}\, ds, \qquad \forall \boldsymbol{w} \in \boldsymbol{C}_h,$$

- $\mathcal{N}_1 : \boldsymbol{X}(h) \to \Sigma_h$ such that

$$\int_\Omega \mathcal{N}_1(\boldsymbol{v}) w\, d\mathbf{x} = \sum_{e\in\mathcal{F}_h} \int_e [\![\boldsymbol{v}]\!]_N \, \{\!\{w\}\!\}\, ds, \qquad \forall w \in \Sigma_h,$$

- $\mathcal{N}_2 : \boldsymbol{C}(h) \to \Sigma_h$ such that

$$\int_\Omega \mathcal{N}_2(\boldsymbol{v}) w\, d\mathbf{x} = \sum_{e\in\mathcal{F}_h} \int_e [\![\boldsymbol{v}]\!]_N \, \{\!\{w\}\!\}\, ds, \qquad \forall w \in \Sigma_h,$$

where Σ_h is the auxiliary space:

$$\Sigma_h = \{w \in L^2(\Omega);\ w|_T \in \mathcal{P}^k(T),\ T \in \mathcal{T}_h\},$$

- $\mathcal{M}_1 : \boldsymbol{X}(h) \to Q_h$ such that

$$\int_\Omega \mathcal{M}_1(\boldsymbol{v}) q\, d\mathbf{x} = \sum_{e\in\mathcal{F}_h} \int_e [\![\boldsymbol{v}]\!]_N \, \{\!\{q\}\!\}\, ds, \qquad \forall q \in Q_h,$$

- $\mathcal{M}_2 : Q(h) \to \boldsymbol{X}_h$ such that

$$\int_\Omega \mathcal{M}_2(q) \cdot \boldsymbol{v}\, d\mathbf{x} = \sum_{e\in\mathcal{F}_h^I} \int_e [\![q]\!] \cdot \{\!\{\boldsymbol{v}\}\!\}\, ds, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_h.$$

So, we rewrite the modified bilinear forms:

$$\begin{aligned}
\widetilde{A_h}(\boldsymbol{u}_h, \boldsymbol{v}_h) := {} & \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \boldsymbol{v}_h \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{u}_h)(\operatorname{div} \boldsymbol{v}_h)\, d\boldsymbol{x} \\
& - \nu\Big(\int_\Omega \mathcal{L}_1(\boldsymbol{v}_h) \cdot \mathbf{curl}\, \boldsymbol{u}_h \, d\boldsymbol{x} + \int_\Omega \mathcal{L}_1(\boldsymbol{u}_h) \cdot \mathbf{curl}\, \boldsymbol{v}_h \, d\boldsymbol{x} \Big) \\
& - \nu\Big(\int_\Omega \mathcal{N}_1(\boldsymbol{v}_h) \operatorname{div} \boldsymbol{u}_h \, d\boldsymbol{x} + \int_\Omega \mathcal{N}_1(\boldsymbol{u}_h) \operatorname{div} \boldsymbol{v}_h \, d\boldsymbol{x} \Big) \\
& + \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_T \cdot [\![\boldsymbol{v}_h]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{u}_h]\!]_N [\![\boldsymbol{v}_h]\!]_N \, ds,
\end{aligned}$$

$$\widetilde{B_h}(\boldsymbol{v}_h, q_h) := -\sum_{T\in\mathcal{T}_h}\int_T (\operatorname{div}\boldsymbol{v}_h)\, q_h\, d\boldsymbol{x} + \int_\Omega \mathcal{M}_1(\boldsymbol{v}_h) q_h\, d\boldsymbol{x},$$

$$\begin{aligned}\widetilde{M_h}(\boldsymbol{b}_h, \boldsymbol{c}_h) := {} & \kappa\mu \sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\boldsymbol{b}_h\cdot\mathbf{curl}\,\boldsymbol{c}_h\, d\boldsymbol{x} + \kappa\mu\sum_{T\in\mathcal{T}_h}\int_T (\operatorname{div}\boldsymbol{b}_h)(\operatorname{div}\boldsymbol{c}_h)\, d\boldsymbol{x}\\ & - \kappa\mu\Big(\int_\Omega \mathcal{L}_2(\boldsymbol{c}_h)\cdot\mathbf{curl}\,\boldsymbol{b}_h\, d\boldsymbol{x} + \int_\Omega \mathcal{L}_2(\boldsymbol{b}_h)\cdot\mathbf{curl}\,\boldsymbol{c}_h\, d\boldsymbol{x}\Big)\\ & - \kappa\mu\Big(\int_\Omega \mathcal{N}_2(\boldsymbol{c}_h)\operatorname{div}\boldsymbol{b}_h\, d\boldsymbol{x} + \int_\Omega \mathcal{N}_2(\boldsymbol{b}_h)\operatorname{div}\boldsymbol{c}_h\, d\boldsymbol{x}\Big)\\ & + \sum_{e\in\mathcal{F}_h^I}\frac{\kappa\mu m_1}{h_e}\int_e [\![\boldsymbol{b}_h]\!]_T\cdot[\![\boldsymbol{c}_h]\!]_T\, ds + \sum_{e\in\mathcal{F}_h}\frac{\kappa\mu m_2}{h_e}\int_e [\![\boldsymbol{b}_h]\!]_N [\![\boldsymbol{c}_h]\!]_N\, ds,\end{aligned}$$

We note that, by integration by parts, the form $\widetilde{B}_h$ can be written equivalently as:

$$\widetilde{B_h}(\boldsymbol{v}_h, q_h) := \sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{v}_h\cdot\nabla q_h\, d\boldsymbol{x} - \int_\Omega \mathcal{M}_2(q_h)\cdot\boldsymbol{v}_h\, d\boldsymbol{x},$$

As a consequence, we have the following equivalent discrete formulation: Find $((\boldsymbol{u}_h, \boldsymbol{b}_h), p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$\widetilde{\mathcal{A}}_h((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{v}_h, \boldsymbol{c}_h)) + \widetilde{\mathcal{B}}_h((\boldsymbol{v}_h, \boldsymbol{c}_h), p_h) = \mathcal{L}((\boldsymbol{v}_h, \boldsymbol{c}_h)), \tag{3.3.3}$$

$$\widetilde{\mathcal{B}}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), q_h) = 0, \tag{3.3.4}$$

for all $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}(\Omega)$ and $q_h \in Q_h$, where $\widetilde{\mathcal{A}}_h$ and $\widetilde{\mathcal{B}}_h$ are defined by

$$\begin{aligned}\widetilde{\mathcal{A}}_h((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{v}_h, \boldsymbol{c}_h)) &= \widetilde{A}_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \widetilde{M}_h(\boldsymbol{b}_h, \boldsymbol{c}_h) + O_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h)\\ &+ C_h(\boldsymbol{d}, \boldsymbol{v}_h, \boldsymbol{b}_h) - C_h(\boldsymbol{d}, \boldsymbol{u}_h, \boldsymbol{c}_h),\end{aligned} \tag{3.3.5}$$

$$\widetilde{\mathcal{B}}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), q_h) = \widetilde{B}_h(\boldsymbol{u}_h, q_h), \tag{3.3.6}$$

We may remark that $\widetilde{A_h}$ (respectively $\widetilde{M_h}$) coincides with A_h (respectively M_h) on $\boldsymbol{X}_h \times \boldsymbol{X}_h$ (respectively on $\boldsymbol{C}_h\times\boldsymbol{C}_h$). Similarly, $\widetilde{B_h}$ coincides with B_h on $\boldsymbol{X}_h\times Q_h$. Moreover, $\widetilde{A_h}$ and $\widetilde{M_h}$ coincide with A and M respectively on $\boldsymbol{H}^1_T(\Omega) \times \boldsymbol{H}^1_T(\Omega)$. Similarly $\widetilde{B_h}$ coincides with B on $\boldsymbol{H}^1_T(\Omega) \times \boldsymbol{L}^2_0(\Omega)$.

3.3.2 Properties of extended forms

In order to obtain the same properties of continuity than those define in section 3.2.2, we give stability estimates for the lifting operators defined above.

Lemma 3.3.2. *Let $\mathcal{L}_i$, $\mathcal{N}_i$ and $\mathcal{M}_i$ with $i = 1, 2$ be the lifting operators defined above. For all $\boldsymbol{v} \in \boldsymbol{X}(h)$, $\boldsymbol{c} \in \boldsymbol{C}(h)$ and for all $q \in Q(h)$ we have*

$$\|\mathcal{L}_1(\boldsymbol{v})\|_{0,\Omega} \leqslant C_{\text{lift}}\Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\,\|[\![\boldsymbol{v}]\!]_T\|^2_{0,e}\Big)^{\frac12} \quad\text{and}\quad \|\mathcal{L}_2(\boldsymbol{c})\|_{0,\Omega} \leqslant C_{\text{lift}}\Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\,\|[\![\boldsymbol{c}]\!]_T\|^2_{0,e}\Big)^{\frac12} \tag{3.3.7}$$

$$\|\mathcal{N}_1(\boldsymbol{v})\|_{0,\Omega} \leqslant C_{\text{lift}}\Big(\sum_{e\in\mathcal{F}_h} h^{-1}\,\|[\![\boldsymbol{v}]\!]_N\|^2_{0,e}\Big)^{\frac12} \quad\text{and}\quad \|\mathcal{N}_2(\boldsymbol{c})\|_{0,\Omega} \leqslant C_{\text{lift}}\Big(\sum_{e\in\mathcal{F}_h} h^{-1}\,\|[\![\boldsymbol{c}]\!]_N\|^2_{0,e}\Big)^{\frac12} \tag{3.3.8}$$

$$\|\mathcal{M}_1(\boldsymbol{v})\|_{0,\Omega} \leqslant C_{\text{lift}}\Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\,\|[\![\boldsymbol{v}]\!]_N\|^2_{0,e}\Big)^{\frac12} \quad\text{and}\quad \|\mathcal{M}_2(q)\|_{0,\Omega} \leqslant C_{\text{lift}}\Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\,\|[\![q]\!]\|^2_{0,e}\Big)^{\frac12} \tag{3.3.9}$$

The constant C_{lift} is independent of h but depends on the shape–regularity of the mesh.

Proof. The proof is very much similar to [78, Proposition 12]. Let $\boldsymbol{v} \in \boldsymbol{X}(h)$. By definition we have $\boldsymbol{v} = \boldsymbol{v}_h + \boldsymbol{v}^c$ with $\boldsymbol{v}_h \in \boldsymbol{X}_h$ and $\boldsymbol{v}^c \in \boldsymbol{H}^1_T(\Omega)$. Then, since $[\![\boldsymbol{v}^c]\!]_T = 0$ for all $e \in \mathcal{F}_h^I$, we have,

$$\|\mathcal{L}_1(\boldsymbol{v})\|_{0,\Omega} = \sup_{\boldsymbol{w}\in \boldsymbol{X}_h} \frac{\displaystyle\int_\Omega \mathcal{L}_1(\boldsymbol{v})\cdot \boldsymbol{w}\, d\boldsymbol{x}}{\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}} = \sup_{\boldsymbol{w}\in \boldsymbol{X}_h} \frac{\sum_{e\in\mathcal{F}_h^I}\displaystyle\int_e [\![\boldsymbol{v}_h]\!]_T \cdot \{\!\{\boldsymbol{w}\}\!\}\, ds}{\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}} \tag{3.3.10}$$

Using the same arguments in the proof of Lemma 3.2.5, we have:

$$\sum_{e\in\mathcal{F}_h^I}\int_e [\![\boldsymbol{v}_h]\!]_T \cdot \{\!\{\boldsymbol{w}\}\!\}\, ds \le \Big(\sum_{e\in\mathcal{F}_h^I} h_e \|\{\!\{\boldsymbol{w}\}\!\}\|_{0,e}^2\Big)^{\frac12}\Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1}\|[\![\boldsymbol{v}_h]\!]_T\|_{0,e}^2\Big)^{\frac12}.$$

Using the trace inequality and a standard scaling argument, we obtain

$$\sum_{e\in\mathcal{F}_h^I} h_e \|\{\!\{\boldsymbol{w}\}\!\}\|_{0,e}^2 \le C\sum_{T\in\mathcal{T}_h}\|\boldsymbol{w}\|_{0,T}^2 = C\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}^2$$

Inserting this estimate in (3.3.10) proves the first estimate in (3.3.7). For the other estimates, we proceed in the same way.

□

The following Lemma gives the continuity of the extended forms $\widetilde{A_h}$, $\widetilde{M_h}$ and $\widetilde{B_h}$.

Lemma 3.3.3. *For all $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{X}(h)$, $p \in Q(h)$, $\boldsymbol{b},\boldsymbol{c} \in \boldsymbol{C}(h)$ we have:*

$$\begin{aligned}
\widetilde{A_h}(\boldsymbol{u},\boldsymbol{v}) &\leqslant C\nu\|\boldsymbol{u}\|_{v,h}\|\boldsymbol{v}\|_{v,h}\\
\widetilde{B_h}(\boldsymbol{v},p) &\leqslant C\|\boldsymbol{v}\|_{v,h}\|p\|_{0,\Omega}\\
\widetilde{M_h}(\boldsymbol{b},\boldsymbol{c}) &\leqslant C\kappa\mu\|\boldsymbol{b}\|_{m,h}\|\boldsymbol{c}\|_{m,h}
\end{aligned}$$

Proof. Let us prove the property for the form $\widetilde{A_h}$

$$\begin{aligned}
\widetilde{A_h}(\boldsymbol{u},\boldsymbol{v}) :=\; &\nu\sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x} + \nu\sum_{T\in\mathcal{T}_h}\int_T(\operatorname{div}\boldsymbol{u})(\operatorname{div}\boldsymbol{v})\,d\boldsymbol{x}\\
&-\nu\Big(\int_\Omega \mathcal{L}_1(\boldsymbol{v})\cdot\mathbf{curl}\,\boldsymbol{u}\,d\boldsymbol{x} + \int_\Omega \mathcal{L}_1(\boldsymbol{u})\cdot\mathbf{curl}\,\boldsymbol{v}\,d\boldsymbol{x}\Big)\\
&-\nu\Big(\int_\Omega \mathcal{N}_1(\boldsymbol{v})\operatorname{div}\boldsymbol{u}\,d\boldsymbol{x} + \int_\Omega \mathcal{N}_1(\boldsymbol{u})\operatorname{div}\boldsymbol{v}\,d\boldsymbol{x}\Big)\\
&+\nu\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h_e}\int_e [\![\boldsymbol{u}]\!]_T\cdot[\![\boldsymbol{v}]\!]_T\,ds + \nu\sum_{e\in\mathcal{F}_h}\frac{\sigma_2}{h_e}\int_e [\![\boldsymbol{u}]\!]_N[\![\boldsymbol{v}]\!]_N\,ds,\\
=\; &A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

The terms A_1 and A_4 can be bounded as for I_1 and I_2 respectively in the proof of Lemma 3.2.7. For A_2, we have by applying Cauchy-Schwarz inequality and trace inequality combining with the bounds for the lifting operators $\mathcal{L}_1$

$$\begin{aligned}
|A_2| \;&\le \nu\|\mathcal{L}_1(\boldsymbol{v})\|_{0,\Omega}\Big(\sum_{e\in\mathcal{F}_h^I} h\|\mathbf{curl}\,\boldsymbol{u}\|_{0,e}^2\Big)^{1/2} + \nu\|\mathcal{L}_1(\boldsymbol{u})\|_{0,\Omega}\Big(\sum_{e\in\mathcal{F}_h^I} h\|\mathbf{curl}\,\boldsymbol{v}\|_{0,e}^2\Big)^{1/2}\\
&\le \nu C_{\text{lift}}\Big\{\Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\|[\![\boldsymbol{v}]\!]_T\|_{0,e}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\mathbf{curl}\,\boldsymbol{u}\|_{0,T}^2\Big)^{\frac12} + \Big(\sum_{e\in\mathcal{F}_h^I} h^{-1}\|[\![\boldsymbol{u}]\!]_T\|_{0,e}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h}\|\mathbf{curl}\,\boldsymbol{v}\|_{0,T}^2\Big)^{\frac12}\Big\}\\
&\le C\nu\|\boldsymbol{u}\|_{v,h}\|\boldsymbol{v}\|_{v,h}.
\end{aligned}$$

Proceeding as above and using the bound for the lifting operator $\mathcal{N}_1$ given in Lemma 3.3.2, we can obtain the following bound for A_3

$$|A_3| \leq C\nu \|\boldsymbol{u}\|_{v,h} \|\boldsymbol{v}\|_{v,h} .$$

For the forms $\widetilde{B_h}$ and $\widetilde{M_h}$, the proof of the bounds follows from Cauchy-Schwarz inequality and the stability estimates for the operators $\mathcal{L}_i$, $\mathcal{N}_i$ and $\mathcal{M}_i$ for $i = 1, 2$ established in Lemma 3.3.2. $\square$

For the continuity of the convection form O_h and the coupling form C_h, we have the following result.

Lemma 3.3.4. *Let $\boldsymbol{u} \in \boldsymbol{X}(h)$, $\boldsymbol{v} \in \boldsymbol{X}(h)$ and $\boldsymbol{c} \in \boldsymbol{C}(h)$. We have:*

$$\begin{aligned} C_h(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{c}) \quad &\leqslant \quad C\kappa \|\boldsymbol{d}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{c}\|_{m,h} \Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + (\sum_{e\in\mathcal{F}_h^I} h_e \|\boldsymbol{v}\|_{0,e}^2)^{\frac{1}{2}} \Big) \\ O_h(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) \quad &\leqslant \quad C \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \end{aligned}$$

Proof. The proof of (3.3.11) is similar to that in Lemma 3.2.8 and can be found in [61, Proposition 4.2]. For the convection form O_h, the estimate (3.3.11) follows immediately by applying Cauchy-Schwarz inequality as in the proof of (3.2.48). $\square$

Furthermore, we define the kernel

$$\mathrm{Ker}(\widetilde{\mathcal{B}}_h) = \{(\boldsymbol{u}, \boldsymbol{b}) \in \boldsymbol{W}_h; \quad \widetilde{\mathcal{B}}_h((\boldsymbol{u}, \boldsymbol{b}), q) = \widetilde{B}_h(\boldsymbol{u}, q) = 0, \quad \forall q \in Q_h\}$$

Then, we have the following Inf-sup for $\widetilde{\mathcal{B}}_h$ and the coercivity result for the forms $\widetilde{\mathcal{A}}_h$.

Lemma 3.3.5. *The following properties hold:*
(i) There exists $C > 0$ independent of h such that

$$\widetilde{\mathcal{A}}_h((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h)) \geq C \min(\nu, \kappa\mu) |||(\boldsymbol{u}_h, \boldsymbol{b}_h)|||^2. \tag{3.3.11}$$

(ii) The inf-sup condition

$$\inf_{q_h\in Q_h} \sup_{(\boldsymbol{v}_h, \boldsymbol{c}_h)\in \boldsymbol{W}_h} \frac{\widetilde{\mathcal{B}}_h((\boldsymbol{v}_h, \boldsymbol{c}_h), q_h)}{|||(\boldsymbol{v}_h, \boldsymbol{c}_h)||| \, \|q\|_{L^2(\Omega)}} \geqslant C, \tag{3.3.12}$$

holds with a constant $C > 0$ independent of h.

3.3.3 Residuals

Let $(\boldsymbol{u}, p, \boldsymbol{b})$ solution of the analytical problem (3.2.1)-(3.2.4). We introduce the residuals: for all $\boldsymbol{v} \in \boldsymbol{X}_h$, $\boldsymbol{c} \in \boldsymbol{C}_h$:

$$\begin{aligned} R_h((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{c}), p) = &\widetilde{A_h}(\boldsymbol{u}, \boldsymbol{v}) + O_h(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) + C_h(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{b}) + \widetilde{B_h}(\boldsymbol{v}, p) + \widetilde{M_h}(\boldsymbol{b}, \boldsymbol{c}) - C_h(\boldsymbol{d}, \boldsymbol{u}, \boldsymbol{c}) \\ &- (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega - (\boldsymbol{g}, \boldsymbol{c}_h)_\Omega \end{aligned} \tag{3.3.13}$$

We set $\boldsymbol{e}^u = \boldsymbol{u} - \boldsymbol{u}_h$, $\boldsymbol{e}^b = \boldsymbol{b} - \boldsymbol{b}_h$ and $e^p = p - p_h$. Then, we have

$$R_h((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{c}), p) = \widetilde{A_h}(\boldsymbol{e}^u, \boldsymbol{v}) + O_h(\boldsymbol{w}, \boldsymbol{e}^u, \boldsymbol{v}) + C_h(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{e}^b) + \widetilde{B_h}(\boldsymbol{v}, e^p) + \widetilde{M_h}(\boldsymbol{e}^b, \boldsymbol{c}) - C_h(\boldsymbol{d}, \boldsymbol{e}^u, \boldsymbol{c}) \tag{3.3.14}$$

Observe that the equation (3.3.4) is consistent. So, the residual $R_h((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{u},\boldsymbol{b}),q) = \widetilde{\mathcal{B}}_h((\boldsymbol{e}^u,\boldsymbol{e}^b),q) = 0$ for all $q \in Q_h$.
We consider the expression:

$$\mathcal{R}_h((\boldsymbol{u},\boldsymbol{b}),p) = \sup_{(\boldsymbol{v},\boldsymbol{c})\in \boldsymbol{W}_h} \frac{R_h((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c}),p)}{|||(\boldsymbol{v},\boldsymbol{c})|||} \tag{3.3.15}$$

The following result gives some bounds on the residuals. The idea of the proof is based on the use of the stability properties of the forms in conforming spaces. The proof techniques given in [60, 59, 77] can be easily modified to prove Proposition 3.3.6 and hence the proof is omitted.

Proposition 3.3.6. *Assume the analytical solution* $(\boldsymbol{u},p,\boldsymbol{b})$ *of* (3.2.1)-(3.2.4) *with boundary conditions* (3.4.4)-(3.4.6) *satisfies the smoothness assumption in* (3.3.1). *Then, we have for all* $\boldsymbol{v} \in \boldsymbol{X}_h$, $\boldsymbol{c} \in \boldsymbol{C}_h$*:*

$$|R_h((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c}),p)| \quad \leqslant |||(\boldsymbol{v},\boldsymbol{c})|||\, \mathcal{E}(\boldsymbol{u},\boldsymbol{b},p) \tag{3.3.16}$$

with $\mathcal{E}(\boldsymbol{u},\boldsymbol{b},p)$ *bounded by*

$$\mathcal{E}(\boldsymbol{u},\boldsymbol{b},p) \quad \leqslant \quad Ch^k\big(\,\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\,\big) \tag{3.3.17}$$

with constants $C > 0$ *independent of the mesh size.*

Remark 3.3.1. *We note that the previous result is given in [60, 59] with* $\mathcal{E}(\boldsymbol{u},\boldsymbol{b},p)$ *bounded by*

$$\begin{aligned}\mathcal{E}(\boldsymbol{u},\boldsymbol{b},p) \quad &\leqslant \quad Ch^{\min\{\sigma,k\}}\big(\,\|\boldsymbol{u}\|_{\boldsymbol{H}^{\sigma+1}(\Omega)} + \|p\|_{H^\sigma(\Omega)}\,\big)\\ &+ \quad Ch^{\min\{\tau,k\}}\big(\,\|\boldsymbol{b}\|_{\boldsymbol{H}^{\tau}(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^{\tau}(\Omega)}\,\big),\end{aligned} \tag{3.3.18}$$

for $\sigma,\tau > 1/2$ *provided that*

$$\boldsymbol{u} \in \boldsymbol{H}^{\sigma+1}(\Omega),\;\; p \in H^\sigma(\Omega),\;\; \boldsymbol{b} \in \boldsymbol{H}^\tau(\Omega),\;\; \mathbf{curl}\,\boldsymbol{b} \in \boldsymbol{H}^\tau(\Omega). \tag{3.3.19}$$

The domain Ω *is assumed to be a bounded simply-connected Lipschitz polyhedron and then the magnetic field may have regularity below* $\boldsymbol{H}^1(\Omega)$. *Indeed, there exists an exponent* $\tau > 1/2$*, such that the space* $\boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}_0(\mathrm{div},\Omega)$ *is continuously imbedded in* $\boldsymbol{H}^\tau(\Omega)$. *In our work,* Ω *is convex polyhedral or of class* $\mathcal{C}^{1,1}$ *and then the space* $\boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}_0(\mathrm{div},\Omega)$ *is continuously imbedded in* $\boldsymbol{H}^1(\Omega)$. *More generally, due to Corollary 1.3.2 in Chapter 1, the space* $\boldsymbol{X}^{\tau,2}(\Omega)$ *defined by*

$$\boldsymbol{X}^{\tau,2}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\; \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^\tau(\Omega),\; \mathrm{div}\boldsymbol{v} \in H^\tau(\Omega),\; \boldsymbol{v}\cdot\boldsymbol{n} = 0 \;\text{ on}\,\Gamma\}$$

is continuously imbedded in $\boldsymbol{H}^{\tau+1}(\Omega)$ *for* $\tau > 0$. *So, if* $\boldsymbol{b}$ *satisfies the regularity in* (3.3.19) *and* Ω *is more regular, we deduce that* $\boldsymbol{b}$ *belongs to* $\boldsymbol{H}^{\tau+1}(\Omega)$ *and then we have*

$$\|\boldsymbol{b}\|_{\boldsymbol{H}^\tau(\Omega)} + \|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^\tau(\Omega)} \leq \|\boldsymbol{b}\|_{\boldsymbol{H}^{\tau+1}(\Omega)}\,. \tag{3.3.20}$$

Hence, the bound (3.3.18) *implies* (3.3.17).

3.3.4 Proof of the error estimates

In this subsection, we prove the error estimates of Theorem 3.3.1. To do that, we proceed in two steps: we first bound the errors in the velocity $\boldsymbol{u}$ and the magnetic field $\boldsymbol{b}$ and second we bound the errors in the pressure p.

We split the errors in $\boldsymbol{u}$, $\boldsymbol{b}$ and p as:

$$\boldsymbol{e}^u = \boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{\eta}_u + \boldsymbol{\chi}_u, \quad \boldsymbol{e}^b = \boldsymbol{b} - \boldsymbol{b}_h = \boldsymbol{\eta}_b + \boldsymbol{\chi}_b \quad \text{and} \quad e^p = p - p_h = \eta_p + \chi_p,$$

where, for all fixed $\boldsymbol{v} \in \boldsymbol{X}_h$, $\boldsymbol{c} \in \boldsymbol{C}_h$ and $q \in Q_h$, we have

$$\begin{aligned} \boldsymbol{\eta}_u &= \boldsymbol{u} - \boldsymbol{v}, & \boldsymbol{\chi}_u &= \boldsymbol{v} - \boldsymbol{u}_h \\ \boldsymbol{\eta}_b &= \boldsymbol{b} - \boldsymbol{c}, & \boldsymbol{\chi}_b &= \boldsymbol{c} - \boldsymbol{b}_h \\ \eta_p &= p - q, & \chi_p &= q - p_h \end{aligned}$$

Specific approximations $\boldsymbol{v}$, $\boldsymbol{c}$ and q will be chosen at the end of the error analysis. So, to bound the errors $\boldsymbol{e}^u$, $\boldsymbol{e}^b$ and e^p, we need only to derive bounds for $\boldsymbol{\chi}_u$, $\boldsymbol{\chi}_b$, χ_p and then we can conclude using triangle inequality.

Abstract error in $\boldsymbol{u}$ and $\boldsymbol{b}$. In the aim to prove the bounds for $\boldsymbol{e}^u$ and $\boldsymbol{e}^b$, we proceed as in the error analysis of the standard conforming mixed methods, taking into account the residual terms. Let us begin by proving the following abstract estimate for $\|\boldsymbol{e}^u\|_{v,h}$ and $\|\boldsymbol{e}^b\|_{m,h}$. The proof is similar to that in [30, Theorem 4.8] for the error analysis of a DG scheme for the Navier-Stokes problem. See also [77, Theorem 4.7] and [59, Theorem 6.1] for an abstract error bound for the Maxwell equations.

Theorem 3.3.7. *There exists positive constant C such that*

$$|||(\boldsymbol{e}^u, \boldsymbol{e}^b)||| \leq C\Big(\inf_{(\boldsymbol{v},\boldsymbol{c})\in \boldsymbol{W}_h} |||(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)||| + \inf_{q\in Q_h} \|p-q\|_{L^2(\Omega)} + \mathcal{R}(\boldsymbol{u}, \boldsymbol{b}, p)\Big). \tag{3.3.21}$$

Proof. Consider first the case of elements $(\boldsymbol{v}, \boldsymbol{c}) \in \mathrm{Ker}(\widetilde{\mathcal{B}}_h)$. By the coercivity property of the form $\widetilde{\mathcal{A}}_h$ in (3.3.11), we obtain:

$$\begin{aligned} C|||(\boldsymbol{\chi}_u, \boldsymbol{\chi}_b)|||^2 = C\Big(\nu \|\boldsymbol{\chi}_u\|^2_{v,h} + \kappa\mu \|\boldsymbol{\chi}_b\|^2_{m,h}\Big) &\leq \widetilde{\mathcal{A}}_h((\boldsymbol{w}, \boldsymbol{d}), (\boldsymbol{\chi}_u, \boldsymbol{\chi}_b), (\boldsymbol{\chi}_u, \boldsymbol{\chi}_b)) \\ &:= \widetilde{A_h}(\boldsymbol{\chi}_u, \boldsymbol{\chi}_{\boldsymbol{u}}) + \widetilde{M_h}(\boldsymbol{\chi}_b, \boldsymbol{\chi}_b) \end{aligned} \tag{3.3.22}$$

Observe that,

$$\widetilde{A_h}(\boldsymbol{\chi}_u, \boldsymbol{\chi}_u) + \widetilde{M_h}(\boldsymbol{\chi}_b, \boldsymbol{\chi}_b) = \widetilde{A_h}(\boldsymbol{e}^u, \boldsymbol{\chi}_u) + \widetilde{M_h}(\boldsymbol{e}^b, \boldsymbol{\chi}_b) - \widetilde{A_h}(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) - \widetilde{M_h}(\boldsymbol{\eta}_b, \boldsymbol{\chi}_b)$$

From (3.3.14), we have for any $\boldsymbol{v} \in \boldsymbol{X}_h$ and $\boldsymbol{c} \in \boldsymbol{C}_h$

$$\begin{aligned} \widetilde{A_h}(\boldsymbol{e}^u, \boldsymbol{v}) + \widetilde{M_h}(\boldsymbol{e}^b, \boldsymbol{c}) = R_h((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{v}, \boldsymbol{c}), p) - O_h(\boldsymbol{w}, \boldsymbol{e}^u, \boldsymbol{v}) - \widetilde{B_h}(\boldsymbol{v}, e^p) \\ + C_h(\boldsymbol{d}, \boldsymbol{e}^u, \boldsymbol{c}) - C_h(\boldsymbol{d}, \boldsymbol{v}, \boldsymbol{e}^b) \qquad . \end{aligned} \tag{3.3.23}$$

Then, choosing $(\boldsymbol{v}, \boldsymbol{c}) = (\boldsymbol{\chi}_u, \boldsymbol{\chi}_b)$ in (3.3.23) and then inserting this in (3.3.22), we obtain

$$\begin{aligned} C\Big(\nu \|\boldsymbol{\chi}_u\|^2_{v,h} + \kappa\mu \|\boldsymbol{\chi}_b\|^2_{m,h}\Big) &\leq -\big(\widetilde{A_h}(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + \widetilde{M_h}(\boldsymbol{\eta}_b, \boldsymbol{\chi}_b)\big) + R_h((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{\chi}_u, \boldsymbol{\chi}_b), p) \\ -\widetilde{B_h}(\boldsymbol{\chi}_u, e^p) - O_h(\boldsymbol{w}, \boldsymbol{e}^u, \boldsymbol{\chi}_u) &+ \big(C_h(\boldsymbol{d}, \boldsymbol{e}^u, \boldsymbol{\chi}_b) - C_h(\boldsymbol{d}, \boldsymbol{\chi}_u, \boldsymbol{e}^b)\big) := T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

Using the continuity of $\widetilde{A}_h$ and $\widetilde{M}_h$ in Lemma 3.3.3, the term T_1 can be estimated as follows

$$\begin{aligned}|T_1| &\leq C\nu \left\|\boldsymbol{\eta}_u\right\|_{v,h} \left\|\boldsymbol{\chi}_u\right\|_{v,h} + C\mu\kappa \left\|\boldsymbol{\eta}_b\right\|_{m,h} \left\|\boldsymbol{\chi}_b\right\|_{m,h} \\ &\leq C \max\left(\nu, \mu\kappa\right) \|\!|\!|(\boldsymbol{\chi}_{\boldsymbol{u}}, \boldsymbol{\chi}_b)\|\!|\!| \, \|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!|\end{aligned}$$

By the definition of $\mathcal{R}_h$, we have

$$R_h((\boldsymbol{u}, \boldsymbol{b}), (\boldsymbol{\chi}_u, \boldsymbol{\chi}_b), p) \leq \mathcal{R}_h((\boldsymbol{u}, \boldsymbol{b}), p)\|\!|\!|(\boldsymbol{\chi}_{\boldsymbol{u}}, \boldsymbol{\chi}_b)\|\!|\!|. \tag{3.3.24}$$

Next, due to (3.2.36) and our assumption that $(\boldsymbol{v}, \boldsymbol{c}) \in \operatorname{Ker}(\widetilde{\mathcal{B}}_h)$, we deduce that $\boldsymbol{v} - \boldsymbol{u}_h$ belongs to Ker $\widetilde{\mathcal{B}}_h$. Then, for any $q \in Q_h$

$$T_3 = -\widetilde{B_h}(\boldsymbol{\chi}_u, e^p) = -\widetilde{B_h}(\boldsymbol{\chi}_u, p-q).$$

It follows from Lemma 3.3.3 that

$$|T_3| \leq C \left\|\boldsymbol{\chi}_u\right\|_{v,h} \left\|p-q\right\|_{L^2(\Omega)} \leq C\|\!|\!|(\boldsymbol{\chi}_u, \boldsymbol{\eta}_b)\|\!|\!| \left\|p-q\right\|_{L^2(\Omega)}$$

To estimate T_4, observe first that $T_4 = -O_h(\boldsymbol{w}, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u)$. Then by the bound in Lemma 3.3.4 and (3.2.42), we obtain

$$|T_4| \leq C \left\|\boldsymbol{\eta}_u\right\|_{v,h} \left\|\boldsymbol{\chi}_u\right\|_{m,h} \leq C\|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!|\|\!|\!|(\boldsymbol{\chi}_u, \boldsymbol{\chi}_b)\|\!|\!|.$$

Similarly, the term T_5 verifies

$$T_5 = C_h(\boldsymbol{d}, \boldsymbol{\eta}_u, \boldsymbol{\chi}_b) - C_h(\boldsymbol{d}, \boldsymbol{\chi}_u, \boldsymbol{\eta}_b).$$

Then, T_5 can be estimated by using Lemma 3.3.4 and applying Young's inequality $(2ab \leq a^2 + b^2)$

$$|T_5| \leq C\kappa \left\|\boldsymbol{\eta}_u\right\|_{v,h} \left\|\boldsymbol{\chi}_b\right\| + \left\|\boldsymbol{\chi}_u\right\|_{v,h} \left\|\boldsymbol{\eta}_b\right\| \leq C\kappa\|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!|\|\!|\!|(\boldsymbol{\chi}_u, \boldsymbol{\chi}_b)\|\!|\!|$$

Combining the above bounds yields

$$\|\!|\!|(\boldsymbol{\chi}_u, \boldsymbol{\chi}_b)\|\!|\!| \leq C\|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!| + C \left\|p-q\right\|_{L^2(\Omega)} + \mathcal{R}(\boldsymbol{u}, \boldsymbol{b}, p).$$

Then, by the triangle inequality, we obtain

$$\|\!|\!|(\boldsymbol{e}^u, \boldsymbol{e}^b)\|\!|\!| \leq C\|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!| + C \left\|p-q\right\|_{L^2(\Omega)} + \mathcal{R}(\boldsymbol{u}, \boldsymbol{b}, p), \tag{3.3.25}$$

for any $(\boldsymbol{v}, \boldsymbol{c}) \in \operatorname{Ker}(\widetilde{\mathcal{B}}_h)$ and $q \in Q_h$, which gives the estimate (3.3.21). Let us prove now that this estimate still holds for an arbitrary $(\boldsymbol{v}, \boldsymbol{c}) \in \boldsymbol{W}_h$. We consider the problem: find $(\boldsymbol{r}, \boldsymbol{s}) \in \boldsymbol{W}_h$ such that

$$\widetilde{\mathcal{B}}_h((\boldsymbol{r}, \boldsymbol{s}), q) = \widetilde{\mathcal{B}}_h((\boldsymbol{\eta}_u, \boldsymbol{\eta}_b), q), \qquad \forall q \in Q_h.$$

Thanks to the inf-sup condition and the continuity for $\widetilde{\mathcal{B}}_h$, there exists a solution $(\boldsymbol{r}, \boldsymbol{s})$ satisfying

$$\|\!|\!|(\boldsymbol{r}, \boldsymbol{s})\|\!|\!| \leq C\|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!|. \tag{3.3.26}$$

Since $\widetilde{\mathcal{B}}_h((\boldsymbol{u}, \boldsymbol{b}), q) = 0$ for any $q \in Q_h$, we have that $(\boldsymbol{r} + \boldsymbol{v}, \boldsymbol{s} + \boldsymbol{c}) \in \operatorname{Ker}(\widetilde{\mathcal{B}}_h)$ and we can insert it in (3.3.27). Applying triangle inequality, we obtain

$$\|\!|\!|(\boldsymbol{e}^u, \boldsymbol{e}^b)\|\!|\!| \leq C\|\!|\!|(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)\|\!|\!| + \|\!|\!|(\boldsymbol{r}, \boldsymbol{s})\|\!|\!| + C \left\|p-q\right\|_{L^2(\Omega)} + \mathcal{R}(\boldsymbol{u}, \boldsymbol{b}, p). \tag{3.3.27}$$

So, the abstract error estimate (3.3.21) follows by taking into account the bound (3.3.26). □

Abstract error in the pressure p. We want to bound the error in p. We begin by proving an abstract error estimate for the pressure by using the same arguments used for the error in $(\boldsymbol{u}, \boldsymbol{b})$.

Theorem 3.3.8. *There exists positive constant C such that*

$$\|e^p\|_{L^2(\Omega)} \leq C\Big(\inf_{(\boldsymbol{v},\boldsymbol{c})\in \boldsymbol{W}_h} |||(\boldsymbol{\eta}_u, \boldsymbol{\eta}_b)||| + \inf_{q\in Q_h} \|p-q\|_{L^2(\Omega)} + \mathcal{R}(\boldsymbol{u}, \boldsymbol{b}, p)\Big). \tag{3.3.28}$$

Proof. Fix $q \in Q_h$. Thanks to the inf-sup condition for $\widetilde{\mathcal{B}}_h$, we obtain

$$\|p_h - q\|_{L^2(\Omega)} = \|\chi_p\|_{L^2(\Omega)} \leq C \sup_{(\boldsymbol{v},\boldsymbol{c})\in \boldsymbol{W}_h} \frac{\widetilde{\mathcal{B}}_h((\boldsymbol{v},\boldsymbol{c}),\eta_p)}{|||(\boldsymbol{v},\boldsymbol{c})|||} + C \sup_{(\boldsymbol{v},\boldsymbol{c})\in \boldsymbol{W}_h} \frac{\widetilde{\mathcal{B}}_h((\boldsymbol{v},\boldsymbol{c}),e^p)}{|||(\boldsymbol{v},\boldsymbol{c})|||},$$

and then by continuity of $\widetilde{\mathcal{B}}_h$,

$$\|p_h - q\|_{L^2(\Omega)} \leq C\,\|\eta_p\| + C \sup_{(\boldsymbol{v},\boldsymbol{c})\in \boldsymbol{W}_h} \frac{\widetilde{\mathcal{B}}_h((\boldsymbol{v},\boldsymbol{c}),e^p)}{|||(\boldsymbol{v},\boldsymbol{c})|||},$$

It remains to bound the first term in the right-hand side of the above estimate. In view of (3.3.14), we have

$$\begin{aligned}\widetilde{\mathcal{B}}_h((\boldsymbol{v},\boldsymbol{c}),e^p) = R_h((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v},\boldsymbol{c}),p) - \widetilde{A_h}(\boldsymbol{e}^u,\boldsymbol{v}) - O_h(\boldsymbol{w},\boldsymbol{e}^u,\boldsymbol{v})\\ -C_h(\boldsymbol{d},\boldsymbol{v},\boldsymbol{e}^b) - \widetilde{M_h}(\boldsymbol{e}^b,\boldsymbol{c}) + C_h(\boldsymbol{d},\boldsymbol{e}^u,\boldsymbol{c})\end{aligned}$$

Proceeding as above, by using the continuity of the forms $\widetilde{A}_h$, $\widetilde{M}_h$, O_h and C_h, we obtain

$$\|p_h - q\|_{L^2(\Omega)} \leq C|||(\boldsymbol{e}^u,\boldsymbol{e}^b)||| + C\,\|\eta_p\|\,.$$

Inserting the abstract error in (3.3.21) and using the triangle inequality, we obtain the estimate (3.3.28). □

Approximation estimates. Now, we want to give the approximations of the magnetic field and the pressure denoted by $\boldsymbol{v}$, $\boldsymbol{c}$ and q respectively. For this, we use the operator I_h to approximate the velocity field $\boldsymbol{u}$ and the magnetic field $\boldsymbol{b}$. So, we have the following approximation result (3.2.55).

We use the L^2-projection of degree $k-1$ onto Q_h to approximate the pressure p. So for any $T \in \mathcal{T}_h$,

$$\forall q \in \mathcal{P}_{k-1}, \quad \int_T q(p - \Pi_Q p)\,d\boldsymbol{x} = 0. \tag{3.3.29}$$

Moreover, we recall the following standard approximation result (see for instance [29])

Lemma 3.3.9. *For any $q \in H^s(\Omega) \cap L^2_0(\Omega)$ with $s \in [0,k]$, we have*

$$\|p - \Pi_Q p\|_{0,T} \leq h_T^s\,\|q\|_{s,T} \tag{3.3.30}$$

So, if we take $\boldsymbol{v} = I_h(\boldsymbol{u})$; $\boldsymbol{c} = I_h\boldsymbol{b}$, $q = \Pi_Q(p)$ with Π_Q the projection of order $k-1$ on Q_h and I_h defined in (3.2.53)-(3.2.54), we deduce from the properties (3.2.55) and (3.3.30) the following estimate

Proposition 3.3.10. *Assume $((\boldsymbol{u},\boldsymbol{b}),p)$ satisfies the smoothness assumption* (3.3.1)*. Then, we have*

$$|||(\boldsymbol{\eta}_u,\boldsymbol{\eta}_b)||| + \|\eta_p\|_{L^2(\Omega)} \leqslant Ch^k\Big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\Big), \tag{3.3.31}$$

with $C > 0$ independent of the mesh size.

We are now ready to give the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. Injecting (3.3.17) and (3.3.31) in (3.3.21) and (3.3.28), we obtain (3.3.2) □

3.4 The discontinuous Galerkin scheme for the nonlinear MHD system

The aim of this section is to extend the fully discontinuous approach of the previous section to the nonlinear MHD problem:

$$-\nu\,\Delta \boldsymbol{u} + (\mathbf{curl}\,\boldsymbol{u}) \times \boldsymbol{u} + \nabla p - \kappa(\mathbf{curl}\,\boldsymbol{b}) \times \boldsymbol{b} = \boldsymbol{f} \qquad \text{in}\,\Omega, \tag{3.4.1}$$

$$\kappa\mu\;\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{b} - \kappa\,\mathbf{curl}(\boldsymbol{u} \times \boldsymbol{b}) = \boldsymbol{g} \qquad \text{in}\,\Omega, \tag{3.4.2}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{b} = 0 \qquad \text{in}\,\Omega, \tag{3.4.3}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on}\,\Gamma, \tag{3.4.4}$$

$$\boldsymbol{b}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{b} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on}\,\Gamma. \tag{3.4.5}$$

and p with zero mean over Ω:

$$\int_\Omega p\,dx = 0. \tag{3.4.6}$$

Analogously to the continuous setting for the nonlinear MHD problem (see Theorem 2.3.1, Chapter 2), we introduce the discrete problem and we apply a fixed-point argument to prove its well posedness. We shall make the additional smoothness assumptions on Ω. Indeed, we suppose that the domain Ω has a boundary of class $\mathcal{C}^{2,1}$

We introduce the following mixed finite element method: Find $((\boldsymbol{u}_h, \boldsymbol{b}_h), p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$\begin{aligned}
\mathcal{A}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{u}_h, \boldsymbol{b}_h), (\boldsymbol{v}_h, \boldsymbol{c}_h)) + \mathcal{B}_h((\boldsymbol{v}_h, \boldsymbol{c}_h), p_h) &= \mathcal{L}((\boldsymbol{v}_h, \boldsymbol{c}_h)), && (3.4.7)\\
\mathcal{B}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), q_h) &= 0, && (3.4.8)
\end{aligned}$$

for all $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$, where the discrete forms $\mathcal{A}_h$ and $\mathcal{B}_h$ are defined in (3.2.35) and (3.2.36).

We introduce the space $\boldsymbol{K}_h$ defined by:

$$\boldsymbol{K}_h = \{(\boldsymbol{u}_h, \boldsymbol{b}_h) \in \boldsymbol{W}_h; \quad \mathcal{B}_h((\boldsymbol{u}_h, \boldsymbol{b}_h), q_h) = B_h(\boldsymbol{u}_h, q_h) = 0, \quad \forall q_h \in Q_h\}.$$

and we consider the discrete operator $\mathcal{F}_h : \boldsymbol{K}_h \to \boldsymbol{K}_h$ defined by: $\mathcal{F}_h(\boldsymbol{w}, \boldsymbol{d}) = (\boldsymbol{u}_h, \boldsymbol{b}_h)$ with $(\boldsymbol{u}_h, \boldsymbol{b}_h)$ the component of the unique solution $(\boldsymbol{u}_h, p_h, \boldsymbol{b}_h, r_h)$ of (3.2.33) – (3.2.34) given by Theorem 3.2.11. Then, the discrete DG scheme (3.4.7)-(3.4.8) can be rewritten, equivalently as the fixed-point problem:

$$\text{Find } (\boldsymbol{u}_h,\, \boldsymbol{b}_h) \in \boldsymbol{K}_h \quad \text{such that} \quad \mathcal{F}_h(\boldsymbol{w}, \boldsymbol{d}) = (\boldsymbol{u}_h,\, \boldsymbol{b}_h). \tag{3.4.9}$$

In what follows, we focus on analysing the existence and uniqueness of such a fixed point. For this purpose, we must verify that the operator $\mathcal{F}_h$ defining the discret problem (3.4.7)-(3.4.8) satisfies the hypothesis of the Brouwer's fixed-point theorem [28, Theorem 9.9-2]. Next, we derive an *a priori* error estimate for the method.

3.4.1 A new L^p discrete Sobolev's inequality on discountinuous spaces

In this subsection, we first establish a regularity result for the Laplace's equation with Navier-type boundary condition. This result plays a central role in the proof of a new discrete Sobolev embedding allowing one to establish the well-posedness and the convergence of the DG scheme (3.4.7)-(3.4.8). In the following result, we demonstrate that it is indeed possible to derive a regularity result similar to those in [13, 15] for the Stokes problem when the divergence constraint is not imposed. This result is presented here in a non Hilbertian setting which is more general than that needed to analyze our model. However, since the result below is of independent interest to analyze other nonlinear problems, we choose to give it in $\boldsymbol{L}^p$ spaces with $p \geq 6/5$.

Proposition 3.4.1. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded simply-connected set of class $\mathcal{C}^{2,1}$. Let us suppose that $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ with $p \geq 6/5$. Then, the following Laplace equation:*

$$\begin{cases} -\Delta \boldsymbol{u} = \boldsymbol{g} \quad \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma \end{cases} \tag{3.4.10}$$

has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ which also satisfies:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leqslant C \|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{3.4.11}$$

Proof. We first consider the case $p = 2$ and prove the existence and uniqueness of solutions in $\boldsymbol{H}^1(\Omega)$. The problem (3.4.10) is equivalent to the following variational formulation: Find $\boldsymbol{u} \in \boldsymbol{H}^1_T(\Omega)$ such that:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega), \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega (\mathrm{div} \boldsymbol{u})\, (\mathrm{div} \boldsymbol{v})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{g} \cdot \boldsymbol{v}\, d\boldsymbol{x}. \tag{3.4.12}$$

Let us introduce the bilinear continuous form: $a(\cdot, \cdot) : \boldsymbol{H}^1_T(\Omega) \times \boldsymbol{H}^1_T(\Omega) \to \mathbb{R}$ defined as follows:

$$a(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\boldsymbol{x} + \int_\Omega (\mathrm{div} \boldsymbol{u})\, (\mathrm{div} \boldsymbol{v})\, d\boldsymbol{x}.$$

Using inequality (3.2.9), the form $a(\cdot, \cdot)$ is coercive on $\boldsymbol{H}^1_T(\Omega)$. Since the right hand-side defines a linear continuous form on $\boldsymbol{H}^1_T(\Omega)$, we deduce by the Lax-Milgram's Lemma that problem (3.4.12) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1_T(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\, \boldsymbol{g} \,\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.4.13}$$

Moreover, we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Then $\boldsymbol{z}$ satisfies the problem

$$\begin{cases} -\Delta\, \boldsymbol{z} = \mathbf{curl}\, \boldsymbol{g} \quad \text{and} \quad \mathrm{div}\, \boldsymbol{z} = 0 & \text{in } \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{3.4.14}$$

which is a problem of type $(\mathcal{E}_\mathcal{N})$ analyzed in Chapter 1. Since $\mathbf{curl}\, \boldsymbol{g}$ belongs to $[\boldsymbol{H}^{2,2}_0(\mathbf{curl}, \Omega)]'$ and satisfies the compatibility condition $\mathrm{div}(\mathbf{curl}\; \boldsymbol{g}) = 0$ in Ω, it follows from Lemma 1.3.4 (see Chapter1) that $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$ and satisfies the estimate

$$\|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = \|\boldsymbol{z}\|_{\boldsymbol{H}^1(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.4.15}$$

As a consequence, $\nabla \,\mathrm{div}\, \boldsymbol{u} = \Delta \boldsymbol{u} + \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ and then $\mathrm{div}\, \boldsymbol{u}$ belongs to $H^1(\Omega)$. Applying Corollary 1.3.2 leads to deduce that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$. Using (3.4.15), we obtain

$$\|\mathrm{div}\, \boldsymbol{u}\|_{H^1(\Omega)} \leq C \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}. \tag{3.4.16}$$

Finally, estimate (3.4.11) for $p = 2$ follows from (3.4.15) and (3.4.16). For the result in the non Hilbert case, we consider two cases.
First case: $p > 2$. We know that problem (3.4.10) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$. Then $\boldsymbol{u}$ belongs to $\boldsymbol{L}^q(\Omega)$ for all $1 \leq q \leq \infty$. Moreover, since $\mathbf{curl}\,\boldsymbol{g} \in [\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'$, it follows again from Lemma 1.3.4 (see Chapter1) that problem (3.4.14) has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$. Similarly to the case $p = 2$, this implies that $\nabla \operatorname{div} \boldsymbol{u}$ belongs to $\boldsymbol{L}^p(\Omega)$ and by [10, Proposition 2.10], we deduce that $\operatorname{div} \boldsymbol{u} \in W^{1,p}(\Omega)$. In summary, we have

$$\boldsymbol{u} \in \boldsymbol{L}^p(\Omega), \quad \operatorname{div} \boldsymbol{u} \in L^p(\Omega), \quad \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma. \tag{3.4.17}$$

Applying again Corollary 1.3.2, we deduce that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (3.4.11).
Second case: $6/5 \leq p < 2$. Observe that $\boldsymbol{g}$ is at least $\boldsymbol{L}^{6/5}(\Omega)$. Since $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^6(\Omega)$, the right-hand side in (3.4.12) still defines a linear continuous form on $\boldsymbol{H}^1(\Omega)$. By the Lax-Milgram lemma, the problem (3.4.10) has a unique solution $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\,\boldsymbol{g}\,\|_{\boldsymbol{L}^{6/5}(\Omega)}.$$

Next, we use the same argument as in the first step in order to prove that $\mathbf{curl}\,\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and then $\nabla \operatorname{div} \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Now, since $\boldsymbol{u} \in \boldsymbol{L}^6(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, the solution $\boldsymbol{u}$ satisfies (3.4.17). Applying again Corollary 1.3.2, we deduce that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (3.4.11). □

To prove that the discrete problem is well posed, we shall require a discrete $\boldsymbol{L}^p$ estimate for functions in $\boldsymbol{X}_h$ with $p \leq 6$, in terms of the norm $\|\cdot\|_{v,h}$. The inequality (3.4.19) given in the next Lemma is the equivalent of the following $\boldsymbol{L}^r$ Sobolev's inequality with $r \in [2,\,\infty)$ proven in [47, Lemma 6.2] in two dimensions: for any $\boldsymbol{v}_h \in \boldsymbol{X}_h$,

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^r(\Omega)} \leq C(r)|||\boldsymbol{v}_h|||_h := C(r)\Big(\sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h\|_{0,T}^2 + \sum_{e\in\mathcal{F}_h} \frac{\gamma}{h_e}\,\|[\![\boldsymbol{v}_h]\!]\|_{0,e}^2\Big)^{1/2}, \tag{3.4.18}$$

where $\gamma > 0$ is a stabilization parameter.
Sobolev's inequality (3.4.19) will be needed in the proof of Theorem 3.4.7 below. More specifically, it is required to bound the forms O_h in terms of the norm $\|\cdot\|_{v,h}$ when deriving the continuity of the operator $\mathcal{F}_h$. This inequality will also be required to handle the coupling form C_h. The proof is similar to (3.4.18) but not elementary because it heavily relies on the new regularity theorem stated in Proposition 3.4.1. Moreover, the proof is given when the tangential and normal jumps norms run over the faces of the triangulation.

Lemma 3.4.2. *For each real number $p \in (1,\,6]$, there exists a constant $C > 0$ independent of h such that :*

$$\forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^p(\Omega)} \leqslant C\,\|\boldsymbol{v}_h\|_{v,h} \tag{3.4.19}$$

Proof. Let $\boldsymbol{H}^1_T(\Omega)$ be the space of functions in $\boldsymbol{H}^1(\Omega)$ with zero normal trace defined in (3.2.8). Following [47], we introduce the lifting $\boldsymbol{v}(h)$ of $\boldsymbol{v}_h$ where $\boldsymbol{v}(h) \in \boldsymbol{H}^1_T(\Omega)$ is the unique solution of

$$\begin{aligned}\forall \boldsymbol{w} \in \boldsymbol{H}^1_T(\Omega), \qquad & \int_\Omega \mathbf{curl}\,\boldsymbol{v}(h)\cdot\mathbf{curl}\,\boldsymbol{w}\,d\boldsymbol{x} + \int_\Omega \operatorname{div}\boldsymbol{v}(h)\operatorname{div}\boldsymbol{w}\,d\boldsymbol{x} \\ &= \sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\boldsymbol{v}_h\cdot\mathbf{curl}\,\boldsymbol{w}\,d\boldsymbol{x} + \sum_{T\in\mathcal{T}_h}\int_T (\operatorname{div}\boldsymbol{v}_h)(\operatorname{div}\boldsymbol{w})\,d\boldsymbol{x}.\end{aligned} \tag{3.4.20}$$

satisfying the estimate

$$\|\operatorname{div} \boldsymbol{v}(h)\|_{L^2(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}(h)\|_{\boldsymbol{L}^2(\Omega)} \leq \Big(\sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{v}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{0,T}^2 \Big)^{1/2} \leq \|\boldsymbol{v}_h\|_{v,h}\,.$$

Moreover, since $\boldsymbol{v}(h)$ belongs to $\boldsymbol{H}^1_T(\Omega)$, we have from (3.2.10) for any $1 \leq p \leq 6$

$$\|\boldsymbol{v}(h)\|_{\boldsymbol{L}^p(\Omega)} \leq C\, \|\boldsymbol{v}_h\|_{v,h}\,. \tag{3.4.21}$$

So, it suffices to prove that (3.4.19) holds for $\boldsymbol{v}_h - \boldsymbol{v}(h)$ and then to use triangle inequality. By duality, we have

$$\|\boldsymbol{v}_h - \boldsymbol{v}(h)\|_{\boldsymbol{L}^p(\Omega)} = \sup_{\boldsymbol{g}\in\boldsymbol{L}^{p'}(\Omega)} \frac{\displaystyle\int_\Omega (\boldsymbol{v}_h - \boldsymbol{v}(h))\cdot \boldsymbol{g}\, d\boldsymbol{x}}{\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)}},$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$. Since $p \leqslant 6$, then $p' \geqslant \frac{6}{5}$ and $\boldsymbol{g}$ always belongs to $\boldsymbol{L}^{6/5}(\Omega)$. Thanks to Proposition 3.4.1, the following problem

$$\begin{cases} -\Delta\boldsymbol{\phi} = \boldsymbol{g} \quad \text{in } \Omega \\ \mathbf{curl}\,\boldsymbol{\phi}\times\boldsymbol{n} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\phi}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma \end{cases}$$

has a unique solution $\boldsymbol{\phi} \in \boldsymbol{W}^{2,\frac{6}{5}}(\Omega)$ satisfying the estimate

$$\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{2,\frac{6}{5}}(\Omega)} \leqslant C\, \|\boldsymbol{g}\|_{\boldsymbol{L}^{\frac{6}{5}}(\Omega)} \leq C\, \|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)} \tag{3.4.22}$$

Then, since $[\![\boldsymbol{v}(h)]\!]_N = 0$ for all $e \in \mathcal{F}_h$, $[\![\boldsymbol{v}(h)]\!]_T = \mathbf{0}$ for all $e \in \mathcal{F}_h^I$ and $(\mathbf{curl}\,\boldsymbol{\phi})\times\boldsymbol{n} = \mathbf{0}$ on Γ, the regularity of $\boldsymbol{\phi}$ and (3.4.20) imply that

$$\begin{aligned} \int_\Omega (\boldsymbol{v}_h - \boldsymbol{v}(h))\cdot\boldsymbol{g}\, d\boldsymbol{x} &= -\int_\Omega \Delta\boldsymbol{\phi}\cdot(\boldsymbol{v} - \boldsymbol{v}(h))\, d\boldsymbol{x} \\ &= -\sum_{e\in\mathcal{F}_h}\int_e \operatorname{div}\boldsymbol{\phi}\, [\![\boldsymbol{v}_h]\!]_N\, ds + \sum_{e\in\mathcal{F}_h^I}\int_e \mathbf{curl}\,\boldsymbol{\phi}\cdot[\![\boldsymbol{v}_h]\!]_T\, ds \end{aligned} \tag{3.4.23}$$

We need to bound the right-hand side of (3.4.23). We give the proof for the first term involving the divergence operator and the normal jumps. A similar bound for the second term, involving the rotational operator and the tangential jumps, can be obtained exactly with the same arguments and obvious modifications. Since $\operatorname{div}\boldsymbol{\phi}$ belongs to $\boldsymbol{W}^{1,6/5}(\Omega)$, the trace of $\operatorname{div}\boldsymbol{\phi}$ on each face e belongs to $\boldsymbol{W}^{1/6,6/5}(e) \hookrightarrow \boldsymbol{L}^{4/3}(e)$. Then, we have

$$\Big|\int_e \operatorname{div}\boldsymbol{\phi}\, [\![\boldsymbol{v}_h]\!]_N\, ds\Big| \leqslant \|\operatorname{div}\boldsymbol{\phi}\|_{\boldsymbol{L}^{4/3}(e)}\, \|[\![\boldsymbol{v}_h]\!]_N\|_{L^4(e)} \tag{3.4.24}$$

Passing to the reference element $\hat{T}$ with the face $\hat{e}$, we have

$$\|\operatorname{div}\boldsymbol{\phi}\|_{\boldsymbol{L}^{4/3}(e)} \leq C|e|^{\frac{3}{4}} \left\|\widehat{\operatorname{div}\boldsymbol{\phi}}\right\|_{L^{4/3}(\hat{e})}$$

Next, using the following trace inequality:

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^p(e)} \leq c h_T^{-\frac{1}{p}+d(\frac{1}{p}-\frac{1}{q})}\, \|\boldsymbol{v}\|_{\boldsymbol{L}^q(T)}\,, \tag{3.4.25}$$

with $p = \frac{4}{3}$, $q = 2$ and $d = 3$ together with the embedding $W^{1,6/5}(\hat{T}) \hookrightarrow L^2(\hat{T})$, we obtain

$$\|\mathrm{div}\boldsymbol{\phi}\|_{L^{4/3}(e)} \leq C|e|^{\frac{3}{4}} \left\|\widehat{\mathrm{div}\,\boldsymbol{\phi}}\right\|_{L^2(\hat{T})} \leq C|e|^{\frac{3}{4}} \left\|\widehat{\mathrm{div}\,\boldsymbol{\phi}}\right\|_{W^{1,6/5}(\hat{T})}$$

Applying the inequality

$$\|\widehat{v}\|_{W^{1,p}(\hat{T})} \leq C \frac{h_T}{|T|^{\frac{1}{p}}} \|v\|_{W^{1,p}(T)}$$

we deduce that

$$\|\mathrm{div}\boldsymbol{\phi}\|_{L^{4/3}(e)} \leq C|e|^{\frac{3}{4}} \frac{h_T}{|T|^{\frac{5}{6}}} \|\mathrm{div}\boldsymbol{\phi}\|_{W^{1,6/5}(T)}$$

Moreover, by virtue of assumption (3.2.32), we have for any $T \in \mathcal{T}_h$

$$\frac{|e|}{\sqrt{T}} \leq C \frac{h_T}{\rho_T} \leq C\varsigma,$$

which gives

$$\|\mathrm{div}\boldsymbol{\phi}\|_{L^{4/3}(e)} \leq C \|\mathrm{div}\boldsymbol{\phi}\|_{W^{1,6/5}(T)}. \tag{3.4.26}$$

On the other hand, we have

$$\|[\![\boldsymbol{v}_h]\!]_N\|_{L^4(e)} \leq C|e|^{1/4} \|[\![\widehat{\boldsymbol{v}_h}]\!]_N\|_{L^2(\hat{e})} \leq C|e|^{-1/2} \|[\![\boldsymbol{v}_h]\!]_N\|_{L^2(e)}, \tag{3.4.27}$$

where a local equivalence of norms is used. Combining (3.4.26) and (3.4.27) in (3.4.24), we obtain

$$\left| \int_e \mathrm{div}\,\boldsymbol{\phi}\, [\![\boldsymbol{v}_h]\!]_N \, ds \right| \leqslant C \|\mathrm{div}\boldsymbol{\phi}\|_{W^{1,6/5}(T)} |e|^{-1/2} \|[\![\boldsymbol{v}_h]\!]_N\|_{L^2(e)}. \tag{3.4.28}$$

Finally, summing over $e \in \mathcal{F}_h$ and using estimate (3.4.22), we obtain

$$\sum_{e\in\mathcal{F}_h} \int_e \mathrm{div}\,\boldsymbol{\phi}\, [\![\boldsymbol{v}_h]\!]_N \, ds \leqslant C \|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)} \Big(\sum_{e\in\mathcal{F}_h} \frac{\sigma_1}{|e|} \|[\![\boldsymbol{v}_h]\!]_N\|^2_{L^2(e)} \Big)^{1/2}. \tag{3.4.29}$$

A very much similar proof gives

$$\sum_{e\in\mathcal{F}_h^I} \int_e \mathbf{curl}\,\boldsymbol{\phi} \cdot [\![\boldsymbol{v}_h]\!]_T \, ds \leqslant C \|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Omega)} \Big(\sum_{e\in\mathcal{F}_h^I} \frac{\sigma_2}{|e|} \|[\![\boldsymbol{v}_h]\!]_T\|^2_{L^2(e)} \Big)^{1/2}. \tag{3.4.30}$$

Collecting the above estimates (3.4.29)-(3.4.30) into (3.4.23), we obtain

$$\int_\Omega (\boldsymbol{v}_h - \boldsymbol{v}(h)) \cdot \boldsymbol{g}\, d\boldsymbol{x} \quad \leqslant C \|\boldsymbol{g}\|_{L^{p'}(\Omega)} \|\boldsymbol{v}_h\|_{1,h}$$

which achieves the proof of the estimate (3.4.19). □

Remark 3.4.3. *Observe that the discrete Sobolev embedding* (3.4.19) *is the counterpart of that valid at the continuous level* (3.2.10). *Indeed, if* $\boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega)$, *we have* $[\![\boldsymbol{v}(h)]\!]_N = 0$ *for all* $e \in \mathcal{F}_h$ *and* $[\![\boldsymbol{v}(h)]\!]_T = \mathbf{0}$ *for all* $e \in \mathcal{F}_h^I$ *and then, we have* (3.2.10).

3.4.2 Well-posedness of the discrete problem

We now aim to show the solvability of (3.4.7)-(3.4.8) by analysing the equivalent fixed-point problem (3.4.9). Our next goal is to establish the continuity properties for the non linear forms O_h and C_h which are needed to prove that the operator $\mathcal{F}_h$ associated with the discrete problem is a contraction. In view of (3.4.19), we have the following result concerning the continuity of the form O_h.

Lemma 3.4.4. *There exists a constant $C > 0$ such that for any $(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) \in \boldsymbol{X}_h \times \boldsymbol{X}_h \times \boldsymbol{X}_h$,*

$$O_h(\boldsymbol{w}, \boldsymbol{u}_h, \boldsymbol{v}_h) \leqslant C \left\|\boldsymbol{w}\right\|_{v,h} \left\|\boldsymbol{u}_h\right\|_{v,h} \left\|\boldsymbol{v}_h\right\|_{v,h}. \tag{3.4.31}$$

Proof. Using Hölder's inequality, we obtain

$$O_h(\boldsymbol{w}, \boldsymbol{u}_h,\, \boldsymbol{v}_h) := \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{u}_h) \cdot \boldsymbol{v}_h \, d\boldsymbol{x} \leq C\Big(\sum_{T\in\mathcal{T}_h} \|\,\mathbf{curl}\, \boldsymbol{w}\|_{0,T}^2\Big)^{1/2} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^4(\Omega)} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}$$

Applying (3.4.19) with $p = 4$, the result is then achieved. □

For the continuity of the coupling form C_h, we require the discrete Sobolev inequality (3.4.19) with $p = 4$ for both the velocity and the magnetic field. Indeed, we consider an $\boldsymbol{L}^4$-$\boldsymbol{L}^4$-$\boldsymbol{L}^2$ argument when applying the Hölder inequality. We note that, in the proof given in [81, Lemma 4.3], an argument $\boldsymbol{L}^3$-$\boldsymbol{L}^6$-$\boldsymbol{L}^2$ is used to handle the form C_h. This approach is realistic because they consider the general case of a bounded Lipschitz domain $\Omega \in \mathbb{R}^3$ and a Dirichlet boundary condition for the veloicty. So, in this case, the discrete inequality given in (3.4.18) is used for the velocity $\boldsymbol{u}$ with $r = 6$. However, this inequality can not be applied for the magnetic field $\boldsymbol{b}$. With boundary conditions of type (3.4.5) or $\boldsymbol{b} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, $\boldsymbol{b}$ has a regularity only in $\boldsymbol{H}^{1/2}(\Omega) \hookrightarrow \boldsymbol{L}^3(\Omega)$. Hence, they establish in [81, Theorem 8.1] the following discrete $\boldsymbol{L}^3$ estimate for functions in $\boldsymbol{C}_h$.

Theorem 3.4.5. [*W. Qiu and Ke. Shi, 2020*] *There is a positive constant C such that for any $\boldsymbol{b}_h \in \boldsymbol{C}_h$ we have*

$$\|\boldsymbol{b}_h\|_{\boldsymbol{L}^3(\Omega)} \leqslant C\big(\sum_{e\in\mathcal{F}_h^I} \|h^{-\frac{1}{2}}[\![\boldsymbol{b}_h]\!]_T\|_{0,e}^2 + \sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{b}_h\|_{0,T}^2 + \|\operatorname{div}_h^N \boldsymbol{b}_h\|_{L^2(\Omega)}\big), \tag{3.4.32}$$

where the discrete divergence $\operatorname{div}_h^N \boldsymbol{b}_h$ is defined to be the unique function in $H^1(\Omega) \cap L_0^2(\Omega) \cap S_h$ satisfying

$$(\operatorname{div}_h^N \boldsymbol{b}_h, s)_{\mathcal{T}_h} = -(\boldsymbol{b}_h, \nabla s)_{\mathcal{T}_h}, \quad \textit{for all } s \in H^1(\Omega) \cap L_0^2(\Omega) \cap S_h,$$

where $S_h := \big\{s_h \in L^2(\Omega);\, s_{h|T} \in \mathcal{P}_{k+1}(T)\,,\, \forall\, T \in \mathcal{T}_h\big\}$.

We can now derive the continuity of the form $\boldsymbol{C}_h$ as in [81, Lemma 4.3]. The main tool different from their proof is that we use an $\boldsymbol{L}^4$-$\boldsymbol{L}^4$-$\boldsymbol{L}^2$ argument instead of an $\boldsymbol{L}^3$-$\boldsymbol{L}^6$-$\boldsymbol{L}^2$ argument when applying the Hölder inequality. So, for our case, we apply the discrete Sobolev inequality (3.4.19) with $p = 4$ instead of (3.4.18) and (3.4.32). The proof is given for the sake of completeness.

Lemma 3.4.6. *There exists a constant $C > 0$ such that for any $(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{C}_h \times \boldsymbol{X}_h \times \boldsymbol{C}_h$,*

$$C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{c}_h) \leqslant C\kappa \left\|\boldsymbol{b}_h\right\|_{m,h} \left\|\boldsymbol{v}_h\right\|_{v,h} \left\|\boldsymbol{c}_h\right\|_{m,h}. \tag{3.4.33}$$

Proof. By the definition of the form $\boldsymbol{C}_h$, we have

$$C_h(\boldsymbol{b}_h,\boldsymbol{v}_h,\boldsymbol{c}_h)=\kappa\sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{v}_h\times\boldsymbol{b}_h)\cdot\mathbf{curl}\,\boldsymbol{c}_h\,d\boldsymbol{x}-\kappa\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{v}_h\times\boldsymbol{b}_h\}\!\}\cdot[\![\boldsymbol{c}_h]\!]_T\,ds \tag{3.4.34}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}C_h(\boldsymbol{b}_h,\boldsymbol{v}_h,\boldsymbol{c}_h) &\leq C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\Big(\sum_{T\in\mathcal{T}_h}\|\,\mathbf{curl}\,\boldsymbol{c}_h\|_{0,T}^2\Big)^{1/2}\\ &+ C\kappa\Big(\sum_{e\in\mathcal{F}_h^I}h\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(e)}^4\Big)^{1/4}\Big(\sum_{e\in\mathcal{F}_h^I}h\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(e)}^4\Big)^{1/4}\Big(\sum_{e\in\mathcal{F}_h^I}h^{-1}\|[\![\boldsymbol{c}_h]\!]_T\|_{\boldsymbol{L}^2(e)}^2\Big)^{1/2}\end{aligned}$$

Thanks to the trace theorem, we have

$$\begin{aligned}C_h(\boldsymbol{b}_h,\boldsymbol{v}_h,\boldsymbol{c}_h) &\leq C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\Big(\sum_{T\in\mathcal{T}_h}\|\,\mathbf{curl}\,\boldsymbol{c}_h\|_{0,T}^2\Big)^{1/2}\\ &+ C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\Big(\sum_{e\in\mathcal{F}_h^I}\frac{\sigma_1}{h}\|[\![\boldsymbol{c}_h]\!]_T\|_{\boldsymbol{L}^2(e)}^2\Big)^{1/2}\\ &\leq C\kappa\|\boldsymbol{b}_h\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^4(\Omega)}\,\|\boldsymbol{c}_h\|_{m,h}\end{aligned}$$

Then, the bound (3.4.33) is a direct application of the Sobolev's inequalities (3.4.32) and (3.4.19) with $p=4$. □

Remark 3.4.1. *It is not possible to reuse the same strategy in [81] with a regularity below $\boldsymbol{H}^1(\Omega)$ for both the velocity and the magnetic field. Indeed, in our work, the velocity and the magnetic field verify the same type of boundary conditions. So, if we consider the case of non smooth domain, we have $\boldsymbol{u}\in\boldsymbol{H}^{1/2}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$ and $\boldsymbol{b}\in\boldsymbol{H}^{1/2}(\Omega)\hookrightarrow\boldsymbol{L}^3(\Omega)$. Then, the argument $\boldsymbol{L}^3$-$\boldsymbol{L}^3$-$\boldsymbol{L}^2(\mathcal{T}_h)$ does not allow to bound $\sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{v}\times\boldsymbol{b})\cdot\mathbf{curl}\,\boldsymbol{b}\,d\boldsymbol{x}$ and then the form C_h can not be bounded.*

In the following result, we provide the assumptions under which problem (3.4.9) is well defined.

Theorem 3.4.7. *Assume that*

$$\nu^{-2}\|\boldsymbol{f}\|_{0,\Omega}\,,\ \mu^{-1}\nu^{-1}\|\boldsymbol{f}\|_{0,\Omega}\,,\ \kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}\nu^{-\frac{3}{2}}\|\boldsymbol{g}\|_{0,\Omega}\,,\ \nu^{-\frac{1}{2}}\mu^{-\frac{3}{2}}\kappa^{-\frac{1}{2}}\|\boldsymbol{g}\|_{0,\Omega} \tag{3.4.35}$$

are small enough. Then, the DG scheme (3.4.7)-(3.4.8) has a unique solution $(\boldsymbol{u}_h,\boldsymbol{b}_h,p_h)\in\boldsymbol{X}_h\times\boldsymbol{C}_h\times Q_h$ which satisfies:

$$\nu\|\boldsymbol{u}_h\|_{v,h}^2+\kappa\mu\|\boldsymbol{b}_h\|_{m,h}^2\ \leqslant\ C(\nu^{-1}\|\boldsymbol{f}\|_{0,\Omega}^2+\kappa^{-1}\mu^{-1}\|\boldsymbol{g}\|_{0,\Omega}^2) \tag{3.4.36}$$

Proof. We begin to prove that the operator $\mathcal{F}_h$ defined in (3.4.9) has a unique fixed-point. Two steps are needed to that.

Step 1: We prove that $\mathcal{F}_h$ maps a closed ball $\boldsymbol{G}_M$ into itself for a real number $M>0$. Let us take $(\boldsymbol{v}_h,\boldsymbol{c}_h)=(\boldsymbol{u}_h,\boldsymbol{b}_h)$ and $q_h=-p_h$ as test functions in (3.4.8)-(3.4.8). We have:

$$\mathcal{A}_h((\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{u}_h,\boldsymbol{b}_h),(\boldsymbol{u}_h,\boldsymbol{b}_h))=A_h(\boldsymbol{u}_h,\boldsymbol{u}_h)+M_h(\boldsymbol{b}_h,\boldsymbol{b}_h)=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}_h\,d\boldsymbol{x}+\int_\Omega\boldsymbol{g}\cdot\boldsymbol{b}_h\,d\boldsymbol{x}. \tag{3.4.37}$$

We get from the coercivity of A_h and M_h in (3.2.43) and (3.2.44) respectively and Cauchy-Schwarz inequality:

$$\nu \|\boldsymbol{u}_h\|_{v,h}^2 + \kappa\mu|\boldsymbol{b}_h|_{m,h}^2 \quad\leqslant\quad C\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{b}_h\|_{\boldsymbol{L}^2(\Omega)} \Big)$$

The Sobolev inequality (3.4.19) with $p = 2$ and Young's inequality imply that

$$\nu \|\boldsymbol{u}_h\|_{v,h}^2 + \kappa\mu \|\boldsymbol{b}_h\|_{m,h}^2 \leqslant C\Big(\nu^{-1} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}^2 + \kappa^{-1}\mu^{-1} \|\boldsymbol{g}\|_{\boldsymbol{L}^2(\Omega)}^2\Big) := M \tag{3.4.38}$$

So, we define $\boldsymbol{G}_M$ as:

$$\boldsymbol{G}_M := \Big\{(\boldsymbol{u}_h, \boldsymbol{b}_h) \in \boldsymbol{K}_h,\ \nu \|\boldsymbol{u}_h\|_{v,h}^2 + \kappa\mu \|\boldsymbol{b}_h\|_{m,h}^2 \leqslant M\Big\}. \tag{3.4.39}$$

It is easy to see that $\boldsymbol{G}_M$ is a closed set of $\boldsymbol{K}_h$ and $\mathcal{F}_h(\boldsymbol{G}_M) \subset \boldsymbol{G}_M$.

Step 2: We prove that when the data $\boldsymbol{f}$ and $\boldsymbol{g}$ are small enough, $\mathcal{F}_h$ is a contraction from $\boldsymbol{G}_M$ into itself. We remark in advance that a combination of the Cauchy-Schwarz and Hölder inequalities with the estimate (3.4.19) plays a key role in the proof.
Let $(\boldsymbol{w}_1, \boldsymbol{d}_1), (\boldsymbol{w}_2, \boldsymbol{d}_2) \in \boldsymbol{G}_h$ and $(\boldsymbol{u}_h^1, \boldsymbol{b}_h^1) := \mathcal{F}(\boldsymbol{w}_1, \boldsymbol{d}_1)$, $(\boldsymbol{u}_h^2, \boldsymbol{b}_h^2) := \mathcal{F}(\boldsymbol{w}_2, \boldsymbol{d}_2)$ be the components of the corresponding solutions of the linearized problems (3.2.33)-(3.2.34).
From this fact, the differences $\boldsymbol{u}_h := \boldsymbol{u}_h^1 - \boldsymbol{u}_h^2$, $\boldsymbol{b}_h := \boldsymbol{b}_h^1 - \boldsymbol{b}_h^2$ and $p_h := p_h^1 - p_h^2$ satisfy: for any $(\boldsymbol{v}_h, \boldsymbol{c}_h, q_h) \in \boldsymbol{W}_h \times Q_h$

$$\begin{aligned}
& A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + M_h(\boldsymbol{b}_h, \boldsymbol{c}_h) + O_h(\boldsymbol{w}_1, \boldsymbol{u}_h^1, \boldsymbol{v}_h) - O_h(\boldsymbol{w}_2, \boldsymbol{u}_h^2, \boldsymbol{v}_h) \\
+\quad & C_h(\boldsymbol{d}_1, \boldsymbol{v}_h, \boldsymbol{b}_h^1) - C_h(\boldsymbol{d}_2, \boldsymbol{v}_h, \boldsymbol{b}_h^2) - C_h(\boldsymbol{d}_1, \boldsymbol{u}_h^1, \boldsymbol{c}_h) + C_h(\boldsymbol{d}_2, \boldsymbol{u}_h^2, \boldsymbol{c}_h) \\
+\quad & B_h(\boldsymbol{v}_h, p) - B_h(\boldsymbol{u}_h, q_h) = 0.
\end{aligned}$$

Taking $((\boldsymbol{u}_h, \boldsymbol{b}_h), p_h)$ as test function in the above relation, adding and substracting $C_h(\boldsymbol{d}_1, \boldsymbol{u}_h, \boldsymbol{b}_h^2)$, we obtain:

$$A_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + M_h(\boldsymbol{b}_h, \boldsymbol{b}_h) = O_h(\boldsymbol{w}_2 - \boldsymbol{w}_1, \boldsymbol{u}_h^2, \boldsymbol{u}_h) - C_h(\boldsymbol{d}_1 - \boldsymbol{d}_2, \boldsymbol{u}_h, \boldsymbol{b}_h^2) + C_h(\boldsymbol{d}_1 - \boldsymbol{d}_2, \boldsymbol{u}_h^2, \boldsymbol{b}_h), \tag{3.4.40}$$

where we have used the fact that

$$O_h(\boldsymbol{w}_2, \boldsymbol{u}_h^2, \boldsymbol{u}_h) - O_h(\boldsymbol{w}_1, \boldsymbol{u}_h^1, \boldsymbol{u}_h) = O_h(\boldsymbol{w}_2 - \boldsymbol{w}_1, \boldsymbol{u}_h^2, \boldsymbol{u}_h).$$

Let us estimate each term in the right-hand side of (3.4.40). The first term can be easily estimated by using Lemma 3.4.4 and Young's inequality. Indeed, it follows that

$$\begin{aligned}
O_h(\boldsymbol{w}_2 - \boldsymbol{w}_1, \boldsymbol{u}_h^2, \boldsymbol{u}_h) \quad &\leqslant\quad C \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{v,h} \left\|\boldsymbol{u}_h^2\right\|_{v,h} \|\boldsymbol{u}_h\|_{v,h} \\
&\leq\quad C\nu^{-1} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{v,h}^2 \left\|\boldsymbol{u}_h^2\right\|_{v,h}^2 + \frac{1}{4}\nu \|\boldsymbol{u}_h\|_{v,h}^2
\end{aligned}$$

Next, by using Corollary 3.4.6, we have

$$\begin{aligned}
C_h(\boldsymbol{d}_1 - \boldsymbol{d}_2, \boldsymbol{u}_h, \boldsymbol{b}_h^2) \quad &\leqslant\quad C\kappa \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{m,h} \|\boldsymbol{u}_h\|_{v,h} \left\|\boldsymbol{b}_h^2\right\|_{m,h} \\
&\leq\quad C\kappa^2\nu^{-1} \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{m,h}^2 \left\|\boldsymbol{b}_h^2\right\|_{m,h}^2 + \frac{1}{4}\nu \|\boldsymbol{u}_h\|_{v,h}^2,
\end{aligned}$$

$$\begin{aligned}
C_h(\boldsymbol{d}_1 - \boldsymbol{d}_2, \boldsymbol{u}_h^2, \boldsymbol{b}_h) \quad &\leqslant\quad C\kappa \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{m,h} \left\|\boldsymbol{u}_h^2\right\|_{v,h} \|\boldsymbol{b}_h\|_{m,h} \\
&\leq\quad C\kappa\mu^{-1} \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{m,h}^2 \left\|\boldsymbol{u}_h^2\right\|_{v,h}^2 + \frac{1}{2}\kappa\mu \|\boldsymbol{b}_h\|_{m,h}^2.
\end{aligned}$$

Collecting the above estimates in (3.4.40), using the coercivity of A_h and M_h, we obtain

$$\begin{aligned}\nu \|\boldsymbol{u}_h\|_{v,h}^2 + \kappa\mu \|\boldsymbol{b}_h\|_{m,h}^2 &\leqslant C\nu \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|_{m,h}^2 \nu^{-2} \left\|\boldsymbol{u}_h^2\right\|_{v,h}^2 \\ &+ C\big(\kappa\mu^{-1}\nu^{-1} \left\|\boldsymbol{b}_h^2\right\|_{m,h}^2 + \kappa\mu^{-2} \left\|\boldsymbol{u}_h^2\right\|_{v,h}^2\big)\kappa\mu \|\boldsymbol{d}_1 - \boldsymbol{d}_2\|_{m,h}^2\end{aligned}$$

So, if $\nu^{-2} \left\|\boldsymbol{u}_h^2\right\|_{1,h} < 1$ and $\kappa\mu^{-1}\nu^{-1} \left\|\boldsymbol{b}_h^2\right\|_C^2 + \kappa\mu^{-2} \left\|\boldsymbol{u}_h^2\right\|_{1,h}^2 < 1$, that is, if the smallness conditions (3.4.35) on $\boldsymbol{f}$ and $\boldsymbol{g}$ are satisfied, the operator $\mathcal{F}_h$ is a contraction on $\boldsymbol{G}_M$.

The above proofs show that $\mathcal{F}_h$ satisfies the hypotheses of Brouwer's fixed-point theorem on $\boldsymbol{G}_M$. Then, $\mathcal{F}_h$ has a unique fixed-point $(\boldsymbol{u}_h, \boldsymbol{b}_h)$ in $\boldsymbol{G}_M$. Therefore, the existence and uniqueness of the solution $(\boldsymbol{u}_h, \boldsymbol{b}_h)$ has been proved. In turn, the a priori estimate (3.4.36) follows directly from (3.4.38). Now that $\boldsymbol{u}_h$ and $\boldsymbol{b}_h$ have been computed, we want to recover the pressure p_h. Observe that p_h is the solution of

$$B_h(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, d\boldsymbol{x} - A_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) \tag{3.4.41}$$

Applying Lemma 3.4.2, Lemma 3.2.7, Lemma 3.4.4 and Corollary 3.4.6 to bound A_h, O_h and C_h respectively, we deduce that the right-hand side of the above equation defines a continuous linear functional on $\boldsymbol{X}_h$. The inf-sup condition in Lemma 3.2.10 gives the existence and uniqueness of $p_h \in Q_h$ to the problem (3.4.41). □

3.4.3 Error estimate for the nonlinear MHD system

In this subsection, we present a priori error bounds for the proposed DG method. The proof uses essentially the same techniques as in [81] where a similar result has been proved for the MHD problem with Dirichlet boundary condition for the velocity together with zero tangential trace for the magnetic field. Let us first note that the exact solution $(\boldsymbol{u}, \boldsymbol{b}, p)$ of the continuous MHD problem (3.4.1)-(3.4.6) satisfies

$$\begin{aligned}\mathcal{A}_h((\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{u},\boldsymbol{b}),(\boldsymbol{v}_h,\boldsymbol{c}_h)) + \mathcal{B}_h((\boldsymbol{v}_h,\boldsymbol{c}_h),p) &= \mathcal{L}((\boldsymbol{v}_h,\boldsymbol{c}_h)), \quad \forall(\boldsymbol{v}_h,\boldsymbol{c}_h) \in \boldsymbol{W}_h, &(3.4.42)\\ \mathcal{B}_h((\boldsymbol{u},\boldsymbol{b}),q_h) &= 0, \quad \forall q_h \in Q_h. &(3.4.43)\end{aligned}$$

As for the linear case, we split the errors in two parts: $\boldsymbol{e}^u = \boldsymbol{u} - \boldsymbol{u}_h$, $\boldsymbol{e}^b = \boldsymbol{b} - \boldsymbol{b}_h$ and $e^p = p - p_h$ with

$$\begin{aligned}\boldsymbol{\chi}_u &= I_h(\boldsymbol{u}) - \boldsymbol{u}_h, & \boldsymbol{\eta}_u &= \boldsymbol{u} - I_h(\boldsymbol{u})\\ \boldsymbol{\chi}_b &= I_h(\boldsymbol{b}) - \boldsymbol{b}_h, & \boldsymbol{\eta}_b &= \boldsymbol{b} - I_h(\boldsymbol{b})\\ \chi_p &= \Pi_Q(p) - p_h, & \eta_p &= p - \Pi_Q(p)\end{aligned}$$

We have the following lemma for the error projection:

Lemma 3.4.8. *Let $(\boldsymbol{u}, \boldsymbol{b}, p) \in \boldsymbol{H}^{k+1}(\Omega) \times \boldsymbol{H}^{k+1}(\Omega) \times H^k(\Omega)$ be a solution of the continuous MHD problem* (3.4.1)-(3.4.6). *Let $(\boldsymbol{u}_h, p_h, \boldsymbol{b}_h)$ be the solution in $\boldsymbol{X}_h \times Q_h \times \boldsymbol{C}_h$ of the DG method* (3.4.7)-(3.4.8). *In addition to the assumptions in Theorem 3.4.7, we assume that $\frac{1}{\min(\nu,\mu)} \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and $\frac{\kappa^{1/2}}{\nu^{1/2}\mu^{1/2}} \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$ are small enough. Then, we have the following estimate on the errors:*

$$\begin{aligned}&\nu \|\boldsymbol{\chi}_u\|_{v,h}^2 + \kappa\mu \|\boldsymbol{\chi}_b\|_{m,h}^2\\ &\leqslant Ch^{2k}\Big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 \big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 + \|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}^2 \big)\Big)\end{aligned} \tag{3.4.44}$$

with C depending on the datas of the problem but not of the mesh size h.

Proof. Thanks to (3.4.42)-(3.4.43), we have for all $(\boldsymbol{v}_h, \boldsymbol{c}_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$:

$$
\begin{aligned}
&A_h(\boldsymbol{u} - I_h(\boldsymbol{u}), \boldsymbol{v}_h) + A_h(I_h(\boldsymbol{u}), \boldsymbol{v}_h) + M_h(\boldsymbol{b} - I_h(\boldsymbol{b}), \boldsymbol{c}_h) + M_h(I_h(\boldsymbol{b}), \boldsymbol{c}_h) + O_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\
&+ C_h(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{b}) - C_h(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{c}_h) + B_h(\boldsymbol{v}_h, p - \Pi_Q(p)) + B_h(\boldsymbol{v}_h, \Pi_Q(p)) = \mathcal{L}(\boldsymbol{v}_h, \boldsymbol{c}_h), \\
&B_h(\boldsymbol{u} - I_h(\boldsymbol{u}), q_h) + B_h(I_h(\boldsymbol{u}), q_h) = 0.
\end{aligned}
\tag{3.4.45}
$$

By the definition of I_h (c.f. (3.2.53) and (3.2.54)) and Π_Q (c.f. (3.3.29)), we have:

$$
\begin{aligned}
B_h(\boldsymbol{v}_h, \Pi_Q(p) - p) &= -\sum_{T \in \mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{v}_h)(\Pi_Q(p) - p)\, dx + \sum_{e \in \mathcal{F}_h} \int_e \{\!\{\Pi_Q(p) - p\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds \\
&= \sum_{e \in \mathcal{F}_h} \int_e \{\!\{\Pi_Q(p) - p\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds \\
B_h(\boldsymbol{u} - I_h(\boldsymbol{u}), q_h) &= -\sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(I_h(\boldsymbol{u}) - \boldsymbol{u}) q_h \, dx + \sum_{e \in \mathcal{F}_h} \int_e \{\!\{q_h\}\!\} [\![I_h(\boldsymbol{u}) - \boldsymbol{u}]\!]_N \, ds \\
&= 0.
\end{aligned}
$$

So, from (3.4.45), we obtain:

$$
\begin{aligned}
&A_h(I_h(\boldsymbol{u}), \boldsymbol{v}_h) + M_h(I_h(\boldsymbol{b}), \boldsymbol{c}_h) + O_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) + C_h(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{b}) - C_h(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{c}_h) + B_h(\boldsymbol{v}_h, \Pi_Q(p)) \\
&= A_h(I_h(\boldsymbol{u}) - \boldsymbol{u}, \boldsymbol{v}_h) + M_h(I_h(\boldsymbol{b}) - \boldsymbol{b}, \boldsymbol{c}_h) + \sum_{e \in \mathcal{F}_h} \int_e \{\!\{\Pi_Q(p) - p\}\!\} [\![\boldsymbol{v}_h]\!]_N \, ds + \mathcal{L}(\boldsymbol{v}_h, \boldsymbol{c}_h), \\
&B_h(I_h(\boldsymbol{u}), q_h) = 0.
\end{aligned}
$$

We next substract from (3.4.7)-(3.4.8) to obtain

$$
\begin{aligned}
&A_h(\boldsymbol{\chi}_u, \boldsymbol{v}_h) + M_h(\boldsymbol{\chi}_b, \boldsymbol{c}_h) + B_h(\boldsymbol{v}_h, \chi_p) \\
&= O_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - O_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) + C_h(\boldsymbol{b}_h, \boldsymbol{v}_h, \boldsymbol{b}_h) - C_h(\boldsymbol{b}, \boldsymbol{v}_h, \boldsymbol{b}) + C_h(\boldsymbol{b}, \boldsymbol{u}, \boldsymbol{c}_h) - C_h(\boldsymbol{b}_h, \boldsymbol{u}_h, \boldsymbol{c}_h) \\
&+ A_h(\underbrace{I_h(\boldsymbol{u}) - \boldsymbol{u}}_{-\boldsymbol{\eta}_u}, \boldsymbol{v}_h) + M_h(\underbrace{I_h(\boldsymbol{b}) - \boldsymbol{b}}_{-\boldsymbol{\eta}_b}, \boldsymbol{c}_h) + \sum_{e \in \mathcal{F}_h} \int_e \underbrace{\{\!\{\Pi_Q(p) - p\}\!\}}_{-\{\!\{\eta_p\}\!\}} [\![\boldsymbol{v}_h]\!]_N \, ds, \\
&B_h(\boldsymbol{\chi}_u, q_h) = 0.
\end{aligned}
\tag{3.4.46}
$$

Combining inequalities of (3.4.46) with test functions $(\boldsymbol{\chi}_u, \boldsymbol{\chi}_b, \chi_p)$ and using (3.2.43)-(3.2.44), we obtain:

$$
\begin{aligned}
C\big(\nu \|\boldsymbol{\chi}_u\|^2_{v,h} + \kappa\mu \|\boldsymbol{\chi}_b\|^2_{m,h}\big) \le{}& A_h(\boldsymbol{\chi}_u, \boldsymbol{\chi}_u) + M_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_b) \\
&= -A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) - M_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_b) - \sum_{e \in \mathcal{F}_h} \int_e \{\!\{\eta_p\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds \\
&- (O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u)) \\
&+ (C_h(\boldsymbol{\eta}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_u, \boldsymbol{b})) - (C_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) - C_h(\boldsymbol{\chi}_b, \boldsymbol{u}, \boldsymbol{\chi}_b)) \\
&+ (C_h(\boldsymbol{b}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_b) - C_h(\boldsymbol{b}_h, \boldsymbol{\chi}_u, \boldsymbol{\eta}_b)).
\end{aligned}
\tag{3.4.47}
$$

Let us bound every term in the right hand side of (3.4.47).
By the definition of A_h, we have

$$
\begin{aligned}
A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) := {} & \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{\eta}_u \cdot \mathbf{curl}\ \boldsymbol{\chi}_u \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{\eta}_u)(\operatorname{div} \boldsymbol{\chi}_u)\, d\boldsymbol{x} \\
& - \nu \sum_{e\in\mathcal{F}_h^I} \Big(\int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds + \int_e \{\!\{\mathbf{curl}\, \boldsymbol{\chi}_u\}\!\} \cdot [\![\boldsymbol{\eta}_u]\!]_T \, ds \Big) \\
& - \nu \sum_{e\in\mathcal{F}_h} \Big(\int_e \{\!\{\operatorname{div}\ \boldsymbol{\eta}_u\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds + \int_e \{\!\{\operatorname{div} \boldsymbol{\chi}_u\}\!\} [\![\boldsymbol{\eta}_u]\!]_N \, ds \Big) \\
& \nu \sum_{e\in\mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_T \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_N [\![\boldsymbol{\chi}_u]\!]_N \, ds
\end{aligned}
\tag{3.4.48}
$$

Applying Cauchy-Schwarz inequality and (3.2.55), we have for the terms in the first row of (3.4.48):

$$
\begin{aligned}
& \nu \sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \boldsymbol{\eta}_u \cdot \mathbf{curl}\ \boldsymbol{\chi}_u \, d\boldsymbol{x} + \nu \sum_{T\in\mathcal{T}_h} \int_T (\operatorname{div} \boldsymbol{\eta}_u)(\operatorname{div} \boldsymbol{\chi}_u)\, d\boldsymbol{x} \\
& \le \nu \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{\eta}_u\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{\eta}_u\|_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\, \boldsymbol{\chi}_u\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\operatorname{div} \boldsymbol{\chi}_u\|_{0,T}^2 \Big)^{1/2} \\
& \le C\nu h^k \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \, \|\boldsymbol{\chi}_u\|_{v,h}
\end{aligned}
$$

Moreover, combining the continuity of $\boldsymbol{u}$ and (3.2.54), we have

$$
\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\, \boldsymbol{\chi}_u\}\!\} \cdot [\![\boldsymbol{\eta}_u]\!]_T \, ds + \nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\operatorname{div} \boldsymbol{\chi}_u\}\!\} [\![\boldsymbol{\eta}_u]\!]_N \, ds = 0.
$$

So, for the second and the third rows of (3.4.48), it remains to bound the terms:

$$
\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \quad \text{and} \quad \nu \sum_{e\in\mathcal{F}_h} \int_e \{\!\{\operatorname{div}\ \boldsymbol{\eta}_u\}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds
$$

Let us give detail proof for the bound of the first term. For this, let us introduce the standard Lagrange interpolation operator of polynomial degree k, denoted Π_h. So, we have

$$
\int_e \{\!\{\mathbf{curl}\ \boldsymbol{\eta}_u\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds = \int_e \big(\{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T + \{\!\{\mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \big) \, ds \tag{3.4.49}
$$

For the first term in (3.4.49), we have

$$
\int_e \{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \le \|\{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\}\|_{0,e} \, \|[\![\boldsymbol{\chi}_u]\!]_T\|_{0,e}
$$

Let $e \in \mathcal{F}_h^I$ such that $e = \partial T_1 \cap \partial T_2$ with $T_1, T_2 \in \mathcal{T}_h$. Using the discrete trace inequality and the approximation property of the Lagrange interpolation, we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{|e|}} \|\{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\}\|_{0,e} \le C \Big(& \frac{1}{h_{T_1}} \|\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{0,T_1} + \frac{1}{h_{T_2}} \|\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{0,T_2} \\
& + |\,\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})|_{1,T_1\cup T_2} \Big) \le C h^{k-1} \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(T_1\cup T_2)}
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\nu \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\mathbf{curl}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds & \le C\nu h^k \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \Big(\sum_{e\in\mathcal{F}_h^I} \frac{1}{|e|} \|[\![\boldsymbol{\chi}_u]\!]_T\| \Big)^{1/2} \\
& \le C\nu h^k \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \, \|\boldsymbol{\chi}_u\|_{v,h}
\end{aligned}
\tag{3.4.50}
$$

For the second term in (3.4.49), since $\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}$ is polynomial, we can write

$$\frac{1}{\sqrt{|e|}} \left\| \{\!\{ \mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}) \}\!\} \right\|_{0,e} \leq C \Big(\frac{1}{h_{T_1}} \left\| \mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}) \right\|_{0,T_1} + \frac{1}{h_{T_2}} \left\| \mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}) \right\|_{0,T_2} \Big)$$

By triangle inequality and (3.2.55), we have for each element T of $\mathcal{T}_h$

$$\left\| \mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}) \right\|_{0,T} \leq \left\| \mathbf{curl}(\Pi_h \boldsymbol{u} - \boldsymbol{u}) \right\|_{0,T} + \left\| \mathbf{curl}(\boldsymbol{u} - I_h \boldsymbol{u}) \right\|_{0,T} \leq C h_T^k \left\| \boldsymbol{u} \right\|_{k+1,\Delta_T}$$

Then, we have

$$\nu \sum_{e \in \mathcal{F}_h^I} \int_e \big(\{\!\{ \mathbf{curl}(\Pi_h \boldsymbol{u} - I_h \boldsymbol{u}) \}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \leq C \nu h^k \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{v,h} \tag{3.4.51}$$

Combining (3.4.50) and (3.4.51), we obtain

$$\nu \sum_{e \in \mathcal{F}_h^I} \int_e \{\!\{ \mathbf{curl}\ \boldsymbol{\eta}_u \}\!\} \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \leq C \nu h^k \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{v,h}. \tag{3.4.52}$$

Similarly, we can obtain

$$\nu \sum_{e \in \mathcal{F}_h} \int_e \{\!\{ \mathrm{div}\ \boldsymbol{\eta}_u \}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds \leq C \nu h^k \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{v,h}. \tag{3.4.53}$$

For the terms in the last row of (3.4.48), we use (3.2.55) and similar arguments as above to obtain

$$\nu \sum_{e \in \mathcal{F}_h^I} \frac{\sigma_1}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_T \cdot [\![\boldsymbol{\chi}_u]\!]_T \, ds \leq C \nu h^k \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{v,h}, \tag{3.4.54}$$

and similarly,

$$\nu \sum_{e \in \mathcal{F}_h} \frac{\sigma_2}{h_e} \int_e [\![\boldsymbol{\eta}_u]\!]_N [\![\boldsymbol{\chi}_u]\!]_N \, ds \leq C \nu h^k \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{v,h}, \tag{3.4.55}$$

Collecting the bounds (3.4.49), (3.4.49)-(3.4.55) yields

$$A_h(\boldsymbol{\eta}_u, \boldsymbol{\chi}_u) \leq C \nu h^k \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{v,h} \leq \frac{\nu}{10} \left\| \boldsymbol{\chi}_u \right\|_{v,h}^2 + C h^{2k} \left\| \boldsymbol{u} \right\|_{\boldsymbol{H}^{k+1}(\Omega)}^2. \tag{3.4.56}$$

Next, in view of the definition of M_h and the fact that $\boldsymbol{C}_h = \boldsymbol{X}_h$, the terms of M_h can be bounded in a similar way as the terms of A_h to obtain:

$$M_h(\boldsymbol{\eta}_b, \boldsymbol{\chi}_b) \leqslant C \kappa \mu h^k \left\| \boldsymbol{\chi}_b \right\|_{m,h} \left\| \boldsymbol{b} \right\|_{\boldsymbol{H}^{k+1}(\Omega)} \leq \frac{\kappa \mu}{8} \left\| \boldsymbol{\chi}_b \right\|_{m,h}^2 + C h^{2k} \left\| \boldsymbol{b} \right\|_{\boldsymbol{H}^{k+1}(\Omega)}^2. \tag{3.4.57}$$

Similarly, using the estimate (3.3.30) for the L^2-projection Π_Q, we obtain:

$$\begin{aligned} \sum_{e \in \mathcal{F}_h} \int_e \{\!\{ \eta_p \}\!\} [\![\boldsymbol{\chi}_u]\!]_N \, ds &\leqslant C \sum_{e \in \mathcal{F}_h} \Big(\int_e h_e \{\!\{ \eta_p \}\!\}^2 \, ds \Big)^{\frac{1}{2}} \Big(\int_e h_e^{-1} [\![\boldsymbol{\chi}_u]\!]_N{}^2 \, ds \Big)^{\frac{1}{2}} \\ &\leqslant C h^k \left\| \boldsymbol{\chi}_u \right\|_{v,h} \left\| p \right\|_{H^k(\Omega)} \leq \frac{\nu}{10} \left\| \boldsymbol{\chi}_u \right\|_{v,h}^2 + C h^{2k} \left\| p \right\|_{H^k(\Omega)}^2. \end{aligned} \tag{3.4.58}$$

For the terms on O_h in the third row of (3.4.47), we use Hölder's inequality to obtain:

$$\begin{aligned} & O_h(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\chi}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) + O_h(\boldsymbol{\eta}_u, \boldsymbol{u}, \boldsymbol{\chi}_u) \\ & \leqslant \Big(\sum_{T \in \mathcal{F}_h} \left\| \mathbf{curl}\, \boldsymbol{u}_h \right\|_{0,T}^2 \Big)^{1/2} \left\| \boldsymbol{\eta}_u \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{\boldsymbol{L}^4(\Omega)} + \Big(\sum_{T \in \mathcal{F}_h} \left\| \mathbf{curl}\, \boldsymbol{\chi}_u \right\|_{0,T}^2 \Big)^{1/2} \left\| \boldsymbol{u} \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{\boldsymbol{L}^4(\Omega)} \\ & + \Big(\sum_{T \in \mathcal{F}_h} \left\| \mathbf{curl}\, \boldsymbol{\eta}_u \right\|_{0,T}^2 \Big)^{1/2} \left\| \boldsymbol{u} \right\|_{\boldsymbol{L}^4(\Omega)} \left\| \boldsymbol{\chi}_u \right\|_{\boldsymbol{L}^4(\Omega)} \end{aligned}$$

Using the Sobolev embedding $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$ together with the L^p discrete Sobolev's inequality (3.4.19) with $p=4$ and (3.2.55), we obtain

$$\begin{aligned}
&O_h(\boldsymbol{u}_h,\boldsymbol{\eta}_u,\boldsymbol{\chi}_u)+O_h(\boldsymbol{\chi}_u,\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{\eta}_u,\boldsymbol{u},\boldsymbol{\chi}_u)\\
&\leqslant Ch^k\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}\Big(\|\boldsymbol{u}_h\|_{v,h}+\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\Big)\|\boldsymbol{\chi}_u\|_{v,h}+C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{\chi}_u\|^2_{v,h}
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
&O_h(\boldsymbol{u}_h,\boldsymbol{\eta}_u,\boldsymbol{\chi}_u)+O_h(\boldsymbol{\chi}_u,\boldsymbol{u},\boldsymbol{\chi}_u)+O_h(\boldsymbol{\eta}_u,\boldsymbol{u},\boldsymbol{\chi}_u)\\
&\leqslant \frac{\nu}{10}\|\boldsymbol{\chi}_u\|^2_{v,h}+Ch^{2k}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)}+C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{\chi}_u\|^2_{v,h}
\end{aligned}\tag{3.4.59}$$

For the two first terms in (3.4.47), we have since $[\![\boldsymbol{b}]\!]_T=\boldsymbol{0}$ on $\mathcal{F}_h^I$:

$$\begin{aligned}
&C_h(\boldsymbol{\eta}_b,\boldsymbol{u},\boldsymbol{\chi}_b)-C_h(\boldsymbol{\eta}_b,\boldsymbol{\chi}_u,\boldsymbol{b})\\
&=\kappa\sum_{T\in\mathcal{T}_h}\Big(\int_T(\boldsymbol{u}\times\boldsymbol{\eta}_b)\cdot\mathbf{curl}\,\boldsymbol{\chi}_b\,d\boldsymbol{x}-\int_T(\boldsymbol{\chi}_u\times\boldsymbol{\eta}_b)\cdot\mathbf{curl}\,\boldsymbol{b}\,d\boldsymbol{x}\Big)-\kappa\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{u}\times\boldsymbol{\eta}_b\}\!\}\cdot[\![\boldsymbol{\chi}_b]\!]_T ds
\end{aligned}\tag{3.4.60}$$

Using (3.4.19) and the fact that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\le C\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}\quad\text{and}\quad\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^4(\Omega)}\le C\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)},$$

the two first terms in (3.4.60) can be bounded as follows:

$$\begin{aligned}
&\kappa\sum_{T\in\mathcal{T}_h}\Big(\int_T(\boldsymbol{u}\times\boldsymbol{\eta}_b)\cdot\mathbf{curl}\,\boldsymbol{\chi}_b\,d\boldsymbol{x}-\int_T(\boldsymbol{\chi}_u\times\boldsymbol{\eta}_b)\cdot\mathbf{curl}\,\boldsymbol{b}\,d\boldsymbol{x}\Big)\\
&\leqslant C\kappa\Big(\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\boldsymbol{\eta}_b\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{\chi}_b\|_{\boldsymbol{L}^2(\mathcal{T}_h)}+\|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{\eta}_b\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{L}^4(\Omega)}\Big)\\
&\leqslant C\kappa\Big(h^k\|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{\chi}_b\|_{m,h}+h^k\|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{\chi}_u\|_{v,h}\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)}\Big)\\
&\le C\kappa h^k\|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}\big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{\chi}_b\|_{m,h}+\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)}\|\boldsymbol{\chi}_u\|_{v,h}\big)
\end{aligned}\tag{3.4.61}$$

For the last term in (3.4.60), by using the discrete inequality and the Lagrange interpolation Π_h, we obtain:

$$\begin{aligned}
&\kappa\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{u}\times\boldsymbol{\eta}_b\}\!\}\cdot[\![\boldsymbol{\chi}_b]\!]_T ds\\
&=\kappa\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{u}\times(I_h(\boldsymbol{b})-\Pi_h(\boldsymbol{b}))\}\!\}\cdot[\![\boldsymbol{\chi}_b]\!]_T\,ds+\kappa\sum_{e\in\mathcal{F}_h^I}\int_e\{\!\{\boldsymbol{u}\times(\Pi_h(\boldsymbol{b})-\boldsymbol{b})\}\!\}\cdot[\![\boldsymbol{\chi}_b]\!]_T\,ds\\
&\le C\kappa\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\Big(\sum_{e\in\mathcal{F}_h^I}h_e\|(I_h(\boldsymbol{b})-\Pi_h(\boldsymbol{b}))\|^2_{0,e}\Big)^{\frac12}\Big(\sum_{e\in\mathcal{F}_h^I}h_e^{-1}\|[\![\boldsymbol{\chi}_b]\!]_T\|^2_{0,e}\Big)^{\frac12}\\
&+C\kappa\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\Big(\sum_{e\in\mathcal{F}_h^I}h_e\|(\Pi_h(\boldsymbol{b})-\boldsymbol{b})\|^2_{0,e}\Big)^{\frac12}\Big(\sum_{e\in\mathcal{F}_h^I}h_e^{-1}\|[\![\boldsymbol{\chi}_b]\!]_T\|^2_{0,e}\Big)^{\frac12}\\
&\le C\kappa h^k\|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{\chi}_b\|_{m,h}
\end{aligned}\tag{3.4.62}$$

Collecting (3.4.61) and (3.4.62) in (3.4.60), we obtain

$$\begin{aligned}
C_h(\boldsymbol{\eta}_b,\boldsymbol{u},\boldsymbol{\chi}_b)-C_h(\boldsymbol{\eta}_b,\boldsymbol{\chi}_u,\boldsymbol{b})\ &\le\ \kappa h^k\|\boldsymbol{b}\|_{\boldsymbol{H}^{k+1}(\Omega)}\big(\|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{\chi}_b\|_{m,h}+\|\mathbf{curl}\,\boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)}\|\boldsymbol{\chi}_u\|_{v,h}\big)\\
&\le\ \frac{\nu}{10}\|\boldsymbol{\chi}_u\|^2_{v,h}+\frac{\kappa\mu}{8}\|\boldsymbol{\chi}_b\|^2_{m,h}+Ch^{2k}\|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)}\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)}\\
&+\ Ch^{2k}\|\boldsymbol{b}\|^4_{\boldsymbol{H}^{k+1}(\Omega)}.
\end{aligned}\tag{3.4.63}$$

In a similar way, we have

$$
\begin{aligned}
&C_h(\boldsymbol{\chi}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) \\
&= \kappa\Big(\sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{u} \times \boldsymbol{\chi}_b) \cdot \mathbf{curl}\, \boldsymbol{\chi}_b \, dx - \sum_{e\in\mathcal{F}_h^I} \int_e \{\!\{\boldsymbol{u} \times \boldsymbol{\chi}_b\}\!\} \cdot [\![\boldsymbol{\chi}_b]\!]_T \, ds - \sum_{T\in\mathcal{T}_h} \int_T (\boldsymbol{\chi}_u \times \boldsymbol{\chi}_b) \cdot \mathbf{curl}\, \boldsymbol{b} \, dx \Big) \\
&\leqslant C\kappa \, \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \, \|\boldsymbol{\chi}_b\|_{\boldsymbol{L}^4(\Omega)} \, \|\mathbf{curl}\, \boldsymbol{\chi}_b\|_{\boldsymbol{L}^2(\mathcal{T}_h)} \\
&+ \kappa\Big(\sum_{e\in\mathcal{F}_h^I} h \|\boldsymbol{u}\|^4_{\boldsymbol{L}^4(e)} \Big)^{1/4} \Big(\sum_{e\in\mathcal{F}_h^I} h \|\boldsymbol{\chi}_b\|^4_{\boldsymbol{L}^4(e)} \Big)^{1/4} \Big(\sum_{e\in\mathcal{F}_h^I} h_e^{-1} \, \|[\![\boldsymbol{\chi}_b]\!]_T\|^2_{0,e} \Big)^{\frac{1}{2}} \\
&+ \kappa \, \|\boldsymbol{\chi}_u\|_{\boldsymbol{L}^4(\Omega)} \, \|\boldsymbol{\chi}_b\|_{\boldsymbol{L}^4(\Omega)} \, \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{L}^2(\Omega)} \, .
\end{aligned}
$$

Then, applying (3.4.19) and discrete trace inequality, we have

$$
\begin{aligned}
C_h(\boldsymbol{\chi}_b, \boldsymbol{u}, \boldsymbol{\chi}_b) - C_h(\boldsymbol{\chi}_b, \boldsymbol{\chi}_u, \boldsymbol{b}) &\leq C\kappa \Big(\, \|\boldsymbol{\chi}_b\|^2_{m,h} \, \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)} \, \|\boldsymbol{\chi}_u\|_{v,h} \, \|\boldsymbol{\chi}_b\|_{m,h} \, \Big) \\
&\leq C\kappa \, \|\boldsymbol{\chi}_b\|^2_{m,h} \, \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \frac{\kappa\mu}{8} \, \|\boldsymbol{\chi}_b\|^2_{m,h} + C\frac{\kappa}{\mu} \, \|\boldsymbol{b}\|^2_{\boldsymbol{H}^1(\Omega)} \, \|\boldsymbol{\chi}_u\|^2_{v,h}
\end{aligned}
\tag{3.4.64}
$$

Finally, the following bound for the two last terms in (3.4.47) can be obtained is a similar way:

$$
\begin{aligned}
&C_h(\boldsymbol{b}_h, \boldsymbol{\eta}_u, \boldsymbol{\chi}_b) - C_h(\boldsymbol{b}_h, \boldsymbol{\chi}_u, \boldsymbol{\eta}_b) \\
&\leqslant C\kappa \, \|\boldsymbol{b}_h\|_{m,h} \, \Big(h^k \, \|\boldsymbol{u}\|_{\boldsymbol{H}^{k+1}(\Omega)} \, \|\boldsymbol{\chi}_b\|_{m,h} + h^k \, \|\mathbf{curl}\, \boldsymbol{b}\|_{\boldsymbol{H}^k(\Omega)} \, \|\boldsymbol{\chi}_u\|_{v,h} \Big) \\
&\leq \frac{\kappa\mu}{8} \, \|\boldsymbol{\chi}_b\|^2_{m,h} + \frac{\nu}{10} \, \|\boldsymbol{\chi}_u\|^2_{v,h} + h^{2k} \, \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + h^{2k} \, \|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} \, .
\end{aligned}
\tag{3.4.65}
$$

Thus, combining all estimates, we obtain the estimate (3.4.44) by assuming that $\frac{1}{\min(\nu,\nu_m)} \, \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ and $\frac{\sqrt{\kappa}}{\sqrt{\nu\nu_m}} \, \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$ are small enough. □

As a consequence, we have the following result

Theorem 3.4.9. *Assume the analytical solution* $(\boldsymbol{u}, \boldsymbol{b}, p)$ *of* (3.4.1)-(3.4.6) *satisfies the smoothness assumption* (3.3.1). *Let* $(\boldsymbol{u}_h, p_h, \boldsymbol{b}_h)$ *be the solution in* $\boldsymbol{X}_h \times Q_h \times \boldsymbol{C}_h$ *of the DG method* (3.4.7)-(3.4.8). *In addition to the assumptions in Theorem 3.4.7, we assume that* $\frac{1}{\min(\nu,\mu)} \, \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}$ *and* $\frac{\kappa^{1/2}}{\nu^{1/2}\mu^{1/2}} \, \|\boldsymbol{b}\|_{\boldsymbol{H}^1(\Omega)}$ *are small enough. Then, we have the following error estimate:*

$$
\begin{aligned}
&\nu \, \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{v,h} + \kappa\mu \, \|\boldsymbol{b} - \boldsymbol{b}_h\|^2_{m,h} \\
&\leqslant Ch^{2k} \Big(\, \|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} \, \big(\, \|\boldsymbol{u}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} + \|\boldsymbol{b}\|^2_{\boldsymbol{H}^{k+1}(\Omega)} \, \big) \Big)
\end{aligned}
\tag{3.4.66}
$$

with C *depending on the datas of the problem but not of the mesh size* h.

Proof. Since $\boldsymbol{u} - \boldsymbol{u}_h = \boldsymbol{\eta}_{\boldsymbol{u}} + \boldsymbol{\chi}_u$ and $\boldsymbol{b} - \boldsymbol{b}_h = \boldsymbol{\eta}_{\boldsymbol{b}} + \boldsymbol{\chi}_b$, then the result is a direct consequence of Lemma 3.4.8, (3.2.55) and the triangle inequality. □

Chapter 4

Instationary Navier-Stokes equations with Navier-type boundary conditions

4.1 Introduction

In this chapter, we study the following Navier-Stokes equations:

$$
(\mathcal{NS})\quad
\begin{cases}
\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \nabla p + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega\times[0,T]\\
\operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega\times[0,T]\\
\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma\times[0,T]\\
\boldsymbol{u}(x,0) = \boldsymbol{u}_0 & \forall x\in\Omega.
\end{cases}
$$

where Ω is a bounded domain of $\mathbb{R}^3$ simply connected, whose boundary Γ is of class $\mathcal{C}^{1,1}$. The functions $\boldsymbol{u}$ and p describe respectively the velocity and the pressure. The functions $\boldsymbol{f}$ and $\boldsymbol{u}_0$ are the external forces and the initial velocity respectively, and $\nu > 0$ the constant kinematic.

The original motivation comes from the MHD problem in Chapter 2 and Chapter 3. The goal is to study the nonstationary case and then to test other strategies to overcome the difficulties to treat the incompressibility constraint $\operatorname{div}\boldsymbol{b} = 0$ in Ω. Since the MHD system is formed by coupling the Navier-Stokes equations and the Maxwell equations, the study of the time-dependent full MHD system may be quite cumbersome. Thus, we first started to analyze the Navier-Stokes system $(\mathcal{NS})$ which is simplest and it is itself an interesting problem to study.

The boundary condition considered here is a Navier-type boundary condition introduced by H. Navier [74]. The system $(\mathcal{NS})$ has been studied by Da Veiga [21] and Xiao and Xin [101] where the convergence to the solutions of the Euler problem when the viscosity tends to zero is established. Also in the work of Mitrea and Monniaux [70, 71], the authors have studied the analyticity of the Stokes semi-group with the Navier-type boundary conditions together with the local existence of mil solutions. In Miyakawa [72], the author aimed to prove a local existence of the solutions. [2] have considered the same boundary conditions and show that the Stokes operator with Navier-type boundary condition generates a bounded analytic semi-group. We also mention the interesting work [1], where the Navier boundary condition with non-smooth slip coefficient α is studied.

One of the difficulties in the study of the Navier-Stokes problem is how to enforce numerically the divergence-free constraint on the velocity, that can be computationally costly. One of the well-known methods to relax this constraint is the pressure stabilization method which was first studied by Chorin [26, 27] and Temam [96]. This method consists to add $\epsilon\Delta p^\epsilon$ to the incompressibility equation :

$$\operatorname{div} \boldsymbol{u}^\epsilon + \epsilon\Delta p^\epsilon = 0 \quad \text{in } \Omega\times]0,T[,$$

where $\epsilon > 0$ is a parameter of perturbation.

Another method with $\epsilon\Delta p^\epsilon$ replaced by ϵp^ϵ, which is a penalty method was used in several works (see [20] and [90]). In addition, the case of the artificial compressibility method where the term $\epsilon\Delta p^\epsilon$ is replaced by ϵp^ϵ_t :

$$\operatorname{div} \boldsymbol{u}^\epsilon + \epsilon p^\epsilon_t = 0 \quad \text{in } \Omega\times]0,T[$$

has been considered in different works, see [40, 95]. Moreover, the pseudo compressibility method, similar to the artificial compressibility method [91], through inserting a pressure stabilizing term in the equation of the mass conservation is given by:

$$\operatorname{div} \boldsymbol{u}^\epsilon - \epsilon\Delta p^\epsilon_t = 0 \quad \text{in } \Omega\times]0,T[\quad \text{and} \quad \frac{\partial p^\epsilon_t}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma,$$

In our work, we are interested in the study of the existence of weak solutions of the Navier-Stokes system $(\mathcal{NS})$ by using Faedo Galerkin method. Next, two different types of the above approximations will be analyzed to relax the incompressibility constraint: the artificial compressibility method and the pseudo compressibility method.

This chapter is organized as follows: Section 2 is devoted to the proof of weak solutions for the Navier-Stokes equations $(\mathcal{NS})$ by using Faedo-Galerkin's method. The techniques used are classic and inspired by the books of J.L. Lions [65] and R. Temam [96]. Section 3 deals with the analysis of the artificial compressibility method. We focus on the existence of solution for the approximate system and the convergence to the incompressible case. Section 4 deals with the study of the pseudo compressibility method. We give an error analysis for the perturbed system with respect to $(\mathcal{NS})$. Section 5 is concerned with the error analysis for second-order time discretizations of a perturbed system of the Navier-Stokes equations $(\mathcal{NS})$.

4.2 Analysis of the problem

In this section, we want to give an equivalent variational problam and to show that it has at least one solution. Let us begin by introducing some notations and relations. We will use the standard notation for Sobolev spaces $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{H}^1(\Omega)$ and $\boldsymbol{H}^2(\Omega)$. We shall use bold characters for the vectors or the vector spaces and the non-bold characters for the scalars. We also use the standard notation for the $L^2(\Omega)$ inner product, that is, $(\cdot,\cdot)$. We note $\|\cdot\|, \|\cdot\|_1$ and $\|\cdot\|_2$ to define respectively the norm of function in $\boldsymbol{L}^2(\Omega)$, $\boldsymbol{H}^1(\Omega)$ and in $\boldsymbol{H}^2(\Omega)$. The norm corresponding to $\boldsymbol{L}^p(\Omega)$ with $p \neq 2$ will be denoted by $\|\cdot\|_{\boldsymbol{L}^p(\Omega)}$.
Moreover, we define the following subspace:

$$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$$

and recall the following subspace

$$\boldsymbol{H}^1_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\},$$

$$\boldsymbol{X}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \operatorname{div} \boldsymbol{v} \in L^2(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

$$\boldsymbol{H}_0(\operatorname{div}, \Omega) = \boldsymbol{H}_0^{2,2}(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\,,\, \operatorname{div} \boldsymbol{v} \in L^2(\Omega)\ ,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\},$$

and

$$\boldsymbol{V}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \ \text{ in } \ \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\}.$$

Moreover, if X is a Banach space, we define by $L^2(0, T; X)$ the time-space function spaces such that

$$\|\boldsymbol{v}\|^2_{L^2(0,T;X)} = \int_0^T \|\boldsymbol{v}(t)\|^2_X \, dt < \infty, \qquad \forall \boldsymbol{v} \in L^2(0, T; X)$$

We define $B(\boldsymbol{u}, \boldsymbol{v})$, $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ as follow :
$B(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \qquad b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = (B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}), \qquad \forall \boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{w} \in \boldsymbol{H}^1_T(\Omega).$

We will denote by C a generic positive constant depending on Ω. We recall the following inequalities:
• Sobolev inequality for any $1 \leq p \leq 6$: it is well known (see for instance [46]) that if Ω is bounded and simply-connected, then there exists a constant $C(\Omega)$ such that

$$\|\boldsymbol{v}\|^2_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\, \mathbf{curl}\, \boldsymbol{v}\|^2 + \|\operatorname{div}\, \boldsymbol{v}\|^2\Big), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega). \tag{4.2.1}$$

Furthermore, we have

$$\|\nabla \boldsymbol{v}\|^2 \leq C\Big(\|\, \mathbf{curl}\, \boldsymbol{v}\|^2 + \|\operatorname{div}\, \boldsymbol{v}\|^2\Big), \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega). \tag{4.2.2}$$

• Poincaré's inequality:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C\, \|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\,, \qquad \forall \boldsymbol{u} \in \boldsymbol{X}_T(\Omega). \tag{4.2.3}$$

• Agmon's inequality:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \leqslant C\, \|\boldsymbol{u}\|^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega)}\, \|\boldsymbol{u}\|^{\frac{1}{2}}_{\boldsymbol{H}^2(\Omega)}\,, \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^2(\Omega). \tag{4.2.4}$$

• Gagliardo-Nirenberg's inequality:
We assume $p < d$, $q < r < p*$, $\theta \in [0, 1]$, $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p*}$ and $\frac{d}{p*} = \frac{d}{p} - 1$ where d is the dimension. Then :

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^r(\Omega)} \leqslant \|\boldsymbol{u}\|^{\theta}_{\boldsymbol{L}^q(\Omega)}\, \|\nabla \boldsymbol{u}\|^{1-\theta}_{\boldsymbol{L}^p(\Omega)}\,. \tag{4.2.5}$$

If we take in (4.2.5): $r = 3$, $p = q = 2$ and $\theta = \frac{1}{2}$, we have:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)} \leqslant \|\boldsymbol{u}\|^{\frac{1}{2}}\, \|\nabla \boldsymbol{u}\|^{\frac{1}{2}}\,. \tag{4.2.6}$$

Let $T > 0$ be given and $\boldsymbol{f}$ and $\boldsymbol{u}_0$ given functions with:

$$\boldsymbol{u}_0 \in \boldsymbol{H}, \qquad \boldsymbol{f} \in L^2(0, T; (\boldsymbol{H}_0(\operatorname{div}, \Omega))'). \tag{4.2.7}$$

In the first proposed method, we assume that $\nu = 1$.
We can easily check that any solution of $(\mathcal{NS})$ is also solution of the variational problem : find $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;\boldsymbol{V}_T(\Omega))$ such that

$$\begin{cases} \dfrac{d}{dt}(\boldsymbol{u},\boldsymbol{v}) + (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v}) + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega, \qquad \forall \boldsymbol{v} \in \mathbf{V}_T(\Omega) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{4.2.8}$$

where the duality on Ω is defined by

$$\langle \cdot,\cdot\rangle_\Omega = \langle \cdot,\cdot\rangle_{(\boldsymbol{H}_0(\mathrm{div}\,,\Omega))'\times \boldsymbol{H}_0(\mathrm{div}\,,\Omega)}.$$

The following Theorem gives the existence of solutions of (4.2.8). The proof is based on the Faedo-Galerkin approximation and *a priori* estimates. See [23] for a similar analysis.

Theorem 4.2.1. *Given $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfying (4.2.7), the Navier-Stokes problem $(\mathcal{NS})$ admits at least one solution such as :*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^1_T(\Omega)) \cap L^\infty(0,T;\boldsymbol{H}\,) \;\;,\;\; p \in L^2(0,T;L^2(\Omega))$$

Proof. We will solve the equivalent formulation (4.2.8) on [0,T]. Let m be a positive integer. We consider an orthonormal basis of $\mathbf{H}^1_T(\Omega)$ constituted of $\boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_m$. We introduce the space $\boldsymbol{V}_m =< \boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_m >$ and we define an approximated solution $\boldsymbol{u}_m$ of (4.2.8) with :

$$\boldsymbol{u}_m(t) = \sum_{i=1}^m g_{im}(t)\boldsymbol{w}_i \tag{4.2.9}$$

such that :

$$\begin{cases} (\boldsymbol{u}'_m(t),\boldsymbol{w}_i) + (\mathbf{curl}\,\boldsymbol{u}_m(t),\mathbf{curl}\,\boldsymbol{w}_i) + b(\boldsymbol{u}_m(t),\boldsymbol{u}_m(t),\boldsymbol{w}_i) = \langle \boldsymbol{f}(t),\boldsymbol{w}_i\rangle_\Omega \\ \forall i = 1,\dots,m\,,\; t\in[0,t_m], \\ \boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \end{cases} \tag{4.2.10}$$

where $\boldsymbol{u}_{0m}$ is the orthogonal projection of $\boldsymbol{u}_0$ on the space $\boldsymbol{V}_m$.
Using the theory of the ordinary differential equations, the problem (4.2.10) admits at least one solution $\boldsymbol{u}_m$ defined on $[0,t_m]$ with $t_m < T$.

In order to pass to the limit in the approximate problem (4.2.10), we need some a priori estimates on $\boldsymbol{u}_m$ independent of m. Multiplying (4.2.10) by $g_{im}(t)$ and adding these equations for $i = 1,\dots,m$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{u}_m(t)\right\|^2 + \left\|\mathbf{curl}\,\boldsymbol{u}_m(t)\right\|^2 = \langle \boldsymbol{f}(t),\boldsymbol{u}_m(t)\rangle_\Omega$$

Note that we have used the fact that the non linear term $b(\boldsymbol{u}_m,\boldsymbol{u}_m,\boldsymbol{u}_m) = 0$.
Using Cauchy-Schwarz, (4.2.1) with div $\boldsymbol{u}_m = 0$ and Young inequalities', we obtain :

$$\langle \boldsymbol{f}(t),\boldsymbol{u}_m(t)\rangle_\Omega \;\leqslant C\left\|\boldsymbol{f}(t)\right\|^2_{(\boldsymbol{H}_0(\mathrm{div}\,,\Omega))'} + \frac{1}{2}\left\|\mathbf{curl}\,\boldsymbol{u}_m(t)\right\|^2.$$

So :

$$\frac{d}{dt}\left\|\boldsymbol{u}_m(t)\right\|^2 + \left\|\mathbf{curl}\,\boldsymbol{u}_m(t)\right\|^2 \leqslant C\left\|\boldsymbol{f}(t)\right\|^2_{(\mathbf{H}_0(\mathrm{div}\,,\Omega))'} \tag{4.2.11}$$

Now, integrating (4.2.11) from 0 to t with $t \in [0, t_m]$, we obtain in particular :

$$\sup_{t\in[0,t_m]} \|\boldsymbol{u}_m(t)\|^2 \leqslant C\int_0^T \|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0(\mathrm{div}\,,\Omega))'}\,dt + \|\boldsymbol{u}_{0m}\|^2 \tag{4.2.12}$$

The estimate (4.2.12) is valid with $t_m = T$ which gives

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_m(t)\|^2 \leqslant C_1, \tag{4.2.13}$$

where

$$C_1 = \|\boldsymbol{u}_0\|^2 + \int_0^T \|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0(\mathrm{div}\,,\Omega))'}\,dt. \tag{4.2.14}$$

The a priori estimate (4.2.13) shows the existence of an element $\boldsymbol{u}^* \in L^\infty(0,T;\boldsymbol{H})$ and a subsequence still denoted $(\boldsymbol{u}_m)_m$ such that:

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}^* \quad \text{weak star in } L^\infty(0,T;\boldsymbol{H}) \tag{4.2.15}$$

Next, integration (4.2.11) over time, from 0 to T, we obtain :

$$\|\boldsymbol{u}_m(T)\|^2 + \int_0^T \|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2\,dt \leqslant \|\boldsymbol{u}_0\|^2 + C\int_0^T \|\boldsymbol{f}(t)\|^2_{(\mathbf{H}_0(\mathrm{div}\,,\Omega))'}\,dt. \tag{4.2.16}$$

So, we have in particular

$$\int_0^T \|\mathbf{curl}\,\boldsymbol{u}_m(t)\|^2\,dt \leqslant C_1, \tag{4.2.17}$$

The estimate (4.2.17) shows the existence of some $\boldsymbol{u}^*$ in $L^2(0,T;\boldsymbol{H}^1(\Omega))$ and some subsequence still denoted $(\boldsymbol{u}_m)_m$ such that :

$$\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}^* \quad \text{weak in } L^2(0,T;\boldsymbol{H}^1(\Omega)) \tag{4.2.18}$$

By uniqueness of the limit, we deduce that $\boldsymbol{u}^* = \boldsymbol{u}$. The two above weak convergences allow to pass to the limit in the linear terms. □

To pass to the limit in the nonlinear term, it is necessary to obtain a strong convergence result. In addition to the previous estimates, we need to prove the following estimate of the fractional derivative with respect to time of the approximate solution:

$$\int_{\mathbb{R}} |\tau|^{2\gamma}\,\|\widehat{\boldsymbol{u}_m}(\tau)\|^2\,d\tau \leqslant C, \quad \text{for some } \gamma \in (0, \frac{1}{4}). \tag{4.2.19}$$

This technique is detailed in the next section, so we omit the proof of (4.2.19).

The estimate (4.2.19) along with (4.2.18) and compactness Theorem [96, Theorem 2.2, p.274] will imply that we can extract a subsequence of $(\boldsymbol{u}_m)$, still denoted $(\boldsymbol{u}_m)_m$, such that

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{strongly in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{4.2.20}$$

We can now pass to the limit in all terms and prove that $\boldsymbol{u}$ satisfies (4.2.8). We note that the proof of the passage to the limit in the nonlinear term is classical. The reader can refer to, for example, [65] and [96] for details. So we omit the proof at this stage.

It remains to recover the pressure. Let $\boldsymbol{v} \in \mathbf{V}_T(\Omega)$ and we introduce the continuous fonctionnal:

$$H(\boldsymbol{v})(t) = \int_0^T [\langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega - (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v}) - (\boldsymbol{u}.\nabla \boldsymbol{u}, \boldsymbol{v})]\, ds - (\boldsymbol{u}(t), \boldsymbol{v}) + (\boldsymbol{u}_0, \boldsymbol{v})$$

Thanks to variational formulation (4.2.8), H vanishes on $\mathbf{V}_T(\Omega)$. We deduce the existence of function $\tilde{p}(t) \in L^2(\Omega)$ and $C > 0$ such as for all $\boldsymbol{v} \in \mathbf{V}_T(\Omega)$:

$$H(\boldsymbol{v})(t) = (\mathrm{div}\, \boldsymbol{v}, \tilde{p}(t))$$

We time derivate H, and when we set $p(t) = \tilde{p}_t(t)$, the pressure p is well defined.

4.3 Approximation of the system by an artificial compressibility method

In this section, we discuss the approximation of the incompressible Navier-Stokes equations $(\mathcal{NS})$ with Navier-type boundary conditions via the artificial compressibility method.

As in the previous section, we prove the existence of a solution for this problem by using the Faedo-Galerkin's method, and we prove that the perturbed system (4.3.3) converges to the initial system $(\mathcal{NS})$. Finally, we will derive an error estimate for the method. We begin with recalling a useful compactness result [96, Theorem 2.2, p.274]

Theorem 4.3.1. *Let X_0, X, X_1 Hilbert spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$, the embeddings being continuous and the embedding $X_0 \hookrightarrow X$ is compact. For a given $\gamma > 0$, we define the Hilbert space :*

$$M^\gamma := M^\gamma(\mathbb{R}; X_0, X_1) := \{\psi \in L^2(\mathbb{R}; X_0), D_t^\gamma \psi \in L^2(\mathbb{R}; X_1)\}$$

with the norm :

$$\|\psi\|_{M^\gamma} = \{\|\psi\|^2_{L^2(\mathbb{R},X_0)} + \left\||\tau|^\gamma \widehat{\psi}(\tau)\right\|^2_{L^2(\mathbb{R},X_1)}\}$$

where $\widehat{\psi}$ denote the Fourier transform. For any set $K \subset \mathbb{R}$, we define

$$M_K^\gamma := M_K^\gamma(\mathbb{R}; X_0, X_1) = \{\psi \in M^\gamma, \mathrm{support}\, \psi \subseteq K\}. \tag{4.3.1}$$

So we have the following compact embedding :

$$M_K^\gamma(\mathbb{R}; X_0, X_1) \hookrightarrow L^2(\mathbb{R}; X) \tag{4.3.2}$$

4.3.1 Study of the perturbed system

Let $\epsilon \in (0, 1]$ be a parameter and we consider the following family of perturbed compressible Navier-Stokes equations:

$$\begin{cases} \boldsymbol{u}_t^\epsilon - \Delta \boldsymbol{u}^\epsilon + \nabla p^\epsilon + \boldsymbol{u}^\epsilon \cdot \nabla \boldsymbol{u}^\epsilon + \dfrac{1}{2}(\mathrm{div}\ \boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon = \boldsymbol{f}, & \text{in } \Omega\times]0, T[\\ \mathrm{div}\ \boldsymbol{u}^\epsilon + \epsilon p_t^\epsilon = 0, & \text{in } \Omega\times]0, T[\\ \boldsymbol{u}^\epsilon \cdot \boldsymbol{n} = 0\,,\ \mathbf{curl}\ \boldsymbol{u}^\epsilon \times \boldsymbol{n} = \boldsymbol{0}, & \text{on } \Gamma\times]0, T[\\ \boldsymbol{u}^\epsilon(0) = \boldsymbol{u}_0,\ \ p^\epsilon(0) = p_0 \in L^2(\Omega) \end{cases} \tag{4.3.3}$$

First, we will write a variational formulation of problem (4.3.3). Next, we derive some a priori estimates as in the previous section and obtain the existence of solutions by means of Galerkin-Faedo approximation.

Multiplying $(4.3.3)_1$ by $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$ and $(4.3.3)_2$ by $q \in \mathcal{D}(\Omega)$, integrating by parts over Ω and then using a continuity argument, for any $\boldsymbol{v} \in \boldsymbol{V}_T(\Omega)$ and $q \in L^2(\Omega)$, we obtain the following variational formulation:

For $\epsilon > 0$ fixed, $\boldsymbol{f}$, $\boldsymbol{u}_0$ given satisfying (4.2.7) and

$$p_0 \in L^2(\Omega), \tag{4.3.4}$$

find $\boldsymbol{u}^\epsilon \in L^2(0,T;\boldsymbol{H}^1_T(\Omega))$ and $p^\epsilon \in L^2(0,T;L^2(\Omega))$ such as, for almost all $t \in [0,T]$:

$$\begin{cases} \dfrac{d}{dt}(\boldsymbol{u}^\epsilon, \boldsymbol{v}) + (\mathbf{curl}\, \boldsymbol{u}^\epsilon, \mathbf{curl}\, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{u}^\epsilon, \operatorname{div} \boldsymbol{v}) - (p^\epsilon, \operatorname{div} \boldsymbol{v}) + \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega) \\ \epsilon \dfrac{\mathrm{d}}{\mathrm{d}t}(p^\epsilon, q) + (\operatorname{div} \boldsymbol{u}^\epsilon, q) = 0, \quad \forall q \in L^2(\Omega) \\ \boldsymbol{u}^\epsilon(0) = \boldsymbol{u}_0, \quad p^\epsilon(0) = p_0. \end{cases} \tag{4.3.5}$$

We note that we use the form $\tilde{b}$ instead of b defined by:

$$\tilde{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2}[b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) - b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})] = (\tilde{B}(\boldsymbol{u},\, \boldsymbol{v}), \boldsymbol{w}) \tag{4.3.6}$$

with $\tilde{B}(\boldsymbol{u},\, \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + \frac{1}{2}(\operatorname{div} \boldsymbol{u})\boldsymbol{v}$. The difference is the additional term $\frac{1}{2}(\operatorname{div} \boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon$ which is a stabilization term. Indeed, if $\operatorname{div} \boldsymbol{u} \neq 0$, $b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) \neq 0$. However, by introducing $\tilde{b}$ as in (4.3.6), this allows to:

$$\tilde{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_T(\Omega). \tag{4.3.7}$$

The following result gives the well-posedness of the variational problem (4.3.5). The proof is similar to [68, Theorem 5.1].

Theorem 4.3.2. *Let ϵ arbitrary in $(0,1]$. Given (4.2.7) and (4.3.4), the problem* (4.3.5) *admits at least one solution. Moreover,*

$$\boldsymbol{u}^\epsilon \in L^2(0,T;\boldsymbol{H}^1_T(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \;,\; p^\epsilon \in L^2(0,T;L^2(\Omega))$$

Proof. To establish the proof of Theorem 4.3.2, we return to the Galerkin approximation and we look for similar a priori estimates with the presence of pressure.

Let $(\boldsymbol{w}_i)$ an orthonormal basis of $\boldsymbol{H}^1_T(\Omega)$ and (v_i) an orthonormal basis of $L^2(\Omega)$. Let $\boldsymbol{W}_m =< \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m >$ and $V_m =< v_1, \ldots, v_m >$. We denote by:

$$\boldsymbol{u}^\epsilon_m(t) = \sum_{i=1}^m g_{im}(t)\boldsymbol{w}_i \qquad \text{and} \qquad p^\epsilon_m(t) = \sum_{j=1}^m \xi_{jm}(t) v_j,$$

approximated solutions which satisfy :

$$\begin{cases} \dfrac{d}{dt}(\boldsymbol{u}^\epsilon_m, \boldsymbol{w}_k) + (\mathbf{curl}\, \boldsymbol{u}^\epsilon_m, \mathbf{curl}\, \boldsymbol{w}_k) + (\operatorname{div} \boldsymbol{u}^\epsilon_m, \operatorname{div} \boldsymbol{w}_k) - (p^\epsilon_m, \operatorname{div} \boldsymbol{w}_k) \\ + \tilde{b}(\boldsymbol{u}^\epsilon_m, \boldsymbol{u}^\epsilon_m, \boldsymbol{w}_k) = \langle \boldsymbol{f}, \boldsymbol{w}_k \rangle_\Omega, \quad k = 1, \ldots, m \\ \epsilon \dfrac{d}{dt}(p^\epsilon_m, v_l) + (\operatorname{div} \boldsymbol{u}^\epsilon_m, v_l) = 0, \quad l = 1, \ldots, m \\ \boldsymbol{u}^\epsilon_m(0) = \boldsymbol{u}_{0m} \;,\; p^\epsilon_m(0) = p_{0m} \end{cases} \tag{4.3.8}$$

for which $\boldsymbol{u}_{0m}$ is the orthogonal projection of $\boldsymbol{u}(0)$ on the space $\boldsymbol{W}_m$ and p_{0m} the orthogonal projection of $p(0)$ on the space V_m.

The existence of at least one solution on $[0, t_m]$ with $t_m \leqslant T$ is guaranteed by the theory of ordinary differential equations. We will prove that in fact $t_m = T$.

Step 1: A priori estimates. As usual, we will make some a priori estimates and then pass to the limit. We multiply $(4.3.8)_1$ by $g_{km}(t)$ $(k = 1, \dots, m)$, and $(4.3.8)_2$ by $\xi_{lm}(t)$ $(l = 1, \dots, m)$ and then adding these resulting equalities, we obtain:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|\boldsymbol{u}_m^\epsilon\|^2 + \|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon\|^2 + \|\mathrm{div}\,\boldsymbol{u}_m^\epsilon\|^2 - (p_m^\epsilon, \mathrm{div}\boldsymbol{u}_m^\epsilon) + \tilde{b}(\boldsymbol{u}_m^\epsilon, \boldsymbol{u}_m^\epsilon, \boldsymbol{u}_m^\epsilon) = \langle \boldsymbol{f}, \boldsymbol{u}_m^\epsilon \rangle_\Omega \\ \dfrac{\epsilon}{2}\dfrac{d}{dt}\|p_m^\epsilon\|^2 + (\mathrm{div}\,\boldsymbol{u}_m^\epsilon, p_m^\epsilon) = 0 \end{cases}$$

Using (4.3.7), and summing the both equations we obtain :

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m^\epsilon\|^2 + \|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon\|^2 + \|\mathrm{div}\,\boldsymbol{u}_m^\epsilon\|^2 + \frac{\epsilon}{2}\frac{d}{dt}\|p_m^\epsilon\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}_m^\epsilon \rangle_\Omega$$

Using Cauchy Schwarz, (4.2.1) and Young inequality

$$\frac{d}{dt}\|\boldsymbol{u}_m^\epsilon\|^2 + \|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon\|^2 + \|\mathrm{div}\,\boldsymbol{u}_m^\epsilon\|^2 + \epsilon\frac{d}{dt}\|p_m^\epsilon\|^2 \leqslant C\,\|\boldsymbol{f}\|^2_{\mathbf{H}_0(\mathrm{div}\,,\Omega)'} \tag{4.3.9}$$

Integrating (4.3.9) from 0 to t with $t \leq t_m$,

$$\sup_{t\in[0,t_m]}\left(\|\boldsymbol{u}_m^\epsilon(t)\|^2 + \epsilon\,\|p_m^\epsilon(t)\|^2\right) \leqslant C\int_0^T \|\boldsymbol{f}(t)\|^2_{\mathbf{H}_0(\mathrm{div}\,,\Omega)'}\,dt + \|\boldsymbol{u}_{0m}^\epsilon\|^2 + \|p_{0m}^\epsilon\|^2. \tag{4.3.10}$$

The right-hand side of (4.3.10) is independent of t_m. As a consequence, the estimate (4.3.10) is valid with $t_m = T$ which gives the estimate

$$\sup_{t\in[0,T]}\left(\|\boldsymbol{u}_m^\epsilon(t)\|^2 + \epsilon\,\|p_m^\epsilon(t)\|^2\right) \leqslant C_2, \tag{4.3.11}$$

where

$$C_2 = \|\boldsymbol{u}_0^\epsilon\|^2 + \|p_0^\epsilon\|^2 + \int_0^T \|\boldsymbol{f}(t)\|^2_{\mathbf{H}_0(\mathrm{div}\,,\Omega)'}\,dt. \tag{4.3.12}$$

We deduce from (4.3.10) that the sequences $\boldsymbol{u}_m^\epsilon$ and p_m^ϵ respectively remain in a bounded set of $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$.

Now integrating (4.3.9) from 0 to T, we obtain easily:

$$\int_0^T \|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon(t)\|^2\,dt + \int_0^T \|\mathrm{div}\,\boldsymbol{u}_m^\epsilon(t)\|^2\,dt \leqslant C\int_0^T \|\boldsymbol{f}(t)\|^2_{\mathbf{H}_0(\mathrm{div}\,,\Omega)'}\,dt + \|\boldsymbol{u}_m^\epsilon(0)\|^2 + \|p_m^\epsilon(0)\|^2,$$

which gives the estimate

$$\int_0^T \|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon(t)\|^2\,dt + \int_0^T \|\mathrm{div}\,\boldsymbol{u}_m^\epsilon(t)\|^2\,dt \leqslant C_2, \tag{4.3.13}$$

and shows according to (4.2.1) that the sequence $\boldsymbol{u}_m^\epsilon$ remains in a bounded set of $L^2(0,T;\boldsymbol{H}_T^1(\Omega))$.

In order to pass to the limit in the nonlinear term in (4.3.8), we need some estimates on the fractional derivative of $\boldsymbol{u}_m^\epsilon$ with respect to time. To this end, we follow [96, Chapter3] and [23, Theorem 4.2.2, Chapter 4]. We extend the functions $\boldsymbol{u}_m^\epsilon$ and p_m^ϵ as follows:

$$\widetilde{\boldsymbol{u}}_m^\epsilon = \begin{cases} \boldsymbol{u}_m^\epsilon(t), & t \in [0,T] \\ 0 & \text{outside,} \end{cases}$$

$$\widetilde{p}_m^\epsilon = \begin{cases} p_m^\epsilon(t), & t \in [0,T] \\ 0 & \text{outside.} \end{cases}$$

We denote by $\widehat{\boldsymbol{u}}_m^\epsilon$ the Fourier transform defined by:

$$\widehat{\boldsymbol{u}}_m^\epsilon(\tau) = \int_{\mathbb{R}} e^{-2i\pi t\tau}\, \widetilde{\boldsymbol{u}}_m^\epsilon(t)\, dt.$$

Since we have discontinuities on 0 and on T, we have the system :

$$\begin{cases} \dfrac{d}{dt}(\widetilde{\boldsymbol{u}}_m^\epsilon, \boldsymbol{w}_k) + (\mathbf{curl}\, \widetilde{\boldsymbol{u}}_m^\epsilon, \mathbf{curl}\, \boldsymbol{w}_k) + (\operatorname{div} \widetilde{\boldsymbol{u}}_m^\epsilon, \operatorname{div} \boldsymbol{w}_k) - (\widetilde{p}_m^\epsilon, \operatorname{div}\boldsymbol{w}_k) \\ +\, \tilde{b}(\widetilde{\boldsymbol{u}}_m^\epsilon, \widetilde{\boldsymbol{u}}_m^\epsilon, \boldsymbol{w}_k) = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{w}_k \rangle + (\boldsymbol{u}_{0m}, \boldsymbol{w}_k)\delta_{(0)} - (\boldsymbol{u}_m^\epsilon(T), \boldsymbol{w}_k)\delta_{(T)}, \\ \epsilon \dfrac{d}{dt}(\widetilde{p}_m^\epsilon, v_l) + (\operatorname{div} \widetilde{\boldsymbol{u}}_m^\epsilon, v_l) = \epsilon(p_{0m}, v_l)\delta_{(0)} - \epsilon(p_m^\epsilon(T), v_k)\delta_{(T)} \end{cases}$$

where $\delta_{(T)}$ and $\delta_{(0)}$ are the Dirac distributions at T and 0, respectively. Taking the Fourier transform and using the fact that

$$\widehat{D_t^\gamma \boldsymbol{u}_m^\epsilon(t)} = (2i\pi\tau)^\gamma \widehat{\boldsymbol{u}}_m^\epsilon(\tau), \quad \text{for some fixed real } \gamma.$$

we obtain :

$$\begin{cases} 2i\pi\tau(\widehat{\boldsymbol{u}}_m^\epsilon, \boldsymbol{w}_k) + (\mathbf{curl}\, \widehat{\boldsymbol{u}}_m^\epsilon, \mathbf{curl}\, \boldsymbol{w}_k) + (\operatorname{div} \widehat{\boldsymbol{u}}_m^\epsilon, \operatorname{div} \boldsymbol{w}_k) - (\widehat{p}_m^\epsilon, \operatorname{div}\boldsymbol{w}_k) \\ +\, \tilde{b}(\widehat{\boldsymbol{u}}_m^\epsilon, \widehat{\boldsymbol{u}}_m^\epsilon, \boldsymbol{w}_k) = \langle \widehat{\boldsymbol{f}}, \boldsymbol{w}_k \rangle_\Omega + (\boldsymbol{u}_{0m}, \boldsymbol{w}_k) - (\boldsymbol{u}_m^\epsilon(T), \boldsymbol{w}_k) e^{-2i\pi\tau T} \\ \epsilon 2i\pi\tau(\widehat{p}_m^\epsilon, v_l) + (\operatorname{div} \widehat{\boldsymbol{u}}_m^\epsilon, v_l) = \epsilon(p_{0m}, v_l) - \epsilon(p_m^\epsilon(T), v_l) e^{-2i\pi\tau T} \end{cases} \tag{4.3.14}$$

We will apply the compactness Theorem 4.3.1 with $X_0 := \boldsymbol{H}_T^1(\Omega)$ and $X = X_1 := \boldsymbol{L}^2(\Omega)$. For this we need to show that $\boldsymbol{u}_m^\epsilon \in M_K^\gamma(\mathbb{R}, \boldsymbol{H}_T^1(\Omega), \mathbf{L}^2(\Omega))$ (see (4.3.1) for the definition of this space). We already have, thanks to (4.3.13), $\boldsymbol{u}_m^\epsilon \in L^2(0,T; \boldsymbol{H}_T^1(\Omega))$. So we just need :

$$D_t^\gamma \boldsymbol{u}_m^\epsilon \in L^2(0,T; \boldsymbol{L}^2(\Omega)).$$

As for $\boldsymbol{u}_m^\epsilon$ and p_m^ϵ, we extend g_{im} and ξ_{jm} in $\mathbb{R}$ with $\widetilde{g}_{im}$ and $\widetilde{\xi}_{jm}$, and we multiply $(4.3.14)_1$ by $\widehat{g}_{im}(t)$ $(i = 1, \ldots, m)$ and $(4.3.14)_2$ by $\widehat{\xi}_{jm}(t)$ $(j = 1, \ldots, m)$, then adding the resulting equalities, we obtain:

$$\begin{aligned} &2i\pi\tau[\|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 + \epsilon \|\widehat{p}_m^\epsilon\|^2] + \|\mathbf{curl}\, \widehat{\boldsymbol{u}}_m^\epsilon\|^2 + \|\operatorname{div} \widehat{\boldsymbol{u}}_m^\epsilon\|^2 = \langle \widehat{\boldsymbol{f}}, \widehat{\boldsymbol{u}}_m^\epsilon \rangle_\Omega + < \boldsymbol{u}_{0m}, \widehat{\boldsymbol{u}}_m^\epsilon(\tau) > \\ &+ \epsilon(p_{0m}, \widehat{p}_m^\epsilon(\tau)) - \Big((\boldsymbol{u}_m^\epsilon(T), \widehat{\boldsymbol{u}}_m^\epsilon(\tau)) + \epsilon < p_m^\epsilon(T), \widehat{p}_m^\epsilon(\tau) > \Big) e^{-2i\pi\tau T} \end{aligned}$$

Thanks to the previous equation and Cauchy-Schwarz inequality :

$$\begin{aligned} 2i\pi\tau \|\widehat{\boldsymbol{u}}_m^\epsilon\|^2 \leqslant & \left\|\widehat{\boldsymbol{f}}\right\| \|\widehat{\boldsymbol{u}}_m^\epsilon\| + \|\boldsymbol{u}_{0m}\| \|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\| + \epsilon \|p_{0m}\| \|\widehat{p}_m^\epsilon(\tau)\| + \|\boldsymbol{u}_m^\epsilon(T)\| \|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\| \\ & + \epsilon \|p_m^\epsilon(T)\| \|\widehat{p}_m^\epsilon(\tau)\| \end{aligned}$$

Using (4.3.11), Minkowski and Young inequalities, we obtain:

$$2\pi\tau \left\|\widehat{\boldsymbol{u}}_m^\epsilon\right\|^2 \leqslant \frac{1}{2}(\left\|\widehat{\boldsymbol{f}}\right\|^2 + \left\|\widehat{\boldsymbol{u}}_m^\epsilon\right\|^2) + 2\sqrt{C_2}(\left\|\widehat{\boldsymbol{u}}_m^\epsilon(\tau)\right\| + \epsilon\left\|\widehat{p}_m^\epsilon(\tau)\right\|) \tag{4.3.15}$$

With this last inequality, we can show as in [96, Theorem 8.1] and in [68, Theorem 5.1] that for $\gamma \in]0, \frac{1}{4}[$:

$$\int_{\mathbb{R}} |\tau|^{2\gamma} \left\|\widehat{\boldsymbol{u}}_m^\epsilon\right\|^2 \leq C. \tag{4.3.16}$$

Step 2: Passage to the limit. At this step, we keep ϵ fixed and we are only concerned with the passage to the limit as $m \to \infty$.

It follows from (4.3.11) and (4.3.13) that there exists a subsequence still denoted $(\boldsymbol{u}_m)_m$ such that:

$$\boldsymbol{u}_m^\epsilon \rightharpoonup \boldsymbol{u}^\epsilon \quad \text{weak in} \quad L^2(0,T;\boldsymbol{H}_T^1(\Omega)), \tag{4.3.17}$$

$$\boldsymbol{u}_m^\epsilon \rightharpoonup \boldsymbol{u}^\epsilon \quad \text{weak star in} \quad L^\infty(0,T;\boldsymbol{L}^2(\Omega)), \tag{4.3.18}$$

$$p_m^\epsilon \rightharpoonup p^\epsilon \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{4.3.19}$$

Using (4.3.16) and (4.3.17) and applying the compactness Theorem 4.3.1, we deduce the existence of a subsequence of $(\boldsymbol{u}_m^\epsilon)_m$ still denoted by $(\boldsymbol{u}_m^\epsilon)_m$ such that

$$\boldsymbol{u}_m^\epsilon \to \boldsymbol{u}^\epsilon \qquad \text{strong in} \quad L^2(0,T;\boldsymbol{L}^2(\Omega)) \tag{4.3.20}$$

It remains to prove that $(\boldsymbol{u}^\epsilon, p^\epsilon)$ is a solution of (4.3.5). For this, we use the same arguments in [96]. Multiply system (4.3.8) by $\psi(t) \in C^\infty(0,T)$ such that $\psi(T) = 0$, and we integrate from 0 to T, we obtain:

$$\begin{cases} -\displaystyle\int_0^T (\boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k)\psi'(t)\,dt + \int_0^T (\mathbf{curl}\,\boldsymbol{u}_m^\epsilon, \psi(t)\,\mathbf{curl}\,\boldsymbol{w}_k)\,dt + \int_0^T (\mathrm{div}\,\boldsymbol{u}_m^\epsilon, \psi(t)\mathrm{div}\,\boldsymbol{w}_k)\,dt \\ -\displaystyle\int_0^T (p_m^\epsilon, \mathrm{div}\psi(t)\boldsymbol{w}_k)\,dt + \int_0^T \tilde{b}(\boldsymbol{u}_m^\epsilon, \boldsymbol{u}_m^\epsilon, \psi(t)\boldsymbol{w}_k)\,dt = \int_0^T \langle \boldsymbol{f}, \psi(t)\boldsymbol{w}_k\rangle_\Omega\,dt \\ +(\boldsymbol{u}_{0m}, \boldsymbol{w}_k)\psi(0) - \epsilon\displaystyle\int_0^T (p_m^\epsilon, \psi'(t)v_l) + \int_0^T (\mathrm{div}\,\boldsymbol{u}_m^\epsilon, \psi(t)v_l)\,dt = \epsilon(p_{0m}, v_l)\psi(0) \end{cases} \tag{4.3.21}$$

It is easy to pass to the limit $m \to \infty$ in the linear terms of (4.3.21). It suffies to use the previous convergence results. We now look at the convergence of the nonlinear term $\tilde{b}$. For $\psi(t) \in C^\infty(0,T)$, if $m \to \infty$, we have

$$\begin{aligned} &\left|\int_0^T \tilde{b}(\boldsymbol{u}_m^\epsilon, \boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k\psi(t) - \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t))dt\right| \\ &\leqslant \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}_m^\epsilon, \boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k\psi(t))dt - \int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t))dt\right| \\ &+ \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k\psi(t), \boldsymbol{u}_m^\epsilon)dt - \int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t), \boldsymbol{u}^\epsilon)dt\right| \\ &\leqslant \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon, \boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k\psi(t))dt\right| + \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t))dt\right| \\ &+ \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t), \boldsymbol{u}_m^\epsilon)dt\right| + \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t), \boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon)dt\right| \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.3.22}$$

We want to bound the terms I_1, I_2, I_3 and I_4 separately as below.

Using (4.2.1) and the strong convergence (4.3.20), we have

$$
\begin{aligned}
I_1 &= \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon, \boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k\psi(t))dt\right| \leqslant \frac{1}{2}\sup_{t\in[0,T]}|\psi(t)|\sup_{x\in\Omega}|\boldsymbol{w}_k(x)|\int_0^T \|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|\,\|\nabla \boldsymbol{u}_m^\epsilon\|\,dt\\
&\leqslant C\int_0^T \|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|\left(\|\mathbf{curl}\,\boldsymbol{u}_m^\epsilon\| + \|\mathrm{div}\,\boldsymbol{u}_m^\epsilon\|\right)dt\\
&\leqslant C\,\|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\,\|\boldsymbol{u}_m^\epsilon\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \to 0 \quad \text{as} \quad m\to\infty
\end{aligned}
$$

Similarly, we have:

$$
\begin{aligned}
I_2 &= \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t))dt\right| = \frac{1}{2}\left|\int_{[0,T]\times\Omega}\sum_{i,j=1}^{3}(\boldsymbol{u}_\epsilon)_i\frac{d(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon)_j}{dx_i}(\psi(t)\boldsymbol{w}_k)_j dx\,dt\right|\\
&\leqslant \frac{1}{2}\int_0^T\sum\left|\int_\Omega(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon)_j\frac{d(\boldsymbol{u}^\epsilon)_i}{dx_i}(\psi(t)\boldsymbol{w}_k)_j\,dx\right|dt\\
&+\frac{1}{2}\int_0^T\sum\left|\int_\Omega(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon)_j(\boldsymbol{u}^\epsilon)_i\psi(t)\frac{d(\boldsymbol{w}_k)_j}{dx_i}\right|dx\,dt\\
&\leqslant \frac{1}{2}\sup_{t\in(0,T)}|\psi(t)|\sup_{x\in\Omega}|\boldsymbol{w}_k|\int_0^T\|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|\,\|\nabla\boldsymbol{u}_\epsilon\|\,dt\\
&+\frac{1}{2}\sup_{t\in(0,T)}|\psi(t)|\sup_{x\in\Omega}|\nabla\boldsymbol{w}_k|\int_0^T\|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|\,\|\boldsymbol{u}^\epsilon\|\,dt \quad \to 0 \quad \text{as} \quad m\to\infty
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t), \boldsymbol{u}_m^\epsilon)dt\right|\\
&\leqslant \frac{1}{2}\sup_{t\in(0,T)}|\psi(t)|\sup_{x\in\Omega}|\nabla\boldsymbol{w}_k|\int_0^T\|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|\,\|\nabla\boldsymbol{u}_m^\epsilon\|\,dt \quad \to 0 \text{ as } \quad m\to\infty,
\end{aligned}
$$

and finally,

$$
\begin{aligned}
I_4 &= \frac{1}{2}\left|\int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t), \boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon)dt\right|\\
&\leqslant \sup_{t\in(0,T)}|\psi(t)|\sup_{x\in\Omega}|\nabla\boldsymbol{w}_k|\int_0^T\|\boldsymbol{u}_m^\epsilon - \boldsymbol{u}^\epsilon\|\,\|\boldsymbol{u}^\epsilon\|\,dt \quad \to 0 \text{ as } \quad m\to\infty.
\end{aligned}
$$

Collecting all the above estimates, we deduce the following convergence result

$$
\int_0^T \tilde{b}(\boldsymbol{u}_m^\epsilon, \boldsymbol{u}_m^\epsilon, \boldsymbol{w}_k\psi(t))dt \to \int_0^T \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi(t))dt \tag{4.3.23}
$$

Now, observe that the nonlinear term is continuous with respect to $\boldsymbol{w}_k$ for any $\boldsymbol{w}_k \in \boldsymbol{H}_T^1(\Omega)$. Indeed, given $\psi \in C^\infty(0,T)$, we have :

$$
\begin{aligned}
&\left|\int_0^T \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi)\,dt\right| \leqslant \frac{1}{2}\left(\left|\int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi)\,dt\right| + \left|\int_0^T b(\boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi, \boldsymbol{u}^\epsilon)\,dt\right|\right)\\
&\leqslant \frac{1}{2}\sup_{t\in(0,T)}|\psi(t)|\int_0^T\|\boldsymbol{u}^\epsilon\|_{\boldsymbol{L}^4(\Omega)}\|\nabla\boldsymbol{u}^\epsilon\|\,\|\boldsymbol{w}_k\|_{\boldsymbol{L}^4(\Omega)}\;dt + \frac{1}{2}\sup_{t\in(0,T)}|\psi(t)|\int_0^T\|\boldsymbol{u}^\epsilon\|^2_{\boldsymbol{L}^4(\Omega)}\|\nabla\boldsymbol{w}_k\|\;dt
\end{aligned}
$$

Using the continuous imbedding $\boldsymbol{H}^1_T(\Omega) \hookrightarrow \boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$, we get:

$$\left| \int_0^T \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{w}_k\psi)\, dt \right| \leqslant C \left\| \boldsymbol{u}^\epsilon \right\|^2_{L^2(0,T;\boldsymbol{H}^1_T(\Omega))} \left\| \boldsymbol{w}_k \right\|_{\boldsymbol{H}^1_T(\Omega)} \tag{4.3.24}$$

As a consequence, by a continuity argument, taking the limit of (4.3.21) as $m \to \infty$, we have that for every $\boldsymbol{w} \in \boldsymbol{H}^1_T(\Omega)$ and $v \in L^2(\Omega)$

$$\begin{aligned}
&- \int_0^T (\boldsymbol{u}^\epsilon, \boldsymbol{w})\psi'\, dt + \int_0^T (\mathbf{curl}\, \boldsymbol{u}^\epsilon, \psi\, \mathbf{curl}\, \boldsymbol{w})\, dt + \int_0^T (\mathrm{div}\, \boldsymbol{u}^\epsilon, \psi \mathrm{div}\, \boldsymbol{w})\, dt \\
&+ \int_0^T (\nabla p^\epsilon, \psi \boldsymbol{w})\, dt + \int_0^T \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \psi \boldsymbol{w})\, dt = \int_0^T \langle \boldsymbol{f}, \psi \boldsymbol{w} \rangle_\Omega\, dt + (\boldsymbol{u}_0, \boldsymbol{w})\psi(0), \\
&- \epsilon \int_0^T (p^\epsilon, \psi' v) + \int_0^T (\mathrm{div}\, \boldsymbol{u}^\epsilon, \psi v)\, dt = \epsilon (p_0, v)\psi(0),
\end{aligned}$$

that is, in the distribution sense, the problem (4.3.5) without the initial conditions. To prove that the solutions $\boldsymbol{u}^\epsilon$ and p^ϵ satisfy the initial conditions $\boldsymbol{u}^\epsilon(0)$ and $p^\epsilon(0)$ respectively, we use the same arguments in [96]. So, we take a test function $\psi \in C^\infty(0,T)$ with $\psi(T) = 0$, we can prove that

$$\begin{cases} (u_0 - \boldsymbol{u}^\epsilon(0), \boldsymbol{w})\psi(0) = 0 \\ (p_0 - p^\epsilon(0), v)\psi(0) = 0 \end{cases}$$

Next, we take ψ such as $\psi(0) = 1$, so we have $\boldsymbol{u}_0 = \boldsymbol{u}^\epsilon(0)$ and $p^\epsilon(0) = p_0$. □

4.3.2 Convergence of solutions of the perturbed system to the initial incompressible Navier-Stokes system

In this subsection, we prove that the solutions of the compressible Navier-Stokes equations (4.3.3) converge to the solutions of the incompressible Navier-Stokes equations $(\mathcal{NS})$. So, we will consider the limit $\epsilon \to 0$. It is useful to establish some a priori estimates for $\boldsymbol{u}^\epsilon$ and p^ϵ, independent of ϵ. Since we are interested in $\epsilon \to 0$, we can always suppose that $\epsilon \leq 1$.

By the lower semi-continuity of the norm for the weak topology, estimates (4.3.11), (4.3.13) and (4.3.16), we have the following estimates which are independent of ϵ :

$$\begin{cases}
\left\| \boldsymbol{u}^\epsilon \right\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leqslant \liminf\limits_{m\to\infty} \left\| \boldsymbol{u}^\epsilon_m \right\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \leq \sqrt{C_2} \\
\left\| \boldsymbol{u}^\epsilon \right\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}^1_T(\Omega))} \leqslant \liminf\limits_{m\to\infty} \left\| \boldsymbol{u}^\epsilon_m \right\|_{L^2(0,T;\boldsymbol{H}^1_T(\Omega))} \leq \sqrt{C_2} \\
\sqrt{\epsilon} \left\| p^\epsilon \right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \liminf\limits_{m\to\infty} \left\| p^\epsilon_m \right\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{C_2} \\
\displaystyle\int_{\mathbb{R}} |\tau|^{2\theta} \left\| \widehat{\boldsymbol{u}^\epsilon}(\tau) \right\|^2 d\tau \leqslant \liminf\limits_{m\to\infty} \int_{\mathbb{R}} |\tau|^{2\theta} \left\| \widehat{\boldsymbol{u}}^\epsilon_m(\tau) \right\|^2 d\tau < C
\end{cases} \tag{4.3.25}$$

The convergence result is given by the following theorem. The proof is similar to that in [96, Theorem 8.4] and [68, Theorem 5.4] (see also [104, Theorem 4.1]).

Theorem 4.3.3. *Let ϵ arbitrary in $(0,1]$. Let us $\boldsymbol{f}$, $\boldsymbol{u}_0$ and p_0 satisfying (4.2.7) and (4.3.4). Then, there exists a sequence $(\epsilon_n)_n \in (0,1]$ such as $\epsilon_n \to 0^+$ as $n \to \infty$, and the solutions $\{\boldsymbol{u}^{\epsilon_n}, p^{\epsilon_n}\}$ are* *convergent. Let $(\boldsymbol{u}^*, p^*)$ be its limit. Then $(\boldsymbol{u}^*, p^*)$ is a solution of the incompressible Navier-Stokes equations $(\mathcal{NS})$. Moreover, we have*

$$\begin{aligned}
&\boldsymbol{u}^{\epsilon_n} \rightharpoonup \boldsymbol{u}^* \quad in \quad L^2(0,T;\boldsymbol{H}^1_T(\Omega))\, weakly\\
&\boldsymbol{u}^{\epsilon_n} \rightharpoonup \boldsymbol{u}^* \quad in \quad L^\infty(0,T;\boldsymbol{L}^2(\Omega))\, weak\ star\\
&\boldsymbol{u}^{\epsilon_n} \to \boldsymbol{u}^* \quad in \quad L^2(0,T;\boldsymbol{L}^2(\Omega))\ strongly\\
&\nabla p^{\epsilon_n} \rightharpoonup \nabla p^* \quad in \quad \boldsymbol{H}^{-1}(Q)\, weakly
\end{aligned} \tag{4.3.26}$$

Proof. Since $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ satisfies the *a priori* estimates in (4.3.25) and these estimates are independent of ε, we deduce the existence of a sequence $(\varepsilon_n)_n \in]0,1]$ ($\epsilon_n \to 0^+$ as $n \to \infty$) and some $\boldsymbol{u}^*$ and p^* such that we have

$$\begin{aligned}
&\boldsymbol{u}^{\epsilon_n} \rightharpoonup \boldsymbol{u}^*\, in\, L^2(0,T;\boldsymbol{H}^1_T(\Omega))\, weak\\
&\boldsymbol{u}^{\epsilon_n} \rightharpoonup \boldsymbol{u}^*\, in\, L^\infty(0,T;\boldsymbol{L}^2(\Omega))\, weak\ star\\
&\sqrt{\epsilon_n} p^{\epsilon_n} \rightharpoonup p^*\, in\, L^\infty(0,T;L^2(\Omega))\, weak\ star
\end{aligned} \tag{4.3.27}$$

Moreover, due to the last bound in (4.3.25) and the compactness theorem 4.3.1, we deduce that

$$\boldsymbol{u}^{\epsilon_n} \to \boldsymbol{u}^*\, in\, L^2(0,T;\boldsymbol{L}^2(\Omega))\ strong \tag{4.3.28}$$

Observe that

$$\sqrt{\epsilon_n}\frac{\mathrm{d}}{\mathrm{d}t}(p^{\epsilon_n}, v) \to \frac{\mathrm{d}}{\mathrm{d}t}(p^*, v),$$

in the distribution sense and then

$$\epsilon_n \frac{\mathrm{d}}{\mathrm{d}t}(p^{\epsilon_n}, v) = \sqrt{\epsilon_n}\sqrt{\epsilon_n}\frac{\mathrm{d}}{\mathrm{d}t}(p^{\epsilon_n}, v) \to 0.$$

Passing to the limit in the second equation of (4.3.5) for the sequence ε_n, we deduce that

$$(\operatorname{div} \boldsymbol{u}^*, q) = 0, \qquad \forall q \in L^2(\Omega),$$

which implies that $\operatorname{div} \boldsymbol{u}^* = 0$.

Next, we prove that $\boldsymbol{u}^*$ verifies the variational formulation (4.2.8). Taking $\boldsymbol{w} \in \boldsymbol{V}_T(\Omega)$, multiplying (4.2.8) with $\psi \in C_0^\infty(0,T)$ and integrating from 0 to T, we obtain :

$$\begin{aligned}
&-\int_0^T (\boldsymbol{u}^{\epsilon_n}(t), \boldsymbol{w})\psi'(t)\, dt + \int_0^T (\mathbf{curl}\, \boldsymbol{u}^{\epsilon_n}(t), \psi(t)\, \mathbf{curl}\, \boldsymbol{w})\, dt + \int_0^T (\operatorname{div} \boldsymbol{u}^{\epsilon_n}(t), \psi(t)\operatorname{div} \boldsymbol{w})\, dt\\
&+ \int_0^T \tilde{b}(\boldsymbol{u}^{\epsilon_n}(t), \boldsymbol{u}^{\epsilon_n}, \psi(t)\boldsymbol{w})\, dt = \int_0^T \langle \boldsymbol{f}, \psi(t)\boldsymbol{w}\rangle_\Omega\, dt, \qquad \forall \boldsymbol{w} \in \boldsymbol{V}_T(\Omega).
\end{aligned} \tag{4.3.29}$$

Using the weak convergence (4.3.27) and the strong convergence (4.3.28), we can pass to the limit in(4.3.29) and we obtain

$$\begin{aligned}
&-\int_0^T (\boldsymbol{u}^*(t), \boldsymbol{w})\psi'(t)\, dt + \int_0^T (\mathbf{curl}\, \boldsymbol{u}^*(t), \psi(t)\, \mathbf{curl}\, \boldsymbol{w})\, dt + \int_0^T b(\boldsymbol{u}^*(t), \boldsymbol{u}^*(t), \psi(t)\boldsymbol{w})\, dt\\
&= \int_0^T \langle \boldsymbol{f}, \psi(t)\boldsymbol{w}\rangle_\Omega\, dt, \qquad \forall \boldsymbol{w} \in \boldsymbol{V}_T(\Omega).
\end{aligned} \tag{4.3.30}$$

Now, writing in particular, (4.3.30) with $\psi = \phi \in \mathcal{D}((0,T))$, we see that $\boldsymbol{u}^*$ satisfies (4.2.8) in the distribution sense. So, with the corresponding pressure p^*, we have $(\boldsymbol{u}^*, p^*)$ is solution of $(\mathcal{NS})$.

The first three convergences in (4.3.26) follow easily from (4.3.27) and (4.3.28). It remains to prove the last convergence result in (4.3.26). Since

$$\nabla p^{\varepsilon_n} = \boldsymbol{f} - \boldsymbol{u}_t^{\varepsilon_n} + \Delta \boldsymbol{u}^{\varepsilon_n} + \tilde{B}(\boldsymbol{u}^{\varepsilon_n}, \boldsymbol{u}^{\varepsilon_n}). \tag{4.3.31}$$

The convergence results for $\boldsymbol{u}^\varepsilon$ shows that the right-hand side of (4.3.31) converges to

$$\boldsymbol{f} - \boldsymbol{u}_t^* + \Delta \boldsymbol{u}^* + \tilde{B}(\boldsymbol{u}^*, \boldsymbol{u}^*) \quad \text{weakly}, \tag{4.3.32}$$

in $\boldsymbol{H}^{-1}(Q)$ as $\varepsilon_n \to 0$. Since (4.3.32) is exactly ∇p^*, we deduce that

$$\nabla p^{\varepsilon_n} \to \nabla p^* \quad \text{in} \quad \boldsymbol{H}^{-1}(Q), \tag{4.3.33}$$

which completes the proof. □

Remark 4.3.1. *For the 2-dimensional case, the solution $\boldsymbol{u}$ of* (4.2.8) *is unique and the whole sequence $\boldsymbol{u}^\varepsilon$, converges to $\boldsymbol{u}$ in the sense* (4.3.27)-(4.3.28). *Moreover, the convergence* (4.3.33) *to the associated pressure p holds for the whole sequence ∇p^ε*

4.3.3 Error estimate

This subsection is devoted to the error analysis with respect to the perturbation parameter ε. For this, we need to suppose the following regularity result.

Proposition 4.3.2. *For $\boldsymbol{u}_0$ and $\boldsymbol{f}$ given satisfying*

$$\boldsymbol{u}_0 \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_T^1(\Omega) \quad \text{and} \quad \boldsymbol{f} \in C([0,T]; \boldsymbol{L}^2(\Omega)), \tag{4.3.34}$$

there exists $T_1 \leq T$ such that the solutions of $(\mathcal{NS})$ satisfy :

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\| + \|p(t)\|_{H^1(\Omega)} \leqslant C, \quad \forall t \in [0, T_1]. \tag{4.3.35}$$

In addition to (4.3.34), *we assume that*

$$\boldsymbol{f}_t, \boldsymbol{f}_{tt} \in C([0,T]; \boldsymbol{L}^2(\Omega)). \tag{4.3.36}$$

Then for any $t_0 \in (0, T_1)$, the solution of $(\mathcal{NS})$ satisfies on $[t_0, T_1]$:

$$\begin{aligned} &\|\boldsymbol{u}_{tt}(t)\|^2 + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|p_t(t)\|^2_{H^1(\Omega)} \\ &+ \int_{t_0}^t \Big(\|\boldsymbol{u}_{ttt}(s)\|^2 + \|\boldsymbol{u}_{tt}(s)\|^2_{\boldsymbol{H}^2(\Omega)} + \|p_{tt}(s)\|^2_{H^1(\Omega)} \Big)\, ds \leqslant C. \end{aligned} \tag{4.3.37}$$

Following [91], we prove the next error estimate for the perturbed system (4.3.3) .

Theorem 4.3.4. *Assume* (4.3.34) *and* (4.3.36) *and $t_0 \in (0, T_1)$. Then, we have:*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2 + \int_{t_0}^t \|\boldsymbol{u}(s) - \boldsymbol{u}^\epsilon(s)\|^2\, ds + \epsilon \|p(t) - p^\epsilon(t)\|^2 \leqslant C\epsilon^2 \ , \ \forall t \in [t_0, T_1]. \tag{4.3.38}$$

Proof. We set $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^\epsilon$ and $q = p - p^\epsilon$.

We substract the approximated problem (4.3.3) from the initial problem ($\mathcal{NS}$), we obtain :

$$\boldsymbol{e}_t - \Delta \boldsymbol{e} + \nabla q + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u}^\epsilon \cdot \nabla)\boldsymbol{u}^\epsilon - \frac{1}{2}\text{div}\,(\boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon = 0 \tag{4.3.39}$$

Remark that :

$$\begin{aligned}\tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}) + \tilde{B}(\boldsymbol{e}, \boldsymbol{u}) &= (\boldsymbol{u}^\epsilon.\nabla)\boldsymbol{e} - \frac{1}{2}\text{div}\,(\boldsymbol{u}^\epsilon)\boldsymbol{e} + (\boldsymbol{e}.\nabla)\boldsymbol{u} - \frac{1}{2}\text{div}\,(\boldsymbol{e})\boldsymbol{u} \\ &= (\boldsymbol{u}.\nabla)\boldsymbol{u} - (\boldsymbol{u}^\epsilon.\nabla)\boldsymbol{u}^\epsilon - \frac{1}{2}\text{div}\,(\boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon\end{aligned}$$

We finally obtain the next system :

$$\begin{cases} \boldsymbol{e}_t - \Delta \boldsymbol{e} + \nabla q + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}) + \tilde{B}(\boldsymbol{e}, \boldsymbol{u}) = 0 \\ \text{div}\,\boldsymbol{e} + \epsilon q_t = \epsilon p_t \\ \boldsymbol{e}(0) = q(0) = 0 \end{cases} \tag{4.3.40}$$

We take the inner product of $(4.3.40)_1$ with $\boldsymbol{e}$ and $(4.3.40)_2$ with q, using (4.2.1) and the fact that

$$(\boldsymbol{e}, \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e})) = \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{e}, \boldsymbol{e}) = 0,$$

we have:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|^2 + \|\text{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2 + \frac{\epsilon}{2}\frac{d}{dt}\|q\|^2 \leqslant \epsilon(p_t, q) + |\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{e})| \tag{4.3.41}$$

Observe that $\tilde{b}$ is an antisymmetric form and then

$$\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{e}) = -\tilde{b}(\boldsymbol{e}, \boldsymbol{e}, \boldsymbol{u})$$

So, applying Hölder's inequality and Agmon's inequality (4.2.4) together with (4.3.35), we have

$$\begin{aligned}\left|\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{e})\right| &\leq \|\boldsymbol{e}\|\,\|\nabla \boldsymbol{e}\|\,\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)} \leq C\,\|\boldsymbol{e}\|\,\|\nabla \boldsymbol{e}\|\,\|\boldsymbol{u}\|_2 \leq C\,\|\boldsymbol{e}\|\,\|\nabla \boldsymbol{e}\| \\ &\leq \frac{1}{4}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\text{div}\,\boldsymbol{e}\|^2) + C\,\|\boldsymbol{e}\|^2\end{aligned}$$

Then, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{e}\|^2 + \|\text{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2 + \frac{\epsilon}{2}\frac{d}{dt}\|q\|^2 \\ &\leqslant \epsilon(p_t, q) + \frac{1}{4}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\text{div}\,\boldsymbol{e}\|^2) + C\,\|\boldsymbol{e}\|^2\end{aligned} \tag{4.3.42}$$

In order to bound the term $\epsilon(p_t, q)$, we recall the following result concerning the stability of the divergence operator (cf. [46]).

Due to (4.3.37), $p_t(t) \in \mathcal{C}([t_0, T_1]; \boldsymbol{L}^2(\Omega))$, then there exists $\boldsymbol{\Phi}(t) \in \boldsymbol{H}_0^1(\Omega)$ such that:

$$\text{div}\,\boldsymbol{\Phi}(t) = p_t(t), \qquad \|\boldsymbol{\Phi}(t)\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\,\|p_t(t)\|\,, \quad \forall t \in [t_0, T]. \tag{4.3.43}$$

Furthermore, have

$$\text{div}\,\boldsymbol{\Phi}_t(t) = p_{tt}(t), \qquad \|\boldsymbol{\Phi}_t(t)\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\,\|p_{tt}(t)\|\,, \quad \forall t \in [t_0, T]. \tag{4.3.44}$$

Applying this result and using the first equation in (4.3.40), we obtain:

$$\begin{aligned}
&\epsilon(p_t, q) = \epsilon(\operatorname{div}\boldsymbol{\Phi}, q) = -\epsilon(\boldsymbol{\Phi}, \nabla q) \\
&= \epsilon(\boldsymbol{\Phi}, \boldsymbol{e}_t) + \epsilon(\mathbf{curl}\,\boldsymbol{\Phi}, \mathbf{curl}\,\boldsymbol{e}) + \epsilon(\operatorname{div}\boldsymbol{\Phi}, \operatorname{div}\boldsymbol{e}) + \epsilon\tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{e}, \boldsymbol{\Phi}) + \epsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\Phi}) \\
&= \varepsilon\frac{\mathrm{d}}{\mathrm{dt}}(\boldsymbol{\Phi}, \boldsymbol{e}) - \epsilon(\boldsymbol{\Phi}_t, \boldsymbol{e}) + \epsilon(\mathbf{curl}\,\boldsymbol{\Phi}, \mathbf{curl}\,\boldsymbol{e}) + \epsilon(\operatorname{div}\boldsymbol{\Phi}, \operatorname{div}\boldsymbol{e}) + \epsilon\tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{e}, \boldsymbol{\Phi}) + \epsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\Phi}) \\
&\leqslant \epsilon\frac{\mathrm{d}}{\mathrm{dt}}(\boldsymbol{\Phi}, \boldsymbol{e}) + \frac{1}{4}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2) + C\epsilon^2(\|\boldsymbol{\Phi}_t\|^2 + \|\boldsymbol{\Phi}\|^2_{\boldsymbol{H}^1(\Omega)}) + \varepsilon\tilde{b}(\boldsymbol{u}^\varepsilon, \boldsymbol{e}, \boldsymbol{\Phi}) + \varepsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\Phi})
\end{aligned}$$

Using (4.2.1), (4.3.35), (4.3.37) and (4.3.43), we have :

$$\begin{aligned}
\epsilon\tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{e}, \boldsymbol{\Phi}) &= \epsilon\tilde{b}(\boldsymbol{u} - \boldsymbol{e}, \boldsymbol{e}, \boldsymbol{\Phi}) \\
&\leqslant C\epsilon\,\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\,\|\nabla\boldsymbol{e}\|\,\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^6(\Omega)} + C\varepsilon\,\|\boldsymbol{e}\|_{\boldsymbol{L}^6(\Omega)}\,\|\nabla\boldsymbol{e}\|\,\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^3(\Omega)} \\
&\leq C\varepsilon\,\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\,\|\nabla\boldsymbol{e}\| + C\varepsilon(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2) \\
&\leqslant \frac{1}{8}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2) + C\epsilon^2\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} + C\varepsilon(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2) \\
&\leq C\varepsilon^2 + (C\epsilon + \frac{1}{8})(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\epsilon\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{\Phi}) &\leq C\epsilon\,\|\nabla\boldsymbol{e}\|\,\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\,\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^6(\Omega)} \\
&\leqslant \frac{1}{8}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2) + C\epsilon^2\,\|\boldsymbol{\Phi}\|^2_{\boldsymbol{H}^1(\Omega)}\,\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} \\
&\leqslant \frac{1}{8}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\operatorname{div}\boldsymbol{e}\|^2) + C\epsilon^2
\end{aligned}$$

Collecting the previous inequalities in (4.3.42) and integrating from t_0 to $t \in [t_0, T_1]$, we derive for ϵ sufficiently small :

$$\begin{aligned}
&\|\boldsymbol{e}(t)\|^2 + \int_{t_0}^t (\|\mathbf{curl}\,\boldsymbol{e}(s)\|^2 + \|\operatorname{div}\boldsymbol{e}(s)\|^2)\mathrm{d}s + \epsilon\,\|q(t)\|^2 \\
&\leqslant C\epsilon(\boldsymbol{\Phi}, \boldsymbol{e}(t)) + C\epsilon^2 \\
&\leqslant \frac{1}{2}\,\|\boldsymbol{e}(t)\|^2 + C\epsilon^2\,\|\boldsymbol{\Phi}(t)\|^2 + C\epsilon^2
\end{aligned}$$

Using (4.3.43) and the regularity of the pressure in (4.3.37), we conclude :

$$\|\boldsymbol{e}(t)\|^2 + \int_{t_0}^t (\|\mathbf{curl}\,\boldsymbol{e}(s)\|^2 + \|\operatorname{div}\boldsymbol{e}(s)\|^2)\mathrm{d}s + \epsilon\,\|q(t)\|^2 \leqslant C\epsilon^2$$

□

4.4 Approximation of the system by a pseudocompressibility method

We want to study a new pseudocompressibility method which is very similar to the projection method. We approximate the solution $(\boldsymbol{u}, p)$ by $(\boldsymbol{u}^\epsilon, p^\epsilon)$ satisfying the following perturbed system :

$$(\mathcal{NS})_\epsilon \quad \begin{cases} \boldsymbol{u}_t^\epsilon - \nu\Delta \boldsymbol{u}^\epsilon + \nabla p^\epsilon + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon) = \boldsymbol{f} \qquad \text{in } \Omega\times]0,T[, \\ \text{div } \boldsymbol{u}^\epsilon - \epsilon \Delta p_t^\epsilon = 0 \qquad \text{in } \Omega\times]0,T[, \\ \boldsymbol{u}^\epsilon \cdot \boldsymbol{n} = 0, \quad \mathbf{curl}\ \boldsymbol{u}^\epsilon \times \boldsymbol{n} = \mathbf{0}, \quad \dfrac{\partial p_t^\epsilon}{\partial \boldsymbol{n}} = 0 \qquad \text{on } \Gamma\times]0,T[, \\ p^\epsilon(0) = p_0^\varepsilon, \quad \boldsymbol{u}^\epsilon(0) = \boldsymbol{u}_0^\varepsilon \quad \text{in } \Omega. \end{cases}$$

4.4.1 A priori estimates

We want to prove some regularity results for the system $(\mathcal{NS})_\varepsilon$. We use similar arguments to [23, 91]. Let us mention that the proof of the next result is obtained by using the Faedo-Galerkin method. We will not introduce the sequence of solutions $\boldsymbol{u}_m^\varepsilon$, p_m^ε, we will just show how to get *a priori* bounds. We will also omit the passage to the limit using the *a priori* estimates.

Theorem 4.4.1. *We assume that*

$$\boldsymbol{u}_0^\epsilon \in \boldsymbol{H}^2(\Omega),\ p_0^\epsilon \in H^1(\Omega)/\mathbb{R},\ \boldsymbol{f} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \text{ and } \boldsymbol{f}_t \in L^2(0,T;[\boldsymbol{H}_0(\text{div}\,,\,\Omega)]'). \tag{4.4.1}$$

Thus, there exists $C_0 = C_0(\boldsymbol{f}, \Omega, \boldsymbol{u}_0^\varepsilon, p_0^\varepsilon)$ *such that for* $T_0 = \min\{T, \varepsilon^2/C_0\}$, *the system* $(\mathcal{NS})_\epsilon$ *has a unique solution* $(\boldsymbol{u}^\epsilon, p^\epsilon)$ *satisfying :*

$$\boldsymbol{u}^\epsilon \in L^\infty(0,T_0;\boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_T^1(\Omega)), \tag{4.4.2}$$

$$\boldsymbol{u}_t^\epsilon \in L^\infty(0,T_0;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T_0;\boldsymbol{H}_T^1(\Omega)), \tag{4.4.3}$$

$$p^\epsilon\,,\ p_t^\epsilon \in L^\infty(0,T_0;H^1(\Omega)/\mathbb{R}). \tag{4.4.4}$$

Proof. To prove this theorem, we establish the necessary *a priori* estimates:

First estimates. We take the inner product of the first equation in $(\mathcal{NS})_\epsilon$ with $\boldsymbol{u}^\epsilon$ and of the second with p^ϵ, we sum the both equations and, we obtain by using (4.3.7), (4.2.1) and Cauchy-Schwarz inequality:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\epsilon\|^2 + \nu\big(\|\mathbf{curl}\,\boldsymbol{u}^\epsilon\|^2 + \|\text{div}\,\boldsymbol{u}^\epsilon\|^2\big) + \frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p^\epsilon\|^2 = \langle \boldsymbol{f}, \boldsymbol{u}^\varepsilon\rangle_\Omega \\ &\leqslant \|\boldsymbol{f}\|_{[\boldsymbol{H}_0(\text{div}\,,\Omega)]'}\|\boldsymbol{u}^\epsilon\|_{\boldsymbol{H}_0(\text{div}\,,\Omega)} \\ &\leqslant \frac{C}{\nu}\|\boldsymbol{f}\|^2_{[\boldsymbol{H}_0(\text{div}\,,\Omega)]'} + \frac{\nu}{2}\big(\|\text{div}\,\boldsymbol{u}^\epsilon\|^2 + \|\mathbf{curl}\,\boldsymbol{u}^\epsilon\|^2\big). \end{aligned}$$

Integrating the above equation from 0 to t for $t \leq T$, we obtain

$$\begin{aligned} &\|\boldsymbol{u}^\epsilon(t)\|^2 + \epsilon\|\nabla p^\epsilon(t)\|^2 + \nu\int_0^t \big(\|\mathbf{curl}\,\boldsymbol{u}^\epsilon(s)\|^2 + \|\text{div}\,\boldsymbol{u}^\epsilon(s)\|^2\big)ds \\ &\leqslant \frac{C}{\nu}\int_0^t \|\boldsymbol{f}(s)\|^2_{[\boldsymbol{H}_0(\text{div}\,,\Omega)]'}\,ds + \|\boldsymbol{u}^\epsilon(0)\|^2 + \varepsilon\|\nabla p^\epsilon(0)\|^2 \\ &\leqslant \frac{C}{\nu}\int_0^T \|\boldsymbol{f}(s)\|^2_{[\boldsymbol{H}_0(\text{div}\,,\Omega)]'}\,ds + \|\boldsymbol{u}_0^\epsilon\|^2 + \|\nabla p_0^\epsilon\|^2 := d_1 \end{aligned} \tag{4.4.5}$$

We deduce that

$$\sup_{t\in[0,T]} \left(\|\boldsymbol{u}^{\varepsilon}(t)\|^2 + \varepsilon \|\nabla p^{\varepsilon}(t)\|^2 \right) \leq d_1, \tag{4.4.6}$$

and then $\boldsymbol{u}^{\epsilon} \in L^{\infty}(0,T;\boldsymbol{L}^2(\Omega))$. Moreover, we deduce from (4.2.2), (4.2.3) and (4.4.5) that

$$\int_0^T \|\boldsymbol{u}^{\varepsilon}(t)\|^2_{\boldsymbol{H}^1(\Omega)}\, dt \leq C \int_0^T \left(\|\mathbf{curl}\, \boldsymbol{u}^{\epsilon}(t)\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\epsilon}(t)\|^2 \right) ds \leqslant \frac{Cd_1}{\nu}, \tag{4.4.7}$$

which shows that $\boldsymbol{u}^{\epsilon} \in L^2(0,T;\boldsymbol{H}^1_T(\Omega))$.

Second estimates. We take the inner product of the first equation of $(\mathcal{NS})_\epsilon$ with $-\Delta \boldsymbol{u}^{\epsilon}$, we have:

$$-(\boldsymbol{u}^{\epsilon}_t, \Delta \boldsymbol{u}^{\epsilon}) + \nu \|\Delta \boldsymbol{u}^{\epsilon}\|^2 = (\nabla p^{\epsilon}, \Delta \boldsymbol{u}^{\epsilon}) + \tilde{b}(\boldsymbol{u}^{\epsilon}, \boldsymbol{u}^{\epsilon}, \Delta \boldsymbol{u}^{\epsilon}) - (\boldsymbol{f}, \Delta \boldsymbol{u}^{\epsilon}),$$

and then,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{curl}\, \boldsymbol{u}^{\epsilon}\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\epsilon}\|^2 \Big) + \nu \|\Delta \boldsymbol{u}^{\epsilon}\|^2 \\ &\leq \frac{2}{\nu}\|\nabla p^{\varepsilon}\|^2 + \frac{\nu}{8}\|\Delta \boldsymbol{u}^{\varepsilon}\| + \frac{2}{\nu}\|\boldsymbol{f}\|^2 + \frac{\nu}{8}\|\Delta \boldsymbol{u}^{\varepsilon}\| + \tilde{b}(\boldsymbol{u}^{\epsilon}, \boldsymbol{u}^{\epsilon}, \Delta \boldsymbol{u}^{\epsilon}). \end{aligned} \tag{4.4.8}$$

Using (4.2.2), (4.2.4) and Young's inequality, the term $\tilde{b}$ can be controled as follows:

$$\begin{aligned} \tilde{b}(\boldsymbol{u}^{\epsilon}, \boldsymbol{u}^{\epsilon}, \Delta \boldsymbol{u}^{\epsilon}) &\leq C \|\Delta \boldsymbol{u}^{\epsilon}\| \, \|\nabla \boldsymbol{u}^{\epsilon}\| \, \|\boldsymbol{u}^{\epsilon}\|_{\boldsymbol{L}^{\infty}(\Omega)} \leq C \|\Delta \boldsymbol{u}^{\varepsilon}\|^{3/2} \|\nabla \boldsymbol{u}^{\varepsilon}\|^{3/2} \\ &\leq \|\Delta \boldsymbol{u}^{\varepsilon}\|^{3/2} \left(\|\mathbf{curl}\, \boldsymbol{u}^{\varepsilon}\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\varepsilon}\|^2 \right)^{3/4} \\ &\leq \frac{\nu}{4}\|\Delta \boldsymbol{u}^{\varepsilon}\|^2 + \frac{C}{\nu}\left(\|\mathbf{curl}\, \boldsymbol{u}^{\varepsilon}\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\varepsilon}\|^2 \right)^3 \end{aligned} \tag{4.4.9}$$

Thanks to (4.4.6), we have from (4.4.8)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{curl}\, \boldsymbol{u}^{\epsilon}\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\epsilon}\|^2 \Big) + \nu \|\Delta \boldsymbol{u}^{\epsilon}\|^2 \leq \frac{C_1}{\varepsilon} + C_2\left(\|\mathbf{curl}\, \boldsymbol{u}^{\varepsilon}\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\varepsilon}\|^2 \right)^3. \tag{4.4.10}$$

So, we have the next differential inequality (with $C_2 \geq 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \leqslant C_2 y(t)^3, \tag{4.4.11}$$

where $y(t) = \frac{C_1}{\epsilon} + \|\mathbf{curl}\, \boldsymbol{u}^{\epsilon}\|^2 + \|\mathrm{div}\, \boldsymbol{u}^{\epsilon}\|^2$ and $y(0) = \frac{C_1}{\epsilon} + \|\mathrm{div}\, \boldsymbol{u}_0\|^2 + \|\mathbf{curl}\, \boldsymbol{u}_0\|^2$.
We solve (4.4.11) and we obtain for $t \leqslant \frac{1}{2C_2 y(0)^2}$:

$$y(t) \leqslant \frac{y(0)}{\sqrt{1 - 2C_2 y(0)^2 t}}.$$

If we take $t \leqslant \frac{3}{8C_2 y(0)^2}$, we have $\frac{1}{2} \leqslant \sqrt{1 - 2C_2 y(0)^2 t}$ and then $y(t) \leqslant 2y(0)$.

Observe that, there exists $C_0 > 0$ such that

$$t \leqslant \frac{3}{8C_2 y(0)^2} = \frac{3}{8C_2(\frac{C_1}{\epsilon} + \|\mathrm{div}\, \boldsymbol{u}_0\|^2 + \|\mathbf{curl}\, \boldsymbol{u}_0\|^2)^2} \leq \frac{\varepsilon^2}{C_0}.$$

We conclude that for $t \leq T_0 = \min\{T, \frac{\varepsilon^2}{C_0}\}$, we have

$$\|\mathrm{div}\, \boldsymbol{u}^{\epsilon}(t)\|^2 + \|\mathbf{curl}\, \boldsymbol{u}^{\epsilon}(t)\|^2 \leqslant 2(\frac{C_1}{\epsilon} + \|\mathrm{div}\, \boldsymbol{u}_0\|^2 + \|\mathbf{curl}\, \boldsymbol{u}_0\|^2) \leqslant \frac{C_3}{\epsilon} \tag{4.4.12}$$

Return to (4.4.10) using (4.4.12), (4.4.7) and the fact that

$$
\begin{aligned}
\int_0^t \|\mathbf{curl}\, \boldsymbol{u}^\epsilon(s)\|^6 + \|\mathrm{div}\, \boldsymbol{u}^\epsilon(s)\|^6 \, ds &\leqslant \int_0^t \|\mathbf{curl}\, \boldsymbol{u}^\epsilon(s)\|^4 \, \|\mathbf{curl}\, \boldsymbol{u}^\epsilon(s)\|^2 \, ds \\
&+ \int_0^t \|\mathrm{div}\, \boldsymbol{u}^\epsilon(s)\|^4 \, \|\mathrm{div}\, \boldsymbol{u}^\epsilon(s)\|^2 \, ds \\
&\leqslant \frac{C}{\epsilon^2} \int_0^t \big(\|\mathbf{curl}\, \boldsymbol{u}^\epsilon(s)\|^2 + \|\mathrm{div}\, \boldsymbol{u}^\epsilon(s)\|^2 \big) ds \\
&\leqslant \frac{C}{\epsilon^2} \frac{d_1}{\nu},
\end{aligned} \tag{4.4.13}
$$

we deduce that:

$$
\int_0^t \|\Delta \boldsymbol{u}^\epsilon(s)\|^2 \, ds \leqslant \frac{C_4}{\epsilon^2}, \qquad \forall t \in [0,\, T_0]. \tag{4.4.14}
$$

Third estimates. We take the time derivative of the perturbed system $(\mathcal{NS})_\varepsilon$, we obtain:

$$
\begin{cases}
\boldsymbol{u}^\epsilon_{tt} - \nu \Delta \boldsymbol{u}^\epsilon_t + \nabla p^\epsilon_t + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon_t) + \tilde{B}(\boldsymbol{u}^\epsilon_t, \boldsymbol{u}^\epsilon) = \boldsymbol{f}_t, & \text{in}\, \Omega \times [0,T] \\
\mathrm{div}\ \boldsymbol{u}^\epsilon_t - \epsilon \Delta p^\epsilon_{tt} = 0, & \text{in}\, \Omega \times [0,T] \\
\dfrac{\partial p^\epsilon_{tt}}{\partial \boldsymbol{n}} = 0, & \text{on}\, \Gamma \times [0,T]
\end{cases} \tag{4.4.15}
$$

with the initial conditions:

$$
\boldsymbol{u}^\varepsilon_t(0) = \boldsymbol{f}(0) + \nu \Delta \boldsymbol{u}^\varepsilon_0 - \nabla p^\varepsilon_0 - \tilde{B}(\boldsymbol{u}^\varepsilon_0,\, \boldsymbol{u}^\varepsilon_0) \tag{4.4.16}
$$

$$
\mathrm{div}\, \boldsymbol{u}^\varepsilon_0 - \varepsilon \Delta p^\varepsilon_t(0) = 0, \qquad \frac{\partial p^\varepsilon_t(0)}{\partial \boldsymbol{n}}\Big|_\Gamma = 0 \tag{4.4.17}
$$

Thanks to assumption on $\boldsymbol{f}$, $\boldsymbol{u}^\epsilon_0$ and p^ϵ_0, we have:

$$
\|\boldsymbol{u}^\epsilon_t(0)\| \leqslant C \ , \quad \|\nabla p^\epsilon_t(0)\| \leqslant \frac{C}{\epsilon} \tag{4.4.18}
$$

We take the inner product of $(4.4.15)_1$ with $\boldsymbol{u}^\epsilon_t$ and of $(4.4.15)_2$ with p^ϵ_t, we obtain by using (4.3.7):

$$
\begin{aligned}
&\frac{\mathrm{d}}{2\mathrm{d}t} \|\boldsymbol{u}^\epsilon_t\|^2 + \nu\big(\|\mathbf{curl}\, \boldsymbol{u}^\epsilon_t\|^2 + \|\mathrm{div}\, \boldsymbol{u}^\epsilon_t\|^2 \big) + \frac{\epsilon}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla p^\epsilon_t\|^2 = \langle \boldsymbol{f}_t, \boldsymbol{u}^\epsilon_t \rangle_\Omega - \tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon_t) \\
&\leqslant C \|\boldsymbol{f}\|_{[\boldsymbol{H}_0(\mathrm{div}\,,\Omega)]'} \|\boldsymbol{u}^\epsilon_t\|_{\boldsymbol{H}_0(\mathrm{div}\,,\Omega)} + |\tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon_t)| \\
&\leqslant \frac{C}{\nu} \|\boldsymbol{f}\|^2_{[\boldsymbol{H}_0(\mathrm{div}\,,\Omega)]'} + \frac{\nu}{4}\big(\|\mathrm{div}\, \boldsymbol{u}^\epsilon_t\|^2 + \|\mathbf{curl}\, \boldsymbol{u}^\epsilon_t\|^2 \big) + |\tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon_t)|
\end{aligned}
$$

Thanks to (4.2.2) and (4.2.4):

$$
\begin{aligned}
\tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon_t) &\leqslant C \|\boldsymbol{u}^\epsilon_t\| \, \|\nabla \boldsymbol{u}^\epsilon_t\| \, \|\boldsymbol{u}^\epsilon\|_{\boldsymbol{L}^\infty(\Omega)} \\
&\leqslant C \|\boldsymbol{u}^\epsilon_t\| \, \|\nabla \boldsymbol{u}^\epsilon_t\| \, \|\boldsymbol{u}^\epsilon\|^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}^\epsilon\|^{\frac{1}{2}}_{\boldsymbol{H}^2(\Omega)} \\
&\leqslant \frac{\nu}{4}(\|\mathrm{div}\, \boldsymbol{u}^\epsilon_t\|^2 + \|\mathbf{curl}\, \boldsymbol{u}^\epsilon_t\|^2) + \frac{C}{2\nu} \|\boldsymbol{u}^\epsilon\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}^\epsilon\|_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{u}^\epsilon_t\|^2
\end{aligned}
$$

So we obtain :

$$
\begin{aligned}
&\frac{\mathrm{d}}{2\mathrm{d}t} \|\boldsymbol{u}^\epsilon_t\|^2 + \nu\big(\|\mathbf{curl}\, \boldsymbol{u}^\epsilon_t\|^2 + \|\mathrm{div}\, \boldsymbol{u}^\epsilon_t\|^2 \big) + \frac{\epsilon}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla p^\epsilon_t\|^2 \\
&\leq \frac{C}{\nu} \|\boldsymbol{f}\|^2_{[\boldsymbol{H}_0(\mathrm{div}\,,\Omega)]'} + \frac{\nu}{2}\big(\|\mathrm{div}\, \boldsymbol{u}^\epsilon_t\|^2 + \|\mathbf{curl}\, \boldsymbol{u}^\epsilon_t\|^2 \big) + \frac{C}{2\nu} \|\boldsymbol{u}^\epsilon\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}^\epsilon\|_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{u}^\epsilon_t\|^2
\end{aligned} \tag{4.4.19}
$$

By using (4.4.12) and (4.4.14), we have:

$$\int_0^{T_0} \|\boldsymbol{u}^\epsilon(s)\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}^\epsilon(s)\|_{\boldsymbol{H}^2(\Omega)}\, ds \leqslant \Big(\int_0^{T_0} \|\boldsymbol{u}^\epsilon(s)\|^2_{\boldsymbol{H}^1(\Omega)}\, ds\Big)^{\frac{1}{2}} \Big(\int_0^{T_0} \|\boldsymbol{u}^\epsilon(s)\|^2_{\boldsymbol{H}^2(\Omega)}\, ds\Big)^{\frac{1}{2}}$$
$$\leqslant \frac{\sqrt{C_3}}{\sqrt{\epsilon}} \frac{\sqrt{C_4}}{\epsilon} \leq \frac{C}{\epsilon^{\frac{3}{2}}}.$$

Using the above estimate and applying the Gronwall's lemma with (4.4.18), we obtain for any $t \in [0, T_0]$:

$$\|\boldsymbol{u}_t^\epsilon(t)\|^2 + \varepsilon \|\nabla p_t^\epsilon(t)\|^2 + \nu \int_0^t \big(\|\mathbf{curl}\, \boldsymbol{u}_t^\epsilon(s)\|^2 + \|\mathrm{div}\, \boldsymbol{u}_t^\epsilon(s)\|^2 \big)\, \mathrm{d}s \leqslant \frac{C}{\epsilon} \exp\Big(\frac{C}{\epsilon^{\frac{3}{2}}}\Big) \tag{4.4.20}$$

So, we deduce from the estimate (4.4.20) that $\boldsymbol{u}_t^\epsilon \in L^\infty(0, T_0; \boldsymbol{L}^2(\Omega)) \cap L^2(0, T_0; \boldsymbol{H}^1_T(\Omega))$ and that $p_t^\epsilon \in L^\infty(0, T;\, H^1(\Omega)/\mathbb{R})$.

Finally, taking the inner product with the first equation of $(\mathcal{NS})_\varepsilon$ with $-\Delta \boldsymbol{u}^\varepsilon(t)$ and using the previous estimate, we obtain that for each $t \in (0, T_0]$,

$$\|\Delta \boldsymbol{u}^\varepsilon(t)\|^2 \leq C(\varepsilon),$$

with $C(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. This achieves the proof.

4.4.2 Error estimates

This subsection is devoted to the proof of the error estimates for the pseudocompressibility method. The proof is different from the analysis given for the artificial compressibility method. We need to derive some ε-independent *a priori* estimates. For this, we fix $t_0 > 0$ and we use $\boldsymbol{u}(t_0)$, $p(t_0)$ as initial condition at $t = t_0$. The main result of this subsection is the following result.

Theorem 4.4.2. *We assume* (4.3.34) *and* (4.3.36). *Then, for t_0 sufficiently small, there exists $T_0 \in (t_0, T_1]$ such that for any $t \in [t_0, T_0]$, we have*

$$\int_{t_0}^t \|\boldsymbol{u}(s) - \boldsymbol{u}^\epsilon(s)\|^2\, \mathrm{d}s + \epsilon^{\frac{1}{2}} \|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2 + \epsilon(\|\boldsymbol{u}(t) - \boldsymbol{u}^\epsilon(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|p(t) - p^\epsilon(t)\|^2) \leqslant C\epsilon^2.$$

4.4.3 ϵ-independent a priori estimates

We want to prove somme ϵ-independent *a priori* estimates. Let us denote $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^\epsilon$ and $p = p - p^\epsilon$. Substracting $(\mathcal{NS})$ from $(\mathcal{NS})_\epsilon$, we have the following error equation

$$\begin{aligned} &\boldsymbol{e}_t - \nu\Delta \boldsymbol{e} + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}) + \tilde{B}(\boldsymbol{e}, \boldsymbol{u}) + \nabla q = 0 \\ &\mathrm{div}\, \boldsymbol{e} - \epsilon \Delta q_t = -\epsilon \Delta p_t, \qquad \frac{\partial p_t^\epsilon}{\partial \boldsymbol{n}} = 0 \,\mathrm{on}\, \Gamma \times [t_0, T] \\ &\boldsymbol{e}(t_0) = \boldsymbol{0} \quad \text{and} \quad q(t_0) = 0 \end{aligned} \tag{4.4.21}$$

We have the following estimates for the perturbed solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$.

Proposition 4.4.1. *We assume* (4.3.34) *and* (4.3.36). *There exists $T_0 \in (t_0,\, T_1]$ such that :*

$$\|\boldsymbol{u}^\epsilon(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|p^\epsilon(t)\|^2_{H^1(\Omega)} \leqslant C, \quad \forall t \in [t_0,\, T_0], \tag{4.4.22}$$

$$\int_{t_0}^t \|\boldsymbol{u}_t^\epsilon(s)\|^2_{\boldsymbol{H}^2(\Omega)}\, ds + \|\boldsymbol{u}_t^\epsilon(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|p_t^\epsilon(t)\|^2_{H^1(\Omega)} \leqslant C, \quad \forall t \in [t_0,\, T_0], \tag{4.4.23}$$

$$\|\boldsymbol{e}_t(t)\|^2 \leqslant C\epsilon, \quad \forall t \in [t_0,\, T_0]. \tag{4.4.24}$$

Proof. We take the inner product of $(4.4.21)_1$ with $\boldsymbol{e}$ and $(4.4.21)_2$ with q, we sum and obtain, with (4.2.1) and (4.3.7):

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}\|^2 + \nu\|\mathrm{div}\,\boldsymbol{e}\|^2 + \nu\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|^2 \\
&= \epsilon(\nabla p_t, \nabla q) - \tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \boldsymbol{e}) \\
&\leqslant \frac{\epsilon}{2}\|\nabla p_t\|^2 + \frac{\epsilon}{2}\|\nabla q\|^2 + C\|\boldsymbol{u}\|_{\mathrm{L}^\infty(\Omega)}\|\boldsymbol{e}\|\|\nabla\boldsymbol{e}\| \\
&\leqslant \frac{\epsilon}{2}\|\nabla p_t\|^2 + \frac{\epsilon}{2}\|\nabla q\|^2 + C\|\boldsymbol{u}\|^{\frac{1}{2}}_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|^{\frac{1}{2}}_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{e}\|\|\nabla\boldsymbol{e}\| \\
&\leqslant \frac{\epsilon}{2}\|\nabla p_t\|^2 + \frac{\epsilon}{2}\|\nabla q\|^2 + \frac{\nu}{2}(\|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2) + C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{e}\|^2
\end{aligned}$$

Using Gronwall's lemma and (4.3.37) :

$$\|\boldsymbol{e}(t)\|^2 + \nu\int_{t_0}^{t}\big(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\mathrm{div}\,\boldsymbol{e}\|^2\big)\,d\boldsymbol{x} + \epsilon\|\nabla q(t)\|^2 \leqslant C\epsilon\int_0^t\|\nabla p_t\|^2 \leqslant C\epsilon \tag{4.4.25}$$

Now, we take the inner product of (4.4.21) with $-\Delta\boldsymbol{e}$, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2\big) + \nu\|\Delta\boldsymbol{e}\|^2 &= (\nabla q, \Delta\boldsymbol{e}) + \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{e}, \Delta\boldsymbol{e}) + \tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \Delta\boldsymbol{e}) \\
&= (\nabla q, \Delta\boldsymbol{e}) + \tilde{b}(\boldsymbol{u}, \boldsymbol{e}, \Delta\boldsymbol{e}) - \tilde{b}(\boldsymbol{e}, \boldsymbol{e}, \Delta\boldsymbol{e}) \\
&\quad + \tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \Delta\boldsymbol{e}).
\end{aligned} \tag{4.4.26}$$

We bound the nonlinear terms separately as follow:

Using Holder's and Young's inequalities and (4.2.1), we have :

$$\begin{aligned}
\tilde{b}(\boldsymbol{u}, \boldsymbol{e}, \Delta\boldsymbol{e}) &\leqslant \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{e}\|\|\Delta\boldsymbol{e}\| \\
&\leqslant \frac{\nu}{8}\|\Delta\boldsymbol{e}\|^2 + C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\mathrm{div}\,\boldsymbol{e}\|^2),
\end{aligned}$$

$$\begin{aligned}
\tilde{b}(\boldsymbol{e}, \boldsymbol{e}, \Delta\boldsymbol{e}) &\leqslant \|\boldsymbol{e}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{e}\|\|\Delta\boldsymbol{e}\| \leq C\|\Delta\boldsymbol{e}\|^{3/2}\|\nabla\boldsymbol{e}\|^{3/2} \\
&\leq \|\Delta\boldsymbol{e}\|^{3/2}\big(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\mathrm{div}\,\boldsymbol{e}\|^2\big)^{3/4} \\
&\leq \frac{\nu}{8}\|\Delta\boldsymbol{e}\|^2 + C\big(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\mathrm{div}\,\boldsymbol{e}\|^2\big)^3
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \Delta\boldsymbol{e}) &\leqslant \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{e}\|\|\Delta\boldsymbol{e}\| \\
&\leqslant \frac{\nu}{8}\|\Delta\boldsymbol{e}\|^2 + C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}(\|\mathbf{curl}\,\boldsymbol{e}\|^2 + \|\mathrm{div}\,\boldsymbol{e}\|^2).
\end{aligned}$$

Collecting the above equations in (4.4.26), we derive :

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2\big) + \nu\|\Delta\boldsymbol{e}\|^2 &\leqslant \frac{C}{\nu}\|\nabla q\|^2 + C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}(\|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2) \\
&\quad + C\big(\|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2\big)^3
\end{aligned}$$

As in (4.4.10), we can write the above relations as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(C_1 + \|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2) \leqslant C_2(C_1 + \|\mathrm{div}\,\boldsymbol{e}\|^2 + \|\mathbf{curl}\,\boldsymbol{e}\|^2)^3$$

With the same arguments in the proof of Theorem 4.4.1 (**Second estimates**), we can derive that for $t \in [t_0, T_0]$ with

$$T_0 = \min\{T, \frac{1}{4C_2C_1^2}\}, \tag{4.4.27}$$

we have for any $t \in [t_0, T_0]$:

$$\big(\|\mathrm{div}\, \boldsymbol{e}(t)\|^2 + \|\mathbf{curl}\, \boldsymbol{e}(t)\|^2 \big) + \int_{t_0}^{t} \|\Delta \boldsymbol{e}(s)\|^2 \, ds \leqslant C. \tag{4.4.28}$$

Now, we take the time derivative of the perturbed system :

$$\begin{cases} \boldsymbol{e}_{tt} - \nu\Delta \boldsymbol{e}_t + \tilde{B}(\boldsymbol{u}^\epsilon_t, \boldsymbol{e}) + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}_t) + \tilde{B}(\boldsymbol{e}_t, \boldsymbol{u}) + \tilde{B}(\boldsymbol{e}, \boldsymbol{u}_t) + \nabla q_t = 0, \\ \mathrm{div}\, \boldsymbol{e}_t - \epsilon\Delta q_{tt} = -\epsilon \Delta p_{tt}, \\ \boldsymbol{e}_t(t_0) = \boldsymbol{0}, \quad q_t(t_0) = p_t(t_0). \end{cases} \tag{4.4.29}$$

Taking the inner product of $(4.4.29)_1$ with $\boldsymbol{e}_t$ ans q_t, we derive:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\boldsymbol{e}_t\|^2 + \nu\big(\|\mathbf{curl}\, \boldsymbol{e}_t\|^2 + \|\mathrm{div}\, \boldsymbol{e}_t\|^2 \big) + \frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}} \|\nabla q_t\|^2 \\ =&\epsilon(\nabla p_{tt}, \nabla q_t) - \tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{e}, \boldsymbol{e}_t) - \tilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \boldsymbol{e}_t) - \tilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \boldsymbol{e}_t) \end{aligned} \tag{4.4.30}$$

Using Holder's and Young's inequalities, we have :

$$\begin{aligned} \tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{e}, \boldsymbol{e}_t) &= \tilde{b}(\boldsymbol{u}_t - \boldsymbol{e}_t, \boldsymbol{e}, \boldsymbol{e}_t) \\ &\leqslant \|\boldsymbol{u}_t\|_{\boldsymbol{L}^6(\Omega)} \|\boldsymbol{e}\|_{\boldsymbol{L}^3(\Omega)} \|\nabla \boldsymbol{e}_t\| + \|\boldsymbol{e}\|_{\boldsymbol{L}^\infty(\Omega)} \|\boldsymbol{e}_t\| \|\nabla \boldsymbol{e}_t\| \\ &\leqslant \|\boldsymbol{u}_t\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|_{\boldsymbol{H}^1(\Omega)} \|\nabla \boldsymbol{e}_t\| + \|\boldsymbol{e}\|^{1/2}_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|^{1/2}_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{e}_t\| \|\nabla \boldsymbol{e}_t\| \\ &\leqslant \frac{\nu}{8}(\|\mathrm{div}\, \boldsymbol{e}_t\|^2 + \|\mathbf{curl}\, \boldsymbol{e}_t\|^2) + C\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|^2_{\boldsymbol{H}^1(\Omega)} \\ &+ \frac{\nu}{8}(\|\mathrm{div}\, \boldsymbol{e}_t\|^2 + \|\mathbf{curl}\, \boldsymbol{e}_t\|^2) + C\|\boldsymbol{e}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{e}_t\|^2 \end{aligned}$$

$$\begin{aligned} \tilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \boldsymbol{e}_t) &\leqslant \|\boldsymbol{e}_t\| \|\nabla \boldsymbol{e}_t\| \|\boldsymbol{u}\|^{1/2}_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}\|^{1/2}_{\boldsymbol{H}^2(\Omega)} \\ &\leqslant \frac{\nu}{8}(\|\mathrm{div}\, \boldsymbol{e}_t\|^2 + \|\mathbf{curl}\, \boldsymbol{e}_t\|^2) + C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{e}_t\|^2, \end{aligned}$$

and similarly

$$\begin{aligned} \tilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \boldsymbol{e}_t) &\leq \|\boldsymbol{u}_t\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|_{\boldsymbol{H}^1(\Omega)} \|\nabla \boldsymbol{e}_t\| \\ &\leqslant \frac{\nu}{8}(\|\mathrm{div}\, \boldsymbol{e}_t\|^2 + \|\mathbf{curl}\, \boldsymbol{e}_t\|^2) + C\|\boldsymbol{e}\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^1(\Omega)}. \end{aligned}$$

Collecting the above equations in (4.4.30), we have :

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} \|\boldsymbol{e}_t\|^2 + \nu\big(\|\mathbf{curl}\, \boldsymbol{e}_t\|^2 + \|\mathrm{div}\, \boldsymbol{e}_t\|^2 \big) + \epsilon\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla q_t\|^2 \\ &\leqslant \epsilon\|p_{tt}\|^2_{H^1(\Omega)} + \epsilon\|q_t\|^2_{H^1(\Omega)} + C\|\boldsymbol{u}_t\|^2_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|^2_{\boldsymbol{H}^1(\Omega)} \\ &+ C\|\boldsymbol{e}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{e}_t\|^2 + C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)} \|\boldsymbol{e}_t\|^2. \end{aligned} \tag{4.4.31}$$

Integrating from t_0 to t, Gronwall's lemma, (4.3.35), (4.3.37) and the fact that $\|q(t_0)\|^2_{H^1(\Omega)} = \|p(t_0)\|^2_{H^1(\Omega)} \leqslant C$ and

$$\int_{t_0}^{T_0} \|\boldsymbol{e}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{e}\|_{\boldsymbol{H}^2(\Omega)} \leqslant \left(\int_{t_0}^{T_0} \|\boldsymbol{e}\|^2_{\boldsymbol{H}^1(\Omega)}\right)^{\frac{1}{2}} \left(\int_{t_0}^{T_0} \|\boldsymbol{e}\|^2_{\boldsymbol{H}^2(\Omega)}\right)^{\frac{1}{2}} < C, \tag{4.4.32}$$

we derive:

$$\|\boldsymbol{e}_t(t)\|^2 + \nu \int_{t_0}^{t} \left(\|\mathrm{div}\, \boldsymbol{e}_t(s)\|^2 \, ds + \|\mathbf{curl}\, \boldsymbol{e}_t(s)\|^2 \right) + \epsilon \|\nabla q_t(t)\|^2 \leqslant C\epsilon \tag{4.4.33}$$

Using estimates (4.3.37) and triangle inequality, the estimate on the pressure derivative in Proposition 4.4.1 follows from (4.4.33). Now we take the inner product of $-\Delta \boldsymbol{e}$ with $-\nu\Delta\boldsymbol{e} + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}) + \tilde{B}(\boldsymbol{e}, \boldsymbol{u}) = -\boldsymbol{e}_t - \nabla q$ to have :

$$\begin{aligned}
\nu \|\Delta \boldsymbol{e}\|^2 &= (\boldsymbol{e}_t, \Delta\boldsymbol{e}) + (\nabla q, \Delta \boldsymbol{e}) + \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{e}, \Delta\boldsymbol{e}) + \tilde{B}(\boldsymbol{e}, \boldsymbol{u}, \Delta\boldsymbol{e}) \\
&\leqslant \frac{5}{\nu}\|\boldsymbol{e}_t\|^2 + \frac{\nu}{10}\|\Delta\boldsymbol{e}\|^2 + \frac{5}{\nu}\|\nabla q\| + \frac{\nu}{5}\|\Delta\boldsymbol{e}\|^2 + \left|\tilde{b}(\boldsymbol{u}-\boldsymbol{e}, \boldsymbol{e}, \Delta\boldsymbol{e})\right| + \left|\tilde{b}(\boldsymbol{e}, \boldsymbol{u}, \Delta\boldsymbol{e})\right|
\end{aligned}$$

Using the same arguments to bound the terms on $\tilde{b}$, thanks to previous estimates (4.4.25) and (4.4.33), we obtain :

$$\|\Delta \boldsymbol{e}(t)\|^2 \leqslant C, \quad \forall t \in [t_0, T_0]. \tag{4.4.34}$$

To conclude, we proof that :

$$\|\boldsymbol{e}_t\|^2_{\boldsymbol{H}^1(\Omega)} + \int_{t_0}^{t} \|\boldsymbol{e}_t(s)\|^2_{\boldsymbol{H}^2(\Omega)} \, ds \leqslant C \ , \ \forall t \in [t_0, T] \tag{4.4.35}$$

We take the inner product of $(4.4.21)_1$ with $-\Delta\boldsymbol{e}_t$, we obtain :

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla\boldsymbol{e}_t\|^2 + \nu\|\Delta\boldsymbol{e}_t\|^2 &= \tilde{b}(\boldsymbol{u}^\epsilon_t, \boldsymbol{e}, \Delta\boldsymbol{e}_t) + \tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{e}_t, \Delta\boldsymbol{e}_t) + \tilde{b}(\boldsymbol{e}_t, \boldsymbol{u}, \Delta\boldsymbol{e}_t) \\
&\quad + \tilde{b}(\boldsymbol{e}, \boldsymbol{u}_t, \Delta\boldsymbol{e}_t) + (\nabla q_t, \Delta\boldsymbol{e}_t)
\end{aligned} \tag{4.4.36}$$

Applying the same procedure to bound the four non linear terms, and using the previous estimates, we obtain the estimate (4.4.35).

Finally, as we know estimates for $\boldsymbol{e}$ (respectively q) and $\boldsymbol{u}$ (respectively p), with triangular inequality and (4.3.35),(4.3.37), we deduce the estimates of this Proposition. □

4.4.4 Error estimates for a linearly perturbed system

In this subsection, we will perform the error analysis introduced by perturbing the linear operator. We consider the linearly perturbed problem:

$$\begin{cases}
\boldsymbol{v}^\epsilon_t - \nu\Delta\boldsymbol{v}^\epsilon + \nabla r^\epsilon = \boldsymbol{f} - B(\boldsymbol{u}, \boldsymbol{u}) \\
\mathrm{div}\, \boldsymbol{v}^\epsilon - \epsilon\Delta r^\epsilon_t = 0 \\
\boldsymbol{v}^\epsilon \cdot \boldsymbol{n} = 0, \ \ \mathbf{curl}\, \boldsymbol{v}^\epsilon \times \boldsymbol{n} = \boldsymbol{0} \, , \ \frac{\partial r^\epsilon_t}{\partial \boldsymbol{n}} = 0 \ \text{ on } \Gamma \\
\boldsymbol{v}^\epsilon(0) = \boldsymbol{u}(0) \ , \ \ r^\epsilon(0) = p(0)
\end{cases} \tag{4.4.37}$$

where $\boldsymbol{u}$ is the solution of $(\mathcal{NS})$. We denotes $\boldsymbol{\xi} = \boldsymbol{u} - \boldsymbol{v}^\epsilon$ and $\psi = p - r^\epsilon$, we substract this system from initial problem :

$$\begin{cases} \boldsymbol{\xi}_t - \nu\Delta\boldsymbol{\xi} + \nabla\psi = 0 \\ \operatorname{div}\boldsymbol{\xi} - \epsilon\Delta\psi_t = -\epsilon\Delta p_t \\ \boldsymbol{\xi}(0) = \boldsymbol{0} \quad \text{and} \quad \psi(0) = 0. \end{cases} \tag{4.4.38}$$

From Proposition 4.4.1, we can deduce in particular, for any $t \in [t_0, T_0]$:

$$\|r^\epsilon(t)\|^2_{H^1(\Omega)} + \|r^\epsilon_t(t)\|^2_{H^1(\Omega)} + \|\boldsymbol{v}^\epsilon(t)\|^2_{\boldsymbol{H}^2(\Omega)} \leqslant C \tag{4.4.39}$$

and

$$\|\boldsymbol{\xi}_t(t)\|^2 \leqslant C\epsilon. \tag{4.4.40}$$

We have the following result.

Lemma 4.4.3. *Under the same assumption of Theorem 4.4.2. We have :*

$$\int_0^t \|\boldsymbol{\xi}(s)\|^2\, ds + \epsilon^{\frac{1}{2}} \|\boldsymbol{\xi}(t)\|^2 + \epsilon(\|\boldsymbol{\xi}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\psi(t)\|^2) \leqslant C\epsilon^2, \qquad \forall t \in [t_0, T_0]. \tag{4.4.41}$$

Proof. We use a parabolic duality argument. Given $t \in [t_0, T_0]$, let $(\boldsymbol{w}, q)$ solution of the dual problem

$$\begin{cases} \boldsymbol{w}_s + \nu\Delta\boldsymbol{w} + \nabla q = \boldsymbol{\xi}(s), \quad \operatorname{div}\boldsymbol{w} = 0, \quad s \in [t_0, t], \\ \boldsymbol{w}\cdot\boldsymbol{n} = 0, \ \mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma \quad \boldsymbol{w}(t) = \boldsymbol{0}. \end{cases} \tag{4.4.42}$$

Let us first establish the following inequality :

$$\int_{t_0}^t (\|\Delta\boldsymbol{w}(s)\|^2 + \|\nabla q(s)\|^2) ds \leqslant C \int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2\, ds. \tag{4.4.43}$$

Taking the inner product of $(4.4.42)_1$ with $\Delta\boldsymbol{w} = -\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w}$, we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{curl}\,\boldsymbol{w}(s)\|^2 + \nu\|\Delta\boldsymbol{w}\|^2 \leqslant \frac{\nu}{2}\|\Delta\boldsymbol{w}\|^2 + \frac{1}{2\nu}\|\boldsymbol{\xi}\|^2$$

Integrating from t_0 to t, we derive :

$$\|\mathbf{curl}\,\boldsymbol{w}(t_0)\|^2 + \nu\int_{t_0}^t \|\Delta\boldsymbol{w}(s)\|^2\, ds \leqslant C\int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2\, ds \tag{4.4.44}$$

Now, we take the inner product of $(4.4.42)_1$ with $\boldsymbol{w}_s$ to obtain :

$$\|\boldsymbol{w}_s\|^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{curl}\,\boldsymbol{w}\|^2 \leqslant \frac{1}{2}\|\boldsymbol{\xi}\|^2 + \frac{1}{2}\|\boldsymbol{w}_s\|^2$$

Then, integrating from t_0 to t, we obtain in particular :

$$\int_{t_0}^t \|\boldsymbol{w}_s(s)\|^2\, ds \leqslant C\int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2\, ds \tag{4.4.45}$$

Taking the inner product of $(4.4.42)_1$ with ∇q and using (4.4.44) and (4.4.45) we obtain the end of (4.4.43) :

$$\int_{t_0}^t \|\nabla q\|^2 \leqslant C\int_{t_0}^t \|\boldsymbol{\xi}(s)\|^2\, ds$$

Now, we take the inner product of $(4.4.42)_1$ with $\boldsymbol{\xi}(s)$ and, by using relations on $\boldsymbol{\xi}(s)$ in (4.4.38) and the fact that $\operatorname{div}\boldsymbol{w}=0$ we obtain :

$$\begin{aligned}\|\boldsymbol{\xi}(s)\|^2 &= (\boldsymbol{w}_s,\boldsymbol{\xi}(s))+\nu(\Delta\boldsymbol{w},\boldsymbol{\xi}(s))-(q,\operatorname{div}\boldsymbol{\xi}(s))\\ &=\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{\xi},\boldsymbol{w})-(\boldsymbol{\xi}_s,\boldsymbol{w})+\nu(\Delta\boldsymbol{\xi},\boldsymbol{w})-\epsilon(\nabla q,\nabla r_s^\epsilon)\\ &=\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{\xi},\boldsymbol{w})+\underbrace{(\nabla\psi,\boldsymbol{w})}_{=0}-\epsilon(\nabla q,\nabla r_s^\epsilon)\\ &=\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{\xi},\boldsymbol{w})-\epsilon(\nabla q,\nabla r_s^\epsilon).\end{aligned}$$

We integrate from t_0 to t to have :

$$\begin{aligned}\int_{t_0}^t\|\boldsymbol{\xi}(s)\|^2\,ds &\leqslant \epsilon\int_{t_0}^t\|\nabla q(s)\|\,\|\nabla r_s^\epsilon(s)\|\,ds\\ &\leqslant \delta\int_{t_0}^t\|\nabla q(s)\|^2\,ds+\epsilon^2C(\delta)\int_{t_0}^t\|\nabla r_s^\epsilon(s)\|^2\,ds\end{aligned}$$

Thanks to (4.4.39) and (4.4.43), we can take δ sufficiently small such as :

$$\int_{t_0}^t\|\boldsymbol{\xi}(s)\|^2\,ds\leqslant C\epsilon^2\ ,\ \forall t\in[t_0,T_0]. \tag{4.4.46}$$

Now, for given $t\in[t_0,T_0]$, we consider the following dual problem:

$$\begin{cases}\boldsymbol{w}_s+\nu\Delta\boldsymbol{w}+\nabla q=\boldsymbol{\xi}_s(s), \quad \text{and} \quad \operatorname{div}\boldsymbol{w}=0, \quad s\in[t_0,t]\\ \boldsymbol{w}\cdot\boldsymbol{n}=0, \quad \mathbf{curl}\,\boldsymbol{w}\times\boldsymbol{n}=\mathbf{0}\ \text{ on } \Gamma \quad \text{and} \quad \boldsymbol{w}(t)=0.\end{cases} \tag{4.4.47}$$

As for (4.4.43) we have :

$$\int_{t_0}^t(\|\Delta\boldsymbol{w}(s)\|^2+\|\nabla q(s)\|^2)ds\leqslant C\int_{t_0}^t\|\boldsymbol{\xi}_s(s)\|^2\,ds \tag{4.4.48}$$

We take the time derivative in $(4.4.38)_2$ so :

$$\operatorname{div}\boldsymbol{\xi}_t-\epsilon\Delta\psi_{tt}=-\epsilon\Delta p_{tt}$$

Now, we take the inner product of $(4.4.38)_1$ with $\boldsymbol{\xi}_t$ and using the previous equation, we obtain:

$$\begin{aligned}\|\boldsymbol{\xi}_t\|^2+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{dt}}\big(\,\|\operatorname{div}\boldsymbol{\xi}\|^2+\|\mathbf{curl}\,\boldsymbol{\xi}\|^2\,\big) &= (\psi,\operatorname{div}\boldsymbol{\xi}_t)\\ &=\epsilon(\nabla\psi,\nabla p_{tt})-\epsilon(\nabla\psi,\nabla\psi_{tt})\\ &=\epsilon(\nabla\psi,\nabla p_{tt})-\epsilon\frac{\mathrm{d}}{\mathrm{dt}}(\nabla\psi,\nabla\psi_t)+\epsilon\,\|\nabla\psi_t\|^2\end{aligned}$$

We integrate this equation and, since $\psi(t_0)=0$, we have $\forall t\in[t_0,T_0]$

$$\begin{aligned}&\int_{t_0}^t\|\boldsymbol{\xi}_t(s)\|^2\,ds+\nu\big(\,\|\operatorname{div}\boldsymbol{\xi}(t)\|^2+\|\mathbf{curl}\,\boldsymbol{\xi}(t)\|^2\,\big)\\ &\leqslant C\epsilon\int_{t_0}^t(\|\nabla p_{tt}(s)\|^2+\|\nabla\psi(s)\|^2)ds+C\epsilon\,\|\nabla\psi_t(t)\|\,\|\nabla\psi(t)\|\\ &+C\epsilon\int_{t_0}^t\|\nabla\psi_t(s)\|^2\,ds\end{aligned}$$

Using (4.3.37),(4.2.1) and (4.4.39), we obtain in particular :

$$\nu \|\boldsymbol{\xi}(t)\|^2_{\boldsymbol{H}^1(\Omega)} \leqslant C\nu(\|\operatorname{div}\boldsymbol{\xi}(t)\|^2 + \|\mathbf{curl}\,\boldsymbol{\xi}(t)\|^2) \leqslant C\epsilon. \tag{4.4.49}$$

Now, we take the inner product of (4.4.47) with $\boldsymbol{\xi}(s)$ to obtain :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\boldsymbol{\xi}(s)\|^2 = \frac{\mathrm{d}}{\mathrm{ds}}(\boldsymbol{\xi}, \boldsymbol{w}) - \epsilon(\nabla q, \nabla r_s^\epsilon).$$

We integrate the previous equation from t_0 to t, using (4.4.43), (4.4.39) and (4.4.40), we have for all $t \in [t_0, T_0]$:

$$\begin{aligned}\|\boldsymbol{\xi}(t)\|^2 &\leqslant \epsilon \int_{t_0}^t \|\nabla q(s)\| \, \|\nabla r_s^\epsilon(s)\| \, ds \\ &\leqslant \epsilon^{\frac{1}{2}} \int_{t_0}^t \|\nabla q(s)\|^2 \, ds + \epsilon^{\frac{3}{2}} \int_{t_0}^t \|\nabla r_s^\epsilon(s)\|^2 \, ds \\ &\leqslant C\epsilon^{\frac{1}{2}} \int_{t_0}^t \|\boldsymbol{\xi}_s(s)\|^2 \, ds + C\epsilon^{\frac{3}{2}} \leqslant C\epsilon^{\frac{3}{2}}\end{aligned}$$

We can conclude, with (4.4.40) and (4.4.48), for $t \in [t_0, T_0]$:

$$\begin{aligned}\|\psi(t)\| \leqslant C \|\nabla\psi\|_{\boldsymbol{H}^{-1}(\Omega)} &\leqslant C \sup_{\boldsymbol{v}\in \boldsymbol{H}_0^1(\Omega)} \frac{(\nabla\psi(t), \boldsymbol{v})}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}} \leqslant C \sup_{\boldsymbol{v}\in \mathrm{H}_0^1(\Omega)} \frac{(\boldsymbol{\xi}_t(t), \boldsymbol{v}) + (\nabla\boldsymbol{\xi}, \nabla\boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathrm{H}^1(\Omega)}} \\ &\leqslant C(\|\boldsymbol{\xi}_t(t)\| + \|\nabla\boldsymbol{\xi}(t)\|) \leqslant C\epsilon^{\frac{1}{2}}.\end{aligned} \tag{4.4.50}$$

□

4.4.5 Error estimates for the nonlinear problem

In this subsection, we perform an error analysis associated with the nonlinear term.
Let $\boldsymbol{\eta} = \boldsymbol{v}^\epsilon - \boldsymbol{u}^\epsilon$ and $\phi = r^\epsilon - p^\epsilon$. We substract $(\mathcal{NS})_\epsilon$ from (4.4.37), then we obtain:

$$\begin{cases}\boldsymbol{\eta}_t - \nu\Delta\boldsymbol{\eta} + \nabla\phi = \tilde{B}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon) - \tilde{B}(\boldsymbol{u}, \boldsymbol{u}), \\ \operatorname{div}\boldsymbol{\eta} - \epsilon\Delta\phi_t = 0, \\ \boldsymbol{\eta}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{\eta}\times\boldsymbol{n} = \mathbf{0}, \quad \dfrac{\partial\phi_t}{\partial\boldsymbol{n}} = 0 \text{ on } \Gamma, \\ \boldsymbol{\eta}(t_0) = \mathbf{0}\ ,\ \phi(t_0) = 0.\end{cases} \tag{4.4.51}$$

Lemma 4.4.4. *Under the same assumptions of Theorem 4.4.2, we have:*

$$\|\boldsymbol{\eta}(t)\|^2 + \epsilon\|\boldsymbol{\eta}(t)\|^2_{\boldsymbol{H}^1(\Omega)} + \int_{t_0}^t (\|\operatorname{div}\boldsymbol{\eta}(s)\|^2 + \|\mathbf{curl}\boldsymbol{\eta}(s)\|^2)ds + \epsilon\|\nabla\phi(t)\|^2 \leqslant C\epsilon^2, \quad \forall t\in[t_0, T_0]$$

Proof. As $\boldsymbol{u} - \boldsymbol{u}^\epsilon = \boldsymbol{\xi} + \boldsymbol{\eta}$, we have :

$$\tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{u}^\epsilon, \boldsymbol{\eta}) - \tilde{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\eta}) = -\tilde{b}(\boldsymbol{u}^\epsilon, \boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{\eta}) - \tilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\eta})$$

We take the inner product of $(4.4.51)_1$ with $\boldsymbol{\eta}$ and $(4.4.51)_2$ with ϕ, we derive:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\boldsymbol{\eta}\|^2+\nu\big(\|\mathrm{div}\,\boldsymbol{\eta}\|^2+\|\mathbf{curl}\,\boldsymbol{\eta}\|^2\big)+\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla\phi\|^2\\
&=-\tilde{b}(\boldsymbol{u}^\epsilon,\boldsymbol{\xi},\boldsymbol{\eta})-\tilde{b}(\boldsymbol{\xi}+\boldsymbol{\eta},\boldsymbol{u},\boldsymbol{\eta})\\
&\leqslant C\|\boldsymbol{u}^\epsilon\|_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{\xi}\|\,\|\boldsymbol{\nabla\eta}\|+C(\|\boldsymbol{\xi}\|+\|\boldsymbol{\eta}\|)\,\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{\nabla\eta}\|\\
&\leqslant\frac{\nu}{4}(\|\mathbf{curl}\,\boldsymbol{\eta}\|^2+\|\mathrm{div}\,\boldsymbol{\eta}\|^2)+C\|\boldsymbol{u}^\epsilon\|^2_{\boldsymbol{H}^2(\Omega)}\|\boldsymbol{\xi}\|^2+\frac{\nu}{4}(\|\mathbf{curl}\,\boldsymbol{\eta}\|^2+\|\mathrm{div}\,\boldsymbol{\eta}\|^2)\\
&+C\|\boldsymbol{u}\|^2_{\boldsymbol{H}^2(\Omega)}(\|\boldsymbol{\xi}\|^2+\|\boldsymbol{\eta}\|^2)
\end{aligned}$$

Using Gronwall inequality, (4.3.35) and Lemma 4.4.3, we obtain:

$$\|\boldsymbol{\eta}(t)\|^2+\nu\int_{t_0}^t(\|\mathbf{curl}\,\boldsymbol{\eta}(s)\|^2+\|\mathrm{div}\,\boldsymbol{\eta}(s)\|^2)ds+\epsilon\|\nabla\phi(t)\|^2\leqslant C\epsilon^2 \tag{4.4.52}$$

To obtain estimate of $\|\boldsymbol{\eta}\|^2_{\boldsymbol{H}^1(\Omega)}\simeq\|\mathrm{div}\,\boldsymbol{\eta}\|^2+\|\mathbf{curl}\,\boldsymbol{\eta}\|^2$, we take the inner product of (4.4.51) with $-\Delta\boldsymbol{\eta}$ and we use the same arguments as previously. $\square$

To conclude the proof of Theorem 4.4.2, we take $\boldsymbol{u}-\boldsymbol{u}^\epsilon=\boldsymbol{\xi}+\boldsymbol{\eta}$ and $p-p^\epsilon=\psi+\phi$ and combine Lemma 4.4.3 and 4.4.4.

4.5 A time discrete approximation

Given $(\boldsymbol{u}^0,p^0)$ in $\boldsymbol{H}^1_T(\Omega)\times H^1(\Omega)/\mathbb{R}$, we define $(\boldsymbol{u}^{n+1},p^{n+1})$ to be the solution of the system:

$$\begin{cases}\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^n}{k}-\frac{\nu}{2}\Delta(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)+\nabla p^n+\tilde{B}(\frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2},\frac{\boldsymbol{u}^{n+1}+\boldsymbol{u}^n}{2})=\boldsymbol{f}(t_{n+\frac{1}{2}})\\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}=0,\quad \mathbf{curl}\,\boldsymbol{u}^{n+1}\times\boldsymbol{n}=0\qquad\text{on }\Gamma,\end{cases} \tag{4.5.1}$$

$$\mathrm{div}\,\boldsymbol{u}^{n+1}-\beta k\Delta(p^{n+1}-p^n)=0,\qquad \frac{\partial p^{n+1}}{\partial\boldsymbol{n}}\Big|_\Gamma=\frac{\partial p^n}{\partial\boldsymbol{n}}\Big|_\Gamma, \tag{4.5.2}$$

where β is a constant to be determined. k is the time step, $t_{n+\frac{1}{2}}=(n+\frac{1}{2})k$.

In the remainder of this document, we study the stability of (4.5.1)-(4.5.2), and we establish an error analysis. Before this, we give the following condition:

Let $t_0\in(0,T_1)$ fixed, following [92] we assume that the initial data $(\boldsymbol{u}^0,p^0)$ satisfies :

$$\begin{cases}\big\|\boldsymbol{u}^0-\boldsymbol{u}(t_0)\big\|\leq Ck^2\\ \big\|\mathrm{div}\,(\boldsymbol{u}^0-\boldsymbol{u}(t_0))\big\|+\big\|\mathbf{curl}(\boldsymbol{u}^0-\boldsymbol{u}(t_0))\big\|+\big\|\nabla(p^0-p(t_0))\big\|\leq Ck\end{cases} \tag{4.5.3}$$

The main result of this section is the following error estimates:

Theorem 4.5.1. *Assume* (4.3.34) *and* (4.3.36). *Given* $t_0\in]0,T_1[$, $\beta>\frac{1}{4}$ *and* $(\boldsymbol{u}^0,p^0)$ *satisfying* (4.5.3), *there exists a positive constant depending on the data and* t_0 *such that:*

$$\begin{aligned}
&k\sum_{n=1}^m\|\boldsymbol{u}(t_n)-\boldsymbol{u}^n\|^2+k^2(\|\mathrm{div}\,(\boldsymbol{u}(t_m)-\boldsymbol{u}^m)\|^2+\|\mathbf{curl}(\boldsymbol{u}(t_m)-\boldsymbol{u}^m)\|^2)\\
&+k^2\|p(t_m)-p^m\|^2\leqslant Ck^4,\ \forall 1\leqslant m\leqslant M=[\frac{T_1-t_0}{k}]
\end{aligned}$$

where $(\boldsymbol{u}(t_n),p(t_n))$ *and* $(\boldsymbol{u}^n,p^n)$ *are respectively solutions of* $(\mathcal{NS})$ *and* (4.5.1)-(4.5.2)

First, we recall the following discrete Gronwall's lemma that we will use in this section.

Lemma 4.5.1. *Let y^n, h^n, g^n, f^n be nonnegative sequences satisfying*

$$y^m + k\sum_{n=0}^{m} h^n \leqslant B + k\sum_{n=0}^{m}(g^n y^n + f^n), \text{ with } k\sum_{n=0}^{[\frac{T}{k}]} g^n \leqslant M, \forall 0 \leqslant m \leqslant \frac{T}{k}$$

Assume $kg^n < 1$ and let $\sigma = \max_{0\leqslant n\leqslant \frac{T}{k}}(1-kg^n)^{-1}$. Thus:

$$y^m + k\sum_{n=1}^{m} h^n \leqslant \exp(\sigma)M(B + k\sum_{n=0}^{m} f^n), \forall m \leqslant \frac{T}{k}$$

We will use also the following algebraic relations :

$$(a-b,a) = \frac{1}{2}(|a|^2 - |b|^2 + |b-a|^2) \tag{4.5.4}$$

$$(a-b,b) = \frac{1}{2}(|a|^2 - |b|^2 - |b-a|^2) \tag{4.5.5}$$

4.5.1 Stability of the scheme

Before proving Theorem 4.5.1, we study the stability for the scheme (4.5.1)-(4.5.2). The proof of the following lemma is inspired by [92].

Lemma 4.5.2. *Let $\beta \geqslant \frac{1}{4}$. Then there exists $C > 0$ such that for all m with $1 \leqslant m \leqslant \frac{T}{k} - 1$, such that:*

$$\begin{aligned}&(1-\frac{1}{4\beta})\left\|\boldsymbol{u}^{m+1}\right\|^2 + k\sum_{n=1}^{m}(\left\|\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2)\\ &\leq C(\left\|\boldsymbol{u}^1\right\|^2 + k^2\left\|\nabla p^0\right\|^2 + k^2\left\|\nabla p^1\right\|^2 + k\|\boldsymbol{f}(t_{n+\frac{1}{2}})\|^2_{\mathcal{C}([0,T];[\boldsymbol{H}_0(\operatorname{div},\Omega)]')}\end{aligned}$$

Proof. Writing (4.5.2) at step n and summing the result with (4.5.2) at step $n+1$, we derive

$$\begin{cases}\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n) - \beta k\Delta(p^{n+1}-p^{n-1}) = 0\\ \frac{\partial p^{n+1}}{\partial \boldsymbol{n}}_{|\Gamma} = \frac{\partial p^{n-1}}{\partial \boldsymbol{n}}_{|\Gamma}\end{cases} \tag{4.5.6}$$

Taking the inner product of (4.5.1) with $k(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)$,this of (4.5.6) with kp^n and summing the relations, we obtain:

$$\begin{aligned}&\left\|\boldsymbol{u}^{n+1}\right\|^2 - \left\|\boldsymbol{u}^n\right\|^2 + \frac{\nu k}{2}(\left\|\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2)\\ &+ \beta k^2(\nabla(p^{n+1}-p^{n-1}), \nabla \mathrm{p}^n) = k\langle \boldsymbol{u}^{n+1}+\boldsymbol{u}^n, \boldsymbol{f}(t_{n+\frac{1}{2}})\rangle_\Omega\end{aligned} \tag{4.5.7}$$

where we recall that $\langle\cdot,\cdot\rangle_\Omega$ defines the duality product between $\boldsymbol{H}_0(\operatorname{div},\Omega)$ and $[\boldsymbol{H}_0(\operatorname{div},\Omega)]'$. We have:

$$\begin{aligned}k\langle \boldsymbol{u}^{n+1}+\boldsymbol{u}^n, \boldsymbol{f}(t_{n+\frac{1}{2}})\rangle_\Omega &\leqslant k\left\|\boldsymbol{u}^{n+1}+\boldsymbol{u}^n\right\|_{\boldsymbol{H}_0(\operatorname{div},\Omega)}\|\boldsymbol{f}(t_{n+\frac{1}{2}})\|_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'}\\ &\leqslant k(\left\|\boldsymbol{u}^{n+1}+\boldsymbol{u}^n\right\|^2 + \left\|\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2)^{\frac{1}{2}}\|\boldsymbol{f}(t_{n+\frac{1}{2}})\|_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'}\\ &\leqslant \frac{\nu k}{4}(\left\|\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2) + Ck\|\boldsymbol{f}(t_{n+\frac{1}{2}})\|^2_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'}\end{aligned} \tag{4.5.8}$$

We derive from (4.5.4)-(4.5.5):

$$\begin{aligned}(\nabla(p^{n+1}-p^{n-1}),\nabla p^n) &= \frac{1}{2}(\left\|\nabla p^{n+1}\right\|^2 - \left\|\nabla p^{n-1}\right\|^2)\\ &+\frac{1}{2}(\left\|\nabla(p^n-p^{n-1})\right\|^2 - \left\|\nabla(p^{n+1}-p^n)\right\|^2)\end{aligned} \tag{4.5.9}$$

Using (4.5.8) and (4.5.9) and summing (4.5.7) for n from 1 to m, we obtain:

$$\begin{aligned}&\left\|\boldsymbol{u}^{m+1}\right\|^2 + \frac{\nu k}{4}\sum_{n=1}^{m}(\left\|\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2) + \frac{\nu k^2}{2}(\left\|\nabla p^{m+1}\right\|^2 + \|\nabla p^m\|^2)\\ &\leqslant \left\|\boldsymbol{u}^1\right\|^2 + Ck\|\boldsymbol{f}\|^2_{\mathcal{C}([0,T];[\boldsymbol{H}_0(\operatorname{div},\Omega)]')} + \frac{\beta k^2}{2}(\left\|\nabla p^1\right\|^2 + \left\|\nabla p^0\right\|^2) + \frac{\beta k^2}{2}\left\|\nabla(p^{m+1}-p^m)\right\|^2\end{aligned} \tag{4.5.10}$$

Using (4.5.2), we derive:

$$\beta k^2\left\|\nabla(p^{m+1}-p^m)\right\|^2 \leqslant \frac{1}{\beta}\left\|\boldsymbol{u}^{m+1}\right\|^2$$

So, by splitting the previous left-hand term:

$$\begin{aligned}\frac{\beta k^2}{2}\left\|\nabla(p^{m+1}-p^m)\right\|^2 &\leqslant \frac{1}{4\beta}\left\|\boldsymbol{u}^{m+1}\right\|^2 + \frac{\beta k^2}{4}\left\|\nabla(p^{m+1}-p^m)\right\|^2\\ &\leqslant \frac{1}{4\beta}\left\|\boldsymbol{u}^{m+1}\right\|^2 + \frac{\beta k^2}{2}(\left\|\nabla p^{m+1}\right\|^2 + \|\nabla p^m\|^2)\end{aligned}$$

Applying the previous inequality in (4.5.10):

$$\begin{aligned}&(1-\frac{1}{4\beta})\left\|\boldsymbol{u}^{m+1}\right\|^2 + \frac{\nu k}{4}\sum_{n=1}^{m}(\left\|\operatorname{div}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)\right\|^2)\\ &\leqslant \left\|\boldsymbol{u}^1\right\|^2 + \frac{\beta k^2}{2}(\left\|\nabla p^1\right\|^2 + \left\|\nabla p^0\right\|^2) + Ck\|\boldsymbol{f}(t_{n+\frac{1}{2}})\|^2_{\mathcal{C}([0,T];[\boldsymbol{H}_0(\operatorname{div},\Omega)]')}\end{aligned}$$

□

4.5.2 Preliminary a priori estimates

In the aim to prove Theorem 4.5.1, we need other k-independent estimates. We note that the result of Lemma 4.5.2 is not sufficient because it does not provide stability result for $\|\nabla p^n\|$. In this subsection, we will use the following notations:

- $t_\alpha = t_0 + \alpha k$
- $\tilde{w}(t_{n+\frac{1}{2}}) = \frac{1}{2}(w(t_{n+1}) + w(t_n))$ for any function $w(t)$
- $\tilde{a}^{n+\frac{1}{2}} = \frac{1}{2}(a^{n+1} + a^n)$ for any sequences $\{a^n\}$
- $M = [\frac{T_1-t_0}{k}]$ is the integer part of $\frac{T_1-t_0}{k}$.

We have the following first *a priori* estimates on $(\boldsymbol{u}^n, p^n)$:

Lemma 4.5.3. *Assuming* (4.3.34) *and* (4.3.36). *For* $\beta > \frac{1}{4}$ *and initial data satisfying* (4.5.3), *we have:*

$$\left\|\operatorname{div}\boldsymbol{u}^{m+1}\right\|^2 + \left\|\mathbf{curl}\,\boldsymbol{u}^{m+1}\right\|^2 + \left\|\Delta(\boldsymbol{u}^{m+1}+\boldsymbol{u}^m)\right\|^2 + \left\|\nabla p^{m+1}\right\|^2 \leqslant C, \quad \forall 0 \leqslant m \leqslant M-1$$

Proof. We denote by $\boldsymbol{e}^n = \boldsymbol{u}(t_n) - \boldsymbol{u}^n$ and $q^n = p(t_n) - p^n$ the errors on the velocity and the pressure. The error equations are given by

$$\frac{\boldsymbol{e}^{n+1} - \boldsymbol{e}^n}{k} - \frac{\nu}{2}\Delta(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n) + \nabla q^n = \mathbf{R}^n + \mathbf{Q}^n \tag{4.5.11}$$

where $\mathbf{R}^n$ is the truncation error defined by:

$$\mathbf{R}^n = \frac{\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}(t_n)}{k} - \frac{\nu}{2}\Delta(\boldsymbol{u}(t_{n+1}) + \boldsymbol{u}(t_n)) + \tilde{B}(\tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) + \nabla p(t_n) + \boldsymbol{f}(t_{n+\frac{1}{2}})$$

and $\mathbf{Q}^n$ is the error related to the non linear terms:

$$\mathbf{Q}^n = \tilde{B}(\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}, \tilde{\boldsymbol{u}}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) = -\tilde{B}(\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}, \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}, \tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})).$$

Moreover, we have:

$$\operatorname{div} \boldsymbol{e}^{n+1} - \beta k\Delta(q^{n+1} - q^{n-1}) = -\beta k\Delta(p(t_{n+1}) - p(t_{n-1}) \tag{4.5.12}$$

and then,

$$\operatorname{div}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n) - \beta k\Delta(q^{n+1} - q^n) = -\beta k\Delta(p(t_{n+1}) - p(t_n)) \tag{4.5.13}$$

Taking the inner product of (4.5.11) with $2k\tilde{e}^{n+\frac{1}{2}}$ and (4.5.13) with kq^n, and summing them, we derive:

$$\begin{aligned} &\left\|\boldsymbol{e}^{n+1}\right\|^2 - \|\boldsymbol{e}^n\|^2 + 2\nu k(\|\operatorname{div} \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2 + \|\,\mathbf{curl}(\tilde{\boldsymbol{e}}^{n+\frac{1}{2}})\|^2) + \beta k^2(\nabla(q^{n+1} - q^{n-1}), \nabla q^n) \\ &= 2k\langle\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}, \mathbf{R}^n\rangle_\Omega - 2k\tilde{b}(\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}, \tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}) + \beta k^2(\nabla(p(t_{n+1}) - p(t_{n-1})), \nabla q^n). \end{aligned} \tag{4.5.14}$$

We derive from (4.5.4)-(4.5.5):

$$\begin{aligned} \beta k^2(\nabla(q^{n+1} - q^{n-1}), \nabla q^n) = &\frac{\beta k^2}{2}\{\left\|\nabla q^{n+1}\right\|^2 - \left\|\nabla q^{n-1}\right\|^2\} \\ &+ \frac{\beta k^2}{2}\{\left\|\nabla(q^n - q^{n-1})\right\|^2 - \left\|\nabla(q^{n+1} - q^n)\right\|^2\}. \end{aligned} \tag{4.5.15}$$

The terms in the right-hand side of (4.5.14) can be controled as follow:

$$\begin{aligned} k\langle\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}, \mathbf{R}^n\rangle_\Omega &\leqslant k\|\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|_{\boldsymbol{H}_0(\operatorname{div},\Omega)}\|\mathbf{R}^n\|_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'} \\ &\leqslant Ck(\|\operatorname{div} \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2 + \|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2)^{\frac{1}{2}}\,\|\mathbf{R}^n\|_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'} \\ &\leqslant \frac{\nu k}{2}(\|\operatorname{div} \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2 + \|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2) + Ck\,\|\mathbf{R}^n\|^2_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'} \end{aligned}$$

and

$$\beta k^2(\nabla(p(t_{n+1}) - p(t_{n-1})), \nabla q^n) \leqslant k^3\,\|\nabla q^n\|^2 + Ck\,\|\nabla(p(t_{n+1}) - p(t_{n-1}))\|^2$$

Using (4.2.1) and the regularity result (4.3.35) for $\boldsymbol{u}$, we have

$$\begin{aligned} 4k\tilde{b}(\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}, \tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}) &\leqslant Ck\|\nabla\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|\|\tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})\|_{\boldsymbol{H}^2(\Omega)}\|\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\| \\ &\leqslant \frac{\nu k}{2}(\|\operatorname{div} \tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2 + \|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2) + Ck\|\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2 \end{aligned}$$

Applying these previous estimates in (4.5.14), and summing for $n=1$ to m:

$$\begin{aligned}&\left\|\boldsymbol{e}^{m+1}\right\|^2+\nu k\sum_{n=1}^{m}(\|\operatorname{div}\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2+\|\operatorname{\mathbf{curl}}\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2)+\frac{\beta k^2}{2}(\left\|\nabla q^{m+1}\right\|^2+\|\nabla q^m\|^2)\\&\leqslant\left\|\boldsymbol{e}^1\right\|^2+\frac{\beta k^2}{2}(\left\|\nabla q^1\right\|^2+\left\|\nabla q^0\right\|^2+\left\|\nabla(q^{m+1}-q^m)\right\|^2)\\&+Ck\sum_{n=1}^{m}(\|\mathbf{R}^n\|^2_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'}+\|\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\|^2+k^2\|\nabla q^n\|^2)+Ck^3\left\|\nabla p_t(\xi_n)\right\|^2,\ \xi_n\in(t_{n-1},t_{n+1})\end{aligned}\tag{4.5.16}$$

Next, we use the following estimates

$$\|\mathbf{R}^n\|\leqslant ck(\max_{t\in[t_0,T_1]}\|\boldsymbol{u}_{tt}(t)\|+\max_{t\in[t_0,T_1]}\|\Delta\boldsymbol{u}_t(t)\|+\max_{t\in[t_0,T_1]}\|\nabla p_t(t)\|),\quad\forall 0\leqslant n\leqslant M-1\tag{4.5.17}$$

and

$$\left\|\boldsymbol{u}(t_i)-\boldsymbol{u}^i\right\|^2+k^2\left\|\nabla(p(t_i)-p^i)\right\|^2\leqslant ck^4,\ i=1,\dots,m\tag{4.5.18}$$

The proof of (4.5.17) and (4.5.18) follows exactly same steps as in [92, Lemma A2] and [92, Lemma A2]. We derive from (4.5.3), (4.3.37) togheter with (4.5.17) and (4.5.18):

$$\begin{aligned}&\left\|\boldsymbol{e}^{m+1}\right\|^2+\nu k\sum_{n=1}^{m}(\left\|\operatorname{div}\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\right\|^2+\left\|\operatorname{\mathbf{curl}}\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\right\|^2)+\frac{\beta k^2}{2}(\left\|\nabla q^{m+1}\right\|^2+\|\nabla q^m\|^2)\\&\leqslant Ck^2+Ck\sum_{n=1}^{m}(\left\|\boldsymbol{e}^{n+1}\right\|^2+k^2\|\nabla q^n\|^2)+\frac{\beta k^2}{2}\left\|\nabla(q^{m+1}-q^{m-1})\right\|^2.\end{aligned}\tag{4.5.19}$$

Now, it stays to estimate the term $\left\|\nabla(q^{m+1}-q^{m-1})\right\|^2$. Let $\delta=\beta-\frac{1}{4}>0$ and $p(t_{m+1})-p(t_m)=kp_t(\xi_m)$ with $\xi_m\in(t_m,t_{m+1})$. Taking the inner product of (4.5.12) with $q^{n+1}-q^n$ and remarking that $\frac{\beta}{2}-\frac{3\delta}{8}>0$, we obtain:

$$\begin{aligned}\beta k\left\|\nabla(q^{m+1}-q^m)\right\|^2&=(\boldsymbol{e}^{m+1},\nabla(q^{m+1}-q^m))+\beta k(\nabla(p(t_{m+1})-p(t_m)),\nabla(q^{m+1}-q^m))\\&\leqslant(\frac{\beta}{2}-\frac{3\delta}{8})k\left\|\nabla(q^{m+1}-q^m)\right\|^2+\frac{1}{4k(\frac{\beta}{2}-\frac{3\delta}{8})}\left\|\boldsymbol{e}^{m+1}\right\|^2\\&+\frac{3\delta k}{8}\left\|\nabla(q^{m+1}-q^m)\right\|^2+Ck\left\|\nabla(p(t_{m+1})-p(t_m))\right\|^2\\&\leqslant\frac{\beta k}{2}\left\|\nabla(q^{m+1}-q^m)\right\|^2+\frac{2}{(\delta+1)k}\left\|\boldsymbol{e}^{m+1}\right\|^2+Ck^3\left\|\nabla p_t(\xi_m)\right\|^2\end{aligned}$$

We derive from this result and the regularity (4.3.37):

$$\beta k^2\left\|\nabla(q^{m+1}-q^m)\right\|^2\leqslant\frac{4}{\delta+1}\left\|\boldsymbol{e}^{m+1}\right\|^2+Ck^4.$$

Thus, by splitting:

$$\begin{aligned}\frac{\beta k^2}{2}\left\|\nabla(q^{m+1}-q^m)\right\|^2&=\frac{(1+\frac{\delta}{2})\beta k^2}{4}\left\|\nabla(q^{m+1}-q^m)\right\|^2+\frac{(1-\frac{\delta}{2})\beta k^2}{4}\left\|\nabla(q^{m+1}-q^m)\right\|^2\\&\leqslant\frac{1+\frac{\delta}{2}}{1-\frac{\delta}{2}}\left\|\boldsymbol{e}^{m+1}\right\|^2+Ck^4+\frac{(1-\frac{\delta}{2})\beta k^2}{2}(\left\|\nabla q^{m+1}\right\|^2+\|\nabla q^m\|^2)\end{aligned}\tag{4.5.20}$$

Applying the previous estimate in (4.5.19), we derive:

$$\begin{aligned}&\frac{\delta}{2(1+\delta)}\left\|\boldsymbol{e}^{m+1}\right\|^2+\nu k\sum_{n=1}^{m}(\left\|\operatorname{div}\tilde{e}^{n+1}\right\|^2+\left\|\operatorname{\mathbf{curl}}\tilde{e}^{n+1}\right\|^2)+\frac{\delta\beta k^2}{4}(\left\|\nabla q^{m+1}\right\|^2+\|\nabla q^m\|^2)\\&\leqslant Ck^2+Ck\sum_{n=1}^{m}(\left\|\boldsymbol{e}^{n+1}\right\|^2+k^2\|\nabla q^n\|^2)\end{aligned}$$

We set $y^n = \left\|\boldsymbol{e}^{n+1}\right\|^2 + k^2\left\|\nabla q^n\right\|^2$ and we apply the discrete Gronwall's lemma 4.5.1, then we obtain:

$$\left\|\boldsymbol{e}^{m+1}\right\|^2 + k\sum_{n=1}^{m}(\|\operatorname{div}\tilde{e}^{n+\frac{1}{2}}\|^2 + \|\operatorname{\mathbf{curl}}\tilde{e}^{n+\frac{1}{2}}\|^2) + k^2\left\|q^{m+1}\right\|^2 \leqslant Ck^2, \quad \forall 1\leqslant m\leqslant M-1 \quad (4.5.21)$$

We deduce from this inequality and (4.3.35):

$$\|\operatorname{div}\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + \|\operatorname{\mathbf{curl}}\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + \|\nabla p^n\|^2 \leqslant C, \quad \forall 1\leqslant n\leqslant M-1. \quad (4.5.22)$$

Considering ∇p^n as a source term in (4.5.1) and taking scalar product of (4.5.1) with $k\Delta\tilde{u}^{n+\frac{1}{2}}$, we obtain:

$$\begin{aligned}
&2(\left\|\operatorname{div}\boldsymbol{u}^{n+1}\right\|^2 - \|\operatorname{div}\boldsymbol{u}^n\|^2) + 2(\left\|\operatorname{\mathbf{curl}}\boldsymbol{u}^{n+1}\right\|^2 - \|\operatorname{\mathbf{curl}}\boldsymbol{u}^n\|^2) + 4k\nu\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2\\
&= 4k(\boldsymbol{f}(t_{n+\frac{1}{2}}) - \nabla p^n, \Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}) + 4k\tilde{b}(\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}, \tilde{\boldsymbol{u}}^{n+\frac{1}{2}}, \Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}})\\
&\leqslant \nu k\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + Ck\|\boldsymbol{f}(t_{n+\frac{1}{2}}) - \nabla p^n\|^2 + Ck\|\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^{1/2}_{\boldsymbol{H}^1(\Omega)}\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^{1/2}\|\nabla\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^{1/2}\\
&\leqslant \nu k\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 + Ck\|\boldsymbol{f}(t_{n+\frac{1}{2}}) - \nabla p^n\|^2 + \frac{ck}{\nu}\|\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^6_{\boldsymbol{H}^1(\Omega)} + \nu k\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2.
\end{aligned}$$

We derive from this inequality that, with (4.5.22)

$$\left\|\operatorname{div}\boldsymbol{u}^{m+1}\right\|^2 + \left\|\operatorname{\mathbf{curl}}\boldsymbol{u}^{m+1}\right\|^2 + k\nu\sum_{n=1}^{m}\|\Delta\tilde{\boldsymbol{u}}^{n+\frac{1}{2}}\|^2 \leqslant C, \quad \forall 1\leqslant m\leqslant M-1. \quad (4.5.23)$$

Now, to achieve the proof, we need to show that $\|\Delta\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}\| \leqslant C$. For this, we take the inner product of (4.5.11) with $-\Delta\tilde{\boldsymbol{e}}^{n+\frac{1}{2}}$ and we use the same techniques used previously. □

Now, we have the following second *a priori* estimates

Lemma 4.5.4. *We consider the same assumptions as in Lemma 4.5.3. Then, we have:*

$$\left\|\boldsymbol{e}^{n+1} - \boldsymbol{e}^n\right\| + k\left\|\nabla(q^{n+1} - q^n)\right\| \leqslant Ck^2, \quad \forall 1\leqslant n\leqslant M-1$$

Proof. We introduce the following notations:

- $\boldsymbol{\epsilon}^n = \boldsymbol{e}^n - \boldsymbol{e}^{n-1}$, $\boldsymbol{w}^n = \boldsymbol{u}^n - \boldsymbol{u}^{n-1}$, $\mathrm{r}^n = q^n - q^{n-1}$
- $\boldsymbol{E}_r^n = \mathbf{R}^n - \mathbf{R}^{n-1}$, $\boldsymbol{E}_q^n = \mathbf{Q}^n - \mathbf{Q}^{n-1}$
- $E_p^n = (p(t_{n+1}) - p(t_n)) - (p(t_{n-1}) - p(t_{n-2}))$

Summing (4.5.11) and (4.5.13) at step $n-1$ and n we obtain:

$$\frac{\boldsymbol{\epsilon}^{n+1} - \boldsymbol{\epsilon}^n}{k} - \nu\Delta\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}} + \nabla\mathrm{r}^n = \mathbf{E}_r^n + \mathbf{E}_q^n \quad (4.5.24)$$

$$2\operatorname{div}\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}} - \beta k\Delta(\mathrm{r}^{n+1} - \mathrm{r}^{n-1}) = -\beta k\Delta E_p^n \quad (4.5.25)$$

Taking the inner product of (4.5.24) with $2k\tilde{\boldsymbol{\epsilon}}$ and (4.5.25) with $k\mathrm{r}^n$, and summing these relations, we obtain:

$$\begin{aligned}
&\left\|\boldsymbol{\epsilon}^{n+1}\right\|^2 - \|\boldsymbol{\epsilon}^n\|^2 + 2\nu k(\|\operatorname{div}\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}}\|^2 + \|\operatorname{\mathbf{curl}}\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}}\|^2) + \beta k^2(\nabla(\mathrm{r}^{n+1} - \mathrm{r}^{n-1}), \nabla\mathrm{r}^n)\\
&= 2k(\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}}, \boldsymbol{E}_r^n + \boldsymbol{E}_q^n) + \beta k^2(\nabla E_p^n, \nabla\mathrm{r}^n) \qquad (4.5.26)\\
&\leqslant \frac{\nu k}{4}(\|\operatorname{div}\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}}\|^2 + \|\operatorname{\mathbf{curl}}\tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}}\|^2) + Ck\left\|\boldsymbol{E}_r^n\right\|^2_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'} + 2k(\boldsymbol{E}_q^n, \tilde{\boldsymbol{\epsilon}}^{n+\frac{1}{2}}) + k\left\|\nabla E_p^n\right\|^2 + Ck^3\left\|\nabla\mathrm{r}^n\right\|^2
\end{aligned}$$

We derive from (4.5.4)-(4.5.5) that:

$$\begin{aligned}&\beta k^2(\nabla(\mathrm{r}^{n+1}-\mathrm{r}^{n-1}),\nabla \mathrm{r}^n)\\&=\frac{\beta k^2}{2}(\left\|\nabla \mathrm{r}^{n+1}\right\|^2-\left\|\nabla \mathrm{r}^{n-1}\right\|^2+\left\|\nabla(\mathrm{r}^n-\mathrm{r}^{n-1})\right\|^2-\left\|\nabla(\mathrm{r}^{n+1}-\mathrm{r}^n)\right\|^2)\end{aligned}\tag{4.5.27}$$

Then, we have:

$$\boldsymbol{E}_q^n=-\tilde{B}(\tilde{\boldsymbol{w}}^{n+\frac12},\tilde{\boldsymbol{e}}^{n+\frac12})-\tilde{B}(\tilde{\boldsymbol{u}}^{\boldsymbol{n}-\frac12},\tilde{\boldsymbol{\epsilon}}^{n+\frac12})-\tilde{B}(\tilde{\boldsymbol{\epsilon}}^{n+\frac12},\tilde{\boldsymbol{u}}(t_{n+\frac12}))-\tilde{B}(\tilde{\boldsymbol{e}}^{n-\frac12},\tilde{\boldsymbol{u}}(t_{n+\frac12})-\tilde{\boldsymbol{u}}(t_{n-\frac12})).$$

So:

$$(\tilde{\boldsymbol{\epsilon}}^{n+\frac12},\boldsymbol{E}_q^n)=-\tilde{b}(\tilde{\boldsymbol{w}}^{n+\frac12},\tilde{\boldsymbol{e}}^{n+\frac12},\tilde{\boldsymbol{\epsilon}}^{n+\frac12})-\tilde{b}(\tilde{\boldsymbol{\epsilon}}^{n+\frac12},\tilde{\boldsymbol{u}}(t_{n+\frac12}),\tilde{\boldsymbol{\epsilon}}^{n+\frac12})-\tilde{b}(\tilde{\boldsymbol{e}}^{n-\frac12},\tilde{\boldsymbol{u}}(t_{n+\frac12})-\tilde{\boldsymbol{u}}(t_{n-\frac12}),\tilde{\boldsymbol{\epsilon}}^{n+\frac12}).$$

Taking $\tilde{\boldsymbol{u}}(t_{n+\frac12})-\tilde{\boldsymbol{u}}(t_{n-\frac12})=k\boldsymbol{u}_t(\xi_n)$ with $\xi_n\in(t_{n-\frac12},t_{n+\frac12})$ and estimating as for perturbed problem error, we obtain:

$$\begin{aligned}&k\tilde{b}(\tilde{\boldsymbol{w}}^{n+\frac12},\tilde{\boldsymbol{e}}^{n+\frac12},\tilde{\boldsymbol{\epsilon}}^{n+\frac12})=k\tilde{b}((\tilde{\boldsymbol{u}}(t_{n+\frac12})-\tilde{\boldsymbol{u}}(t_{n-\frac12}))-\tilde{\boldsymbol{\epsilon}}^{n+\frac12},\tilde{\boldsymbol{e}}^{n+\frac12},\tilde{\boldsymbol{\epsilon}}^{n+\frac12})\\&\leqslant Ck^2\,\|\nabla\boldsymbol{u}_t(\xi_n)\|\,\|\nabla\tilde{\boldsymbol{e}}^{n+\frac12}\|\|\nabla\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|+Ck\|\nabla\tilde{\boldsymbol{e}}^{n+\frac12}\|\|\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^{\frac12}\|\nabla\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^{\frac32}\\&\leqslant\frac{\nu k}{4}(\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2)+Ck^3\,\|\nabla\boldsymbol{u}_t(\xi_n)\|\,(\|\mathrm{div}\,\tilde{\boldsymbol{e}}^{n+\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n+\frac12}\|^2)\\&+Ck\|\nabla\tilde{\boldsymbol{e}}^{n+\frac12}\|^4\|\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2\end{aligned}$$

We use (4.3.37) to bound $\|\nabla\boldsymbol{u}_t(\xi_n)\|$ and Lemma 4.5.3 to bound the term $\|\nabla\tilde{\boldsymbol{e}}^{n+\frac12}\|^4$, then we obtain:

$$\begin{aligned}k\tilde{b}(\tilde{\boldsymbol{w}}^{n+\frac12},\tilde{\boldsymbol{e}}^{n+\frac12},\tilde{\boldsymbol{\epsilon}}^{n+\frac12})\leqslant{}&\frac{\nu k}{4}(\left\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\right\|^2+\left\|\mathbf{curl}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\right\|^2)\\&+Ck^3(\left\|\mathrm{div}\,\tilde{\boldsymbol{e}}^{n+\frac12}\right\|^2+\left\|\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n+\frac12}\right\|^2)+Ck\left\|\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\right\|^2\end{aligned}$$

Similarly, we have:

$$\begin{aligned}k\tilde{b}(\tilde{\boldsymbol{e}}^{n-\frac12},\tilde{\boldsymbol{u}}(t_{n+\frac12})-\tilde{\boldsymbol{u}}(t_{n-\frac12}),\tilde{\boldsymbol{\epsilon}}^{n+\frac12})\leqslant{}&\frac{\nu k}{4}(\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2)\\&+Ck^3(\|\mathrm{div}\,\tilde{\boldsymbol{e}}^{n-\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n-\frac12}\|^2)\end{aligned}$$

$$k\tilde{b}(\tilde{\boldsymbol{\epsilon}}^{n+\frac12},\tilde{\boldsymbol{u}}(t_{n+\frac12}),\tilde{\boldsymbol{\epsilon}}^{n+\frac12})\leqslant\frac{\nu k}{4}(\|\mathrm{div}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2)+Ck\|\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2$$

Summing (4.5.27) for $n=2$ to m, and applying the previous inequalities, we derive:

$$\begin{aligned}&\left\|\boldsymbol{\epsilon}^{m+1}\right\|^2+\nu k\sum_{n=1}^{m}(\left\|\mathrm{div}\,\boldsymbol{\epsilon}^{n+1}\right\|^2+\left\|\mathbf{curl}\,\boldsymbol{\epsilon}^{n+1}\right\|^2)+\frac{\beta k^2}{2}(\left\|\nabla \mathrm{r}^{m+1}\right\|^2+\|\nabla \mathrm{r}^m\|^2)\\&\leqslant\left\|\boldsymbol{\epsilon}^2\right\|^2+\frac{\beta k^2}{2}(\left\|\nabla \mathrm{r}^2\right\|^2+\left\|\nabla \mathrm{r}^1\right\|^2+\left\|\nabla(\mathrm{r}^{m+1}-\mathrm{r}^m)\right\|^2)\\&+Ck\sum_{n=2}^{m}\Big(\|\tilde{\boldsymbol{\epsilon}}^{n+\frac12}\|^2+k^2(\|\mathrm{div}\,\tilde{\boldsymbol{e}}^{n+\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n+\frac12}\|^2+\|\mathrm{div}\,\tilde{\boldsymbol{e}}^{n-\frac12}\|^2+\|\,\mathbf{curl}\,\tilde{\boldsymbol{e}}^{n-\frac12}\|^2)\Big)\\&+Ck\sum_{n=2}^{m}(\|\boldsymbol{E}_r^n\|_\Omega^2+\left\|\nabla E_p^n\right\|^2)+Ck^3\sum_{n=2}^{m}(\|\nabla \mathrm{r}^n\|^2+\left\|\nabla \mathrm{r}^{n-1}\right\|^2)\end{aligned}\tag{4.5.28}$$

As in (4.5.20), we have

$$\frac{\beta k^2}{2}\left\|\nabla(\mathrm{r}^{m+1}-\mathrm{r}^m)\right\|^2 \leqslant \frac{1+\frac{\delta}{2}}{1+\delta}\left\|\boldsymbol{\epsilon}^{m+1}\right\|^2 + Ck^4\left\|\nabla p_t(\xi_n)\right\|^2 + \frac{1-\frac{\delta}{2}}{2}\beta k^2(\left\|\nabla\mathrm{r}^{m+1}\right\|^2 + \left\|\nabla\mathrm{r}^m\right\|^2)$$

According to (4.5.18), we have

$$\left\|\boldsymbol{\epsilon}^2\right\|^2 \leqslant Ck^4$$

Taking the inner product of (4.5.12) with $q^{n+1}-q^n$ for $n=0,1$, we obtain:

$$\left\|\nabla(q^{n+1}-q^n)\right\| \leqslant C(\frac{1}{k}\left\|\boldsymbol{e}^{n+1}\right\| + \left\|\nabla(p(t_{n+1})-p(t_n)\right\|) \leqslant Ck$$

Combining these estimates in (4.5.28), (4.5.21), (4.3.37) and the following estimate:

$$k\sum_{n=2}^{M-1}\left\|\nabla E_p^n\right\|^2 \leqslant Ck^4\int_{t_0}^{T_1}\left\|\nabla p_{tt}(s)\right\|^2 \mathrm{ds} \leqslant Ck^4,$$

we derive

$$\frac{\delta}{2(\delta+1)}\left\|\boldsymbol{\epsilon}^{m+1}\right\|^2 + \frac{\nu k}{4}\sum_{n=2}^{m}(\left\|\mathrm{div}\,(\boldsymbol{\epsilon}^{n+1}+\boldsymbol{\epsilon}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{\epsilon}^{n+1}+\boldsymbol{\epsilon}^n)\right\|^2) + \delta\beta k^2(\left\|\nabla\mathrm{r}^{m+1}\right\|^2 + \left\|\nabla\mathrm{r}^m\right\|^2)$$
$$\leqslant Ck^4 + Ck\sum_{n=2}^{m}(\left\|\boldsymbol{\epsilon}^{n+1}\right\|^2 + k^2\left\|\nabla\mathrm{r}^n\right\|^2)$$

Applying Gronwall's lemma 4.5.1, we obtain for $2 \leqslant m \leqslant M-1$:

$$\left\|\boldsymbol{\epsilon}^{m+1}\right\|^2 + k\sum_{n=2}^{m}(\left\|\mathrm{div}\,(\boldsymbol{\epsilon}^{n+1}+\boldsymbol{\epsilon}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{\epsilon}^{n+1}+\boldsymbol{\epsilon}^n)\right\|^2) + k^2(\left\|\nabla\mathrm{r}^{m+1}\right\|^2 + \left\|\nabla\mathrm{r}^m\right\|^2) \leqslant Ck^4.$$

□

Now, we are able to give a sketch of the proof of Theorem 4.5.1. It is the same reasoning than the proof of Theorem 4.4.2. The idea is to split the errors into a first part associated with the linear Stokes operator and a second part associated with the nonlinear term.

Proof of Theorem 4.5.1. We study the error estimates for a linear auxiliary problem, and then for a non linear problem.

Firstly, we define $(\boldsymbol{v}^{n+1}, \mathrm{r}^{n+1})$ the solution of the following auxiliary problem:

$$\begin{cases} \frac{\boldsymbol{v}^{n+1}-\boldsymbol{v}^n}{k} - \frac{\nu}{2}\Delta(\boldsymbol{v}^{n+1}+\boldsymbol{v}^n) + \nabla\mathrm{r}^n = \boldsymbol{f}(t_{n+\frac{1}{2}}) - \tilde{B}(\tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}}), \tilde{\boldsymbol{u}}(t_{n+\frac{1}{2}})) \\ \boldsymbol{v}^{n+1}\cdot\boldsymbol{n}_{|\Gamma} = 0, \quad \mathbf{curl}\,\boldsymbol{v}^{n+1}\times\boldsymbol{n}_{|\Gamma} = 0 \end{cases} \tag{4.5.29}$$

and

$$\left\{\mathrm{div}\,\boldsymbol{v}^{n+1} - \beta k\Delta(\mathrm{r}^{n+1}-\mathrm{r}^n) = 0, \quad \frac{\partial\mathrm{r}^{n+1}}{\partial\boldsymbol{n}}_{|\Gamma} = \frac{\partial\mathrm{r}^{n-1}}{\partial\boldsymbol{n}}_{|\Gamma}\right. \tag{4.5.30}$$

with $(\boldsymbol{v}^0, \mathrm{r}^0) = (\boldsymbol{u}^0, p^0)$.

We denote $\boldsymbol{\xi}^n = \boldsymbol{u}(t_n) - \boldsymbol{v}^n$ and $\phi^n = p(t_n) - \mathrm{r}^n$.

In the case of linear problem, we need to bound $\boldsymbol{u}(t_n) - \boldsymbol{v}^n$. Then, with nonlinear problem, we prove the same thing with $\boldsymbol{v}^n - \boldsymbol{u}^n$. We conclude by using triangular inequality.

Using the lemmas proved in the previous section apply for the linear problem, we deduce:

$$\left\|\operatorname{div} \boldsymbol{v}^{m+1}\right\|^2 + \left\|\mathbf{curl}\, \boldsymbol{v}^{m+1}\right\|^2 + k\sum_{n=1}^{m} \left\|\Delta(\boldsymbol{v}^{n+1} + \boldsymbol{v}^n)\right\|^2 + \left\|\nabla \mathrm{r}^{m+1}\right\|^2 \leqslant C$$

$$\frac{1}{k^2}\left\|\boldsymbol{\xi}^{m+1} - \boldsymbol{\xi}^m\right\|^2 + \left\|\nabla(\mathrm{r}^{m+1} - \mathrm{r}^m)\right\|^2 + \left\|\nabla(\phi^{m+1} - \phi^m)\right\|^2 \leqslant Ck, \quad \forall 1 \leqslant m \leqslant M-1.$$

Thanks to these results, we can prove, with the same arguments in Theorem 4.4.3 the following estimates (see also [92, Lemma 3.1] and [92, Lemma 3.2])

$$\begin{aligned} & k\sum_{n=1}^{m} \|\boldsymbol{u}(t_n) - \boldsymbol{v}^n\|^2 \leqslant Ck^4 \\ & \|\operatorname{div}(\boldsymbol{u}(t_n) - \boldsymbol{v}^n)\|^2 + \|\mathbf{curl}(\boldsymbol{u}(t_n) - \boldsymbol{v}^n)\|^2 + \|p(t_n) - \mathrm{r}^n\|^2 \leqslant Ck^2, \quad \forall 1 \leqslant m \leqslant M \end{aligned} \tag{4.5.31}$$

Concerning the error estimates for the nonlinear problem, we denote $\boldsymbol{\eta} = \boldsymbol{v}^n - \boldsymbol{u}^n$ and $\psi^n = \mathrm{r}^n - p^n$. Substracting (4.5.1)-(4.5.2) from (4.5.29)-(4.5.30), we derive:

$$\begin{cases} \frac{\boldsymbol{\eta}^{n+1} - \boldsymbol{\eta}^n}{k} - \frac{\nu}{2}\Delta(\boldsymbol{\eta}^{n+1} + \boldsymbol{\eta}^n) + \nabla\psi^n = \mathbf{Q}^n \\ \operatorname{div}(\boldsymbol{\eta}^{n+1} + \boldsymbol{\eta}^n) - \beta k \Delta(\psi^{n+1} - \psi^{n-1}) = 0 \\ \frac{\partial \psi^{n+1}}{\partial \boldsymbol{n}}_{|\Gamma} = \frac{\partial \psi^{n-1}}{\partial \boldsymbol{n}}_{|\Gamma} \end{cases} \tag{4.5.32}$$

with $\boldsymbol{\eta}^0 = 0$ and $\psi^0 = 0$. Again using the same arguments in the proof of Theorem 4.4.4 (or [92, Lemma 3.3]), we can derive the next estimate:

$$\|\boldsymbol{\eta}\|^2 + k\sum_{n=1}^{m}(\left\|\operatorname{div}(\boldsymbol{\eta}^{n+1} + \boldsymbol{\eta}^n)\right\|^2 + \left\|\mathbf{curl}(\boldsymbol{\eta}^{n+1} + \boldsymbol{\eta}^n)\right\|^2 + k^2\left\|\nabla\psi^{m+1}\right\|^2 \leqslant Ck^4, \quad \forall 1 \leqslant m \leqslant M-1$$

Then, we can achieve the proof of the theorem by denoting $\boldsymbol{u}(t_n) - \boldsymbol{u}^n = \boldsymbol{\xi}^n + \boldsymbol{\eta}^n$ and $p(t_n) - p^n = \phi^n + \psi^n$ and by combining (4.5.31) and (4.5.33).

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