BPZ equations and integrability of Gaussian multiplicative chaos in Liouville conformal field theory

Tunan Zhu

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Tunan Zhu
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Antti KUPIAINEN University of Helsinki	*Rapporteur*
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Vincent VARGAS Université Paris 13	*Directeur de thèse*

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Tunan Zhu[1]

[1]Département de mathématiques et applications, École normale supérieure, CNRS, PSL Research University, 75005 Paris, France.

Résumé

Cette thèse de doctorat porte sur l'étude de deux objets probabilistes, les mesures de chaos multiplicatif gaussien (GMC) et la théorie conforme des champs de Liouville (LCFT). Le GMC fut introduit par Kahane en 1985 et il s'agit aujourd'hui d'un objet extrêmement important en théorie des probabilités et en physique mathématique. Très récemment le GMC a été utilisé pour définir les fonctions de corrélation de la LCFT, une théorie qui est apparue pour la première fois en 1981 dans le célèbre article de Polyakov, "Quantum geometry of bosonic strings".

Grâce à ce lien établi entre GMC et LCFT, nous pouvons traduire les techniques de la théorie conforme des champs dans un langage probabiliste pour effectuer des calculs exacts sur les mesures de GMC. Nous partons des équations de BPZ pour LCFT, introduite par Belavin, Polyakov et Zamolodchikov en 1983. Le méchanisme de ces équations sera étudié vers la fin de cette thèse et nous prouvrons les équations de BPZ d'ordre supérieure avec un formalisme général.

En développant les méthodes probabilistes établies par Kupiainen-Rhodes-Vargas pour la résolution des équations de BPZ et en traitant plusieurs difficultés majeures, nous obtiendront des relations non triviales sur les objets fondamentaux du LCFT. Plus précisément, nous prouvrons les formules exactes pour toutes les constantes de structure du LCFT sur le disque avec la constante cosmologique nulle à l'intérieur du disque, dont une parmi ces quatre a été résolue par Remy en 2017. Comme cas particulier, nous trouverons la distribution de la masse totale du GMC sur l'intervalle avec des log-singularités placées en deux extrémités, qui a été conjecturé indépendamment par Ostrovsky et par Fyodorov, Le Doussal, et Rosso en 2009. Une autre conséquence directe est la loi de la masse totale du GMC sur le cercle avec log-singularité en 1, conjecturé par Ostrovsky en 2016.

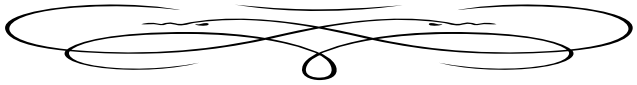
Abstract

Throughout this PhD thesis we will study two probabilistic objects, Gaussian multiplicative chaos (GMC) measures and Liouville conformal field theory (LCFT). The GMC measures were first introduced by Kahane in 1985 and have grown into an extremely important field of probability theory and mathematical physics. Very recently GMC has been used to give a probabilistic definition of the correlation functions of LCFT, a theory that first appeared in Polyakov's 1981 seminal work, "Quantum geometry of bosonic strings".

Once the connection between GMC and LCFT is established, one can hope to translate the techniques of conformal field theory in a probabilistic framework to perform exact computations on the GMC measures. We start from the BPZ equations for LCFT, introduced by Belavin, Polyakov and Zamolodchikov in 1983. The mechanism of these equations is studied in the last part of this thesis and we prove the higher order BPZ equations with a general formalism.

Following the probabilistic methods established by Kupiainen-Rhodes-Vargas for the resolution of the BPZ equations and after overcoming several major difficulties, we obtain non trivial relations for some fundamental objects of LCFT. More precisely, we prove the exact formulas for all the four structure constants of LCFT on the disk with null cosmological constant in the bulk, one of which was solved by Remy in 2017. As a special case, we find the distribution of the total mass of GMC on the interval with log-singularities put on both ends, a conjecture that has been independently predicted by Ostrovsky and by Fyodorov, Le Doussal, and Rosso in 2009. Another direct consequence is the law of the total mass of GMC on the unit circle with a log-singularity, conjectured by Ostrovsky in 2016.

Remerciements



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Contents

CHAPTER 1

An overview of Gaussian chaos and Liouville theory

In collaboration with Guillaume Remy. A great thanks to him for sharing with me his texts.

1.1 Introduction

The present chapter is destined to give an overview of the results concerning two probabilistic objects, the Gaussian multiplicative chaos (GMC) measures and Liouville conformal field theory (LCFT). We will present the definitions, the most important properties and the required information the reader needs to understand the following chapters. The theory of Gaussian multiplicative chaos was first introduced heuristically by B. Mandelbrot in 1972 in [65, 66] and then defined rigorously by J.-P. Kahane in 1985 in [56] and has ever since been extensively studied in many problems of probability theory and mathematical physics with applications including random geometry, 2d quantum gravity, statistical physics, 3d turbulence and mathematical finance. See [92] for a review.

On the other hand Liouville conformal field theory was only very recently understood as a probabilistic object. It first appeared in the physics literature in the seminal paper "Quantum geometry of bosonic strings" by A. Polyakov in 1981 [82]. Considerable efforts were then devoted by theoretical physicists to try to solve Liouville theory - in other words to compute its correlation functions - motivated by its importance in the study of non-critical string theory and of random geometry in two dimensions. A major step in this direction was taken by Belavin, Polyakov, and Zamolodchikov (BPZ) in 1984 in the paper [10] which laid down the foundations of conformal field theory (CFT).[1] Liouville field theory is indeed a CFT and using these techniques the celebrated DOZZ formula was conjectured independently by Dorn and Otto in [25] and Zamolodchikov and Zamolodchikov in [108]. This formula gives the value of the most fundamental quantity in Liouville theory: the three-point correlation function of the theory on the Riemann sphere. However, the major problem behind this study of Liouville theory was that it completely lacked mathematical rigour and the derivation of the DOZZ formula was seen more as a guess even on a physics level of rigour. The biggest problem came from the fact that it appeared unclear how to construct a consistent theory based on the so-called path integral formalism in the case of Liouville theory. Many reviews on LCFT abandon this path integral definition right from the start and replace it by purely algebraic definitions. This is the famous conformal bootstrap approach to LCFT, see [91].

[1] The original motivation of BPZ in [10] was precisely to obtain an exact value for the correlation function of LCFT although CFT has now grown into a huge field of theoretical physics with countless applications.

A decisive step to solve this problem was made by David-Kupiainen-Rhodes-Vargas in the paper [20] where a probabilistic definition is given for the Liouville field theory. The authors find a way to interpret the path integral of LCFT: using a regularization and renormalization procedure they obtain a rigorous probabilistic definition of the theory. The correlation functions of LCFT are thus expressed in terms of moments of GMC measures with log-singularities. We emphasize that this establishes a link that was unknown to two major communities of physicists, the statistical physics community working with GMC related models and the theoretical physics community working on LCFT.

Once a probabilistic content has been given to the correlation functions of LCFT, the next natural step is to try to prove all the properties one expects for a CFT. The correlation functions of LCFT are shown to behave as conformal tensors (KPZ formula) and obey the Weyl anomaly (behaviour under a change of background metric). Next we can introduce the stress-energy tensor of the theory and prove the Ward identities. Lastly the BPZ equations are established for a correlation function where at least one of the point in the correlation has a so-called degenerate weight. These differential equations are extremely important as they translate the constraints imposed by the local conformal invariance (by opposition to the KPZ formula which just takes into account global conformal invariance).

Using these BPZ equations CFT tells us that it should be possible to perform exact computations of certain correlation functions with a small number of points. In the case of Liouville theory the precise way to extract information out of the BPZ equations to obtain non-trivial relations on the correlation functions goes back to Teschner [101]. On the Riemann sphere the simplest correlation is the three-point function and the BPZ equations thus lead to relations that completely determine this function, this is the content of the celebrated DOZZ formula. Implementing all of the above in a probabilistic framework allowed Kupiainen-Rhodes-Vargas in 2017 to give a proof of the DOZZ formula [60, 61]. Very shortly after, the same procedure was implemented by Remy to prove the Fyodorov-Bouchaud formula [41] that can also be interpreted as a bulk one point function of boundary LCFT on the disk with $\mu_{bulk} = 0$, see 1.3.2 for the Liouville potential on the disk. The above mentioned works are both based on BPZ equations with a degenerate insertion in the bulk (in the sphere case there is no boundary), the general mechanism of this type of BPZ equations will be explained in chapter 4.

This thesis continues to investigate the other three building blocks [37, 52, 84] of boundary LCFT, in the case where $\mu_{bulk} = 0$. The BPZ equations in this case are also discussed in chapter 4. One of the main reasons for working in this regime is that the BPZ equations for boundary LCFT are not completely understood. Writing BPZ equations with a degenerate insertion on the boundary adds some special constraints on boundary cosmological constants around the degenerate insertion and it is still a work in progress. However, when $\mu_{bulk} = 0$, the condition simplifies to a phase change and we can obtain a stronger holomorphic version for the BPZ equations on the boundary. We illustrate in chapter 3 how these differential equations can be directly obtained without prior knowledge of LCFT. In some special cases of our building blocks, we recover several results of GMC measures: the distribution of the total mass of GMC on the interval with two insertions, and the distribution of the total mass of GMC on the circle with one insertion, the latter also known as the Fyodorov-Bouchaud formula with insertion. Let us take a small detour in the world of statistical physics. In this world GMC measures were first considered on the interval in the work of Bacry-Muzy [9] followed by the work of Fyodorov-Bouchaud [41] on a random energy model, which they studied on the circle. Fyodorov and Bouchaud conjectured the law of the total mass on the circle using a heuristic analytic continuation from integer to complex moments which we refer to as the Fyodorov-Bouchaud formula. They also conjectured the distribution of the maximum of underlying gaussian field. The Fyodorov-Bouchaud formula was extended by Ostrovsky [78] to include a single insertion point, which is the result we obtain from rigorous mathematical arguments. The case of the distribution of the total mass of the GMC measure on the interval was first considered by Ostrovsky [74] using his theory of intermittency differentiation resulting in explicit conjectures for the negative moments [75] and for the fractional moments of the total mass [76]. The problem of the total mass on the interval with two insertion points was independently considered by Fyodorov et al. [45] using the technique of analytic continuation from integer to complex moments. Fyodorov et al. conjectured the fractional moments of the total mass as well as the Laplace transform of the distribution of the maximum of the underlying log-correlated field, see also [44]. The two approaches to the problem with two insertion points were unified in [77], see also [79] for a detailed review. We will prove these conjectures by using the connection with LCFT which allows us to introduce the correct auxiliary functions corresponding to holomorphic observables

of CFT.

We now briefly summarize the main results of this thesis.

◇ **Main result 1:** *Content of chapter 2, in collaboration with Guillaume Remy.*
Consider the log-correlated field X on $[0,1]$ with covariance:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|}. \tag{1.1.1}$$

Then for $\gamma \in (0,2)$ and for real numbers a, b, p satisfying suitable bounds we will give the exact value of the following quantity:

$$M(\gamma,p,a,b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{1.1.2}$$

The strategy of proof adapts the proof of the DOZZ formula [61] to dimension 1, but it does not require any knowledge of CFT. Subsection 1.5.2 summarizes this result.

◇ **Main result 2:** *Content of chapter 3, in collaboration with Guillaume Remy.*
Consider the log-correlated field $X_{\mathbb{D}}$ on the unit circle $\partial\mathbb{D}$ with covariance:

$$\mathbb{E}[X_{\mathbb{D}}(e^{i\theta})X_{\mathbb{D}}(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}. \tag{1.1.3}$$

For $\gamma \in (0,2)$ and suitable parameters α, β, we will prove the exact formula for

$$\overline{G}(\alpha,\beta) := \mathbb{E}\left[\left(\int_0^{2\pi}\frac{1}{|1-e^{i\theta}|^{\frac{\gamma\beta}{2}}}e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{i\theta})}d\theta\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\right], \tag{1.1.4}$$

where $Q = \frac{\gamma}{2}+\frac{2}{\gamma}$. This formula generalizes the Fyodorov-Bouchaud formula which corresponds to the special case where $\beta = 0$. The proof consists in finding shift equations on β while keeping α constant. Therefore the result is constructed on the Fyodorov-Bouchaud formula and there is no overlap between the two results although the new one is more general.

Additionally, for $\gamma \in (0,2)$ and suitable β_k, μ_k, $1 \le k \le 3$ we will prove the exact value of

$$\mathbb{E}\left[\left(\int_0^{2\pi}\frac{1}{\prod_{k=1}^3|e^{i\frac{2k\pi}{3}}-e^{i\theta}|^{\frac{\gamma\beta_k}{2}}}(\sum_{k=1}^3\mu_k\mathbf{1}_{\theta\in[\frac{2(k-1)\pi}{3},\frac{2k\pi}{3})})e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{i\theta})}d\theta\right)^{\frac{1}{\gamma}(2Q-\sum_{k=1}^3\beta_k)}\right].$$

The three singularity points $e^{i\frac{2k\pi}{3}}$, $1 \le k \le 3$, are arbitrary on the circle and can be related by conformal identities. It also appears that taking $\mu_1 = 1$ and $\mu_2 = \mu_3 = 0$ and after a change of metric, the above result will recover the previous result (1.1.2) on the interval, see subsection **??**. However, we emphasis that there is no overlap and the proof of the results in chapter 3 requires prior knowledge of the formulas for the interval case. Subsection 1.5.3 summarizes this chapter with more details, where we choose to represent the expressions on the real axis equipped with an appropriate metric.

◇ **Main result 3:** *Content of chapter 4.*
In all the results mentioned above, we start by introducing some auxiliary functions and study their second order differential equations. The general form of this type of differential equations are known as the BPZ equations. Our third result proves higher order BPZ equations, based on some intrinsic relations between correlation functions in CFT and Coulomb gas integrals. We summarize this result in more details in subsection 1.4.3.

The remaining part of this overview is divided as follows. First we will give the definitions and basic properties of all the objects that we will use, the Gaussian free field, the Gaussian multiplicative chaos measures and the correlation functions of LCFT. Then we will explain in more details all the properties Liouville field theory inherits from the framework of CFT. Exploiting the constraints imposed by the BPZ equations leads to the derivation of exact formulas for GMC measures and Liouville theory. This is what one should understand by the word "integrability" that we have used in the title, the existence of exact formulas for fractional moments of certain GMC measures or equivalently for the corresponding correlation functions of LCFT. Applications to related problems in probability theory are also included. Lastly we will state some open problems for future directions of research as well as a list of additional exact formulas that we expect to be able to prove using similar techniques.

1.2 Gaussian free fields and Gaussian multiplicative chaos

1.2.1 Gaussian free fields and log-correlated fields

The fundamental Gaussian processes that we will always work with are log-correlated fields which can be defined on $\mathbb{R}^d$ in any dimension d. However for our purpose we will only need the cases $d = 1$ or $d = 2$. In the case of dimension two the field is then known as the Gaussian free field (GFF). In dimension one a log-correlated field can be seen as the restriction to a 1d domain of a Gaussian free field in two-dimensions. For the purpose of this overview and for the coming chapters we will consider many different domains, the Riemann sphere $\mathbb{S}^2$, the unit disk $\mathbb{D}$, the upper-half plane $\mathbb{H}$ for the two-dimensional cases and the unit interval $[0, 1]$, the unit circle $\partial\mathbb{D}$, the real axis $\mathbb{R}$ for the one-dimensional case. Although the general idea is always the same, defining the field on each domain will be slightly different based on the topology of the domain. We will try to treat all the above cases in the most concise way possible.

Let us start with the case of the full plane GFF $\overline{X}$ that will be used to define the GFF on the Riemann phere $\mathbb{S}^2$. One should keep in mind that a very natural way to represent the sphere $\mathbb{S}^2$ is simply to add a point at infinity to the complex plane, i.e. $\mathbb{C}\cup\{\infty\}$. We start by introducing the space $\mathcal{S}(\mathbb{C})$ of smooth test functions on $\mathbb{C}$ with compact support and the subspace $\mathcal{S}_0(\mathbb{C}) \subset \mathcal{S}(\mathbb{C})$ of test functions that have zero average over $\mathbb{C}$. We denote by $\mathcal{S}'(\mathbb{C})$ the space of distributions associated to $\mathcal{S}(\mathbb{C})$ and by $\mathcal{S}'(\mathbb{C})/\mathbb{R}$ the space of distributions associated to $\mathcal{S}_0(\mathbb{C})$ which can be seen as a space of distributions modulo constants.

We can now define the full plane GFF $\overline{X}$ as a Gaussian random variable living in $\mathcal{S}'(\mathbb{C})/\mathbb{R}$ with the following covariance for all $f, h \in \mathcal{S}_0(\mathbb{C})$:

$$\mathbb{E}[(\int_{\mathbb{C}} f(x)\overline{X}(x)d^2x)(\int_{\mathbb{C}} h(y)\overline{X}(y)d^2y)] = \int_{\mathbb{C}^2} f(x)h(y)\ln\frac{1}{|x-y|}d^2xd^2y.^{2} \tag{1.2.2}$$

Note that the above definition can be extended to the case where f, h are generalized functions. Therefore we can also view the filed $\overline{X}$ as a centered Gaussian process indexed by a certain signed measure space. With this full plane GFF we can define a GFF living in $\mathcal{S}'(\mathbb{C})$ by prescribing the value of $\overline{X}$ against a probability measure on $\mathbb{C}$. This will fix the undetermined constant and thus X will be defined against any test function in $\mathcal{S}'(\mathbb{C})$. Anticipating the coming constructions of LCFT it is natural to write $g(x)d^2x$ for the measure on $\mathbb{C}$ with which we define:

$$X_g(x) = \overline{X}(x) - \int_{\mathbb{C}} \overline{X}(x)g(x)d^2x. \tag{1.2.3}$$

We will say that X_g is the GFF on the Riemann sphere $\mathbb{S}^2$ of zero average with respect to the metric g. A natural choice is to chose,

$$\hat{g}(x) := \frac{4}{(1+|x|^2)^2}, \tag{1.2.4}$$

[2] This is a suitable definition for the covariance of a Gaussian process as one can prove by Fourier analysis the following identity,

$$\int_{\mathbb{C}^2} f(x)h(y)\ln\frac{1}{|x-y|}d^2xd^2y = c\int_{\mathbb{C}}\frac{\hat{f}(\xi)\overline{\hat{h}(\xi)}}{|\xi|^2}d^2\xi > 0, \tag{1.2.2}$$

where $c > 0$ is a constant.

as this is the canonical spherical metric on $\mathbb{S}^2$. More explicitly with this choice $\hat{g}$ the field $X_{\hat{g}}$ then has the covariance: for $x, y \in \mathbb{C}$,

$$\mathbb{E}[X_{\hat{g}}(x)X_{\hat{g}}(y)] = \ln \frac{1}{|x-y|} - \frac{1}{4}\ln \hat{g}(x) - \frac{1}{4}\ln \hat{g}(y) + \ln 2 - \frac{1}{2}. \tag{1.2.5}$$

We will see in subsection 1.4.1 that in LCFT it suffices to study one particular metric and all the results under other conformally equivalent metrics can be obtained by using Weyl anomaly. By conformally equivalent we mean that $g(x) = e^{\varphi(x)}\hat{g}(x)$ with $\varphi \in \mathcal{C}^1(\mathbb{C} \cup \{\infty\})$ such that $\int_{\mathbb{C}} |\partial\varphi(x)|^2 g(x) d^2x < \infty$. This completes the description of the different GFF on the plane and on the sphere.

We move on to the boundary case and start by looking at the unit disk $\mathbb{D}$. When one works on a domain with boundary there is of course a choice to be made of which boundary conditions to work with. The most natural choice one can think of are the Dirichlet boundary conditions where one imposes the GFF $X_{\mathbb{D}}$ to be equal to 0 on the boundary. It turns out that this is not the right choice to construct LCFT as this boundary condition is much too strong and many of the expected properties of CFT - see section 1.4 - would fail to hold. Hence we will rather choose Neumann boundary conditions, also sometimes called free boundary conditions. The covariance of the Neumann GFF $X_{\mathbb{D}}$ is given for $x, y \in \mathbb{D}$ by:

$$\mathbb{E}[X_{\mathbb{D}}(x)X_{\mathbb{D}}(y)] = \ln \frac{1}{|x-y||1-x\overline{y}|}. \tag{1.2.6}$$

Of course just like for the case of the full plane GFF, $X_{\mathbb{D}}$ lives in the space of distributions so the equation (1.2.6) is to be understood by,

$$\mathbb{E}[(\int_{\mathbb{D}} f(x)X_{\mathbb{D}}(x)d^2x)(\int_{\mathbb{D}} h(y)X_{\mathbb{D}}(y)d^2y)] = \int_{\mathbb{D}^2} f(x)h(y)\ln\frac{1}{|x-y||1-x\overline{y}|}d^2xd^2y, \tag{1.2.7}$$

where again f and h are smooth test functions defined on $\mathbb{D}$. We also define for different metric g conformally equivalent to the canonical metric on $\mathbb{D}$ the field $X_{\mathbb{D},g}$ with vanishing mean on the circle, i.e. $\int_{\partial\mathbb{D}} X_{\mathbb{D},g}(x)g(x)^{1/2}dx = 0$. The field $X_{\mathbb{D},g}$ can be obtained by

$$X_{\mathbb{D},g} = X_{\mathbb{D}} - \int_{\partial\mathbb{D}} X_{\mathbb{D}}(x)g(x)^{1/2}dx. \tag{1.2.8}$$

With the covariance (1.2.6) established one can then give the covariance for the GFF defined on $\mathbb{H}$ by using the conformal mapping $z \to \frac{z-i}{z+i}$ linking $\mathbb{H}$ and $\mathbb{D}$. This provides the expression:

$$\mathbb{E}[X_{\mathbb{H}}(x)X_{\mathbb{H}}(y)] = \ln\frac{1}{|x-y||x-\overline{y}|} + \ln|x+i|^2 + \ln|y+i|^2 - 2\ln 2. \tag{1.2.9}$$

Thed fileds $X_{\mathbb{H},g}$ are then defined as $X_{\mathbb{H}} - \int_{\mathbb{R}} X_{\mathbb{H}}(x)g(x)^{1/2}dx$. For some technical reasons, when computing exact formulas in chapter 2 and 3, we will consider another log correlated field X defined on $\mathbb{H}$:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|} + 2\ln|x|_+ + 2\ln|y|_+, \quad \text{where } |x|_+ = \max(|x|, 1). \tag{1.2.10}$$

The field X can be constructed from $X_{\mathbb{H}}$ using:

$$X = X_{\mathbb{H}} - \frac{1}{\pi}\int_0^\pi X_{\mathbb{H}}(e^{i\theta})d\theta. \tag{1.2.11}$$

We will explain how results calculated using the field X can be related to those using the field $X_{\mathbb{H}}$.

Lastly we conclude this discussion with the one dimensional cases that we need. For the case of the circle $\partial\mathbb{D}$ we will simply restrict the Neumann boundary GFF $X_{\mathbb{D}}$ defined on the disk $\mathbb{D}$ to the circle. The covariance we thus obtain is given for two points $e^{i\theta}$, $e^{i\theta'}$ by:

$$\mathbb{E}[X_{\mathbb{D}}(e^{i\theta})X_{\mathbb{D}}(e^{i\theta'})] = 2\ln\frac{1}{|e^{i\theta}-e^{i\theta'}|}. \tag{1.2.12}$$

For the case of the real axis, by analogy with (1.2.12) we will simply chose for our field X defined on $\mathbb{R}$ the covariance given for $x, y \in \mathbb{R}$:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|} + 2\ln|x|_+ + 2\ln|y|_+. \tag{1.2.13}$$

Particularly, for the interval case where $x, y \in [0,1]$, the covariance equals $2\ln\frac{1}{|x-y|}$.

1.2.2 Gaussian multiplicative chaos measures

We will now introduce the Gaussian multiplicative chaos measures, a fundamental building block of LCFT. Our goal is to define a random measure with density with respect to the Lebesgue measure given by the exponential of our log-correlated fields. The major problem that must be overcome comes from the fact that the exponential of a distribution is ill-defined. Thus this requires to use a regularization and renormalization procedure. The same method will work to define GMC measures associated to all the different cases of log-correlated field that we have described above.

The first step is to provide a regularization procedure of X, a log-correlated field defined on $\mathbb{C}$ or $\mathbb{H}$. Let η be a smooth function on $\mathbb{R}_+$ supported in $[\frac{1}{2}, 1]$, such that $\pi\int_{\mathbb{R}_+}\eta(t)dt = 1$ and define $\eta_\delta(x) := \frac{1}{\delta^2}\eta(|x|^2/\delta^2)$. We will work with the regularization

$$X_\delta(x) := \int_{\mathbb{C}} X(x-y)\eta_\delta(y)d^2y. \tag{1.2.14}$$

When X is a Gaussian field on $\mathbb{H}$, we should consider an extension of the field $X(x) = X(\overline{x})$ for $x \in \overline{\mathbb{H}}$, then the above definition makes sense. Note that our regularization is indeed a smooth version of the circle average (it is slightly differnt when x is close to the boundary for the case of $\mathbb{H}$). This choice makes it possible to take derivatives of the field X_δ. There are also other regularizations as long as certain conditions are satisfied, and they are all equivalent for defining the GMC measures, see [12] for more details. When X is a log-correlated field on $\mathbb{D}$, a possible method to do this is to use circle average regularization although many other paths are possible. For $\epsilon > 0$ we call $l_\delta(x)$ the length of the arc $A_\delta(x) = \{z \in \mathbb{D}; |z-x| = \delta\}$ and we set:

$$X_\delta(x) = \frac{1}{l_\delta(x)}\int_{A_\delta(x)} X(x+s)ds. \tag{1.2.15}$$

We now state the following proposition-definition of Gaussian multiplicative chaos for dimension 1 and 2:

Proposition 1.2.1. *(Definition of GMC) Let $\gamma \in (0,2)$ and X a log-correlated field on a domain D of dimension 2. We then define the random measure $e^{\gamma X(x)}d^2x$ as the following limit in probability*

$$e^{\gamma X(x)}d^2x := \lim_{\delta\to 0} e^{\gamma X_\delta(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\delta(x)^2]}d^2x \tag{1.2.16}$$

in the sense of weak convergence of measures. More precisely this means that for all continuous compactly supported test functions $f : D \to \mathbb{R}$, the following convergence holds in probability:

$$\int_D f(x)e^{\gamma X(x)}d^2x = \lim_{\delta\to 0}\int_D f(x)e^{\gamma X_\delta(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\delta(x)^2]}d^2x. \tag{1.2.17}$$

We also define for the one dimensional case where X is a 1d log-correlated field defined: for $\gamma \in (0,2)$,

$$e^{\frac{\gamma}{2}X(x)}dx := \lim_{\delta\to 0} e^{\frac{\gamma}{2}X_\delta(x) - \frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx. \tag{1.2.18}$$

Remark 1.2.2. *The above definition can be extended to d-dimensional case, see [92]. Note that there is a slight difference when we talk about 1d log-correlated field. Here we are considering 1d fields with kernel $2\ln\frac{1}{|x-y|} + h(x,y)$ instead of $\ln\frac{1}{|x-y|} + h(x,y)$, as described in the previous subsection. This difference in convention changes the domain of existence for γ.*

We always integrate the GMC measure with a background metric g, in which case we can integrate a much larger family of test functions. Actually in the later LCFT constructions, we will integrate against the following measures on different geometries:

- On $\mathbb{S}^2$: $e^{\gamma X_g(x)}g(x)d^2x$.
- On $\mathbb{D}$: $e^{\gamma X_{\mathbb{D},g}(x)}\frac{g(x)}{(1-|x|^2)^{\frac{\gamma^2}{2}}}d^2x$, and on $\partial\mathbb{D}$: $e^{\frac{\gamma}{2}X_{\mathbb{D},g}(e^{i\theta})}g(e^{i\theta})^{\frac{1}{2}}d\theta$.
- On $\mathbb{H}$: $e^{\gamma X_{\mathbb{H},g}(x)}\frac{g(x)}{|x-\bar{x}|^{\frac{\gamma^2}{2}}}d^2x$, and on $\mathbb{R}$: $e^{\frac{\gamma}{2}X_{\mathbb{H},g}(x)}g(x)^{\frac{1}{2}}dx$.

Remark that we can easily relate the GMC on $\mathbb{D}$ with GMC on $\mathbb{H}$ using conformal invariance of GFF. For example, starting from $X_{\mathbb{H}}$ with covariance given by (1.2.9), we take the conformal map $\psi(z)=\frac{z-i}{z+i}$ and $f:\mathbb{D}\to\mathbb{R}$ a test function, then

$$\int_{\mathbb{D}} f(x)e^{\gamma X_{\mathbb{D}}(x)}d^2x=\int_{\mathbb{H}} f(\psi(x))e^{\gamma X_{\mathbb{H}}(x)}|\psi'(x)|^2d^2x. \tag{1.2.19}$$

Therefore we are free to present the results of GMC on $\mathbb{D}$ or on $\mathbb{H}$.

Now we give a list of results on the existence of moments of GMC on $\mathbb{S}^2$, $\mathbb{D}$, and $\partial\mathbb{D}$. We also include the generalization where certain fractional moments are added in the GMC measure. The bounds obtained will be extremely important for the definitions of the correlations of LCFT. We start with the existence of moments:

Proposition 1.2.3. *(Moments of GMC on $\mathbb{S}^2$) Let $\hat{g}$ be the canonical spherical metric on $\mathbb{S}^2$ and let $\gamma\in(0,2)$. Then we have,*

$$\mathbb{E}[(\int_{\mathbb{C}} e^{\gamma X_{\hat{g}}(x)}\hat{g}(x)d^2x)^p]<+\infty, \tag{1.2.20}$$

if and only if $p<\frac{4}{\gamma^2}$.

We also provide a similar result for $\mathbb{D}$ and $\partial\mathbb{D}$:

Proposition 1.2.4. *(Moments of GMC on $\mathbb{D}$) Let $\gamma\in(0,2)$, then we have*

$$\mathbb{E}[(\int_{\mathbb{D}}\frac{e^{\gamma X_{\mathbb{D}}(x)}}{(1-|x|^2)^{\frac{\gamma^2}{2}}}d^2x)^p]<+\infty \tag{1.2.21}$$

if and only if $p<\frac{2}{\gamma^2}$ and

$$\mathbb{E}[(\int_0^{2\pi} e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{i\theta})}d\theta)^p]<+\infty \tag{1.2.22}$$

if and only if $p<\frac{4}{\gamma^2}$.

We have a different result for the moments of the bulk measure because of singularity $\frac{1}{(1-|x|^2)^{\frac{\gamma^2}{2}}}$ and because of the behaviour of the GFF X near the boundary. To study LCFT we also need the same type of results but with insertion points. The bounds for non-triviality of a GMC moment with insertion have been obtained in [20, 53], they are:

Proposition 1.2.5. *(Moments of GMC with insertions) Let $\gamma\in(0,2)$, $Q=\frac{\gamma}{2}+\frac{2}{\gamma}$, $\alpha,\beta\in\mathbb{R}$, $z\in\mathbb{C}$ with $|z|<1$, and $s\in\partial\mathbb{D}$. Then we have,*

$$0<\mathbb{E}[(\int_{\mathbb{C}}\frac{1}{|z-x|^{\gamma\alpha}}e^{\gamma X_{\hat{g}}(x)}\hat{g}(x)d^2x)^p]<+\infty, \tag{1.2.23}$$

if and only if $\alpha<Q$ and $p<\frac{4}{\gamma^2}\wedge\frac{2}{\gamma}(Q-\alpha)$. Similarly,

$$0<\mathbb{E}[(\int_{\mathbb{D}}\frac{1}{|z-x|^{\gamma\alpha}}\frac{1}{|s-x|^{\frac{\gamma\beta}{2}}}\frac{e^{\gamma X_{\mathbb{D}}(x)}}{(1-|x|^2)^{\frac{\gamma^2}{2}}}d^2x)^p]<+\infty, \tag{1.2.24}$$

if and only if[3] $\alpha < Q$, $\beta < Q$, *and* $p < \frac{2}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha) \wedge \frac{1}{\gamma}(Q-\beta)$. *Lastly,*

$$0 < \mathbb{E}[(\int_0^{2\pi} \frac{1}{|s-e^{i\theta}|^{\frac{\gamma\beta}{2}}} e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{i\theta})} d\theta)^p] < +\infty, \tag{1.2.25}$$

if and only if $\beta < Q$ *and* $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\beta)$.

1.3 Liouville conformal field theory

We now move on to the study of Liouville conformal field theory (LCFT). As we mentioned in introduction this theory first appeared in Polyakov's seminal 1981 paper [82] out of the need to understand what is a canonical random Riemannian metric on a surface of given topology. More precisely Polyakov tells us that for a fixed reference metric g on a given surface M one should consider the random metric $e^{\gamma\phi}g$ where ϕ is the Liouville field.

This section will provide the construction of LCFT on different surfaces. Based on the topology of the surface M there will be some differences in the construction. We will start by the simplest case, LCFT on the Riemann sphere following [20]. Next will come the case of the unit disk $\mathbb{D}$ first studied in [53] where one has to cope with the presence of a boundary. Let us mention that LCFT has also been constructed on other domains, see [87] for the annulus case, [21] for the torus and [49] for compact surfaces of higher genus.

1.3.1 LCFT on the Riemann sphere

The case of the Riemann sphere $\mathbb{S}^2$ studied in detail in [20] is the simplest to define the theory of LCFT as it is a compact simply connected boundaryless surface. Our choice of coordinates to represent the sphere $\mathbb{S}^2$ will be the complex plane with a point added at infinity $\mathbb{C} \cup \{\infty\}$.

To define LCFT, physicists use what is referred to as the path integral formalism.[4] Informally it tells us that our Liouville field ϕ will be given in terms of an infinite measure on a suitable functional space. Until the very recent work [20] giving a rigorous probabilistic content to LCFT had remained an open problem. We now sketch the physicists' heuristic definitions before explaining how to make them rigorous. Consider the following space of maps:

$$\Sigma = \{X : \mathbb{S}^2 \to \mathbb{R}\}. \tag{1.3.1}$$

The Liouville field ϕ is then given by the following formal definition, for any background metric g on $\mathbb{S}^2$,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}} \int_\Sigma F(X) e^{-S_L(X,g)} DX, \tag{1.3.2}$$

where $S_L(X,g)$ is the so-called Liouville action:

$$S_L(X,g) = \frac{1}{4\pi} \int_{\mathbb{C}} (|\partial^g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}) g(x) d^2x. \tag{1.3.3}$$

Let us comment on all our notations. The three real parameters γ, Q, μ that appear obey $\gamma \in (0,2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, and $\mu > 0$. DX is the formal uniform measure on the space Σ and g is the background Riemannian metric used to define the theory. To avoid going too deep into the framework of Riemannian geometry we restrict

[3] For $\gamma \in (\sqrt{2}, 2)$ it remains to be proved that the bound written above on p is optimal although it is strongly believed to be the case, see [53] for an explanation of this subtlety.

[4] It is instructive to note that the path integral formalism we explain can also be used to heuristically define Brownian motion. Consider the space of path $\Sigma' = \{\sigma : [0,1] \to \mathbb{R},\ \sigma(0) = 0\}$ and the action $S_{BM} = \frac{1}{2}\int_0^1 |\sigma'(t)|^2 dt$. Then for all suitable F,

$$\mathbb{E}[F((B_s)_{0\le s\le 1})] = \frac{1}{\mathcal{Z}} \int_{\Sigma'} F(\sigma) e^{-S_{BM}(\sigma)} D\sigma,$$

where $D\sigma$ is a formal uniform measure on Σ'.

ourselves to the case where the background metric g is a diagonal tensor, meaning that g will simply be a positive function defined on $\mathbb{C} \cup \{\infty\}$. In this simple case we have $\partial^g X = g^{-1}\partial X$ and $R_g = -g^{-1}\Delta \ln g$. As a matter of fact it turns out that on the Riemman sphere up to a change of coordinates any metric tensor can be written under this form. Lastly $\mathcal{Z}$ is a formal normalization constant.

Let us now explain the three terms appearing in the Liouville action. The gradient term $|\partial^g X|^2$ of (1.3.3) is the free field or kinetic energy term. If this was the only term present, up to a global constant, the law of ϕ would be that of our Gaussian free field X_g on $\mathbb{S}^2$. But the action (1.3.3) also contains a non-linear interaction term, the exponential term $e^{\gamma X}$. At first glance it may seem unclear why it is interesting to consider this specific interaction. The first motivation comes from theoretical physics - in particular string theory and 2d quantum gravity, see [19, 23, 24, 82] - where as we have said earlier $e^{\phi}g$ formally defines the correct random Riemannian metric on $\mathbb{S}^2$. The second reason is that, out of all possible interaction terms, the exponential term is the simplest term that defines a conformal field theory. The implications of this will be explained thoroughly in sections 1.4 and 1.5. Finally, the term QR_gX is the linear coupling of X to the background metric g. It is required to get a consistent theory but does not pose any mathematical problems as it is a linear term in X.

One may wonder why the Liouville action is the correct action to define canonical random metrics. A first answer comes from physics, in particular from Polyakov in [82]. An easier answer comes from the study of classical Liouville theory, meaning that we look for the functions X minimizing the Liouville action. It is a well known fact of classical geometry that such a minimum $X_{\min}$ is unique if it exists and the new metric $g' = e^{\gamma X_{\min}} g$ is of constant negative curvature provided that $Q_c = \frac{2}{\gamma}$[5]. In other words, the minimum of the Liouville action uniformizes the surface (M, g) and it is therefore natural to look at quantum fluctuations of the uniformized metric $e^{\gamma X_{\min}} g$. This is precisely the meaning of (1.3.2).

As we have written it the path integral (1.3.2) diverges for any surface of genus 0 or 1, including therefore the case of the sphere. To see this we can write the Gauss-Bonnet formula, given here for a boundaryless surface M of genus h equipped with a metric g, $\int_M R_g(x)g(x)d^2x = 8\pi(1-h)$. When $h = 0$ or 1, this implies that it is impossible to define on the surface a metric of constant negative curvature, meaning that $S_L(X, g)$ will have no minimum and therefore the path integral (1.3.2) diverges. To solve this problem we proceed as in [20] and add insertion points. We consider the new expression,

$$\mathbb{E}[F(\phi)] = \frac{1}{\mathcal{Z}} \int_{\Sigma} F(X) e^{\sum_{i=1}^{N} \alpha_i X(z_i)} e^{-S_L(X,g)} DX, \tag{1.3.4}$$

where we have chosen N insertion points $z_i \in \mathbb{S}^2$ with weights $\alpha_i \in \mathbb{R}$. By choosing $F = 1$ in the above expression we define the N-point correlation function of LCFT, the most fundamental observable of the theory:

$$\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \rangle_{\mathbb{S}^2, g} = \frac{1}{\mathcal{Z}} \int_{\Sigma} e^{\sum_{i=1}^{N} \alpha_i X(z_i)} e^{-S_L(X,g)} DX, \tag{1.3.5}$$

where $V_{\alpha_i}(z_i) = e^{\alpha_i \phi(z_i)}$ are the so-called vertex operators.

We show that the following conditions known as the Seiberg bounds must be satisfied in order for (1.3.4) and (1.3.5) to exist:

$$\sum_{i=1}^{N} \alpha_i > 2Q \quad \text{and} \quad \forall i,\ \alpha_i < Q. \tag{1.3.6}$$

The minimum number of insertion points needed to satisfy these bounds is three. This is precisely the requirement to entirely determine a conformal automorphism of the sphere, the so-called Mobius transformations. From a geometric standpoint, we can also view insertion points as conical singularities of the metric which allow hyperbolic metrics to be defined on the surface.

We now move on to the probabilistic definition of (1.3.5). This will require to use both the Gaussian free field and the Gaussian multiplicative chaos measures. The GFF will appear out of the need to make sense

[5] Here we write $Q_c = \frac{2}{\gamma}$ as this is the correct value in the classical theory where the goal is to minimize the Liouville action. In the quantum theory we will always have $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

of the formal density $e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x}DX$ which we will interpret as the density of a Gaussian vector in infinite dimensions. The key observation is that by performing an integration by parts we have:

$$\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x = \frac{1}{2}\int_{\mathbb{C}} X\left(-\frac{\Delta_g}{2\pi}\right)Xg(x)d^2x. \tag{1.3.7}$$

Therefore we are formally constructing a field with covariance given by the inverse of the Laplacian. To give a more precise mathematical meaning to this observation we will replace the abstract space Σ by the following L^2 space,

$$L^2(\mathbb{S}^2) := \{X : \mathbb{S}^2 \to \mathbb{R}, \int_{\mathbb{C}} X(x)^2 g(x)d^2x < +\infty\}, \tag{1.3.8}$$

on which we can diagonalize the Laplacian. Now let $(e_j)_{j\geq 1}$ be a basis of eigenvectors for $-\Delta_g$ meaning that,

$$-g(x)^{-1}\Delta e_j(x) = \lambda_j e_j(x), \tag{1.3.9}$$

where the e_j are normalized to have an L^2 norm equal to 1: $\int_{\mathbb{C}} e_j(x)^2 g(x)d^2x = 1$. With this basis any function $X \in L^2(\mathbb{S}^2)$ can be written,

$$X = c + \sum_{j=1}^{\infty} c_j e_j, \tag{1.3.10}$$

where the coefficient c_j are obtained by $c_j = \int_{\mathbb{C}} X(x)e_j(x)g(x)d^2x$. Such a decomposition tells us that we have,

$$\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x = \frac{1}{4\pi}\sum_{j=1}^{\infty} c_j^2\lambda_j, \tag{1.3.11}$$

and so heuristically it is natural to want to write:

$$\int_{L^2(\mathbb{S}^2)} F(X)e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x}DX = \int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}} F(c+\sum_{j=1}^{\infty} c_j e_j)dc\prod_{j=1}^{\infty} e^{-\frac{c_j^2\lambda_j}{4\pi}}dc_j. \tag{1.3.12}$$

Here dc and each dc_i are Lebesgue measures on $\mathbb{R}$. The sum $\sum_{j\geq 1}\sqrt{\frac{2\pi}{\lambda_j}}\epsilon_j e_j(x)$ with $(\epsilon_j)_{j\geq 1}$ being a sequence of i.i.d. standard Gaussians converges in the space of distributions towards X_g and thus we can make sense of the above formal expression by writing that:

$$\int_{L^2(\mathbb{S}^2)} F(X)e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X_g|^2 g(x)d^2x}DX := \int_{\mathbb{R}}\mathbb{E}[F(X_g+c)]dc. \tag{1.3.13}$$

In the above expression, the left hand side is a formal expression where X is an integration variable. On the right hand side X_g is the GFF that we have constructed. By construction of our GFF X_g on $\mathbb{S}^2$ we have $\int_{\mathbb{C}} X_g(x)R_g(x)g(x)d^2x = 0$. Therefore:

$$\int_{L^2(\mathbb{S}^2)} F(X)e^{-\frac{1}{4\pi}\int_{\mathbb{C}}|\partial^g X|^2 g(x)d^2x - \frac{1}{4\pi}\int_{\mathbb{C}} X(x)R_g(x)g(x)d^2x}DX := \int_{\mathbb{R}} e^{-2Qc}\mathbb{E}[F(X_g+c)]dc. \tag{1.3.14}$$

Thus using the Gaussian free field we have given a probabilistic meaning to (1.3.2) in the very special case where $\mu = 0$. We now tackle the problem of handling the exponential interaction term, this is where we will require the theory of Gaussian multiplicative chaos. To define the correlations (1.3.5) we want to pick in (1.3.14) a functionnal F of the type,

$$F(X) = \prod_{i=1}^{N} e^{\alpha_i X(z_i)}e^{-\mu\int_{\mathbb{C}} e^{\gamma X(x)}g(x)d^2x}, \tag{1.3.15}$$

but since X_g is not defined pointwise but lives in the space of distributions this will require a regularization and renormalization procedure. We now state the theorem that insures that the correlations of LCFT on $\mathbb{S}^2$ are well-defined by using the regularization-renormalization procedure.

Theorem 1.3.1. *(Correlation functions of LCFT on $\mathbb{S}^2$) Choose N points $z_i \in \mathbb{S}^2$ of weights $\alpha_i \in \mathbb{R}$ such that the Seiberg bounds (1.3.6) are satisfied. Then the N-point correlation function of LCFT is defined by the following limit,*

$$\langle\prod_{l=1}^{N} V_{\alpha_l}(z_l)\rangle_{\mathbb{S}^2,\hat{g}} := \lim_{\delta\to 0}\langle\prod_{l=1}^{N} V_{\alpha_l,\delta}(z_l)\rangle_{\mathbb{S}^2,\hat{g}} \tag{1.3.16}$$

where the expresssion of the regularized correlation is given by:

$$\begin{aligned}\langle\prod_{i=1}^{N} V_{\alpha_i}(z_I)\rangle_{\mathbb{S}^2,\hat{g},\delta} :=& 2e^{(\ln 2-\frac{1}{2})(\frac{1}{2}\sum_{l=1}^{N}\alpha_l^2-\frac{\gamma}{2}\sum_{l=1}^{N}\alpha_l-\frac{4Q}{\gamma})}\int_{\mathbb{R}} e^{-2Qc}\,\mathbb{E}\Big[\prod_{l=1}^{N}\hat{g}_\delta(z_l)^{\Delta_{\alpha_l}} e^{\alpha_l(X_{\hat{g},\delta}(z_l)+c)-\frac{\alpha_l^2}{2}\mathbb{E}[X_{\hat{g},\delta}(z_l)^2]}\\ &\times e^{-\mu e^{\gamma c}\int_{\mathbb{C}} e^{\gamma X_{\hat{g},\delta}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\hat{g},\delta}(x)^2]}\hat{g}_\delta(x)d^2x}\Big]dc,\end{aligned} \tag{1.3.17}$$

where $\Delta_\alpha := \frac{\alpha}{2}(Q-\frac{\alpha}{2})$ is the conformal weight. The limit written above exists and is different from 0 and $+\infty$. By simple manipulations on the above definition, one obtains the more compact expression,

$$\langle\prod_{i=1}^{N} V_{\alpha_i}(z_i)\rangle_{\mathbb{S}^2,\hat{g}} = Z(\boldsymbol{\alpha})\prod_{1\le i<j\le N}\frac{1}{|z_i-z_j|^{\alpha_i\alpha_j}}\mathbb{E}\left[\left(\int_{\mathbb{C}}\frac{e^{\gamma X_{\hat{g}}(x)}\hat{g}(x)^{1-\frac{\gamma}{4}\sum_{i=1}^{N}\alpha_i}d^2x}{\prod_{k=1}^{N}|x-z_k|^{\gamma\alpha_k}}\right)^{-\frac{\sum_{i=1}^{N}\alpha_i-2Q}{\gamma}}\right], \tag{1.3.18}$$

where

$$Z(\boldsymbol{\alpha}) := 2e^{-\frac{(\ln 2-1/2)}{2}(\sum_{i=1}^{N}\alpha_i-2Q)(\sum_{i=1}^{N}\alpha_i-\frac{4}{\gamma})}\gamma^{-1}\Gamma\left(\frac{\sum_{i=1}^{N}\alpha_i-2Q}{\gamma},\mu\right). \tag{1.3.19}$$

Remark 1.3.2. *g_δ is defined as $g_\delta := g * \eta_\delta$. In the expression (1.3.17), we manually add the multiplier term $\prod_{i=1}^{N} g_\delta(z_l)^{\Delta_{\alpha_l}}$ to have the conformal invariance consistent with the literature, and we add the constant prefactor term that is independent of z_i to stay in the same convention as the DOZZ formula in the literature. Both added terms play no important roles and can be easily analyzed. We chose to present in this way as it is easier for the proof of the BPZ equations. Of course we can define the correlation functions for any metric g conformally equivalent to $\hat{g}$ by using X_g in the above definition instead of $X_{\hat{g}}$, and we only need to change the prefactor accordingly. More details can be found in [20].*

There are two simple steps to go from (1.3.17) to (1.3.18). The first is to apply the Girsanov theorem (also called the complete the square trick in physics) to the insertions $e^{\alpha_i X_{\hat{g},\delta}(z_i)-\frac{\alpha_i^2}{2}\mathbb{E}[X_{\hat{g},\delta}(z_i)^2]}$, which produces a shift on the field $X_{\hat{g},\delta}(x) \to X_{\hat{g},\delta}(x) + \sum_i \alpha_i \mathbb{E}[X_{\hat{g},\delta}(x)X_{\hat{g},\delta}(z_i)]$. By then using the explicit expression of the covariance this creates the fractional powers in the GMC measure. The second step is simply to perform a change of variable on the integral over c by setting $u = \mu e^{\gamma c}\int_{\mathbb{C}} e^{\gamma X_{\hat{g},\delta}(x)-\frac{\gamma^2}{2}\mathbb{E}[X_{\hat{g},\delta}(x)^2]}d^2x$. The integral over u then gives a Gamma function and we are left with (1.3.18). This last expression we obtain for the value of the N-point correlation of LCFT on $\mathbb{S}^2$ is fundamental. It tells us that we have a definition for the correlation function in terms of a relatively simple probabilistic object, a moment of a GMC measure with prescribed log-singularities. With this expression well-established one can now hope to implement rigorously in a probabilistic framework all the techniques of conformal field theory.

1.3.2 LCFT on a domain with boundary

Following [53] LCFT can of course also be defined on a domain with a boundary. We start by explaining the construction on the unit disk $\mathbb{D}$ and we denote by $\partial\mathbb{D}$ its boundary. By conformal mapping this will provide as well the construction of LCFT on $\mathbb{H}$. The biggest difference with the case of the Riemann sphere $\mathbb{S}^2$ is that we can now have two exponential interaction terms, a bulk interaction $\mu e^{\gamma X}d^2x$ and a boundary interaction $\mu_\partial e^{\frac{\gamma}{2}X}d\theta$. Therefore the Liouville action will contain in this case boundary terms:

$$S_\partial(X,dx^2) = \frac{1}{4\pi}\int_{\mathbb{D}}(|\partial X|^2+4\pi\mu e^{\gamma X})d^2x + \frac{1}{2\pi}\int_{\partial\mathbb{D}}(QX+2\pi\mu_\partial e^{\frac{\gamma}{2}X})d\theta. \tag{1.3.20}$$

In the same way as before we can write a heuristic path integral definition for the correlation functions of the theory:

$$\langle\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B_{\beta_j}(s_j)\rangle_{\mathbb{D}}=\frac{1}{\mathcal{Z}}\int_{\Sigma}e^{\sum_i\alpha_iX(z_i)+\frac{1}{2}\sum_j\beta_jX(s_j)}e^{-S_\partial(X,dx^2)}DX \tag{1.3.21}$$

In our presentation we have restricted ourselves to the case of the Euclidean metric dx^2 on $\mathbb{D}$ but of course one could also define the theory in any background metric g. We note that there are two types of insertion points, bulk insertions $z_i\in\mathbb{D}$ with weights $\alpha_i\in\mathbb{R}$ that are inside the disk and boundary insertions $s_j\in\partial\mathbb{D}$ with weights $\beta_j\in\mathbb{R}$ that are placed on the unit circle. Similar techniques as the ones explained for the case of $\mathbb{S}^2$ allow one to give a probabilistic meaning to the above path integral. The Seiberg bounds for this case are given by:

$$\sum_{i=1}^{N}\alpha_i+\sum_{j=1}^{M}\frac{\beta_j}{2}>Q\quad\text{and}\quad\forall i,\ \alpha_i<Q,\ \forall j,\ \beta_j<Q. \tag{1.3.22}$$

Again we notice that these bounds impose a minimum number of points that corresponds exactly to completely determining a conformal automorphism of the disk $\mathbb{D}$. We can now state:

Theorem 1.3.3. *(Correlation functions of LCFT on $\mathbb{D}$) Choose N points $z_i\in\mathbb{D}$ and M points $s_j\in\partial\mathbb{D}$ such that the bounds (1.3.22) hold. We replace μ_∂ by different boundary cosmological constants μ_j between s_j and s_{j+1} with s_j well ordered on the circle and the corresponding boundary measure is denoted by $d\mu_\partial(\theta)$. Then the correlation function has an expression given by,*

$$\begin{aligned}
&\langle\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{D}}\\
&=\prod_{i=1}^{N}(1-|z_i|^2)^{-\frac{\alpha_i^2}{2}}\prod_{1\le i<i'\le N}|z_i-z_{i'}|^{-\alpha_i\alpha_{i'}}|1-z_i\bar{z}_{i'}|^{-\alpha_i\alpha_{i'}}\prod_{1\le i\le N,1\le j\le M}|z_i-s_j|^{-\alpha_i\beta_j}\prod_{1\le j<j'\le M}|s_j-s_{j'}|^{-\frac{\beta_j\beta_{j'}}{2}}\\
&\int_{\mathbb{R}}dc\,e^{(\sum_i\alpha_i+\sum_j\frac{\beta_j}{2}-Q)c}\mathbb{E}\Bigg[\exp\Bigg(-\mu e^{\gamma c}\int_{\mathbb{D}}\frac{1}{\prod_{i=1}^{N}|x-z_i|^{\gamma\alpha_i}|1-x\bar{z}_i|^{\gamma\alpha_i}\prod_{j=1}^{M}|x-s_j|^{\gamma\beta_j}}\frac{e^{\gamma X(x)}}{(1-|x|^2)^{\frac{\gamma^2}{2}}}d^2x\\
&-e^{\frac{\gamma c}{2}}\int_0^{2\pi}\frac{1}{\prod_{i=1}^{N}|e^{i\theta}-z_i|^{\gamma\alpha_i}\prod_{j=1}^{M}|e^{i\theta}-s_j|^{\frac{\gamma\beta_j}{2}}}e^{\frac{\gamma}{2}X(e^{i\theta})}d\mu_\partial(\theta)\Bigg)\Bigg].
\end{aligned}$$

Remark 1.3.4. *The boundary measure $d\mu_\partial(\theta)$ is defined as:*

$$d\mu_\partial(\theta)/d\theta=\sum_{j=1}^{M-1}\mu_j\mathbf{1}_{e^{i\theta}\ \text{between}\ s_j\ \text{and}\ s_{j+1}}+\mu_M\mathbf{1}_{e^{i\theta}\ \text{between}\ s_M\ \text{and}\ s_1}. \tag{1.3.23}$$

The difference with $\mathbb{S}^2$ is that if both interaction terms are present, i.e. $\mu>0$, $\mu_j>0$, then we will not land on an expression as simple as (1.3.18). Indeed, it is not possible to perform the same change of variable on c and we must stick to the expression of Theorem 1.3.3. On the other hand if one chooses $\mu=0$ or $\mu_\partial=0$ then the above can be simplified to obtain an expression involving a moment of GMC on the circle $\partial\mathbb{D}$ or respectively on the disk $\mathbb{D}$. For example, in the case $\mu=0$ we have

$$\begin{aligned}
&\langle\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{D},\mu=0}\\
&=\prod_{i=1}^{N}(1-|z_i|^2)^{-\frac{\alpha_i^2}{2}}\prod_{1\le i<i'\le N}|z_i-z_{i'}|^{-\alpha_i\alpha_{i'}}|1-z_i\bar{z}_{i'}|^{-\alpha_i\alpha_{i'}}\prod_{1\le i\le N,1\le j\le M}|z_i-s_j|^{-\alpha_i\beta_j}\prod_{1\le j<j'\le M}|s_j-s_{j'}|^{-\frac{\beta_j\beta_{j'}}{2}}\\
&\frac{2}{\gamma}\Gamma(\frac{2\sum_i\alpha_i+\sum_j\beta_j-2Q}{\gamma})\mathbb{E}\Bigg[\Bigg(\int_0^{2\pi}\frac{1}{\prod_{i=1}^{N}|e^{i\theta}-z_i|^{\gamma\alpha_i}\prod_{j=1}^{M}|e^{i\theta}-s_j|^{\frac{\gamma\beta_j}{2}}}e^{\frac{\gamma}{2}X(e^{i\theta})}d\mu_\partial(\theta)\Bigg)^{-\frac{2\sum_i\alpha_i+\sum_j\beta_j-2Q}{\gamma}}\Bigg].
\end{aligned}$$

We give the equivalent representation on $\mathbb{H}$ equipped with the metric $\hat{g}(x) = \frac{4}{|x+i|^4}$,

$$\begin{aligned}
&\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j) \rangle_{\mathbb{H}} \\
&= \prod_{i=1}^{N} |z_i - \bar{z}_i|^{-\frac{\alpha_i^2}{2}} \prod_{1\le i<i'\le N} |z_i - z_{i'}|^{-\alpha_i\alpha_{i'}} |z_i - \bar{z}_{i'}|^{-\alpha_i\alpha_{i'}} \prod_{1\le i\le N, 1\le j\le M} |z_i - s_j|^{-\alpha_i\beta_j} \prod_{1\le j<j'\le M} |s_j - s_{j'}|^{-\frac{\beta_j\beta_{j'}}{2}} \\
&\int_{\mathbb{R}} dc\, e^{\frac{\gamma p}{2}c} \mathbb{E}\Bigg[\exp\Bigg(-\mu e^{\gamma c} \int_{\mathbb{H}} \frac{\hat{g}(x)^{\frac{\gamma^2}{4}(-p-1)} / |x-\bar{x}|^{\frac{\gamma^2}{2}}}{\prod_{i=1}^{N} |x-z_i|^{\gamma\alpha_i} |x-\bar{z}_i|^{\gamma\alpha_i} \prod_{j=1}^{M} |x-s_j|^{\gamma\beta_j}} e^{\gamma X(x)} d^2x \\
&- e^{\frac{\gamma c}{2}} \int_{\mathbb{R}} \frac{\hat{g}(r)^{\frac{\gamma^2}{8}(-p-1)}}{\prod_{i=1}^{N} |r-z_i|^{\gamma\alpha_i} \prod_{j=1}^{M} |r-s_j|^{\frac{\gamma\beta_j}{2}}} e^{\frac{\gamma}{2}X(r)} d\mu_\partial(r) \Bigg) \Bigg].
\end{aligned}$$

where

$$p = \frac{2\sum_{i=1}^{N}\alpha_i + \sum_{j=1}^{M}\beta_j - 2Q}{\gamma}, \tag{1.3.24}$$

and the boundary measure is defined by

$$d\mu_\partial(r)/dr = \sum_{j=1}^{M-1} \mu_j \mathbf{1}_{s_j<r<s_{j+1}} + \mu_M \mathbf{1}_{r\notin(s_1,s_M)}. \tag{1.3.25}$$

1.4 The techniques of conformal field theory

Now that we have at our disposal a probabilistic construction of LCFT we can verify that it satisfies all the different properties expected of a conformal field theory. This will provide extremely powerful techniques to perform exact computations on the Liouville theory. In particular the celebrated equations of Belavin-Polyakov-Zamolodchikov (BPZ) translate the constraints imposed by the local conformal invariance of CFT. By using these equations we will be able to obtain exact computations on certain correlation functions of the theory. We insist on the fact that all the properties we will consider - Weyl anomaly, KPZ formula and BPZ equations - are theorems that are proved starting from the definition given by Theorem 1.3.1. They are not imposed as axioms as in other approaches [91]. In the following three subsections there will always be two cases, the case of the Riemann sphere $\mathbb{S}^2$ and the case of a domain with boundary.

1.4.1 Weyl anomaly and KPZ formula

In section 1.3 we gave a definition of LCFT for any background metric g defined on the Riemann sphere $\mathbb{S}^2$. It is thus natural to wonder how the correlation functions of LCFT depend on this choice of background metric. The answer to this question is given by the Weyl anomaly, a property expected of all CFT's. Following [20]:

Theorem 1.4.1. *(Weyl anomaly on $\mathbb{S}^2$) Given a metric $g = e^{\varphi}\hat{g}$ conformally equivalent to the spherical metric $\hat{g}$ we have,*

$$\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \rangle_{\mathbb{S}^2,g} = \exp\left(\frac{c_L}{96\pi} \Big(\int_{\mathbb{C}} |\partial\varphi(x)|^2 d^2x + 4\int_{\mathbb{C}} \varphi(x)\hat{g}(x)d^2x \Big) \right) \langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \rangle_{\mathbb{S}^2,\hat{g}}, \tag{1.4.1}$$

where $c_L = 1 + 6Q^2$. This constant c_L is the so-called central charge of the Liouville conformal field theory.

We can also write down the exact same theorem for a domain with boundary. We write here the result for the unit disk $\mathbb{D}$ although of course a similar formula is true for the upper half plane $\mathbb{H}$ or for other geometries where LCFT has been constructed. Following [53]:

Theorem 1.4.2. *(Weyl anomaly on $\mathbb{D}$) Given a metric $g = e^{\varphi}dx^2$ conformally equivalent to the Euclidean metric dx^2 on $\mathbb{D}$ we have,*

$$\langle\prod_{i=1}^{N} V_{\alpha_i}(z_i)\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{D},g} = \exp\left(\frac{c_L}{96\pi}(\int_{\mathbb{D}}|\partial\varphi(x)|^2d^2x + 4\int_{\partial\mathbb{D}}\varphi(e^{i\theta})d\theta)\right)\langle\prod_{i=1}^{N} V_{\alpha_i}(z_i)\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{D}}, \tag{1.4.2}$$

where again $c_L = 1 + 6Q^2$.

We now move on to the KPZ formula which describes the behaviour of the Liouville theory when one applies a conformal automorphism to the domain. In other words it tells us that the correlations of LCFT behave as conformal tensors. We start with $\mathbb{S}^2$, see [20]:

Theorem 1.4.3. *(KPZ formula on $\mathbb{S}^2$) For any conformal automorphism ψ of $\mathbb{S}^2$, the following holds:*

$$\langle\prod_{i=1}^{N} e^{\alpha_i\phi(\psi(z_i))}\rangle_{\mathbb{S}^2,\hat{g}} = \prod_{i=1}^{N}|\psi'(z_i)|^{-2\Delta_{\alpha_i}}\langle\prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2,\hat{g}}. \tag{1.4.3}$$

The $\Delta_{\alpha_i} := \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ in the above theorem are the so-called conformal weights of CFT. Similarly for a domain with boundary such as the unit disk $\mathbb{D}$, see [53]:

Theorem 1.4.4. *(KPZ formula on $\mathbb{D}$) For any conformal automorphism ψ of $\mathbb{D}$, the following holds:*

$$\langle\prod_{i=1}^{N} V_{\alpha_i}(\psi(z_i))\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(\psi(s_j))\rangle_{\mathbb{D}} = \prod_{i=1}^{N}|\psi'(z_i)|^{-2\Delta_{\alpha_i}}\prod_{j=1}^{M}|\psi'(s_j)|^{-\Delta_{\beta_j}}\langle\prod_{i=1}^{N} V_{\alpha_i}(z_i)\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{D}}. \tag{1.4.4}$$

Another very similar result that we have at our disposal is the conformal change of domain formula. This tells us how the theory behaves when we change domains by a conformal map ψ (for instance we can map the unit disk $\mathbb{D}$ to upper half plane $\mathbb{H}$). The result is again that the theory behaves as a conformal tensor.

Theorem 1.4.5. *(Conformal change of domain) Let D be a domain of $\mathbb{C}$ with a smooth boundary and conformally equivalent to the unit disk $\mathbb{D}$. Let $\psi : D \mapsto \mathbb{D}$ be a conformal map between D and $\mathbb{D}$ and let $g_\psi = |\psi'|^2 g(\psi)$ be the pull-back of a metric g on $\mathbb{D}$ by ψ. Then we have:*

$$\langle\prod_{i=1}^{N} V_{\alpha_i}(z_i)\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{D,g_\psi} = \prod_{i=1}^{N}|\psi'(z_i)|^{2\Delta_{\alpha_i}}\prod_{j=1}^{M}|\psi'(s_j)|^{\Delta_{\beta_j}}\langle\prod_{i=1}^{N} V_{\alpha_i}(\psi(z_i))\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(\psi(s_j))\rangle_{\mathbb{D},g}.$$

1.4.2 Degenerate fields and BPZ equations

In this last subsection we explain the BPZ equations of Belavin-Polyakov-Zamolodchikov that will give us the constraints we need to compute certain correlation functions of LCFT. We start by explaining the case of the Riemann sphere $\mathbb{S}^2$. Consider the following function,

$$z \to \langle V_{-\chi}(z)\prod_{i=1}^{N} V_{\alpha_i}(z_i)\rangle_{\mathbb{S}^2,\hat{g}}, \tag{1.4.5}$$

where the points $z, z_i \in \mathbb{S}^2$ have respective weights $-\chi, \alpha_i$ chosen so that the Seiberg bounds (1.3.6) hold. The reason why we distinguish the point z of weight $-\chi$ is that we will choose this $-\chi$ to be equal to a very special value, either $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$. $V_{-\chi}(z)$ is then called a degenerate field insertion and the value of $-\chi$ is the degenerate weight. More generally, the degenerate weights of order $(r,s) \in \mathbb{N}^{*2}$ are given by $-(\frac{(r-1)\gamma}{2} + \frac{2(s-1)}{\gamma})$. Here we are considering degenerate insertions with order $(2,1)$ and $(1,2)$, which are the most important cases for the integrability program of LCFT. Under this very specific condition the function of z given by (1.4.5) will be solution to a second order differential equation. Indeed, it is shown in [60] using probabilistic techniques:

Theorem 1.4.6. *Consider points $z, z_i \in \mathbb{S}^2$ with respective weights $-\chi, \alpha_i \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.6) hold. Then we have the following PDE,*

$$(\frac{1}{\chi^2}\partial_{zz} + \sum_{i=1}^{N} \frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{1}{z-z_i}\partial_{z_i})\langle V_{-\chi}(z)\prod_{i=1}^{N} V_{\alpha_i}(z_i)\rangle_{\mathbb{S}^2,\hat{g}} = 0,$$

under the condition that χ is worth $\frac{\gamma}{2}$ or $\frac{2}{\gamma}$.

Again we see the conformal weight $\Delta_{\alpha_i} := \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ appear. For the purposes of proving the DOZZ formula the authors of [61] have applied the differential equation more specifically to the case of a four-point function, see subsection 1.5.1. Naturally a similar story can be told for the case of a domain with boundary. It will be more convenient to write the differential equation on $\mathbb{H}$ although by conformal mapping we can easily transform it into an equation on $\mathbb{D}$. For this boundary case we consider correlations of the type $\langle\prod_{i=1}^{N} V_{\alpha_i}(z_i)\prod_{j=1}^{M} B_{\beta_j}(s_j)\rangle_{\mathbb{H}}$ where we have bulk insertions $z_i \in \mathbb{H}$ with $\alpha_i \in \mathbb{R}$ and boundary insertions $s_j \in \partial\mathbb{H} = \mathbb{R}$ with $\beta_j \in \mathbb{R}$ and again we choose the weights so that the bounds (1.3.22) hold. Now in the boundary case there are two possibilities for the position of the degenerate insertion, either in the bulk $\mathbb{H}$ or on the boundary $\mathbb{R}$. Both of these cases will lead to BPZ equations. We start by giving the theorem corresponding to the case where the degenerate insertion is placed in the bulk:

Theorem 1.4.7. *Consider points $z, z_i \in \mathbb{H}$ with respective weights $-\chi$, $\alpha_i \in \mathbb{R}$ and points $s_j \in \mathbb{R}$ with respective weights $\beta_j \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.22) hold. Then we have the following PDE,*

$$\begin{aligned}&(\frac{1}{\chi^2}\partial_{zz} + \frac{\Delta_{-\chi}}{(z-\overline{z})^2} + \sum_{i=1}^{N} \frac{\Delta_{\alpha_i}}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{\Delta_{\alpha_i}}{(z-\overline{z_i})^2} + \sum_{j=1}^{M} \frac{\Delta_{\beta_j}}{(z-s_j)^2}\\ &+ \frac{1}{z-\overline{z}}\partial_{\overline{z}} + \sum_{i=1}^{N} \frac{1}{z-z_i}\partial_{z_i} + \sum_{i=1}^{N} \frac{1}{z-\overline{z_i}}\partial_{\overline{z_i}} + \sum_{j=1}^{M} \frac{1}{z-s_j}\partial_{s_j})\langle V_{-\chi}(z)\prod_{i=1}^{N} V_{\alpha_i}(z_i)\prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{H}} = 0,\end{aligned}$$

where χ needs to be equal to $\frac{\gamma}{2}$ or $\frac{2}{\gamma}$.

The proof of this differential equation is very similar to the sphere case and there is no additional major difficulties. We only need to pay attention to the boundary measure $d\mu_\partial$ since we put different boundary cosmological constants μ_j, and the technical problem is the same as in the subsection 2.3 where we use a ϵ-cut off around each boundary insertion s_j and the ϵ singularities regrouped together can be shown to converge to 0.

Moving on to the second possibility where we place the degenerate insertion on the boundary, we also expect to have a BPZ equation of order 2 but it turns out that for such an equation to hold we impose a very special condition on the μ_∂ values around the degenerate insertion. More precisely, for each μ_j we can write

$$\mu_j = \sqrt{\frac{\mu}{\sin(\pi\frac{\gamma^2}{4})}}\cos(\pi\gamma(\sigma_j - \frac{Q}{2})). \tag{1.4.6}$$

When the degenerate insertion s is added between s_j and s_{j+1} (by convention $s_0 = -\infty$ and $s_{M+1} = \infty$), the BPZ equations hold true when the degenrate vertex operator $B_{-\chi}^{\mu_j,\mu_j'}(s)$ satisfies $\mu_j' = \sqrt{\frac{\mu}{\sin(\pi\frac{\gamma^2}{4})}}\cos(\pi\gamma(\sigma_j \pm \frac{\chi}{2} - \frac{Q}{2}))$. The statement is as follows:

Conjecture 1. *Consider points $z_i \in \mathbb{H}$ with weights $\alpha_i \in \mathbb{R}$ and points $s, s_j \in \mathbb{R}$ with respective weights $-\chi, \beta_j \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.22) hold, where $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. We order the boundary*

insertion points accordingly : $s_1 < \cdots < s_M$. Then we have the following PDE for $s > s_M$:

$$(\frac{1}{\chi^2}\partial_{ss} + \sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(s-z_i)^2} + \sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(s-\overline{z}_i)^2} + \sum_{j=1}^{M}\frac{\Delta_{\beta_j}}{(s-s_j)^2} + \sum_{i=1}^{N}\frac{1}{s-z_i}\partial_{z_i} + \sum_{i=1}^{N}\frac{1}{s-\overline{z}_i}\partial_{\overline{z}_i} + \sum_{j=1}^{M}\frac{1}{s-s_j}\partial_{s_j})$$
$$\langle B_{-\chi}^{\mu_M,\mu_M'}(s)\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{H}} = 0,$$

under the condition that there exists $\sigma_M \in \mathbb{C}$ such that

$$\mu_M = \sqrt{\frac{\mu}{\sin(\pi\frac{\gamma^2}{4})}}\cos(\pi\gamma(\sigma_M - \frac{Q}{2})), \quad \mu_M' = \sqrt{\frac{\mu}{\sin(\pi\frac{\gamma^2}{4})}}\cos(\pi\gamma(\sigma_M \pm \frac{\chi}{2} - \frac{Q}{2})).$$

We also have similar equations when s is inserted elsewhere on $\mathbb{R}$.

This phenomenon is partially understood in the special case where $\mu = 0$, where the condition becomes simply $\mu_j' = e^{\pm i\frac{\gamma\chi}{2}}\mu_j$. This is done in chapter 3 for the four point correlations and discussed in 4 for the higher order BPZ equations (see also the next subsection). To understand the constraint in the general case where $\mu > 0$, we need to do some finer analysis of the potential singularities and this would hopefully come out in an upcoming work. We also expect the some other conditions for the BPZ equations to hold:

Conjecture 2. *Consider points $z_i \in \mathbb{H}$ with weights $\alpha_i \in \mathbb{R}$ and points $s, s_j \in \mathbb{R}$ with respective weights $-\chi, \beta_j \in \mathbb{R}$ chosen so that the Seiberg bounds (1.3.22) hold, where $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. Suppose that μ_∂ is a constant (instead of piecewise constant) and verifies $\mu = \mu_\partial^2\tan(\pi\frac{\gamma^2}{8})$, then we have the following PDE:*

$$(\frac{1}{\chi^2}\partial_{ss} + \sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(s-z_i)^2} + \sum_{i=1}^{N}\frac{\Delta_{\alpha_i}}{(s-\overline{z}_i)^2} + \sum_{j=1}^{M}\frac{\Delta_{\beta_j}}{(s-s_j)^2} + \sum_{i=1}^{N}\frac{1}{s-z_i}\partial_{z_i} + \sum_{i=1}^{N}\frac{1}{s-\overline{z}_i}\partial_{\overline{z}_i} + \sum_{j=1}^{M}\frac{1}{s-s_j}\partial_{s_j})$$
$$\langle B_{-\chi}(s)\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B_{\beta_j}(s_j)\rangle_{\mathbb{H}} = 0.$$

1.4.3 Higher order BPZ equations, *Summary of chapter 4*

In fact, CFT predicts there to be BPZ equations of all orders. To illustrate the idea, let us consider the sphere case in this subsction. We can index the BPZ equations by two parameters (r, s), with r, s positive integers. This index corresponds to the parameter $-(\frac{(r-1)\gamma}{2} + \frac{2(s-1)}{\gamma})$ for the degenerate insertion. The equation associated with the parameter (r, s) is a partial differential equation of order rs in several complex variables. There is no general combinatorial formula for the BPZ equations of general orders (r, s). Nevertheless, in 1988, Benoît and Saint-Aubin (BSA, [11]) found an explicit formula for the BPZ equations of order $(r, 1)$ and $(1, r)$. To present the combinatorial formula, we will need the notations for some differential operators:

Definition 1.4.8. *We introduce the differential operators $(L_{-n})_{n\geq 1}$ defined as*

$$L_{-1} := \partial_z, \quad L_{-n} := \sum_{l=1}^{N}\left(-\frac{1}{(z_l - z)^{n-1}}\partial_{z_l} + \frac{\Delta_{\alpha_l}(n-1)}{(z_l - z)^n}\right) \quad n \geq 2. \tag{1.4.7}$$

Now let us state the BSA formulas for the BPZ equations:

Theorem 1.4.9. *Let $r \geq 2$ an integer and*

$$\chi = \frac{\gamma}{2} \text{ or } \frac{2}{\gamma}. \tag{1.4.8}$$

The BPZ equations of order r hold true for $\gamma \in (\sqrt{\frac{2(r-2)}{r-1}}, 2)$ when $\chi = \frac{\gamma}{2}$ and for $\gamma \in (0,2)$ when $\chi = \frac{2}{\gamma}$:

$$\mathcal{D}_r \langle V_{-(r-1)\chi}(z) \prod_{l=1}^{N} V_{\alpha_l}(z_l) \rangle = 0, \tag{1.4.9}$$

where the differential operator $\mathcal{D}_r$ is given by the Benoît and Saint-Aubin's formula:

$$\mathcal{D}_r = \sum_{k=1}^{r} \sum_{\substack{(n_1,\dots,n_k)\in(\mathbb{N}^*)^k \\ n_1+\cdots+n_k=r}} \frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(\sum_{i=j+1}^{k} n_i)} L_{-n_1}\dots L_{-n_k}, \tag{1.4.10}$$

with L_{-n} defined in definition 4.1.8.

Remark 1.4.10. *The constraint on γ when $\chi = \frac{\gamma}{2}$ is a purely technical condition and is only required by Proposition 4.2.9. In particular, we have BPZ equations of all orders for $\gamma \in [\sqrt{2}, 2)$ when $\chi = \frac{\gamma}{2}$.*

It can be easily checked that when $r = 2$, the above equations are coherent with the results in Theorem 1.4.6.

In the boundary LCFT case we can have a boundary degenerate insertion or a bulk degenerate insertion. When it comes to a boundary degenerate insertion $B^{\pm}_{-(r-1)\chi}(s)$ defined as below, we will work with an extended definition where s lives in the upper-half plane:

Definition 1.4.11. *Let $\mu_1, \dots \mu_M > 0$, $-\infty < s_1 < \cdots < s_M < \infty$ and $s \in \mathbb{H}\backslash\{z_i, 1 \le i \le N\} \cup \mathbb{R}$. We define the extended correlation function $\langle B^{+}_{-(r-1)\chi}(s) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B^{\mu_{j-1},\mu_j}_{\beta_j}(s_j) \rangle_{\mathbb{H}}$ by*

$$\begin{aligned}
&\prod_{i=1}^{N} ((z_i - s)(\overline{z}_i - s))^{\frac{(r-1)\chi\alpha_i}{2}} \prod_{j=1}^{M} (s_i - s)^{\frac{(r-1)\chi\beta_j}{2}} \prod_{1\le i<i'\le N} |z_i - z_{i'}|^{-\alpha_i\alpha_{i'}} \prod_{1\le i\le i'\le N} |z_i - \overline{z}_{i'}|^{-\alpha_i\alpha_{i'}} \prod_{1\le i\le N, 1\le j\le M} |z_i - s_j|^{-\alpha_i\beta_j} \\
&\prod_{1\le j<j'\le M} |s_j - s_{j'}|^{-\beta_j\beta_{j'}/2} \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(s-u)^{\frac{(r-1)\gamma\chi}{2}}}{|u - z_i|^{\gamma\alpha_i} |u - s_j|^{\gamma\beta_j/2}} e^{\frac{\gamma}{2} X_{\mathbb{H}}(u)} \hat{g}_{\mathbb{H}}(u)^{-\frac{\gamma^2}{8}(p - \frac{(r-1)\chi}{\gamma} + 1)} d\mu_{\partial}(u) \right)^{-p + \frac{(r-1)\chi}{\gamma}} \right].
\end{aligned} \tag{1.4.11}$$

Similarly, we define $\langle B^{-}_{-(r-1)\chi}(s) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B^{\mu_{j-1},\mu_j}_{\beta_j}(s_j) \rangle_{\mathbb{H}}$ by replacing the term $(s-u)^{\frac{(r-1)\gamma\chi}{2}}$ in the above integral by $(u-s)^{\frac{(r-1)\gamma\chi}{2}}$.

Remark 1.4.12. *We can take a larger family of values for μ_j, as long as the moment of GMC in the above expectation is well defined without ambiguity. See chapter 3 for more details.*

Now we state the BPZ equations for boundary LCFT, where we can prove the result without constraint on γ.

Theorem 1.4.13. *Let $r \ge 2$ an integer and $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. Let $\mu_1, \dots, \mu_M > 0$, $s_1 < \cdots < s_M$ and $s \in \mathbb{H}\backslash\{z_i, 1 \le i \le N\}$. The BPZ equations of order r for a boundary degenerate insertion hold true for $\gamma \in (0,2)$:*

$$\mathcal{D}^{\mathbb{H}}_r \langle B^{\pm}_{-(r-1)\chi}(s) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B^{\mu_{j-1},\mu_j}_{\beta_j}(s_j) \rangle_{\mathbb{H}} = 0, \tag{1.4.12}$$

where the expression of the differential operator $\mathcal{D}^{\mathbb{H}}_r$ is given by (1.4.9), where we replace the operators L_{-n} by $L^{\mathbb{H}}_{-n}$ defined as $L^{\mathbb{H}}_{-1} := \partial_s$, and for $n \ge 2$:

$$\begin{aligned}
L^{\mathbb{H}}_{-n} :=& \sum_{l=1}^{N} \left(-\frac{1}{(z_l - s)^{n-1}} \partial_{z_l} - \frac{1}{(\overline{z}_l - s)^{n-1}} \partial_{\overline{z}_l} + \frac{\Delta_{\alpha_l}(n-1)}{(z_l - s)^n} + \frac{\Delta_{\alpha_l}(n-1)}{(\overline{z}_l - s)^n} \right) \\
&+ \sum_{l=1}^{M} \left(-\frac{1}{(s_l - s)^{n-1}} \partial_{s_l} + \frac{\Delta_{\beta_l}(n-1)}{(s_l - s)^n} \right).
\end{aligned} \tag{1.4.13}$$

The BPZ equations of order r also hold true when we insert a bulk degenerate insertion: for $\gamma \in (0,2)$,

$$\mathcal{D}_r^{\mathbb{H},z}\langle V_{-(r-1)\chi}(z)\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B_{\beta_j}^{\mu_{j-1},\mu_j}(s_j)\rangle_{\mathbb{H}}=0, \tag{1.4.14}$$

where $\mathcal{D}_r^{\mathbb{H},z}$ is defined by the expression (1.4.9) where we replace L_{-n} by $L_{-n}^{\mathbb{H},z}$ defined as $L_{-1}^{\mathbb{H},z}:=\partial_z$, and for $n\geq 2$,

$$\begin{aligned}L_{-n}^{\mathbb{H},z}:=&\sum_{l=1}^{N}\left(-\frac{1}{(z_l-z)^{n-1}}\partial_{z_l}-\frac{1}{(\overline{z}_l-z)^{n-1}}\partial_{\overline{z}_l}+\frac{\Delta_{\alpha_l}(n-1)}{(z_l-z)^n}+\frac{\Delta_{\alpha_l}(n-1)}{(\overline{z}_l-z)^n}\right)\\&-\frac{1}{(\overline{z}-z)^{n-1}}\partial_{\overline{z}}+\frac{\Delta_{-(r-1)\chi}(n-1)}{(\overline{z}-z)^n}+\sum_{l=1}^{M}\left(-\frac{1}{(s_l-z)^{n-1}}\partial_{s_l}+\frac{\Delta_{\beta_l}(n-1)}{(s_l-z)^n}\right).\end{aligned} \tag{1.4.15}$$

1.5 Integrability of GMC and of Liouville theory

1.5.1 The DOZZ formula on the Riemann sphere

In this section we will sketch the proof of the celebrated DOZZ formula which gives the value of the three-point correlation function of LCFT on $\mathbb{S}^2$. We state the main result of [61]:

Theorem 1.5.1. *(DOZZ formula) Choose three points $z_1,z_2,z_3\in\mathbb{S}^2$ of respective weights $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ such that the bounds (1.3.6) are satisfied. Then we have the following expression for the three-point correlation function,*

$$\langle\prod_{i=1}^{3}V_{\alpha_i}(z_i)\rangle_{\mathbb{S}^2,\hat{g}}=|z_1-z_2|^{2\Delta_{12}}|z_2-z_3|^{2\Delta_{23}}|z_1-z_3|^{2\Delta_{13}}C_\gamma(\alpha_1,\alpha_2,\alpha_3), \tag{1.5.1}$$

where we have:

$$C_\gamma(\alpha_1,\alpha_2,\alpha_3)=(\pi\mu l(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{2-\frac{\gamma^2}{2}})^{\frac{2Q-\overline{\alpha}}{\gamma}}\frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-Q)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\frac{\overline{\alpha}}{2}-\alpha_3)}. \tag{1.5.2}$$

In the above expressions we have $l(x)=\frac{\Gamma(x)}{\Gamma(1-x)}$, $\overline{\alpha}=\alpha_1+\alpha_2+\alpha_3$, $\Delta_{12}=\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}$, $\Delta_{\alpha_i}=\frac{\alpha_i}{2}(Q-\frac{\alpha_i}{2})$, $Q=\frac{\gamma}{2}+\frac{2}{\gamma}$ and Υ_γ is a special function expressed in terms of the double gamma function Γ_γ,

$$\Upsilon_\gamma(x)=\frac{1}{\Gamma_\gamma(x)\Gamma_\gamma(Q-x)}, \tag{1.5.3}$$

where:

$$\ln\Gamma_\gamma(x)=\int_0^\infty\frac{dt}{t}\left[\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})}-\frac{(\frac{Q}{2}-x)^2}{2}e^{-t}+\frac{x-\frac{Q}{2}}{t}\right]. \tag{1.5.4}$$

We will now briefly mention the main arguments leading to the DOZZ formula. The first part of the theorem that gives the dependence on the positions z_1, z_2, z_3 is a consequence of the KPZ formula (1.4.3). Indeed one simply needs to apply (1.4.3) to the Mobius map ψ that satisfies $\psi(z_1)=0$, $\psi(z_2)=1$, $\psi(z_3)=\infty$. The structure constant $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$ is then recovered by taking the following limit:

$$C_\gamma(\alpha_1,\alpha_2,\alpha_3)=\lim_{z_3\to\infty}|z_3|^{4\Delta_3}\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\mathbb{S}^2,\hat{g}}. \tag{1.5.5}$$

The difficult part of the theorem is thus to give the exact value for the constant $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$. This is where we need to use the two BPZ equations of Theorem 1.4.6 for a four-point function. There are two such

equations as the degenerate weight can be equal either to $-\frac{\gamma}{2}$ or to $-\frac{2}{\gamma}$. One can then deduce two non-trivial relations on $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$ which are,

$$\frac{C_\gamma(\alpha_1+\frac{\gamma}{2},\alpha_2,\alpha_3)}{C_\gamma(\alpha_1-\frac{\gamma}{2},\alpha_2,\alpha_3)}=-\frac{1}{\pi\mu}\frac{l(-\frac{\gamma^2}{4})l(\frac{\gamma\alpha_1}{2})l(\frac{\alpha_1\gamma}{2}-\frac{\gamma^2}{4})l(\frac{\gamma}{4}(\bar{\alpha}-2\alpha_1-\frac{\gamma}{2}))}{l(\frac{\gamma}{4}(\bar{\alpha}-\frac{\gamma}{2}-2Q))l(\frac{\gamma}{4}(\bar{\alpha}-2\alpha_3-\frac{\gamma}{2}))l(\frac{\gamma}{4}(\bar{\alpha}-2\alpha_2-\frac{\gamma}{2}))}, \tag{1.5.6}$$

and

$$\frac{C_\gamma(\alpha_1+\frac{2}{\gamma},\alpha_2,\alpha_3)}{C_\gamma(\alpha_1-\frac{2}{\gamma},\alpha_2,\alpha_3)}=-\frac{1}{\pi\tilde{\mu}}\frac{l(-\frac{4}{\gamma^2})l(\frac{2\alpha_1}{\gamma})l(\frac{2\alpha_1}{\gamma}-\frac{4}{\gamma^2})l(\frac{1}{\gamma}(\bar{\alpha}-2\alpha_1-\frac{2}{\gamma}))}{l(\frac{1}{\gamma}(\bar{\alpha}-\frac{2}{\gamma}-2Q))l(\frac{1}{\gamma}(\bar{\alpha}-2\alpha_3-\frac{2}{\gamma}))l(\frac{1}{\gamma}(\bar{\alpha}-2\alpha_2-\frac{2}{\gamma}))}, \tag{1.5.7}$$

with $\tilde{\mu}=\frac{(\mu\pi l(\frac{\gamma^2}{2}))^{\frac{4}{\gamma^2}}}{\pi l(\frac{4}{\gamma^2})}$ being the dual cosmological constant. Of course by symmetry similar relations are verified for a shift on α_2 or on α_3. It turns out that these two shift equations completely determine the dependence on the α_i if $\gamma^2\notin\mathbb{Q}$. Then one can extent the formula to all values of γ by a standard continuity argument.

Let us give more comments on the shift equations. One of the major difficulties in the rigorous mathematical proof is the second shift equation (1.5.7). In physics this second shift equation is predicted by replacing $\frac{\gamma}{2}$ by $\frac{2}{\gamma}$ in the first shift equation. However they are very different in nature. While the first equation comes from a Taylor expansion, the second equation will have a fractional shift on the moment of the GMC, which cannot be simply obtained by an expansion. This requires a careful study of the local behavior of GFF around singularities and we will need to compute the value of the so-called reflection coefficient before we can obtain the second shift equation. For a much more complete summary of the proof of the DOZZ formula one can take a look at the lecture notes [103]. We will see that in certain cases such as GMC on the interval or the Fyodorov-Bouchaud formula there will be some simplifications, and in the case of the boundary Liouville structure constants there will be some complications.

1.5.2 GMC on the unit interval, *Summary of chapter 2*

We study the GMC measure associated to the log-correlated field X_I defined on the unit interval $[0,1]$. We will prove an exact formula for the moments of the total mass of the GMC measure. The log-correlated field X that we work with has a covariance given by:

$$\mathbb{E}[X(x)X(y)]=2\ln\frac{1}{|x-y|}. \tag{1.5.8}$$

Note that this is a restriction of the filed X defined on $\overline{\mathbb{H}}$ with covariance given by (1.2.1). For this model of GMC the most general quantity that we will provide an expression for is defined for $\gamma\in(0,2)$ and for real p, a, b obeying the bounds (1.5.10) written below:

$$M(\gamma,p,a,b):=\mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{1.5.9}$$

This is the moment p of the total mass of our GMC measure with two "insertion points" in 0 and 1 of weight a and b. As explained in subsection 1.2.2, the theory of Gaussian multiplicative chaos tells us that these moments are non-trivial, i.e. different from 0 and $+\infty$, if and only if:

$$a>-\frac{\gamma^2}{4}-1,\quad b>-\frac{\gamma^2}{4}-1,\quad p<\frac{4}{\gamma^2}\wedge(1+\frac{4}{\gamma^2}(1+a))\wedge(1+\frac{4}{\gamma^2}(1+b)). \tag{1.5.10}$$

The techniques of proof of this formula on $[0,1]$ are similar in spirit to the techniques used to prove the DOZZ formula. First remark that $M(\gamma,p,a,b)$ can be related to $\langle B^{0,1}_{\beta_1}(0)B^{1,0}_{\beta_2}(1)B^{0,0}_{\beta_3}(\infty)\rangle_{\mathbb{H},g}$, where we take $\mu_\partial=0$ on $[0,1]^c$ and $\mu_\partial=1$ on $[0,1]$ and the background metric is given by $g(x)=\frac{1}{|x|_+^4}$. We are free to choose the metric thanks to Theorem 1.4.2. Now by applying (1.4.12) with $r=2$ we have the differential equations for the four point auxiliary function $\langle B^{\pm}_{-\chi}(s)B^{0,1}_{\beta_1}(0)B^{1,0}_{\beta_2}(1)B^{0,0}_{\beta_3}(\infty)\rangle_{\mathbb{H},g}$. On the other hand, the way we choose

μ_∂ makes the parameter β_3 independent of the total mass of GMC, and this parameter gives a degree of freedom to the moment p. Therefore we can work with fixed p and obtain two shift equations on a and b, which completely determine the dependence of (1.5.9) on a and b. For the dependence of p we can do as in the proof of Fyodorov-Bouchaud formula, where the author calculates simply the negative integer moments of the total mass of GMC and shows that this suffices to characterize the law. However in the interval case, as can be observed from the final exact formula, it is much less trivial to exploit its distribution. A self-contained way is to again use the reflection techniques introduced in the proof of DOZZ formula, where this time the simplification is that we do not need the exact value of the reflection coefficient.

We state here the main result of chapter 2:

Theorem 1.5.2. *For $\gamma \in (0,2)$ and for p, a, b satisfying (1.5.10)[6], $M(\gamma,p,a,b)$ is given by,*

$$\frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p}\frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})},$$

where the function $\Gamma_\gamma(x)$ is defined by equation (1.5.4).

As a corollary by choosing $a=b=0$ we obtain the value of the moments of the GMC measure without insertions:

Corollary 1.5.3. *For $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$:*

$$\mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}X_I(x)}dx)^p] = \frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p}\frac{\Gamma_\gamma(\frac{2}{\gamma}-(p-1)\frac{\gamma}{2})^2\Gamma_\gamma(\frac{4}{\gamma}-(p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}+\frac{\gamma}{2})^2\Gamma_\gamma(\frac{4}{\gamma}-(2p-2)\frac{\gamma}{2})}. \tag{1.5.11}$$

Finally we state that Theorem 1.5.2 can be used to obtain the following tail expansion for a GMC measure in dimension one. In the result appears $\overline{R}_1^\partial(\alpha)$, a one-dimensional reflection coefficient that can be calculated based on the value of $M(\gamma,p,a,b)$.

Proposition 1.5.4. *For $\gamma \in (0,2)$ and $\eta \in (0,1)$ define:*

$$I_{1,\eta}^\partial(\alpha) := \int_0^\eta x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X_I(x)}dx. \tag{1.5.12}$$

Then for $\alpha \in (\frac{\gamma}{2}, Q)$ we have the following tail expansion for $I_{1,\eta}^\partial(\alpha)$ as $u \to \infty$ and for some $\nu > 0$,

$$\mathbb{P}(I_{1,\eta}^\partial(\alpha) > u) = \frac{\overline{R}_1^\partial(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\alpha)+\nu}}), \tag{1.5.13}$$

where the value of $\overline{R}_1^\partial(\alpha)$ is given by:

$$\overline{R}_1^\partial(\alpha) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)-\frac{1}{2}}}{(Q-\alpha)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}}\frac{\Gamma_\gamma(\alpha-\frac{\gamma}{2})}{\Gamma_\gamma(Q-\alpha)}. \tag{1.5.14}$$

1.5.3 Integrability of boundary Liouville theory, *Summary of chapter 3*

In this subsection we work in the probabilistic framework of boundary LCFT with $\mu = 0$. We present exact formulas for the basic correlation functions of the theory, i.e., the bulk one point function, the bulk-boundary correlator, the boundary two-point and the boundary three-point functions. These four correlations should be seen as the fundamental building blocks of boundary Liouville theory, playing the analogue role of the DOZZ formula in the case of the Riemann sphere.

[6] Again the result also holds for all complex p such that $\mathrm{Re}(p)$ satisfies the bounds (1.5.10).

We suppose that the following Seiberg bounds hold:

$$\sum_{i=1}^{N}\alpha_i+\sum_{j=1}^{M}\frac{\beta_j}{2}>Q,\quad \forall j,\ \beta_j<Q. \tag{1.5.15}$$

Notice here that we do not have the condition $\alpha_i<Q$ as we do not have a bulk potential. Using conformal invariance one can thus reduce computing the following basic correlation functions to computing a single constant called the structure constant:

- **Bulk one-point function.** For $z\in\mathbb{H},\quad \alpha>Q$:

$$\langle V_\alpha(z)\rangle_{\mathbb{H}}=\frac{U(\alpha)}{|z-\overline{z}|^{2\Delta_\alpha}}. \tag{1.5.16}$$

- **Bulk-boundary correlator.** For $z\in\mathbb{H},\ s\in\mathbb{R},\quad \beta<Q,\quad \alpha+\frac{\beta}{2}>Q$:

$$\langle V_\alpha(z)B_\beta(s)\rangle_{\mathbb{H}}=\frac{G(\alpha,\beta)}{|z-\overline{z}|^{2\Delta_\alpha-\Delta_\beta}|z-s|^{2\Delta_\beta}}. \tag{1.5.17}$$

- **Boundary two-point function.** For $s_1,s_2\in\mathbb{R},\quad \beta\in(\frac{\gamma}{2},Q)$:

$$\left\langle B_\beta^{\mu_1,\mu_2}(s_1)B_\beta^{\mu_2,\mu_1}(s_2)\right\rangle_{\mathbb{H}}=\frac{R(\beta,\mu_1,\mu_2)}{|s_1-s_2|^{2\Delta_\beta}}. \tag{1.5.18}$$

- **Boundary three-point function.** For $i\in\{1,2,3\}$, $s_i,\beta_i\in\mathbb{R}$ satisfying $\beta_i<Q$ and $\sum_i\beta_i>2Q$:

$$\left\langle B_{\beta_1}^{\mu_3,\mu_1}(s_1)B_{\beta_2}^{\mu_1,\mu_2}(s_2)B_{\beta_3}^{\mu_2,\mu_3}(s_3)\right\rangle_{\mathbb{H}}=\frac{H_{(\mu_1,\mu_2,\mu_3)}^{(\beta_1,\beta_2,\beta_3)}}{|s_1-s_2|^{\Delta_1+\Delta_2-\Delta_3}|s_1-s_3|^{\Delta_1+\Delta_3-\Delta_2}|s_2-s_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{1.5.19}$$

We have used the notations $\Delta_\alpha=\frac{\alpha}{2}(Q-\frac{\alpha}{2})$, $\Delta_\beta=\frac{\beta}{2}(Q-\frac{\beta}{2})$, and $\Delta_i=\frac{\beta_i}{2}(Q-\frac{\beta_i}{2})$. Each of the four structure constants U,G,R,H will have a definition involving Gaussian multiplicative chaos.

With this at hand one can now give a probabilistic definition to the four structure constants U,G,R,H using moments of GMC on $\mathbb{H}$. We study the log-correlated field X defined on $\overline{\mathbb{H}}$ with covariance given by

$$\mathbb{E}[X(x)X(y)]=2\ln\frac{1}{|x-y|}+2\ln|x|_++2\ln|y|_+,\quad \text{where } |x|_+=\max(|x|,1). \tag{1.5.20}$$

The background metric that we work with is $g(x)=\frac{1}{|x|_+^4}$. In order to define the boundary two-point and three-point functions we will consider parameters μ_1,μ_2,μ_3 in $\mathbb{C}$. To be able to choose a suitable branch cut to define the probabilistic expressions below, we introduce the following conditions we will refer to as the half-space conditions.

Definition 1.5.5. *(Half-space condition for μ_i) Consider $\mu_1,\mu_2,\mu_3\in\mathbb{C}$. We say that $(\mu_i)_{i=1,2,3}$ satisfies the half-space condition if there exists a half-space $\mathcal{H}$ of $\mathbb{C}$ whose boundary is a line passing through the origin not equal to the real axis and satisfying the following. The half-space $\mathcal{H}$ does not contain the half-line $(-\infty,0)$. Each μ_i is contained in $\overline{\mathcal{H}}$ (the half-space with its boundary included) and the sum $\mu_1+\mu_2+\mu_3$ is strictly contained in $\mathcal{H}$. We will also refer to the half-space condition for a pair $\mu_1,\mu_2\in\mathbb{C}$ which will be the condition above with μ_3 set to 0.*

Definition 1.5.6. *(Correlation functions of Liouville theory on $\mathbb{H}$) Fix $\gamma\in(0,2)$. Consider parameters $\alpha,\beta,\beta_1,\beta_2,\beta_3\in\mathbb{R}$, $\mu_\partial\in(0,+\infty)$, and $\mu_1,\mu_2,\mu_3\in\mathbb{C}$. The four correlation functions U,G,R,H have the following probabilistic definitions:*

- $U(\alpha) = \frac{2}{\gamma}\Gamma(\frac{2(\alpha-Q)}{\gamma})\left(\mu_\partial^{\frac{2(Q-\alpha)}{\gamma}}\right)\overline{U}(\alpha)$ *where for* $\alpha > \frac{\gamma}{2}$*:*

$$\overline{U}(\alpha) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha)}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}X(x)}dx\right)^{\frac{2(Q-\alpha)}{\gamma}}\right]. \tag{1.5.21}$$

- $G(\alpha,\beta) = \frac{2}{\gamma}\Gamma(\frac{2\alpha+\beta-2Q}{\gamma})\left(\mu_\partial^{\frac{2Q-2\alpha-\beta}{\gamma}}\right)\overline{G}(\alpha,\beta)$ *where for* $\beta < Q$, $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$*:*

$$\overline{G}(\alpha,\beta) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha-\frac{\beta}{2})}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}X(x)}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\right]. \tag{1.5.22}$$

- $H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \frac{2}{\gamma}\Gamma(\frac{\beta_1+\beta_2+\beta_3-2Q}{\gamma})\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ *where in the following range of parameters,*

$$(\mu_i)_{i=1,2,3} \text{ satisfies Definition 1.5.5}, \quad \beta_i < Q, \quad \frac{1}{\gamma}(2Q - \sum_{i=1}^3 \beta_i) < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q-\beta_i), \tag{1.5.23}$$

one can define:

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{8}(\frac{4}{\gamma}-\sum_{i=1}^3\beta_i)}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} e^{\frac{\gamma}{2}X(x)}d\mu(x)\right)^{\frac{1}{\gamma}(2Q-\sum_{i=1}^3\beta_i)}\right]. \tag{1.5.24}$$

The dependence on the parameters μ_1,μ_2,μ_3 *appears through the measure:*

$$d\mu(x) = \mu_1\mathbf{1}_{(-\infty,0)}(x)dx + \mu_2\mathbf{1}_{(0,1)}(x)dx + \mu_3\mathbf{1}_{(1,\infty)}(x)dx. \tag{1.5.25}$$

The GMC integral inside the expectation is a complex number avoiding $(-\infty,0)$*. To define its fractional power we choose its argument in* $(-\pi,\pi)$*.*

- $R(\beta,\mu_1,\mu_2) = -\Gamma(1-\frac{2(Q-\beta)}{\gamma})\overline{R}(\beta,\mu_1,\mu_2)$*, where* $\overline{R}(\beta,\mu_1,\mu_2)$ *is defined for* $\beta \in (\frac{\gamma}{2},Q)$ *and* μ_1,μ_2 *obeying the constraint of Definition 1.5.5 by the following limiting procedure. Consider* $\frac{\gamma}{2} < \beta_2 < \beta < Q$ *and* $\beta - \beta_2 < \beta_3 < Q$*. Then the following limits exists and we set:*

$$\overline{R}(\beta,\mu_1,\mu_2) := \frac{1}{2(Q-\beta)} \lim_{\beta_3 \downarrow \beta-\beta_2} (\beta_2+\beta_3-\beta_1)\overline{H}^{(\beta,\beta_2,\beta_3)}_{(\mu_1,\mu_2,1)}. \tag{1.5.26}$$

In order to state our main results, we introduce the function $S_{\frac{\gamma}{2}}(x)$ defined for $\gamma \in (0,2)$ and $\mathrm{Re}(x) \in (0,Q)$ by:

$$S_{\frac{\gamma}{2}}(x) = \frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(Q-x)}, \tag{1.5.27}$$

and the double functions are defined in (1.5.4). Both functions $\Gamma_{\frac{\gamma}{2}}(x)$ and $S_{\frac{\gamma}{2}}(x)$ admit meromorphic extensions to all $x \in \mathbb{C}$ with a known pole structure, see subsection 3.5.4 for more details. Using these two functions one can define the following special function introduced in [84]. For $i \in \{1,2,3\}$, define σ_i through the relation $\mu_i := e^{i\pi\gamma(\sigma_i - \frac{Q}{2})}$ with the convention that for positive μ_i one has $\mathrm{Re}(\sigma_i) = \frac{Q}{2}$. Denote $\overline{\beta} = \beta_1+\beta_2+\beta_3$. Then

define:

$$
\begin{aligned}
&I_{PT}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix} \qquad (1.5.28)\\
&=\frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\beta}}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}}\Gamma(\frac{\overline{\beta}-2Q}{\gamma})}\frac{\Gamma_{\frac{\gamma}{2}}(2Q-\frac{\overline{\beta}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_2+\beta_3-\beta_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(Q-\beta_3)}\\
&\times\frac{e^{i\frac{\pi}{2}(-(2Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)(Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)+(Q+\frac{\beta_2}{2}-\sigma_2-\sigma_3)(\frac{\beta_2}{2}-\sigma_2-\sigma_3)+(Q+\frac{\beta_3}{2}-\sigma_1-\sigma_3)(\frac{\beta_3}{2}-\sigma_1-\sigma_3)-2\sigma_3(2\sigma_3-Q))}}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1)}\\
&\times\int_{-i\infty}^{i\infty}\frac{S_{\frac{\gamma}{2}}(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)}{S_{\frac{\gamma}{2}}(Q+\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q+r)}e^{i\pi(-\frac{\beta_2}{2}+\sigma_2-\sigma_3)r}\frac{dr}{i}.
\end{aligned}
$$

The contour of the integral is to the right of the poles at $r=-(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$, $r=-(\frac{\beta_3}{2}+\sigma_3-\sigma_1)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$, $r=-(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$ and to the left of the poles at $r=-(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)+n\frac{\gamma}{2}+m\frac{2}{\gamma}$, $r=-(Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)+n\frac{\gamma}{2}+m\frac{2}{\gamma}$, $r=n\frac{\gamma}{2}+m\frac{2}{\gamma}$ with $m,n\in\mathbb{N}^2$. We can now state our main results. For the sake of completeness we recall:

Theorem 1.5.7. *(Bulk one-point function, R. 2017 [88]) For $\gamma\in(0,2)$, $\alpha>\frac{\gamma}{2}$, one has:*

$$\overline{U}(\alpha)=\left(\frac{2^{-\frac{\gamma\alpha}{2}}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\alpha)}\Gamma(\frac{\gamma\alpha}{2}-\frac{\gamma^2}{4}). \qquad (1.5.29)$$

Now the main results of the present work is to provide expressions for the remaining three structure constants. We will indeed prove the following theorems:

Theorem 1.5.8. *(Bulk-boundary correlator) For $\gamma\in(0,2)$, $\beta<Q$, $\frac{\gamma}{2}-\alpha<\frac{\beta}{2}<\alpha$, one has:*

$$\overline{G}(\alpha,\beta)=\left(\frac{2^{\frac{\gamma}{2}(\frac{\beta}{2}-\alpha)}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\frac{\Gamma(\frac{\gamma\alpha}{2}+\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma_{\frac{\gamma}{2}}(\alpha-\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(\alpha+\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2})^2}{\Gamma_{\frac{\gamma}{2}}(Q-\beta)\Gamma_{\frac{\gamma}{2}}(\alpha)^2\Gamma_{\frac{\gamma}{2}}(Q)}. \qquad (1.5.30)$$

Theorem 1.5.9. *(Boundary two-point and three-point functions) Consider $\gamma\in(0,2)$, $\beta\in(\frac{\gamma}{2},Q)$, and μ_1,μ_2 obeying the condition of Definition 1.5.5. Then one has:*

$$\overline{R}(\beta,\mu_1,\mu_2)=\frac{(2\pi)^{\frac{2}{\gamma}(Q-\beta)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\beta)-\frac{1}{2}}}{(Q-\beta)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\beta)}}\frac{\Gamma_{\frac{\gamma}{2}}(\beta-\frac{\gamma}{2})e^{i\pi(\sigma_1+\sigma_2-Q)(Q-\beta)}}{\Gamma_{\frac{\gamma}{2}}(Q-\beta)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_2-\sigma_1)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_1-\sigma_2)}. \qquad (1.5.31)$$

Similarly, for β_1,β_2,β_3 and μ_1,μ_2,μ_3 satisfying the set of conditions (3.1.18),

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}=I_{PT}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}. \qquad (1.5.32)$$

Let us now state how the value of $\overline{R}(\beta,\mu_1,\mu_2)$ provides a very general first order tail expansion for the probability of a one-dimensional GMC measure to be large. For this discussion we choose $\mu_1,\mu_2\in[0,\infty)$ with at most one of the two parameters being 0, and we introduce the notation:

$$I_{\eta_1,\eta_2}(\beta):=\int_{-\eta_1}^{\eta_2}\frac{1}{|x|^{\frac{\beta\gamma}{2}}}e^{\frac{\gamma}{2}X(x)}\left(\mu_1\mathbf{1}_{\{x<0\}}+\mu_2\mathbf{1}_{\{x>0\}}\right)dx. \qquad (1.5.33)$$

In the above $\eta_1,\eta_2\in(0,1)$. Now the tail expansion result is the following:

Proposition 1.5.10. *For $\beta \in (\frac{\gamma}{2}, Q)$ and any $\eta_1, \eta_2 \in (0,1)$, we have the following tail expansion for $I_{\eta_1,\eta_2}(\beta)$ as $u \to \infty$ and for some $\nu > 0$:*

$$\mathbb{P}(I_{\eta_1,\eta_2}(\beta) > u) = \frac{\overline{R}(\beta,\mu_1,\mu_2)}{u^{\frac{2}{\gamma}(Q-\beta)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\beta)+\nu}}). \tag{1.5.34}$$

Although the proof strategy of boundary LCFT structure constants follows the same lines as the DOZZ formula, there are additional technical difficulties that arise because of the presence of complex valued quantities included with the GMC measures. Performing OPE in this case will require some care and extra estimates. Furthermore, the computation of the reflection coefficient $\overline{R}(\beta,\mu_1,\mu_2)$ is not direct and will require the specific value of $\overline{R}(\beta,1,0) = \overline{R}_1^{\partial}(\beta)$ calculated in the interval case (1.5.14).

1.6 Applications and perspectives

Finally, this section provides a list of applications of GMC and of Liouville theory to other problems in probability theory and in mathematical physics. We also provide perspectives on future research.

1.6.1 The maximum of X and random matrix theory

We now turn to the applications of the exact formulas for the GMC measures. It turns out that the Fyodorov-Bouchaud formula will give us some precise information on the behaviour of the maximum of our field X on the unit circle. Characterizing the behaviour of the maximum of X requires to compute the law of the total mass of the derivative martingale,

$$Y' := -\frac{1}{2}\int_0^{2\pi} X(e^{i\theta})e^{X(e^{i\theta})}d\theta, \tag{1.6.1}$$

which following [6] can be characterized by the convergence in law:

$$2Y' = \lim_{\gamma\to 2}\frac{1}{2-\gamma}Y_\gamma. \tag{1.6.2}$$

Therefore using Fyodorov-Bouchaud formula we can easily compute the density for $2Y'$:

$$f_{2Y'}(y) = y^{-2}e^{-y^{-1}}\mathbf{1}_{[0,\infty[}(y).$$

We observe that $\ln 2Y'$ is distributed like a standard Gumbel law. Recall that an impressive series of works (see [16, 18] for the latest results) have proven that for suitable sequences of cut-off approximations X_ϵ the following convergence in law holds,

$$\max_{\theta\in[0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G} + \ln Y' + C, \tag{1.6.3}$$

where $\mathcal{G}$ is a Gumbel law independent from Y' and C is a non universal constant that depends on the cut-off procedure. From this convergence and previous considerations, one can deduce the following convergence in law,

$$\max_{\theta\in[0,2\pi]} X_\epsilon(e^{i\theta}) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C, \tag{1.6.4}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are two independent Gumbel laws and where we have absorbed the factor $\ln 2$ in the constant C. This convergence was conjectured in Fyodorov-Bouchaud [41]. As a matter of fact, Fyodorov-Bouchaud state (1.6.4) as their main result. Mathematically, it is the first occurrence of an explicit formula for the limit density of the properly recentered maximum of a GFF.

A similar story can be told for unitary random matrices. Let U_N denote the $N \times N$ random matrices distributed according to the Haar probability measure on the unitary group $U(N)$. Denoting by $(e^{i\theta_1},\dots,e^{i\theta_n})$

the eigenvalues of U_N, we consider its characteristic polynomial $p_N(\theta)$ evaluated on the unit circle at a point $e^{i\theta}$:

$$p_N(\theta) = \det(1 - e^{-i\theta}U_N) = \prod_{k=1}^{N}(1 - e^{i(\theta_k - \theta)}). \tag{1.6.5}$$

Recently, the following convergence in law has been obtained in [104] for a real $\alpha \in (-\frac{1}{2}, \sqrt{2})$:

$$\frac{|p_N(\theta)|^\alpha}{\mathbb{E}[|p_N(\theta)|^\alpha]}d\theta \underset{N\to\infty}{\to} e^{\frac{|\alpha|}{2}X(e^{i\theta})}d\theta. \tag{1.6.6}$$

This convergence seems to indicate that $2\ln|p_N(\theta)|$ should be seen as a cut-off of X just like our X_ϵ with N corresponding to $\frac{1}{\epsilon}$. Based on this analogy, it is reasonable that the properly shifted maximum of $2\ln|p_N(\theta)|$ should converge to the same limit as the (properly shifted) maximum of X on the unit circle. Indeed it has been recently conjectured by Fyodorov, Hiary and Keating [42] (and further developed in [43]) that the following convergence in law should hold,

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + C, \tag{1.6.7}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are again two independent Gumbel laws and C a real constant. On the mathematical side the most recent result [17] establishes that

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \tag{1.6.8}$$

is tight. Just like for the GFF it is natural to expect that the following convergence is easier to establish directly

$$\max_{\theta\in[0,2\pi]} \ln|p_N(\theta)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \ln Y' + C. \tag{1.6.9}$$

Our result could then prove instrumental in precisely identifying the limit in the conjectured convergence (1.6.7).

In the case of the unit interval it is also possible with our exact formula to study the behaviour of the maximum of X_I on $[0,1]$ and to establish a link with random Hermitian matrices. The behaviour of the maximum is given by the following convergence in law, first conjectured in [78],

$$\max_{x\in[0,1]} X_{I,\epsilon}(x) - 2\ln\frac{1}{\epsilon} + \frac{3}{2}\ln\ln\frac{1}{\epsilon} \underset{\epsilon\to 0}{\to} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln X_2 + \ln X_3 + C, \tag{1.6.10}$$

with again as before $\mathcal{G}_1$, $\mathcal{G}_2$ two Gumbel laws and C a non-universal constant that depends on the cut-off procedure. On the other hand we see there are three additional terms, a Gaussian law and two additional laws $\ln X_2$ and $\ln X_3$ (see chapter 2 for a precise definition). Moving on to the link with random matrices, to obtain as limit a GMC on the interval the correct matrix ensemble is an ensemble of Hermitian matrices H_N. Their eigenvalues are on the real line but with the right rescaling the limit of the characteristic polynomial will be a GMC on $[0,1]$. We can thus conjecture:

$$\max_{x\in[0,1]} \ln|\det(H_N - x)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln X_2 + \ln X_3 + C. \tag{1.6.11}$$

1.6.2 Relation with the one-point toric conformal blocks

Very recently in [48], a GMC expression has been proposed for the one-point conformal blocks for LCFT on the torus. The main result of [48] is that this probabilistic definition matches the known expression in physics

given by Zamolodchikov's recursion. More precisely, the expression of [48] for conformal blocks is given by, for parameters $\beta \in (-\frac{4}{\gamma}, Q)$,[7] $P \in \mathbb{R}$, $q \in (0,1)$,

$$\mathcal{G}^q_{\gamma,P}(\beta) := \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\beta\gamma}{2}}e^{\pi\gamma Px}dx\right)^{-\frac{\beta}{\gamma}}\right] \tag{1.6.12}$$

where Z is a normalization constant, Θ_τ is the Jacobi theta function, and Y_τ is a log-correlated field which can be thought of as the restriction of a 2d GFF on the torus to one of the loops of the torus (see [48] for more details). Both Θ_τ and Y_τ depend on the parameter q related to the moduli τ of the torus by $q = e^{i\pi\tau}$. The proof strategy of [48] to show that (1.6.12) matches known definitions still relies on the BPZ equations, operator product expansions, and this time a whole system of coupled shift equations. Therefore one needs again to perform the OPE with reflection and in order to obtain an explicit answer use the formula of Theorem 1.5.9 for the boundary two-point function. The precise quantity required in [48] evaluated using Theorem 1.5.9 is

$$\frac{\overline{R}(\beta,1,e^{-i\pi+\pi\gamma P})}{\overline{R}(\beta+\frac{2}{\gamma}-\frac{\gamma}{2},1,e^{-i\pi\frac{\gamma^2}{4}+\pi\gamma P})} = \frac{2(2\pi)^{\frac{4}{\gamma^2}-1}\Gamma(\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma})}{\gamma(Q-\beta)\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}-1}\Gamma(\frac{\gamma\beta}{2}-\frac{\gamma^2}{2})\Gamma(1-\frac{\gamma\beta}{2}+\frac{\gamma^2}{4})}\frac{1-e^{\frac{4\pi P}{\gamma}-\frac{4i\pi}{\gamma^2}+i\pi\frac{2\beta}{\gamma}}}{1+e^{\pi\gamma P-\frac{i\pi\gamma^2}{2}+i\pi\frac{\gamma\beta}{2}}}.$$

Furthermore, the normalization Z of the conformal block $\mathcal{G}^q_{\gamma,P}(\beta)$ is explicitly given by

$$Z = q^{\frac{1}{12}(\frac{\beta\gamma}{2}+\frac{\beta^2}{2}-1)}\eta(q)^{\beta^2+1-\frac{\beta\gamma}{2}}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{2\pi ix})}[-2\sin(\pi x)]^{-\frac{\beta\gamma}{2}}e^{\pi\gamma Px}dx\right)^{-\frac{\beta}{\gamma}}\right] \tag{1.6.13}$$

where $\eta(q)$ is the Dedekind eta function. As an output of the proof of [48], the GMC expectation above is explicitly evaluated, for $\beta \in (-\frac{4}{\gamma}, Q)$, $P \in \mathbb{R}$:

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\infty(x)}[-2\sin(\pi x)]^{-\frac{\beta\gamma}{2}}e^{\pi\gamma Px}dx\right)^{-\frac{\beta}{\gamma}}\right] \\ &= \left(\frac{\gamma}{2}\right)^{\frac{\gamma\beta}{4}}e^{i\pi\frac{\beta^2}{2}-\frac{\pi\beta P}{2}}\Gamma(1-\frac{\gamma^2}{4})^{\frac{\beta}{\gamma}}\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}+\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2}-iP)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2}+iP)}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q-iP)\Gamma_{\frac{\gamma}{2}}(Q+iP)\Gamma_{\frac{\gamma}{2}}(Q-\beta)}.\end{aligned} \tag{1.6.14}$$

The proof of (1.6.14) follows very closely the techniques used to prove Theorem 1.5.8 and relies again of the exact formula for the boundary two-point function given by Theorem 1.5.9. Lastly an observation is that both (1.5.30) and (1.6.14) degenerate to the same formula (up to a prefactor due to the mapping of $\mathbb{D}$ to $\mathbb{H}$ and to a global phase) if one choose $\alpha = Q$ in (1.5.30) and $P = 0$ in (1.6.14). Both (1.5.30) and (1.6.14) up to trivial prefactors are thus a special case of the following GMC moment, written here with the field $X_{\mathbb{D}}$ for $P \in \mathbb{R}$, $\beta < Q$, $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$

$$\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{2\pi ix})}|e^{2\pi ix}-1|^{-\frac{\beta\gamma}{2}}e^{\pi\gamma Px}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\right]. \tag{1.6.15}$$

This quantity is also expected to have an explicit evaluation using $\Gamma_{\frac{\gamma}{2}}$ which can be conjectured form Selberg integrals. We leave repeating our methods to prove a formula for (1.6.15) for a future work.

[7] In [48] this parameter β is called α, but we use here the notation β in order to keep the convention of this paper for insertions on the boundary.

1.6.3 Stochastic variance model

The 1d GMC measures with kernel $2\ln^+ \frac{T}{|x-y|}$ with $\ln^+(x) = \max(\ln x, 0)$ provide a model for the stochastic variance of an financial asset. GMC is a multifractal measure and is used in the modeling of intermittent time signals characterized by an alternation of small fluctuations and large fluctuations (hence the terminology "intermittence"). Financial assets (stocks, currencies, indices, etc.) are precisely intermittent signals and in particular their stochastic variance [8]. Numerous empirical studies show that the GMC with kernel $2\ln^+ \frac{T}{|x-y|}$, also called lognormal Multifractal Random measure (MRM), verifies most of the properties and statisties observed on the variance of the assets: lognormality of the volatility, and long range correlations, we refer to [7, 8] for a study and a review. Volatility modeling and forecasting is an important field of finance since it is related to option pricing and risk forecasting. We refer to [32] for the problem of forecasting volatility with this choice of GMC.

Here we give an application of our results on the unit interval. A call option of the variance starting from $t = 0$ (current time) with exercise date $t = T$ offers the buyer the opportunity to buy the variance of a financial product over the future period $[0, T]$, with a fixed price K called the strike. If we model the variance with the total mass of GMC over $[0, T]$, denoted by $M_\gamma([0, T])$, then the price of the call option at $t = 0$ is given by

$$C(K, T) = e^{-rT}\mathbb{E}[(M_\gamma([0, T]) - K)_+|\mathcal{F}_0], \tag{1.6.16}$$

where r is the interest rate, $(x)_+ = \max(x, 0)$ and $\mathcal{F}_0$ is the sigma-algebra that contains all the past information. Take X the log-correlated field defined on $[0, 1]$ with covariance given by $2\ln \frac{1}{|x-y|}$. We observe that

$$M_\gamma([0, T]) \overset{(d)}{=} \int_0^T e^{\frac{\gamma}{2}X(x/T)}dx = T\int_0^1 e^{\frac{\gamma}{2}X(x)}dx = TM_\gamma([0, 1]). \tag{1.6.17}$$

Therefore Theorem 1.5.2 also characterizes the law of $M_\gamma([0, T])$. Thanks to the computations done by Ostrovski (see [79] for a review), an explicit formula for the Fourier transform of $\ln M_\gamma([0, T])$ has been established. It is also deduced that $\ln M_\gamma([0, T])$ has a density f_T. Let us denote by ϕ_T its Fourier transform and $k = \ln K$. Without prior knowledge of the past, i.e. $\mathcal{F}_0 = \emptyset$, the value of the option can be priced as

$$C_T(k) := C(K, T) = e^{-rT}\int_k^\infty (e^x - e^k)f_1(x)dx. \tag{1.6.18}$$

Here C_1 is not a square integrable function since when $k \to 0$, $C_1(k) \to \mathbb{E}[M_\gamma([0, T])]$. Hence we consider the function

$$\psi_T(v) = \int_{\mathbb{R}} e^{ivk}e^{\alpha k}C_T(k)dk \tag{1.6.19}$$

for a suitable $\alpha > 0$. The value of α affects the speed of convergence. We can express ψ_T in function of ϕ_T after some computations:

$$\psi_T(v) = \frac{e^{-rT}\phi_T(v - i(\alpha + 1))}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}.$$

The expression of $\psi_T(v)$ is explicit and we can do a fast Fourier transform to get the approximated value of $C_T(k)$.

Finally as a remark, Theorem 1.5.2 provides the Levy-Kintchine decomposition of the infinite divisible distribution $\ln M_\gamma([0, T])$, hence we can use the simulation of a Levy process in order to apply Monte Carlo methods. This will enable the computation of many other exotic option values.

[8] The stochastic variance of an asset is the variance of its log-return; this variance is a random measure in the mathematical sense. Besides, the standard error of its log- Yield is called volatility.

1.6.4 Additional applications

A major motivation for probabilists to study Liouville theory is that it provides the conjectured scaling limit of random planar maps potentially weighted by a statistical physics model. To every integrability formula in Liouville theory there should correspond an observable on random planar maps; see [20] for the conjectures in the case of the Riemann sphere. Let us mention a slightly different approach to Liouville theory developed in [33, 71]. The strength of this approach is that it bridges the gap between the discrete world of random planar maps and the continuum description of Liouville quantum gravity. Major progress has been made in the case of pure gravity $\gamma = \sqrt{\frac{8}{3}}$ corresponding to uniform random planar maps. The same framework has also allowed to derive rigorously the so-called KPZ relation of [58]; see [36]. Establishing links between the approach of [33] and the path integral construction we propose could prove very useful to obtain a better understanding of Liouville theory. Finally, we mention the widespread approach to the probabilistic study of CFT using the celebrated SLE curves. BPZ type equations can be written for these curves [28] and precise links with the Virasoro algebra of CFT have been uncovered [30]. Coupling between the SLE and Liouville theory has also been widely developed in [33].

We can briefly mention a few additional problems linked to Liouville theory. First, the integrability formulas we present in section 1.5 are only a very small part of what can be found in the review [72]. For instance there is another whole set of formulas for the ZZ-branes of Liouville theory. Also, the problem of random geometry has been studied with a modified action where an additional Mabuchi term is added to (1.3.2), see [15]. This new Mabuchi term can be seen as a perturbation of the Liouville CFT. We mention as well the celebrated AGT conjecture [1] linking the Liouville theory to the Nekrasov partition function of a four-dimensional gauge theory. This has been recently studied on the mathematical side in the work [68], where the reflection operator of Liouville theory appears. Finally there is a link between the zeros of the Riemann zeta function and GMC theory, see for instance [95] and references therein.

1.6.5 Other exact formulas for Liouville theory

In this section we give some perspectives on a series of formulas and results that we expect to obtain with similar techniques.

Let us start by a generalization of the formula (1.5.30) for the case $\mu = 0$. In fact the Selberg integral on the circle - the so-called Morris integral - predicts that one can go even furthermore and hope to prove the following formula.

Conjecture 3. *For $\gamma \in (0,2)$ and suitable a, b, $p \in \mathbb{R}$ give a meaning to and prove the following formula:*

$$\mathbb{E}[(\int_0^{2\pi} \frac{e^{\frac{i\theta\gamma}{4}(b-a)}}{|e^{i\theta}-1|^{\frac{\gamma}{2}(a+b)}} e^{\frac{\gamma}{2}X(e^{i\theta})} d\theta)^p] = (2\pi)^p \frac{\Gamma(1-\frac{\gamma^2 p}{4})\Gamma_\gamma(Q-a-b-\frac{\gamma p}{2})\Gamma_\gamma(Q-a)\Gamma_\gamma(Q-b)\Gamma_\gamma(Q-\frac{\gamma p}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p\Gamma_\gamma(Q-a-\frac{\gamma p}{2})\Gamma_\gamma(Q-b-\frac{\gamma p}{2})\Gamma_\gamma(Q-a-b)\Gamma_\gamma(Q)}. \quad (1.6.20)$$

This formula is interesting because it characterizes the Fourier transform of GMC on the unit circle. The first difficulty is to well define this object to be a single valued function. The second difficulty is that it is no more covered by correlation functions of LCFT because of the term $e^{i\frac{\theta\gamma}{4}(b-a)}$, although certain non-hypergeometric differential equations are expected for the auxiliary functions.

We will now state a list of formulas that generalizes the formulas of subsection 1.5.3. We keep using the notations $U(\alpha)$, $G(\alpha,\beta)$, $R(\beta,\mu_1,\mu_2)$ and $H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ but we now work with $\mu > 0$.

Conjecture 4. *For $\mu > 0$, $\mu_\partial > 0$, $z \in \mathbb{H}$, and suitable α*

$$U(\alpha) = \frac{4}{\gamma}(\pi\mu l(\frac{\gamma^2}{4}))^{\frac{Q-2\alpha}{\gamma}}\Gamma(\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4})\Gamma(\frac{2}{\gamma}\alpha-\frac{4}{\gamma^2}-1)\cos(2\pi(\alpha-Q)(\sigma-\frac{Q}{2})) \quad (1.6.21)$$

with σ defined by the relation $\cos^2(\pi\gamma(\sigma-\frac{Q}{2})) = \frac{\mu_\partial^2}{\mu}\sin\frac{\pi\gamma^2}{4}$.

For suitable β, μ_1, μ_2,

$$R(\beta,\mu_1,\mu_2) = -\frac{(2\pi)^{\frac{2}{\gamma}(Q-\beta)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\beta)-\frac{1}{2}}}{(Q-\beta)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\beta)}}\frac{1}{S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_1+\sigma_2-Q)S_{\frac{\gamma}{2}}(\frac{\beta}{2}-\sigma_1-\sigma_2+Q)}$$
$$\frac{\Gamma_{\frac{\gamma}{2}}(\beta-\frac{\gamma}{2})\Gamma(1-\frac{2(Q-\beta)}{\gamma})}{\Gamma_{\frac{\gamma}{2}}(Q-\beta)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_2-\sigma_1)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_1-\sigma_2)}. \tag{1.6.22}$$

We can also generalize the three point structure constant:

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} \tag{1.6.23}$$
$$=\frac{2}{\gamma}\Gamma(\frac{\beta_1+\beta_2+\beta_3-2Q}{\gamma})\frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\beta}}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}}\Gamma(\frac{\overline{\beta}-2Q}{\gamma})}$$
$$\times\frac{\Gamma_{\frac{\gamma}{2}}(2Q-\frac{\overline{\beta}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_2+\beta_3-\beta_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(Q-\beta_3)}$$
$$\times\frac{1}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1)S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1+\sigma_2-Q)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}-\sigma_1-\sigma_3+Q)}$$
$$\times\int_{-i\infty}^{i\infty}\frac{S_{\frac{\gamma}{2}}(-\frac{\beta_2}{2}+\sigma_2+\sigma_3+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)}{S_{\frac{\gamma}{2}}(Q+\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2\sigma_3+r)S_{\frac{\gamma}{2}}(Q+r)}\frac{dr}{i}.$$

For the expression of $G(\alpha,\beta)$, see the review [72].

The algebra of the proof for $R(\beta,\mu_1,\mu_2)$ and $H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ is exactly the same as the $\mu = 0$ case proved in chapter 3. There are some additional difficulties that we need to understand: the conjecture 1 for BPZ equations on the boundary, and all the reflection techniques should be reproved in this case where both bulk and boundary potential are present. We also expect the exact formula of $R(\beta,\mu_1,\mu_2)$ to give the distribution of the total mass of the quantum disk. The calculus of $U(\alpha)$ seems to have some technical problems as the domain of definition is not sufficient to write shift equations.

Finally let us note that formulas also exist in the case of other topologies (torus [50] or annulus [67]). Proving these formulas is a whole new adventure as one needs to perform the so-called "modular bootstrap".

1.6.6 More on BPZ equations

Higher order BPZ equations can also be used to deduce exact formulas of certain correlation functions of LFCT, such as the integral forms introduced by Fateev-Litvinov [38, 39]. The idea is to first show that the solution space of higher order BPZ equations is of dimension 1 using monodromy arguments [26, 27]. It is not hard to verify that the integral forms of Fateev-Litvinov satisfy higher order BPZ equations using its relation with Coulomb gas integrals and analycity of its parameters, especially analycity in γ. Therefore we can identify the LCFT correlation functions to Fateev-Litvinov integrals. See the conjecture below. However this conjecture only works for the paramters $\chi = \frac{\gamma}{2}$ and $\gamma < \frac{2}{\sqrt{r-1}}$, while the proof of higher order BPZ equations works for $\gamma > \sqrt{\frac{2(r-2)}{r-1}}$. As long as $r \geq 4$, the intersection is empty.

Conjecture 5. *Four point Liouville correlation functions can be expressed as follows (in the domain of*

existence for the integral):

$$\begin{aligned}&\langle V_{-\frac{(r-1)\gamma}{2}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\mathbb{S}^2}\\ =&\frac{\langle V_{\alpha_1-\frac{(r-1)\gamma}{2}}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle}{S_{r-1}\left(\frac{\gamma}{2}(\frac{(r-1)\gamma}{2}-\alpha_1),\frac{\gamma}{2}(\Lambda-\alpha_2),-\frac{\gamma^2}{2}\right)}|z|^{\frac{(r-1)\gamma\alpha_1}{2}}|z-1|^{\frac{(r-1)\gamma\alpha_2}{2}}\\ &\int_{\mathbb{C}^{r-1}}\prod_{s=1}^{r-1}|w_s|^{\gamma(\Lambda-\alpha_1)}|w_s-1|^{\gamma(\Lambda-\alpha_2)}|w_s-z|^{\gamma(-\Lambda+\frac{(r-1)\gamma}{2})}\prod_{1\le s<s'\le r-1}|w_s-w_{s'}|^{-\gamma^2}d^2\mathbf{w},\end{aligned}\tag{1.6.24}$$

where $\Lambda=\frac{1}{2}(\sum_{i=1}^3\alpha_i-2Q+\frac{(r-1)\gamma}{2})$*, and* S_{r-1} *is the complex Selberg integral defined by*

$$S_{r-1}(a,b,g)=\int_{\mathbb{C}^{r-1}}\prod_{s=1}^{r-1}|w_s|^{2a}|w_s-1|^{2b}\prod_{1\le s<s'\le r-1}|w_s-w_{s'}|^{2g}d^2\mathbf{w}.\tag{1.6.25}$$

The integral is well defined for the following range of parameters:

$$\gamma^2<\frac{4}{r-1},\quad \Lambda<\frac{\gamma}{2}+\frac{2}{\gamma},\quad \forall i\ \ \Lambda-\alpha_i>\frac{(r-2)\gamma}{2}-\frac{2}{\gamma}.\tag{1.6.26}$$

Remark 1.6.1. *Note that there is no conjecture for the degenerate insertion with weight* $-\frac{2(r-1)}{\gamma}$*, in which case the domain of definition is empty for the integral on the right hand side.*

The renormalizing factors in equation (1.6.24) are explicit. The value of the three point correlation function $\langle V_{\alpha_1-\frac{(r-1)\gamma}{2}}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle$ is given by the DOZZ formula and the value of complex Selberg integrals can be written explicitly in terms of Gamma functions.

Another series of higher order differential equations, known as equations of motion, are conjectured in [109]:

Conjecture 6 (higher equations of motion)**.** *Define* $V'_\alpha=\phi V_\alpha$*,*

$$\bar{\mathcal{D}}_r\mathcal{D}_r\langle V'_{-\frac{(r-1)\gamma}{2}}(z)\prod_l V_{\alpha_l}(z_l)\rangle_{\mathbb{S}^2}=B_r\langle V_{\frac{(r+1)\gamma}{2}}(z)\prod_l V_{\alpha_l}(z_l)\rangle_{\mathbb{S}^2},\tag{1.6.27}$$

where $l(x)=\frac{\Gamma(x)}{\Gamma(1-x)}$ *and* B_r *is defined by*

$$B_r=(\pi\mu\, l(\frac{\gamma^2}{4}))^r\frac{(-1)^{r-1}(\frac{\gamma}{2})^{4r-3}((r-1)!)^2}{l(\frac{r\gamma^2}{4})}.\tag{1.6.28}$$

The operators $\mathcal{D}_r$ *are the same as in higher order BPZ equations,* $\bar{\mathcal{D}}_r$ *are defined by replacing* L_{-n} *by* $\bar{L}_{-n}$ *in the expresion of* $\mathcal{D}_r$*, with* $\bar{L}_{-n}$ *given by*

$$\bar{L}_{-1}=\partial_{\bar{z}},\tag{1.6.29}$$

$$\bar{L}_{-n}=\sum_{l=1}^N\left(-\frac{1}{(\bar{z}_l-\bar{z})^{n-1}}\partial_{\bar{z}_l}+\frac{\Delta_l(n-1)}{(\bar{z}_l-\bar{z})^n}\right)\quad n\ge 2.\tag{1.6.30}$$

Remark 1.6.2. *We have an algebraical proof for the* $r=2$ *case using the methods proposed in chapter 4. The major difficulty is the existence of the certain objects coming from higher order derivatives.*

CHAPTER 2

Gaussian multiplicative chaos on the unit interval

In collaboration with Guillaume Remy.

We consider a sub-critical Gaussian multiplicative chaos (GMC) measure defined on the unit interval $[0,1]$ and prove an exact formula for the fractional moments of the total mass of this measure. Our formula includes the case where log-singularities (also called insertion points) are added in 0 and 1, the most general case predicted by the Selberg integral. The idea to perform this computation is to introduce certain auxiliary functions resembling holomorphic observables of conformal field theory that will be solutions of hypergeometric equations. Solving these equations then provides non-trivial relations that completely determine the moments we wish to compute. We also include a detailed discussion of the so-called reflection coefficients appearing in tail expansions of GMC measures and in Liouville theory. Our theorem provides an exact value for one of these coefficients. Lastly we mention some additional applications to small deviations for GMC measures, to the behavior of the maximum of the log-correlated field on the interval and to random hermitian matrices.

2.1 Introduction and main result

Starting from a log-correlated field X one can define the associated Gaussian multiplicative chaos (GMC) measure which has a density with respect to the Lebesgue measure formally given by the exponential of X. This definition is formal as X lives in the space of distributions but since the pioneering work of Kahane [56] in 1985 it is well understood how to give a rigorous probabilistic definition to these GMC measures by using a limiting procedure. Ever since GMC has been extensively studied in probability theory and mathematical physics with applications including 3d turbulence, statistical physics, mathematical finance, random geometry and 2d quantum gravity. See for instance [92] for a review.

Despite the importance of GMC measures in many active fields of research, rigorous computations have remained until very recently completely out of reach. A large number of exact formulas have been conjectured by the physicists' trick of analytic continuation from positive integers to real numbers (see the explanations below) but with no indication of how to rigorously prove such formulas. A decisive step was made in [20] where a connection is uncovered between GMC measures and the correlation functions of Liouville conformal field theory (LCFT). By implementing the techniques of conformal field theory (CFT) in a probabilistic setting one can hope to perform rigorous computations on GMC.

Indeed, in 2017 a proof was given by Kupiainen-Rhodes-Vargas of the celebrated DOZZ formula [60, 61] first conjectured independently by Dorn and Otto in [25] and by Zamolodchikov and Zamolodchikov in [108]. This formula gives the value of the three-point correlation function of LCFT on the Riemann sphere and it can

also be seen as the first rigorous computation of fractional moments of a GMC measure. Very shortly after, the study of LCFT on the unit disk by the first author led in [88] to the proof of a probability density for the total mass of the GMC measure on the unit circle. This result proves the conjecture of Fyodorov and Bouchaud stated in [41] and it is the first explicit probability density for a GMC measure obtained in the mathematical literature.

The present paper presents a third case where exact computations are tractable using CFT-inspired techniques which is the case of GMC on the unit interval $[0,1]$ with X of covariance written below (2.1.1). This model was studied by Bacry-Muzy in [9] where they prove existence of moments and other properties of GMC. Five years after exact formulas for this model on the interval were conjectured independently by Fyodorov-Le Doussal-Rosso in [45, 44] and by Ostrovsky in [75, 76]. In [45, 44] the exact formulas are found using an analytic continuation from integers to real numbers but in his papers Ostrovsky went a step further and showed that the formulas did correspond to a valid probability distribution. He also performs the computation of the derivatives of all order in γ of (2.1.4) at $\gamma = 0$ which is referred to as the intermittency differentiation. However a crucial analycity argument is missing for this approach to prove rigorously an exact formula. See [79] for a beautiful review on all the known results and conjectures for the GMC on the interval (and also for the similar model on the circle) as well as for many additional references.

The main result of our work is precisely the proof of these conjectures for the GMC measure on $[0,1]$. The major input of our paper is the introduction of two auxiliary functions that will be solutions to hypergeometric equations, see Proposition 2.1.4. This observation was to the best of our knowledge unknown to the statistical physics community although an analogous statement was known in the case of the Selberg integral, see [57] and the explanations of subsection 2.1.1. By studying the solution space of these differential equations we obtain non-trivial relations on the GMC that allow us to rigorously prove the formulas conjectured by physicists.

Let us now introduce the framework of our paper. We consider the log-correlated field X on the interval $[0,1]$ with covariance given for $x, y \in [0,1]$ by:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|}.^{1} \tag{2.1.1}$$

Because of the singularity of its covariance X is not defined pointwise and lives in the space of distributions. We define the associated GMC measure on the interval $[0,1]$ by the standard regularization procedure for $\gamma \in (0,2)$,

$$e^{\frac{\gamma}{2}X(x)}dx := \lim_{\delta\to 0} e^{\frac{\gamma}{2}X_\delta(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx, \tag{2.1.2}$$

where X_δ stands for any reasonable cut-off of X that converges to X as δ goes to 0. The convergence in (2.1.2) is in probability with respect to the weak topology of measures, meaning that for all continuous test functions $f : [0,1] \mapsto \mathbb{R}$ the following holds in probability:

$$\int_0^1 f(x)e^{\frac{\gamma}{2}X(x)}dx = \lim_{\delta\to 0}\int_0^1 f(x)e^{\frac{\gamma}{2}X_\delta(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx. \tag{2.1.3}$$

For an elementary proof of this convergence see [12]. We now introduce the main quantity of interest of our paper, for $\gamma \in (0,2)$ and for real p, a, b:

$$M(\gamma,p,a,b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{2.1.4}$$

This quantity is the moment p of the total mass of our GMC measure with two "insertion points" in 0 and 1 of weight a and b. The theory of Gaussian multiplicative chaos tells us that these moments are non-trivial, i.e. different from 0 and $+\infty$, if and only if:

$$a > -\frac{\gamma^2}{4} - 1, \quad b > -\frac{\gamma^2}{4} - 1, \quad p < \frac{4}{\gamma^2} \wedge (1 + \frac{4}{\gamma^2}(1+a)) \wedge (1 + \frac{4}{\gamma^2}(1+b)). \tag{2.1.5}$$

[1] Our normalization differs from the $\ln\frac{1}{|x-y|}$ usually found in the literature.

The first two conditions are required for the GMC measure to integrate the fractional powers x^a and $(1-x)^b$. Notice that this condition is weaker than the one we would get with the Lebesgue measure, $a > -1$ and $b > -1$.[2] We then have a bound on the moment p, the first part $p < \frac{4}{\gamma^2}$ is the standard condition for the existence of a moment of GMC without insertions. The additional condition on p, $p < (1 + \frac{4}{\gamma^2}(1+a)) \wedge (1 + \frac{4}{\gamma^2}(1+b))$, comes from the presence of the insertions. A proof of the bounds (2.1.5) can be found in [93, 53].

Now the goal of our paper is simply to prove the following exact formula for $M(\gamma, p, a, b)$:

Theorem 2.1.1. *For $\gamma \in (0,2)$ and for p, a, b satisfying (2.1.5)[3], $M(\gamma, p, a, b)$ is given by,*

$$\frac{(2\pi)^p \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+1) - (p-1)\frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(b+1) - (p-1)\frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+b+2) - (p-2)\frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} - p\frac{\gamma}{2})}{(\frac{\gamma}{2})^{p\frac{\gamma^2}{4}} \Gamma(1-\frac{\gamma^2}{4})^p \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+1) + \frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(b+1) + \frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+b+2) - (2p-2)\frac{\gamma}{2})},$$

where the function $\Gamma_{\frac{\gamma}{2}}(x)$ is defined for $x > 0$ and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ by:

$$\ln \Gamma_{\frac{\gamma}{2}}(x) = \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2}-x)^2}{2} e^{-t} + \frac{x - \frac{Q}{2}}{t} \right]. \tag{2.1.6}$$

As a corollary by choosing $a = b = 0$ we obtain the value of the moments of the GMC measure without insertions:

Corollary 2.1.2. *For $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$:*

$$\mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}X(x)} dx)^p] = \frac{(2\pi)^p (\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p} \frac{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} - (p-1)\frac{\gamma}{2})^2 \Gamma_{\frac{\gamma}{2}}(\frac{4}{\gamma} - (p-2)\frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} - p\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} + \frac{\gamma}{2})^2 \Gamma_{\frac{\gamma}{2}}(\frac{4}{\gamma} - (2p-2)\frac{\gamma}{2})}.$$

Thanks to the computations performed by Ostrovsky [77], we can also state our main result in the following equivalent way:

Corollary 2.1.3. *The following equality in law holds,*

$$\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx = 2\pi 2^{-(3(1+\frac{\gamma^2}{4})+2(a+b))} L Y_\gamma X_1 X_2 X_3, \tag{2.1.7}$$

where $L, Y_\gamma, X_1, X_2, X_3$ are five independent random variables in $\mathbb{R}_+$ with the following laws:

$$\begin{aligned}
L &= \exp(\mathcal{N}(0, \gamma^2 \ln 2)), \\
Y_\gamma &= \frac{1}{\Gamma(1-\frac{\gamma^2}{4})} \mathcal{E}(1)^{-\frac{\gamma^2}{4}}, \\
X_1 &= \beta_{2,2}^{-1}(1, \frac{4}{\gamma^2}; 1 + \frac{4}{\gamma^2}(1+a), \frac{2(b-a)}{\gamma^2}, \frac{2(b-a)}{\gamma^2}), \\
X_2 &= \beta_{2,2}^{-1}(1, \frac{4}{\gamma^2}; 1 + \frac{2}{\gamma^2}(2+a+b), \frac{1}{2}, \frac{2}{\gamma^2}), \\
X_3 &= \beta_{2,2}^{-1}(1, \frac{4}{\gamma^2}; 1 + \frac{4}{\gamma^2}, \frac{1}{2} + \frac{2}{\gamma^2}(1+a+b), \frac{1}{2} + \frac{2}{\gamma^2}(1+a+b)).
\end{aligned}$$

Here $\mathcal{E}(1)$ is an exponential law of parameter 1 and $\beta_{2,2}$ is a special beta law defined in appendix 2.4.5. It satisfies $\beta_{2,2} \in [0,1]$.

[2] Proving Theorem 2.1.1 for $-1 - \frac{\gamma^2}{4} < a \leq -1$ will require a lot of technical work as precise estimates on GMC measures are required to show that Proposition 2.1.4 holds in this case.

[3] The result also holds for all complex p such that $\mathrm{Re}(p)$ satisfies the bounds (2.1.5).

The advantage of this formulation is that it is more transparent than the large formula of Theorem 2.1.1. The log-normal law L is a global mode coming from the fact that X is not of zero average on $[0,1]$, see the discussion of subsection 2.1.3. The random variable Y_γ is actually the law of the total mass of the GMC measure defined on the unit circle - see [88] - and it will play a crucial role in understanding the small deviations of GMC, see again subsection 2.1.3. Lastly the generalized beta laws studied in [78] have a complicated definition but take values in $[0,1]$ just like the standard beta law.

2.1.1 Strategy of the proof

We start off with the well known observation that a formula can be given for $M(\gamma,p,a,b)$ in the very special case where $p\in\mathbb{N}$, $a>-1$, $b>-1$ and p satisfying (2.1.5). Indeed, in this case the computation reduces to a real integral - the famous Selberg integral - whose value is known, see for instance [40]. This is because for a positive integer moment we can write p integrals and exchange them with the expectation $\mathbb{E}[\cdot]$. More precisely for $a,b>-1$, p satisfying (2.1.5) and $p\in\mathbb{N}$ we have, using any suitable regularization procedure:

$$\begin{aligned}
&\mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]\\
&=\lim_{\delta\to 0}\mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X_\delta(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx)^p]\\
&=\lim_{\delta\to 0}\int_{[0,1]^p}\prod_{i=1}^p x_i^a(1-x_i)^b\mathbb{E}[\prod_{i=1}^p e^{\frac{\gamma}{2}X_\delta(x_i)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x_i)^2]}]dx_1\dots dx_p\\
&=\int_{[0,1]^p}\prod_{i=1}^p x_i^a(1-x_i)^b e^{\frac{\gamma^2}{4}\sum_{i<j}\mathbb{E}[X(x_i)X(x_j)]}dx_1\dots dx_p\\
&=\int_{[0,1]^p}\prod_{i=1}^p x_i^a(1-x_i)^b\prod_{i<j}\frac{1}{|x_i-x_j|^{\frac{\gamma^2}{2}}}dx_1\dots dx_p\\
&=\prod_{j=1}^p\frac{\Gamma(1+a-(j-1)\frac{\gamma^2}{4})\Gamma(1+b-(j-1)\frac{\gamma^2}{4})\Gamma(1-j\frac{\gamma^2}{4})}{\Gamma(2+a+b-(p+j-2)\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})}.
\end{aligned}\tag{2.1.8}$$

The last line is precisely given by the Selberg integral. It is then natural to look for an analytic continuation of this expression from positive integer p to any real p satisfying (2.1.5). Notice that giving the analytic continuation of a such a quantity is a highly non-trivial problem as p appears both in the argument of the Gamma functions as well as in the number of terms in the product. To find the right candidate for the analytic continuation we start by writing down the following relations that we will refer to as the shift equations. They are deduced by simple algebra from (2.1.8) again for $p\in\mathbb{N}$ and under the bounds (2.1.5),

$$\frac{M(\gamma,p,a+\frac{\gamma^2}{4},b)}{M(\gamma,p,a,b)}=\frac{\Gamma(1+a+\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})}{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})},\tag{2.1.9}$$

$$\frac{M(\gamma,p,a+1,b)}{M(\gamma,p,a,b)}=\frac{\Gamma(\frac{4}{\gamma^2}(1+a)+1)\Gamma(\frac{4}{\gamma^2}(2+a+b)-(2p-2))}{\Gamma(\frac{4}{\gamma^2}(1+a)-(p-1))\Gamma(\frac{4}{\gamma^2}(2+a+b)-(p-2))},\tag{2.1.10}$$

and for $p\in\mathbb{N}^*$ under the bounds (2.1.5),

$$\frac{M(\gamma,p,a,b)}{M(\gamma,p-1,a,b)}=\frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(1+b-(p-1)\frac{\gamma^2}{4})\Gamma(1-p\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a+b-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})}.\tag{2.1.11}$$

Of course similar shift equations hold for b but as there is a symmetry $M(\gamma,p,a,b)=M(\gamma,p,b,a)$ we will write everything only for a. The reason why the function $\Gamma_{\frac{\gamma}{2}}(x)$ introduced in Theorem 2.1.1 appears is that

it verifies the following two relations, for $\gamma \in (0,2)$ and $x > 0$,

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x+\frac{\gamma}{2})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{\gamma x}{2})(\frac{\gamma}{2})^{-\frac{\gamma x}{2}+\frac{1}{2}}, \tag{2.1.12}$$

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x+\frac{2}{\gamma})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{2x}{\gamma})(\frac{\gamma}{2})^{\frac{2x}{\gamma}-\frac{1}{2}}. \tag{2.1.13}$$

See appendix 2.4.5 for more details on $\Gamma_{\frac{\gamma}{2}}(x)$. Therefore we can use $\Gamma_{\frac{\gamma}{2}}(x)$ to construct a candidate function that will verify all the shift equations (2.1.9), (2.1.10), (2.1.11) not only for $p \in \mathbb{N}$ but for any real p satisfying the bounds (2.1.5). More precisely for any function $C(p)$ of p (and γ) the following quantity,

$$C(p)\frac{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})}, \tag{2.1.14}$$

is a solution to the shift equations (2.1.9), (2.1.10). Notice that for $\frac{\gamma^2}{4} \notin \mathbb{Q}$ these two shift equations completely determine the dependence on a (and on b by symmetry) of $M(\gamma,p,a,b)$. Then by a standard continuity argument in γ we will be able to extend the expression (2.1.14) to all $\gamma \in (0,2)$. Next the equation (2.1.11) translates into a constraint on the unknown function $C(p)$:

$$\frac{C(p)}{C(p-1)} = \sqrt{2\pi}(\frac{\gamma}{2})^{(p-1)\frac{\gamma^2}{4}-\frac{1}{2}}\frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}. \tag{2.1.15}$$

We see that (2.1.15) is not enough to fully determine the function $C(p)$. An additional shift equation that is a priori not predicted by the Selberg integral (2.1.8) is required. We will indeed prove that we have,

$$\frac{C(p)}{C(p-\frac{4}{\gamma^2})} = f(\gamma)(\frac{\gamma}{2})^{-p}\Gamma(\frac{4}{\gamma^2}-p), \tag{2.1.16}$$

where $f(\gamma)$ is an unknown positive function of γ. Now combining (2.1.15) and (2.1.16) completely determines the function $C(p)$ again up to an unknown constant c_γ of γ:

$$C(p) = c_\gamma\frac{(2\pi)^p}{\Gamma(1-\frac{\gamma^2}{4})^p}(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}-p\frac{\gamma}{2}). \tag{2.1.17}$$

This last constant c_γ is evaluated by choosing $p=0$ and thus we arrive at the function of Theorem 2.1.1 giving the expression of $M(\gamma,p,a,b)$.

Now the major difficulty that must be overcome is to find a way to prove all the shift equations (2.1.9), (2.1.10), (2.1.11) as well as the additional equation (2.1.16) for all real values of p,a,b satisfying (2.1.5) and not just for positive integer p. To achieve this the key ingredient of our proof is to introduce the following two auxiliary functions for $t \leq 0$,

$$U(t) := \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p], \tag{2.1.18}$$

and

$$\tilde{U}(t) := \mathbb{E}[(\int_0^1 (x-t)\, x^a(1-x)^b\, e^{\frac{\gamma}{2}X(x)}dx)^p], \tag{2.1.19}$$

and to show using probabilistic techniques that the following holds:

Proposition 2.1.4. *For $\gamma \in (0,2)$, a,b,p satisfying (2.1.5) and $t<0$, $U(t)$ is solution of the hypergeometric equation:*

$$t(1-t)U''(t) + (C-(A+B+1)t)U'(t) - ABU(t) = 0. \tag{2.1.20}$$

The parameters A, B, C are given by:

$$A = -\frac{p\gamma^2}{4}, B = -(a+b+1) - (2-p)\frac{\gamma^2}{4}, C = -a - \frac{\gamma^2}{4}. \tag{2.1.21}$$

Similarly $\tilde{U}(t)$ is solution of the hypergeometric equation but with parameters $\tilde{A}, \tilde{B}, \tilde{C}$ given by:

$$\tilde{A} = -p, \tilde{B} = -\frac{4}{\gamma^2}(a+b+2) + p - 1, \tilde{C} = -\frac{4}{\gamma^2}(a+1). \tag{2.1.22}$$

Let us make a few comments on the meaning of $U(t)$ and $\tilde{U}(t)$. These auxiliary functions are very similar to the correlation functions of LCFT with a degenerate field insertion - see [60, 61] for the case of the sphere and [88] for the unit disk - which also obey differential equations known as the BPZ equations. What is mysterious in our present case is that it is not clear whether there exists an actual CFT where $U(t)$ and $\tilde{U}(t)$ correspond to correlations with degenerate insertions which would explain why the differential equations of Proposition 2.1.4 hold. Furthermore if we replace the real t by a complex variable $t \in \mathbb{C}\backslash[0,\infty]$, it is not hard to see that $U(t)$ is a holomorphic function and Proposition 2.1.4 will hold if we replace the ordinary derivative by a complex derivative ∂_t. In the conformal bootstrap approach of CFT initiated by Belavin-Polyakov-Zamolodchikov in [10], a correlation function with a degenerate insertion can be decomposed into combinations of the structure constants and of the conformal blocks. A conformal block is a locally holomorphic function and it is always accompanied by its complex conjugate in the decomposition. What is mysterious with $U(t)$ and $\tilde{U}(t)$ is that we only see the holomorphic part. At this stage we have no CFT-based explanation for this observation although a possible path could be to look at boundary LCFT with multiple boundary cosmological constants, see for instance [72]. On the other hand let us mention that again in the very special case where $p \in \mathbb{N}$, $U(t)$ and $\tilde{U}(t)$ reduce to Selberg-type integrals and the equations of Proposition 2.1.4 were known in this case, see [57].

Proposition 2.1.4 will be established in section 2.3 by performing direct computations on $U(t)$ and $\tilde{U}(t)$. We then write the solutions of the hypergeometric equations in two different bases. One solution corresponds to a power series expansion in $|t|$ and the other to an expansion in $|t|^{-1}$. The change of basis formula (2.4.56) written in appendix 2.4.5 given by the theory of hypergeometric equations then provides non-trivial relations which are precisely the shift equations that we wish to prove. This is performed in detail in section 2.2 where Proposition 2.2.1 completely determines the dependence in a and b of $M(\gamma, p, a, b)$ and Proposition 2.2.2 establishes (2.1.17). Thus we have proved Theorem 2.1.1.

2.1.2 Tail expansion for GMC and the reflection coefficients

Before moving into the proof of our main result, we provide in this subsection and in the following some applications of Theorem 2.1.1. The first application we will consider deals with tail expansions for GMC measures, in other words the probability for a GMC measure to be large. We choose to include here a very general discussion about these tail expansions of GMC with an arbitrary insertion both in one and in two dimensions. For each tail expansion result there will appear a universal coefficient known as the reflection coefficient.

The first case that was studied is the tail expansion of a GMC in dimension two and a precise asymptotic was given in [61] in terms of the reflection coefficient $\overline{R}_2(\alpha)$,[4] see Proposition 2.1.6 below.[5] Let us mention that it was recently discovered in [103] that $\overline{R}_2(\alpha)$ corresponds to the partition function of the α-quantum sphere introduced by Duplantier-Miller-Sheffield in [33]. Now our exact formula on the unit interval will allow us to write a similar tail expansion for GMC in dimension one. Following [33] we use the standard radial decomposition of the covariance (2.1.1) of X around the point 0, i.e. we write for $s \geq 0$,

$$X(e^{-s/2}) = B_s + Y(e^{-s/2}), \tag{2.1.23}$$

[4] In [61] or [103] this coefficient is actually called $\overline{R}(\alpha)$ but for the needs of our discussion we introduce the 2 to indicate the dimension. Furthermore the bar stands for the fact that it is the unit volume coefficient.

[5] $\overline{R}_2(\alpha)$ is the bulk reflection coefficient in dimension two, a boundary reflection coefficient $\overline{R}_2^{\partial}(\alpha)$ also exists but its value remains unknown, see the figure below.

where B_s is a standard Brownian motion and Y is an independent Gaussian process that can be defined on the whole plane with covariance given for $x, y \in \mathbb{C}$ by:

$$\mathbb{E}[Y(x)Y(y)] = 2\ln\frac{|x|\vee|y|}{|x-y|}. \tag{2.1.24}$$

Motivated by the Williams decomposition of Theorem 2.4.3, we introduce for $\lambda > 0$ the process that will be used in the definitions below,

$$\mathcal{B}_s^\lambda := \begin{cases} \hat{B}_s - \lambda s & s \geq 0 \\ \bar{B}_{-s} + \lambda s & s < 0, \end{cases} \tag{2.1.25}$$

where $(\hat{B}_s - \lambda s)_{s\geq 0}$ and $(\bar{B}_s - \lambda s)_{s\geq 0}$ are two independent Brownian motions with negative drift conditioned to stay negative. We can now give the definitions of the two coefficients in dimension one $\overline{R}_1^\partial(\alpha)$ and $\overline{R}_1(\alpha)$ along with the associated GMC measures with insertion $I_{1,\eta}^\partial(\alpha)$ and $I_{1,\eta}(\alpha)$ whose tail behavior will be governed by the corresponding coefficient:

$$\begin{aligned}
\overline{R}_1^\partial(\alpha) &:= \mathbb{E}[(\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})} ds)^{\frac{2}{\gamma}(Q-\alpha)}], \\
\overline{R}_1(\alpha) &:= \mathbb{E}[(\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} (e^{\frac{\gamma}{2}Y(e^{-s/2})} + e^{\frac{\gamma}{2}Y(-e^{-s/2})}) ds)^{\frac{2}{\gamma}(Q-\alpha)}], \\
I_{1,\eta}^\partial(\alpha) &:= \int_0^\eta x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)} dx, \\
I_{1,\eta}(\alpha) &:= \int_{v-\eta}^{v+\eta} |x-v|^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)} dx.
\end{aligned}$$

Let us make some comments on these definitions. Here $\alpha \in (\frac{\gamma}{2}, Q)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, and η is an arbitrary positive real number chosen small enough. To match the conventions of the study of LCFT we have written the fractional power $x^{-\frac{\gamma\alpha}{2}}$, so in these notations we have $a = -\frac{\gamma\alpha}{2}$. Notice that the difference between $I_{1,\eta}^\partial(\alpha)$ and $I_{1,\eta}(\alpha)$ lies in the position of the insertion. For $I_{1,\eta}^\partial(\alpha)$ the insertion is placed in 0 (by symmetry we could have placed it in 1). Our Theorem 2.1.1 will give us the value of the associated coefficient $\overline{R}_1^\partial(\alpha)$. The other case corresponds to placing the insertion at a point v inside the interval, $v \in (0,1)$, and gives the quantity $I_{1,\eta}(\alpha)$. The computation of the associated $\overline{R}_1(\alpha)$ will be done in a future work. We now claim:

Proposition 2.1.5. *For $\alpha \in (\frac{\gamma}{2}, Q)$ we have the following tail expansion for $I_{1,\eta}^\partial(\alpha)$ as $u \to \infty$ and for some $\nu > 0$,*

$$\mathbb{P}(I_{1,\eta}^\partial(\alpha) > u) = \frac{\overline{R}_1^\partial(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\alpha)+\nu}}), \tag{2.1.26}$$

where the value of $\overline{R}_1^\partial(\alpha)$ is given by:

$$\overline{R}_1^\partial(\alpha) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)-\frac{1}{2}}}{(Q-\alpha)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma_{\frac{\gamma}{2}}(\alpha-\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}. \tag{2.1.27}$$

The proof of this proposition is done in appendix 2.4.4. Notice that we impose the condition $\alpha \in (\frac{\gamma}{2}, Q)$. This is crucial for the tail behavior of $I_{1,\eta}^\partial(\alpha)$ (or similarly for $I_{1,\eta}(\alpha)$) to be dominated by the insertion and this is precisely why the asymptotic expansion is independent of the choice of η. It also explains why the radial decomposition (2.1.23) is natural as it is well suited to study X around a particular point. If one is interested in the case where $\alpha < \frac{\gamma}{2}$ (or simply $\alpha = 0$), a different argument known as the localization trick is required to obtain the tail expansion, see [94] for more details. For the sake of completeness of our discussion we also recall the tail expansion in dimension two that was obtained in [61]. The normalizations in this case are slightly

different as we do not include a factor 2 in the covariance. We work with a Gaussian process $\tilde{X}$ defined on the unit disk $\mathbb{D}$ with covariance $\ln \frac{1}{|x-y|}$. Instead of Y we use $\tilde{Y}$ with covariance:

$$\mathbb{E}[\tilde{Y}(x)\tilde{Y}(y)] = \ln \frac{|x| \vee |y|}{|x-y|}. \tag{2.1.28}$$

For an insertion placed in z, $|z| < 1$ we now define,

$$\overline{R}_2(\alpha) := \mathbb{E}[(\int_{-\infty}^{\infty} e^{\gamma \mathcal{B}_s^{Q-\alpha}} \int_0^{2\pi} e^{\gamma \tilde{Y}(e^{-s}e^{i\theta})} ds)^{\frac{2}{\gamma}(Q-\alpha)}],$$
$$I_{2,\eta}(\alpha) := \int_{B(z,\eta)} |x-z|^{-\gamma\alpha} e^{\gamma \tilde{X}(x)} d^2x,$$

and we state the result obtained in [61]:

Proposition 2.1.6. *(Kupiainen-Rhodes-Vargas [61]) For $\alpha \in (\frac{\gamma}{2}, Q)$ we have the following tail expansion for $I_{2,\eta}(\alpha)$ as $u \to \infty$ and for some $\nu > 0$,*

$$\mathbb{P}(I_{2,\eta}(\alpha) > u) = \frac{\overline{R}_2(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\alpha)+\nu}}), \tag{2.1.29}$$

where the value of $\overline{R}_2(\alpha)$ is given by:

$$\overline{R}_2(\alpha) = -\frac{\gamma}{2(Q-\alpha)} \frac{(\pi\Gamma(\frac{\gamma^2}{4}))^{\frac{2}{\gamma}(Q-\alpha)}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma(-\frac{\gamma}{2}(Q-\alpha))}{\Gamma(\frac{\gamma}{2}(Q-\alpha))\Gamma(\frac{2}{\gamma}(Q-\alpha))}. \tag{2.1.30}$$

A similar proposition is also expected for $\overline{R}_2^{\partial}(\alpha)$, the boundary reflection coefficient in dimension two, whose expression and computation are left for a future paper. One notices that $\overline{R}_1^{\partial}(\alpha)$ has a more convoluted expression than $\overline{R}_2(\alpha)$ as the special function $\Gamma_{\frac{\gamma}{2}}$ appears in its expression. Such expressions have already appeared in the study of Liouville theory for instance in [85] where a general formula for the reflection amplitude is given. We now summarize the four different cases that we have discussed in the following figure. For each coefficient the number 1 or 2 stands for the dimension and the partial ∂ symbol stands for the boundary cases, no ∂ corresponds to the bulk cases.

2.1.3 Small deviations for GMC

We now turn to the problem of determining the universal behavior of the probability for a GMC to be small. Both the exact formulas of Theorem 2.1.1 and the one proven on the unit circle in [88] will provide crucial insight. For this subsection only we will use the following shorthand notation:

$$I_{\gamma,a,b} := \int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx. \tag{2.1.31}$$

In the following we will rely extensively on the decomposition

$$I_{\gamma,a,b} = \tilde{c} L Y_\gamma X_1 X_2 X_3$$

coming from Corollary 2.1.3 with $\tilde{c}$ being a positive constant. First L is a log-normal law, so one has $\mathbb{P}(L \leq \epsilon) \leq c_1 \exp(-c_2(\ln \epsilon)^2)$ for some $c_1, c_2 > 0$. On the other hand the probability for Y_γ to be small is much smaller since $\mathbb{P}(Y_\gamma \leq \epsilon) \leq \exp(-c\epsilon^{-\frac{4}{\gamma^2}})$ for some $c > 0$. From the above and since $X_1, X_2, X_3 \geq 1$ the probability to be small for $I_{\gamma,a,b}$ will be of log-normal type. By comparison in the case of the total mass of the GMC on the unit circle it was shown in [88] that it is distributed according to Y_γ and so its probability to be small is of order $\exp(-c\epsilon^{-\frac{4}{\gamma^2}})$.

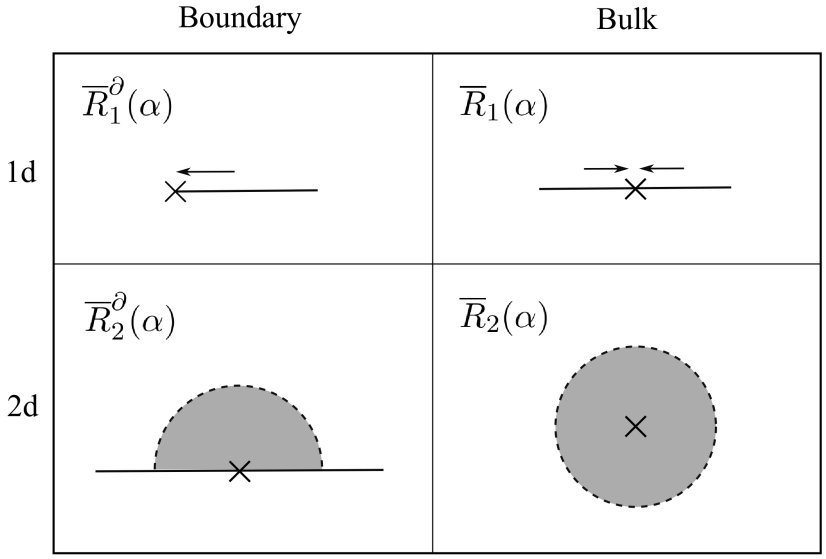

Figure 2.1: Four types of reflection coefficients

Thus it appears that GMC on the unit interval and the unit circle have completely different small deviations. However this difference comes from the fact that the log-correlated field on the circle is of average zero while in the case of the interval there is a non-zero global mode producing the log-normal variable L. Therefore on the interval if one subtracts the average of X with respect to the correct measure (see below) one can remove the log-normal law L appearing in the decomposition of Corollary 2.1.3. The probability for the resulting GMC to be small will then be bounded by $\exp(-c\epsilon^{-\frac{4}{\gamma^2}})$ for some $c > 0$ just like for the case of the circle. We expect this to be the correct universal behavior.

Let us make the above more precise. We start by writing down the decomposition of the covariance of our field in terms of the Chebyshev polynomials. For all $x, y \in [0,1]$ with $x \neq y$ we have:

$$-2\ln|x-y| = 4\ln 2 + \sum_{n=1}^{+\infty} \frac{4}{n} T_n(2x-1)T_n(2y-1). \tag{2.1.32}$$

We recall that the Chebyshev polynomial of order n is the unique polynomial verifying $T_n(\cos\theta) = \cos(n\theta)$. This basis of polynomials is also orthogonal with respect to dot product given by the integration against $\frac{1}{\sqrt{1-x^2}}dx$, i.e.

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{for} \quad n \neq m \\ \pi & \text{for} \quad n = m = 0 \\ \frac{\pi}{2} & \text{for} \quad n = m \neq 0 \end{cases} \tag{2.1.33}$$

From the above our field $X(x)$ can be constructed by the series:

$$X(x) = 2\sqrt{\ln 2}\alpha_0 + \sum_{n=1}^{+\infty} \frac{2\alpha_n}{\sqrt{n}} T_n(2x-1). \tag{2.1.34}$$

Here $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. standard Gaussians. This of course only makes sense if one integrates both sides against a test function. We now introduce:

$$\overline{X} := \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-(2x-1)^2}} X(x)dx = 2\sqrt{\ln 2}\alpha_0 \quad \text{and} \quad X_\perp(x) := X(x) - \overline{X}.$$

We easily check that $e^{\frac{\gamma}{2}\overline{X}} \overset{\text{law}}{=} \exp(\mathcal{N}(\gamma^2 \ln 2))$. The probability to be small for the GMC associated to $X_\perp(x)$ is now given by,

$$\mathbb{P}(\int_0^1 e^{\frac{\gamma}{2}X_\perp(x)}dx \leq \epsilon) \leq \exp(-c\epsilon^{-\frac{4}{\gamma^2}}). \tag{2.1.35}$$

This result can be easily obtained from Corollary 2.1.3 by noticing that since we removed $L = \exp(\mathcal{N}(\gamma^2 \ln 2))$ the probability to be small is now governed by Y_γ which gives the bound written above. The argument we have just described is expected to work for any GMC in any dimension, a result of this nature can be found in [63].

There is also a direct application of these observations to determining the law of the random variable $I_{\gamma,a,b}$. This is linked to how the strategy of the proof of the present paper differs from the one used in [88] to prove the Fyodorov-Bouchaud formula. In subsection 2.2.2 we first use the differential equation (2.1.20) on $U(t)$ to obtain a relation between $M(\gamma, p, a, b)$ and $M(\gamma, p-1, a, b)$. Thus from this relation and knowing that $M(\gamma, 0, a, b) = 1$ one can compute recursively all the negative moments of the random variable $I_{\gamma,a,b}$. As it was emphasized in many papers (see the review [79] by Ostrovsky and references therein), the negative moments of $I_{\gamma,a,b}$ do not determine its law as the growth of the negative moments is too fast. This is why we must derive a second relation between $M(\gamma, p, a, b)$ and $M(\gamma, p - \frac{4}{\gamma^2}, a, b)$ which gives enough information to complete the proof. By contrast in the case of the total mass of the GMC on the unit circle the negative moments do capture uniquely the probability distribution and so the proof of the Fyodorov-Bouchaud formula given in [88] only requires one shift equation (in a similar fashion one obtains a relation between the moment p and the moment $p-1$ of the total mass of the GMC).

But the negative moments of $I_{\gamma,a,b}$ do not determine its law only because of the log-normal law L in the decomposition of Corollary 2.1.3. By using Corollary 2.1.3 and by independence of $X_\perp(x)$ and $\overline{X}$ one can factor out $e^{\frac{\gamma}{2}\overline{X}} \overset{\text{law}}{=} L$ and the computation of the negative moments is now sufficient to uniquely determine the distribution. Thus the negative moments of a GMC measure always determine its law if one removes the global Gaussian coming from the average of the field with respect to an appropriate measure. From this observation the relation between $M(\gamma, p, a, b)$ and $M(\gamma, p - \frac{4}{\gamma^2}, a, b)$ could be omitted in the proof of Theorem 2.1.1. Nonetheless if one only computes the negative moments it is not clear that the analytic continuation given by the Γ_γ functions does correspond to the fractional moments of a random variable, this fact has been checked by Ostrovsky in [76]. Thus in order to keep the proof of our theorem self-contained we choose to keep both shift equations.

2.1.4 Other applications

Similarly as in [88] we will write the applications of our Theorem 2.1.1 to the behavior of the maximum of X and to random matrix theory. We refer to [88] for more detailed explanations and for additional references on these problems.

Characterizing the behavior of the maximum of X requires to compute the law of the total mass of the derivative martingale,

$$M' = -\frac{1}{2}\int_0^1 X(x)e^{X(x)}dx := -\frac{1}{2}\lim_{\delta\to 0}\int_0^1 (X_\delta(x) - \mathbb{E}[X_\delta(x)^2])e^{X_\delta(x)-\frac{1}{2}\mathbb{E}[X_\delta(x)^2]}dx,$$

which following [6] can be characterized by the convergence in law:

$$2M' = \lim_{\gamma\to 2}\frac{1}{2-\gamma}\int_0^1 e^{\frac{\gamma}{2}X(x)}dx. \tag{2.1.36}$$

Therefore from our Theorem 2.1.1 we can easily compute the moments of this quantity,

$$\begin{aligned}\mathbb{E}[(2M')^p] &= (2\pi)^p \frac{\Gamma_1(1-p)\Gamma_1(2-p)^2\Gamma_1(4-p)}{\Gamma_1(2)^2\Gamma_1(4-2p)} \\ &= \frac{G(4-2p)}{G(1-p)G(2-p)^2G(4-p)},\end{aligned}$$

where $G(x)$ is the so-called Barnes G function, see appendix 2.4.5 for more details. Just like in Corollary 2.1.3 an explicit description of the resulting law has been found in [78],

$$2M' \overset{law}{=} \frac{\pi}{32}\tilde{L}\tilde{X}_1\tilde{X}_2\tilde{X}_3, \tag{2.1.37}$$

where $\tilde{L}, \tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ are four independent random variables on $\mathbb{R}_+$ with the following laws:

$$\begin{aligned}
\tilde{L} &= \exp(\mathcal{N}(0, 4\ln 2)),\\
\tilde{X}_1 &= \frac{1}{y^2}e^{-1/y},\ y > 0\\
\tilde{X}_2 &= \beta_{2,2}^{-1}(1,1;2,\frac{1}{2},\frac{1}{2}),\\
\tilde{X}_3 &= \frac{2}{y^3}dy,\ y > 1.
\end{aligned}$$

Then for a suitable regularization X_δ of X the following convergence holds in law:

$$\begin{aligned}
\max_{x\in[0,1]} X_\delta(x) - 2\ln\frac{1}{\delta} + \frac{3}{2}\ln\ln\frac{1}{\delta} &\underset{\delta\to 0}{\to} \mathcal{G}_1 + \ln M' + c\\
&= \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln\tilde{X}_2 + \ln\tilde{X}_3 + c.
\end{aligned}$$

All the random variables appearing above are independent, $\mathcal{G}_1$ and $\mathcal{G}_2$ are two independent Gumbel laws, and c is a non-universal real constant that depends on the regularization procedure. We have also used the fact that $\ln\tilde{X}_1 \overset{law}{=} \mathcal{G}_2$.

Lastly we briefly mention that in the case of the interval it is also possible to see the GMC measure as the limit of the characteristic polynomial of random Hermitian matrices, the connection in this case was established in [13]. The main result of [13] is that for suitable random Hermitian matrices H_N, the quantity

$$\frac{|\det(H_N - x)|^\gamma}{\mathbb{E}|\det(H_N - x)|^\gamma}dx$$

converges in law to the GMC measure on the unit interval $[0,1]$.[6] Therefore the same applications as the ones given in [88] hold and in particular one can conjecture that the following convergence in law holds:

$$\max_{x\in[0,1]} \ln|\det(H_N - x)| - \ln N + \frac{3}{4}\ln\ln N \underset{N\to\infty}{\to} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4\ln 2) + \ln\tilde{X}_2 + \ln\tilde{X}_3 + c.$$

This conjecture first appeared in [46] although it was written on $[-1,1]$ instead of $[0,1]$.

2.2 The shift equations on a and p

To prove Theorem 2.1.1 we proceed in two steps. We first completely determine the dependence of $M(\gamma, p, a, b)$ on the parameters a and b, see the result of Proposition 2.2.1 just below. We are then left with an unknown function $C(p)$ of p (and γ) and give its value in Proposition 2.2.2. Throughout this section we extensively use the fact that $U(t)$ and $\tilde{U}(t)$ are solutions of the hypergeometric equations of Proposition 2.1.4 proven in section 2.3.

[6] Actually in [13] the limiting GMC measure is defined on $[-1,1]$ but of course by a change of variable we can write everything on $[0,1]$.

2.2.1 The shifts in a

The goal of this subsection is to prove the shift equations (2.1.9), (2.1.10) on a and b to completely determine the dependence of $M(\gamma,p,a,b)$ on these two parameters. By symmetry we will write everything only for a. We will thus prove that:

Proposition 2.2.1. *For $\gamma\in(0,2)$ and a,b,p satisfying the bounds (2.1.5), $M(\gamma,p,a,b)$ is given by the expression,*

$$C(p)\frac{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})}, \tag{2.2.1}$$

where $C(p)$ is the function that contains the remaining dependence on p (and γ). It will be computed in subsection 2.2.2.

$\diamond$ *The $+\frac{\gamma^2}{4}$ shift equation*
Here we start with the first auxiliary function, for $\gamma\in(0,2)$ and a,b,p satisfying (2.1.5):

$$U(t)=\mathbb{E}[(\int_0^1(x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{2.2.2}$$

From the result of Proposition 2.1.4, $U(t)$ is solution to a hypergeometric equation. As explained in appendix 2.4.5 we can write the solutions of this hypergeometric equation for $t\in(-\infty,0)$ in two different bases, one corresponding to an expansion in powers of $|t|$ and one to an expansion in power of $|t|^{-1}$. Since the solution space is a two-dimensional real vector space, each basis will be parametrized by two real constants. Let C_1,C_2 and D_1,D_2 stand for these constants. The theory of hypergeometric equations then gives an explicit change of basis formula (2.4.56) linking C_1,C_2 and D_1,D_2. Thus we can write, when $A-B$ and C are not integers,

$$\begin{aligned}U(t)&=C_1F(A,B,C,t)+C_2|t|^{1-C}F(1+A-C,1+B-C,2-C,t) &(2.2.3)\\ &=D_1|t|^{-A}F(A,1+A-C,1+A-B,t^{-1})+D_2|t|^{-B}F(B,1+B-C,1+B-A,t^{-1}), &(2.2.4)\end{aligned}$$

where F is the hypergeometric function. We recall that the parameters A,B,C are given by:

$$A=-\frac{p\gamma^2}{4},B=-(a+b+1)-(2-p)\frac{\gamma^2}{4},C=-a-\frac{\gamma^2}{4}. \tag{2.2.5}$$

The values of A,B,C left out corresponding to $A-B$ or C being integers will be recovered at the level of the shift equation (2.2.11) by continuity. The idea is now to identify the constants C_1,C_2,D_1,D_2 by performing asymptotic expansions on $U(t)$. Two of the above constants are easily obtained by evaluating $U(t)$ in $t=0$ and by taking the limit $t\to-\infty$:

$$\begin{aligned}C_1&=M(\gamma,p,a+\frac{\gamma^2}{4},b), &(2.2.6)\\ D_1&=M(\gamma,p,a,b). &(2.2.7)\end{aligned}$$

By performing a more detailed asymptotic expansion in $t\to-\infty$ we claim that:

$$D_2=0. \tag{2.2.8}$$

We sketch a short proof. For $t<-2$ (arbitrary) and $x\in[0,1]$,

$$(x-t)^{\frac{\gamma^2}{4}}-|t|^{\frac{\gamma^2}{4}}\leq c|t|^{\frac{\gamma^2}{4}-1},$$

for some constant $c > 0$. By interpolating, for $t < -2$,

$$\begin{aligned}
|U(t) - D_1|t|^{\frac{p\gamma^2}{4}}| &= \Big|\mathbb{E}[(\int_0^1 \big(u(x-t)^{\frac{\gamma^2}{4}} + (1-u)|t|^{\frac{\gamma^2}{4}}\big)x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]\,|_{u=1} \\
&\quad - \mathbb{E}[(\int_0^1 \big(u(x-t)^{\frac{\gamma^2}{4}} + (1-u)|t|^{\frac{\gamma^2}{4}}\big)x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]\,|_{u=0}\Big| \\
&\leq |p|\int_0^1 dx_1\big((x_1-t)^{\frac{\gamma^2}{4}} - |t|^{\frac{\gamma^2}{4}}\big)x_1^a(1-x_1)^b\Big(\mathbb{E}[(\int_0^1 \frac{(x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b}{|x_1-x|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\
&\quad + \mathbb{E}[(\int_0^1 \frac{|t|^{\frac{\gamma^2}{4}}x^a(1-x)^b}{|x_1-x|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]\Big) \\
&\leq c'|t|^{\frac{p\gamma^2}{4}-1}M(\gamma,p,a,b) \underset{t\to-\infty}{=} O(|t|^{\frac{p\gamma^2}{4}-1}),
\end{aligned}$$

where in both steps we have used the Girsanov theorem (see appendix 2.4.1) and $c' > 0$ is some constant. However, by using the bound (2.1.5) over p:

$$(-A) - (-B) = -(a+b+1+(2-2p)\frac{\gamma^2}{4}) < 1. \tag{2.2.9}$$

This implies that $D_2 = 0$. We then use the following identity coming from the theory of hypergeometric functions (2.4.56):

$$C_1 = \frac{\Gamma(1-C)\Gamma(A-B+1)}{\Gamma(A-C+1)\Gamma(1-B)}D_1. \tag{2.2.10}$$

This leads to the first shift equation (2.1.9):

$$\frac{M(\gamma,p,a+\frac{\gamma^2}{4},b)}{M(\gamma,p,a,b)} = \frac{\Gamma(1+a+\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})}{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})}. \tag{2.2.11}$$

$\diamond$ *The $+1$ shift equation*
We now write everything with the second auxiliary function, for $\gamma \in (0,2)$ and a, b, p satisfying (2.1.5):

$$\tilde{U}(t) = \mathbb{E}[(\int_0^1 (x-t)x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]. \tag{2.2.12}$$

Again we write the solutions of the hypergeometric equation around $t = 0_-$ and $t = -\infty$, when $\tilde{C}$ and $\tilde{A} - \tilde{B}$ are not integers,

$$\tilde{U}(t) = \tilde{C}_1 F(\tilde{A},\tilde{B},\tilde{C},t) + \tilde{C}_2|t|^{1-\tilde{C}}F(1+\tilde{A}-\tilde{C},1+\tilde{B}-\tilde{C},2-\tilde{C},t) \tag{2.2.13}$$

$$= \tilde{D}_1|t|^{-\tilde{A}}F(\tilde{A},1+\tilde{A}-\tilde{C},1+\tilde{A}-\tilde{B},t^{-1}) + \tilde{D}_2|t|^{-\tilde{B}}F(\tilde{B},1+\tilde{B}-\tilde{C},1+\tilde{B}-\tilde{A},t^{-1}). \tag{2.2.14}$$

As before we have introduced four real constants $\tilde{C}_1, \tilde{C}_2, \tilde{D}_1, \tilde{D}_2$ and $\tilde{A}, \tilde{B}, \tilde{C}$ are given by:

$$\tilde{A} = -p, \tilde{B} = -\frac{4}{\gamma^2}(a+b+2)+p-1, \tilde{C} = -\frac{4}{\gamma^2}(a+1). \tag{2.2.15}$$

Two of our constants are again easily obtained,

$$\tilde{C}_1 = M(\gamma,p,a+1,b), \tag{2.2.16}$$

$$\tilde{D}_1 = M(\gamma,p,a,b), \tag{2.2.17}$$

and we can proceed as previously to obtain:

$$\tilde{D}_2 = 0. \tag{2.2.18}$$

The relation between $\tilde{C}_1$ and $\tilde{D}_1$ (2.4.56) then leads to the shift equation (2.1.10):

$$\frac{M(\gamma,p,a+1,b)}{M(\gamma,p,a,b)} = \frac{\Gamma(\frac{4}{\gamma^2}(1+a)+1)\Gamma(\frac{4}{\gamma^2}(2+a+b)-(2p-2))}{\Gamma(\frac{4}{\gamma^2}(1+a)-(p-1))\Gamma(\frac{4}{\gamma^2}(2+a+b)-(p-2))}. \tag{2.2.19}$$

Therefore for $\frac{\gamma^2}{4} \notin \mathbb{Q}$, (2.2.11) and (2.2.19) prove the formula of Proposition 2.2.1. The result for the other values of γ follows from the well known fact that $\gamma \mapsto M(\gamma,p,a,b)$ is a continuous function.

2.2.2 The shifts in p

We now tackle the problem of determining two shift equations on p, (2.1.15) and (2.1.16), to completely determine the function $C(p)$ of Proposition 2.2.1. We will work only with $U(t)$. The idea is to perform a computation at the next order in the expressions of the previous subsection. This will give the desired result:

Proposition 2.2.2. *For $\gamma \in (0,2)$ and $p < \frac{4}{\gamma^2}$:*

$$C(p) = \frac{(2\pi)^p}{\Gamma(1-\frac{\gamma^2}{4})^p}(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}\frac{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})}. \tag{2.2.20}$$

$\diamond$ *The $+1$ shift equation*
Since we have completely determined the dependence of M on a,b by equation (2.2.1) we are free to choose a and b as we wish. To find the next order in $t \to 0_-$, the most natural idea is to take a such that $0 < 1-C = 1+a+\frac{\gamma^2}{4} < 1$, and then it suffices to study the equivalent of $U(t)-U(0)$ when $t \to 0_-$. For technical reasons this only gives the expression of C_2 when $\gamma < \sqrt{2}$. To obtain C_2 for all $\gamma \in (0,2)$, we will need to go one order further in the asymptotic expansion and we make the choice $0 < a < 1-\frac{\gamma^2}{4}$ and $b = 0$. In this case, we have $p < \frac{4}{\gamma^2}$, $1 < 1-C < 2$. We perform a Taylor expansion around $t = 0_-$,

$$U(t) = U(0) + tU'(0) + t^2\int_0^1 U''(tu)(1-u)du,$$

with

$$\begin{aligned} U''(tu) &\overset{(\star)}{=} -\frac{p\gamma^2}{4}\int_0^1 dx_1(x_1-tu)^{\frac{\gamma^2}{4}-1}x_1^a\frac{a}{x_1}\mathbb{E}[(\int_0^1\frac{(x-tu)^{\frac{\gamma^2}{4}}x^a}{|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\ &= -\frac{p\gamma^2 a}{4}|tu|^{-1+a+\frac{\gamma^2}{4}}\int_0^{-\frac{1}{tu}}dy(y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}\mathbb{E}[(\int_0^1\frac{(x-tu)^{\frac{\gamma^2}{4}}x^a}{|x+tuy|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]. \end{aligned}$$

$(\star)$ comes from multiple applications of the Girsanov theorem (see appendix 2.4.1) and symmetrization tricks. One may refer to (2.3.5) where we calculate rigorously the derivatives of $U(t)$. Next we have the following bound for $y \in [0,-\frac{1}{tu}]$, $u \in [0,1]$, and $t \in [-1,0]$:

$$\begin{aligned} &\mathbb{E}[(\int_0^1\frac{(x-tu)^{\frac{\gamma^2}{4}}x^a}{|x+tuy|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\ &\leq \sup_{x_1\in[0,1]}\{\mathbb{E}[(\int_0^1\frac{x^{a+\frac{\gamma^2}{4}}}{|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + \mathbb{E}[(\int_0^1\frac{(x+1)^{\frac{\gamma^2}{4}}x^a}{|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]\} \\ &< \infty. \end{aligned}$$

Then we get by dominant convergence that,

$$U''(tu) \overset{t\to 0_-}{\sim} -\frac{p\gamma^2 a}{4}|tu|^{-1+a+\frac{\gamma^2}{4}}\int_0^\infty dy(y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}M(\gamma,p-1,a-\frac{\gamma^2}{4},0),$$

and again by dominant convergence:

$$U(t)-U(0)-tU'(0) = -\frac{p\gamma^2 a}{4}\frac{\Gamma(a+\frac{\gamma^2}{4})}{\Gamma(2+a+\frac{\gamma^2}{4})}|t|^{1+a+\frac{\gamma^2}{4}}\int_0^\infty dy(y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}M(\gamma,p-1,a-\frac{\gamma^2}{4},0)+o(|t|^{1+a+\frac{\gamma^2}{4}}).$$

The value of the integral above is given by (2.4.64). We arrive at the expression for C_2:

$$C_2 = p\frac{\Gamma(a+1)\Gamma(-a-\frac{\gamma^2}{4}-1)}{\Gamma(-\frac{\gamma^2}{4})}M(\gamma,p-1,a-\frac{\gamma^2}{4},0). \tag{2.2.21}$$

The theory of hypergeometric equations (2.4.56) gives this time the relation:

$$C_2 = \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)}D_1. \tag{2.2.22}$$

By identifying the above two expressions of C_2, we get

$$M(\gamma,p-1,a-\frac{\gamma^2}{4},0) = \frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a-(p-2)\frac{\gamma^2}{4})}{\Gamma(1+a)\Gamma(2+a-(2p-3)\frac{\gamma^2}{4})}M(\gamma,p,a,0).$$

By using the shift equation (2.2.11) on a, we can drop the $-\frac{\gamma^2}{4}$ after a in the expression $M(\gamma,p-1,a-\frac{\gamma^2}{4},0)$ and we obtain for $0<a<1-\frac{\gamma^2}{4}$ and $b=0$,

$$\frac{M(\gamma,p,a,0)}{M(\gamma,p-1,a,0)} = \frac{\Gamma(1-\frac{p\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})\Gamma(2+a-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a-(2p-2)\frac{\gamma^2}{4})}.$$

Combined with (2.2.1), this leads to a first relation on our constant $C(p)$, for $p<\frac{4}{\gamma^2}$,

$$\frac{C(p)}{C(p-1)} = \sqrt{2\pi}(\frac{\gamma}{2})^{(p-1)\frac{\gamma^2}{4}-\frac{1}{2}}\frac{\Gamma(1-p\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}. \tag{2.2.23}$$

Reversely, (2.2.23) and (2.2.1) show that for all a,b,p satisfying the bounds (2.1.5):

$$\frac{M(\gamma,p,a,b)}{M(\gamma,p-1,a,b)} = \frac{\Gamma(1-\frac{p\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(1+b-(p-1)\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a+b-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})}. \tag{2.2.24}$$

$\diamond$ *The $+\frac{4}{\gamma^2}$ shift equation*

Since the relation (2.2.23) is not enough to completely determine the function $C(p)$, we seek another relation on $C(p)$ that is not predicted by the Selberg integral. The techniques of this subsection are a little more involved, they lead to a relation between $C(p)$ and $C(p-\frac{4}{\gamma^2})$. Again we can pick a and b as we wish so we choose $b=0$ and $-1-\frac{\gamma^2}{4}<a<-1-\frac{\gamma^2}{4}+a_0$ where $a_0>0$ is a constant introduced in lemma 2.4.9 of appendix 2.4.3. The asymptotic in $t\to 0_-$ of the following quantity is then given by the lemma 2.4.9,

$$\begin{aligned}&\mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}}x^a e^{\frac{\gamma}{2}X(x)}dx)^p] - \mathbb{E}[(\int_0^1 x^{a+\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}X(x)}dx)^p]\\ &= g(\gamma,a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)}|t|^{1+a+\frac{\gamma^2}{4}}M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0)+o(|t|^{1+a+\frac{\gamma^2}{4}}),\end{aligned}$$

where $g(\gamma,a)$ is a real function that only depends on γ and a. Comparing with the expansion (2.2.3), we have:

$$C_2 = g(\gamma,a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)}M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0). \tag{2.2.25}$$

With the identity (2.4.56) coming from hypergeometric equations:

$$\begin{aligned} C_2 &= \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)} D_1 \\ &= \frac{\Gamma(-1-a-\frac{\gamma^2}{4})\Gamma(2+a-(2p-2)\frac{\gamma^2}{4})}{\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})} M(\gamma,p,a,0). \end{aligned}$$

Comparing the above two expressions of C_2 yields:

$$g(\gamma,a) = \frac{M(\gamma,p,a,0)}{M(\gamma,p-1-\frac{4}{\gamma^2}(a+1),-2-a-\frac{\gamma^2}{4},0)} \frac{\Gamma(-p)\Gamma(-1-a-\frac{\gamma^2}{4})\Gamma(2+a-(2p-2)\frac{\gamma^2}{4})}{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}. \quad (2.2.26)$$

A crucial remark is that from (2.2.1) and analycity of the function Γ_γ, $M(\gamma,p,a,b)$ is analytic in a,b. Thus the right hand side of (2.2.26) is analytic in a. We can then do analytic continuation simultaneously for both sides in the above equation. This shows that the expression of the right hand side does not depend on p not only for $-1-\frac{\gamma^2}{4} < a < -1-\frac{\gamma^2}{4}+a_0$ but for all appropriate a where the expression is well-defined, i.e. $-1-\frac{\gamma^2}{4} < a < -1$.

In the following computations $f(\gamma)$ stands for a real function depending only on γ and we will use the abuse of notation that it could be a different function of γ every time it appears. Consider the case where $\frac{4}{k+1} < \gamma^2 < \frac{4}{k}$ for a $k \in \mathbb{N}^*$. For this range of γ we make the special choice $a = -\frac{(k+1)\gamma^2}{4}$ and thus the bounds $-1-\frac{\gamma^2}{4} < a < -1$ on a are satisfied. In the previous paragraph we have shown that for $a = -\frac{(k+1)\gamma^2}{4}$:

$$\frac{M(\gamma,p,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2}+k,\frac{k\gamma^2}{4}-2,0)} = f(\gamma)\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}{\Gamma(-p)\Gamma(\frac{k\gamma^2}{4}-1)\Gamma(2-(2p+k-1)\frac{\gamma^2}{4})}. \quad (2.2.27)$$

By the shift equations (2.2.11) and (2.2.19):

$$\begin{aligned} \frac{M(\gamma,p-\frac{4}{\gamma^2}+k,\frac{k\gamma^2}{4}-2,0)}{M(\gamma,p-\frac{4}{\gamma^2}+k,-\frac{(k+1)\gamma^2}{4},0)} &= f(\gamma)\prod_{j=0}^{1}\frac{\Gamma(j\frac{4}{\gamma^2}+1-p)\Gamma((1+j)\frac{4}{\gamma^2}+2-p)}{\Gamma((2+j)\frac{4}{\gamma^2}-k+2-2p)} \\ &\quad \times \prod_{i=0}^{2k}\frac{\Gamma(4-(2p+3k-i-1)\frac{\gamma^2}{4})}{\Gamma(2-(p+2k-i)\frac{\gamma^2}{4})\Gamma(3-(p+2k-i-1)\frac{\gamma^2}{4})}. \end{aligned}$$

Then by (2.2.24),

$$\begin{aligned} &\frac{M(\gamma,p-\frac{4}{\gamma^2}+k,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)} \\ &= f(\gamma)\prod_{i=0}^{k-1}\frac{\Gamma(2-(p+k+i+1)\frac{\gamma^2}{4})\Gamma(2-(p+i)\frac{\gamma^2}{4})\Gamma(2-(p+1+i)\frac{\gamma^2}{4})\Gamma(3-(p+k+i)\frac{\gamma^2}{4})}{\Gamma(4-(2p+k+2i)\frac{\gamma^2}{4})\Gamma(4-(2p+k+2i+1)\frac{\gamma^2}{4})}, \end{aligned}$$

and the product of the above two equations gives:

$$\begin{aligned} \frac{M(\gamma,p-\frac{4}{\gamma^2}+k,\frac{k\gamma^2}{4}-2,0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)} &= f(\gamma)\frac{\Gamma(4-(2p+k-1)\frac{\gamma^2}{4})}{\Gamma(3-(p-1)\frac{\gamma^2}{4})\Gamma(3-p\frac{\gamma^2}{4})\prod_{i=0}^{k-2}(2-(p+1+i)\frac{\gamma^2}{4})} \\ &\quad \times \prod_{j=0}^{1}\frac{\Gamma(j\frac{4}{\gamma^2}+1-p)\Gamma((1+j)\frac{4}{\gamma^2}+2-p)}{\Gamma((2+j)\frac{4}{\gamma^2}-k+2-2p)}. \end{aligned}$$

Combining this relation with the previous shift equations (2.2.27):

$$\begin{aligned}
&\frac{M(\gamma,p,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)}\\
&= f(\gamma)\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})\Gamma(4-(2p+k-1)\frac{\gamma^2}{4})}{\Gamma(-p)\Gamma(\frac{k\gamma^2}{4}-1)\Gamma(2-(2p+k-1)\frac{\gamma^2}{4})\Gamma(3-(p-1)\frac{\gamma^2}{4})}\\
&\times\frac{\Gamma(1-p)\Gamma(\frac{4}{\gamma^2}+1-p)\Gamma(\frac{4}{\gamma^2}+2-p)\Gamma(\frac{8}{\gamma^2}+2-p)}{\Gamma(3-p\frac{\gamma^2}{4})\prod_{i=0}^{k-2}(2-(p+1+i)\frac{\gamma^2}{4})\Gamma(\frac{8}{\gamma^2}-k+2-2p)\Gamma(\frac{12}{\gamma^2}-k+2-2p)}\\
&= f(\gamma)\frac{\Gamma(-p\frac{\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})\Gamma(4-(2p+k-1)\frac{\gamma^2}{4})\Gamma(1-p)}{\Gamma(3-p\frac{\gamma^2}{4})\Gamma(3-(p-1)\frac{\gamma^2}{4})\Gamma(2-(2p+k-1)\frac{\gamma^2}{4})\Gamma(-p)}\\
&\times\frac{\Gamma(\frac{8}{\gamma^2}+2-p)\Gamma(\frac{4}{\gamma^2}+2-p)\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{4}{\gamma^2}+1-p)}{\prod_{i=0}^{k-2}(\frac{8}{\gamma^2}-(p+1+i))\Gamma(\frac{12}{\gamma^2}-k+2-2p)\Gamma(\frac{8}{\gamma^2}-k+2-2p)}\\
&= f(\gamma)\Gamma(\frac{4}{\gamma^2}-p)\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{4}{\gamma^2}+1-p)\Gamma(\frac{8}{\gamma^2}-k+1-p)}{\Gamma(\frac{12}{\gamma^2}-k+1-2p)\Gamma(\frac{8}{\gamma^2}-k+1-2p)}.
\end{aligned}$$

By (2.2.1), the same ratio of M can also be written as,

$$\frac{M(\gamma,p,-\frac{(k+1)\gamma^2}{4},0)}{M(\gamma,p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)}=\frac{C(p)}{C(p-\frac{4}{\gamma^2})}f(\gamma)(\frac{\gamma}{2})^p\frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{4}{\gamma^2}+1-p)\Gamma(\frac{8}{\gamma^2}-k+1-p)}{\Gamma(\frac{12}{\gamma^2}-k+1-2p)\Gamma(\frac{8}{\gamma^2}-k+1-2p)},$$

thus we obtain for $\frac{4}{k+1}<\gamma^2<\frac{4}{k}$:

$$\frac{C(p)}{C(p-\frac{4}{\gamma^2})}=f(\gamma)(\frac{\gamma}{2})^{-p}\Gamma(\frac{4}{\gamma^2}-p). \tag{2.2.28}$$

This proves the second shift equation (2.1.16) on $C(p)$. Then for every fixed γ such that $\frac{4}{\gamma^2}\notin\mathbb{Q}$ both shift equations (2.2.23) and (2.2.28) completely determine the value $C(p)$ up to a constant c_γ of γ. To see this, take another continuous function $\mathfrak{C}(p)$ that satisfies both shift equations (2.2.23) and (2.2.28). Then the ratio $\mathfrak{R}(p):=\frac{C(p)}{\mathfrak{C}(p)}$ is a 1-periodic and $\frac{4}{\gamma^2}$-periodic continuous function. Combining this with the fact that $\frac{4}{\gamma^2}\notin\mathbb{Q}$ implies that the ratio $\mathfrak{R}(p)$ is constant and $C(p)$ is determined up to a constant c_γ of γ by the two shift equations on p.

The constant c_γ is then evaluated by choosing $p=0$ and by using the known value $M(\gamma,0,a,b)=1$. Thus we arrive at the formula of Proposition 2.2.2. Finally by the continuity of $\gamma\to M(\gamma,p,a,b)$, we can extend the formula to the values of γ that were left out. This completes the proof of Proposition 2.2.2.

2.3 Proof of the differential equations

We now move to the proof of Proposition 2.1.4. In order to show that $U(t)$ and $\tilde{U}(t)$ satisfy these differential equations we will need to introduce a regularization procedure. We will work with two small parameters $\delta>0$ and $\epsilon>0$ which will be sent to 0 at the appropriate places in the proof. The first parameter δ controls the cut-off procedure used to smooth X. A convenient smoothing procedure can be written by seeing X as the restriction of the centered Gaussian field defined on the disk $\mathbb{D}+(\frac{1}{2},0)$, i.e. the unit disk centered in $(\frac{1}{2},0)$. X still has a covariance given by:

$$\mathbb{E}[X(x)X(y)]=2\ln\frac{1}{|x-y|}. \tag{2.3.1}$$

Then for any smooth function $\theta \in \mathcal{C}^\infty([0,\infty),\mathbb{R}_+)$ with support in $[0,1]$ and satisfying $\int_0^\infty \theta = \frac{1}{\pi}$, we write $\theta_\delta := \frac{1}{\delta^2}\theta(\frac{|\cdot|^2}{\delta^2})$ and define the regularized field $X_\delta := X * \theta_\delta$. Similarly we introduce:

$$\frac{1}{(x)_\delta} := \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{1}{x+y_1+y_2}\theta_\delta(y_1)\theta_\delta(y_2)d^2y_1 d^2y_2. \tag{2.3.2}$$

This quantity will appear when we take the derivative of $\mathbb{E}[X_\delta(x)X_\delta(y)]$. Now since we have the singularities x^a and $(1-x)^b$ that appear in $U(t)$ and $\tilde{U}(t)$, we will also need to restrict the integration from $[0,1]$ to the smaller interval $[\epsilon, 1-\epsilon]$ for some small ϵ that will be sent to 0. Finally we introduce some more compact notations for various expressions that depend on both δ and ϵ:

$$\begin{aligned}
G_\delta(x,y) &:= \mathbb{E}[X_\delta(x)X_\delta(y)]\\
D(x;t) &:= (x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b\\
U_{\epsilon,\delta}(t) &:= \mathbb{E}[(\int_\epsilon^{1-\epsilon} D(x;t)e^{\frac{\gamma}{2}X_\delta(x)}dx)^p]\\
V^{(1)}_{\epsilon,\delta}(x_1;t) &:= \mathbb{E}[(\int_\epsilon^{1-\epsilon} D(x;t)e^{\frac{\gamma}{2}X_\delta(x)+\frac{\gamma^2}{4}G_\delta(x,x_1)}dx)^{p-1}]\\
V^{(2)}_{\epsilon,\delta}(x_1,x_2;t) &:= \mathbb{E}[(\int_\epsilon^{1-\epsilon} D(x;t)e^{\frac{\gamma}{2}X_\delta(x)+\frac{\gamma^2}{4}(G_\delta(x,x_1)+G_\delta(x,x_2))}dx)^{p-2}]\\
E_{0,\epsilon,\delta}(t) &:= D(\epsilon;t)V^{(1)}_{\epsilon,\delta}(\epsilon;t)\\
E_{1,\epsilon,\delta}(t) &:= D(1-\epsilon;t)V^{(1)}_{\epsilon,\delta}(1-\epsilon;t).
\end{aligned}$$

The terms $V^{(1)}_{\epsilon,\delta}$ and $V^{(2)}_{\epsilon,\delta}$ will appear when we compute respectively the first and second order derivatives of $U_{\epsilon,\delta}$. The terms $E_{0,\epsilon,\delta}$ and $E_{1,\epsilon,\delta}$ are the boundary terms of the integration by parts performed below. We will also use $U_\epsilon(t)$, $V^{(1)}_\epsilon(x_1;t)$, $V^{(2)}_\epsilon(x_1,x_2;t)$, $E_{0,\epsilon}(t)$, $E_{1,\epsilon}(t)$ for the limit of the above quantities as δ goes to 0.

Proof. First we prove the equation for $U(t)$. We recall the definition,

$$U(t) = \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p], \tag{2.3.3}$$

and we calculate the derivatives with the help of the Girsanov theorem of appendix 2.4.1:

$$\begin{aligned}
U'_{\epsilon,\delta}(t) =& p\int_\epsilon^{1-\epsilon} dx_1\, \partial_t D(x_1;t)V^{(1)}_{\epsilon,\delta}(x_1;t)\\
=& -p\int_\epsilon^{1-\epsilon} dx_1 \partial_{x_1}((x_1-t)^{\frac{\gamma^2}{4}})x_1^a(1-x_1)^b V^{(1)}_{\epsilon,\delta}(x_1;t)\\
=& -p\Big(E_{1,\epsilon,\delta}(t) - E_{0,\epsilon,\delta}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_{\epsilon,\delta}(x_1;t)(\frac{a}{x_1}-\frac{b}{1-x_1})\\
& - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)\,\partial_{x_1}V^{(1)}_{\epsilon,\delta}(x_1;t)\Big).
\end{aligned}$$

We claim that the last term in the sum equals zero. Indeed,

$$\begin{aligned}
\int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)\,\partial_{x_1}V^{(1)}_{\epsilon,\delta}(x_1;t) =& (p-1)\frac{\gamma^2}{2}\int_\epsilon^{1-\epsilon}\int_\epsilon^{1-\epsilon} dx_1dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_2-x_1)_\delta}e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)}V^{(2)}_{\epsilon,\delta}(x_1,x_2;t)\\
=& 0 \qquad \text{by symmetry.}
\end{aligned}$$

Thus, by sending δ to 0,

$$U'_\epsilon(t) = -p\Big(E_{1,\epsilon}(t) - E_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_\epsilon(x_1;t)(\frac{a}{x_1}-\frac{b}{1-x_1})\Big). \tag{2.3.4}$$

In the same spirit, we calculate:

$$U''_{\epsilon,\delta}(t) = \frac{p\gamma^2}{4}\Big[-\int_\epsilon^{1-\epsilon} dx_1 \partial_t\big(\frac{D(x_1;t)}{(x_1-t)}\big)V^{(1)}_{\epsilon,\delta}(x_1;t) \\ + \frac{(p-1)\gamma^2}{4}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-t)} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V^{(2)}_{\epsilon,\delta}(x_1,x_2;t)\Big].$$

An integration by parts gives:

$$\begin{aligned} &-\int_\epsilon^{1-\epsilon} dx_1 \partial_t\big(\frac{D(x_1;t)}{(x_1-t)}\big)V^{(1)}_{\epsilon,\delta}(x_1;t) \\ =&\int_\epsilon^{1-\epsilon} dx_1 \partial_{x_1}\big(\frac{(x_1-t)^{\frac{\gamma^2}{4}}}{x_1-t}\big)x_1^a(1-x_1)^b V^{(1)}_{\epsilon,\delta}(x_1;t) \\ =&\frac{1}{1-t-\epsilon}E_{1,\epsilon,\delta}(t) + \frac{1}{t-\epsilon}E_{0,\epsilon,\delta}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_{\epsilon,\delta}(x_1;t)\frac{1}{x_1-t}\big(\frac{a}{x_1}-\frac{b}{1-x_1}\big) \\ &-\frac{(p-1)\gamma^2}{2}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V^{(2)}_{\epsilon,\delta}(x_1,x_2;t). \end{aligned}$$

By symmetry of the expression under the exchange of x_1 and x_2,

$$\begin{aligned} &\frac{(p-1)\gamma^2}{2}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V^{(2)}_{\epsilon,\delta}(x_1,x_2;t) \\ =&\frac{(p-1)\gamma^2}{4}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 D(x_1;t)D(x_2;t) e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} \\ &\times\big(\frac{1}{(x_1-t)(x_2-x_1)_\delta}+\frac{1}{(x_2-t)(x_1-x_2)_\delta}\big)V^{(2)}_{\epsilon,\delta}(x_1,x_2;t) \\ =&\frac{(p-1)\gamma^2}{4}\int_\epsilon^{1-\epsilon} dx_1 \int_\epsilon^{1-\epsilon} dx_2 \frac{D(x_1;t)D(x_2;t)}{(x_1-t)(x_2-t)}\frac{x_2-x_1}{(x_2-x_1)_\delta} e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)} V^{(2)}_{\epsilon,\delta}(x_1,x_2;t). \end{aligned}$$

Since $\frac{x_2-x_1}{(x_2-x_1)_\delta} \le c$ for some constant $c>0$ independent of δ, by sending δ to 0,

$$U''_\epsilon(t) = \frac{p\gamma^2}{4}\Big(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t) + \frac{1}{t-\epsilon}E_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_\epsilon(x_1;t)\frac{1}{x_1-t}\big(\frac{a}{x_1}-\frac{b}{1-x_1}\big)\Big). \quad (2.3.5)$$

A further calculation shows that,

$$\begin{aligned} &\int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_\epsilon(x_1;t)\frac{1}{x_1-t}\big(\frac{a}{x_1}-\frac{b}{1-x_1}\big) \\ &= \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_\epsilon(x_1;t)\big(\frac{a}{t}\big(\frac{1}{x_1-t}-\frac{1}{x_1}\big) - \frac{b}{1-t}\big(\frac{1}{x_1-t}+\frac{1}{1-x_1}\big)\big) \\ &= -\int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_\epsilon(x_1;t)\big(\frac{a}{tx_1}+\frac{b}{(1-t)(1-x_1)}\big) - \frac{4}{p\gamma^2}\big(\frac{a}{t}-\frac{b}{1-t}\big)U'_\epsilon(t), \end{aligned}$$

and as a consequence,

$$\begin{aligned} U''_\epsilon(t) =&\frac{p\gamma^2}{4}\Big(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t-\epsilon) + \frac{1}{t}E_{0,\epsilon}(t) + \int_\epsilon^{1-\epsilon} dx_1 D(x_1;t)V^{(1)}_\epsilon(x_1;t)\big(\frac{a}{tx_1}+\frac{b}{(1-t)(1-x_1)}\big)\Big) \\ &+\big(\frac{a}{t}-\frac{b}{1-t}\big)U'_\epsilon(t). \end{aligned} \quad (2.3.6)$$

We can also write $U_{\epsilon,\delta}(t)$ in a similar form, by doing an integration by parts:

$$\begin{aligned}
&(1-t-\epsilon)E_{1,\epsilon,\delta}(t)+(t-\epsilon)E_{0,\epsilon,\delta}(t)-\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon,\delta}(x_1;t)(x_1-t)(\frac{a}{x_1}-\frac{b}{1-x_1})\\
=&(1+\frac{\gamma^2}{4})\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon,\delta}(x_1;t)\\
&+(p-1)\frac{\gamma^2}{2}\int_\epsilon^{1-\epsilon}\int_\epsilon^{1-\epsilon}dx_1dx_2D(x_1;t)D(x_2;t)e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)}\frac{x_1-t}{(x_2-x_1)_\delta}V^{(2)}_{\epsilon,\delta}(x_1,x_2;t)\\
=&(1+\frac{\gamma^2}{4})\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon,\delta}(x_1;t)\\
&-(p-1)\frac{\gamma^2}{4}\int_\epsilon^{1-\epsilon}dx_1\int_\epsilon^{1-\epsilon}dx_2D(x_1;t)D(x_2;t)e^{\frac{\gamma^2}{4}G_\delta(x_2,x_1)}\frac{x_2-x_1}{(x_2-x_1)_\delta}V^{(2)}_{\epsilon,\delta}(x_1,x_2;t).
\end{aligned}$$

By sending δ to 0 and by applying the Girsanov theorem of appendix 2.4.1, we obtain,

$$\begin{aligned}
-(B+a+b)U_\epsilon(t)=&(1-t-\epsilon)E_{1,\epsilon}(t)+(t-\epsilon)E_{0,\epsilon}(t)\\
&-\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon}(x_1;t)(x_1-t)(\frac{a}{x_1}-\frac{b}{1-x_1}),
\end{aligned}$$

where we recall that $B=-(a+b+1)-(2-p)\frac{\gamma^2}{4}$. We also note that,

$$\begin{aligned}
&\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon}(x_1;t)(x_1-t)(\frac{a}{x_1}-\frac{b}{1-x_1})\\
&\quad=(a+b)U_{\epsilon,\delta}(t)-\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon}(x_1;t)(\frac{at}{x_1}+\frac{b(1-t)}{1-x_1}),
\end{aligned}$$

and hence,

$$-BU_\epsilon(t)=(1-t-\epsilon)E_{1,\epsilon}(t)+(t-\epsilon)E_{0,\epsilon}(t)+\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon}(x_1;t)(\frac{at}{x_1}+\frac{b(1-t)}{1-x_1}). \tag{2.3.7}$$

Combining this with the expressions for U'_ϵ and U''_ϵ, equations (2.3.4) and (2.3.6),

$$\begin{aligned}
U'_\epsilon(t)=&-p\Big(E_{1,\epsilon}(t)-E_{0,\epsilon}(t)-\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon}(x_1;t)(\frac{a}{x_1}-\frac{b}{1-x_1})\Big),\\
U''_\epsilon(t)=&\frac{p\gamma^2}{4}\Big(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t)+\frac{1}{t-\epsilon}E_{0,\epsilon}(t)+\int_\epsilon^{1-\epsilon}dx_1D(x_1;t)V^{(1)}_{\epsilon}(x_1;t)(\frac{a}{tx_1}+\frac{b}{(1-t)(1-x_1)})\Big)\\
&+(\frac{a}{t}-\frac{b}{1-t})U'_\epsilon(t),
\end{aligned}$$

we finally arrive at:

$$\begin{aligned}
&t(1-t)U''_\epsilon(t)+(C-(A+B+1)t)U'_\epsilon(t)-ABU_\epsilon(t)\\
&\qquad=\epsilon(1-\epsilon)\frac{p\gamma^2}{4}(\frac{1}{1-t-\epsilon}E_{1,\epsilon}(t)+\frac{1}{t-\epsilon}E_{0,\epsilon}(t)).
\end{aligned} \tag{2.3.8}$$

From this expression we see that the last thing we need to check is that as ϵ goes to zero the right hand side of the above expression converges to 0 in a suitable sense. Indeed we will prove that, for t in a fixed compact set $K\subseteq(-\infty,0)$, $\epsilon E_{1,\epsilon}(t)$ and $\epsilon E_{0,\epsilon}(t)$ converge uniformly to 0 for a well chosen sequence of ϵ. Let us consider $\epsilon E_{0,\epsilon}(t)$ as $\epsilon E_{1,\epsilon}(t)$ can be treated in a similar fashion:

$$\epsilon E_{0,\epsilon}(t)=(\epsilon-t)^{\frac{\gamma^2}{4}}\epsilon^{a+1}(1-\epsilon)^b\mathbb{E}[(\int_\epsilon^{1-\epsilon}\frac{(x-t)^{\frac{\gamma^2}{4}}x^a(1-x)^b}{|x-\epsilon|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}].$$

In the following we will discuss three disjoint cases based on the value of a. They are $a > -1 + \frac{\gamma^2}{4}$, $-1 < a \leq -1 + \frac{\gamma^2}{4}$, and $-1 - \frac{\gamma^2}{4} < a \leq -1$.

i) $a > -1 + \frac{\gamma^2}{4}$

This is the simplest case as we have for ϵ sufficiently small and for some $c_0 > 0$,

$$\epsilon E_{0,\epsilon}(t) \leq c_0 \epsilon^{a+1}(1-\epsilon)^b \mathbb{E}[(\int_0^1 \frac{x^a(1-x)^b}{|x-\epsilon|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]$$
$$\overset{\epsilon\to 0}{\sim} c_0 \epsilon^{a+1} M(\gamma, p, a - \frac{\gamma^2}{2}, b),$$

which converges to 0 as $\epsilon \to 0$ uniformly over $t \in K$.

ii) $-1 < a \leq -1 + \frac{\gamma^2}{4}$.

In this case we have $p - 1 < 1$ and $\epsilon^{a+1} \underset{\epsilon\to 0}{\longrightarrow} 0$. If $p - 1 \leq 0$,

$$\mathbb{E}[(\int_\epsilon^{1-\epsilon} \frac{(x-t)^{\frac{\gamma^2}{4}} x^a (1-x)^b}{|x-\epsilon|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}]$$

is uniformly bounded thus it is immediate to obtain the convergence to 0. Hence it suffices to consider the case $0 < p - 1 < 1$. We choose $\epsilon_N = \frac{1}{2^N}$. Using the sub-additivity of the function $x \mapsto x^{p-1}$, we have for some $c_0, c' > 0$ independent of K:

$$\begin{aligned}\epsilon_N E_{0,\epsilon_N}(t) \leq & c_0 \epsilon_N^{a+1} \mathbb{E}[(\int_{\epsilon_N}^{\frac{1}{2}} \frac{x^a}{|x-\epsilon_N|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + c'\epsilon_N^{a+1} \\ \leq & c_0 \epsilon_N^{a+1} \sum_{n=1}^{N-1} \mathbb{E}[(\int_{\epsilon_{n+1}}^{\epsilon_n} \frac{x^a}{|x-\epsilon_{n+1}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + c'\epsilon_N^{a+1}.\end{aligned}$$

Then by the scaling property of GMC,

$$\begin{aligned}&\mathbb{E}[(\int_{\epsilon_{n+1}}^{\epsilon_n} \frac{x^a}{|x-\epsilon_{n+1}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\ =& 2^{\frac{\gamma^2}{4}(p-1)(p-2) - (a-\frac{\gamma^2}{2}+1)(p-1)} \mathbb{E}[(\int_{\epsilon_n}^{\epsilon_{n-1}} \frac{u^a}{|u-\epsilon_n|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(u)}du)^{p-1}] \\ =& 2^{\frac{\gamma^2}{4}p^2 - (\frac{\gamma^2}{4}+a+1)p+a+1} \mathbb{E}[(\int_{\epsilon_n}^{\epsilon_{n-1}} \frac{u^a}{|u-\epsilon_n|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(u)}du)^{p-1}].\end{aligned}$$

We can deduce that,

$$\begin{aligned}\epsilon_N E_{0,\epsilon_N}(t) &\leq c_1 2^{-N(a+1)} 2^{(N-1)(\frac{\gamma^2}{4}p^2 - (\frac{\gamma^2}{4}+a+1)p+a+1)} \mathbb{E}[(\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x^a}{|x-\frac{1}{4}|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + c'\epsilon_N^{a+1} \\ &\leq c 2^{N(\frac{\gamma^2}{4}p - \frac{\gamma^2}{4} - a - 1)p} + c'\epsilon_N^{a+1} \overset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

for some constants $c_1, c, c' > 0$. The convergence holds since $p > 0$ and $\frac{\gamma^2}{4}p - \frac{\gamma^2}{4} - a - 1 < 0$ (this inequality comes from (2.1.5)), and it holds uniformly over t in K.

iii) $-1 - \frac{\gamma^2}{4} < a \leq -1$

In this case $p - 1 < 0$ so we are always dealing with negative moments. This implies that for t in K, we can bound $\epsilon E_{0,\epsilon}(t)$ by,

$$\epsilon E_{0,\epsilon}(t) \leq c_0 \epsilon^{a+1} \mathbb{E}[(\int_\epsilon^{\frac{1}{2}} x^{a-\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}],$$

simply by restricting the integral over $[\epsilon, 1-\epsilon]$ to $[\epsilon, 1/2]$. An estimation of the resulting GMC moment is given by lemma 2.4.4 in appendix 2.4.2. For ϵ sufficiently small, there exists a constant $c > 0$ such that,

$$\mathbb{E}[(\int_\epsilon^{\frac{1}{2}} x^{a-\frac{\gamma^2}{2}} e^{\frac{\gamma}{2}X(x)} dx)^{p-1}] \leq \begin{cases} c\,\epsilon^{(\frac{\gamma}{4}-\frac{1}{\gamma}(a+1))^2}, & 1+a+\frac{\gamma^2}{4}-\frac{p\gamma^2}{2} > 0 \\ c\,\epsilon^{(p-1)(1+a-\frac{\gamma^2}{4})-\frac{(p-1)^2\gamma^2}{4}}, & 1+a+\frac{\gamma^2}{4}-\frac{p\gamma^2}{2} \leq 0 \end{cases}$$

This suffices to show the convergence to 0 of $\epsilon E_{0,\epsilon}(t)$.

Indeed, in the first case, a basic inequality shows that $(\frac{\gamma}{4} - \frac{1}{\gamma}(a+1))^2 \geq -(a+1)$ with equality when $-(a+1) = \frac{\gamma^2}{4}$. Since the condition cannot be satisfied, we have the strict inequality. In the second case where $1+a+\frac{\gamma^2}{4}-\frac{p\gamma^2}{2} \leq 0$, we can easily show that under this condition together with the bound (2.1.5) for p, $(p-1)(1+a-\frac{\gamma^2}{4}) - \frac{(p-1)^2\gamma^2}{4} > -(a+1)$. Hence in both cases, $\epsilon E_{0,\epsilon}(t) \longrightarrow 0$, where the convergence is again uniform over t in K.

Combining the cases (i), (ii) and (iii), we have proven the differential equation 2.1.20 in the weak sense (in the sense of distributions). Since it is a hypoelliptic equation (the dominant operator is a Laplacian) with analytic coefficients, $U(t)$ is analytic and the equation holds in the strong sense.
Let us now briefly mention the case of $\tilde{U}(t)$. In a similar manner, we calculate,

$$\begin{aligned}
-\tilde{B}\tilde{U}(t) =& \frac{4}{\gamma^2}\Big((1-t-\epsilon)\tilde{E}_{1,\epsilon}(t) + (t-\epsilon)\tilde{E}_{0,\epsilon}(t) + \int_\epsilon^{1-\epsilon} dx_1 \tilde{D}(x_1;t)\tilde{V}_\epsilon^{(1)}(x_1;t)(\frac{at}{x_1} + \frac{b(1-t)}{1-x_1})\Big), \\
\tilde{U}'_\epsilon(t) =& -p\Big(\tilde{E}_{1,\epsilon}(t) - \tilde{E}_{0,\epsilon}(t) - \int_\epsilon^{1-\epsilon} dx_1 \tilde{D}(x_1;t)\tilde{V}_\epsilon^{(1)}(x_1;t)(\frac{a}{x_1} - \frac{b}{1-x_1})\Big), \\
\tilde{U}''_\epsilon(t) =& \frac{4p}{\gamma^2}\Big(\frac{1}{1-t-\epsilon}\tilde{E}_{1,\epsilon}(t) + \frac{1}{t-\epsilon}\tilde{E}_{0,\epsilon}(t) + \int_\epsilon^{1-\epsilon} dx_1 \tilde{D}(x_1;t)\tilde{V}_\epsilon^{(1)}(x_1;t)(\frac{a}{tx_1} + \frac{b}{(1-t)(1-x_1)})\Big) \\
&+ \frac{4}{\gamma^2}(\frac{a}{t} - \frac{b}{1-t})\tilde{U}'_\epsilon(t),
\end{aligned}$$

where $\tilde{D}(x;t) := (x-t)\,x^a(1-x)^b$ and where $\tilde{V}_\epsilon^{(1)}(x_1;t)$, $\tilde{E}_{0,\epsilon}(t)$, $\tilde{E}_{1,\epsilon}(t)$ are defined as functions of $\tilde{D}(x;t)$, the same as their definitions without the tilde. We verify easily that,

$$\begin{aligned}
t(1-t)\tilde{U}''_\epsilon(t) + (\tilde{C} - (\tilde{A}+\tilde{B}+1)t)\tilde{U}'_\epsilon(t) - \tilde{A}\tilde{B}\tilde{U}_\epsilon(t) \\
= \epsilon(1-\epsilon)\frac{p\gamma^2}{4}(\frac{1}{1-t-\epsilon}\tilde{E}_{1,\epsilon}(t) + \frac{1}{t-\epsilon}\tilde{E}_{0,\epsilon}(t)),
\end{aligned} \tag{2.3.9}$$

and the right hand side of the above expression converges again to zero uniformly for t in any compact set of $(-\infty, 0)$, which finishes the proof of the Proposition 2.1.4. □

One may wonder if other differential equations can be obtained for similar observables. If instead of $U(t)$ and $\tilde{U}(t)$ one introduces the more general function

$$t \to \mathbb{E}[(\int_0^1 (x-t)^\chi x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx)^p] \tag{2.3.10}$$

for some arbitrary real number χ, then this function will be solution to a second order differential equation if and only if $\chi = \frac{\gamma^2}{4}$ or $\chi = 1$ (except for some special cases where "non-trivial" relations hold for instance for $p = 0$). This fact can be obtained by similar computations as the ones performed above. On the other hand conformal field theory predicts that differential equations of any order are expected to be verified by suitable observables although it is not clear to us at this stage what information can be extracted from these higher order differential equations.

2.4 Appendix

2.4.1 Reminder on some useful theorems

We recall some theorems in probability that we will use without further justification. In the following, D is a compact subset of $\mathbb{R}^d$.

Theorem 2.4.1 (Girsanov theorem)**.** *Let $(Z(x))_{x\in D}$ be a continuous centered Gaussian process and Z a Gaussian variable which belongs to the L^2 closure of the vector space spanned by $(Z(x))_{x\in D}$. Let F be a real continuous bounded function from $\mathcal{C}(D,\mathbb{R})$ to $\mathbb{R}$. Then we have the following identity:*

$$\mathbb{E}[e^{Z-\frac{\mathbb{E}[Z^2]}{2}}F((Z(x))_{x\in D})] = \mathbb{E}[F((Z(x)+\mathbb{E}[Z(x)Z])_{x\in D})]. \tag{2.4.1}$$

When applied to our case, although the log-correlated field X is not a continuous Gaussian process, we can still make the arguments rigorous by using a regularization procedure. Let us illustrate the idea by a simple example that is used in section 2.3. We introduce three cut-off parameters, δ to smooth the log-correlated field X, ϵ to avoid the singularities in 0 and 1, and $A>0$ to apply (2.4.1) to a bounded functional F. Hence the following computation:

$$\begin{aligned}
\mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p] &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\lim_{A\to+\infty}\int_\epsilon^{1-\epsilon} dx_1\, x_1^a(1-x_1)^b\mathbb{E}\Big[\mathbf{1}_{\|X_\delta\|_\infty\le A}e^{\frac{\gamma}{2}X_\delta(x_1)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x_1)^2]} \\
&\qquad (\int_\epsilon^{1-\epsilon} x^a(1-x)^b e^{\frac{\gamma}{2}X_\delta(x)-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx)^{p-1}\Big] \\
&\overset{(2.4.1)}{=} \lim_{\epsilon\to 0}\lim_{\delta\to 0}\lim_{A\to+\infty}\int_\epsilon^{1-\epsilon} dx_1\, x_1^a(1-x_1)^b\mathbb{E}\Big[\mathbf{1}_{\|X_\delta\|_\infty\le A} \\
&\qquad (\int_\epsilon^{1-\epsilon} x^a(1-x)^b e^{\frac{\gamma}{2}X_\delta(x)+\frac{\gamma^2}{4}\mathbb{E}[X_\delta(x)X_\delta(x_1)]-\frac{\gamma^2}{8}\mathbb{E}[X_\delta(x)^2]}dx)^{p-1}\Big] \\
&= \int_0^1 dx_1\, x_1^a(1-x_1)^b\mathbb{E}[(\int_0^1 \frac{x^a(1-x)^b}{|x_1-x|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx)^{p-1}].
\end{aligned}$$

The next theorem is a comparison result due to Kahane [56]:

Theorem 2.4.2 (Convexity inequality)**.** *Let $(Z_1(x))_{x\in D}$, $(Z_2(x))_{x\in D}$ be two continuous centered Gaussian processes such that for all $x,y\in D$:*

$$\mathbb{E}[Z_1(x)Z_1(y)] \le \mathbb{E}[Z_2(x)Z_2(y)].$$

Then for all convex function (resp. concave) F with at most polynomial growth at infinity, and σ a positive finite measure over D,

$$\mathbb{E}[F(\int_D e^{Z_1(x)-\frac{1}{2}\mathbb{E}[Z_1(x)^2]}\sigma(dx))] \le (\ \textit{resp.}\ \ge\ )\,\mathbb{E}[F(\int_D e^{Z_2(x)-\frac{1}{2}\mathbb{E}[Z_2(x)^2]}\sigma(dx))]. \tag{2.4.2}$$

To apply this theorem to log-correlated fields, one needs again to use a regularization procedure. Finally, we provide the Williams decomposition theorem, see for instance [106]:

Theorem 2.4.3. *Let $(B_s-vs)_{s\ge 0}$ be a Brownian motion with negative drift, i.e. $v>0$ and let $M=\sup_{s\ge 0}(B_s-vs)$. Then conditionally on M the law of the path $(B_s-vs)_{s\ge 0}$ is given by the joining of two independent paths:*
1) A Brownian motion $(B_s^1+vs)_{0\le s\le\tau_M}$ with positive drift v run until its hitting time τ_M of M.
2) $(M+B_t^2-vt)_{t\ge 0}$ where $(B_t^2-vt)_{t\ge 0}$ is a Brownian motion with negative drift conditioned to stay negative.

Moreover, one has the following time reversal property for all $C > 0$ (where τ_C denotes the hitting time of C),

$$(B^1_{\tau_C - s} + v(\tau_C - s) - C)_{0\leq s\leq \tau_C} \overset{law}{=} (\tilde{B}_s - vs)_{0\leq s\leq L_{-C}}, \tag{2.4.3}$$

where $(\tilde{B}_s - vs)_{s\geq 0}$ is a Brownian motion with drift $-v$ conditioned to stay negative and L_{-C} is the last time $(\tilde{B}_s - vs)_{s\geq 0}$ hits $-C$.

2.4.2 An estimate on GMC

We now move on to the proof of some technical lemmas required in the previous sections. Lemma 2.4.4 written below will be used in section 2.3 to show that the boundary terms obtained in the derivation of the differential equations converge to 0. Just like in section 2.1.2 for $s \geq 0$ we write $X(e^{-s/2}) = B_s + Y(e^{-s/2})$ where B_s is a standard Brownian motion and Y is an independent centered Gaussian field on $\mathbb{C}$ with covariance:

$$\mathbb{E}[Y(x)Y(y)] = 2\ln\frac{|x|\vee|y|}{|x-y|}. \tag{2.4.4}$$

Denote the GMC measure associated to $Y(e^{-s/2})$ by $\mu_Y(ds) := e^{\frac{\gamma}{2}Y(e^{-s/2})}ds$. The goal of this subsection is to prove the following lemma:

Lemma 2.4.4. *For $q > 0$, $a < -1 - \frac{\gamma^2}{4}$, and a fixed constant $A > 0$, there exists $\epsilon_1 < A$ sufficiently small such that for all $\epsilon \leq \epsilon_1$,*

$$\mathbb{E}[(\int_\epsilon^A x^a e^{\frac{\gamma}{2}X(x)}dx)^{-q}] \leq \begin{cases} c\,\epsilon^{(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1))^2}, & 1+a+\frac{\gamma^2}{4}+\frac{q\gamma^2}{2} > 0, \\ c\,\epsilon^{-q(1+a+\frac{\gamma^2}{4})-\frac{q^2\gamma^2}{4}}, & 1+a+\frac{\gamma^2}{4}+\frac{q\gamma^2}{2} \leq 0 \end{cases} \tag{2.4.5}$$

where $c > 0$ is a constant that depends on A, γ, a and q.

By using the decomposition described above, we can transform this lemma into another equivalent form,

$$\begin{aligned}\mathbb{E}[(\int_\epsilon^A x^a e^{\frac{\gamma}{2}X(x)}dx)^{-q}] =& 2^q\mathbb{E}[(\int_{-2\ln A}^{-2\ln\epsilon} e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds))^{-q}] \\ =& 2^q\mathbb{E}[(\int_{-2\ln A}^{-2\ln\epsilon} e^{\frac{\gamma}{2}(B_s+\alpha s)}\mu_Y(ds))^{-q}],\end{aligned}$$

where again $(B_s)_{s\geq 0}$ is a standard Brownian motion independent from Y, and $\alpha = -\frac{\gamma}{4} - \frac{1}{\gamma}(a+1)$. Therefore lemma 2.4.4 is equivalent to the following lemma:

Lemma 2.4.5. *For $q > 0$, $\alpha > 0$, a fixed constant r_0, there exists $r_1 > r_0$ sufficiently large such that for all $r \geq r_1$,*

$$\mathbb{E}[(\int_{r_0}^r e^{\frac{\gamma}{2}(B_s+\alpha s)}\mu_Y(ds))^{-q}] \leq \begin{cases} c\,e^{-\frac{\alpha^2}{2}r}, & \alpha - \frac{q\gamma}{2} < 0 \\ c\,e^{(\frac{q^2\gamma^2}{8} - \frac{q\gamma\alpha}{2})r}, & \alpha - \frac{q\gamma}{2} \geq 0 \end{cases} \tag{2.4.6}$$

where $c > 0$ is a constant that depends on r_0, γ, α and q.

A similar result for 2d GMC has been proved in [60] (proposition 5.1). A slight difference is that in [60] the power q depends on a.

We start by proving three intermediate results. We denote $y_s = B_s + \alpha s$, and we introduce for $\beta \geq 1$ the stopping time $T_\beta = \inf\{s \geq 0, y_s = \beta - 1\}$. Recall the density of T_β for $\beta > 1, u > 0$:

$$\mathbb{P}(T_\beta \in (u, u+du)) = \frac{\beta-1}{\sqrt{2\pi}u^{3/2}}e^{-\frac{(\beta-1-\alpha u)^2}{2u}}du. \tag{2.4.7}$$

Lemma 2.4.6. *For $\alpha, A > 0$, we have:*

$$\mathbb{P}(\sup_{s\leq t} y_s \leq A) \leq e^{\alpha A - \frac{\alpha^2 t}{2}}. \tag{2.4.8}$$

Proof. We know the density of $\sup_{s\leq t} y_s$:

$$\mathbb{P}(\sup_{s\leq t} y_s \leq A) = \mathbb{P}(T_{A+1} \geq t) = \frac{A}{\sqrt{2\pi}} \int_t^\infty \frac{e^{-\frac{(A-\alpha s)^2}{2s}}}{s^{3/2}} ds \leq \frac{A e^{\alpha A - \frac{\alpha^2 t}{2}}}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-\frac{A^2}{2s}}}{s^{3/2}} ds = e^{\alpha A - \frac{\alpha^2 t}{2}}.$$

□

Lemma 2.4.7. *We set for $t > 0$:*

$$I(t) = \int_t^{t+1} e^{\frac{\gamma}{2}(y_s - y_t)} \mu_Y(ds). \tag{2.4.9}$$

For $q > 0$, we have the following inequality,

$$\mathbb{E}[I(t)^{-q} | y_{t+1} - y_t] \leq c_1 (e^{-\frac{\gamma}{2} q (y_{t+1} - y_t)} + 1) \quad a.s., \tag{2.4.10}$$

where c_1 depends on γ, q.

Proof. Conditioning on $y_{t+1} - y_t = y$, $(B_s - B_t)_{t\leq s\leq t+1}$ has the law of a Brownian bridge between 0 and $y - \alpha$. Hence it has the law of $(B'_s - sB'_1 + s(y-\alpha))_{0\leq s\leq 1}$, where B' is an independent Brownian motion. We have:

$$\mathbb{E}[I(t)^{-q} | y_{t+1} - y_t = y] = \mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}(B'_s - sB'_1 + sy)} \mu_Y(ds))^{-q}].$$

Notice that $e^{\frac{\gamma}{2} sy} \geq e^{\frac{\gamma}{2} y} \wedge 1$, and a classic result on the moments of Gaussian multiplicative chaos shows that,

$$\mathbb{E}[(\mu_Y([0,1]))^{-q}] < \infty,$$

thus:

$$\begin{aligned}\mathbb{E}[(\int_0^1 e^{\frac{\gamma}{2}(B'_s - sB'_1)} \mu_Y(ds))^{-q}] &\leq \mathbb{E}[e^{-\frac{q\gamma}{2} \inf_{0\leq s\leq 1}(B'_s - sB'_1)}] \mathbb{E}[(\mu_Y([0,1]))^{-q}] \\ &=: c_1 < \infty.\end{aligned}$$

We can now derive that:

$$\mathbb{E}[I(t)^{-q} | y_{t+1} - y_t = y] \leq c_1 (e^{-\frac{\gamma}{2} qy} \vee 1) \leq c_1 (e^{-\frac{\gamma}{2} qy} + 1) \quad a.s.$$

□

Lemma 2.4.8. *Define for $\beta > 1$, $\alpha > 0$, $q > 0$ and $r \geq 2$:*

$$J_{r,\beta} := \mathbb{E}[\frac{\mathbf{1}_{\{\sup_{s\in[0,r]} y_s \in [\beta-1, \beta]\}}}{(\int_0^r e^{\frac{\gamma}{2} y_s} \mu_Y(ds))^q}]. \tag{2.4.11}$$

Then there exists $c_2 > 0$ depending on γ, α, q such that:

$$J_{r,\beta} \leq c_2 e^{-\frac{\alpha^2}{2} r} e^{(\alpha - \frac{q\gamma}{2})\beta}. \tag{2.4.12}$$

Proof.

$$J_{r,\beta} \leq e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}[\mathbf{1}_{\{T_\beta\leq r-1\}}\frac{\mathbf{1}_{\{\sup_{s\in[0,r]}y_s\in[\beta-1,\beta]\}}}{I(T_\beta)^q}]+\mathbb{E}[\mathbf{1}_{\{T_\beta>r-1\}}\frac{\mathbf{1}_{\{\sup_{s\in[0,r]}y_s\in[\beta-1,\beta]\}}}{e^{\frac{q\gamma y_{r-1}}{2}}I(r-1)^q}]=:A+B.$$

We first bound A. By using the strong Markov property of $(y_s)_{s\geq 0}$ with respect to $\mathcal{F}_{T_\beta+1}$:

$$\begin{aligned}A\leq& e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}[\mathbf{1}_{\{T_\beta+1\leq r\}}I(T_\beta)^{-q}\mathbf{1}_{\{\sup_{s\in[T_\beta+1,r]}y_s-y_{T_\beta+1}\leq\beta-y_{T_\beta+1}\}}]\\ =&e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}[\mathbf{1}_{\{T_\beta+1\leq r\}}I(T_\beta)^{-q}\mathbb{E}[\mathbf{1}_{\{\sup_{s\in[0,r-T_\beta-1]}y'_s\leq\beta-y_{T_\beta+1}\}}|\mathcal{F}_{T_\beta+1}]]\\ =&e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}\Big[\mathbf{1}_{\{T_\beta+1\leq r\}}\mathbb{E}[I(T_\beta)^{-q}|\mathcal{F}_{T_\beta},\beta-y_{T_\beta+1}]\mathbb{E}[\mathbf{1}_{\{\sup_{s\in[0,r-T_\beta-1]}y'_s\leq\beta-y_{T_\beta+1}\}}|\mathcal{F}_{T_\beta},\beta-y_{T_\beta+1}]\Big].\end{aligned}$$

By lemma 2.4.7,

$$\mathbb{E}[I(T_\beta)^{-q}|\mathcal{F}_{T_\beta},\beta-y_{T_\beta+1}]\leq c_1(e^{-\frac{\gamma}{2}q(y_{T_\beta+1}-\beta)}+1)\quad a.s.$$

By lemma 2.4.6,

$$\mathbb{E}[\mathbf{1}_{\{\sup_{s\in[0,r-T_\beta-1]}y'_s\leq\beta-y_{T_\beta+1}\}}|\mathcal{F}_{T_\beta},\beta-y_{T_\beta+1}]\leq e^{\alpha(\beta-y_{T_\beta+1})-\frac{\alpha^2(r-T_\beta-1)}{2}}\quad a.s.$$

Therefore:

$$A\leq c_1e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}\Big[\mathbf{1}_{\{T_\beta+1\leq r\}}(e^{-\frac{\gamma}{2}q(y_{T_\beta+1}-\beta)}+1)e^{\alpha(\beta-y_{T_\beta+1})-\frac{\alpha^2(r-T_\beta-1)}{2}}\Big].$$

Conditioning on $\mathcal{F}_{T_\beta}$, $y_{T_\beta+1}-\beta$ has the law of $N+\alpha$ where $N\sim\mathcal{N}(0,1)$. Hence,

$$\begin{aligned}A\leq&c_1e^{-\frac{q\gamma(\beta-1)}{2}}\mathbb{E}[(e^{-\frac{\gamma}{2}q(N+\alpha)}+1)e^{-\alpha(N+\alpha)}]\mathbb{E}[\mathbf{1}_{\{T_\beta+1\leq r\}}e^{-\frac{\alpha^2(r-T_\beta-1)}{2}}]\\ =&c_1e^{-\frac{q\gamma(\beta-1)}{2}}(e^{-\frac{\alpha^2}{2}+\frac{\gamma^2q^2}{8}}+e^{-\frac{\alpha^2}{2}})\mathbb{E}[\mathbf{1}_{\{T_\beta+1\leq r\}}e^{-\frac{\alpha^2(r-T_\beta-1)}{2}}]\\ \leq&c_1e^{-\frac{q\gamma(\beta-1)}{2}}(e^{\frac{\gamma^2q^2}{8}}+1)e^{-\frac{\alpha^2r}{2}}\mathbb{E}[\mathbf{1}_{\{T_\beta\leq r-1\}}e^{\frac{\alpha^2T_\beta}{2}}].\end{aligned}$$

We calculate with the density of T_β:

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{T_\beta\leq r-1\}}e^{\frac{\alpha^2T_\beta}{2}}]&=\int_0^{r-1}\frac{\beta-1}{\sqrt{2\pi}u^{3/2}}e^{-\frac{(\beta-1-\alpha u)^2}{2u}}e^{\frac{\alpha^2u}{2}}du\\ &=e^{\alpha(\beta-1)}\sqrt{\frac{2}{\pi}}\int_{\frac{\beta-1}{\sqrt{r-1}}}^{\infty}e^{-\frac{x^2}{2}}dx\\ &\leq e^{\alpha(\beta-1)}.\end{aligned}$$

Combining the elements above we get,

$$A\leq c_1'e^{-\frac{\alpha^2r}{2}}e^{(\alpha-\frac{q\gamma}{2})\beta},\tag{2.4.13}$$

for some constant $c_1'>0$ of γ,α and q. We proceed similarly for B, using again the Markov property:

$$\begin{aligned}B=&\mathbb{E}\Big[\mathbf{1}_{\{T_\beta>r-1\}}\frac{\mathbf{1}_{\{\sup_{s\in[r-1,r]}(y_s-y_{r-1})\in[\beta-1-y_{r-1},\beta-y_{r-1}]\}}}{e^{\frac{q\gamma y_{r-1}}{2}}I(r-1)^q}\Big]\\ \leq&\mathbb{E}\Big[\mathbf{1}_{\{T_\beta>r-1\}}\frac{1}{e^{\frac{q\gamma}{2}(\beta-1-\sup_{s\in[r-1,r]}(y_s-y_{r-1}))}I(r-1)^q}\Big]\\ =&e^{-\frac{q\gamma}{2}(\beta-1)}\mathbb{P}(T_\beta>r-1)\mathbb{E}\Big[e^{\frac{q\gamma}{2}\sup_{s\in[r-1,r]}(y_s-y_{r-1})}I(r-1)^{-q}\Big]\end{aligned}$$

We show that the expectation term can be easily bounded: let us denote $(y'_s)_s$ an independent process which has the same law as $(y_s)_s$,

$$\begin{aligned}\mathbb{E}\Big[e^{\frac{q\gamma}{2}\sup_{s\in[r-1,r]}(y_s-y_{r-1})}I(r-1)^{-q}\Big] \leq& \mathbb{E}\Big[e^{q\gamma\sup_{s\in[0,1]}y'_s}\Big]^{\frac{1}{2}}\mathbb{E}\Big[I(r-1)^{-2q}\Big]^{\frac{1}{2}}\\ \leq& c_1^{\frac{1}{2}}\mathbb{E}\Big[e^{q\gamma\sup_{s\in[0,1]}y'_s}\Big]^{\frac{1}{2}}\cdot\mathbb{E}[e^{-\gamma q y'_1}+1]^{\frac{1}{2}},\end{aligned}$$

where in the last inequality we have used lemma 2.4.7. We see that this whole expression is a constant that depends on γ, α and q.

Now it suffices to compute:

$$\begin{aligned}\mathbb{P}(T_\beta > r-1) &= \int_{r-1}^{\infty}\frac{\beta-1}{\sqrt{2\pi}u^{3/2}}e^{-\frac{(\beta-1-\alpha u)^2}{2u}}du\\ &\leq \frac{\beta-1}{\sqrt{2\pi}}e^{\alpha(\beta-1)-\frac{\alpha^2(r-1)}{2}}\int_{r-1}^{\infty}u^{-3/2}e^{-\frac{(\beta-1)^2}{2u}}du\\ &\leq e^{\alpha(\beta-1)-\frac{\alpha^2(r-1)}{2}}.\end{aligned}$$

Hence

$$B \leq c_1'' e^{-\frac{\alpha^2 r}{2}}e^{(\alpha-\frac{q\gamma}{2})\beta}. \tag{2.4.14}$$

Equations (2.4.13) and (2.4.14) together finish the proof of the lemma. □

Now we can prove the main lemma:

Proof of lemma 2.4.5. Define for $n \geq 1$:

$$M_n = \{\sup_{s\in[r_0,r]}(y_s - y_{r_0}) \in [n-1,n]\}. \tag{2.4.15}$$

We can write,

$$\begin{aligned}\mathbb{E}[(\int_{r_0}^{r}e^{\frac{\gamma}{2}y_s}\mu_Y(ds))^{-q}] =& e^{(\frac{q^2\gamma^2}{8}-\frac{q\gamma\alpha}{2})r_0}\sum_{n\geq 1}\mathbb{E}[\mathbf{1}_{M_n}(\int_{r_0}^{r}e^{\frac{\gamma}{2}(y_s-y_{r_0})}\mu_Y(ds))^{-q}]\\ =& e^{(\frac{q^2\gamma^2}{8}-\frac{q\gamma\alpha}{2})r_0}\sum_{n\geq 1}J_{r-r_0,n},\end{aligned}$$

and by lemma 2.4.8 when $r - r_0 \geq 2$:

$$J_{r-r_0,n} \leq c_2 e^{-\frac{\alpha^2 r}{2}}e^{(\alpha-\frac{q\gamma}{2})n}.$$

In the case where $\alpha - \frac{q\gamma}{2} < 0$, it is then straightforward that there exists c depending on r_0, γ, α, q such that:

$$\mathbb{E}[(\int_{r_0}^{r}e^{\frac{\gamma}{2}y_s}\mu_Y(ds))^{-q}] \leq c\, e^{-\frac{\alpha^2 r}{2}}.$$

The other case where $\alpha - \frac{q\gamma}{2} \geq 0$ is actually very direct to prove, since we then have:

$$\mathbb{E}[(\int_{r_0}^{r}e^{\frac{\gamma}{2}y_s}\mu_Y(ds))^{-q}] \leq \mathbb{E}[e^{-\frac{q\gamma}{2}y_{r-1}}]\mathbb{E}[I(r-1)^{-q}] \leq c\, e^{(\frac{q^2\gamma^2}{8}-\frac{q\gamma\alpha}{2})r}.$$

In the last inequality we have used the fact that $y_{r-1} = B_{r-1} + \alpha(r-1)$ and that $\mathbb{E}[I(r-1)^{-q}]$ is a constant independent of r that we can absorb in c. Notice this argument actually works whenever $\alpha > 0$. This finishes the proof of lemma 2.4.4. □

2.4.3 Fusion estimation and the reflection coefficient

In this subsection we will prove the asymptotic expansion result that is used in subsection 2.2.2 to obtain the shift equation (2.1.16) on p with a shift $\frac{4}{\gamma^2}$. In this expansion will appear the reflection coefficient introduced in section 2.1.2 which will also be discussed in the next subsection. Here we will thus show:

Lemma 2.4.9. *For $-1-\frac{\gamma^2}{4} < a < -1-\frac{\gamma^2}{4}+a_0$ with $a_0 > 0$ a constant chosen small enough, $p < 1+\frac{4}{\gamma^2}(a+1)$, as $t \to 0_-$,*

$$\begin{aligned} U(t) =& M(\gamma, p, a+\frac{\gamma^2}{4}, 0) + g(\gamma,a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)}|t|^{1+a+\frac{\gamma^2}{4}} M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0) \\ &+ o(|t|^{1+a+\frac{\gamma^2}{4}}), \end{aligned} \tag{2.4.16}$$

where $g(\gamma,a)$ is defined as:

$$g(\gamma,a) = -\Gamma(-\frac{4}{\gamma^2}(a+1))\mathbb{E}[(\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)}} \mu_Y(ds))^{1+\frac{4}{\gamma^2}(a+1)}]. \tag{2.4.17}$$

The process $\mathcal{B}^{\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)}$ is defined by (2.1.25) and $\mu_Y(ds) = e^{\frac{\gamma}{2}Y(e^{-s/2})}ds$ is the notation introduced in section 2.4.2.

Notice that in the expression of $g(\gamma,a)$ we recognize the reflection coefficient $\overline{R}_1^{\partial}(-\frac{2a}{\gamma})$ of section 2.1.2. We emphasize that we only need the result for a in a small open set, it is not necessary to obtain an explicit value for a_0.

Remark 2.4.10. *From the conditions on a and p in the lemma, we have $-2-a-\frac{\gamma^2}{4} > -1-\frac{\gamma^2}{4}$ and $p-1-\frac{4}{\gamma^2}(a+1) < 0$, thus the bounds (2.1.5) are satisfied and $M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0)$ is well defined. We also want to mention that a similar result holds for $\tilde{U}(t)$ and the proof is almost the same.*

Proof. We adapt the arguments in [61] for the proof of this lemma. We introduce the notation

$$K_I(t) := \int_I (x-t)^{\frac{\gamma^2}{4}} x^a e^{\frac{\gamma}{2}X(x)} dx \tag{2.4.18}$$

for a borel set $I \subseteq [0,1]$. Recall that we work with $-1-\frac{\gamma^2}{4} < a < -1-\frac{\gamma^2}{4}+a_0$ with a_0 small, hence $p < 1+\frac{4}{\gamma^2}(a+1) < 1$. We want to study the asymptotic of

$$\mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[0,1]}(0)^p] =: T_1 + T_2, \tag{2.4.19}$$

where we defined:

$$T_1 := \mathbb{E}[K_{[|t|,1]}(t)^p] - \mathbb{E}[K_{[0,1]}(0)^p], \quad T_2 := \mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[|t|,1]}(t)^p]. \tag{2.4.20}$$

$\diamond$ First we consider T_1. The goal is to show that $T_1 = o(|t|^{1+a+\frac{\gamma^2}{4}})$. By interpolation,

$$\begin{aligned} |T_1| \leq& |p| \int_0^1 du \mathbb{E}[|K_{[|t|,1]}(t) - K_{[0,1]}(0)|(uK_{[|t|,1]}(t) + (1-u)K_{[0,1]}(0))^{p-1}] \\ \leq& |p|\mathbb{E}[|K_{[|t|,1]}(t) - K_{[0,1]}(0)|K_{[|t|,1]}(0)^{p-1}] \leq |p|(A_1+A_2), \end{aligned} \tag{2.4.21}$$

where

$$A_1 = \mathbb{E}[|K_{[|t|,1]}(t) - K_{[|t|,1]}(0)|K_{[|t|,1]}(0)^{p-1}]$$

and

$$A_2 = \mathbb{E}[|K_{[|t|,1]}(0) - K_{[0,1]}(0)|K_{[|t|,1]}(0)^{p-1}].$$

We start by estimating A_1. Using the sub-additivity of the function $x \mapsto x^{\frac{\gamma^2}{4}}$,

$$\begin{aligned} A_1 =& \mathbb{E}[|K_{[|t|,1]}(t) - K_{[|t|,1]}(0)|K_{[|t|,1]}(0)^{p-1}] \\ \leq& |t|^{\frac{\gamma^2}{4}} \int_{|t|}^1 dx_1\, x_1^a \mathbb{E}[(\int_{|t|}^1 \frac{x^{a+\frac{\gamma^2}{4}}}{|x-x_1|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\ \leq& |t|^{\frac{\gamma^2}{4}} \int_{|t|}^{t_0} dx_1\, x_1^a \mathbb{E}[(\int_{x_1}^1 x^{a-\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] + c|t|^{\frac{\gamma^2}{4}}, \end{aligned}$$

where t_0 is a constant in $(0,1)$ to be fixed. Note that in this subsection we will use $c>0$ to denote a positive constant with the abuse of notation that it can be a different constant every time it appears. Here we now need to apply lemma 2.4.4. We check that the bounds of (2.1.5) on p imply that $1+a+(1-p)\frac{\gamma^2}{2} > 0$. Therefore we are in the first case of lemma 2.4.4 which implies there exists $\epsilon_1 > 0$ such that for all $x_1 < \epsilon_1$:

$$\mathbb{E}[(\int_{x_1}^1 x^{a-\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \leq c\, x_1^{\frac{1}{\gamma^2}(a+1)^2}. \tag{2.4.22}$$

Taking $t_0 = \epsilon_1$ we obtain:

$$\begin{aligned} A_1 \leq& c|t|^{\frac{\gamma^2}{4}} \int_{|t|}^{\epsilon_1} dx_1\, x_1^{a+\frac{1}{\gamma^2}(a+1)^2} + c|t|^{\frac{\gamma^2}{4}} \\ \leq& c\,|t|^{1+\frac{\gamma^2}{4}+a+\frac{1}{\gamma^2}(a+1)^2} + c|t|^{\frac{\gamma^2}{4}} = o\big(|t|^{1+a+\frac{\gamma^2}{4}}\big). \end{aligned} \tag{2.4.23}$$

On the other hand:

$$\begin{aligned} A_2 &= \mathbb{E}[K_{[0,|t|]}(0)K_{[|t|,1]}(0)^{p-1}] \qquad (2.4.24) \\ &= \int_0^{|t|} dx_1\, x_1^{a+\frac{\gamma^2}{4}} \mathbb{E}[(\int_{|t|}^1 \frac{x^{a+\frac{\gamma^2}{4}}}{|x-x_1|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\ &\leq \int_0^{|t|} dx_1\, x_1^{a+\frac{\gamma^2}{4}} \mathbb{E}[(\int_{|t|}^1 x^{a-\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}X(x)}dx)^{p-1}] \\ &\overset{(2.4.22)}{\leq} c|t|^{1+a+\frac{\gamma^2}{4}+\frac{1}{\gamma^2}(a+1)^2} = o(|t|^{1+a+\frac{\gamma^2}{4}}). \qquad (2.4.25) \end{aligned}$$

Hence we have shown that $T_1 = o(|t|^{1+a+\frac{\gamma^2}{4}})$.

$\diamond$ Now we focus on T_2. The goal is to restrict K to the complementary of $[|t|^{1+h}, |t|]$, with $h>0$ a constant to be fixed, and then on the two parts the GMC's are weakly correlated. The same computation as (2.4.21) together with the technique we used for T_1 show that for $|t|$ sufficiently small:

$$\begin{aligned} \mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[|t|^{1+h},|t|]^c}(t)^p] &\leq |p|\mathbb{E}[K_{[|t|^{1+h},|t|]}(t)K_{[|t|,1]}(0)^{p-1}] \\ &\leq c|t|^{\frac{\gamma^2}{4}} \int_{|t|^{1+h}}^{|t|} dx_1\, x_1^{a+\frac{1}{\gamma^2}(a+1)^2} \\ &\leq c|t|^{\frac{\gamma^2}{4}+(1+h)\left(1+a+\frac{1}{\gamma^2}(a+1)^2\right)} \end{aligned}$$

By taking $h < -\frac{1+a}{1+a+\gamma^2}$, we have

$$\frac{\gamma^2}{4} + (1+h)(1+a+\frac{1}{\gamma^2}(a+1)^2) > 1+a+\frac{\gamma^2}{4}, \tag{2.4.26}$$

hence

$$\mathbb{E}[K_{[0,1]}(t)^p] - \mathbb{E}[K_{[|t|^{1+h},|t|]^c}(t)^p] = o(|t|^{1+a+\frac{\gamma^2}{4}}). \tag{2.4.27}$$

This means that it suffices to evaluate $\mathbb{E}[K_{[|t|^{1+h},|t|]^c}(t)^p] - \mathbb{E}[K_{[|t|,1]}(t)^p]$. We will use the radial decomposition of X with the notations introduced in the first paragraph of section 2.4.2,

$$K_1(t) := K_{[|t|,1]}(t) = \frac{1}{2}\int_0^{2\ln\frac{1}{|t|}} (e^{-s/2}-t)^{\frac{\gamma^2}{4}}\, e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds), \tag{2.4.28}$$

$$K_2(t) := K_{[0,|t|^{1+h}]}(t) = \frac{1}{2}\int_{2(1+h)\ln\frac{1}{|t|}}^{\infty} (e^{-s/2}-t)^{\frac{\gamma^2}{4}}\, e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds). \tag{2.4.29}$$

From (2.4.4), we deduce that for $s \le 2\ln\frac{1}{|t|}$ and $s' \ge 2(1+h)\ln\frac{1}{|t|}$,

$$0 \le \mathbb{E}[Y(e^{-s/2})Y(e^{-s'/2})] = \ln\frac{1}{|1-e^{-(s'-s)/2}|} \le 2|t|^h, \tag{2.4.30}$$

where we used the inequality $\ln\frac{1}{1-x} \le 2x$ for $x \in [0,\frac{1}{2}]$. Define the processes,

$$P(e^{-s/2}) := Y(e^{-s/2})\mathbf{1}_{\{s\le 2\ln\frac{1}{|t|}\}} + Y(e^{-s/2})\mathbf{1}_{\{s\ge 2(1+h)\ln\frac{1}{|t|}\}},$$

$$\tilde{P}(e^{-s/2}) := Y(e^{-s/2})\mathbf{1}_{\{s\le 2\ln\frac{1}{|t|}\}} + \tilde{Y}(e^{-s/2})\mathbf{1}_{\{s\ge 2(1+h)\ln\frac{1}{|t|}\}},$$

where $\tilde{Y}$ is a gaussian field independent from everything and has the same law as Y. Then we have the inequality over the covariance:

$$\mathbb{E}[\tilde{P}(e^{-s/2})\tilde{P}(e^{-s'/2})] \le \mathbb{E}[P(e^{-s/2})P(e^{-s'/2})] \le \mathbb{E}[\tilde{P}(e^{-s/2})\tilde{P}(e^{-s'/2})] + 2|t|^h. \tag{2.4.31}$$

The function $x \mapsto x^p$ is convex when $p \le 0$ and concave when $0 < p < 1$. We will only work with the case $p \le 0$ since the case $0 < p < 1$ can be treated in the same way. By applying Kahane's inequality of Theorem 2.4.2,

$$\mathbb{E}[(K_1(t)+\tilde{K}_2(t))^p] \le \mathbb{E}[(K_1(t)+K_2(t))^p] \le e^{\frac{\gamma^2}{4}(p^2-p)|t|^h}\mathbb{E}[(K_1(t)+\tilde{K}_2(t))^p], \tag{2.4.32}$$

where $\tilde{K}_2(t) := \frac{1}{2}\int_{2(1+h)\ln\frac{1}{|t|}}^{\infty}(e^{-s/2}-t)^{\frac{\gamma^2}{4}}\, e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_{\tilde{Y}}(ds)$. By the Markov property of Brownian motion and stationarity of $\mu_{\tilde{Y}}$, we have

$$\tilde{K}_2(t) := \frac{1}{2}|t|^{(1+h)(1+a+\frac{\gamma^2}{4})+\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/|t|)}}\int_0^{\infty}(|t|^h e^{-s/2}+1)^{\frac{\gamma^2}{4}}\, e^{\frac{\gamma}{2}(\tilde{B}_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_{\tilde{Y}}(ds), \tag{2.4.33}$$

with $\tilde{B}$ an independent Brownian motion. We denote

$$\sigma_t := |t|^{(1+h)(1+a+\frac{\gamma^2}{4})+\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/|t|)}}, \quad V := \frac{1}{2}\int_0^{\infty} e^{\frac{\gamma}{2}(\tilde{B}_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_{\tilde{Y}}(ds), \tag{2.4.34}$$

then:

$$\begin{aligned}\mathbb{E}[(K_1(t)+(1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t V)^p] &\le \mathbb{E}[(K_1(t)+K_2(t))^p] \\ &\le e^{\frac{\gamma^2}{4}(p^2-p)|t|^h}\mathbb{E}[(K_1(t)+\sigma_t V)^p].\end{aligned} \tag{2.4.35}$$

By the Williams path decomposition of Theorem 2.4.3 we can write,

$$V = e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-L_M}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}^{\lambda}_s}\mu_{\tilde{Y}}(ds), \tag{2.4.36}$$

where $\lambda = \frac{\gamma}{4}+\frac{1}{\gamma}(a+1)$, $M = \sup_{s\geqslant 0}(\tilde{B}_s - \lambda s)$ and L_M is the last time $\left(\mathcal{B}^{\lambda}_{-s}\right)_{s\ge 0}$ hits $-M$. Recall that the law of M is known, for $v \ge 1$,

$$\mathbb{P}(e^{\frac{\gamma}{2}M} > v) = \frac{1}{v^{\frac{4\lambda}{\gamma}}}. \tag{2.4.37}$$

For simplicity, we introduce the notations:

$$\rho_A(\lambda) := \frac{1}{2}\int_{-L_A}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^\lambda}\mu_{\tilde{Y}}(ds), \quad \rho(\lambda) := \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^\lambda}\mu_{\tilde{Y}}(ds). \tag{2.4.38}$$

Now we discuss the lower and upper bound separately.

◇ Lower bound: Since we work with $p \le 0$,

$$\begin{aligned}
&\mathbb{E}[(K_1(t)+K_2(t))^p] - \mathbb{E}[K_1(t)^p] \\
&\ge \mathbb{E}[(K_1(t)+(1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t e^{\frac{\gamma}{2}M}\rho(\lambda))^p] - \mathbb{E}[K_1(t)^p] \\
&= \frac{4\lambda}{\gamma}\mathbb{E}\Big[\int_1^\infty \frac{dv}{v^{\frac{4\lambda}{\gamma}+1}}\Big((K_1(t)+(1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda)v)^p - K_1(t)^p\Big)\Big] \\
&= \frac{4\lambda}{\gamma}\mathbb{E}\Big[\int_{\frac{(1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda)}{K_1(t)}}^\infty \frac{du}{u^{\frac{4\lambda}{\gamma}+1}}((u+1)^p-1)((1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}\Big] \\
&\overset{(2.4.63)}{\ge} \frac{4\lambda}{\gamma}\frac{\Gamma(-p+\frac{4\lambda}{\gamma})\Gamma(-\frac{4\lambda}{\gamma})}{\Gamma(-p)}\mathbb{E}[((1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}].
\end{aligned}$$

By the Girsanov theorem,

$$\begin{aligned}
&\mathbb{E}[((1+|t|^h)^{\frac{\gamma^2}{4}}\sigma_t\rho(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}] \\
&= \big(|t|(1+|t|^h)\big)^{1+a+\frac{\gamma^2}{4}}\mathbb{E}[\rho(\lambda)^{\frac{4\lambda}{\gamma}}]\mathbb{E}\big[\big(\frac{1}{2}\int_0^{2\ln\frac{1}{|t|}}(e^{-s/2}-t)^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}(B_s+s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds)\big)^{p-\frac{4\lambda}{\gamma}}\big] \\
&\underset{t\to 0_-}{\sim} |t|^{1+a+\frac{\gamma^2}{4}}\mathbb{E}[\rho(\lambda)^{\frac{4\lambda}{\gamma}}]M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0).
\end{aligned} \tag{2.4.39}$$

This completes the proof for lower bound.

◇ Upper bound: we start with an inequality:

$$\begin{aligned}
&\mathbb{E}[\big((K_1(t)+K_2(t))^p\big] - \mathbb{E}[K_1(t)^p] \\
&\le \mathbb{E}[(K_1(t)+\sigma_t V)^p] - \mathbb{E}[K_1(t)^p] + (e^{\frac{\gamma^2}{4}(p^2-p)|t|^h}-1)\mathbb{E}[K_1(0)^p] &(2.4.40)\\
&= \mathbb{E}[(K_1(t)+\sigma_t V)^p] - \mathbb{E}[K_1(t)^p] + O(|t|^h). &(2.4.41)
\end{aligned}$$

To get rid of the big O term, we will need an h such that

$$h > 1 + a + \frac{\gamma^2}{4}. \tag{2.4.42}$$

Together with the condition 2.4.26, we have

$$1 + a + \frac{\gamma^2}{4} < h < -\frac{1+a}{1+a+\gamma^2}. \tag{2.4.43}$$

There exists such an h when a is sufficiently close to $-1-\frac{\gamma^2}{4}$.

For $A > 0$ fixed, since $p \le 0$ we have,

$$\begin{aligned}
&\mathbb{E}[(K_1(t)+\sigma_t V)^p - K_1(t)^p] \le \mathbb{E}[\big((K_1(t)+\sigma_t V)^p - K_1(t)^p\big)\mathbf{1}_{\{M>A\}}] \\
&\le \mathbb{E}[\big((K_1(t)+\sigma_t\, e^{\frac{\gamma}{2}M}\rho_A(\lambda))^p - K_1(t)^p\big)\mathbf{1}_{\{M>A\}}] \\
&\overset{(2.4.37)}{=} \frac{4\lambda}{\gamma}\mathbb{E}\Big[\int_{\frac{e^{\gamma A/2}\sigma_t\rho_A(\lambda)}{K_1(t)}}^\infty \frac{du}{u^{\frac{4\lambda}{\gamma}+1}}((u+1)^p-1)(\sigma_t\rho_A(\lambda))^{\frac{4\lambda}{\gamma}}K_1(t)^{p-\frac{4\lambda}{\gamma}}\Big] \\
&\overset{\text{Girsanov}}{=} \frac{4\lambda}{\gamma}|t|^{1+a+\frac{\gamma^2}{4}}\mathbb{E}\Big[\int_{\frac{e^{\gamma A/2}\hat{\sigma}_t\rho_A(\lambda)}{\hat{K}_1(t)}}^\infty \frac{du}{u^{\frac{4\lambda}{\gamma}+1}}((u+1)^p-1)\rho_A(\lambda)^{\frac{4\lambda}{\gamma}}\hat{K}_1(t)^{p-\frac{4\lambda}{\gamma}}\Big],
\end{aligned}$$

where

$$\hat{K}_1(t) = \mathbb{E}\big[\big(\frac{1}{2}\int_0^{2\ln\frac{1}{|t|}} (e^{-s/2}-t)^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}(B_s+s(\frac{\gamma}{4}+\frac{1}{\gamma}(a+1)))}\mu_Y(ds)\big)^{p-\frac{4\lambda}{\gamma}}\big]$$
$$\overset{t\to 0^-}{\sim} M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0),$$

and for $a < -1-\frac{h\gamma^2}{4(1+h)}$,

$$\hat{\sigma}_t = |t|^{-(1+h)(1+a+\frac{\gamma^2}{4})+\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/|t|)}} \overset{t\to 0^-}{\longrightarrow} 0 \quad \text{a.s.}$$

Hence $\mathbb{E}[(K_1(t)+\sigma_t V)^p - K_1(t)^p]$ is smaller than a term equivalent to:

$$\frac{4\lambda}{\gamma}\frac{\Gamma(-p+\frac{4\lambda}{\gamma})\Gamma(-\frac{4\lambda}{\gamma})}{\Gamma(-p)}|t|^{1+a+\frac{\gamma^2}{4}}\mathbb{E}[\rho_A(\lambda)^{\frac{4\lambda}{\gamma}}]M(\gamma, p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0).$$

We can conclude by sending A to ∞. □

2.4.4 Computation of the reflection coefficient

The goal of this subsection is to prove the tail expansion result for GMC given by Proposition 2.1.5. In the first step we give a proof of the tail expansion (2.1.26) where the coefficient $\overline{R}_1^{\partial}$ is expressed in terms of the processes Y and $\mathcal{B}_s^{\alpha}$ as defined in the section 2.1.2. The proof is almost the same as in [61]. In the second step we provide the exact value (2.1.27) for $\overline{R}_1^{\partial}$ by using Theorem 2.1.1. Before proving the proposition, we provide a useful lemma. The proof can be found in [61] (see lemma 2.8).

Lemma 2.4.11. *Let $\alpha \in (\frac{\gamma}{2}, Q)$ with $Q = \frac{\gamma}{2}+\frac{2}{\gamma}$, then for $p < \frac{4}{\gamma^2}$ and all non trivial interval $I \subseteq \mathbb{R}$:*

$$\mathbb{E}[(\frac{1}{2}\int_I e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds)^p] < \infty. \tag{2.4.44}$$

This lemma tells us that the additional term $e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}}$ behaves nicely and the bound on p is the same as in the case of GMC moments.

Proof of Proposition 2.1.5. Using the decomposition $X(e^{-s/2}) = B_s + Y(e^{-s/2})$ we have,

$$I_{1,\eta}^{\partial}(\alpha) = \int_0^{\eta} x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)}dx = \frac{1}{2}\int_{-2\ln\eta}^{\infty} e^{\frac{\gamma}{2}(B_s - s(\frac{\gamma}{4}+\frac{1}{\gamma}-\frac{\alpha}{2}))} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds$$
$$\overset{\text{Theorem 2.4.3}}{=} e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-2\ln\eta-L_M}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds,$$

where $M = \sup_{s\geqslant 0}(B_s - \frac{Q-\alpha}{2}s)$ and L_M is the last time $\left(\mathcal{B}_{-s}^{\frac{Q-\alpha}{2}}\right)_{s\geq 0}$ hits $-M$. The law of M is given by:

$$\mathbb{P}(e^{\frac{\gamma}{2}M} > v) = \frac{1}{v^{\frac{2(Q-\alpha)}{\gamma}}} \quad (v \geq 1). \tag{2.4.45}$$

We denote

$$\rho_A(\frac{Q-\alpha}{2}) = \frac{1}{2}\int_{-L_A}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds,$$
$$\rho(\frac{Q-\alpha}{2}) = \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds,$$

and study the upper and lower bounds for $\mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u)$.
$\diamond$ Upper bound:

$$\mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u) \leq \mathbb{P}(e^{\frac{\gamma}{2}M}\rho(\frac{Q-\alpha}{2}) > u) \leq \frac{\mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}]}{u^{\frac{2(Q-\alpha)}{\gamma}}} = \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2}{\gamma}(Q-\alpha)}}.$$

$\diamond$ Lower bound: we first show that the tail behavior is concentrated at $x = 0$ and that the value of η does not matter. Consider $h, \epsilon > 0$ sufficiently small,

$$\begin{aligned}\mathbb{P}(I^{\partial}_{1,1}(\alpha) > u + u^{1-h}) - \mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u) &\leq \mathbb{P}(\int_{\eta}^{1} x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)}dx > u^{1-h}) \\ &\leq \frac{\mathbb{E}[(\int_{\eta}^{1} x^{-\frac{\gamma\alpha}{2}} e^{\frac{\gamma}{2}X(x)}dx)^{\frac{4}{\gamma^2}-\epsilon}]}{u^{(1-h)(\frac{4}{\gamma^2}-\epsilon)}} = O_{u\to\infty}(\frac{1}{u^{\frac{2(Q-\alpha)}{\gamma}+\nu}}),\end{aligned} \tag{2.4.46}$$

where $\nu > 0$ can be any constant that satisfies $\nu \leq (1-h)(\frac{4}{\gamma^2}-\epsilon) - \frac{2(Q-\alpha)}{\gamma^2}$. Thus it suffices to study the tail behavior of $I^{\partial}_{1,1}(\alpha)$. Take $A = \frac{2\nu}{\gamma}\ln u$,

$$\begin{aligned}\mathbb{P}(I^{\partial}_{1,1}(\alpha) > u) &\geq \mathbb{P}(e^{\frac{\gamma}{2}M}\rho_A(\frac{Q-\alpha}{2}) > u, M > A) \\ &= \mathbb{P}\left(e^{\frac{\gamma}{2}M} > \max\left\{\frac{u}{\rho_A(\frac{Q-\alpha}{2})}, e^{\frac{\gamma}{2}A}\right\}\right) \\ &= \mathbb{E}\left[\min\left\{\frac{\rho_A(\frac{Q-\alpha}{2})}{u}, \frac{1}{u^{\nu}}\right\}^{\frac{2(Q-\alpha)}{\gamma}}\right] \\ &\geq u^{-\frac{2(Q-\alpha)}{\gamma}}\left(\mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] - \mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}\mathbf{1}_{\rho_A(\frac{Q-\alpha}{2})>u^{1-\nu}}]\right)\end{aligned}$$

Take $h' > 1$ a constant such that $h'\frac{2(Q-\alpha)}{\gamma} < \frac{4}{\gamma^2}$, by Hölder's inequality and Markov inequality:

$$\begin{aligned}&E[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}\mathbf{1}_{\rho_A(\frac{Q-\alpha}{2})>u^{1-\nu}}] \\ \leq&\mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{h'\frac{2(Q-\alpha)}{\gamma}}]^{\frac{1}{h'}}\mathbb{P}(\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}} > u^{1-\nu})^{\frac{h'-1}{h'}} \\ \leq&\mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{h'\frac{2(Q-\alpha)}{\gamma}}]u^{-(1-\nu)(h'-1)} = O(u^{-(1-\nu)(h'-1)}).\end{aligned}$$

We impose additionally that ν satisfies $\nu < (1-\nu)(h'-1)$, then

$$\mathbb{P}(I^{\partial}_{1,1}(\alpha) > u) \geq u^{-\frac{2(Q-\alpha)}{\gamma}}\mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] + O(u^{-\frac{2(Q-\alpha)}{\gamma}-\nu}). \tag{2.4.47}$$

We claim that for $u > 1$ and for some $c > 0$,

$$\mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] - \mathbb{E}[\rho_A(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] \leq cu^{-\nu}. \tag{2.4.48}$$

This shows that:

$$\mathbb{P}(I^{\partial}_{1,1}(\alpha) > u) = \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2(Q-\alpha)}{\gamma}}} + O(\frac{1}{u^{\frac{2(Q-\alpha)}{\gamma}+\nu}}). \tag{2.4.49}$$

By applying the tail result to (2.4.46) we deduce,

$$\mathbb{P}(I^{\partial}_{1,\eta}(\alpha) > u) = \frac{\overline{R}^{\partial}_1(\alpha)}{u^{\frac{2(Q-\alpha)}{\gamma}}} + O(\frac{1}{u^{\frac{2(Q-\alpha)}{\gamma}+\min(\nu,h)}}), \tag{2.4.50}$$

which finishes the proof for the first part. For the second part let $\epsilon > 0$, the value of $\overline{R}_1^{\partial}(\alpha)$ is then determined by the following limit, with $p = \frac{2(Q-\alpha)}{\gamma}$,

$$\lim_{\epsilon \to 0} \epsilon \mathbb{E}[I_{1,1}^{\partial}(\alpha)^{p-\epsilon}] = p\overline{R}_1^{\partial}(\alpha). \tag{2.4.51}$$

With our Theorem 2.1.1 we can compute this limit and get:

$$\begin{aligned} p\overline{R}_1^{\partial}(\alpha) &= \frac{(2\pi)^p (\frac{2}{\gamma})^{p\frac{\gamma^2}{4}} \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} - p\frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} - (p-1)\frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{4}{\gamma} - \alpha - (p-2)\frac{\gamma}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} - \alpha + \frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma} + \frac{\gamma}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{4}{\gamma} - \alpha - (2p-2)\frac{\gamma}{2})} \lim_{\epsilon \to 0} \epsilon \Gamma_{\frac{\gamma}{2}}(\frac{\gamma\epsilon}{2}) \\ &= \frac{(2\pi)^p (\frac{2}{\gamma})^{p\frac{\gamma^2}{4}} \Gamma_{\frac{\gamma}{2}}(\alpha - \frac{\gamma}{2})}{\Gamma(1-\frac{\gamma^2}{4})^p \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}) \Gamma_{\frac{\gamma}{2}}(Q-\alpha)} \frac{1}{\sqrt{2\pi}} (\frac{\gamma}{2})^{-\frac{1}{2}} \Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}) \\ &= \frac{1}{\sqrt{\gamma\pi}} \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)} (\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma_{\frac{\gamma}{2}}(\alpha - \frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}. \end{aligned}$$

It remains to show (2.4.48). By (2.4.3) of the Williams decomposition theorem of appendix 2.4.1, the process $\hat{\mathcal{B}}_s^{\frac{Q-\alpha}{2}}$ defined for $s \leq 0$ by

$$\hat{\mathcal{B}}_s^{\frac{Q-\alpha}{2}} = \mathcal{B}^{\frac{Q-\alpha}{2}}_{s-L_{\frac{2\nu}{\gamma}\ln u}} + \frac{2\nu}{\gamma} \ln u$$

is independent from everything and has the same law as $(\mathcal{B}_s^{\frac{Q-\alpha}{2}})_{s \leq 0}$. We can then write,

$$\rho(\frac{Q-\alpha}{2}) = A_1 + u^{-\nu} A_2, \tag{2.4.52}$$

where:

$$\begin{aligned} A_1 &= \frac{1}{2} \int_{-L_{\frac{2\nu}{\gamma}\ln u}}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})} ds, \\ A_2 &= \frac{1}{2} \int_{-\infty}^{0} e^{\frac{\gamma}{2}\hat{\mathcal{B}}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})} ds. \end{aligned} \tag{2.4.53}$$

By interpolation (see (2.4.21) for example),

$$\begin{aligned} &\mathbb{E}[(A_1 + u^{-\nu}A_2)^{\frac{2(Q-\alpha)}{\gamma}} - A_1^{\frac{2(Q-\alpha)}{\gamma}}] \\ &\quad \leq \frac{2(Q-\alpha)}{\gamma} u^{-\nu} \mathbb{E}[A_2 \max\{\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}-1}, A_1^{\frac{2(Q-\alpha)}{\gamma}-1}\}]. \end{aligned}$$

If $\frac{2(Q-\alpha)}{\gamma} \leq 1$,

$$\begin{aligned} \mathbb{E}[(A_1 + u^{-\nu}A_2)^{\frac{2(Q-\alpha)}{\gamma}} - A_1^{\frac{2(Q-\alpha)}{\gamma}}] &\leq u^{-\nu} \mathbb{E}[A_2 A_1^{\frac{2(Q-\alpha)}{\gamma}-1}] \\ &\overset{\text{Hölder}}{\leq} u^{-\nu} \mathbb{E}[A_2^p]^{1/p} \mathbb{E}[A_1^{\frac{p}{p-1}(\frac{2(Q-\alpha)}{\gamma}-1)}]^{(p-1)/p} < cu^{-\nu}, \end{aligned}$$

where $1 < p < \frac{4}{\gamma^2}$ to ensure that $\mathbb{E}[A_2^p]$ is finite, and we know that

$$A_1 \geq \frac{1}{2} \int_0^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})} ds$$

has negative moments. On the other hand, if $\frac{2(Q-\alpha)}{\gamma} > 1$, then:

$$\mathbb{E}[(A_1 + u^{-\nu}A_2)^{\frac{2(Q-\alpha)}{\gamma}} - A_1^{\frac{2(Q-\alpha)}{\gamma}}] \leq \frac{2(Q-\alpha)}{\gamma} u^{-\nu} \mathbb{E}[\rho(\frac{Q-\alpha}{2})^{\frac{2(Q-\alpha)}{\gamma}}] < cu^{-\nu}.$$

This last upper bound comes from the fact that the moment of $\rho(\frac{Q-\alpha}{2})$ is finite thanks to Lemma 2.4.11 and since $\frac{2(Q-\alpha)}{\gamma} < \frac{4}{\gamma^2}$. □

2.4.5 Special functions

Lastly we include here a detailed discussion on hypergeometric functions and on the special functions $\Gamma_{\frac{\gamma}{2}}$ and G that we have used in our paper. First, let us discuss the theory of hypergeometric equations and the so-called connection formulas between the different bases of their solutions. For $A > 0$ let $\Gamma(A) = \int_0^\infty t^{A-1}e^{-t}dt$ denote the standard Gamma function and let $(A)_n := \frac{\Gamma(A+n)}{\Gamma(A)}$. For A, B, C, and x real numbers we define the hypergeometric function F by:

$$F(A,B,C,x) := \sum_{n=0}^{\infty} \frac{(A)_n(B)_n}{n!(C)_n} x^n. \tag{2.4.54}$$

This function can be used to solve the following hypergeometric equation:

$$(t(1-t)\frac{d^2}{dt^2} + (C-(A+B+1)t)\frac{d}{dt} - AB)U(t) = 0. \tag{2.4.55}$$

For our purposes we will always work with the parameter $t \in (-\infty, 0)$ and we can give the following two bases of solutions, under the assumption that C and $A-B$ are not integers,

$$\begin{aligned} U(t) &= C_1 F(A,B,C,t) \\ &\quad + C_2|t|^{1-C}F(1+A-C,1+B-C,2-C,t) \\ &= D_1|t|^{-A}F(A,1+A-C,1+A-B,t^{-1}) \\ &\quad + D_2|t|^{-B}F(B,1+B-C,1+B-A,t^{-1}), \end{aligned}$$

where the first expression is an expansion in power of $|t|$ and the second is an expansion in powers of $|t|^{-1}$. For each basis we have two real constants that parametrize the solution space, C_1, C_2 and D_1, D_2. We thus expect to have an explicit change of basis formula that will give a link between C_1, C_2 and D_1, D_2. This is precisely what give the so-called connection formulas:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1-C)\Gamma(A-B+1)}{\Gamma(A-C+1)\Gamma(1-B)} & \frac{\Gamma(1-C)\Gamma(B-A+1)}{\Gamma(B-C+1)\Gamma(1-A)} \\ \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)} & \frac{\Gamma(C-1)\Gamma(B-A+1)}{\Gamma(B)\Gamma(C-A)} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}. \tag{2.4.56}$$

This relation comes from the theory of hypergeometric equations and we will extensively use it to deduce our shift equations. We will apply it for both hypergeometric equations of Proposition 2.1.4.

We will now provide some explanations on the function $\Gamma_{\frac{\gamma}{2}}(x)$ that we have introduced as well as its connection with the so-called G Barnes' function. Our function $\Gamma_{\frac{\gamma}{2}}(x)$ is equal to the function $\Gamma_b(x)$ defined in the appendix of [72] with $b = \frac{\gamma}{2}$. [7] For all $\gamma \in (0,2)$ and for $x > 0$, $\Gamma_{\frac{\gamma}{2}}(x)$ is defined by the integral formula written in Theorem 2.1.1,

$$\ln \Gamma_{\frac{\gamma}{2}}(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2}-x)^2}{2}e^{-t} + \frac{x-\frac{Q}{2}}{t}\right], \tag{2.4.57}$$

where we have $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Since the function $\Gamma_{\frac{\gamma}{2}}(x)$ is continuous it is completely determined by the following two shift equations,

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x+\frac{\gamma}{2})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{\gamma x}{2})(\frac{\gamma}{2})^{-\frac{\gamma x}{2}+\frac{1}{2}}, \tag{2.4.58}$$

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x+\frac{2}{\gamma})} = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{2x}{\gamma})(\frac{\gamma}{2})^{\frac{2x}{\gamma}-\frac{1}{2}}, \tag{2.4.59}$$

[7] In [79] Ostrovsky uses a slightly different special function $\Gamma_2(x|\tau)$, the relation with our $\Gamma_{\frac{\gamma}{2}}(x)$ is:

$$\Gamma_{\frac{\gamma}{2}}(x) = (\frac{2}{\gamma})^{\frac{1}{2}(x-\frac{Q}{2})^2}\frac{\Gamma_2(\frac{2x}{\gamma}|\tau)}{\Gamma_2(\frac{Q}{\gamma}|\tau)}.$$

and by its value in $\frac{Q}{2}$, $\Gamma_{\frac{\gamma}{2}}(\frac{Q}{2}) = 1$. We mention that $\Gamma_{\frac{\gamma}{2}}(x)$ is an analytic function of x. In the case where $\gamma = 2$ the function $\Gamma_{\frac{\gamma}{2}}(x)$ reduces to,

$$\Gamma_1(x) = (2\pi)^{\frac{x}{2}-\frac{1}{2}}G(x)^{-1}, \tag{2.4.60}$$

where $G(x)$ is the so-called Barnes G function. This function is useful when we study the limit $\gamma \to 2$ in section 2.1.4. Finally in our Corollary 2.1.3 we have used a special $\beta_{2,2}$ distribution defined in [79]. Here we recall the definition:

Definition 2.4.12 (Existence theorem)**.** *The distribution $-\ln \beta_{2,2}(a_1, a_2; b_0, b_1, b_2)$ is infinitely divisible on $[0, \infty)$ and has the Lévy-Khintchine decomposition for $\mathrm{Re}(p) > -b_0$:*

$$\mathbb{E}[\exp(p \ln \beta_{2,2}(a_1, a_2; b_0, b_1, b_2))] = \exp\Big(\int_0^\infty (e^{-pt} - 1)e^{-b_0 t} \frac{(1 - e^{-b_1 t})(1 - e^{-b_2 t})}{(1 - e^{-a_1 t})(1 - e^{-a_2 t})} \frac{dt}{t} \Big). \tag{2.4.61}$$

Furthermore, the distribution $\ln \beta_{2,2}(a_1, a_2; b_0, b_1, b_2)$ is absolutely continuous with respect to the Lebesgue measure.

We only work with the case $(a_1, a_2) = (1, \frac{4}{\gamma^2})$. Then $\beta_{2,2}(1, \frac{4}{\gamma^2}; b_0, b_1, b_2)$ depends on 4 parameters γ, b_0, b_1, b_2 and its real moments $p > -b_0$ are given by the formula:

$$\mathbb{E}[\beta_{2,2}(1, \frac{4}{\gamma^2}; b_0, b_1, b_2)^p] = \frac{\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(p + b_0))\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(b_0 + b_1))\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(b_0 + b_2))\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(p + b_0 + b_1 + b_2))}{\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}b_0)\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(p + b_0 + b_1))\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(p + b_0 + b_2))\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2}(b_0 + b_1 + b_2))}. \tag{2.4.62}$$

Of course we have $\gamma \in (0, 2)$ and the real numbers p, b_0, b_1, b_2 must be chosen so that the arguments of all the $\Gamma_{\frac{\gamma}{2}}$ are positive. We conclude this section with a few computations that we need that also involve hypergeometric functions.

Lemma 2.4.13. *For $p < 0$ and $-1 < a < 0$ or for $0 < p < 1$ and $-1 < a < -p$ we have the identity:*

$$\int_0^\infty ((u + 1)^p - 1)u^{a-1}du = \frac{\Gamma(a)\Gamma(-a - p)}{\Gamma(-p)}. \tag{2.4.63}$$

Proof. Denote by $(x)_n := x(x + 1) \dots (x + n - 1)$.

$$\begin{aligned}
\int_0^\infty ((u + 1)^p - 1)u^{a-1}du &= \sum_{n=0}^\infty \frac{(-1)^n}{n!}(-p)_n \frac{1}{n + a} - \sum_{n=0}^\infty \frac{(-1)^n}{n!}(-p)_n \frac{1}{a + p - n} \\
&= \frac{1}{a} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{(-p)_n (a)_n}{(a + 1)_n} - \frac{1}{a + p} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{(-p)_n (-a - p)_n}{(-a - p + 1)_n} \\
&= \frac{1}{a} F(-p, a, a + 1, -1) - \frac{1}{a + p} F(-p, -a - p, -a - p + 1, -1) \\
&= \frac{\Gamma(a)\Gamma(-a - p)}{\Gamma(-p)},
\end{aligned}$$

where in the last line we used the formula, for suitable $\bar{a}, \bar{b} \in \mathbb{R}$,

$$\bar{b}F(\bar{a} + \bar{b}, \bar{a}, \bar{a} + 1, -1) + \bar{a}F(\bar{a} + \bar{b}, \bar{b}, \bar{b} + 1, -1) = \frac{\Gamma(\bar{a} + 1)\Gamma(\bar{b} + 1)}{\Gamma(\bar{a} + \bar{b})}.$$

□

Lemma 2.4.14. *For $0 < a < 1 - \frac{\gamma^2}{4}$ we have:*

$$\frac{\gamma^2}{4} \int_0^\infty (y + 1)^{\frac{\gamma^2}{4} - 1} y^{a-1} dy = (a + \frac{\gamma^2}{4}) \frac{\Gamma(a)\Gamma(-a - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}. \tag{2.4.64}$$

Proof. By the previous lemma,

$$\int_0^\infty ((y+z)^{\frac{\gamma^2}{4}} - 1)y^{a-1}dy = z^{a+\frac{\gamma^2}{4}} \frac{\Gamma(a)\Gamma(-a-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

We take the derivative in z in the above equation and evaluate it at $z = 1$ to get:

$$\frac{\gamma^2}{4}\int_0^\infty (y+1)^{\frac{\gamma^2}{4}-1}y^{a-1}dy = (a+\frac{\gamma^2}{4})\frac{\Gamma(a)\Gamma(-a-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

□

CHAPTER 3

Integrability of boundary Liouville theory

In collaboration with Guillaume Remy.

Liouville conformal field theory (LCFT) is considered on a simply connected domain with boundary, specializing to the case where the Liouville potential is integrated only over the boundary of the domain. We work in the probabilistic framework of boundary LCFT introduced by Huang-Rhodes-Vargas (2015). Building upon the known proof of the bulk one-point function by the first author, exact formulas are rigorously derived for the remaining basic correlation functions of the theory, i.e., the bulk-boundary correlator, the boundary two-point and the boundary three-point functions. These four correlations should be seen as the fundamental building blocks of boundary Liouville theory, playing the analogue role of the DOZZ formula in the case of the Riemann sphere. Our study of boundary LCFT also provides the general framework to understand the integrability of one-dimensional Gaussian multiplicative chaos measures as well as their tail expansions. Finally this work sets the stage for studying the more general case of boundary LCFT with both bulk and boundary Liouville potentials.

3.1 Introduction and main results

Liouville conformal field theory - LCFT henceforth - first appeared in Polyakov's seminal 1981 paper [82] where he introduces a theory of summation over the space of Riemannian metrics on a given two-dimensional surface. As a fundamental building block of non-critical string theory, the necessity to solve Liouville theory lead Belavin, Polyakov, and Zamolodchikov (BPZ) to introduce in [10] conformal field theory (CFT), a powerful framework to study quantum field theories possessing conformal symmetry. On the Riemann sphere, solving Liouville theory amounts to computing the three-point function - which is given by the DOZZ formula proposed in [25, 108] - and arguing that higher order correlation functions can be obtained from it using the conformal bootstrap method of [10]. A similar program can be pursued for surfaces with boundary, where the basic correlations have been derived in the physics literature in [37, 52, 84] and the conformal bootstrap is also applicable.

We work here in the probabilistic framework of LCFT first introduced by David-Kupiainen-Rhodes-Vargas on the Riemann sphere in [20], and later followed by companion works for the boundary case [53] and in higher genus [21, 49, 87]. The strength of this framework is it allows to put Liouville theory on solid mathematical grounds and to rigorously carry out the program of solving the theory as described above. Indeed, in the case of the Riemann sphere, the BPZ differential equations expressing the constraints of the local conformal invariance of CFT were shown to hold in [60]. Building on this work a proof of the DOZZ formula was then

given in [61]. Very shortly after, the same procedure was implemented by the first author [88] in the case of boundary LCFT to prove the Fyodorov-Bouchaud formula proposed in [41] that can also be interpreted as a bulk one-point function of boundary LCFT.

The purpose of the present work is to pursue solving Liouville theory on a domain with boundary, in the special case where the Liouville potential is only present on the boundary, see the Liouville action (3.1.2) below. In the study of boundary LCFT there are four basic correlation functions that must be computed: the bulk one-point function, the bulk-boundary correlator, and the boundary two-point and three-point functions. For the last two correlations we allow the freedom to choose different cosmological constants for each connected component of the boundary, see again (3.1.2). Taking as an input our previous works [88, 89], we will thus compute all the basic correlations of boundary LCFT. In a future work we plan to address the same problem in the more general setting where there is also a bulk Liouville potential in the action. Lastly for finding higher order correlations one needs in principle to apply the conformal bootstrap method, although at the level of probability this remains a challenge even in the case of the Riemann sphere.

The key probabilistic object required to define LCFT is the Gaussian multiplicative chaos (GMC) measure, which formally corresponds to exponentiating a log-correlated Gaussian field. Since the pioneering work of Kahane [56], it is well understood how to define this object using a suitable regularization procedure [12, 92]. GMC measures are now an extremely well studied object in probability theory and appear in many apparently unrelated problems such as 3d turbulence, mathematical finance, statistical physics, two-dimensional random geometry and probabilistic LCFT. One illustration is the Fyodorov-Bouchaud formula giving the law of the total mass of the GMC measure on the unit circle that was first proposed in statistical physics [41] in the context of random energy models. It was proved in [88] by viewing it as the bulk one-point function of boundary LCFT derived in [37] and by using the BPZ equations. This connection between [41] and [37] was unknown to physicists. Furthermore, our previous work [89] studies GMC on the unit interval making again rigorous predictions of statistical physicists [45] (see also the related set of works [76, 78, 79]) using once more the BPZ equations coming from CFT.

In the present paper we further uncover these connections between the theory of GMC measures and Liouville CFT. We show how the law of the total mass of GMC on the unit interval studied in [89] can be derived from a special case of the boundary three-point function of boundary LCFT. Lastly we explain how the boundary two-point function gives a very general result on the tail expansion of one-dimensional GMC measures. The study of boundary LCFT with boundary Liouville potential is thus the most general framework to understand the integrability of one-dimensional GMC measures.

Let us now introduce the framework of our paper. By conformal invariance we can work equivalently on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ or on the unit disk $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ but for almost all of this paper we will work on $\mathbb{H}$. We use notations $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$, $\partial\mathbb{D}$ for the unit circle and similarly $\overline{\mathbb{D}} = \mathbb{D} \cup \partial\mathbb{D}$. In theoretical physics Liouville theory is defined using the path integral formalism. Let us fix N bulk insertion points $z_i \in \mathbb{H}$ of associated weights $\alpha_i \in \mathbb{R}$ and M boundary insertions points $s_j \in \mathbb{R}$ with weight $\beta_j \in \mathbb{R}$. In physics the correlation function of LCFT at these points is defined using the following infinite dimensional integral on the space of maps $X : \mathbb{H} \mapsto \mathbb{R}$,

$$\left\langle \prod_{i=1}^{N} e^{\alpha_i \phi(z_i)} \prod_{j=1}^{M} e^{\frac{\beta_j}{2}\phi(s_j)} \right\rangle = \int_{X:\mathbb{H}\mapsto\mathbb{R}} DX \prod_{i=1}^{N} e^{\alpha_i X(z_i)} \prod_{j=1}^{M} e^{\frac{\beta_j}{2}X(s_j)} e^{-S_L(X)}, \tag{3.1.1}$$

where DX is a formal uniform measure on the maps X and $S_L(X)$ is the Liouville action given by:

$$S_L(X) = \frac{1}{4\pi}\int_{\mathbb{H}} \left(|\partial^g X|^2 + QR_g X\right) d\lambda_g + \frac{1}{2\pi}\int_{\mathbb{R}} \left(QK_g X + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right) d\lambda_{\partial g}. \tag{3.1.2}$$

Here $\gamma \in (0,2)$, and one has $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.[1] For a choice g of background metric on $\mathbb{H}$, ∂^g, R_g, K_g, $d\lambda_g$, $d\lambda_{\partial g}$ respectively stand for the gradient, Ricci curvature, geodesic curvature of the boundary, volume form and line element in the metric g. The precise choice of g is irrelevant thanks to the Weyl anomaly proven in [53], see also Lemma 3.5.7 in appendix. μ_∂ is the cosmological constant tuning the interaction strength of

[1] The central of the theory is then given by $c = 1 + 6Q^2$.

the Liouville potential. It will be chosen either to be a fixed positive number or more generally a function $\mu_\partial : \mathbb{R} \mapsto \mathbb{C}$ constraint to be constant in between two consecutive insertion points s_j on $\mathbb{R}$. Of course since the path integral (3.1.1) does not make rigorous sense we will rely on the construction of [53] to obtain a valid probabilistic definition for these correlation functions. A requirement for a correlation to be well-defined is that the following Seiberg bounds must hold:

$$\sum_{i=1}^{N} \alpha_i + \sum_{j=1}^{M} \frac{\beta_j}{2} > Q, \quad \forall j,\ \beta_j < Q. \tag{3.1.3}$$

Notice here that we do not have the condition $\alpha_i < Q$ as we do not have a bulk potential. One of the key properties of a CFT is that its correlations behave as conformal tensors under conformal automorphism. This has indeed been checked for the probabilistic definitions in [20, 53]. Using conformal invariance one can thus reduce computing the following basic correlation functions to computing a single constant called the structure constant. We perform this reduction for the four basic correlations that will be at the heart of our work:

- **Bulk one-point function.** For $z \in \mathbb{H}, \quad \alpha > Q$:

$$\left\langle e^{\alpha\phi(z)} \right\rangle = \frac{U(\alpha)}{|z-\overline{z}|^{2\Delta_\alpha}}. \tag{3.1.4}$$

- **Bulk-boundary correlator.** For $z \in \mathbb{H},\ s \in \mathbb{R}, \quad \beta < Q, \quad \alpha + \frac{\beta}{2} > Q$:

$$\left\langle e^{\alpha\phi(z)} e^{\frac{\beta}{2}\phi(s)} \right\rangle = \frac{G(\alpha,\beta)}{|z-\overline{z}|^{2\Delta_\alpha - \Delta_\beta}|z-s|^{2\Delta_\beta}}. \tag{3.1.5}$$

- **Boundary two-point function.** For $s_1, s_2 \in \mathbb{R}, \quad \beta \in (\frac{\gamma}{2}, Q)$:

$$\left\langle e^{\frac{\beta}{2}\phi(s_1)} e^{\frac{\beta}{2}\phi(s_2)} \right\rangle = \frac{R(\beta,\mu_1,\mu_2)}{|s_1-s_2|^{2\Delta_\beta}}. \tag{3.1.6}$$

- **Boundary three-point function.** For $i \in \{1,2,3\}$, $s_i, \beta_i \in \mathbb{R}$ satisfying $\beta_i < Q$ and $\sum_i \beta_i > 2Q$:

$$\left\langle e^{\frac{\beta_1}{2}\phi(s_1)} e^{\frac{\beta_2}{2}\phi(s_2)} e^{\frac{\beta_3}{2}\phi(s_3)} \right\rangle = \frac{H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}}{|s_1-s_2|^{\Delta_1+\Delta_2-\Delta_3}|s_1-s_3|^{\Delta_1+\Delta_3-\Delta_2}|s_2-s_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{3.1.7}$$

  We have used the notations $\Delta_\alpha = \frac{\alpha}{2}(Q-\frac{\alpha}{2})$, $\Delta_\beta = \frac{\beta}{2}(Q-\frac{\beta}{2})$, and $\Delta_i = \frac{\beta_i}{2}(Q-\frac{\beta_i}{2})$. Each of the four structure constants U, G, R, H will have a definition involving Gaussian multiplicative chaos.

3.1.1 Probabilistic definitions

In this section we will introduce the probabilistic objects with which we can rigorously define the four correlations (3.1.4) through (3.1.7). We will define all of the probabilistic objects on $\mathbb{H}$. We view $\mathbb{H}$ as being equipped with the following background metric g, written here in diagonal form $g = g(x)dx^2$,

$$g(x) = \frac{1}{|x|_+^4}, \quad \text{where} \quad |x|_+ := \max(|x|, 1). \tag{3.1.8}$$

This choice is convenient to work with because it will make some computations work in the same way as in [61] and [89]. We now need to define the Gaussian free field (GFF) we will be working with.

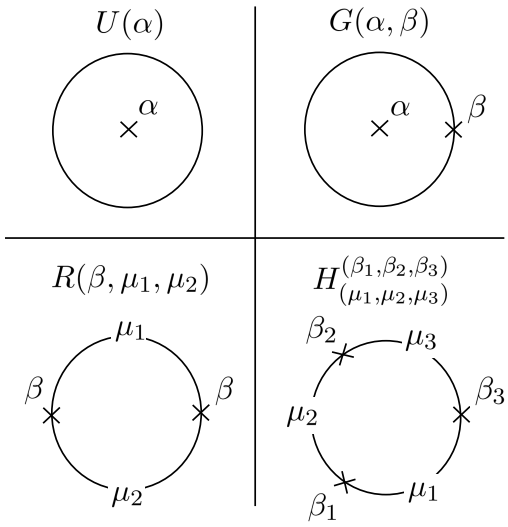

Figure 3.1: Structure constants for boundary Liouville theory

Definition 3.1.1. *(Gaussian free field on $\mathbb{H}$) The Gaussian free field X is the centered Gaussian process on $\mathbb{H}$ with covariance given by:*

$$\mathbb{E}[X(x)X(y)] = \ln \frac{1}{|x-y||x-\bar{y}|} - \frac{1}{2}\ln g(x) - \frac{1}{2}\ln g(y). \tag{3.1.9}$$

Since the variance at each point is infinite, X is not defined pointwise and exists as a random distribution. It also satisfies:

$$\int_0^\pi X(e^{i\theta})d\theta = 0. \tag{3.1.10}$$

To construct the above GFF X one can perform the following. Consider $X_{\mathbb{D}}$ the Neumann boundary (also called free boundary) GFF on $\mathbb{D}$. This field has a covariance given by, for $x, y \in \mathbb{D}$,

$$\mathbb{E}[X_{\mathbb{D}}(x)X_{\mathbb{D}}(y)] = \ln \frac{1}{|x-y||1-x\bar{y}|}. \tag{3.1.11}$$

The field $X_{\mathbb{D}}$ has zero average on the unit circle. One can then conformally map the disk $\mathbb{D}$ equipped with the Euclidean metric to the upper-half plane $\mathbb{H}$ equipped with the metric $\hat{g}(x) = \frac{4}{|x+i|^4}$. By this map from the field $X_{\mathbb{D}}$ we obtain the field $X_{\hat{g}}$ defined on $\mathbb{H}$ which has covariance,

$$\mathbb{E}[X_{\hat{g}}(x)X_{\hat{g}}(y)] = \ln \frac{1}{|x-y||x-\bar{y}|} - \frac{1}{2}\ln \hat{g}(x) - \frac{1}{2}\ln \hat{g}(y), \tag{3.1.12}$$

and zero average on $\mathbb{R}$ in the metric $\hat{g}$. Finally the above field X can be obtained from the field $X_{\hat{g}}$ by simply setting:

$$X(x) = X_{\hat{g}}(x) - \frac{1}{\pi}\int_0^\pi X_{\hat{g}}(e^{i\theta})d\theta. \tag{3.1.13}$$

We now define the Gaussian multiplicative chaos measure on $\mathbb{R}$.

Definition 3.1.2. *(Gaussian multiplicative chaos) Fix a $\gamma \in (0,2)$. The GMC measure associated to the field X is defined by the following limit,*

$$e^{\frac{\gamma}{2}X(x)}dx = \lim_{\epsilon\to 0} e^{\frac{\gamma}{2}X_\epsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[X_\epsilon(x)^2]}dx, \tag{3.1.14}$$

where the convergence is in probability and in the sense of weak convergence of measures on $\mathbb{R}$. *Here* X_ϵ *is a suitable regularization of the field. More precisely, for a continuous compactly supported function* f *on* $\mathbb{R}$, *the following convergence holds in probability:*

$$\int_{\mathbb{R}} f(x) e^{\frac{\gamma}{2} X(x)} dx = \lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x) e^{\frac{\gamma}{2} X_\epsilon(x) - \frac{\gamma^2}{8} \mathbb{E}[X_\epsilon(x)^2]} dx. \tag{3.1.15}$$

For an elementary proof of this convergence and examples of smoothing of the field X_ϵ, see for instance [12]. With this at hand one can now give a probabilistic definition to the four structure constants U, G, R, H using moments of GMC on $\mathbb{H}$. The reason why the following definitions are the correct interpretation of (3.1.1) has been performed in [53]. We will also work with the four quantities $\overline{U}, \overline{G}, \overline{R}, \overline{H}$ which will be purely defined as moment of GMC on $\mathbb{H}$ and be each related to the corresponding U, G, R, H by an explicit prefactor.

In order to define the boundary two-point and three-point functions we will consider parameters μ_1, μ_2, μ_3 in $\mathbb{C}$ corresponding to the values taken by μ_∂ in the Liouville action (3.1.2). To be able to choose a suitable branch cut to define the probabilistic expressions below, we introduce the following conditions we will refer to as the half-space conditions.

Definition 3.1.3. *(Half-space condition for* μ_i*) Consider* $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$. *We say that* $(\mu_i)_{i=1,2,3}$ *satisfies the half-space condition if there exists a half-space* $\mathcal{H}$ *of* $\mathbb{C}$ *whose boundary is a line passing through the origin not equal to the real axis and satisfying the following. The half-space* $\mathcal{H}$ *does not contained the half-line* $(-\infty, 0)$. *Each* μ_i *is contained in* $\overline{\mathcal{H}}$ *(the half-space with its boundary included) and the sum* $\mu_1 + \mu_2 + \mu_3$ *is strictly contained in* $\mathcal{H}$. *We will also refer to the half-space condition for a pair* $\mu_1, \mu_2 \in \mathbb{C}$ *which will be the condition above with* μ_3 *set to* 0.

Definition 3.1.4. *(Correlation functions of Liouville theory on* $\mathbb{H}$*) Fix* $\gamma \in (0, 2)$. *Consider parameters* $\alpha, \beta, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$, $\mu_\partial \in (0, +\infty)$, *and* $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$. *The four correlation functions* U, G, R, H *have the following probabilistic definitions:*

- $U(\alpha) = \frac{2}{\gamma}\Gamma(\frac{2(\alpha-Q)}{\gamma}) \left(\mu_\partial^{\frac{2(Q-\alpha)}{\gamma}}\right) \overline{U}(\alpha)$ *where for* $\alpha > \frac{\gamma}{2}$*:*

$$\overline{U}(\alpha) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha)}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}X(x)} dx\right)^{\frac{2(Q-\alpha)}{\gamma}}\right]. \tag{3.1.16}$$

- $G(\alpha, \beta) = \frac{2}{\gamma}\Gamma(\frac{2\alpha+\beta-2Q}{\gamma}) \left(\mu_\partial^{\frac{2Q-2\alpha-\beta}{\gamma}}\right) \overline{G}(\alpha, \beta)$ *where for* $\beta < Q$, $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$*:*

$$\overline{G}(\alpha, \beta) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha-\frac{\beta}{2})}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}X(x)} dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\right]. \tag{3.1.17}$$

- $H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \frac{2}{\gamma}\Gamma(\frac{\beta_1+\beta_2+\beta_3-2Q}{\gamma}) \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ *where in the following range of parameters,*

$$(\mu_i)_{i=1,2,3} \text{ satisfies Definition 3.1.3}, \quad \beta_i < Q, \quad \frac{1}{\gamma}(2Q - \sum_{i=1}^3 \beta_i) < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q - \beta_i), \tag{3.1.18}$$

one can define:

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{8}(\frac{4}{\gamma}-\sum_{i=1}^3 \beta_i)}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^{\frac{1}{\gamma}(2Q-\sum_{i=1}^3 \beta_i)}\right]. \tag{3.1.19}$$

The dependence on the parameters μ_1, μ_2, μ_3 appears through the measure:

$$d\mu(x) = \mu_1 \mathbf{1}_{(-\infty,0)}(x)dx + \mu_2 \mathbf{1}_{(0,1)}(x)dx + \mu_3 \mathbf{1}_{(1,\infty)}(x)dx. \tag{3.1.20}$$

The GMC integral inside the expectation is a complex number avoiding $(-\infty, 0)$. To define its fractional power we choose its argument in $(-\pi, \pi)$.

- *$R(\beta, \mu_1, \mu_2) = -\Gamma(1 - \frac{2(Q-\beta)}{\gamma})\overline{R}(\beta, \mu_1, \mu_2)$, where $\overline{R}(\beta, \mu_1, \mu_2)$ is defined for $\beta \in (\frac{\gamma}{2}, Q)$ and μ_1, μ_2 obeying the constraint of Definition 3.1.3 by the following limiting procedure. Consider $\frac{\gamma}{2} < \beta_2 < \beta < Q$ and $\beta - \beta_2 < \beta_3 < Q$. Then the following limits exists and we set:*

$$\overline{R}(\beta, \mu_1, \mu_2) := \frac{1}{2(Q-\beta)} \lim_{\beta_3 \downarrow \beta - \beta_2} (\beta_2 + \beta_3 - \beta)\overline{H}^{(\beta,\beta_2,\beta_3)}_{(\mu_1,\mu_2,1)}. \tag{3.1.21}$$

The proof of why the above moments of GMC are well-defined under the written parameter ranges has been performed in [53].

3.1.2 Main theorems

In order to state our main results, we need to introduce the following special functions. For all $\gamma \in (0, 2)$ and for $\mathrm{Re}(x) > 0$, $\Gamma_{\frac{\gamma}{2}}(x)$ is defined by the following integral formula:

$$\ln \Gamma_{\frac{\gamma}{2}}(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt} - e^{-\frac{Qt}{2}}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - x)^2}{2}e^{-t} + \frac{x - \frac{Q}{2}}{t}\right]. \tag{3.1.22}$$

Furthermore consider the function $S_{\frac{\gamma}{2}}(x)$ defined for $\gamma \in (0, 2)$ and $\mathrm{Re}(x) \in (0, Q)$ by:

$$S_{\frac{\gamma}{2}}(x) = \frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(Q - x)}. \tag{3.1.23}$$

Both functions $\Gamma_{\frac{\gamma}{2}}(x)$ and $S_{\frac{\gamma}{2}}(x)$ admit meromorphic extensions to all $x \in \mathbb{C}$ with a known pole structure, see Section 3.5.4 for more details. Using these two functions one can define the following special function introduced in [84]. For $i \in \{1, 2, 3\}$, define σ_i through the relation $\mu_i := e^{i\pi\gamma(\sigma_i - \frac{Q}{2})}$ with the convention that for positive μ_i one has $\mathrm{Re}(\sigma_i) = \frac{Q}{2}$. Denote $\overline{\beta} = \beta_1 + \beta_2 + \beta_3$. Then define:

$$\begin{aligned}
&I_{PT}\begin{pmatrix}\beta_1, \beta_2, \beta_3\\ \sigma_1, \sigma_2, \sigma_3\end{pmatrix} \qquad (3.1.24)\\
&= \frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1}(\frac{2}{\gamma})^{(\frac{\gamma}{2}-\frac{2}{\gamma})(Q-\frac{\overline{\beta}}{2})-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}}\Gamma(\frac{\overline{\beta}-2Q}{\gamma})} \frac{\Gamma_{\frac{\gamma}{2}}(2Q-\frac{\overline{\beta}}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\beta_1+\beta_3-\beta_2}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_1+\beta_2-\beta_3}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta_2+\beta_3-\beta_1}{2})}{\Gamma_{\frac{\gamma}{2}}(Q)\Gamma_{\frac{\gamma}{2}}(Q-\beta_1)\Gamma_{\frac{\gamma}{2}}(Q-\beta_2)\Gamma_{\frac{\gamma}{2}}(Q-\beta_3)}\\
&\times \frac{e^{i\frac{\pi}{2}(-(2Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)(Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)+(Q+\frac{\beta_2}{2}-\sigma_2-\sigma_3)(\frac{\beta_2}{2}-\sigma_2-\sigma_3)+(Q+\frac{\beta_3}{2}-\sigma_1-\sigma_3)(\frac{\beta_3}{2}-\sigma_1-\sigma_3)-2\sigma_3(2\sigma_3-Q))}}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1)}\\
&\times \int_{-i\infty}^{i\infty} \frac{S_{\frac{\gamma}{2}}(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)}{S_{\frac{\gamma}{2}}(Q+\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q+r)} e^{i\pi(-\frac{\beta_2}{2}+\sigma_2-\sigma_3)r}\frac{dr}{i}.
\end{aligned}$$

The contour of the integral is to the right of the poles at $r = -(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$, $r = -(\frac{\beta_3}{2}+\sigma_3-\sigma_1)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$, $r = -(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1)-n\frac{\gamma}{2}-m\frac{2}{\gamma}$ and to the left of the poles at $r = -(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)+n\frac{\gamma}{2}+m\frac{2}{\gamma}$, $r = -(Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)+n\frac{\gamma}{2}+m\frac{2}{\gamma}$, $r = n\frac{\gamma}{2}+m\frac{2}{\gamma}$ with $m, n \in \mathbb{N}^2$. We can now state our main results. For the sake of completeness we recall:

Theorem 3.1.5. *(Bulk one-point function, R. 2017 [88]) For $\gamma \in (0,2)$, $\alpha > \frac{\gamma}{2}$, one has:*

$$\overline{U}(\alpha) = \left(\frac{2^{-\frac{\gamma\alpha}{2}} 2\pi}{\Gamma(1-\frac{\gamma^2}{4})} \right)^{\frac{2}{\gamma}(Q-\alpha)} \Gamma(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{4}). \tag{3.1.25}$$

Now the main results of the present work is to provide expressions for the remaining three structure constants. We will indeed prove the following theorems:

Theorem 3.1.6. *(Bulk-boundary correlator) For $\gamma \in (0,2)$, $\beta < Q$, $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$, one has:*

$$\overline{G}(\alpha,\beta) = \left(\frac{2^{\frac{\gamma}{2}(\frac{\beta}{2}-\alpha)} 2\pi}{\Gamma(1-\frac{\gamma^2}{4})} \right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})} \frac{\Gamma(\frac{\gamma\alpha}{2} + \frac{\gamma\beta}{4} - \frac{\gamma^2}{4}) \Gamma_{\frac{\gamma}{2}}(\alpha - \frac{\beta}{2}) \Gamma_{\frac{\gamma}{2}}(\alpha + \frac{\beta}{2}) \Gamma_{\frac{\gamma}{2}}(Q - \frac{\beta}{2})^2}{\Gamma_{\frac{\gamma}{2}}(Q-\beta) \Gamma_{\frac{\gamma}{2}}(\alpha)^2 \Gamma_{\frac{\gamma}{2}}(Q)}. \tag{3.1.26}$$

Theorem 3.1.7. *(Boundary two-point and three-point functions) Consider $\gamma \in (0,2)$, $\beta \in (\frac{\gamma}{2}, Q)$, and μ_1, μ_2 obeying the condition of Definition 3.1.3. Then one has:*

$$\overline{R}(\beta,\mu_1,\mu_2) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\beta)-\frac{1}{2}} (\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\beta)-\frac{1}{2}}}{(Q-\beta)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\beta)}} \frac{\Gamma_{\frac{\gamma}{2}}(\beta - \frac{\gamma}{2}) e^{i\pi(\sigma_1+\sigma_2-Q)(Q-\beta)}}{\Gamma_{\frac{\gamma}{2}}(Q-\beta) S_{\frac{\gamma}{2}}(\frac{\beta}{2} + \sigma_2 - \sigma_1) S_{\frac{\gamma}{2}}(\frac{\beta}{2} + \sigma_1 - \sigma_2)}. \tag{3.1.27}$$

Similarly, for $\beta_1, \beta_2, \beta_3$ and μ_1, μ_2, μ_3 satisfying the set of conditions (3.1.18),

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = I_{PT} \begin{pmatrix} \beta_1, \beta_2, \beta_3 \\ \sigma_1, \sigma_2, \sigma_3 \end{pmatrix}. \tag{3.1.28}$$

Before moving on to the proof of these results we will first explain how the boundary two-point function can be viewed as a reflection coefficient and also present an outline of our proof strategy.

3.1.3 The reflection coefficient

In this section we explain how the boundary two-point function $\overline{R}(\beta,\mu_1,\mu_2)$ can also be seen as a reflection coefficient providing a tail expansion for one-dimensional Gaussian multiplicative chaos measures on the real line $\mathbb{R}$. A more detailed discussion of this phenomenon is provided in [89]. We start by explaining how we can give a direct probabilistic definition to $\overline{R}(\beta,\mu_1,\mu_2)$ without using the limit of (3.1.21). Following [33] we use the standard radial decomposition of the covariance (3.1.9) of X around the point 0, i.e. we write for $s \geq 0$,

$$X(e^{-s/2}) = B_s + Y(e^{-s/2}), \quad X(-e^{-s/2}) = B_s + Y(-e^{-s/2}), \tag{3.1.29}$$

where B_s is a standard Brownian motion and Y is an independent Gaussian process that can be defined on the whole plane with covariance given for $x, y \in \mathbb{C}$ by:

$$\mathbb{E}[Y(x)Y(y)] = 2 \ln \frac{|x| \vee |y|}{|x-y|}. \tag{3.1.30}$$

We introduce for $\lambda > 0$ the process that will be used in the definition below,

$$\mathcal{B}^\lambda_s := \begin{cases} \hat{B}_s - \lambda s & s \geq 0 \\ \bar{B}_{-s} + \lambda s & s < 0, \end{cases} \tag{3.1.31}$$

where $(\hat{B}_s - \lambda s)_{s\geq 0}$ and $(\bar{B}_s - \lambda s)_{s \geq 0}$ are two independent Brownian motions with negative drift conditioned to stay negative. Now for $\beta \in (\frac{\gamma}{2}, Q)$ and μ_1, μ_2 satisfying the constraint of Definition 3.1.3 we can give an alternative definition of $\overline{R}(\beta,\mu_1,\mu_2)$:

$$\overline{R}(\beta,\mu_1,\mu_2) = \mathbb{E}\left[\left(\frac{1}{2} \int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta}{2}}} \left(\mu_2 e^{\frac{\gamma}{2} Y(e^{-s/2})} + \mu_1 e^{\frac{\gamma}{2} Y(-e^{-s/2})} \right) \right)^{\frac{2}{\gamma}(Q-\beta)} \right]. \tag{3.1.32}$$

We now provide a lemma proven in Section 3.5.2.3 that shows that both definitions (3.1.21) and (3.1.32) of $\overline{R}(\beta,\mu_1,\mu_2)$ are equivalent.

Lemma 3.1.8. *Assume that $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$ obey the constraint of Definition 3.1.3. Consider $\frac{\gamma}{2} \vee \beta_2 < \beta_1 < Q$ and $\beta_1 - \beta_2 < \beta_3 < Q$. Taking (3.1.32) as the definition of $\overline{R}(\beta, \mu_1, \mu_2)$ the following limit holds:*

$$\lim_{\beta_3 \downarrow \beta_1 - \beta_2} (\beta_2 + \beta_3 - \beta_1) \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = 2(Q - \beta_1) \overline{R}(\beta_1, \mu_1, \mu_2). \tag{3.1.33}$$

A similar result holds when $\beta_1 = \beta_2$ and $0 < \beta_3 < Q$:

$$\lim_{\beta_3 \downarrow 0} \beta_3 \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = 2(Q - \beta_1) \left(\overline{R}(\beta_1, \mu_1, \mu_2) + \overline{R}(\beta_1, \mu_2, \mu_3) \right). \tag{3.1.34}$$

Let us now state how the value of $\overline{R}(\beta, \mu_1, \mu_2)$ provides a very general first order tail expansion for the probability of a one-dimensional GMC measure to be large. For this discussion we choose $\mu_1, \mu_2 \in [0, \infty)$ with at most one of the two parameters being 0, and we introduce the notation:

$$I_{\eta_1,\eta_2}(\beta) := \int_{-\eta_1}^{\eta_2} \frac{1}{|x|^{\frac{\beta\gamma}{2}}} e^{\frac{\gamma}{2} X(x)} \left(\mu_1 \mathbf{1}_{\{x<0\}} + \mu_2 \mathbf{1}_{\{x>0\}} \right) dx. \tag{3.1.35}$$

In the above $\eta_1, \eta_2 \in (0, 1)$. Now the tail expansion result is the following:

Proposition 3.1.9. *For $\beta \in (\frac{\gamma}{2}, Q)$ and any $\eta_1, \eta_2 \in (0, 1)$, we have the following tail expansion for $I_{\eta_1,\eta_2}(\beta)$ as $u \to \infty$ and for some $\nu > 0$:*

$$\mathbb{P}(I_{\eta_1,\eta_2}(\beta) > u) = \frac{\overline{R}(\beta, \mu_1, \mu_2)}{u^{\frac{2}{\gamma}(Q-\beta)}} + O(\frac{1}{u^{\frac{2}{\gamma}(Q-\beta)+\nu}}). \tag{3.1.36}$$

The proof of this proposition follows exactly the same steps as for the case of $\mu_1 = 0, \mu_2 > 0$ considered in [89]. Notice that we impose the condition $\beta \in (\frac{\gamma}{2}, Q)$. This is crucial for the tail behavior of $I_{\eta_1,\eta_2}(\beta)$ to be dominated by the insertion and this is precisely why the asymptotic expansion is independent of the choice of η_1 and η_2. It also explains why the radial decomposition (3.1.29) is natural as it is well suited to study X around a particular point. If one is interested in the case where $\beta < \frac{\gamma}{2}$ (or simply $\beta = 0$), a different argument known as the localization trick is required to obtain the tail expansion, see [94] for more details.

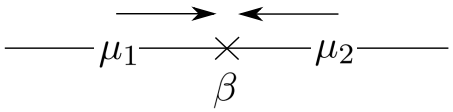

Figure 3.2: $\overline{R}(\beta, \mu_1, \mu_2)$

The above picture summarizes what the reflection coefficient computes. In the range $\beta \in (\frac{\gamma}{2}, Q)$, the tail expansion of the GMC is dominated by the insertion. The parameters μ_1, μ_2 tune the weights of both sides as we approach the insertion. For more details and results on tail expansions of GMC measures with the reflection coefficients see the works [63, 94, 107].

3.1.4 Outline of the proof

We summarize here the main steps of the proof and the intermediate results that will lead us to Theorems 3.1.6 and 3.1.7. Our proof strategy follows closely the one of the previous works [60, 88, 89] but there are many novel difficulties that must be resolved due to the fact that we are forced to work with complex valued quantities (instead of positive as in the cited works).

- **BPZ differential equations.** Since LCFT is a conformal field theory, correlation functions containing a field with a degenerate insertion are predicted to obey a differential equation known as the BPZ equation. Therefore if one considers a correlation function where one of the boundary insertion points has a weight

$\beta = -\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$, then the whole correlation will obey the BPZ equation.[2] More precisely, for $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$ and $t \in \mathbb{H}$, we will consider the following observables,

$$G_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}} g(x)^{\frac{\gamma}{8}(p-1)} e^{\frac{\gamma}{2}X(x)} dx\right)^p\right] \text{ where } p = \frac{2}{\gamma}(Q - \alpha - \frac{\beta}{2} + \frac{\chi}{2}),$$

$$H_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu_\chi(x)\right)^q\right] \text{ where } q = \frac{1}{\gamma}(2Q - \beta_1 - \beta_2 - \beta_3 + \chi).$$

The functions $G_\chi(t)$ and $H_\chi(t)$ will be used respectively to prove Theorem 3.1.6 and Theorem 3.1.7. In Section 3.4 we show that $H_\chi(t)$ obeys a hypergeometric equation and similarly for $G_\chi(t)$ after an extra change of variable. It is then possible to write down explicitly a solution space, writing it here to illustrate the discussion for $H_\chi(t)$,

$$\begin{aligned} H_\chi(t) &= C_1 F(A,B,C,t) + C_2 t^{1-C} F(1+A-C, 1+B-C, 2-C, t) \\ &= B_1 F(A,B,1+A+B-C,1-t) + B_2(1-t)^{C-A-B} F(C-A, C-B, 1+C-A-B, 1-t), \end{aligned}$$

where A, B, C are known parameters depending on $\gamma, \beta_1, \beta_2, \beta_3$ and the C_1, C_2, B_1, B_2 are parameters that parametrize the solution space of the hypergeometric equation. These last four parameters are unknown at this stage of the proof.

- **Operator product expansion (OPE).** The next step is to perform an asymptotic analysis directly on the probabilistic definition of $H_\chi(t)$ (and similarly for $G_\chi(t)$) to identify the constants C_1, C_2, B_1, B_2 in terms $\overline{H}$, the quantity we are interested in computing. For instance by sending t to 0, one immediately obtains the result,

$$C_1 = \overline{H}^{(\beta_1 - \frac{\gamma}{2}, \beta_2, \beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.1.37}$$

In the case where $\chi = \frac{\gamma}{2}$ and for a suitable range of β_i in which $\beta_1 \in (\frac{\gamma}{2}, \frac{2}{\gamma})$, one can obtain by a straightforward analysis of a real integral on $\mathbb{R}$ that:

$$C_2 = q\frac{\Gamma(-1 + \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right)\overline{H}^{(\beta_1+\frac{\gamma}{2}, \beta_2, \beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.1.38}$$

- **OPE with reflection.** The method described above only works for the first degenerate weight $\chi = \frac{\gamma}{2}$, and only in a very specific domain of parameters. In the case of $\chi = \frac{2}{\gamma}$, or for $\chi = \frac{\gamma}{2}$ but with β_1 chosen close to Q, the asymptotic analysis required to identified C_2 will be much more involved. It is called the OPE with reflection as the boundary two-point function - also called the reflection coefficient - will always appear in the answer. Carrying this out one finds the answer:

$$C_2 = \frac{2(Q-\beta_1)}{\gamma}\frac{\Gamma(\frac{2}{\gamma}(\beta_1 - Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1) - q)}{\Gamma(-q)}\overline{R}(\beta_1, \mu_1, \mu_2)\overline{H}^{(2Q-\beta_1-\frac{\gamma}{2}, \beta_2, \beta_3)}_{(\mu_1, e^{i\pi\frac{\gamma^2}{4}}\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.1.39}$$

This phenomenon was known to physicists and its probabilistic description is one of the major achievements of [61].

- **Shift equations and analytic continuation.** Once we have derived expressions for the coefficients C_1, C_2, B_1, B_2, the theory of hypergeometric equation will imply a non trivial relation on our quantity of interest. For instance one has the following relation between C_1, C_2, B_1 in the case of the hypergeometric equation satisfied by the function $H_\chi(t)$:

$$B_1 = \frac{\Gamma(\chi(\beta_1 - \chi))\Gamma(1 - \chi\beta_2 + \chi^2)}{\Gamma(\chi(\beta_1 - \chi + q\frac{\gamma}{2})\Gamma(1 - \chi\beta_2 + \chi^2 - q\frac{\gamma\chi}{2})}C_1 + \frac{\Gamma(2 - \chi\beta_1 + \chi^2)\Gamma(1 - \chi\beta_2 + \chi^2)}{\Gamma(1 + \frac{q\gamma\chi}{2})\Gamma(2 - \chi(\beta_1 + \beta_2 - 2\chi + q\frac{\gamma}{2}))}C_2. \tag{3.1.40}$$

[2] It is also possible to consider degenerate insertions in the bulk but they will not be used in the present paper.

These equations will then translate to functional equations on $\overline{G}$ and $\overline{H}$ that we will refer to as shift equations because they will involve our functions of interest at shifted values of the insertion weight, the shift being $\pm\chi$ for $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. A key observation is that the shift equation obtained for $\chi = \frac{\gamma}{2}$ allows to analytically continue our probabilistic definitions of $\overline{H}$ and $\overline{G}$ to meromorphic functions defined in a complex neighborhood of the real line. The procedure is analogue to the well-known example of the Gamma function where the functional equation $\Gamma(x+1) = x\Gamma(x)$ can be used to extend the gamma function to a meromorphic function of $\mathbb{C}$ with prescribed poles. In our case the poles will also be prescribed by the shift equations. Once established analytic continuation will then be used to derive a second shift equation corresponding to $\chi = \frac{2}{\gamma}$.

- **Shift equations imply the result.** The final step is simply to check that the two shift equations obtained for a specific correlation function completely specify its value. Let us explain this for $\overline{G}(\alpha,\beta)$. Assume $\gamma^2 \notin \mathbb{Q}$. The shift equations imply a relation between the correlation at β and $\beta+\gamma$ and between the correlation at β and $\beta+\frac{4}{\gamma}$. Since the ratio of the two periods is not in $\mathbb{Q}$, the shift equations uniquely specify the function up to the knowledge of one value which can be taken to be when $\beta = 0$. One then has $\overline{G}(\alpha,0) = \overline{U}(\alpha)$ which is known from the previous work [88]. By using the special functions $\Gamma_{\frac{\gamma}{2}}$, $S_{\frac{\gamma}{2}}$ introduced in appendix, it is also possible to explicitly construct an analytic function satisfying the same shift equations. Therefore the correlation function must be equal to this analytic function, and we can extend the result to the case where $\gamma^2 \in \mathbb{Q}$ by continuity in γ.

Although the above proof strategy follows the same lines as the previous works [60, 88, 89], there are additional technical difficulties that arise because of the presence of complex valued quantities included with the GMC measures. Performing OPE in this case will require some care and extra estimates and is the purpose of Lemmas 3.5.4 and 3.5.5.

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3.2 The bulk-boundary correlator

In this section we will prove Theorem 3.1.6. To compute our quantity of interest $\overline{G}(\alpha,\beta)$ we will show it obeys two functional equations that will completely specify its value. We thus need to show:

Proposition 3.2.1. *(Shift equations for $\overline{G}(\alpha,\beta)$) For every fixed $\alpha > Q$, the function $\beta \to \overline{G}(\alpha,\beta)$ originally defined for $\beta \in (\gamma - 2\alpha, Q)$ admits a meromorphic extension in a complex neighborhood of the real line and this extension satisfies the following two equations,*

$$\overline{G}(\alpha,\beta+\gamma) = \frac{\Gamma(1-\frac{\gamma^2}{4})}{2^{\frac{\gamma\beta}{2}}\pi}\frac{\Gamma(\frac{\gamma\alpha}{2}-\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta}{4})^2}{\Gamma(\frac{\gamma\alpha}{2}+\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta}{2})\Gamma(1-\frac{\gamma\beta}{2}-\frac{\gamma^2}{4})}\overline{G}(\alpha,\beta), \tag{3.2.1}$$

$$\overline{G}(\alpha,\beta+\frac{4}{\gamma}) = \frac{\gamma^2\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}{2^{\frac{2\beta}{\gamma}+1}(2\pi)^{\frac{4}{\gamma^2}}}\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{\beta}{\gamma})^2}{\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}\overline{G}(\alpha,\beta), \tag{3.2.2}$$

viewed as equalities of meromorphic functions.

Using Proposition 3.2.1 and the fact that $\overline{U}(\alpha)$ is known from the previous work [88], it is easy to prove the value of $\overline{G}(\alpha,\beta)$.

Proof of Theorem 3.1.6. The two shift equations of Proposition 3.2.1 completely specify the dependence in β of $\overline{G}(\alpha,\beta)$ up to a real constant depending only on α. Since the value $\overline{G}(\alpha,0)$ is given by $\overline{U}(\alpha)$, we can write:

$$\overline{G}(\alpha,\beta) = \left(\frac{2^{\frac{\gamma}{2}(-\alpha+\frac{\beta}{2})}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\frac{\Gamma(\frac{\gamma\alpha}{2}+\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma_{\frac{\gamma}{2}}(\alpha-\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(\alpha+\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2})^2}{\Gamma_{\frac{\gamma}{2}}(Q-\beta)\Gamma_{\frac{\gamma}{2}}(\alpha)^2\Gamma_{\frac{\gamma}{2}}(Q)}. \tag{3.2.3}$$

□

To show Proposition 3.2.1, we will use the solvability coming from the BPZ equations of Liouville theory. For $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$, we denote:

$$p = \frac{2}{\gamma}(Q - \alpha - \frac{\beta}{2} + \frac{\chi}{2}). \tag{3.2.4}$$

We now introduce two auxiliary functions corresponding to the two values of $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$ and for $t \in \mathbb{H}$:

$$G_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}} g(x)^{\frac{\gamma}{8}(p-1)} e^{\frac{\gamma}{2}X(x)} dx\right)^p\right]. \tag{3.2.5}$$

The parameter range where $G_\chi(t)$ is well-defined is:

$$\beta < Q, \quad \text{and} \quad p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\beta). \tag{3.2.6}$$

Let us justify why $G_\chi(t)$ is well-defined under these conditions. First for $\chi = \frac{\gamma}{2}$ since $(t-x)$ is always contained in the upper half plane we can define $(t-x)^{\frac{\gamma^2}{4}}$ by choosing the argument to be in $[0,\pi]$. This means for $t \in \mathbb{H}$ and for either value of χ, the GMC integral

$$\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}} g(x)^{\frac{\gamma}{8}(p-1)} e^{\frac{\gamma}{2}X(x)} dx \tag{3.2.7}$$

is a random complex number almost surely contained in $\mathbb{H}$. We can thus define its p power again by choosing an argument in $[0,\pi]$. Finally we must argue why the moment itself is finite. Since $(t-x)^{\frac{\gamma\chi}{2}}$ is strictly contained in $\mathbb{H}$, there can be no cancellation of the imaginary part and thus the condition of existence of the moment of this GMC is equivalent to the positive case which gives the condition we have written on p.

Assume $t \in \{re^{i\theta} \mid r > 0,\ \theta \in (0, \frac{\pi}{2})\}$ and perform the change of variable $s = \frac{1}{1+t^2}$. The variable s then belongs to the set $s \in -\mathbb{H}$. We choose the argument of s to be in $(-\pi, 0)$ and define $\sqrt{1-s} = t\sqrt{s}$. Now set:

$$\tilde{G}_\chi(s) = s^{p\frac{\gamma\chi}{4}} G_\chi(t). \tag{3.2.8}$$

Then one has,

$$\tilde{G}_\chi(s) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(\sqrt{1-s} - \sqrt{s}x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}X(x)} g(x)^{\frac{\gamma^2}{8}(p-1)} dx\right)^p\right], \tag{3.2.9}$$

where the argument of the GMC integral can be chosen in $(-\pi, \pi)$. We will introduce a dual set of auxiliary functions corresponding to, for $t \in \{re^{i\theta} \mid r > 0,\ \theta \in (-\frac{\pi}{2}, 0)\}$, $s = \frac{1}{1+t^2}$ with argument this time in $(0,\pi)$, $\sqrt{1-s} = t\sqrt{s}$, and:

$$\hat{G}_\chi(s) = s^{p\frac{\gamma\chi}{4}} G_\chi(-t). \tag{3.2.10}$$

One lands on the expression,

$$\hat{G}_\chi(s) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(-\sqrt{1-s} - \sqrt{s}x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}X(x)} g(x)^{\frac{\gamma^2}{8}(p-1)} dx\right)^p\right]. \tag{3.2.11}$$

The above GMC integral in the expectation avoids the cut $(0,\infty)$ and its argument is chosen to be in $(0, 2\pi)$. We prove in Section 3.4.1 that $\tilde{G}_\chi(s)$ obeys the following hypergeometric equation,

$$s(1-s)\partial_s^2 \tilde{G}_\chi(s) + (C - (A+B+1)s)\partial_s \tilde{G}_\chi(s) - AB\tilde{G}_\chi(s) = 0, \tag{3.2.12}$$

with parameters given by:

$$A = -p\frac{\gamma\chi}{4}, B = 1 + \chi(\chi - \alpha - p\frac{\gamma}{4}), C = \frac{3}{2} + \chi(\chi - \alpha - p\frac{\gamma}{2}). \tag{3.2.13}$$

The exact same equation also holds for $\hat{G}_\chi(s)$. As detailed in Section 3.5.4, one can explicitly write the solution space of the equation around $s=0$ and $s=1$, under the assumption that C and $C-A-B$ are not integers:[3]

$$\begin{aligned}\tilde{G}_\chi(s) &= \tilde{C}_1 F(A,B,C,s) + \tilde{C}_2 s^{1-C} F(1+A-C,1+B-C,2-C,s) \\ &= \tilde{B}_1 F(A,B,1+A+B-C,1-s) + \tilde{B}_2 (1-s)^{C-A-B} F(C-A,C-B,1+C-A-B,1-s),\end{aligned} \tag{3.2.14}$$

$$\begin{aligned}\hat{G}_\chi(s) &= \hat{C}_1 F(A,B,C,s) + \hat{C}_2 s^{1-C} F(1+A-C,1+B-C,2-C,s) \\ &= \hat{B}_1 F(A,B,1+A+B-C,1-s) + \hat{B}_2 (1-s)^{C-A-B} F(C-A,C-B,1+C-A-B,1-s).\end{aligned} \tag{3.2.15}$$

Here $\tilde{C}_1, \tilde{C}_2, \tilde{B}_1, \tilde{B}_2, \hat{C}_1, \hat{C}_2, \hat{B}_1, \hat{B}_2$ are all real constants that parametrize the different basis of solutions. Since the solution space is two-dimensional, there is a change of basis formula (3.5.105) that relates $\tilde{C}_1, \tilde{C}_2$ with $\tilde{B}_1, \tilde{B}_2$ and similarly for $\hat{C}_1, \hat{C}_2, \hat{B}_1, \hat{B}_2$. In the following we will relate several of these coefficients to $\overline{G}$ and it is precisely the change of basis that will lead us to the shift equations of Proposition 3.2.1.

3.2.1 First shift equation

In this section we prove the first shift equation (3.2.1) in a restricted range of parameters where $\overline{G}(\alpha,\beta)$ is well-defined probabilistically without analytic continuation.

Lemma 3.2.2. *For α, β satisfying $\beta < \frac{2}{\gamma} - \frac{\gamma}{2}$ and $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha - \frac{\gamma}{2}$, the following equation holds:*

$$\overline{G}(\alpha,\beta+\gamma) = \frac{1}{2^{\frac{\gamma\beta}{2}}\pi} \frac{\Gamma(1-\frac{\gamma^2}{4})\Gamma(\frac{\gamma\alpha}{2}-\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta}{4})^2}{\Gamma(\frac{\gamma\alpha}{2}+\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta}{2})\Gamma(1-\frac{\gamma\beta}{2}-\frac{\gamma^2}{4})} \overline{G}(\alpha,\beta). \tag{3.2.16}$$

Proof. We start off with the following parameter choices:

$$\chi = \frac{\gamma}{2}, \quad \alpha > Q, \quad \frac{\gamma}{2} - \frac{2}{\gamma} < \beta < \frac{2}{\gamma}. \tag{3.2.17}$$

In the case of $\chi = \frac{\gamma}{2}$ we can actually assume $t \in (0,+\infty)$ which means that $s \in (0,1)$. By sending s to 0 one automatically gets that:

$$\tilde{C}_1 = \overline{G}(\alpha,\beta-\frac{\gamma}{2}), \quad \hat{C}_1 = e^{i\pi p \frac{\gamma^2}{4}} \overline{G}(\alpha,\beta-\frac{\gamma}{2}). \tag{3.2.18}$$

Although we cannot express $\tilde{B}_1$ and $\hat{B}_1$ in terms of the bulk-boundary correlator $\overline{G}$, by setting $s=1$ one obtains the equality:

$$\tilde{B}_1 = \hat{B}_1. \tag{3.2.19}$$

In order to derive an expression for $\tilde{C}_2$, we have to expand $\tilde{G}_{\frac{\gamma}{2}}(s)$ up to the order s^{1-C}. In this case $C = \frac{1}{2} - \frac{\gamma^2}{8} + \frac{\gamma\beta}{4}$. The parameter choice (3.2.17) implies that $0 < 1-C < 1$. Thus we have to get the leading asymptotic of the difference $\tilde{G}_{\frac{\gamma}{2}}(s) - \tilde{G}_{\frac{\gamma}{2}}(0)$ as $s \to 0$. Following the analysis of [89] and applying Theorem

[3] The values excluded here are recovered by an easy continuity argument.

3.5.1 we obtain:

$$\begin{aligned}
&\tilde{G}_{\frac{\gamma}{2}}(s)-\tilde{G}_{\frac{\gamma}{2}}(0) \qquad (3.2.20)\\
&=p\int_{\mathbb{R}}dx_1\frac{(\sqrt{1-s}-\sqrt{s}x_1)^{\frac{\gamma^2}{4}}-1}{|x_1-i|^{\gamma\alpha}}\mathbb{E}\left[e^{\frac{\gamma}{2}X(x_1)}\left(\int_{\mathbb{R}}\frac{g(x)^{\frac{\gamma^2}{8}(p-2)}}{|x-i|^{\gamma\alpha}}e^{\frac{\gamma}{2}X(x)}dx\right)^{p-1}\right]+o(s^{1-C})\\
&=p\int_{\mathbb{R}}dx_1\frac{(\sqrt{1-s}-\sqrt{s}x_1)^{\frac{\gamma^2}{4}}-1}{|x_1-i|^{\gamma\alpha}}\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{g(x)^{\frac{\gamma^2}{8}(p-2)}}{|x-i|^{\gamma\alpha}|x-x_1|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx\right)^{p-1}\right]+o(s^{1-C})\\
&\overset{x_1=\frac{u}{\sqrt{s}}}{=}p\int_{\mathbb{R}}dx_1\frac{(\sqrt{1-s}-u)^{\frac{\gamma^2}{4}}-1}{|\frac{u}{\sqrt{s}}-i|^{\gamma\alpha}}\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{g(x)^{\frac{\gamma^2}{8}(p-2)}}{|x-i|^{\gamma\alpha}|x-\frac{u}{\sqrt{s}}|^{\frac{\gamma^2}{2}}}e^{\frac{\gamma}{2}X(x)}dx\right)^{p-1}\right]+o(s^{1-C})\\
&=s^{1-C}p\left(\int_{\mathbb{R}}du\frac{(1-u)^{\frac{\gamma^2}{4}}-1}{|u|^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}\right)\overline{G}(\alpha,\beta+\frac{\gamma}{2})+o(s^{1-C}).
\end{aligned}$$

The way to obtain the bound in $o(s^{1-C})$ on the error terms is the same as [89]. The correct way to interpret the above integral over $\mathbb{R}$ is by writing,

$$\int_{\mathbb{R}}du\frac{(1-u)^{\frac{\gamma^2}{4}}-1}{|u|^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}=\int_{\mathbb{R}_+}du\frac{(1+u)^{\frac{\gamma^2}{4}}-1}{u^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}-e^{i\pi(\gamma\alpha+(p-1)\frac{\gamma^2}{2})}\int_{\mathbb{R}_+e^{i\pi}}du\frac{(1+u)^{\frac{\gamma^2}{4}}-1}{u^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}, \qquad (3.2.21)$$

where here $\mathbb{R}_+e^{i\pi}$ means that the integral should be understood as a contour integral on $(-\infty,0)$ passing just above the point $u=-1$. Notice also this integral converges because of the condition $\beta<\frac{2}{\gamma}$. In Section 3.5.4.3 we give the exact value of this integral in terms of the gamma function (3.5.112). Putting everything together we have shown,

$$\begin{aligned}
&\tilde{G}_{\frac{\gamma}{2}}(s)-\tilde{G}_{\frac{\gamma}{2}}(0)=s^{1-C}p\frac{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}(1-e^{i\pi(3-2C)})\overline{G}(\alpha,\beta+\frac{\gamma}{2})+o(s^{1-C})\\
&\Rightarrow\ \tilde{C}_2=p\frac{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}(1-e^{i\pi(3-2C)})\overline{G}(\alpha,\beta+\frac{\gamma}{2}). \qquad (3.2.22)
\end{aligned}$$

$\hat{C}_2$ can be calculated in a similar manner, using this time:

$$\begin{aligned}
\int_{\mathbb{R}}du\frac{(-1-u)^{\frac{\gamma^2}{4}}-(-1)^{\frac{\gamma^2}{4}}}{|u|^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}&=e^{i\pi\frac{\gamma^2}{4}}\left(\int_{\mathbb{R}_+}du\frac{(1+u)^{\frac{\gamma^2}{4}}-1}{u^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}-e^{-i\pi(\gamma\alpha+(p-1)\frac{\gamma^2}{2})}\int_{\mathbb{R}_+e^{-i\pi}}du\frac{(1+u)^{\frac{\gamma^2}{4}}-1}{u^{\gamma\alpha+(p-1)\frac{\gamma^2}{2}}}\right)\\
&=p\frac{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}e^{i\pi p\frac{\gamma^2}{4}}(1-e^{-i\pi(3-2C)}).
\end{aligned}$$

Hence:

$$\hat{C}_2=p\frac{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}e^{i\pi p\frac{\gamma^2}{4}}(1-e^{-i\pi(3-2C)})\overline{G}(\alpha,\beta+\frac{\gamma}{2}). \qquad (3.2.23)$$

Now we have the connection formula (3.5.105) expressing $\tilde{B}_1$ in terms of $\tilde{C}_1,\tilde{C}_2$ and similarly for $\hat{B}_1,\hat{C}_1,\hat{C}_2$.

Using the fact that $\tilde{B}_1=\hat{B}_1$ and our expressions for $\tilde{C}_1,\tilde{C}_2,\hat{C}_1,\hat{C}_2$ in terms of $\overline{G}$ we deduce:

$$\begin{aligned}
&\frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}\overline{G}(\alpha,\beta-\frac{\gamma}{2})(1-e^{-i2\pi A})\\
&+p\frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)}\frac{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}(1-e^{i\pi(3-2C)})(1+e^{i\pi(2B-2)})\overline{G}(\alpha,\beta+\frac{\gamma}{2})=0.
\end{aligned}$$

We thus land on the following shift equation, which is simplified using (3.5.98):

$$\begin{aligned}\frac{\overline{G}(\alpha,\beta+\frac{\gamma}{2})}{\overline{G}(\alpha,\beta-\frac{\gamma}{2})} &= \frac{\Gamma(C)\Gamma(1-A)\Gamma(1-B)}{p\Gamma(2-C)\Gamma(C-A)\Gamma(C-B)}\frac{\Gamma(-\frac{\gamma^2}{4})}{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}\frac{\sin(\pi A)}{2\cos(\pi C)\cos(\pi B)}\\ &= \frac{\Gamma(C)\Gamma(1-A)\Gamma(1-B)}{p\Gamma(2-C)\Gamma(\frac{1}{2}+A)\Gamma(\frac{1}{2}+B)}\frac{\Gamma(-\frac{\gamma^2}{4})}{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}\frac{\sin(\pi C)\sin(\pi A)}{\sin(2\pi C)\sin(\pi(\frac{1}{2}+B))}\\ &= \frac{\Gamma(\frac{1}{2}-B)\Gamma(1-B)}{p\Gamma(1-C)\Gamma(2-C)\Gamma(A)\Gamma(\frac{1}{2}+A)}\frac{\Gamma(3-2C)\Gamma(-\frac{\gamma^2}{4})}{\Gamma(2-2C-\frac{\gamma^2}{4})}\\ &= \frac{\Gamma(1-\frac{\gamma^2}{4})\Gamma(1-2B)\Gamma(\frac{3}{2}-C)^2}{\Gamma(1+2A)\Gamma(2-2C)\Gamma(2-2C-\frac{\gamma^2}{4})}\frac{1}{2^{2C-1}\pi}\\ &= \frac{\Gamma(1-\frac{\gamma^2}{4})\Gamma(\frac{\gamma\alpha}{2}-\frac{\gamma\beta}{4}-\frac{\gamma^2}{8})\Gamma(1-\frac{\gamma\beta}{4}+\frac{\gamma^2}{8})^2}{\Gamma(\frac{\gamma\alpha}{2}+\frac{\gamma\beta}{4}-\frac{3\gamma^2}{8})\Gamma(1-\frac{\gamma\beta}{2}+\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta}{2})}\frac{1}{2^{\frac{\gamma\beta}{2}-\frac{\gamma^2}{4}}\pi}.\end{aligned}$$

Then by replacing β by $\beta+\frac{\gamma}{2}$ one lands on the equation of Lemma 3.2.2. To extend it to the wider range of validity in α and β one uses the analycity of $\overline{G}$ with respect to these parameters shown in Lemma 3.5.6. □

One consequence of Lemma 3.2.2 is that it allows to analytically continue $\overline{G}$ as a meromorphic function defined in a complex neighborhood of the real line.

Lemma 3.2.3. *Fix $\alpha > Q$. The function $\beta \mapsto \overline{G}(\alpha,\beta)$ originally defined for $\beta < Q$, $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$ admits a meromorphic extension in a complex neighborhood of the real line.*

Proof. Lemma 3.5.6 shows that $\overline{G}(\alpha,\beta)$ is complex analytic in a complex neighborhood of the real line where it is defined probabilistically. The shift equation of Lemma 3.2.2 then shows $\beta \mapsto \overline{G}(\alpha,\beta)$ can be meromorphically continued to a complex neighborhood of the whole real line, the pole structure being prescribed by the Gamma functions in the shift equation. □

3.2.2 Second shift equation

We will now derive an expression of $\tilde{C}_2$ in a different manner corresponding to the so-called operator product expansion (OPE) with reflection. This computation will be valid for $\frac{2}{\gamma} < \beta < Q$. For $\chi = \frac{\gamma}{2}$ this will give us the reflection principle and for $\chi = \frac{2}{\gamma}$ it will allow us to obtain the second shift equation on β. A complete proof of the following steps can be found in [61]. We first perform a change of variable $x \to \frac{1}{x}$ on the expression of $\tilde{G}_\chi(s)$:

$$\tilde{G}_\chi(s) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{\left(\sqrt{1-s}\,x-\sqrt{s}\right)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}|x|^{\frac{\gamma\beta}{2}}} e^{\frac{\gamma}{2}X(x)} g(x)^{\frac{\gamma^2}{8}(p-1)}dx\right)^p\right]. \tag{3.2.24}$$

Note that $C = \frac{1}{2} - \frac{\chi^2}{2} + \frac{\chi\beta}{2}$. For all $\beta \in (Q-\beta_0, Q)$ where β_0 is a small positive number, the following asymptotic is then shown in Lemma 3.5.4 for the case $\chi = \frac{2}{\gamma}$ and in Lemma 3.5.5 for the case of $\chi = \frac{\gamma}{2}$:

$$\tilde{G}_\chi(s) - \tilde{G}_\chi(0) = -s^{1-C}\frac{\Gamma(1-\frac{2(Q-\beta)}{\gamma})\Gamma(-p+\frac{2}{\gamma}(Q-\beta))}{\Gamma(-p)}\overline{R}(\beta,1,e^{i\pi\frac{\gamma\chi}{2}})\overline{G}(\alpha,2Q-\beta-\chi)+o(s^{1-C}). \tag{3.2.25}$$

In the above s is chosen in $(0,1)$ for $\chi = \frac{\gamma}{2}$ and in $(-1,0)$ for $\chi = \frac{2}{\gamma}$. From this we can deduce the expression of $\tilde{C}_2$, still for $\beta \in (Q-\beta_0, Q)$,

$$\tilde{C}_2 = -\frac{\Gamma(1-\frac{2(Q-\beta)}{\gamma})\Gamma(-p+\frac{2}{\gamma}(Q-\beta))}{\Gamma(-p)}\overline{R}(\beta,1,e^{i\pi\frac{\gamma\chi}{2}})\overline{G}(\alpha,2Q-\beta-\chi). \tag{3.2.26}$$

The range of validity of the above expression can be extended from $\beta \in (Q-\beta_0, Q)$ to the range $\beta \in (\frac{2}{\gamma}, Q)$ by using analycity in the parameter β. Indeed, Lemma 3.5.6 implies the analycity in β in a complex neighborhood of $(\frac{2}{\gamma}, Q)$ of both $\tilde{G}_\chi(s)$ and $\overline{G}(\alpha, 2Q-\beta-\chi)$. The analycity of $\tilde{G}_\chi(s)$ then implies the analycity of $\tilde{C}_2$ and the analycity of $\overline{R}(\beta, 1, e^{i\pi\frac{\gamma\chi}{2}})$ is known from the exact formula for $\overline{R}$ proved in Section 3.3. Thus we extend the equality to $\beta \in (\frac{2}{\gamma}, Q)$. From this we can deduce:

Lemma 3.2.4 (Reflection principle for $\overline{G}(\alpha,\beta)$)**.** *We can analytically continue the definition of $\overline{G}(\alpha,\beta)$ in β beyond the point $\beta = Q$ by the following formula, valid for α, β satisfying $\beta \in (\frac{\gamma}{2}, Q)$ and $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$:*

$$\overline{G}(\alpha,\beta) = -\frac{\Gamma(\frac{2\beta}{\gamma} - \frac{4}{\gamma^2})\Gamma(\frac{2\alpha}{\gamma} - \frac{\beta}{\gamma})}{\Gamma(-1 + \frac{2\alpha}{\gamma} + \frac{\beta}{\gamma} - \frac{4}{\gamma^2})}\overline{R}(\beta,1,1)\overline{G}(\alpha, 2Q-\beta). \tag{3.2.27}$$

Proof. We work with $\chi = \frac{\gamma}{2}$. We have seen two ways of calculating $\tilde{C}_2$ based on the value of β:

$$\tilde{C}_2 = \begin{cases} p\frac{\Gamma(2C-2)\Gamma(2-2C-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}(1-e^{i\pi(3-2C)})\overline{G}(\alpha,\beta+\frac{\gamma}{2}), & \beta < \frac{2}{\gamma}, \\ -\frac{\Gamma(1-\frac{2(Q-\beta)}{\gamma})\Gamma(-p+\frac{2}{\gamma}(Q-\beta))}{\Gamma(-p)}\overline{R}(\beta,1,e^{i\pi\frac{\gamma^2}{4}})\overline{G}(\alpha,2Q-\beta-\frac{\gamma}{2}), & \frac{2}{\gamma} < \beta < Q. \end{cases} \tag{3.2.28}$$

Since $\tilde{G}_\chi(s)$ is complex analytic in β in a complex neighborhood of $\frac{2}{\gamma}$, this implies the analyticity of $\tilde{C}_2$ around $\beta = \frac{2}{\gamma}$. This implies that there is an equality between the two expressions for $\tilde{C}_2$ viewed as meromorphic functions of β in a neighborhood of $\frac{2}{\gamma}$. Lastly from equation (3.3.22) of Section 3.3 we have a shift equation that relates $\overline{R}(\beta+\frac{\gamma}{2},1,1)$ and $\overline{R}(\beta,1,e^{i\pi\frac{\gamma^2}{4}})$. Therefore we can rewrite the relation in the desired way claimed in the lemma. □

With both Lemma 3.2.3 and Lemma 3.2.4 we can now finish the proof of Proposition 3.2.1.

Proof of Proposition 3.2.1. We switch to $\chi = \frac{2}{\gamma}$ to deduce the second shift equation. Using again the known shift equations for $\overline{R}$ we first write:

$$\frac{\overline{R}(\beta,1,e^{i\pi})}{\overline{R}(\beta+\frac{2}{\gamma},1,1)} = -\frac{2}{\gamma(Q-\beta)}(2\pi)^{\frac{4}{\gamma^2}-1}\frac{\Gamma(\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma})}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(1-e^{-i\pi(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}). \tag{3.2.29}$$

By applying Lemma 3.2.4 with $\beta + \frac{2}{\gamma}$, we obtain:

$$\begin{aligned}\tilde{C}_2 &= \frac{\frac{4}{\gamma^2}(2\pi)^{\frac{4}{\gamma^2}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}\frac{\Gamma(-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{2}{\gamma^2})}{\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{6}{\gamma^2})}(1-e^{-i\pi(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})})\overline{G}(\alpha,\beta+\frac{2}{\gamma}), \\ &= \frac{\frac{4}{\gamma^2}(2\pi)^{\frac{4}{\gamma^2}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}\frac{\Gamma(2C-2)\Gamma(1-\frac{2\beta}{\gamma})\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{2}{\gamma^2})}{\Gamma(2A)}(1+e^{i\pi\frac{2}{\gamma}(Q-\beta)})\overline{G}(\alpha,\beta+\frac{2}{\gamma}).\end{aligned}$$

We can find easily the other coefficients:

$$\tilde{C}_1 = \overline{G}(\alpha,\beta-\frac{2}{\gamma}), \quad \hat{C}_1 = e^{i\pi p}\overline{G}(\alpha,\beta-\frac{2}{\gamma}), \quad \hat{C}_2 = e^{i\pi(p-\frac{2}{\gamma}(Q-\beta))}\tilde{C}_2, \quad \tilde{B}_1 = \hat{B}_1.$$

As in the previous subsection we can thus write:

$$\frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}\tilde{C}_1(1-e^{i\pi p}) + \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)}\tilde{C}_2(1-e^{i\pi(p-\frac{2}{\gamma}(Q-\beta))}) = 0.$$

Then we can deduce the shift equation,

$$\begin{aligned}\frac{\overline{G}(\alpha,\beta+\frac{2}{\gamma})}{\overline{G}(\alpha,\beta-\frac{2}{\gamma})} &= \frac{\frac{\gamma^2}{4}(2\pi)^{1-\frac{4}{\gamma^2}}\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}{\Gamma(2C-2)\Gamma(1-\frac{2\beta}{\gamma})\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{2}{\gamma^2})}\frac{\Gamma(2A)\Gamma(C)\Gamma(1-A)\Gamma(1-B)}{\Gamma(2-C)\Gamma(A+\frac{1}{2})\Gamma(B+\frac{1}{2})}\\ &\quad\times\frac{\sin(\pi A)}{2\sin(\pi(B+\frac{1}{2}))\sin(\pi(C-\frac{1}{2}))}\\ &= \frac{\frac{\gamma^2}{4}\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}{(2\pi)^{\frac{4}{\gamma^2}-1}2^{2C-1}\pi}\frac{\Gamma(1-2B)\Gamma(\frac{3}{2}-C)^2}{\Gamma(2-2C)\Gamma(1-\frac{2\beta}{\gamma})\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{2}{\gamma^2})}\\ &= \frac{\frac{\gamma^2}{4}\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}{(2\pi)^{\frac{4}{\gamma^2}}2^{-1+\frac{2\beta}{\gamma}-\frac{4}{\gamma^2}}}\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{\beta}{\gamma}-\frac{2}{\gamma^2})\Gamma(1-\frac{\beta}{\gamma}+\frac{2}{\gamma^2})^2}{\Gamma(1-\frac{2\beta}{\gamma}+\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma}-\frac{2}{\gamma^2})},\end{aligned}$$

and finally:

$$\frac{\overline{G}(\alpha,\beta+\frac{4}{\gamma})}{\overline{G}(\alpha,\beta)} = \frac{\frac{\gamma^2}{4}\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}{(2\pi)^{\frac{4}{\gamma^2}}2^{-1+\frac{2\beta}{\gamma}}}\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{\beta}{\gamma})^2}{\Gamma(1-\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(-1+\frac{2\alpha}{\gamma}+\frac{\beta}{\gamma})}. \tag{3.2.30}$$

Hence we have proven Proposition 3.2.1. □

3.3 The boundary two-point and three-point functions

The goal of this section is to prove Theorem 3.1.7. We follow roughly the same steps as in the previous section, except we will derive explicitly the expression for the boundary two-point function used in the proof of Theorem 3.1.6. Again we will rely on the hypergeometric equations shown in Section 3.4 to obtain shift equations on $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$. The difference here is that the functional equation obtained will contain 3 terms instead of 2, see for instance equation (3.3.10) below. Throughout this section we will use:

$$q = \frac{1}{\gamma}(2Q-\beta_1-\beta_2-\beta_3+\chi). \tag{3.3.1}$$

We introduce the auxiliary function for $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$ and $t\in\mathbb{H}$,

$$H_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}}g(x)^{\frac{\gamma^2}{8}(q-1)}e^{\frac{\gamma}{2}X(x)}d\mu(x)\right)^q\right], \tag{3.3.2}$$

where:

$$d\mu(x) = \mu_1\mathbf{1}_{(-\infty,0)}(x)dx+\mu_2\mathbf{1}_{(0,1)}(x)dx+\mu_3\mathbf{1}_{(1,\infty)}(x)dx. \tag{3.3.3}$$

To start the range of parameters we want to work with is:

$$\beta_i<Q,\ \ \mu_1\in(0,\infty),\ \ \mu_2,\mu_3\in-\overline{\mathbb{H}}\quad\text{and}\quad q<\frac{4}{\gamma^2}\wedge\min_i\frac{2}{\gamma}(Q-\beta_i). \tag{3.3.4}$$

By $\mu_2,\mu_3\in-\overline{\mathbb{H}}$ we mean that their argument is chosen in $[-\pi,0]$. With this choice the GMC integral,

$$\int_{\mathbb{R}}\frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}}g(x)^{\frac{\gamma^2}{8}(q-1)}e^{\frac{\gamma}{2}X(x)}d\mu(x), \tag{3.3.5}$$

never hits the line $(-\infty,0)$ and so its argument can be chosen to be in $(-\pi,\pi)$ and its q power is thus well-defined. Now $t\mapsto H_\chi(t)$ is holomorphic in $\mathbb{H}$ and it is shown in Section 3.4 that $H_\chi(t)$ satisfies the hypergeometric equation,

$$t(1-t)\partial_t^2H_\chi(t)+(C-(A+B+1)t)\partial_tH_\chi(t)-ABH_\chi(t)=0, \tag{3.3.6}$$

with parameters:

$$A = -q\frac{\gamma\chi}{2}, \quad B = -1 + \chi(\beta_1 + \beta_2 - 2\chi + q\frac{\gamma}{2}), \quad C = \chi(\beta_1 - \chi). \tag{3.3.7}$$

We will also use the auxiliary function $\tilde{H}_\chi(t)$,

$$\tilde{H}_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(x-t)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^q\right], \tag{3.3.8}$$

which is defined with the following parameter choices:

$$t \in -\mathbb{H}, \ \ \beta_i < Q, \ \ \mu_1, \mu_2 \in -\overline{\mathbb{H}}, \ \ \mu_3 \in (0,\infty), \quad \text{and} \quad q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q - \beta_i). \tag{3.3.9}$$

With these choices of parameters the GMC integral is again a complex number which is avoiding the half-line $(-\infty, 0)$ and whose argument can be chosen again in $(-\pi, \pi)$. $\tilde{H}_\chi(t)$ obeys the exact same hypergeometric equation as $H_\chi(t)$. As in the previous section the key idea will be that we can explicitly write the solution of the hypergeometric equation.

3.3.1 First shift equation for the three-point function

We start again by proving the first shift equation on $\overline{H}$ by setting $\chi = \frac{\gamma}{2}$ and working with the functions $H_{\frac{\gamma}{2}}(t)$ and $\tilde{H}_{\frac{\gamma}{2}}(t)$. For this first lemma the parameter range on the β_i and μ_i is such that each $\overline{H}$ appearing is defined probabilistically (without analytic continuation) meaning the bounds (3.1.18) are satisified.

Lemma 3.3.1. *($\frac{\gamma}{2}$-shift equations for probabilistic $\overline{H}$) The following two shift equations for $\overline{H}$ hold,*

$$\begin{aligned}\overline{H}^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}_{(\mu_1,\mu_2,\mu_3)} &= \frac{\Gamma(\frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_2}{2} + \frac{\gamma^2}{4})}{\Gamma(\frac{\gamma\beta_1}{2} + (q-1)\frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma}{2}\beta_2 - (q-1)\frac{\gamma^2}{4}))}\overline{H}^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2,\mu_3)} \\ &+ \frac{q\Gamma(2 + \frac{\gamma^2}{4} - \frac{\gamma\beta_1}{2})\Gamma(1 - \frac{\gamma\beta_2}{2} + \frac{\gamma^2}{4})\Gamma(-1 + \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_1}{2})}{\Gamma(1 + \frac{q\gamma^2}{4})\Gamma(2 - \frac{\gamma}{2}(\beta_1+\beta_2) - (q-2)\frac{\gamma^2}{4}))\Gamma(-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right)\overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2,\mu_3)},\end{aligned} \tag{3.3.10}$$

and,

$$\begin{aligned}\Gamma(1 - \frac{\gamma\beta_2}{2})(\mu_3 - \mu_2 e^{\frac{i\pi\gamma\beta_2}{2}})\overline{H}^{(\beta_1,\beta_2+\frac{\gamma}{2},\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma^2}{4}}\mu_2,\mu_3)} &= -\frac{4}{q\gamma^2}\frac{\Gamma(1 - \frac{\gamma^2}{4})\Gamma(\frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{q\gamma^2}{4})\Gamma(-1 + \frac{\gamma\beta_1}{2} + \frac{\gamma\beta_2}{2} - \frac{\gamma^2}{2} + q\frac{\gamma^2}{4})}\overline{H}^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} \\ &+ \frac{\Gamma(2 - \frac{\gamma\beta_1}{2} + \frac{\gamma^2}{4})\Gamma(1 - \frac{\gamma\beta_1}{2})\Gamma(\frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4} - 1)}{\Gamma(1 - \frac{\gamma\beta_1}{2} + \frac{\gamma^2}{4} - q\frac{\gamma^2}{4})\Gamma(\frac{\gamma\beta_2}{2} - \frac{\gamma^2}{4} + q\frac{\gamma^2}{4})}(\mu_1 - \mu_2 e^{i\pi(\frac{\gamma^2}{4} - \frac{\gamma\beta_1}{2})})\overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)},\end{aligned} \tag{3.3.11}$$

provided that for every function $\overline{H}$ appearing, its parameters obey the constraint (3.1.18) required for $\overline{H}$ to be defined probabilistically. Furthermore, for both shift equations, given a fixed $\beta_1 \in (\frac{\gamma}{2}, \frac{2}{\gamma})$, there exists an open set $U \subset \mathbb{R}^2$ such that for $(\beta_2, \beta_3) \in U$, the parameters of each $\overline{H}$ of the equation obeys (3.1.18).

Proof. We first choose the parameters $\beta_1, \beta_2, \beta_3$ and μ_1, μ_2, μ_3 so that they obey the constraint (3.3.4) plus the following extra constraint on β_1:

$$\frac{\gamma}{2} < \beta_1 < \frac{2}{\gamma}. \tag{3.3.12}$$

The function $t \mapsto H_{\frac{\gamma}{2}}(t)$ is holomorphic on $\mathbb{H}$ and extends continuously on $\overline{\mathbb{H}}$. Using the basis of solutions of the hypergeometric equation recalled in Section 3.5.4.1, we can write the following solutions around $t = 0$,

$t=1$ and $t=\infty$, under the assumption that neither C, $C-A-B$, or $A-B$ are integers:[4]

$$\begin{aligned} H_{\frac{\gamma}{2}}(t) &= C_1 F(A,B,C,t) + C_2 t^{1-C} F(1+A-C,1+B-C,2-C,t) \\ &= B_1 F(A,B,1+A+B-C,1-t) + B_2(1-t)^{C-A-B} F(C-A,C-B,1+C-A-B,1-t) \\ &= D_1 e^{i\pi A} t^{-A} F(A,1+A-C,1+A-B,t^{-1}) + D_2 e^{i\pi B} t^{-B} F(B,1+B-C,1+B-A,t^{-1}). \end{aligned} \tag{3.3.13}$$

The constants $C_1, C_2, B_1, B_2, D_1, D_2$ are again the real constants that parametrize the solution space around the different points. As was performed in Section 3.2 we will identify them by Taylor expansion. First we note that by setting $t=0$:

$$C_1 = H_{\frac{\gamma}{2}}(0) = \overline{H}^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.3.14}$$

Next to find C_2 we go at higher order in the $t\to 0$ limit. We then expand the increment $H_{\frac{\gamma}{2}}(t) - H_{\frac{\gamma}{2}}(0)$ at first order following the same step as for (3.2.20):

$$\begin{aligned} &H_{\frac{\gamma}{2}}(t) - H_{\frac{\gamma}{2}}(0) \\ &= q\int_{\mathbb{R}} d\mu(x_1) \frac{(t-x_1)^{\frac{\gamma^2}{4}} - (-x_1)^{\frac{\gamma^2}{4}}}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}} \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma^2}{4}} g(x)^{\frac{\gamma^2}{8}(q-1)}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}|x-x_1|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^{q-1}\right] + o(t^{1-C}) \\ &= qt^{1-C}\left(\int_{\mathbb{R}} d\mu(x_1) \frac{(t-x_1)^{\frac{\gamma^2}{4}} - (-x_1)^{\frac{\gamma^2}{4}}}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}}\right) \overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)} + o(t^{1-C}). \end{aligned} \tag{3.3.15}$$

The integral in front of $\overline{H}$ converges thanks to the condition (3.3.12). Also notice with our conventions the argument of $(-x_1)$ is either 0 or π. Hence one obtains:

$$C_2 = q\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right)\overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.3.16}$$

Similarly by setting $t=1$ we get:

$$B_1 = \overline{H}^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}_{(\mu_1,\mu_2, e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.3.17}$$

The connection formula (3.5.105) between C_1, C_2, and B_1 then implies the shift equation (3.3.10) in the range of parameters constraint by (3.3.4) and (3.3.12), after performing furthermore the replacement $\mu_3 \to e^{-\frac{i\pi\gamma^2}{4}}\mu_3$ (which also rotates the domain where μ_3 belongs). To lift these constraint we then invoque the analycity of $\overline{H}$ as a function of its parameters given by Lemma 3.5.6. We have thus shown that (3.3.10) holds whenever all three $\overline{H}$ appearing are well-defined. The claim on the existence of the interval U for every fixed β_1 can be easily checked. Now we repeat these steps with $\tilde{H}_{\frac{\gamma}{2}}$ to obtain the shift equation with the opposite phase. We expand $\tilde{H}_{\frac{\gamma}{2}}(t)$,

$$\begin{aligned} \tilde{H}_{\frac{\gamma}{2}}(t) &= \tilde{C}_1 F(A,B,C,t) + \tilde{C}_2 t^{1-C} F(1+A-C,1+B-C,2-C,t) \\ &= \tilde{B}_1 F(A,B,1+A+B-C,1-t) + \tilde{B}_2(1-t)^{C-A-B} F(C-A,C-B,1+C-A-B,1-t) \\ &= \tilde{D}_1 e^{i\pi A} t^{-A} F(A,1+A-C,1+A-B,t^{-1}) + \tilde{D}_2 e^{i\pi B} t^{-B} F(B,1+B-C,1+B-A,t^{-1}), \end{aligned} \tag{3.3.18}$$

and compute in the same way the values of $\tilde{C}_1, \tilde{C}_2, \tilde{B}_1$:

$$\tilde{C}_1 = \overline{H}^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(e^{\frac{i\pi\gamma^2}{4}}\mu_1,\mu_2,\mu_3)}, \quad \tilde{B}_1 = \overline{H}^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}_{(e^{\frac{i\pi\gamma^2}{4}}\mu_1, e^{i\pi\frac{\gamma^2}{4}}\mu_2,\mu_3)}, \tag{3.3.19}$$

$$\tilde{C}_2 = q\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi(\frac{\gamma^2}{4}-\frac{\gamma\beta_1}{2})}\right)\overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(e^{\frac{i\pi\gamma^2}{4}}\mu_1,\mu_2,\mu_3)}. \tag{3.3.20}$$

Then the connection formula (3.5.105) implies the shift equation (3.3.11). □

[4] Again the values excluded here are recovered by a continuity argument.

3.3.2 Solving the two-point function

At this point we will postpone computing the boundary three-point function $\overline{H}$ and focus on determining shift equations that will completely specify $\overline{R}$. Once we have proved the exact formula for $\overline{R}$, it will be then be possible to finish computing $\overline{H}$. In a similar way the value of $\overline{R}$ is also required in the proof of the value of $\overline{G}$ in Section 3.2.

3.3.2.1 First shift equation on the reflection coefficient

We start again by proving a first shift equation for $\overline{R}(\beta_1, \mu_1, \mu_2)$ restricted to the case where $\overline{R}$ is defined probabilistically.

Lemma 3.3.2. *Consider $\gamma \in (0,2)$, $\beta_1 \in (\frac{\gamma}{2}, \frac{2}{\gamma})$, $\mu_1, \mu_2 \in \mathbb{C}$ such that both pairs (μ_1, μ_2) and $(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2)$ both obey the condition of Definition 3.1.3. Then $\overline{R}(\beta, \mu_1, \mu_2)$ obeys,*

$$\overline{R}(\beta_1, \mu_1, \mu_2) = -\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(2-\frac{\gamma\beta_1}{2})}{\Gamma(1-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right)\overline{R}(\beta_1+\frac{\gamma}{2}, \mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2). \tag{3.3.21}$$

Similarly for $\beta_1 \in (0, \frac{2}{\gamma}-\frac{\gamma}{2})$ and the same constraint on μ_1, μ_2 as before,

$$\overline{R}(\beta_1+\frac{\gamma}{2}, \mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2) = -\frac{\Gamma(-1+\frac{\gamma\beta_1}{2})\Gamma(2-\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{-i\pi\frac{\gamma\beta_1}{2}}\right)\overline{R}(\beta_1+\gamma, \mu_1, \mu_2). \tag{3.3.22}$$

Proof. The key idea to derive the shift equations for $\overline{R}$ is to take suitable limits of the shift equations of Lemma 3.3.1 to make $\overline{R}$ appear from $\overline{H}$. We will use extensively the Lemma 3.1.8 of Section 3.1.3 which provides this limit. Fix a $\beta_1 \in (\frac{\gamma}{2}, \frac{2}{\gamma})$. Consider two parameters $\epsilon, \eta > 0$ and set $\beta_2 = \beta_1 - \epsilon$, $\beta_3 = \beta_1 - \beta_2 + \frac{\gamma}{2} + \eta = \frac{\gamma}{2} + \epsilon + \eta$. Notice that for this parameter choice the three $\overline{H}$ functions appearing in the shift equation (3.3.10) are well-defined. The idea now is to match the poles of (3.3.10) as η goes to 0 or in other words as β_3 goes to $\beta_1 - \beta_2 + \frac{\gamma}{2}$. By applying Lemma 3.1.8 we get:

$$\begin{aligned}
&\lim_{\beta_3 \downarrow \beta_1 - \beta_2 + \frac{\gamma}{2}} (\beta_2+\beta_3-\beta_1-\frac{\gamma}{2}) H^{(\beta_1, \beta_2-\frac{\gamma}{2}, \beta_3)}_{(\mu_1,\mu_2,\mu_3)} = 2(Q-\beta_1)\overline{R}(\beta_1,\mu_1,\mu_2)\\
&\lim_{\beta_3 \downarrow \beta_1 - \beta_2 + \frac{\gamma}{2}} (\beta_2+\beta_3-\beta_1-\frac{\gamma}{2}) \frac{\Gamma(\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_2}{2}+\frac{\gamma^2}{4})}{\Gamma(\frac{\gamma\beta_1}{2}+(q-1)\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma}{2}\beta_2-(q-1)\frac{\gamma^2}{4}))} \overline{H}^{(\beta_1-\frac{\gamma}{2}, \beta_2, \beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, \mu_3)} = 0\\
&\lim_{\beta_3 \downarrow \beta_1 - \beta_2 + \frac{\gamma}{2}} (\beta_2+\beta_3-\beta_1-\frac{\gamma}{2}) \left[\frac{q\Gamma(2+\frac{\gamma^2}{4}-\frac{\gamma\beta_1}{2})\Gamma(1-\frac{\gamma\beta_2}{2}+\frac{\gamma^2}{4})\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(1+\frac{q\gamma^2}{4})\Gamma(2-\frac{\gamma}{2}(\beta_1+\beta_2)-(q-2)\frac{\gamma^2}{4}))\Gamma(-\frac{\gamma^2}{4})}\right.\\
&\qquad\qquad \left. \times \left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right) \overline{H}^{(\beta_1+\frac{\gamma}{2}, \beta_2, \beta_3)}_{(\mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2, \mu_3)}\right]\\
&= \frac{\frac{2}{\gamma}(Q-\beta_1)\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right) 2(\frac{2}{\gamma}-\beta_1)\overline{R}(\beta_1+\frac{\gamma}{2}, \mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2).
\end{aligned}$$

This leads to a relation on the reflection coefficient:

$$\overline{R}(\beta_1, \mu_1, \mu_2) = -\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(2-\frac{\gamma\beta_1}{2})}{\Gamma(1-\frac{\gamma^2}{4})}\left(\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right)\overline{R}(\beta_1+\frac{\gamma}{2}, \mu_1, e^{\frac{i\pi\gamma^2}{4}}\mu_2). \tag{3.3.23}$$

By using the alternative auxiliary function $\tilde{H}_{\frac{\gamma}{2}}(t)$ along the same lines we obtain a relation between $\overline{R}(\beta_1+\frac{\gamma}{2},\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2)$ and $\overline{R}(\beta_1+\gamma,\mu_1,\mu_2)$:

$$\overline{R}(\beta_1+\frac{\gamma}{2},\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2)=-\frac{\Gamma(-1+\frac{\gamma\beta_1}{2})\Gamma(2-\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\left(\mu_1-\mu_2e^{-i\pi\frac{\gamma\beta_1}{2}}\right)\overline{R}(\beta_1+\gamma,\mu_1,\mu_2). \tag{3.3.24}$$

Hence this implies the claim of the lemma. □

At this point in the proof we need to show $\overline{R}(\beta_1,\mu_1,\mu_2)$ is analytic in β_1 in the interval $(\frac{\gamma}{2},Q)$. For this we will again take a limit from the first shift equation.

Lemma 3.3.3. *(Analycity of $\overline{R}(\beta_1,\mu_1,\mu_2)$ in β_1 and μ_1,μ_2) For all μ_1,μ_2 obeying Definition 3.1.3, the function $\beta_1\mapsto\overline{R}(\beta_1,\mu_1,\mu_2)$ is complex analytic on a complex neighborhood of any compact set $K\subset(\frac{\gamma}{2},Q)$. For all $\beta_1\in(\frac{\gamma}{2},Q)$, the function $(\mu_1,\mu_2)\mapsto\overline{R}(\beta_1,\mu_1,\mu_2)$ is complex analytic on any compact set $\tilde{K}$ contained in the open set of pairs (μ_1,μ_2) obeying Definition 3.1.3.*

Proof. In the shift equation (3.3.10), set $\beta_1=\frac{\gamma}{2}+\eta$, $\frac{\gamma}{2}<\beta_2=\beta_3<Q$. We multiply the shift equation (3.3.10) by η, exchange μ_2 and μ_3, and let $\eta\to0_+$, this yields:

$$\begin{aligned}2(Q-\beta_2)\overline{R}(\beta_2,\mu_1,\mu_2)&=(Q-\beta_2)\left(\overline{R}(\beta_2,\mu_1,\mu_2)+\overline{R}(\beta_2,\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_3)\right)\\&\quad+\frac{2}{\gamma}\left(\mu_1-\mu_3e^{i\pi\frac{\gamma^2}{4}}\right)\overline{H}^{(\gamma,\beta_2,\beta_2)}_{(\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_3,\mu_2)}\\\Rightarrow\overline{R}(\beta_2,\mu_1,\mu_2)&=\overline{R}(\beta_2,\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_3)+\frac{2}{\gamma(Q-\beta_2)}\left(\mu_1-\mu_3e^{i\pi\frac{\gamma^2}{4}}\right)\overline{H}^{(\gamma,\beta_2,\beta_2)}_{(\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_3,\mu_2)}.\end{aligned} \tag{3.3.25}$$

Take $\mu_3=0$ in the previous equation and fix a compact $K\subset(\frac{\gamma}{2},Q)$. In our previous work [89] we have calculated the expression of $\overline{R}(\beta_2,\mu_1,0)$ and it is complex analytic in β_2 in a complex neighborhood of K. By the result of Lemma 3.5.6 we know the function $\overline{H}^{(\gamma,\beta_2,\beta_2)}_{(\mu_1,0,\mu_2)}$ is also complex analytic in β_2 in a complex neighborhood of K. Therefore the above equation with $\mu_3=0$ implies the claim of analycity for $\beta_2\mapsto\overline{R}(\beta_2,\mu_1,\mu_2)$. The exact same reasoning implies the analycity of $(\mu_1,\mu_2)\mapsto\overline{R}(\beta_2,\mu_1,\mu_2)$. □

3.3.2.2 OPE with reflection and the reflection principle

We now move to performing the OPE with reflection. We rely extensively on Lemma 3.5.4 and Lemma 3.5.5 giving the Taylor expansion using the reflection coefficient. As in Section 3.2.2 we first use OPE with reflection for $\chi=\frac{\gamma}{2}$ to obtain the reflection principle.

Lemma 3.3.4 (Reflection principle for $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$)**.** *Consider parameters μ_1,μ_2,μ_3, β_1,β_2,β_3 such that $\beta_1\in(\frac{\gamma}{2},Q)$ and satisfying the parameter range (3.1.18) for $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ and $\overline{R}(\beta_1,\mu_1,\mu_2)$ to be well-defined. Then one can meromorphically extend $\beta_1\mapsto\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ beyond the point $\beta_1=Q$ by the following relation:*

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}=-\frac{\Gamma(\frac{2\beta_1}{\gamma}-\frac{4}{\gamma^2})\Gamma(\frac{\beta_2+\beta_3-\beta_1}{\gamma})}{\Gamma(\frac{\beta_1+\beta_2+\beta_3-2Q}{\gamma})}\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}. \tag{3.3.26}$$

The quantity $\overline{H}^{(2Q-\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ is thus well-defined as long as $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ and $\overline{R}(\beta_1,\mu_1,\mu_2)$ are well-defined. Similarly, for (μ_1,μ_2) satisfying the constraint of Definition 3.1.3, we can analytically extend $\beta_1\mapsto\overline{R}(\beta_1,\mu_1,\mu_2)$ to the range $(\frac{\gamma}{2},Q+\frac{2}{\gamma})$ thanks to the relation:

$$\overline{R}(\beta_1,\mu_1,\mu_2)\overline{R}(2Q-\beta_1,\mu_1,\mu_2)=\frac{1}{\Gamma(1-\frac{2(Q-\beta_1)}{\gamma})\Gamma(1+\frac{2(Q-\beta_1)}{\gamma})}. \tag{3.3.27}$$

Proof. Throughout the proof we keep the same notations as used in the proof of Lemma 3.3.1 for the solution space of the hypergeometric equation satisfied by $H_{\frac{\gamma}{2}}(t)$. The first step is to assume $\beta_1 \in (Q-\beta_0, Q)$ so that we can apply the result of Lemma 3.5.5 and identify the value of C_2 to be:

$$C_2 = \frac{2(Q-\beta_1)}{\gamma}\frac{\Gamma(\frac{2}{\gamma}(\beta_1-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1)-q)}{\Gamma(-q)}\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma^2}{4}}\mu_2,e^{\frac{i\pi\gamma^2}{4}}\mu_3)}. \tag{3.3.28}$$

The key argument is to observe that since by Lemma 3.5.6 $\beta_1 \mapsto H_{\frac{\gamma}{2}}(t)$ is complex analytic so is the coefficient C_2. By using this combined with the analycity of $\overline{R}$ and $\overline{H}$, we can extend the range of validity of equation (3.3.28) from $\beta_1 \in (Q-\beta_0, Q)$ to $\beta_1 \in (\frac{2}{\gamma}, Q)$. Now equation (3.3.16) derived in the the proof of Lemma 3.3.1 gives us an alternative expression for C_2, which is valid for $\beta_1 \in (\frac{\gamma}{2}, \frac{2}{\gamma})$. The analycity of $\beta_1 \mapsto C_2$ in a complex neighborhood of $\frac{2}{\gamma}$ then implies that one can "glue" together the two expressions for C_2. More precisely the equality,

$$\begin{aligned}&\frac{2(Q-\beta_1)}{\gamma}\frac{\Gamma(\frac{2}{\gamma}(\beta_1-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1)-q)}{\Gamma(-q)}\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma^2}{4}}\mu_2,e^{\frac{i\pi\gamma^2}{4}}\mu_3)} \\ &= q\frac{\Gamma(-1+\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma\beta_1}{2})}{\Gamma(-\frac{\gamma^2}{4})}\left(\mu_1-\mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right)\overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2,e^{\frac{i\pi\gamma^2}{4}}\mu_3)},\end{aligned} \tag{3.3.29}$$

provides the desired analytic continuation of $\overline{H}$. To land on the form of the reflection equation given in the lemma one needs to replace β_1 by $\beta_1 - \frac{\gamma}{2}$. This transforms $\overline{R}(\beta_1,\mu_1,\mu_2)$ into $\overline{R}(\beta_1-\frac{\gamma}{2},\mu_1,\mu_2)$ which we can shift back to $\overline{R}(\beta_1,\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2)$ using the shift equation (3.3.21). Lastly we perform the parameter replacement $e^{\frac{i\pi\gamma^2}{4}}\mu_2$ to μ_2 and $e^{\frac{i\pi\gamma^2}{4}}\mu_3$ to μ_3. Therefore this implies the claim of the reflection principle for $\overline{H}$. The claim for $\overline{R}$ is then an immediate consequence. □

3.3.2.3 Analytic continuation of $\overline{H}$ and $\overline{R}$

At this stage we will use the shift equations we have derived to analytically continue $\overline{H}$ and $\overline{R}$ both in the parameters β_i and μ_i. The analytic continuations will be defined in a larger range of parameters than the one required for the GMC expression to be well-defined.

Lemma 3.3.5. *(Analytic continuation of $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$) Fix μ_1,μ_2,μ_3 obeying the condition of Definition 3.1.3. Then the function $(\beta_1,\beta_2,\beta_3) \mapsto \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ originally defined in the parameter range given by (3.1.18) extends to a meromorphic function of the three variables in a small complex neighborhood of $\mathbb{R}^3$. Now fix β_1,β_2,β_3 in this complex neighborhood of $\mathbb{R}^3$ and write $\mu_i := e^{i\pi\gamma(\sigma_i-\frac{Q}{2})}$ with the convention that $\mathrm{Re}(\sigma_i) = \frac{Q}{2}$ when $\mu_i > 0$. The function*

$$(\sigma_1,\sigma_2,\sigma_3) \mapsto \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(e^{i\pi\gamma(\sigma_1-\frac{Q}{2})},e^{i\pi\gamma(\sigma_2-\frac{Q}{2})},e^{i\pi\gamma(\sigma_3-\frac{Q}{2})})} \tag{3.3.30}$$

then extends to a meromorphic function of $\mathbb{C}^3$.

Proof. We first work at fixed μ_1,μ_2,μ_3 obeying the constraint of Definition 3.1.3 and perform the analytic continuation in the parameters β_1,β_2,β_3. First notice that for any triple (μ_1,μ_2,μ_3) obeying Definition 3.1.3, then either $(\mu_1,e^{\frac{i\pi\gamma^2}{4}}\mu_2,\mu_3)$ or $(\mu_1,e^{-\frac{i\pi\gamma^2}{4}}\mu_2,\mu_3)$ can be obtained by turning μ_2 without crossing the branch $(-\infty,0)$ and is such that it obeys Definition 3.1.3. Let us assume that the triplet satisfying this is $(\mu_1,e^{-\frac{i\pi\gamma^2}{4}}\mu_2,\mu_3)$. Then in the shift equation (3.3.10) one can choose the parameters β_1,β_2,β_3 such that $\overline{H}^{(\beta_1,\beta_2-\frac{\gamma}{2},\beta_3)}_{(\mu_1,e^{-\frac{i\pi\gamma^2}{4}}\mu_2,\mu_3)}$ is defined probabilistically and that out of the two remaining terms,

$$\overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} \quad \text{and} \quad \overline{H}^{(\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)},$$

one of them is defined probabilistically and the other has parameters $\beta_1, \beta_2, \beta_3$ that do not obey (3.1.18). The shift equation thus provides the definition of the third term. By using Lemma 3.5.6 giving the analycity of $\overline{H}$ in a complex neighborhood of the domain where it is defined probabilistically combined with the iteration of the above procedure provides a meromorphic extension of $(\beta_1, \beta_2, \beta_3) \mapsto \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ in a complex neighborhood of $\mathbb{R}^3$. The locations of the poles of the meromorphic extension are prescribed by the shift equation.

Now we move to the analytic continuation in the μ_i. We rewrite these parameters as $\mu_i := e^{i\pi\gamma(\sigma_i - \frac{Q}{2})}$ with the convention that $\mathrm{Re}(\sigma_i) = \frac{Q}{2}$ when $\mu_i > 0$. Then one can analytic continue $\overline{H}$ in the parameters σ_i to a meromorphic function of $\mathbb{C}^3$ by using again the shift equations of Lemma 3.3.1. As a concrete example, for $\mu_1, \mu_2, \mu_3 \in (0, \infty)$, in order to define $\overline{H}^{(\beta_1,\beta_2-\frac{2}{\gamma},\beta_3)}_{(\mu_1,e^{-i\pi}\mu_2,\mu_3)}$, one can express it using the first shift equation of Lemma 3.3.1 in terms of:

$$\overline{H}^{(\beta_1-\frac{\gamma}{2},\beta_2+\frac{\gamma}{2}-\frac{2}{\gamma},\beta_3)}_{(\mu_1,e^{i\pi(\frac{\gamma^2}{4}-1)}\mu_2,\mu_3)}, \quad \text{and} \quad \overline{H}^{(\beta_1+\frac{\gamma}{2},\beta_2+\frac{\gamma}{2}-\frac{2}{\gamma},\beta_3)}_{(\mu_1,e^{i\pi(\frac{\gamma^2}{4}-1)}\mu_2,\mu_3)}.$$

The μ_i appearing above are now obeying the constraint of Definition 3.1.3, and therefore the $\overline{H}$ functions are well-defined. The poles of $\overline{H}$ as a function of the σ_i are again prescribed by the shift equations.

□

Lemma 3.3.6. *(Analytic continuation of $\overline{R}(\beta_1, \mu_1, \mu_2)$) For all μ_1, μ_2 obeying the constraint of Definition 3.1.3, the function $\beta_1 \mapsto \overline{R}(\beta_1, \mu_1, \mu_2)$ originally defined on the interval $(\frac{\gamma}{2}, Q)$ extends to a meromorphic function defined in a complex neighborhood of $\mathbb{R}$ and satisfying the shift equation:*

$$\overline{R}(\beta_1, \mu_1, \mu_2) = -\frac{\Gamma(-1 + \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})\Gamma(2 - \frac{\gamma\beta_1}{2} - \frac{\gamma^2}{4})}{\Gamma(1 - \frac{\gamma^2}{4})^2} \frac{\pi}{\sin(\pi\frac{\gamma\beta_1}{2})} \left|\mu_1 - \mu_2 e^{i\pi\frac{\gamma\beta_1}{2}}\right|^2 \overline{R}(\beta_1 + \gamma, \mu_1, \mu_2). \quad (3.3.31)$$

Furthermore, for a fixed β_1 in the above complex neighborhood of $\mathbb{R}$, the function $\overline{R}(\beta_1, e^{i\pi\gamma(\sigma_1-\frac{Q}{2})}, e^{i\pi\gamma(\sigma_2-\frac{Q}{2})})$ extends to a meromorphic function of (σ_1, σ_2) on $\mathbb{C}^2$.

Proof. $\beta_1 \mapsto \overline{R}(\beta_1, \mu_1, \mu_2)$ is originally defined on an interval of size $\frac{2}{\gamma}$, but using (3.3.27) we can analytically extend its definition to an interval of size $\frac{4}{\gamma}$, i.e. the interval $\beta_1 \in (\frac{\gamma}{2}, Q+\frac{2}{\gamma})$. This gives us a large enough interval to successively apply both shift equations of Lemma 3.3.2 to extend $\beta_1 \mapsto \overline{R}(\beta_1, \mu_1, \mu_2)$ to a meromorphic function defined in a complex neighborhood of the real line and to get the shift equation stated above. The analytic continuation in (σ_1, σ_2) follows the exact same steps as for $\overline{H}$. □

3.3.2.4 Second shift equation on the reflection coefficient

Finally we will derive the second shift on $\overline{R}(\beta_1, \mu_1, \mu_2)$ that will completely specify its value.

Lemma 3.3.7. *(Second shift equation for $\overline{R}(\beta_1, \mu_1, \mu_2)$). For all μ_1, μ_2 obeying the constraint of Definition 3.1.3, the meromorphic function $\beta_1 \mapsto \overline{R}(\beta_1, \mu_1, \mu_2)$ defined in a complex neighborhood of $\mathbb{R}$ satisfies the following shift equation:*

$$\overline{R}(\beta, \mu_1, \mu_2) = \frac{(2\pi)^{\frac{8}{\gamma^2}}}{\gamma^2(Q-\beta)(\frac{\gamma}{2}-\beta)} \frac{1}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{8}{\gamma^2}} \sin(\pi\frac{2\beta}{\gamma})\sin(\pi(\frac{2\beta}{\gamma}+\frac{4}{\gamma^2}))} \left|\mu_1^{\frac{4}{\gamma^2}} - \mu_2^{\frac{4}{\gamma^2}} e^{i\pi\frac{2\beta}{\gamma}}\right|^2 \overline{R}(\beta + \frac{4}{\gamma}, \mu_1, \mu_2). \quad (3.3.32)$$

Proof. We are now working exclusively with the choice $\chi = \frac{2}{\gamma}$. There will be several steps that will successively require to choose different ranges of parameters. We first place ourselves in the following range of parameters:

$$t \in \mathbb{H}, \ \epsilon \in (0, \beta_0), \ \beta_1 = \beta_2 = Q - \epsilon, \ \beta_3 = \frac{2}{\gamma} + \epsilon \ \mu_1 \in (0, +\infty), \ \mu_2, \mu_3 \in (-\infty, 0). \quad (3.3.33)$$

In the above ϵ is chosen small enough, smaller than the constant β_0 required to apply Lemma 3.5.4. Notice also that in this range $q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q - \beta_i)$. Furthermore in the above the choice of μ_i is such that we can

apply Proposition 3.4.3 giving that $H_{\frac{2}{\gamma}}(t)$ obeys the hypergeometric equation. We can thus expand $H_{\frac{2}{\gamma}}(t)$ on the basis,

$$\begin{aligned} H_{\frac{2}{\gamma}}(t) &= C_1 F(A,B,C,t) + C_2 t^{1-C} F(1+A-C,1+B-C,2-C,t) \\ &= B_1 F(A,B,1+A+B-C,1-t) + B_2(1-t)^{C-A-B} F(C-A,C-B,1+C-A-B,1-t) \\ &= D_1 e^{i\pi A} t^{-A} F(A,1+A-C,1+A-B,t^{-1}) + D_2 e^{i\pi B} t^{-B} F(B,1+B-C,1+B-A,t^{-1}), \end{aligned} \tag{3.3.34}$$

where again $C_1, C_2, B_1, B_2, D_1, D_2$ are parametrizing the solution space around the points 0, 1, and ∞. As before by sending t to 0 and to 1 one obtains:

$$C_1 = H_{\frac{2}{\gamma}}(0) = \overline{H}^{(\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi}\mu_2,e^{i\pi}\mu_3)}, \quad B_1 = H_{\frac{2}{\gamma}}(1) = \overline{H}^{(\beta_1,\beta_2-\frac{2}{\gamma},\beta_3)}_{(\mu_1,\mu_2,e^{i\pi}\mu_3)}. \tag{3.3.35}$$

Let us make some comments on the values of the μ_i appearing in C_1 and B_1. For C_1 since μ_2, μ_3 are negative, μ_1, $e^{i\pi}\mu_2$, $e^{i\pi}\mu_3$ are all positive numbers and the function $\overline{H}$ appearing is thus well-defined as a GMC quantity. For B_1 now μ_1 and $e^{i\pi}\mu_3$ are positive while μ_2 is negative, so we are no longer under the constraint of Definition 3.1.3, but rather in a limiting case. Since the moment of the GMC of the $\overline{H}$ appearing in B_1 is positive, i.e. the moment is equal to $\frac{\epsilon}{\gamma}$, we can still make sense of this GMC by a simple continuity argument. Since the condition required for Lemma 3.5.4, $\beta_1 \in (Q-\beta_0, Q)$, is satisfied one then derives:

$$C_2 = \frac{2(Q-\beta_1)}{\gamma} \frac{\Gamma(\frac{2}{\gamma}(\beta_1-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1)-q)}{\Gamma(-q)} \overline{R}(\beta_1,\mu_1,\mu_2) \overline{H}^{(2Q-\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi}\mu_2,e^{i\pi}\mu_3)}. \tag{3.3.36}$$

Now we write the connection formula (3.5.105) linking C_1, B_1, C_2, setting $\chi = \frac{2}{\gamma}$ in the equation below:

$$B_1 = \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(\chi(\beta_1-\chi+q\frac{\gamma}{2})\Gamma(1-\chi\beta_2+\chi^2-q\frac{\gamma\chi}{2})} C_1 + \frac{\Gamma(2-\chi\beta_1+\chi^2)\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(1+\frac{q\gamma\chi}{2})\Gamma(2-\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))} C_2. \tag{3.3.37}$$

In the range of parameters we have been working with, all three constants C_1, B_1, C_2 are well-defined probabilistic quantities through a function $\overline{H}$. But now by analytic continuation of Lemma 3.3.5 we can view the above identity as an identity of the analytic function $\overline{H}$. By repeating the above strategy in the range of parameters,

$$t \in \mathbb{H}, \ \ \epsilon \in (0,\beta_0), \ \ \beta_1 = \beta_2 = Q-\epsilon, \ \ \beta_3 = \frac{2}{\gamma} + \epsilon \ \ \mu_1, \mu_2 \in (0,+\infty), \ \ \mu_3 \in (-\infty, 0), \tag{3.3.38}$$

one can identify B_1, B_2, C_1. Then again we can write the connection formula (3.5.105) linking B_1, B_2, C_1 and extend the identity to an identity of analytic functions. We can proceed similarly for all the triples (B_1, B_2, D_1), (B_1, D_1, D_2), (C_1, C_2, D_1), and (C_1, D_1, D_2). At the level of analytic functions, the values of these remaining constants are as follows:

$$D_1 = \overline{H}^{(\beta_1,\beta_2,\beta_3-\frac{2}{\gamma})}_{(e^{i\pi}\mu_1,e^{i\pi}\mu_2,e^{i\pi}\mu_3)}, \tag{3.3.39}$$

$$B_2 = \frac{2(Q-\beta_2)}{\gamma} \frac{\Gamma(\frac{2}{\gamma}(\beta_2-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_2)-q)}{\Gamma(-q)} \overline{R}(\beta_2,e^{i\pi}\mu_2,e^{i\pi}\mu_3) \overline{H}^{(\beta_1,2Q-\beta_2-\frac{2}{\gamma},\beta_3)}_{(\mu_1,\mu_2,e^{i\pi}\mu_3)}, \tag{3.3.40}$$

$$D_2 = \frac{2(Q-\beta_3)}{\gamma} \frac{\Gamma(\frac{2}{\gamma}(\beta_3-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_3)-q)}{\Gamma(-q)} \overline{R}(\beta_3,\mu_1,e^{i\pi}\mu_3) \overline{H}^{(\beta_1,\beta_2,2Q-\beta_3-\frac{2}{\gamma})}_{(e^{i\pi}\mu_1,e^{i\pi}\mu_2,e^{i\pi}\mu_3)}. \tag{3.3.41}$$

We now know that the analytic function $\overline{H}$ appears in all the constants $C_1, C_2, B_1, B_2, D_1, D_2$ and is related to the probabilistic definition of $\overline{H}$ in the appropriate range of parameters. With this at hand we apply the connection formulas coming from the hypergeometric equation in the following way. We use the relation (3.5.105) expressing B_2 in terms of C_1 and C_2, as well as

$$D_2 = \frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)} C_1 + e^{i\pi(1-C)} \frac{\Gamma(2-C)\Gamma(A-B)}{\Gamma(1-B)\Gamma(A-C+1)} C_2 \tag{3.3.42}$$

coming from (3.5.104) to eliminate C_1 and obtain the following relation:

$$\frac{\Gamma(B)}{\Gamma(A+B-C)}B_2-\frac{\Gamma(C-B)}{\Gamma(A-B)}D_2=\frac{\Gamma(2-C)}{\Gamma(A-C+1)}\left(\frac{\Gamma(B)}{\Gamma(B-C+1)}-\frac{e^{i\pi(1-C)}\Gamma(C-B)}{\Gamma(1-B)}\right)C_2. \tag{3.3.43}$$

Let us now take $\beta_1=\beta\in(\frac{\gamma}{2},\frac{2}{\gamma})$, $\beta_2=\frac{\gamma}{2}+\eta$, $\beta_3=Q-\beta$, $\mu_3=0$ and study the asymptotic as $\eta\to 0$. In this case,

$$q=\frac{4}{\gamma^2}-\frac{\eta}{\gamma},\quad A=-\frac{4}{\gamma^2}+\frac{\eta}{\gamma},\quad B=\frac{2\beta}{\gamma}-\frac{4}{\gamma^2}+\frac{\eta}{\gamma},\quad C=\frac{2\beta}{\gamma}-\frac{4}{\gamma^2}, \tag{3.3.44}$$

and

$$\lim_{\eta\to 0}\eta D_2=-2(\frac{2}{\gamma}-\beta)\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})}{\Gamma(-\frac{4}{\gamma^2})}\overline{R}(Q-\beta,\mu_1,0)\overline{R}(\beta+\frac{\gamma}{2},e^{i\pi}\mu_1,0), \tag{3.3.45}$$

$$\lim_{\eta\to 0}\eta C_2=-2(\beta-\frac{\gamma}{2})\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})}{\Gamma(-\frac{4}{\gamma^2})}\overline{R}(\beta,\mu_1,\mu_2)\overline{R}(2Q-\beta-\frac{2}{\gamma},\mu_1,e^{i\pi}\mu_2), \tag{3.3.46}$$

$$\lim_{\eta\to 0}\eta^2 B_2=-\frac{8}{\gamma}\lim_{\eta\to 0}\eta\overline{R}(\frac{\gamma}{2}+\eta,e^{i\pi}\mu_2,0). \tag{3.3.47}$$

To obtain the above limits one simply needs to apply Lemma 3.1.8, but there is again a subtlety coming from the fact that the μ_i appearing above does not strictly obey the condition of Definition 3.1.3 and our proof of Lemma 3.1.8 does not cover this case. But to remedy this, one can simply apply again the shift equations of Lemma 3.3.1 to rotate the μ_i that is on the line $(-\infty,0)$. Then it is possible to obtain $\overline{R}$ from $\overline{H}$ using Lemma 3.1.8 and finally one can rotate back the μ_i to its original value by using Lemma 3.3.2. In the limit of B_2 we also use a limit calculated in [61]:

$$\lim_{\eta\to 0}\overline{H}^{(\beta,Q-\eta,Q-\beta)}_{(\mu_1,\mu_2,0)}=2. \tag{3.3.48}$$

The moment of the GMC defining $\overline{H}$ in this limit is $\frac{\eta}{\gamma}$ and tends to 0, this gives a contribution 1 to the limit. But in this case there is also a concentration behavior at the insertion with parameter $Q-\eta$, this adds 1 to the final limit.

We will need the result from the interval case [89] where we have found the reflection coefficient $\overline{R}_1^{\partial}(\alpha)$ with one of the μ_i set to 0:

$$\overline{R}(\alpha,1,0)=\overline{R}_1^{\partial}(\alpha)=\frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)-\frac{1}{2}}}{(Q-\alpha)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}}\frac{\Gamma_{\frac{\gamma}{2}}(\alpha-\frac{\gamma}{2})}{\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}. \tag{3.3.49}$$

The rest of the proof is now direct algebraic computations. Together with (3.3.27) we have:

$$\begin{aligned}\lim_{\eta\to 0}\eta D_2&=-2(\frac{2}{\gamma}-\beta)\mu_1^{\frac{4}{\gamma^2}}\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma})}{\Gamma(-\frac{4}{\gamma^2})}\frac{e^{i\pi(\frac{4}{\gamma^2}-\frac{2\beta}{\gamma})}}{\Gamma(1-\frac{2\beta}{\gamma})\Gamma(1+\frac{2\beta}{\gamma})}\frac{\overline{R}(\beta+\frac{\gamma}{2},1,0)}{\overline{R}(\beta+Q,1,0)},\\&=\frac{4}{\gamma}(2\pi)^{\frac{4}{\gamma^2}-1}\mu_1^{\frac{4}{\gamma^2}}e^{i\pi(\frac{4}{\gamma^2}-\frac{2\beta}{\gamma})}\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})\Gamma(-\frac{2\beta}{\gamma})}{\Gamma(-\frac{4}{\gamma^2})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}},\end{aligned} \tag{3.3.50}$$

$$\lim_{\eta\to 0}\eta C_2=\gamma\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}{\Gamma(-\frac{4}{\gamma^2})\Gamma(\frac{2\beta}{\gamma})}\frac{\overline{R}(\beta,\mu_1,\mu_2)}{\overline{R}(\beta+\frac{2}{\gamma},\mu_1,e^{i\pi}\mu_2)}, \tag{3.3.51}$$

$$\lim_{\eta\to 0}\eta^2 B_2=-4(2\pi)^{\frac{4}{\gamma^2}-1}\mu_2^{\frac{4}{\gamma^2}}e^{i\pi\frac{4}{\gamma^2}}\frac{1}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}. \tag{3.3.52}$$

Putting all these into (3.3.43), we get:

$$4(2\pi)^{\frac{4}{\gamma^2}-1}e^{i\pi\frac{4}{\gamma^2}}\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}{\Gamma(-\frac{4}{\gamma^2})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(\mu_1^{\frac{4}{\gamma^2}}e^{-i\pi\frac{2\beta}{\gamma}}-\mu_2^{\frac{4}{\gamma^2}}) \tag{3.3.53}$$
$$=-2\gamma(Q-\beta)e^{i\pi(-\frac{2\beta}{\gamma}+\frac{4}{\gamma^2})}\frac{\Gamma(\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}{\Gamma(-\frac{4}{\gamma^2})\Gamma(\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma})}\frac{\overline{R}(\beta,\mu_1,\mu_2)}{\overline{R}(\beta+\frac{2}{\gamma},\mu_1,e^{i\pi}\mu_2)}.$$

After simplification:

$$\frac{\overline{R}(\beta,\mu_1,\mu_2)}{\overline{R}(\beta+\frac{2}{\gamma},\mu_1,e^{i\pi}\mu_2)}=-\frac{2}{\gamma(Q-\beta)}(2\pi)^{\frac{4}{\gamma^2}-1}\frac{\Gamma(\frac{2\beta}{\gamma})\Gamma(1-\frac{2\beta}{\gamma})}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(\mu_1^{\frac{4}{\gamma^2}}-\mu_2^{\frac{4}{\gamma^2}}e^{i\pi\frac{2\beta}{\gamma}}). \tag{3.3.54}$$

Similarly by working with auxiliary function $\tilde{H}_\chi(t)$ yields the shift equation:

$$\frac{\overline{R}(\beta+\frac{2}{\gamma},\mu_1,e^{i\pi}\mu_2)}{\overline{R}(\beta+\frac{4}{\gamma},\mu_1,\mu_2)}=-\frac{2}{\gamma(\frac{\gamma}{2}-\beta)}(2\pi)^{\frac{4}{\gamma^2}-1}\frac{\Gamma(\frac{2\beta}{\gamma}+\frac{4}{\gamma^2})\Gamma(1-\frac{2\beta}{\gamma}-\frac{4}{\gamma^2})}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(\mu_1^{\frac{4}{\gamma^2}}-\mu_2^{\frac{4}{\gamma^2}}e^{-i\pi\frac{2\beta}{\gamma}}). \tag{3.3.55}$$

Hence we arrive at (3.3.31). □

3.3.2.5 Solution of the shift equation on $\overline{R}$

Proof of Theorem 3.1.7, equation (3.1.27). We introduce σ_1,σ_2 defined through the relation $\mu_i := e^{i\pi\gamma(\sigma_i-\frac{Q}{2})}$ with the convention that for positive μ_i one has $\mathrm{Re}(\sigma_i)=\frac{Q}{2}$. We can thus write for $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$ that:

$$\left|\mu_1^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{i\pi\chi\beta}\right|^2=4e^{2i\pi\chi(\sigma_1+\sigma_2-Q)}\sin\left(\frac{\pi\chi}{2}(\beta+2(\sigma_1-\sigma_2))\right)\sin\left(\frac{\pi\chi}{2}(\beta+2(\sigma_2-\sigma_1))\right). \tag{3.3.56}$$

One can then rewrite the two shift equations under the following form,

$$\frac{\overline{R}(\beta,\mu_1,\mu_2)}{\overline{R}(\beta+\gamma,\mu_1,\mu_2)}=-\frac{\Gamma(-1+\frac{\gamma\beta}{2}-\frac{\gamma^2}{4})\Gamma(2-\frac{\gamma\beta}{2}-\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})^2}\frac{\pi}{\sin(\pi\frac{\gamma\beta}{2})} \tag{3.3.57}$$
$$\times 4e^{i\pi\gamma(\sigma_1+\sigma_2-Q)}\sin\left(\frac{\pi\gamma}{4}(\beta+2(\sigma_1-\sigma_2))\right)\sin\left(\frac{\pi\gamma}{4}(\beta+2(\sigma_2-\sigma_1))\right),$$

$$\frac{\overline{R}(\beta,\mu_1,\mu_2)}{\overline{R}(\beta+\frac{4}{\gamma},\mu_1,\mu_2)}=\frac{(2\pi)^{\frac{8}{\gamma^2}}}{\gamma^2(Q-\beta)(\frac{\gamma}{2}-\beta)}\frac{1}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{8}{\gamma^2}}\sin(\pi\frac{2\beta}{\gamma})\sin(\pi(\frac{2\beta}{\gamma}+\frac{4}{\gamma^2}))} \tag{3.3.58}$$
$$\times 4e^{\frac{4i\pi}{\gamma}(\sigma_1+\sigma_2-Q)}\sin\left(\frac{\pi}{\gamma}(\beta+2(\sigma_1-\sigma_2))\right)\sin\left(\frac{\pi}{\gamma}(\beta+2(\sigma_2-\sigma_1))\right).$$

These two shift equation completely specify the function $\overline{R}(\beta,\mu_1,\mu_2)$ as a function of the parameter β up to one value. Since we know that $\overline{R}(Q,\mu_1,\mu_2)=1$, the function $\overline{R}$ is thus uniquely specified and can be identified to be the function written in equation (3.1.27) since it obeys the same shift equations and has the same value at $\beta=Q$. □

3.3.3 Solving the three-point function

With the value of $\overline{R}$ completely specified, we complete the proof of the expression for $\overline{H}$. The first step is to derive the additional shift equation in $\frac{2}{\gamma}$.

3.3.3.1 The shift equations for $\overline{H}$

Proposition 3.3.8 (Shift equations for $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$)**.** *Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. We have the following functional equations for $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$, viewed as a meromorphic function of all of its parameters,*

$$\begin{aligned}\overline{H}^{(\beta_1,\beta_2-\chi,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} &= \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(\chi(\beta_1-\chi+q\frac{\gamma}{2}))\Gamma(1-\chi\beta_2+\chi^2-q\frac{\gamma\chi}{2})}\overline{H}^{(\beta_1-\chi,\beta_2,\beta_3)}_{(\mu_1,e^{\frac{i\pi\gamma\chi}{2}}\mu_2,\mu_3)} \\ &- \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\pi\Gamma(-q+\frac{2\chi}{\gamma})\Gamma(1-\chi\beta_1)\Gamma(1-\chi\beta_2+\chi^2)(\mu_1^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{2i\pi\chi\beta_1})}{\sin(\pi\chi(\beta_1-\chi))\Gamma(-q)\Gamma(1+\frac{q\gamma\chi}{2})\Gamma(2-\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))}\overline{H}^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\mu_1,e^{\frac{i\pi\gamma\chi}{2}}\mu_2,\mu_3)},\end{aligned} \tag{3.3.59}$$

and:

$$\begin{aligned}&\frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\Gamma(-q+\frac{2\chi}{\gamma})\Gamma(1-\chi\beta_2)}{\Gamma(-q)}(\mu_3^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{i\pi\chi\beta_2})\overline{H}^{(\beta_1,\beta_2+\chi,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma\chi}{2}}\mu_2,\mu_3)} \\ &= \frac{\Gamma(\chi(\beta_1-\chi))}{\Gamma(-q\frac{\gamma\chi}{2})\Gamma(-1+\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))}\overline{H}^{(\beta_1-\chi,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} \\ &+ \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\Gamma(2-\chi\beta_1+\chi^2)\Gamma(-q+\frac{2\chi}{\gamma})\Gamma(1-\chi\beta_1)\Gamma(\chi(\beta_1-Q))}{\Gamma(-q)\Gamma(1-\chi(\beta_1-\chi+q\frac{\gamma}{2})\Gamma(\chi\beta_2-\chi^2+q\frac{\gamma\chi}{2})}(\mu_1^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{i\pi\chi(\chi-\beta_1)})\overline{H}^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}.\end{aligned} \tag{3.3.60}$$

Proof. These shift equations all come from applying (3.5.105). The first comes from the relation,

$$B_1 = \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(\chi(\beta_1-\chi+q\frac{\gamma}{2})\Gamma(1-\chi\beta_2+\chi^2-q\frac{\gamma\chi}{2})}C_1 + \frac{\Gamma(2-\chi\beta_1+\chi^2)\Gamma(1-\chi\beta_2+\chi^2)}{\Gamma(1+\frac{q\gamma\chi}{2})\Gamma(2-\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))}C_2, \tag{3.3.61}$$

and the second can be deduced by replacing β_1 by $\beta_1+\frac{\gamma}{2}$ in the following relation:

$$\tilde{B}_2 = \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(-1+\chi\beta_2-\chi^2)}{\Gamma(-q\frac{\gamma\chi}{2})\Gamma(-1+\chi(\beta_1+\beta_2-2\chi+q\frac{\gamma}{2}))}\tilde{C}_1 + \frac{\Gamma(2-\chi\beta_1+\chi^2)\Gamma(-1+\chi\beta_2-\chi^2)}{\Gamma(1-\chi(\beta_1-\chi+q\frac{\gamma}{2})\Gamma(\chi\beta_2-\chi^2+q\frac{\gamma\chi}{2})}\tilde{C}_2. \tag{3.3.62}$$

The case of interest $\chi=\frac{2}{\gamma}$ requires a little bit more effort than for deriving the $\chi=\frac{\gamma}{2}$ shift equations. For instance the expression of C_2 is expressed as $\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi}\mu_2,e^{i\pi}\mu_3)}$. To transform it into $\overline{H}^{(\beta_1+\frac{2}{\gamma},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi}\mu_2,e^{i\pi}\mu_3)}$ we will need to apply the shift equation of $\frac{\overline{R}(\beta_1,\mu_1,\mu_2)}{\overline{R}(\beta_1+\frac{2}{\gamma},\mu_1,e^{i\pi}\mu_2)}$ and then the reflection principle of Lemma 3.3.4. The same strategy has to be applied to $\tilde{C}_2$ and $\tilde{B}_2$. This allows us to write:

$$C_2 = \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\Gamma(-q+\frac{2\chi}{\gamma})\Gamma(1-\chi\beta_1)\Gamma(\chi(\beta_1-Q))}{\Gamma(-q)}(\mu_1^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{i\pi\chi\beta_1})\overline{H}^{(\beta_1+\chi,\beta_2,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma\chi}{2}}\mu_2,e^{i\pi\frac{\gamma\chi}{2}}\mu_3)}, \tag{3.3.63}$$

$$\tilde{C}_2 = \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\Gamma(-q+\frac{2\chi}{\gamma})\Gamma(1-\chi\beta_1)\Gamma(\chi(\beta_1-Q))}{\Gamma(-q)}(\mu_1^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{i\pi\chi(\chi-\beta_1)})\overline{H}^{(\beta_1+\chi,\beta_2,\beta_3)}_{(e^{i\pi\frac{\gamma\chi}{2}}\mu_1,\mu_2,\mu_3)}, \tag{3.3.64}$$

$$\tilde{B}_2 = \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\Gamma(-q+\frac{2\chi}{\gamma})\Gamma(1-\chi\beta_2)\Gamma(\chi(\beta_2-Q))}{\Gamma(-q)}(\mu_3^{\frac{2\chi}{\gamma}}-\mu_2^{\frac{2\chi}{\gamma}}e^{i\pi\chi\beta_2})\overline{H}^{(\beta_1,\beta_2+\chi,\beta_3)}_{(e^{i\pi\frac{\gamma\chi}{2}}\mu_1,e^{i\pi\frac{\gamma\chi}{2}}\mu_2,\mu_3)}. \tag{3.3.65}$$

Putting all these into the crossing relation proves the shift equations stated in the proposition. □

3.3.3.2 The exact formula I_{PT} satisfies the shift equations and the reflection principle

Take again $\mu_i := e^{i\pi\gamma(\sigma_i-\frac{Q}{2})}$ with the convention that $\mathrm{Re}(\sigma_i)=\frac{Q}{2}$ when $\mu_i>0$. Recall also that $\overline{\beta}=\beta_1+\beta_2+\beta_3$.

To show that $\overline{H}$ is equal to the exact formula I_{PT} given by (3.1.24), there are three steps that remain to be shown. 1) The function I_{PT} satisfies the shift equations of Lemma 3.3.8. 2) A solution of the shift equations of Lemma 3.3.8 is completely specified up to one global constant. 3) I_{PT} and $\overline{H}$ are equal at one particular value of parameters. In the following we successively show these three claims. We introduce the following notation:

$$I_{PT}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix} := \int_{-i\infty}^{i\infty} \varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)dr. \tag{3.3.66}$$

Lets start by showing the lemma:

Lemma 3.3.9. *The function* $I_{PT}\begin{pmatrix}\beta_1,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}$ *satisfies the shift equations satisfied by* $\overline{H}$.

Proof. Checking that I_{PT} satisfies the shift equations of Lemma 3.3.8 is equivalent to checking the following shift equations,

$$\begin{aligned}\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} &= \frac{\Gamma(\chi(\beta_1-\chi))\Gamma(1-\chi\beta_2)}{\Gamma(1-\chi(\beta_2+\beta_3-\beta_1))\Gamma(\frac{\chi}{2}(\beta_1+\beta_3-\beta_2-2\chi))}\overline{H}^{(\beta_1-\chi,\beta_2+\chi,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma\chi}{2}}\mu_2,\mu_3)} \\ &- \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\pi\Gamma(\frac{1}{\gamma}(\overline{\beta}-\frac{2}{\chi}))\Gamma(1-\chi\beta_1)\Gamma(1-\chi\beta_2)2ie^{i\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1+\sigma_2)}\sin(\pi\chi(\frac{\beta_1}{2}-\sigma_1+\sigma_2))}{\sin(\pi\chi(\beta_1-\chi))\Gamma(\frac{1}{\gamma}(\overline{\beta}-2Q))\Gamma(1+\chi(Q-\frac{\overline{\beta}}{2}))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\overline{H}^{(\beta_1+\chi,\beta_2+\chi,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma\chi}{2}}\mu_2,\mu_3)},\end{aligned} \tag{3.3.67}$$

and:

$$\begin{aligned}&\frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\Gamma(\frac{1}{\gamma}(\overline{\beta}-\frac{2}{\chi}))\Gamma(1-\chi\beta_2)}{\Gamma(\frac{1}{\gamma}(\overline{\beta}-2Q))}2ie^{i\frac{\pi\gamma}{2}(\frac{\beta_2}{2}-\chi+\sigma_2+\sigma_3)}\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3))\overline{H}^{(\beta_1+\chi,\beta_2+\chi,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma\chi}{2}}\mu_2,\mu_3)} \\ &= \frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} - \frac{\chi^2(2\pi)^{\frac{2\chi}{\gamma}-1}}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\frac{\pi\Gamma(\frac{1}{\gamma}(\overline{\beta}-\frac{2}{\chi}))\Gamma(1-\chi\beta_1-\chi^2))}{\sin(\pi\chi\beta_1)\Gamma(\frac{1}{\gamma}(\overline{\beta}-2Q))} \\ &\times\frac{2ie^{i\pi\chi(-\frac{\beta_1}{2}-\chi+\sigma_1+\sigma_2)}\sin(\pi\chi(-\frac{\beta_1}{2}-\sigma_1+\sigma_2))}{\Gamma(\frac{\chi}{2}(\beta_2+\beta_3-\beta_1-2\chi))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_3-\beta_2))}\overline{H}^{(\beta_1+2\chi,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}.\end{aligned} \tag{3.3.68}$$

We calculate the ratios of the integrands,

$$\begin{aligned}\frac{\varphi^{(\beta_1-\chi,\beta_2+\chi,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)}(s)}{\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(s)} =& \frac{\Gamma(\frac{\chi}{2}(\beta_1+\beta_3-\beta_2-2\chi))\Gamma(1-\frac{\chi}{2}(\beta_2+\beta_3-\beta_1))\Gamma(1-\chi\beta_1+\chi^2)}{\pi\Gamma(1-\chi\beta_2)} \\ &\times\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1-\sigma_2))\frac{\sin(\pi\chi(-\frac{\beta_1}{2}+\frac{\beta_2}{2}+\sigma_1-\sigma_3-s))}{\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-s))},\end{aligned} \tag{3.3.69}$$

$$\begin{aligned}\frac{\varphi^{(\beta_1+\chi,\beta_2+\chi,\beta_3)}_{(\sigma_1,\sigma_2+\frac{\chi}{2},\sigma_3)}(s)}{\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(s)} =& -\frac{\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}\Gamma(\frac{1}{\gamma}(\overline{\beta}-2Q))\Gamma(1+\chi(Q-\frac{\overline{\beta}}{2}))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))ie^{i\pi\chi(Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)}}{\chi^2(2\pi)^{\frac{2\chi}{\gamma}}\Gamma(\frac{1}{\gamma}(\overline{\beta}-\frac{2}{\chi}))\Gamma(1-\chi\beta_1)\Gamma(1-\chi\beta_2)} \\ &\times\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-s))}{\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-s))}.\end{aligned} \tag{3.3.70}$$

If we plug I_{PT} into equation (3.3.67) and regroup terms on one side, we will get:

$$\begin{aligned}\int_{-i\infty}^{i\infty} dr\,\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)\Bigg[&\frac{\sin(\pi\chi(\frac{\beta_1}{2}-\chi+\sigma_1-\sigma_2))\sin(\pi\chi(-\frac{\beta_1}{2}+\frac{\beta_2}{2}+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\beta_1-\chi))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}-1 \\ &+\frac{\sin(\pi\chi(\frac{\beta_1}{2}-\sigma_1+\sigma_2))\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\beta_1-\chi))\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}\Bigg].\end{aligned} \tag{3.3.71}$$

We can verify with some algebra that the integrand of the above integral equals 0, hence I_{PT} satisfies (3.3.67). To check the second shift equation, we will need additionally the ratio:

$$\frac{\varphi^{(\beta_1+2\chi,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)}{\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)} = -\frac{\pi\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}\Gamma(\frac{1}{\gamma}(\overline{\beta}-2Q))\Gamma(1+\chi(Q-\frac{\overline{\beta}}{2}))\Gamma(1-\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}{\chi^2(2\pi)^{\frac{2\chi}{\gamma}}\Gamma(\frac{1}{\gamma}(\overline{\beta}-\frac{2}{\chi}))\Gamma(\frac{\chi}{2}(\beta_1+\beta_3-\beta_2))\Gamma(1-\frac{\chi}{2}(\beta_2+\beta_3-\beta_1-2\chi))} \frac{ie^{i\pi\chi(Q-\frac{\beta_1}{2}-\sigma_1-\sigma_2)}\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\Gamma(1-\chi\beta_1-\chi^2)\Gamma(1-\chi\beta_1)\sin(\pi\chi(\frac{\beta_1}{2}+\sigma_1-\sigma_2))\sin(\pi\chi(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi+\sigma_3-\sigma_1+r))}. \tag{3.3.72}$$

If we plug I_{PT} into equation (3.3.68) and regroup things on one side, we will get:

$$\begin{aligned}
&\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\int_{-i\infty}^{i\infty} dr\,\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)\\
&\Bigg[\frac{\sin(\pi\chi\beta_1)\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3))e^{i\pi\chi(-\frac{\beta_1}{2}+\frac{\beta_2}{2}-\sigma_1+\sigma_3)}}{\sin(\pi\chi(\frac{\overline{\beta}}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))}\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}\\
&-1+\frac{\sin(\frac{\pi\chi}{2}(\beta_1+\beta_3-\beta_2))\sin(\frac{\pi\chi}{2}(\beta_2+\beta_3-\beta_1-2\chi))e^{-i\pi\chi\beta_1}}{\sin(\pi\chi(\frac{\overline{\beta}}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))}\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi+\sigma_3-\sigma_1+r))}\Bigg].
\end{aligned}$$

After some algebra we will be able to write it in the form,

$$\begin{aligned}
&\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\frac{\sin(\pi\chi\beta_1)e^{i\pi\chi(-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi-\sigma_1+\sigma_3)}}{\sin(\pi\chi(\frac{\overline{\beta}}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))}\int_{-i\infty}^{i\infty} dr\,\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2,\sigma_3)}(r)e^{i\pi\chi r}\\
&\Bigg[\frac{\sin(\pi\chi(\frac{\beta_1}{2}+\frac{\beta_2}{2}-\chi+\sigma_1-\sigma_3-r))\sin(\pi\chi r)}{\sin(\pi\chi(\frac{\beta_2}{2}+\sigma_2-\sigma_3-r))}e^{i\pi\chi(\frac{\beta_2}{2}-\chi-\sigma_2+\sigma_3)}\\
&+\frac{\sin(\pi\chi(\frac{\beta_3}{2}-\sigma_1+\sigma_3+r))\sin(\pi\chi(\frac{\beta_3}{2}-\chi+\sigma_1-\sigma_3-r))}{\sin(\pi\chi(\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi+\sigma_3-\sigma_1+r))}\Bigg]\\
=&\frac{\Gamma(\chi\beta_1)}{\Gamma(\frac{\chi}{2}(\overline{\beta}-2Q))\Gamma(\frac{\chi}{2}(\beta_1+\beta_2-\beta_3))}\frac{\sin(\pi\chi\beta_1)e^{i\pi\chi(-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\chi-\sigma_1+\sigma_3)}}{\sin(\pi\chi(\frac{\overline{\beta}}{2}-\chi))\sin(\frac{\pi\chi}{2}(\beta_1+\beta_2-\beta_3))}\\
&\times\frac{1}{2}\int_{-i\infty}^{i\infty} ds\,(T_{-\chi}-1)\varphi^{(\beta_1,\beta_2,\beta_3)}_{(\sigma_1,\sigma_2+\chi,\sigma_3+\chi)}(s)e^{i\pi\chi r},
\end{aligned}$$

where $T_{-\chi}f(r)=f(r-\chi)$ for any function f. The integral term should be understood as a contour integral and it equals 0 thanks to Cauchy's theorem. □

Next we move on to showing:

Lemma 3.3.10. *The function I_{PT} satisfies the following two properties,*

$$I_{PT}\begin{pmatrix}2Q-\beta_2-\beta_3,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}=1, \tag{3.3.73}$$

and the reflection principle of Lemma 3.3.4.

Proof. It is rather direct to observe that it satisfies the reflection principle, since the integrand of the Barnes type integral is not changed when applying the transform $\beta_1\to 2Q-\beta_1$. The rest is an easy algebra using the shift equations of $\Gamma_{\frac{\gamma}{2}}$ and $S_{\frac{\gamma}{2}}$. To see the special value at $\beta_1=2Q-\beta_2-\beta_3$ equals 1, we will need to apply the residue theorem. When β_1 approaches $2Q-\beta_2-\beta_3$ from the right hand side, we have a preceding

term $\frac{1}{\Gamma(\frac{\overline{\beta}-2Q}{\gamma})}$ that goes to 0. Additionally in the Barnes type integral, the two poles at $r=-(\frac{\beta_3}{2}+\sigma_3-\sigma_1)$ and $r=-(Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)$ will collapse. To extract the divergent term that can be compensated with the preceding term, we can slightly modify the contour to let it go from the right hand side of $r=-(Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1)$, this allows us to pick up the divergent term by residue theorem:

$$\int_{-i\infty}^{i\infty}\frac{S_{\frac{\gamma}{2}}(Q-\frac{\beta_2}{2}+\sigma_3-\sigma_2+r)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_3-\sigma_1+r)}{S_{\frac{\gamma}{2}}(Q+\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(2Q-\frac{\beta_1}{2}-\frac{\beta_2}{2}+\sigma_3-\sigma_1+r)S_{\frac{\gamma}{2}}(Q+r)}e^{i\pi(-\frac{\beta_2}{2}+\sigma_2-\sigma_3)r}\frac{dr}{i}$$
$$\underset{\beta_1\to 2Q-\beta_2-\beta_3}{\sim}\frac{1}{2\pi(\frac{\overline{\beta}}{2}-Q)}\frac{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(Q-\beta_3)}{S_{\frac{\gamma}{2}}(\beta_1)S_{\frac{\gamma}{2}}(Q-\frac{\beta_3}{2}+\sigma_1-\sigma_3)}e^{i\pi(\frac{\beta_2}{2}-\sigma_2+\sigma_3)(\frac{\beta_3}{2}+\sigma_3-\sigma_1)}$$

We can check that when $\beta_1\to 2Q-\beta_2-\beta_3$, the preceding term is equivalent to

$$2\pi(\frac{\overline{\beta}}{2}-Q)S_{\frac{\gamma}{2}}(\beta_1)S_{\frac{\gamma}{2}}(\beta_3)\frac{e^{-i\pi(\frac{\beta_2}{2}-\sigma_2+\sigma_3)(\frac{\beta_3}{2}+\sigma_3-\sigma_1)}}{S_{\frac{\gamma}{2}}(\frac{\beta_1}{2}+\sigma_1-\sigma_2)S_{\frac{\gamma}{2}}(\frac{\beta_3}{2}+\sigma_3-\sigma_1)} \tag{3.3.74}$$

This proves that $I_{PT}\begin{pmatrix}2Q-\beta_2-\beta_3,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}=1$.

□

3.3.3.3 Uniqueness of the shift equations on $\overline{H}$

We will now finish the proof of Theorem 3.1.7.

Proof of Theorem 3.1.7, equation (3.1.28). The information to extract from (3.3.59) and (3.3.60) is that we have a three term shift equation on β_1. If we fix the parameters $\beta_2,\beta_3,\mu_1,\mu_2,\mu_3$, then we know that $\beta_1\mapsto H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ is a solution to the following shift equations (here we denote $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$):

$$(A_{2\chi}(x)T_{2\chi}+B_{2\chi}(x)+C_{2\chi}(x)T_{-2\chi})g(x)=0, \tag{3.3.75}$$

with $T_a g(x)=g(x+a)$. Note that we have analytically extended the domain of definition for $H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ to all values of $\beta_1\in\mathbb{R}$. If we further impose that g satisfies the reflection principle, i.e.

$$g(x)=-\frac{\Gamma(1-\frac{2(Q-x)}{\gamma})\Gamma(\frac{\beta_2+\beta_3-x}{\gamma})\overline{R}(x,\mu_1,\mu_2)}{\Gamma(\frac{x+\beta_2+\beta_3-2Q}{\gamma})}g(2Q-x)=:S(x)g(2Q-x), \tag{3.3.76}$$

it can be shown that the dimension of the solution space is at most 1.

In fact suppose that g is a solution to the system of equations (3.3.75) and (3.3.76). The equation (3.3.75) with $\chi=\frac{\gamma}{2}$ tells that g is characterized by its values in $[Q-\gamma,Q+\gamma]$. Together with (3.3.76), g can be characterized by its values in $[Q-\gamma,Q)$. Since we already know a solution I_{PT} to this system, we can define c_1 a γ-periodic function by

$$c_1(x)=\frac{g(x)}{I_{PT}(x)},x\in[Q-\gamma,Q). \tag{3.3.77}$$

Then it is easy to see that $c_1(x)I_{PT}(x)$ is still a solution to

$$(A_\gamma(x)T_\gamma+B_\gamma(x)+C_\gamma(x)T_{-\gamma})g(x)=0, \tag{3.3.78}$$

and it has the same value as $g(x)$ on the interval $[Q-\gamma,Q)$, hence we should have $g(x)=c_1(x)I_{PT}(x)$ for $x\in\mathbb{R}$. In the same manner we can show that there exists a $\frac{4}{\gamma}$-periodic function $c_2(x)$ such that $g(x)=c_2(x)I_{PT}(x)$ for $x\in\mathbb{R}$. Indeed we should have $c_1(x)=c_2(x)$ which is γ-periodic and $\frac{4}{\gamma}$-periodic. This implies that $g(x)=cI_{PT}(x)$ with c a constant and that the solution space is of dimension 1. We can fix the constant c to be 1 with the special value $g(2Q-\beta_2-\beta_3)=1$. This proves uniqueness of the solution. We prove in the following that I_{PT} indeed satisfies the reflection principle and $I_{PT}\begin{pmatrix}2Q-\beta_2-\beta_3,\beta_2,\beta_3\\ \sigma_1,\sigma_2,\sigma_3\end{pmatrix}=1$.

□

3.3.3.4 Consistency with the interval GMC

Finally we include here the consistency check that our formula matches the one of [89]. To check that the result is coherent with the interval case, we look at the limit when $\mathrm{Im}(\sigma_1), \mathrm{Im}(\sigma_3) \to +\infty$ and $\sigma_2 = \frac{Q}{2}$ and use the asymptotic of $S_{\frac{\gamma}{2}}$ function provided in the appendix. First let us do a change of variable $r \to r - \frac{\beta_3}{2} + \sigma_1 - \sigma_3$, the Barnes type integral will become

$$\int_{-i\infty}^{i\infty} \frac{S_{\frac{\gamma}{2}}(Q - \frac{\beta_2}{2} - \frac{\beta_3}{2} + \sigma_1 - \sigma_2 + r) S_{\frac{\gamma}{2}}(r) S_{\frac{\gamma}{2}}(Q - \beta_3 + r) e^{i\pi(-\frac{\beta_2}{2} + \sigma_2 - \sigma_3)(-\frac{\beta_3}{2} + \sigma_1 - \sigma_3)}}{S_{\frac{\gamma}{2}}(Q + \frac{\beta_1}{2} - \frac{\beta_2}{2} - \frac{\beta_3}{2} + r) S_{\frac{\gamma}{2}}(2Q - \frac{\overline{\beta}}{2} + r) S_{\frac{\gamma}{2}}(Q - \frac{\beta_3}{2} + \sigma_1 - \sigma_3 + s)} e^{i\pi(-\frac{\beta_2}{2} + \sigma_2 - \sigma_3) r} \frac{dr}{i}. \tag{3.3.79}$$

We send $\mathrm{Im}(\sigma_1) \to +\infty$ and do the change $r \to -r$:

$$e^{i\pi(-\frac{\beta_2}{2} + \sigma_2 - \sigma_3)(-\frac{\beta_3}{2} + \sigma_1 - \sigma_3)} e^{i\frac{\pi}{2}(-(Q - \frac{\beta_2}{2} - \frac{\beta_3}{2} + \sigma_1 - \sigma_2)(-\frac{\beta_2}{2} - \frac{\beta_3}{2} + \sigma_1 - \sigma_2) + (Q - \frac{\beta_3}{2} + \sigma_1 - \sigma_3)(-\frac{\beta_3}{2} + \sigma_1 - \sigma_3))}$$
$$\times \int_{-i\infty}^{i\infty} \frac{S_{\frac{\gamma}{2}}(\frac{\beta_2}{2} + \frac{\beta_3}{2} - \frac{\beta_1}{2} + s) S_{\frac{\gamma}{2}}(\frac{\overline{\beta}}{2} - Q + s)}{S_{\frac{\gamma}{2}}(Q + s) S_{\frac{\gamma}{2}}(\beta_3 + s)} e^{-i2\pi(\sigma_2 - \sigma_3)s} \frac{ds}{i}. \tag{3.3.80}$$

From the result in [83] on the b-hypergeometric functions, when $\mathrm{Im}(\sigma_3) \to +\infty$, the above integral (excluding the preceding term) converges to:

$$\frac{S_{\frac{\gamma}{2}}(\frac{\beta_2}{2} + \frac{\beta_3}{2} - \frac{\beta_1}{2}) S_{\frac{\gamma}{2}}(\frac{\beta}{2} - Q)}{S_{\frac{\gamma}{2}}(\beta_3)}. \tag{3.3.81}$$

The rest of the terms in I_{PT} are much easier to analyse. Putting everything together and taking $\mathrm{Im}(\sigma_1), \mathrm{Im}(\sigma_3) \to +\infty$, $\sigma_2 = \frac{Q}{2}$ will yield after simplification:

$$H^{(\beta_1,\beta_2,\beta_3)}_{(0,1,0)} = \frac{(2\pi)^{\frac{2Q-\overline{\beta}}{\gamma}+1} (\frac{2}{\gamma})^{(\frac{\gamma}{2} - \frac{2}{\gamma})(Q - \frac{\overline{\beta}}{2}) - 1}}{\Gamma(1 - \frac{\gamma^2}{4})^{\frac{2Q-\overline{\beta}}{\gamma}} \Gamma(\frac{\overline{\beta} - 2Q}{\gamma})} \frac{\Gamma_{\frac{\gamma}{2}}(\frac{\overline{\beta}}{2} - Q) \Gamma_{\frac{\gamma}{2}}(\frac{\beta_1 + \beta_3 - \beta_2}{2}) \Gamma_{\frac{\gamma}{2}}(\frac{\beta_2 + \beta_3 - \beta_1}{2}) \Gamma_{\frac{\gamma}{2}}(Q - \frac{\beta_1 + \beta_2 - \beta_3}{2})}{\Gamma_{\frac{\gamma}{2}}(Q) \Gamma_{\frac{\gamma}{2}}(Q - \beta_1) \Gamma_{\frac{\gamma}{2}}(Q - \beta_2) \Gamma_{\frac{\gamma}{2}}(\beta_3)}. \tag{3.3.82}$$

It can be easily checked that this formula is exactly the same as what the authors have found in [89].

3.4 Proof of the BPZ differential equations

The goal of this section is to check the BPZ differential equations - reducing in our case to the standard hypergeometric equations - that have been used extensively in the previous two sections.

3.4.1 Bulk-boundary case

We prove here the differential equation used in Section 3.2.

Proposition 3.4.1. *Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$, $p = \frac{2}{\gamma}(Q - \alpha - \frac{\beta}{2} + \frac{\chi}{2})$. Consider in the following parameter range,*

$$\beta < Q, \quad p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q - \beta), \quad t \in \mathbb{H}, \tag{3.4.1}$$

the auxiliary function,

$$G_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x - i|^{\gamma\alpha}} g(x)^{\frac{\gamma^2}{8}(p-1)} e^{\frac{\gamma}{2} X(x)} dx \right)^p \right]. \tag{3.4.2}$$

Consider furthermore for $t \in \{re^{i\theta} \,|\, r > 0, \theta \in (0, \frac{\pi}{2})\}$ *the change of variable* $s = \frac{1}{1+t^2}$ *and* $\tilde{G}_\chi(s) = s^{p\frac{\gamma\chi}{4}} G_\chi(t)$. *Then the function* $\tilde{G}_\chi(s)$ *obeys the hypergeometric equation,*

$$s(1-s)\partial_s^2 \tilde{G}_\chi(s) + (C - (A+B+1)s)\partial_s \tilde{G}_\chi(s) - AB\tilde{G}_\chi(s) = 0, \tag{3.4.3}$$

with the parameters defined by:

$$A = -p\frac{\gamma\chi}{4}, B = 1 + \chi(\chi - \alpha - p\frac{\gamma}{4}), C = \frac{3}{2} + \chi(\chi - \alpha - p\frac{\gamma}{2}). \tag{3.4.4}$$

Remark 3.4.2. *As explained in Section 3.2, in the change of variable from* t *to* s *the argument of* s *is in* $(-\pi, 0)$ *and one has* $\sqrt{1-s} = t\sqrt{s}$. *Furthermore, the exact same hypergeometric equation holds for the dual function* $\hat{G}_\chi(s)$ *introduced in Section 3.2.*

Proof. For simplicity, we introduce the notations,

$$V_1(x_1;t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}|x-x_1|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} g(x)^{\frac{\gamma^2}{8}(p-2)} dx\right)^{p-1}\right], \tag{3.4.5}$$

$$V_2(x_1,x_2;t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x-i|^{\gamma\alpha}|x-x_1|^{\frac{\gamma^2}{2}}|x-x_2|^{\frac{\gamma^2}{2}}} e^{\frac{\gamma}{2}X(x)} g(x)^{\frac{\gamma^2}{8}(p-3)} dx\right)^{p-2}\right]. \tag{3.4.6}$$

We will not be bothered here with the regularization procedure of the log-correlated field X that must in principle be used to perform the computations. A fully rigorous proof implementing the regularization method can be found in [89]. Let us compute the derivatives of the function $G_\chi(t)$ with the help of the Girsanov theorem 3.5.1:

$$\begin{aligned}\partial_t G_\chi =& p\int_{\mathbb{R}} dx_1 \frac{\partial_t (t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}} V_1(x_1;t) = -p\int_{\mathbb{R}} dx_1 \frac{\partial_{x_1}(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}} V_1(x_1;t) \\ =& -p\frac{\gamma}{2}\int_{\mathbb{R}} dx_1 \left(\frac{\alpha}{x_1-i} + \frac{\alpha}{x_1+i}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}} V_1(x_1;t) + p\int_{\mathbb{R}} dx_1 \frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}} \partial_{x_1} V_1(x_1;t).\end{aligned} \tag{3.4.7}$$

We compute the last term:

$$\begin{aligned}&\int_{\mathbb{R}} dx_1 \frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}} \partial_{x_1} V_1(x_1;t) \\ &= -(p-1)\frac{\gamma^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dx_1 dx_2 \frac{1}{x_1-x_2} \frac{(t-x_1)^{\frac{\gamma\chi}{2}}(t-x_2)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}|x_2-i|^{\gamma\alpha}|x_1-x_2|^{\frac{\gamma^2}{2}}} V_2(x_1,x_2;t) \\ &= 0 \qquad \text{by the symmetry} \quad x_1 \leftrightarrow x_2.\end{aligned} \tag{3.4.8}$$

Note that integrability problem of $\frac{1}{x_1-x_2}$ can be handled by taking the regularized version of X, see [89] for more details. We can also compute the first order derivative directly without doing integration by parts:

$$\partial_t G_\chi = p\frac{\gamma\chi}{2}\int_{\mathbb{R}} dx_1 \frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}} V_1(x_1;t). \tag{3.4.9}$$

Then we have,

$$\begin{aligned}\partial_t^2 G_\chi =& -p\frac{\gamma\chi}{2}\int_{\mathbb{R}} dx_1 \frac{\partial_{x_1}(t-x_1)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}} V_1(x_1;t) + p\frac{\gamma\chi}{2}\int_{\mathbb{R}} dx_1 \frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}} \partial_t V_1(x_1;t) \\ =& -p\frac{\gamma^2\chi}{4}\int_{\mathbb{R}} dx_1 \left(\frac{\alpha}{x_1-i} + \frac{\alpha}{x_1+i}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}} V_1(x_1;t) \\ &+ p\frac{\gamma\chi}{2}\int_{\mathbb{R}} dx_1 \frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}} (\partial_{x_1} + \partial_t) V_1(x_1;t).\end{aligned} \tag{3.4.10}$$

We can compute the last term by using the symmetry between x_1 and x_2:

$$\begin{aligned}
&\int_{\mathbb{R}} dx_1 \frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}}(\partial_{x_1}+\partial_t)V_1(x_1;t) \qquad (3.4.11)\\
&\quad = (p-1)\frac{\gamma}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dx_1dx_2\left(-\frac{\gamma}{x_1-x_2}+\frac{\chi}{t-x_2}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}(t-x_2)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}|x_2-i|^{\gamma\alpha}|x_1-x_2|^{\frac{\gamma^2}{2}}}V_2(x_1,x_2;t)\\
&\quad = (p-1)\frac{\gamma}{2}\left(-\frac{\gamma}{2}+\chi\right)\int_{\mathbb{R}}\int_{\mathbb{R}} dx_1dx_2\frac{(t-x_1)^{\frac{\gamma\chi}{2}-1}(t-x_2)^{\frac{\gamma\chi}{2}-1}}{|x_1-i|^{\gamma\alpha}|x_2-i|^{\gamma\alpha}|x_1-x_2|^{\frac{\gamma^2}{2}}}V_2(x_1,x_2;t).
\end{aligned}$$

Then we have when $\chi=\frac{\gamma}{2}$,

$$\begin{aligned}
\partial_t^2 G_\chi &= -p\frac{\gamma^2\chi}{4}\int_{\mathbb{R}} dx_1\left(\frac{1}{t-x_1}\left(\frac{\alpha}{t-i}+\frac{\alpha}{t+i}\right)+\frac{\alpha}{(t-i)(x_1-i)}+\frac{\alpha}{(t+i)(x_1+i)}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t)\\
&= -\frac{\gamma}{2}\left(\frac{\alpha}{t-i}+\frac{\alpha}{t+i}\right)\partial_t G - p\frac{\gamma^3}{8}\int_{\mathbb{R}} dx_1\left(\frac{\alpha}{(t-i)(x_1-i)}+\frac{\alpha}{(t+i)(x_1+i)}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t).
\end{aligned} \qquad (3.4.12)$$

The expression of $\partial_t^2 G_\chi$ in the case where $\chi=\frac{2}{\gamma}$ uses the fact that $\partial_{x_1}(t-x_1)^{\frac{\gamma\chi}{2}-1}=0=\frac{4}{\gamma^2}\partial_{x_1}(t-x_1)^{\frac{\gamma\chi}{2}-1}$. We repeat the previous computation and obtain when $\chi=\frac{2}{\gamma}$,

$$\partial_t^2 G_\chi = -\frac{2}{\gamma}\left(\frac{\alpha}{t-i}+\frac{\alpha}{t+i}\right)\partial_t G_\chi - p\frac{2}{\gamma}\int_{\mathbb{R}} dx_1\left(\frac{\alpha}{(t-i)(x_1-i)}+\frac{\alpha}{(t+i)(x_1+i)}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t). \qquad (3.4.13)$$

We can also write G_χ in a similar form. An integration by parts together with the symmetry shows that:

$$\begin{aligned}
(\frac{\gamma\chi}{2}+1)G_\chi &= -\int_{\mathbb{R}} dx_1\frac{\partial_{x_1}(t-x_1)^{\frac{\gamma\chi}{2}+1}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t) \qquad (3.4.14)\\
&= -\frac{\gamma}{2}\int_{\mathbb{R}} dx_1\left(\frac{\alpha}{x_1-i}+\frac{\alpha}{x_1+i}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}+1}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t)+(p-1)\frac{\gamma^2}{4}G_\chi\\
&= -\frac{\gamma}{2}\int_{\mathbb{R}} dx_1\left(\frac{\alpha(t-i)}{x_1-i}+\frac{\alpha(t+i)}{x_1+i}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t)+((p-1)\frac{\gamma^2}{4}+\gamma\alpha)G_\chi.
\end{aligned}$$

Now we summarize the expressions of the derivatives,

$$(2\chi+\frac{1}{\chi}-p\frac{\gamma}{2}-2\alpha)G_\chi = -\int_{\mathbb{R}} dx_1\left(\frac{\alpha(t-i)}{x_1-i}+\frac{\alpha(t+i)}{x_1+i}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t), \qquad (3.4.15)$$

$$\partial_t G_\chi = -p\frac{\gamma}{2}\int_{\mathbb{R}} dx_1\left(\frac{\alpha}{x_1-i}+\frac{\alpha}{x_1+i}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t), \qquad (3.4.16)$$

and when $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$,

$$\partial_t^2 G_\chi = -\chi\left(\frac{\alpha}{t-i}+\frac{\alpha}{t+i}\right)\partial_t G_\chi - p\frac{\gamma\chi^2}{2}\int_{\mathbb{R}} dx_1\left(\frac{\alpha}{(t-i)(x_1-i)}+\frac{\alpha}{(t+i)(x_1+i)}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1-i|^{\gamma\alpha}}V_1(x_1;t). \qquad (3.4.17)$$

Combining everything implies that G_χ satisfies a differential equation:

$$(t^2+1)\partial_t^2 G_\chi + 2\chi(\alpha-\chi)t\partial_t G_\chi + p\chi^2(\gamma\chi+\frac{\gamma}{2\chi}-p\frac{\gamma^2}{4}-\gamma\alpha)G_\chi = 0. \qquad (3.4.18)$$

Now consider $s = \frac{1}{1+t^2}$ and take the function,

$$\tilde{G}_\chi(s) = s^{p\frac{\gamma\chi}{4}} G_\chi(t). \tag{3.4.19}$$

One then has:

$$\partial_s \tilde{G}_\chi(s) = -\frac{1}{2} s^{-\frac{3}{2}} (1-s)^{-\frac{1}{2}} s^{p\frac{\gamma\chi}{4}} \partial_t G_\chi(t) + p\frac{\gamma\chi}{4} s^{-1} \tilde{G}_\chi(s), \tag{3.4.20}$$

$$\begin{aligned}\partial_s^2 \tilde{G}_\chi(s) =& \frac{1}{4}\left((3 - p\frac{\gamma\chi}{2})s^{-1} - (1-s)^{-1}\right) s^{-\frac{3}{2}} (1-s)^{-\frac{1}{2}} s^{p\frac{\gamma\chi}{4}} \partial_t G_\chi(t) \\ &+ \frac{1}{4} s^{-3} (1-s)^{-1} s^{p\frac{\gamma\chi}{4}} \partial_t^2 G_\chi(t) + p\frac{\gamma\chi}{4} s^{-1} \partial_s \tilde{G}_\chi(s) - p\frac{\gamma\chi}{4} s^{-2} \tilde{G}_\chi(s).\end{aligned} \tag{3.4.21}$$

Then,

$$t\partial_t G_\chi = -2s(1-s)\partial_s \tilde{G}_\chi + p\frac{\gamma\chi}{2}(1-s)\tilde{G}_\chi, \tag{3.4.22}$$

$$\begin{aligned}(t^2+1)\partial_t^2 G_\chi =& 4s^2(1-s)\partial_s^2 \tilde{G}_\chi - p\gamma\chi s(1-s)\partial_s \tilde{G}_\chi + p\gamma\chi(1-s)\tilde{G}_\chi \\ &+ 2s\left((p\frac{\gamma\chi}{2} - 4)s + 3 - p\frac{\gamma\chi}{2}\right)\partial_s \tilde{G}_\chi - p\frac{\gamma\chi}{2}\left((p\frac{\gamma\chi}{2} - 4)s + 3 - p\frac{\gamma\chi}{2}\right)\tilde{G}_\chi \\ =& 4s^2(1-s)\partial_s^2 \tilde{G}_\chi + 2s((p\gamma\chi - 4)s + 3 - p\gamma\chi)\partial_s \tilde{G}_\chi + p\frac{\gamma\chi}{2}\left((2 - p\frac{\gamma\chi}{2})s + p\frac{\gamma\chi}{2} - 1\right)\tilde{G}_\chi.\end{aligned} \tag{3.4.23}$$

This allows to transform the equation of G_χ into a hypergeometric equation of $\tilde{G}_\chi$,

$$s(1-s)\partial_s^2 \tilde{G}_\chi + (C - (A+B+1)s)\partial_s \tilde{G}_\chi - AB\tilde{G}_\chi = 0, \tag{3.4.24}$$

with the parameters defined by

$$A = -p\frac{\gamma\chi}{4}, B = 1 + \chi(\chi - \alpha - p\frac{\gamma}{4}), C = \frac{3}{2} + \chi(\chi - \alpha - p\frac{\gamma}{2}). \tag{3.4.25}$$

□

3.4.2 Boundary three-point case

Moving on to the equation used in Section 3.3.

Proposition 3.4.3. *Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$ and $q = \frac{1}{\gamma}(2Q - \beta_1 - \beta_2 - \beta_3 + \chi)$. In the parameter range,*

$$\beta_i < Q, \quad \mu_1 \in (0,\infty), \ \mu_2, \mu_3 \in -\overline{\mathbb{H}}, \quad q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q - \beta_i), \quad t \in \mathbb{H}, \tag{3.4.26}$$

we define the function,

$$H_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^q\right]. \tag{3.4.27}$$

Then $H_\chi(t)$ obeys the hypergeometric equation,

$$t(1-t)\partial_t^2 H + (C - (A+B+1)t)\partial_t H - ABH = 0, \tag{3.4.28}$$

with parameters:

$$A = -q\frac{\gamma\chi}{2}, B = -1 + \chi(\beta_1 + \beta_2 - 2\chi + q\frac{\gamma}{2}), C = \chi(\beta_1 - \chi). \tag{3.4.29}$$

Furthermore the exact same hypergeometric equation holds for the dual function,

$$\tilde{H}_\chi(t) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(x-t)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^q\right], \tag{3.4.30}$$

this time in the range of parameters:

$$t \in -\mathbb{H},\ \beta_i < Q,\ \mu_1, \mu_2 \in -\overline{\mathbb{H}},\ \mu_3 \in (0,\infty), \quad \textit{and} \quad q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q-\beta_i). \tag{3.4.31}$$

Proof. We denote for a small $\epsilon > 0$,

$$R_\epsilon = \mathbb{R}\backslash\{(-\epsilon,\epsilon)\cup(1-\epsilon,1+\epsilon)\}. \tag{3.4.32}$$

Consider

$$H_{\chi,\epsilon}(t) = \mathbb{E}\left[\left(\int_{R_\epsilon} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^q\right]. \tag{3.4.33}$$

For simplicity, we introduce the notations,

$$V_\epsilon(x_1;t) = \mathbb{E}\left[\left(\int_{R_\epsilon} \frac{(t-x)^{\frac{\gamma\chi}{2}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}|x-x_1|^{\frac{\gamma^2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-2)} e^{\frac{\gamma}{2}X(x)} d\mu(x)\right)^{q-1}\right], \tag{3.4.34}$$

$$E^-_{0,\epsilon}(t) = \mu_1 \frac{(t+\epsilon)^{\frac{\gamma\chi}{2}}}{\epsilon^{\frac{\gamma\beta_1}{2}}(1+\epsilon)^{\frac{\gamma\beta_2}{2}}} V_\epsilon(-\epsilon;t), \qquad E^+_{0,\epsilon}(t) = \mu_2 \frac{(t-\epsilon)^{\frac{\gamma\chi}{2}}}{\epsilon^{\frac{\gamma\beta_1}{2}}(1-\epsilon)^{\frac{\gamma\beta_2}{2}}} V_\epsilon(\epsilon;t), \tag{3.4.35}$$

$$E^-_{1,\epsilon}(t) = \mu_2 \frac{(t-1+\epsilon)^{\frac{\gamma\chi}{2}}}{(1-\epsilon)^{\frac{\gamma\beta_1}{2}}\epsilon^{\frac{\gamma\beta_2}{2}}} V_\epsilon(1-\epsilon;t), \quad E^+_{1,\epsilon}(t) = \mu_3 \frac{(t-1-\epsilon)^{\frac{\gamma\chi}{2}}}{(1+\epsilon)^{\frac{\gamma\beta_1}{2}}\epsilon^{\frac{\gamma\beta_2}{2}}} V_\epsilon(1+\epsilon;t). \tag{3.4.36}$$

The proof follows the same step as the previous case, the only difference is that here we have additional boundary terms when performing integration by parts due to the presence of the insertions in 0 and 1. Similarly we compute,

$$\begin{aligned}(2\chi + \frac{1}{\chi} - q\frac{\gamma}{2} - \beta_1 - \beta_2)H_{\chi,\epsilon} = &-\int_{R_\epsilon} d\mu(x_1)\left(\frac{\beta_1 t}{x_1} + \frac{\beta_2(t-1)}{x_1-1}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}} V_\epsilon(x_1;t)\\ &+\frac{2}{\gamma}\left(-(t+\epsilon)E^-_{0,\epsilon}(t) + (t-\epsilon)E^+_{0,\epsilon}(t) - (t-1+\epsilon)E^-_{1,\epsilon}(t) + (t-1-\epsilon)E^+_{1,\epsilon}(t)\right),\end{aligned} \tag{3.4.37}$$

$$\begin{aligned}\partial_t H_{\chi,\epsilon} = &-q\frac{\gamma}{2}\int_{R_\epsilon} d\mu(x_1)\left(\frac{\beta_1}{x_1} + \frac{\beta_2}{x_1-1}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}} V_\epsilon(x_1;t)\\ &+q\left(-E^-_{0,\epsilon}(t) + E^+_{0,\epsilon}(t) - E^-_{1,\epsilon}(t) + E^+_{1,\epsilon}(t)\right),\end{aligned} \tag{3.4.38}$$

$$\begin{aligned}\partial_t^2 H_{\chi,\epsilon} = &-\chi\left(\frac{\beta_1}{t} + \frac{\beta_2}{t-1}\right)\partial_t H_{\chi,\epsilon} - q\frac{\gamma\chi^2}{2}\int_{R_\epsilon} d\mu(x_1)\left(\frac{\beta_1}{tx_1} + \frac{\beta_2}{(t-1)(x_1-1)}\right)\frac{(t-x_1)^{\frac{\gamma\chi}{2}}}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}} V_\epsilon(x_1;t)\\ &+q\chi^2\left(-\frac{1}{t+\epsilon}E^-_{0,\epsilon}(t) + \frac{1}{t-\epsilon}E^+_{0,\epsilon}(t) - \frac{1}{t-1+\epsilon}E^-_{1,\epsilon}(t) + \frac{1}{t-1-\epsilon}E^+_{1,\epsilon}(t)\right).\end{aligned} \tag{3.4.39}$$

Then we have

$$\begin{aligned}&t(1-t)\partial_t^2 H_{\chi,\epsilon} + (C-(A+B+1)t)\partial_t H_{\chi,\epsilon} - ABH_{\chi,\epsilon}\\ &= q\chi^2\left(\frac{\epsilon(1+\epsilon)}{t+\epsilon}E^-_{0,\epsilon}(t) + \frac{\epsilon(1-\epsilon)}{t-\epsilon}E^+_{0,\epsilon}(t) - \frac{\epsilon(1-\epsilon)}{t-1+\epsilon}E^-_{1,\epsilon}(t) - \frac{\epsilon(1+\epsilon)}{t-1-\epsilon}E^+_{1,\epsilon}(t)\right),\end{aligned} \tag{3.4.40}$$

with the parameters given by:

$$A = -q\frac{\gamma\chi}{2}, B = -1 + \chi(\beta_1 + \beta_2 - 2\chi + q\frac{\gamma}{2}), C = \chi(\beta_1 - \chi). \tag{3.4.41}$$

To complete the proof the only thing left is to argue that the boundary terms $\epsilon E^{\pm}_{\cdot,\epsilon}(t)$ converge to 0 as ϵ goes to 0 locally uniformly in t. This has been done in [89]. Thus we have proved the differential equation for H_χ in a weak sense, which is equivalent to its strong sense in this case. Finally the exact same argument works for $\tilde{H}_\chi(t)$. □

3.5 Appendix

3.5.1 Some useful theorems

We recall some theorems in probability that we will use without further justification. In the following D is a compact subset of $\mathbb{R}^d$.

Theorem 3.5.1 (Girsanov theorem)**.** *Let $(Z(x))_{x\in D}$ be a continuous centered Gaussian process and Z a Gaussian variable which belongs to the L^2 closure of the vector space spanned by $(Z(x))_{x\in D}$. Let F be a real continuous bounded function from $\mathcal{C}(D,\mathbb{R})$ to $\mathbb{R}$. Then we have the following identity:*

$$\mathbb{E}\left[e^{Z-\frac{\mathbb{E}[Z^2]}{2}}F((Z(x))_{x\in D})\right] = \mathbb{E}[F((Z(x)+\mathbb{E}[Z(x)Z])_{x\in D})]. \tag{3.5.1}$$

When applied to our case, although the log-correlated field X is not a continuous Gaussian process, we can still make the arguments rigorous by using a regularization procedure (see [89] for a more detailed explanation). Next we recall Kahane's inequality:

Theorem 3.5.2 (Kahane's inequality)**.** *Let $(Z_0(x))_{x\in D}$, $(Z_1(x))_{x\in D}$ be two continuous centered Gaussian processes such that for all $x, y \in D$:*

$$|\mathbb{E}[Z_0(x)Z_0(y)] - \mathbb{E}[Z_1(x)Z_1(y)]| \leq C. \tag{3.5.2}$$

Define for $u \in [0,1]$:

$$Z_u = \sqrt{1-u}Z_0 + \sqrt{u}Z_1, \quad W_u = \int_D e^{Z_u(x)-\frac{1}{2}\mathbb{E}[Z_u(x)^2]}\sigma(dx). \tag{3.5.3}$$

Then for all smooth function F with at most polynomial growth at infinity, and σ a complex Radon measure over D,

$$\left|\mathbb{E}\left[F\left(\int_D e^{Z_0(x)-\frac{1}{2}\mathbb{E}[Z_0(x)^2]}\sigma(dx)\right)\right] - \mathbb{E}\left[F\left(\int_D e^{Z_1(x)-\frac{1}{2}\mathbb{E}[Z_1(x)^2]}\sigma(dx)\right)\right]\right| \leq \sup_{u\in[0,1]}\frac{C}{2}\mathbb{E}[|W_u|^2|F''(W_u)|]. \tag{3.5.4}$$

The same remark as for Theorem 3.5.1 is valid to justify one can use this inequality in the case where Z_0 and Z_1 are log-correlated fields. Finally we provide the Williams decomposition theorem, see for instance [106] for a reference:

Theorem 3.5.3. *Let $(B_s - vs)_{s\geq 0}$ be a Brownian motion with negative drift, i.e. $v > 0$ and let $M = \sup_{s\geq 0}(B_s - vs)$. Then conditionally on M the law of the path $(B_s - vs)_{s\geq 0}$ is given by the joining of two independent paths:*
1) A Brownian motion $(B^1_s + vs)_{0\leq s\leq \tau_M}$ with positive drift v run until its hitting time τ_M of M.
2) $(M + B^2_t - vt)_{t\geq 0}$ where $(B^2_t - vt)_{t\geq 0}$ is a Brownian motion with negative drift conditioned to stay negative.
Moreover, one has the following time reversal property for all $C > 0$ (where τ_C denotes the hitting time of C),

$$(B^1_{\tau_C - s} + v(\tau_C - s) - C)_{0\leq s\leq \tau_C} \overset{law}{=} (\tilde{B}_s - vs)_{0\leq s\leq L_{-C}}, \tag{3.5.5}$$

where $(\tilde{B}_s - vs)_{s\geq 0}$ is a Brownian motion with drift $-v$ conditioned to stay negative and L_{-C} is the last time $(\tilde{B}_s - vs)_{s\geq 0}$ hits $-C$.

3.5.2 Technical estimates on GMC

We repeat here several proofs found in [61, 89] that must be adapted because our objects are complex valued.

3.5.2.1 OPE with reflection

We want to compute the asymptotic expansion of the functions $\tilde{G}_\chi$ and H_χ in the case where there will be reflection. This has been performed in the previous works [61, 89] but it is not straightforward to adapt the proofs as we are working with complex valued quantities so there are many inequalities that need to be adapted. We will treat separately the cases where $\chi = \frac{\gamma}{2}$ and $\chi = \frac{2}{\gamma}$. Starting with the case where $\chi = \frac{2}{\gamma}$:

Lemma 3.5.4. *(OPE with reflection for $\chi = \frac{2}{\gamma}$) Recall $p = \frac{2}{\gamma}(Q - \alpha - \frac{\beta}{2} + \frac{1}{\gamma})$ and consider $s \in (-1, 0)$. There exits a small parameter $\beta_0 > 0$ such that for $\beta \in (Q - \beta_0, Q)$ and α such that $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q - \beta)$, the following asymptotic expansion holds:*

$$\tilde{G}_{\frac{2}{\gamma}}(s) - \tilde{G}_{\frac{2}{\gamma}}(0) = -s^{\frac{1}{2}+\frac{2}{\gamma^2}-\frac{\beta}{\gamma}} \frac{\Gamma(1 - \frac{2}{\gamma}(Q-\beta))\Gamma(\frac{2}{\gamma}(Q-\beta) - p)}{\Gamma(-p)} \overline{R}(\beta, 1, -1)\overline{G}(\alpha, 2Q - \beta - \frac{2}{\gamma}) + o(|s|^{\frac{1}{2}+\frac{2}{\gamma^2}-\frac{\beta}{\gamma}}). \tag{3.5.6}$$

Similarly, recall $q = \frac{1}{\gamma}(2Q - \beta_1 - \beta_2 - \beta_3 + \frac{2}{\gamma})$ and consider $t \in (0, 1)$. Then in the following parameter range,

$$\beta_1 \in (Q - \beta_0, Q), \quad q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q - \beta_i), \quad \mu_1 \in (0, +\infty), \quad \mu_2, \mu_3 \in (-\infty, 0), \tag{3.5.7}$$

the following asymptotic also holds:

$$\begin{aligned} H_{\frac{2}{\gamma}}(it) - H_{\frac{2}{\gamma}}(0) = &-(it)^{1-\frac{2\beta_1}{\gamma}+\frac{4}{\gamma^2}} \frac{\Gamma(1 - \frac{2}{\gamma}(Q-\beta_1))\Gamma(\frac{2}{\gamma}(Q-\beta_1) - q)}{\Gamma(-q)} \overline{R}(\beta_1, \mu_1, \mu_2)\overline{H}^{(2Q-\beta_1-\frac{2}{\gamma}, \beta_2, \beta_3)}_{(\mu_1, -\mu_2, -\mu_3)} \\ &+ o(|t|^{1-\frac{2\beta_1}{\gamma}+\frac{4}{\gamma^2}}). \end{aligned} \tag{3.5.8}$$

Proof. We will prove only the case of $H_{\frac{2}{\gamma}}$, the case of $\tilde{G}_{\frac{2}{\gamma}}$ can be treated in a similar fashion. For a Borel set $I \subseteq \mathbb{R}$, we introduce the notation,

$$K_I(it) := \int_I \frac{it - x}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu(x), \tag{3.5.9}$$

where as always $d\mu(x) = \mu_1 \mathbf{1}_{(-\infty,0)}(x)dx + \mu_2 \mathbf{1}_{(0,1)}(x)dx + \mu_3 \mathbf{1}_{(1,\infty)}(x)dx$. In the following it is convenient to use $d|\mu|(x)$ to denote the measure $\mu_1 \mathbf{1}_{(-\infty,0)}(x)dx - \mu_2 \mathbf{1}_{(0,1)}(x)dx - \mu_3 \mathbf{1}_{(1,\infty)}(x)dx$ which is a positive measure thanks to our choice $\mu_1 \in (0, +\infty)$, $\mu_2, \mu_3 \in (-\infty, 0)$. The signs of the parameters μ_i allows to separate $K_I(it)$ into a positive real part $K_I(0)$ and an imaginary part. This remark is used to bound $|K_I(it)|^{q-1}$ by $|K_I(0)|^{q-1}$ and in several other similar cases (remark that necessarily $q - 1 < 0$). Now we want to study the asymptotic of,

$$\mathbb{E}[K_{\mathbb{R}}(it)^q] - \mathbb{E}[K_{\mathbb{R}}(0)^q] =: T_1 + T_2, \tag{3.5.10}$$

where we defined:

$$T_1 := \mathbb{E}[K_{(-t,t)^c}(it)^q] - \mathbb{E}[K_{\mathbb{R}}(0)^q], \quad T_2 := \mathbb{E}[K_{\mathbb{R}}(it)^q] - \mathbb{E}[K_{(-t,t)^c}(it)^q]. \tag{3.5.11}$$

$\diamond$ First we consider T_1. The goal is to show that $T_1 = o(|t|^{1-\frac{2\beta_1}{\gamma}+\frac{4}{\gamma^2}}) = o(|t|^{\frac{2}{\gamma}(Q-\beta_1)})$. By interpolation,

$$\begin{aligned} |T_1| \leq &|q| \int_0^1 du \mathbb{E}\left[|K_{(-t,t)^c}(it) - K_{\mathbb{R}}(0)||uK_{(-t,t)^c}(it) + (1-u)K_{\mathbb{R}}(0)|^{q-1}\right] \\ \leq &|q| \mathbb{E}\left[\left(|K_{(-t,t)^c}(it) - K_{(-t,t)^c}(0)| + |K_{(-t,t)^c}(0) - K_{\mathbb{R}}(0)|\right)|K_{(-t,t)^c}(0)|^{q-1}\right] \\ = &|q|(A_1 + A_2), \end{aligned} \tag{3.5.12}$$

with:

$$A_1 := \mathbb{E}\left[|K_{(-t,t)^c}(it) - K_{(-t,t)^c}(0)||K_{(-t,t)^c}(0)|^{q-1}\right], \quad A_2 := \mathbb{E}\left[|K_{(-t,t)^c}(0) - K_{\mathbb{R}}(0)||K_{(-t,t)^c}(0)|^{q-1}\right]. \tag{3.5.13}$$

We have

$$\begin{aligned} A_1 &\leq t\int_{(-t,t)^c} d|\mu|(x_1)\,\frac{1}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}}\mathbb{E}\left[\left(\int_{(-t,t)^c}\frac{g(x)^{\frac{\gamma^2}{8}(q-2)}e^{\frac{\gamma}{2}X(x)}d|\mu|(x)}{|x|^{\frac{\gamma\beta_1}{2}-1}|x-1|^{\frac{\gamma\beta_2}{2}}|x-x_1|^{\frac{\gamma^2}{2}}}\right)^{q-1}\right] \qquad (3.5.14)\\ &\leq t\int_{\mathbb{R}} d|\mu|(x_1)\,\frac{2|x_1|\mathbf{1}_{(-\frac{1}{2},\frac{1}{2})^c}+\mathbf{1}_{(-\frac{1}{2},-t)\cup(t,\frac{1}{2})}}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}}\mathbb{E}\left[\left(\int_{(-t,t)^c}\frac{g(x)^{\frac{\gamma^2}{8}(q-2)}e^{\frac{\gamma}{2}X(x)}d|\mu|(x)}{|x|^{\frac{\gamma\beta_1}{2}-1}|x-1|^{\frac{\gamma\beta_2}{2}}|x-x_1|^{\frac{\gamma^2}{2}}}\right)^{q-1}\right]\\ &\leq 2t\mathbb{E}\left[\left(\int_{(-t,t)^c}\frac{g(x)^{\frac{\gamma^2}{8}(q-1)}e^{\frac{\gamma}{2}X(x)}d|\mu|(x)}{|x|^{\frac{\gamma\beta_1}{2}-1}|x-1|^{\frac{\gamma\beta_2}{2}}}\right)^{q}\right]+O(t^{2-\frac{\gamma\beta_1}{2}})=O(t^{2-\frac{\gamma\beta_1}{2}}). \end{aligned}$$

In the last equality we have ignored the first term since it is a $O(t)$ and we will take $\beta_1 > \frac{2}{\gamma}$. On the other hand,

$$A_2 \leq c_1\int_{(-t,t)} d|\mu|(x_1)\,\frac{1}{|x_1|^{\frac{\gamma\beta_1}{2}-1}|x_1-1|^{\frac{\gamma\beta_2}{2}}} = O(t^{2-\frac{\gamma\beta_1}{2}}), \tag{3.5.15}$$

for some constant $c_1 > 0$. When $\beta_1 > \frac{\frac{4}{\gamma^2}-1}{\frac{\gamma}{2}-\frac{2}{\gamma}}$ is satisfied, i.e., $\beta_0 < \frac{1-\frac{\gamma^2}{4}}{\frac{\gamma}{2}-\frac{2}{\gamma}}$, we have $O(t^{2-\frac{\gamma\beta_1}{2}}) = o(t^{\frac{2}{\gamma}(Q-\beta_1)})$. This proves that

$$T_1 = o(t^{\frac{2}{\gamma}(Q-\beta_1)}). \tag{3.5.16}$$

$\diamond$ Now we focus on T_2. The goal is to restrict K to $(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)$, with $h>0$ a small positive constant to be fixed, and then the GMC on the three disjoint parts will be weakly correlated. We have by interpolation and by dropping the imaginary part,

$$\begin{aligned} &\left|\mathbb{E}[K_{\mathbb{R}}(it)^q] - \mathbb{E}[K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(it)^q]\right| \qquad (3.5.17)\\ &\leq |q|\int_0^1 du\mathbb{E}\left[|K_{(-t,-t^{1+h})\cup(t^{1+h},t)}(it)||uK_{\mathbb{R}}(0)+(1-u)K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(0)|^{q-1}\right]\\ &\leq c_2|q|\int_{(-t,-t^{1+h})\cup(t^{1+h},t)} d|\mu|(x_1)\,\frac{t+|x_1|}{|x_1|^{\frac{\gamma\beta_1}{2}}|x_1-1|^{\frac{\gamma\beta_2}{2}}}\\ &= O(t^{1+(1+h)(1-\frac{\gamma\beta_1}{2})}), \end{aligned}$$

for some constant $c_2 > 0$. By taking h satisfying the condition,

$$h < \frac{1+(\frac{2}{\gamma}-\frac{\gamma}{2})\beta_1-\frac{4}{\gamma^2}}{\frac{\gamma\beta_1}{2}-1}, \tag{3.5.18}$$

we have:

$$\mathbb{E}[K_{\mathbb{R}}(it)^q] - \mathbb{E}[K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(it)^q] = o(t^{\frac{2}{\gamma}(Q-\beta_1)}). \tag{3.5.19}$$

It remains to evaluate $\mathbb{E}[K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(it)^q] - \mathbb{E}[K_{(-t,t)^c}(it)^q]$. We now introduce the radial decomposition of the field X,

$$X(x) = B_{-2\ln|x|} + Y(x), \tag{3.5.20}$$

where B, Y are independent Gaussian processes with $(B_s)_{s\in\mathbb{R}}$ a Brownian motion starting from 0 for $s \geq 0$, $B_s = 0$ when $s < 0$, and Y is a centered Gaussian process with covariance,

$$\mathbb{E}[Y(x)Y(y)] = \begin{cases} 2\ln\frac{|x|\vee|y|}{|x-y|}, & |x|,|y|\leq 1,\\ 2\ln\frac{1}{|x-y|} - \frac{1}{2}\ln g(x) - \frac{1}{2}\ln g(y), & \text{else.}\end{cases} \tag{3.5.21}$$

One can wonder why the process Y with the above covariance is well-defined. To construct Y, starting from X set:

$$Y(x) = \begin{cases} X(x) - \frac{1}{\pi}\int_0^\pi X(|x|e^{i\theta})d\theta, & |x| \le 1, \\ X(x), & |x| \ge 1. \end{cases} \tag{3.5.22}$$

Now with this decomposition one can write:

$$K_I(it) = \int_I \frac{it-x}{|x|^{\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}B_{-2\ln|x|}} e^{\frac{\gamma}{2}Y(x)} d\mu(x). \tag{3.5.23}$$

From (3.5.21), we deduce that for $|x'| \le t^{1+h}$ and $|x| \ge t$,

$$|\mathbb{E}[Y(x)Y(x')]| = \left|2\ln\left|1-\frac{x'}{x}\right|\right| \le 4t^h, \tag{3.5.24}$$

where we used the inequality $|\ln|1-x|| \le 2|x|$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Define the processes,

$$P(x) := Y(x)\mathbf{1}_{|x|\le t^{1+h}} + Y(x)\mathbf{1}_{|x|\ge t}, \tag{3.5.25}$$

$$\tilde{P}(x) := \tilde{Y}(x)\mathbf{1}_{|x|\le t^{1+h}} + Y(x)\mathbf{1}_{|x|\ge t}, \tag{3.5.26}$$

where $\tilde{Y}$ is an independent copy of Y. Then we have the inequality over the covariance:

$$\left|\mathbb{E}[P(x)P(y)] - \mathbb{E}[\tilde{P}(x)\tilde{P}(y)]\right| \le 4t^h. \tag{3.5.27}$$

Consider now for $u \in [0,1]$:

$$P_u(x) = \sqrt{1-u}P(x) + \sqrt{u}\tilde{P}(x), \tag{3.5.28}$$

$$K_I(it,u) = \int_I \frac{it-x}{|x|^{\frac{\gamma\beta_1}{2}-\frac{\gamma^2}{4}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}B_{-2\ln|x|}} e^{\frac{\gamma}{2}P_u(x)} d\mu(x). \tag{3.5.29}$$

By applying Kahane's inequality of Theorem 3.5.2,

$$\begin{aligned} &\left|\mathbb{E}\left[K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(it)^q\right] - \mathbb{E}\left[\left(K_{(-\infty,-t)\cup(t,\infty)}(it) + \tilde{K}_{(-t^{1+h},t^{1+h})}(it)\right)^q\right]\right| \\ &\le 2|q(q-1)|t^h \sup_{u\in[0,1]} \mathbb{E}\left[|K_I(it,u)|^q\right] \\ &\le c_3\, t^h, \end{aligned} \tag{3.5.30}$$

for some constant $c_3 > 0$, and where in $\tilde{K}_{(-t^{1+h},t^{1+h})}(it)$ we simply use the field $\tilde{Y}$ instead of Y. When $h > \frac{2}{\gamma}(Q-\beta_1)$, we can bound the previous term by $o(t^{\frac{2}{\gamma}(Q-\beta_1)})$.

Consider now the change of variable $x = t^{1+h}e^{-s/2}$ for the field $\tilde{K}_{(-t^{1+h},t^{1+h})}(it)$. By the Markov property of the Brownian motion and stationarity of

$$d\mu_{\tilde{Y}}(s) := \mu_1\tilde{Y}(-e^{-s/2})ds + \mu_2\tilde{Y}(e^{-s/2})ds, \tag{3.5.31}$$

we have

$$\tilde{K}_{(-t^{1+h},t^{1+h})}(it) = \frac{1}{2}it^{1+(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})}e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/t)}}\int_0^\infty \frac{(1+it^he^{-s/2})}{|t^{1+h}e^{-s/2}-1|^{\frac{\gamma\beta_2}{2}}} e^{\frac{\gamma}{2}(\tilde{B}_s-\frac{s}{2}(Q-\beta_1))}d\mu_{\tilde{Y}}(s), \tag{3.5.32}$$

with $\tilde{B}$ an independent Brownian motion. We denote

$$\sigma_t := t^{1+(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})}e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/t)}}, \quad V := \frac{1}{2}\int_0^\infty e^{\frac{\gamma}{2}(\tilde{B}_s-\frac{s}{2}(Q-\beta_1))}d\mu_{\tilde{Y}}(s). \tag{3.5.33}$$

By interpolation, we can prove that for some constant $c_4 > 0$:

$$\begin{aligned}&\left|\mathbb{E}[(K_{(-t,t)^c}(it)+\tilde{K}_{(-t^{1+h},t^{1+h})}(it))^q]-\mathbb{E}\left[\left(K_{(-t,t)^c}(it)+i\sigma_t V\right)^q\right]\right| \\ &\leq c_4|q|t^{1+h+(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})}\mathbb{E}\left[e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/t)}}\int_0^\infty e^{\frac{\gamma}{2}(\tilde{B}_s-\frac{s}{2}(Q-\beta_1))}d\mu_{\tilde{Y}}(s)|K_{(1,2)}(0)|^{q-1}\right].\end{aligned} \tag{3.5.34}$$

Since $B_{2(1+h)\ln(1/t)}$, $(\tilde{B}_s)_{s\geq 0}$, $(\tilde{Y}(x))_{|x|\leq 1}$, and $K_{(1,2)}(0)$ are independent, we can easily bound the last term by, for some $c_5 > 0$,

$$c_5\, t^{(1+h)(2-\frac{\gamma\beta_1}{2})} = o(t^{\frac{2}{\gamma}(Q-\beta_1)}). \tag{3.5.35}$$

By the Williams path decomposition of Theorem 3.5.3 we can write,

$$V = e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-L_M}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}}\mu_{\tilde{Y}}(ds), \tag{3.5.36}$$

where $M = \sup_{s\geqslant 0}(\tilde{B}_s - \frac{Q-\beta_1}{2}s)$ and L_M is the last time $\left(\mathcal{B}_{-s}^{\frac{Q-\beta_1}{2}}\right)_{s\geq 0}$ hits $-M$. Recall that the law of M is known, for $v \geq 1$,

$$\mathbb{P}(e^{\frac{\gamma}{2}M} > v) = \frac{1}{v^{\frac{2}{\gamma}(Q-\beta_1)}}. \tag{3.5.37}$$

For simplicity, we introduce the notation:

$$\rho(\beta_1) := \frac{1}{2}\int_{-\infty}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}}\mu_{\tilde{Y}}(ds). \tag{3.5.38}$$

Again by interpolation and then independence we can show that

$$\begin{aligned}&\left|\mathbb{E}\left[\left(K_{(-t,t)^c}(it)+i\sigma_t V\right)^q\right]-\mathbb{E}\left[\left(K_{(-t,t)^c}(it)+i\sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1)\right)^q\right]\right| \\ &\leq \frac{1}{2}|q|t^{1+(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})}\mathbb{E}\left[e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/t)}}\int_{-\infty}^0 e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}}\mu_{\tilde{Y}}(ds)\left|K_{(1,2)}(0)\right|^{q-1}\right] \\ &= O(t^{1+(1+h)(1-\frac{\gamma\beta_1}{2})}) = o(t^{\frac{2}{\gamma}(Q-\beta_1)}).\end{aligned} \tag{3.5.39}$$

In summary,

$$T_2 = \mathbb{E}[(K_{(-t,t)^c}(it)+i\sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q] - \mathbb{E}[K_{(-t,t)^c}(it)^q] + o(t^{\frac{2}{\gamma}(Q-\beta_1)}). \tag{3.5.40}$$

Finally, we evaluate the above difference at first order explicitly using the fact that density of $e^{\frac{\gamma}{2}M}$ is known:

$$\begin{aligned}&\mathbb{E}[(K_{(-t,t)^c}(it)+i\sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q] - \mathbb{E}[K_{(-t,t)^c}(it)^q] \\ &= \frac{2}{\gamma}(Q-\beta_1)\mathbb{E}\left[\int_1^\infty \frac{dv}{v^{\frac{2}{\gamma}(Q-\beta_1)+1}}\left(\left(K_{(-t,t)^c}(it)+i\sigma_t\rho(\beta_1)v\right)^q - K_{(-t,t)^c}(it)^q\right)\right] \\ &= t^{\frac{2}{\gamma}(Q-\beta_1)}\frac{2}{\gamma}(Q-\beta_1)\mathbb{E}\left[\int_{\frac{\bar{\sigma}_t\rho(\beta_1)}{\bar{K}_{(-t,t)^c}(it)}}^\infty \frac{du}{u^{\frac{2}{\gamma}(Q-\beta_1)+1}}((iu+1)^q-1)\rho(\beta_1)^{\frac{2}{\gamma}(Q-\beta_1)}\hat{K}_{(-t,t)^c}(it)^{q-\frac{2}{\gamma}(Q-\beta_1)}\right].\end{aligned} \tag{3.5.41}$$

In the last equality we have applied Theorem 3.5.1. Next,

$$\hat{K}_{(-t,t)^c}(it) = \int_{(-t,t)^c}\frac{it-x}{|x|^{\frac{\gamma}{2}(2Q-\beta_1-\frac{2}{\gamma})}|x-1|^{\frac{\gamma\beta_2}{2}}}g(x)^{\frac{\gamma^2}{8}(q-1)}e^{\frac{\gamma}{2}X(x)}d\mu(x) \xrightarrow[\text{a.s.}]{t\to 0_+} \hat{K}_{\mathbb{R}}(0), \tag{3.5.42}$$

and for $h < \frac{2}{\gamma(Q-\beta_1)} - 1$,

$$\hat{\sigma}_t = t^{1-(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})} e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/t)}} \xrightarrow[\text{a.s.}]{t\to 0_+} 0. \tag{3.5.43}$$

With some simple arguments of uniform integrability, we conclude that:

$$\begin{aligned}
&\mathbb{E}\left[\left(K_{(-t,t)^c}(it) + i\sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1)\right)^q\right] - \mathbb{E}[K_{(-t,t)^c}(it)^q] \qquad (3.5.44)\\
&\overset{t\to 0_+}{\sim} t^{\frac{2}{\gamma}(Q-\beta_1)}\frac{2}{\gamma}(Q-\beta_1)\left(\int_0^\infty \frac{du}{u^{\frac{2}{\gamma}(Q-\beta_1)+1}}((iu+1)^q-1)\right)\mathbb{E}\left[\rho(\beta_1)^{\frac{2}{\gamma}(Q-\beta_1)}\right]\mathbb{E}\left[\hat{K}_{\mathbb{R}}(0)^{q-\frac{2}{\gamma}(Q-\beta_1)}\right]\\
&= (it)^{\frac{2}{\gamma}(Q-\beta_1)}\frac{2}{\gamma}(Q-\beta_1)\frac{\Gamma(\frac{2}{\gamma}(\beta_1-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1)-q)}{\Gamma(-q)}\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1-\frac{2}{\gamma},\beta_2,\beta_3)}_{(\mu_1,-\mu_2,-\mu_3)}.
\end{aligned}$$

The power of i comes from the evaluation of the integral. Inspecting the proof we see that the conditions on β_0 and h indeed allow us to find small values of these parameters that make the arguments work. Therefore we have proved the claim. □

Now the analogue result for $\chi = \frac{\gamma}{2}$:

Lemma 3.5.5. *(OPE with reflection for $\chi = \frac{\gamma}{2}$) Recall $p = \frac{2}{\gamma}(Q - \alpha - \frac{\beta}{2} + \frac{\gamma}{4})$ and consider $s \in (0,1)$. There exists a small parameter $\beta_0 > 0$ such that for $\beta \in (Q-\beta_0, Q)$ and α such that $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\beta)$, the following asymptotic expansion holds:*

$$\begin{aligned}
\tilde{G}_{\frac{\gamma}{2}}(s) - \tilde{G}_{\frac{\gamma}{2}}(0) = &-s^{\frac{1}{2}+\frac{\gamma^2}{8}-\frac{\gamma\beta}{4}}\frac{\Gamma(1-\frac{2(Q-\beta)}{\gamma})\Gamma(-p+\frac{2}{\gamma}(Q-\beta))}{\Gamma(-p)}\overline{R}(\beta,1,e^{i\pi\frac{\gamma^2}{4}})\overline{G}(\alpha,2Q-\beta-\frac{\gamma}{2})\\
&+ o(|s|^{\frac{1}{2}+\frac{\gamma^2}{8}-\frac{\gamma\beta}{4}}). \qquad (3.5.45)
\end{aligned}$$

Similarly, recall $q = \frac{1}{\gamma}(2Q-\beta_1-\beta_2-\beta_3+\frac{\gamma}{2})$ and consider $t \in (0,1)$. Then for $\mu_1,\mu_2,\mu_3 \in (0,+\infty)$, $\beta_1 \in (Q-\beta_0, Q)$ and β_2,β_3 chosen so that $q < \frac{4}{\gamma^2} \wedge \min_i \frac{2}{\gamma}(Q-\beta_i)$, the following asymptotic also holds:

$$\begin{aligned}
H_{\frac{\gamma}{2}}(t) - H_{\frac{\gamma}{2}}(0) = &\, t^{1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4}}\frac{2(Q-\beta_1)}{\gamma}\frac{\Gamma(\frac{2}{\gamma}(\beta_1-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1)-q)}{\Gamma(-q)}\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,e^{i\pi\frac{\gamma^2}{4}}\mu_2,e^{i\pi\frac{\gamma^2}{4}}\mu_3)}\\
&+ o(|t|^{1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4}}). \qquad (3.5.46)
\end{aligned}$$

Proof. We will keep the notations in the proof of Lemma 3.5.4 although there are some slight differences. This time K is defined with the $\chi = \frac{\gamma}{2}$ insertion:

$$K_I(t) := \int_I \frac{(t-x)^{\frac{\gamma^2}{4}}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(q-1)} e^{\frac{\gamma}{2}X(x)} d\mu(x). \tag{3.5.47}$$

To deal with the complex phase we will simply use the following inequality. For a fixed $p < 1$ and $\varphi \in [0,\pi)$, there exists a constant $c > 0$ such that for all $x_1, x_2, y_1, y_2 \in (0,+\infty)$:

$$|(x_1 + e^{i\varphi}y_1)^p - (x_2 + e^{i\varphi}y_2)^p| \le c(|x_1^p - x_2^p| + |y_1^p - y_2^p|). \tag{3.5.48}$$

This inequality can be proved by studying the derivative of the function $(x,y) \mapsto (x^{1/p} + e^{i\varphi}y^{1/p})^p$. With the help of this inequality we will be able to perform the same proof as in the case of the previous lemma. Following the same steps as in [89], we have:

$$\mathbb{E}[|K_{(-\infty,-t)}(t)^q - K_{(-\infty,0)}(0)^q|] = o(t^{\frac{\gamma}{2}(Q-\beta_1)}), \tag{3.5.49}$$

$$\mathbb{E}[||K_{(t,\infty)}(t)|^q - |K_{(0,\infty)}(0)|^q|] = o(t^{\frac{\gamma}{2}(Q-\beta_1)}). \tag{3.5.50}$$

Applying (3.5.48) implies that:

$$\begin{aligned}\mathbb{E}[|K_{(-t,t)^c}(t)^q - K_{\mathbb{R}}(0)^q|] &= \mathbb{E}[|(K_{(-\infty,-t)}(t) + e^{i\pi\frac{\gamma^2}{4}}|K_{(t,\infty)}(t)|)^q - (K_{(-\infty,0)}(0) + e^{i\pi\frac{\gamma^2}{4}}|K_{(0,\infty)}(0)|)^q|] \\ &\leq c\mathbb{E}[|K_{(-\infty,-t)}(t)^q - K_{(-\infty,0)}(0)^q|] + c\mathbb{E}[||K_{(t,\infty)}(t)|^q - |K_{(t,\infty)}(0)|^q|] \\ &\leq o(t^{\frac{\gamma}{2}(Q-\beta_1)}). \end{aligned} \tag{3.5.51}$$

Next we repeat the step where we introduce a small $h > 0$ and want to compare $K_{\mathbb{R}}(t)$ and $K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(t)$. Following again the steps of [89], under the constraint on h,

$$h < \frac{\frac{\gamma\beta_1}{2} - 1}{1 - \frac{\gamma\beta_1}{2} + \gamma^2}, \tag{3.5.52}$$

one can show that:

$$\mathbb{E}[|K_{(-\infty,t)}(t)^q - K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})}(t)^q|] = o(t^{\frac{\gamma}{2}(Q-\beta_1)}). \tag{3.5.53}$$

By applying again (3.5.48) one obtains,

$$\mathbb{E}[|K_{\mathbb{R}}(t)^q - K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(t)^q|] \leq c\mathbb{E}[|K_{(-\infty,t)}(t)^q - K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})}(t)^q|] = o(t^{\frac{\gamma}{2}(Q-\beta_1)}). \tag{3.5.54}$$

Therefore as in the previous lemma we have now reduced the problem to studying the difference:

$$\mathbb{E}[K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(t)^q] - \mathbb{E}[K_{(-t,t)^c}(t)^q]. \tag{3.5.55}$$

We proceed exactly in the same way as the case $\chi = \frac{2}{\gamma}$, using Kahane's inequality of Theorem 3.5.2 to obtain:

$$\mathbb{E}[K_{(-\infty,-t)\cup(-t^{1+h},t^{1+h})\cup(t,\infty)}(t)^q] - \mathbb{E}[(K_{(-t,t)^c}(t) + \sigma_t V)^q] = O(t^h) + O(t^{(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})}). \tag{3.5.56}$$

When $h > \frac{\gamma}{2}(Q-\beta_1)$ this term is also a $o(t^{\frac{\gamma}{2}(Q-\beta_1)})$. Here the expression of σ_t is slightly different:

$$\sigma_t = t^{\frac{\gamma^2}{4}+(1+h)(1-\frac{\gamma\beta_1}{2}+\frac{\gamma^2}{4})} e^{\frac{\gamma}{2}B_{2(1+h)\ln(1/t)}}. \tag{3.5.57}$$

As in our previous work [89], we can show that

$$\begin{aligned}&\mathbb{E}[(K_{(-\infty,-t)}(t) + \sigma_t V)^q] - \mathbb{E}[K_{(-\infty,-t)}(t)^q] \\ &= \mathbb{E}[(K_{(-\infty,-t)}(t) + \sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q] - \mathbb{E}[K_{(-\infty,-t)}(t)^q] + o(t^{\frac{\gamma}{2}(Q-\beta_1)})\end{aligned} \tag{3.5.58}$$

This result is proved using inequalities, the lower bound and upper bound are equivalent to a term with order $t^{\frac{\gamma}{2}(Q-\beta_1)}$. As a consequence,

$$\mathbb{E}[(K_{(-\infty,-t)}(t) + \sigma_t V)^q] - \mathbb{E}[(K_{(-\infty,-t)}(t) + \sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q] = o(t^{\frac{\gamma}{2}(Q-\beta_1)}). \tag{3.5.59}$$

However, we can write V as

$$\sigma_t V = \sigma_t e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-L_M}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}}\mu_{\tilde{Y}}(ds) \leq \sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1). \tag{3.5.60}$$

This allows us to put an absolute value in expectation:

$$\mathbb{E}[|(K_{(-\infty,-t)}(t) + \sigma_t V)^q - (K_{(-\infty,-t)}(t) + \sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q|] = o(t^{\frac{\gamma}{2}(Q-\beta_1)}) \tag{3.5.61}$$

We can conclude by using (3.5.48) that:

$$\mathbb{E}[(K_{(-t,t)^c}(t)+\sigma_t V)^q]-\mathbb{E}[(K_{(-t,t)^c}(t)+\sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q]=o(t^{\frac{\gamma}{2}(Q-\beta_1)}) \tag{3.5.62}$$

We estimate as in the case $\chi=\frac{2}{\gamma}$:

$$\begin{aligned}&\mathbb{E}[(K_{(-t,t)^c}(t)+\sigma_t e^{\frac{\gamma}{2}M}\rho(\beta_1))^q]-\mathbb{E}[K_{(-t,t)^c}(t)^q] \\ &=t^{\frac{\gamma}{2}(Q-\beta_1)}\frac{2(Q-\beta_1)}{\gamma}\frac{\Gamma(\frac{2}{\gamma}(\beta_1-Q))\Gamma(\frac{2}{\gamma}(Q-\beta_1)-q)}{\Gamma(-q)}\overline{R}(\beta_1,\mu_1,\mu_2)\overline{H}^{(2Q-\beta_1-\frac{\gamma}{2},\beta_2,\beta_3)}_{(\mu_1,\mu_2e^{i\pi\frac{\gamma^2}{4}},\mu_3e^{i\pi\frac{\gamma^2}{4}})}+o(t^{\frac{\gamma}{2}(Q-\beta_1)}).\end{aligned} \tag{3.5.63}$$

Finally it is again possible to choose suitable small $h>0$ and $\beta_0>0$. This concludes the proof of the lemma. □

3.5.2.2 Analytic continuation

In this section we prove the lemma of analyticity of the moments of GMC that we have used repetitively throughout the paper. This fact has been first shown in [61] in the case of the correlation functions on the sphere. The main idea is that starting from the range of real parameters of β_i or α where a given GMC expression is defined, one can find a small neighborhood in $\mathbb{C}$ of the parameter range where the quantity will still be well-defined and is complex analytic in this parameter range. We also use in Section 3.3 the fact that the three-point function is complex analytic in the μ_i. This fact is obtain directly just be differentiating with respect to μ_i.

Lemma 3.5.6. *(Analycity in insertions weights and in μ_i of moments of GMC) Consider the following functions defined in the given parameter range:*

- $(\alpha,\beta)\mapsto\overline{G}(\alpha,\beta)$ *for* $\beta<Q$, $\frac{\gamma}{2}-\alpha<\frac{\beta}{2}<\alpha$.
- $(\alpha,\beta)\mapsto G_\chi(t)$ *for* $t\in\mathbb{H}$, $\beta<Q$, $\frac{2}{\gamma}\left(Q-\alpha-\frac{\beta}{2}+\frac{\chi}{2}\right)<\frac{4}{\gamma^2}\wedge\frac{2}{\gamma}(Q-\beta)$.
- $(\beta_1,\beta_2,\beta_3)\mapsto\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ *for:*

$$(\mu_i)_{i=1,2,3}\ \text{ satisfies Definition 3.1.3},\quad \beta_i<Q,\quad \frac{1}{\gamma}(2Q-\sum_{i=1}^3\beta_i)<\frac{4}{\gamma^2}\wedge\min_i\frac{2}{\gamma}(Q-\beta_i).$$

- $(\beta_1,\beta_2,\beta_3)\mapsto H_\chi(t)$ *for:*

$$\beta_i<Q,\quad \mu_1\in(0,\infty),\ \mu_2,\mu_3\in-\overline{\mathbb{H}},\quad q<\frac{4}{\gamma^2}\wedge\min_i\frac{2}{\gamma}(Q-\beta_i),\quad t\in\mathbb{H}.$$

Then for each function above, and for each of the function's variables, it is complex analytic in a small complex neighborhood of any compact set K contained in the domain of definition of the function for real parameters. Furthermore the function $\overline{H}$ now viewed as a function of μ_1,μ_2,μ_3 is complex analytic in any compact $\tilde{K}$ contained in the range of parameters written above.

Proof. We briefly adapt the proof of [61] for the function $H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ as the other cases can be treated in a similar manner. The first step performed in [61] is to apply the Girsanov theorem to pull out the insertions outside of the GMC expectation. It will be convenient to assume the three insertions are not located at 0, 1 and ∞ but rather at three points s_1, s_2, s_3 all in $\mathbb{R}$ and obeying the extra constraints $|s_i|>2$ and $|s_i-s_{i'}|>2$ respectively for all $i\in\{1,2,3\}$ and for all $i\neq i'$. The reason it is possible to assume this is that the Liouville correlations are conformally invariant in the sense of the KPZ formula of [53]. It will be convenient to use the notations $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)$ and $\mathbf{s}=(s_1,s_2,s_3)$. Our starting point is thus that it is possible to write,

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}=\mathcal{P}\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{s}), \tag{3.5.64}$$

for $\mathcal{P} \in \mathbb{R}$ an explicit prefactor that is analytic in the β_i and hence can be ignored and where we have introduced:

$$\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{s}) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{8}(\frac{4}{\gamma}-\sum_{i=1}^3\beta_i)}}{\prod_{i=1}^3|x-s_i|^{\frac{\gamma\beta_i}{2}}} e^{\frac{\gamma}{2}X(x)}d\mu(x)\right)^{\frac{1}{\gamma}(2Q-\sum_{i=1}^3\beta_i)}\right]. \tag{3.5.65}$$

Now by applying Theorem 3.5.1 we can obtain $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{s})$ from the following limit,

$$H^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}(\mathbf{s}) = \lim_{r\to\infty}\mathcal{F}_r(\boldsymbol{\beta}), \tag{3.5.66}$$

where we have introduced,

$$\mathcal{F}_r(\boldsymbol{\beta}) = \mathbb{E}\left[\prod_{i=1}^3 e^{\beta_i X_r(s_i)-\frac{\beta_i^2}{2}\mathbb{E}[X_r(s_i)^2]}\left(\int_{\mathbb{R}_r} g(x)^{\frac{1}{2}}e^{\frac{\gamma}{2}X(x)}d\mu(x)\right)^{p_0}\right], \tag{3.5.67}$$

$p_0 = \frac{1}{\gamma}(2Q-\sum_{i=1}^3\beta_i)$ and:

$$\mathbb{R}_r := \mathbb{R}\backslash \cup_{i=1}^3 (s_i - e^{-r/2}, s_i + e^{-r/2}). \tag{3.5.68}$$

The fields $X_r(s_i)$ are radial parts of $X(s_i)$ obtained by taking the mean of $X(s_i)$ over the upper-half circles of radius $e^{-r/2}$, $\partial B(s_i, e^{-r/2})_+$.

Now when β_i are complex numbers, we write $\beta_i = a_i + ib_i$. We want to prove there exists a complex neighborhood V in $\mathbb{C}^3$ containing the domain of definition for real β_i such that for all compact sets contained in V, $\mathcal{F}_r(\boldsymbol{\beta})$ converges uniformly as $r\to+\infty$ over the compact set. It is known that $X_{r+t}(s_i) - X_r(s_i)$ are independent Brownian motions for different s_i. Hence,

$$\begin{aligned}
&|\mathcal{F}_{r+1}(\boldsymbol{\beta}) - \mathcal{F}_r(\boldsymbol{\beta})| \qquad (3.5.69)\\
&= \left|\mathbb{E}\left[\prod_{i=1}^3 e^{ib_iX_{r+1}(s_i)+\frac{b_i^2}{2}\mathbb{E}[X_{r+1}(s_i)^2]}\left(\left(\int_{\mathbb{R}_{r+1}} e^{\frac{\gamma}{2}X(x)}f(x)d\mu(x)\right)^{p_0} - \left(\int_{\mathbb{R}_r} e^{\frac{\gamma}{2}X(x)}f(x)d\mu(x)\right)^{p_0}\right)\right]\right|\\
&\le c\, e^{\frac{r+1}{2}\sum_{i=1}^3 b_i^2}\mathbb{E}\left[\left|\left(\int_{\mathbb{R}_{r+1}} e^{\frac{\gamma}{2}X(x)}f(x)d\mu(x)\right)^{p_0} - \left(\int_{\mathbb{R}_r} e^{\frac{\gamma}{2}X(x)}f(x)d\mu(x)\right)^{p_0}\right|\right],
\end{aligned}$$

where we denote $f(x) = \frac{g(x)^{\frac{\gamma^2}{8}(p_0-1)}}{\prod_{i=1}^3|x-s_i|^{\frac{\gamma a_i}{2}}}$. Set $Z_r := \int_{\mathbb{R}_r} e^{\frac{\gamma}{2}X(x)}f(x)d\mu(x)$ and $Y_r := Z_{r+1} - Z_r$. We want to estimate

$$\mathbb{E}[|(Z_r+Y_r)^{p_0} - Z_r^{p_0}|] \le \mathbb{E}[\mathbf{1}_{|Y_r|<\epsilon}|(Z_r+Y_r)^{p_0} - Z_r^{p_0}|] + \mathbb{E}[\mathbf{1}_{|Y_r|\ge\epsilon}|(Z_r+Y_r)^{p_0} - Z_r^{p_0}|], \tag{3.5.70}$$

where $\epsilon > 0$ will be fixed later. By interpolation,

$$\mathbb{E}[\mathbf{1}_{|Y_r|<\epsilon}|(Z_r+Y_r)^{p_0} - Z_r^{p_0}|] \le |p_0|\epsilon \sup_{u\in[0,1]}\mathbb{E}[|(1-u)Z_r + uY_r|^{\mathrm{Re}(p_0)-1}] \le c\,\epsilon. \tag{3.5.71}$$

For the other term, we use the Hölder inequality with $\lambda > 1$ such that $\frac{\lambda}{\lambda-1}\mathrm{Re}(p_0) < \min_{i=1}^3 \frac{2}{\gamma}(Q-a_i) \wedge \frac{4}{\gamma^2}$, and $0 < m < \frac{4}{\gamma^2}$,

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{|Y_r|\ge\epsilon}|(Z_r+Y_r)^{p_0} - Z_r^{p_0}|] &\le c\,\mathbb{P}(|Y_r|\ge\epsilon)^{\frac{1}{\lambda}} \le c\epsilon^{-\frac{m}{\lambda}}\mathbb{E}[|Y_r|^m]^{\frac{1}{\lambda}} \qquad (3.5.72)\\
&\le c\epsilon^{-\frac{m}{\lambda}}\mathbb{E}\left[\left|\sum_{i=1}^3\int_{(s_i-e^{-r/2},s_i+e^{r/2})} e^{\frac{\gamma}{2}X(x)}f(x)d\mu(x)\right|^m\right]^{\frac{1}{\lambda}}\\
&\le c'\epsilon^{-\frac{m}{\lambda}}\left(\max_i e^{-\frac{r}{2}((1+\frac{\gamma^2}{2}-\frac{\gamma a_i}{2})m-\frac{\gamma^2m^2}{2})}\right)^{\frac{1}{\lambda}} =: c'\epsilon^{-\frac{m}{\lambda}}e^{-\frac{\theta}{\lambda}r},
\end{aligned}$$

where in the last step $\theta \in \mathbb{R}$ is defined by the last equality and we have used the multifractional scaling property of the GMC. We can choose a suitable m such that $\theta > 0$. Now take $\epsilon = e^{-\eta r}$ with $\eta = \frac{\theta}{\lambda+m}$, then:

$$\mathbb{E}[|(Z_r + Y_r)^{p_0} - Z_r^{p_0}|] \leq c\, e^{\frac{r+1}{2}\sum_{i=1}^3 b_i^2}(\epsilon + \epsilon^{-\frac{m}{\lambda}} e^{-\frac{\theta}{\lambda}r}) \leq c'\, e^{-(\eta - \frac{1}{2}\sum_{i=1}^3 b_i^2)r}. \tag{3.5.73}$$

Hence if one choose the open set V in such a way that $\frac{1}{2}\sum_{i=1}^3 b_i^2 < \eta$ always holds, all the inequalities we have done before hold true and hence we have shown that $\mathcal{F}_r(\boldsymbol{\beta})$ converges locally uniformly. This proves the analycity result.

Lastly we very briefly justify all the other cases. The analycity of $\overline{G}(\alpha, \beta)$ can be proved in the exact same way as done above for $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$. Furthermore adding the dependence t to get the functions $G_\chi(t)$ and $H_\chi(t)$ also changes nothing to the above argument and so the same claim also holds in this case. Lastly for the analycity in μ_i of $\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$, one simply needs to notice the complex derivatives are well-defined. For instance for μ_1 one can write,

$$\begin{aligned}\partial_{\mu_1}\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} &= \partial_{\mu_1}\mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{8}(\frac{4}{\gamma}-\sum_{i=1}^3\beta_i)}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} e^{\frac{\gamma}{2}X(x)}d\mu(x)\right)^{\frac{1}{\gamma}(2Q-\sum_{i=1}^3\beta_i)}\right] \\ &= \int_{-\infty}^0 dx_1 \frac{g(x_1)^{\frac{\gamma}{8}(\frac{4}{\gamma}-\sum_{i=1}^3\beta_i)}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}}\mathbb{E}\left[e^{\frac{\gamma}{2}X(x_1)}\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{8}(\frac{4}{\gamma}-\sum_{i=1}^3\beta_i)}}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} e^{\frac{\gamma}{2}X(x)}d\mu(x)\right)^{\frac{1}{\gamma}(2Q-\sum_{i=1}^3\beta_i)-1}\right],\end{aligned} \tag{3.5.74}$$

where the last expression is clearly well-defined. Furthermore one can check that $\partial_{\overline{\mu}_1}\overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)} = 0$. Therefore $\mu_1 \mapsto \overline{H}^{(\beta_1,\beta_2,\beta_3)}_{(\mu_1,\mu_2,\mu_3)}$ is complex analytic in the claimed domain. □

3.5.2.3 The limit of $\overline{H}$ recovers $\overline{R}$

Here we will prove Lemma 3.1.8. With our choice of μ_i satisfying Definition 3.1.3 this is an easy adaptation of the positive case.

Proof. We prove the lemma in the first case where $\beta_2 < \beta_1$. Let us denote $\epsilon = \frac{\beta_3-(\beta_1-\beta_2)}{\gamma}$, $p_1 = \frac{2}{\gamma}(Q-\beta_1)$. For $I \subseteq \mathbb{R}$ a Borel set, we introduce the notation:

$$K_I = \int_I \frac{1}{|x|^{\frac{\gamma\beta_1}{2}}|x-1|^{\frac{\gamma\beta_2}{2}}} g(x)^{\frac{\gamma^2}{8}(p-1-\epsilon)} e^{\frac{\gamma}{2}X(x)}dx. \tag{3.5.75}$$

In our previous paper [89] it is proved that:

$$\epsilon\mathbb{E}[K_{[0,1]}^{p_1-\epsilon}] \xrightarrow{\epsilon\to 0} p_1\overline{R}(\beta_1, 0, 1). \tag{3.5.76}$$

Using the density of $e^{\frac{\gamma}{2}M}$, we have by definition of the reflection coefficient,

$$\epsilon\mathbb{E}\left[\left(e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p_1-\epsilon}\right] \xrightarrow{\epsilon\to 0} p_1\overline{R}(\beta_1, 0, 1), \tag{3.5.77}$$

where:

$$\rho_\pm(\beta_1) := \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}} e^{\frac{\gamma}{2}Y(\pm e^{-s/2})}ds. \tag{3.5.78}$$

On the other hand, by the William's path decomposition of Theorem 3.5.3 we can write:

$$K_{[0,1]} = e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-L_M}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}} e^{\frac{\gamma}{2}Y(e^{-s/2})}ds \leq e^{\frac{\gamma}{2}M}\rho_+(\beta_1). \tag{3.5.79}$$

Therefore, the result from [89] implies that:

$$\mathbb{E}\left[\left|K_{[0,1]}^{p_1-\epsilon} - (e^{\frac{\gamma}{2}M}\rho_+(\beta_1))^{p_1-\epsilon}\right|\right] = o(\epsilon^{-1}). \tag{3.5.80}$$

Similarly we also have

$$\mathbb{E}\left[\left|K_{[-1,0)}^{p_1-\epsilon} - \left(e^{\frac{\gamma}{2}M}\rho_-(\beta_1)\right)^{p_1-\epsilon}\right|\right] = o(\epsilon^{-1}). \tag{3.5.81}$$

We will use these results to prove the complex μ_i case. Consider first the case $p_1 > 1$. Using interpolation and Hölder's inequality, for $\lambda > 1$,

$$\begin{aligned}
&\mathbb{E}\left[\left|(\mu_1 K_{(-\infty,0)} + \mu_2 K_{[0,1]} + \mu_3 K_{(1,\infty)})^{p_1-\epsilon} - (\mu_1 K_{[-1,0)} + \mu_2 K_{[0,1]})^{p_1-\epsilon}\right|\right] \qquad (3.5.82)\\
&\leq \mathbb{E}\left[|\mu_1 K_{(-\infty,-1)} + \mu_3 K_{(1,\infty)}|^{\lambda}\right]^{\frac{1}{\lambda}}\\
&\quad \times \sup_{u\in[0,1]} \mathbb{E}\left[\left|(1-u)(\mu_1 K_{(-\infty,0)} + \mu_2 K_{[0,1]} + \mu_3 K_{(1,\infty)}) + u(\mu_1 K_{[-1,0)} + \mu_2 K_{[0,1]})\right|^{(p_1-1-\epsilon)\frac{\lambda}{\lambda-1}}\right]^{\frac{\lambda-1}{\lambda}}.
\end{aligned}$$

Take $p_1 < \lambda < \min\{\frac{4}{\gamma^2}, \frac{2}{\gamma}(Q - \beta_2 \vee \beta_3)\}$, then both expectations can be bounded by $O(1)$. By the same techniques with $\lambda = p - \epsilon$ we prove:

$$\begin{aligned}
&\mathbb{E}\left[\left|\left(\mu_1 K_{[-1,0)} + \mu_2 K_{[0,1]}\right)^{p_1-\epsilon} - \left(\mu_1 e^{\frac{\gamma}{2}M}\rho_-(\beta_1) + \mu_2 e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p_1-\epsilon}\right|\right] \qquad (3.5.83)\\
&\leq \mathbb{E}\left[\left(e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-\infty}^{-L_M} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}}\left(|\mu_1| e^{\frac{\gamma}{2}Y(-e^{-s/2})} + |\mu_2| e^{\frac{\gamma}{2}Y(e^{-s/2})}\right) ds\right)^{p_1-\epsilon}\right]^{\frac{1}{p_1-\epsilon}}\\
&\times \mathbb{E}\left[\left(|\mu_1| e^{\frac{\gamma}{2}M}\rho_-(\beta_1) + |\mu_2| e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p_1-\epsilon}\right]^{\frac{p_1-1-\epsilon}{p_1-\epsilon}}.
\end{aligned}$$

The second expectation is a $O(\epsilon^{-1})$. For the first expectation, we use the inequality that for $x, y > 0$ one has $x^{p_1-\epsilon} + y^{p_1-\epsilon} < (x+y)^{p_1-\epsilon}$. This shows that:

$$\begin{aligned}
&\mathbb{E}\left[\left(e^{\frac{\gamma}{2}M}\frac{1}{2}\int_{-\infty}^{-L_M} e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\beta_1}{2}}}\left(|\mu_1| e^{\frac{\gamma}{2}Y(-e^{-s/2})} + |\mu_2| e^{\frac{\gamma}{2}Y(e^{-s/2})}\right) ds\right)^{p_1-\epsilon}\right] \qquad (3.5.84)\\
&\leq \mathbb{E}\left[\left(|\mu_1| e^{\frac{\gamma}{2}M}\rho_-(\beta_1) + |\mu_2| e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p-\epsilon}\right] - \mathbb{E}\left[\left(|\mu_1| K_{[-1,0)} + |\mu_2| K_{[0,1]}\right)^{p_1-\epsilon}\right] = o(\epsilon^{-1}).
\end{aligned}$$

The last inequality comes from the fact that the two expectations are equivalent when $\epsilon \to 0$ to a term $O(\epsilon^{-1})$. Therefore:

$$\mathbb{E}\left[\left|\left(\mu_1 K_{[-1,0)} + \mu_2 K_{[0,1]}\right)^{p_1-\epsilon} - \left(\mu_1 e^{\frac{\gamma}{2}M}\rho_-(\beta_1) + \mu_2 e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p_1-\epsilon}\right|\right] = o(\epsilon^{-1}). \tag{3.5.85}$$

Now consider the case $p_1 \leq 1$. Since $p_1 = \frac{2}{\gamma}(Q - \beta_1) > 0$, we are in the case $0 < p_1 \leq 1$. By studying the first order derivatives of the function,

$$(\mathbb{R}_+^*)^3 \ni (x_1, x_2, x_3) \mapsto \left(\mu_1 x_1^{\frac{1}{p_1}} + \mu_2 x_2^{\frac{1}{p_1}} + \mu_3 x_3^{\frac{1}{p_1}}\right)^{p_1}, \tag{3.5.86}$$

we can prove the following inequality with a constant $c > 0$ depending only on the μ_i. For $x_i, x_i' > 0$,

$$\left|(\sum_{i=1}^3 \mu_i x_i)^{p_1} - (\sum_{i=1}^3 \mu_i x_i')^{p_1}\right| \leq c\sum_{i=1}^3 |x_i^{p_1} - x_i'^{p_1}|. \tag{3.5.87}$$

Applying the inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left|\left(\mu_1 K_{[-\infty,0)}+\mu_2 K_{[0,1]}+\mu_3 K_{(1,\infty)}\right)^{p_1-\epsilon}-\left(\mu_1 e^{\frac{\gamma}{2}M}\rho_-(\beta_1)+\mu_2 e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p_1-\epsilon}\right|\right] \\
&\leq c\mathbb{E}\left[\left|K_{(-\infty,0)}^{p_1-\epsilon}-(e^{\frac{\gamma}{2}M}\rho_-(\beta_1))^{p_1-\epsilon}\right|\right]+c\mathbb{E}\left[\left|K_{[0,1]}^{p_1-\epsilon}-(e^{\frac{\gamma}{2}M}\rho_+(\beta_1))^{p_1-\epsilon}\right|\right]+O(1) \\
&\leq c\mathbb{E}\left[\left|K_{(-\infty,0)}^{p_1-\epsilon}-(e^{\frac{\gamma}{2}M}\rho_-(\beta_1))^{p_1-\epsilon}\right|\right]+o(\epsilon^{-1}).
\end{aligned} \tag{3.5.88}$$

Moreover, by sub-additivity,

$$\mathbb{E}[|K_{(-\infty,0)}^{p_1-\epsilon}-K_{(-1,0)}^{p_1-\epsilon}|]=\mathbb{E}[K_{(-\infty,0)}^{p_1-\epsilon}-K_{(-1,0)}^{p_1-\epsilon}]\leq\mathbb{E}[K_{(-\infty,-1)}^{p_1-\epsilon}]=O(1). \tag{3.5.89}$$

Therefore we can bound

$$\mathbb{E}\left[\left|K_{(-\infty,0)}^{p_1-\epsilon}-(e^{\frac{\gamma}{2}M}\rho_-(\beta_1))^{p_1-\epsilon}\right|\right]\leq\mathbb{E}\left[\left|K_{(-1,0)}^{p_1-\epsilon}-(e^{\frac{\gamma}{2}M}\rho_-(\beta_1))^{p_1-\epsilon}\right|\right]+o(\epsilon^{-1})=o(\epsilon^{-1}). \tag{3.5.90}$$

In conclusion,

$$\begin{aligned}
\lim_{\epsilon\to 0}\epsilon\mathbb{E}\left[\left(\mu_1 K_{(-\infty,0)}+\mu_2 K_{[0,1]}+\mu_3 K_{(1,\infty)}\right)^{p_1-\epsilon}\right]&=\lim_{\epsilon\to 0}\epsilon\mathbb{E}\left[\left(\mu_1 e^{\frac{\gamma}{2}M}\rho_-(\beta_1)+\mu_2 e^{\frac{\gamma}{2}M}\rho_+(\beta_1)\right)^{p_1-\epsilon}\right] \\
&=p_1\overline{R}(\beta_1,\mu_1,\mu_2).
\end{aligned} \tag{3.5.91}$$

This finishes the proof of the lemma. □

3.5.3 Mapping GMC moments from $\mathbb{D}$ to $\mathbb{H}$

We prove here a lemma providing a very concrete computation linking the moment of GMC on $\mathbb{D}$ to the moment on $\mathbb{H}$. This will be used to relate the moment formula for GMC on the circle of [88] to the $\overline{U}(\alpha)$ defined in our paper.

Lemma 3.5.7. *Consider $\alpha>\frac{\gamma}{2}$ and let X and $X_{\mathbb{D}}$ be the GFF respectively on $\mathbb{H}$ and $\mathbb{D}$ with covariance given by equations (3.1.9) and (3.1.11). Then the following equality holds,*

$$\mathbb{E}\left[\left(\int_0^{2\pi}e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{i\theta})}d\theta\right)^{\frac{2Q-2\alpha}{\gamma}}\right]=2^{\alpha(Q-\alpha)}\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{e^{\frac{\gamma}{2}X(x)}}{|x-i|^{\gamma\alpha}}g(x)^{\frac{1}{2}-\frac{\alpha\gamma}{4}}dx\right)^{\frac{2Q-2\alpha}{\gamma}}\right], \tag{3.5.92}$$

where both GMC measures are defined by a renormalization according to variable as performed in Definition 3.1.2.

Proof. Take $\psi:z\mapsto i\frac{1+z}{1-z}$ the conformal map that maps the unit disk $\mathbb{D}$ equipped with the Euclidean metric to the upper-half plane $\mathbb{H}$ equipped with the metric $\hat{g}(x)=\frac{4}{|x+i|^4}$. This also maps the field $X_{\mathbb{D}}$ to the field $X_{\hat{g}}$ with covariance given by (3.1.12). This coordinate change applied to the GMC implies the following relation:

$$\mathbb{E}\left[\left(\int_0^{2\pi}e^{\frac{\gamma}{2}X_{\mathbb{D}}(e^{i\theta})-\frac{\gamma^2}{8}\mathbb{E}[X_{\mathbb{D}}(e^{i\theta})^2]}d\theta\right)^{\frac{2Q-2\alpha}{\gamma}}\right]=2^{\alpha(Q-\alpha)}\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{e^{\frac{\gamma}{2}X_{\hat{g}}(x)-\frac{\gamma^2}{8}\mathbb{E}[X_{\hat{g}}(e^{i\theta})^2]}}{|x-i|^{\gamma\alpha}}\hat{g}(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha)}dx\right)^{\frac{2Q-2\alpha}{\gamma}}\right]. \tag{3.5.93}$$

Notice in the above expression we explicitly wrote the renormalization of the GMC to emphasize the formula holds when the GMC is renormalized by variance. Now lets momentarily assume $\alpha>Q$ and write the integral over c:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{e^{\frac{\gamma}{2}X_{\hat{g}}(x)}}{|x-i|^{\gamma\alpha}}\hat{g}(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha)}dx\right)^{\frac{2Q-2\alpha}{\gamma}}\right] \\
&=\frac{\gamma}{2}\frac{1}{\Gamma(\frac{2}{\gamma}(\alpha-Q))}e^{\frac{\alpha}{2}(Q-\alpha)\ln\hat{g}(i)}\int_{\mathbb{R}}dce^{(\alpha-Q)c}\mathbb{E}\left[e^{\alpha X_{\hat{g}}(i)-\frac{\alpha^2}{2}\mathbb{E}[X_{\hat{g}}(i)^2]}e^{-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}e^{\frac{\gamma}{2}X_{\hat{g}}(x)-\frac{\gamma^2}{8}\mathbb{E}[X_{\hat{g}}(x)^2]}\hat{g}(x)^{\frac{1}{2}}dx}\right].
\end{aligned} \tag{3.5.94}$$

To go from the field $X_{\hat{g}}$ to the field X we must perform the change of variable $X = X_{\hat{g}} - Y$ with $Y = \frac{1}{\pi}\int_0^\pi X_{\hat{g}}(e^{i\theta})d\theta$. We perform this replacement and at the same time shift the integration over c by $-Y$ to obtain:

$$\begin{aligned}&\int_{\mathbb{R}} dc e^{(\alpha-Q)c}\mathbb{E}\left[e^{QY}e^{\alpha X(i)-\frac{\alpha^2}{2}\mathbb{E}[X_{\hat{g}}(i)^2]}e^{-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}e^{\frac{\gamma}{2}X(x)-\frac{\gamma^2}{8}\mathbb{E}[X_{\hat{g}}(x)^2]}\hat{g}(x)^{\frac{1}{2}}dx}\right] \\ &= \int_{\mathbb{R}} dc e^{(\alpha-Q)c}\mathbb{E}\left[e^{\frac{Q^2}{2}\mathbb{E}[Y^2]}e^{\alpha X(i)+\alpha Q\mathbb{E}[X(i)Y]-\frac{\alpha^2}{2}\mathbb{E}[X_{\hat{g}}(i)^2]}e^{-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}e^{\frac{\gamma}{2}X(x)+\frac{\gamma Q}{2}\mathbb{E}[X(x)Y]-\frac{\gamma^2}{8}\mathbb{E}[X_{\hat{g}}(x)^2]}\hat{g}(x)^{\frac{1}{2}}dx}\right].\end{aligned} \tag{3.5.95}$$

In the last line we have applied the Girsanov Theorem 3.5.1 to e^{QY}. Record the following easy computations:

$$\begin{aligned}&\mathbb{E}[Y^2] = -\frac{1}{\pi}\int_0^\pi \ln\hat{g}(e^{i\theta})d\theta, \qquad \mathbb{E}[YX_{\hat{g}}(x)] = \frac{1}{2}\ln\frac{g(x)}{\hat{g}(x)} + \frac{1}{2}\mathbb{E}[Y^2], \\ &\mathbb{E}[YX(x)] = \frac{1}{2}\ln\frac{g(x)}{\hat{g}(x)} - \frac{1}{2}\mathbb{E}[Y^2], \qquad \mathbb{E}[X_{\hat{g}}(x)^2] = \mathbb{E}[X(x)^2] + \ln\frac{g(x)}{\hat{g}(x)}.\end{aligned}$$

Then:

$$\begin{aligned}&\int_{\mathbb{R}} dc e^{(\alpha-Q)c}\mathbb{E}\left[e^{\frac{Q^2}{2}\mathbb{E}[Y^2]}e^{\alpha X(i)+\alpha Q\mathbb{E}[X(i)Y]-\frac{\alpha^2}{2}\mathbb{E}[X_{\hat{g}}(i)^2]}e^{-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}e^{\frac{\gamma}{2}X(x)+\frac{\gamma Q}{2}\mathbb{E}[X(x)Y]-\frac{\gamma^2}{8}\mathbb{E}[X_{\hat{g}}(x)^2]}\hat{g}(x)^{\frac{1}{2}}dx}\right] \\ &= \int_{\mathbb{R}} dc e^{(\alpha-Q)c}\mathbb{E}\left[e^{\frac{Q(Q-\alpha)}{2}\mathbb{E}[Y^2]}e^{\frac{\alpha}{2}(Q-\alpha)\ln\frac{g(i)}{\hat{g}(i)}}e^{\alpha X(i)-\frac{\alpha^2}{2}\mathbb{E}[X(i)^2]}e^{-e^{\frac{\gamma c}{2}-\frac{\gamma Q}{4}\mathbb{E}[Y^2]}\int_{\mathbb{R}}e^{\frac{\gamma}{2}X(x)-\frac{\gamma^2}{8}\mathbb{E}[X(x)^2]}g(x)^{\frac{1}{2}}dx}\right] \\ &= e^{\frac{\alpha}{2}(Q-\alpha)\ln\frac{g(i)}{\hat{g}(i)}}\int_{\mathbb{R}} dc e^{(\alpha-Q)c}\mathbb{E}\left[e^{\alpha X(i)-\frac{\alpha^2}{2}\mathbb{E}[X(i)^2]}e^{-e^{\frac{\gamma c}{2}}\int_{\mathbb{R}}e^{\frac{\gamma}{2}X(x)-\frac{\gamma^2}{8}\mathbb{E}[X(x)^2]}g(x)^{\frac{1}{2}}dx}\right] \\ &= \frac{2}{\gamma}\Gamma\left(\frac{2}{\gamma}(\alpha-Q)\right)e^{\frac{\alpha}{2}(\alpha-Q)\ln\hat{g}(i)}\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{e^{\frac{\gamma}{2}X(x)-\frac{\gamma^2}{8}\mathbb{E}[X(x)^2]}}{|x-i|^{\gamma\alpha}}g(x)^{\frac{1}{2}-\frac{\alpha\gamma}{4}}dx\right)^{\frac{2Q-2\alpha}{\gamma}}\right].\end{aligned} \tag{3.5.96}$$

To obtain the third line we have shifted the integral over c by $\frac{Q}{2}\mathbb{E}[Y^2]$ and to obtain the last one we have computed the integral over c. The conclusion of the above is thus that:

$$\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{e^{\frac{\gamma}{2}X_{\hat{g}}(x)}}{|x-i|^{\gamma\alpha}}\hat{g}(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha)}dx\right)^{\frac{2Q-2\alpha}{\gamma}}\right] = \mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{e^{\frac{\gamma}{2}X(x)-\frac{\gamma^2}{8}\mathbb{E}[X(x)^2]}}{|x-i|^{\gamma\alpha}}g(x)^{\frac{1}{2}-\frac{\alpha\gamma}{4}}dx\right)^{\frac{2Q-2\alpha}{\gamma}}\right]. \tag{3.5.97}$$

To lift the constraint $\alpha > Q$ we have introduced to write the c integrals we can simply use analycity in α of both sides of the above equation. Then combining this equation with (3.5.93) implies the claim of the lemma. □

3.5.4 Special functions

3.5.4.1 Hypergeometric equations

Here we recall some facts we have used on the hypergeometric equation and its solution space. For $A > 0$ let $\Gamma(A) = \int_0^\infty t^{A-1}e^{-t}dt$ denote the standard Gamma function which can then be analytically extended to $\mathbb{C}\setminus\{-\mathbb{N}\}$. We recall the following useful properties:

$$\Gamma(A)\Gamma(1-A) = \frac{\pi}{\sin(\pi A)}, \quad \Gamma(A)\Gamma(A+\frac{1}{2}) = \sqrt{\pi}2^{1-2A}\Gamma(2A). \tag{3.5.98}$$

Let $(A)_n := \frac{\Gamma(A+n)}{\Gamma(A)}$. For A, B, C, and x real numbers we define the hypergeometric function F by:

$$F(A,B,C,t) := \sum_{n=0}^\infty \frac{(A)_n(B)_n}{n!(C)_n}t^n. \tag{3.5.99}$$

This function can be used to solve the following hypergeometric equation:

$$\left(t(1-t)\frac{d^2}{dt^2}+(C-(A+B+1)t)\frac{d}{dt}-AB\right)f(t)=0. \tag{3.5.100}$$

We can give the following three bases of solutions corresponding respectively to a power series expansion around $t=0$, $t=1$, and $t=\infty$. Under the assumption that C is not an integer:

$$f(t)=C_1F(A,B,C,t)+C_2t^{1-C}F(1+A-C,1+B-C,2-C,t). \tag{3.5.101}$$

Under the assumption that $C-A-B$ is not an integer:

$$\begin{aligned} f(t)&=B_1F(A,B,1+A+B-C,1-t) \\ &\quad+B_2(1-t)^{C-A-B}F(C-A,C-B,1+C-A-B,1-t). \end{aligned} \tag{3.5.102}$$

Under the assumption that $A-B$ is not an integer:

$$\begin{aligned} f(t)&=D_1t^{-A}F(A,1+A-C,1+A-B,t^{-1}) \\ &\quad+D_2t^{-B}F(B,1+B-C,1+B-A,t^{-1}). \end{aligned} \tag{3.5.103}$$

For each basis we have two real constants that parametrize the solution space, C_1,C_2, B_1,B_2, and D_1,D_2. We thus expect to have an explicit change of basis formula that will give a link between C_1,C_2, B_1,B_2, and D_1,D_2. This is precisely what gives the so-called connection formulas,

$$\begin{pmatrix}C_1\\C_2\end{pmatrix}=\begin{pmatrix}\frac{\Gamma(1-C)\Gamma(A-B+1)}{\Gamma(A-C+1)\Gamma(1-B)} & \frac{\Gamma(1-C)\Gamma(B-A+1)}{\Gamma(B-C+1)\Gamma(1-A)}\\ \frac{\Gamma(C-1)\Gamma(A-B+1)}{\Gamma(A)\Gamma(C-B)} & \frac{\Gamma(C-1)\Gamma(B-A+1)}{\Gamma(B)\Gamma(C-A)}\end{pmatrix}\begin{pmatrix}D_1\\D_2\end{pmatrix}, \tag{3.5.104}$$

$$\begin{pmatrix}B_1\\B_2\end{pmatrix}=\begin{pmatrix}\frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)}\\ \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} & \frac{\Gamma(2-C)\Gamma(A+B-C)}{\Gamma(A-C+1)\Gamma(B-C+1)}\end{pmatrix}\begin{pmatrix}C_1\\C_2\end{pmatrix}. \tag{3.5.105}$$

These relations come from the theory of hypergeometric equations and we will extensively use them in Section 3.2 and Section 3.3 to deduce our shift equations.

3.5.4.2 The double gamma function

We will now provide some explanations on the function $\Gamma_{\frac{\gamma}{2}}(x)$ and $S_{\frac{\gamma}{2}}(x)$ that we have introduced. For all $\gamma\in(0,2)$ and for $\mathrm{Re}(x)>0$, $\Gamma_{\frac{\gamma}{2}}(x)$ is defined by the integral formula,

$$\ln\Gamma_{\frac{\gamma}{2}}(x)=\int_0^\infty\frac{dt}{t}\left[\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})}-\frac{(\frac{Q}{2}-x)^2}{2}e^{-t}+\frac{x-\frac{Q}{2}}{t}\right], \tag{3.5.106}$$

where we have $Q=\frac{\gamma}{2}+\frac{2}{\gamma}$. Since the function $\Gamma_{\frac{\gamma}{2}}(x)$ is continuous it is completely determined by the following two shift equations,

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x+\frac{\gamma}{2})}=\frac{1}{\sqrt{2\pi}}\Gamma(\frac{\gamma x}{2})(\frac{\gamma}{2})^{-\frac{\gamma x}{2}+\frac{1}{2}}, \tag{3.5.107}$$

$$\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(x+\frac{2}{\gamma})}=\frac{1}{\sqrt{2\pi}}\Gamma(\frac{2x}{\gamma})(\frac{\gamma}{2})^{\frac{2x}{\gamma}-\frac{1}{2}}, \tag{3.5.108}$$

and by its value in $\frac{Q}{2}$, $\Gamma_{\frac{\gamma}{2}}(\frac{Q}{2})=1$. Furthermore $x\mapsto\Gamma_{\frac{\gamma}{2}}(x)$ admits a meromorphic extension to all of $\mathbb{C}$ with single poles at $x=-n\frac{\gamma}{2}-m\frac{2}{\gamma}$ for any $n,m\in\mathbb{N}$ and $\Gamma_{\frac{\gamma}{2}}(x)$ is never equal to 0. We have also used the double sine function defined by:

$$S_{\frac{\gamma}{2}}(x)=\frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(Q-x)}. \tag{3.5.109}$$

It obeys the following two shift equations:

$$\frac{S_{\frac{\gamma}{2}}(x+\frac{\gamma}{2})}{S_{\frac{\gamma}{2}}(x)} = 2\sin(\frac{\gamma\pi}{2}x), \quad \frac{S_{\frac{\gamma}{2}}(x+\frac{2}{\gamma})}{S_{\frac{\gamma}{2}}(x)} = 2\sin(\frac{2\pi}{\gamma}x). \tag{3.5.110}$$

The double sine function admits a meromorphic extension to $\mathbb{C}$ with poles at $x = -n\frac{\gamma}{2} - m\frac{2}{\gamma}$ and with zeros at $x = Q + n\frac{\gamma}{2} + m\frac{2}{\gamma}$ for any $n, m \in \mathbb{N}$. Lastly we will need the following asymptotic for $S_{\frac{\gamma}{2}}(x)$:

$$S_{\frac{\gamma}{2}}(x) \sim \begin{cases} e^{-i\frac{\pi}{2}x(x-Q)} & \text{as} \quad \mathrm{Im}(x) \to \infty, \\ e^{i\frac{\pi}{2}x(x-Q)} & \text{as} \quad \mathrm{Im}(x) \to -\infty. \end{cases} \tag{3.5.111}$$

3.5.4.3 Some useful integrals

Lemma 3.5.8. *For $\theta_0 \in [-\pi, \pi]$, $-1 < g < 1$ and $1 \vee (1+g) < b < 2$ we have the identity:*

$$\int_{\mathbb{R}_+ e^{i\theta_0}} \frac{(1+u)^g - 1}{u^b} du = \frac{\Gamma(1-b)\Gamma(-1+b-g)}{\Gamma(-g)}. \tag{3.5.112}$$

By $\mathbb{R}_+ e^{i\theta_0}$ we mean a complex contour that is obtained by rotating the half-line $(0, +\infty)$ by an angle $e^{i\theta_0}$. In particular for $\theta_0 = \pi$ it is passing above -1 and for $\theta_0 = -\pi$ it is passing below.

Proof. Denote by $(x)_n := x(x+1)\dots(x+n-1)$. We start by the case $\theta_0 = 0$:

$$\begin{aligned} \int_0^\infty \frac{(1+u)^g - 1}{u^b} du =& \sum_{n=0}^\infty \frac{(-1)^n}{n!}(-g)_n \frac{1}{n+1-b} - \sum_{n=0}^\infty \frac{(-1)^n}{n!}(-g)_n \frac{1}{1-b+g-n} \\ =& \frac{1}{1-b}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{(-g)_n(1-b)_n}{(2-b)_n} - \frac{1}{1-b+g}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{(-g)_n(-1+b-g)_n}{(b-g)_n} \\ =& \frac{1}{1-b}\,{}_2F_1(-g, 1-b, 2-b, -1) - \frac{1}{1-b+g}\,{}_2F_1(-g, -1+b-g, b-g, -1) \\ =& \frac{\Gamma(1-b)\Gamma(-1+b-g)}{\Gamma(-g)}, \end{aligned} \tag{3.5.113}$$

where in the last line we used the formula, for suitable $\bar{a}, \bar{b} \in \mathbb{R}$,

$$\bar{b}\,{}_2F_1(\bar{a}+\bar{b}, \bar{a}, \bar{a}+1, -1) + \bar{a}\,{}_2F_1(\bar{a}+\bar{b}, \bar{b}, \bar{b}+1, -1) = \frac{\Gamma(\bar{a}+1)\Gamma(\bar{b}+1)}{\Gamma(\bar{a}+\bar{b})}. \tag{3.5.114}$$

Then by rotating the contour, it is easy to observe that the value of the integral is the same for all $\theta_0 \in [-\pi, \pi]$, which finishes the proof. □

A direct consequence by a change of variable is the following identity:

Lemma 3.5.9. *For $\theta_0 \in [-\pi, \pi]$, $-1 < g < 1$ and $g < b < 1 \wedge (1+g)$ we have the identity:*

$$\int_{\mathbb{R}_+ e^{i\theta_0}} \frac{(1+u)^g - u^g}{u^b} du = \frac{\Gamma(1-b)\Gamma(-1+b-g)}{\Gamma(-g)}. \tag{3.5.115}$$

CHAPTER 4

Higher order BPZ equations

Inspired by some intrinsic relations between Coulomb gas integrals and Gaussian multiplicative chaos, this article introduces a general mechanism to prove BPZ equations of order $(r,1)$ and $(1,r)$ in the setting of probabilistic Liouville conformal field theory, a family of conformal field theory which depends on a parameter $\gamma \in (0,2)$. The method consists in regrouping singularities on the degenerate insertion, and transforming the proof into an algebraic problem. With this method we show that BPZ equations hold on the sphere for the parameter $\gamma \in [\sqrt{2},2)$ in the case $(r,1)$ and for $\gamma \in (0,2)$ in the case $(1,r)$. The same technique applies to the boundary Liouville field theory when the bulk cosmological constant $\mu_{\text{bulk}} = 0$, where we prove BPZ equations of order $(r,1)$ and $(1,r)$ for $\gamma \in (0,2)$.

4.1 Introduction

Liouville conformal field theory (LCFT) falls within the general framework of conformal field theory (CFT). One of the main goals of the theory is to characterize the correlation functions, which can be considered as probability amplitudes for some interacting particle system. A direct relevance with probability theory is their conjectured relation to the scaling limit of large planar maps via the so-called KPZ relation [58].

The purpose of this paper is to show that certain correlation functions of LCFT satisfy the Belavin-Polyakov-Zamolodchikov (BPZ) equations, which were first proposed in 1984 [10] in the general context of CFT. The BPZ equations are indexed by two parameters (r,s), with r,s positive integers. The equation associated with the parameter (r,s) is a partial differential equation of order rs in several complex variables. There is no general combinatorial formula for the BPZ equations of all orders (r,s). Nevertheless, in 1988, Benoît and Saint-Aubin (BSA, [11]) found an explicit formula for the BPZ equations of order $(r,1)$ and $(1,r)$. The approach that the authors employed is based on the theory of representations. Despite the simplicity, this approach lacks rigorous definitions of the objects involved.

Recently, in a rigorous mathematical framework, a probabilistic approach to LCFT has been proposed in David-Kupiainen-Rhodes-Vargas [20]: the authors construct the correlation functions of LCFT on the sphere using Gaussian multiplicative chaos (GMC). The challenge is to show that the probabilistic setting allows to prove the conjectures made in the physics literature. In this direction, the BPZ equations of order $(2,1)$ and $(1,2)$ have been proved in [60], which constitutes an important step in proving the remarkable DOZZ formula [61], first proposed by Dorn-Otto-Zamolodchikov-Zamolodchikov [25, 108].

Other results on GMC are also proved in different geometries based on the BPZ equations, such as the Fyodorov Bouchaud's formula [88], the probabilistic distribution of GMC on the unit interval [89] and exact formulas for the boundary Liouville structure constants [37, 84] in an upcoming work. All these series of

projects prove the BPZ equations of order $(2,1)$ and $(1,2)$ in a different setting and use this to deduce non trivial shift equations of the object in question, which corresponds to the conformal bootstrap method in physics.

As a matter of fact, higher order BPZ equations can also be used to deduce exact formulas of certain correlation functions of LFCT, such as the integral forms introduced by Fateev-Litvinov [38, 39]. The idea is to first show that the solution space of higher order BPZ equations is of dimension 1 using monodromy arguments [26, 27]. It is not hard to verify that the integral forms of Fateev-Litvinov satisfy higher order BPZ equations using its relation with Coulomb gas integrals and analycity of its parameters, especially analycity in γ. The hard part is to show higher order BPZ equations for Liouville correlation functions where the analycity in γ is an open problem.

In this article, we investigate the intrinsic problem lying in the BPZ equations for LCFT on the sphere and on the unit disk, and we prove that the BSA formula for BPZ equations of order $(r,1)$ and $(1,r)$ holds true for these two cases under some constraints.

4.1.1 Basic notions

The Gaussian free field with vanishing mean on the Riemannian sphere $(\mathbb{C},\hat{g})$, with $\hat{g}(x) := \frac{4}{(|x|^2+1)^2}$, has covariance given by [20]:

$$\mathbb{E}[X(x)X(y)] = \ln\frac{1}{|x-y|} - \frac{1}{4}\left(\ln\hat{g}(x) + \ln\hat{g}(y)\right) + \ln 2 - \frac{1}{2}. \tag{4.1.1}$$

Because of the singularity of its covariance, X is not defined pointwise and lives in the space of distributions. We use a regularization for the Gaussian free field $X_\epsilon = X * \eta_\epsilon$, where the function η_ϵ is defined by $\eta_\epsilon = \frac{1}{\epsilon^2}\eta(\frac{|x|^2}{\epsilon^2})$, and $\eta \in \mathcal{C}^\infty$ is a non-negative smooth function defined on $\mathbb{R}_+$ with compact support in $[\frac{1}{2},1]$ that satisfies $\pi\int_0^\infty \eta(t)dt = 1$. The variance of the regularized field X_ϵ is given by:

$$\mathbb{E}[X_\epsilon(x)^2] = -\frac{1}{2}\ln\hat{g}_\epsilon(x) + \ln 2 - \frac{1}{2}, \tag{4.1.2}$$

where $\hat{g}_\epsilon = \hat{g} * \eta_\epsilon$.

We define the associated GMC measure [56] by a standard regularization procedure: for $\gamma \in (0,2)$,

$$e^{\gamma X(x)}\hat{g}(x)d^2x := \lim_{\epsilon\to 0} e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}\hat{g}_\epsilon(x)d^2x. \tag{4.1.3}$$

The above convergence is in probability in the weak topology of measures, i.e. for any continuous test function $f : \mathbb{C}\cup\{\infty\} \to \mathbb{R}$, the following limit holds in probability:

$$\int_{\mathbb{C}} f(x)e^{\gamma X(x)}\hat{g}(x)d^2x := \lim_{\epsilon\to 0}\int_{\mathbb{C}} f(x)e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}\hat{g}_\epsilon(x)d^2x. \tag{4.1.4}$$

For an elementary proof of this, see [12].

Denote $\mathbf{z} = (z_1,\dots,z_N)$, and

$$U_N := \{(z_1',\dots,z_N') \in \mathbb{C}^N : \forall i \neq j, z_i' \neq z_j'\}. \tag{4.1.5}$$

We define

$$Q = \frac{\gamma}{2} + \frac{2}{\gamma}, \tag{4.1.6}$$

which is related to the central charge of the LCFT by the formula $c = 1+6Q^2$. Let us introduce the probabilistic Liouville correlation functions first defined in [20]. The definition we give here is coherent with the physics literature and is different from the definition in [20] by a multiplicative factor that is of no importance in the setting of this paper.

Definition 4.1.1 (Liouville correlations). *For $N \in \mathbb{N}^*$, $\boldsymbol{\alpha} \in \mathbb{R}^N$ and $\mathbf{z} \in U_N$, the correlation functions are defined as follows:*

$$\begin{aligned}\langle\prod_{l=1}^{N} V_{\alpha_l}(z_l)\rangle &:= \lim_{\epsilon\to 0}\langle\prod_{l=1}^{N} V_{\alpha_l,\epsilon}(z_l)\rangle \\ &= \lim_{\epsilon\to 0} 2e^{(\ln 2-\frac{1}{2})(\frac{1}{2}\sum_{l=1}^{N}\alpha_l^2-\frac{\gamma}{2}\sum_{l=1}^{N}\alpha_l-\frac{4Q}{\gamma})}\int_{\mathbb{R}} e^{-2Qc}\,\mathbb{E}\Big[\prod_{l=1}^{N} g_\epsilon(z_l)^{\Delta_{\alpha_l}} e^{\alpha_l(X_\epsilon(z_l)+c)-\frac{\alpha_l^2}{2}\mathbb{E}[X_\epsilon(z_l)^2]} \\ &\quad\times e^{-\mu e^{\gamma c}\int_{\mathbb{C}} e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}\hat{g}_\epsilon(x)d^2x}\Big]dc,\end{aligned} \tag{4.1.7}$$

where $\Delta_\alpha := \frac{\alpha}{2}(Q-\frac{\alpha}{2})$ is the conformal weight, and $\mu > 0$ is the cosmological constant.

When the Seiberg bounds $\sum_{i=1}^{N}\alpha_i > 2Q$ and $\forall i, \alpha_i < Q$ are satisfied, the limit above exists and converges to the following expression:

$$Z(\boldsymbol{\alpha})\prod_{1\le i<j\le N}\frac{1}{|z_i-z_j|^{\alpha_i\alpha_j}}\mathbb{E}\left[\left(\int_{\mathbb{C}}\frac{e^{\gamma X(x)}\hat{g}(x)^{1-\frac{\gamma}{4}\sum_{i=1}^{N}\alpha_i}d^2x}{\prod_{k=1}^{N}|x-z_k|^{\gamma\alpha_k}}\right)^{-\frac{\sum_{i=1}^{N}\alpha_i-2Q}{\gamma}}\right], \tag{4.1.8}$$

where

$$Z(\boldsymbol{\alpha}) := 2e^{-\frac{(\ln 2-1/2)}{2}(\sum_{i=1}^{N}\alpha_i-2Q)(\sum_{i=1}^{N}\alpha_i-\frac{4}{\gamma})}\gamma^{-1}\Gamma\left(\frac{\sum_{i=1}^{N}\alpha_i-2Q}{\gamma},\mu\right). \tag{4.1.9}$$

Note that with different conventions, the constant term $Z(\boldsymbol{\alpha})$ can differ, but this will not have any impact on the differential equations. In the notation of $Z(\boldsymbol{\alpha})$ we ignore the dependence on γ because the parameter γ should be fixed at first to define the background geometry of Liouville fields. The constraint $\sum_{i=1}^{N}\alpha_i > 2Q$ is actually subject to the pole at 0 of the Gamma function. If we remove the gamma function, the domain of existence can be extended [20]:

Lemma 4.1.2 (Existence). *$\langle\prod_{l=1}^{N} V_{\alpha_l}(z_l)\rangle/Z(\boldsymbol{\alpha})$ is non trivial if and only if*

$$(\forall i, \alpha_i < Q)\wedge\left(-\frac{\sum_{i=1}^{N}\alpha_i-2Q}{\gamma} < \frac{4}{\gamma^2}\wedge\min_i\left\{\frac{2}{\gamma}(Q-\alpha_i)\right\}\right). \tag{4.1.10}$$

This bound is actually the constraint on the moment of total mass of GMC with log singularities. It allows to have positive moments in the expectation.

Now let us discuss briefly about the regularity of each parameter. It is not hard to show that $\langle\prod_{l=1}^{N} V_{\alpha_l}(z_l)\rangle$ is continuous in γ, $\boldsymbol{\alpha}$ and $\mathbf{z}$ respectively, but we can go further. Correlation functions are actually analytic in $\boldsymbol{\alpha}$, proved in [61] for α_i with small imaginary part (the domain of analyticity was then extended in [54]). The correlation functions are also smooth in $\mathbf{z}$, as proved in a recent work by Oikarinen [73]:

Lemma 4.1.3 (Smoothness). *$U_N \ni \mathbf{z} \mapsto \langle\prod_{l=1}^{N} V_{\alpha_l}(z_l)\rangle$ is $\mathcal{C}^\infty$.*

Let us give the definition for boundary Liouville correlations on the unit disk represented by $(\mathbb{H}, \hat{g}_{\mathbb{H}})$, where $\hat{g}_{\mathbb{H}}(x) = \frac{4}{|x+i|^4}$ is the background metric. It was studied in [53] and here we give a version that adds different boundary cosmological constants μ_i

Definition 4.1.4 (Boundary Liouville correlations). *Define the Gaussian free field with Newmann boundary conditions and vanishing mean on the boundary:*

$$\mathbb{E}[X_{\mathbb{H}}(x)X_{\mathbb{H}}(y)] = \ln\frac{1}{|x-y||x-\bar{y}|} - \frac{1}{2}\ln\hat{g}_{\mathbb{H}}(x) - \frac{1}{2}\ln\hat{g}_{\mathbb{H}}(y). \tag{4.1.11}$$

Let $z_1,\dots,z_N\in\mathbb{H}$ pairwise distinct, $-\infty<t_1<\cdots<t_M<\infty$ and $\mu_1,\dots,\mu_M>0$ with $\mu_0=\mu_M$ by convention, then the boundary Liouville correlations with $\mu_{bulk}=0$ are defined as

$$\begin{aligned}&\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}^{\mu_{j-1},\mu_j}(t_j)\rangle_{\mathbb{H}}\\ :=&Z(\boldsymbol{\alpha};\boldsymbol{\beta})\prod_{1\le i<i'\le N}(|z_i-z_{i'}||z_i-\overline{z}_{i'}|)^{-\alpha_i\alpha_{i'}}\prod_{1\le i\le N,1\le j\le M}|z_i-t_j|^{-\alpha_i\beta_j}\prod_{1\le j<j'\le M}|t_j-t_{j'}|^{-\beta_j\beta_{j'}/2}\\ &\times\mathbb{E}\left[\left(\int_{\mathbb{R}}\frac{1}{\prod_{i=1}^N|u-z_i|^{\gamma\alpha_i}\prod_{j=1}^M|u-t_j|^{\gamma\beta_j/2}}e^{\frac{\gamma}{2}X_{\mathbb{H}}(u)}\hat{g}_{\mathbb{H}}(u)^{\frac{\gamma^2}{8}(p+1)}d\mu_\partial(u)\right)^{-p}\right],\end{aligned}\tag{4.1.12}$$

where $p=\frac{2(\sum_{i=1}^N\alpha_i+\frac12\sum_{j=1}^M\beta_j-Q)}{\gamma}$ and

$$\frac{d\mu_\partial(u)}{du}=\sum_{j=1}^{M-1}\mu_j\mathbf{1}_{t_j<u<t_{j+1}}+\mu_M\mathbf{1}_{u\notin(t_1,t_M)}.\tag{4.1.13}$$

Remark 4.1.5. *$\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}^{\mu_{j-1},\mu_j}(t_j)\rangle_{\mathbb{H}}/Z(\boldsymbol{\alpha};\boldsymbol{\beta})$ is well defined if and only if*

$$(\forall i,\alpha_i<Q)\wedge(\forall j,\beta_j<Q)\wedge\left(-p<\frac{4}{\gamma^2}\wedge\min_j\left\{\frac{2}{\gamma}(Q-\beta_j)\right\}\right).\tag{4.1.14}$$

The different values of μ_j represent boundary cosmological constants on each piece of the boundary. We can send some of the μ_j to 0 as long as $d\mu_\partial$ is non trivial. The expression of normalization factor $Z(\boldsymbol{\alpha};\boldsymbol{\beta})$ is of no importance since we are only interested in differential equations in $\mathbf{z}$.

Let us also introduce Coulomb gas integrals that will be useful for proving the BPZ equations. We will explain later in section 4.1.3 how these integrals are related to Liouville correlation functions.

Definition 4.1.6 (Coulomb gas integrals)**.** *Let $\mathbf{z}\in U_N$, $l\in\mathbb{N}^*$. Define the complex Coulomb gas integrals*

$$\mathfrak{C}_{\boldsymbol{\alpha}}^{(l)}(\mathbf{z}):=\prod_{1\le i<j\le N}|z_i-z_j|^{-\alpha_i\alpha_j}\int_{\mathbb{C}^l}\prod_{1\le i\le N,1\le s\le l}|y_s-z_i|^{-\gamma\alpha_i}\prod_{1\le s<s'\le l}|y_s-y_{s'}|^{-\gamma^2}d^2\mathbf{y}.\tag{4.1.15}$$

The integral converges when $\gamma^2<\frac{4}{l}$, $\forall i\ \alpha_i<\frac{2}{\gamma}-\frac{(l-1)\gamma}{2}$ and $\sum_{i=1}^N\alpha_i>\frac{2}{\gamma}-\frac{(l-1)\gamma}{2}$. In the proof of the BPZ equations, we will need real Coulomb gas integrals: for $t_0<t_1<\cdots<t_N(N\ge2)$,

$$C_{\alpha_0,\boldsymbol{\alpha}}^{(l)}(t_0,\mathbf{t}):=\prod_{0\le i<j\le N}(t_j-t_i)^{-\frac{\alpha_i\alpha_j}{2}}\int_{t_{N-1}<x_1<\cdots<x_l<t_N}\prod_{0\le i\le N,1\le s\le l}(x_s-t_i)^{-\frac{\gamma\alpha_i}{2}}\prod_{1\le s'<s\le l}(x_s-x_{s'})^{-\frac{\gamma^2}{2}}d\mathbf{x},\tag{4.1.16}$$

where $(-1)^\alpha$ depends on the choice of contour and is set to be $e^{i\alpha\pi}$. Especially, when $(\gamma,\alpha_{N-1},\alpha_N)\in(i\mathbb{R}_+)^3$, $C_{\alpha_0,\boldsymbol{\alpha}}^{(l)}(t_0,\mathbf{t})$ is always well defined.

Remark 4.1.7. *In section 4.2.5, we will consider $C_{-(r-1)\chi,\boldsymbol{\alpha}}^{(l)}(t,\mathbf{t})$ for $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$, and we will work with $\alpha_{N-1},\alpha_N\in i\mathbb{R}_+$ sufficiently large in absolute value to have enough differentiability.*

4.1.2 Main results

Definition 4.1.8. *Denote $\mathcal{L}_-(z;\mathbf{z})$ the algebra generated by the differential operators $(L_{-n})_{n\ge1}$ and the identity operator id, where*

$$L_{-1}:=\partial_z,\quad L_{-n}:=\sum_{l=1}^N\left(-\frac{1}{(z_l-z)^{n-1}}\partial_{z_l}+\frac{\Delta_{\alpha_l}(n-1)}{(z_l-z)^n}\right)\quad n\ge2.\tag{4.1.17}$$

In the literature, V_α are called local fields, we also call it an insertion. A field V_α is degenerate if $\alpha = -\frac{(r-1)\gamma}{2} - \frac{2(s-1)}{\gamma}$ for $r,s \in \mathbb{N}^*$, in this case we call it a $(r,s)-$degenerate insertion. When there is a degenerate field, physicists have predicted that correlation functions of CFT satisfy certain partial differential equations with highest order ∂_z^{rs} known as the BPZ equations. Although it is theoretically possible to construct this differential equation from operators of $\mathcal{L}_-(z;\mathbf{z})$, there is no general formula to achieve this. Only in the case when $r=1$ or $s=1$, Benoît and Saint-Aubin [11] found an explicit and compact formula:

Theorem 4.1.9. *Let $r \geq 2$ an integer and*

$$\chi = \frac{\gamma}{2} \text{ or } \frac{2}{\gamma}. \tag{4.1.18}$$

The BPZ equations of order r hold true for $\gamma \in (\sqrt{\frac{2(r-2)}{r-1}}, 2)$ when $\chi = \frac{\gamma}{2}$ and for $\gamma \in (0,2)$ when $\chi = \frac{2}{\gamma}$:

$$\mathcal{D}_r \langle V_{-(r-1)\chi}(z) \prod_{l=1}^N V_{\alpha_l}(z_l)\rangle = 0, \tag{4.1.19}$$

where the differential operator $\mathcal{D}_r$ is given by the Benoît and Saint-Aubin's formula:

$$\mathcal{D}_r = \sum_{k=1}^r \sum_{\substack{(n_1,\dots,n_k)\in(\mathbb{N}^*)^k \\ n_1+\cdots+n_k=r}} \frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^j n_i)(\sum_{i=j+1}^k n_i)} L_{-n_1}\dots L_{-n_k}, \tag{4.1.20}$$

with L_{-n} defined in definition 4.1.8.

Remark 4.1.10. *In this article we use BPZ of order r to represent the two cases of order $(r,1)$ and $(1,r)$. We remark that BPZ equations of order 2 has been well investigated by Kupiainen-Rhodes-Vargas in [60]. The theorem above generalizes their result to all $r \geq 2$, with a constraint on γ when $\chi = \frac{\gamma}{2}$. With some slight efforts, the order 3 BPZ equations can be proved for all $\gamma \in (0,2)$. This will be shown in section 4.2.4.*

Remark 4.1.11. *The constraint on γ when $\chi = \frac{\gamma}{2}$ is a purely technical condition and is only required by Proposition 4.2.9. In particular, we have BPZ equations of all orders for $\gamma \in [\sqrt{2}, 2)$ when $\chi = \frac{\gamma}{2}$.*

In the boundary LCFT case we can have a boundary degenerate insertion or a bulk degenerate insertion. When it comes to a boundary degenerate insertion $B^{\pm}_{-(r-1)\chi}(t)$ defined as below, we will work with an extended definition where t lives in the upper-half plane:

Definition 4.1.12. *Let $\mu_1,\dots\mu_M > 0$, $-\infty < t_1 < \cdots < t_M < \infty$ and $t \in \mathbb{H}\backslash\{z_i, 1\leq i\leq N\}\cup\mathbb{R}$. We define the extended correlation function $\langle B^+_{-(r-1)\chi}(t)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j}(t_j)\rangle_{\mathbb{H}}$ by*

$$\prod_{i=1}^N ((z_i-t)(\overline{z}_i-t))^{\frac{(r-1)\chi\alpha_i}{2}} \prod_{j=1}^M (t_i-t)^{\frac{(r-1)\chi\beta_j}{2}} \prod_{1\leq i<i'\leq N} (|z_i-z_{i'}||z_i-\overline{z}_{i'}|)^{-\alpha_i\alpha_{i'}} \prod_{1\leq i\leq N, 1\leq j\leq M} |z_i-t_j|^{-\alpha_i\beta_j}$$
$$\prod_{1\leq j<j'\leq M} |t_j-t_{j'}|^{-\beta_j\beta_{j'}/2} \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{(t-u)^{\frac{(r-1)\gamma\chi}{2}}}{|u-z_i|^{\gamma\alpha_i}|u-t_j|^{\gamma\beta_j/2}} e^{\frac{\gamma}{2}X_{\mathbb{H}}(u)} \hat{g}_{\mathbb{H}}(u)^{\frac{\gamma^2}{8}(p-\frac{(r-1)\chi}{\gamma}+1)} d\mu_\partial(u)\right)^{-p+\frac{(r-1)\chi}{\gamma}}\right]. \tag{4.1.21}$$

Similarly, we define $\langle B^-_{-(r-1)\chi}(t)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j}(t_j)\rangle_{\mathbb{H}}$ by replacing the term $(t-u)^{\frac{(r-1)\gamma\chi}{2}}$ in the above integral by $(u-t)^{\frac{(r-1)\gamma\chi}{2}}$.

Remark 4.1.13. *When $t\in\mathbb{R}$ satisfies $t_{i_0} < t < t_{i_0+1}$ for certain $1\leq i_0\leq M-1$, we have*

$$\langle B^{\pm}_{-(r-1)\chi}(t)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j}(t_j)\rangle_{\mathbb{H}} = \langle B^{\mu_{i_0},\mu_{i_0}e^{\pm i\pi\frac{\gamma\chi}{2}}}_{-(r-1)\chi}(t)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j}(t_j)\rangle_{\mathbb{H}}.$$

We also have similar results when $t<t_1$ and $t>t_M$. This explains the reason that we call it an extended correlation function.

Now we state the BPZ equations for boundary LCFT, where we can prove the result without constraint on γ.

Theorem 4.1.14. *Let $r \geq 2$ an integer and $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. Let $\mu_1, \dots, \mu_M > 0$, $t_1 < \cdots < t_M$ and $t \in \mathbb{H} \backslash \{z_i, 1 \leq i \leq N\}$. The BPZ equations of order r for a boundary degenerate insertion hold true for $\gamma \in (0,2)$:*

$$\mathcal{D}_r^{\mathbb{H}} \langle B^{\pm}_{-(r-1)\chi}(t) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(t_j) \rangle_{\mathbb{H}} = 0, \tag{4.1.22}$$

where the expression of the differential operator $\mathcal{D}_r^{\mathbb{H}}$ is given by (4.1.20), where we replace the operators L_{-n} by $L_{-n}^{\mathbb{H}}$ defined as $L_{-1}^{\mathbb{H}} := \partial_t$, and for $n \geq 2$:

$$\begin{aligned} L_{-n}^{\mathbb{H}} :=& \sum_{l=1}^{N} \left(-\frac{1}{(z_l - t)^{n-1}} \partial_{z_l} - \frac{1}{(\overline{z}_l - t)^{n-1}} \partial_{\overline{z}_l} + \frac{\Delta_{\alpha_l}(n-1)}{(z_l - t)^n} + \frac{\Delta_{\alpha_l}(n-1)}{(\overline{z}_l - t)^n} \right) \\ &+ \sum_{l=1}^{M} \left(-\frac{1}{(t_l - t)^{n-1}} \partial_{t_l} + \frac{\Delta_{\beta_l}(n-1)}{(t_l - t)^n} \right). \end{aligned} \tag{4.1.23}$$

The BPZ equations of order r also hold true when we insert a bulk degenerate insertion: for $\gamma \in (0,2)$,

$$\mathcal{D}_r^{\mathbb{H},z} \langle V_{-(r-1)\chi}(z) \prod_{i=1}^{N} V_{\alpha_i}(z_i) \prod_{j=1}^{M} B_{\beta_j}^{\mu_{j-1},\mu_j}(t_j) \rangle_{\mathbb{H}} = 0, \tag{4.1.24}$$

where $\mathcal{D}_r^{\mathbb{H},z}$ is defined by the expression (4.1.20) where we replace L_{-n} by $L_{-n}^{\mathbb{H},z}$ defined as $L_{-1}^{\mathbb{H},z} := \partial_z$, and for $n \geq 2$,

$$\begin{aligned} L_{-n}^{\mathbb{H},z} :=& \sum_{l=1}^{N} \left(-\frac{1}{(z_l - z)^{n-1}} \partial_{z_l} - \frac{1}{(\overline{z}_l - z)^{n-1}} \partial_{\overline{z}_l} + \frac{\Delta_{\alpha_l}(n-1)}{(z_l - z)^n} + \frac{\Delta_{\alpha_l}(n-1)}{(\overline{z}_l - z)^n} \right) \\ &- \frac{1}{(\overline{z} - z)^{n-1}} \partial_{\overline{z}} + \frac{\Delta_{-(r-1)\chi}(n-1)}{(\overline{z} - z)^n} + \sum_{l=1}^{M} \left(-\frac{1}{(t_l - z)^{n-1}} \partial_{t_l} + \frac{\Delta_{\beta_l}(n-1)}{(t_l - z)^n} \right). \end{aligned} \tag{4.1.25}$$

In the boundary LCFT case, the proof of BPZ equations is very similar to the sphere case but there is no more technical difficulties, see section 4.2.6.

4.1.3 Strategy of the proof

Let us start by explaining the motivation of introducing Coulomb gas integrals and how it relates to Liouville correlations in a natural way. We consider $-\frac{\sum_{i=1}^{N} \alpha_i - 2Q}{\gamma} = n \in \mathbb{N}^*$. Under this condition, the moment of Liouville correlations can be expanded by Fubini (the rigorous way is to take a regularization for X):

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{\mathbb{C}} \frac{e^{\gamma X(x)} \hat{g}(x)^{1 - \frac{\gamma}{4} \sum_{i=1}^{N} \alpha_i} d^2x}{\prod_{k=1}^{N} |x - z_k|^{\gamma \alpha_k}} \right)^n \right] = \int_{\mathbb{C}^n} \prod_{j=1}^{n} \frac{\hat{g}(x_j)^{1 - \frac{\gamma}{4} \sum_{i=1}^{N} \alpha_i}}{\prod_{k=1}^{N} |x_j - z_k|^{\gamma \alpha_k}} \prod_{i<j} e^{\gamma^2 \mathbb{E}[X(x_i) X(x_j)]} d^2\mathbf{x} \\ &= e^{\frac{n(n-1)\gamma^2}{2}(\ln 2 - \frac{1}{2})} \int_{\mathbb{C}^n} \prod_{j=1}^{n} \frac{\hat{g}(x_j)^{1 - \frac{\gamma}{4} \sum_{i=1}^{N} \alpha_i - \frac{(n-1)\gamma^2}{4}}}{\prod_{k=1}^{N} |x_j - z_k|^{\gamma \alpha_k}} \prod_{i<j} \frac{1}{|x_i - x_j|^{\gamma^2}} d^2\mathbf{x} \\ &= e^{\frac{n(n-1)\gamma^2}{2}(\ln 2 - \frac{1}{2})} \int_{\mathbb{C}^n} \prod_{j=1}^{n} \frac{1}{\prod_{k=1}^{N} |x_j - z_k|^{\gamma \alpha_k}} \prod_{i<j} \frac{1}{|x_i - x_j|^{\gamma^2}} d^2\mathbf{x}. \end{aligned}$$

Together with the expression of $Z(\boldsymbol{\alpha})$ (4.1.9), we deduce that when $-\frac{\sum_{i=1}^N \alpha_i - 2Q}{\gamma} = n$,

$$\langle \prod_{l=1}^N V_{\alpha_l}(z_l) \rangle = 2\mathfrak{C}_{\boldsymbol{\alpha}}^{(n)}(\mathbf{z}). \tag{4.1.26}$$

It is explained in [103] how physicists use this relation to predict exact formulas on correlation functions of LCFT.

Now we explain the strategy. Consider the Liouville correlation function on the sphere with a degenerate insertion: $\langle V_{-(r-1)\chi}(z) \prod_{l=1}^N V_{\alpha_l}(z_l) \rangle$. By taking successive derivatives following the operators L_{-n} (see section 4.2.1), we will have integrals that have singularities at z and z_l. Using integration by parts and some identities we can regroup all the singularities on z. By doing so we observe some repeating terms $P_{\mathbf{n}}Q_{\mathbf{q}}$ (Definition 4.2.6). This allows us to transform the proof of the BPZ equation into an algebraic problem where we search to cancel the coefficients before each $P_{\mathbf{n}}Q_{\mathbf{q}}$.

On the other hand, we can prove directly that real Coulomb gas integrals satisfy BPZ equations. This is based on the fact that the integrand satisfies BPZ equations (see section 4.3). Furthermore, real Coulomb gas integrals have the same algebraic development into $P_{\mathbf{n}}Q_{\mathbf{q}}$ as Liouville correlations, but with a different definition for the quantities $P_{\mathbf{n}}$ and $Q_{\mathbf{q}}$. This is not a surprising fact from the previous explanation of their relations. A study of linear independence of this family allows to show that all the coefficients are actually zero, which means that Liouville correlations satisfy BPZ equations. Remark that we use real Coulomb gas integrals instead of complex ones in order to avoid the problem of integrating against the singularities.

For the organization of this paper, we will present a detailed proof for Theorem 4.1.9 in section 4.2, and in 4.2.6 we give the proof of Theorem 4.1.14. Section 4.3 provides an original and elementary proof showing that the integrand of Coulomb gas integrals satisfy BPZ equations, which implies as a consequence that real Coulomb gas integrals also satisfy the BPZ equations.

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4.2 Proof of the BPZ equations

The subsections 4.2.1 to 4.2.5 are devoted to the proof of the BPZ equations on the sphere. In section 4.2.6 we will see that the BPZ equations for boundary LCFT can be proved in exactly the same manner as the sphere case, but without constraint on γ since the technical problem is avoided by taking $\mu_{bulk} = 0$.

4.2.1 Derivatives of correlation functions

We shall first understand how to derive the correlation functions. A proof for the derivative rule is recalled in the appendix 4.4.

In this subsection, we will consider the correlation functions $\langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l) \rangle$ with $z_0 = z$, $\alpha_0 = -(r-1)\chi$. This is to stay consistent in notations with the later proof of the BPZ equations, but all the results in this subsection hold true for general values of α_0 and z_0. Let $\theta : \mathbb{R}_+ \to [0,1]$ be a smooth function that equals 0 in $[0, \frac{1}{2}]$ and 1 in $[1, \infty)$ and define $\theta_\delta = \theta(\frac{|\cdot|}{\delta})$ a regularization function. We introduce the notations:

Definition 4.2.1. *Define for $\delta > 0$, and $(z, \mathbf{z}) \in U_{N+1}$:*

$$\begin{aligned}\langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l) \rangle_\delta =& 2e^{(\ln 2 - \frac{1}{2})(\frac{1}{2}\sum_{l=0}^N \alpha_l^2 - \frac{\gamma}{2}\sum_{l=0}^N \alpha_l - \frac{4Q}{\gamma})} \int_{\mathbb{R}} e^{-2Qc}\, \mathbb{E}\Big[\prod_{l=0}^N g_\epsilon(z_l)^{\Delta_{\alpha_l}} e^{\alpha_l(X_\epsilon(z_l)+c) - \frac{\alpha_l^2}{2}\mathbb{E}[X_\epsilon(z_l)^2]} \\ & \times e^{-\mu e^{\gamma c}\int_{\mathbb{C}} \theta(x-z_0) e^{\gamma X_\epsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]} \hat{g}_\epsilon(x) d^2x} \Big] dc, \end{aligned} \tag{4.2.1}$$

where we add a regularization around z_0 while integrating the GMC measure compared to the expression of $\langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l) \rangle$ defined in (4.1.7).

We denote $\langle\prod_{l=0}^N V_{\alpha_l}(z_l)\rangle_\delta$ the limit of $\langle\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta$ when ϵ goes to 0, which equals

$$Z(\alpha_0,\boldsymbol{\alpha})\prod_{0\le i<j\le N}|z_i-z_j|^{-\alpha_i\alpha_j}\mathbb{E}[(\int_{\mathbb{C}}\frac{\theta_\delta(x-z_0)e^{\gamma X(x)}\hat{g}(x)^{1-\frac{\gamma}{4}\sum_{i=0}^N\alpha_i}d^2x}{\prod_{k=0}^N|x-z_k|^{\gamma\alpha_k}})^{-\frac{\sum_{i=0}^N\alpha_i-2Q}{\gamma}}]. \tag{4.2.2}$$

The notation $\boldsymbol{\alpha}$ still stands for $(\alpha_1,\dots,\alpha_N)$. Note that $\langle\prod_{l=0}^N V_{\alpha_l}(z_l)\rangle_\delta$ converges in the weak topology to $\langle\prod_{l=0}^N V_{\alpha_l}(z_l)\rangle$.

Lemma 4.2.2 (Derivative rule)**.** *For $(z_0,\mathbf{z})\in U_{N+1}$ and $0\le i\le N$,*

$$\begin{aligned}\partial_{z_i}\langle\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta =&\sum_{\substack{j=0\\ j\neq i}}^N\frac{\alpha_i\alpha_j}{2(z_j-z_i)_\epsilon}\langle\prod_{l=1}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta-\frac{\mu\gamma\alpha_i}{2}\int_{\mathbb{C}}\frac{\theta_\delta(y-z_0)}{(y-z_i)_\epsilon}\langle V_{\gamma,\epsilon}(y)\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y\\ &+\mathbf{1}_{\{i=0\}}\mu\int_{\mathbb{C}}\partial_z\theta_\delta(y-z_0)\langle V_{\gamma,\epsilon}(y)\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y,\end{aligned} \tag{4.2.3}$$

where

$$\frac{1}{(z)_\epsilon}:=\int_{\mathbb{C}}\int_{\mathbb{C}}\frac{1}{z-x_1+x_2}\eta_\epsilon(x_1)\eta_\epsilon(x_2)d^2x_1d^2x_2. \tag{4.2.4}$$

Remark 4.2.3. *The functions $\frac{1}{(z)_\epsilon}$ and $\langle\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta$ are smooth. The only difference in this derivative rule with [60] is that we take the regularization θ_δ.*

Let us explain briefly how to understand this derivative rule from the expression (4.2.2). There are three terms. The first comes from the preceding term $\prod_{i<j}|z_i-z_j|^{-\alpha_i\alpha_j}$ with regularization. The other two terms appear whenever we take derivatives on moment of Gaussian multiplicative chaos. We can consider them as a simple derivative under expectation and then an application of the Girsanov's theorem. Finally we state an identity that will be useful:

Lemma 4.2.4 (KPZ Identity)**.** *For $\delta,\epsilon\ge 0$, the integral $\int_{\mathbb{C}}\theta_\delta(y-z)\langle V_{\gamma,\epsilon}(y)\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y$ is well defined and*

$$\mu\gamma\int_{\mathbb{C}}\theta_\delta(y-z)\langle V_{\gamma,\epsilon}(y)\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y=(\sum_{l=0}^N\alpha_l-2Q)\langle\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta. \tag{4.2.5}$$

Remark 4.2.5. *When δ or ϵ equal 0, it simply means that there is no regularization. By applying the lemma multiple times, we obtain in particular that for $p\ge 1$, the integral*

$$\int_{\mathbb{C}^p}\prod_{j=1}^p\theta_\delta(y_j-z)\langle\prod_{j=1}^p V_{\gamma,\epsilon}(y_j)\prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2\mathbf{y}$$

is well defined. An important information to extract from this is the integrability at infinity of the above integral.

Proof. For $\epsilon>0$ and $\delta\ge 0$, by a change of variable $c'=\frac{\ln\mu}{\gamma}+c$, we have

$$\begin{aligned}\langle\prod_l V_{\alpha_l,\epsilon}(z_l)\rangle_\delta =&\mu^{-\frac{\sum_l\alpha_l-2Q}{\gamma}}2e^{(\ln 2-\frac{1}{2})(\frac{1}{2}\sum_{l=1}^N\alpha_l^2-\frac{\gamma}{2}\sum_{l=1}^N\alpha_l-\frac{4Q}{\gamma})}\\ &\int_{\mathbb{R}}e^{-2Qc'}\mathbb{E}\Big[\prod_{l=1}^N g_\epsilon(z_l)^{\Delta_{\alpha_l}}e^{\alpha_l(X_\epsilon(z_l)+c')-\frac{\alpha_l^2}{2}\mathbb{E}[X_\epsilon(z_l)^2]}e^{-e^{\gamma c'}\int_{\mathbb{C}}e^{\gamma X_\epsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2]}\hat{g}_\epsilon(x)d^2x}\Big]dc'.\end{aligned}$$

We obtain the lemma by taking the derivative with respect to μ on both sides. The case $\epsilon=0$ can be obtained by sending $\epsilon\to 0$. □

In [60], the authors proved that the functions

$$y \to \sup_{\epsilon} \langle V_{\gamma,\epsilon}(y) \prod_{l=1}^{N} V_{\alpha_l,\epsilon}(z_l) \rangle \text{ and } (x,y) \to \sup_{\epsilon} \langle V_{\gamma,\epsilon}(x) V_{\gamma,\epsilon}(y) \prod_{l=1}^{N} V_{\alpha_l,\epsilon}(z_l) \rangle$$

are integrable. The results generalize easily to the case with θ_δ. This fact will be useful later to justify the convergences.

4.2.2 Repeating patterns

Let us introduce some notations for the terms that will play a central role in the proof of BPZ equations.

Definition 4.2.6. *For $n \in \mathbb{N}^*$, we define*

$$P_n(z, \mathbf{z}) := \sum_{l=1}^{N} \frac{\alpha_l}{2(z_l - z)^n} \tag{4.2.6}$$

with $\mathbf{z} = (z_1, \dots, z_N)$. For $\mathbf{n} = (n_1, \dots, n_m)$, note $P_{\mathbf{n}} = \prod_{i=1}^{m} P_{n_i}$.

Let $p \in \mathbb{N}$, and $\mathbf{q} = (q_1, \dots, q_p) \in (\mathbb{N}^*)^p$, we define

$$Q_{\mathbf{q}}(z, \mathbf{z}) := (\frac{\mu\gamma}{2})^p \int_{\mathbb{C}^p} \prod_{j=1}^{p} \frac{\theta_\delta(y_j - z)}{(y_j - z)^{q_j}} \langle V_{-(r-1)\chi}(z) \prod_{i=1}^{p} V_\gamma(y_i) \prod_{l=1}^{N} V_{\alpha_l}(z_l) \rangle_\delta \, d^2\mathbf{y} \tag{4.2.7}$$

We also provide an operator T_k on $Q_{\mathbf{q}}$, with $k \in \mathbb{N}^*$:

$$T_k Q_{\mathbf{q}} = Q_{q_1, \dots, q_k + 1, \dots, q_p} \tag{4.2.8}$$

Later we will show that proving the BPZ equations is equivalent to a combinatorial problem in the algebra generated by $P_{\mathbf{n}}$ and $Q_{\mathbf{q}}$.

Definition 4.2.7. *Denote $\mathfrak{R}_\delta$ for an arbitrary term in the functional vectorial space:*

$$Vect\Bigg((z, \mathbf{z}) \mapsto D \left[P_{\mathbf{n}} \int_{\mathbb{C}^p} \frac{\partial_z \theta_\delta(y_1 - z)}{(y_1 - z)^{q_1}} \prod_{i=2}^{p} \frac{\theta_\delta(y_i - z)}{(y_i - z)^{q_i}} \langle V_{-(r-1)\chi}(z) \prod_{i=1}^{p} V_\gamma(y_i) \prod_{l=1}^{N} V_{\alpha_l}(z_l) \rangle_\delta \, d^2\mathbf{y} \right], \\ D \in \mathcal{L}_-(z; \mathbf{y}), \mathbf{n} \in \mathbb{N}_*^m (m \geq 0), \mathbf{q} \in \mathbb{N}_*^p (p \geq 1), \sum_{j=1}^{p} q_j \leq r - 1 \Bigg). \tag{4.2.9}$$

Remark 4.2.8. *$D\mathfrak{R}_\delta = \mathfrak{R}_\delta$ for all $D \in \mathcal{L}_-(z; \mathbf{y})$.*

The reason of introducing $\mathfrak{R}_\delta(z, \mathbf{z})$ is that they appear in the calculus of derivatives as perturbation terms and we want to control these terms. We show that they do not have contribution:

Proposition 4.2.9. *When $\chi = \frac{\gamma}{2}$ and $\gamma \in (\sqrt{\frac{2(r-2)}{r-1}}, 2)$ or when $\chi = \frac{2}{\gamma}$ and $\gamma \in (0, 2)$, $\mathfrak{R}_\delta(z, \mathbf{z})$ converges weakly to 0 in the sense of distributions as $\delta \to 0$.*

Proof. We can set D to identity since if we have weak convergence to 0 when $D = \mathrm{id}$, then applying differential operators from $\mathcal{L}_-(z; \mathbf{z})$ will not affect its weak convergence to 0. Without loss of generality, we take $P_{\mathbf{n}} = 1$. It is easy to see that $\partial_z \theta_\delta$ is supported in $B(0, \delta) \backslash B(0, \frac{\delta}{2})$, and $\|\partial_z \theta_\delta\|_\infty \leq \frac{c}{\delta}$ for a constant $c > 0$. Then it

suffices to control

$$\begin{aligned}
&\left|\int_{\mathbb{C}^p}\frac{\partial_z\theta_\delta(y_1-z)}{(y_1-z)^{q_1}}\prod_{i=2}^{p}\frac{\theta_\delta(y_i-z)}{(y_i-z)^{q_i}}\langle V_{-(r-1)\chi}(z)\prod_{i=1}^{p}V_\gamma(y_i)\prod_{l=1}^{N}V_{\alpha_l}(z_l)\rangle_\delta\, d^2\mathbf{y}\right|\\
&\le c'\delta^{-1-\sum_{i=1}^p q_i}\int_{B(z,\delta)\setminus B(z,\frac{\delta}{2})}\int_{\mathbb{C}^{p-1}}\prod_{i=2}^{p}\theta_\delta(y_i-z)\langle V_{-(r-1)\chi}(z)\prod_{i=1}^{p}V_\gamma(y_i)\prod_{l=1}^{N}V_{\alpha_l}(z_l)\rangle_\delta\, d^2\mathbf{y}\\
&\overset{(4.2.5)}{\le} c(\boldsymbol{\alpha},\gamma,\mu)\delta^{-r}\int_{B(z,\delta)\setminus B(z,\frac{\delta}{2})}\langle V_{-(r-1)\chi}(z)V_\gamma(y_1)\prod_{l=1}^{N}V_{\alpha_l}(z_l)\rangle_\delta d^2y_1.
\end{aligned}$$

Consider $(z,\mathbf{z})$ in a compact of U_{N+1}. We take $\delta_0<1\wedge\min_{i\neq j}|z_i-z_j|\wedge\min_i|z_i-z|$, then

$$\begin{aligned}
\langle V_{-(r-1)\chi}(z)V_\gamma(y_1)\prod_{l=1}^{N}V_{\alpha_l}(z_l)\rangle_\delta &\le |y_1-z|^{(r-1)\gamma\chi}Z(-(r-1)\chi,\gamma,\boldsymbol{\alpha})\prod_{i=1}^{N}|z_i-z|^{(r-1)\chi\alpha_i}\prod_{i<j}\frac{1}{|z_i-z_j|^{\alpha_i\alpha_j}}\\
&\sup_{y'\in B(z,\delta_0)}\mathbb{E}[(\int_{B(0,1)^c}\frac{|x-z|^{(r-1)\chi}e^{\gamma X(x)}\hat g(x)^{1-\frac{\gamma}{4}\sum_{i=1}^N\alpha_i}d^2x}{|x-y'|^{\gamma^2}\prod_{k=1}^N|x-z_k|^{\gamma\alpha_k}})^{-\frac{\sum_{i=1}^N\alpha_i-2Q}{\gamma}}]\\
&\le c(\boldsymbol{\alpha},\gamma,\mu)|y_1-z|^{(r-1)\gamma\chi}.
\end{aligned}$$

Then we can bound the whole term by $c(\boldsymbol{\alpha},\gamma,\mu)\delta^{(r-1)\gamma\chi+2-r}$, which converges to 0 when the condition on γ is satisfied.

□

4.2.3 Recursive formulas

This subsection is devoted to proving a recursive formula that allows to transform the higher BPZ equations into a combinatorial form. The main result is the following proposition.

Proposition 4.2.10. *The following relation holds when $n+|\mathbf{n}|+|\mathbf{q}|\le r$:*

$$\begin{aligned}
L_{-n}P_{\mathbf{n}}Q_{\mathbf{q}} =&\Bigg[\sum_i n_i\frac{P_{n_i+n}}{P_{n_i}}-\sum_{i=1}^{n-1}P_iP_{n-i}+((n-1)Q-(r-1)\chi)P_n+2\sum_{i=1}^{n-1}P_iT_{p+1}^{n-i}+((r-1)\chi-\frac{2(n-1)}{\gamma})T_{p+1}^n\\
&-\sum_{i=1}^{n-1}T_{p+2}^{n-i}T_{p+1}^i+\sum_{j=1}^{p}\Bigg(-\gamma\sum_{i=1}^{n-1}P_iT_j^{n-i}+(\frac{(n-1)\gamma Q}{2}-\frac{(r-1)\gamma\chi}{2}+q_j)T_j^n\\
&+\gamma\sum_{i=1}^{n-1}T_{p+1}^{n-i}T_j^i-\frac{\gamma^2}{4}\sum_{j'=1}^{p}\sum_{i=1}^{n-1}T_{j'}^{n-i}T_j^i\Bigg)\Bigg]P_{\mathbf{n}}Q_{\mathbf{q}}+\mathfrak{R}_\delta
\end{aligned}\tag{4.2.10}$$

The recursive relation seems complicated but we will not use directly this expression, what we need is only the homogeneity of $P_{\mathbf{n}}Q_{\mathbf{q}}$. The proposition shows that $\mathcal{D}_rQ_0$ can be expressed as

$$\mathcal{D}_rQ_0=\sum_{\mathbf{n},\mathbf{q}:|\mathbf{n}|+|\mathbf{q}|=r}\lambda_{\mathbf{n},\mathbf{q}}(\gamma)P_{\mathbf{n}}Q_{\mathbf{q}}+\mathfrak{R}_\delta,\tag{4.2.11}$$

where $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ are rational fractions in γ and are independent of other parameters (the expression of $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ is different when χ takes the value $\frac{\gamma}{2}$ or $\frac{2}{\gamma}$). To avoid ambiguity of the definition, we proceed as if the family $(P_{\mathbf{n}}Q_{\mathbf{q}})_{\mathbf{n},\mathbf{q}}$ is linearly independent and regroup the coefficients to obtain the above equation. In section 4.2.5, we will prove that: every coefficient $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ equals 0. Then by sending δ to 0, we have $\mathcal{D}_r\langle V_{-\frac{(r-1)\gamma}{2}}(z)\prod_l V_{\alpha_l}(z_l)\rangle=0$ in the weak sense. The smoothness of correlation functions allows to conclude the proof for Theorem 4.1.9.

Now we discuss the first step of proving Proposition 4.2.10. It is easy to see that

$$L_{-n}P_{\mathbf{n}}Q_{\mathbf{q}}=(\sum_i n_i\frac{P_{n_i+n}}{P_{n_i}})P_{\mathbf{n}}Q_{\mathbf{q}}+P_{\mathbf{n}}L_{-n}Q_{\mathbf{q}}.\tag{4.2.12}$$

Without loss of generality, we can consider $P_{\mathbf{n}} = 1$. Let us first prove an intermediate lemma, which is the special case where $\mathbf{q} = 0$:

Lemma 4.2.11. *The following relation holds when $n \leq r$:*

$$L_{-n}Q_0 = \Big(-\sum_{i=1}^{n-1} P_i P_{n-i} + ((n-1)Q - (r-1)\chi)P_n \Big) Q_0 + 2\sum_{i=1}^{n-1} P_i Q_{n-i} + ((r-1)\chi - \frac{2(n-1)}{\gamma})Q_n$$
$$- \sum_{i=1}^{n-1} Q_{i,n-i} + \mu \int_{\mathbb{C}} \frac{\partial_z \theta_\delta(y-z)}{(y-z)^{n-1}} \langle z, \mathbf{z}; y \rangle_\delta \, d^2y. \tag{4.2.13}$$

Remark 4.2.12. *For $\epsilon \geq 0$, the notation $\langle z, \mathbf{z}; \mathbf{y} \rangle_{\delta,\epsilon}$ stands for $\langle V_{-(r-1)\chi,\epsilon}(z) \prod_{i=1}^{p} V_{\gamma,\epsilon}(y_i) \prod_{l=1}^{N} V_{\alpha_l,\epsilon}(z_l) \rangle_\delta$.*

Proof. We will need to use Lemma 4.2.2 for taking derivatives of $\langle V_{-(r-1)\chi,\epsilon}(z) \prod_{l=1}^{N} V_{\alpha_l,\epsilon}(z_l) \rangle_\delta$, and then tend ϵ to 0 to get the desired relation. Remark that all the convergences in this proof are locally uniform convergences for $(z, \mathbf{z})$ when $\epsilon \to 0$, and it will not be specified.

Let us work with the case $n \geq 2$,

$$L_{-n}\langle z, \mathbf{z} \rangle_{\delta,\epsilon} = \left(-\sum_j \sum_{l \neq j} \frac{\alpha_j \alpha_l}{2(z_j - z)^{n-1}(z_l - z_j)_\epsilon} - \sum_j \frac{(r-1)\chi\alpha_j}{2(z_j - z)^{n-1}(z_j - z)_\epsilon} + \sum_j \frac{(n-1)\Delta_{\alpha_j}}{(z_j - z)^n} \right) \langle z, \mathbf{z} \rangle_{\delta,\epsilon}$$
$$+ \sum_j \frac{\mu\gamma\alpha_j}{2(z_j - z)^{n-1}} \int_{\mathbb{C}} \frac{\theta_\delta(y-z)}{(y-z_j)_\epsilon} \langle z, \mathbf{z}; y \rangle_{\delta,\epsilon} \, d^2y$$
$$=: A_\epsilon \langle z, \mathbf{z} \rangle_{\delta,\epsilon} + B_\epsilon. \tag{4.2.14}$$

We use a simple identity to calculate $\lim_{\epsilon\to 0} A_\epsilon$ and $\lim_{\epsilon\to 0} B_\epsilon$:

$$\frac{1}{(x_1 - x_2)(x_2 - z)^{n-1}} - \frac{1}{(x_1 - x_2)(x_1 - z)^{n-1}} = \sum_{i=1}^{n-1} \frac{1}{(x_1 - z)^i} \frac{1}{(x_2 - z)^{n-i}}. \tag{4.2.15}$$

By symmetry and the above identity,

$$\sum_j \sum_{l \neq j} \frac{\alpha_j \alpha_l}{2(z_l - z_j)(z_j - z)^{n-1}} = \frac{1}{2} \sum_j \sum_{l \neq j} \frac{\alpha_j \alpha_l}{2(z_l - z_j)(z_j - z)^{n-1}} - \frac{\alpha_j \alpha_l}{2(z_l - z_j)(z_l - z)^{n-1}}$$
$$= \sum_{i=1}^{n-1} P_i P_{n-i} - \sum_j \frac{(n-1)\alpha_j^2}{4(z_j - z)^n}.$$

Therefore taking the limit for A_ϵ yields

$$\lim_{\epsilon\to 0} A_\epsilon = -\sum_{i=1}^{n-1} P_i P_{n-i} + ((n-1)Q - (r-1)\chi) P_n.$$

For B_ϵ, note that

$$B_\epsilon - \int_{\mathbb{C}} \sum_j \frac{\mu\gamma\alpha_j \theta_\delta(y-z)}{2(y - z_j)_\epsilon (y-z)^{n-1}} \langle z, \mathbf{z}; y \rangle_{\delta,\epsilon} \, d^2y$$
$$= \sum_{i=1}^{n-1} \sum_j \frac{\mu\gamma\alpha_j}{2(z_j - z)^i} \int_{\mathbb{C}} \frac{y - z_j}{(y - z_j)_\epsilon} \frac{\theta_\delta(y-z)}{(y-z)^{n-i}} \langle z, \mathbf{z}; y \rangle_{\delta,\epsilon} \, d^2y \xrightarrow{\epsilon\to 0} 2\sum_{i=1}^{n-1} P_i Q_{n-i}.$$

Here we have used dominant convergence, where we can bound $\frac{x}{(x)_\epsilon} < c$ with c a constant independent of ϵ, and the function $y \mapsto \sup_\epsilon \langle z, \mathbf{z}; y \rangle_{\delta,\epsilon}$ is integrable.

An integration by parts formula (or Stokes formula) gives the following identity:

$$\begin{aligned}
&\int_{\mathbb{C}} \sum_j \alpha_j \frac{\mu\gamma\theta_\delta(y-z)}{2(y-z_j)_\epsilon (y-z)^{n-1}} \langle z, \mathbf{z}; y\rangle_{\delta,\epsilon}\, d^2y \\
=&-\frac{2(n-1)}{\gamma}\int_{\mathbb{C}} \frac{\mu\gamma\theta_\delta(y-z)}{2(y-z)^n}\langle z,\mathbf{z};y\rangle_{\delta,\epsilon}\, d^2y + (r-1)\chi\int_{\mathbb{C}} \frac{\mu\gamma\theta_\delta(y-z)}{2(y-z)^{n-1}(y-z)_\epsilon}\langle z,\mathbf{z};y\rangle_{\delta,\epsilon}\, d^2y \\
&+\mu\int_{\mathbb{C}} \frac{\partial_z\theta_\delta(y-z)}{(y-z)^{n-1}}\langle z,\mathbf{z};y\rangle_{\delta,\epsilon}\, d^2y - 2\int_{\mathbb{C}^2} \frac{(\mu\gamma)^2\theta_\delta(x-z)\theta_\delta(y-z)}{4(x-y)_\epsilon (y-z)^{n-1}}\langle z,\mathbf{z};x,y\rangle_{\delta,\epsilon}\, d^2x d^2y.
\end{aligned}$$

The first two terms in the sum converge to $((r-1)\chi - \frac{2(n-1)}{\gamma})Q_n$. For the last term, using the integrability of $(x,y)\mapsto \sup_\epsilon \langle z,\mathbf{z};x,y\rangle_{\delta,\epsilon}$ and by symmetry,

$$\begin{aligned}
&2\int_{\mathbb{C}^2} \frac{(\mu\gamma)^2\theta_\delta(x-z)\theta_\delta(y-z)}{4(x-y)_\epsilon(y-z)^{n-1}}\langle z,\mathbf{z};x,y\rangle_{\delta,\epsilon}\, d^2x d^2y \\
=&\sum_{i=1}^{n-1}\int_{\mathbb{C}^2}\frac{(\mu\gamma)^2}{4}\frac{x-y}{(x-y)_\epsilon}\frac{\theta_\delta(x-z)\theta_\delta(y-z)}{(y-z)^i(x-z)^{n-i}}\langle z,\mathbf{z};x,y\rangle_{\delta,\epsilon} d^2x d^2y \overset{\epsilon\to 0}{\longrightarrow} \sum_{i=1}^{n-1} Q_{i,n-i}.
\end{aligned}$$

From the above calculus, we deduce that

$$\lim_{\epsilon\to 0} B_\epsilon = 2\sum_{i=1}^{n-1} P_i Q_{n-i} + ((r-1)\chi - \frac{2(n-1)}{\gamma})Q_n - \sum_{i=1}^{n-1} Q_{i,n-i} + \mu\int_{\mathbb{C}} \frac{\partial_z\theta_\delta(y-z)}{(y-z)^{n-1}}\langle z,\mathbf{z};y\rangle_\delta\, d^2y.$$

Sending ϵ to 0 in $L_{-n}\langle z,\mathbf{z}\rangle_{\delta,\epsilon} = A_\epsilon\langle z,\mathbf{z}\rangle_{\delta,\epsilon} + B_\epsilon$ proves the lemma for $n\geq 2$ in the weak derivative sense. Then it suffices to conclude with the smoothness of correlations. It is not hard to verify the validity for the case $n=1$, which concludes the proof. □

We have shown Proposition 4.2.10 in the special case $\mathbf{q}=0$ in the previous lemma. The proof for the general case can then be deduced from this result.

Proof of Proposition 4.2.10. Consider the case $n\geq 2$. Let $\mathbf{y}=(y_1,\dots,y_p)$ and $R>0$. Note that the operator L_{-n} commutes with the integral sign in the following expression:

$$L_{-n}(\frac{\mu\gamma}{2})^p\int_{B(0,R)^p}\prod_{j=1}^p \frac{\theta_\delta(y_j-z)}{(y_j-z)^{q_j}}\langle z,\mathbf{z};\mathbf{y}\rangle_{\delta,\epsilon}\, d^2\mathbf{y} = (\frac{\mu\gamma}{2})^p\int_{B(0,R)^p}\prod_{j=1}^p \frac{\theta_\delta(y_j-z)}{(y_j-z)^{q_j}} L_{-n}\langle z,\mathbf{z};\mathbf{y}\rangle_{\delta,\epsilon}\, d^2\mathbf{y}.$$

We introduce the notation

$$L_{-n}^{\langle p\rangle} = \sum_{l=1}^N\left(-\frac{\partial_{z_l}}{(z_l-z)^{n-1}} + \frac{(n-1)\Delta_{\alpha_l}}{(z_l-z)^n}\right) + \sum_{i=1}^p\left(-\frac{\partial_{y_i}}{(y_i-z)^{n-1}} + \frac{(n-1)\Delta_\gamma}{(y_i-z)^n}\right).$$

This newly defined operator considers $\mathbf{y}$ as insertions and applies the corresponding differential operators. Remark that the value of Δ_γ is 1 and we can write

$$\begin{aligned}
&(\frac{\mu\gamma}{2})^p\int_{B(0,R)^p}\prod_{j=1}^p\frac{\theta_\delta(y_j-z)}{(y_j-z)^{q_j}} L_{-n}\langle z,\mathbf{z};\mathbf{y}\rangle_{\delta,\epsilon}\, d^2\mathbf{y} \\
=&(\frac{\mu\gamma}{2})^p\int_{B(0,R)^p}\prod_{j=1}^p\frac{\theta_\delta(y_j-z)}{(y_j-z)^{q_j}} L_{-n}^{\langle p\rangle}\langle z,\mathbf{z};\mathbf{y}\rangle_{\delta,\epsilon}\, d^2\mathbf{y} \\
&+(\frac{\mu\gamma}{2})^p\int_{B(0,R)^p}\prod_{j=1}^p\frac{\theta_\delta(y_j-z)}{(y_j-z)^{q_j}}\sum_{i=1}^p(\frac{\partial_{y_i}}{(y_i-z)^{n-1}} - \frac{n-1}{(y_i-z)^n})\langle z,\mathbf{z};\mathbf{y}\rangle_{\delta,\epsilon}\, d^2\mathbf{y} \\
=:&\tilde{A}_{R,\epsilon} + \tilde{B}_{R,\epsilon}.
\end{aligned} \tag{4.2.16}$$

By the previous lemma, when $\epsilon \to 0$, $\tilde{A}_{R,\epsilon}$ converges to

$$\mathbf{z} \mapsto (\frac{\mu\gamma}{2})^p \int_{B(0,R)^p} \prod_{j=1}^{p} \frac{\theta_\delta(y_j - z)}{(y_j - z)^{q_j}} L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta \, d^2\mathbf{y}$$

in the sense of distributions. This is because the integral can be regarded as integrating $L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_{\delta,\epsilon}$ against a test function of $\mathbf{y}$. From the expression of $L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta$ (see Lemma 4.2.11) we can see that it does not introduce any singularity for the integral. Consider for example the integral below:

$$(\frac{\mu\gamma}{2})^p \left| \int_{B(0,R)^c} \int_{B(0,R)^{p-1}} \prod_{j=1}^{p} \frac{\theta_\delta(y_j - z)}{(y_j - z)^{q_j}} L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta \, d^2\mathbf{y} \right|.$$

We can bound it simply by

$$\frac{1}{R^{q_1}} (\frac{\mu\gamma}{2})^p \int_{\mathbb{C}^p} \theta_\delta(y_1 - z) \prod_{j=2}^{p} \frac{\theta_\delta(y_j - z)}{|y_j - z|^{q_j}} \left| L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta \right| \, d^2\mathbf{y}.$$

The above term is well defined and by sending $R \to \infty$ it converges to 0 for fixed δ. Therefore we can write in the weak sense:

$$\tilde{A} := \lim_{R \to \infty} \tilde{A}_{R,0} = (\frac{\mu\gamma}{2})^p \int_{\mathbb{C}^p} \prod_{j=1}^{p} \frac{\theta_\delta(y_j - z)}{(y_j - z)^{q_j}} L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta \, d^2\mathbf{y}. \tag{4.2.17}$$

On the other hand, an integration by parts shows that

$$\tilde{B}_{R,\epsilon} = \sum_{i=1}^{p} (\frac{\mu\gamma}{2})^p \int_{B(0,R)^p} \prod_{j:j\neq i} \frac{\theta_\delta(y_j - z)}{(y_j - z)^{q_j}} \left(\frac{q_i \theta_\delta(y_i - z)}{(y_i - z)^{q_i + n}} - \frac{\partial_z \theta_\delta(y_i - z)}{(y_i - z)^{q_i + n - 1}} \right) \langle z, \mathbf{z}; \mathbf{y} \rangle_{\delta,\epsilon} \, d^2\mathbf{y} + O_{R\to\infty}(R^{-1}),$$

where the $O_{R\to\infty}(R^{-1})$ comes from the boundary term and can be bounded independently of ϵ. Therefore by first sending ϵ to 0 and then $R \to \infty$, we obtain the limit which equals:

$$\tilde{B} := \sum_{i=1}^{p} q_i T_i^n Q_{\mathbf{q}} + \mathfrak{R}_\delta. \tag{4.2.18}$$

The above arguments show that

$$L_{-n} (\frac{\mu\gamma}{2})^p \int_{\mathbb{C}^p} \prod_{j=1}^{p} \frac{\theta_\delta(y_j - z)}{(y_j - z)^{q_j}} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta \, d^2\mathbf{y} = \tilde{A} + \tilde{B}.$$

In the rest of this proof we do not need to take regularizations with η_ϵ. With Lemma 4.2.11, we calculate $\tilde{A}$:

$$\begin{aligned} L_{-n}^{\langle p \rangle} \langle z, \mathbf{z}; \mathbf{y} \rangle_\delta = & \Big(- \sum_{i=1}^{n-1} P_i^{\langle p \rangle} P_{n-i}^{\langle p \rangle} + ((n-1)Q - (r-1)\chi) P_n^{\langle p \rangle} \Big) Q_0^{\langle p \rangle} + 2 \sum_{i=1}^{n-1} P_i^{\langle p \rangle} Q_{n-i}^{\langle p \rangle} \\ & + ((r-1)\chi - \frac{2(n-1)}{\gamma}) Q_n^{\langle p \rangle} - \sum_{i=1}^{n-1} Q_{i,n-i}^{\langle p \rangle} + \mu \int_{\mathbb{C}} \frac{\partial_z \theta_\delta(y_{p+1} - z)}{(y_{p+1} - z)^{n-1}} \langle z, \mathbf{z}; \mathbf{y}, y_{p+1} \rangle_\delta \, d^2 y_{p+1}, \end{aligned}$$

where

$$P_k^{\langle p \rangle}(z, \mathbf{z}, \mathbf{y}) = \sum_{l=1}^{N} \frac{\alpha_l}{2(z_l - z)^k} + \sum_{j=1}^{p} \frac{\gamma}{2(y_j - z)^k} = P_k + \sum_{j=1}^{p} \frac{\gamma}{2(y_j - z)^k},$$

$$Q_0^{\langle p\rangle}(z,\mathbf{z},\mathbf{y}) = \langle V_{-(r-1)\chi}(z)\prod_{i=1}^{p}V_\gamma(y_i)\prod_{l=1}^{N}V_{\alpha_l}(z_l)\rangle_\delta.$$

Hence

$$\tilde{A} = \Bigg[((r-1)Q-(r-1)\chi)P_n + 2\sum_{i=1}^{n-1}P_iT_{p+1}^{n-i} + ((r-1)\chi - \frac{2(n-1)}{\gamma})T_{p+1}^n - \sum_{i=1}^{n-1}T_{p+2}^{n-i}T_{p+1}^i$$
$$+\sum_{j=1}^{p}\Bigg(-\gamma\sum_{i=1}^{n-1}P_iT_j^{n-i} + (\frac{(n-1)\gamma Q}{2} - \frac{(r-1)\gamma\chi}{2})T_j^n + \gamma\sum_{i=1}^{n-1}T_{p+1}^{n-i}T_j^i - \frac{\gamma^2}{4}\sum_{j'=1}^{p}\sum_{i=1}^{n-1}T_{j'}^{n-i}T_j^i\Bigg)\Bigg]P_{\mathbf{n}}Q_{\mathbf{q}} + \mathfrak{R}_\delta.$$

This allows to prove the statement when $P_{\mathbf{n}} = 1$. And as remarked previously, this suffices to prove the statement for any $P_{\mathbf{n}}$.

Otherwise, one can verify the validity of the formula for the case $n = 1$. This finishes the proof. □

4.2.4 Illustration with order 2 and 3

We give the commutation relation for L_{-n} $(n \geq 1)$, which can be easily verified with Definition 4.1.8:

Lemma 4.2.13. *For $n, m \geq 1$*

$$[L_{-n}, L_{-m}] = (m-n)L_{-(n+m)} \tag{4.2.19}$$

Now we check BPZ equations of order $r = 2$ and $r = 3$ with $\gamma \in (0, 2)$. We will see that the proof of the BPZ equations becomes rather simple and involves only algebraic simplifications.

◇ $r = 2$: By definition,

$$\mathcal{D}_2 = \chi^2 L_{-2} + L_{-1}^2. \tag{4.2.20}$$

With the help of Proposition 4.2.10, we calculate:

$$L_{-1}Q_0 = -\chi P_1 Q_0 + \chi Q_1 + \mathfrak{R}_\delta.$$

By applying the operator L_{-1} to the above equation, we obtain

$$L_{-1}^2Q_0 = (-\chi P_2 + \chi^2 P_1^2)Q_0 - 2\chi^2 P_1 Q_1 + \chi^2 Q_{1,1} + \chi(-\frac{\gamma\chi}{2}+1)Q_2 + \mathfrak{R}_\delta.$$

Again by Proposition 4.2.10, we calculate:

$$L_{-2}Q_0 = (-P_1^2 + \frac{1}{\chi}P_2)Q_0 + 2P_1Q_1 + (\chi - \frac{2}{\gamma})Q_2 - Q_{1,1} + \mathfrak{R}_\delta.$$

We can verify easily that in $\mathcal{D}_2Q_0 = \chi^2 L_{-2}Q_0 + L_{-1}^2Q_0$, all the coefficients before $P_{\mathbf{n}}Q_{\mathbf{q}}$ cancel and therefore $\mathcal{D}_2Q_0 = \mathfrak{R}_\delta$. This allows to show BPZ equations in the weak sense, we can then conclude with the smoothness of correlation functions that the equation holds in the strong sense. □

◇ $r = 3$: By definition,

$$\mathcal{D}_3 = \chi^4 L_{-3} + \frac{\chi^2}{2}L_{-1}L_{-2} + \frac{\chi^2}{2}L_{-2}L_{-1} + \frac{1}{4}L_{-1}^3. \tag{4.2.21}$$

We have by lemma 4.2.19:

$$L_{-2}L_{-1} = L_{-1}L_{-2} - L_{-3}.$$

Then we can write

$$\mathcal{D}_3 = (\chi^4 - \frac{\chi^2}{2})L_{-3} + L_{-1}(\chi^2 L_{-2} + \frac{1}{4}L_{-1}^2). \tag{4.2.22}$$

Using Proposition 4.2.10:

$$L_{-1}^2Q_0 = (-2\chi P_2 + 4\chi^2 P_1^2)Q_0 - 8\chi^2 P_1Q_1 + 4\chi^2 Q_{1,1} + 2\chi(-\gamma\chi+1)Q_2 + \mathfrak{R}_\delta,$$

$$L_{-2}Q_0 = (-P_1^2 + (\frac{1}{\chi} - \chi)P_2)Q_0 + 2P_1Q_1 + (2\chi - \frac{2}{\gamma})Q_2 - Q_{1,1} + \mathfrak{R}_\delta.$$

Hence

$$(\chi^2 L_{-2} + \frac{1}{4}L_{-1}^2)Q_0 = (\frac{\chi}{2} - \chi^3)P_2Q_0 + (\chi^3 - \frac{\chi}{2})Q_2 + \mathfrak{R}_\delta.$$

We can then write

$$\mathcal{D}_3Q_0 = (\chi^4 - \frac{\chi^2}{2})L_{-3}Q_0 + (\frac{\chi}{2} - \chi^3)L_{-1}P_2Q_0 + (\chi^3 - \frac{\chi}{2})L_{-1}Q_2 + \mathfrak{R}_\delta.$$

We mention that when $\chi = \frac{\gamma}{2}$ and $\gamma \in (0, 1]$, there is a type of $\mathfrak{R}_\delta$ that does not vanish when $\delta \to 0$:

$$\int_{\mathbb{C}} \frac{\partial_z \theta_\delta(y-z)}{(y-z)^2} \langle V_{-(r-1)\chi}(z) V_\gamma(y) \prod_{l=1}^N V_{\alpha_l}(z_l)\rangle_\delta \, d^2y. \tag{4.2.23}$$

For this kind of non-vanishing perturbation term, we will write it directly with its expression instead of writing $\mathfrak{R}_\delta$. In general, there is another type

$$\int_{\mathbb{C}^2} \frac{\partial_z \theta_\delta(y_1-z)}{y_1-z} \frac{\theta_\delta(y_2-z)}{y_2-z} \langle V_{-(r-1)\chi}(z) V_\gamma(y_1) V_\gamma(y_2) \prod_{l=1}^N V_{\alpha_l}(z_l)\rangle_\delta \, d^2y_1 d^2y_2$$

that does not vanish. But thanks to the specific replication we use for $\mathcal{D}_3$ (4.2.22), this term will not appear in the final expression, .

For the other types of perturbation terms, they still converge to 0, and we will keep the notation $\mathfrak{R}_\delta$. With our calculus, in the expression of $\mathcal{D}_3Q_0$ the term (4.2.23) appears only in $(\chi^3 - \frac{\chi}{2})L_{-1}Q_2$ and in $(\chi^4 - \frac{\chi^2}{2})L_{-3}Q_0$. We can find the exact form of the perturbation term in the proof of Proposition 4.2.10. More precisely, we have

$$\begin{aligned}
&(\frac{\chi}{2} - \chi^3)L_{-1}P_2Q_0 + (\chi^3 - \frac{\chi}{2})L_{-1}Q_2 \\
=&-(\chi^4 - \frac{\chi^2}{2})\Big((\frac{2}{\chi}P_3 - 2P_1P_2)Q_0 + 2P_2Q_1 + 2P_1Q_2 - 2Q_{2,1} + (\gamma - \frac{2}{\chi})Q_3\Big) \\
&-\mu\frac{\gamma}{2}(\chi^3 - \frac{\chi}{2}) \int_{\mathbb{C}} \frac{\partial_z \theta_\delta(y-z)}{(y_1-z)^2} \langle V_{-(r-1)\chi}(z) V_\gamma(y) \prod_{l=1}^N V_{\alpha_l}(z_l)\rangle_\delta \, d^2\mathbf{y} + \mathfrak{R}_\delta,
\end{aligned}$$

and

$$\begin{aligned}
L_{-3}Q_0 =&(\frac{2}{\chi}P_3 - 2P_1P_2)Q_0 + 2P_2Q_1 + 2P_1Q_2 - 2Q_{2,1} + (2\chi - \frac{2}{\gamma})Q_3 \\
&+\mu \int_{\mathbb{C}} \frac{\partial_z \theta_\delta(y-z)}{(y_1-z)^2} \langle V_{-(r-1)\chi}(z) V_\gamma(y) \prod_{l=1}^N V_{\alpha_l}(z_l)\rangle_\delta \, d^2\mathbf{y} + \mathfrak{R}_\delta.
\end{aligned}$$

When $\chi = \frac{\gamma}{2}$, we can verify that all the terms cancel and $\mathcal{D}_3Q_0 = \mathfrak{R}_\delta$. Especially, the perturbations that we cannot control cancel among them. When $\chi = \frac{2}{\gamma}$, the term

$$\int_{\mathbb{C}} \frac{\partial_z \theta_\delta(y-z)}{(y_1-z)^2} \langle V_{-(r-1)\chi}(z) V_\gamma(y) \prod_{l=1}^N V_{\alpha_l}(z_l)\rangle_\delta \, d^2\mathbf{y}$$

converges to 0 by Proposition 4.2.9 and we can keep using the notation $\mathfrak{R}_\delta$ for it. Hence we also have $\mathcal{D}_3Q_0 = \mathfrak{R}_\delta$. This finishes the proof for BPZ equations of order 3. □

4.2.5 Proof of the BPZ equations of order r with real Coulomb gas

According to the previous discussions, the proof of the BPZ equations of order r has been reduced to an algebraic problem. Interestingly, real coulomb gas integrals with a degenerate insertion satisfy the same recursive relations, but without perturbation terms.

Definition 4.2.14. *For $x_1 < \cdots < x_l$ $(l \geq 1)$ and $t < t_1 < \cdots < t_N$ $(N \geq 2)$, we denote the integrand of real Coulomb gas integrals with degenerate insertions as*

$$f^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t};\mathbf{x}) := \prod_{0\leq i<j\leq N}(t_j-t_i)^{-\frac{\alpha_i\alpha_j}{2}} \prod_{0\leq i\leq N,1\leq s\leq l}(x_s-t_i)^{-\frac{\gamma\alpha_i}{2}} \prod_{1\leq s'<s\leq l}(x_s-x_{s'})^{-\frac{\gamma^2}{2}}, \tag{4.2.24}$$

where we denote $\alpha_0 = -(r-1)\chi, t_0 = t$ and by convention $(-1)^\alpha = e^{i\pi\alpha}$.

For real Coulomb gas integrals, we will always work with the condition

$$(\gamma,\alpha_{N-1},\alpha_N) \in (i\mathbb{R}_+)^3, \min\{-\frac{\gamma\alpha_{N-1}}{2},-\frac{\gamma\alpha_N}{2}\} \geq r. \tag{4.2.25}$$

It is easy to see that under this condition,

$$C^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t}) = \int_{t_{N-1}<x_1<\cdots<x_l<t_N} f^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t};\mathbf{x})d\mathbf{x}$$

is well defined and at least $\mathcal{C}^r$.

Next we define the equivalent of $Q_{\mathbf{q}}$ for Coulomb gas integrals.

Definition 4.2.15. *Let $p \in \mathbb{N}$, and $\mathbf{q} = (q_1,\ldots,q_p) \in (\mathbb{N}^*)^p$, we define*

$$Q^{(l)}_{\mathbf{q}}(t,\mathbf{t}) := \int_{t_{N-1}<x_1<\cdots<x_l<t_N} R^{(l)}_{\mathbf{q}}(t;\mathbf{x}) f^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t};\mathbf{x})d^2\mathbf{x}. \tag{4.2.26}$$

where

$$R^{(l)}_{\mathbf{q}}(t;\mathbf{x}) := (-\frac{\gamma}{2})^p \sum_{1\leq s_1<\cdots<s_p\leq l}\prod_{j=1}^{p}\frac{1}{(x_{s_j}-t)^{q_j}}. \tag{4.2.27}$$

The operator T_k on $Q^{(l)}_{\mathbf{q}}$ with $k \in \mathbb{N}^*$ is defined as:

$$T_k Q^{(l)}_{\mathbf{q}} = Q^{(l)}_{q_1,\ldots,q_k+1,\ldots,q_p}. \tag{4.2.28}$$

Remark 4.2.16. *By convention $Q^{(l)}_{\mathbf{q}} = 0$ if $p > l$. Note that in the expression of $Q^{(l)}_{\mathbf{q}}$ there is no need for regularization θ_δ around $x_i = t$, since we are considering $x_i > t_{N-1} > t$ so that t is not a singularity.*

By abuse of notation, when dealing with real variables, L_{-n} is defined as a real differential operator:

$$L_{-1} = \partial_t, \quad L_{-n} = \sum_{l=1}^{N}\left(-\frac{1}{(t_l-t)^{n-1}}\partial_{t_l} + \frac{\Delta_{\alpha_l}(n-1)}{(t_l-t)^n}\right) \quad n \geq 2. \tag{4.2.29}$$

Then the same recursive relation holds for real Coulomb gas integrals:

Proposition 4.2.17. *$(P_{\mathbf{n}}(t,\mathbf{t})Q^{(l)}_{\mathbf{q}}(t,\mathbf{t}))_{\mathbf{n},\mathbf{q}}$ satisfies Proposition 4.2.10 with no perturbation terms $\mathfrak{R}_\delta$.*

Remark 4.2.18. *There is no $\mathfrak{R}_\delta$ since there is no regularization θ_δ. The proof follows exactly the same steps as the proof of Proposition 4.2.10 and there is no need for the regularization η_ϵ to help calculate the derivatives.*

The proposition tells in particular that

$$\mathcal{D}_r Q_0^{(l)} = \sum_{\mathbf{n},\mathbf{q}:|\mathbf{n}|+|\mathbf{q}|=r} \lambda_{\mathbf{n},\mathbf{q}}(\gamma) P_{\mathbf{n}} Q_{\mathbf{q}}^{(l)}, \tag{4.2.30}$$

with the same coefficients $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ as introduced in (4.2.11).

To prove BPZ equations, we only need to show that all the $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ are equal to 0. We know that they are rational fractions of γ, so it suffices to prove it for an infinity number of values for $\gamma \in i\mathbb{R}_+$. We show in the following that real Coulomb gas integrals actually satisfy BPZ equations, which allows to solve the combinatorial problem, see Proposition 4.2.22.

Lemma 4.2.19. *For $r \in \mathbb{N}^*$, the following differential equation holds:*

$$\mathcal{D}_r f^{(0)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t}) = 0. \tag{4.2.31}$$

Remark 4.2.20. *Here D_r is composed of real differential operators L_{-n}, see (4.2.29). The proof of this lemma can be found in section 4.3. Note that this result has been proved by Kytola-Peltola [62] with a fusion technique in [31]. The fusion technique requires some non trivial manipulations of the Virasoro algebra. We would like to mention that our proof is purely combinatorial and elementary.*

Proposition 4.2.21. *Take $(\gamma,\boldsymbol{\alpha})$ such that (4.2.25) is satisfied. Then the real Coulomb gas integrals verify BPZ equations of order r. More precisely,*

$$\mathcal{D}_r C^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t}) = 0. \tag{4.2.32}$$

Proof. By applying derivation under the integral sign and Stokes Theorem,

$$\begin{aligned}\mathcal{D}_r C^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t}) &= \int_{t_{N-1}<x_1<\cdots<x_l<t_N} \mathcal{D}_r f^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t};\mathbf{x})d\mathbf{x} \\ &= \int_{t_{N-1}<x_1<\cdots<x_l<t_N} \mathcal{D}_r^{\langle l\rangle} f^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t};\mathbf{x})d\mathbf{x},\end{aligned}$$

with $\mathcal{D}_r^{\langle l\rangle}$ the operator $\mathcal{D}_r$ where we replace in its expression the operators L_{-n} by

$$L_{-n}^{\langle l\rangle} := L_{-n} + \sum_{s=1}^{l}\left(-\frac{\partial_{x_s}}{(x_s-t)^n} + \frac{n-1}{(x_s-t)^n}\right) \quad n \geq 2.$$

By Lemma 4.2.19, $\mathcal{D}_r^{\langle l\rangle} f^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t};\mathbf{x}) = 0$, this shows that $\mathcal{D}_r C^{(l)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t}) = 0$. □

Proposition 4.2.22. *For $\mathbf{n},\mathbf{q}$ such that $|\mathbf{n}|+|\mathbf{q}| = r$, the rational function $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ equals 0.*

Proof. We will work under the condition (4.2.25) with N sufficiently large. From the above proposition together with the discussion from the previous subsection, we deduce that

$$\sum_{\mathbf{n},\mathbf{q}:|\mathbf{n}|+|\mathbf{q}|=r} \lambda_{\mathbf{n},\mathbf{q}}(\gamma) P_{\mathbf{n}} Q_{\mathbf{q}}^{(r)} = \mathcal{D}_r C^{(r)}_{-(r-1)\chi,\boldsymbol{\alpha}}(t,\mathbf{t}) = 0.$$

For simplicity, let us denote $t_0 = t$ and $\alpha_0 = -(r-1)\chi$. We can divide the left hand side of the equation by the common term $\prod_{0\leq i<j\leq N}(t_j-t_i)^{-\frac{\alpha_i\alpha_j}{2}}$, then we have

$$\sum_{\mathbf{n}:|\mathbf{n}|\leq r} P_{\mathbf{n}} \sum_{\mathbf{q}:|\mathbf{q}|=r-|\mathbf{n}|} \lambda_{\mathbf{n},\mathbf{q}}(\gamma) \int_{t_{N-1}<x_1<\cdots<x_r<t_N} R_{\mathbf{q}}^{(r)}(t;\mathbf{x}) \prod_{0\leq i\leq N,1\leq s\leq r}(x_s-t_i)^{-\frac{\gamma\alpha_i}{2}} \prod_{1\leq s<s'\leq r}(x_s-x_{s'})^{-\frac{\gamma^2}{2}} d\mathbf{x} = 0.$$

Denote the left hand side by $(\star)$. We claim that the function $g_{\mathbf{n}}((t_i)_{r+1\le i\le N})$ defined by

$$g_{\mathbf{n}}((t_i)_{r+1\le i\le N}) := \sum_{\mathbf{q}:|\mathbf{q}|=r-|\mathbf{n}|} \lambda_{\mathbf{n},\mathbf{q}}(\gamma) \int_{t_{N-1}<x_1<\cdots<x_r<t_N} R^{(r)}_{\mathbf{q}}(t,\mathbf{x}) \prod_{r+1\le i\le N,1\le s\le r} (x_s-t_i)^{-\frac{\gamma\alpha_i}{2}} \prod_{1\le s\le r} (x_s-t)^{-\frac{\gamma\sum_{i=0}^r\alpha_i}{2}} \prod_{1\le s<s'\le r} (x_s-x_{s'})^{-\frac{\gamma^2}{2}} d\mathbf{x} \tag{4.2.33}$$

equals 0 for all $|\mathbf{n}|\le r$. To see this, suppose that for all $1\le l\le r$, $\alpha_l\neq 0$. We study the asymptotic when $t_1,t_2,\dots,t_r$ tend to t simultaneously:

$$\begin{aligned}(\star) &= \left(\sum_{\mathbf{n}:|\mathbf{n}|\le r}\prod_{j=1}^m\left(\sum_{l=1}^r\frac{\alpha_l}{2(t_l-t)^{n_j}}\right) g_{\mathbf{n}}((t_i)_{r+1\le i\le N})\right)(1+o(1))\\ &= \left(\sum_{\mathbf{n}:|\mathbf{n}|\le r}\left(\sum_{\mathbf{n}':\mathbf{n}\subseteq\mathbf{n}'} c_{\mathbf{n}'}((\alpha_l)_{1\le l\le r}) g_{\mathbf{n}'}((t_i)_{r+1\le i\le N})\right)\sum_{1\le i_1<\cdots<i_m\le r}\frac{m!}{\prod_{j=1}^m(t_{i_j}-t)^{n_j}}\right)(1+o(1)),\end{aligned}$$

where $\mathbf{n}\subseteq\mathbf{n}'$ means that $\mathbf{n}$ is a sub-tuple of $\mathbf{n}'$ and in particular, when $\mathbf{n}'=\mathbf{n}$,

$$c_{\mathbf{n}'}((\alpha_l)_{1\le l\le r}) = \frac{\prod_{l=1}^r\alpha_l}{2^r}.$$

Since $(\star)$ also equals 0, it is not hard to show (it should be done in a certain order) that

$$\forall|\mathbf{n}|\le r,\ \sum_{\mathbf{n}':\mathbf{n}\subseteq\mathbf{n}'} c_{\mathbf{n}'}((\alpha_l)_{1\le l\le r}) g_{\mathbf{n}'}((t_i)_{r+1\le i\le N}) = 0.$$

The equations above form a linear system with a triangular coefficient matrix with non null values on the diagonal, hence we will be able to conclude that $g_{\mathbf{n}}((t_i)_{r+1\le i\le N})=0$ for all $|\mathbf{n}|\le r$.

Now we have

$$\sum_{\mathbf{q}:|\mathbf{q}|=r-|\mathbf{n}|}\lambda_{\mathbf{n},\mathbf{q}}(\gamma)\int_{t_{N-1}<x_1<\cdots<x_r<t_N} R^{(r)}_{\mathbf{q}}(t,\mathbf{x})\prod_{r+1\le i\le N,1\le s\le r}(x_s-t_i)^{-\frac{\gamma\alpha_i}{2}} \prod_{1\le s\le r}(x_s-t)^{-\frac{\gamma\sum_{i=0}^r\alpha_i}{2}}\prod_{1\le s<s'\le r}(x_s-x_{s'})^{-\frac{\gamma^2}{2}}d\mathbf{x}=0. \tag{4.2.34}$$

Let us take $\alpha_{r+1}=\cdots=\alpha_{N-2}=-\frac{2}{\gamma}$, note that we can sum over a finite set E_n of values of t_i for each $r+1\le i\le N-2$ to obtain

$$\sum_{t_i\in E_n}\prod_{1\le s\le r}(x_s-t_i) = \sum_{1\le i_1<\cdots<i_n\le r} x_{i_1}\dots x_{i_n} = e_n(x_1,\dots,x_r). \tag{4.2.35}$$

Hence with a sum over $(t_{r+1},\dots,t_{N-2})\in E_{n_{r+1}}\times\cdots\times E_{n_{N-2}}$, we can obtain a product of fundamental symmetric polynomials: $\prod_{j=r+1}^{N-2}e_{n_j}(x_1,\dots,x_r)$. Since $\sum_{\mathbf{q}:|\mathbf{q}|=r-|\mathbf{n}|}\lambda_{\mathbf{n},\mathbf{q}}(\gamma)R^{(r)}_{\mathbf{q}}(t,\mathbf{x})\prod_{s=1}^r(x_s-t)^r$ is a symmetric polynomial in $(x_s)_{1\le s\le r}$, by the fundamental theorem of symmetric polynomials, when N is sufficiently large we can sum up different values of t_i $(r+1\le i\le N-2)$ to get

$$\int_{t_{N-1}<x_1<\cdots<x_r<t_N}\left|\sum_{\mathbf{q}:|\mathbf{q}|=r-|\mathbf{n}|}\lambda_{\mathbf{n},\mathbf{q}}(\gamma)R^{(r)}_{\mathbf{q}}(t,\mathbf{x})\right|^2\prod_{s=1}^r(x_s-t)^{r-\frac{\gamma\sum_{i=0}^r\alpha_i}{2}} \prod_{N-1\le i\le N,1\le s\le r}(x_s-t_i)^{-\frac{\gamma\alpha_i}{2}}\prod_{1\le s<s'\le r}(x_s-x_{s'})^{-\frac{\gamma^2}{2}}d\mathbf{x}=0. \tag{4.2.36}$$

This implies that for all $|\mathbf{n}| \leq r$,

$$\sum_{\mathbf{q}:|\mathbf{q}|=r-|\mathbf{n}|} \lambda_{\mathbf{n},\mathbf{q}}(\gamma) R^{(r)}_{\mathbf{q}}(t,\mathbf{x}) = 0. \tag{4.2.37}$$

We can easily extend the above equations to all x_s different from t. Then by a study of asymptotic when x_s tend simultaneously to t, we conclude that $\lambda_{\mathbf{n},\mathbf{q}}(\gamma) = 0$. □

This finishes the proof of Theorem 4.1.9.

4.2.6 BPZ equations for boundary Liouville theory

Let us illustrate the idea with the degenerate insertion $B^+_{-(r-1)\chi}(t)$. The correlation function (4.1.21) is holomorphic in t in the upper half plane except the points z_i. The smoothness in $(\mathbf{t},\mathbf{z})$ in this case has not been proved, but the method in [73] applies to this case and we will assume this property. Note that the derivative in $L^{\mathbb{H}}_{-1}$ should be understood as a complex derivative with respect to t.

If we think heuristically $V_{\alpha_i}(z_i) = B_{\alpha_i}(z_i)B_{\alpha_i}(\overline{z}_i)$, we can observe that the things behave very similar to the sphere case: firstly the form of $L^{\mathbb{H}}_{-n}$ is nothing but L_{-n} written for the points $z_i, \overline{z}_i, t_j$, secondly we can observe the same form of derivative rule that we illustrate below. We use the regularization η_ϵ for the Gaussian free field X: for $x \in \mathbb{R}$

$$X_\epsilon(x) = 2\int_{y\in\mathbb{H}} X(y)\eta_\epsilon(x-y)d^2y,$$

and for $z_i, 1 \leq i \leq N$, we define for ϵ sufficiently small

$$X_\epsilon(z_i) = X * \eta_\epsilon.$$

By abuse of notation, for $1 \leq i \neq j \leq N$ and $x, x' \in \mathbb{R}$, we denote

$$\begin{aligned}
\frac{1}{(z_i-z_j)_\epsilon} &= \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{1}{z_i-z_j-x_1+x_2}\eta_\epsilon(x_1)\eta_\epsilon(x_2)d^2x_1d^2x_2,\\
\frac{1}{(x-x')_\epsilon} &= 4\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{1}{x-x'-x_1+x_2}\eta_\epsilon(x_1)\eta_\epsilon(x_2)d^2x_1d^2x_2,\\
\frac{1}{(x-z_i)_\epsilon} &= 2\int_{\mathbb{H}}\int_{\mathbb{C}} \frac{1}{x-z_i-x_1+x_2}\eta_\epsilon(x_1)\eta_\epsilon(x_2)d^2x_1d^2x_2.
\end{aligned}$$

For the derivative rules, we have

$$\begin{aligned}
&\partial_{z_k}\langle\prod_{i=1}^N V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}\\
&= \left(\sum_{i:i\neq k}\frac{\alpha_i\alpha_k}{2(z_i-z_k)_\epsilon} + \sum_i \frac{\alpha_i\alpha_k}{2(\overline{z}_i-z_k)_\epsilon} + \sum_j \frac{\beta_j\alpha_k}{2(t_j-z_k)_\epsilon}\right)\langle\prod_{i=1}^N V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}\\
&\quad - \mu\int_{\mathbb{R}}\frac{\gamma\alpha_k}{2(y-z_k)_\epsilon}\langle B_{\gamma,\epsilon}(y)\prod_{i=1}^N V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^M B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}dy,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{\overline{z}_k}\langle\prod_{i=1}^{N}V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}\\
&=\left(\sum_i\frac{\alpha_i\alpha_k}{2(z_i-\overline{z}_k)_\epsilon}+\sum_{i:i\neq k}\frac{\alpha_i\alpha_k}{2(\overline{z}_i-\overline{z}_k)_\epsilon}+\sum_j\frac{\beta_j\alpha_k}{2(t_j-\overline{z}_k)_\epsilon}\right)\langle\prod_{i=1}^{N}V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}\\
&\quad-\mu\int_{\mathbb{R}}\frac{\gamma\alpha_k}{2(y-\overline{z}_k)_\epsilon}\langle B_{\gamma,\epsilon}(y)\prod_{i=1}^{N}V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}dy.
\end{aligned}$$

Here the notation $B_\gamma(y)$ simply means that inserting y between any t_j and t_{j+1} will keep the same boundary constant s_j on both sides of y.

Similarly, when deriving with respect to t_j:

$$\begin{aligned}
\partial_{t_k}\langle\prod_{i=1}^{N}V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}&=\sum_i\frac{\alpha_i\beta_k}{2(z_i-t_k)_\epsilon}+\sum_i\frac{\alpha_i\beta_k}{2(\overline{z}_i-t_k)_\epsilon}+\sum_{j:j\neq k}\frac{\beta_j\beta_k}{2(t_j-t_k)_\epsilon}\\
&\quad-\mu\int_{\mathbb{R}}\frac{\gamma\beta_k}{2(y-t_k)_\epsilon}\langle B_{\gamma,\epsilon}(y)\prod_{i=1}^{N}V_{\alpha_i,\epsilon}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j,\epsilon}(t_j)\rangle_{\mathbb{H}}dy.
\end{aligned}$$

We observe that the derivative rules behave exactly as if we have insertions $z_i,\overline{z}_i,t_j$ on the sphere.

Now we present the analogies of $P_{\mathbf{n}}Q_{\mathbf{q}}$:

Definition 4.2.23. *For $n\in\mathbb{N}^*$, we define*

$$\mathcal{P}_n(t,\mathbf{t};\mathbf{z}):=\sum_{i=1}^{N}\left(\frac{\alpha_i}{2(z_i-t)^n}+\frac{\alpha_i}{2(\overline{z}_i-t)^n}\right)+\sum_{j=1}^{M}\frac{\beta_j}{2(t_j-t)^n}. \tag{4.2.38}$$

Let $p\in\mathbb{N}$, and $\mathbf{q}=(q_1,\dots,q_p)\in(\mathbb{N}^*)^p$, we define

$$\mathcal{Q}_{\mathbf{q}}(z,\mathbf{z}):=(\frac{\mu\gamma}{2})^p\int_{\mathbb{R}^p}\prod_{l=1}^{p}\frac{1}{(y_l-t)^{q_l}}\langle B^+_{-(r-1)\chi}(t)\prod_{l=1}^{p}B_\gamma(y_l)\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j}(t_j)\rangle_{\mathbb{H}}\,d\mathbf{y} \tag{4.2.39}$$

We keep using the notation T_k for $T_kQ_{\mathbf{q}}=Q_{q_1,\dots,q_k+1,\dots,q_p}$.

Remark 4.2.24. *There is no need for the regularization θ_δ around t because $y_l=t$ is not a singularity when t is in the upper half plane. Therefore $\mathcal{Q}_{\mathbf{q}}$ are well defined objects.*

From the observations above, one can easily notice the following result:

Proposition 4.2.25. *$L^{\mathbb{H}}_{-n}$ with $\mathcal{P}_{\mathbf{n}}\mathcal{Q}_{\mathbf{q}}$ satisfy Proposition 4.2.10 without perturbation terms $\mathfrak{R}_\delta$.*

As a consequence, we can write

$$D^{\mathbb{H}}_r\langle B^+_{-(r-1)\chi}(t)\prod_{i=1}^{N}V_{\alpha_i}(z_i)\prod_{j=1}^{M}B^{\mu_{j-1},\mu_j}_{\beta_j}(t_j)\rangle_{\mathbb{H}}=\sum_{|\mathbf{n}|+|\mathbf{q}|=r}\lambda_{\mathbf{n},\mathbf{q}}(\gamma)\mathcal{P}_{\mathbf{n}}\mathcal{Q}_{\mathbf{q}}=0.$$

Since there is no perturbation term, there is no constraint on γ. This finishes the proof of Theorem 4.1.14 with the degenerate insertion $B^+_{-(r-1)\chi}(t)$. The case with $B^-_{-(r-1)\chi}(t)$ is exactly the same, and the case of BPZ equation with the degenerate insertion $V_{-(r-1)\chi}(z)$ can also be proved in the same manner. This allows to conclude the proof for Theorem 4.1.14.

4.3 BPZ equations for the integrand

Here we provide an elementary proof to show that the integrand of real Coulomb gas integrals satisfy BPZ equations. This result is the key element to show that real Coulomb gas integrals satisfy BPZ equations.

Proposition 4.3.1. *For $r \in \mathbb{N}^*$, the following differential equation holds:*

$$\mathcal{D}_r f(t, \mathbf{t}) = 0, \tag{4.3.1}$$

where

$$f(t, \mathbf{t}) := f^{(0)}_{-(r-1)\chi, \boldsymbol{\alpha}}(t, \mathbf{t}) = \prod_{1 \le i < j \le N} (t_j - t_i)^{-\frac{\alpha_i \alpha_j}{2}} \prod_{i=1}^{N} (t_i - t)^{\frac{(r-1)\chi \alpha_i}{2}}$$

and recall that L_{-n} for real variables are defined by

$$L_{-1} = \partial_t, \quad L_{-n} = \sum_{l=1}^{N} \left(-\frac{\partial_{t_l}}{(t_l - t)^{n-1}} + \frac{\Delta_l (n-1)}{(t_l - t)^n} \right) \quad n \ge 2. \tag{4.3.2}$$

$\Delta_l := \frac{\alpha_l}{2}(Q - \frac{\alpha_l}{2})$ is the conformal weight.

Proof. First let us transform it into a combinatorial problem as what we have done in section 4.2.3. Let $P_n(t, \mathbf{t}) = \sum_{l=1}^{N} \frac{\alpha_l}{2(t_l - t)^n}$, we can show easily

$$L_{-n} P_{\mathbf{n}} f = \left(\sum_i n_i \frac{P_{n_i + n}}{P_{n_i}} - \sum_{i=1}^{n-1} P_i P_{n-i} + ((n-1)Q - (r-1)\chi) P_n \right) P_{\mathbf{n}} f \tag{4.3.3}$$

As a consequence, $\mathcal{D}_r f = \left(\sum_{|\mathbf{n}| = r} \lambda_{\mathbf{n}}(\gamma) P_{\mathbf{n}} \right) f$, where $\lambda_{\mathbf{n}}(\gamma)$ corresponds to $\lambda_{\mathbf{n},\mathbf{q}}(\gamma)$ with $\mathbf{q} = 0$ (see (4.2.11)). Thus we will need to show that all the coefficients $\lambda_{\mathbf{n}}(\gamma)$ are zero.

Without loss of generality, we can restrict ourselves to the case $N \ge r$. Interestingly, we can largely simplify the problem if we take another point of view by treating $P_{\mathbf{n}}(t, \mathbf{t})$ as a polynomial of $(t, \mathbf{t})$ with values in the algebra $\mathbb{C}[\boldsymbol{\alpha}]$. If we quotient by the relation

$$\forall 1 \le i \le N, \ \alpha_i^2 = 0, \tag{4.3.4}$$

then we can write

$$P_{\mathbf{n}} = \prod_{i=1}^{m} P_{n_i} = \sum_{1 \le i_1 < \dots i_m \le N} \frac{N!}{2^m (N-m)!} \frac{\alpha_{i_1} \dots \alpha_{i_m}}{(t_{i_1} - t)^{n_{i_1}} \dots (t_{i_m} - t)^{n_{i_m}}},$$

and

$$\mathcal{D}_r f = \left(\sum_{m=1}^{r} \sum_{\substack{(n_1, \dots, n_k) \in \mathbb{N}^{*k} \\ n_1 + \dots + n_k = r}} \lambda_{\mathbf{n}}(\gamma) \sum_{1 \le i_1 < \dots i_m \le N} \frac{N!}{2^m (N-m)!} \frac{\alpha_{i_1} \dots \alpha_{i_m}}{(t_{i_1} - t)^{n_{i_1}} \dots (t_{i_m} - t)^{n_{i_m}}} \right) f.$$

Thus if we can show $\mathcal{D}_r f = 0$ under the quotient relation (4.3.4), then by linear independence of functions

$$(\boldsymbol{\alpha}, t, \mathbf{t}) \mapsto \frac{\alpha_{i_1} \dots \alpha_{i_m}}{(t_{i_1} - t)^{n_{i_1}} \dots (t_{i_m} - t)^{n_{i_m}}},$$

we have that $\lambda_{\mathbf{n}}(\gamma) = 0$ for all $|\mathbf{n}| = r$.

Now we prove $\mathcal{D}_r f = 0$ under the condition (4.3.4). In this setting, the operators L_{-n} can be rewritten as

$$L_{-n} := \sum_{l=1}^{N} \left(-\frac{\partial_{t_l}}{(t_l - t)^{n-1}} + \frac{Q(n-1)\alpha_l}{2(t_l - t)^n} \right). \tag{4.3.5}$$

By first developing $L_{-n_1}f$ with the formula (4.3.3), we have

$$\mathcal{D}_r f = \sum_{l_1=1}^{N}\sum_{k=1}^{r}\sum_{n_1+\cdots+n_k=r}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}L_{-n_k}\dots L_{-n_2}$$
$$\left[\left(-\frac{(r-n_1)\chi}{2}+\frac{(n_1-1)}{2\chi}\right)\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1}}-\sum_{l_2:l_2\neq l_1}\sum_{i=1}^{n_1-1}\frac{\alpha_{l_1}\alpha_{l_2}}{4}\frac{1}{(t_{l_1}-t)^i(t_{l_2}-t)^{n_1-i}}\right]f. \tag{4.3.6}$$

Certain terms with order k and $k+1$ (here the order means the number of n_i) cancel among themselves, for example, for fixed $(n_1,\dots,n_k)$ and l_1, we consider the following term with order $k+1$:

$$\sum_{n_1'+n_1''=n_1}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}\frac{1}{\chi^2 n_1'(r-n_1')}L_{-n_k}\dots L_{-n_2}L_{-n_1''}(-\frac{(r-n_1')\chi}{2})\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1'}}f. \tag{4.3.7}$$

If we extract the term that depends on t_{l_1} in the operator $L_{-n_1''}$, we have

$$\left(-\frac{1}{(t_{l_1}-t)^{n_1''-1}}\partial_{t_{l_1}}+\frac{Q(n_1''-1)\alpha_{l_1}}{2(t_{l_1}-t)^{n_1''}}\right)(-\frac{(r-n_1')\chi}{2})\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1'}}f=-\frac{n_1'(r-n_1')\chi}{2}\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1}}f,$$

In this equation we have eliminated all the terms that contain $\alpha_{l_1}^2$. For example, it is not hard to see that $\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1'+n_1''-1}}\partial_{t_{l_1}}f=0$. The previous calculus shows that if we extract the term $-\frac{1}{(t_{l_1}-t)^{n_1''-1}}\partial_{t_{l_1}}+\frac{Q(n_1''-1)\alpha_{l_1}}{2(t_{l_1}-t)^{n_1''}}$ from $L_{-n_1''}$ in (4.3.7), we can simplify as follows to obtain a term of order k:

$$\begin{aligned}&-\sum_{n_1'+n_1''=n_1}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}L_{-n_k}\dots L_{-n_2}L_{-n_1''}\frac{1}{2\chi}\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1'}}f\\ =&-\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}L_{-n_k}\dots L_{-n_2}\frac{(n_1-1)}{2\chi}\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1}}f.\end{aligned}$$

The last line that we extract from (4.3.7) cancels a term of order k in (4.3.6). Thus, after such cancellations,

$$\mathcal{D}_r f=\sum_{l_1=1}^{N}\sum_{k=1}^{r}\sum_{n_1+\cdots+n_k=r}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}\left[(-\frac{(r-n_1)\chi}{2})L_{-n_k}\dots L_{-n_2}^{\langle\not l\rangle}\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1}}\right.$$
$$\left.-L_{-n_k}\dots L_{-n_2}\sum_{l_2:l_2\neq l_1}\sum_{i=1}^{n_1-1}\frac{\alpha_{l_1}\alpha_{l_2}}{4}\frac{1}{(t_{l_1}-t)^i(t_{l_2}-t)^{n_1-i}}\right]f \tag{4.3.8}$$

where $L_{-n}^{\langle l\rangle}=\sum_{l'\neq l}\left(-\frac{\partial_{t_{l'}}}{(t_{l'}-t)^{n-1}}+\frac{Q(n-1)\alpha_{l'}}{2(t_{l'}-t)^n}\right)$.

Again by fixing $(n_1,\dots,n_k)$, we consider the following term of order $k+1$:

$$\sum_{n_1'+n_1''=n_1}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}\frac{1}{\chi^2 n_1'(r-n_1')}(-\frac{(r-n_1')\chi}{2})L_{-n_k}\dots L_{-n_2}L_{-n_1''}^{\langle\not l\rangle}\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1'}}f$$
$$=-\sum_{l_2:l_2\neq l_1}\sum_{n_1'+n_1''=n_1}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}\frac{1}{2\chi n_1'}L_{-n_k}\dots L_{-n_2}$$
$$\left[(-\frac{(r-n_1'')\chi}{2}+\frac{(n_1''-1)}{2\chi})\frac{\alpha_{l_1}\alpha_{l_2}}{(t_{l_1}-t)^{n_1'}(t_{l_2}-t)^{n_1''}}-\sum_{l_3:l_3\notin\{l_1,l_2\}}\sum_{i=1}^{n_1''-1}\frac{\alpha_{l_1}\alpha_{l_2}\alpha_{l_3}}{4}\frac{1}{(t_{l_1}-t)^{n_1'}(t_{l_2}-t)^i(t_{l_3}-t)^{n_1''-i}}\right]f,$$

where in the equality we interchange $L_{-n_1''}^{\langle \not 1 \rangle}$ and $\frac{\alpha_{l_1}}{(t_{l_1}-t)^{n_1'}}$ and then apply (4.3.3) to calculate $L_{-n_1''}^{\langle \not 1 \rangle} f$. The terms with $\alpha_{l_1}^2$ or $\alpha_{l_2}^2$ were eliminated. Then we get

$$\begin{aligned}\mathcal{D}_r f = &\sum_{l_1\neq l_2}\sum_{k=2}^{r}\sum_{n_1+\cdots+n_k=r}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}(-\frac{(r-n_1)\chi}{2})L_{-n_k}\dots L_{-n_3}\\ &\Bigg[(-\frac{(r-n_1-n_2)\chi}{2}+\frac{(n_2-1)}{2\chi})\frac{\alpha_{l_1}\alpha_{l_2}}{(t_{l_1}-t)^{n_1}(t_{l_2}-t)^{n_2}}\\ &-\sum_{l_3:l_3\notin\{l_1,l_2\}}\sum_{i=1}^{n_2-1}\frac{\alpha_{l_1}\alpha_{l_2}\alpha_{l_3}}{4}\frac{1}{(t_{l_1}-t)^{n_1}(t_{l_2}-t)^{i}(t_{l_3}-t)^{n_2-i}}\Bigg]f\end{aligned}\tag{4.3.9}$$

Comparing the expression above to the expression (4.3.6), we can proceed with a recurrence. Suppose that we arrive at the following expression, with $1\leq K\leq r-1$:

$$\begin{aligned}\mathcal{D}_r f = &\sum_{\substack{l_1,\dots,l_K:\\ \forall i\neq j, l_i\neq l_j}}\sum_{k=K}^{r}\sum_{n_1+\cdots+n_k=r}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}\prod_{j=1}^{K-1}(-\frac{(r-\sum_{i=1}^{j}n_i)\chi}{2})L_{-n_k}\dots L_{-n_{K+1}}\\ &\Bigg[(-\frac{(r-\sum_{i=1}^{K}n_i)\chi}{2}+\frac{(n_K-1)}{2\chi})\prod_{j=1}^{K}\frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\\ &-\sum_{l_{K+1}:l_{K+1}\notin\{l_1,\dots,l_K\}}\sum_{i=1}^{n_K-1}\left(\prod_{j=1}^{K-1}\frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right)\frac{\alpha_{l_K}\alpha_{l_{K+1}}}{4(t_{l_K}-t)^{i}(t_{l_{K+1}}-t)^{n_K-i}}\Bigg]f.\end{aligned}\tag{4.3.10}$$

We will repeat what we have done from (4.3.6) to (4.3.9) in this general setting. For fixed tuple of $(n_1,\dots,n_k,l_1,\dots,l_K)$, consider the following configurations with $1\leq L\leq K$

$$(n_1,\dots,n_{L-1},n_K',n_L,\dots,n_{K-1},n_K'',n_{K+1},\dots,n_k)$$

Remark that if $L=K$, the decomposition is simply $(n_1,\dots,n_{K-1},n_K',n_K'',n_{K+1},\dots,n_k)$. The new configuration has $k+1$ terms and we can find some similar cancellations as previously by investigating the following term of order $k+1$:

$$\begin{aligned}&\sum_{L=1}^{K}\sum_{n_K'+n_K''=n_K}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}\frac{1}{\chi^2}\frac{\prod_{j=L}^{K-1}(\sum_{i=1}^{j}n_i)(r-\sum_{i=1}^{j}n_i)}{\prod_{j=L-1}^{K-1}(\sum_{i=1}^{j}n_i+n_K')(r-\sum_{i=1}^{j}n_i-n_K')}\\ &\prod_{j=1}^{L-1}(-\frac{(r-\sum_{i=1}^{j}n_i)\chi}{2})\prod_{j=L-1}^{K-2}(-\frac{(r-\sum_{i=1}^{j}n_i-n_K')\chi}{2})L_{-n_k}\dots L_{-n_{K+1}}\\ &(-\frac{(r-\sum_{i=1}^{K-1}n_i-n_K')\chi}{2})\left(-\frac{\partial_{t_{l_L}}}{(t_{l_L}-t)^{n_K''-1}}+\frac{\alpha_{l_L}}{(t_{l_L}-t)^{n_K''}}\right)\sum_{\substack{l_1,\dots,l_K:\\ \forall i\neq j, l_i\neq l_j}}\left(\prod_{\substack{j=1\\ j\neq L}}^{K-1}\frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right)\frac{\alpha_{l_L}}{(t_{l_L}-t)^{n_K'}}f\end{aligned}$$

Remark that this corresponds to the term extracted from (4.3.7). After some simplifications, this equals

$$\begin{aligned}&-\sum_{n_K'+n_K''=n_K}\frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j}n_i)\prod_{j=K}^{k-1}(r-\sum_{i=1}^{j}n_i)}\frac{1}{2\chi}(-\frac{\chi}{2})^{K-1}\left(\sum_{L=1}^{K}\frac{n_K'\prod_{j=L}^{K-1}(\sum_{i=1}^{j}n_i)}{\prod_{j=L-1}^{K-1}(\sum_{i=1}^{j}n_i+n_K')}\right)\\ &L_{-n_k}\dots L_{-n_{K+1}}\sum_{\substack{l_1,\dots,l_K:\\ \forall i\neq j, l_i\neq l_j}}\left(\prod_{j=1}^{K}\frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right)f.\end{aligned}$$

We claim that

$$\sum_{L=1}^{K} \frac{n'_K \prod_{j=L}^{K-1}(\sum_{i=1}^{j} n_i)}{\prod_{j=L-1}^{K-1}(\sum_{i=1}^{j} n_i + n'_K)} = 1, \tag{4.3.11}$$

which is not difficult to prove by induction on K: the case $K = 1$ is trivial. Suppose that the identity holds for $K = K_0$, by induction we consider $K = K_0 + 1$:

$$\sum_{L=1}^{K} \frac{n'_K \prod_{j=L}^{K-1}(\sum_{i=1}^{j} n_i)}{\prod_{j=L-1}^{K-1}(\sum_{i=1}^{j} n_i + n'_K)} = \frac{\sum_{i=1}^{K_0} n_i}{\sum_{i=1}^{K_0} n_i + n'_{K_0+1}} + \frac{n'_{K_0+1}}{\sum_{i=1}^{K_0} n_i + n'_{K_0+1}} = 1, \tag{4.3.12}$$

where we separate the sum in to $\sum_{L=1}^{K_0} + \sum_{L=K_0+1}$. Therefore we can further simplify the expression above for the term of order $k+1$, which equals now

$$-\sum_{\substack{l_1,\dots,l_K:\\ \forall i\neq j, l_i\neq l_j}} \frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(r-\sum_{i=1}^{j} n_i)} \prod_{j=1}^{K-1}(-\frac{(r-\sum_{i=1}^{j} n_i)\chi}{2}) L_{-n_k}\dots L_{-n_{K+1}} \frac{(n_K-1)}{2\chi} \left(\prod_{j=1}^{K} \frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right) f.$$

The last line cancels a term of order k in (4.3.10). Therefore, after canceling the terms, we arrive at an expression which is the analogue of (4.3.8)

$$\begin{aligned}\mathcal{D}_r f = &\sum_{\substack{l_1,\dots,l_K:\\ \forall i\neq j, l_i\neq l_j}} \sum_{k=K}^{r} \sum_{n_1+\dots+n_k=r} \frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(r-\sum_{i=1}^{j} n_i)} \prod_{j=1}^{K-1}(-\frac{(r-\sum_{i=1}^{j} n_i)\chi}{2}) \\ &\left[L_{-n_k}\dots L_{-n_{K+1}}^{\langle \cancel{l_1},\dots,\cancel{l_K}\rangle}(-\frac{(r-\sum_{i=1}^{K} n_i)\chi}{2}) \left(\prod_{j=1}^{K} \frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right) - L_{-n_k}\dots L_{-n_{K+1}}\right. \\ &\left.\sum_{l_{K+1}:l_{K+1}\notin\{l_1,\dots,l_K\}} \sum_{i=1}^{n_K-1} \left(\prod_{j=1}^{K-1} \frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right) \frac{\alpha_{l_K}\alpha_{l_{K+1}}}{4(t_{l_K}-t)^i (t_{l_{K+1}}-t)^{n_K-i}}\right] f\end{aligned} \tag{4.3.13}$$

The last step is to develop the operator $L_{-n_{K+1}}^{\langle \cancel{l_1},\dots,\cancel{l_K}\rangle}$ exactly as what we did to (4.3.8) and we will obtain

$$\begin{aligned}\mathcal{D}_r f = &\sum_{\substack{l_1,\dots,l_{K+1}:\\ \forall i\neq j, l_i\neq l_j}} \sum_{k=K+1}^{r} \sum_{n_1+\dots+n_k=r} \frac{(\chi^2)^{r-k}}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(r-\sum_{i=1}^{j} n_i)} \prod_{j=1}^{K}(-\frac{(r-\sum_{i=1}^{j} n_i)\chi}{2}) \\ &L_{-n_k}\dots L_{-n_{K+2}}\left[(-\frac{(r-\sum_{i=1}^{K+1} n_i)\chi}{2} + \frac{(n_{K+1}-1)}{2\chi}) \left(\prod_{j=1}^{K} \frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right)\right. \\ &\left.- \sum_{l_{K+2}:l_{K+2}\notin\{l_1,\dots,l_{K+1}\}} \sum_{i=1}^{n_{K+1}-1} \left(\prod_{j=1}^{K} \frac{\alpha_{l_j}}{(t_{l_j}-t)^{n_j}}\right) \frac{\alpha_{l_{K+1}}\alpha_{l_{K+2}}}{4(t_{l_{K+1}}-t)^i (t_{l_{K+2}}-t)^{n_{K+1}-i}}\right] f\end{aligned} \tag{4.3.14}$$

This allows us to go from K to $K+1$ in the statement (4.3.10). When K grows to r, as $n_i = 1$ for all $1 \le i \le r$ and $\sum_{i=1}^{r} n_i = r$, we obtain

$$\mathcal{D}_r f = 0 \tag{4.3.15}$$

under the condition (4.3.4). By discussions at the beginning of the proof, this allows to conclude that $\lambda_{\mathbf{n}}(\gamma) = 0$ for all $|\mathbf{n}| = r$, hence $\mathcal{D}_r f = 0$ in the general setting. □

4.4 Appendix: Proof of the derivative rule

Let us first recall the Gaussian integration by parts formula:

Lemma 4.4.1 (Gaussian integration by parts)**.** *Let $(Y_1, Y_2) \in \mathbb{R} \times \mathbb{R}^d, d \geq 1$ a centered Gaussian vector, and $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ a function that decays faster than any polynomials at infinity. Then,*

$$\mathbb{E}[Y_1 \phi(Y_2)] = \mathbb{E}[Y_1 Y_2]\mathbb{E}[\nabla \phi(Y_2)] \tag{4.4.1}$$

Proof. Let $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^d$, we calculate

$$\mathbb{E}[e^{\lambda Y_1 + \mu \cdot Y_2}] = e^{\frac{\lambda^2}{2}\mathbb{E}[Y_1^2] + \lambda\mu \cdot \mathbb{E}[Y_1 Y_2] + \frac{1}{2}{}^t\mu \mathrm{Var}(Y_2)\mu}$$

Taking the derivative with respect to λ and evaluate at $\lambda = 0$, we obtain:

$$\mathbb{E}[Y_1 e^{\mu \cdot Y_2}] = \mu \cdot \mathbb{E}[Y_1 Y_2] e^{\frac{1}{2}{}^t\mu \mathrm{Var}(Y_2)\mu}$$

This proves the forumla for the function $\phi(y) = e^{\mu \cdot y}$. We then conclude with an argument of density. □

To calculate derivatives of the correlation functions, we will need a "continuous" version of Gaussian integration by parts, where Y_2 is now of infinite dimension:

Lemma 4.4.2. *Let $\epsilon > 0$ and f a smooth test function with compact support. Denote $X(f) = (X, f)$, we have*

$$\begin{aligned} \langle X(f) \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta =& \sum_{l=0}^N \alpha_l \mathbb{E}[X(f) X_\epsilon(z_l)] \langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta \\ & - \mu\gamma \int_{\mathbb{C}} \theta_\delta(y - z_0)\mathbb{E}[X(f)X_\epsilon(y)]\langle V_{\gamma,\epsilon}(y) \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y \end{aligned} \tag{4.4.2}$$

This result can be obtained from the previous Gaussian integration by parts formula by discretizing the Gaussian multiplicative chaos measure. Consider a function f such that $\int_{\mathbb{C}} f = 0$. By definition, we have for $\epsilon > 0$ and $x \in \mathbb{C}$,

$$\mathbb{E}[X(f)X_\epsilon(x)] = f * C_\epsilon(x) - \frac{1}{4}\int_{\mathbb{C}} \ln \hat{g}(y) f(y) d^2y$$

where $C_\epsilon = \ln \frac{1}{|z|} * \eta_\epsilon$. Recall that $\eta_\epsilon = \frac{1}{\epsilon^2}\eta(\frac{|x|^2}{\epsilon^2})$ with η supported in $[\frac{1}{2}, 1]$. Then by the Gaussian integration by parts formula,

$$\begin{aligned} & \langle (X + \frac{Q}{2}\ln \hat{g})(f) \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta \\ =& \sum_{l=0}^N \alpha_l\, f * C_\epsilon(z_l) \langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta - \mu\gamma \int_{\mathbb{C}} \theta_\delta(y - z_0) f * C_\epsilon(y) \langle V_{\gamma,\epsilon}(y) \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y \\ & + \frac{1}{4}\int_{\mathbb{C}} \ln \hat{g}(y) f(y) d^2y \Big((2Q - \sum_{l=0}^N \alpha_l)\langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta + \mu\gamma \int_{\mathbb{C}} \theta_\delta(y' - z_0)\langle V_{\gamma,\epsilon}(y') \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y' \Big) \\ =& \sum_{l=0}^N \alpha_l\, f * C_\epsilon(z_l) \langle \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta - \mu\gamma \int_{\mathbb{C}} \theta_\delta(y - z_0) f * C_\epsilon(y) \langle V_{\gamma,\epsilon}(y) \prod_{l=0}^N V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y, \end{aligned}$$

where in the last equality we used the KPZ identity (4.2.5) to help the cancellation. Now we take $f = \partial_z \eta_\epsilon(z_i - \cdot)$. Note that for $1 \leq i \leq N$,

$$\begin{aligned} \partial_{z_i}\left(g_\epsilon(z_i)^{\Delta_{\alpha_i}} e^{\alpha_i X_\epsilon(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_\epsilon(z_i)^2]} \right) &= \alpha_i \partial_{z_i}(X_\epsilon(z_i) + \frac{Q}{2}\ln \hat{g}_\epsilon(z_i))\, g_\epsilon(z_i)^{\Delta_{\alpha_i}} e^{\alpha_i X_\epsilon(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_\epsilon(z_i)^2]} \\ &= \alpha_i (X + \frac{Q}{2}\ln \hat{g})(f)\, g_\epsilon(z_i)^{\Delta_{\alpha_i}} e^{\alpha_i X_\epsilon(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_\epsilon(z_i)^2]}. \end{aligned}$$

Hence for $1 \leq i \leq N$,

$$\partial_{z_i}\langle\prod_{l=0}^{N} V_{\alpha_l,\epsilon}(z_l)\rangle_\delta = \sum_{\substack{j=0\\ j\neq i}}^{N} \frac{\alpha_i\alpha_j}{2(z_j-z_i)_\epsilon}\langle\prod_{l=1}^{N} V_{\alpha_l,\epsilon}(z_l)\rangle_\delta - \frac{\mu\gamma\alpha_i}{2}\int_{\mathbb{C}} \frac{\theta_\delta(y-z_0)}{(y-z_i)_\epsilon}\langle V_{\gamma,\epsilon}(y)\prod_{l=0}^{N} V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y.$$

When $i = 0$, there is an additional term coming from the derivative of the regularization $\theta_\delta(y - z_0)$, which gives

$$\mu\int_{\mathbb{C}} \partial_z\theta_\delta(y-z_0)\langle V_{\gamma,\epsilon}(y)\prod_{l=0}^{N} V_{\alpha_l,\epsilon}(z_l)\rangle_\delta d^2y,$$

this concludes the proof for the derivative rule 4.2.2.

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RÉSUMÉ

Cette thèse de doctorat porte sur l'étude de deux objets probabilistes, les mesures de chaos multiplicatif gaussien (GMC) et la théorie conforme des champs de Liouville (LCFT). Le GMC fut introduit par Kahane en 1985 et il s'agit aujourd'hui d'un objet extrêmement important en théorie des probabilités et en physique mathématique. Très récemment le GMC a été utilisé pour définir les fonctions de corrélation de la LCFT, une théorie qui est apparue pour la première fois en 1981 dans le célèbre article de Polyakov, "Quantum geometry of bosonic strings".

Grâce à ce lien établi entre GMC et LCFT, nous pouvons traduire les techniques de la théorie conforme des champs dans un langage probabiliste pour effectuer des calculs exacts sur les mesures de GMC. Nous partons des équations de BPZ pour LCFT, introduite par Belavin, Polyakov et Zamolodchikov en 1983. Le méchanisme de ces équations sera étudié vers la fin de cette thèse et nous prouvrons les équations de BPZ d'ordre supérieure avec un formalisme général.

En développant les méthodes probabilistes établies par Kupiainen-Rhodes-Vargas pour la résolution des équations de BPZ et en traitant plusieurs difficultés majeures, nous obtiendront des relations non triviales sur les objets fondamentaux du LCFT. Plus précisément, nous prouvrons les formules exactes pour toutes les constantes de structure du LCFT sur le disque avec la constante cosmologique nulle à l'intérieur du disque, dont une parmi ces quatre a été résolue par Remy en 2017. Comme cas particulier, nous trouverons la distribution de la masse totale du GMC sur l'intervalle avec des log-singularités placées en deux extrémités, qui a été conjecturé indépendamment par Ostrovsky et par Fyodorov, Le Doussal, et Rosso en 2009. Une autre conséquence directe est la loi de la masse totale du GMC sur le cercle avec log-singularité en 1, conjecturé par Ostrovsky en 2016.

MOTS CLÉS : Chaos multiplicatif gaussien, Théorie de Liouville, Théorie conforme des champs, Equations de BPZ

ABSTRACT

Throughout this PhD thesis we will study two probabilistic objects, Gaussian multiplicative chaos (GMC) measures and Liouville conformal field theory (LCFT). The GMC measures were first introduced by Kahane in 1985 and have grown into an extremely important field of probability theory and mathematical physics. Very recently GMC has been used to give a probabilistic definition of the correlation functions of LCFT, a theory that first appeared in Polyakov's 1981 seminal work, "Quantum geometry of bosonic strings".

Once the connection between GMC and LCFT is established, one can hope to translate the techniques of conformal field theory in a probabilistic framework to perform exact computations on the GMC measures. We start from the BPZ equations for LCFT, introduced by Belavin, Polyakov and Zamolodchikov in 1983. The mechanism of these equations is studied in the last part of this thesis and we prove the higher order BPZ equations with a general formalism.

Following the probabilistic methods established by Kupiainen-Rhodes-Vargas for the resolution of the BPZ equations and after overcoming several major difficulties, we obtain non trivial relations for some fundamental objects of LCFT. More precisely, we prove the exact formulas for all the four structure constants of LCFT on the disk with null cosmological constant in the bulk, one of which was solved by Remy in 2017. As a special case, we find the distribution of the total mass of GMC on the interval with log-singularities put on both ends, a conjecture that has been independently predicted by Ostrovsky and by Fyodorov, Le Doussal, and Rosso in 2009. Another direct consequence is the law of the total mass of GMC on the unit circle with a log-singularity, conjectured by Ostrovsky in 2016.

KEYWORDS : Gaussian multiplicative chaos, Liouville field theory, Conformal field theory, BPZ equations