

Applications of the renormalisation group equations: from Grand Unification theories to epidemiology

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Applications of the Renormalisation Group equations From Grand Unification theories to epidemiology

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Applications of the Renormalisation Group equations :

From Grand Unification theories to epidemiology

 ${\rm Corentin}~{\rm Cot}$

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Publications

Publications related to the thesis :

- → Minimal SU(5) Asymptotic Grand Unification G. Cacciapaglia, A. S. Cornell, C. Cot and A. Deandrea https://arxiv.org/abs/2012.14732
- → Second wave COVID-19 pandemics in Europe : a temporal playbook G. Cacciapaglia, C. Cot and F. Sannino https://www.nature.com/articles/s41598-020-72611-5
- → Mining Google and Apple mobility data : Temporal Anatomy for COVID-19 Social Distancing C. Cot, G. Cacciapaglia and F. Sannino https://www.nature.com/articles/s41598-021-83441-4
- → Multiwave pandemic dynamics explained : How to tame the next wave of infectious diseases G. Cacciapaglia, C. Cot and F. Sannino https://www.nature.com/articles/s41598-021-85875-2
- → Impact of US vaccination strategy on COVID-19 wave dynamics G. Cacciapaglia, C. Cot, A. Sigridur Islind, M. Óskarsdóttir, F. Sannino https://www.nature.com/articles/s41598-021-90539-2
- → The field theoretical ABC of epidemic dynamics G. Cacciapaglia, C. Cot, M. Della Morte, S. Hohenegger, F. Sannino, S. Vatani https://arxiv.org/abs/2101.11399
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Other publications :

→ Naturalness of lepton non-universality and muon g-2 Giacomo Cacciapaglia, Corentin Cot, Francesco Sannino https://arxiv.org/abs/2104.08818

Applications des équations du groupe de renormalisation : Des théories de Grande Unification à l'épidémiologie.

Résumé :

Depuis son développement il y a presque cent ans, le principe du Groupe de Renormalisation a permis de repousser les limites de notre compréhension de la physique au delà de l'échelle atomique et d'en prédire les aspects qu'elle présenterait en allant vers l'infiniment petit. Il a aidé à la réalisation d'importantes améliorations de notre compréhension de divers domaines de la physique, de la théorie quantique des champs à la physique du solide. Le Groupe de Renormalisation décrit l'invariance des observables par rapport à l'échelle d'énergie considérée et permet d'étudier l'évolution que cela induit sur les constantes de couplages en fonction de cette échelle. Une des plus importantes prédictions qu'il amène est la théorie de Grande Unification dans laquelle les interactions fondamentales (électromagnétisme, interactions faible et forte) ne devienne plus qu'une unique force émanant à haute énergie. Cette possibilité permettrait de décrire la physique des premiers instants après le Big Bang selon des lois simples se basant sur une unique symétrie. Bien que les théories de grande unification standards voient cette unification comme un point d'intersection entre les courbes d'évolution des couplages. ma thèse propose de la voir comme un procédé asymptotique pour lequel les couplages de jauge tendent à s'unifier en étant amenés vers un point fixe commun dans l'ultra-violet. La grande unification asymptotique introduit de nouvelles particules permettant de s'affranchir de la désintégration du proton, d'avoir un candidat à la matière noire et de reproduire l'asymétrie baryonique observée dans l'Univers. Le modèle minimal en SU(5) avec une dimension supplémentaire compactifiée sera présenté en détail, mais de plus grand groupes de symétrie permettraient d'introduire de nouveaux aspects prometteurs à la théorie.

En parallèle, le groupe de renormalisation a aussi été employé pour étudier la pandémie de COVID-19 qui a débuté à la fin de l'année 2019 et qui est toujours un sujet d'inquiétude au moment où cette thèse est rédigée. Grâce à une nouvelle méthode appelée Groupe de Renormalisation épidémique inspirée par le formalisme de la théorie quantique des champs, il permet d'intégrer des procédés complexes, comme la distanciation sociale ou l'interaction entre régions grâce à l'utilisation d'équations différentielles simples. Alors que les derniers modèles d'épidémiologie développés deviennent de plus en plus précis, leur complexité croissante les rend plus gourmands en temps d'exécution et moins transparents pour comprendre les scénarios de propagation, de son côté, l'équation du groupe de renormalisation épidémique n'a besoin que de deux paramètres dans une unique équation différentielle pour reproduire l'évolution d'une unique vague épidémique. Bien que cette méthode soit assez simple pour pouvoir réaliser une analyse détaillée sur un ordinateur portable en peu de temps, elle a montré une grande précision même pour des prédictions sur plusieurs mois. De nouveaux ajouts peuvent, de plus, y être intégrés aisément permettant de décrire par exemple des vagues multiples, l'apparition de variants mais aussi la vaccination.

Les points de vues originaux offerts par les équations du groupe de renormalisation permettent donc d'ouvrir de nouvelles frontières à l'étude de la théorie quantique des champs pour laquelle elle a été développée à la base, mais aussi pour l'épidémiologie. Ces méthodes pourraient même être étendues à des domaines bien plus variés tels que l'économie ou la chimie.

Applications of the Renormalisation Group equations : From Grand Unification theories to epidemiology

Summary :

Since the introduction in the early fifties of the Renormalisation Group (RG) scheme, our comprehension of physics below the atomic scale has been greatly advanced with important improvements in multiple domains of physics, from quantum field theory to solid-state physics. The Renormalisation Group scheme describes the invariance of observables with respect to the energy scale of the process considered and reveals constraints on the evolution of the coupling strength. One of the main ideas arising from their study are Grand Unification Theories (GUT) in which fundamental interactions (electromagnetism, weak and strong forces) become one unique force at very high energy, this therefore advocates for the existence of a simple symmetry description of physics just after the Big Bang. While standard GUT see unification as a crossing between the running coupling constants, my thesis proposes to view it as an asymptotic process where gauge couplings tend to unify by running toward a common limit being an attractive UV fixed point. The asymptotic Grand Unification Theories introduce new particles removing issues concerning the proton decay, proposing dark matter candidates, and reproducing the baryon asymmetry observed in the Universe. The minimal model on SU(5)with one compactified extra-dimension will be presented, but larger symmetry groups could be involved to introduce promising features to the model.

In addition, the RG has allowed us to investigate the COVID-19 pandemic, that began towards the end of 2019 and is still a major concern at the time when this thesis was written. With a totally new framework called the epidemic Renormalisation Group (eRG) inspired by the quantum field theory formalism, an integration of complex features has been possible, such as social distancing and interactions between countries in a simple modeling scheme. While the latest classical epidemiological models tend to become more precise, their increasing complexity make them computationally time consuming and less transparent when trying to understand the scenario of spreading. On the other hand, the simplest epidemic Renormalisation Group equation needs only two parameters in one differential equation to reproduce a simple wave. Despite its simplicity allowing it to run on a simple personal computer even for complex analysis, our new framework has shown an important level of precision for predictions even a few months into the future. Adding new features to the equations is also straightforward, just like interactions between fields or fixed point can be integrated in quantum field theory.

These original viewpoints offered by the RG equations open new frontiers not only for the study of quantum field theory and for epidemiology, but also extend to others various fields ranging from chemistry to economies.

Table des matières

1	Intr	oduction	8
2	Ren	normalisation group	11
	2.1	The Standard Model	11
		2.1.1 Symmetries and particle content	11
		2.1.2 Standard Model Lagrangian	12
		2.1.3 Electroweak symmetry breaking	15
	2.2	Renormalisation theory	17
		2.2.1 Feynman rules	17
		2.2.2 Renormalisation principle	20
	2.3	Renormalisation Group and beta function	27
		2.3.1 Renormalisation Group Equations	27
		2.3.2 Standard Model Renormalisation Group equations	30
		2.3.3 Fixed points	32
3	Gra	and Unification theories	38
	3.1	Classical Grand Unification	38
		3.1.1 $SU(5)$ model description	38
		3.1.2 Running couplings and unification interpretation	41
	3.2	Extra-dimensional theories	45
		3.2.1 Extra-dimensions principle	45
		3.2.2 Mathematical tools	47
		3.2.3 An example of an extra-dimensional model : Gauge-Higgs Unification	50
	3.3	Asymptotic Grand Unification	52
		3.3.1 Model description	52
		3.3.2 Running of the couplings	58
		3.3.3 Baryogenesis	72
4	Epi	demiology and Renormalisation Group	78
	4.1	Epidemiologic models	78
		4.1.1 Compartmental models (deterministic approach)	78
		4.1.2 Percolation models (stochastic approach)	82
	4.2	Epidemic Renormalisation Group equations	85
		4.2.1 eRG basics	85
		4.2.2 Correlation with mobility data	89
		4.2.3 Interacting eRG	97
		4.2.4 Complex fixed points in epidemiolgy	110
	4.3	Vaccination, mutations and other disease	116
		4.3.1 Vaccination	116
		4.3.2 Mutations	123
		4.3.3 Others pandemics	132
5	Dise	cussion and conclusion	136
A	open	dices	138

\mathbf{A}	Mathematical formulas	138
	A.1 Group generators	138
	A.2 Group factors formulas	139
	A.3 Dimensional regularisation	139
в	One loop Yukawa beta-function diagrams calculations	140
	B.1 Feynman rules	140
	B.2 Relevant diagrams for the Yukawa calculation	143
	B.3 Calculation in the Standard Model	145
\mathbf{C}	Epidemic Renormalisation Group	146
	C.1 Monte-Carlo simulation	146
	C.2 $2 - \gamma$ function fit parameters vs mobility data	147
	C.3 US divisions composition	149
	C.4 Second European waves	149
	C.5 Multiwave and CeRG	152

Références

1 Introduction

Following the revolution of quantum mechanics in the beginning of the twentieth century, the development of subatomic physics and its description of the nature of matter began a rich journey of discovery for science. Quantum Field Theory (QFT) [1, 2] and its framework to describe the known particles and interactions, the Standard Model (SM) [3, 4, 5], lead to an unprecedented precision of measurements in the history of physics. Within this description of the Universe through symmetries, it allows for the prediction of the observable quantities values such as decay rates and cross sections, but also the effective estimations of the Lagrangian parameters within the theory, such as particle masses or coupling constants. It has provided numerous theoretical predictions, later measured experimentally such as the discovery of the massive gauge bosons W and Z [6, 7] or more recently the Higgs boson [8, 9]. Agreement between theoretical predictions and experimental observations has never been so great than within the SM. For example, the most precise quantities computed have yielded agreement up to the twelfth decimal between theory [10, 11] and experiment [12]. The model is so robust that any tiny deviation from its predictions [13, 14, 15] creates keen interest in the scientific community as proposals for extensions to the SM [16] to explain the discrepancy are made.

One of the predictions offered by QFT is the Renormalisation Group (RG) [17] and the associated Equations (RGE) which give rise to the beta functions [18, 19, 20]. Their purpose is to show through differential equations, the evolution of the Lagrangian parameters as dependant on the energy at which the theory is considered. This allows for predictions of the value of quantities for different energy scales in experiments ranging from the early Compton scattering at keV scale experiments [21] to the record breaking 13 TeV reached at the Large Hadron Collider (LHC) [22]. RGE are also useful in understanding the limits in which theories are well defined [23, 24]. For example, this can show that Quantum Chromo-Dynamics (QCD) is non-perturbative below a certain energy scale, thus hadronic models are needed to replace QCD there. In a similar way, above a certain scale, Quantum Electro-Dynamics (QED) needs to be embed within the electroweak model to describe the physics accurately at higher scales.

Among all its achievements, the SM suffers from yet unexplained features which may change our understanding of the Universe. One of the main problems comes from the behavior of the SM at very high energies. The SM can explain physics processes accurately up to the ElectroWeak (EW) scale, but then predicts that no new phenomenology would arise until we reach eighteen orders of magnitudes above the EW symmetry breaking scale that is the Planck scale at which the strength of gravity would become comparable to the others interactions and therefore could not be neglected [25]. This is equivalent to not knowing what is happening in the length range of 10^{-17} m to 10^{-35} m. The ratio between these bounds is equivalent to comparing the size of the Earth with the size of an atom. The "void" predicted by the SM in this very large discrepancy between these two scales creates what is called the Hierarchy problem, and suggests that something may exist between these scales. Moreover, the evolution of the coupling constants in the SM, as given by the RGE, shows that some of them would go to infinite values before reaching the Planck scale. This divergence is the sign of an inconsistency in the theory at large energy scales [26] as was argued to advocate that QED is an effective theory. Fortunately, this unpleasant problem could be removed if the gauge couplings run toward an unique point. Supposing that the three interactions unify at this point, their unified coupling evolution being described by another group could avoid dangerous infinities up to the Planck scale. Such models [27, 28] are called Grand Unification Theories (GUT) [29] and drove important interest in deriving a unified description of the Universe. However, this unification cannot precisely occur at one unique point in the SM unless new ingredients such as SuperSymmetry (SUSY) are added [30, 31]. No experiments have yet observed any supersymmetric particles, something which has created concerns and dampered hopes about SUSY in the scientific community [32, 33, 34]. Moreover, GUT cannot be disentangled from the prediction of the proton decay [35], a process that has never been observed experimentally [36], and so it considerably constrains models by pushing the unification to very high energies.

One of the other main issue in the Standard Model is Dark Matter (DM). This first arose due to a disagreement between the dynamical evolution of large structures in the Universe and the estimation of the visible matter within these structures [37], as was first pointed out in 1884 and measured more precisely later [38]. This could be explained by admitting the existence of a non-visible matter interacting through gravitation with visible matter, as first proposed by Oort in 1932 [39] and consolidated in 1970 by the work of Rubin and Ford [40]. These particles could also explain gravitational lensing effects [41, 42], structures in the cosmic microwave background [43] and multiple other observations but these are yet to be measured directly experimentally [44].

Therefore, there is a need for new theories involving description of both unification and dark matter candidates. Among these possible theories, extra-dimensional theories have been of great interest, as they were developed first to unify gravitation and the other interactions [45, 46, 47]. Through a different unification paradigm, this thesis presents extra-dimensional models seeing unification as an asymptotic effect instead of a crossing point. It primarily removes the above problems that were systematically encountered in standard GUT, forbidding the proton decay and proposing a DM candidate, but it also gives the opportunity to think the unification in a different way.

The framework employed to describe the Renormalisation Group can also be applied in completely different domains. Along with the spreading of the coronavirus SARS-CoV-2 and its related disease the COVID-19 pandemic [48], a renewed interest in epidemiological models has emerged in the scientific community from 2020. The century-old SIR compartmental model [49, 50] developed after the Spanish Influenza pandemic that had spread during and after the first World War had been intensively used and modified to take into account various aspects of the spreading of diseases [51, 52]. For complex study these models require a lot of input parameters in order to improve their accuracy at reproducing data, thus their complexity gives rise to a lack of predictability and leads to longer computation times in order to reach a usable result. Therefore, a simpler and more comprehensive framework is needed to study the spread of viruses. Surprisingly, Renormalisation Group has brought an original point of view to epidemiology and the mathematical tools developed within QFT through the newly developed epidemic Renormalisation Group (eRG) [53] served this purpose. In a single master equation the method can be employed to simulate [54, 55], predict [56, 57] and understand [58, 59] the evolution of the pandemic across different geographic scales. The eRG has brought about a simple visualisation of the interaction between regions just like an interaction between fields could be modeled. Similarly, the multiple waves and the endemic phases of the pandemic can be thought in terms of fixed points of the RGE. A careful description of the effects of vaccination campaigns, or of the emergence of new variants, can also be added to the master equation. While the eRG was developed along with the COVID-19 pandemic, it could also be adapted to other past or future pandemics that may threaten humanity.

This thesis is organised as follows: in chapter 2 we will introduce the Renormalisation Group approach. We start by presenting the main features of the Standard Model, then by showing the problems related to infinities in the calculations, we will see how renormalisation and regularisation methods have to be employed to tame these infinities. We will also investigate the implication of the renormalisation scheme on the evolution of the parameters of the theory using the Renormalisation Group framework through the β -function. We will finally study the equations in the Standard Model and see the consequences of having fixed points. In chapter 3 we introduce Grand Unification theories, first by showing how and why standard Grand Unification had been developed and what its limits are. We then investigate extra-dimensional theories and the mathematical description of such models. Finally, through the new asymptotic Grand Unification paradigm, we present the minimal model and how it addresses the problems of standard unification. In chapter 4 we introduce the process used to study the behavior of pandemics through the Renormalisation Group equations. We first introduce the simplest epidemiological models, focusing on their similarities and limits. We then introduce the epidemic Renormalisation Group framework on simple waves first seeing how correlation with mobility can be extracted, then adding interaction and multiple waves features in it. We finally deal with other add-on, such as vaccination, mutations or other diseases. We conclude with a discussion of possible extensions to the two directions of the thesis in chapter 5.

2 Renormalisation group

2.1 The Standard Model

2.1.1 Symmetries and particle content

The **Standard Model** (SM) [2, 1, 60] is probably the most successful theory in the history of physics. It is defined within the Quantum Field Theory mathematical formalism and proposes to describe all the particles detected and, in the scope of symmetries, three of the four known fundamental interactions of physics within a coherent framework. The forces it describes are the **electromagnetism**, responsible for the interaction between electrically charged particles, allowing the existence of the stable structure of atoms and interactions between them in molecules, the **weak force**, responsible for some nuclear reactions such as β decay and for flavour violating processes in nature and finally the **strong interaction**, accountable for the stability of the atom nucleus and of its components but also for most of the mass of matter. Including the gravitational interaction in this description is one of the main challenges of physics and while several theories such as string theories [61] or quantum loop gravity try [62] to integrate it in the SM, their phenomenological predictions are pushed to very high energies. We will limit the range of the theories presented in this thesis to energies lower than the Planck mass which represents the scale for which gravitational strength becomes comparable to subatomic forces.

In the Hamilton's description, any physical theory is associated to a quantity allowing to derive the equations of motion associated to the fields considered in the theory called the action. In the SM, this quantity S_{SM} is invariant under the following symmetry group :

$$\mathcal{G}_{SM} = \mathbf{P}_4 \otimes SU(3)_c \otimes SU(2)_L \otimes U(1)_Y ,$$

where \mathbf{P}_4 is the 4-dimensional Poincaré group related to the invariance under special relativity transformations. The three other groups are gauge groups associated with internal symmetry of the fields. $SU(3)_c$ refers to the Quantum Chromo-Dynamic (QCD) group, representing the strong interaction, $SU(2)_L$ represents the group in which left-handed fermions and the Higgs boson are coupled and $U(1)_Y$ refers to the hypercharge group. The non-simple group $SU(2) \times U(1)$ is also called the ElectroWeak (EW) group as it contains the electromagnetism and weak interactions that would appear through spontaneous symmetry breaking. Those three groups give rise to a set of gauge fields that are the mediators of the three subatomic forces. Gauge fields are described by vector-transforming mathematical objects of spin 1 and are called gauge bosons, the term **boson** referring to spin-integer particles as opposed to **fermions** with half-integer spin particles.

The **photon** γ is the gauge boson associated with **electromagnetism**. In the SM, it has no mass and is described by the electromagnetic field A_{μ} whose Abelian characteristics allow it to commute with himself. The three **massive gauge bosons** W^{\pm} and Z^{0} are associated with the **weak interaction**. They are given masses due to the spontaneous symmetry breaking of the $SU(2) \times U(1)$ group and are described by the non-Abelian fields W^{\pm}_{μ} , W^{-}_{μ} and Z^{0}_{μ} . Finally, the eight **gluons** fields g^{a} are gauge bosons responsible for the **strong force** stemming directly from the Quantum Chromodynamics (QCD) group $SU(3)_c$ and are massless.

In the SM, it exists another boson whose spin is zero and can therefore be described by a scalar field. This particle is called the Higgs boson h and is at the origin of the EW symmetry breaking. Its interactions with fundamental particles give rise to mass terms in the Lagrangian thanks to the Higgs mechanism. The Higgs boson is also the latest fundamental particle discovered by the scientific community in 2012 and is therefore heavily studied.

The Standard Model also contains fermions composing what we call matter. Fermions can be separated into two sub-groups having different interactions in the SM : the **leptons** that are only sensitive to the electromagnetic and weak forces and the **quarks** that are sensitive to all interactions including the strong force. Each fermion category is composed by three copies of two particles as shown in Table 1, each copy being called generation or family.

Generation	Quarks		Leptons	
	up-type quark	down-type quark	charged lepton	neutrinos
1	up u	down d	electron e	electronic ν_e
2	charm c	strange s	muon μ	muonic ν_{μ}
3	top t	bottom b	tau τ	tau ν_τ

TABLE 1 – Fermion content of the Standard Model.

However, most of the matter we interact with everyday is composed by only the three charged fermions of the first generation, saying the up and down quarks embedded into the protons and neutrons composing atomic nucleus and the electrons interacting with the nuclei to form atoms. The other particles are either weakly interacting with visible matter (as all the neutrinos) either unstable, decaying into lighter particles. Their short life time forces us to investigate their properties only in high energy processes such as cosmic rays or high energy physics experiments. All these particles can be easily integrated within the Standard Model by deriving the associated **Lagrangian** representing all the interactions and properties of the particles in a single equation.

2.1.2 Standard Model Lagrangian

The Lagrangian \mathcal{L}_{SM} associated with the SM can be written :

$$\mathcal{L}_{SM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} G^b_{\mu\nu} G^{b\mu\nu} + i \overline{l_L} \not D l_L + i \overline{e_R} \not D e_R + i \overline{q_L} \not D q_L + i \overline{u_R} \not D u_R + i \overline{d_R} \not D d_R + \left(\overline{l_L} \phi^c\right) Y_l e_R + \left(\overline{q_L} \phi\right) Y_d d_R + \left(\overline{q_L} \phi^c\right) Y_u u_R + h.c + \left| D_\mu \phi \right|^2 + \mu^2 \phi^{\dagger} \phi + \lambda \left(\phi^{\dagger} \phi\right)^2 ,$$

$$(1)$$

where the repeated indices are summed according to the Einstein summation rule explained in the appendix A. a, b are gauge indices being summed over the dimension of the adjoint space of the groups, $\not{D} = \partial_{\mu}\gamma^{\mu} - ig_{C}T^{a}C^{a}_{\mu}\gamma^{\mu}$ is the covariant derivative, where the γ_{μ} are the Dirac gamma matrices, g_{A} is the coupling constant associated with the gauge field C^{a}_{μ} belonging to any of the gauge group. F, W, G are the field strength tensors defined by :

$$F_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\mu}B_{\nu} ,$$

$$W^{a}_{\mu\nu} = \partial_{\mu}W^{a}_{\nu} - \partial_{\mu}W^{a}_{\nu} + ig_{2}f^{abc}_{SU(2)}W^{b}_{\mu}W^{c}_{\nu} ,$$

$$W^{a}_{\mu\nu} = \partial_{\mu}W^{a}_{\nu} - \partial_{\mu}W^{a}_{\nu} + ig_{3}f^{abc}_{SU(3)}W^{b}_{\mu}W^{c}_{\nu} ,$$

(2)

where the f^{abc} are the structure constants and g_1, g_2 and g_3 are the coupling constants associated with the groups U(1), SU(2) and SU(3) respectively. l, q, e, u and d are the fermion fields and ϕ , the scalar field in which the Higgs boson is embedded. All the fields are functions of space and time and takes their values in irreducible representations of the SM group \mathcal{G}_{SM} .

Each field is associated to local gauge operators U(x) leaving the Lagrangian invariant according to the transformations :

$$\psi(x) \to U(x)\psi(x) . \tag{3}$$

The transformations can be rewritten in terms of the gauge group generators such that :

$$U(x) = e^{iq_{\psi}T^a} , \qquad (4)$$

where q is the charge of the field ψ associated with the SM subgroup and T^a are the subgroup generators which are defined in Appendix A. The charges of the fermions fields under the three SM groups are listed in 2.

Name	Multiplets	Fields	$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$
Lepton doublet	l_L	e_L	$\left(1,2,-rac{1}{2} ight)$
		$ u_L $	
Charged lepton singlet		e_R	(1,1,-1)
Quark doublet	q_L	u_L	$\left(3,2,rac{1}{6} ight)$
		d_L	
Up-type quark singlet		u_R	$(3, 1, \frac{2}{3})$
Down-type quark singlet		d_R	$\left(3,1,-rac{1}{3} ight)$

TABLE 2 – Quantum charges of the fermion fields in the SM group.

As a naive first guess, one can think that the fermion kinetic term could be written $\overline{\psi} \partial \!\!\!/ \psi$. However, this expression is not invariant under the gauge transformations. Therefore, we need to add a term coupling the gauge bosons to the fermion to restore the symmetry such that $\partial \!\!\!/$ is replaced by $\not D = \partial - ig_A T^a A^a_\mu$ that is what we called the covariant derivative.

Discovered in 2012 [8, 9] after being predicted by Brout, Englert and Higgs [63, 64], the **Higgs boson** is an major piece of the Standard Model. It is accountable for the electroweak symmetry breaking giving rise to fundamental particle masses through Lagrangian terms. The Higgs boson is integrated within the Lagrangian via gauge invariant operators among which the terms $\mu^2 \phi^{\dagger} \phi$ and $\lambda (\phi^{\dagger} \phi)^2$ are known as the **Higgs potential** and will be studied in details in section 2.1.3.

The Higgs field comes with interaction terms in what is called the Yukawa sector where the fermions and the Higgs boson are coupled. In these operators, the Higgs that is an SU(2)doublet is always contracted with a fermion doublet and a fermion singlet. These parts give rise to mass terms for the fermions after the Higgs develop a non-zero vacuum expectation value due to its potential, thus inducing the EW symmetry breaking.

The fermion families are embedded in flavor multiplets such that :

$$e_R = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}, \quad l_L = \begin{pmatrix} l_L^e \\ l_L^\mu \\ l_L^\tau \end{pmatrix}, \quad u_R = \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}, \quad d_R = \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix}, \quad q_L = \begin{pmatrix} q_L^u \\ q_L^c \\ q_L^\tau \\ q_L \end{pmatrix}.$$
(5)

The subscripts L/R refers to the chirality of the field where the right-handed (R) fields are singlets and left-handed (L) doublets can be decomposed for each family as :

$$l_L^e = \begin{pmatrix} \nu_L^e \\ e_L^- \end{pmatrix}, \quad q_L^u = \begin{pmatrix} u_L \\ d_L \end{pmatrix}.$$
(6)

Therefore, in flavor space, fermions can be rotated without changing the Lagrangian, using orthogonal matrices U and V (where orthogonality is defined in Appendix A) and following the transformations :

$$l_L \to U_l l_L, \quad e_R \to U_e e_R, q_L \to V_q q_L, \quad u_R \to U_u u_R, \quad d_R \to U_d d_R.$$

$$\tag{7}$$

Yukawa matrices as shown in the Lagrangian are just generic matrices, but using these transformations, the lepton Yukawa matrix can be diagonalised, so we can consider it directly as : (W_{ij})

$$Y_l = \begin{pmatrix} Y_e & & \\ & Y_\mu & \\ & & Y_\tau \end{pmatrix} , \tag{8}$$

where each eigenvalue correspond to a lepton flavor (electron, muon and tau). For the quark sector, the two matrices cannot be diagonalised simultaneously. The convention in the Standard Model is to take Y_d diagonal and to keep Y_u standard. It is possible then to choose V_q, V_u, V_d and an appropriate orthogonal matrix V such that :

$$Y_{d} = V_{q} \hat{Y}_{d} V_{d} \qquad , \quad Y_{u} = V \hat{Y}_{u} V_{u} ,$$

$$\hat{Y}_{d} = \begin{pmatrix} Y_{d} & & \\ & Y_{s} & \\ & & Y_{b} \end{pmatrix} \qquad , \quad \hat{Y}_{u} = \begin{pmatrix} Y_{u} & & \\ & Y_{c} & \\ & & Y_{t} \end{pmatrix} .$$
(9)

Doing the transformation for the Yukawa quark terms in the Lagrangian, we can finally write the quark Yukawa terms as :

$$\left(\overline{q_L}\phi\right)\hat{Y_d}d_R + \left(\overline{q_L}\phi^c\right)\mathcal{V}\hat{Y_u}u_R , \qquad (10)$$

where $\mathcal{V} = V_q^T V$ is called the **Cabibbo-Kobayashi-Maskawa** (CKM) **matrix** [65, 66] that is responsible for the flavor-mixing charged current in the SM. This matrix is the reason why mass-states of up-type quarks are different from weak interacting states.

2.1.3 Electroweak symmetry breaking

A crucial part of the Standard Model is the electroweak symmetry breaking. In the SM Lagrangian, mass terms for the gauge bosons are forbidden because they do not conserve the gauge symmetries, however, the gauge bosons W and Z are massive which seems to contradict the previous affirmation. Their mass terms can actually be derived thanks to the non-zero Vacuum Expectation Value (VEV) the Higgs boson gets at low energy which causes the electroweak symmetry to spontaneously break. This VEV can be derived by looking at the Higgs potential :

$$V(\phi) = -\mu^2 |\phi|^2 + \lambda |\phi|^4 .$$
(11)

At low energy, the theory will be driven to its ground state that is obtained at a minimum of this potential. The expression in equation (11) is shown in Figure 1.



FIGURE 1 – Higgs potential as a function of $|\phi|$. The red dot shows that gauge invariant value while the green dot shows the global minimum that breaks the explicitly the symmetry.

One can see that the minimum is different from $|\phi| = 0$, therefore the potential drives the theory to a state that is not gauge invariant anymore. One can remark that even if a minimum does not respect the gauge symmetry, the set of minimums of the potential is gauge invariant. This is the principle of spontaneous symmetry breaking, while the potential is still symmetric with respect to SU(2), the theory will be pushed to a set of solutions that is gauge-invariant but for which each solution isn't. In the case of the Higgs potential, the minimum is reached when :

$$|\phi|^{2} = \frac{\mu^{2}}{2\lambda} = \frac{v^{2}}{2} , \qquad (12)$$

giving the field a non-zero vacuum expectation value within a set of solutions that are just different up to a complex phase. Therefore, a perturbative state around the ground state for the Higgs SU(2) doublet can be written as :

$$\phi = e^{i\pi_a(x)\cdot\sigma^a} \begin{pmatrix} 0\\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} , \qquad (13)$$

where σ are the three generators of SU(2), $\pi(x)$ are the Goldstone fields that are unphysical massless scalar fields appearing thanks to the symmetry breaking. Finally, h(x) is the physical Higgs field. Replacing this expression in the Higgs potential gives :

$$-\mu^{2} \left(\frac{v+h(x)}{\sqrt{2}}\right)^{2} + \lambda \left(\frac{v+h(x)}{\sqrt{2}}\right)^{4} = \lambda v^{2} h(x)^{2} + \lambda v h(x)^{3} + \frac{\lambda}{4} h(x)^{4} - \frac{\lambda v^{2}}{4}.$$
 (14)

These terms can be identified respectively as a mass term for the Higgs boson associated to $m_h = \sqrt{\lambda}v$, a trilinear and a quartic couplings between Higgs fields and a constant term that shows the change of the ground state but is unphysical so can be neglected.

In the same way, the dynamical term associated to the gauge fields W^a and B associated with the electroweak group $SU(2) \times U(1)$ can be expanded such that :

$$\begin{split} |D_{\mu}\phi|^{2} &= \frac{1}{2} \left(\begin{pmatrix} 0\\ \partial_{\mu}h \end{pmatrix} - i \begin{pmatrix} 0\\ h+v \end{pmatrix} \left(g_{1}B_{\mu} + g_{2}\sigma^{a}W_{\mu}^{a} \right) \right) \left(\begin{pmatrix} 0\\ \partial^{\mu}h \end{pmatrix} + i \begin{pmatrix} 0\\ h+v \end{pmatrix} \left(g_{1}B^{\mu} + g_{2}\sigma^{a}W^{a,\mu} \right) \right) \\ &= \frac{1}{2}\partial_{\mu}h\partial^{\mu}h + \frac{1}{8}\left(h+v\right)^{2} \left(g_{2}^{2} \left(W_{\mu}^{1} - iW_{\mu}^{2} \right) \left(W^{\mu 1} + iW^{\mu 2} \right) + \left(g_{2}W_{\mu}^{3} - g_{1}B_{\mu} \right)^{2} \right) \\ &= \frac{1}{2}\partial_{\mu}h\partial^{\mu}h + \frac{g_{2}^{2}v^{2}}{4}W_{\mu}^{+}W_{-}^{\mu} + \left(g_{2}^{2} + g_{1}^{2} \right) \frac{v^{2}}{8}Z_{\mu}Z^{\mu} + \text{interactions-terms} \,, \end{split}$$
(15)

where the last line was obtained by defining the fields W^{\pm} and Z such that :

$$W_{\mu}^{\pm} = \frac{W_{\mu}^{1} \mp i W_{\mu}^{2}}{\sqrt{2}} , \quad Z_{\mu} = \frac{g_{2} W_{\mu}^{3} - g_{1} B_{\mu}}{\sqrt{g_{2}^{2} + g_{1}^{2}}} . \tag{16}$$

In the last line of equation (15), we derived the Higgs boson kinetic term and the mass terms for the Z and W bosons associated to the weak interaction where :

$$m_W = \frac{g_2}{2}v$$
, $m_Z = \frac{\sqrt{g_2^2 + g_1^2}}{2}v$. (17)

The photon field can also be added by defining :

$$A_{\mu} = \frac{g_1 W_{\mu}^3 + g_2 B_{\mu}}{\sqrt{g_2^2 + g_1^2}} .$$
 (18)

No mass term is associated to this field leaving the photon with its standard massless description. The transformation of W^3 and B in A and Z fields can be seen as a rotation via the following relation :

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \begin{pmatrix} \cos(\theta_W) & \sin(\theta_W) \\ -\sin(\theta_W) & \cos(\theta_W) \end{pmatrix} \begin{pmatrix} B \\ W^3 \end{pmatrix},$$
(19)

where :

$$\cos(\theta_W) = \frac{g_2}{\sqrt{g_2^2 + g_1^2}}, \quad \sin(\theta_W) = \frac{g_1}{\sqrt{g_2^2 + g_1^2}}.$$
 (20)

 θ_W is known as the **weak mixing angle** or **Weinberg angle** and is an active sector of the experimental research because it can constrain theoretical models beyond the Standard Model. The Weinberg angle is also involved in the relations between the massive bosons masses and between the gauge group coupling constant g_2 and the electric charge e:

$$\frac{m_W}{m_Z} = \cos(\theta_W) , \quad e = g_2 \sin(\theta_W) . \tag{21}$$

The most precise measure obtained is $\cos(\theta_W) = 0.88147 \pm 0.00013$ [67] which is one of the most precise measures in particle physics.

The charge associated to the photon field is the standard electric charge \mathcal{Q} defined by :

$$\mathcal{Q} = \frac{y}{2} + \sigma_3 , \qquad (22)$$

where y is the charge related to the U(1) group also called **hypercharge**. Finally, the electroweak symmetry breaking has an effect on the Yukawa sector giving mass to the fermions. The Yukawa terms in the Lagrangian can be rewritten :

$$\mathcal{L}_{Y} = \left(\overline{l_{L}}\phi^{c}\right)\hat{Y}_{l}e_{R} + \left(\overline{q_{L}}\phi\right)\hat{Y}_{d}d_{R} + \left(\overline{\tilde{q}_{L}}\phi^{c}\right)\mathcal{V}\hat{Y}_{u}\tilde{u}_{R} + h.c$$

$$= \frac{vY_{l}}{\sqrt{2}}\left(\overline{e_{L}}e_{R} + \overline{e_{R}}e_{L}\right) + \frac{vY_{d}}{\sqrt{2}}\left(\overline{d_{L}}d_{R} + \overline{d_{R}}d_{L}\right) + \frac{vY_{u}}{\sqrt{2}}\left(\overline{u_{L}}u_{R} + \overline{u_{R}}u_{L}\right) + \mathcal{O}\left(h(x)\right) .$$

$$(23)$$

The last equality is obtained because the three \hat{Y} -matrices are diagonal in flavour space and considering that the mass states of up-type quarks fields are defined by the matrices u_L and u_R such that :

$$\tilde{u}_L = \mathcal{V}^{\dagger} u_L , \quad \tilde{u}_R = u_R .$$
⁽²⁴⁾

The last line of equation (23) shows that mass terms can be generated thanks to the electroweak symmetry breaking and fermions are thus associated to the masses :

$$m_{f_i} = \frac{vY_f^i}{\sqrt{2}} \ . \tag{25}$$

We saw that the electroweak symmetry breaking is essential to derive the Standard Model particles and their characteristics. We will now investigate how to predict observable values theoretically and why Renormalisation Group is needed in order to understand the energy dependence of theories such as the Standard Model.

2.2 Renormalisation theory

2.2.1 Feynman rules

Using symmetries and action principles, Quantum Field Theory (QFT) is yet the simplest way to describe particle physics models. It has been shown that quantities and observables in QFT can be calculated by using diagrams describing contributions to the full processes considered : The Feynman diagrams. Most of the quantities derived in QFT can be predicted calculating transition amplitudes $\mathcal{A}_{i\to f}$, whose norm corresponds to the probability for a process transforming an initial state *i* into the final state *f* to occur. Initial and final states are called asymptotic states symbolising the fact that states do not interact anymore and are well separated. In order to calculate these transition amplitudes, the quantum operator *S* is needed, such that :

$$\mathcal{A}_{i \to f} = \langle i | S | f \rangle , \qquad (26)$$

where $\langle | \text{ and } | \rangle$ are the **bra** and **ket** notations where the bra corresponds to a linear form transforming a vector into a complex number and the ket is acting as a vector. Therefore, $\langle i|S|f \rangle$ can be thought as a matrix element of S.

The S-matrix cannot be computed in general but can be estimated using perturbation theory. The principle of perturbation theory is to develop from an analytical interaction-free theory a model in which an interaction term is added whose contribution is small. The theory will then be expanded in terms of powers of the small interaction around the bare theory as we did for the Higgs field next to its minimum in equation (13). This expansion can be derived thanks to the **LSZ theorem** allowing to link elements of the S-matrix with time-ordered correlation functions. As a general example, taking n particles in the initial state and m in the final states for scalar quantum fields $\phi(x)$ of defined momentum $p_1, ... p_{n+m}$, states $|i\rangle$ and $\langle f|$ can be written :

$$|i\rangle = \sqrt{2|p_1}...\sqrt{2|p_n}a_{p_1}^{\dagger}(-\infty)...a_{p_n}^{\dagger}(-\infty)|\Omega\rangle , \qquad (27)$$

$$\langle f | = \sqrt{2} | p_{n+1} ... \sqrt{2} | p_{n+m} \langle \Omega | a_{p_{n+1}} (\infty) ... a_{p_{n+m}} (\infty) , \qquad (28)$$

where $|\Omega\rangle$ is the vacuum of the interacting theory (which can be different from the vacuum of the free theory written $|0\rangle$) and the $a_p^{\dagger}(t)$ (respectively $a_p(t)$) are operators creating (destroying) a particle of momentum p at the time t and are thus called creation (annihilation) operators. Then, the matrix element can be written :

$$\langle i|S|f\rangle = \prod_{k=1}^{n+m} \left[i \int \mathrm{d}^4 x_k e^{-ip_k x_k} \left(\Box_k + m^2 \right) \right] \langle \Omega|T\{\phi(x_1)...\phi(x_{n+m})\} |\Omega\rangle , \qquad (29)$$

where $\Box_i = \left(\frac{\partial}{\partial x_i^{\mu}}\right)^2$ and $T\{\ldots\}$ refers to the time-ordering product putting all operators from the later times to the left of operators from earlier time. Finally, this expression can be written in terms of product of two-point functions thanks to the **Wick theorem** :

$$T\{\phi(x_1)\phi(x_2)...\phi(x_n)\} =: \phi(x_1)\phi(x_2)...\phi(x_n) + \text{ all possible contractions }:, \qquad (30)$$

where : ... : represents the normal ordering moving all the creation operators a^{\dagger} to the left and all annihilation operators a to the right and the term **contraction** refers to the process of associating all fields together by pairs to get a Feynman propagator for each pair. Feynman propagators are related to Green's functions linking two point of space-time together and will be seen later when showing Feynman rules. Normal ordering allows to make some of the terms of the matrix element vanish thanks to the free theory vacuum relation such that :

$$a \left| \Omega \right\rangle = 0 \,, \tag{31}$$

and leaves us only with the fully contracted terms giving Feynman propagators. These propagators allow to show the connection between space-time points as lines and therefore to show the sum of all possible contractions using diagrams.

In the end, all the theorems we have seen allow to derive rules that can be simply explained in terms of diagrams. Each piece of the diagrams can be expressed by a mathematical expression giving the contribution to the quantity of interest. As a simple example, we show here the Feynman rules in Quantum Electro-Dynamics (QED) :

,

where p corresponds to the momentum associated with the propagator, $\tilde{\epsilon}$ is a parameter allowing to avoid poles when integrating over the momentum and which should be brought to 0 at the end of the calculation. $g_{\mu\nu}$ is the metric tensor defined as Diag(+, -, -, -), ξ is the gauge parameter from which the final result should be independent as any choice should lead to the same result for the S-matrix due to its gauge invariance, the slashed notation p corresponds to the contraction with the Dirac matrices γ^{μ} , defined in Appendix A, such that $p = p_{\mu}\gamma^{\mu}$.

This set of Feynman rules is sufficient to derive all the relevant quantities associated within QED. The Feynman diagrams associated to a physical quantity are an infinite set of diagrams that allows to approximate it using perturbation theory. For example, when trying to evaluate the interactions between two electrons, we have to compute the following diagrams :



where ... refers to the infinite number of diagrams that can be added to calculate the interaction, while here we just represent some diagrams up to the one-loop order.

The process shown by each diagram alone shouldn't be taken as a real occurring process, because it exhibits virtual particles with nonphysical characteristics. This can be seen by looking, for example, at the diagram in equation (33).



It is possible to compute the four-vector for each particle and find that in the center-of-mass reference frame, we have :

$$\tilde{k}^2 = -4E^2 \neq m^2 = 0 , \qquad (34)$$

where \tilde{k} is the photon four-vector and E is the energy of each electron in the center-of-mass reference frame. This result shows that the photon four-vector does not respect the on-shell condition $\tilde{k}^2 = E^2 - p^2 = m^2$, thus we call it an off-shell particle. This result can be understood in the light of the Heisenberg uncertainty principle $\Delta t \Delta E > \frac{\hbar}{2}$, where Δt represents the time of the process and ΔE the uncertainty on the energy associated. In the diagram considered before, the virtual photon appear and disappear so fast that Δt is really small forcing ΔE to be higher than the energy of each electron. Such particle cannot be observed as it clearly violates the laws of relativity, thus, virtual particles shouldn't be considered as real particles and Feynman diagrams shouldn't be considered as real processes. They are just practical mathematical tools to study a process through perturbation theory.

2.2.2 Renormalisation principle

Calculating diagrams sometimes exhibits infinite quantities that seems to imply that the theory doesn't work. This is the case for most of the loop diagrams such as a vacuum polarisation diagram :

$$\gamma \sim \gamma \qquad (35)$$

Neglecting fermion masses and taking only the divergent part, this diagram behaves as :

$$i\mathcal{M} \approx \int_0^{+\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^2}{\left(k^2 - \Delta + i\tilde{\epsilon}\right)^2} \approx \int_0^{+\infty} k \mathrm{d}k \to \infty \,. \tag{36}$$

This integral is, indeed, infinite. This result was first found by Weisskopf in 1934 [68] and used by Bethe in 1947 [69] when trying to calculate the diagram contributions to the Lamb Shift [70]. In Weisskopf calculation, infinities came from very short wavelength photons, which can be attributed to very large momentum k in our diagram calculation. It exists, in fact, methods to get rid of the infinities and give finite results to the diagram calculations. Different methods can be used and are called regularisation schemes. As a first example, one can put an UV cut-off regulator Λ as an upper boundary for the integral over d^4k instead of the ∞ boundary in equation (36). This can be interpreted as supposing that the model considered is valid until an upper energy scale. This brings for the diagram calculation a finite value that depends on Λ . However, summing all the contributions participating to any observable up to a certain order gives a result that doesn't depend anymore on Λ if the symmetries of the theory are respected.

Other techniques can be used to pass through the infinity problem, among which dimensional regularisation will be often used throughout the thesis. This regularisation consists in considering that the theory is no longer four-dimensional but instead described within d dimensions. The final trick would be to make d tend to the standard 4 dimensions. In a more practical way, one can consider that $d = 4 - \epsilon$ and drive ϵ to 0 at the end of the calculation. One shouldn't mistake the dimensional parameter ϵ and the term $\tilde{\epsilon}$ used to avoid the mass poles in the propagators. As calculations don't depend on the parameter Λ in the UV cut-off regularisation, the final result in dimensional regularisation will not depend on ϵ . This fundamental result has profound consequences on the interpretation of the theory and will be the reason why the Renormalisation Group and the associated equations emerge and can give us important predictions on the energy scale dependence of the theory.

Before discussing the Renormalisation Group, let's take a simple example with the calculation of the photon propagator correction diagrams to illustrate the dimensional regularisation method. The full expression of the vacuum polarisation diagram shown in equation (35) is :

$$i\Pi_{\mu\nu} = (-ie)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_{\mu} \left(-\not p + \not k + m\right)\gamma_{\nu} \left(\not k + m\right)\right]}{\left((p-k)^2 - m^2 + i\tilde{\epsilon}\right)\left(k^2 - m^2 + i\tilde{\epsilon}\right)}.$$
(37)

We first use trace formulas (as shown in Appendix A), drop terms linear in k as they would give 0 contributions because of the oddness of the integral and keep only terms proportional to $g_{\mu\nu}$ as we also have a term proportional to $p_{\mu}p_{\nu}$ that will be added later for simplicity. Introducing the Feynman parameter x and changing the integration variable such that $k_{\mu} \rightarrow k_{\mu} + p_{\mu}(1-x)$, we get :

$$i\Pi_{\mu\nu} = 4ie^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{2k_\mu k_\nu - g_{\mu\nu} \left[k^2 - x(1-x)p^2 - m^2\right]}{\left(k^2 + p^2 x(1-x) - m^2 + i\tilde{\epsilon}\right)^2} \,. \tag{38}$$

Considering now that instead of 4 dimensions, the calculation is held in arbitrary d dimensions and using the formula $k_{\mu}k_{\nu} = \frac{1}{d}k^2g_{\mu\nu}$, we obtain :

$$i\Pi_{\mu\nu} = 4ie^2 \mu^{4-d} g_{\mu\nu} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{2k_\mu k_\nu - g_{\mu\nu} \left[k^2 - x(1-x)p^2 - m^2\right]}{\left(k^2 + p^2 x(1-x) - m^2 + i\tilde{\epsilon}\right)^2} , \qquad (39)$$

where μ is the **dimensionful energy scale** factor with mass dimension 1 allowing to keep the same dimension of the diagram calculation for any d as will be explained in the case of extra-dimensional theory in equation (139). Now, taking $\Delta = m^2 - p^2 x (1 - x)$ and using formulas in Appendix A, we get :

$$i\Pi_{\mu\nu} = -8p^2 g_{\mu\nu} \frac{e^2}{(2\pi)^{\frac{d}{2}}} \mu^{4-d} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 \mathrm{d}x x (1-x) \left(\frac{1}{m^2 - p^2 x (1-x)}\right)^{2-\frac{a}{2}} .$$
(40)

Finally, by replacing $d = 4 - \epsilon$ and adding the missing $p_{\mu}p_{\nu}$ term, we obtain :

$$i\Pi_{\mu\nu} = -\frac{e^2}{2\pi^2} \left(p^2 g_{\mu\nu} - p_{\mu} p_{\nu} \right) \int_0^1 \mathrm{d}x x (1-x) \left[\frac{2}{\epsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2 x (1-x)}\right) + \mathcal{O}(\epsilon) \right].$$
(41)

Multiple information can be extracted from this result. First, we see that the infinity seen before comes now from the term $2/\epsilon$ when ϵ is pushed to 0. Then, the tensor can be separated from the infinite part by considering the following definition :

$$i\Pi_{\mu\nu} = i\left(-p^2 g_{\mu\nu} + p_{\mu} p_{\nu}\right) e^2 \Pi\left(p^2\right) , \qquad (42)$$

where :

$$\Pi\left(p^{2}\right) = \frac{1}{2\pi^{2}} \int_{0}^{1} \mathrm{d}xx \left(1-x\right) \left[\frac{2}{\epsilon} + \ln\left(\frac{\tilde{\mu}^{2}}{m^{2}-p^{2}x(1-x)}\right)\right].$$
(43)

Now that we have seen the calculation, we can use the vacuum polarisation diagram result to see how renormalisation works to define a quantity. The diagram we considered participate to the correction to the photon propagator such that if at tree level, we have :

$$iG_{\mu\nu}^{\text{tree}} = \cdots = \frac{-i\left(g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}\right)}{p^2 + i\tilde{\epsilon}}, \qquad (44)$$

then, considering the value of the propagator at 1-loop would give :

Then, we can clearly see the correction to the tree level propagator. Unfortunately, this correction is still infinite, which is non-sense as perturbation theory should take into account small quantities in comparison with the free theory. In order to see how to get rid of this inconsistency, we need to introduce the corrected Fourier transformed Coulomb potential $\tilde{V}(p)$ defined through :

$$iG_{\mu\nu}^{\text{one-loop}} = \frac{-i\left(g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}\right)}{e^2}\tilde{V}(p) .$$

$$\tag{46}$$

 \tilde{V} is such that at tree-level we obtain the classical Fourier transform Coulomb potential and at one-loop, we find :

$$\tilde{V}(p) = e^2 \frac{1 - e^2 \Pi(p^2)}{p^2} .$$
(47)

In order to renormalise the theory, we need to redefine the coupling e by setting a condition at a reference scale. As a general case, by taking a reference momentum p_0 , we can define the renormalised coupling as :

$$e_R^2 \equiv \frac{\tilde{V}(p_0^2)}{p_0^2} \,. \tag{48}$$

Replacing in equation (47), we obtain :

$$e_R^2 = e^2 - e^4 \Pi(p_0^2) .$$
⁽⁴⁹⁾

Solving this equation for the bare coupling e as a function of e_R up to order e_R^4 gives :

$$e^{2} = e_{R}^{2} + e_{R}^{4} \Pi(p_{0}^{2}) + \mathcal{O}\left(e_{R}^{6}\right) .$$
(50)

 $\Pi(p_0^2)$ is infinite, so e is infinite as well, but because e is a Lagrangian parameter and is not an observable, we don't need it to be finite and physical, while e_R should be as it represents the macroscopic electron charge we measure in experiment. Taking now the definition of the Coulomb potential in (47) and replacing the equation (50) in it would give up to order e_R^4 :

$$p^{2}\tilde{V}(p^{2}) = e_{R}^{2} - e_{R}^{4} \left[\Pi(p^{2}) - \Pi(p_{0}^{2}) \right] + \mathcal{O}\left(e_{R}^{4}\right) \,. \tag{51}$$

In order to reproduce the macroscopic electric charge, let's take the momentum $p_0 = 0$ corresponding to a length scale of $r_0 = +\infty$. Replacing Π by the expression obtained in equation (43), we then have :

$$\Pi(p^2) - \Pi(0) = -\frac{1}{2\pi^2} \int_0^1 \mathrm{d}x x (1-x) \ln\left(1 - \frac{p^2}{m^2} x (1-x)\right) \,. \tag{52}$$

First, we can see that the infinity stemming from the term $1/\epsilon$ disappeared and so ϵ can be put to 0, keeping the integral in equation (52) finite and giving us therefore a finite prediction for the Coulomb potential :

$$\tilde{V}(p^2) = \frac{e_R^2}{p^2} \left[1 + \frac{e_R^2}{2\pi^2} \int_0^1 \mathrm{d}x x (1-x) \ln\left(1 - \frac{p^2}{m^2} x (1-x)\right) + \mathcal{O}(e_R^4) \right].$$
(53)

We then have a finite perturbative expansion for the Coulomb potential and for the renormalised coupling e_R which is finite as well. This computation shows that the measured electric charge is e_R and is clearly different from the bare coupling constant defined in the Lagrangian of the theory e due to equation (49).

Now that we have a finite expression for the Coulomb potential, we will see how it can give predictions and how it was proven to work on the specific example of Lamb shift. Let's first take the small-momentum limit such that $|p^2| \ll m^2$, we then obtain for the integral in (52):

$$\int_0^1 \mathrm{d}x x (1-x) \ln\left(1 - \frac{p^2}{m^2} x (1-x)\right) \approx \int_0^1 \mathrm{d}x x (1-x) \left[-\frac{p^2}{m^2} x (1-x)\right] = -\frac{p^2}{20m^2} , \quad (54)$$

implying :

$$\tilde{V}(p^2) = \frac{e_R^2}{p^2} - \frac{e_R^4}{60\pi^2 m^2} + \dots .$$
(55)

By Fourier transform this result, we obtain :

$$V(r) = -\frac{e_R^2}{4\pi r} - \frac{e_R^4}{60\pi^2 m^2} \delta(r) .$$
 (56)

We then obtain the classical Coulomb potential corresponding to the free theory and a small correction known as the Uehling term. This correction factor contributes to the degeneracy of the $2S_{1/2}$ and $2P_{1/2}$ levels in the energy spectrum of the hydrogen atom due to the fact that only the L = 0 atomic orbitals have a non-zero value at r = 0. This creates a degeneracy between the two levels, because in the tree-level description of the hydrogen atom, their energy depends only on the value of the quantum number n. The energy difference between the two levels is known as the Lamb shift measured in 1947 by Wallis Lamb [70] and the Uehling term reduces the $2S_{1/2}$ frequency by 27 MHz which is a small but measurable contribution to the-total Lamb shift of 1057 MHz. This effect and the reliability with theory was one of the major points that convinced people of the accuracy of quantum field theory and renormalisation.

The other limit is to consider, now, large momenta such that $m^2 \ll |p^2|$ giving :

$$\tilde{V}(p^{2}) = \frac{e_{R}^{2}}{p^{2}} + \frac{e_{R}^{4}}{2\pi^{2}p^{2}} \int_{0}^{1} \mathrm{d}xx(1-x)\ln\left(1-\frac{p^{2}}{m^{2}}x(1-x)\right) + \mathcal{O}(e_{R}^{6})$$

$$\approx \frac{e_{R}^{2}}{p^{2}} + \frac{e_{R}^{4}}{2\pi^{2}p^{2}}\ln\left(-\frac{p^{2}}{m^{2}}\right) \int_{0}^{1} \mathrm{d}xx(1-x) + \mathcal{O}(e_{R}^{6})$$

$$= \frac{e_{R}^{2}}{p^{2}}\left(1 + \frac{e_{R}^{2}}{12\pi^{2}}\ln\left(-\frac{p^{2}}{m^{2}}\right)\right) + \mathcal{O}(e_{R}^{6}) .$$
(57)

In this specific example, $-p^2$ is positive, so the logarithm is real. This expression can then be rewritten in term of an effective coupling constant such that :

$$\tilde{V}(p^2) = \frac{e_{\rm eff}^2(p^2)}{p^2} , \qquad (58)$$

with :

$$e_{\text{eff}}^2(p^2) = e_R^2 \left(1 + \frac{e_R^2}{12\pi^2} \ln\left(\frac{-p^2}{m^2}\right) \right) \,, \tag{59}$$

where for simplicity, we took the renormalisation condition $e_R \equiv e_{\text{eff}}(m)$.

Then, we see that for any scale p, the corrected potential looks like a Coulomb potential with a charge $e_{\text{eff}}(p^2)$ instead of e_R and that this effective coupling evolves with the momentum. The higher the $|p^2|$ is (or the smaller distances are) the higher the effective coupling becomes. One of the interpretation of this evolution of the QED coupling constant with respect to the energy (or the distance) can be explained as due to the screening of the charge by virtual particles pairs constantly created in the vacuum. Therefore, the closer we are to an

electron, the more you see it "naked" from its shell of charged virtual particles which influences the value of e_{eff} that can be measured at this scale. This interpretation means that the "real" charge of the electron $e = e_{\text{eff}} \left(p^2 = \infty \right)$ is non-accessible. The charge observed at low energy scale can be obtained only by calculating its effective value taking into account the screening of other particles.

In order to show the scale dependence, it is interesting to see what happen by merging several one-loop diagrams together and summing the different diagrams such that :

Then, the effective coupling constant can be written :

$$e_{\rm eff}^2(p^2) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln\left(\frac{-p^2}{m^2}\right)},$$
(61)

which is known as a running coupling due to its dependence on the parameter p^2 . The last equation can also be rewritten as :

$$\frac{1}{e_{\rm eff}^2(p^2)} = \frac{1}{e_R^2} - \frac{1}{12\pi^2} \ln\left(\frac{-p^2}{m^2}\right) \,. \tag{62}$$

One of the interesting feature about the Renormalisation Group is that the result obtained in equation (62) would have been identical by changing the renormalisation condition for any other reference scale μ , instead of the reference scale m we were using here, such that :

$$\frac{1}{e_{\rm eff}^2(p^2)} = \frac{1}{e_{\rm eff}^2(\mu^2)} - \frac{1}{12\pi^2} \ln\left(\frac{-p^2}{\mu^2}\right).$$
(63)

Knowing any scale value allows to predict the whole evolution of the parameter with energy. This property is one of the major advantages of the Renormalisation Group in studying the asymptotic behavior of a theory from a single value. Returning to equation (62), the left-hand side is independent from μ , we can derive both sides of the equation with respect to μ , giving :

$$0 = -\frac{2}{e_{\text{eff}}^3} \frac{de_{\text{eff}}}{d\mu} + \frac{1}{12\pi^2} \frac{2}{\mu} \,. \tag{64}$$

Reorganising it, we finally obtain :

$$\beta\left(e_{\text{eff}}\right) = \mu \frac{\mathrm{d}e_{\text{eff}}}{\mathrm{d}\mu} = \frac{e_{\text{eff}}^3}{12\pi^2} , \qquad (65)$$

which is the first example of what we will call a **Renormalisation Group equation** (RGE). The function β (e_{eff}) shown in the right hand-side is what is called a **beta-function** and will have an important role throughout the rest of the thesis as it is a scale-invariant function that does not depend on the energy scale μ directly.

The β -function can always be extracted from the calculation of the relevant diagrams contributing to the the perturbation series for the parameter we want to renormalise. In order to simplify the method to extract β , we will explain how to use the formalism of counterterms. As a basic example, we take the QED Lagrangian that can be written :

$$\mathcal{L}_{QED}^{0} = -\frac{1}{4} \left(\partial_{\mu} A_{\nu}^{0} - \partial_{\nu} A_{\mu}^{0} \right)^{2} + \overline{\psi}^{0} \left(i \partial \!\!\!/ - e_{0} A^{0} - m_{0} \right) \psi^{0} , \qquad (66)$$

where the superscript 0 refers to the bare quantities (parameters and fields) while the superscript R will be used for renormalised ones. Renormalised quantities can be define thanks to renormalisation factors Z through the following relations :

$$\psi^0 = \sqrt{Z_2} \psi^R , \quad A^0_\mu = \sqrt{Z_3} A^R_\mu , \quad m_0 = Z_m m_R , \quad e_0 = Z_e e_R .$$
 (67)

Indeed, the renormalisation factors Z should be equal to 1 at the tree-level in order to reproduce the Feynman rules with bare parameters. Going further in the perturbation orders, it is thus natural to consider perturbation terms to Z such that :

$$Z = 1 + \delta , \tag{68}$$

where δ is called a counterterm. Thanks to this definition, the QED Lagrangian can be rewritten showing the contribution from the counterterms :

$$\mathcal{L}_{QED} = \mathcal{L}_{QED}^{0} - \frac{\delta_3}{4} \left(\partial_{\mu} A_{\nu}^R - \partial_{\nu} A_{\mu}^R \right)^2 + \delta_2 \overline{\psi}^R \left(i \partial \!\!\!/ - \delta_m m_R \right) \psi^R - \delta_1 e_R \overline{\psi}^R A^R \psi^R , \qquad (69)$$

where $\delta_1 = \delta_e \delta_2 \sqrt{\delta_3}$. Therefore, counterterms can be taken into account by adding new Feynman rules associated to their Lagrangian terms giving the following rules :

$$\longrightarrow \otimes \longrightarrow = \delta_3 \frac{-i\left(g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}\right)}{k^2 + i\tilde{\epsilon}} ,$$

$$\longrightarrow = \frac{i\left(\delta_2 \not p - \left(\delta_m + \delta_2\right)m_R\right)}{p^2 + i\tilde{\epsilon}} ,$$

$$(70)$$

$$\longrightarrow = -i\delta_1 e_R \gamma_\mu .$$

These counterterms are undefined unless we specify renormalisation conditions thanks to what the infinite contributions from loop diagrams and a finite part depending on the subtraction scheme will be absorbed by the counterterms. Deducing from the conditions the expression for each δ , we are now able to compute the Renormalisation Group equations. One just have to express the bare parameter in terms of the renormalised one. Then, deriving both sides of the equation with respect to the dimensionful energy scale μ , we obtain the RGE. For example, if we want the RGE for the mass of the fermion, we have :

$$m_0 = Z_m m_R ,$$

$$\Rightarrow \mu \frac{\mathrm{d}m_0}{\mathrm{d}\mu} = 0 = \mu \frac{\mathrm{d}m_R}{\mathrm{d}\mu} + \mu \frac{\mathrm{d}\delta_m}{\mathrm{d}\mu} ,$$

$$\Rightarrow \beta_m = \mu \frac{\mathrm{d}m_R}{\mathrm{d}\mu} = -\mu \frac{\mathrm{d}\delta_m}{\mathrm{d}\mu} .$$
(71)

It is now straightforward to derive the Renormalisation Group equation by replacing the value of the counterterm calculated thanks to the renormalisation condition and the substraction scheme.

2.3 Renormalisation Group and beta function

2.3.1 Renormalisation Group Equations

One of the major differential equation stemming from the Renormalisation Group is the Callan-Symanzik equation [18, 19]. To show how to derive it in a simple example, let's study a theory called the ϕ^4 – theory built with a massive scalar and a four-leg interaction given by the following Lagrangian :

$$\mathcal{L}_{\phi^4} = -\frac{1}{2}\phi\left(\Box + m^2\right)\phi - \frac{1}{4!}\lambda\phi^4 .$$
(72)

In the massless version of this theory (m = 0), shifting the energy scale μ yields the following transformations for the Lagrangian quantities and the *n*-leg Green's functions $G^{(n)}$ via renormalisation :

$$\mu \to \mu + \delta \mu \qquad , \quad \lambda \to \lambda + \delta \lambda ,$$

$$\phi = Z\phi \approx (1 - \delta_1) \phi \qquad , \quad G_0^{(n)} = Z^n G^{(n)} \approx (1 - n\delta_1) G^{(n)} .$$
(73)

Since the bare Green's function does not depend on the energy scale, deriving it with respect to μ gives the following relation between the couplings shifts :

$$0 = \frac{\mathrm{d}G_0^{(n)}}{\mathrm{d}\mu} = \frac{\mathrm{d}Z^n G^{(n)}}{\mathrm{d}\mu} = -n\frac{\partial\delta_1}{\partial\mu}G^{(n)} + \frac{\partial G^{(n)}}{\partial\mu} + \frac{\partial G^{(n)}}{\partial\lambda}\frac{\partial\lambda}{\partial\mu} \,. \tag{74}$$

Multiplying each side of the equation by μ and defining the quantities :

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu}, \quad \gamma(\lambda) = -\mu \frac{\partial \delta_1}{\partial \mu},$$
(75)

yields the Callan-Symanzik equation :

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\left(\lambda\right)\frac{\partial}{\partial\lambda} + n\gamma\left(\lambda\right)\right)G^{(n)} = 0, \qquad (76)$$

where β is the beta-function (we already saw an example in equation (65)) and γ is called the anomalous dimension showing the scale dependence of the theory. In a scale-invariant description, space-time dilation such that $x \to Cx$ rescale any operator by the value C^{γ_0} , where γ_0 is the scaling dimension of the operator. This is not true anymore in scale-dependent schemes, where the rescaling is achieved instead using $C^{\gamma_0+\gamma(\lambda)}$ showing the scale-dependence of the operator clearly through the anomalous dimension. Equation (75) also shows that β and γ are dimensionless because λ and δ_1 are as well. Therefore, the only scale in the theory being μ , they do not depend on it and are thus scale-invariant functions depending only on λ . β and γ can be seen respectively as showing the energy-scale dependence of the coupling λ and respectively of the renormalised field. Therefore, equation (75) allows to see in a general expression what are the dependence of any renormalised quantity on the energy-scale.

The Callan-Symanzik equation was one of the first historical differential equation derived in the light of the RG. It was found later that it can be closely related to a growing bacteria experiments [71, 1]. To illustrate this analogy, let's take a unidimensional pipe containing a fluid whose velocity is v(x) inhabited by bacteria cells whose growth rate $\rho(x)$ is dependent on the space coordinate, for example because of more light or more food deposed at some specific points. The bacteria density is given for any time t by D(x, t) and is solution to the following differential equation :

$$\left(\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} - \rho(x)\right)D(x,t) = 0.$$
(77)

Equations (75) and (77) can be related by considering the following replacements :

$$\ln\left(\frac{\mu}{\mu_0}\right) \leftrightarrow \frac{t}{t_0} , \quad \lambda \leftrightarrow x , \quad \beta(\lambda) \leftrightarrow v(x) ,$$

$$n\gamma(\lambda) \leftrightarrow -\rho(x) , \quad G^{(n)}(\lambda,\mu) \leftrightarrow D(x,t) .$$
(78)

Through these relations, we can see that the evolution of the parameters of the theory with respect to the energy scale can be understood as a moving fluid element of bacteria whose growth rate influence its value. Indeed, we will see in section 2.3.3 that the Renormalisation Group equations can be visualised using flows diagrams just like we could do with hydro-dynamical experiments and that a similar analogy was performed to relate QFT with epidemiology in section 4.2.1.

As we saw before, the independence of any observable \mathcal{O} on both the dimensionful energyscale μ (or the cut-off regulator Λ or the dimensional regularisation parameter ϵ) implies that it exists a scale-invariant function $\beta_{\mathcal{A}}$ for any Lagrangian parameter \mathcal{A} associated with the calculation of \mathcal{O} such that :

$$\mu \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}\mu} = \frac{\mathrm{d}\mathcal{A}}{\mathrm{dln}(\mu)} = \beta_{\mathcal{A}}(\mathcal{A}) \ . \tag{79}$$

The most important property here is to see that β depends only on \mathcal{A} and not on the energy scale μ . Therefore, for any Lagrangian parameter in the theory a differential equation exists at any perturbation order. The beta function is fundamental in QFT as it allows to see the evolution of the parameters depending on the energy scale. This is of prime importance, for example, when we want to predict the value of the coupling constant depending on the energy of the center of mass in a collider experiment like in the LHC. Most of the time, this

 β -function will be a polynomial in the parameter \mathcal{A} .

Some famous examples of the β -function are shown hereafter. For the one-loop factor α_e in Quantum Electro Dynamics (QED), defined by $\alpha_e = \frac{e_{\text{eff}}^2}{4\pi}$ at first order, the β -function is :

$$2\pi\beta_{\alpha_e} = 2\pi \frac{\mathrm{d}\alpha_e}{\mathrm{d}\ln(\mu)} = \frac{4}{3}\alpha_e^2 \,. \tag{80}$$

For the one-loop factor α_s in Quantum Chromo-Dynamics (QCD) at first order :

$$2\pi\beta_{\alpha_s} = 2\pi \frac{\mathrm{d}\alpha_s}{\mathrm{dln}(\mu)} = \left(\frac{1}{3}n_q - \frac{11}{2}\right)\alpha_s^2 \,, \tag{81}$$

where n_q corresponds to the number of quarks accessible at the scale considered. Solving these equations for $\alpha_e(m_Z) = 0.0073$ and $\alpha_s(m_Z) = 0.1179$ [67], we obtain the running as shown in Figure 2.



FIGURE 2 – Running of the QED and QCD one-loop factors at first order.

Figure 2 shows that the two running couplings have really different behaviors. First, looking at the QED one-loop factor, we can see that it goes from 0 in the low-energy part (called InfraRed or IR part) to infinity when approaching high energies (Ultra-Violet or UV part) next to the value $t \approx 630$ corresponding to $\mu = 10^{286}$ eV. This scale is clearly not accessible in any experiments and is, in fact, even higher than the Planck mass. This infinity value reached for a finite energy scale is called a **Landau pole** and tends to say that the theory would not be consistent after this scale. This is a hint to see that QED is not the correct model to study processes at very high energy.

Looking now at the QCD one-loop factor, we can see an opposed trend. While there exists a Landau pole in the IR now, the coupling is going to 0 in the UV. The latter behaviour is what we call **asymptotic freedom**. This explains why high-energy processes are less coupled in QCD and that the theory is perturbative for high energies. This is no longer the case when going to the IR as the one-loop factor becomes big. For values of α_s higher than one, we consider that the theory is no longer perturbative. Such condition is obtained at a typical scale corresponding to the mass of the lightest hadron : the π^0 . This accounts for the impossibility to use QCD below the hadronisation scale, where quarks and gluons confine into bigger structures called hadrons and can no longer be considered as separated objects but studied instead in hadronic models.

2.3.2 Standard Model Renormalisation Group equations

Now that we saw some examples of the Renormalisation Group equations, we will see how to derive them for the Standard Model couplings. In order to do that, one needs first to compute the RGE for a general non-Abelian gauge theory, also called Yang-Mills theory. As we saw before, to find the RGE, we need to calculate the contributions of all the diagrams involved up to a defined perturbation order. In the most general theory on a group SU(N)the Lagrangian can be written :

$$\mathcal{L}_{YM}^{0} = -\frac{1}{4} \left(F_{\mu\nu}^{a0} \right)^{2} - \frac{1}{2\xi} \left(\partial_{\mu} A_{\mu}^{a0} \right)^{2} + \overline{\psi}_{i}^{0} \left(i D \!\!\!/ - m_{0} \right) \psi_{i}^{0} + \left| D_{\mu} \phi^{0} \right|^{2} - M_{0}^{2} \left| \phi^{0} \right|^{2} .$$
(82)

To renormalise the gauge coupling g at one loop, we need to consider the following diagrams :

$$\mathcal{M}_{1}: \mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{2}: \mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{3}: \mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{4}: \mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{5}: \bullet \bullet \bullet \bullet$$

$$\mathcal{M}_{6}: \mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{7}: \mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{7}: \mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}$$

$$\mathcal{M}_{7}: \mathfrak{u}\mathfrak{u}\mathfrak{u}\mathfrak{u}$$

For simplicity, we wish to consider the renormalised Lagrangian, in the Feynman gauge $\xi = 1$ but as we saw before, this doesn't change the final result. Using the renormalisation factors Z, related with the counterterms by the relation $Z = 1 + \delta$, the Lagrangian can be written :

$$\mathcal{L}_{\rm YM}^{R} = \mathcal{L}_{\rm YM}^{0} - \frac{1}{4} \delta_3 \left(\partial_{\mu} A_{\nu}^{aR} - \partial_{\nu} A_{\mu}^{aR} \right)^2 + \delta_2 \overline{\psi}_i^{R} \left(i \not\!\!\!\!\partial - \delta_m m_R \right) \psi_i^{R} + \delta_1 g_R A_{\mu}^{aR} \overline{\psi}_i^{R} \gamma^{\mu} T_{ij}^{a} \psi_j^{R} .$$

$$\tag{84}$$

Then, using the one-loop diagrams listed in equation (83), we can extract the values of the counterterms in dimensional regularisation $(d = 4 - \epsilon)$ that will be useful to define the Renormalisation Group equation :

$$\delta_{1} = \frac{1}{\epsilon} \left(\frac{g^{2}}{16\pi^{2}} \right) \left[-2C_{F} - 2C_{2} \left(G \right) \right] ,$$

$$\delta_{2} = \frac{1}{\epsilon} \left(\frac{g^{2}}{16\pi^{2}} \right) \left[-2C_{F} \right] ,$$

$$\delta_{3} = \frac{1}{\epsilon} \left(\frac{g^{2}}{16\pi^{2}} \right) \left[\frac{10}{3}C_{2} \left(G \right) - \frac{8}{3}n_{f}T \left(R_{f} \right) - \frac{2}{3}n_{s}T \left(R_{s} \right) \right] ,$$
(85)

where we consider n_f fermions in the representation R_f of SU(N), n_s scalars in representation R_s and the factors C(G), C_F and T(R) are the group factors defined in Appendix A.

Now, we need to specify the relation between the bare charge g_0 and the renormalised charge g_R . In dimensional regularisation, we then have, for the gauge coupling Lagrangian term :

$$\mu^{\frac{\epsilon}{2}}g_{R}Z_{1}A^{aR}_{\mu}\overline{\psi}^{R}_{i}\gamma^{\mu}T^{a}_{ij}\psi^{R}_{j} = \mu^{\frac{\epsilon}{2}}g_{R}\frac{Z_{1}}{Z_{2}\sqrt{Z_{3}}}A^{a0}_{\mu}\overline{\psi}^{0}_{i}\gamma^{\mu}T^{a}_{ij}\psi^{0}_{j} = g_{0}A^{a}_{\mu0}\overline{\psi}^{0}_{i}\gamma^{\mu}T^{a}_{ij}\psi^{0}_{j} , \qquad (86)$$

implying :

$$g_{0} = g_{R} \frac{Z_{1}}{Z_{2}\sqrt{Z_{3}}} \mu^{\frac{\epsilon}{2}} ,$$

$$\Rightarrow \mu \frac{\mathrm{d}g_{0}}{\mathrm{d}\mu} = 0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \left[g_{R} \frac{Z_{1}}{Z_{2}\sqrt{Z_{3}}} \mu^{\frac{\epsilon}{2}} \right] ,$$

$$\Rightarrow \beta \left(g_{R} \right) = \mu \frac{\mathrm{d}g_{R}}{\mathrm{d}\mu} = g_{R} \left[-\frac{\epsilon}{2} - \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \left(\delta_{1} - \delta_{2} - \frac{1}{2} \delta_{3} \right) \right] .$$
(87)

Since the counterterms δ depend on μ only through g_R , we find :

$$\beta(g_R) = -\frac{\epsilon}{2}g_R - g_R\mu \frac{\mathrm{d}g_R}{\mathrm{d}\mu} \frac{\mathrm{d}}{\mathrm{d}g_R} \left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right) = -\frac{\epsilon}{2}g_R + \frac{\epsilon}{2}g_R^2 \frac{\mathrm{d}}{\mathrm{d}g_R} \left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right). \tag{88}$$

Replacing the counterterms by their values in equation (85) and taking the limit $\epsilon \to 0$, we finally obtain :

$$\beta(g_R) = -\frac{g_R^3}{16\pi^2} \left[\frac{11}{3} C_2(G) - \frac{1}{3} n_s T(R_s) - \frac{4}{3} n_f T(R_f) \right].$$
(89)

For the Standard Model subgroups, we can derive these coefficients by considering that fermions and scalars are only in the fundamental representation of SU(N) groups and gauge vectors are always in the adjoint representation. Therefore, we have $T(R_f) = T(R_s) = \frac{1}{2}$ and $C_2(G) = N$.

For SU(3), the Higgs is not color-charged while 6 fermions are colored, the quarks. Thus, $n_s = 0$ and $n_f = 6$ which gives :

$$-\left(\frac{11}{3}C_2(G) - \frac{1}{3}n_sT(R_s) - \frac{4}{3}n_fT(R_f)\right) = -7.$$
(90)

For SU(2), the Higgs field is a weak-doublet and there are 6 left-handed fermion doublets. Then, $n_s = 1$ and $n_f = 6$:

$$-\left(\frac{11}{3}C_2(G) - \frac{1}{3}n_sT(R_s) - \frac{4}{3}n_fT(R_f)\right) = -\frac{19}{6}.$$
(91)

The two coefficients derived here allow to derive the beta function expressions for the Standard Model subgroups gauge coupling constants. Unfortunately, the hypercharge U(1) group coefficient is defined up to a normalisation factor that depends on the group they can

be embedded in. For example, in an SU(5) Grand Unified Theory, that will be seen in section 3.1.1, the coefficient for U(1) is 41/10. Therefore, the Renormalisation Group equations for the Standard Model couplings in an SU(5) Grand Unified theory are :

$$16\pi^{2}\beta_{1}(g_{1}) = \frac{41}{10}g_{1}^{3}, \quad 16\pi^{2}\beta_{2}(g_{2}) = -\frac{19}{6}g_{2}^{3}, \quad 16\pi^{2}\beta_{3}(g_{2}) = -7g_{3}^{3}.$$
(92)

We can immediately see that the one-loop gauge beta-functions in the SM depends on g only as g^3 , meaning that other couplings will not influence the running. This is no longer the case by going to higher perturbation order and for the Yukawa sector.

Considering now the heavy fermions Yukawa couplings (i.e top, bottom and tau) in the Standard model, by a similar calculation, we obtain at one-loop :

$$16\pi^{2}\beta_{t}(y_{t}) = \left[\frac{9}{2}y_{t}^{2} + \frac{3}{2}y_{b}^{2} + y_{\tau}^{2} - \frac{17}{20}g_{1}^{2} - \frac{9}{4}g_{2}^{2} - 8g_{3}^{2}\right]y_{t}, \qquad (93)$$

$$16\pi^{2}\beta_{b}(y_{b}) = \left[\frac{9}{2}y_{b}^{2} + \frac{3}{2}y_{t}^{2} + y_{\tau}^{2} - \frac{1}{4}g_{1}^{2} - \frac{9}{4}g_{2}^{2} - 8g_{3}^{2}\right]y_{b}, \qquad (94)$$

$$16\pi^{2}\beta_{\tau}(y_{\tau}) = \left[\frac{5}{2}y_{\tau}^{2} + 3y_{t}^{2} + 3y_{b}^{2} - \frac{9}{4}(g_{1}^{2} + g_{2}^{2})\right]y_{\tau}.$$
(95)

This shows that, even at one-loop, Yukawa running coupling constants depend on the other couplings unlike the gauge beta-functions. It is then mandatory to study first the running of the gauge couplings then using the solutions to derive the Yukawa running couplings.

2.3.3 Fixed points

One can remark that for some values \mathcal{A}^* the β -function vanishes such that

$$\beta_{\mathcal{A}}\left(\mathcal{A}^{*}\right) = 0 \tag{97}$$

Those values are called fixed-points and are the sign of a scale invariant behavior of the model. A theory featuring an UV fixed-point may not be an effective field theory and could thus well explain the physics up to any scale.

As a trivial example of a fixed point, taking all parameters to be equal to 0, solutions to the RGE are all constants and are thus equal to 0. Such solution is called a **Gaussian fixed point**, because it represents the Lagrangian of a free theory with a massless field and therefore the path integral of the action is a Gaussian. Solving the equations (96) for positive initial conditions, we can see from the sign of the coefficient that g_2 and g_3 will run to 0 in the UV showing that the Gaussian fixed point is UV-attractive. On the other hand, g_1 will be driven to infinity because of the UV-repulsiveness of the Gaussian fixed point leading to a Landau pole.

Indeed, in the one-loop SM RGE, only the Gaussian fixed point exists, but other models show non-trivial fixed points (i.e with at least one non-zero fixed point). We will see that theories showing a beta-function for a parameter \mathcal{A} of the type :

$$\beta_{\mathcal{A}}\left(\mathcal{A}\right) = -b_0\mathcal{A} + b_1\mathcal{A}^2 \tag{98}$$

presents interesting features concerning their fixed point. As a simple example, let's study the ϕ^4 -theory Lagrangian as considered in equation (72). The RGE for the coupling λ and the dimensionless mass $\tilde{m}_R(\mu) \equiv \frac{1}{\mu} m_R(\mu)$ are given at second order in dimensional regularisation by :

$$\beta_{\lambda} (\lambda) = -\epsilon \lambda + \frac{3}{16\pi^{2}} \lambda^{2} = \frac{3}{16\pi^{2}} \lambda \left(\lambda - \frac{16\pi^{2}\epsilon}{3} \right) ,$$

$$\beta_{\tilde{m}^{2}} \left(\tilde{m}^{2} \right) = \left(-2 + \frac{\lambda}{16\pi^{2}} \right) \tilde{m}^{2} .$$
(99)

The equations admit the standard Gaussian fixed point but also a non-trivial fixed point for :

$$\lambda_1^* = \frac{16\pi^2 \epsilon}{3} , \quad \tilde{m}^{2*} = 0 .$$
 (100)

Such non-trivial fixed point is called a Wilson-Fisher (WF) fixed point and can be obtained by going to extra-dimensions such that $\epsilon \neq 0$. This gives an interesting behavior for the running of the parameters λ and m^2 depending on their initial conditions as can be seen in Figure 3 for d = 5. An RG flow plot is shown there where the arrows indicate the running the parameters would have if their initial conditions was along this line, following increasing μ . This figure can be understood through the hydrodynamical bacteria growth analogy performed in equation (77) where the increasing μ can be seen as going forward in time. Each line represents the trajectory of a fluid element due to the current.

We can see that for any initial value $\lambda(t_0) < \lambda_1^*$, the parameters will run to the Gaussian fixed points. On the other-hand, for $\lambda(t_0) > \lambda_1^*$, the system will be driven to infinity hitting a Landau pole. Therefore, we call the WF fixed point UV-repulsive (or IR-attractive), while the gaussian fixed point is UV-attractive (or IR-repulsive).

This attractiveness can be studied by looking directly at the evolution of the beta-function with respect to the associated parameter. By looking at the beta-function in the case of the ϕ^4 -theory as shown in the bottom-left plot of Figure 3, we can see that there are three regions. The first green region for λ below 0 shows positive values for the β -function. This positiveness drives the system up toward the Gaussian fixed point. In the red part, β becomes negative, running down the system again to the Gaussian fixed point. Finally, in the second green part above the Wilson-Fischer fixed point, the β -function is again positive, but this time, the system will be driven to infinity. This shows the UV-attractiveness of the Gaussian fixed point and the UV-repulsiveness of the WF fixed point.

Therefore, one can look at the derivative of the β -function at the fixed points. These values θ are called scaling exponents and represent the strength of the attractiveness of the fixed point. If θ is negative (positive) at the fixed point, the latter is UV-attractive (UV-repulsive). For example, in the case of the ϕ^4 -theory, we have :

$$\frac{\partial \beta_{\lambda}(\lambda)}{\partial \lambda} = -\epsilon + \frac{6}{16\pi^{2}}\lambda , \qquad (101)$$

which gives for the fixed points :

$$\theta_0 = \frac{\partial \beta_\lambda(0)}{\partial \lambda} = -\epsilon , \quad \theta_{\lambda_1^*} = \frac{\partial \beta_\lambda(\lambda_1^*)}{\partial \lambda} = \epsilon .$$
 (102)



FIGURE 3 – Top plot : Renormalisation Group flow for the ϕ^4 – theory in dimension 5 (i.e $\epsilon = 1$) in the space of parameters (\tilde{m}^2, λ) . The two fixed points are represented by a dot, where the gaussian is in green and the Wilson-Fischer in red. Bottom-right plot : β -function for the parameter λ . The red (green) part corresponds to the negative (positive) values of the β -function. Bottom-left plot : Running of the parameter λ for the condition $\lambda(0) = 1$.

Then, we see directly that in extra-dimensions (i.e $\epsilon > 0$), the derivative is negative for the Gaussian fixed point and positive for the WF fixed point. The trend is opposite when going to lower dimensions than 4 (i.e $\epsilon < 0$). In this case, the Gaussian fixed point becomes UV-repulsive and the WF one is UV-attractive.

Another example of non trivial fixed point can be seen in Banks-Zaks theories [72]. Such theory is obtained when, if going further in the perturbation theory, the Renormalisation Group equation for a parameter α can be written :

$$\beta_{\alpha}(\alpha) = -b_0 \alpha^3 + b_1 \alpha^5 + \mathcal{O}\left(\alpha^7\right) \,. \tag{103}$$

Such β -function exhibits multiple fixed points that are :

$$\alpha_0^* = 0, \quad (\alpha_{\pm}^*)^2 = \frac{b_0}{b_1}.$$
(104)
We see directly that the non trivial fixed points are reals only if the coefficients b_0 and b_1 are of the same sign. In the case where the non-trivial fixed points are real and smaller than 1 (i.e for which the perturbation theory holds), they are called Banks-Zaks (BZ) fixed points. Looking at the derivative of the beta function, we obtain the following scaling exponents at the fixed points :

$$\frac{\partial \beta \left(\alpha_{0}^{*}\right)}{\partial \alpha} = 0 , \quad \frac{\partial \beta \left(\alpha_{\pm}^{*}\right)}{\partial \alpha} = 2 \frac{b_{0}^{2}}{b_{1}} . \tag{105}$$

Therefore, the UV-attractiveness of the non-trivial fixed points depends only on the sign of the coefficient b_1 , while for the Gaussian fixed point, this cannot be seen from the derivative. In Figure 4, the evolution of the β -function in the specific example of $b_0 = 1$ and $b_1 = 2$ is shown. For $b_1 > 0$ (and $b_0 > 0$), we can see that the non-trivial fixed points are UV-repulsive, while the Gaussian fixed point is UV-attractive. The attractiveness of each fixed points is opposite by taking $b_1 < 0$ (and $b_0 < 0$).



FIGURE 4 $-\beta$ -function of the BZ theory for $b_0 = 1$ and $b_1 = 2$.

One of the most important example of BZ theory is obtained for a general non-abelian gauge theory in $SU(N_c)$ with N_f massless charged fermions. At the second-order in perturbation theory, the gauge coupling RGE coefficients are :

$$b_0 = \frac{1}{16\pi^2} \frac{1}{3} \left(11N_c - 2N_f \right) , \quad b_1 = -\frac{1}{\left(16\pi^2\right)^2} \left(\frac{34}{3}N_c^2 - \frac{1}{2}N_f \left(2\frac{N_c^2 - 1}{N_c} + \frac{20}{3}N_c \right) \right) . \quad (106)$$

If one wants to reproduce the asymptotic freedom of the QCD and avoid Landau poles, the theory needs to be driven from an IR-attractive fixed point to an UV-attractive gaussian fixed point, therefore requires $b_1 > 0$ and $b_0 > 0$. The two conditions gives boundaries on N_f and N_c such that :

$$\frac{34N_c^3}{12N_c^2 - 3} < N_f < \frac{11}{2}N_c .$$
(107)

In the specific case of massless fermions in QCD, for which $N_c = 3$, we should then have $8.7 < N_f < 16.5$. As $N_f = 6$, this condition is not fulfilled, then the theory hits a Landau pole in the IR.

Such behavior shows that the parameter can simply run from one fixed point to another and this will be of prime importance in our applications of the Renormalisation Group equations in the next parts.

The last type of fixed points needed to be introduce for the following parts are the complex fixed points. In a general technicolor $SU(N_c)$ gauge theory [73, 74], the β -function at two-loops can be written :

$$\beta(\alpha) = -b\alpha^2 - c\alpha^3 - d\alpha^4 + \dots, \qquad (108)$$

where the coefficients b, c and d depend again on N_c the number of technicolors and N_f the number of fermions sensible to the technicolors. For some values of N_f and N_c next to a critical condition, the β -function can be rewritten :

$$\beta(\alpha) = -\alpha^2 \left((\alpha - 1)^2 - \delta \right) , \qquad (109)$$

where $\delta = 1 - b$. If δ is positive, equation (109) shows that it exists two non-trivial fixed points that are :

$$\alpha_{\pm}^* = 1 \pm \sqrt{\delta} . \tag{110}$$

One can remark also that if $\delta = 0$, the two non-trivial merge into a unique fixed point for $\alpha^* = 1$. Finally, if $\delta < 0$, the equation (109) can be rewritten :

$$\beta(\alpha) = -\alpha^2 \left(\alpha - 1 + i\sqrt{|\delta|}\right) \left(\alpha - 1 - i\sqrt{|\delta|}\right) , \qquad (111)$$

where we can see that there are, now, no real non-trivial fixed points anymore. However, it exists two complex values that are :

$$\alpha_{\pm}^* = 1 \pm i\sqrt{\delta} \ . \tag{112}$$

The complex fixed points are part of the walking dynamics introduced in [75, 76]. The term "walking" is related to the behavior of the solution of the RGE next to the real part of the fixed points. Figure 5 presents the solution of equation (109) for different choices of δ and for an initial condition $\alpha(0) = 2$.



FIGURE 5 – Solutions of equation (109) for different values of δ where $t = \ln(\mu/\mu_0)$.

We can directly see that, when $\alpha = 0$, the system cannot go through the real non-trivial fixed point $\alpha^* = 1$ to reach the Gaussian one. However, in the case where δ is negative, we can see that the complex fixed point cannot be reached because of the function α being always real. In the case $\delta = -0.01$, we can see that the system is sensible to the complex fixed points because they slow the running next to their real part making them "walk" instead. At the real part, we have :

$$\beta \left(\operatorname{Re} \left(\alpha_{+}^{*} \right) \right) = \delta \implies \alpha \left(t \right) \approx \delta t , \qquad (113)$$

whose value is clearly reduced in comparison to values away from the fixed point. After being slowed down, the system will start running again being driven toward the Gaussian fixed point. However, the system is sensible to the complex fixed points only if the value is smaller than a critical value. Equation (109) shows that a local minimum for β is reached $\alpha = \frac{1}{4}(3 - \sqrt{1 + 8\delta})$ and a local maximum for $\alpha = \frac{1}{4}(3 + \sqrt{1 + 8\delta})$. Therefore, these maximas can be obtained only if $\delta > -1/8 = \delta_c$ and the complex fixed points effects are visible only in this case. The cases at the limit and below it can be seen in Figure 5 for $\delta = -1/8 = 0.125$ and $\delta = -0.3$. There, the system is driven toward the Gaussian fixed without being slowed down much by the complex fixed points.

Walking dynamics allows technicolor models to have power-scaling properties [77], to develop scale-invariant behavior [78] and suppressed flavor-changing neutral currents [79]. Complex fixed points will be studied in more details in section 4.2.4 following their use in the epidemiological Renormalisation Group framework.

Now that we have seen different types of RGE and their associated fixed points, we will see how it can be used to improve Grand Unification theories through a new paradigm.

3 Grand Unification theories

3.1 Classical Grand Unification

3.1.1 SU(5) model description

Unification consists in considering that separated symmetries can be embedded into an unique group describing an unique coupling constant such that the three subgroups responsible for strong, weak and electromagnetic interactions weren't separated in the first moments of the Universe.

As a first example of an unification, the electroweak symmetry breaking proposed by Glashow, Weinberg and Salam [3, 4, 5] was aiming at explaining the masses of the W^{\pm} and Z^{0} gauge vectors and to take into account the interactions between the leptons and their masses within the $SU(2) \times U(1)$ gauge group as explained in section 2.1.3. This mechanism allows to embed different leptons within multiplets of SU(2). Therefore, the electroweak model could be though as a unification of the weak and electromagnetic forces but also of the left-handed fermions. However, the non-simple gauge group $SU(2) \times U(1)$ still describe two independents coupling constants while a true unification should involve only one.

To achieve a true unification, Georgi and Glashow [80] first tried to integrate the EW group in a larger group having only one coupling constant. Alas, none of the possible groups (SU(3), $SU(3) \times SU(3)$ or SU(6) involves the fractional charges of the quarks. Therefore, any true unification should also embed the QCD group SU(3) such that the larger group \mathcal{G} contains the whole SM symmetry $SU(3) \times SU(2) \times U(1)$. In such model, leptons and quarks must lie together in the same irreducible representations of the unified group. This imply that some of the gauge fields carry both lepton and baryon number. As \mathcal{G} includes the unbroken group $SU(3) \times U(1)$, it needs to be at least of rank 4 and only seven rank 4 Lie groups involving only one coupling and containing SU(3) exist. As the fermion content of the Standard Model is a complex representation whose complex conjugate is not equivalent, the group \mathcal{G} should have the same property. Among all the groups, only $SU(3) \times SU(3)$ and SU(5) are of this type. Moreover, $SU(3) \times SU(3)$ is excluded according to the non-existence of fractional charges for the quarks as was explained before for the electroweak unification. In the end, only SU(5) can be considered as a minimal group containing the Standard Model group and having a unique coupling constant. Indeed, we focused here only on the rank 4 groups, but larger groups could be considered as well and will be discussed later.

The SU(5) model is the minimal **Grand Unification Theory** (GUT) possible. One can write the representations of the unified group as a sum of the representations of the subgroups. For example, the anti-fundamental $\overline{\mathbf{5}}$ representation of SU(5) transforms under the subgroup $SU(3) \times SU(2) \times U(1)$ like :

$$\overline{\mathbf{5}} = \left(\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3}\right) \oplus \left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right) \,. \tag{114}$$

Therefore, this decomposition can represent the right handed down-type antiquark d_R^c and the left-handed lepton doublet l_L . In the same way, the antisymmetric **10** representation transforms like :

$$\mathbf{10} = \left(\mathbf{3}, \mathbf{2}, \frac{1}{6}\right) \oplus \left(\overline{\mathbf{3}}, \mathbf{1}, \frac{2}{3}\right) \oplus \left(\mathbf{1}, \mathbf{1}, 1\right) , \qquad (115)$$

thus including the colored left handed quark doublet q_L , the right handed up-type antiquark u_R and the right handed antilepton singlet e_R^c . In addition, the singlet representation **1** transforming as :

$$\mathbf{1} = (\mathbf{1}, \mathbf{1}, 0) , \qquad (116)$$

can also be added to the model, representing the right handed antineutrino ν_R^c . Specifying the SU(5) representations in terms of the SM fields, one can write :

$$\psi_{1} = (\nu) , \quad \psi_{\overline{5}} = \begin{pmatrix} d_{1}^{c} \\ d_{2}^{c} \\ d_{3}^{c} \\ e^{-} \\ -\nu_{e} \end{pmatrix}, \quad \psi_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_{3}^{c} & -u_{2}^{c} & -u_{1} & -d_{1} \\ -u_{3}^{c} & 0 & u_{1}^{c} & -u_{2} & -d_{2} \\ u_{2}^{c} & -u_{1}^{c} & 0 & -u_{3} & -d_{3} \\ u_{1} & u_{2} & u_{3} & 0 & -e^{c} \\ d_{1} & d_{2} & d_{3} & e^{c} & 0 \end{pmatrix}, \quad (117)$$

where the quarks subscripts refers to their QCD SU(3) charge. Therefore, it is possible to embed the entire SM fermion content by creating a copy of a 10, a $\overline{5}$ and a 1 representations of SU(5) for each fermion family. This integration leads to what is called a fermion unification.

The Higgs boson can also be embedded in a fundamental **5** representation giving rise to a new colored multiplet H and the standard Higgs doublet ϕ such that :

$$\phi_{5} = \begin{pmatrix} H_{1} \\ H_{2} \\ H_{3} \\ \phi_{+} \\ -\phi_{0} \end{pmatrix} .$$
(118)

Going to the gauge sector, the bosons can be integrated within the 24 adjoint representation of SU(5) that transforms under the SM group as :

24 = (**8**, **1**, 0)
$$\oplus$$
 (**1**, **3**, 0) \oplus (**1**, **1**, 0) \oplus (**3**, **2**, $-\frac{5}{6}$) \oplus ($\overline{\mathbf{3}}$, **2**, $\frac{5}{6}$), (119)

representing the eight generators of SU(3) (i.e the gluons g^a), the three generators of SU(2) W^a , the generator B of U(1) and two new colored off-diagonal doublets X and Y. In an adapted generator basis, the **24** representation matrix can then be written :

$$A_{24} = \frac{1}{\sqrt{2}} \begin{pmatrix} G_{11} - \frac{2B}{\sqrt{30}} & G_{12} & G_{13} & X_1^c & Y_1^c \\ G_{21} & G_{22} - \frac{2B}{\sqrt{30}} & G_{23} & X_2^c & Y_2^c \\ G_{31} & G_{32} & G_{33} - \frac{2B}{\sqrt{30}} & X_3^c & Y_3^c \\ X_1 & X_2 & X_3 & \frac{W_3}{\sqrt{2}} + \frac{2B}{\sqrt{30}} & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W_3}{\sqrt{2}} + \frac{2B}{\sqrt{30}} \end{pmatrix}.$$
(120)

This decomposition includes the whole gauge content of the Standard Model but also includes the off-diagonal bosons X and Y. These fields are called "lepto-quarks" due to the

fact that they carry both lepton and baryon number and violate the conservation of both of them. This characteristic lead to the proton decay prediction that will constrain a lot the GUT models.

The first phenomenological implication of the SU(5) Grand Unification is that it could explain the baryon asymmetry observed in the Universe. Due to the abundance of the matter over the antimatter in our actual Universe, an excess should have been produced just after the Big Bang. As pointed by Sakharov [81], a matter excess is possible only if the baryon number B is not conserved, if both C and CP symmetries are violated, and if the process happens during a phase of non-equilibrium. In the case of the SU(5) model, we already saw that B is not conserved due to lepto-quarks. Moreover, at the high energy density of the early Universe, the masses of the lepto-quark X and Y were accessible. Then, when the Universe cooled down to temperatures lower than the lepto-quark masses, their only fate was to decay into pure matter via processes such as $X \to uu$ or $Y \to ud$. The excess of matter created should be small, because if most of the matter annihilated with antimatter, this leaves an enormous quantity of photons in comparison to matter as is observed experimentally :

$$\frac{N_{\gamma}}{N_b} \approx 10^{10} . \tag{121}$$

Another important implication of the Georgi-Glashow model is the relation between the charges of the quarks and the leptons. As the charge operator Q is constructed thanks to the U(1) generator and because all generators must have a vanishing trace, we have for the $\overline{5}$ representation :

$$\operatorname{Tr} (Q\psi_{\overline{5}}) = 3Q(d^{c}) + Q(e^{-}) + Q(\nu_{e}) = 0 ,$$

$$\Rightarrow Q(d) = -Q(d^{c}) = \frac{1}{3}Q(e^{-}) = -\frac{1}{3}e .$$
(122)

In the same way for the **10** representation :

Tr
$$(Q\psi_{10}) = 3Q(u) + 3Q(d) + Q(e^{-}) = 0$$
,
 $\Rightarrow Q(u) = \frac{2}{3}e$. (123)

These results imply that the charge of the quarks must be 2/3 and -1/3 of that of the electron, quantizing the charges of all fermion particles through this relation.

Unfortunately, Grand Unification also imply unwanted phenomenological features. In the classical SM group, isolated protons cannot decay due to the conservation of the baryon number and to the fact that they are the lightest hadron existing [82]. The β^+ radiation converting the proton in a neutron by emitting a positron and a neutrino is not a proton decay as it emerges from the interaction with the nucleus and therefore the proton is not isolated in this case. In the Georgi-Glashow SU(5) GUT, the baryon number is no longer conserved and processes involving a proton decay appear. This would be a sign that matter is unstable and more precisely that the hydrogen atom should decay on a sufficiently long time. At tree-level, the simplest Feynman diagram involving a proton decay consists in the annihilation of a left-handed and a right-handed up quarks, yielding a X^+ boson, decaying then into a positron and a left-handed anti-down quark of opposite chiralities following the process :

$$u_L + u_R \to X^+ \to e_{L/R}^+ + \overline{d}_{R/L} . \qquad (124)$$

Considering that the anti-down quark and the remaining down quark from the proton confine together to form a neutral pion π^0 , the overall process of a proton decay can be written :

$$p \to e^+ + \pi^0 \,. \tag{125}$$

Other decaying processes are allowed for the proton, but none as strong as the $e^+\pi^0$ branch. By decaying through the lepto-quark X^+ , the lifetime of the proton can be derived using the naive formula :

$$\tau_p \approx \frac{M_X^4}{\alpha_5^2 m_p^5} \,, \tag{126}$$

where M_X is the lepto-quark mass, $\alpha_5 = \frac{g_5^2}{4\pi}$ is the one-loop factor in SU(5) and m_p is the proton mass. However, the proton decay was never observed experimentally, driving the lower limit of the proton lifetime to $\tau_p > 10^{34} - 10^{35}$ years [83]. Considering that the preferred value for the one-loop factor is $\alpha_5 \approx 1/24$ as will be seen later, we finally find that :

$$M_X > 10^{16} \text{ GeV}$$
 (127)

Thus, the proton decay phenomenology drives the lepto-quark mass to very high energies, which constrains significantly the model. The experimental bounds make the simplest SU(5) GUT models, including the one described here, to be completely excluded [84].

In addition to this, the masses of the X and Y bosons induces big contributions from gauge loops diagram to radiative corrections to the Higgs mass. This leads to corrections of the order $\Delta m_h^2 = \mathcal{O}(M_X^2)$ that are of multiple orders of magnitude higher than the actual Higgs mass, which is unlikely to happen in a perturbation theory. Therefore, these corrections would need to be compensate by "fine-tuning" relations between them. This problem can be solved considering different additional features to the GUT model among which supersymmetry that will be seen later.

3.1.2 Running couplings and unification interpretation

We saw previously in section 2.3.2 that the RGE for the three couplings in the SM can be written :

$$16\pi^2 \frac{\mathrm{d}g_i}{\mathrm{d}t} = b_i g_i^3 , \qquad (128)$$

where *i* can take values from 1 to 3, $b_i = (41/10, -19/6, -7)$ and $t = \ln(\mu/m_Z)$ where μ is the scale of energy density and m_Z is the Z boson mass.

Taking as initial conditions for the coupling constants the measured values at the electroweak scale : $g_i(t = 0) = (0.45, 0.66, 1.2)$ [67], the runnings of the couplings constants show an interesting behavior. It can be seen in Figure 6 that they become closer as t increases and that they seem to cross at specific energy scales. These crossings are identified, in GUT models, as the unification scales.

However, one can remark that the three couplings don't all cross at the same point and that the two couplings of the electroweak group g_2 and g_1 meet before they reach the QCD coupling g_3 . This is in contradiction with the possibility to create a unified group for the



FIGURE 6 – Evolution of the SM couplings constants g_3 (full line), g_2 (small dashed line) and g_1 (dashed line) solutions of the RGE as function of the energy density.

 $SU(2) \times U(1)$ electroweak group as discussed in subsection 3.1.1. The only possibility allowed is to unify the three couplings at the exact same scale, therefore to have the three running couplings to cross at a unique point, which is called **exact unification**. We could argue that going at higher orders in the perturbation theory, additional contributions could make the crossing points closer from each other and achieve an exact unification. Unfortunately, adding higher orders do not change the results, a single unification point is excluded by more than 8 standard deviations in the SM [85]. Theoretically, α_3 meets the crossing point of α_1 and α_3 for a starting value close to $\alpha_3(m_Z) = 0.07$, while the measured value is 0.119 ± 0.002. This means that unification can only be obtained if new physics enters between the electroweak and the Planck scale. This pushed forward the idea of supersymmetrical models.

Supersymmetry (SUSY) consists in introducing a symmetry transforming particles of spin $j \pm \frac{1}{2}$, adding therefore a complete copy of the SM particle content, transforming bosons into fermions and conversely. The new set of particles are called the superpartners of the SM particles or sparticles. It has been shown that the SUSY is the only additional symmetry of the S-matrix allowed by a weaker set of assumptions [82]. While SUSY particles have yet no experimental evidence, their appealing properties have rushed the scientific community to study it abundantly.

Among these interesting features, SUSY solves the fine-tuning problem discussed in section 3.1.1 due to the fact that loop corrections contain both fermions and bosons in the loops, which, according to the Feynman rules, contribute with an opposite sign giving the following correction term :

$$\Delta m_h^2 \approx \mathcal{O}\left(\alpha\right) \left| M_F^2 - M_B^2 \right| \approx \mathcal{O}\left(10^{-2}\right) M_{\text{SUSY}}^2 , \qquad (129)$$

where M_{SUSY} represents the typical mass scale at which sparticles start to appear. The masses of the superpartners would not be the same for each sparticle, but we have considered in first approximation that there is no mass splitting between them. This implies that the fine-tuning problem would vanish if the superpartners are not too heavy compared to their associated particles. Requiring that the radiative corrections are not much larger than the Higgs mass, this would imply that $M_{\rm SUSY} < 10^3$ GeV.

Therefore, at the SUSY scale (around 1 TeV), the superpartners would start contributing to the renormalisation of the gauge couplings and then have an effect on their running above this scale. The RGE in the Minimal SuperSymmetrical Model (MSSM) version for the gauge couplings of the SU(5) GUT are at one-loop [86]:

$$\frac{\mathrm{d}g_i}{\mathrm{d}t} = \begin{cases} b_i^{\mathrm{SM}} \frac{g_i^3}{16\pi^2} & \text{for } \mu < M_{\mathrm{SUSY}} \\ \\ b_i^{\mathrm{SUSY}} \frac{g_i^3}{16\pi^2} & \text{for } \mu > M_{\mathrm{SUSY}} \end{cases},$$
(130)

where we can see that the values of the coefficients b_i change if the energy scale is below or above the SUSY scale. Indeed, for $\mu > M_{\text{SUSY}}$, SUSY effects appear and they start contributing to the RGE. The coefficients b_i^{SM} and b_i^{SUSY} are defined as :

$$b^{\rm SM} = \begin{pmatrix} 0\\ -22/3\\ -11 \end{pmatrix} + N_g \begin{pmatrix} 4/3\\ 4/3\\ 4/3 \end{pmatrix} + N_h \begin{pmatrix} 1/10\\ 1/6\\ 0 \end{pmatrix} , \qquad (131)$$

$$b^{\text{SUSY}} = \begin{pmatrix} 0\\ -6\\ -9 \end{pmatrix} + N_g \begin{pmatrix} 2\\ 2\\ 2 \end{pmatrix} + N_h \begin{pmatrix} 3/10\\ 1/2\\ 0 \end{pmatrix}, \qquad (132)$$

where N_g is the number of fermion generation and N_h the number of Higgs doublets in the theory. In the case of three fermion generations and one Higgs doublet, we then have

$$b^{\text{SM}} = \begin{pmatrix} 41/10 \\ -19/6 \\ -7 \end{pmatrix}, \quad b^{\text{SUSY}} = \begin{pmatrix} 63/10 \\ 1/2 \\ -3 \end{pmatrix}.$$
 (133)

Thanks to the appearance of SUSY particles contribution in the RGE, an exact unification is now possible as can be seen in Figure 7. The kink visible for t = 3.3 corresponds to the SUSY scale where the parameters b_i changes. The value of this scale needs to be tuned in order to allow the exact unification. We can see that for a value of $M_{SUSY} = 10^{3.4}$ GeV, we can find almost the exact unification scale close to $M_{SU(5)} = 10^{17}$ GeV. Almost, because at one-loop, it is not possible to find the exact unification for any value of the SUSY scale. A more accurate calculation done at two-loops [87] (and even at three-loop [88]), including mixed terms between different gauge and Yukawa couplings, can find an exact unification and gives the following results :

$$M_{\rm SUSY} = 10^{3.4 \pm 0.9 \pm 0.4} \text{GeV} ,$$

$$M_{SU(5)} = 10^{15.8 \pm 0.3 \pm 0.1} \text{GeV} .$$
(134)

The first error originates from the uncertainty related to the coupling constants experimental values, while the second error is due to the uncertainty in the mass splittings between the SUSY particles.



FIGURE 7 – Evolution of the gauge couplings constants for the Minimal supersymmetric Model of SU(5) GUT. Gray lines represent the scale of SUSY and the scale of unification, where $M_{\rm SUSY} = 10^{3.4}$ GeV and $M_{SU(5)} \approx 10^{17}$ GeV.

We can see first that M_{SUSY} is close to the range limit we found for the fine-tuning problem. This result gave a new interest in supersymmetrical models in the scientific community. One of the other major point to advocate the SUSY importance is that exact unification can also be reached only for three generations of fermions or less, showing that it is not easy to obtain it. Nevertheless, exact unification does not prove supersymmetry, while sparticles observation should. However, for now, there are no experimental sign of SUSY. Supersymmetry introduces also another difficulty called the triplet-doublet splitting problem. Just like standard GUT, in SUSY GUT, the proton decay can be mediated via the colored Higgs triplet H. As we saw before for the lepto-quarks, this decay imposes that its mass has to be higher than 10^{16} GeV. Unlike lepto-quarks, the colored Higgs H is embedded in a fundamental multiplet 5 of SU(5)with the standard Higgs doublet ϕ_h whose mass remains close to the electroweak scale, so 14 orders lower than the H mass. This difference is then another problem of fine-tuning that can be related to the hierarchy problem asking why the ratio between GUT scale and ordinary physics scale is so big. Such problem is solved through interesting proposals extending the model to new physics [89, 90, 91, 92, 93], but in most SUSY models, this persists.

Nevertheless, the most important constraints concern the coupling constant unification combined with the proton lifetime lower limits and their agreement with experimental data. These requirements exclude the Standard Model [94], the Georgi-Glashow SU(5) GUT model, the minimal SU(5) SUSY GUT model and other more complicated models. Adding also the fact that lightest sparticles are still not observed in LHC data, all the possible models presented above are not realistic in their simplest definitions. Moreover, the triplet-doublet splitting problem is still not solved in MSSM GUT. One can argue that to save GUT models, we need to involve bigger groups of unification such as SO(10). Another possibility is to imagine a new unification paradigm in which the theory is free of both constraining conditions. That is the case of the asymptotic Grand Unification we will present in section 3.3, but before introducing it, we need to show the characteristics of extra-dimensional models.

3.2 Extra-dimensional theories

3.2.1 Extra-dimensions principle

Quantum Field Theory is mainly developed in four dimensions (three spatial and one temporal dimensions), allowing to describe our world as we perceive it in our daily experience. However, it is also possible to consider that the four dimensions we feel are just an effective description of a higher dimensional description that can only be seen at higher energies. The simplest way to understand why these extra-dimensions are invisible to us is to imagine one compactified extra-dimension. The term compactified means that the dimension is rolled on itself. A simple analogy to this is to imagine a paper sheet, which represents a two-dimensional space. This paper can be fold back on itself to give it the shape of a cylinder. The folded dimension is called compactified and we denote R, the radius of compactification, the radius of the circular section of the cylinder. If we were able to make R smaller and smaller until we reach a scale for which our eyes cannot distinguish it anymore (for example $R = 1 \mu m$), we could just see a small line, that we could interpret as a one-dimensional space. In that sense, the compactified dimension still exists but we cannot see it with our eyes. To be able to make it visible, we would need to have experiments reaching an energy high enough to see physical processes affected by the compactified dimension.

The compactification can be realised on different mathematical objects called orbifolds. This latter refers to topological spaces that are locally identified as Euclidian space and that can be modelled by a quotient of open subsets of \mathbb{R}^n over finite group actions. As a basic example, a circle, denoted S^1 , define a compactified unidimensional space where the boundary condition are periodical. This imply that for any field χ , we have $\chi (y = y_0 + R) = U\chi (y = y_0)$ where the matrix U is unitary (Unitary matrices are explained in Appendix A) and y refers to the fifth-dimensional space coordinate. While the S^1 orbifold is simple, it will not be sufficient to generate the chirality for the SM fields or to allows for a symmetry breaking from the unified group into the Standard Model group. This is why we need to introduce more complicated orbifolds.

The $S^1/(\mathbb{Z}_2 \times \mathbb{Z}'_2)$ orbifold can be defined by taking a simple S^1 orbifold of compactification radius R and adding two parity boundary conditions at two different points y_0 and y_1 as can be seen in Figure 8. The \mathbb{Z}_2 denominator acts as if we were to glue a copy (mirrored or not) of an arc of S^1 with itself at one of the boundary. Therefore, the two \mathbb{Z}_2 confer two copies with different eigenvalues glued at the two boundary points. The complete fifth-dimension can thus be fully characterised by only studying the interval $[y_0; y_1]$. The parities are such that fields obey to the following boundary conditions :

$$\chi (y = y_0 - y) = \mathcal{P}_0 \chi (y = y_0 + y) , \qquad (135)$$

$$\chi (y = y_1 - y) = \mathcal{P}_1 \chi (y = y_1 + y) , \qquad (136)$$

where \mathcal{P}_0 and \mathcal{P}_1 are called the parity operator. Their eigenvalues can only be +1 or -1 due to the fact that $\mathcal{P}^2 = I$. Therefore, in this orbifold, every field is defined by its two parity



eigenvalues at the boundary points that will be written (\pm_0, \pm_1) where the subscripts will be abandoned in the following parts for simplicity. By taking the two conditions in equation (136) at their specific points y_0 and y_1 , we can see that the parity eigenvalues define the boundary conditions on the fields or on their derivative values :

$$\chi (y = y_1) = \pm \chi (y = y_1) \implies \chi (y = y_1) = 0 \text{ (Dirichlet)} \quad \text{if } \pm = -,$$

$$\frac{\partial \chi}{\partial y}_{|y=y_1} = - \pm \frac{\partial \chi}{\partial y}_{|y=y_1} \implies \frac{\partial \chi}{\partial y}_{|y=y_1} = 0 \text{ (Neumann)} \quad \text{if } \pm = +.$$
(137)

We can see here that each eigenvalue can be identified as either Dirichlet or Neumann boundary conditions. These relations will be necessary to derive the expression of the Fourier modes as will be seen in section 3.2.2.

As in any dimension, the action should always be adimensional, it implies that the Lagrangian should be of dimension m^d , where d is the number of dimensions considered in the model. Therefore, the mass dimensions of the Lagrangian parameters and the field should differ in extra-dimensions. First, let's see what happens in 4D and especially for the mass dimensions in the SM. Starting from the Lagrangian of mass dimension 4 in equation (1), we see directly that :

$$\begin{bmatrix} \phi^{4D} \end{bmatrix} = 1 , \quad \begin{bmatrix} A^{4D} \end{bmatrix} = 1 , \quad \begin{bmatrix} \psi^{4D} \end{bmatrix} = \frac{3}{2} ,$$

$$\begin{bmatrix} \lambda^{4D} \end{bmatrix} = 0 , \quad \begin{bmatrix} \mu^{4D} \end{bmatrix} = 1 , \quad \begin{bmatrix} y^{4D} \end{bmatrix} = 0 , \quad \begin{bmatrix} g^{4D} \end{bmatrix} = 0 .$$

$$(138)$$

Considering now any d-dimensional theory, the mass dimensions become :

$$\begin{bmatrix} \phi^{ED} \end{bmatrix} = \frac{d-2}{2} , \quad \begin{bmatrix} A^{ED} \end{bmatrix} = \frac{d-2}{2} , \quad \begin{bmatrix} \psi^{ED} \end{bmatrix} = \frac{d-1}{2} ,$$

$$\begin{bmatrix} \lambda^{ED} \end{bmatrix} = 4 - d , \quad \begin{bmatrix} \mu^{ED} \end{bmatrix} = 1 , \quad \begin{bmatrix} y^{ED} \end{bmatrix} = \frac{4 - d}{2} , \quad \begin{bmatrix} g^{ED} \end{bmatrix} = \frac{4 - d}{2} .$$

$$(139)$$

The factor $\mu^{\frac{1}{2}}$ needed in dimensional regularisation in equation (86) is a consequence of the change in mass dimensions for the gauge couplings g due to the relation $d = 4 - \epsilon$. Other

changes are expected to happen in the mathematics behind the Lagrangian terms. For example in five-dimensions, the slashed operator is replaced by an extra-dimensional version such that :

$$p = p_{\mu} \gamma^{\mu} \to p_M \gamma^M , \qquad (140)$$

where M goes from 1 to d. One needs to be careful with the extra-dimensional indices here as the range 0 to 3 is often used for the four dimensions description but then the extra-dimensional indices start at 5, representing the fifth-dimension for clarity. Therefore, the fourth index is often not considered and we will use in the following for the indices the range 0, 1, 2, 3, 5, ..., d. γ^5 is the fifth-Dirac matrix which is defined as $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. The momentum is also extended with a fifth-component that is the projection of the total momentum on the fifth dimension space. Similarly, the derivative operator ∂_5 with respect to the fifth spatial coordinate is added in ∂_M .

In the same way, the gauge tensors are rewritten as :

$$F^a_{\mu\nu} \to F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + igf^{abc} A^b_M A^c_N , \qquad (141)$$

where again A_5 corresponds to the fifth component of the gauge vector. A_5 will play an important roles in Gauge-Higgs Unification theories that will be shown later in section 3.2.3.

3.2.2 Mathematical tools

Extra-dimensional theories are difficult to understand in a direct way and to simplify their interpretation, the best approach is often to make them look like four-dimensional ones. Mathematical tools have been developed to understand the features of extra-dimensional theories and will be presented hereafter.

Among all of them, Kaluza-Klein (KK) modes are the most commonly used in the case of a compactified extra-dimension. Theorised by Kaluza and Klein [46, 47] in their attempt to unify electromagnetism and gravity, KK modes can be seen as Fourier modes in the decomposition over the fifth component of any field and this has consequences on the particles mass spectrum. In an $S^1/(Z_2 \times Z'_2)$, the Fourier decomposition depends on the eigenvalues of the parity operators. Following the different eigenvalues, the fields can then be decomposed as :

$$(+,+): \hat{\chi} = \sqrt{\frac{2}{\pi R}} \left(\chi_0 \left(x_\mu \right) + \sqrt{2} \sum_{n=1}^{+\infty} \chi_n \left(x_\mu \right) \cos \left(2n \frac{y}{R} \right) \right) ,$$

$$(-,-): \hat{\chi} = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} \chi_n \left(x_\mu \right) \sin \left(2n \frac{y}{R} \right) ,$$

$$(-,+): \hat{\chi} = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} \chi_n \left(x_\mu \right) \sin \left((2n-1) \frac{y}{R} \right) ,$$

$$(+,-): \hat{\chi} = \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} \chi_n \left(x_\mu \right) \cos \left((2n-1) \frac{y}{R} \right) ,$$

$$(142)$$

where the x_{μ} are the four-dimensional Minkowski coordinates, y is the fifth dimension and n is what is called the KK number corresponding to the Fourier number. We can directly see

from the expansion that only (+, +) fields admit a non-vanishing n = 0 mode. Such modes are called zero-modes.

We want know to replace the expansions in equation (142) in the Lagrangian terms and integrate over the fifth-component to have a four-dimensional expression in order to see the consequences of the KK modes on the theory interpretation. Taking first the fifth component kinetic Lagrangian term for the fermion fields and remembering that fermions with opposed chiralities have opposed boundary conditions, we obtain :

$$\int_{0}^{\frac{\pi R}{2}} \mathrm{d}y \overline{\psi}_{(+,+)} \partial_5 \widehat{\psi}_{(-,-)} = \sum_{n=1}^{+\infty} \frac{2n}{R} \overline{\psi}_n(x_\mu) \psi_n(x^\mu) , \qquad (143)$$

$$\int_{0}^{\frac{\pi R}{2}} \mathrm{d}y \overline{\hat{\psi}}_{(+,-)} \partial_5 \hat{\psi}_{(-,+)} = \sum_{n=1}^{+\infty} \frac{2n-1}{R} \overline{\psi}_n(x_\mu) \psi_n(x^\mu) .$$
(144)

These results show that the 5-dimensional field kinetic term can be rewritten as a sum of massive 4D fields that are the KK modes. The lightest non-zero mode has a mass of the order 1/R. This explains why a small compactification radius, will push the KK modes to very high energy and therefore only zero-modes will be accessible at scale below the compactification scale. The same result is obtained for gauge and scalar fields in 5-dimensions.

KK modes also obey to selection rules at vertices. These rules can be extracted again from the Lagrangian interaction terms. For example, expanding the KK fields in the gauge-fermion Lagrangian term gives :

$$\int_{0}^{\frac{\pi R}{2}} \mathrm{d}y i g_{5} \hat{\psi}_{(+,+)}^{\dagger} \gamma^{\mu} \hat{A}_{\mu}^{a} T^{a} \hat{\psi}_{(+,+)} = i g_{5} \sqrt{\frac{2}{\pi R}} \psi_{0}^{\dagger} \gamma^{\mu} A_{\mu}^{0} \psi_{0}
+ i g_{5} \sqrt{\frac{2}{\pi R}} \sum_{n=1}^{+\infty} \left(\psi_{0}^{\dagger} \gamma^{\mu} A_{\mu}^{an} T^{a} \psi_{n} + \psi_{n}^{\dagger} \gamma^{\mu} A_{\mu}^{an} T^{a} \psi_{0} + \psi_{n}^{\dagger} \gamma^{\mu} A_{\mu}^{a0} T^{a} \psi_{n} \right)
+ i g_{5} \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} \sum_{k=1}^{+\infty} \psi_{n}^{\dagger} \gamma^{\mu} A_{\mu}^{am} T^{a} \psi_{k} \Delta_{3}^{c}(n, m, k) , \qquad (145)$$

where $\Delta_3^c(n, m, k) = \delta_{n, -m-k} + \delta_{n, m+k} + \delta_{m, n+k} + \delta_{k, m+n}$ is the selection rule for three non-zero KK modes. This can be interpreted as and energy conservation rule. This conservation imply that either n = m + k, either m = n + k, either k = n + m meaning that the energy associated with the masses of the three fields is conserved if we consider that the mass of the n^{th} -KK mode can be written n/R.

All these results show that having a compactified extra-dimension can be seen has adding an infinite tower of massive copies of Standard Model particles. This interpretation will simplify the calculations for extra-dimensional model predictions.

Another mathematical tool frequently used is the mixed-momentum space propagator. While KK modes were derived thanks to a fully momenta-description expression, we will now see propagators where the coordinates considered are split between momentum and space phase. Momentum-space propagators consider the four standard dimensions in the momentum space and the fifth in coordinate space. In this framework, a generic scalar propagator in the orbifold S^1 takes the form :

$$G_{S^1}^{5D}\left(\chi, y, y'\right) = \frac{\cos\left(\chi\left(\pi R - |y - y'|\right)\right)}{2\chi \sin\left(\chi \pi R\right)} \quad \text{where} \quad \chi = \sqrt{k^2 - m^2 + i\epsilon} \;, \tag{146}$$

where k is the 4-dimensional momentum, y is the fifth space coordinate and m is the 5D mass of the scalar field. In the orbifold $S^1/(Z_2 \times Z'_2)$, due to the identifications $y \iff -y$ and $y \iff \pi R - y$, we would have instead the following expression respecting the symmetries :

$$G(\chi, y, y') = G_{S^{1}}^{5D}(\chi, y - y') \pm_{0} G_{S^{1}}^{5D}(\chi, y + y') \pm_{1} G_{S^{1}}^{5D}(\chi, y + y' - \pi R) \pm_{0} \pm_{1} G_{S^{1}}^{5D}(\chi, y - y' + \pi R) ,$$
(147)

where \pm_0 and \pm_1 corresponds to the parity eigenvalues of the field at the boundary points $y_0 = 0$ and $y_1 = \pi R/2$. Mixed momentum-space propagators [95] have been used for radiative corrections in finite-temperature field theories [96].

One other possibility to look at 5D fields is to think about the connection between two points in a topological way. With a compactified extra-dimension, there are multiple paths to connect two points in space. By considering that the starting point and the ending point are identical, the connection can be established by turning around the compactified dimension or not. Such connections can be decomposed in a series over n_w the number of rotations around the compactified dimension. The modes in this decomposition are called winding modes and n_w is the winding number. Therefore, for any 5D space coordinate y, we have the identification :

$$y = y + 2n_w \pi R . \tag{148}$$

In order to derive the propagators decomposed in winding modes, let's take the simple example of a massless scalar field whose green-function is described in mixed momentumspace [97] such that :

$$\left(p^{2}-\partial_{y}^{2}\right)\tilde{G}\left(p,\left|y-y'\right|\right)=\delta\left(y-y'\right),$$
(149)

where $y, y' \in [0, 2\pi R]$. The Green's function solution of this equation is then :

$$\tilde{G}\left(p, |y-y'|\right) = \frac{e^{-p|y-y'|}}{p+i\tilde{\epsilon}} .$$
(150)

As the dimension is compactified, we can identify the coordinates y' and $y' + 2\pi n_w R$. Therefore, the real propagator should take into account any winding around the compactified dimension. By summing over the winding modes, we have the following full propagator :

$$\tilde{G}_{\text{Full}}\left(p, |y-y'|\right) = \sum_{n_w = -\infty}^{+\infty} \frac{e^{-p|y-y'+2n_w\pi R|}}{p+i\tilde{\epsilon}} \,. \tag{151}$$

In this decomposition, we see that for $n_w \neq 0$, the propagator is exponentially shrinked for high momentum, therefore loop integrals over momenta will be finite. This is no longer the case for $n_w = 0$ were an infinite contribution appears due to the condition y = y' and because the propagator behave as 1/p. Therefore, the winding modes technology allows to separate the infinite contribution, that is completely carried by the zero-mode, from the finite parts. This is due to the fact that zero-mode are the only modes taking into account small distances connections that are responsible for UV divergences. This is one of the main advantages of the winding modes, while in the Kaluza-Klein description, each mode carries a part of the infinities.

Now that we have seen some of the mathematical tools, we will use them to study the couplings evolution in extra-dimensional theories. As they are simpler to employ, KK modes will allow us to understand the extra-dimensional derivation of the RGE. The equations will depend on the coefficients derived at the SM scale, but once the compactification scale is reached, the first KK mode will start to contribute to the beta function as we have seen for SUSY in section 3.1.2. While the contributions in the SM are logarithmic, the KK contributions exhibit a power-law behavior as explained in [98, 99]. In the case where $\mu R \gg 1$, in good approximation, the calculation consists in computing the contribution from one KK state and then define the number of states contributing up to the scale μ . For one extra dimension, we saw in equations (143) and (144) that the masses of the KK modes contained in the tower are multiples of the scale R^{-1} . Therefore, up to μ the number of KK modes is $S(\mu) = \mu/R^{-1} = \mu R$. This contribution is taken into account by adding to the beta function, above the compactification scale $\mu = R^{-1}$, the term :

$$\beta_{SM} \to \beta_{SM} + (S(\mu) - 1) \beta_{KK} , \qquad (152)$$

where β_{KK} is the contribution from one KK mode. We have already seen that for one extradimension, we have $S(t) = \mu R$ above the compactification scale and S(t) = 1 below it. Therefore, S is continuous and consequently the β -function is as well. As argued before, at the compactification scale R^{-1} , we start to see the fifth dimension and its effects thanks to the first KK mode appearing. The higher we go in the energy scale, the larger the $S(\mu)$ term becomes as more KK modes are involved. Finally, the asymptotic behavior of the beta function will only give the term :

$$\beta \equiv S(\mu) \,\beta_{KK} \,. \tag{153}$$

Indeed, it seems that the beta function explodes as it is proportional to μ . Fortunately, as we saw in section 3.2.1, when going to higher extra-dimensions, renormalised couplings will acquire mass dimensions and will need to be redefined. They will often behaves as μ^{-1} in 5-dimensions and still give the possibility to be driven towards a non-trivial UV fixed point as we will see in the case of the asymptotic Grand Unification in section 3.3. Before looking at it, let's see one example of extra-dimensional model through the Gauge-Higgs Unification.

3.2.3 An example of an extra-dimensional model : Gauge-Higgs Unification

In extra-dimensional models, polarisations of the gauge vectors aligned with higher dimensions can be considered as independent scalar fields. If we identify one of these fields as being the Higgs boson, its interactions with the fermions would be related to the gauge couplings giving the opportunity to achieve a unification with the Yukawa sector. Such theories are called Gauge-Higgs Unification theories (GHU) and have been heavily studied [100, 101, 102]. We will present in this section the main features and implications of GHU on the simplest model [103].

In this model, we consider a five-dimensional theory where the extra-dimension is compactified on a S^1/Z_2 orbifold with a gauge group $\mathcal{G}_{GHU} = SU(3)$ containing the electroweak group. \mathcal{G}_{GHU} is broken to $SU(2) \times U(1)$ thanks to the Z_2 parity matrix defined as :

$$P = \begin{pmatrix} -1 & \\ & -1 \\ & & 1 \end{pmatrix} . \tag{154}$$

P allows us to decompose the generators of SU(3) into generators of $SU(2) \times U(1)$, where the *W*'s, the *B* and the additional off-diagonal fields can be defined as :

$$A_{M} = \begin{pmatrix} W_{M}^{3} & W_{M}^{1} + iW_{M}^{2} & H_{M}^{1} + iH_{M}^{2} \\ W_{M}^{1} - iW_{M}^{2} & -W_{M}^{3} & H_{M}^{3} + iH_{M}^{4} \\ H_{M}^{1} - iH_{M}^{2} & H_{M}^{3} - iH_{M}^{4} & B_{M} \end{pmatrix} .$$
(155)

After the $SU(3) \rightarrow SU(2) \times U(1)$ symmetry breaking, the extra-dimensional component of the 5D gauge fields, A_5 , appears as a scalar [104]. Thus, the off-diagonal components of the A_5 gauge vector, $H_5^1 + iH_5^2$ and $H_5^3 + iH_5^4$, form a charged scalar doublet sharing the same quantum numbers as the standard Higgs boson. Moreover, on a S^1/Z_2 orbifold, with the correct boundary conditions, the doublet radiatively generates a finite potential [105, 106] that could lead to the spontaneous symmetry breaking of the electroweak symmetry. Therefore, the zero-mode of this doublet could be identified to the Higgs boson.

These models are of interest because they could resolve the hierarchy problem by forbidding mass term corrections for the Higgs, due to its gauge invariance. Moreover, GHU models can create gauge-Yukawa asymptotic unification [107]. In the specific example of the $\mathcal{G}_{GHU} = SU(3)$ model, the Renormalisation Group equations for the gauge couplings are given by :

$$16\pi^2 \frac{\mathrm{d}g_i}{\mathrm{d}t} = b_i^{\mathrm{SM}} g_i^3 + (S(t) - 1) b_i^{\mathrm{GHU}} g_i^3 , \qquad (156)$$

where we recall that $t = \ln(\mu/m_Z)$, m_Z is the Z boson mass and S(t) = 1 if $\mu < 1/R$ and $S(t) = \mu R = m_Z R e^t$ above the compactification scale. The coefficients are defined such that :

$$b^{\rm SM} = \begin{pmatrix} 41/10 \\ -19/6 \\ -7 \end{pmatrix}, \quad b^{\rm GHU} = \begin{pmatrix} -17/6 \\ -17/2 \\ -17/2 \end{pmatrix}.$$
 (157)

In addition, the top-Yukawa can also be added to the theory, where its running is given by :

$$\frac{\mathrm{d}y_t}{\mathrm{d}t} = \left(c_t^{\mathrm{SM}} + (S(t) - 1)c_t^{\mathrm{GHU}}\right)y_t^3 + \left(d_i^{\mathrm{SM}} + (S(t) - 1)d_i^{\mathrm{GHU}}\right)y_tg_i^2,$$
(158)

where :

$$c_t^{\rm SM} = 9/2 , \quad c_t^{\rm GHU} = 21/2 , \quad d^{\rm SM} = \begin{pmatrix} -5/12 \\ -9/4 \\ -8 \end{pmatrix} , \quad d^{\rm GHU} = \begin{pmatrix} -35/24 \\ -39/8 \\ -4 \end{pmatrix} .$$
 (159)



FIGURE 9 – Running of the gauge couplings and top Yukawa in the SU(3) GHU model with $R^{-1} = 5$ TeV. The compactification scale is represented by the vertical gray line.

The top-Yukawa running is plotted in Figure 9. We can see here an example of asymptotic unification. Couplings are not unified by crossing at an unique point, but are instead attracted to the same fixed point in the UV. This model is not a Grand Unification Theory as the couplings are not embedded in an unique group but in $SU(3)_{\text{GHU}} \times SU(3)_c$. In order to have asymptotic Grand Unification, we will have to imagine a different framework.

3.3 Asymptotic Grand Unification

3.3.1 Model description

We propose in this part a new definition for a Grand Unified Theory, where unification is achieved by an asymptotic process instead of a crossing of the running coupling constants. In order to do that, we need to consider a model with one extra dimension. This extra-dimension is compactified on an $S^1/(Z_2 \times Z'_2)$ orbifold of radius R and with boundary points $y_0 = 0$ and $y_1 = \pi R/2$. Each one of the two boundary points is associated with the parity matrices P_0 and P_1 . In addition to the extra-dimension, we also consider that the Standard Model gauge group, $\mathcal{G}_{\text{SM}} = SU(3) \times SU(2) \times U(1)$, is contained within the unified group $\mathcal{G}_{\text{aGUT}}$. The parity matrices imply the following relations [108] for the gauge fields A_M^a :

$$(P_0) \Rightarrow \begin{cases} A^a_\mu(x, -y) = P_0 A^a_\mu(x, y) P^{\dagger}_0, \\ A^a_5(x, -y) = -P_0 A^a_5(x, y) P^{\dagger}_0, \end{cases}$$
(160)

$$(P_1) \Rightarrow \begin{cases} A^a_\mu(x, \pi R - y) = P_1 A^a_\mu(x, y) P_1^{\dagger}, \\ A^a_5(x, \pi R - y) = -P_1 A^a_5(x, y) P_1^{\dagger}, \end{cases}$$
(161)

where M is the 5D Lorentz index, a is the generator index of the \mathcal{G}_{aGUT} group and A_5 is the polarisation along the fifth compact dimension.

As a minimal model, we will consider first that the unifying group is SU(5). This group can break into \mathcal{G}_{SM} by considering a symmetry breaking caused by the parity matrices eigenvalues at the two boundary points. The symmetry breaking should occur at y = 0 to give the SM group. Therefore, one needs the following parity matrix expressions :

$$P_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$
(162)

According to the definition of the parity matrices, we see that the symmetry breaking occurs because the boundary conditions on the y_0 brane are not SU(5) invariant. This can be seen from the relations for the generators of SU(5):

$$P_0 T_{\rm SM}^a P_0^{-1} = T_{\rm SM}^a , \quad P_0 T_X^a P_0^{-1} = -T_X^a , \qquad (163)$$

where the $T_{\rm SM}^a$ are the $\mathcal{G}_{\rm SM}$ generators and the T_X^a are the off-diagonal generators. Therefore, the **24**-representation multiplet of SU(5) can be decomposed in the same way as in equation (120). We recall that parity-odd fields respect Dirichlet boundary conditions (i.e. the vanishing of the field at the boundary), while parity-even fields respect Neumann boundary conditions (i.e. the vanishing of the derivative of the field at the boundary). As discussed in section 3.2.2, fields in five dimensions can be decomposed into towers of KK modes, whose characteristics depend on the parities under the \mathbb{Z}_2 and \mathbb{Z}'_2 symmetries, which we denote (\pm, \pm) . Of the four combinations, only the (+, +) case features a zero mode, implying that it can be associated with a SM field. Only the unbroken vector gauge fields of $SU(3) \times SU(2) \times U(1)$ have double-Neumann boundary conditions, and can thus develop a zero-mode, in agreement with the fact that they are the only gauge fields visible at the SM scale.

The Higgs boson is embedded in a bulk scalar field ϕ_5 , transforming as a fundamental 5 of SU(5), with parities :

$$(P_0) \Rightarrow \qquad \phi_5(x, -y) = -P_0\phi_5(x, y), (P_1) \Rightarrow \qquad \phi_5(x, \pi R - y) = +P_1\phi_5(x, y).$$

$$(164)$$

As in standard SU(5) GUT models, the Higgs, ϕ_h , is accompanied by a QCD-triplet scalar H:

$$\phi_{\mathbf{5}} = \begin{pmatrix} H\\ \phi_h \end{pmatrix},\tag{165}$$

where H has no zero mode due to the Dirichlet boundary conditions on the y = 0 boundary. At this stage, the particle content of the multiplets is identical to the standard GUT SU(5) theory, except for the origin of the symmetry breaking.

Models of Grand Unification with one extra-dimension have already been studied. In these models unification was achieved by having an unique crossing point for the running coupling constants like standard GUT [109, 110, 111]. However, if the gauge symmetry is broken due to the parity boundary conditions, it is no longer possible to have the SM fermions embedded into bulk multiplets. This is due to the fact that the SM fermions should have a zero-mode, thus they must have the (+,+) boundary conditions. However, because of the P_0 operator,

the boundary conditions of the fields in the **5** and **10** multiplets of SU(5) cannot be identical. Therefore, multiplets cannot be filled as in equation (117), and bulk fermion unification cannot be achieved in extra-dimensional models. One way to resolve this is to localise the fields on a brane [109, 112, 113, 114] or to include a supersymmetrical sector [115]. However, in the most simple extra-dimensional models, the requirements to have the unification of the gauge couplings at a single point and to the embed of SM fermions in the same multiplets cannot be fulfilled at the same time. In the new model we will present here, we do not initially address this requirements, and try to understand the unification in a new way.

To start with, we will describe the fermion content of the model. For fermion fields with unspecified chirality, the parity eigenvalues notation will always be related to the left-handed version, remembering that opposite chiralities have opposite eigenvalues. The SM fermions will be incorporated within 5 different representations for the fermion multiplets : $\mathbf{1}$, $\mathbf{5}$, $\mathbf{\overline{5}}$, $\mathbf{10}$ and $\mathbf{\overline{10}}$, having the following parity conditions :

$$(P_{0}) \Rightarrow \begin{cases} \psi_{1}(x,-y) = -\gamma_{5}\psi_{1}(x,y), \\ \psi_{5}(x,-y) = +P_{0}\gamma_{5}\psi_{5}(x,y), \\ \psi_{5}(x,-y) = +P_{0}^{\dagger}\gamma_{5}\psi_{5}(x,y), \\ \psi_{10}(x,-y) = +P_{0}^{\dagger}\gamma_{5}\psi_{10}(x,y)P_{0}^{T}, \\ \psi_{\overline{10}}(x,-y) = +P_{0}^{\dagger}\gamma_{5}\psi_{\overline{10}}(x,y)P_{0}^{*}; \end{cases}$$
(166)
$$(P_{1}) \Rightarrow \begin{cases} \psi_{1}(x,\pi R - y) = -\gamma_{5}\psi_{1}(x,y), \\ \psi_{5}(x,\pi R - y) = +P_{1}\gamma_{5}\psi_{5}(x,y), \\ \psi_{5}(x,\pi R - y) = -P_{1}^{\dagger}\gamma_{5}\psi_{\overline{5}}(x,y), \\ \psi_{10}(x,\pi R - y) = -P_{1}\gamma_{5}\psi_{10}(x,y)P_{1}^{T}, \\ \psi_{\overline{10}}(x,\pi R - y) = +P_{1}^{\dagger}\gamma_{5}\psi_{\overline{10}}(x,y)P_{1}^{*}, \end{cases}$$
(167)

where γ_5 is the fifth-Dirac matrix in the diagonal Weyl representation.

These relations imply that fermion multiplets are built with fields having zero modes, corresponding to the SM fermions and new fields without zero modes, due to their parity eigenvalues and with masses of the order of the compactification scale. These components fields can be seen by writting the multiplets' content as :

$$\psi_{\mathbf{1}_{L/R}} = N_{L/R}, \quad \psi_{\mathbf{5}_{L/R}} = \begin{pmatrix} b \\ L^c \end{pmatrix}_{L/R}, \quad \psi_{\mathbf{\overline{5}}_{L/R}} = \begin{pmatrix} B^c \\ l \end{pmatrix}_{L/R},$$

$$\psi_{\mathbf{10}_{L/R}} = \frac{1}{\sqrt{2}} \begin{pmatrix} T^c & q \\ & \mathcal{T}^c \end{pmatrix}_{L/R}, \quad \psi_{\mathbf{\overline{10}}_{L/R}} = \frac{1}{\sqrt{2}} \begin{pmatrix} t & Q^c \\ & \tau \end{pmatrix}_{L/R},$$
(168)

where the superscript ^c indicates the 4D charge conjugate, and capitalised letters indicate the additional fields that do not have a zero mode. We will call these components "Indalo" fields and denoted them as $\widehat{\P}$ -fields. Indalo means "Creation" in the Zulu language and the symbol is also called "Indalo", but refers to a geometric figure found in Spain from prehistoric drawings. $\widehat{\P}$ -fields are written with the capital letter corresponding to the SM field with whom they share their SM quantum numbers. The charges of all the multiplets components are listed in Table 3. In the last two columns we indicate the presence of a zero mode and the mass of the lightest non-zero KK mode.

Field	$(\mathbb{Z}_2,\mathbb{Z}_2')$	SM	Zero mode	First KK mass
l	(+,+)	(1, 2, -1/2)	\checkmark	2/R
L	(+, -)	(-,-, -, -, -, -, -, -, -, -, -, -, -, -,	_	1/R
τ	(-, -)	$(1 \ 1 \ -1)$	\checkmark	2/R
$ \mathcal{T} $	(-, +)	(1, 1, -1)	—	1/R
N	(-,-)	(1,1,0)	\checkmark	2/R
\overline{q}	(+, +)	(3 2 1/6)	\checkmark	2/R
\overline{Q}	(+, -)	(0, 2, 1/0)	—	1/R
t	(-, -)	(9 1 9/2)	\checkmark	2/R
T	(-, +)	(3, 1, 2/3)	_	1/R
b	(-,-)	(3, 1, −1/3)	\checkmark	2/R
\overline{B}	(-, +)		_	1/R
ϕ_h	(+,+)	(1, 2, 1/2)	\checkmark	2/R
H	(-,+)	(3, 1, − 1/3)	—	1/R
B_{μ}	(+,+)	(1,1,0)	\checkmark	2/R
W^a_μ	(+,+)	(1,3 ,0)	\checkmark	2/R
G^b_μ	(+,+)	(8,1 ,0)	\checkmark	2/R
X_{μ}	(-,+)	(3 , 2 , −5/6)	—	1/R

TABLE 3 - Quantum numbers and parities of the fields. Fields with zero modes, appearing in the Standard Model, are within boxes, and for the fermions we indicate the parities of the left-handed chirality.

The lightest tier of KK modes, with mass $m_{\rm KK} = 1/R$, is populated by the fields without a zero mode. They constitute a complete copy of the SM fermion families, the QCD-charged Higgs H and the off-diagonal SU(5) vector X_{μ} . As will be seen later, all these fields play a special role in this model, thus we collectively name them Indalo also.

As seen earlier, due to the parity at the y = 0 boundary, it is not possible to embed all the SM fermions in the same bulk fermion multiplet. Therefore, we introduced five multiplets. This model naturally produces a "fake GUT" [116] structure in the fermion sector, where gauge couplings unify but fermions do not. An important consequence of this property is that the Yukawa couplings are not required to unify as we obtained in Gauge-Higgs Unification in section 3.2.3.

With the set of fields introduced above, we can write down the most general bulk Lagran-

gian as follows :

$$\mathcal{L}_{SU(5)} = -\frac{1}{4} F_{MN}^{(a)} F^{(a)MN} - \frac{1}{2\xi} (\partial_{\mu} A^{\mu} - \xi \partial_{5} A_{y})^{2} + i \overline{\psi_{5}} \mathcal{D} \psi_{5} + i \overline{\psi_{5}} \mathcal{D} \psi_{\overline{5}} + i \overline{\psi_{10}} \mathcal{D} \psi_{10} + i \overline{\psi_{\overline{10}}} \mathcal{D} \psi_{\overline{10}} + i \overline{\psi_{1}} \partial \psi_{1} - \left(\sqrt{2} Y_{\tau} \overline{\psi_{\overline{5}}} \psi_{\overline{10}} \phi_{5}^{*} + \sqrt{2} Y_{b} \overline{\psi_{5}} \psi_{10} \phi_{5}^{*} + \frac{1}{2} Y_{t} \epsilon_{5} \overline{\psi_{\overline{10}}} \psi_{10} \phi_{5} + Y_{\nu} \overline{\psi_{1}} \psi_{\overline{5}} \phi_{5} + \text{h.c.} \right) + \left| D_{M} \phi_{5} \right|^{2} - V(\phi_{5}),$$
(169)

where $D_M = \partial_M - ig^a T^a A^a_M$ with T^a being the SU(5) generators in the appropriate representation, and ϵ_5 is the 5-dimensional Levi-Civita tensor with gauge indices. The $V(\phi_5)$ term is a generic potential for the scalar field, which is responsible for generating the non-zero vacuum expectation value for the Higgs zero mode, involved in electroweak symmetry breaking. We also recall that gauge and Yukawa couplings, via dimensional analysis in 5D, have a scaling mass dimension of -1/2.

The normalisation of the Yukawa couplings is chosen to reproduce the SM definitions for the zero modes. Expanding the SU(5) multiplets with respect to their components, the Yukawa couplings yield the following terms :

$$\overline{\psi_{1}}\psi_{\overline{5}}\phi_{5} = \overline{N}\phi_{h}l + \overline{N}H\mathcal{B}^{c},$$

$$\sqrt{2} \ \overline{\psi_{5}}\psi_{10}\phi_{5}^{*} = \overline{b}\phi_{h}^{*}q - \overline{L}^{c}H^{*}q - \overline{L}^{c}\phi_{h}^{*}\mathcal{T}^{c} + \epsilon_{3}\overline{b}H^{*}T^{c},$$

$$\sqrt{2} \ \overline{\psi_{10}}\psi_{\overline{5}}\phi_{\overline{5}}^{*} = -\overline{\tau}\phi_{h}^{*}l - \overline{Q}^{c}H^{*}l + \overline{Q}^{c}\phi_{h}^{*}B^{c} - \epsilon_{3}\overline{t}H^{*}B^{c},$$

$$\frac{1}{2} \ \epsilon_{5}\overline{\psi_{\overline{10}}}\psi_{10}\phi_{5} = \overline{t}\phi_{h}q + \overline{t}H\mathcal{T}^{c} + \overline{\tau}HT^{c} + \overline{Q}^{c}\phi_{h}T^{c} + \epsilon_{3}\overline{Q}^{c}Hq,$$
(170)

where ϵ_3 is the Levi-Civita tensor which contracts with the QCD SU(3) indices. It can be seen from the above equations that each Yukawa coupling contains only one term between pure SM fields, corresponding to the standard Yukawa terms in the SM. Once normalised to dimensionless 4D couplings such that $y_f = Y_f / \sqrt{2\pi R}$, they correspond exactly to the SM Yukawa couplings. By extension, we can generate the full flavour structure of the SM by adding two more copies for each fermion generation. The only source of flavour violation source in this model is from the SM CKM matrix, and we thus avoid the strong flavour bounds [117, 118].

As seen in standard GUT models, the strong constraints usually arise from proton decay considerations, such as baryon violating couplings of the QCD-charged scalar H and of the lepto-quark bosons X and Y [27, 119]. However, the violation of baryon and lepton number in standard GUT models occurs because the SM fields are embedded in the same SU(5) multiplet, while in our model they are not by construction.

In order to show baryon and lepton number conservation, we will show how to extract their values for each field. Let us first assign to each field in equation (169) a U(1) charge, such that :

$$\begin{split} \psi_{\mathbf{1}} &\to e^{i\alpha_{1}}\psi_{\mathbf{1}}, \qquad \psi_{\mathbf{5}} \to e^{i\alpha_{5}}\psi_{\mathbf{5}}, \qquad \psi_{\overline{\mathbf{5}}} \to e^{i\alpha_{\overline{5}}}\psi_{\overline{\mathbf{5}}}, \\ \psi_{\mathbf{10}} &\to e^{i\alpha_{10}}\psi_{\mathbf{10}}, \qquad \psi_{\overline{\mathbf{10}}} \to e^{i\alpha_{\overline{10}}}\psi_{\overline{\mathbf{10}}}, \\ \phi_{\mathbf{5}} \to e^{i\alpha_{\phi}}\phi_{\mathbf{5}}, \qquad A_{X} \to e^{i\alpha_{X}}A_{X}. \end{split}$$
(171)

As all the kinetic terms involve quadratic contributions for each field, they automatically respect this global U(1) symmetry. Focusing on the Yukawa terms, it is possible to extract relations between the different charges defined here by imposing the invariance of the terms under this global transformation. Therefore, we obtain :

$$Y_{\tau}\overline{\psi_{10}}\psi_{\overline{5}}\phi_{\overline{5}}^{*} \rightarrow e^{-\alpha_{\phi}-\alpha_{\overline{10}}+\alpha_{\overline{5}}}Y_{\tau}\overline{\psi_{10}}\psi_{\overline{5}}\phi_{\overline{5}} \implies \alpha_{\overline{5}} = \alpha_{\overline{10}} + \alpha_{\phi} ,$$

$$Y_{b}\overline{\psi_{5}}\psi_{10}\phi_{\overline{5}}^{*} \rightarrow e^{-\alpha_{\phi}-\alpha_{5}+\alpha_{10}}Y_{b}\overline{\psi_{5}}\psi_{10}\phi_{\overline{5}} \implies \alpha_{10} = \alpha_{5} + \alpha_{\phi} ,$$

$$Y_{t}\overline{\psi_{\overline{10}}}\psi_{10}\phi_{\overline{5}} \rightarrow e^{\alpha_{\phi}-\alpha_{\overline{10}}+\alpha_{10}}Y_{t}\overline{\psi_{\overline{10}}}\psi_{10}\phi_{\overline{5}} \implies \alpha_{\overline{10}} = \alpha_{10} + \alpha_{\phi} ,$$

$$Y_{\nu}\overline{\psi_{1}}\psi_{\overline{5}}\phi_{\overline{5}} \rightarrow e^{\alpha_{\phi}-\alpha_{1}+\alpha_{5}}Y_{\nu}\overline{\psi_{1}}\psi_{\overline{5}}\phi_{\overline{5}} \implies \alpha_{1} = \alpha_{5} + \alpha_{\phi} .$$

$$(172)$$

The charges for the off-diagonal bosons can be constrained also with the covariant derivative terms in the Lagrangian, where :

$$-ig\overline{\psi_5}A_X\psi_5 \to -ige^{-\alpha_5+\gamma_X+\alpha_5}\overline{\psi_5}A_X\psi_5 \Longrightarrow \alpha_X = 0.$$
(173)

Relating all the equations, we finally find :

$$\alpha_{\overline{5}} = \alpha_{\overline{10}} + \alpha_{\phi} = \alpha_{10} + 2\alpha_{\phi} = \alpha_5 + 3\alpha_{\phi} = \alpha_1 + 2\alpha_{\phi} . \tag{174}$$

Thus, without α_X , for which we already know its value, we are left with 6 unknown variables and 4 independent equations. It is therefore possible to express all the charges as functions of only two parameters. We decide for simplicity to define $\alpha_{\overline{5}} = \alpha$ and $\alpha_{\phi} = \beta$. Adding the charges of each multiplet component to the other global U(1) symmetry : the hypercharges Y, to degenerate the charges in the multiplets components, we obtain the values gathered in Table 4. Since the Higgs doesn't have any baryon or lepton number, we need $\beta = -1/2$ in order to have a vanishing charge. Thus, we directly replace the value of $\beta = -1/2$ in the last column of the table showing the sum of the U(1) charges.

It can be seen from Table 4 that there exist similar charges for the different field types, such that they can be placed into four categories :

- → SM lepton charge : $\alpha 1/2$.
- → SM quark charge : $\alpha + 7/6$.
- → \widehat{P} -quark charge : $\alpha + 1/3$.
- → $\widehat{+}$ -lepton charge : $\alpha + 2$.

In order to reproduce the standard baryon (B) and lepton (L) numbers, we have to constrain α . For B, we need to ensure that the SM leptons have a zero baryon number, this can be done by taking $\alpha = 1/2$ and normalising the charges by 1/5 to give a 1/3 charge for the SM quark fields. Similarly, by taking $\alpha = -7/6$ and rescaling by a factor -3/5, we obtain a zero charge for the SM quarks, and a charge 1 for SM lepton fields corresponding to L. Finally, the values B and L for each field are summarised in the first two columns of Table 5.

The existence of the charges B and L ensures that no proton decay is allowed in this aGUT model. Therefore the unification scale and the compactification scale are free to be at low energy in comparison to the standard GUTs. Furthermore, all $\hat{\mathbb{T}}$ -states carry both B and L charges, in values that are half of the SM charges. This implies that it is not possible for them to decay into pure SM fields. This property makes the lightest $\hat{\mathbb{T}}$ -field stable, and a potential candidate for dark matter, as we will study in section 3.3.3.

U(1) symmetries									
SU(5) fields	SM fields	<i>U</i> (1)	Y	U(1) + Y					
$\psi_{\overline{5}}$	B_R^c	α	1/3	lpha+1/3					
	l_L		-1/2	lpha - $1/2$					
ψ_{5}	b_R	lpha - $3eta$	-1/3	lpha+7/6					
	L_L^c		1/2	lpha+2					
ψ_{10}	T_R^c	lpha - $2eta$	- 2/3	lpha+1/3					
	\mathcal{T}_{R}^{c}		1	lpha+2					
	q_L		1/6	lpha+7/6					
$\psi_{\overline{10}}$	t_R	α - β	2/3	lpha+7/6					
	τ_R		-1	lpha - $1/2$					
	$\overline{Q_L^c}$		-1/6	lpha+1/3					
ψ_1	N	$\alpha - 2\beta$	0	$\alpha - 1/2$					
ϕ_{5}	H	β	-1/3	-5/6					
	ϕ_h		1/2	0					
A_X	A_X	0	-5/6	-5/6					

TABLE 4 – Global U(1) charges for the SU(5) fields. The last column is obtained by summing the additional U(1) charge and the hypercharges by replacing β with -1/2 in order to have a zero charge for the Higgs.

3.3.2 Running of the couplings

In traditional GUT model building, the gauge couplings are supposed to run towards a common point at a given high scale. Therefore, the interesting feature of the Renormalisation Group equations is the relative evolution of the couplings. The same approach has been considered in GUT models in extra dimensions [109, 120, 111, 121].

Here we will see a different scenario where it is the SU(5) invariant running, provided by the bulk KK modes, that drives the unification of the gauge couplings asymptotically at high energy. This possibility was first noted in [122], and applied to gauge-Higgs unification models in [123, 107, 124, 125], as seen before in section 3.2.3. We recall that the RGEs for the gauge couplings are SM-like up to the compactification scale R^{-1} , where the effects of the KK modes enter. As in standard SU(5) models, we will follow the evolution of the SM sub-groups $SU(3) \times SU(2) \times U(1)$ couplings $g_i = \{g_1, g_2, g_3\}$, where the hypercharge coupling is normalised as $g_1 = \sqrt{\frac{5}{3}g'}$. The RGEs can be written as [126] :

$$2\pi \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = b_i^{\mathrm{SM}} \alpha_i^2 + (S(t) - 1) \ b^{\mathrm{aGUT}} \alpha_i^2 \,, \tag{175}$$

where again the coupling strength $\alpha_i = g_i^2/4\pi$, $t = \ln(\mu/m_Z)$, m_Z is the Z boson mass, and

Multiplets	Fields	L	В	Q	Q_3
$\psi_{\overline{5}}$	B_R^c	1/2	1/6	1/3	0
	τ_L	1	0	-1	-1
	ν_L	1	0	0	1
ψ_{5}	b_R	0	1/3	-1/3	0
	$\overline{\mathcal{T}_L^c}$	-1/2	1/2	1	1
	\mathcal{N}_L^c	-1/2	1/2	0	-1
ψ_{10}	T_R^c	1/2	1/6	-2/3	0
	\mathcal{T}_R^c	-1/2	1/2	1	0
	t_L	0	1/3	2/3	1
	b_L	0	1/3	-1/3	-1
$\psi_{\overline{10}}$	t_R	0	1/3	2/3	0
	τ_R	1	0	-1	0
	T_L^c	1/2	1/6	-2/3	-1
	B_L^c	1/2	1/6	1/3	1
ψ_1	N	1	0	0	0
ϕ_{5}	H	1/2	-1/6	-1/3	0
	ϕ^+	0	0	1	1
	ϕ_0	0	0	0	-1
A_X	X	1/2	-1/6	-4/3	-1
	Y	1/2	-1/6	-1/3	1

TABLE 5 – Lepton and baryon numbers for the components of the SU(5) multiplets. We also indicate their electromagnetic charge Q and weak isospin Q_3 .

the SM coefficients can be read :

$$b^{\rm SM} = \begin{pmatrix} 41/10\\ -19/6\\ -7 \end{pmatrix} \,. \tag{176}$$

The second term in equation (175) includes the contributions of the KK states contained in the function S(t) defined, as we have seen in section 3.2.2, by :

$$S(t) = \begin{cases} \mu R = m_Z R e^t & \text{for } \mu > 1/R, \\ 1 & \text{for } m_Z < \mu < 1/R. \end{cases}$$
(177)

As the KK modes tend to appear in complete multiplets of SU(5), all the gauge couplings receive the same five-dimensional beta function, given by using the relation derived in equation (89). In the minimal model with SU(5), we consider n_g fermion generations of one **5** and $\overline{5}$ representation associated to $T(R_F) = 1/2$ and one **10** and $\overline{10}$ with $T(R_A) = 3/2$. We also take into account one Higgs multiplet in the **5**-representation $(T(R_s) = 1/2)$ and 5 fifth-component of the gauge vectors that can be considered as scalars in the fundamental representation as well. Therefore, we have $n_s = 6$. Finally, the Casimir of the adjoint in SU(5) gives $C_2(G) = 5$. Using the equation (89), we thus obtain :

$$b^{\text{aGUT}} = -\frac{11}{3}C_2(G) + \frac{1}{3}n_sT(R_s) + \frac{4}{3}2n_g(T(R_F) + T(R_A)) = -\frac{52}{3} + \frac{16}{3}n_g, \quad (178)$$

For 3 families, as we will consider in the following parts, we find $b^{\text{aGUT}} = -4/3$. As explained in section 3.2.1, the 4D one-loop factor α will develop a mass dimension when going to 5D. In order to keep it dimensionless, we need to redefine it such that :

$$\tilde{\alpha}_i = \begin{cases} \alpha_i(t) & \text{for } \mu < 1/R, \\ \alpha_i(t)S(t) & \text{for } \mu > 1/R. \end{cases}$$
(179)

For 3 families of fermions and taking the Z mass scale values $\{g_1^0, g_2^0, g_3^0\} = \{0.45, 0.66, 1.2\}$, we show in Figure 10 the solutions for the $\tilde{\alpha}_i$. The appearance of the 5D effects is visible at the scale 1/R, where the running behavior changes sharply. Thanks to the baryon and lepton number conservation, this scale can be low, so it was chosen to be $R^{-1} = 10$ TeV as a benchmark in this plot.



FIGURE 10 – Running of the gauge couplings through the dimensionless one-loop factors $\tilde{\alpha}_i$, with $R^{-1} = 10$ TeV. The range of t corresponds to the Z mass (t = 0) and the reduced 5D Planck mass.

One can see that the couplings never cross like in standard GUTs. However, they get very close and tend to the same value asymptotically in the UV. This value corresponds to the UV safe fixed point of the 5D theory [127, 123]. A clearer approach to study the asymptotic behavior of the gauge couplings is to rewrite the RGE in terms of the dimensionless one-loop factors $\tilde{\alpha}_i$ at large energies. There, the RGEs are the same for all gauge couplings and are written :

$$2\pi \frac{\mathrm{d}\tilde{\alpha}_{i}}{\mathrm{d}t} = 2\pi \frac{\mathrm{d}\left(\alpha_{i}S(t)\right)}{\mathrm{d}t} = 2\pi\alpha_{i}S(t) + b_{i}^{\mathrm{SM}}\alpha_{i}^{2}S(t) + (S(t) - 1) b^{\mathrm{aGUT}}\alpha_{i}^{2}S(t)$$

$$= 2\pi\tilde{\alpha}_{i} + \left(b^{\mathrm{aGUT}} + \frac{1}{S(t)}\left(b_{i}^{\mathrm{SM}} - b^{\mathrm{aGUT}}\right)\right)\tilde{\alpha}_{i}^{2}.$$
(180)

For very high energies, we have $\mu \gg 1/R$, thus $S(t) \gg 1$. Keeping only the leading terms in $S(t)^{-1}$, the RGEs become :

$$2\pi \frac{\mathrm{d}\tilde{\alpha}_i}{\mathrm{d}t} \approx 2\pi \tilde{\alpha}_i + b^{\mathrm{aGUT}} \tilde{\alpha}_i^2 = 2\pi \tilde{\alpha}_i \left(1 + \frac{b^{\mathrm{aGUT}}}{2\pi} \tilde{\alpha}_i\right).$$
(181)

The beta-function here looks like a Wilson-Fisher RGE as was derived in equation (99) where extra-dimensions create a non-trivial fixed point. There, the beta function vanishes at the fixed points :

$$\tilde{\alpha}_i^*(IR) = 0, \quad \tilde{\alpha}_i^*(UV) = -\frac{2\pi}{b^{\text{aGUT}}}, \quad (182)$$

respectively the IR and the UV fixed points. The positiveness of the one-loop gauge factor imply that the UV fixed point only exists for $b^{aGUT} < 0$. Therefore, using the result in equation (178), we see that the existence of the fixed point requires $n_g \leq 3$. For 4 or more bulk generations, the asymptotic unification would fail because the fixed point would be negative and thus unreachable.

For 3 bulk generations, using the value $b^{aGUT} = -4/3$, we find :

$$\tilde{\alpha}_i^* = \frac{3\pi}{2} \,. \tag{183}$$

The coupling fixed point being greater than 1, its value is apparently non-perturbative. However, the extra-dimensional loop factor definition differs from the 4D one and one needs to use the following definition :

$$\tilde{\alpha}_d = \frac{\Omega(d)}{(2\pi)^d} 4\pi \tilde{\alpha} \,, \tag{184}$$

where Ω_d is the *d*-dimensional solid angle defined by :

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \,. \tag{185}$$

For d = 5 and 3 fermion generations, we find :

$$\tilde{\alpha}_5 = \frac{\tilde{\alpha}_i^*}{3\pi^2} = \frac{1}{2\pi} < 1.$$
(186)

Therefore, the theory remains perturbative at high energies once the extra-dimensional nature of the theory is fully taken into account. Increasing R^{-1} does not change the picture qualitatively.

For the running, we take as an upper bound the 5D reduced Planck mass M_5^{Pl} [128], which corresponds to the largest value of t shown in the plot. M_5^{Pl} is related to the 4D reduced Planck Mass M_4^{Pl} by the following relation :

$$\left(M_4^{\rm Pl}\right)^2 \approx R \left(M_5^{\rm Pl}\right)^3 \,. \tag{187}$$

The plot in Figure 10 is realised taken shows that the couplings become indistinguishable at the scale $t \approx 10$. This scale is well below the 5D reduced Planck mass. As the couplings seem to behave as one above it, the scale at which their differences cannot be seen anymore can be thought as an unification scale. As the unification is asymptotic, one could argue that this scale is arbitrary and that true unification never really occurs. The aim of the aGUT philosophy is to consider that unification happens, in fact, differently and to see it as an approximation. The expression in Equation (181) shows the same behavior for any gauge coupling and was obtained thanks to the approximation $\mu \gg R^{-1}$. Therefore, considering fully the extra-dimension allows the gauge couplings to evolve in a similar way, thus showing an unified behavior. This also forces any unification to happen well above the compactification scale.

Considering now the Yukawa sector, as argued before, the Yukawa couplings do not unify. The study of Yukawa running is still important to make sure that no Landau pole is reached below the 5D Planck mass, ensuring the consistency of the theory.

Due to the fact that the gauge couplings enter in the Yukawa beta function at one loop, the calculation of the Yukawa running is more difficult to achieve than the gauge one. Moreover, the coupled terms concerns not only the SM gauge couplings, but also the couplings associated with the massive \mathbb{T} -gauge bosons A_X . The running of these couplings is totally independent from the SM gauge couplings below the unification scale and therefore we don't know anything about how it would behave. To take into account this uncertainty in the Yukawa running, we compute the mixed gauge-Yukawa contribution between the compactification and the unification scales using a single 5D coupling g_5 whose values are taken between the extreme values in the SM $g_5 = g_1$ and $g_5 = g_3$. The variation between the two will be considered as a systematic uncertainty in the Yukawa running. In the following, we will only consider the third generation of Yukawa couplings, as they feature the largest strengths.

First, in the SM the four Yukawa couplings run according to the following RGEs :

$$2\pi \frac{\mathrm{d}\alpha_t}{\mathrm{d}t}\Big|_{\mathrm{SM}} = \left[\frac{9}{2}\alpha_t + \frac{3}{2}\alpha_b + \alpha_\tau + \alpha_\nu - \frac{17}{20}\alpha_1 - \frac{9}{4}\alpha_2 - 8\alpha_3\right]\alpha_t, \qquad (188)$$

$$2\pi \frac{\mathrm{d}\alpha_b}{\mathrm{d}t}\Big|_{\mathrm{SM}} = \left[\frac{9}{2}\alpha_b + \frac{3}{2}\alpha_t + \alpha_\tau + \alpha_\nu - \frac{1}{4}\alpha_1 - \frac{9}{4}\alpha_2 - 8\alpha_3\right]\alpha_b, \qquad (189)$$

$$2\pi \frac{\mathrm{d}\alpha_{\tau}}{\mathrm{d}t}\Big|_{\mathrm{SM}} = \left[\frac{5}{2}\alpha_{\tau} + 3\alpha_t + 3\alpha_b - \frac{1}{2}\alpha_{\nu} - \frac{9}{4}(\alpha_1 + \alpha_2)\right]\alpha_{\tau}, \qquad (190)$$

$$2\pi \frac{\mathrm{d}\alpha_{\nu}}{\mathrm{d}t}\Big|_{\mathrm{SM}} = \left[\frac{5}{2}\alpha_{\nu} + 3\alpha_t + 3\alpha_b - \frac{1}{2}\alpha_{\tau} - \frac{9}{20}\alpha_1 - \frac{9}{4}\alpha_2\right]\alpha_{\nu}, \qquad (191)$$

where $\alpha_f = y_f^2/4\pi$. These equations will be used for the running of the Yukawa couplings between the EW scale and the compactification scale.

In order to derive the beta functions above the compactification scale, we will first use the formalism of counterterms similarly as what was seen in section 2.3.2, to derive the betafunction. We will see as a general example, a Yukawa coupling between a scalar, a fermion and an anti-fermion field. Therefore, the renormalised propagator of the fermion called ψ_1 can be written :

$$iG_{\psi_1}^R = \frac{1}{Z_1} \frac{i}{\not p - m_1^0} + \text{loops} = \frac{1}{1 + \delta_1} \frac{i}{\not p - m_1^R - \delta_{m_1} m_1^R} + \text{loops} , \qquad (192)$$

where "loops" refers to loop-diagrams contributing to the renormalisation of the propagator. Expanding at first order, we have :

The one-loop diagram contributions can always be written such as :

$$\text{loops} = \frac{i}{\not p - m_1^R} \left(i \Sigma_1 \not p - i \Sigma_1^m m_1^R \right) \frac{i}{\not p - m_1^R} , \qquad (194)$$

with Σ_1 representing the loops diagram contributions proportional to p and Σ_1^m the contributions proportional to m_1 . Replacing the loops expression in the renormalised propagator, we finally have :

$$iG_{\psi_1}^R = \frac{i}{\not p - m_1^R} + \frac{i}{\not p - m_1^R} \left(i \left((\delta_1 + \Sigma_1) \not p - (\delta_1 + \delta_{m_1} + \Sigma_1^m) m_1^R \right) \right) \frac{i}{\not p - m_1^R} .$$
(195)

By requiring to have only the first term and therefore making the second term in Equation (195) vanishing, we need :

$$\delta_1 = -\Sigma_1 , \quad \delta_{m_1} = \Sigma_1 - \Sigma_1^m .$$
 (196)

Using similar methods for the anti-fermion ψ_2 and scalar ϕ propagators counterterms for and for the coupling, we finally have the set of equations :

$$\delta_1 = -\Sigma_1 , \quad \delta_2 = \Sigma_2 , \quad \delta_\phi = -\Sigma_\phi , \quad \delta_{\text{Coupling}} = \Sigma_{\text{Coupling}} , \qquad (197)$$

where Σ_1 refers to contributions proportional to p in the renormalisation of the fermion propagator, Σ_2 to contributions proportional to p in the renormalisation of the anti-fermion propagator, Σ_{ϕ} , contributions proportional to p^2 in the renormalisation of the scalar propagator, and Σ_{coupling} contributions to the renormalisation of the Yukawa vertex.

Now, using the relations between bare and renormalised parameters, we have :

$$\psi_1^0 = \sqrt{Z_1}\psi_1^R , \quad \overline{\psi_2^0} = \sqrt{Z_2}\overline{\psi_2^R} , \quad \phi^0 = \sqrt{Z_\phi}\phi^R ,$$

$$y^0\overline{\psi_2^0}\psi_1^0\phi^0 = Z_{\text{Coupling}}y^R\overline{\psi_2^R}\psi_1^R\phi^R .$$
 (198)

Therefore, focusing on the renormalisation factor of the Yukawa coupling and using dimensional regularisation where $d = 4 - \epsilon$, we have :

$$y^{0} = \frac{Z_{\text{Coupling}}}{\sqrt{Z_{1}Z_{2}Z_{\phi}}}y^{R}\mu^{\frac{\epsilon}{2}} = Z_{y}y^{R}\mu^{\frac{\epsilon}{2}} ,$$

$$\Rightarrow 0 = \frac{\mathrm{d}}{\mathrm{d}\mu} \left(\frac{Z_{\text{Coupling}}}{\sqrt{Z_{1}Z_{2}Z_{\phi}}}y^{R}\mu^{\frac{\epsilon}{2}}\right) .$$
(199)

Replacing $Z_i = 1 + \delta_i$, taking the first order in the counterterms and neglecting to superscript R for simplicity, we obtain the expression of the Yukawa β -function function of the different diagram contributions :

$$\beta_{y} = -\frac{\epsilon}{2}y - \frac{\epsilon}{2}y^{2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\delta_{\mathrm{Coupling}} - \frac{1}{2}\left(\delta_{1} + \delta_{\phi} + \delta_{2}\right)\right)$$

$$= -\frac{\epsilon}{2}y - \frac{\epsilon}{2}y^{2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\Sigma_{\mathrm{Coupling}} + \frac{1}{2}\left(\Sigma_{1} + \Sigma_{\phi}\right) - \frac{1}{2}\Sigma_{2}\right).$$
(200)

Now we need to consider the diagrams calculations in order to find the expressions for the Σ_i . The Feynman rules of the minimal SU(5) aGUT are specified in Appendix B.1 and allow to compute the diagram contributions to the beta-function as listed in Appendix B.2 for the top-Yukawa.

Adding all the parts for the bulk top Yukawa beta function above the compactification scale, we obtain the following RGE with respect to $t = \ln(\mu/m_Z)$:

$$16\pi^{2} \frac{\mathrm{d}Y_{t}}{\mathrm{d}t}\Big|_{\mathrm{aGUT}} = Y_{t}^{3} \left(\frac{1}{2}3!2! + \frac{1}{2}3!2! + 2 \times 4!1!\right) + Y_{t}Y_{\nu}^{2} (2) + Y_{t}Y_{\tau}^{2} \left(\frac{1}{2} + 2 \times 2 + 2\right) + Y_{t}Y_{b}^{2} \left(\frac{1}{2} + 2 \times 2 + 2\right) + Y_{t}g_{5}^{2} \left(-6 \times \frac{2 \times 6}{5} - 3\frac{12}{5} - 2 \times \frac{12}{5} + \frac{1}{2}\frac{18}{5} + \frac{1}{2}\frac{18}{5}\right) + Y_{t}g_{5}^{2} \xi \left(-2 \times \frac{12}{5} + \frac{18}{5} + \frac{18}{5} + \frac{12}{5} + 2\frac{-6}{5} - 2\frac{6}{5}\right),$$

$$(201)$$

where, as explained before, we computed the coefficients of the gauge contribution with an unified gauge coupling g_5 . The sum of the contributions shows that the gauge-parameter ξ vanishes as expected by the gauge-invariance. Then, the one loop RGE gives :

$$16\pi^2 \frac{\mathrm{d}Y_t}{\mathrm{d}t}\Big|_{\mathrm{aGUT}} = 60Y_t^3 + \frac{13}{2}Y_tY_\tau^2 + \frac{13}{2}Y_tY_b^2 + 2Y_tY_\nu^2 - \frac{114}{5}Y_tg_5^2 \,. \tag{202}$$

Rescaling, using the correspondence for the coupling in the SM $Y_t = y_t/2$, $Y_b = \sqrt{2}y_b$, $Y_\tau = \sqrt{2}y_\tau$ and $Y_\nu = y_\nu$ as seen in Equation (170), we find :

$$16\pi^2 \frac{\mathrm{d}y_t}{\mathrm{d}t} \bigg|_{\mathrm{aGUT}} = 15(y_t)^3 + 13y_t(y_\tau)^2 + 13y_t(y_b)^2 + 2y_t y_\nu^2 - \frac{114}{5}y_t g_5^2 , \qquad (203)$$

Similarly, for the other Yukawas couplings in the bulk, written in terms of α_f , we obtain :

$$2\pi \frac{d\alpha_{t}}{dt} = 2\pi \frac{d\alpha_{t}}{dt}\Big|_{SM} + (S(t) - 1)\left(15\alpha_{t} + 13\alpha_{\tau} + 13\alpha_{b} + 2\alpha_{\nu} - \frac{114}{5}\alpha_{5}\right)\alpha_{t},$$

$$2\pi \frac{d\alpha_{b}}{dt} = 2\pi \frac{d\alpha_{b}}{dt}\Big|_{SM} + (S(t) - 1)\left(11\alpha_{b} + \frac{39}{2}\alpha_{t} + 8\alpha_{\tau} + 2\alpha_{\nu} - \frac{93}{5}\alpha_{5}\right)\alpha_{b},$$

$$2\pi \frac{d\alpha_{\tau}}{dt} = 2\pi \frac{d\alpha_{\tau}}{dt}\Big|_{SM} + (S(t) - 1)\left(11\alpha_{\tau} + \frac{39}{2}\alpha_{t} + 8\alpha_{b} + \frac{9}{2}\alpha_{\nu} - \frac{93}{5}\alpha_{5}\right)\alpha_{\tau},$$

$$2\pi \frac{d\alpha_{\nu}}{dt} = 2\pi \frac{d\alpha_{\nu}}{dt}\Big|_{SM} + (S(t) - 1)\left(5\alpha_{\nu} + 12\alpha_{t} + 8\alpha_{b} + 18\alpha_{\tau} - 6\alpha_{5})\alpha_{\nu}.$$
(204)

To keep a dimensionless definition for the running coupling, we will express it in terms of $\tilde{\alpha}_f$, as defined in Equation (179). To check first for the possibility to have an UV fixed point for the Yukawa running, we start by looking at the presence of zeros for the beta function for large energies. Like in the gauge case, we can express the RGEs in terms of $\tilde{\alpha}_f$ and expand at the leading order in $1/R\mu$. This gives :

$$2\pi \frac{d\tilde{\alpha}_{t}}{dt} \approx 2\pi \tilde{\alpha}_{t} + \left(15\tilde{\alpha}_{t} + 13\tilde{\alpha}_{\tau} + 13\tilde{\alpha}_{b} + 2\tilde{\alpha}_{\nu} - \frac{114}{5}\tilde{\alpha}_{5}\right)\tilde{\alpha}_{t} ,$$

$$2\pi \frac{d\tilde{\alpha}_{b}}{dt} \approx 2\pi \tilde{\alpha}_{b} + \left(11\tilde{\alpha}_{b} + \frac{39}{2}\tilde{\alpha}_{t} + 8\tilde{\alpha}_{\tau} + 2\tilde{\alpha}_{\nu} - \frac{93}{5}\tilde{\alpha}_{5}\right)\tilde{\alpha}_{b} ,$$

$$2\pi \frac{d\tilde{\alpha}_{\tau}}{dt} \approx 2\pi \tilde{\alpha}_{\tau} + \left(11\tilde{\alpha}_{\tau} + \frac{39}{2}\tilde{\alpha}_{t} + 8\tilde{\alpha}_{b} + \frac{9}{2}\tilde{\alpha}_{\nu} - \frac{93}{5}\tilde{\alpha}_{5}\right)\tilde{\alpha}_{\tau} ,$$

$$2\pi \frac{d\alpha_{\nu}}{dt} \approx 2\pi \alpha_{\nu} + (5\alpha_{\nu} + 12\alpha_{t} + 8\alpha_{b} + 18\alpha_{\tau} - 6\alpha_{5})\alpha_{\nu} .$$

$$(205)$$

Solving these equations thanks to the gauge fixed point value in Equation (182), we find for the non-trivial fixed points in the general case of n_g generations :

$$\tilde{\alpha}_{t}^{*} = -\frac{14(2777 - 140n_{g})\pi}{3585(13 - 4n_{g})}, \quad \tilde{\alpha}_{b}^{*} = \frac{(55889 - 1160n_{g})\pi}{2390(13 - 4n_{g})}, \\ \tilde{\alpha}_{\tau}^{*} = -\frac{(42671 - 1640n_{g})\pi}{2390(13 - 4n_{g})}, \quad \tilde{\alpha}_{\nu}^{*} = \frac{336(176 - 5n_{g})\pi}{1195(13 - 4n_{g})}.$$
(206)

The fact that the zeroes for the top and tau Yukawas are always negative for $n_g \leq 3$ implies the absence of a completely safe fixed point in the case of bulk-described Yukawa couplings. Thus, the only physical possibility to keep all Yukawa couplings in the bulk is that they may run to zero in the UV. Because of the small neutrino mass, we will ignore the running of y_{ν} when solving the RGE in the following parts. Taking the EW scale values for :

$$y_t(0) = 0.991$$
, $y_b(0) = 0.023$, $y_\tau(0) = 0.0102$, (207)

the running is pushed toward the asymptotically free point only if the compactification scale agrees to :

$$R^{-1} \gtrsim R_c^{-1} = 3 \cdot 10^5 \text{ TeV}$$
 (208)

The Yukawa running for two compactification scales above and two below the critical value are shown in Figure 11. We recall that the bands correspond to the systematic uncertainties derived from the ignorance of the running of the off-diagonal gauge boson couplings.

For smaller values of the compactification scale, the UV behavior of the bulk Yukawa couplings cannot be determined, because of the systematic uncertainties. In these cases, anything can happens, either the running hits a Landau pole, either it is driven to an asymptotic free description.

This is well illustrated by looking at the RGE of the top Yukawa alone, assuming that the other Yukawas are negligible. In this case, the top RGE can be approximated at very high energies by :

$$2\pi \frac{\mathrm{d}\tilde{\alpha}_t}{\mathrm{d}t} \approx \left(2\pi + 15\tilde{\alpha}_t - \frac{114}{5}\tilde{\alpha}_5\right)\tilde{\alpha}_t \,. \tag{209}$$



FIGURE 11 – Running of the bulk Yukawas as compared to the gauge running couplings. The couplings are solved up from the EW scale to the 5D reduced Planck mass. In the top row, we show two sample values of compactification scales below the critical value while in the bottom row, the two choices of R^{-1} shown are above the critical value. The bands indicate the systematic uncertainty from the gauge couplings, while the SM value of the top Yukawa at the EW scale is indicated by the blue tick at t = 0.

Then, alone, the top Yukawa has a non-trivial UV fixed point defined by :

$$\tilde{\alpha}_t^* = \frac{1}{15} \left(\frac{114}{5} \tilde{\alpha}_5^* - 2\pi \right) = \frac{(41 + 40n_g)\pi}{75(13 - 4n_g)} = \frac{161}{75}\pi,$$
(210)

where the value in the last equality is obtained for 3 fermion families. Therefore, solving the RGE with the UV fixed point as a boundary condition at the reduced 5D Planck mass, the range predicted by the uncertainties for the top Yukawa at the EW scale always includes its SM value. This can be seen in Figure 12 where small blue lines at t = 0 shows the EW value. Therefore, the top Yukawa may run to the fixed point or to zero if the RGE trajectory lies below the critical one. This shows that the theory could be consistent even for R^{-1} below the threshold in Equation (208) but that we cannot know.

However, solving first the top-Yukawa alone down from its fixed point then running up the bottom and tau Yukawa from their SM values with the top mixed terms gives the behavior as seen in Figure 13. In fact, neglecting first all Yukawas except the top, and taking it and the gauge couplings at their UV fixed point, the high-energy beta functions for the other Yukawas



FIGURE 12 - Top bulk Yukawa as compared to the gauge couplings solved down from the UV fixed point imposed at the 5D Planck scale for four sample values of the compactification scales.

can be approximated by :

$$2\pi \frac{\mathrm{d}\tilde{\alpha}_b}{\mathrm{d}t} \approx \frac{399\pi}{25} \tilde{\alpha}_b \,, \quad 2\pi \frac{\mathrm{d}\tilde{\alpha}_\tau}{\mathrm{d}t} \approx \frac{399\pi}{25} \tilde{\alpha}_\tau \,, \quad 2\pi \frac{\mathrm{d}\tilde{\alpha}_\nu}{\mathrm{d}t} \approx \frac{469\pi}{25} \tilde{\alpha}_\nu \,. \tag{211}$$

This shows that no non-trivial fixed point can be found as only the Gaussian fixed point is allowed for them. It shows also that if the top Yukawa is solved from its UV fixed point alone, the other Yukawas would increase too fast at high energies. Therefore, the top Yukawa UV fixed point is excluded, and the only feasible possibility is that all bulk Yukawa couplings run to zero.

After having explored the possibility of having all the Yukawa in the bulk, we will now focus on the effect of localising them on a brane. Localising the Yukawa consists in forcing the interaction to occur at a specific boundary brane (i.e for a specific fifth coordinate) by adding a delta-function in the Lagrangian term. This modification is equivalent for the Yukawa running as considering that the fermions are localised on the same brane. For simplicity, and to preserve the GUT principle, we will consider the case where the localisation is held on the $y = \pi R/2$ brane, where the SU(5) symmetry is not broken. Thus, the localised Yukawa couplings have the same form than the bulk ones in Equation (169) with an additional delta factor.

In the case of a localised coupling, the KK selections rules are different from the ones



FIGURE 13 – Top Yukawa run down from its fixed point then running up bottom and tau Yukawa from their SM values for four sample values of the compactification scales.

derived in Equation (145) for the bulk case. This is due to the delta factor that will change the result of the integral over the extra-dimension. Therefore, integrating the localised Yukawa Lagrangian term for all-(+, +) fields gives :

$$\int_{0}^{\frac{\pi R}{2}} dyy \delta\left(y - \frac{\pi R}{2}\right) \hat{\psi}_{(+,+)}^{\dagger} \hat{\phi}_{(+,+)} \hat{\psi}_{(+,+)} = y \sqrt{\frac{2}{\pi R}} \psi_{0}^{\dagger} \phi_{0} \psi_{0} \\
+ y \frac{2}{\sqrt{\pi R}} \sum_{n=1}^{+\infty} (-1)^{n} \left(\psi_{n}^{\dagger} \phi_{0} \psi_{0} + \psi_{0}^{\dagger} \phi_{n} \psi_{0} + \psi_{0}^{\dagger} \phi_{0} \psi_{n}\right) \\
+ y \frac{2\sqrt{2}}{\sqrt{\pi R}} \sum_{n,m=1}^{+\infty} (-1)^{n+m} \left(\psi_{n}^{\dagger} \phi_{m} \psi_{0} + \psi_{n}^{\dagger} \phi_{0} \psi_{m} + \psi_{0}^{\dagger} \phi_{n} \psi_{m}\right) \\
+ y \frac{4}{\sqrt{\pi R}} \sum_{n,m,k=1}^{+\infty} (-1)^{n+m+k} \psi_{n}^{\dagger} \phi_{m} \psi_{k} .$$
(212)

We see here, that in the case of a localised interaction, a vertex can couple three fields with any KK number. So there is no selection rules anymore in this case.

Investigating whether or not the different couplings in the Lagrangian will enter in the beta-functions, we need to focus on the linear divergences as they are the only one contributing to the β -functions above the compactification scale. We can see in Equation (212) that the

addition of the δ -function factors changes the divergences. Comparing with Equation (145), we see that there is one more sum over the KK number for every term. Therefore, localising a coupling adds one more power of divergence. Due to this fact, it can be shown that localised interactions will not contribute to the beta function of neither bulk couplings, neither other localised interactions.

These new selection rules will also change the contributions of the KK modes in the diagrams involved in the Yukawa beta-function as listed in Tables 13 and 14. In order to show the differences with the bulk case, we will take the example of the self-energy diagram for the fermion propagator. We denote in the diagrams y_B for bulk coupling and y_L for the localised case and write explicitly the KK number of the propagators under parenthesis. In the case of the renormalisation of a bulk coupling, the only possibility was to have :



showing that the KK numbers of the internal propagators can be any integer n in the loop. The other internal propagator is forced to have a KK number of 0 due to the KK conservation at the final vertex. In the case of the beta-function of a localised Yukawa, there are now two different possibilities for the coupling with localised fields :



Here, we see that the contribution of such diagrams to the beta function of a localised Yukawa is twice the contribution obtained in the bulk case. This result is applicable for all the other diagrams in the case where all the Yukawa couplings are localised and only the gauged are in the bulk. Moreover, in this specific case, the linear running, contributing to the beta functions, is due to loops involving only bulk fields, namely the gauge ones. Therefore, we find that the gauge contribution is the same as for the bulk Yukawas in Equation (204) with an



FIGURE 14 - Running of the localised Yukawa couplings compared to the bulk gauge ones for two extreme sample values of the compactification scale.

additional factor of two. Thus giving :

$$2\pi \frac{\mathrm{d}\alpha_t}{\mathrm{d}t} = 2\pi \left. \frac{\mathrm{d}\alpha_t}{\mathrm{d}t} \right|_{\mathrm{SM}} - 2(S(t) - 1) \frac{114}{5} \alpha_5 \alpha_t ,$$

$$2\pi \frac{\mathrm{d}\alpha_b}{\mathrm{d}t} = 2\pi \left. \frac{\mathrm{d}\alpha_b}{\mathrm{d}t} \right|_{\mathrm{SM}} - 2(S(t) - 1) \frac{93}{5} \alpha_5 \alpha_b ,$$

$$2\pi \frac{\mathrm{d}\alpha_\tau}{\mathrm{d}t} = 2\pi \left. \frac{\mathrm{d}\alpha_\tau}{\mathrm{d}t} \right|_{\mathrm{SM}} - 2(S(t) - 1) \frac{93}{5} \alpha_5 \alpha_\tau ,$$

$$2\pi \frac{\mathrm{d}\alpha_\nu}{\mathrm{d}t} = 2\pi \left. \frac{\mathrm{d}\alpha_\nu}{\mathrm{d}t} \right|_{\mathrm{SM}} - 2(S(t) - 1) 6\alpha_5 \alpha_\nu .$$
(215)

Numerically, the localised Yukawas always run to the asymptotically free fixed point in the UV, for any value of R^{-1} as shown for two extreme choices of the compactification scale in Figure 14. Thus, the "all-localised configuration" always leads to a consistent Yukawa sector for the SU(5) aGUT model.

Investigating now the possibility of having both bulk and localised Yukawas, we will list the different possibilities. Starting first with only the top in the bulk and all the other localised, we have the following RGE :

$$2\pi \frac{d\alpha_t}{dt} = 2\pi \frac{d\alpha_t}{dt}\Big|_{\rm SM} + (S(t) - 1)\left(15\alpha_t - \frac{114}{5}\alpha_5\right)\alpha_t$$

$$2\pi \frac{d\alpha_b}{dt} = 2\pi \frac{d\alpha_b}{dt}\Big|_{\rm SM} + 2\left(S(t) - 1\right)\left(\frac{39}{2}\alpha_t - \frac{93}{5}\alpha_5\right)\alpha_b$$

$$2\pi \frac{d\alpha_\tau}{dt} = 2\pi \frac{d\alpha_\tau^{SM}}{dt}\Big|_{\rm SM} + 2\left(S(t) - 1\right)\left(\frac{39}{2}\alpha_t - \frac{93}{5}\alpha_5\right)\alpha_\tau$$

$$2\pi \frac{d\alpha_\nu}{dt} = 2\pi \frac{d\alpha_\nu}{dt}\Big|_{\rm SM} + 2\left(S(t) - 1\right)\left(12\alpha_t - 6\alpha_5\right)\alpha_\nu$$
(216)

This leads to the same asymptotic behavior as the one considered by neglecting all the Yukawas coupling except the top and specified in Equation (211) leaving us with the only possibility to have all couplings driven to 0.
We scrutinise now the case where only the ν -Yukawa is localised gives the following asymptotic behavior :

$$\tilde{\alpha}_{t}^{*} = \frac{14(10n_{g} - 19)\pi}{555(13 - 4n_{g})}, \quad \tilde{\alpha}_{b}^{*} = \frac{3(73 + 20n_{g})\pi}{370(13 - 4n_{g})},$$

$$\tilde{\alpha}_{\tau}^{*} = \frac{3(73 + 20n_{g})\pi}{370(13 - 4n_{g})}, \quad \tilde{\alpha}_{\nu}^{*} = 0.$$
(217)

Here, we see that for $1 < n_g < 3$, it exists a non-trivial fixed point for both the top, bottom and tau Yukawas. In the case of 3 fermion generations, solving the heavy fermion Yukawa RGEs from their UV fixed points gives the running as shown in Figure 15.



FIGURE 15 – Bulk top, bottom and tau Yukawa run down from their fixed point for four sample values of the compactification scale. The small lines on the right of the plots specify the EW values of the couplings.

Even if the fixed points are positives for all the heavy fermions Yukawas couplings, we see that the running cannot account for the electroweak values. Moreover, we see that the top is always smaller than bottom and tau Yukawas. This hierarchy is conserved if we take the other choices for n_g such that :

$$n_{g} = 3: \quad \tilde{\alpha}_{t}^{*} = \frac{154}{555}\pi, \quad \tilde{\alpha}_{b}^{*} = \frac{399}{270}\pi, \qquad \tilde{\alpha}_{\tau}^{*} = -\frac{399}{270}\pi, \quad \tilde{\alpha}_{\nu}^{*} = 0, \quad (218)$$
$$n_{g} = 2: \quad \tilde{\alpha}_{t}^{*} = \frac{14}{2775}\pi, \quad \tilde{\alpha}_{b}^{*} = \frac{339}{1850}\pi, \qquad \tilde{\alpha}_{\tau}^{*} = -\frac{339}{1850}\pi, \quad \tilde{\alpha}_{\nu}^{*} = 0.$$

Therefore, reaching the non-trivial UV fixed points gives again scenarios that are not consistent. Without going into more details, in a more general way, no other scenario non-discussed in this part is consistent and in order to summarise the coherent possibilities, we list hereafter the allowed Yukawa scenarios :

- → All-bulk Yukawas : For $R^{-1} \gtrsim R_c^{-1}$, all the couplings run to 0. For $R^{-1} \lesssim R_c^{-1}$, anything can happen.
- → All-localised Yukawas : For any R^{-1} , all Yukawas are driven to 0.

This shows that realistic scenarios exist for the Yukawa couplings, but they do not give much information on the compactification scale. They only allows to exclude non-trivial fixed points for the Yukawas couplings.

3.3.3 Baryogenesis

As discussed in the previous sections, the \mathbb{R} -partners of the SM particles have non-standard baryon number assignments as is recapped in Table 5, preventing them from decaying into SM fields. Therefore, the lightest stable \mathbb{R} -state would be a stable massive particle that is a natural candidate for dark matter. Because the mass splitting between the lowest KK modes is induced by loops [129] and due to the Higgs, the lightest state should naturally be the \mathbb{R} left-handed electron neutrino \mathcal{N} .

However, \mathcal{N} is embedded in a doublet, so its annihilation and co-annihilation cross-sections suppress the thermal relic density, which is thus not enough to generate the required relic density for dark matter. Furthermore, the interactions with the Z boson are excluded by the null outcome of Direct Detection experiments [130, 131, 132]. Therefore, the left-handed \mathcal{A} -neutrino can account only for a small fraction of the total actual dark matter relic density. In our case, the direct detection bounds creates a strong limit on this production, thus providing conservative bounds on the radius of the extra dimension. This requires the \mathcal{A} relic density to be below one part in 10¹¹ when compared to the baryon one. For this value, one cannot neglect the thermal relic density anymore and the densities would be above the experimental excluded bounds. This analysis shows that the minimal model, as presented so far, is disfavored by cosmology.

A well known way out of this problem was first used in supersymmetry [133] and consists of generating a Majorana mass for the Dirac field, so that the two components are split into two Majorana mass eigenstates. However, this route is forbidden in this model because Majorana fields cannot exist in 5 dimensions.

Therefore, we will investigate the possibility of adding a new field taking the place of \mathcal{N} as lightest $\hat{\mathcal{L}}$ -state. In our model building, the simplest way we used to generate the neutrino masses was to introduce of a singlet field, ψ_1 , corresponding to the SM right-handed neutrino. For a complete fermion symmetry, we could also introduce a second singlet, ψ'_1 . Note that having a second singlet allows us to combine all the fermion fields into a **16** and a **16** of an SO(10) symmetry group for each family through the decomposition in $SU(5) \times U(1)$:

$$\mathbf{16} \to (\mathbf{10}, 1) \oplus (\overline{\mathbf{5}}, -3) \oplus (\mathbf{1}, 5) . \tag{219}$$

ϕ_{5}		$\psi_{\overline{5}}$		ψ_{5}		
H:	μ_H	B_R^c :	$-\mu_{B_R}$	b_R :	μ_{b_R}	
ϕ^+ :	μ_+	$ au_L$:	$\mu_{ au_L}$	\mathcal{T}_L^c :	$-\mu_{\mathcal{T}_L}$	
ϕ_0 :	μ_0	$ u_L$:	$\mu_{ u_L}$	\mathcal{N}_L^c :	$-\mu_{\mathcal{N}_L}$	
ψ_{10}		$\psi_{\overline{10}}$		A^a_μ		
T_R^c :	$-\mu_{T_R}$	t_R :	μ_{t_R}	W^+ :	μ_W	
\mathcal{T}_{R}^{c} :	$-\mu_{\mathcal{T}_R}$	$ au_R$:	$\mu_{ au_R}$	X:	μ_X	
t_L :	μ_{t_L}	T_L^c :	$-\mu_{T_L}$	Y:	μ_Y	
b_L :	μ_{b_L}	B_L^c :	$-\mu_{B_L}$			

TABLE 6 – Chemical potentials associated to the relevant fields, where the SM ones correspond to the zero modes and the $\frac{1}{2}$ ones to the lowest KK tier.

Such decomposition shows that SO(10) could integrate the fields studied in the minimal SU(5) model in a simpler and more natural way. The parities for the additional field are the same than the ones for the original singlet ψ_1 . This new field allows us to add also the following bulk Yukawa couplings to the Lagrangian :

$$\Delta \mathcal{L} = Y_{\nu'} \overline{\psi}_1' \psi_5 \phi_5^* + \text{h.c.}$$
(220)

The field ψ_1 contained a right-handed zero mode, which corresponds to a right-handed neutrino N_R . Thus, the coupling Y_{ν} generates a Dirac mass for the right-handed neutrinos, and is thus bound to be extremely small. On the other hand, ψ'_1 has no zero mode, and the lightest KK mode corresponds to an \mathcal{F} -singlet denoted \mathcal{S} . The Yukawa coupling $Y_{\nu'}$ ensures that \mathcal{S} has the same baryon and lepton numbers as the \mathcal{F} -neutrino \mathcal{N} that were seen in Table 3.

Thanks to the $\hat{\mathbb{T}}$ -states carrying both baryon and lepton number, a relic density can be generated at the EW phase transition together with the baryon asymmetry. This mechanism can be used to generate an asymmetric dark matter relic density [134].

To estimate the \mathbb{P} -particles induced relic density produced during baryogenesis, we will employ the usual calculation using the chemical potential equilibrium for every species of particles active at the EW phase transition [135]. The precise dynamics of the phase transition in extra-dimensions are complicated [136], therefore, we will only provide, here, an estimate.

The states we consider here are the lowest tier of the KK modes and their associated chemical potentials, are listed in Table 6. All other states are heavier and their contributions are thus negligible. Furthermore, we assume that the three families of fermions share the same chemical potentials. The relationships between them are extracted from the interaction terms in the Lagrangian, which allow to express all the chemical potentials with only four of them, chosen to be μ_{t_L} , μ_W , μ_H and μ_0 . In order to keep the Yukawa Lagrangian terms in Equation (169) invariant under the global U(1) charge related with the chemical potentials, we need the following relationships :

$$\mu_{H} = -\mu_{T_{R}} - \mu_{b_{R}} = \mu_{b_{L}} + \mu_{\mathcal{N}_{L}} = \mu_{t_{L}} + \mu_{\mathcal{T}_{L}} = -\mu_{B_{R}} - \mu_{t_{R}} = \mu_{\tau_{L}} + \mu_{T_{L}}$$

$$= \mu_{\nu_{L}} + \mu_{B_{L}} = \mu_{\tau_{R}} + \mu_{T_{R}} = \mu_{t_{R}} + \mu_{\mathcal{T}_{R}} = -\mu_{T_{L}} - \mu_{b_{L}} = -\mu_{B_{L}} - \mu_{t_{L}} ,$$

$$\mu_{0} = \mu_{b_{L}} - \mu_{b_{R}} = \mu_{\mathcal{T}_{L}} - \mu_{\mathcal{T}_{R}} = \mu_{T_{R}} - \mu_{T_{L}} = \mu_{t_{R}} - \mu_{t_{L}} = \mu_{B_{L}} - \mu_{B_{R}} = \mu_{\tau_{L}} - \mu_{\tau_{R}} ,$$

$$\mu_{+} = \mu_{\mathcal{N}_{L}} - \mu_{\mathcal{T}_{R}} = \mu_{t_{R}} - \mu_{b_{L}} = \mu_{t_{L}} - \mu_{b_{R}} = \mu_{\tau_{L}} - \mu_{\tau_{R}} .$$
(221)

We also have, for the gauge terms :

$$\mu_W = \mu_{T_L} - \mu_{B_L} = \mu_{\nu_L} - \mu_{\tau_L} = \mu_{\mathcal{N}_L} - \mu_{\mathcal{T}_L} = \mu_{t_L} - \mu_{b_L} = \mu_{+} - \mu_0 ,$$

$$\mu_X = \mu_{\tau_L} + \mu_{B_R} = \mu_{b_R} + \mu_{\mathcal{T}_L} = -\mu_{t_L} - \mu_{T_R} = \mu_{b_L} + \mu_{\mathcal{T}_R} = -\mu_{t_R} - \mu_{T_L} = \mu_{\tau_R} + \mu_{B_L} ,$$

$$\mu_Y = \mu_{\nu_L} + \mu_{B_R} = \mu_{b_R} + \mu_{\mathcal{N}_L} = -\mu_{b_L} - \mu_{\mathcal{T}_R} = \mu_{t_L} + \mu_{\mathcal{T}_R} = -\mu_{t_R} - \mu_{B_L} = \mu_{\tau_R} + \mu_{T_L} .$$

$$(222)$$

At the freeze-out temperature T_f , the matter-antimatter asymmetry for each field of mass M can be written as [137] :

$$n = n_{+} - n_{-} = d_{\rm dof} T_{f}^{3} \frac{\mu}{T_{f}} \frac{\sigma\left(\frac{M}{T_{f}}\right)}{6} , \qquad (223)$$

where μ is the energy scale, d_{dof} is the degrees of freedom multiplicity and :

$$\sigma(z) = \begin{cases} \frac{6}{4\pi^2} \int_0^\infty dx x^2 \cosh^{-2} \left(\frac{1}{2}\sqrt{x^2 + z^2}\right) & \text{for fermions}, \\ \frac{6}{4\pi^2} \int_0^\infty dx x^2 \sinh^{-2} \left(\frac{1}{2}\sqrt{x^2 + z^2}\right) & \text{for bosons}. \end{cases}$$
(224)

The σ function is normalised such that it is equal to 1 for massless fermions and to 2 for massless bosons. Considering now that only the 2-particles and the top quark have a non-negligible mass, in order to keep the particle density to 1 for the others SM fermions and 2 for the SM bosons, the total density of each species is given in Table 7.

Field	Density	Field	Density
t	$3(2+\sigma_t)(\mu_{t_L}+\mu_{t_R})$	b	$9(\mu_{b_L} + \mu_{b_R})$
ν	$3(\mu_{\nu_L})$	ϕ^-	$2\mu_{\phi^-}$
τ	$3(\mu_{\tau_L} + \mu_{\tau_R})$	h	$2\mu_h$
T	$18\sigma_T(\mu_{T_L} + \mu_{T_R})$	B	$18\sigma_B(\mu_{B_L}+\mu_{B_R})$
\mathcal{N}	$6\sigma_{\mathcal{N}}\mu_{\mathcal{N}_L}$	\mathcal{T}	$6\sigma_{\mathcal{T}}(\mu_{\mathcal{T}_L} + \mu_{\mathcal{T}_R})$
X	$3\sigma_X\mu_X$	Y	$3\sigma_Y \mu_Y$
H	$3\sigma_H \mu_H$		

TABLE 7 – Normalised particle densities for each field.

Using the charges listed in Table 5, the total charge density of the Universe and the total weak isospin density can be written as :

$$Q^{\text{Tot}} = \frac{2}{3}3(2 + \sigma_t)(\mu_{t_L} + \mu_{t_R}) - \frac{1}{3}(3)(3)(\mu_{b_L} + \mu_{b_R}) - 3(\mu_{\tau_L} + \mu_{\tau_R}) - 2(2)\mu_W + 2\mu_+ - 1(3\sigma_T)(\mu_{\tau_L} + \mu_{\tau_R}) - \frac{1}{3}(3)(3\sigma_B)(\mu_{B_L} + \mu_{B_R}) + \frac{2}{3}(3)(3\sigma_T)(\mu_{T_L} + \mu_{T_R}) - \frac{4}{3}(3)\sigma_X\mu_X - \frac{1}{3}(3)\sigma_Y\mu_Y - \frac{1}{3}(3)\sigma_H\mu_H, Q_3^{\text{Tot}} = 3(2 + \sigma_t)(\mu_{t_L}) - 9(\mu_{b_L}) - 3(\mu_{\tau_L} - \mu_{\nu_L}) - 4\mu_W + 2(\mu_+ - \mu_0) + 6(\mu_{\tau_L}\sigma_T - \mu_{\mathcal{N}_L}\sigma_\mathcal{N}) + 18(\mu_{T_L}\sigma_T - \mu_{B_L}\sigma_B) + 3(\sigma_Y\mu_Y - \sigma_X\mu_X).$$
(225)

All these parts also allow to calculate the total baryon number stored separately in the SM and in the $\widehat{\Lambda}$ -sectors. They will depend on the total densities of each species given in Table 7 at any temperature T. Once the $\widehat{\Lambda}$ -particles emerge from thermal equilibrium, they will decay directly into \mathcal{S} , because it is the lightest one, and will release a part of their baryon number to the SM sector again. For example, it can be seen in Table 5 that the $\widehat{\Lambda}$ -quarks T and B have a -1/6 baryon number, while \mathcal{S} has baryon number -1/2. Therefore, a 1/3 baryon number will be released in the SM sector through the decay of T in \mathcal{S} . This applies as well to the $\widehat{\Lambda}$ -bosons H, X and Y, while \mathcal{T} and \mathcal{S} share the same baryon number. Consequently, after freeze-out, the baryon numbers in the SM-sector and in \mathcal{S} can be expressed as :

$$B_{\rm SM} = \frac{1}{3}3(2+\sigma_t)(\mu_{t_L}+\mu_{t_R}) + \frac{1}{3}9(\mu_{b_L}+\mu_{b_R}) + \frac{1}{3}3(\sigma_H\mu_H+\sigma_X\mu_X+\sigma_Y\mu_Y) + \frac{1}{3}18\sigma_B(\mu_{B_L}+\mu_{B_R}) + \frac{1}{3}18\sigma_T(\mu_{T_L}+\mu_{T_R}), \qquad (226)$$
$$B_{\mathcal{S}} = -\frac{1}{2}(3\sigma_{\mathcal{N}}(\mu_{\mathcal{N}_L}) + 3\sigma_T(\mu_{\mathcal{T}_L}+\mu_{\mathcal{T}_R}) + 3(\sigma_H\mu_H+\sigma_X\mu_X+\sigma_Y\mu_Y) + 9\sigma_B(\mu_{B_L}+\mu_{B_R}) + 9\sigma_T(\mu_{T_L}+\mu_{T_R})).$$

Using the relations in Equation (222) and assuming for simplicity that all \mathbb{A} -states have the same mass (i.e. $\sigma_T = \sigma_B = \sigma_T = \sigma_N \equiv \sigma_F$ and $\sigma_X = \sigma_Y = \sigma_H$), the total baryon numbers can be written :

$$B_{\rm SM} = (10 + 2\sigma_t - 24\sigma_F)\mu_{t_L} + (12\sigma_F - 6 - \sigma_X)\mu_W + (3\sigma_X - 24\sigma_F)\mu_H + (\sigma_t - 1)\mu_0 ,$$

$$B_{\mathcal{S}} = \frac{3}{2} \left(30\sigma_F \mu_{t_L} + (\sigma_X - 14\sigma_F)\mu_W + (18\sigma_F - 3\sigma_X)\mu_H + 2\sigma_F \mu_0 \right) .$$
(227)

Now, the relic density can be express using the mass density of S divided by the SM mass density. This can can now be expressed using the baryon number density such that :

$$\frac{\Omega_{\mathcal{S}}}{\Omega_b} = \frac{2m_{\mathcal{S}}}{m_p} \frac{B_{\mathcal{S}}}{B_{\rm SM}} , \qquad (228)$$

where the factor 2 is caused by the half baryon number of S and m_p is the proton mass representing the typical scale of the lightest baryons. In the same way, the total electric charge and the total weak isospin as written in equation (225) can be simplified using the same relations. Consequently, we can write them :

m

$$Q^{\text{Tot}} = 4(\sigma_t - 1)\mu_{t_L} + (10 + 24\sigma_F + 4\sigma_X)\mu_W - (12 + 2\sigma_F + 4\sigma_X)\mu_H + (12 + 2\sigma_t + 24\sigma_F - 3\sigma_X)\mu_0 , \qquad (229)$$
$$Q^{\text{Tot}}_3 = 3(\sigma_t - 1)\mu_{t_L} + (10 + 12\sigma_F + 3\sigma_X)\mu_W - 6\sigma_X\mu_0 .$$

Now, in order to reduce the number of degrees of freedom for the chemical potentials, we need to consider relations due to the EW phase transition, where we assume that it can be of first or second order. The first relation to take into account is due to the electroweak B + L anomaly. This violation allows the existence of exchange processes between two minimas of the EW potential that correspond to the creation of a $n\nu$ state from each generation out of the vacuum. Such vacuum transitions are called "sphaleron processes" and impose relations among the chemical potentials. All \hat{A} -particles are vector-like, therefore they do not contribute to the sphaleron rate, so that the same relationship holds as in the SM [135] :

$$3(\mu_{t_L} + 2\mu_{b_L}) + 3\mu_{\nu_L} = 3\mu_{t_L} - 2\mu_W + 2\mu_H = 0.$$
(230)

In addition, the order of the EW phase transition gives two more equations leading, together with the sphaleron condition, to the possibility to describes the baryon numbers with an unique chemical potential that will vanish in the relic density expression. Let's start with a first order phase transition. It is characterised by the vanishing of the total charge and total weak isospin, such that $Q^{\text{Tot}} = Q_3^{\text{Tot}} = 0$. Considering that the mass of all \P -particles particles are the same in first approximation (i.e $\sigma_X = \sigma_F$), the potential can thus be written :

$$\mu_{H} = -\mu_{t} \frac{3(96 + 392\sigma + 483\sigma^{2} + 40\sigma_{t} + 88\sigma\sigma_{t} + 4\sigma_{t}^{2})}{2(120 + 378\sigma + 447\sigma^{2} + 20\sigma_{t} + 30\sigma\sigma_{t})},$$

$$\mu_{W} = -\mu_{t} \frac{3(-12 + 7\sigma + 18\sigma^{2} + 10\sigma_{t} + 29\sigma\sigma_{t} + 2\sigma_{t}^{2})}{120 + 378\sigma + 447\sigma^{2} + 20\sigma_{t} + 30\sigma\sigma_{t}},$$

$$\mu_{0} = -\mu_{t} \frac{134 + 366\sigma + 135\sigma^{2} + 46\sigma_{t} - 6\sigma\sigma_{t}}{120 + 378\sigma + 447\sigma^{2} + 20\sigma_{t} + 30\sigma\sigma_{t}}.$$
(231)

Replacing the relations in equation (228), the relic density ratio is now given by :

$$\frac{\Omega_{\mathcal{S}}}{\Omega_b}\Big|_{1^{\text{st}}} = 2\frac{m_{\mathcal{N}}}{m_p} 2\frac{2236 + 9624\sigma + 7785\sigma^3 + 36\sigma\sigma_t (92 + 7\sigma_t) + 4\sigma_t (266 + 15\sigma_t) + 36\sigma^2 (443 + 103\sigma_t)}{3\sigma \left(5949\sigma^2 + 18\sigma (229 + 7\sigma_t) - 4\left(-352 + \sigma_t + 6\sigma_t^2\right)\right)}$$
(232)

Similarly, in a second order phase transition, the vanishing of the total charge is required but the weak isospin condition is replaced by the vanishing of the Higgs chemical potential, such that $Q^{\text{Tot}} = \mu_0 = 0$. This gives the following relic density ratio :

$$\frac{\Omega_{\mathcal{S}}}{\Omega_b}\Big|_{2^{\text{nd}}} = 2\frac{m_{\mathcal{N}}}{m_p} \frac{128 + 40\sigma_t + 6\sigma \left(133 + 85\sigma + 28\sigma_t\right)}{3\sigma \left(-43 + 147\sigma - 8\sigma_t\right)} \,. \tag{233}$$

We can see that the ratio only depends on the mass of the Indalo S and the temperature of the phase transition T^* . The results are shown in Figure 16, where the relic density ratio is plot



FIGURE 16 – 1st order (left panel) and 2nd order (right panel) phase transition results. Values of the S relic density as a function of the mass for $T^* = v_{\rm SM}$ (solid black), $v_{\rm SM}/2$ (dashed red) and $2v_{\rm SM}$ (dashed blue). The green shaded region is excluded by the over-closure of the Universe.

as a function of the S mass for three different values of the EW phase transition temperature. The region in green is excluded by the over-closure of the Universe [138], generating a lower bound on the KK mass scale. This pushes the mass of the lightest \mathcal{F} -fermion above 1 to 10 TeV, depending on the temperature of the phase transition. At such large values of the masses it can be seen from the figure that there are only minor differences between the two types of phase transitions. Taking the lower bounds for the S relic density, its mass should lie between 1 and 6 TeV depending on the temperature of the EW phase transition, where the central value is obtained for :

$$m_{\rm KK} = 2.4 \text{ TeV for } T^* = v_{\rm SM}$$
. (234)

This favoured mass was therefore used for one of the compactification scale samples shown in the previous Yukawas coupling scenarios.

We saw in this part coherent scenarios for the minimal model of asymptotic Grand Unification in SU(5). While the model has appealing features, forbidding the proton decay, proposing a tangible dark matter candidate and having reasonable layout for the Yukawa content, it lacks phenomenology predictions and natural behaviors for the Yukawa couplings. As discussed briefly before, models embedded in larger group such as SO(10) or flipped SU(5) could be of interest to solve the inconsistencies and propose a more natural unification of the fermion fields. However, this minimal model allowed us to illustrate the principle of asymptotic Grand Unification and to see how it can address problems encounters in standard Grand Unification.

4 Epidemiology and Renormalisation Group

4.1 Epidemiologic models

Mathematical modeling are of great interest to study the evolution of transmissible diseases across human population. Following the historical journey of epidemiology [139] started with the first model simulation an epidemic evolution proposed by Daniel Bernoulli in 1760 [140] aiming at measuring the increase in life expectancy of the population allowed by the elimination of a major threat at the time : the smallpox. The discipline took a significant leap a hundred years later when Pasteur and Koch find out that infectious diseases are caused by living organisms creating a crucial analogy between epidemics and population evolution [141]. Therefore, inspired by the chemical kinetics models developed earlier [142, 143, 144] and the law of mass action, the pioneering work of Ross and Hudson [49, 50] gave rise to the first compartmental models for epidemiology proposed in 1927 [145]. Explaining pandemics was of prime importance at this time after the outbreak of the Spanish flu pandemic that started in 1918. Compartmental models are consider to be the starting point of the field of epidemiology. The purpose of this discipline is to understand the spreading of a disease, in order to model, in a temporal and spatial viewpoint, the scenario of an epidemic event and to prevent a new episode. In this part, we will first list different examples of the main models used in epidemiology and the relations between them. Then, in section 4.2, we will introduce the newly conceived epidemic Renormalisation Group (eRG) and see how this new model improves the understanding of the data related to the COVID-19 pandemic.

4.1.1 Compartmental models (deterministic approach)

Among the models used in epidemiology, the compartmental ones have become the main framework in which simulation are made by epidemiologists. Inspired by chemical reactions kinetics [146, 147], compartmental models propose to study different type of populations through differential equations. To illustrate the principle of compartmental models, we focus on the simplest example : the SI model. In this model, we consider a constant population of N individuals who can be either Susceptible (S) either Infected (I). Therefore, the total population fulfill the equation N = S(t) + I(t) for any time t, where S(t) and I(t) correspond respectively to the total number of susceptible and infected individuals among the total population. The two sub-populations are also coupled thanks to the following differential equations :

$$\begin{cases} \frac{dS}{dt} = -\gamma S(t)I(t) ,\\ \frac{dI}{dt} = \gamma S(t)I(t) . \end{cases}$$
(235)

where γ is called the **infection rate**, with a dimension of inverse time and represents how fast the susceptible individuals can turn into infectious through the influence of ill neighbours. The conservation of the population is ensured by the fact that the sum of the two differential equations yields 0. The population conservation rule allows also to rewrite the second line in equation (235) to make it look like a Wilson-Fischer Renormalisation Group equation as shown in Equation (99) :

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \gamma I(t) \left(N - I(t) \right) \,. \tag{236}$$

We can clearly see two fixed points for I = 0 and I = N, corresponding to the invariant periods before and after the epidemic episode. The SI model is one of the few epidemiological models where the solutions are analytical functions. In this case, they are given by :

$$I(t) = \frac{Ne^{\gamma t}}{b + e^{\gamma t}}, \quad S(t) = \frac{Nb}{b + e^{\gamma t}}, \quad (237)$$

where b is the integration constant. b can be seen as a time-shifting of the function following the relation :

$$\frac{Ne^{\gamma(t-t_0)}}{1+e^{\gamma(t-t_0)}} = \frac{Ne^{\gamma t}}{e^{\gamma t_0}+e^{\gamma t}} \Rightarrow b = e^{\gamma t_0} .$$
(238)

The function for I(t) is called a **logistic function** and is commonly used in several domains of physics and biophysics as it shows transitional behavior between two scale invariant domains. The evolution of S(t) and I(t) is shown in the top left plot of Figure 17.

While the SI model is the simplest compartmental one, the most used nowadays by epidemiological teams is the SIR model, adding to the SI model a Recovered (R) individuals compartment. R symbolise individuals who recovered from the disease after being infected but we can also integrate the ones who died from it. Therefore, the R compartment is also sometimes called Removed. Each sub-population evolution can be expressed now by the following equations :

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = -\gamma S(t)I(t) ,\\ \frac{\mathrm{d}I}{\mathrm{d}t} = \gamma S(t)I(t) - \epsilon I(t) ,\\ \frac{\mathrm{d}R}{\mathrm{d}t} = \epsilon I(t) . \end{cases}$$
(239)

where ϵ is called the **recovery rate** and quantify the speed at which infected individuals recovers (or dies). It represents also the inverse time during which an individual stay infected.

This model allows to see the "wave" of infected by looking at the compartment I(t) directly, where I(t) represents now the number of "active" infected at the time t. The solutions for a sample of parameters are shown in the top-right panel of Figure 17.

The SIR yields two different regimes for the number of infected depending on the values of γ and ϵ . This can be seen by looking at the differential equation for I(t) in equation (239). Its derivative is positive for :

$$\gamma S(t)I(t) > \epsilon I(t) \Rightarrow \frac{\gamma}{\epsilon}S(t) > 1$$
. (240)

Equation (240) shows that the maximum of the number of active infected is reached when $S(t) = \frac{\epsilon}{\gamma}$. Moreover, in the beginning of the epidemic episode, we can consider that $S(0) \approx 1$. Therefore, the condition (240) to have an increase of the number of active infected becomes :

$$\sigma = \frac{\gamma}{\epsilon} > 1 , \qquad (241)$$

where σ is also called the **basic reproduction number** denoted R_0 . σ can be interpreted as the average number of susceptible individuals that will be infected by a single infectious one. Such definition can be derived by remembering that γ represents the inverse of the average time before a susceptible person gets infected by an infectious neighbour $\tau_{\gamma} = \gamma^{-1}$ and ϵ represents the inverse of the average time during which an infectious person stays infected $\tau_{\epsilon} = \epsilon^{-1}$. Thus, the mean number of infected neighbours by one individual during the average period of infection τ_{ϵ} is :

$$R_0 = \frac{\tau_\epsilon}{\tau_\gamma} = \frac{\gamma}{\epsilon} = \sigma .$$
(242)

Depending on the value of σ , it is possible to extract two distinct behaviors from the initial conditions of the model, considering that $S(0) \approx 1$:

- → For $\sigma < 1$: the pandemic will die out shortly after it started, I(t) shrinking from I(0) to 0 monotonously.
- → For $\sigma > 1$: the pandemic rises to a maximum higher than I(0) then goes down to 0 giving to the pandemic a wave pattern through the function I(t).



FIGURE 17 – Top panel : SI model (left) and SIR model (right) for the following parameters $\gamma = 0.2$, N = 1, I(0) = 0.01 and $\epsilon = 0.1$ for the SIR model. Bottom panel : Asymptotic values of the number of susceptibles and the cumulative number of infected in the SIR model for different values of σ and I(0) = 0.01.

The reproduction number R_0 is often used in epidemiology to quantify the efficiency of the social distancing measures [148]. In order to contain a pandemic, the main goal of models simulating the effects of measures is to maintain R_0 under 1 to decrease the number of newly infected persons.

Studying now the asymptotic behavior of the model, we can see that there are numerous differences between the SI and the SIR model. However, it is possible to write the SIR

equations in (239) in a similar way as for the SI model in (235) by considering another compartment $I_c(t) = I(t) + R(t)$ which corresponds to the cumulative number of infected individuals and allows to rewrite the SIR equation to make it look almost like the SI equation.

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = -\gamma S(t)I(t) ,\\ \frac{\mathrm{d}I_c}{\mathrm{d}t} = \gamma S(t)I(t) . \end{cases}$$
(243)

However, because I_c and I become more and more different as the epidemic evolves, the asymptotic behavior will differ anyway between SI and SIR. The asymptotic value of I(t) is always N in the SI model while it is always 0 in the SIR model. On the other hand, the asymptotic value of the I_c population is not unique and depends on the parameter σ (through the parameters γ and ϵ) and on I(0) as shown in the bottom plot in Figure 17. It is possible to compute the dependence of the final value of I_c by noticing that it exist a conserved quantity G(t) defined as :

$$G(t) = S(t)e^{\sigma R(t)}, \qquad (244)$$

such that :

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \left(\frac{\mathrm{d}S}{\mathrm{d}t} + \sigma \frac{\mathrm{d}R}{\mathrm{d}t}S(t)\right)e^{\sigma R(t)} = (-\gamma + \sigma\epsilon)S(t)I(t)e^{\sigma R(t)} = 0.$$
(245)

This invariance implies that G is constant and that for any time t, there is a relation between S(t) and S(0). By considering that the initial number of recovered is R(0) = 0, for any time t, we have G(t) = G(0) = S(0) and we have :

$$S(t) = S(0)e^{-\sigma(1 - I(t) - S(t))}.$$
(246)

Taking a sufficiently large time such that I(t) = 0, we can solve the last equation to express $S(\infty)$, and thus $I_c(\infty)$, in terms of I(0) and σ :

$$I_c(\infty) = 1 - S(\infty) = 1 + \frac{1}{\sigma} W \left[\sigma e^{-\sigma} \left(I(0) - 1 \right) \right], \qquad (247)$$

where W is the W-Lambert function defined such that :

If
$$y = xe^x$$
, then $W(y) = x$. (248)

The dependence of the asymptotic value of $I_c(t)$ on σ can be seen in the bottom plot of Figure 17. As argued before, we can see that the total number of infected starts being quantitative when $\sigma > 1$ and grows to any values from 0 to N for different choices of σ .

The SIR model can also be extended by allowing for reinfection of the recovered population. This would model a disease where immunity exists only for a certain time which is the case, for example, considering seasonal flu. We will focus on the case of flu in section 4.3.3. The reinfection can be modeled using the equations :

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = -\gamma S(t)I(t) + \zeta R(t) ,\\ \frac{\mathrm{d}I}{\mathrm{d}t} = \gamma S(t)I(t) - \epsilon I(t) ,\\ \frac{\mathrm{d}R}{\mathrm{d}t} = \epsilon I(t) - \zeta R(t) , \end{cases}$$
(249)

where ζ can be seen as the inverse of the average time of immunity before becoming susceptible again.

Other more complex models have been investigated, adding for example the compartments of Deaths (D) associated with a "death rate" different from the recovery rate, Exposed (E)representing susceptible who have been infected but are not yet infectious, Quarantined (Q)where an infected get quarantined and cannot infect anymore and many more other compartments [149].

4.1.2 Percolation models (stochastic approach)

Even though compartmental models reproduce the behavior of a pandemic accurately, it doesn't give any information on the microscopic scale of the process. This is why models simulating the spread of the infection at the unit level have been investigated. They are the most direct way, yet more computer time-consuming, to look at a pandemic from the fundamental scale of individuals. Among them are the lattice simulations, in which individuals are represented by the lattice nodes of a grid. The sites can be in different states such as susceptible or infected as we saw for compartmental models. The disease is spread with a certain probability to the neighbours of infected nodes, following an established set of rules. Introducing probabilities, lattice models, therefore, have to be employed using discretised time steps and any general result is extracted after taking the average over several simulations to reduce stochastic effects. As will be seen in the following, even simple models of this type show features such as criticality and time-scaling symmetries.

A larger class of stochastic processes are the percolation models, which generally speaking consist of sites scattered in space that can be connected together through links. One can distinguish two types of percolation models :

- → Bond percolation models : Points have a fixed space coordinate and the links between them are created randomly. Lattices simulations in various spatial dimensions and shapes are the most common example.
- → Site percolation models : The position of the points is random, while the links between different points are created based on rules related to their coordinates.

It also exists complex models involving both characteristics, but we will not consider them. In any percolation model, the probability that two points are connected to each other through a chain of links is called the pair connectedness and is of prime importance to know how a system could evolve. Assuming there are infinitely many sites, it is possible to distinguish whether the structure is made of only local clusters in which finitely many sites are connected or whether it is in a percolating state where infinitely many sites are connected. Going from one state to the other usually depends on the value of the probability p that a link exists between two neighbouring sites. Therefore, the transition from local to percolation state can be described as a phase transition. Close to the critical value p_c , the system lies in the same universality class of several other models, implying that analog quantities will have the same characteristic power law behaviour in the different theories. As an example, the probability P(p) for a system to be in the percolating state takes the form :

$$\lim_{p \to p_c} P(p) \approx \left(p - p_c \right)^{\nu}, \qquad (250)$$

where ν is called a critical exponent. Models within the same universality class share the same critical exponents despite the fact that the concrete details of the theory and in particular the meaning of the quantity P may be very different. This connection makes clear the similarities between percolation and other models and explain why many of them have been studied extensively.

The simplest possibility for a stochastic model is to consider that each site represents only one individual within a population scattered on a finite two-dimensional squared lattice Γ . The individuals are generated at the knots of the grid and can be either Susceptible (S), Infected (I)or Recovered (R). They follow the rules through an algorithm driven by two basic mechanisms that simulate the evolution of the disease in discretised time steps. The laws represents the infection by an infectious neighbour and the recovery from the disease. In the following, we will point out some of the key-features of this model as functions of two parameters :

- → The infection probability $0 < p_{\gamma} < 1$ for an infectious individual to infect a susceptible neighbouring site. At each step of time t, if a susceptible site is next to, at least, one infected knot, a random number is generated between 0 and 1. If this number is smaller than p_{γ} , the site becomes infected at the next time step t + 1. If this number is higher, the site remains susceptible.
- → The recovery probability $0 < p_{\epsilon} < 1$ for an infectious individual to recover from the disease. At each time step t, if a site is infected, a random number is generated. If this number is smaller than p_{ϵ} , then the site becomes recovered. If not, it remains infected.



FIGURE 18 – Top plots : Lattice simulation at different times (t = 0, 50, 100, 150, 200) with a unique initial infected in the middle of the (100 × 100) lattice as initial condition and $p_{\gamma} = 0.3$ and $p_{\epsilon} = 0.2$. White dots are susceptible individuals, black dots are infected ones and brown are recovered. Bottom plot : Evolution of the total number of susceptible, infected, recovered and cumulative infected for each step of time in the simulation.

We will consider in this simple model that infectious individuals can infect only their closest neighbours. In Figure 18, we can see the evolution of the quantities generated in the simulation described before considering only one infectious individual in the center of the lattice as an initial condition. The cumulative number of infected I_c as show in the bottom plot seems to have a similar behavior as for SIR model. For example, at large times t, the asymptotic number of infected, which is a function of $(p_{\gamma}, p_{\epsilon})$ can be substantially different [55]. This was also obtained by varying γ and ϵ in SIR model. However, multiple differences should be taken into account between stochastic and deterministic models. First, we can see in this simulation that boundary conditions were imposed. Here the spreading stops when it reaches the borders and this influences the evolution of the virus. Indeed, if we had considered periodic boundary condition, the final solution would be different. In a second time, we can argue that taking a squared lattice has an effect on the evolution of the virus. Triangle or hexagonal lattices would have given different results due to the different number of neighbours for one knot. Finally, we have considered here that only the closest neighbours can be infected by a ill knot and that the infection probability doesn't depend on the number of infected neighbours. It would be possible to define a radius in which a site can affect its neighbours and to make the probability dependent on the number of ill neighbours or their distance. The model presented above was the simplest one, but not representative of what can be done for lattice models. To have a more general point of view, one needs to look at clear connections between percolation models and compartmental model.

To make the connection with compartmental models clearer, we will now study a new model where sub-populations S, I and R can be stacked at space-time points. We then show the time evolution of a disease by dividing the individuals that are present at a given lattice site $x \in \Gamma$ into three classes [150] such that :

- → Susceptible where n_x^S is the number of susceptible individuals at x.
- → Infectious where n_x^I is the number of infectious individuals at x.
- → Recovered where n_x^R is the number of removed individuals at x.

The processes for nearest neighbour x and x' in the model are :

$$n_x^S + n_{x'}^I \to n_x^I + n_{x'}^I, \quad \text{Infection rate } \hat{\gamma} ,$$

$$n_x^I \to n_x^R , \quad \text{Recovery rate } \hat{\epsilon} .$$
(251)

By considering the process as deterministic and the variables n_x^S , n_x^I and n_x^R as continuous functions of time, the processes can be expressed with first order differential equations :

$$\begin{cases} \frac{dn_x^S}{dt}(t) = -\hat{\gamma}n_x^S(t)\sum_{x'\in\mathbf{e}} n_{x'}^I(t) ,\\ \frac{dn_x^I}{dt}(t) = -\hat{\epsilon}n_x^I(t) + \hat{\gamma}n_x^S(t)\sum_{x'\in\mathbf{e}} n_{x'}^I(t) ,\\ \frac{dn_x^R}{dt}(t) = \hat{\epsilon}n_x^I(t) . \end{cases}$$
(252)

where the sums extend over the nearest neighbours of site $x : \mathbf{e}(x)$. This set of equation implies that the total quantity of individuals is conserved for any time t such that :

$$N = \sum_{x \in \Gamma} \left(n_x^S(t) + n_x^I(t) + n_x^R(t) \right) \,. \tag{253}$$

It is then natural to define the percentage of representation of each compartment over the total population using the following definitions :

$$S(t) = \frac{1}{N} \sum_{x \in \Gamma} n_x^S(t) , \quad I(t) = \frac{1}{N} \sum_{x \in \Gamma} n_x^I(t) , \quad R(t) = \frac{1}{N} \sum_{x \in \Gamma} n_x^R(t) , \quad (254)$$

such that S(t) + I(t) + R(t) = 1 for any time t. Taking the mean field approximation such that the sums in Equation (252) can be rewritten in terms of I(t), we finally obtain the equations :

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = -\hat{\gamma}qS(t)I(t) ,\\ \frac{\mathrm{d}I}{\mathrm{d}t} = \hat{\gamma}qS(t)I(t) - \hat{\epsilon}I(t) ,\\ \frac{\mathrm{d}R}{\mathrm{d}t} = \hat{\epsilon}I(t) , \end{cases}$$
(255)

where q is the number of nearest neighbours. Thus, by redefining $\gamma = \hat{\gamma}q$ and $\epsilon = \hat{\epsilon}$, we exactly obtain the equations of the standard *SIR* model shown in equation (239). Therefore, taking a deterministic approach and the mean-field approximation is sufficient to go from percolation models to compartmental models.

We saw here that multiple models can be used represent the outbreak of an epidemic from a microscopic to a macroscopic point of view or from a deterministic or a stochastic approach. Each model described above can be related with one another but each one has its particularities. We will now see a new point of view given through the Renormalisation Group equations scope.

4.2 Epidemic Renormalisation Group equations

4.2.1 eRG basics

The time dependence of the data for the cumulative number of infected reported cases shows the same characteristic behaviour for different epidemic and various regions of the world. Looking at the Hong Kong SARS 2003 outbreak, as well as the COVID-19 pandemic during the spring of 2020, it has been shown that logistic functions are of prime importance when studying the cumulative number of infected [53]. The same observation has been done for the total number of hospitalisation, people admitted in reanimation and number of deaths, but we will follow mostly the cumulative number of infected in the next parts. As an example, data for the first wave of COVID-19 in four sample countries from four different continents are shown in Figure 19.

We can see from the figure that the evolution for the first wave is similar in each country, no matter what continent they belong to. The cumulative number of infected cases seems to follow the opposite trend shown in Figure 3 in the case of Wilson-Fischer fixed point.



FIGURE 19 – Cumulative number of infected cases of the first wave of COVID-19 for 4 different countries (United Kingdom, South Africa, Brazil and Japan) from 4 different continents, showing the universality of the evolution of the disease.

The two periods before and after the wave are showing a quasi-constant number of cumulative infected corresponding to what we called a fixed point in QFT as explained in section 2.3.3. This evolution can be easily understood by going back shortly to compartmental models.

Starting from the equation we extracted before in the case of the SI model in Equation (236), we can redefine $N\gamma_{SI} = \gamma$ and N = a giving :

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \gamma I(t) \left(1 - \frac{I(t)}{a} \right) \,. \tag{256}$$

This equation will be the starting point of the study of an epidemic episode through our new framework. To show its relation with Renormalisation Group, we can expand this equation to obtain :

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \gamma I(t) - \frac{\gamma}{a} I(t)^2 \,. \tag{257}$$

Comparing this to the general beta function obtained in equation (98) showing non-trivial fixed points, one can identify the following quantities :

$$\ln\left(\frac{\mu}{\mu_0}\right) \leftrightarrow -\frac{t}{t_0} \quad , \quad \mathcal{A}(\mu) \leftrightarrow I(t) \; , \qquad \beta_{\mathcal{A}} \leftrightarrow -\frac{\mathrm{d}I}{\mathrm{d}t} \; , \qquad (258)$$
$$b_0 \leftrightarrow \gamma \qquad , \quad b_1 \leftrightarrow \frac{\gamma}{a} \; .$$

These relations allow us to see the connection between epidemiology and QFT via the Wilson-Fischer or Banks-Zaks fixed points we studied in section 2.3.3. It also gives the opportunity to link it with the asymptotic Grand Unification through equation (181), showing the universality of the RGE to describe different processes.

As we saw in the case of compartmental models, the solution for equation (256) is analytical and has the form of a logistic function defined by the equation :

$$I(t) = \frac{ae^{\gamma t}}{b + e^{\gamma t}} .$$
(259)

Therefore, the parameter a corresponds to the asymptotic number of infected cases at the end of the epidemic wave, because any logistic function has the following asymptotic values :

$$\lim_{t \to -\infty} I(t) = 0 , \quad \lim_{t \to +\infty} I(t) = a , \qquad (260)$$

corresponding to the zeros of the derivative. The parameter γ , with the dimension of a rate, measures how fast the number of infections increases, while *b* corresponds to a time shifting of the entire curve as was seen in equation (238). We can also notice that the parameters γ and *a* can be removed from the differential equation by rescaling the function and the time variable such that :

$$\tilde{\beta} = \frac{\mathrm{d}\tilde{f}}{\mathrm{d}\tau}, \quad \tau = \gamma t, \quad \tilde{f}(\tau) = \frac{f\left(\frac{\tau}{\gamma}\right)}{a},$$
(261)

while a is incorporated by a mere normalisation giving the fixed points 0 and 1, γ can be thought of as a "time dilation" parameter. Once the normalised solutions are shown in the local time τ , all epidemic waves reveal the same temporal shape. In order to compare the different regions, we will, however, keep a and γ for the data analysis.

The logistic function stemming from equation (256) allows to fit the actual data and to extract γ and a for any country. Looking at the cumulative number of infected for Italy in Figure 20, where full lines are fits and dots are daily data, we can see that the logistic function is enough to fit it, but seems to depend strongly on the earlier and later times of the wave.



FIGURE 20 – Data for Italy's first wave for the cumulative number of infected (left plot) and daily number of new infected (right plot) and logistic function fit for different range of data (lightest blue for t from 1 to 60, then darker fits for 80, 100 and 120.

In the figure, the fits for different range are shown where the lightest curve is performed for t from 1 to 60, then the darker curves goes to 80, 100 and 120. The fits parameters obtained

for the first wave in Italy for the different range of fits are :

$$a = \begin{pmatrix} 129663\\ 179960\\ 213769\\ 225753 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0.184\\ 0.133\\ 0.105\\ 0.095 \end{pmatrix}.$$
(262)

a and γ depend strongly on the range of the data studied. This is caused by an interplay between the parameters a and γ , as shown in [151]. However, it seems from the left plot of Figure 20 that the last fit is better at reproducing the data. Moreover, we can see from the same figure that the derivative of the logistic function is symmetrical with respect to the peak timing, while it is not the case for the data. Pushing the end of the fit further make the logistic function fit away from the peak, losing some accuracy on the parameter γ but gaining in the parameter a accuracy. The asymmetry observed in the data can be due to the fact that the situation within a country evolve thanks to social distancing measures or change in behavior. This possibility has been analysed by taking into account possible effects of social distancing measures, such as lockdown, on the data and will be seen in section 4.2.2.

We saw already that the features shown by the data could be studied thanks to an equation similar to Renormalisation Group equation. Therefore, in order to relate the spread of a disease to the Wilsonian RG approach, in the same way we related the QFT parameters to the hydrodynamic evolution of bacteria in a pipe in equation (78), a dictionary was developed linking the parameters from both models :

→ The time variable is associated to the negative logarithm of the energy scale μ , with reference time t_0 and reference scale μ_0 , such that we have :

$$\frac{t}{t_0} \leftrightarrow -\ln\left(\frac{\mu}{\mu_0}\right) \,. \tag{263}$$

For simplicity, we will consider $t_0 = 1$ in the following. For such identification, the differential equation presented in (256) is similar to the gauge coupling RGE in a theory with an interactive fixed point in the IR [72] such as BZ fixed points, as seen in section 2.3.3.

→ The quantity studied in epidemiology, mostly the cumulative number of infected I_c , can be associated to a coupling constant in the RGE. The epidemic coupling strength is defined as a monotonic, derivable and bijective function ϕ of the epidemic quantity such that :

$$\alpha(t) \leftrightarrow \phi(I_c(t)) . \tag{264}$$

We will mostly consider the case $\phi(x) = x$ throughout the thesis but in [53, 56, 58, 59], we considered it as being the natural logarithm $\phi(x) = \ln(x)$. The logarithm definition will be explicitly shown by the mention (log) next to the fit parameters a in the following tables. The different choices of ϕ are justified by a more or less better fit to the actual data of the COVID-19 pandemic.

→ The beta function, that is function of the coupling strength, is defined as the timederivative of the epidemic coupling strength :

$$\beta \equiv \frac{\mathrm{d}\alpha}{\mathrm{dln}\left(\frac{\mu}{\mu_0}\right)} = -\frac{\mathrm{d}\alpha}{\mathrm{d}t} \leftrightarrow -\frac{\mathrm{d}\phi\left(I_c\right)}{\mathrm{d}t} = -\frac{\mathrm{d}\phi}{\mathrm{d}I_c}\frac{\mathrm{d}I_c(t)}{\mathrm{d}t} \ . \tag{265}$$

Since ϕ is a monotonic function, fixed points of the β -function correspond to zeroes of the derivative of I_c , denoted I_c^* and $\alpha^* = \phi(I_c^*)$ is the epidemic coupling at the fixed point. Any fixed point can be characterised through the scaling exponents :

$$\theta = \left. \frac{\mathrm{d}\beta}{\mathrm{d}\alpha} \right|_{\alpha^*} \,, \tag{266}$$

which are negative for a UV-repulsive fixed point and positive for an UV-attractive one.

Thanks to this dictionary the features of epidemiology have clear relations with Renormalisation Group equation in QFT and will allow us to study the spread of epidemics in the scope of RG through the **epidemic Renormaisation Group** (eRG).

4.2.2 Correlation with mobility data

In order to look for any correlation between social distancing measures and effects on the epidemic evolution, we used open-source mobility data from Google and Apple accessible on the following websites : google.com/covid19/mobility/ and covid19.apple.com/mobility as explained in [58]. The sets show different data collected in a different way. Google data shows movement trends by region, across different categories of places including working places, residential locations, grocery shops, parks ... The data represents how time spent in categorised places by visitors changed using geotracking from the Google maps application. It shows the evolution of mobility as compared to a reference period defined as the median value of the five weeks interval from the 3rd of January to the 6th of February 2020, predating the spread of the virus in Europe and in the US. As categories, we will only use the "Residential" and "Workplaces" percentage changes, that best describe the change in people's behaviour after the implementation of social distancing measures. On the other hand, Apple data represents a relative volume of direction requests on the Apple maps application per country, region, sub-region or city, as compared to a baseline volume defined on the 13th of January 2020. We will only use, from Apple, information about "Driving" and "Walking" percentage change, because "Transit" data is not accessible for all regions. We assume that the two categories represent the time spent by people away from home. For the US states, only "Driving" data is available, so we will not use the "Walking" category for them.

We decided to focus our analysis on several European countries and on all the US states, in order to have comparable data. Looking at the raw mobility shown in Figure 21 for the extremes cases of mobility change for Europe (Spain and Sweden) and in the US (New York state and Wyoming), we can see directly that the regions had a drop in workplace, driving and walking percentages and a rise in the residential percentage just after the first wave started growing exponentially. This can be related to social distancing measures taken at the region level such as the lockdowns imposed for example in Spain or in New York, but also to changes



FIGURE 21 – Raw Google and Apple mobility data during the first wave of COVID-19 in 2020 for Europe (left) and the US (right). The time scale is shifted so that the beginning of the social distancing measures, defined by a 20% drop in Google's Workplace indicator shown by the first vertical grey line, coincides for the two sampled countries or states. The other two vertical lines mark the end of the 6 and 8 week averaging periods respectively. The Highest Mobility regions are shown in orange and the Lowest Mobility one in blue : for Europe, Sweden (orange) and Spain (blue); for the US, Wyoming (orange) and New York (blue).

in population self-behavior, even if no measures were taken. This is the case for Sweden and Wyoming where a similar change is visible even if no lockdowns were imposed.

Google mobility data shows also the week periodicity caused by weekends. This is easily explained by the fact that before any social distancing measures were taken, people were staying more at home and going less to work during the weekend, reducing the percentage of modification during the COVID-19 first wave as compared to the reference period. In the same way, Apple shows periodicity for the data before the social distancing measures. This is due to the fact that the reference day was taken on a Monday, showing strong differences with weekend days after social distancing measures have been taken. To avoid such artifacts, we decided to study the mobility reduction by averaging it on several weeks. Another particularity is that the mobility data rises (or decreases for the residential data) shortly after it dropped. Unfortunately, the increase is much slower than the reduction making impossible to define an clear ending date to the measures.

For all the other countries and states studied, a similar drop (or rise for the residential percentage) is observed. This drop can be used to define a starting date for the social distancing effects that we considered more accurate than governmental social measures imposition dates for two reasons. First, as said before, some countries didn't impose any measures, so this drop is the only way to define a beginning to mobility changes. Secondly, some countries imposed social distancing measures but gave several days before applying them completely to let people coming back to their family home or to decide where to spend the lockdown. So, days when measures had an effect are more easily accessible using mobility data. In order to have a universal and trustful reference for the starting point, we decided to specify that this timing was reached whenever the working places data dropped for the first time below -20%. As can be seen from Figure 21, at this specific reference time, all the other mobility data have a similar change comforting us in the relevance of this choice. While defining a starting point of the social distancing measures is easy, defining an ending point is much more difficult. From Figure 21, we can see that, after reaching a minimum, the mobility data is rising slowly and really differently for the various counties and states of the study.

To improve our confidence in the mobility data relevance, we decided to compare the different sets of mobility data. In order to get rid of the week periodicity and as it is almost impossible to define a social distancing measures period because the ending point is not well defined, we decided to average the mobility data over 6 and 8 weeks after the beginning of the measures. The two choices are defined to be of the order of the official lockdown duration imposed in Europe and in the US and to show the stability over the arbitrary choice of the averaging period. Comparisons between the different mobility indicators are shown in Figure 22. In these "tadpoles" plots, the head of the tadpole represents the averaged mobility value over 6 weeks after the beginning of social distancing measures, while the tail of the tadpole represents the averaged value over 8 weeks. Looking at the first row of plots, we can clearly see that within the same dataset (from Google or Apple), the indicators we choose are strongly correlated for both Europe and the US. This tendency is still visible for the second and third rows showing correlation between Google and Apple mobility categories. Moreover, we can clearly see that the US states mobility averaging distribution is more compact than for Europe showing a more universal response to the pandemic in America than in Europe. Finally, these plots show that even if datasets are obtained from different methods, they are in agreement and this strengthen our confidence in the coherence of the mobility categories we choose.

To compare the mobility of different regions, it will be useful to combine the different data into a single value that will allow us to classify the states and countries. Such value is called



FIGURE 22 – Tadpole plots showing correlations between the four mobility reduction categories : Residential and Workplace from Google, Driving and Walking from Apple (Walking is not available for the US). The head of the tadpoles corresponds to the averaged value over 6 weeks after the social distancing measures begins, while the tail indicates the 8 week average. The colour code in the tadpole plots reflects the immobility indicator range defined in Figure 23

the "immobility indicator" (noted M) and is defined by :

$$\mathcal{M}(\text{region}) = \sum_{j \in \text{cat.}} \frac{|p_j(\text{region})|}{\max[|p_j|]}, \qquad (267)$$

where the sum is over the category of mobility data : "Workingplaces", "Residential", "Walking" (only for Europe) and "Driving". $|p_j(\text{region})|$ is the absolute value of the average percentage variation in each category labelled by j. For each mobility type, we divide by the maximal value observed in the pool of regions. Note that for European countries we have four categories, so that $0 < \mathcal{M} < 4$, while for the US states we have 3 categories, so that $0 < \mathcal{M} < 3$. We use this indicator to rank the European countries and the American states from the ones with high mobility (HM) (i.e small \mathcal{M}) to the one with low mobility (LM) (i.e large \mathcal{M}). The values of the immobility indicator we obtain for the European countries under study and US states are shown in Figure 23. The colour code ranges from the highest mobility region in bright red to the lowest one in cyan, with gradient proportional to the value of the immobility indicator and was also used for Figure 22.



FIGURE 23 – Top panel : The two maps represent respectively the European and US states with different shades of mobility from the highest (HM) in bright red to the lowest (LM) in cyan. Bottom panel : Immobility indicator for the European countries and the US states. Values of M are shown for Europe on the top and the US at the bottom. The colour code corresponds to the ranking of each European country and each US state.

Now that we are able to define and classify the different mobility in the regions, one can wonder if it is possible to see any correlation between the mobility reduction intensity and the parameters of the logistic functions used to fit the epidemiological data. This could show if the strength of the measures has an impact on the spread of the virus. To this end, we fit the data for the cumulative number of infected cases for the countries studied for the first wave from the website "OurWorldInData" accessible here : https://ourworldindata.org/coronavirus. Then, we compare the fit parameter γ of the logistic function with the mobility data, giving the "race-cars" plots in Figure 24.

Unfortunately, we can see that no clear correlation exists between mobility data and the parameters γ . This may have multiple explanations. First, we can expect that mobility data



FIGURE 24 – Infection rate compared to the mobility data. Race-car plots showing the fitted infection rates γ versus the Google and Apple mobility categories. The vertical segment indicates the difference between 6 weeks (dot) and 8 weeks averages while the horizontal bars indicate the fit error on γ .

only represents a small part of the total change in behavior adopted to curb the evolution of the virus. For example, it does not take into account mask wearing, cultural and shopping places closures, hand-washing habits, measures against big gatherings... while these variations may have influence the virus evolution [152, 153]. Secondly, γ represents an integrated infection rate over the whole wave period. It may not represent clearly the effects of social distancing measures as these effects could be hidden within the fitting period. In order to test the last possibility, we will study the hypothesis of a variation of the infection rate after the social distancing measures were imposed.

In this part, we suppose that the social distancing measures had an effect on the evolution of the pandemic, but only on the parameter γ . For simplicity, we consider that the Social Distancing (SD) measures are taken at the time $t_0 = 0$. Supposing that the effects due to SD are delayed, we represent this delay by arguing that the change in the infection rate happens at a specific time Δt after the measures are taken. Therefore, the wave can be split into three different time periods represented in Figure 25. Period **A** represents the period before SD measures are taken, period **B** corresponds to the period after SD measures are taken but before they start to have an effect on the evolution of the epidemic and finally C is the period after the effects are fully acting on the infection rate.



FIGURE 25 – Evolution of $\alpha_{2\gamma}$ function in comparison with the standard logistic function. Region A is taking place before any social distancing measures are imposed, region B corresponds to the period between the beginning of SD measures and when they start having an effect on the infection rate and finally region C corresponds to the period after the effects are fully considered.

The assumption of a change in γ can be achieved by supposing different equations for the epidemic strength α for the region $\mathbf{A} + \mathbf{B}$ and for the region \mathbf{C} such that :

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \begin{cases} \gamma_B \alpha(t) \left(1 - \frac{\alpha(t)}{a}\right) & \text{for } t < \Delta t ,\\ \gamma_C \alpha(t) \left(1 - \frac{\alpha(t)}{a}\right) & \text{for } t > \Delta t , \end{cases}$$
(268)

where γ_B represents the infection rate in the period $\mathbf{A} + \mathbf{B}$, before social distancing measures effects and γ_C in the period \mathbf{C} after the effects are taken into account. A differential equation valid on different domains is common as what was seen in the Renormalisation Group in the case of supersymmetry in equation (130) or for the asymptotic Grand Unification in equation (175). Therefore, one can interpret this equation as involving a phase transition between a time period without any SD effects and one where they are fully considered and for which the coupling γ changes. As shown in equation (259), each part of the differential equation has an analytical solution which happens to be a logistic function. By connecting the two solutions such that the reconstructed function is continuous, we obtain the following solution :

$$\alpha_{2\gamma}(t) = \begin{cases} a \frac{e^{\gamma B^{t}}}{b + e^{\gamma B t}} & \text{for } t < \Delta t , \\ a \frac{e^{\gamma C t}}{b e^{(\gamma C - \gamma B)\Delta t} + e^{\gamma C t}} & \text{for } t > \Delta t . \end{cases}$$
(269)

 $\alpha_{2\gamma}$ is defined to be a continuous function, but cannot be derived at the time $t = \Delta t$. More complex models could investigate a smoother variation, by introducing a transitional function

driving γ_B to γ_C smoothly during period **B**. However, the choice of the transitional function would be arbitrary and introduce more parameters. On the other hand, $\alpha_{2\gamma}$ is defined by five parameters $(a, \gamma_B, \gamma_C, b \text{ and } \Delta t)$, so two more than a standard logistic function. As γ_B and γ_C should differ, we define the percentage of variation in the infection rate $\Delta\gamma$ as :

$$\Delta \gamma = \frac{\gamma_C - \gamma_B}{\gamma_B} \,. \tag{270}$$

Before using the function $\alpha_{2\gamma}$ on real data, it needs to be tried on a Monte Carlo (MC) simulation to show its effectiveness. MC data are randomly generated around an average value following the function $\alpha_{2\gamma}$ within one standard deviation $\left[N_i - \sqrt{N_i}, N_i + \sqrt{N_i}\right]$, where N_i is the cumulative number of cases per day as predicted by the generating function. We generated 500 independent sets of mock data, where one is shown in Figure 26. Then, fitting the MC data using a simple logistic function, we find it is not enough to extract the parameters used to generate the data as can be seen in Table 8. We see that the parameters a and b are underestimated and that the fitted γ seems to be an average of the values for γ_B and γ_C . On the other hand, a fit realised with the function $\alpha_{2\gamma}$ seems to recreate accurately the generation parameters. It can be shown that the accuracy depends on the value of a due to the increased uncertainties in the generation of the data. For a = 100, we can determine the value of Δt within a range of one week and $\Delta \gamma$ with an accurate precision below 2%. Moreover, generating data where $\gamma_B = \gamma_C$, equivalent to a simple logistic function generation, the function $\alpha_{2\gamma}$ fails to predict Δt and $\Delta \gamma$ by fitting random values as can be seen in Appendix C.1. Therefore, if the fits on real data show values narrowly distributed around a certain average, this would show that the data may certainly behave as in a 2- γ scenario. Having acquired confidence in the method, we now apply it to the real data.



FIGURE 26 – Sample of the Monte-Carlo generation (black dots) and the associated fits for a logistic function (orange) and for the function $\alpha_{2\gamma}$ (blue). The vertical gray line represents the time $t = \Delta t$.

In order to extract Δt in a straightforward way, we decided to fit only the data starting at the day defined to be the starting of social distancing measures (i.e. when "workingplaces"

	generated	$\alpha_{2\gamma}$ fit	logistic fit
a	100	100.0(4)	98.4(3)
γ_B	0.1	0.100(2)	ø
γ_C	0.05	0.050(2)	ø
γ	ø	ø	0.077(1)
b	100	100(7)	41(2)
Δt	50	50(1)	ø
$\Delta \gamma$	-0.5	-0.50(2)	ø

TABLE 8 – Parameters obtained for the MC data fits using a logistic function and the function $\alpha_{2\gamma}$ defined in equation (269). Data generation parameters are shown in the second column.

data is first reduced by 20%). The results for the pool of countries and US states are shown in Figure 27.

As can be seen from the top histograms, most regions display a similar Δt . By fitting the distributions to a Gaussian function, we find that to the two sigma level we have $\Delta t = 2.7 \pm 1.7$ weeks for Europe and $\Delta t = 3.3 \pm 1.6$ weeks for the US. The high compatibility of the two ranges shows the emergence of a universal time scale before social distancing measures start being effective on the data. Another important result is the general and strong reduction of the infection rate measured within and after Δt both for Europe and the US, as shown in Figure 27. As expected, $\Delta \gamma$ is negative, following a reduction in γ caused by the social distancing measures. Here also, reductions are concentrated mostly between -20% and -40% for European countries and between -20% and -70% for the US states. These results comfort the idea that social distancing measures had an effect on the evolution of the pandemic, however, the comparison between the averaging mobility data and the infection rates still doesn't show any correlation as can be seen in Appendix C.2. Then, the $2 - \gamma$ description was useful to extract a typical time delay for SD measures to have an effect and to show that there is a significant reduction in the infection rate, but is not enough to show any effect related to the strength of the mobility reductions.

Now that we saw that the simplest eRG equation can be used to explain the evolution of the pandemics, we will try to embed in it other epidemiological features such as the interaction between different countries.

4.2.3 Interacting eRG

While we have only considered yet the evolution of the disease for isolated regions, the interactions between them allow to understand the spreading scenario or the timing of the waves. For any country other than the one in which the first infection appeared, the emergence of the COVID-19 was caused by an interaction with an infected country through travelling passengers. In order to take into account this interaction, we consider an eRG equation for each region considered and add interaction term between them. Associating the evolution for each country thanks to the subscript i, the interaction eRG equation can be written :

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} = \gamma_i I_i(t) \left(1 - \frac{I_i(t)}{a_i} \right) + \sum_j \frac{k_{ij}}{N_i} \left(\frac{I_j(t)}{I_i(t)} - 1 \right) \,, \tag{271}$$



FIGURE 27 – SD effects delay Δt (top) and percentage variation in the infection rate $\Delta \gamma$ (bottom) for the European countries (left) and US states considered (right). Colors follows the ones used through the immobility indicator in the maps in Figure 23.

where N_i is the population in region *i* and the terms k_{ij} are the number of travelers from region *j* to region *i* per time unit. The matrix k_{ij} is considered symmetric here such that there are as



FIGURE 28 – SD effects delay Δt (top) and percentage variation in the infection rate $\Delta \gamma$ (bottom for the European countries (left) and US states considered (right) in the form of histograms, for Europe and the US separately, highlighting that Δt clusters around similar values.

much travellers from j to i than there are from i to j, in order to impose the invariance of the population within each region. The interaction term shows that if the situation is identical in two different regions, then the term would vanish, seeing them as a unique isolated country. In any other case, the most infected region will inevitably inject more infectious individual in the less infected region. This model has been studied in details in [154] showing the importance of border controls before any infection happens within the country, while border controls become inefficient once the country is sufficiently infected. In this situation, only social distancing measures can efficiently curb the disease propagation. One of the important property of the interactive eRG equation is that the values of the k_{ij} matrix control mainly the timing of the beginning of the wave but doesn't change much the following curve that can still be fitted with a logistic function. This is due to the fact that once the exponential phase of the wave has started the second term in equation (271) becomes negligible.

As a first step to check the validity of this model to represent real data, we try to use it to simulate the spread of the first wave of COVID-19 in the United States [59]. The US are considered to be sufficiently large to show delayed evolution of the epidemics across its states and sufficiently homogeneous in the way data is collected. We also decided to split the 50 states of

the US into the 9 census divisions which are : New-England (NE), Mid-Atlantic (MA), South Atlantic (SA), East South Central (ESC), West South Central (WSC), East North Central (ENC), West North Central (WNC), Mountains (M) and Pacific (P). The division locations are shown in the top-left panel of Figure 29 and their state composition is shown in table 16 in the Appendix C.3. The choice to study the spread at the census divisions scale is motivated by the fact that states have very different demographic distribution. A state-by-state mathematical modeling, therefore, is challenged by statistical artifacts.

Looking at the situation in the whole US in the top-right panel of Figure 29, we can see different waves whose peaks happen in the beginning of April, in the middle of July and in the period from November to January.



FIGURE 29 - Top-left panel : Location of each census division considered. Number of new infected cases per day in the whole United States (top-right panel) and in each division (bottom panels).

The two first bumps cannot be considered as two separate waves because they didn't happen in the same locations. Indeed, as can be seen in the bottom panels of Figure 29, the wave in April was mostly located in the New England and South Atlantic divisions which are located next to each other in the North-East part of the country. SA contains the city of New York that was the main place affected by the pandemic at the beginning of March. The virus came in the US mainly in New York where flights connections with abroad destinations are very important. Moreover, the states in the NE division are very well connected to the city of New York, as it as at the limit between the divisions. Some people are also living in NE states but working in NY, allowing the virus to spread easily there. We can also see a small bump in ENC and WNC divisions around the same time that were maybe related to Chicago infections, but as they didn't involved as many cases as in NE and MA, we didn't consider them as a first wave.

Then, we can see that another peak happens in the middle of July and that for this wave, while nothing can be seen for the two divisions previously cited (MA and NE), it is well visible in SA, ESC, WSC, M and P. Those divisions compose the south and the west part of the country that are holidays destinations during summer. The scenario suspected there is that the virus came in New York, infected the city, then spread across the country thanks to the holiday travelling.

In order to corroborate this scenario, we first try to find what values of k_{ij} are needed to reproduce the first wave if the pandemic was due only to the interaction between the Mid-Atlantic division and any other division (i.e. $k_{ij} \neq 0$ only if i = 2 is the Mid-Atlantic division). In order to do that, we start by fitting the first wave happening in each division with a logistic function to extract their parameters a and γ as shown in Table 9. Then, replacing them in equation (271) and solving it for each division j by taking as a seed region the MA data and just changing the value of k_{2j} , the most accurate values defined by using a χ^2 method are listed in the first row of Table 10. The k_{ij} values represent, here, the number of travellers per week in million. We can clearly see that the values for SA, ESC, WSC, ENC, WNC, M and P are between 5×10^{-4} and 5.3×10^{-3} , in agreement with the values obtained for a similar simulation for Europe in [56]. Therefore, the number of travellers from MA to other divisions reproduced with this method is between 500 and 5300 per week, which is a reasonable estimate. Moreover, one can see that the k value between New England and Mid-Atlantic is 0.72, corresponding to 720,000 travellers per week so a 100 times larger than the ones for the other divisions. This is in agreement with what we argued before about the strong link between states from NE next to New York city where workers lives while working in NY. Similarly, we can see that the second strongest coupling is with the Pacific division containing California states with which New York has strong flight connections. However, the presence of a small peak of infections for ENC and WNC, around March is problematic. This feature cannot originate from the MA division, as that would imply a k-value of order 10, which is clearly unrealistic. We therefore decided to fit the value with the second bump of the division data.

While this scenario is interesting, it does not reproduce well the reality as it doesn't take into account the couplings between other divisions than MA. Now that we gained confidence in the fact that the k_{ij} matrix can be generated with a reasonable number of travellers, we will investigate the possibility to derive it with mobility data. We saw that the second bump of the first wave was happening in southern and western divisions, that are also holidays locations. Because of the long distance involved to reach it from NE and MA divisions, we expect that travels are made essentially by planes. Therefore, we decided to use open access flight data to quantify the matrix k_{ij} . The data is available on the website *opensky-network.org* and gives the number of flights between the different states of the US on a daily basis. It is straightforward to deduce the number of flights between the census divisions, by just summing over the states composing them. However, this data doesn't give the actual number of passengers. Therefore, we will use an average number of person per flights \overline{n} to compute the k_{ij} matrix.

With the help of data, we can reduce the 81 parameters required to fill the k_{ij} matrix to a single one : \overline{n} . To have a realistic matrix for k_{ij} , we first take the mean number of flights from

Division	First wave par	ameters (fitted)	Second wave	e parameters		
	$a (\log)$	γ	a	γ		
NE	9.397(7)	0.416(7)	11.006	0.214		
MA	9.496(7)	0.516(9)	10.882	0.206		
SA	9.691(5)	0.370(3)	10.885	0.185		
ESC	9.63(2)	0.331(6)	11.201	0.207		
WSC	9.720(6)	0.340(3)	10.713	0.213		
ENC	8.88(3)	0.300(8)	11.074	0.250		
WNC	8.66(2)	0.342(6)	11.060	0.263		
M	9.478(9)	0.330(4)	11.089	0.213		
P	9.33(1)	0.291(5)	11.535	0.171		

TABLE 9 – Logistic parameters used for the simulations. The left column shows the parameters obtained through fitting the first wave while the right column features the parameters determined in December 2020 for the second wave.

division *i* to division *j* during the period from April 1st to May 31st for the first wave. Then, we multiply the number of flights by \overline{n} , and normalise it by 10⁶ to have it in millions following the definition of k_{ij} . We again use the value a_i and γ_i already derived by fitting the data with a simple logistic function. Then, we consider the number of cases at the end of March as initial conditions for the seed region MA and 0 for the others. Then, solving the interactive eRG equations using these initial conditions, the values of a_i , γ_i and varying the number \overline{n} involved in the data-derived matrix elements k_{ij} , we check which value of \overline{n} gives the most accurate scenario. We also decided to keep the value of $k_{ij} = 0.72$ for the coupling between NE and MA as there the travels are mostly done by car and cannot be reproduced using flight data.

For the first wave, the optimal average number of passengers is found to be $\overline{n} = 10$. Note that this value do not correspond to the real number of passengers in the flights. In fact, the values of the couplings k_{ij} also take into account the probability of the passengers to carry the infection as compared to the average in the division of origin. When the value is low it might suggest that the sample of passengers in a flight is less infectious than average, as people with symptoms tend not to travel. Controls at airports may also contribute to this reduction by refusing infected persons. The key information we extract from the flight data is the relative flux of travellers among different divisions. The values k_{ij} obtained for this choice of \overline{n} are shown in the middle section of Table 10. We can clearly see that the couplings are, again, of the order $10^{-4} - 10^{-3}$ and the solutions of the eRG equations are shown in Figure 30 where dots represent the data and lines show the solutions of the interactive eRG equations.

We can see that, except for the closest divisions SA, ESC, ENC and WNC, the behaviors are well reproduced giving confidence in the possibilities offered by the interactive eRG. The difference for ENC and WNC can be related to the fact that the wave was already there before as discussed previously. For SA and ESC, it could be caused by car travels from the east coast to the south of the country that are not represented by the flight data. On the other hand, it is astonishing to see that the divisions far from MA divisions (WSC, M and P) are well reproduced as the mobility interactions are mostly due to flights for them.

k_{ij} values (first wave fits)									
Division	NE	MA	SA	ESC	WSC	ENC	WNC	М	Р
MA	0.72	0	0.0014	0.00075	0.0017	0.0023	0.0005	0.002	0.0053
	First wave k_{ii} values (Flight data, from April 1 st to May 31 st)								
Division	NE	MA	SA	ESC	WSC	ENC	WNC	М	Р
NE	0	0.72	0.0045	0.00088	0.00087	0.0024	0.00052	0.00067	0.00091
MA	0.72	0	0.019	0.0056	0.0041	0.012	0.0025	0.0031	0.0059
SA	0.0043	0.018	0	0.0085	0.013	0.019	0.0057	0.0050	0.0067
ESC	0.00092	0.0053	0.0093	0	0.0051	0.0068	0.0023	0.0035	0.0065
WSC	0.00095	0.0038	0.014	0.0055	0	0.0092	0.0054	0.011	0.010
ENC	0.0025	0.012	0.018	0.0063	0.0086	0	0.0082	0.0079	0.0099
WNC	0.00038	0.0022	0.0056	0.0019	0.0046	0.0070	0	0.0055	0.0027
M	0.00050	0.0020	0.0042	0.0026	0.011	0.0072	0.0043	0	0.028
P	0.00084	0.0055	0.0063	0.0050	0.010	0.0092	0.0033	0.030	0
S	econd wa	ve k_{ij} va	lues (Fli	ght data,	from Sep	tember 1	l st to Oct	ober 31^{st})
Division	NE	MA	SA	ESC	WSC	ENC	WNC	М	Р
Reg-X	0.0066	0.028	0.029	0.013	0.019	0.027	0.014	0.03	0.03
NE	0	0.72	0.0028	0.00046	0.00031	0.0015	0.00026	0.00041	0.00082
MA	0.72	0.	0.011	0.002	0.0017	0.0064	0.0013	0.0021	0.0029
SA	0.0026	0.011	0	0.005	0.005	0.0096	0.003	0.0033	0.0035
ESC	0.00041	0.0019	0.0051	0	0.0019	0.0028	0.00087	0.0012	0.0015
WSC	0.00028	0.0015	0.0049	0.0018	0	0.0028	0.0016	0.004	0.0034
ENC	0.0014	0.0062	0.0089	0.0028	0.003	0	0.0039	0.0043	0.0045
WNC	0.00024	0.0013	0.0028	0.0009	0.0017	0.0038	0	0.0028	0.0016
M	0.00032	0.0017	0.0029	0.0011	0.0054	0.004	0.0026	0	0.014
P	0.00074	0.0028	0.0032	0.0014	0.0046	0.0041	0.0018	0.015	0

TABLE 10 – Values of the k_{ij} entries among the US divisions. Top section : Values between Mid-Atlantic (MA) and the other divisions obtained from fits of the first wave timing. Central and bottom sections : Complete matrix obtained using flight data (except the entries between MA and NE) for the first wave (from April 1st to May 31st 2020) and the second (from September 1st to October 31st 2020), respectively.

We then tried to apply the same method for the second wave which happened in the beginning of October 2020 in the US. For that wave, the fit parameters weren't accessible when the study was realised in December 2020 as the wave wasn't over, thus we couldn't have the correct parameters a and γ from a fit. Therefore, we tried different possibilities for the parameters to predict the evolution of that wave before taking the best values as listed in the right column of Table 9. We also decided to take as a seed region an average sum of all the divisions with a wave peak occurring in the July-August period (i.e. SA, ESC, NSC, ENC, WNC, M and P) normalised by the total population within these divisions that we call Region-X. Finally, we



FIGURE 30 – Simulation of the spread of the first wave (left plots) and the second wave (right plots) using flight-data-derived kappa matrix as gathered in Table 10. For the first wave, the MA division is used as a seed region, while for the second wave a combination of the first waves among divisions acts as the seed region (Region-X)

used the data of the average number of flights per day from September 1st to October 31st and choose $\overline{n} = 5$ giving the values of the k_{ij} matrix in the table at the bottom of Table 10. Again, we can see that the values are of the order $10^{-4} - 10^{-3}$. The data was later actualised as of the date of the 28th of February 2021 (while we recall that the study was realised in the middle of December 2020) at the time when the peak of the waves already happened. The updated data is shown in the right plots of Figure 30. We can see that the parameters predicted before were accurate for almost all the divisions, with a problem for the Pacific division. Once again, the interactive eRG shows its accuracy at forecasting the evolution of future waves.

One other point is that the waves for the period between October 2020 and February 2021

seem to appear almost at the same time in opposition to what we saw for the first wave. This observation can be explained by looking at the geographical distribution of the pandemic. Such distribution can be studied by looking at an indicator of the uniformity of the epidemic using the new cases incidence for each division. This indicator can be followed week by week via a χ^2 -like variable defined by the following expression :

$$\chi^{2}(t) = \frac{1}{9} \sum_{i=1}^{9} \left(\frac{I_{i}'(t)}{\langle I'(t) \rangle} - 1 \right)^{2} , \qquad (272)$$

where $I'_i(t)$ is the number of new cases per day averaged within 7 days around the time tin division i and $\langle I'(t) \rangle$ is the mean value of the same quantity over the 9 divisions. The parameter χ^2 quantifies the geographical diffusion of the SARS-CoV-2 virus in the US : the smaller its value, the more uniform the pandemic is spread across the whole country. The evolution of the indicator is shown in Figure 31. Due to statistical uncertainties, the value of the indicator between the wave doesn't mean much. We will thus focus on the value it takes during the waves represented by the grey bands. We can see that during the first peak in April (light gray shade), the value of χ^2 is large, signalling that the epidemic diffusion is localised in few divisions, which is in agreement with the localisation of the first wave mainly in New York. During the second peak of the first wave (gray shade), the value has dropped, signalling that the epidemic has been spreading to all divisions. Finally, the data for the second wave (dark gray shade) shows that χ^2 has shrink towards zero, as expected for a more diffuse incidence of infections.



FIGURE 31 – Evolution of the uniformity indicator χ^2 over time on a daily basis averaged over 7 days around. The shaded bands indicate the period when epidemic peaks are recorded.

The study of the χ^2 shows that the epidemic is now well spread within the whole country and may explain why peaks appear at similar time for the different divisions for the second and third waves. A similar study of the geographical uniformity indicator has been realised for several other countries showing the same behavior as shown in [57]. Then, we have seen that the interactive-eRG model is good enough to reproduce the spreading scenario within the United States using flight data and could also be extended using boat, train or car transit data to make the study more precise.

The interactive-eRG equation has been also used to predict the second-wave of COVID-19 that happened in September 2020 in Europe as is shown in [56]. Different scenarios were investigated by playing with the couplings (i.e. the k_{ij} matrix elements) for the countries considered in the study, but also for an abroad infected region called "region-X" from where the infected cases during the summer were coming, starting the second wave. Simulations of an infection coming from region-X to all the countries have been investigated, but also for an infection from region-X to only some countries in Europe, infecting in a second step the rest of Europe. This last scenario is represented in Figure 32 where in this scenario, France was taken as the intermediate seed country.



FIGURE 32 – Illustration of the connections k_{ij} between Region-X, France and the other countries considered in this study. Each line represents an exchange of infected cases. The line pointing outside the map represents the connection to Region-X, modeling an inflow from a source outside the pool of countries in the simulation.

As we didn't use the mobility data for this study, random value generations for k_{ij} were needed. Consequently, we propose first a scenario where the Region-X, denoted by the subscript 0, is represented by the number of cases of the rest of the world. Then, Region-X and the countries considered in the study are connected using random generations for the k matrix in the range $10^{-3} < k_{ij} < 10^{-2}$ (scenario **a**). We also propose another scenario where the couplings k_{0j} , between Region-X and the others countries, are reduced by a factor 10 (scenario **b**) and by a factor 100 (scenario **c**), while the other couplings don't change. At the time the study was held, there was no sign of a second wave in most European countries. Therefore, we decide to use, for the eRG equations simulating a coming wave, the same parameters *a* and γ obtained by fitting the first wave. As argued before, the timing of the wave generated by the interactive eRG equation depends strongly on the values of the *k* matrix and therefore the timing of the peak as well. Solving the equations for 100 generations of the *k* matrix and averaging the timing of the next peak over the simulations, we obtain the following values in Figure 33 for the timing versus the infection rate of the countries considered.

The results from the three scenarios show fairly small error bars related to the uncertainty on our knowledge of the k_{ij} couplings. Recalling that we vary the couplings within a


FIGURE 33 – Peak timing, in calendar weeks, versus the infection rate γ for scenario **a**, **b** and **c**. These results were obtained by averaging the outcome over 100 simulation of the k_{ij} values, while the error bars indicate one standard deviation.

factor of 10, it is impressive to see that values of the matrix doesn't change much the peak timing. However, the peak position also depends crucially on the value of the infection rates. Comparing the results in each set of simulations, we discover a clear correlation between the timing of the peak and the infection rate γ of each country. The higher is the infection rate the sooner the peak is reached, as expected. The timing spans over 20 weeks between all the countries. Furthermore, comparing the results for the three scenarios, we see that reducing the couplings with Region-X systematically delay the peaks, in accordance with previous results for the US. Quantitatively, a reduction of a factor ten in the coupling to Region-X delays the peaks by about three weeks. Following the possible interpretations of Region-X, a reduction of the couplings to this region can be seen as the effect of travel bans. Overall, the peak timing ranges from end of July 2020 to beginning 2021. As an additional result, no clear correlation between the peak timing and the value of a_i across the countries we studied was observed.

The three scenarios studied are simple, therefore, they do not reproduce the situation as was observed as of the date of the 5^{th} of August 2020. In reality, multiple countries were already showing signs of a second wave at this time as can be seen from Figure 34 for Croatia.

Such countries will be called "seed countries" and are Belgium, Bosnia, Croatia, Czech Republic, Greece, Netherlands, Serbia, Slovakia, Slovenia and Spain, as can be seen from Figure 62 in Appendix C.4. There, the second wave was already well installed. The k_{i0} and other parameters a_i and γ_i for each of these country were tuned in order to fit the available data from the second wave. As an example, for Croatia we fixed $k_{0i} = 0.1$ and rescaled the first wave parameters γ_i and a_i respectively by the factors 0.6 and 1.06, to obtain the result shown in 34. This implies that Croatia had a high number of travellers during summer, as indicated by the large value of the coupling k_{i0} with Region-X. Many countries, after reaching the peak of the first wave, featured also a period with a linear growth of the infected cases. This phase is called **endemic period** and is common in epidemiology when having multiple waves. To better fit the second wave data by including this period, we added the following term to the



FIGURE 34 – Total number infected cases for Croatia updated to the 5^{th} of August with respect to the theoretical curve (orange line) used to calibrate the simulation for the countries having signs of a second wave at the time. The vertical line shows where the second wave simulation begins.

solution at the time when the first wave reached the plateau :

$$\delta I\left(t > t_0\right) = \theta_i\left(t - t_0\right) , \qquad (273)$$

where θ_i is the constant number of new cases per time unit during the endemic period and is equal to 0 for Croatia. t_0 represents the time at which the plateau starts. This term will be discussed in section 4.2.4, as the linear growth can be understood more deeply using complex fixed points. We have repeated the same tuning for the other countries cited before and the values fitted are listed in Appendix C.4. As in the scenario **a-c**, we decided to take the same parameters a_i and γ_i fitted for the first wave in the other countries considered, allowing them to vary within 10%. The parameters are listed in Table 17 in Appendix C.4 as well. Once the parameters are defined, we finally couple the seed countries to the rest of the European countries by taking $k_{ij} = 0.5 \times 10^{-3}$ for all $i, j \neq 0$, allowing them to vary within 10% and $k_{0j} = 0$ for the non-seed countries. We will call this more realistic scenario "e" in agreement with notation in [56].

The results showing the evolution for France, Italy and the UK in this scenario are presented in Figure 35 while solutions for others countries are shown in Figure 63 in Appendix C.4.

The results show different trends among the countries. For example, the first wave happened first in Italy, then in France, while for the second wave, our predictions shows the inverse. Moreover, because we took the same γ_i and a_i than during the first wave for each non-seed country, we already suspected at that time that the height of the second wave would not be the same. As social distancing measures and testing campaigns would be very different from the unprepared period at the beginning of 2020, both parameters would have different values than the first wave. As argued before, the most important information extracted from the interactive-eRG is the timing of the next wave, which was well anticipated. Looking at the updated data as of the date of 30th of August 2020 in Figure 36, we see that the beginning of the second wave was well anticipated, but not the long term behavior.



FIGURE 35 – Solutions of the interactive-eRG for scenario **e** for France, Italy and the UK. We show the time evolution of the total number of infected cases in the top panel and the new infected in the bottom panel. The bands are generated by varying the infection rates γ_i within 10%.



FIGURE 36 – Epidemiological data (red), adjourned to the 30th of August 2020, for six sample countries compared to the simulation. For all countries, except Croatia, the second wave from case **e** simulation is anticipated by 4 weeks, thus in agreement with the results of case **a**. The bands are generated by varying the infection rates γ_i within 10%.

We can see directly from these results that France second wave seems to appear 1 week earlier than what was expected earlier in the predicted scenario while it happens 1 week later for Finland. In any case, the simulations seem to reproduce the starting of the second wave data with a good timing.

Therefore, we saw that adding the interactive term to the eRG equation allows us to simulate scenarios of infections at a country but also at a continent level. Nevertheless, multiple waves and endemic periods cannot yet be described using the interactive eRG equations. We will need to add new features to the eRG formalism in order to describe them

4.2.4 Complex fixed points in epidemiolgy

Now that the behavior of a single wave has been modeled in the eRG formalism, we can push further to see what happens when having multiple waves. First, we saw in most countries that between two waves, there is a period called "**endemic**", where the number of new cases remains constant, giving a linear evolution for the cumulative number of cases. This behavior can be explained by modeling it with the additional term in equation (273). While this term was introduced to simplify the modeling for the second wave in Europe, it was found in [57] that it can also be generated by introducing complex fixed points, as discussed in section 2.3.3, within the eRG using the following equation :

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \gamma_i \alpha(t) \left(\left(1 - \frac{I(t)}{a} \right)^2 + \delta \right)^{p_0} \\
= \gamma \alpha(t) \left(\left(1 - \frac{\alpha(t)}{a} - i\sqrt{\delta} \right) \left(1 - \frac{\alpha(t)}{a} + i\sqrt{\delta} \right) \right)^{p_0} ,$$
(274)

where $\delta > 0$ is the parameter controlling the endemic period intensity and p_0 is the power acting on the steepness of the wave whose value is $p_0 = 1/2$ in the case of the simplest eRG seen before. We see directly from the second line of equation (274) that two fixed points are generated and both are complex. These fixed points are conjugate of each other and are defined by :

$$\alpha_{\pm}^* = a \left(1 \pm i \sqrt{\delta} \right) \,. \tag{275}$$

We will often refer to the endemic period by calling it **strolling phase** as it can be related to walking dynamics. Complex fixed points are the cause of the strolling period when the number of new cases remains constant. Due to the fact that the number of infected cases has to be real, if the complex fixed point is UV-attractive, the coupling will run to the real part of the fixed point. Taking equation (274) for the real part of the fixed points a, we obtain :

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} \left(\mathrm{Re} \left(\alpha_{\pm}^{*} \right) \right) = \gamma a \delta^{p_{0}} .$$
(276)

We see that this value is positive, therefore it creates new cases even if we are close to a fixed point. As α will never reach any complex fixed point, it will go through them to reach the next fixed point, or infinity if there are no other fixed point. This generate the strolling phase followed by a new wave, which is exactly what happens in epidemic scenarios of multiple waves seen in the data. Such scenario is shown in Figure 37 for a unique couple of complex fixed points described by the equation (274). In top panel, the function $-|\beta(\alpha)|$ is shown as function of the real and imaginary parts of α , where a trajectory from the initial condition is shown in red. In the bottom panels, the solutions $\alpha(t)$ and $\alpha'(t)$ are plotted for different values of δ . We can see that fixed points give a period of linear evolution for the solutions.

In order to be free from the dependence on the parameters γ and a that we already studied before and focus in more details on the impact of the parameters δ and p_0 , we redefine equation (274) using the normalisation explained in equation (261) to obtain :

$$-\beta_{\text{CeRG}}\left(\alpha\right) = \frac{1}{a\gamma} \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = \frac{\alpha}{a} \left[\left(1 - \frac{\alpha}{a}\right)^2 + \delta_0 \right]^{p_0} .$$
(277)



FIGURE 37 – Top panel : $-|\beta(\alpha)|$ function of $Re(\alpha)$ and $Im(\alpha)$ showing the initial fixed point (blue), the two complex fixed points (black) and the trajectory of a fully real solution. Bottom panel : Evolution of solutions and their derivative for $\alpha(t) = I(t)$ and a = 1, $\gamma = 0.1$, $p_0 = 0.65$ and different values of $\delta : 0$ (blue), 0.05 (yellow), 0.1 (green) and 0.18 (red).

Such equation is called a CeRG β -function for "Complex epidemic Renormalisation Group". In this redefinition, the number of new cases during the strolling phase is given only by δ^{p_0} . For now, it has only three fixed points : 0 and the two complex fixed points. Therefore, after going through the complex fixed points, the solutions will be driven to infinity. To avoid this, one can add other fixed points (complex or not) to the equation to account for multiple waves. Redefining the CeRG equation to integrate more fixed points, we have :

$$-\beta_{\text{Multiwave}}\left(\alpha\right) = \frac{I}{a} \left[\left(1 - \frac{I}{a}\right)^2 + \delta_0 \right]^{p_0} \prod_{k=1}^{w} \left[\left(1 - \zeta_k \frac{I}{a}\right)^2 + \delta_k \right]^{p_k} , \qquad (278)$$

where w + 1 is the number of waves and each intermediate wave (i.e for any k smaller than w) presents also a parameter $\delta_k > 0$ which creates a strolling phase between each wave and $\delta_w = 0$ such that the w^{th} wave is the last one to happen. ζ_k is the factor between the real part of the k-th wave fixed point and of the first wave fixed point. Therefore, the consecutive fixed points are defined by :

$$\alpha_{k,\pm}^* = \frac{a}{\zeta_k} \left(1 \pm i\sqrt{\delta_k} \right) \,. \tag{279}$$

In order to sort the waves by their temporal appearances and to ensure that the next fixed point is higher than the previous one, we should have $0 < \zeta_{k+1} < \zeta_k < 1$ for any k.



FIGURE 38 – Top-left plot : Dependence of the solutions on ζ_1 . Top-right plot : Dependence on δ_0 . Bottom-left plot : Dependence on p_0 . Bottom-right plot : Dependence on p_1 . If nothing is specified, the parameters used are : $\gamma = 0.1, a = 1, I_0 = 0.001, \zeta = 0.5, \delta = 0.001$ and $p_0 = p_1 = 0.5$.

The effects of the parameters ζ_1, δ_0, p_0 and p_1 on the solutions are shown in Figure 38 for the specific case of only two waves. We can see, first, that changing ζ_1 (top-left panel) changes the height of the second wave peak, in agreement with equation (279). We can also see that it changes slightly the timing of the second peak and the height of the first wave. Secondly, changing δ_0 (top-right panel) changes the constant number of cases during the strolling phase as expected. The higher δ_0 is, the higher the constant number of cases becomes. Moreover, we can see that δ_0 has also an impact on the timing of the next wave, delaying it when going to smaller values. This effect will be of prime importance to understand why controlling the endemic phase allows to control the timing of the next wave. Finally, we can see that changing p_0 or p_1 (bottom panels) have very different consequences. It doesn't change the total number of cases nor the constant number of new cases during the strolling phases, but while p_0 seems to drive the timing of the next wave, p_1 affects the height of the next wave. One can see that all the parameters have an impact on the timing of the peak of the next wave, but not with the same intensity. Moreover, while ζ_1 and p_1 seems to flatten the curve, δ_0 and p_0 delay the beginning of the exponential growth.

We want now to see how multiple waves can appear when looking at the derivative of the solutions. In the case of two waves, fixing the parameters $p_1 = 1/2$ and $\zeta_1 = 0.5$, the maximas

of the solutions are given by :

$$\alpha_{\max}^{\pm} = 1 \pm \sqrt{\frac{p_0 - \delta_0}{1 + p_0}}, \quad \alpha_{\max}^0 = 1.$$
 (280)

The values of the maximum as functions of δ_0 are shown in the left plot of Figure 39 where $p_0 = 1/2$. It clearly shows that, because of the square root function in equation (280), there are two different regimes depending on the value of δ_0 . Below the critical value $\delta_0 = p_0$, there are three different solutions, while above, only the solution $\alpha_{\max}^0 = 1$ survives as can be seen in the right plot of Figure 39 showing the β -function for two sample values of δ_0 below and above the critical value. Therefore, no intermediate wave is observed for a too high value of δ_0 , because the strolling phase becomes more important than the real part of the intermediate fixed point. Similar phase transition are observed for any other set of values p_0, p_1 and ζ_1 .



FIGURE 39 – Left panel : Maximas of the solutions versus δ_0 . Right plot : β -function for two different sample values of δ_0 below and above the critical value. Plots obtained for $p_0 = p_1 = \zeta_1 = 1/2$.

We are now interested to know the time difference between the peaks of each wave in order to know how to control it. The delay Δt_{peaks} can be obtained thanks to the following equation :

$$\Delta t_{\text{peaks}} = \int_{\bar{\alpha_{\text{max}}}}^{\alpha_{\text{max}}^{+}} \frac{\mathrm{d}x}{-\beta_{\text{multiwave}}(x)} \,. \tag{281}$$

In order to see the dependence of the delay on the strolling phase parameters, one other interesting quantity to calculate is the constant number of new cases during the strolling phase S_t . This can be obtained by approximating the beta-function to the real part of the complex fixed point. We then have the relation for a two-wave description :

$$S_t \equiv \frac{\mathrm{d}\alpha}{\mathrm{d}t}_{|\mathrm{strolling}} = -\beta_{\mathrm{multiwave}} \left(1\right) = \delta^{p_0} \left(1 - \zeta_1\right)^{p_1} \,. \tag{282}$$

Plotting now the time delay between the two peaks as a function of S_t as shown in Figure 40, we can see that the strolling number S_t is crucial in order to delay the next wave. In a CeRG scenario, it is needed to lower S_t as much as possible to prepare for the next epidemic episode. Delaying a peak can be an effective way to increase the time allocated for a vaccination campaign for example. It is therefore of prime importance to control the strolling period.



FIGURE 40 – Dependence of the time delay between the two peaks of new infections Δt_{peak} , as a function of the strolling number S_t for different values of p_0 , where $p_0 = 0.5$ (blue), $p_0 = 0.55$ (green) and $p_0 = 0.6$ (red). The CeRG parameters are fixed to the following values : $p_1 = 0.65$, $\zeta_1 = 0.5$.

The CeRG model was put to use on real data in [57] aiming at forecasting the situation for several countries around the world in Fall 2020. As can be seen for the case of Japan in Figure 41 where $\alpha(t) = I(t)$ corresponding to the cumulative number of cases, the CeRG-equation has been solved for the two waves happening between June and December 2020. This solution was obtained thanks to the parameters a = 63150, $\gamma = 0.103$, $\delta_0 = 4.9 \times 10^{-2}$, $p_0 = 0.7$, $p_1 = 0.6$ and $\zeta_1 = 0.4$.



FIGURE 41 – Evolution of the cumulative and number of new infected cases in Japan (red dots) as of the end of November 2020 and CeRG solution for parameters $a = 63150, \gamma = 0.103, \delta_0 = 4.9 \times 10^{-2}, p_0 = 0.7, p_1 = 0.6$ and $\zeta_1 = 0.4$ (blue curves).

This study has been held for multiple other countries for which some are shown in Appendix C.5. However, updating the data as of the beginning of may 2021, it seems that the forecasting were not as accurate as expected. This can be seen in the top panel of Figure 42 for Japan. We can see that the number of cases has been underestimated because of an unexpected additional peak happening in the middle of January 2021. This may be caused by change in social distancing measures in Japan or to new year eve celebrations.



FIGURE 42 – Updated evolution of the number of new cases (red dots) in Japan (top panel) and in the UK (bottom panel) as of the date of 12^{th} of May 2021 and their CeRG solutions (blue curves).

Another interesting case of a deviation from our forecasting needs to be discussed. Indeed, United Kingdom had a really interesting behavior from the end of 2021. We can see in the bottom panel of Figure 42 that until the end of November 2020, our forecasting was in agreement with the data, but in the beginning of December, a new unexpected wave appeared being higher and steeper than the previous ones. As we will see later in section 4.3.2, the December's wave was caused by a new variant of the virus, the B.1.1.7 or Alpha variant, that was driving most of the European waves in the beginning of 2021. This ruled out our predictions computed only for one species of the virus. Therefore, in order to explain the latest waves, we will need to embed the variant description in our equations.

The other factor that was not considered at that time is the vaccination effect. Indeed, by the end of November 2020, some vaccine started to be distributed across the world, changing the number of cases consequently. In order to take into account these effects, we will study how it affects the data in section 4.3.1.

4.3 Vaccination, mutations and other disease

4.3.1 Vaccination

The emergence of the COVID-19 pandemics and the fear to live periodical episodes of infection for a long time has driven the motivation to prepare an effective vaccine and fast vaccination campaigns [155]. The 9th of November 2020, Pfizer announced that their vaccine was 90% effective [156]. This was followed by the announcement of the other vaccines along the month [157] launching the race to buy and administrate vaccines as soon as possible. The first person vaccinated was a 90-year old woman in the United Kingdom the 8th of December 2020. Then, the United States started their vaccination the 20th of December 2020. Anticipating the effects of a vaccination campaign is important to avoid new waves before a sufficient proportion of the population is immune to the COVID-19.

In order to have a rough idea of the effects of a vaccination campaign on the population, we will explore first the introduction of the vaccine through lattice simulation. This lattice simulation was ruled by the same laws explained in section 4.1.2 including susceptible, infected and recovered knots to which we add a new compartment called vaccinated (V). Susceptible individuals can turn into vaccinated and remains in this category until the end, like for recovery states. The vaccination campaign can be implemented in different ways, but we decided to look at the effect of a vaccination on a certain percentage of the initial population on the evolution of the pandemic. This is done by putting randomly a defined percentage R_v of the total population in the vaccinated compartment as initial condition on a 100×100 grid. Then, we let the simulation runs with the same rules as before and try to fit the evolution of the cumulative number of infected with a simple logistic function. Some of the fits for different values of R_v are shown in the top panel of Figure 43. Then comparing the fits parameters values versus the R_v taken for the simulation, we obtain the plots in the bottom panels of Figure 43.

These results show that for $R_v < 24\%$, *a* and γ are reduced linearly with R_v . Fitting the data with a linear function gives the following parameters as shown by the red lines in bottom panels of Figure 43 with respect to the percentage R_v :

$$a(R_v) = 9592 - 10600R_v \approx 10000 (1 - R_v) ,$$

$$\gamma(R_v) = 0.07921 - 0.092R_v \approx 0.08 (1 - R_v) .$$
(283)

Therefore, we find that $a(R_v)$ and $\gamma(R_v)$ are obtained by reducing their values as compared to the one they have without vaccination $a_0 = a(0)$ and $\gamma_0 = \gamma(0)$ by a factor R_v such that :

$$a(R_v) = a_0 (1 - R_v) , \quad \gamma(R_v) = \gamma_0 (1 - R_v) .$$
(284)

On the other hand, b doesn't seem to be strongly affected by the percentage of vaccinated nodes on this span of R_v . For $R_v > 24\%$, we observe that the fits cannot be realised because the infectious wave will not be able to affect an important number of nodes as can be seen in the plots in Figure 44 showing the lattice simulation for different times and as can be compared to the non-vaccine simulation shown previously in Figure 18. One can think that the value 24% represents the required vaccination percentage of the population needed to forbid any future wave within a country to happen. However, this limit is obtained because the epidemic spread is stuck within a structure of static vaccinated person. In reality, individuals should



FIGURE 43 – Top panel : Cumulative number of infected nodes for some of the simulations (dots) and their logistic function fits (plain lines). Bottom panel : a, γ and b fit parameters versus the initial percentage of vaccinated nodes R_v on a 100 × 100 grid. Plain lines shows the linear fits over the data obtained. The lattice parameters are $p_{\gamma} = 0.6$ and $p_{\epsilon} = 0.4$.

move allowing the disease to spread by jumping over the vaccinated nodes. If this limit exists, it should be well above the one found here.



FIGURE 44 – Lattice simulation for $R_v = 39\%$ at different times of the simulation. White dots are susceptible nodes, green are infected, brown are recovered and black are vaccinated.

The results obtained in the lattice simulation can be used to allow the parameters γ and a to evolve in the eRG equations along with the vaccination campaign. Assuming now that the

vaccination process is time-dependent and that $R_v(t)$ is a function of t, we can describe an infinitesimal process of vaccination. Let's compute the variation in γ for an infinitesimal time delay dt. Thanks to the vaccination, γ will evolve according to an equation depending on the percentage of vaccination realised during the time interval that is equal to $\frac{dR_v}{dt}dt$. Therefore, thanks to the lattice study, we have :

$$\gamma(t + dt) = \gamma(t) + d\gamma(t) = \gamma(t_v) \left(1 - \frac{dR_v}{dt} dt \right) ,$$

$$\Rightarrow \frac{d\gamma(t)}{dt} = -\gamma(t_v) \frac{dR_v(t)}{dt} .$$
 (285)

where $\gamma(t_v)$ is the infection rate before the start of the vaccination campaign. Assuming for simplicity that the population is vaccinated linearly in a time interval Δt starting from the date t_v until the percentage reaches R_v^{max} , we define the "vaccination rate" as being the parameter $c = R_v^{max}/\Delta t$. The solution for the time-dependent effective infection rate is then :

$$\gamma(t) = \begin{cases} \gamma(t_v) & \text{for } t < t_v ,\\ \gamma(t_v) \left(1 - c(t - t_v)\right) & \text{for } t_v < t < t_v + \Delta t ,\\ \gamma(t_v) \left(1 - R_v\right) & \text{for } t_v + \Delta t < t . \end{cases}$$
(286)

To find the variation for a(t) within the vaccination interval t_v to $t_v + \Delta t$, we assume that the not-yet-infected individuals are vaccinated at the same rate c as the total population, including the ones that have been already infected. Indeed, the evolution of the parameter in the lattice simulation suppose that the initial population is either susceptible either vaccinated, therefore through an eRG equation a(0) in equation (284) should be thought as the total population minus the cumulative number of infected. Thus, at any given time, the variation in the number of individuals that will be exposed to the infection is proportional to the difference $\tilde{a}(t) = a(t) - I(t)$. This leads to the following differential equation in the general case :

$$\tilde{a}(t+dt) = a(t) + da(t) - I(t) - dI(t) = (a(t) - I(t)) \left(1 - \frac{dR_v(t)}{dt}dt\right),$$

$$\Rightarrow \frac{da(t)}{dt} = -(a(t) - I(t)) \frac{dR_v(t)}{dt}.$$
(287)

Therefore, in the case of a linear vaccination, we have :

$$\frac{\mathrm{d}a(t)}{\mathrm{d}t} = -c\left(a(t) - I(t)\right) \ . \tag{288}$$

Unlike for γ , this equation depends on I(t) and thus needs to be solved in a coupled system with the eRG one. Note that without vaccination, this derivative is zero, so outside of the time interval $[t_v, t_v + \Delta t]$, a is constant just like γ .

We then make use of this new set of coupled equations to see the effects of a vaccination campaign in the scope of the United States vaccination campaign [59]. We have solved the eRG equations for the second wave in the US divisions as what was done in section 4.2.3, with the addition of the reduction of a_i and γ_i , as detailed in equations (285) and (288). We show the result for two sample divisions in Figure 45 for different weekly vaccination rates c (dashed curves) as compared to the same solutions without vaccines (solid curves).



FIGURE 45 – Evolution of the number of infections without vaccination (c = 0) and with a vaccination rate of 0.64%/week, 1%/week and 2%/week starting on December 14th and stopping at 20% of the population vaccinated. We show the results for two sample divisions : South Atlantic and West North Central.

We can see from the figure that any vaccination rate c creates an important flattening of the epidemic curve for South Atlantic, where the vaccination started earlier than the peak of the infection wave. In the other extreme case, represented by WNC, the vaccine is ineffective at changing the current wave because the peak has already been reached before the vaccination campaign started.

The assumption of a constant vaccination rate does not agree, in fact, with the data on the vaccination in the US given by the New York Times website https://www.nytimes.com. One needs to distinguish in the data two kinds of scenarios for the vaccination. Some vaccines, as Astra-Zeneca and Pfizer, need two doses in order to be fully efficient, we can study the possibility that at least one dose is efficient to lead to an immunisation, that we call partial vaccination, or the possibility that only the persons with two doses, or one dose for the "oneinjection" vaccines, such as the Johnson and Johnson's one, are fully immune. Such possibility is called full vaccination. These two cases are shown in Figure 46 where darker points are associated with partial vaccination and lighter dots with full vaccination. These two scenarios give uncertainty bands on the real situation, as one injection can lead to partial immunisation.

We can see in the data that the vaccination rate is not constant but instead can be more or less represented by a linear function. Then, we consider the case in which the vaccination rate c(t) is time-dependent and described by the function :

$$c(t) = u(t - t_v)$$
, (289)

where t_v is, again, the time at which the vaccination campaign started and u is the vaccination rate slope in days⁻². Such evolution would, indeed, lead to a percentage of vaccinated individuals that is quadratic with time, such that :

$$R_v(t) = \frac{u}{2}(t - t_v)^2 .$$
(290)

This equation allows us to estimate a rough value of u such that, if we have a percentage



FIGURE 46 - Number of new vaccinations per million inhabitants in each division. The darker curves represents the number of new partially vaccinated persons and the lighter curve shows the new fully vaccinated ones.

of vaccinated person within a division at the actual date $t_1 > t_v$ of $R_v^{\max} = R_v(t_1)$, we can consider that :

$$u = \frac{2R_v^{\max}}{(t_1 - t_v)^2} , \qquad (291)$$

allowing us to evaluate the slope of the vaccination rate by just using the updated percentage of vaccinated person. Once u is found, we can apply it to the time-dependent parameters in the simulation such that :

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = -u(t-t_v)\gamma(t_v) , \qquad (292)$$

$$\frac{da(t)}{dt} = -u(t - t_v) (a(t) - I(t)) .$$
(293)

Using these equations and the vaccination values for the US divisions listed in Table 11 as of the date of 24^{th} of march 2021 and redoing the simulation for the second wave, as the one realised in Figure 30, gives the behavior shown in Figure 47. The full vaccination scenario is considered to start four weeks after the partial vaccination as the standard duration between the two doses.

These simulations show that the vaccination campaigns have no strong effect on the wave happening in the beginning of 2021, even more if they started after the peak. On the contrary,

		R_v^{\max}		Slope u	
Division	Code	Partial	Full	Partial	Full
New England	NE	30.1	16.4	0.00450	0.00641
Mid-Atlantic	MA	27.8	14.1	0.00415	0.00554
South Atlantic	SA	24.3	13.5	0.00363	0.00530
East South Central	ESC	23.1	12.6	0.00345	0.00492
West South Central	WSC	23.0	11.9	0.00344	0.00467
East North Central	ENC	25.6	14.7	0.00382	0.00575
West North Central	WNC	26.3	14.9	0.00393	0.00584
Mountain	M	25.7	14.8	0.00386	0.00579
Pacific	Р	26.6	14.0	0.00397	0.00548

TABLE 11 – Percentage of the population vaccinated with at least one dose and with two doses in each US division as of the date of 24^{th} of march 2021.



FIGURE 47 - Simulation of the second wave with a quadratic vaccination campaign based on the data listed in Table 11.

to maximise the effect of the vaccination campaign, one needs to start it between two waves. This could also reduce the chances of infection in vaccination places, because less persons can be infected within these periods. In any case, the next wave following the campaign will be strongly reduced by the campaign.

Another interesting features to look when using the eRG formalism are the vaccination strategies. Different policies have been investigated before starting the campaigns [158] and adopted later. The strategies were concerning the vaccination prioritisation of an age group or job categories within the population. The main decisions were aiming at vaccinating first the most vulnerable categories as was done in most European countries, to vaccination of workers and exposed individuals as was achieved in the US [159]. Due to the availability of the data, we will mostly focus on the method adopted relative to the age of the target population. Without any assumption on the vaccination, looking at the age categories brings interesting observations. Let's first look at the case of Italy in the top panels of Figure 48.



FIGURE 48 – Left panels : Number of new infections per 100,000 people within the age categories in Italy (top) and Denmark (bottom). Right panels : Weighted percentage of new infected cases per age group stacked on top of each others in Italy and Denmark.

Are shown in the figure the number of new infected cases per 100,000 persons within the age group and the weighted percentage for six different age groups that are under 15, 15/24, 25/49, 50/64, 65/79 and above 80 years as can be open accessed on the website *ecdc.europa.eu*. The age groups are made comparable thanks to the normalisation by the population within each group. Therefore, we can directly see from the top-left plot that the number of cases for the 80+ group was really important during the first wave. This is also observed in a more subtle way in the bottom-left panel for Denmark and can have multiple explanations. Italy had a very important number of cases in March 2020 within retirement houses in the north of the country. Therefore an important number of tests has been realised in these places that may

show a more accurate number of the number of infected cases for the 80+ than within the rest of the population, increasing the representation from this age group in the data. Moreover, when looking at the second wave in November 2020 in Italy, we can see that the height of the wave of the 80+ is of the same order of the one for 15/24, 25/49 and 50/64 indicating that the tests were more widely performed and that there are no clear age group being more infected than the others in this wave.

Finally, looking at the last wave in March 2021, it seems that the number of new infections for 80+ seems to be small in comparison to the other categories. This, again, could have multiple explanations. As we will see later, the last wave in Italy was mostly caused by a new variant of the virus first identified in UK. Effects on younger people has been a controversial issue [160], but it was suggested that they were more affected by variants. The most accurate explanation, however, is that this is the effect of the vaccination campaign. As a comparison, looking at the situation in Denmark in the bottom panels of Figure 48, we see that a wave occur in March-April 2021, but not for 65/79 nor 80+, suggesting that the vaccination campaign was effective enough to contain the wave for these age groups.

Indeed, when comparing the vaccination percentage in Italy and Denmark in Figure 49 as given on the website *ecdc.europa.eu*, we see that Denmark was faster at vaccinating individuals. This allowed the last wave to be fully contained for the elder groups. Similar results have been obtained for other European countries and could be used to study through the eRG formalism the effects of different strategies on the behavior of the next waves expected.

This kind of study could also be achieved for countries where the vaccination campaigns were faster such as United Kingdom or Israel, as the effects on the epidemiological data could be easier to see. Moreover, one can study the campaign in other countries with different strategies like in the United States to see if the effects are the same. However, the vaccination is not completely a salvation for the World as the virus mutates rapidly creating more resistant copies that can delay the end of the crisis by being more resistant to the vaccine. We will investigate this property in the next part.

4.3.2 Mutations

The SARS-CoV-2 virus is part of the Coronaviruses family that are known to mutate frequently. Mutations are due to random modifications in the RNA nucleotide chain of the virus when it replicates within a cell. These modifications can change the proteins encoded by the RNA and therefore can change the behavior of the virus. Most of the time, the mutations change nothing or create a weaker version of the virus that will disappear as it is disfavored by natural selection. However, it can also create more efficient mutations that can represent a threat for the future. For example, mutations has led to the opening of the spike protein of the virus, that is responsible for the interaction between a human cell and the virus [161]. This opening allowed faster spreading within the human cells and thus higher infection rates of the virus or different sensibilities to tests and vaccines. Therefore, any mutation can delay the end of the global pandemics further [162] even if the vaccination for previous variants is achieved worldwide.

As saw in the bottom panel of Figure 42 for the case of UK, variants of the original virus



FIGURE 49 – Percentage of the population within each age groups vaccinated with one dose (left) or two doses (right) for Italy (top) and Denmark (bottom).

also lead to changes in the epidemic behavior. The mutations that persist and become dominants over the others are either called Variants of Interest (VoI) (also called Variants Under Investigation (VUI)) if they don't show yet any change in infectiousness or Variant of Concern (VoC) if studies have shown that their transmission is stronger than previous variants. Several of them have been identified by the month of June 2021, among which, there are the variant B.1.1.7 also called Alpha, first described in England and whose infectiousness has been evaluated to be 50 to 70 % more than preexisting ones [163], the variant B.1.351 or Beta, first observed in South Africa, which seems to be resistant to some versions of the vaccine [164, 165], the variant P.1 or Gamma, first sequenced in Japan but originating from Brazil, the variant B.1.429 and B.1.427 or Epsilon, first discovered in California, the variant B.1.617 or Delta originating from India and other variants that will not be discussed in details here.

Variants are a really crucial aspect of the pandemics as they can drive a faster spread of the disease and curb the effects of social distancing that are adapted only to the preexisting variant. In a more dangerous way, it can also resist to vaccines [166] and therefore postpone the end of the crisis to the next campaign of vaccination. However, while data on the positiveness of tests to all kind of variant of the virus are open-sources, details on the evolution of a specific variant across the population are not easily accessible. This is due to the fact that to have the information on a variant, a sequencing of the nucleotide chain has to be done, which is longer and more expensive than a test. Therefore, there are less data on sequencing than on testing and that will give problems of statistics for some countries when studying the evolution of the variants. Fortunately, data on the sequencing are accessible on the GISAID website gisaid.org/. As the number of sequencings is smaller than the number of tests, it is not possible to have access to the type of variant for each test. Instead, we decided to use a naive derivation of the actual number of variant-associated tests over the total number of tests. In order to do this, we need first to calculate the ratio of representation of the variant of concern among the total sequences $p_{\text{VoC}}(t)$ realised at any date t. $p_{\text{VoC}}(t)$ is defined by the number of sequences associated to the variant $N_{\text{VoC}}^{\text{Seq}}(t)$ divided by the total number of sequences $N_{\text{All}}^{\text{Seq}}(t)$ such that :

$$p_{\rm VoC}(t) = \frac{N_{\rm VoC}^{\rm Seq}(t)}{N_{\rm All}^{\rm Seq}(t)} .$$
(294)

Once we have access to the time-dependant percentage, we assume that the ratio of variant among all the tests is the same as the ratio among all the sequencing and then consider that the actual number of new cases associated to the variant $I'_{\text{VoC}}(t)$ is defined by :

$$I'_{\rm VoC}(t) = I'(t) \times p_{\rm VoC}(t)$$
 (295)

This naive definition gives an accurate result only if the number of sequences is comparable with the number of tests. This is why we will first concentrate our study on the case of the United Kingdom where the number of sequencing available on GISAID is the highest in the world by June 2021. The results for the analysis are shown in Figure 50.

In the top-left panel, we show in gray dashed the number of sequences realised per day in UK averaged over seven days around together with the sequences associated to the variant B.1.1.7 in orange and the variant B.1.617 in brown. We can see that before September 2020, the number of sequencing was below 1000 per day, while after it oscillates between 1000 and 3000 sequences per day which is more acceptable. Because of the statistical uncertainties, we show the percentage of the variant from September together with its errors bands. In the top right panel, the percentage of representation of the B.1.1.7 variant using equation (294) is shown in orange with errors bars defined by :

$$\Delta p_{\rm VoC}(t) = 1/\sqrt{N_{\rm All}^{\rm Seq}(t)} .$$
⁽²⁹⁶⁾

Looking at the evolution of the percentage, it presents a transitional evolution and becomes dominant in December 2020. Due to the shape of the curve, we decided to fit it with a logistic function. The fit is shown in the same panel as a orange plain line and has the following parameters a = 0.97, $\gamma = 0.076$. This transitional behavior using logistic function is typical of a natural selection process. The variant B.1.1.7 being more infectious than its partners, it dominates more and more the infections and therefore becomes the main reproducing sequence as the time goes. This behavior will allow us to create a hierarchy of dominance between the variants thanks to their interactions with one another. Another feature to point out is the fact that by the end of April 2021, the percentage decreases. This is due to the appearance of the variant B.1.617 (Delta) within the country that seems to take over the B.1.1.7 (Alpha). The number of sequences associated to this variant is also shown in the top-left panel. Fitting the percentage associated to the Delta variant is higher than the one for the alpha seems to show that its spreading would be faster than any previous variant. This is in agreement with biological results that tend to indicate that the Delta variant is 60% more infectious than the Alpha.



FIGURE 50 – Evolution of the number of the variants B.1.1.7 (Alpha) and B.1.617 (Delta) cases in the United Kingdom. Top-left panel : Number of sequences realised per day average over seven days. Top-right panel : Percentage of the VoC over the total sequences using equation (294). Bottom-left panel : Number of cases per day using the equation (295). Bottom-right panel : Number of new cases (dots) in the United Kingdom and fits (lines) associated with each wave. Gray dashed curves are showing the values for all the variants integrated, yellow curves are for the B.1.1.7 variant, brown curves for the B.1.617 variant and green curves for all variants except B.1.1.7.

The bottom-left panel shows the number of new cases for all variants in dashed gray, for the B.1.1.7 variant as defined with the naive method in equation (295) in orange, for the B.1.617 variant in brown and for the rest of the variants in green. The last curve is obtained by subtracting to the data on all the cases, the data for the variants of concern. Looking at the number of new cases per day derived with our naive method, we can clearly see that the peak occurring from the end of December 2020 to February 2021 and that was totally missed in the CeRG analysis shown in Figure 42, is mostly caused by the B.1.1.7 variant and that the others variants are shrinking in the same period. Moreover, the orange curve seems to be really smooth and add more confidence in the naive method we used to derive it. Looking at the green curve now, we can see that the wave has a strange behavior. It created the wave in October-November 2020, but also a second bump in December-January. Therefore, B.1.1.7 cannot be fully taken as responsible for the second bump. One explanation to this is that UK imposed a lockdown from October to November 2020 and measures were lifted mid-November before being reset in the beginning of December 2020. This could be related also to the celebrations of the end of the year, reuniting families and spreading the virus. The strange behavior of the green curve can thus be explained by a change in social distancing that may have restarted the infections of preexisting variants. Another possibility is that the second bump may have

been caused by a single variant that is integrated out over the others variants and may have his own wave in December 2020. Another interesting point to discuss is the situation by the date at which the thesis is written in June 2021. One can see that the number of new cases is rising again suggesting a new coming wave, mostly caused by the Delta variant. If this is true, it would mean that the vaccination campaign in the UK will not be efficient enough to curb a new episode. Studying the evolution of variants through the eRG is therefore crucial for the future of the crisis.

It is possible to fit each wave easily with a logistic function to compare the parameters obtained. The comparison between the fits (plain lines) and the data (dots) can be seen in the bottom-right panel of Figure 50 for the number of new cases, while the fit parameters for the three waves are listed in Table 12.

Wave		Fit parameters		
Number	VoC	a	γ	
1	Others	2137(12)	0.0668(5)	
2	Others	1153(15)	0.097(2)	
3	B.1.1.7	2534(10)	0.0993(7)	

TABLE 12 - Parameters of the three logistic function fits of the number of new cases for the variant B.1.1.7 and the other variants.

These results show directly that the UK variant has an higher infection rate than the other variants. It is even possible to compare their percentage of difference, such that comparing the first wave of other variant and the Alpha variant wave, we have :

$$\Delta \gamma_{\rm VoC} = \frac{\gamma_{\rm VoC} - \gamma_{\rm Others-1}}{\gamma_{\rm Others-1}} = 0.488 .$$
⁽²⁹⁷⁾

This 49% of difference can be compared to the 50% - 70% obtained in biological studies [163].

Similar interesting results have been realised for South Africa, Brazil, California state and India and are shown in Figure 51. It can be seen from it that the different waves can be understood as coming from only one variant. The case of South Africa is really interesting as the two waves are clearly originating from different variants and the B.1.351 (beta) variant seems to have infected more persons. In the cases of California and India, waves from different variant seem to happen at the same time, similarly to what was observed for UK. In the case of Brazil, as the country didn't impose any social distancing measures, waves cannot be clearly identified, but it can be seen that the appearance of the P.1 (gamma) variant increased the infectiousness of the virus by the end of January 2021.

Another way to improve our understanding on the variants is to see how they interact with each other. Multiple countries have already faced three different classes of variant of concern since the beginning of the pandemic. This is the case for UK, California or India. We already



FIGURE 51 – Evolution of the number of new cases per 100,000 inhabitants for the most dominant variant using Equation (294) for 4 samples countries : South Africa, California, Brazil and India.



FIGURE 52 – Evolution of the percentage for the main variants of concerns for California (left) and India (right).

saw in the top-right panel of Figure 50 that the Alpha variant was taken over by the Delta variant. This percentage evolution is also shown for California and India in Figure 52.

These plots seem to show that the Alpha variant is stronger than the Epsilon variant in California, while the Delta was taking over the Alpha in UK. However, due to statistical uncertainty for India, it is not possible to classify the Alpha and Delta variant. These observations are in agreement with the biological data showing that Epsilon is weaker than Alpha who is weaker than Delta.

From the results obtained by data analysis, we can wonder how to integrate the variants

to the eRG formalism. Two scenarios can be envisioned : One with an interaction between the variants (scenario (i)) and one without (scenario (ii)). For the first scenario, the derivation can be extracted by looking naively at what would happen in an SI model having two different infected compartments, calling it an SII model, such that :

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\gamma_1 S(t) I_1(t) - \gamma_2 S(t) I_2(t) ,$$

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = \gamma_1 S(t) I_1(t) ,$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = \gamma_2 S(t) I_2(t) .$$
(298)

Considering that the population is conserved, we have the condition $N = S(t) + I_1(t) + I_2(t)$ for any time t and therefore we would have for $I_1(t)$ by replacing S(t):

$$\frac{dI_1}{dt} = \gamma_1 I_1(t) \left(N - I_1(t) - I_2(t) \right)
= N \gamma_1 I_1(t) \left(1 - \frac{I_1(t)}{N} \right) - \gamma_1 I_1(t) I_2(t) .$$
(299)

A similar equation can be extracted for $I_2(t)$, by just exchanging $1 \Leftrightarrow 2$. This equation looks like the original eRG equation in equation (256) with an additional interaction term between I_1 and I_2 . The most important effect of this interaction term is to contain the less infectious variant and to make him disappear faster. While this scenario seems interesting, the interaction term is, in fact not realistic in the case of the COVID-19 as due to the limited population that is accessible to the infection. In a pandemic like COVID-19, a small percentage of the population is infected in reality and thus this exchange term would not be interesting enough.

The other scenario consists in considering that the population is big enough so that the two compartments can infect independent populations. This can be seen from equation (299) by taking $I_2(t) \ll N$ where this approximation make the interaction term negligible. Indeed, once $I_2(t) \approx N$ in the SII model, the interaction term will start contributing again and this is not what we want. Therefore, we need to introduce new equations. The independence of each compartment would mean that the parameter a in the eRG equation would be different from the total population and, therefore, the mutations can be described as two independent eRG equations such that :

$$\frac{\mathrm{d}I_1}{\mathrm{d}t} = \gamma_1 I_1(t) \left(1 - \frac{I_1(t)}{a_1} \right) ,$$

$$\frac{\mathrm{d}I_2}{\mathrm{d}t} = \gamma_2 I_2(t) \left(1 - \frac{I_2(t)}{a_2} \right) .$$
(300)

As it was investigated in the Wilson-Fischer case in Figure 3, we can study the flow diagrams associated with the two variables I_1 and I_2 for each scenario. This is shown in Figure 53. One can see there the difference of paradigm between the two possibilities. In scenario (i) the limiting condition is the population and is defined already at the beginning of the outbreak forbidding the pandemic to reach the top-right part of the flow chart corresponding to $I_1 + I_2 > N$. On the other hand, for scenario (ii), the total number of cases for each compartment is defined only at the end of the pandemic.



FIGURE 53 – Flow diagrams for each scenario with the values $N = a_1 = a_2 = 100$, $\gamma_1 = 0.1$ and $\gamma_2 = 0.2$. The attractive fixed points are shown in green, as for scenario (i) it is the line $I_2 = N - I_1$ and the repulsive are shown in red.

The scenario (ii) seems, at first, less interesting as we have already investigated the case of the simple eRG equation and its logistic function solution, but looking at what the sum of two well separated logistic function gives, we may wonder if the mutations were not driving the pandemic before the apparition of the variants of concern. In a really simple example shown in Figure 54, we can clearly see that the period between the two waves shows a number of new infected that is not going to 0 such as during a strolling phase. A similar behavior was seen in the case of complex fixed points in the CeRG model in section 4.2.4.



FIGURE 54 – Solutions of two simple eRG equations such that $a_1 = 1$, $a_2 = 1.2$, $\gamma_1 = 0.1$, $\gamma_2 = 0.05$, $b_1 = 100$ and $b_2 = 1000$ and their sum, showing a strolling phase between the two waves.

Another interesting point of view to investigate this possibility is to see how real world data behave in flow diagrams according to scenario (ii). The results are shown in Figure 55.

Fixed points of interest which are $p_0 = (0,0)$, $p_1 = (a_1,0)$, $p_2 = (0,a_2)$ and $p_3 = (a_1,a_2)$ are specified in the plots. The four fixed points are represented by a dot whose color depends



FIGURE 55 – Flow diagrams for scenario (ii) and for the four countries considered : South Africa, California, Brazil and UK. The flow diagram parameters are obtained by fitting the data.

on its attractiveness : green if attractive in all directions, Orange if attractive in one direction and repulsive for the other and red if repulsive for all directions. For South Africa and Brazil, as the waves are well separated, the flow goes first toward the fixed point p_1 but close before reaching it, it is deviated toward another fixed point that is p_3 . In the case of United Kingdom and even more for California, the waves of different variants happen at the same time, so the flow is pushed toward the center of the diagram. Therefore, in the case of well separated waves, time spent next to the fixed point p_1 can be seen as a strolling phase.

This could be an explanation for the endemic phase if we imagine that this period is driven by mutations that are not distinguished from the other variants. Such mutants could have created small waves keeping the strolling period constant before a new VoC would have driven a much higher wave. Such possibility will certainly be investigated later when more sequencing data will be available, but allows to imagine scenarios to integrate variant in the eRG formalism that are investigated in [167].

Now that we have seen the potential threat represented by the variants and how they can be modeled within the framework, one can wonder if the eRG method could have been employed to other pandemics that happened before and that would happen in the future. We will investigate this possibility in the next subsection.

4.3.3 Others pandemics

Before 2020 and the COVID-19 outbreak, the World was more concerned about seasonal Influenza pandemics, HIV spreading or other more localised viruses. We will therefore try to study the application of the eRG framework for other types of pandemics.

First, let's look at the specific case of seasonal influenza, also called flu. Flu is studied since decades and countries don't really take social distancing measures to curb its evolution. Therefore, data is less affected by social behavior than for COVID-19, giving precious information on the intrinsic properties of the virus. The flu pandemics are periodical, happening often during winter in the Northern Hemisphere and disappearing as soon as the spring is there. This is due to the sensibility of the influenza virus to weather conditions within the country. The UV rays depleting the virus shell once exposed, their light concentration in winter cannot stop viruses from circulating. Moreover, the wind and the cold winter temperature bring people to stay more into confined places and thus the viruses spread in the air are more stable, infecting other individuals more easily than during summer.



FIGURE 56 – Top panel : Weekly number of Influenza-Like Ill patients (ILI) admitted in US hospitals from 1997 to 2021. Bottom panel : Percentage of ILI patients over all admitted patients in the US.

In Figure 56 are shown the weekly number of patients admitted in the US hospital because of a disease related to flu ("Influenza-Like Illness" or ILI) in the top panel and the percentage they represent over the entire people admitted to hospitals from 1997 in the bottom panel. The seasonal periodicity is clearly visible. As this was done in the northern hemisphere, we can expect that a similar trend would happen in southern hemisphere countries during the "southern winters" from June to September. This aspect could also be the reason why the pandemics restart every year. The flu could migrate from the north to the south around April and come back in October, thanks to the interaction between the two hemispheres. The mutations that may happen during a wave within one of the hemisphere could also explain why immunised persons from a past wave can become susceptible again for the next wave. In this sense, simulating the interaction between the northern hemisphere and southern one using the formalism seen in section 4.2.3 could be an asset to see how travels influence the appearance of a new wave.

We can also see in the bottom plot of Figure 56 that the height of the waves seen through the percentage of ILI in US hospitals from one year to another can be very different. This effect can be studied by fitting the waves using a logistic function and compare the fit parameters. The results are shown in Figure 57.



FIGURE 57 – Top panel : Evolution of the cumulative number of ILI patients in the US (dots) for each year and the fits associated to them (plain lines). Bottom panel : Logistic fit parameters a and γ obtained for each year.

One can observe from the top panel that the fits seem to be accurate, except for the 2009 period due to the appearance of two waves during this year. We can see from the bottom plots that while the total number of ILI patients a for each wave increases with time, the infection rate γ decreases to stabilise around 0.15. This could be explained by the improvement in the treatment of the disease in the hospitals, allowing more people to be accepted, thus increasing a and reducing the infection rate thanks to the vaccination.

One other point to discuss is the behavior during the winter 2020-2021. Looking at Figure 56, it seems that no pandemic happened in the US related to flu. This could have multiple explanations. First, we can think that this could be related to the social distancing measures imposed thanks to the COVID-19 crisis allowing the flu to be contained this year. However, this could be also related to the fact that hospitals might have been filled by COVID-19 patients in the winter and people infected with flu could not have been accepted in hospitals, due to the limited the number of patients allowed during this period.

Looking now at the evolution of the pandemic at the state level within the US as shown in the top panel of Figure 58 for the specific example of the 2014-2015 pandemic, we can see that the flu epidemic seems to start in states from the south east and then spread in the north. This scenario happens for almost all years since 2011. This can be illustrated even more by comparing the situation in Texas and in the New York state in the bottom panel of Figure 58. In this figure, both the exponential growth and the peak timing happen always few weeks earlier for Texas than for New York. This scenario was also observed in [168]. Therefore, the evolution of the flu epidemic within the USA could be simulated using an interactive eRG between the states as was achieved in section 4.2.3.



FIGURE 58 – Top panel : Weekly evolution of the percentage of ILI patients in states hospitals for the 2014-2015 flu epidemic episode (from left to right then top to down). The color represents the percentage of ILI from a scale of 0 % in white to 15 % in red. Bottom panel : Weekly evolution of the percentage of ILI patients for New York and Texas states.

Another pandemic of interest that has a really different timescale behavior is caused by the

HIV (for Human immunodeficiency Virus) and its related disease AIDS (Acquired Immuno-Deficiency Syndrome) [169]. From its first description in 1981, the disease has rapidly created a global concern as approximately 75 millions of persons have been infected by the virus and around 30 millions of them died because of it since the beginning of the pandemic according to the website *unaids.org*. Looking at the data from South Africa in Figure 59, we can clearly see the shape of a wave for the number of new HIV cases, as compared to what was obtained for the COVID-19 pandemic as shown in Figure 20. One can remark that for AIDS the wave span on multiple years instead of weeks for COVID-19. Therefore, the HIV spread is slower due to its propagation mode. Indeed, AIDS can be transmitted by sexual contacts or blood transfusions but not with airborne infectious material [170, 171]. This time scale can be seen also by fitting the data for South Africa with a logistic function. We obtain $a = 1.12 \times 10^7$ and $\gamma - > 0.24$ years⁻¹. Therefore, studying the evolution of AIDS and its transmission across the World using eRG models could be straightforward and will certainly be investigated in the future.



FIGURE 59 – Number of new HIV cases and cumulative number of HIV cases for South Africa (dots) and the fits associated (plain lines).

As we have seen, the eRG analysis can be performed for other pandemics, but going a step further, one could also imagine to simulate the emergence of a new pandemic somewhere and model how it would spread over the world. Human activities have led to the multiplication of epidemics from a direct impact (Deforestation, development of cities...) leading to more transmission from wild animals, such as bats, to human due to increasing interactions between them. This could be caused also by an indirect impact due to global warming causing the thawing of the permafrost crust mostly in Russia and Canada [172]. This melting could release new pathogens [173] that were imprisoned in ice. By supposing that a new pandemic would emerge from forest or permafrost edges, we could then predict how it would infect the world and prepare an answer to curb its possible evolution before it's too late.

Using the eRG formalism have led to various study of the dynamics of the COVID-19 and other pandemics and we saw that it can be extended in a simple way to new features. Besides the simplicity of the framework, it allows to extract precious information from the evolution of the pandemics and to predict what would happen in the future.

5 Discussion and conclusion

Throughout this thesis, we have seen the importance of the Renormalisation Group equations through two main axes. It allowed us to go further in our interpretation of physics beyond the Standard Model but also to study fields beyond the scope of particles, investigating how it could model the COVID-19 outbreak.

Defining a new paradigm for the Grand Unification seeing it as an asymptotic process instead of a crossing of the couplings is one the major achievement shown in the thesis. Through extra-dimensional models introducing new particles, the $Indalo(\mathcal{P})$ -fields, the running gauge coupling constants were pushed toward a unique non-trivial fixed point as was illustrated through the minimal SU(5) description. The main advantage of aGUT models is that they are free from proton decay due to the conservation of the baryon and lepton number unlike standard GUT, because lepto-quarks bosons are replaced by Indalo-SM bosons. This allows the model to have a low-energy compactification scale, granting more freedom on the phenomenology of the model. Moreover, a dark matter candidate naturally stems from the models, allowing to reproduce the baryogenesis and to put constraints on the mass of the P-particles. Scenarios in which the Yukawa sector is free from Landau poles exist in the SU(5) model, however they do not propose clear phenomenological bounds. In the case where all Yukawa couplings are in the bulk, no phenomenology can be extracted up to a critical compactification scale $R_c^{-1} = 3 \cdot 10^5$ TeV. Above this scale, any Yukawa couplings is driven to an asymptotically free point. For Yukawa couplings all localised, any scenario would give an asymptotically free behavior. Eventhough the Yukawa sector doesn't give much information, the SU(5) model is just the minimal version of aGUT and can be extended to larger groups such as SO(10)that could allow to embed the six fermion multiplets used in SU(5) in only two representations of SO(10) giving a more appealing shape to the model. Moreover, aGUT models could bring explanation to experimental disagreements with the SM such as the muon g-2 anomaly. Asymptotic Grand Unification is a new field and will definitely bring more insights on possible models for the physics beyond the Standard Model.

Following the specific case of the COVID-19 crisis spreading around the world at the end of 2019, we also saw that the Renormalisation Group formalism can bring a new framework called the epidemic Renormalisation Group (eRG). Compared to compartmental or lattice models, the eRG concept granted, through a single differential equation, a better adaptability to additional features. From the basic one-wave in a single region, it allowed to understand the impact of the mobility reductions and of border controls on the spreading of the COVID-19 pandemic in Europe and in the US. Thanks to the addition of an interaction term, the eRG also provided a working scenario of how waves can spread thanks to mobility transports in a big region. Using this, the timing of the coming waves has been forecasted using the interactions between the countries modeled thanks to mobility data. In a similar way they allowed us to change the paradigm of Grand Unification, the fixed points technology granted a new point of view on the multiple waves behavior of the epidemic. Successive waves and endemic periods can be modeled through successive fixed points being complex for intermediate waves or not for the final wave. Simulating the spread with fixed points allows to forecast the evolution of the pandemic on a longer period. Then, variants and vaccination campaigns effects were introduced via modifications of the original equation explaining the behavior of the COVID-19 crisis in the beginning of 2021 at which time the vaccination campaign velocity and mutations spread will determine when the end of the crisis will happen. We finally saw that all the features developed specifically for the COVID-19 can be also used to study other viruses such as seasonal influenza or AIDS and could allow to simulate the outbreak and the spread of a future pandemic accross the World.

The improvements brought by the RG equations are mainly due to the strong adaptation of the differential equations to new features. The fixed points technology brought a new scope to see the evolution of models. From a unique non-trivial point providing the asymptotic Grand Unification to several complex fixed points indicating multiple waves in the evolution of a pandemic, the RGE allowed a renewing interested in interdisciplinary projects. Such framework could even be applied to other fields based on spreading dynamics such as chemistry, social science or economy.

Appendices

A Mathematical formulas

A.1 Group generators

Throughout this thesis, the Einstein notation is used where the sum over indices is implicitly used when superscripts and subscripts are repeated such as :

$$x_a y^a \equiv \sum_a x_a y^a . \tag{301}$$

The set of $n \times n$ orthogonal matrices is a group called the orthogonal group and denoted O(n). Orthogonal matrices Q are such that :

$$QQ^T = Q^T Q = I_n . aga{302}$$

The set of $n \times n$ unitary matrices is a group called the unitary group and denoted U(n). Unitary matrices U are such that :

$$UU^{\dagger} = U^{\dagger}U = I_n . \tag{303}$$

The groups SO(n) and SU(n) are called "special" if we add the condition on the determinant of any matrix M from the group such that :

$$Det M = +1. (304)$$

Symmetry groups can be fully generated by a minimal set of matrix T which are called group generators and allows any matrix A in the group to be written :

$$A = \alpha_a T^a , \qquad (305)$$

where any $\alpha_a \in \mathbb{R}$.

For SU(2), the generators σ , called the Pauli matrices, can be written :

$$\sigma_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(306)

For SU(3), the generators τ , called the Gell-Mann matrices, can be written :

$$\begin{aligned} \tau_1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_2 &= \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_3 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_4 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \tau_5 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \tau_6 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_7 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \tau_8 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$
(307)

We can also derive the Dirac matrices γ that are essential to understand the spinor structures of the fermion fields. The γ matrices are defined such that :

$$\gamma_0 = \begin{pmatrix} I_2 & 0\\ 0 & I_2 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i\\ -\sigma_i & 0 \end{pmatrix}, \quad (308)$$

where I_2 is the 2 × 2 identity matrix and σ_i are the Paul matrices.

In addition, the fifth-gamma matrix γ_5 is defined as :

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{309}$$

The gamma matrices satisfy the following relations :

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} , \quad \{\gamma^{5}, \gamma^{\nu}\} = 0 , \qquad (310)$$

$$\gamma^{\mu}\gamma_{\mu} = 4 , \quad \gamma^{\mu}\gamma^{\nu}\gamma_{\mu} = -2\gamma^{\nu} , \quad \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} = 4g^{\nu\rho} .$$
(311)

The relations involving the trace over the spinor space are :

$$\operatorname{Tr}\left[\gamma^{\mu}\right] = \operatorname{Tr}\left[\gamma^{5}\right] = \operatorname{Tr}\left[\operatorname{odd} \operatorname{number} \operatorname{of} \gamma\right] = 0, \qquad (312)$$

$$\operatorname{Tr}\left[\gamma^{\mu}\gamma^{\nu}\right] = 4g^{\mu\nu} , \qquad (313)$$

$$\operatorname{Tr}\left[\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}\right] = 4\left(g^{\mu\nu}g^{\alpha\beta} - \gamma^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right) \,. \tag{314}$$

A.2 Group factors formulas

For the generators of a group representation R_F , T^a , the structure constants f are defined such that :

$$\left[T^{a}, T^{b}\right] = T^{a}T^{b} - T^{b}T^{a} = igf_{c}^{ab}T^{c} .$$

$$(315)$$

Similarly, Casimir invariants are defined by :

$$\operatorname{Tr}\left(T^{a}T^{b}\right) = T\left(R_{F}\right)\delta^{ab}, \quad T^{a}T^{a} = C_{F}\mathbf{1}, \quad f_{abc}f_{abd} = C(G)\delta_{cd}.$$
(316)

A.3 Dimensional regularisation

The following integrals are needed to achieve the diagram calculations in dimensional regularisation :

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{\left(k^2 - \Delta + i\tilde{\epsilon}\right)^2} = -\frac{d}{2} \frac{i}{(2\pi)^{\frac{d}{2}}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma\left(\frac{2-d}{2}\right) \,, \tag{317}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left(k^2 - \Delta + i\tilde{\epsilon}\right)^2} = \frac{i}{(2\pi)^{\frac{d}{2}}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right) \,. \tag{318}$$

B One loop Yukawa beta-function diagrams calculations

B.1 Feynman rules

We compile here the Feynman rules in the SU(5) a GUT required in order to calculate the diagrams :

For the ϕ propagator :

$$\phi_i \cdots \phi_j \quad \leftrightarrow \ i\delta^{ij} \sum_{n=1}^{+\infty} \frac{1}{k^2 - m_n^2 + i\tilde{\epsilon}}$$
 (319)

For the ψ fermion propagator :

$$\psi_i \longrightarrow \psi_j \quad \leftrightarrow \ i\delta^{ij} \left(p_\mu \gamma^\mu + m_n \right) \sum_{n=1}^{+\infty} \frac{1}{p^2 - m_n^2 + i\tilde{\epsilon}} . \tag{320}$$

For the $\overline{\psi}$ anti-fermion propagator :

$$\overline{\psi}^{i} \longrightarrow \overline{\psi}^{j} \leftrightarrow i\delta_{ij} \left(-p_{\mu}\gamma^{\mu} + m_{n}\right) \sum_{n=1}^{+\infty} \frac{1}{p^{2} - m_{n}^{2} + i\tilde{\epsilon}} .$$
(321)

For the A_{μ} propagator :

$$A^{a}_{\mu} \sim A^{b}_{\nu} \leftrightarrow -i\delta_{ab} \left[g_{\mu\nu} - (1-\xi) \frac{k_{\mu}k_{\nu}}{k^{2}} \right] \sum_{n=1}^{+\infty} \frac{1}{k^{2} - m_{n}^{2} + i\tilde{\epsilon}} .$$
(322)

For the A_5 propagator :

$$A_5^a \cdots A_5^b \leftrightarrow i\delta_{ab} \sum_{n=1}^{+\infty} \frac{1}{k^2 - \xi (m_n)^2 + i\tilde{\epsilon}} .$$
(323)

For the Y_τ vertex :

$$\begin{array}{cccc}
\psi_{\overline{5}}^{i} & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \leftrightarrow iY_{\tau}^{kl}{}_{ij} = i\frac{y_{\tau}}{2} \left(\delta_{i}^{k}\delta_{j}^{l} - \delta_{i}^{l}\delta_{j}^{k}\right) .
\end{array}$$
(324)

For the Y_b vertex :

$$\psi_{10kl} \longrightarrow \psi_{10kl} \longrightarrow \psi_{5}^{\dagger j} \leftrightarrow iY_{b}^{kl}{}_{ij} = i\frac{y_{b}}{2} \left(\delta_{i}^{k}\delta_{j}^{l} - \delta_{i}^{l}\delta_{j}^{k}\right) .$$
(325)

For the Y_t vertex :

$$\psi_{10jk}$$

$$Y_t \longrightarrow iY_t^{ijklm} = iy_t \epsilon^{ijklm} , \qquad (326)$$

$$\overline{\psi_{10}}_{lm}$$

where ϵ^{ijklm} is the Levi-Civita tensor of order 5.

For the Y_ν vertex :

$$\psi_{1}$$

$$\downarrow_{Y_{\nu}} \qquad \leftrightarrow iY_{\nu j}^{i} = iy_{\nu}\delta_{j}^{i}. \qquad (327)$$

$$\overline{\psi_{5}}^{j}$$

For the $\overline{\psi_R}A_\mu\psi_R$ vertex :

For the $\overline{\psi_R} A_5 \psi_R$ vertex :

For the $\phi^{\dagger}\phi A_{\mu}$ vertex :

For the $\phi^{\dagger}\phi A_{\mu}A_{\nu}$ vertex :

$$\phi_{i} \qquad A^{a}_{\mu} \\ g^{2}_{a} \qquad \leftrightarrow 2ig_{a}g_{b}\delta_{ij}g_{\mu\nu} .$$

$$\phi^{\dagger j} \qquad A^{b}_{\nu}$$

$$(331)$$

For the $\phi^{\dagger}\phi A_5$ vertex :

For the $\phi^{\dagger}\phi A_5 A_5$ vertex :
Feynman diagram	Momentum contribution	SU(5) generator factor			
$\begin{array}{c c} \psi_1 \\ \\ \hline $	$i\mathcal{M} \approx -i\frac{1}{16\pi^2}\frac{2}{\epsilon}$	$-y_t y_b^2 \epsilon^{ijklm}$, $-y_t y_\tau^2 \epsilon^{ijklm}$.			
$\psi_1 \psi_1$	$i\mathcal{M} \approx \frac{1}{2} \times i \frac{1}{16\pi^2} \frac{1}{\epsilon}$	$12y_t^3\epsilon^{ijklm}$, $y_ty_b^2\epsilon^{ijklm}$.			
$\psi_2 $	$i\mathcal{M} \approx -\frac{1}{2} \times \left(-i\frac{1}{16\pi^2}\frac{1}{\epsilon}\right)$	$12y_t^3\epsilon^{ijklm}$, $y_ty_\tau^2\epsilon^{ijklm}$.			
φ φ	$i\mathcal{M} \approx \frac{1}{2} \times i \frac{1}{16\pi^2} \frac{4}{\epsilon}$	$24y_t^3 \epsilon^{ijklm} , \ 2y_t y_b^2 \epsilon^{ijklm} , \ 2y_t y_{\nu}^2 \epsilon^{ijklm} , \ y_t y_{\nu}^2 \epsilon^{ijklm} , \ y_t y_{\nu}^2 \epsilon^{ijklm} , \ y_t y_{\nu'}^2 \epsilon^{ijklm} .$			
$\begin{array}{c} \psi_1 \\ \\ \\ \hline \\ $	$i\mathcal{M} \approx i \frac{g^2}{16\pi^2} \left(3 + \xi\right) \frac{2}{\epsilon}$	$-\frac{12}{5}y_t\epsilon^{ijklm}\;.$			
$\psi_1 \psi_1$	$i\mathcal{M} \approx \frac{1}{2} \times i \frac{g^2}{16\pi^2} \xi \frac{2}{\epsilon}$	$\frac{18}{5}y_t\epsilon^{ijklm} \ .$			

B.2 Relevant diagrams for the Yukawa calculation

TABLE 13 - Relevant brane top-Yukawa renormalisation diagrams - 1.

Feynman diagram	Momentum contribution	SU(5) generator factor
$\overline{\psi_2}$	$i\mathcal{M} \approx -\frac{1}{2} \times \left(-i\frac{g^2}{16\pi^2}\xi\frac{2}{\epsilon}\right)$	$\frac{18}{5} y_t \epsilon^{ijklm} \; .$
φξφ	$i\mathcal{M} \approx \frac{1}{2} \times \left(-i \frac{g^2}{16\pi^2} \left(3 - \xi \right) \frac{2}{\epsilon} \right)$	$\frac{12}{5}y_t\epsilon^{ijklm}\;.$
$\begin{array}{c} \psi_1 \\ & & \\ \hline \\ \hline \\ \hline \\ \psi_2 \end{array} \end{pmatrix} \rightarrow \begin{array}{c} & & \\ \phi \end{array}$	$i\mathcal{M} \approx i \frac{g^2}{16\pi^2} \xi \frac{2}{\epsilon}$	$-rac{6}{5}y_t\epsilon^{ijklm}$.
$\begin{array}{c} \psi_1 \\ \\ \hline \\ \psi_2 \end{array} \\ \end{array} $	$i\mathcal{M} \approx -i\frac{g^2}{16\pi^2}\xi\frac{2}{\epsilon}$	$\frac{6}{5}y_t\epsilon^{ijklm}\;.$
$\begin{array}{c c} \psi_1 \\ A_5 \\ \hline \\ \hline \\ \hline \\ \psi_2 \end{array}$	$i\mathcal{M} \approx -i \frac{g^2}{16\pi^2} \frac{2}{\epsilon}$	$2rac{6}{\overline{5}}y_t\epsilon^{ijklm}$.
$\psi_1 \xrightarrow{A_5} \psi_1$	$i\mathcal{M} \approx \frac{1}{2} \times \left(i \frac{g^2}{16\pi^2} \frac{1}{\epsilon} \right)$	$\frac{24}{5}y_t\epsilon^{ijklm}\;.$
$\overline{\psi_2}$	$i\mathcal{M} \approx -\frac{1}{2} \times \left(-i\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\right)$	$\frac{24}{5}y_t\epsilon^{ijklm}\;.$

TABLE 14-Relevant brane top-Yukawa renormalisation diagrams - 2.

B.3 Calculation in the Standard Model

For the renormalisation of the top Yukawa beta function in the Standard Model group $SU(3) \times SU(2) \times U(1)$ with latin letters for SU(2) indices and greek letters for SU(3) indices, we have :

$$16\pi^{2} \frac{dY_{t}^{i\gamma}{}_{j\delta}}{dt} = -2Y_{b}^{k\epsilon}{}_{j\delta}Y_{b}^{l\zeta}{}_{i\epsilon}\epsilon_{m}^{k}Y_{t}^{m\gamma}{}_{l\zeta} + \frac{1}{2} \left(Y_{t}^{i\epsilon}{}_{j\delta}Y_{t}^{l\zeta}{}_{k\epsilon}Y_{t}^{k\gamma}{}_{l\zeta} + Y_{t}^{k\epsilon}{}_{j\delta}Y_{t}^{l\zeta}{}_{k\epsilon}Y_{t}^{i\gamma}{}_{l\zeta} + Y_{b}^{k\epsilon}{}_{j\delta}Y_{b}^{l\zeta}{}_{k\epsilon}Y_{t}^{i\gamma}{}_{l\zeta} \right) + \left(Y_{t}^{k\gamma}{}_{j\delta}Y_{t}^{l\zeta}{}_{k\epsilon}Y_{t}^{i\epsilon}{}_{l\zeta} + Y_{t}^{k\gamma}{}_{j\delta}Y_{b}^{l\zeta}{}_{k\epsilon}Y_{b}^{i\epsilon}{}_{l\zeta} + Y_{t}^{k\gamma}{}_{j\delta}Y_{t}^{l\zeta}{}_{k}Y_{t}^{i}{}_{l\zeta}\right) + g_{2}^{2} \left((1 - \alpha_{2})T_{2}^{ak}{}_{j}T_{2}^{al}{}_{k}Y_{t}^{i\gamma}{}_{l\delta} - (2 + \alpha_{2})Y_{t}^{k\gamma}{}_{j\delta}T_{2}^{al}{}_{k}T_{2}^{ai}{}_{l} - 2(1 - \alpha_{2})T_{2}^{al}{}_{j}Y_{t}^{k\gamma}{}_{l\delta}T_{2}^{ai}{}_{k}\right) + g_{3}^{2} \left(-2(4 - \alpha_{3})T_{3}^{b\epsilon}{}_{\delta}Y_{t}^{i\zeta}{}_{j\epsilon}T_{3}^{b\gamma}{}_{\zeta} + (1 - \alpha_{3})Y_{t}^{i\epsilon}{}_{j\delta}T_{3}^{b\zeta}{}_{\epsilon}T_{3}^{b\gamma}{}_{\zeta} + (1 - \alpha_{3})T_{3}^{b\epsilon}{}_{\delta}T_{3}^{b\zeta}{}_{\epsilon}Y_{t}^{i\gamma}{}_{j\zeta}\right) + g_{1}^{2} \left(-\frac{1}{9}2(4 - \alpha_{0}) + \frac{4}{9}(1 - \alpha_{0}) + \frac{1}{36}(1 - \alpha_{0}) - \frac{1}{4}(2 + \alpha_{0}) + \frac{-1}{3}2(1 - \alpha_{0}) - \frac{-1}{12}2(1 - \alpha_{0})\right)Y_{t}^{i\gamma}{}_{j\delta}$$

$$(334)$$

where the $\alpha_i = 1 - \xi_i$ are the gauge parameters. Using the explicit expression of the Yukawa tensors such that :

$$Y_t^{i\gamma}{}_{j\delta} = y_t \delta_j^i \delta_\delta^\gamma , \quad Y_b^{i\gamma}{}_{j\delta} = y_b \delta_j^i \delta_\delta^\gamma , \quad Y_\tau^{i\gamma}{}_{j\delta} = y_\tau \delta_j^i \delta_\delta^\gamma , \tag{335}$$

taking $\beta_t = \frac{dy_t}{dt}$ and factorising the α_i terms, we find :

$$16\pi^{2}\beta_{t} = \frac{1}{2} \left(y_{t}^{3} \operatorname{Tr}\left(\delta_{2}\right) + y_{t}^{3} + y_{t}y_{b}^{2} + 2y_{t}^{3} \operatorname{Tr}\left(\delta_{3}\right) + 2y_{t}y_{b}^{2} \operatorname{Tr}\left(\delta_{3}\right) + 2y_{t}y_{\tau}^{2} - 4y_{t}y_{b}^{2} \right) + g_{2}^{2}y_{t} \left(T_{2}^{a}T_{2}^{a} - 2T_{2}^{a}T_{2}^{a} - 2T_{2}^{a}T_{2}^{a} \right) + \alpha_{2}g_{2}^{2}y_{t} \left(-T_{2}^{a}T_{2}^{a} - T_{2}^{a}T_{2}^{a} + 2T_{2}^{a}T_{2}^{a} \right) + g_{3}^{2}y_{t} \left(-8T_{3}^{b}T_{3}^{b} + T_{3}^{b}T_{3}^{b} + T_{3}^{b}T_{3}^{b} \right) + \alpha_{3}g_{3}^{2}y_{t} \left(2T_{3}^{b}T_{3}^{b} - T_{3}^{b}T_{3}^{b} - T_{3}^{b}T_{3}^{b} \right) + g_{1}^{2}y_{t} \left(\frac{4}{9} + \frac{1}{36} - \frac{8}{9} - \frac{2}{3} + \frac{1}{6} - \frac{1}{2} \right) + \alpha_{0}g_{1}^{2} \left(-\frac{4}{9} - \frac{1}{36} + \frac{2}{9} + \frac{2}{3} - \frac{1}{6} - \frac{1}{4} \right) .$$

$$(336)$$

We see that terms proportional to the α_i disappear and we finally have :

$$16\pi^{2}\beta_{t}\Big|_{\rm SM} = \frac{9}{2}y_{t}^{3} + \frac{3}{2}y_{b}^{2}y_{t} + y_{\tau}^{2}y_{t} - \frac{9}{4}g_{2}^{2}y_{t} - 8g_{3}^{2}y_{t} - \frac{17}{12}g_{1}^{2}y_{t} .$$
(337)

Using the same method for y_b and y_{τ} , we can find :

$$16\pi^{2}\beta_{b}\Big|_{\mathrm{SM}} = \frac{9}{2}y_{b}^{3} + \frac{3}{2}y_{t}^{2}y_{b} + y_{\tau}^{2}y_{b} - \frac{1}{4}g_{1}^{2}y_{b} - \frac{9}{4}g_{2}^{2}y_{b} - 8g_{3}^{2}y_{b}, \qquad (338)$$

$$16\pi^2 \beta_\tau \Big|_{\rm SM} = \frac{9}{2} y_\tau^3 + 3y_t^2 y_\tau + 3y_b^2 y_\tau - \frac{9}{4} g_1^2 y_\tau - \frac{9}{4} g_2^2 y_\tau \,. \tag{339}$$

C Epidemic Renormalisation Group

C.1 Monte-Carlo simulation

Generating the MC data via a logistic function with a unique γ now and fitting it with the function $\alpha_{2\gamma}$ gives the following result for the fit parameters :

	Generated	$\alpha_{2\gamma}$ fit	Logistic fit
a	100	100.0(3)	100.0(3)
γ_B	Ø	0.1(2)	Ø
γ_C	Ø	0.100(1)	Ø
γ	0.1	Ø	0.100(1)
b	100	115(47)	100(6)
Δt	ø	3(3)	ø
$\Delta\gamma$	ø	0(45)	Ø

TABLE 15 – Parameters obtained for the generated data fits using a logistic function and the function $\alpha_{2\gamma}$ defined in equation (269). Data generation parameters are : $a = 100, \gamma = 0.1, b = 100$.

C.2 $2 - \gamma$ function fit parameters vs mobility data

Plotting the values of the fit parameters obtained using the function $\alpha_{2\gamma}$ along with the mobility parameters averaged values shows no clear correlation :



FIGURE 60 – Mobility categories versus Δt for European countries and US states.



FIGURE 61 – Mobility categories versus $\Delta \gamma$ for European countries and US states.

C.3 US divisions composition

Division composition					
Division names	Code	States within the division			
New England	NE	Massachusetts, Connecticut, New Hampshire, Maine, Rhode Island and Vermont			
Mid-Atlantic	MA	New York, Pennsylvania, New Jersey, Maryland and Delaware			
South Atlantic	SA	Florida, Georgia, North Carolina, Virginia, South Carolina and West Virginia			
East South Central	ESC	Tennessee, Alabama, Kentucky and Mississippi			
West South Central	WSC	Texas, Louisiana, Oklahoma and Arkansas			
East North Central	ENC	Illinois, Ohio, Michigan, Indiana and Wisconsin			
West North Central	WNC	Missouri, Minnesota, Iowa, Kansas, Nebraska, South Dakota and North Dakota			
Mountains	М	Arizona, Colorado, Utah, Nevada, New Mexico, Idaho, Montana and Wyoming			
Pacific	Р	California, Washington, Oregon, Hawaii and Alaska			

TABLE 16 – States of the US integrated into the 9 census divisions. Maryland and Delaware are moved from South Atlantic to Mid-Atlantic.

C.4 Second European waves

The values of the k_{i0} , the rescaling factors for γ , a, and the parameters of the linear growth for the 10 seed countries are as follow :

	BEL	BOS	HRV	CZE	GRE	NLD	SER	SLK	SLN	SPA
k_{i0}	0.01	0.12	0.3	0.1	0.01	0.01	0.5	0.02	0.035	0.3
γ_i scaling	1	0.7	0.6	0.6	1	1	0.7	0.8	0.6	0.85
a_i scaling	1	1.3	1.06	1.06	1	1	1.05	1	0.95	1
θ_i	60	0	0	40	7	35	35	1	1	30
$t_{ m pl}$	20	_	—	20	20	20	20	20	20	20



FIGURE 62 – Total number infected cases (updated to the 5^{th} of August) with respect to the theoretical curve (orange line) used to calibrate the simulation for the countries having signs of a second wave at the time. The vertical line shows where the second wave simulation begins.



FIGURE 63 – Time evolution for the cumulative number of infected and new cases for the nordic European countries, from the simulation starting at week 25. The bands are generated by varying the infection rates γ_i within 10%.

First wave parameters			Second wave simulation peak timing			
	$a \ (\log)$	γ	case a	case b	case c	case e
Austria	7.463(7)	0.99(25)	30.4(4)	32.4(5)	34.7(5)	34.8 + 0.9 - 0.7
Belgium	8.53(2)	0.55(2)	34.6(7)	37.9(6	41.4(5	$33.8^* + 0.8 - 0.7$
Bosnia	7.88(24)	0.41(2)	33.9(6)	37.4(5)	41.0(5)	$33.4^* + 0.7 - 0.6$
Croatia	6.268(7)	0.71(2)	30.9(6)	33.5(7)	36.6(7)	$27.7^* + 0.1 - 0.1$
Czechia	9.085(14)	0.56(3)	33.5(7)	36.8(6)	40.2(5)	$32.7^* + 0.6 - 0.5$
Denmark	7.667(8)	0.40(1)	35.6(6)	39.2(5)	42.7(5)	39.9 + 1.3 - 1.1
Finland	7.190(5)	0.385(6)	35.4(6)	39.0(5)	42.5(5)	39.8 + 1.3 - 1.1
France	7.711(6)	0.58(12)	36.0(6)	39.3(5)	42.6(4)	41.1 + 1.4 - 1.2
Germany	7.679(7)	0.62(2)	35.8(6)	39.0(5)	42.3(5)	39.9 + 1.4 - 1.1
Greece	5.537(9)	0.57(2)	32.5(6)	35.8(5)	39.2(4)	$31.7^* + 0.6 - 0.5$
Hungary	6.022(9)	0.47(1)	33.9(6)	37.4(5)	40.9(5)	38.1 + 1.1 - 0.9
Ireland	8.580(8)	0.60(2)	33.1(7)	36.1(7)	39.5(6)	36.6 + 1.0 - 0.9
Italy	8.304(4)	0.429(8)	39.1(6)	42.8(4)	46.1(5)	43.9 + 1.7 - 1.4
Netherlands	7.904(5)	0.525(8)	35.2(7)	38.6(6)	42.1(5)	$34.3^* + 0.9 - 0.7$
Norway	7.356(6)	0.58(2)	32.7(6)	35.8(7)	39.3(6)	36.4 + 1.0 - 0.8
Poland	7.13(3)	0.182(7)	45.7(5)	49.3(6)	52.6(6)	54.6 + 2.7 - 2.2
Portugal	10.323(14)	0.517(22)	34.7(7)	38.1(6)	41.6(5)	38.7 + 1.2 - 1.0
Serbia	9.323(12)	0.628(17)	32.6(6)	35.6(6)	38.9(5)	$29.2^* + 0.3 - 0.3$
Slovakia	5.67(2)	0.59(4)	31.7(7)	34.8(7)	38.2(6)	$30.7^* + 0.5 - 0.4$
Slovenia	7.299(7)	0.656(17)	30.7(6)	33.5(7)	36.7(6)	$29.7^* + 0.3 - 0.3$
Spain	8.747(8)	0.46(1)	38.2(7)	41.8(5)	45.2(5)	$33.8^* + 0.8 - 0.7$
Sweden	11.56(4)	0.162(8)	47.8(5)	51.3(5)	54.6(6)	55.8 + 2.9 - 2.4
Switzerland	8.196(3)	0.72(1)	32.4(6)	35.1(7)	38.1(7)	35.8 + 1.0 - 0.8
UK	8.353(7)	0.368(7)	41.0(6)	44.6(4)	48.0(5)	46.2 + 2.0 - 1.6

TABLE 17 – Left block : parameters fitted from the first wave. Right block : median peak time of the second wave in 2020 calendar weeks for the four simulation scenarios (cases \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{e}), with 1 standard deviation. For cases \mathbf{a} , \mathbf{b} and \mathbf{c} , the median and errors only take into account the 100 simulations, differing by randomly generated matrices k_{ij} . For case \mathbf{e} , we include the interval spanned by varying the γ_i within 10% from the fitted values, where the results marked with an asterisk correspond to the seed countries with a beginning of second wave, as of the 5th of August 2020.

C.5 Multiwave and CeRG

Following the analysis using the CeRG model, the plots hereafter show the forecast for the starting waves for various countries :



FIGURE 64 – Evolution of the cumulative and number of new cases for France, Italy, Germany and UK (red dots) and the solutions of the CeRG equations using parameters accessible in Table 18.



FIGURE 65 – Evolution of the total and number of new cases for Spain, Denmark, Saudi Arabia and South Africa (red dots) and the solutions of the CeRG equations using parameters accessible in Table 18.

Countries	CeRG parameters							
	a	γ	δ_0	p_0	p_1	ζ_1		
France	61×10^{3}	0.058	1.0×10^{3}	0.55	0.6	0.5		
Italy	41×10^{3}	0.091	1.0×10^{3}	0.55	0.6	0.5		
UK	26×10^{3}	0.082	1.0×10^{3}	0.55	0.6	0.5		
Germany	22×10^{3}	0.070	1.0×10^{3}	0.55	0.6	0.5		
Spain	36×10^{3}	0.078	1.0×10^{3}	0.55	0.6	0.5		
Switzerland	40×10^{3}	0.127	1.0×10^{3}	0.55	0.6	0.5		
Netherlands	31×10^{3}	0.092	1.0×10^{3}	0.55	0.6	0.5		
Belgium	45×10^{3}	0.157	1.0×10^{3}	0.55	0.6	0.5		
Denmark	15×10^{3}	0.071	1.0×10^{3}	0.55	0.6	0.5		
Iceland	11×10^{3}	0.114	1.0×10^{3}	0.55	0.6	0.5		
Canada	17×10^3	0.055	1.0×10^{3}	0.55	0.6	0.5		
South Africa	12×10^{3}	0.102	2.2×10^{3}	0.56	0.6	0.45		
Bolivia	13×10^{3}	0.057	1.2×10^4	0.51	0.6	0.45		
Saudi Arabia	10×10^{3}	0.058	1.3×10^{3}	0.51	0.6	0.5		
Australia	772	0.122	8.4×10^{6}	0.52	0.6	0.5		
Japan	500	0.103	4.9×10^{2}	0.7	0.6	0.4		
South Korea	180	0.191	5.7×10^{3}	0.6	0.6	0.45		

TABLE $\mathbf{18}$ – Parameters for the CeRG for ecast for the last waves.

Références

- [1] Michael Peskin. An introduction to quantum field theory. CRC press, 2018.
- [2] Matthew D Schwartz. <u>Quantum field theory and the standard model</u>. Cambridge University Press, 2014.
- [3] S.L. Glashow. Partial Symmetries of Weak Interactions. Nucl. Phys., 22:579–588, 1961.
- [4] Steven Weinberg. A Model of Leptons. Phys. Rev. Lett., 19:1264–1266, 1967.
- [5] Abdus Salam. Weak and Electromagnetic Interactions. <u>Conf. Proc. C</u>, 680519 :367–377, 1968.
- [6] G. Arnison et al. Experimental observation of isolated large transverse energy electrons with associated missing energy at s=540 GeV. Physics Letters B, 122(1) :103–116, 1983.
- [7] M. Banner et al. Observation of single isolated electrons of high transverse momentum in events with missing transverse energy at the CERN pp collider. <u>Physics Letters B</u>, 122(5):476–485, 1983.
- [8] G. Aad, T. Abajyan, B. Abbott, J. Abdallah, S. Abdel Khalek, A.A. Abdelalim, O. Abdinov, R. Aben, B. Abi, M. Abolins, and et al. Observation of a new particle in the search for the standard model higgs boson with the ATLAS detector at the LHC. <u>Physics</u> Letters B, 716(1) :1–29, Sep 2012.
- [9] S. Chatrchyan, V. Khachatryan, A.M. Sirunyan, A. Tumasyan, W. Adam, E. Aguilo, T. Bergauer, M. Dragicevic, J. Erö, C. Fabjan, and et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. <u>Physics Letters B</u>, 716(1):30–61, Sep 2012.
- [10] Tatsumi Aoyama, Masashi Hayakawa, Toichiro Kinoshita, and Makiko Nio. Tenthorder QED contribution to the electron g-2 and an improved value of the fine structure constant. Physical Review Letters, 109(11), Sep 2012.
- [11] Tatsumi Aoyama, Masashi Hayakawa, Toichiro Kinoshita, and Makiko Nio. Tenth-order electron anomalous magnetic moment : Contribution of diagrams without closed lepton loops. Physical Review D, 91(3), Feb 2015.
- [12] D. Hanneke, S. Fogwell Hoogerheide, and G. Gabrielse. Cavity control of a single-electron quantum cyclotron : Measuring the electron magnetic moment. <u>Physical Review A</u>, 83(5), May 2011.
- [13] G. W. Bennett et al. Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL. Phys. Rev. D, 73 :072003, 2006.
- [14] LHCb collaboration. Test of lepton universality in beauty-quark decays, 2021.
- [15] B. Abi, T. Albahri, S. Al-Kilani, D. Allspach, L.P. Alonzi, A. Anastasi, A. Anisenkov, F. Azfar, K. Badgley, S. Baeßler, and et al. Measurement of the positive muon anomalous magnetic moment to 0.46 ppm. Physical Review Letters, 126(14), Apr 2021.
- [16] Giacomo Cacciapaglia, Corentin Cot, and Francesco Sannino. Naturalness of lepton non-universality and muon g-2, 2021.
- [17] Murray Gell-Mann and Francis E Low. Quantum electrodynamics at small distances. Physical Review, 95(5) :1300, 1954.

- [18] Curtis G Callan Jr. Broken scale invariance in scalar field theory. <u>Physical Review D</u>, 2(8) :1541, 1970.
- [19] Kurt Symanzik. Small distance behaviour in field theory and power counting. Communications in Mathematical Physics, 18(3) :227–246, 1970.
- [20] Kurt Symanzik. Small-distance-behaviour analysis and Wilson expansions. Communications in mathematical physics, 23(1):49–86, 1971.
- [21] Arthur H Compton. A quantum theory of the scattering of x-rays by light elements. Physical review, 21(5) :483, 1923.
- [22] Morad Aaboud, G Aad, B Abbott, J Abdallah, B Abeloos, R Aben, OS AbouZeid, NL Abraham, H Abramowicz, H Abreu, et al. Measurement of the inelastic protonproton cross section at s= 13 TeV with the ATLAS detector at the LHC. <u>Physical</u> review letters, 117(18) :182002, 2016.
- [23] H. David Politzer. Reliable Perturbative Results for Strong Interactions? <u>Phys. Rev.</u> Lett., 30 :1346–1349, 1973.
- [24] D. J. Gross and Frank Wilczek. Asymptotically Free Gauge Theories I. Phys. Rev. D, 8 :3633–3652, 1973.
- [25] Max Planck. Über irreversible strahlungsvorgänge. In <u>Von Kirchhoff bis Planck</u>, pages 166–171. Springer, 1978.
- [26] LD Landau, AA Abrikosov, and IM Khalatnikov. Dokl. akad. <u>Nauk SSSR 95</u>, 497(773) :1177, 1954.
- [27] H. Georgi and S.L. Glashow. Unity of All Elementary Particle Forces. <u>Phys. Rev. Lett.</u>, 32 :438–441, 1974.
- [28] H. Georgi, Helen R. Quinn, and Steven Weinberg. Hierarchy of Interactions in Unified Gauge Theories. Phys. Rev. Lett., 33:451–454, 1974.
- [29] John Ellis. Physics gets physical. Nature, 415(6875) :957–957, 2002.
- [30] Hironari Miyazawa. Baryon Number Changing Currents. <u>Progress of Theoretical</u> Physics, 36(6) :1266–1276, 12 1966.
- [31] J-L Gervais and Bunji Sakita. Field theory interpretation of supergauges in dual models. Nuclear Physics B, 34(2) :632–639, 1971.
- [32] M. Shifman. Reflections and impressionistic portrait at the conference "frontiers beyond the standard model," ftpi, oct. 2012, 2012.
- [33] Howard Baer, Vernon Barger, and Dan Mickelson. How conventional measures overestimate electroweak fine-tuning in supersymmetric theory. <u>Physical Review D</u>, 88(9), Nov 2013.
- [34] Howard Baer, Vernon Barger, Peisi Huang, Dan Mickelson, Azar Mustafayev, and Xerxes Tata. Radiative natural supersymmetry : Reconciling electroweak fine-tuning and the higgs boson mass. Physical Review D, 87(11), Jun 2013.
- [35] Dimitri V Nanopoulos. Protons are not forever. In <u>High-Energy Physics in the Einstein</u> Centennial Year, pages 91–114. Springer, 1979.
- [36] Borut Bajc, Junji Hisano, Takumi Kuwahara, and Yuji Omura. Threshold corrections to dimension-six proton decay operators in non-minimal SUSY SU(5) GUTs. <u>Nuclear</u> Physics B, 910 :1–22, Sep 2016.

- [37] Gianfranco Bertone and Dan Hooper. History of dark matter. <u>Reviews of Modern</u> Physics, 90(4) :045002, 2018.
- [38] E. Corbelli and P. Salucci. The extended rotation curve and the dark matter halo of M33. Monthly Notices of the Royal Astronomical Society, 311(2) :441–447, Jan 2000.
- [39] Jan H Oort. The force exerted by the stellar system in the direction perpendicular to the galactic plane and some related problems. <u>Bulletin of the Astronomical Institutes</u> of the Netherlands, 6 :249, 1932.
- [40] V C Rubin and W K Ford, Jr. Rotation of the Andromeda nebula from a spectroscopic survey of emission regions. Astrophys. J. 159 : 379-403(Feb 1970).
- [41] A. N. Taylor, S. Dye, T. J. Broadhurst, N. Benitez, and E. van Kampen. Gravitational lens magnification and the mass of Abell 1689. <u>The Astrophysical Journal</u>, 501(2):539–553, Jul 1998.
- [42] Xiang-Ping Wu, Tzihong Chiueh, Li-Zhi Fang, and Yan-Jie Xue. A comparison of different cluster mass estimates : consistency or discrepancy? <u>Monthly Notices of the Royal</u> Astronomical Society, 301(3) :861–871, Dec 1998.
- [43] P. A. R. Ade, N. Aghanim, M. Arnaud, M. Ashdown, J. Aumont, C. Baccigalupi, A. J. Banday, R. B. Barreiro, J. G. Bartlett, and et al. Planck2015 results. <u>Astronomy</u> <u>Astrophysics</u>, 594 :A13, Sep 2016.
- [44] E. Aprile, J. Aalbers, F. Agostini, M. Alfonsi, L. Althueser, F.D. Amaro, M. Anthony, F. Arneodo, L. Baudis, B. Bauermeister, and et al. Dark matter search results from a one ton-year exposure of XENON1T. <u>Physical Review Letters</u>, 121(11), Sep 2018.
- [45] Gunnar Nordström. On the possibility of unifying the electromagnetic and the gravitational fields. arXiv preprint physics/0702221, 2007.
- [46] Theodor Kaluza. Zum unitätsproblem der physik. <u>Sitzungsber. Preuss. Akad. Wiss.</u> Berlin.(Math. Phys.), 1921(966972) :45–46, 1921.
- [47] O. Klein. The Atomicity of Electricity as a Quantum Theory Law. <u>Nature</u>, 118 :516, 1926.
- [48] Thirumalaisamy P Velavan and Christian G Meyer. The COVID-19 epidemic. <u>Tropical</u> medicine & international health, 25(3) :278, 2020.
- [49] Ronald Ross. An application of the theory of probabilities to the study of a priori pathometry; part i. <u>Proceedings of the Royal Society of London. Series A, Containing</u> Papers of a Mathematical and Physical Character, 92(638) :204–230, 1916.
- [50] Ronald Ross and Hilda P. Hudson. An application of the theory of probabilities to the study of a priori pathometry; part ii. <u>Proceedings of the Royal Society of London. Series</u> <u>A</u>, Containing Papers of a Mathematical and Physical Character, 93(650) :212–225, 1917.
- [51] Tiberiu Harko, Francisco S.N. Lobo, and M.K. Mak. Exact analytical solutions of the susceptible-infected-recovered (SIR) epidemic model and of the SIR model with equal death and birth rates. Applied Mathematics and Computation, 236 :184–194, Jun 2014.
- [52] Wuyue Yang, Dongyan Zhang, Liangrong Peng, Changjing Zhuge, and Liu Hong. Rational evaluation of various epidemic models based on the COVID-19 data of China, 2020.

- [53] Michele Della Morte, Domenico Orlando, and Francesco Sannino. Renormalization group approach to pandemics : The COVID-19 case. Frontiers in Physics, 8 :144, 2020.
- [54] Giacomo Cacciapaglia and Francesco Sannino. Evidence for complex fixed points in pandemic data. arXiv preprint arXiv :2009.08861, 2020.
- [55] Giacomo Cacciapaglia, Corentin Cot, Michele Della Morte, Stefan Hohenegger, Francesco Sannino, and Shahram Vatani. The field theoretical ABC of epidemic dynamics, 2021.
- [56] Giacomo Cacciapaglia, Corentin Cot, and Francesco Sannino. Second wave COVID-19 pandemics in Europe : a temporal playbook. Scientific reports, 10(1) :1–8, 2020.
- [57] Giacomo Cacciapaglia, Corentin Cot, and Francesco Sannino. Multiwave pandemic dynamics explained : How to tame the next wave of infectious diseases. <u>Scientific reports</u>, 11(1) :1–8, 2021.
- [58] Corentin Cot, Giacomo Cacciapaglia, and Francesco Sannino. Mining Google and Apple mobility data : temporal anatomy for COVID-19 social distancing. <u>Scientific reports</u>, 11(1) :1–8, 2021.
- [59] Corentin Cot, Giacomo Cacciapaglia, Anna Sigridur Islind, María Óskarsdóttir, and Francesco Sannino. Impact of US vaccination strategy on COVID-19 wave dynamics. Scientific Reports, 11(1):1–11, 2021.
- [60] H. Leutwyler. On the history of the strong interaction, 2012.
- [61] Katrin Becker, Melanie Becker, and John H Schwarz. <u>String theory and M-theory : A</u> modern introduction. Cambridge university press, 2006.
- [62] Carlo Rovelli. Loop quantum gravity. Living reviews in relativity, 11(1):1–69, 2008.
- [63] François Englert and Robert Brout. Broken symmetry and the mass of gauge vector mesons. Physical Review Letters, 13(9):321, 1964.
- [64] Peter W Higgs. Broken symmetries and the masses of gauge bosons. <u>Physical Review</u> Letters, 13(16) :508, 1964.
- [65] Nicola Cabibbo. Unitary symmetry and leptonic decays. <u>Physical Review Letters</u>, 10(12):531, 1963.
- [66] Makoto Kobayashi and Toshihide Maskawa. CP-violation in the renormalizable theory of weak interaction. Progress of theoretical physics, 49(2):652–657, 1973.
- [67] Particle Data Group. Review of Particle Physics. <u>Progress of Theoretical and</u> Experimental Physics, 2020(8), 08 2020. 083C01.
- [68] Victor F Weisskopf. On the self-energy and the electromagnetic field of the electron. Physical Review, 56(1) :72, 1939.
- [69] Hans A Bethe. The electromagnetic shift of energy levels. <u>Physical Review</u>, 72(4) :339, 1947.
- [70] Willis E Lamb Jr and Robert C Retherford. Fine structure of the hydrogen atom by a microwave method. Physical Review, 72(3) :241, 1947.
- [71] Sidney Coleman. <u>Aspects of symmetry : selected Erice lectures</u>. Cambridge University Press, 1988.

- [72] T. Banks and A. Zaks. On the phase structure of vector-like gauge theories with massless fermions. Nuclear Physics B, 196(2) :189–204, 1982.
- [73] VA Miransky and K Yamawaki. On gauge theories with additional four-fermion interaction. Modern Physics Letters A, 4(02) :129–135, 1989.
- [74] VA Miransky and Koichi Yamawaki. Conformal phase transition in gauge theories. Physical Review D, 55(8) :5051, 1997.
- [75] Bob Holdom. Raising the sideways scale. Phys. Rev. D, 24 :1441–1444, Sep 1981.
- [76] Bob Holdom. Techniodor. Physics Letters B, 150(4) :301–305, 1985.
- [77] FRANCESCO SANNINO. Jumping dynamics. <u>Modern Physics Letters A</u>, 28(30) :1350127, Sep 2013.
- [78] Thomas Appelquist, Dimitra Karabali, and L. C. R. Wijewardhana. Chiral hierarchies and flavor-changing neutral currents in Hypercolor. <u>Phys. Rev. Lett.</u>, 57 :957–960, Aug 1986.
- [79] Koichi Yamawaki, Masako Bando, and Ken-iti Matumoto. Scale Invariant Technicolor Model and a Technidilaton. Phys. Rev. Lett., 56 :1335, 1986.
- [80] Howard Georgi and S. L. Glashow. Unity of all elementary-particle forces. <u>Phys. Rev.</u> Lett., 32 :438–441, Feb 1974.
- [81] A. D. Sakharov. Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe. Pisma Zh. Eksp. Teor. Fiz., 5 :32–35, 1967.
- [82] W De Boer. Grand unified theories and supersymmetry in particle physics and cosmology. Progress in Particle and Nuclear Physics, 33 :201–301, Jan 1994.
- [83] H. Nishino, S. Clark, K. Abe, Y. Hayato, T. Iida, M. Ikeda, J. Kameda, K. Kobayashi, Y. Koshio, M. Miura, and et al. Search for proton decay via $p \rightarrow e + \pi_0$ and $p \rightarrow \mu + \pi_0$ in a large water Cherenkov detector. Physical Review Letters, 102(14), Apr 2009.
- [84] Pran Nath and Pavel Fileviez Pérez. Proton stability in grand unified theories, in strings and in branes. Physics Reports, 441(5-6) :191–317, Apr 2007.
- [85] W. de Boer and C. Sander. Global electroweak fits and gauge coupling unification. Physics Letters B, 585(3-4) :276–286, Apr 2004.
- [86] M.B. Einhorn and D.R.T. Jones. The weak mixing angle and unification mass in supersymmetric SU(5). Nuclear Physics B, 196(3) :475–488, 1982.
- [87] Ugo Amaldi, Wim de Boer, Paul H. Frampton, Hermann Furstenau, and James T. Liu. Consistency checks of grand unified theories. Phys. Lett. B, 281 :374–382, 1992.
- [88] W. Martens, L. Mihaila, J. Salomon, and M. Steinhauser. Minimal supersymmetric SU(5) and gauge coupling unification at three loops. <u>Physical Review D</u>, 82(9), Nov 2010.
- [89] Edward Witten. Mass hierarchies in supersymmetric theories. <u>Physics Letters B</u>, 105(4) :267–271, 1981.
- [90] A. Masiero, D.V. Nanopoulos, K. Tamvakis, and T. Yanagida. Naturally massless higgs doublets in supersymmetric SU(5). Physics Letters B, 115(5) :380–384, 1982.
- [91] Benjamin Grinstein. A supersymmetric SU(5) gauge theory with no gauge hierarchy problem. Nuclear Physics B, 206(3) :387–396, 1982.

- [92] Kenzo Inoue, Akira Kakuto, and Hiroshi Takano. Higgs as (Pseudo-) Goldstone Particles. Progress of Theoretical Physics, 75(3):664–676, 03 1986.
- [93] Ignatios Antoniadis, John R. Ellis, J. S. Hagelin, and Dimitri V. Nanopoulos. The Flipped SU(5) x U(1) String Model Revamped. Phys. Lett. B, 231 :65–74, 1989.
- [94] John Ellis, S. Kelley, and D.V. Nanopoulos. A detailed comparison of LEP data with the predictions of the minimal supersymmetric SU(5) GUT. <u>Nuclear Physics B</u>, 373(1):55– 72, 1992.
- [95] Martin Puchwein and Zoltan Kunszt. Radiative corrections with 5D mixed position-/momentum-space propagators. Annals of Physics, 311(2) :288–313, Jun 2004.
- [96] Ashok Das. Finite temperature field theory. World scientific, 1997.
- [97] Leandro Da Rold. Radiative corrections in 5-D and 6-D expanding in winding modes. Phys. Rev. D, 69 :105015, 2004.
- [98] S. Hossenfelder. Running coupling with minimal length. <u>Physical Review D</u>, 70(10), Nov 2004.
- [99] Gautam Bhattacharyya, Anindya Datta, Swarup Kumar Majee, and Amitava Raychaudhuri. Power law scaling in universal extra dimension scenarios. <u>Nuclear Physics B</u>, 760(1-2) :117–127, Jan 2007.
- [100] Yutaka Hosotani. Dynamical Mass Generation by Compact Extra Dimensions. <u>Phys.</u> Lett. B, 126 :309–313, 1983.
- [101] HISAKI HATANAKA, TAKEO INAMI, and C. S. LIM. The gauge hierarchy problem and higher-dimensional gauge theories. <u>Modern Physics Letters A</u>, 13(32) :2601–2611, Oct 1998.
- [102] G. Dvali, S. Randjbar-Daemi, and R. Tabbash. Origin of spontaneous symmetry breaking in theories with large extra dimensions. Physical Review D, 65(6), Feb 2002.
- [103] Claudio A. Scrucca, Marco Serone, and Luca Silvestrini. Electroweak symmetry breaking and fermion masses from extra dimensions. <u>Nuclear Physics B</u>, 669(1-2) :128–158, Oct 2003.
- [104] N.S. Manton. A new six-dimensional approach to the Weinberg-Salam model. <u>Nuclear</u> Physics B, 158(1) :141–153, 1979.
- [105] A. Masiero, M. Serone, C. A. Scrucca, and L. Silvestrini. Nonlocal symmetry breaking in Kaluza-Klein theories. Physical Review Letters, 87(25), Dec 2001.
- [106] I Antoniadis, K Benakli, and M Quirós. Finite Higgs mass without supersymmetry. <u>New</u> Journal of Physics, 3 :20–20, Nov 2001.
- [107] Ammar Abdalgabar, Mohammed Omer Khojali, Alan S. Cornell, Giacomo Cacciapaglia, and Aldo Deandrea. Unification of gauge and Yukawa couplings. <u>Phys. Lett. B</u>, 776:231– 235, 2018.
- [108] Naoyuki Haba, Masatomi Harada, Yutaka Hosotani, and Yoshiharu Kawamura. Dynamical rearrangement of gauge symmetry on the orbifold S1 / Z(2). <u>Nucl. Phys. B</u>, 657 :169–213, 2003. [Erratum : Nucl.Phys.B 669, 381–382 (2003)].
- [109] Yoshiharu Kawamura. Triplet doublet splitting, proton stability and extra dimension. Prog. Theor. Phys., 105 :999–1006, 2001.

- [110] Yoshiharu Kawamura. Split multiplets, coupling unification and extra dimension. Prog. Theor. Phys., 105 :691–696, 2001.
- [111] R. Contino, L. Pilo, R. Rattazzi, and Enrico Trincherini. Running and matching from five-dimensions to four-dimensions. Nucl. Phys. B, 622 :227–239, 2002.
- [112] Guido Altarelli and Ferruccio Feruglio. SU(5) grand unification in extra dimensions and proton decay. Phys. Lett. B, 511 :257–264, 2001.
- [113] Lawrence J. Hall and Yasunori Nomura. Gauge unification in higher dimensions. <u>Phys.</u> Rev. D, 64 :055003, 2001.
- [114] N. Haba, Y. Shimizu, Tomoharu Suzuki, and Kazumasa Ukai. Fermion Mass Hierarchy in the Grand Unified Theory on S₁/Z₂ x Zl₂ Orbifold. <u>Prog. Theor. Phys.</u>, 107:151–162, 2002.
- [115] Arthur Hebecker and John March-Russell. A Minimal S1 / (Z2 x Z2') orbifold GUT. Nucl. Phys. B, 613 :3–16, 2001.
- [116] Masahiro Ibe, Satoshi Shirai, Motoo Suzuki, and Tsutomu T. Yanagida. Novel GUT with apparently complete SU(5) multiplets. Phys. Rev. D, 100(5) :055024, 2019.
- [117] A.J. Buras, P. Gambino, M. Gorbahn, S. Jager, and L. Silvestrini. Universal unitarity triangle and physics beyond the standard model. Phys. Lett. B, 500 :161–167, 2001.
- [118] Christoph Bobeth, Marcella Bona, Andrzej J. Buras, Thorsten Ewerth, Maurizio Pierini, Luca Silvestrini, and Andreas Weiler. Upper bounds on rare K and B decays from minimal flavor violation. Nucl. Phys. B, 726 :252–274, 2005.
- [119] Eugene Golowich. Scalar mediated proton decay. Phys. Rev. D, 24 :2899, 1981.
- [120] N. Haba, T. Kondo, Y. Shimizu, Tomoharu Suzuki, and Kazumasa Ukai. Model with gauge coupling unification on S1 / (Z2 x Z2') orbifold. <u>Prog. Theor. Phys.</u>, 106 :1247– 1254, 2001.
- [121] Lawrence J. Hall and Yasunori Nomura. Gauge coupling unification from unified theories in higher dimensions. Phys. Rev. D, 65 :125012, 2002.
- [122] Keith R. Dienes, Emilian Dudas, and Tony Gherghetta. GUT precursors and nontrivial fixed points in higher dimensional gauge theories. Phys. Rev. Lett., 91 :061601, 2003.
- [123] Tim R. Morris. Renormalizable extra-dimensional models. JHEP, 01 :002, 2005.
- [124] Mohammed Omer Khojali, A.S. Cornell, Aldo Deandrea, and Giacomo Cacciapaglia. Evolution of the gauge couplings and Weinberg angle in 5-dimensions for an SU(5) and flipped SU(5) gauge group. J. Phys. Conf. Ser., 878(1) :012024, 2017.
- [125] Mohammed Omer Khojali, A.S. Cornell, Aldo Deandrea, and Giacomo Cacciapaglia. Evolution of the gauge couplings and Weinberg angle in 5-dimensions for a G₂ gauge group. J. Phys. Conf. Ser., 889(1) :012012, 2017.
- [126] Gautam Bhattacharyya, Anindya Datta, Swarup Kumar Majee, and Amitava Raychaudhuri. Power law blitzkrieg in universal extra dimension scenarios. <u>Nucl. Phys.</u> B, 760 :117–127, 2007.
- [127] Holger Gies. Renormalizability of gauge theories in extra dimensions. <u>Physical Review</u> D, 68(8), Oct 2003.

- [128] Nima Arkani-Hamed, Savas Dimopoulos, and G.R. Dvali. Phenomenology, astrophysics and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity. Phys. Rev. D, 59 :086004, 1999.
- [129] Hsin-Chia Cheng, Konstantin T. Matchev, and Martin Schmaltz. Radiative corrections to Kaluza-Klein masses. Phys. Rev. D, 66 :036005, 2002.
- [130] Xiangyi Cui et al. Dark Matter Results From 54-Ton-Day Exposure of PandaX-II Experiment. Phys. Rev. Lett., 119(18) :181302, 2017.
- [131] D.S. Akerib et al. Results from a search for dark matter in the complete LUX exposure. Phys. Rev. Lett., 118(2) :021303, 2017.
- [132] E. Aprile et al. Dark Matter Search Results from a One Ton-Year Exposure of XE-NON1T. Phys. Rev. Lett., 121(11) :111302, 2018.
- [133] Lawrence J. Hall, Takeo Moroi, and Hitoshi Murayama. Sneutrino cold dark matter with lepton number violation. Phys. Lett. B, 424 :305–312, 1998.
- [134] S. Nussinov. Technocosmology : could a Technibaryon excess provide a "natural" missing mass candidate? Phys. Lett. B, 165 :55–58, 1985.
- [135] Jeffrey A. Harvey and Michael S. Turner. Cosmological baryon and lepton number in the presence of electroweak fermion number violation. <u>Phys. Rev. D</u>, 42 :3344–3349, 1990.
- [136] Kaustubh Agashe, Peizhi Du, Majid Ekhterachian, Soubhik Kumar, and Raman Sundrum. Phase Transitions from the Fifth Dimension. 10 2020.
- [137] Sven Bjarke Gudnason, Chris Kouvaris, and Francesco Sannino. Dark Matter from new Technicolor Theories. Phys. Rev. D, 74 :095008, 2006.
- [138] Mark W. Goodman and Edward Witten. Detectability of Certain Dark Matter Candidates. Phys. Rev. D, 31 :3059, 1985.
- [139] Hans Heesterbeek. The law of mass-action in epidemiology : a historical perspective. Ecological paradigms lost : routes of theory change, pages 81–104, 2005.
- [140] Daniel Bernoulli. Essai d'une nouvelle analyse de la mortalité causée par la petite vérole, et des avantages de l'inoculation pour la prévenir. <u>Histoire de l'Acad., Roy. Sci.(Paris)</u> avec Mem, pages 1–45, 1760.
- [141] Zhen Wang, Chris T. Bauch, Samit Bhattacharyya, Alberto d'Onofrio, Piero Manfredi, Matjaž Perc, Nicola Perra, Marcel Salathé, and Dawei Zhao. Statistical physics of vaccination. Physics Reports, 664 :1–113, 2016. Statistical physics of vaccination.
- [142] Cato M Guldberg and Peter Waage. Studies concerning affinity. <u>CM Forhandlinger</u>: Videnskabs-Selskabet i Christiana, 35(1864) :1864, 1864.
- [143] Peter Waage. Experiments for determining the affinity law. Forhandlinger : Videnskabs-Selskabet i Christiana, 92 :1864, 1864.
- [144] Cato Maximilian Guldberg. Concerning the laws of chemical affinity. <u>CM Forhandlinger</u> : Videnskabs-Selskabet i Christiana, 111 :1864, 1864.
- [145] William Ogilvy Kermack and Anderson G McKendrick. A contribution to the mathematical theory of epidemics. Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character, 115(772) :700–721, 1927.

- [146] Cory Simon. The SIR dynamic model of infectious disease transmission and its analogy with chemical kinetics. 2020.
- [147] Estefanía Sucre-Rosales, Ricardo Fernández-Terán, David Carvajal, Lorenzo Echevarría, and Florencio E. Hernández. Experience-based learning approach to chemical kinetics : Learning from the COVID-19 pandemic. <u>Journal of Chemical Education</u>, 97(9) :2598– 2605, 2020.
- [148] Kristian Soltesz, Fredrik Gustafsson, Toomas Timpka, Joakim Jaldén, Carl Jidling, Albin Heimerson, Thomas B Schön, Armin Spreco, Joakim Ekberg, Örjan Dahlström, et al. The effect of interventions on COVID-19. Nature, 588(7839) :E26–E28, 2020.
- [149] Norman TJ Bailey et al. <u>The mathematical theory of infectious diseases and its</u> <u>applications</u>. Charles Griffin Company Ltd, 5a Crendon Street, High Wycombe, Bucks <u>HP13 6LE.</u>, 1975.
- [150] P. Grassberger. On the critical behavior of the general epidemic process and dynamical percolation. Mathematical Biosciences, 63(2) :157–172, 1983.
- [151] Michele Della Morte and Francesco Sannino. Renormalisation Group approach to pandemics as a time-dependent SIR model. 7 2020.
- [152] Jeremy Howard, Austin Huang, Zhiyuan Li, Zeynep Tufekci, Vladimir Zdimal, Helene-Mari van der Westhuizen, Arne von Delft, Amy Price, Lex Fridman, Lei-Han Tang, et al. An evidence review of face masks against COVID-19. <u>Proceedings of the National</u> Academy of Sciences, 118(4), 2021.
- [153] Nicholas G Davies, Adam J Kucharski, Rosalind M Eggo, Amy Gimma, W John Edmunds, Thibaut Jombart, Kathleen O'Reilly, Akira Endo, Joel Hellewell, Emily S Nightingale, et al. Effects of non-pharmaceutical interventions on COVID-19 cases, deaths, and demand for hospital services in the uk : a modelling study. <u>The Lancet Public</u> Health, 5(7) :e375–e385, 2020.
- [154] Giacomo Cacciapaglia and Francesco Sannino. Interplay of social distancing and border restrictions for pandemics via the epidemic renormalisation group framework. <u>Scientific</u> reports, 10(1) :1–10, 2020.
- [155] Mangalakumari Jeyanathan, Sam Afkhami, Fiona Smaill, Matthew S Miller, Brian D Lichty, and Zhou Xing. Immunological considerations for COVID-19 vaccine strategies. Nature Reviews Immunology, 20(10) :615–632, 2020.
- [156] Fernando P Polack, Stephen J Thomas, Nicholas Kitchin, Judith Absalon, Alejandra Gurtman, Stephen Lockhart, John L Perez, Gonzalo Pérez Marc, Edson D Moreira, Cristiano Zerbini, et al. Safety and efficacy of the BNT162b2 mRNA Covid-19 vaccine. New England Journal of Medicine, 383(27) :2603–2615, 2020.
- [157] Merryn Voysey, Sue Ann Costa Clemens, Shabir A Madhi, Lily Y Weckx, Pedro M Folegatti, Parvinder K Aley, Brian Angus, Vicky L Baillie, Shaun L Barnabas, Qasim E Bhorat, et al. Safety and efficacy of the ChAdOx1 nCoV-19 vaccine (AZD1222) against SARS-CoV-2 : an interim analysis of four randomised controlled trials in Brazil, South Africa, and the UK. The Lancet, 397(10269) :99–111, 2021.
- [158] Kate M Bubar, Kyle Reinholt, Stephen M Kissler, Marc Lipsitch, Sarah Cobey, Yonatan H Grad, and Daniel B Larremore. Model-informed COVID-19 vaccine prioritization strategies by age and serostatus. <u>Science</u>, 371(6532) :916–921, 2021.

- [159] Thomas Hale, Noam Angrist, Rafael Goldszmidt, Beatriz Kira, Anna Petherick, Toby Phillips, Samuel Webster, Emily Cameron-Blake, Laura Hallas, Saptarshi Majumdar, et al. A global panel database of pandemic policies (Oxford COVID-19 Government Response Tracker). Nature Human Behaviour, 5(4) :529–538, 2021.
- [160] Sarah Brookman, James Cook, Mark Zucherman, Simon Broughton, Katharine Harman, and Atul Gupta. Effect of the new SARS-CoV-2 variant B.1.1.7 on children and young people. The Lancet Child & Adolescent Health, 5(4) :e9–e10, 2021.
- [161] Yang C.-Xu Xf. et al. Huang, Y. Structural and functional properties of SARS-CoV-2 spike protein : potential antivirus drug development for COVID-19.
- [162] Arnaud Fontanet, Brigitte Autran, Bruno Lina, Marie Paule Kieny, Salim S Abdool Karim, and Devi Sridhar. SARS-CoV-2 variants and ending the COVID-19 pandemic. The Lancet, 397(10278) :952–954, 2021.
- [163] Nicholas G. et al. Davies. Estimated transmissibility and impact of SARS-CoV-2 lineage B.1.1.7 in England. Science, 2021.
- [164] Shabir A. et al. Madhi. Efficacy of the ChAdOx1 nCoV-19 Covid-19 Vaccine against the B.1.351 Variant. New England Journal of Medicine.
- [165] Jon Cohen. South Africa suspends use of AstraZeneca's COVID-19 vaccine after it fails to clearly stop virus variant. Science, 10, 2021.
- [166] Delphine Planas, Timothée Bruel, Ludivine Grzelak, Florence Guivel-Benhassine, Isabelle Staropoli, Françoise Porrot, Cyril Planchais, Julian Buchrieser, Maaran Michael Rajah, Elodie Bishop, et al. Sensitivity of infectious SARS-CoV-2 B.1.1.7 and B.1.351 variants to neutralizing antibodies. Nature medicine, pages 1–8, 2021.
- [167] Adele de Hoffer Stefan Hohenegger Francesco Sannino Giacomo Cacciapaglia, Corentin Cot and Shahram Vatani. Epidemiological theory of virus variants. To appear.
- [168] Vivek Charu, Scott Zeger, Julia Gog, Ottar N. Bjørnstad, Stephen Kissler, Lone Simonsen, Bryan T. Grenfell, and Cécile Viboud. Human mobility and the spatial transmission of influenza in the United States. PLOS Computational Biology, 13(2):1–23, 02 2017.
- [169] Françoise Barré-Sinoussi, Jean-Claude Chermann, Fran Rey, Marie Therese Nugeyre, Sophie Chamaret, Jacqueline Gruest, Charles Dauguet, Charles Axler-Blin, Françoise Vézinet-Brun, Christine Rouzioux, et al. Isolation of a T-lymphotropic retrovirus from a patient at risk for acquired immune deficiency syndrome (AIDS). <u>Science</u>, 220(4599) :868–871, 1983.
- [170] Robin A Weiss. How does HIV cause AIDS? Science, 260(5112) :1273–1279, 1993.
- [171] Daniel C Douek, Mario Roederer, and Richard A Koup. Emerging concepts in the immunopathogenesis of AIDS. Annual review of medicine, 60 :471–484, 2009.
- [172] Frederick Nelson, O. Anisimov, and Nikolay Shiklomanov. Subsidence risk from thawing permafrost. Nature, 410 :889–90, 05 2001.
- [173] Joanne B Emerson, Simon Roux, Jennifer R Brum, Benjamin Bolduc, Ben J Woodcroft, Ho Bin Jang, Caitlin M Singleton, Lindsey M Solden, Adrian E Naas, Joel A Boyd, et al. Host-linked soil viral ecology along a permafrost thaw gradient. <u>Nature microbiology</u>, 3(8):870–880, 2018.