Systems of PDEs and reflected BSDEs with interconnected obstacles and related optimal switching problems

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Contents

Chapter 1

Introduction

The aim of this thesis is to study different classes of non-linear partial differential equations (PDEs in short) where the monotonicity condition does no longer hold. In particular cases, these PDEs are the Hamilton-Jacobi-Bellman equations associated with optimal switching problems. To study this type of equations, we are often led to study specific classes of backward stochastic differential equations (BSDEs in short).
The object of this chapter is to introduce and motivate the problems we studied and to summarize the main obtained results .

1.1 General results on BSDEs

1.1.1 BSDEs in the classical framework

The theory of backward stochastic equations was initiated by Bismut [7] and later developed by Pardoux and Peng [62] to a non-linear case. Since then, it has attracted steadily a lot of work and has been applied to various fields, such as mathematical finance, stochastic optimal control, stochastic differential games, etc. A non-linear BSDE is defined as follows:

Let $B = (B_t)_{0\leq t\leq T}$ be a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the completed natural filtration of B and T a fixed finite time horizon.

We denote by $\mathcal{P}$ the σ-algebra of $\mathbb{F}$-progressively measurable sets on $\Omega \times$

$[0,T]$ and we use the following notation:

- L^2 is the set of $\mathcal{F}_T$-measurable random variables ξ such that $\mathbb{E}[|\xi|^2] < \infty$;
- $\mathcal{S}_c^2$ is the set of $\mathcal{P}$-measurable, continuous and $\mathbb{R}$-valued processes $Y = (Y_s)_{s\leq T}$ such that $\mathbb{E}[\sup_{s\leq T}|Y_s|^2] < \infty$;
- $\mathcal{H}^{2,l}$ $(l \geqslant 1)$ is the set of $\mathcal{P}$-measurable and $\mathbb{R}^l$-valued processes $Z := (Z_s)_{s\leq T}$ such that $\mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty$.

Next, given a pair (ξ, f), called respectively the terminal condition and generator (or driver) satisfying:

Assumption (H1)

(i) $\xi \in \mathrm{L}^2$ and $\mathbb{R}$-valued;

(ii) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that:

- $f(.,t,y,z)$, written for simplicity $f(t,y,z)$, is $\mathbb{F}$-progressively measurable for all y, z;
- $(f(t,0,0))_{t\leq T} \in \mathcal{H}^{2,1}$;
- f satisfies a uniform Lipschitz condition in (ω, y, z), i.e. there exists a positive constant C such that:

$$|f(t,y,z) - f(t,y',z')| \leq C(|y-y'| + |z-z'|), \ \forall\, y, y', \forall\, z, z' \quad dt \otimes d\mathbb{P}\ a.e.$$

We now consider the following BSDE of one-dimensional type:

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T Z_r dB_r, \ \ s \leq T. \tag{1.1.1}$$

Definition 1.1.1 *A solution of the BSDE* (1.1.1) *is a pair* $(Y,Z) \in \mathcal{S}_c^2 \times \mathcal{H}^{2,d}$ *satisfying* (1.1.1).

The following result is related to existence and uniqueness of the solution of (1.1.1).

Theorem 1.1.2 *(Pardoux-Peng [62]). Under Assumption **(H1)**, there exists a unique solution (Y, Z) of the BSDE (1.1.1).*

Since then, some improvements have been made. In the sense that, many efforts have been made to relax the assumptions on the generator and terminal condition to obtain a solution of (1.1.1). More precisely, and mainly, it is shown that when:

- f is continuous in (y, z) and satisfies a linear growth condition on (y, z), then, there exists a minimal solution to the BSDEs (1.1.1) (see Lepeltier and San Martin [51]).
- f is continuous in (y, z), linear in y, quadratic in z, and ξ is bounded, then, there exists a unique bounded solution to the BSDEs (1.1.1) (see Kobylanski [58]).

One of the great achievements of the theory of BSDEs is the comparison theorem due to S. Peng [65] at first and then generalized by several authors, see e.g. N. El Karoui, S. Peng and M.-C. Quenez [30]. It allows to compare the solutions of two BSDEs whenever we can compare the terminal conditions and the generators.

Theorem 1.1.3 *(El Karoui, Peng and Quenez [30]). Let ξ and ξ' be two terminal conditions for BSDEs (1.1.1) driven respectively by the generators f and f' where only f is assumed to satisfy Assumption **(H1)-(ii)**. Denote by (Y, Z) and (Y', Z') the respective solutions. Then we have:*
If $\xi \leq \xi'$ and $f(., t, Y_t, Z_t) \leq f'(., t, Y_t, Z_t)$, then, $\mathbb{P}$-a.s. $\forall t \leq T$, $Y_t \leq Y'_t$.

1.1.2 BSDEs with jumps

Tang and Li [76] added into the BSDEs, a jump term which is driven by a Poisson random measure independent of the Brownian motion. (Generally speaking, for example, if the Brownian motion stands for the noise from the financial market, then the Poisson random measure can be interpreted as the randomness of the insurance claims). The setting of BSDEs with jumps is the following:

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$, whose filtration satisfies the usual hypotheses of completeness and right-continuity. We suppose that it is generated by a d-dimensional standard Brownian motion $B := (B_t)_{0\leq t\leq T}$ and an independent Poisson random measure μ on $\mathbb{R}^+ \times E$, where $E := \mathbb{R}^l - \{0\}$ $(l \geq 1)$ is equipped with its Borel σ-field $\mathcal{B}(E)$. Let $\nu(dt, de) = dt\lambda(de)$ be its compensated process such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t\leq T}$ is a martingale for every $A \in \mathcal{B}$ satisfying $\lambda(A) < \infty$. The measure λ is assumed to be σ-finite on $(E, \mathcal{B}(E))$ such that $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$.

We denote by $\mathbf{P}$ the σ-algebra of $\mathbb{F}$-predictable subsets on $\Omega \times [0,T]$ and we use the following notations:

- $\mathcal{S}^2$ is the set of $\mathcal{P}$-measurable RCLL processes $Y = (Y_s)_{s\leq T}$ valued in $\mathbb{R}$ such that $\mathbb{E}[\sup_{s\leq T} |Y_s|^2] < \infty$;
- $\mathcal{H}^{2,d}$ is the set of $\mathbf{P}$-measurable processes $Z = (Z_s)_{s\leq T}$ valued in $\mathbb{R}^d$ such that $\mathbb{E}[\int_0^T |Z_s|^2 ds] < \infty$;
- $\mathcal{L}^2(\lambda)$ is the set of $\mathcal{B}(E)$-measurable functions $(\varphi(e))_{e\in E}$ from E into $\mathbb{R}$ such that $\int_E |\varphi(e)|^2 \lambda(de) < \infty$;
- $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ is the set of $\mathbf{P}$-measurable and $\mathcal{L}^2(\lambda)$-valued processes $V = (V_s)_{s\leq T}$ such that $\mathbb{E}[\int_0^T \int_E |V_s(e)|^2 \lambda(de)ds] < \infty$.

We now consider a pair (ξ, f) satisfying:

Assumption (H2)

(i) $\xi \in \mathrm{L}^2$ and $\mathbb{R}$-valued;

(ii) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(\lambda) \to \mathbb{R}$ such that:

- $f(\omega, t, y, z, v) = f(t, y, z, v)$ is $\mathbf{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{L}^2(\lambda))$-measurable;
- $(f(t,0,0,0))_{t\leq T} \in \mathcal{H}^{2,1}$;
- f satisfies a uniform Lipschitz condition in (y,z,v), i.e. there exists a constant C such that, $\forall\, y, y'$, $\forall\, z, z'$ and $\forall\, v, v'$,

$$|f(t,y,z,v) - f(t,y',z',v')| \leq C(|y-y'| + |z-z'| + ||v-v'||).$$

Let us now consider the following BSDE with jumps associated with (ξ, f):

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r, V_r)dr - \int_s^T Z_r dB_r - \int_s^T \int_E V_r(e)\tilde{\mu}(dr, de), \ \ s \leq T. \tag{1.1.2}$$

Definition 1.1.4 *A solution of the BSDE with jumps* (1.1.2) *is a triple* $(Y, Z, V) \in \mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{H}^2(\mathcal{L}^2(\lambda))$ *satisfying* (1.1.2).

Theorem 1.1.5 *(Tang and Li [76]). Under Assumption* ***(H2)****, there exists a unique solution* (Y, Z, V) *to the BSDEs with jumps* (1.1.2).

This result has been later improved by Pardoux in [60], where the author proved the existence of a unique solution to the BSDE with jumps (1.1.2) in the k-dimensional case, $(k \in \mathbb{N}^*)$ under the following assumption:

Assumption (H3)

(i) f is Lipschitz with respect to (z, v);

(ii) f is continuous with respect to y and there exist an $\mathbb{R}^+$-valued adapted process $(\phi_t)_{0\leq t\leq T}$ and a constant $K > 0$ such that,

$$\mathbb{E}\big[\int_0^T \phi_s^2 ds\big] < \infty \text{ and } |f(t, y, z, v)| \leq \phi_t + C\Big(|y| + |z| + \big(\int_E |v(e)|^2 \lambda(de)\big)^{\frac{1}{2}}\Big);$$

(iii) f is monotonic with respect to y:

$$\exists\, \alpha \in \mathbb{R} \text{ such that } \forall t \geq 0, \forall y, y' \in \mathbb{R}, \forall z \in \mathbb{R}^d, \forall v \in \mathcal{L}^2(\lambda)$$
$$(y - y')(f(t, y, z, v) - f(t, y', z, v)) \leq \alpha |y - y'|^2, \quad \mathbb{P} - a.s.$$

The comparison theorem is also an important property of BSDEs with jumps. In [3], a counterexample is given, which shows that in the jump case the conditions $\xi \leq \xi'$ and $f \leq f'$ are not sufficient to guarantee $Y \leq Y'$. Some supplementary conditions should be added to f. In fact, they deal with the case when f has the following form:

$$f(\omega, t, y, z, v) := h(\omega, t, y, z, \int_E v(e)\gamma(\omega, t, e)\lambda(de)), \tag{1.1.3}$$

where $\gamma : \Omega \times [0,T] \times E \to \mathbb{R}$ is $\mathbf{P} \otimes \mathcal{B}(E)$-measurable and satisfies $0 \leq \gamma(.,t,e) \leq C(1 \wedge |e|)$, for any $e \in E$ and $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is $\mathbf{P} \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R})$-measurable and satisfies:

Assumption (H4)

(i) $(h(\omega,t,0,0,0))_{t \leq T} \in \mathcal{H}^{2,1}$;

(ii) h is Lipschitz w.r.t (y,z,q);

(iii) h is non-decreasing w.r.t q.

Then, in order to weaken the previous conditions (i.e. (H4)) on the generator and obtain the comparison result, Royer has introduced in [72] the following condition:

$(\mathbf{A}_\gamma)$ There exist $-1 < C_1 \leq 0$ and $C_2 \geq 0$ such that for all $t,y,z,v,v' \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{L}^2(\lambda)$,

$$f(t,y,z,v) - f(t,y,z,v') \leq \int_E (v(e) - v'(e))\gamma^{y,z,v,v'}(t,e)\lambda(de),$$

where $\gamma^{y,z,v,v'} : \Omega \times [0,T] \times E \to \mathbb{R}$ is $\mathbf{P} \otimes \mathcal{B}(E)$-measurable and satisfies

$$C_1(1 \wedge |e|) \leq \gamma^{y,z,v,v'}(t,e) \leq C_2(1 \wedge |e|).$$

The main difference between the mapping $\gamma^{y,z,v,v'}$ and γ in (1.1.3) is that $\gamma^{y,z,v,v'}$ is allowed to depend on y,z,v,v' and it can be negative as soon as it remains larger than $C_1(1 \wedge |e|)$.

Then, Royer consider the following list of assumptions which remain weaker than **(H4)**:

Assumption (H5)

(i) $(f(\omega,t,0,0,0))_{t \leq T} \in \mathcal{H}^{2,1}$;

(ii) f is Lipschitz w.r.t (y,z);

(iii) f satisfies $(\mathbf{A}_\gamma)$.

Theorem 1.1.6 *(Royer [72]). We consider a generator f satisfying **(H3)** and we require that f' verifies **(H5)**. Let ξ and ξ' be two terminal conditions*

for BSDEs with jumps (1.1.2) *driven respectively by the generators* f *and* f'. *Denote by* (Y, Z, V) *and* (Y', Z', V') *the respective solutions. Then, if* $\xi \leq \xi'$ *and* $f(t, Y, Z, V) \leq f'(t, Y, Z, V)$, *it follows*

$$Y_t \leq Y'_t, \quad \forall t \leq T.$$

1.1.3 The Markovian case: Feynman-Kac representation

One of the important applications of BSDEs occurs in PDEs theory. In fact, in the Markovian case, these equations are linked to PDEs. A BSDE is called Markovian when the randomness of the terminal condition and the generator are generated by a diffusion process $\{(X_s^{t,x})_{s\leq T}, (t, x) \in [0, T] \times \mathbb{R}^k\}$.

1.1.3.1 The continuous setting

In this setting, the process $(X_s^{t,x})_{s\leq T}$ is the solution of the following standard SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r, \quad t \leq s \leq T, \tag{1.1.4}$$

where b and σ are two functions on $[0, T] \times \mathbb{R}^k$ with values in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$ respectively, satisfying the standard Lipschitz continuity and linear growth condition, i.e.

$$|b(s, x)| + |\sigma(s, x)| \leq C(1 + |x|), \quad \text{for some } C > 0.$$

In order to establish the connection between BSDEs and PDEs, we consider the following Markovian BSDE: for all $s \leq T$,

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dB_r, \tag{1.1.5}$$

where g and f are as follows.

Assumption (H6)

(i) g is a continuous function on $\mathbb{R}^k$ with polynomial growth, i.e. for some positive constants C and p,

$$|g(x)| \leq C(1 + |x|^p);$$

(ii) f is a continuous function on $[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$, satisfying a Lipschitz condition in (y,z) uniformly in (t,x) and such that

$$|f(t,x,y,z)|\leq C(1+|x|^p+|y|+|z|).$$

The Markovian BSDE (1.1.5) gives a Feynman-Kac representation for the solution of the following system of non-linear parabolic PDE: $\forall(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\begin{cases} -\partial_t u(t,x)-\mathcal{L}u(t,x)-f(t,x,u(t,x),(\sigma\partial_x u)(t,x))=0;\\ u(T,x)=g(x), \end{cases} \tag{1.1.6}$$

where $\mathcal{L}$ is the second order differential operator associated with the diffusion process $(X_s^{t,x})_{s\leq T}$ defined by:

$$\mathcal{L}u(t,x):=\sum_{i=1}^{k}b_i(t,x)\frac{\partial}{\partial x_i}u(t,x)+\frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma_{ij}^{\top})(t,x)\frac{\partial^2}{\partial x_i\partial x_j}u(t,x).$$

Then, by applying Itô's formula to $u(s,X_s^{t,x})$ between s and T, with u smooth solution to PDE (1.1.6), we obtain :

$$\begin{aligned} u(s,X_s^{t,x})=g(X_T^{t,x})&+\int_s^T f(r,X_r^{t,x},u(r,X_r^{t,x}),(\sigma\partial_x u)(r,X_r^{t,x}))dr\\ &-\int_s^T(\sigma\partial_x u)(r,X_r^{t,x})dB_r. \end{aligned}$$

It follows that the pair $(Y_s^{t,x},Z_s^{t,x})_{s\leq T}=(u(s,X_s^{t,x}),(\sigma\partial_x u)(s,X_s^{t,x}))_{s\leq T}$ is a solution of Markovian BSDE (1.1.5), which is a generalization of the Feynman-Kac formula.

Several generalizations to investigate more general non-linear PDEs have been developed following different approaches of the notion of weak solutions, namely viscosity solution. The notion of viscosity solution for PDEs was introduced firstly by Crandall, Ishii and Lions [48]. Nowdays this theory becomes an important tool in many fields, especially in optimal control theory. A few years later, Pardoux and Peng ([63], [64]) were the first to give a probabilistic representation of the viscosity solution for PDE (1.1.6), using solution of BSDE (1.1.5).

Now, let us introduce the definition of a viscosity solution as in [63].

Definition 1.1.7 *Let $u \in \mathrm{C}([0,T] \times \mathbb{R}^k)$.*

(i) u is called a viscosity sub-solution (resp. super-solution) of PDE (1.1.6), if $u(T,x) \leq$ (resp. $\geq$) $g(x)$, $\forall x \in \mathbb{R}^k$ and moreover for any test function $\phi \in \mathrm{C}^{1,2}([0,T] \times \mathbb{R}^k)$ and $(t,x) \in [0,T) \times \mathbb{R}^k$ which is a local maximum (resp. global minimum) of $u - \phi$, it holds that

$$-\partial_t \phi(t,x) - \mathcal{L}\phi(t,x) - f(t,x,u(t,x),(\sigma \partial_x \phi)(t,x)) \leq (\text{ resp. } \geq)\, 0.$$

(ii) u is called a viscosity solution of PDE (1.1.6) if it is both a sub-solution and super-solution of (1.1.6).

After extending the notion of viscosity solution to PDEs, Pardoux and Peng ([63], [64]) proved the existence of the viscosity solution and obtained the Feynmann-Kac representation. Here, we will recall briefly their main results.

Theorem 1.1.8 *(Pardoux and Peng [63]). Under the Assumption **(H6)**, the function $u(t,x) := Y_t^{t,x}$ is continuous on $[0,T] \times \mathbb{R}^k$ with polynomial growth and is a viscosity solution of PDE (1.1.6).*

1.1.3.2 The discontinuous setting

In this setting, the process $(X_s^{t,x})_{s \leq T}$ is the solution of the following SDE of diffusion-jump type: $\forall t \leq s \leq T$,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r + \int_t^s \int_E \beta(X_{r^-}^{t,x}, e)\tilde{\mu}(dr, de), \tag{1.1.7}$$

where $\beta : \mathbb{R}^k \times E \to \mathbb{R}^k$ is a measurable function such that for some real c and for all $e \in E$,

$$|\beta(x,e)| \leq c(1 \wedge |e|), \quad x \in \mathbb{R}^k;$$
$$|\beta(x,e) - \beta(x',e)| \leq c|x - x'|(1 \wedge |e|), \quad x, x' \in \mathbb{R}^k.$$

In the Markovian case, analogously to standard BSDEs, the BSDE with jumps has the following form: for all $s \leq T$,

$$\begin{aligned} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \int_E V_r^{t,x}(e)\gamma(X_r^{t,x}, e)\lambda(de))dr \\ - \int_s^T Z_r^{t,x} dB_r - \int_s^T \int_E V_r^{t,x}(e)\tilde{\mu}(dr, de), \end{aligned} \tag{1.1.8}$$

where g, f and γ are as follows.

Assumption (H7)

(i) g is a continuous function on $\mathbb{R}^k$ satisfying a polynomial growth condition, i.e.

$$|g(x)| \leq C(1 + |x|^p), \quad \text{for some } C, p > 0;$$

(ii) f is a continuous function on $[0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, satisfying a Lipschitz condition in (y, z, q), uniformly in (t, x) and

$$\begin{aligned} &|f(t, x, 0, 0, 0)| \leq C(1 + |x|^p); \\ &q \to f(t, x, y, z, q) \text{ is non-decreasing ;} \end{aligned}$$

(iii) γ is a measurable function on $\mathbb{R}^k \times E$ such that,

$$\begin{aligned} 0 \leq \gamma(x, e) \leq C(1 \wedge |e|), \quad x \in \mathbb{R}^k, e \in E; \\ |\gamma(x, e) - \gamma(x', e)| \leq C|x - x'|(1 \wedge |e|), \quad x, x' \in \mathbb{R}^k, e \in E. \end{aligned}$$

Now, we consider the following system of integral-PDE (IPDE in short) of parabolic type: $\forall (t, x) \in [0, T] \times \mathbb{R}^k$,

$$\begin{cases} -\partial_t u(t,x) - \mathcal{L}u(t,x) - \mathcal{K}u(t,x) - f(t, x, u(t,x), (\sigma\partial_x u)(t,x), Bu(t,x)) = 0; \\ u(T,x) = g(x). \end{cases} \tag{1.1.9}$$

Here $\mathcal{L}$ is a local second-order differential operator and $\mathcal{K}$, $\mathcal{B}$ are two integral-differential operators defined as follows:

$$\mathcal{L}u(t,x) := b(t,x)D_x u(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x)D^2_{xx}u(t,x)];$$
$$\mathcal{K}u(t,x) := \int_E (u(t,x+\beta(x,e)) - u(t,x) - \beta(x,e)D_x u(t,x))\lambda(de) \quad \text{and}$$
$$Bu(t,x) := \int_E \gamma(x,e)(u(t,x+\beta(x,e)) - u(t,x))\lambda(de).$$

This IPDE (1.1.9) has been studied, using viscosity solutions theory by Barles et al. [3]. Either a different class of solutions or a different type of integral-differential terms have been also considered (see e.g. [1], [73], [74]).

For such a system (1.1.9), we introduce the notion of viscosity solution as in [3].

Definition 1.1.9 *We say that* $u \in \mathrm{C}([0,T]\times\mathbb{R}^k)$*is*

(i) a viscosity sub-solution (resp. super-solution) of IPDE (1.1.9), if $u(T,x) \leq$ *(resp.* $\geq$*)* $g(x)$, $\forall x \in \mathbb{R}^k$ *and for any test function* $\phi \in \mathrm{C}^2([0,T]\times\mathbb{R}^k)$, *wherever* $(t,x) \in [0,T)\times\mathbb{R}^k$ *is a global maximum (resp. global minimum) point of* $u-\phi$,

$$-\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{K}\phi(t,x) - f(t,x,u(t,x),$$
$$(\sigma\partial_x\phi)(t,x), B\phi(t,x)) \leq (\ \textit{resp.}\ \geq)\ 0.$$

(ii) a viscosity solution of IPDE (1.1.9) if it is both a sub-solution and super-solution of (1.1.9).

Now, we give the probabilistic interpretation of the viscosity solution of IPDE (1.1.9) using solution of the BSDE (1.1.8).

Theorem 1.1.10 *(Barles et al. [3]). Under Assumption* ***(H7)****, the function* $u(t,x) := Y_t^{t,x}$ *is a continuous function on* $[0,T]\times\mathbb{R}^k$ *satisfying the polynomial growth condition, and is a viscosity solution of IPDE (1.1.9).*

1.2 Obstacle problem for BSDEs

1.2.1 Single obstacle Reflected BSDEs and optimal stopping problem

1.2.1.1 General results

In this part, we consider a class of BSDEs where the solution Y is forced to stay above a given process L, called obstacle. An increasing process K is introduced to push the solution upwards, above the obstacle. This leads to the notion of Reflected BSDEs.

Standard Reflected BSDEs

When the filtration is generated only by a Brownian motion, the notion of Reflected BSDEs has been first introduced by El-Karoui et al. [28]. A solution to a Reflected BSDE consists of a triple of $\mathcal{P}$-measurable processes $(Y_s, Z_s, K_s)_{s\leq T}$ taking values in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^+$, respectively and satisfying:

$$\begin{cases} Y \in \mathcal{S}_c^2, Z \in \mathcal{H}^{2,d} \text{ and } K \in \mathcal{S}_c^2, (K_0 = 0) \text{ and is increasing ;} \\ Y_s = \xi + \int_s^T f(r, Y_r, Z_r)dr + K_T - K_s - \int_s^T Z_r dB_r, \quad s \leq T; \\ Y_s \geq L_s \text{ and } \int_0^T (Y_s - L_s)dK_s = 0, \end{cases} \tag{1.2.1}$$

where ξ, f and L satisfy the following assumptions:

Assumption (H8)

(i) $\xi \in \mathbb{L}^2$ and $\mathbb{R}$-valued;

(ii) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that

- $f(., y, z) \in \mathcal{H}^{2,1}$;
- f is Lipschitz with respect (y, z);

(iii) $(L_t)_{t\leq T}$ is an obstacle which is a continuous $\mathcal{P}$-measurable, $\mathbb{R}$-valued process such that

$$L_T \leq \xi \quad \text{and} \quad \mathbb{E}\big[\sup_{t\leq T}(L_t^+)^2\big] < \infty.$$

These types of equations are connected with a wide range of applications especially the pricing of American options in markets, constrained or not, mixed

control, partial differential variational inequalities, real options (see e.g. [19], [25], [42], [52], [29], [30], etc. and the references therein).

Under some assumptions on the generator f, the authors show the existence and uniqueness of the solution. Actually, in [28], the authors showed the existence and uniqueness of the solution by both a fixed point argument and approximation via penalization. Moreover, they proved that the solution $(Y_s)_{s\leq T}$ of (1.2.1) is the value function of an optimal stopping problem. Here, we will recall briefly their main results.

Theorem 1.2.1 *Under Assumption **(H8)**, there exists a unique solution $(Y_s, Z_s, K_s)_{s\leq T}$ to the Reflected BSDE (1.2.1), and Y_s has the explicit optimal stopping time representation, for all $s \leq T$,*

$$Y_s = esssup_{\tau\geq s}\mathbb{E}\Big[\int_s^\tau f(r, Y_r, Z_r)dr + L_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}\big|\mathcal{F}_s\Big].$$

Since then, there were several extensions for relaxing the Lipschitz condition on the generator of BSDE. We cite in particular:

- In [59], Matoussi showed the existence of a maximal and a minimal solution for Reflected BSDEs, when the generator f is only continuous and has linear growth in variables y and z.
- In [50], Kobylanski and al. proved the existence of a maximal and minimal bounded solution for the Reflected BSDE when the generator f is super linear increasing in y and quadratic in z.

Similarly to standard BSDEs, there is a comparison result for Reflected BSDEs. It allows to compare the solutions of two Reflected BSDEs once one can compare the terminal conditions, the generators and the obstacles.

Theorem 1.2.2 *(El Karoui et al. [28]). Let (Y, Z, K) and (Y', Z', K') be two solutions of Reflected BSDE associated with parameters (ξ, f, L), and (ξ', f', L'). Only f is assumed to satisfy the Lipshitz condition. Then, we have:*
If $\xi \leq \xi'$ a.s., $f(t, Y_t, Z_t) \leq f'(t, Y_t, Z_t)\, d\mathcal{P}\otimes dt$ a.e. and $L_t \leq L'_t$ a.s., then $Y_t \leq Y'_t, \quad \forall t \leq T$, a.e.

Reflected BSDEs with jumps

The extension to the case of Reflected BSDEs with jumps, which is a standard Reflected BSDEs driven by a Brownian motion and an independent Poisson random measure, has been studied by Hamadène and Ouknine [40]. A solution of such an equation consists of a quadruple of $\mathcal{P}$-measurable processes $(Y_s, Z_s, V_s, K_s)_{s\leq T}$ taking values in $\mathbb{R}, \mathbb{R}^d, \mathcal{L}^2(\lambda)$ and $\mathbb{R}^+$, respectively and satisfying: $\forall s \leq T$,

$$\begin{cases} Y \in \mathcal{S}^2, Z \in \mathcal{H}^{2,d}, V \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), K \in \mathcal{S}_c^2, (K_0 = 0) \text{ and is increasing ;} \\ Y_s = \xi + \int_s^T f(r, Y_r, Z_r, V_r)dr + K_T - K_s - \int_s^T Z_r dB_r \\ \qquad\quad - \int_s^T \int_E V_r(e)\tilde{\mu}(dr, de); \\ Y_s \geq L_s \text{ and } \int_0^T (Y_s - L_s)dK_s = 0. \end{cases} \tag{1.2.2}$$

Here $(L_s)_{s\leq T}$ is an obstacle which is right continuous left limits (rcll in short) whose jumping times are inaccessible stopping times.

In [40], the authors proved the existence and uniqueness of the solution by using penalization argument and Snell envelope theory. Let us recall briefly their main result

Theorem 1.2.3 *Under the Lipschitz Assumption **(H2)**, there exists a unique solution $(Y_s, Z_s, K_s, V_s)_{s\leq T}$ to the Reflected BSDEs with jumps (1.2.2).*

1.2.1.2 Connection with variational inequalities in the Markovian case

We put the Reflected BSDE in a Markovian framework in the sense that the terminal condition, the generator and the obstacle depend on a diffusion process $\{(X_s^{t,x})_{s\leq T}, (t,x) \in [0,T] \times \mathbb{R}^k\}$. More precisely, given a diffusion

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r, \quad t \leq s \leq T,$$

with Lipschitz coefficients b and σ on $[0,T]\times\mathbb{R}^k$, and we consider the following Markovian Reflected BSDEs: for all $s \leq T$,

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + K_T^{t,x} - K_s^{t,x} - \int_s^T Z_r^{t,x} dB_r; \\ Y_s^{t,x} \geq L(s, X_s^{t,x}) \text{ and } \int_0^T (Y_s^{t,x} - L(s, X_s^{t,x}))dK_s^{t,x} = 0, \end{cases} \tag{1.2.3}$$

where g, f and L are as follows: first g is continuous on $\mathbb{R}^k$ with polynomial growth, second, f is continuous on $[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$ and satisfies a Lipschitz condition in (y,z) uniformly in (t,x) and finally, L is continuous on $[0,T]\times\mathbb{R}^k$ with polynomial growth.

El Karoui et al. [28] introduce Markovian Reflected BSDEs (1.2.3) to give a probabilistic interpretation to the viscosity solution of the following variational partial differential inequalities: $\forall(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\begin{cases} \min\{u(t,x)-L(t,x); \\ \qquad -\partial_t u(t,x)-\mathcal{L}u(t,x)-f(t,x,u(t,x),(\sigma\partial_x u)(t,x))\}=0; \\ u(T,x)=h(x), \end{cases} \tag{1.2.4}$$

where

$$\mathcal{L}u(t,x):=\sum_{i=1}^{k} b_i(t,x)\frac{\partial}{\partial x_i}u(t,x)+\frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma_{ij}^{\top})(t,x)\frac{\partial^2}{\partial x_i\partial x_j}u(t,x).$$

Theorem 1.2.4 *The function $u(t,x):=Y_t^{t,x}$ is continuous on $[0,T]\times\mathbb{R}^k$ and is a viscosity solution of (1.2.4).*

1.2.2 Reflected BSDEs with interconnected obstacles and optimal switching problems

In this part, we introduce an optimal switching problems, and their connections with Reflected BSDEs with interconnected obstacles.

1.2.2.1 Optimal switching problems

The theory of optimal switching is an important and classical field of stochastic control, which knows a renewed increasing interest due to its numerous and various applications in economy and finance (see e.g. [2], [6], [9], [10], [15], [21], [35], [46], [14], [16], [17], [19], [21], [20], [42], [79], [78], etc. and the references therein). In fact, it provides a suitable modeling framework for the evaluation of optimal investment decisions in capital for firms. Therefore, it permits to capture the value of managerial flexibility to adapt decisions in response to unexpected markets developments.

We start by describing a two modes switching problem as bellow:
Suppose, for example, that a power plant can be operating or closed. A management strategy is then defined as an increasing sequence of stopping times $\delta = (\tau_n)_{n\geq 0}$, where for $n \geq 1$, $\tau_n \leq \tau_{n+1}$ and τ_{2n-1} (resp. τ_{2n}) are the times when the manager decides to turn on (resp. off). On the other hand, there are expenditures when the plant is in the operating mode as well as in the closed one . In addition, switching from a mode to another is not free and generates sunk costs. Then, mathematically, the plant's expected profits can be modelled as

$$J(\delta) = \mathbb{E}\Big[\int_0^T f(t, X_t, \mathcal{I}_t)dt + h(X_T, \mathcal{I}_T) - \sum_{n\geq 0}\big[C_1 1_{\{\tau_{2n-1}<T\}} + C_2 1_{\{\tau_{2n}<T\}}\big]\Big],$$

where X is the price process of electricity, f is the running profit over time, h is the terminal profit, $\mathcal{I}_t = 1$ (resp. $\mathcal{I}_t = 0$) if the production is open (resp. is closed), and C_1 (resp. C_2) is the cost when the plant is switched from the on (resp. off) mode to the off (resp. on) one.

An optimal management strategy and the value of the power are obtained by solving the following problem:

$$J(\delta^*) = \sup_{\delta} J(\delta),$$

where the supremum is taken over the set of admissible strategies.

More generally, suppose that the power plant runs with m different modes of production. In this case, a management strategy consists, on the one hand, of the choice of an increasing sequence of stopping times $(\tau_n)_{n\geq 1}$, where the manager switches the system across the different operating modes. On the other hand, it consists of the choice of the mode $(\alpha_n)_{n\geq 1}$, which are random variables, $\mathcal{F}_{\tau_n}$-measurable with values in $\{1, ..., m\}$.
Next, with a given strategy $\delta := (\tau_n, \alpha_n)$, we associate a stochastic process $(\eta_t)_{t\leq T}$ which is the indicator of the production activity's mode at time $t \in [0, T]$ and which is given by:

$$\eta_s := \alpha_0 \mathbf{1}_{\{\tau_0\}}(t) + \sum_{n\geq 1} \alpha_n \mathbf{1}_{[\theta_n \leq t < \theta_{n+1})}.$$

Finally, the optimal switching problem is then

$$J(\delta^*) = \sup_{\delta=(\tau_n,\alpha_n)} \mathbb{E}\Big[\int_0^T f^\delta(t,X_t)dt + h^\delta(X_T) - \sum_{n\geq 1}\big[g_{\alpha_n\alpha_{n+1}}1_{\{\tau_n\leq T\}}\big]\Big],$$

where g_{ij} is the cost for switching from mode i to mode j.

In the previous example, we have considered electricity production. However, there are many real cases where this problem intervenes. Among others, we can quote the management of oil tankers, oil fields, etc. In order to tackle those problems, the authors use mainly two approaches. Either a probabilistic one, i.e BSDEs or an approach which uses PDEs.

1.2.2.2 Connection with Reflected BSDEs with interconnected obstacles

Optimal switching problem is closely related to the study of Reflected BSDEs with interconnected obstacles. It turns out that their solution $(Y_s)_{s\leq T}$ is in fact the value of a specific optimal switching problem.

The growing interest in optimal switching problem is then a motivation to study Reflected BSDEs with interconnected obstacles of the following form: $\forall i \in \{1,...,m\}$ and $s \in [t,T]$,

$$\begin{cases} Y_s^i = \xi_i + \int_s^T f_i(r,Y_r^1,...Y_r^m,Z_r^i)dr + K_T^i - K_s^i - \int_s^T Z_r^i dB_r, \\ Y_s^i \geqslant \max_{j\neq i}(Y_s^j - g_{ij}(s)), \\ \int_0^T [Y_s^i - \max_{j\neq i}(Y_s^j - g_{ij}(s))]dK_s^i = 0. \end{cases} \tag{1.2.5}$$

Here $\xi := (\xi_i)_{i=1,m}$ is an m-dimensional random variable known at time T and K^i is an increasing process.

As far as we know, these equations appear for the first time in [42] in the particular case of two modes, i.e. $m = 2$, and a generator not depending on (Y,Z). The general case is then studied by Hu and Tang in [46], where the existence and uniqueness of solutions is proved in the case when the cost process g_{ij} is constant and the generator $f_i(s,y^1,...y^m,z^i) = f_i(s,y^i,z^i)$, for $i \in \{1,...,m\}$. While existence is obtained by a penalization argument, uniqueness is obtained via a verification argument, identifying the solution with a formal switching problem involving switched BSDEs. Unfortunately, this approach

could not be extended to the case of a more general generator. Then, the result is generalized in [43], where the authors allow the generator to depend on all the component of Y but with the following monotonicity condition:

- The function $y^k \mapsto f_i(t, x, y^1, ..., y^{k-1}, y^k, y^{k+1}, ..., y^m, z^i)$ is increasing for all $(y^1, ..., y^{k-1}, y^{k+1}, ..., y^m) \in \mathbb{R}^{m-1}$, $z^i \in \mathbb{R}^d$ and $i \in \{1, ..., m\}$,

Next, Chassagneux, Elie and Kharroubi [13] were able to get rid of this monotonicity condition and have provided existence and uniqueness of solutions of (1.2.5) in the general case. They used an approach that applies a fixed point procedure for Lipschitz generators. This is done via the introduction of a convenient one dimensional switched BSDE and the use of a standard comparaison theorem.

1.3 Interconnected obstacles problem for PDEs

In the first part of this thesis, we study a system of PDEs with interconnected obstacles for which we establish a new existence and uniqueness result of continuous viscosity solution. The novelty is that we relax the so-called monotonicity condition on the generator. The main tool is the notion of system of Reflected BSDEs with interconnected obstacles. First, we start by an overview of the existent literature.

1.3.1 Motivations and state of the art

The interconnected obstacle problem for PDEs and their connection with optimal switching problems have been studied by many authors ([6], [9], [25], [38], [44], [24], [23], [75], [47], ect and the references therein). They model such problems by means of variational inequalities.

Our main interest is to study the following system of variational inequalities with interconnected obstacles: $\forall i \in \mathcal{I} := \{1, ..., m\}$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j \in \mathcal{I}^{-i}} \{u^j(t,x) - g_{ij}(t,x)\}; \\ \qquad -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) - f_i(t, x, (u^k(t,x))_{k=1,...,m}, (\sigma^\top D_x u^i)(t,x))\} = 0 \ ; \\ u^i(T,x) = h_i(x), \end{cases} \tag{1.3.1}$$

where $(g_{ij})_{i\in\mathcal{I}}$, $(f_i)_{i\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ are continuous functions, $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$ and $\mathcal{L}$ is an infinitesimal generator associated with a diffusion process $X^{t,x}$.

In the special case where, for any $i \in \mathcal{I}$, the function f_i does not depend on $(u^k)_{k=1,m}$ and $D_x u^i$, the system (1.3.1) reduces to the Hamilton-Jacobi-Bellman associated with the optimal switching problem whose value function is defined by

$$u^i(t,x) = \sup_{\delta=(\tau_n,\alpha_n)_n} \mathbb{E}\Big[\int_0^T f^\delta(t, X_t^{t,x})dt + h^\delta(X_T^{t,x}) - \sum_{n\geq 0}\big[g_{\alpha_n\alpha_{n+1}}(\tau_n, X_{\tau_n}^{t,x})1_{\{\tau_n\leq T\}}\big]\Big].$$

As for the general case, if the functions $(f_i)_{i\in\mathcal{I}}$ depend on both $(u^k)_{k=1,m}$ and $D_x u^i$, the system (1.3.1) can be related to switching problems with utility functions, Knightian uncertainty, recursive utilities, i.e., the present utility depends also on the future utility.

Actually, in [38], the authors have shown that if, $(f_i)_{i\in\mathcal{I}}$ verify the following monotonicity condition:

- The function $y^k \mapsto f_i(t,x,y^1,...,y^{k-1},y^k,y^{k+1},...,y^m,z^i)$ is increasing for all $(y^1,...,y^{k-1},y^{k+1},...,y^m) \in \mathbb{R}^{m-1}$, $z^i \in \mathbb{R}^d$ and $i \in \{1,...,m\}$,

then, system (1.3.1) has a unique viscosity solution $(u^i)_{i\in\mathcal{I}}$ in the class of continuous functions with polynomial growth which is given by:

$$\forall i \in \mathcal{I},\ u^i(t,x) = Y_t^{i;t,x},\ (t,x) \in [0,T]\times\mathbb{R}^k, \tag{1.3.2}$$

where $(Y^{i;t,x})_{i\in\mathcal{I}}$ is the first component of the solution for Reflected BSDEs system with interconnected obstacles (1.2.5) once one is in the Markovian setting.

We note that the previous monotonicity condition is not only used to compare the sub- and the super-solution of PDEs (1.3.1) but also to construct the viscosity solution itself. Indeed, this viscosity solution is constructed thanks to an increasing approximation scheme for the Reflected BSDEs system, which is essentially obtained with the use of a comparison result that holds under the above assumption. Otherwise, this remains impossible.

1.3.2 New results

The novelty of this part is to prove the existence of a unique continuous viscosity solution for PDEs system with interconnected obstacles (1.3.1) without assuming the monotonicity condition on the generator function. Our method relies basically on its connection with a Reflected BSDEs system with interconnected obstacles for which the solution exists and is unique for general generators $(f_i)_{i\in\mathcal{I}}$, that are merely Lipschitz in (y,z) and nothing more. In the following, we present the main results obtained in the already published paper [36].

The main tool to tackle the PDEs system (1.3.1) is to use the following system of Reflected BSDEs with interconnected obstacles: for all $i \in \mathcal{I}$ and $s \in [t,T]$,

$$\begin{cases} Y_s^i = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, (Y_r^k)_{k=1,,m}, Z_r^i)dr + K_T^i - K_s^i - \int_s^T Z_r^i dB_r, \\ Y_s^i \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^j - g_{ij}(s, X_s^{t,x})), \\ \int_0^T [Y_s^i - \max_{j\in\mathcal{I}^{-i}}(Y_s^j - g_{ij}(s, X_s^{t,x}))]dK_s^i = 0. \end{cases} \tag{1.3.3}$$

In a non-Markovian framework, Chassagneux, Elie and Kharoubi [13] prove that this system admit a unique solution $(Y^i, Z^i, K^i)_{i\in\mathcal{I}}$ without assuming the previous monotonicity condition on the generators $(f_i)_{i\in\mathcal{I}}$.

Now, in a Marokovian framework, we extend this solution $(Y^i, Z^i, K^i)_{i\in\mathcal{I}}$ to a solution of PDEs system (1.3.1). More precisely, we are able to give a Feynman-Kac's formula for the solution of PDEs system (1.3.1) via the Reflected BSDEs system (1.3.3).

Proposition 1.3.1 *Let $(Y^i, Z^i, K^i)_{i\in\mathcal{I}}$ be the unique solution of the Reflected BSDEs system (1.3.3). Then, there exist deterministic continuous functions $(u^i)_{i\in\mathcal{I}}$ of polynomial growth, defined on $[0,T]\times\mathbb{R}^k$ such that*

$$\forall i \in \mathcal{I} \text{ and } s \leq T, \quad Y_s^i = u^i(s, X_s^{t,x}). \tag{1.3.4}$$

The proof is given in two steps. In the first step, we suppose that the functions $(f_i(t,x,0,...,0,0))_{i\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ are bounded. We construct a Pi-

card iterations process $(Y_k^i)_{i\in\mathcal{I}} = \Theta((Y_{k-1}^i)_{i\in\mathcal{I}})$, where Θ is a mapping from $\mathcal{H}^{2,d}$ onto it self. As a result we show the uniform convergence of $(Y_k^i)_{k\geq 0}$ in $\mathcal{S}_c^2$ and the existence of deterministic continuous functions of polynomial growth $(u^{i,k})_{k\geq 0}$ such that $Y_k^i(s) = u^{i,k}(s, X_s^{t,x})$, $\forall s \in [t,T]$. Then, we prove that $(u^{i,k})_{k\geq 0}$ converge uniformly to u^i which implies that u^i is continuous. In the second step, we relax the boundedness condition of $(f_i(t,x,0,...,0,0)_{i\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$, i.e. we suppose that they are of polynomial growth. By applying Ito's formula with $\tilde{Y}^i := Y^i(1+|X^{t,x}|^2)^{-p}$, with $p \geq 1$, we fall in the previous framework. Therefore, we deduce that Y^i has the previous representation (1.3.4) with u^i continuous and of polynomial growth.

Then, we can state our main result as follows.

Theorem 1.3.2 *The functions $(u^i)_{i\in\mathcal{I}}$, defined by (1.3.4), are the unique viscosity solution of PDEs (1.3.1) in the class of continuous functions with polynomial growth.*

The proof divides in two steps. Firstly, thanks to Theorem (1.2.4), we prove that $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of system (1.3.1). Secondly, we show the uniqueness of this solution by using the uniqueness of $(Y^i)_{i\in\mathcal{I}}$, the solution to the system (1.3.3).

1.4 Interconnected obstacle problem for integral-PDEs

In the second part of this thesis, we study a system of second-order integral-partial differential equations (IPDEs for short) with interconnected obstacles and non-local terms for which we establish a new existence and uniqueness result of continuous viscosity solution. The novelty is that we relax the so-called monotonicity condition on the generator with respect to the jump component which is classic assumption in the literature of viscosity solutions of equation with non-local terms. The main tool is the notion of system of Reflected BSDEs with jumps and interconnected obstacles.

1.4.1 Motivations and formulation

Our main interest is the following system of IPDEs with interconnected obstacles and non-local terms: $\forall i \in \mathcal{I} := \{1, ..., m\}$ and $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j \in \mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) - Ku^i(t,x) \\ \qquad -f_i(t,x,(u^k(t,x))_{k=1,m}, (\sigma^T D_x u^i)(t,x), B_i u^i(t,x))\} = 0; \\ u^i(T,x) = h_i(x). \end{cases} \tag{1.4.1}$$

Here $\mathcal{L}$ is the second-order local operator

$$\mathcal{L}u^i(t,x) := b(t,x)^\top D_x u^i(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x) D^2_{xx} u^i(t,x)],$$

K is the non-local operator

$$Ku^i(t,x) := \int_E (u^i(t, x + \beta(x,e)) - u^i(t,x) - \beta(x,e)^\top D_x u^i(t,x))\lambda(de)$$

and $\mathcal{B}_i$ is an integral operator

$$B_i u^i(t,x) := \int_E \gamma_i(x,e)(u^i(t, x + \beta(x,e)) - u^i(t,x))\lambda(de).$$

We note that, due to the presence of $B_i u^i$ and Ku^i in equation (1.4.1), such an IPDE is called of non-local type. The non-local setting has been considered by several authors (see e.g. [3], [5], [34], [39], [45], [55], etc. and the references therein).

In a particular case, the system (1.4.1) is the Hamilton-Jacobi-Bellman equation related to multi-modes switching problems where the state is given by jump-diffusion process. The viscosity solution of (1.4.1) is then the value function of multi-modes switching problem from which the sought optimal strategy follows.

This type of IPDEs is related to the following system of Reflected BSDEs

with jumps and interconnected obstacles: $\forall i \in \mathcal{I}$ and $s \in [t,T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x},e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr,de), \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,t,x} = 0, \end{cases} \tag{1.4.2}$$

where

(a) $B = (B_s)_{s\leq T}$ is a Brownian motion, μ an independant Poison random measure with compensator $ds\lambda(de)$ and $\tilde{\mu}(ds,de) = \mu(ds,de) - ds\lambda(de)$ its compensated random measure;

(b) for any $(t,x) \in [0,T]\times\mathbb{R}^k$, $(X_s^{t,x})_{s\leq T}$ is the solution of the following SDE of diffusion-jump type, $\forall s \in [t,T]$,

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dB_r + \int_t^s \int_E \beta(X_{r^-}^{t,x}, e)\tilde{\mu}(dr,de).$$

Actually, in [45], Hamadène and Zhao have shown that if, $(f_i)_{i\in\mathcal{I}}$ verify the following conditions:

(i) $\gamma_i \geqslant 0$;

(ii) $q \in \mathbb{R} \mapsto \bar{f}_i(t,x,(y_k)_{k=1,m},z,q)$ is non-decreasing, when the other components (t,x,y,z) are fixed;

then, the system (1.4.2) has a unique solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. Moreover, thanks to the Markovian framework, they have made the link between this Reflected BSDEs with the IPDEs (1.4.1) through the representation of Feynman-Kac i.e.,

$$\forall i \in \mathcal{I},\ u^i(t,x) = Y_t^{i,t,x},\ (t,x) \in [0,T]\times\mathbb{R}^k. \tag{1.4.3}$$

Finally, they have proved that the functions $(u^i)_{i\in\mathcal{I}}$ are the unique continuous viscosity solution of IPDEs system (1.4.1) in the class of functions with polynomial growth. Conditions (i)-(ii), which will be referred as the monotonicity

conditions, are needed in order to have the comparison result and to treat the operator $B_i u^i$ which is not well-defined for any arbitrary u^i. Therefore, to our knowledge, without the conditions (i)-(ii), neither the IPDEs system (1.4.1) nor the Reflected BSDEs with jumps and interconnected obstacles system (1.4.2) have been considered.

1.4.2 New results

The novelty of this part is to prove the existence of a unique continuous viscosity solution for IPDEs system with interconnected obstacles (1.4.1) without assuming the previous monotonicity conditions on the generator. Our method relies basically on its connection with a Reflected BSDEs system with jumps and interconnected obstacles for which we show the existence of a unique Markovian solution.

We first study the problem of existence of a solution for Reflected BSDEs with jumps and interconnected obstacles system (1.4.2) without assuming the two points (i)-(ii) mentioned above. Actually we show that when the measure $\lambda(.)$ is finite, the system of RBSDEs (1.4.2) has a solution which is unique among the Markovian solutions, that is to say, which have the representation (1.4.3). Our method relies mainly on the characterization of the jump part of the RBSDEs (1.4.2) by means of the functions $(u^i)_{i=1,m}$ defined in (1.4.3) and the jump-diffusion process $X^{t,x}$. In the second part, we deal with the problem of existence and uniqueness in viscosity sense of the solution of system (3.1.1). We show that the functions $(u^i)_{i=1,m}$ defined in (1.4.3), through the unique solution of (1.4.2), is the unique solution of system (1.4.1).

Let us consider the following assumptions:

(A1) For any $i \in \{1, ..., m\}$,

(a) The function $(t, x) \mapsto \bar{f}_i(t, x, \vec{y}, z, q)$ is continuous, uniformly w.r.t. the variables $(\vec{y}, z, q)$, $\vec{y} = (y^1, ..., y^m)$;

(b) The function $\bar{f}_i$ is Lipschitz continuous w.r.t. the variables $(\vec{y}, z, q)$ uniformly in (t, x);

(c) For any $i \in \mathcal{I}$ and $j \in \mathcal{I}^{-i}$, the mapping
$y^j \mapsto \bar{f}_i(t, x, y^1, ..., y^{j-1}, y^j, y^{j+1}, ..., y^m, z, q)$ is non-decreasing whenever the other components $(t, x, y^1, ..., y^{j-1}, y^{j+1}, ..., y^m, z, q)$ are fixed;

(d) The mapping $(t, x) \mapsto \bar{f}_i(t, x, 0, 0, 0)$ has polynomial growth in x;

(e) The function γ_i is $\mathbb{B}(\mathbb{R}^k) \otimes \mathbb{B}(E)$-measurable such that for some constant $C > 0$:

$$|\gamma_i(x, e)| \leq C(1 \wedge |e|), \quad \forall (x, e) \in \mathbb{R}^k \times E.$$

(A2) For any $i, j \in \{1, ..., m\}$, $g_{ii} = 0$ and for $i \neq j$, $g_{ij}(t, x)$ is non-negative, jointly continuous in (t, x) with polynomial growth and satisfies the following non free loop property: For any $(t, x) \in [0, T] \times \mathbb{R}^k$, for any sequence of indices $i_1, ..., i_k$ such that $i_1 = i_k$ and $card\{i_1, ..., i_k\} = k - 1$ $(k \geq 3)$ we have

$$g_{i_1 i_2}(t, x) + g_{i_2 i_3}(t, x) + ... + g_{i_k i_1}(t, x) > 0.$$

(A3) For any $i \in \{1, ..., m\}$, the function h_i is continuous with polynomial growth and satisfies the following consistency condition:

$$\forall x \in \mathbb{R}^k, \ h_i(x) \geqslant \max_{j \in \mathcal{I}^{-i}} (h_j(x) - g_{ij}(T, x)).$$

We are now in a position to state the existence result. First, we suppose that the functions $\bar{f}_i(t, x, \vec{0}, 0, 0)$ and $h_i(x)$ are bounded. Later on we get rid of those latter boundedness conditions.

Proposition 1.4.1 *Assume that:*
*(i) The functions $(\bar{f}_i)_{i \in \mathcal{I}}, (g_{ij})_{i,j \in \mathcal{I}}$ and $(h_i)_{i \in \mathcal{I}}$ verify Assumptions **(A1)-(A3)**.*
(ii) There exist a constant $\bar{C}$ such that, for any $i \in \mathcal{I}$,

$$|h_i(x)| + |\bar{f}_i(t, x, \vec{0}, 0, 0)| \leq \bar{C}.$$

Then the system (1.4.2) has a solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i \in \mathcal{I}}$. Moreover there exist bounded continuous functions $(u^i)_{i \in \mathcal{I}}$ such that for any $i \in \mathcal{I}$, $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}), \ \forall s \in [t, T].$$

The proof divides in four steps. In Step 1, we construct a Picard iteration process $(V^{i,n})_{i\in\mathcal{I}} = \Theta((V^{i,n-1})_{i\in\mathcal{I}})$, where Θ is a mapping from $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ onto itself. By induction, we prove the existence of deterministic continuous functions of polynomial growth $(u^{i,n})_{n\geq 0}$ such that the Feynman-Kac representation holds, i.e. $Y_s^{i,n} = u^{i,n}(s, X_s^{t,x})$, $\forall s \in [t,T]$. Moreover, as the measure λ is finite, we can characterize the jump part $V^{i,n}$ by means of $(u^{i,n})_{n\geq 0}$ and the jump-diffusion process $X^{t,x}$, i.e. $V_s^{i,n}(e) = u^{i,n}(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^{i,n}(s, X_{s^-}^{t,x})$. In Step 2, we introduce an optimal switching problem and we represent the first component $(Y^{i,n})_{i\in\mathcal{I}}$ of the solution $(Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{i\in\mathcal{I}}$ for the iterative scheme as the value of the switching problem. In Step 3, we show the boundedness of $(u^{i,n})_{i\in\mathcal{I}}$. Next, via the introduction of a switched BSDEs and the use of a comparison argument, we prove the uniform convergence of $(u^{i,n})_{n\geq 0}$. Finally, in Step 4, we prove the convergence of the sequences $((Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ and we verify that the Skorohod condition holds. □

Next, we study the system (1.4.2) in the general case i.e., without assuming the boundedness of the functions $\bar{f}_i(t,x,\vec{0},0,0)$ and $h_i(x)$. To do this, we are going to transform the system (1.4.2) in such a way to fall in the same framework as the one of Proposition 1.4.1. But this transformation induces some perturbations of the Assumptions **(A1)-(a)**. To remedy to this situation we need to assume additional hypothesis on the functions $(\bar{f}_i)_{i\in\mathcal{I}}$ which is weaker than the previous one.

(A4) For any $i \in \{1, ..., m\}$,

(a) For any Φ, a bounded continuous function from $[0,T]\times\mathbb{R}^k$ to $\mathbb{R}$, the function $\bar{f}_i(t,x,\vec{y},z,\Phi(t,x))$ is continuous in $(t,x,\vec{y},z)$.

(b) For any $R>0$, there exists a continuous function m_R from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $m_R(0)=0$ and for any $|x|\leq R$, $|x'|\leq R$ and $|\vec{y}|\leq R$ we have:

$$|\bar{f}_i(t,x,\vec{y},z,\Phi(t,x)) - \bar{f}_i(t,x',\vec{y},z,\Phi(t,x'))| \leq m_R(|x-x'|(1+|z|)). \tag{1.4.4}$$

Theorem 1.4.2 *Assume that the functions $(\bar{f}_i)_{i\in\mathcal{I}}$ verify Assumptions **(A1)-(b)-(c)-(d)** and **(A4)**, $(g_{ij})_{i,j\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ verify Assumptions **(A2)-(A3)** respectively. Then the system (1.4.2) has a solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. Moreover there exist continuous functions $(u^i)_{i\in\mathcal{I}}$ of polynomial growth such that for any $i \in \mathcal{I}$, $(t,x) \in [0,T]\times\mathbb{R}^k$,*

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ \forall s \in [t,T].$$

As a by-product of the Proposition 1.4.1 and Theorem 1.4.2 we have the following

Corollary 1.4.3 *For any $i \in \mathcal{I}$ and $(t,x) \in [0,T]\times\mathbb{R}^k$,*

$$V_s^{i,t,x}(e) = 1_{\{s\geq t\}}\{u^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x})\},$$
$$ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E.$$

Next, we provide the uniqueness of the Markovian solution to Reflected BSDEs with jumps and interconnected obstacles (1.4.2) which is needed in order to establish a uniqueness result for the solution of the IPDEs system.

Proposition 1.4.4 *Let $(\tilde{u}^i)_{i\in\mathcal{I}}$ be the deterministic continuous functions of polynomial growth such that $Y_s^{i,t,x} = \tilde{u}^i(s, X_s^{t,x})$, $\forall s \in [t,T]$. Then, for any $i \in \mathcal{I}$, $\tilde{u}^i = u^i$.*

The proof is given in two cases. In the first one, we suppose that the functions u^i and $\tilde{u}^i$, $i \in \mathcal{I}$, are bounded. In the second case, we deal with the general case, i.e., without assuming the boundedness of the functions u^i and $\tilde{u}^i$, $i \in \mathcal{I}$, but only polynomial growth. □

Finally, we turn to study the existence and uniqueness of a viscosity solution to the IPDEs system (1.4.1). Before doing so, we precise our meaning of the definition of the viscosity solution of this system. It is not exactly the same as the one of Hamadène and Zhao's paper (see Definition (3.4.4) below).

Definition 1.4.5 *We say that a family of deterministic continuous functions $\vec{u} := (u^i)_{i\in\mathcal{I}}$ is a viscosity super-solution (resp. sub-solution) of (1.4.1) if: $\forall i \in \{1,...,m\}$,*

a) $u^i(T,x) \geq (resp.\ \leq)\ h_i(x),\ \forall x \in \mathbb{R}^k$;

b) if $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ *is such that* $(t,x) \in [0,T)\times\mathbb{R}^k$ *a global minimum (resp. maximum) point of* $u^i - \phi$

then

$$\min\Big\{u^i(t,x) - \max_{j\in\mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t\phi(t,x)$$
$$-\mathcal{L}\phi(t,x) - \tilde{f}_i(t,x,(u^k(t,x))_{k=1,,m},(\sigma^\top D_x\phi)(t,x),\mathcal{B}_i u^i(t,x))\Big\} \geq\ (resp.\ \leq)\ 0.$$

We say that $\vec{u} := (u^i)_{i\in\mathcal{I}}$ *is a viscosity solution of* (1.4.1) *if it is both a super-solution and sub-solution of* (1.4.1).

We now compare the two Definitions (1.4.5) and (3.4.4) of viscosity solutions.

Remark 1.4.6 *i) If for any* $i \in \mathcal{I}$, *the function* $\bar{f}_i(t,x,\vec{y},z,q)$ *does not depend on its last component* q *then Definitions* (1.4.5) *and* (3.4.4) *coincide.*

ii) In our definition, we have used $\mathcal{B}_i u^i(t,x)$ *instead* $\mathcal{B}_i\phi(t,x)$, *where* ϕ *is the test function. Indeed,* $\mathcal{B}_i u^i(t,x)$ *is well defined since* u^i *has a polynomial growth,* β *is bounded and the measure* λ *is finite while it is replaced by* $\mathcal{B}_i\phi(t,x)$ *in Definition* (3.4.4) *because, when* λ *is not finite, the lack of regularity of* u^i *makes that* $\mathcal{B}_i u^i(t,x)$ *could be ill-posed.*

We are now able to state the main result of this part.

Theorem 1.4.7 *The functions* $(u^i)_{i\in\mathcal{I}}$ *is the unique viscosity solution of the system* (1.4.1), *according to Definition* (1.4.5), *in the class of continuous functions with polynomial growth.*

The proof relies basically on three points. First the fact that the jump process in the Reflected BSDEs with jumps and interconnected obstacles (1.4.2) system has a representation in terms of some deterministic function. Then, the existence and uniqueness result of a solution to the IPDEs system (1.4.1) when the generator $\bar{f}_i$ does not depend on the jump component. Finally, the uniqueness of the Markovian solution to the system (1.4.2).

1.5 Systems of Markovian Obliquely Reflected BSDEs with Jumps: The case of Infinite Lévy measure

In the last part of this thesis, we are concerned with systems of Markovian reflected backward stochastic differential equations (RBSDEs for short) with jumps and interconnected obstacles. The novelty is that we deal with the case when the Lévy measure is not finite and integrates the function $(1 \wedge |e|)_{e\in E}$. Under appropriate assumptions and using a truncation of the Lévy measure, we show that the system of RBSDEs has a Markovian solution. The Feynman-Kac representation holds true and provides a solution for the associated system of IPDEs with interconnected obstacles. The content of this part is from a work in progress, in collaboration with Saïd Hamadène and Mohamed Mnif.

1.5.1 Motivations

In the second work, we have dealt with a system of Markovian Obliquely Reflected BSDEs with Jumps without assuming the so-called monotonicity conditions on the generator and a finite Lévy measure. Unfortunately, this is not the case in real-life problems, indeed the measures usually encountered are not necessarily finite. Thus the need to consider the problem with an arbitrary measure and not necessarily finite.

We consider the following system of Reflected BSDEs with Jumps and interconnected obstacles: $\forall i \in \mathcal{I} := \{1, ..., m\}$ and $s \in [0, T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)), \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0. \end{cases} \tag{1.5.1}$$

where $(t, x) \in [0, T] \times \mathbb{R}^k$, $ds\lambda(de)$ is the compensator of μ and $\tilde{\mu}(ds, de) := \mu(ds, de) - ds\lambda(de)$ its compensated random measure, and finally $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$.

The process $X^{t,x}$ is the solution of the following standard differential equation:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s + \int_E \beta(s, X_{s^-}^{t,x}, e)\tilde{\mu}(ds, de), \quad s \in [t, T]; \\ X_s^{t,x} = x \in \mathbb{R}^k, \quad 0 \leq s \leq t. \end{cases} \tag{1.5.2}$$

This system of reflected BSDEs (1.5.1) is termed of Marovian type since randomness stems from the process $X^{t,x}$ which is a Markovian. On the other hand, it is deeply related to the optimal stochastic switching problem.

With the system (1.5.1), is associated the following system of integral-partial differential equations (IPDEs for short) with interconnected obstacles: $\forall i \in \mathcal{I}$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j \in \mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) \\ \quad -Ku^i(t,x) - f_i(t,x,(u^k(t,x))_{k=1,m}, (\sigma^T D_x u^i)(t,x), B_i u^i(t,x))\} = 0, \\ u^i(T,x) = h_i(x), \end{cases} \tag{1.5.3}$$

where the operators $\mathcal{L}$, $\mathcal{K}$ and B_i are defined as follows:

$$\begin{aligned} &\mathcal{L}u^i(t,x) := b(t,x)^\top D_x u^i(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x)D_{xx}^2 u^i(t,x)]; \\ &\mathcal{K}u^i(t,x) := \int_E (u^i(t, x+\beta(x,e)) - u^i(t,x) - \beta(x,e)^\top D_x u^i(t,x))\lambda(de) \quad \text{and} \\ &B_i u^i(t,x) := \int_E \gamma_i(x,e)(u^i(t,x+\beta(x,e)) - u^i(t,x))\lambda(de). \end{aligned} \tag{1.5.4}$$

In the previous work, we have shown that the RBSDEs system (1.5.1) has a solution $(Y^{i,t,x}, Z^{i,t,x}, K^{i,t,x}, V^{i,t,x})_{i\in\mathcal{I}}$, without assuming the monotonicity condition on the generator with respect the jump component and when the Lévy measure $\lambda(.)$ associated with the Poisson random measure μ is finite, i.e. $\lambda(E) < \infty$. Moreover the Feynman-Kac representation of the processes $(Y^{i,t,x})_{i\in\mathcal{I}}$ holds true, i.e., there exist deterministic continuous functions $(u^i)_{i\in\mathcal{I}}$ defined on $[0,T] \times \mathbb{R}^k$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and $i \in \mathcal{I}$,

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ \forall s \in [t,T] \text{ and then } u^i(t,x) = Y_t^{i,t,x}. \tag{1.5.5}$$

Finally, it is proved that the functions $(u^i)_{i\in\mathcal{I}}$ are the unique continuous viscosity solution of IPDEs system (1.5.3) in the class of functions with polynomial growth. A property which plays an important role is the representation of the process $(V^{i,t,x})_{i\in\mathcal{I}}$ via the continuous functions $(u^i)_{i\in\mathcal{I}}$ and the process $X^{t,x}$ and which reads:

$$V_s^{i,t,x}(e) = 1_{\{s\geq t\}}\{u^i(s, X_{s^-}^{t,x}+\beta(X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x})\},$$
$$ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E. \qquad (1.5.6)$$

Therefore the main objective of this work is to deal with system (1.5.1) in the case when $\lambda(.)$ is not finite, i.e., $\lambda(E)=\infty$.

1.5.2 New results

In this work, we show that if $\lambda(.)$ is infinite and integrates the function $(1\wedge|e|)_{e\in E}$, in combination with other regularity properties on the data $(\bar{f}_i)_{i\in\mathcal{I}}$, $(h_i)_{i\in\mathcal{I}}$ and $(g_{ij})_{i,j\in\mathcal{I}}$, then the system (1.5.1) has a Markovian solution, moreover the Feynman-Kac representation (1.5.5) holds true. Finally we show that those functions $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of (1.5.3). The relation (1.5.6) which binds the processes $(V^{i,t,x})_{i\in\mathcal{I}}$, the functions $(u^i)_{i\in\mathcal{I}}$ and the process $X^{t,x}$ is also valid. Once more it plays an important role in the proof of our result.

To begin with, we look for a solution to the Reflected BSDEs (1.5.1) without the finiteness of the Lévy measure. To do this, we consider firstly the following system of Reflected BSDEs with jumps and interconnected obstacles: $\forall i\in\mathcal{I}$ and $s\in[t,T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \widehat{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x})dr + K_T^{i,t,x} - K_s^{i,t,x} \\ \qquad\qquad - \int_s^T Z_r^{i,t,x}dB_r - \int_s^T\int_E V_r^{i,t,x}(e)\tilde{\mu}(dr,de), \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)), \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0. \end{cases} \qquad (1.5.7)$$

Note that in this system, the generator $\widehat{f}_i$ does not depend on the jump component and the costs functions $(g_{ij})_{j\in\mathcal{I}}$ does not depend on the variable x. On the other hand, we assume that they satisfy the following assumptions:

(H1) For any $i \in \mathcal{I}$, the function

(a) $\widehat{f}_i$ is Lipschitz continuous w.r.t. the variables $(\vec{y}, z)$ uniformly in (t, x),

(b) $\forall i \in \mathcal{I}$ and $j \in \mathcal{I}^{-i}$, the mapping $y^j \mapsto \widehat{f}_i(t, x, y^1, ..., y^{j-1}, y^j, y^{j+1}, ..., y^m, z)$ is non-decreasing whenever the other components $(t, x, y^1, ..., y^{j-1}, y^{j+1}, ..., y^m, z)$ are fixed.

(c) $\widehat{f}_i$, $i \in \mathcal{I}$, verifies: There exists a continuous concave function Φ_i, from $\mathbb{R}^k$ into $\mathbb{R}$, such that $\Phi_i(0) = 0$ and

$$\forall x, x', \vec{y}, \text{ and } z, \ |\widehat{f}_i(t, x, \vec{y}, z) - \widehat{f}_i(t, x', \vec{y}, z)| \leq \Phi_i(|x - x'|). \quad (1.5.8)$$

(H2) $\forall i, j \in \{1, ..., m\}$, $g_{ii} = 0$ and for $i \neq j$, $g_{ij}(t)$ is non-negative, continuous in t and satisfies the following non free loop property :

For any $t \in [0, T]$, for any sequence of indices $i_1, ..., i_k$ such that $i_1 = i_k$ and $card\{i_1, ..., i_k\} = k - 1$ $(k \geq 3)$ we have

$$g_{i_1 i_2}(t) + g_{i_2 i_3}(t) + ... + g_{i_{k-1} i_1}(t) > 0. \quad (1.5.9)$$

(H3) For $i \in \{1, ..., m\}$, the function h_i is uniformly continuous and satisfies the following consistency condition:

$$h_i(x) \geqslant \max_{j \in \mathcal{I}^{-i}} (h_j(x) - g_{ij}(T)), \quad \forall x \in \mathbb{R}^k. \quad (1.5.10)$$

(H4) The functions h_i and $\widehat{f}_i(t, x, \vec{0}, 0)$, $i \in \mathcal{I}$, are bounded, i.e., there exists a constant $\overline{C}$ such that

$$\forall (t, x) \in [0, T] \times \mathbb{R}^k \text{ and } i \in \mathcal{I}, \ |h_i(x)| + |\widehat{f}_i(t, x, \vec{0}, 0)| \leq \overline{C}.$$

We then have the following first result.

Theorem 1.5.1 *Assume that the functions* $(\widehat{f}_i)_{i\in\mathcal{I}}$, $(g_{ij})_{i,j\in\mathcal{I}}$ *and* $(h_i)_{i\in\mathcal{I}}$ *verify Assumptions (H1)-(H4). Then, we have:*

a) The system (1.5.7) *has a unique solution* $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$.

b) There exist deterministic continuous bounded functions $(u^i)_{i\in\mathcal{I}}$, *defined on* $[0,T]\times\mathbb{R}^k$, *such that:*

$$\forall s\in[t,T],\ Y_s^{i,t,x}=u^i(s,X_s^{t,x}).$$

c) For any $i\in\mathcal{I}$,

$$V_s^{i,t,x}(e)=1_{\{s\geq t\}}(u^i(s,X_{s^-}^{t,x}+\beta(X_{s^-}^{t,x},e))-u^i(s,X_{s^-}^{t,x})),\quad ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E. \tag{1.5.11}$$

The proof is divided into five steps. First, we begin by truncating at the origin of the measure $\lambda(.)$ in such a way to fall in the framework of a finite Lévy measure and we introduce the approximation scheme associated with the BSDEs system (1.5.7), we denote by $({}^nY^{i,t,x}, {}^nZ^{i,t,x}, {}^nV^{i,t,x}, {}^nK^{i,t,x})_{i\in\mathcal{I}})_{n\geq1}$ its solution which exists and is unique. In addition, since the setting is Markovian, there exists $({}^nu^i)_{n\geq1}$ a family of deterministic continuous functions such that ${}^nY_s^{i,t,x}={}^nu^i(s,{}^nX_s^{t,x})$, $\forall s\in[t,T]$. Moreover, as the new measure λ_n is finite, we can characterize the jump part ${}^nV^{i,t,x}$ by means of ${}^nu^i$, i.e. ${}^nV_s^{i,t,x}(e)={}^nu^i(s,{}^nX_{s^-}^{t,x}+\beta({}^nX_{s^-}^{t,x},e))-{}^nu^i(s,{}^nX_{s^-}^{t,x})$. Then, we introduce the associated optimal switching problem and we represent the first component $({}^nY^{i,t,x})_{i\in\mathcal{I}}$ as its value function and we show the uniform convergence of $({}^nu^i)_{n\geq1}$. Finally, we prove the uniform convergence of the sequences $({}^nY^{i,t,x}, {}^nZ^{i,t,x}, {}^nV^{i,t,x}, {}^nK^{i,t,x})_{i\in\mathcal{I}})_{n\geq1}$ and then we show that the jump appearing in the approximation converges to the jump part in the solution. To state this result we need to assume that the functions $g_{ij}, i,j\in\mathcal{I}$ do not depend on x. □

We are now in position to show existence of a solution for system (1.5.1) (the generators depend on the jump components) in the case when $\lambda(.)$ is not

finite and integrates $(1 \wedge |e|)_{e \in E}$. For that, we need to assume additionally the following hypotheses on the functions $(\bar{f}_i)_{i \in \mathcal{I}}$.

(H5) For any $i \in \{1, ..., m\}$,

(a) $\bar{f}_i$ is Lipschitz continuous w.r.t. the variables $(\vec{y}, z, q)$ uniformly in (t, x).

(b) $\forall i \in \mathcal{I}$ and $j \in \mathcal{I}^{-i}$, the mapping $y^j \mapsto \bar{f}_i(t, x, y^1, ..., y^{j-1}, y^j, y^{j+1}, ..., y^m, z, q)$ is non-decreasing whenever the other components $(t, x, y^1, ..., y^{j-1}, y^{j+1}, ..., y^m, z, q)$ are fixed.

(c) For any Φ, a bounded continuous function from $[0, T] \times \mathbb{R}^k$ to $\mathbb{R}$, the function $\bar{f}_i(t, x, \vec{y}, z, \Phi(t, x))$ is continuous in $(t, x, \vec{y}, z)$. Moreover there exists a continuous concave function Ψ_i, from $\mathbb{R}^k$ into $\mathbb{R}$, such that $\Psi_i(0) = 0$ and

$$|\bar{f}_i(t, x, \vec{y}, z, \Phi(t, x)) - \bar{f}_i(t, x', \vec{y}, z, \Phi(t, x'))| \leq \Psi_i(|x - x'|). \quad (1.5.12)$$

(d) The functions $(\bar{f}_i(t, x, \vec{0}, 0, 0))_{i \in \mathcal{I}}$ are bounded.

Remark 1.5.2 *(i) Note that in Assumption (H5)-(c), the function Ψ_i can depend on the function Φ. This assumption is satisfied if, e.g., for any $i \in \mathcal{I}$, $\bar{f}_i(t, x, \vec{y}, z, q) = \bar{g}_i(t, x, \vec{y}, z, q\varphi(x))$ where $\bar{g}_i(t, x, \vec{y}, z, \zeta)$ is Lipschitz in $(x, \vec{y}, z, \zeta)$ uniformly in t and $\varphi(x)$ is a continuous function such that $\lim_{|x| \to \infty} \varphi(x) = 0$. (ii) We should point out that the obtained results are a preliminary step in studying the problem in the general case that we leave as a topic of future research.*

Theorem 1.5.3 *Assume that the functions $(\bar{f}_i)_{i \in \mathcal{I}}$, $(g_{ij})_{i,j \in \mathcal{I}}$ and $(h_i)_{i \in \mathcal{I}}$ verify Assumptions (H2)-(H5). Then the system (1.5.1) has a solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i \in \mathcal{I}}$. Moreover there exist bounded continuous functions $(u^i)_{i \in \mathcal{I}}$ such that for any $i \in \mathcal{I}$, $(t, x) \in [0, T] \times \mathbb{R}^k$,*

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}), \ \forall s \in [t, T].$$

The proof follows the same steps as in the proof of Proposition 3.2 in the previous work (see [37]), except that in our framework we should take into account of the non-boundedness of $\lambda(.)$. This difficulty is tackled in using the fact that for any $i \in \mathcal{I}$, $|\gamma_i(x,e)| \leq c(1 \wedge |e|)$ and $\int_E (1 \wedge |e|)\lambda(de) < \infty$ as well. Actually, we introduce a recursive scheme which we show that is convergent and its limit provides a solution for system (1.5.1). □

As a by-product of Theorem 1.5.3, we obtain the following result:

Corollary 1.5.4 *For any $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times \mathbb{R}^k$,*

$$V_s^{i,t,x}(e) = 1_{\{s \geq t\}} u^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x}),$$
$$ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [0,T] \times \Omega \times E. \quad (1.5.13)$$

With the help of the previous results, mainly the characterization of $V^{i,t,x}$ in terms of u^i , we show the second main result of this part.

Theorem 1.5.5 *The functions $(u^i)_{i \in \mathcal{I}}$ is a viscosity solution of the system (1.5.3), according to Definition (1.4.5).*

The proof of existence is relies mainly on the existence and uniqueness result of a solution for the IPDEs system (1.5.3) when the generator does not depend on the component of jump and the representation of the jump process to the BSDEs sytem (1.5.1). □

Chapter 2

Viscosity Solutions of Systems of PDEs with Interconnected Obstacles and Switching Problem without Monotonicity Condition

The content of this chapter is from an article in collaboration with Saïd Hamadène and Mohamed Mnif [36], published in Asymptotic Analysis.

2.1 Introduction

The main objective of this paper is to study the problem of existence and uniqueness of a solution in viscosity sense $(u^i)_{i=1,m}$ of the following system of partial differential equations with obstacles which depend on the solution: $\forall i \in \mathcal{I} := \{1, ..., m\}$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}\{u^j(t,x) - g_{ij}(t,x)\}; \\ \qquad -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) - f_i(t,x,(u^k(t,x))_{k=1,...,m},(\sigma^\top D_x u^i)(t,x))\} = 0 \ ; \\ u^i(T,x) = h_i(x) \end{cases} \tag{2.1.1}$$

where $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$ and $\mathcal{L}$ is an infinitesimal generator which has the following form

$$\mathcal{L}\varphi(t,x) := b(t,x)^\top . D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x)D^2_{xx}\varphi(t,x)] \tag{2.1.2}$$

and which is associated with the stochastic process $X^{t,x}$ solution of the SDE (2.1.3).

As pointed out previously, in (2.1.1), the obstacle of u^i is the function $\max_{j\in\mathcal{I}^{-i}}\{u^j(t,x)-g_{ij}(t,x)\}$ which actually depends on the solution $(u^i)_{i=1,m}$, which means that the obstacles are interconnected.

This problem is related to the optimal stochastic switching control problem which can be described, through an example, as follows: Let us consider a power plant which has several modes of production and which the manager puts in a specific mode according to its profitability which depends on the electricity price in the energy market evolving according to the following stochastic differential equation

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \quad s \geq t \text{ and } X_t^{t,x} = x. \tag{2.1.3}$$

The aim of the manager is to maximize her global profit over an horizon $[0, T]$ by optimally choosing controls of the form $\delta := (\theta_k, \alpha_k)_{k\geqslant 0}$ where $(\theta_k)_{k\geqslant 0}$ is an increasing sequence of stopping times at which the manager switches the system across the different operating modes and $(\alpha_k)_{k\geqslant 0}$ is a sequence of random variables with values in $\{1, ..., m\}$ which stand for the modes to which the production is switched. Namely for any $k \geq 1$, at θ_k, the manager switches the production from θ_{k-1} to θ_k (θ_0 and α_0 are the starting time and mode respectively). However, switching the plant from the mode i to the mode j is not free. It generates expenditures, which amount to $g_{ij}(s, X_s^{t,x})$ at time s. When the plant is run under a strategy δ, its yield is given by

$$J(\delta; t, x) := \mathbb{E}\Big[\int_t^T f^\delta(s, X_s^{t,x})ds - A_T^\delta + h^\delta(X_T^{t,x})\Big]$$

where:

a) $f^\delta(s, X_s^{t,x})$ is the instantaneous payoff of the plant when run under δ and $h^\delta(X_T^{t,x})$ is the terminal payoff ;

b) the quantity A_T^δ stands for the total switching cost when the strategy δ is implemented.

The problem is to find an optimal management strategy δ^*, i.e., which satisfies

$J(\delta^*;t,x) = \sup_\delta J(\delta;t,x)$. This latter quantity is nothing but the fair price of the power plant in the energy market.

In (2.1.1), if for any $i \in \mathcal{I}$, f_i does not depend on $(u^k)_{k=1,m}$ and $D_x u^i$, the system reduces to the Hamilton-Jacobi-Bellman one associated with the switching problem and it is shown in [38, 25], etc. that it has a unique solution $(u^i)_{i=1,m}$ which satisfies

$$u^i(t,x) = \sup\{J(\delta;t,x), \delta \in \mathcal{A}_t^i\},$$

where $\mathcal{A}_t^i$ is the set of admissible strategies which from mode i at time t.

The main tool to tackle system (2.1.1) is to use with the following system of reflected backward stochastic differential equations (RBSDEs for short) with interconnected obstacles: $\forall i \in \{1,...,m\}$ and $s \in [t,T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k=1,,m}, Z_r^{i,t,x})dr + K_T^{i,t,x} - K_s^{i,t,x} \\ \qquad\qquad - \int_s^T Z_r^{i,t,x} dB_r, \\ Y_s^{i,t,x} \geqslant \max\limits_{j \in \mathcal{I}^{-i}} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T [Y_s^{i,t,x} - \max_{j \in \mathcal{I}^{-i}} (Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))] dK_s^{i,t,x} = 0. \end{cases} \tag{2.1.4}$$

Note that the generators $(f_i)_{i\in\mathcal{I}}$ of the RBSDE system (2.1.4) have a general form, i.e., depend on $(y_i)_{i\in\mathcal{I}}$ and z_i. More precisely, this system can be related to switching problems with utility functions, knightian uncertainty, recursive utilities, i.e., the present utility depends also on the future utility, etc. The notion of recursive utility was first introduced by Duffie and Epstein (see [22]) to allow a separation between risk aversion and intertemporal substitution. In 1997, El Karoui et al. (see e.g. [28]) considered the case when the standard generators f_i can depend on z_i.

This system of RBSDEs has been investigated in several papers including ([13, 43, 38, 46], etc.). In [13], the authors proved that it has a unique solution $(Y^{i,t,x}, Z^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ if the functions $(f_i)_{i\in\mathcal{I}}$ are merely Lipschitz w.r.t $((y_l)_{l=1,m}, z)$.

Concerning now the system of PDEs (2.1.1), Hamadene et al. proved in [38] that, additionally to the Lipschitz property mentioned above, if for any

$i \in \mathcal{I}$ and $k \in \mathcal{I}^{-i}$, $f_i(t,x,(y_l)_{l=1,m},z)$ is increasing w.r.t y_k (see assumption (H4)-(i) below), then system (2.1.4) has a unique solution $(u^i)_{i=1,m}$ in the class of continuous functions with polynomial growth and which is given by:

$$\forall i \in \mathcal{I},\ u^i(t,x) = Y_t^{i,t,x},\ (t,x) \in [0,T] \times \mathbb{R}^k, \tag{2.1.5}$$

where $(Y^{i,t,x})_{i\in\mathcal{I}}$ is the first component of the solution of the system of reflected BSDEs (2.1.4). The same result is obtained if, instead of $(f_i)_{i\in\mathcal{I}}$, their opposites $(-f_i)_{i\in\mathcal{I}}$ verify the previous monotonicity property (see (H4)-(ii)). However without assuming one of either monotonicity conditions on the drivers $(f_i)_{i\in\mathcal{I}}$, the problem of existence and uniqueness of the solution in viscosity sense of system (2.1.4) remains open. In this paper, we show that system (2.1.4) has a unique solution without assuming the previous monotonicity properties on the drivers $(f_i)_{i\in\mathcal{I}}$. This is the main novelty of this work. As a consequence, we fill in the gap between the probabilistic framework and the PDEs one. Our method relies on the link between reflected BSDEs and PDEs with obstacles in the Markovian framework of randomness.

The paper is organized as follows. In Section 2, we formulate accurately the problem. In section 3, we show that Feynman-Kac formula holds for the components $(Y^{i;t,x})_{i\in\mathcal{I}}$ of the solution of (2.1.4), i.e., the representation (2.1.5) holds true. In Section 4, we show that the functions $(u^i)_{i\in\mathcal{I}}$ are continuous and are the unique viscosity solution of (2.1.1) in the class of functions with polynomial growth. The proof is deeply related to the fact that system (2.1.4) of RBSDEs has a unique solution. □

2.2 Preliminaries and notations

Let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma(B_s, s \leqslant t))_{t\leq T}$ and $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is its augmentation with the $\mathbb{P}$-null sets of $\mathcal{F}$. Then $(\mathcal{F}_t)_{0\leq t\leq T}$ is right continuous and complete.

We now introduce the following spaces :

a) $\mathcal{P}$ is the σ-algebra of $\mathbb{F}$-progressively measurable sets on $[0,T]\times\Omega$;

b) $\mathcal{S}^2$ is the set of $\mathcal{P}$-measurable, continuous, $\mathbb{R}$-valued processes $Y = (Y_s)_{s\leq T}$ such that $\mathbb{E}[\sup_{s\leq T}|Y_s|^2] < \infty$;

c) $\mathcal{A}^2$ is the subset of $\mathcal{S}^2$ of non decreasing processes $K = (K_t)_{t\leq T}$ such that $K_0 = 0$;

d) $\mathcal{H}^{2,l}$ $(l \geqslant 1)$ is the set of $\mathcal{P}$-measurable and $\mathbb{R}^l$-valued processes $Z := (Z_s)_{s\leq T}$ such that $\mathbb{E}[\int_0^T |Z_s|^2_{\mathbb{R}^l} ds] < \infty$.

Next, for any given $(t,x) \in [0,T]\times\mathbb{R}^k$ (k is a positive integer), we consider the following standard stochastic differential equation (SDE) :

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \ \ s \in [t,T] \\ X_s^{t,x} = x, \ \ 0 \leq s \leq t \end{cases} \tag{2.2.1}$$

where $b : [0,T]\times\mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : [0,T]\times\mathbb{R}^k \to \mathbb{R}^{k\times d}$ are two continuous functions and Lipschitz w.r.t x, i.e., there exists a positive constant C such that

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq C|x - x'|, \ \ \forall (t,x,x') \in [0,T]\times\mathbb{R}^{k+k}. \tag{2.2.2}$$

Note that the continuity of b, σ and (2.2.2) imply the existence of a constant C such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \ \ \forall (t,x) \in [0,T]\times\mathbb{R}^k. \tag{2.2.3}$$

Conditions (2.2.2) and (2.2.3) ensure, for any $(t,x) \in [0,T]\times\mathbb{R}^k$, the existence and uniqueness of a solution $\{X_s^{t,x}, t \leq s \leq T\}$ to the SDE (2.2.1) (see [70] for more details). Moreover, it satisfies the following estimate: $\forall p \geqslant 1$,

$$\mathbb{E}[\sup_{s\leq T}|X_s^{t,x}|^p] \leqslant C(1 + |x|^p). \tag{2.2.4}$$

Next let us introduce the following deterministic functions $(f_i)_{i=1,\dots,m}$, $(h_i)_{i=1,\dots,m}$

and $(g_{ij})_{i,j=1,...,m}$ defined as follows : for any $i, j \in \{1, ..., m\}$,

a) $f_i : (t, x, \vec{y}, z) \in [0, T] \times \mathbb{R}^{k+m+d} \longmapsto f_i(t, x, \vec{y}, z) \in \mathbb{R}$ $(\vec{y} := (y^1, ..., y^m))$;

b) $g_{ij} : (t, x) \in [0, T] \times \mathbb{R}^k \longmapsto g_{ij}(t, x) \in \mathbb{R}$;

c) $h_i : x \in \mathbb{R}^k \longmapsto h_i(x) \in \mathbb{R}$.

Additionally we assume that they satisfy:

(H1) For any $i \in \{1, ..., m\}$,

(i) The function $(t, x) \mapsto f_i(t, x, \vec{y}, z)$ is continuous, uniformly w.r.t. the variables $(\vec{y}, z)$.

(ii) The function f_i is Lipschitz continuous with respect to the variables $(\vec{y}, z)$ uniformly in (t, x), i.e., there exists a positive constant C_i such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$, $(\vec{y}, z)$ and $(\vec{y}_1, z_1)$ elements of $\mathbb{R}^{m+d}$:

$$|f_i(t, x, \vec{y}, z) - f_i(t, x, \vec{y}_1, z_1)| \leq C_i(|\vec{y} - \vec{y}_1| + |z - z_1|). \quad (2.2.5)$$

(iii) The mapping $(t, x) \mapsto f_i(t, x, 0, ..., 0)$ has polynomial growth in x, i.e., there exist two constants $C > 0$ and $p \geqslant 1$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$|f_i(t, x, 0, ..., 0)| \leq C(1 + |x|^p). \quad (2.2.6)$$

(H2) For all $i, j \in \{1, ..., m\}$, $g_{ii} = 0$, and if $i \neq j$ then $g_{ij}(t, x)$ is non-negative, jointly continuous in (t, x) with polynomial growth and satisfy the following non free loop property :

For any $(t, x) \in [0, T] \times \mathbb{R}^k$, for any sequence of indices $i_1, ..., i_k$ such that $i_1 = i_k$ and $card\{i_1, ..., i_k\} = k - 1$ $(k \geq 3)$ we have

$$g_{i_1 i_2}(t, x) + g_{i_2 i_3}(t, x) + ... + g_{i_k i_1}(t, x) > 0. \quad (2.2.7)$$

(H3) For $i \in \{1, ..., m\}$, the function h_i, which stands for the terminal condition, is continuous with polynomial growth and satisfies the following consistency condition:

$$\forall x \in \mathbb{R}^k,\ h_i(x) \geqslant \max_{j \in \mathcal{I}^{-i}} \{h_j(x) - g_{ij}(T, x)\}. \tag{2.2.8}$$

(H4)-(i) For any $i \in \mathcal{I}$ and $j \in \mathcal{I}^{-i}$,
the mapping $w \in \mathbb{R} \longmapsto f_i(t, x, y^1, ..., y^{j-1}, w, y^{j+1}, ..., y^m, z)$ is non-decreasing whenever the other components $(t, x, y^1, ..., y^{j-1}, y^{j+1}, ..., y^m, z)$ are fixed.

(H4)-(ii) The functions $(-f_i)_{i \in \mathcal{I}}$ verify (H4)-(i). □

The main objective of this paper is to study the following system of PDEs with interconnected obstacles: For any $i \in \mathcal{I} := \{1, ..., m\}$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j \in \mathcal{I}^{-i}} (u^j(t,x) - g_{ij}(t,x)); \\ \qquad -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) - f_i(t, x, (u^k(t,x))_{k=1,...,m}, (\sigma^\top D_x u^i)(t,x))\} = 0\ ; \\ u^i(T,x) = h_i(x) \end{cases} \tag{2.2.9}$$

where the operator $\mathcal{L}$ is the infinitesimal generator associated with $X^{t,x}$, i.e.,

$$\mathcal{L}\varphi(t,x) := b(t,x)^\top D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x) D^2_{xx}\varphi(t,x)] \tag{2.2.10}$$

for any $\mathbb{R}$-valued function $\varphi(t,x)$ such that $D_x\varphi(t,x)$ and $D^2_{xx}\varphi(t,x)$ are defined.

We now define the notion of a solution $(u^i)_{i \in \mathcal{I}}$ of system (2.2.9), in viscosity sense, which is the following:

Definition 2.2.1 *Let $\vec{u} := (u^i)_{i \in \mathcal{I}}$ be a function of $C([0,T] \times \mathbb{R}^k; \mathbb{R}^m)$. We say that $\vec{u}$ is a viscosity super-solution (resp. sub-solution) of (2.2.9) if: $\forall i \in \{1, ..., m\}$,*

a) $u^i(T,x) \geq$ (resp. $\leq$) $h_i(x)$, $\forall x \in \mathbb{R}^k$;

b) if $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ is such that $(t,x) \in [0,T) \times \mathbb{R}^k$ a global minimum (resp. maximum) point of $u^i - \phi$

then

$$\min\Big\{u^i(t,x)-\max_{j\in\mathcal{I}^{-i}}(u^j(t,x)-g_{ij}(t,x));$$
$$-\partial_t\phi(t,x)-\mathcal{L}\phi(t,x)-f_i(t,x,(u^k(t,x))_{k=1,\dots,m},(\sigma^\top D_x\phi)(t,x))\Big\}\geq\ (resp.\leq)\ 0.$$

(ii) We say that $\vec{u}:=(u^i)_{i\in\mathcal{I}}$ *is a viscosity solution of* (2.2.9) *if it is both a super-solution and sub-solution of* (2.2.9).

2.3 Connection with Systems of Reflected BSDEs with Oblique Reflection

The viscosity solution of system (2.2.9) is deeply connected (one can see [38] for more details) with the following system of reflected BSDEs with interconnected obstacles (or oblique reflection) associated with
$((f_i)_{i\in\mathcal{I}},(g_{ij})_{i,j\in\mathcal{I}},(h_i)_{i\in\mathcal{I}}):\forall i=1,...,m$ and $s\in[t,T]$,

$$\begin{cases}Y^{i,t,x}\in\mathcal{S}^2,\ Z^{i,t,x}\in\mathcal{H}^{2,d}\text{ and }K^{i,t,x}\in\mathcal{A}^2;\\ Y_s^{i,t,x}=h_i(X_T^{t,x})+\int_s^T f_i(r,X_r^{t,x},(Y_r^{k,t,x})_{k=1,m},Z_r^{i,t,x})dr+K_T^{i,t,x}-K_s^{i,t,x}\\ \qquad\qquad-\int_s^T Z_r^{i,t,x}dB_r,\\ Y_s^{i,t,x}\geqslant\max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x}-g_{ij}(s,X_s^{t,x})),\\ \int_0^T(Y_s^{i,t,x}-\max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x}-g_{ij}(s,X_s^{t,x})))dK_s^{i,t,x}=0.\end{cases}\tag{2.3.1}$$

This system (2.3.1) of reflected BSDEs is considered in several works (see e.g. [43, 38, 46, 13], etc.). Under (H1)-(H3) and (H4)-(i) as well, this system has been considered first in [43] where issues of existence and uniqueness of the solution, and comparison of the solutions, are considered (see Theorem 3.2, Theorem 4.2 for point i) and Corollary 3.4. in [43] or Remark 1, pp.190 in [38] for point ii)). Actually it is shown:

Theorem 2.3.1 *i) Assume that the deterministic functions* $(f_i)_{i\in\mathcal{I}},(g_{ij})_{i,j\in\mathcal{I}}$ *and* $(h_i)_{i\in\mathcal{I}}$ *verify Assumptions (H1)-(H3) and (H4)-(i). Then system* (2.3.1) *has a unique solution* $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$.
ii) If $(\bar{f}_i)_{i\in\mathcal{I}},(\bar{g}_{ij})_{i,j\in\mathcal{I}}$ *and* $(\bar{h}_i)_{i\in\mathcal{I}})$ *are other functions satisfying (H1)-(H3)*

and (H4)-(i) and, moreover, for any $i, j \in \mathcal{I}$,

$$f_i \leq \bar{f}_i,\ h_i \leq \bar{h}_i \text{ and } g_{ij} \geq \bar{g}_{ij}.$$

Then for any $i \in \mathcal{I}$, $Y^i \leq \bar{Y}^i$ *where* $(\bar{Y}^i, \bar{Z}^i, \bar{K}^i)_{i\in\mathcal{I}}$ *is the solution of the system associated with* $(\bar{f}_i)_{i\in\mathcal{I}}, (\bar{g}_{ij})_{i,j\in\mathcal{I}}$ *and* $(\bar{h}_i)_{i\in\mathcal{I}}$. □

In [13], Chassagneux et al. have also considered system (2.3.1) without assuming Assumption (H4)-(i). Mainly their idea is the following:

Let $\vec{\Gamma} := (\Gamma^i)_{i=1,\dots,m} \in \mathcal{H}^{2,m}$ and let us consider the following mapping:

$$\begin{aligned} \Theta : \mathcal{H}^{2,m} &\to \mathcal{H}^{2,m} \\ \vec{\Gamma} &\mapsto \Theta(\vec{\Gamma}) := (Y^{\Gamma,i})_{i=1,\dots,m} \end{aligned} \tag{2.3.2}$$

where $(Y^{\Gamma,i}, Z^{\Gamma,i}, K^{\Gamma,i})_{i\in\mathcal{I}} \in (\mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{A}^2)^m$ (we omit the dependence on t, x of $Y^{\Gamma,i}, Z^{\Gamma,i}, K^{\Gamma,i}$ as no confusion is possible) is the unique solution of the following system of reflected BSDEs with interconnected obstacles (or oblique reflection): $\forall i \in \mathcal{I}$, and $s \leq T$,

$$\begin{cases} Y_s^{\Gamma,i} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, \vec{\Gamma}_r, Z_r^{\Gamma,i})dr + K_T^{\Gamma,i} - K_s^{\Gamma,i} - \int_s^T Z_r^{\Gamma,i} dB_r, \\ Y_s^{\Gamma,i} \geqslant \max\limits_{j\in\mathcal{I}^{-i}} (Y_s^{\Gamma,j} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T (Y_s^{\Gamma,i} - \max\limits_{j\in\mathcal{I}^{-i}} (Y_s^{\Gamma,j} - g_{ij}(s, X_s^{t,x})))dK_s^{\Gamma,i} = 0. \end{cases} \tag{2.3.3}$$

Next for $\alpha \in \mathbb{R}$, let us introduce the following norm on $\mathcal{H}^{2,m}$: $\forall\ \vec{y} \in \mathcal{H}^{2,m}$

$$\|\vec{y}\|_\alpha := \{\mathbb{E}\Big[\int_0^T e^{\alpha s}|\vec{y}_s|^2 ds\Big]\}^{\frac{1}{2}}.$$

The main result in [13], is to show that Θ has a fixed point, i.e.,

Theorem 2.3.2 *(see [13]) Assume that the deterministic functions* $(f_i)_{i\in\mathcal{I}}$, $(h_i)_{i\in\mathcal{I}}$ *and* $(g_{ij})_{i,j\in\mathcal{I}}$ *verify Assumptions (H1)-(H3). Then there exists some appropriate positive constant* α_0 *(which depends on* m, T *and the Lipschitz constants of* $(f_i)_{i=1,m}$*) such that* Θ *is contraction on* $(\mathcal{H}^{2,m}, \|.\|_{\alpha_0})$. *Therefore it has a unique fixed point* $(Y^i)_{i\in\mathcal{I}}$ *which, combined with the associated processes*

$(Z^i, K^i)_{i\in\mathcal{I}}$, makes that $(Y^i, Z^i, K^i)_{i\in\mathcal{I}}$ is the unique solution of system (2.3.1).

Moreover the following estimate holds true: for any ${}^1\vec{\Gamma}$, ${}^2\vec{\Gamma} \in \mathcal{H}^{2,m}$,

$$\forall\ s \leq T\ \mathbb{E}[e^{\alpha s}|Y_s^{{}^1\Gamma,i} - Y_s^{{}^2\Gamma,i}|^2] \leq \tfrac{2C}{\alpha}\mathbb{E}[\textstyle\int_s^T e^{\alpha r}|{}^1\Gamma_r - {}^2\Gamma_r|^2 dr] \tag{2.3.4}$$

where C is a common Lipschitz constant of the functions $(f_i)_{i=1,m}$ w.r.t $(\vec{y}, z)$ and $\alpha \geq C$. □

As a remark, note that in [13], the assumptions on the deterministic functions $(f_i)_{i\in\mathcal{I}}$ etc., are not exactly the same as (H1)-(H3). In our work, we suppose that the switching costs $(g_{i,j}(t,x))_{i,j\in\mathcal{I}}$ are non-negative, i.e., $g_{i,j}(t,x) \geq 0$, $\forall (t,x) \in [0,T]\times\mathbb{R}^k$ and verify the non free loop property. While in [13], the authors suppose that the switching costs are strictly positive and satisfy the triangle inequality. However the ideas of [13] can be applied under (H1)-(H3) to show the existence and uniqueness of the solution of system (2.3.1).

We next provide some properties of the solution of system (2.3.1) which will be useful later.

Proposition 2.3.3 *Assume (H1)-(H3). Then:*

i) There exist deterministic functions $(u^i)_{i\in\mathcal{I}}$ of polynomial growth, defined on $[0,T]\times\mathbb{R}^k$, such that:

$$\forall\, i \in \mathcal{I}, Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ ds\times d\mathbb{P} \text{ on } [t,T]\times\mathbb{R}^k.$$

ii) Assume moreover that $f_i(t,x,0,0)$ and $h_i(x)$ are bounded. Then the processes $Y^{i,t,x}$ and functions u^i, $i \in \mathcal{I}$, are also bounded.

Proof: First let us focus on the first point. Let $(\bar{Y}, \bar{Z})$ be the solution of the following standard BSDE:

$$\begin{cases} \bar{Y} \in \mathcal{S}^2, \bar{Z} \in \mathcal{H}^{2,d}; \\ \bar{Y}_s = \Phi(X_T^{t,x}) + \int_s^T \Psi(r, X_r^{t,x}, \bar{Y}_r, \bar{Z}_r)dr - \int_s^T \bar{Z}_r dB_r,\ \forall\, s \leq T \end{cases}$$

where for any $(s,x,y,z) \in [0,T] \times \mathbb{R}^{k+1+d}$,

$$\Psi(s,x,y,z) := \bar{C}m|y| + \bar{C}|z| + \sum_{i=1,m} |f_i(s,x,0,\dots,0)| \text{ and } \Phi(x) := \sum_{i=1,m} |h_i(x)|.$$

The constant $\bar{C} := C_1 + ... + C_m$ with, for any $i \in \mathcal{I}$, C_i is the Lipschitz constant of f_i w.r.t $(\vec{y}, z)$. Note that the solution of this BSDE exists and is unique by Pardoux-Peng's result [62].

First note that since $\Psi \geq 0$ and $\Phi \geq 0$ then $\bar{Y} \geq 0$. Next as we are in the Markovian framework of randomness and since Φ and $\Psi(t,x,0,0)$ are of polynomial growth, then there exists a deterministic function $v(t,x)$ of polynomial growth (see e.g. [30]) such that:

$$\forall\, s \in [t,T], \bar{Y}_s^{t,x} = v(s, X_s^{t,x}).$$

Next let us set, for $i \in \mathcal{I}$,

$$\underline{\mathrm{Y}}_i = \bar{Y}, \underline{\mathrm{Z}}_i = \bar{Z} \text{ and } \underline{\mathrm{K}}_i = 0.$$

Therefore, since $g_{ij} \geq 0$ for any $i,j \in \mathcal{I}$, $(\underline{\mathrm{Y}}_i, \underline{\mathrm{Z}}_i, \underline{\mathrm{K}}_i)_{i\in\mathcal{I}}$ is a solution of the following system: for any $i \in \mathcal{I}$ and $s \leq T$,

$$\begin{cases} \underline{\mathrm{Y}}_i(s) = \Phi(X_T^{t,x}) + \int_s^T \Psi(r, X_r^{t,x}, \underline{\mathrm{Y}}_i(r), \underline{\mathrm{Z}}_i(r))dr + \underline{\mathrm{K}}_i(T) - \underline{\mathrm{K}}_i(s) - \int_s^T \underline{\mathrm{Z}}_i(r)B_r \ ; \\ \underline{\mathrm{Y}}_i(s) \geq \max_{j\neq i}\{\underline{\mathrm{Y}}_j(s) - g_{ij}(s)\}; \\ \int_0^T (\underline{\mathrm{Y}}_i(s) - \max_{j\neq i}\{\underline{\mathrm{Y}}_j(s) - g_{ij}(s)\})d\underline{\mathrm{K}}_i(s) = 0. \end{cases} \tag{2.3.5}$$

In the same way let us set for any $i \in \mathcal{I}$,

$$\hat{Y}_i = -\bar{Y}, \hat{Z}_i = -\bar{Z} \text{ and } \hat{K}_i = 0,$$

then $(\hat{Y}_i, \hat{Z}_i, \hat{K}_i)_{i\in\mathcal{I}}$ is a solution of the following system: for any $i \in \mathcal{I}$ and $s \leq T$,

$$\begin{cases} \hat{Y}_i(s) = -\Phi(X_T^{t,x}) - \int_s^T \Psi(r, X_r^{t,x}, -\bar{Y}_r, -\hat{Z}_r)dr + \hat{K}_i(T) - \hat{K}_i(s) - \int_s^T \hat{Z}_i(r)B_r \ ; \\ \hat{Y}_i(s) \geq \max_{j\neq i}\{\hat{Y}_j(s) - g_{ij}(s)\}; \\ \int_0^T (\hat{Y}_i(s) - \max_{j\neq i}\{\hat{Y}_j(s) - g_{ij}(s)\})d\hat{K}_i(s) = 0. \end{cases}$$

Next let us consider the following sequence of processes $((\tilde{Y}_k^i, \tilde{Z}_k^i, \tilde{K}_k^i)_{i\in\mathcal{I}})_{k\geq 0}$:

$$\tilde{Y}_0^i = 0, \text{ for all } i \in \mathcal{I} \text{ and for } k \geq 1, \ (\tilde{Y}_k^i)_{i\in\mathcal{I}} = \Theta((\tilde{Y}_{k-1}^i)_{i\in\mathcal{I}})$$

where Θ is the mapping defined in (2.3.2) and $\tilde{Z}_k^i$, $\tilde{K}_k^i$ are associated with $\tilde{Y}_k^i$, $i \in \mathcal{I}$, through equation (2.3.3). Therefore, as Θ is a contraction (Theorem (2.3.2)), the sequence $((\tilde{Y}_k^i)_{i\in\mathcal{I}})_{k\geq 0}$ converges to $(Y^i)_{i\in\mathcal{I}}$ in $(\mathcal{H}^{2,m}, \|.\|_{\alpha_0})$. On the other hand by an induction argument on k and by using the comparison result of Theorem 2.3.1-ii), we have that:

$$\forall k \geq 0, \forall i \in \mathcal{I}, \ -\bar{Y} = \hat{Y}_i \leq \tilde{Y}_k^i \leq \underline{Y}_i = \bar{Y}. \tag{2.3.6}$$

Indeed for $k = 0$, this obviously holds since $\bar{Y} \geq 0$. Next suppose that (2.3.6) holds for some $k-1$ with $k \geq 1$. Then by a linearization procedure of f_i, which is possible since it is Lipschitz w.r.t $(\vec{y}, z)$, we have: for any $i \in \mathcal{I}$,

$$f_i(s, X_s^{t,x}, (\tilde{Y}_{k-1}^i(s))_{i\in\mathcal{I}}, z) = f_i(s, X_s^{t,x}, 0, 0) + \sum_{l=1,m} a_s^{k,i,l}\tilde{Y}_{k-1}^l(s) + b_s^{k,i,l}z$$

where $a^{k,i,l} \in \mathbb{R}$ and $b^{k,i,l} \in \mathbb{R}^d$ are $\mathcal{P}$-measurable processes, bounded by the Lipschitz constant of f_i. Therefore, using the induction hypothesis, we obtain:

$$|f_i(s, X_s^{t,x}, (\tilde{Y}_{k-1}^i(s))_{i\in\mathcal{I}}, z)| \leq \Psi(s, X_s^{t,x}, \bar{Y}_s, z).$$

Finally by the comparison argument of Theorem 2.3.1-ii) (see also [43], Corollary 3.4, pp.411), we get:
$\forall i \in \mathcal{I}$, $\tilde{Y}_k^i \leq \underline{Y}_i'$ where $(\underline{Y}_i', \underline{Z}_i', \underline{K}_i')_{i\in\mathcal{I}}$ is the unique solution of the system of type (2.3.1) associated with $((\underline{f}_i = \Psi(s, X_s^{t,x}, \bar{Y}_s, z))_{i\in\mathcal{I}}, (\underline{h}_i = \Phi(x))_{i\in\mathcal{I}}$, $(g_{ij}(s, X_s^{t,x}))_{i,j\in\mathcal{I}})$. But the solution of this latter system is unique (Theorem 2.3.2) and by (2.3.5), $(\underline{Y}_i, \underline{Z}_i, \underline{K}_i)_{i\in\mathcal{I}}$ is also a solution. Therefore for any $i \in \mathcal{I}$, $\underline{Y}_i' = \underline{Y}_i$ and then $\forall i \in \mathcal{I}$, $\tilde{Y}_k^i \leq \underline{Y}_i = \bar{Y}$. In the same way one can show that $\forall i \in \mathcal{I}$, $\tilde{Y}_k^i \geq \hat{Y}_i = -\bar{Y}$. Therefore (2.3.6) holds true for any $k \geq 0$.

Next, once more, since we are in the Markovian framework of randomness, and using an induction argument on k we deduce the existence of deterministic continuous functions of polynomial growth $u^{i,k}(t, x)$ (see e.g. [38], Corollary 2, pp.182), $i \in \mathcal{I}$, such that for any $i \in \mathcal{I}$, $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$\tilde{Y}_k^i(s) = u^{i,k}(s, X_s^{t,x}), \ \forall s \in [t, T]. \tag{2.3.7}$$

By (2.3.6), in taking $s = t$, we obtain: for any $k \geq 0$, $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$|u^{i,k}(t,x)| \leq v(t,x). \tag{2.3.8}$$

Next by using the inequality (2.3.4) at $s = t$, we deduce that for any $i \in \mathcal{I}$, $k, p \geq 1$

$$\begin{aligned} |u^{i,k}(t,x) - u^{i,p}(t,x)|^2 & \\ &\leq \frac{2C}{\alpha_0}\mathbb{E}[\int_t^T e^{\alpha_0(r-t)} \sum_{j=1,m} |\tilde{Y}^j_{k-1}(r) - \tilde{Y}^j_{p-1}(r)|^2 dr] \\ &\leq \frac{2C}{\alpha_0}\mathbb{E}[\int_t^T e^{\alpha_0(r-t)} \sum_{j=1,m} |u^{j,k-1}(r, X^{t,x}_r) - u^{j,p-1}(r, X^{t,x}_r)|^2 dr]. \end{aligned} \tag{2.3.9}$$

As $((\tilde{Y}^i_k)_{i\in\mathcal{I}})_k$ is a Cauchy sequence in $(\mathcal{H}^{2,m}, \|.\|_{\alpha_0})$, then $((u^{i,k})_{i\in\mathcal{I}})_k$ is a Cauchy sequence pointwisely. This implies the existence of deterministic functions $(u^i)_{i\in\mathcal{I}}$ such that for any $i \in \mathcal{I}$ and $(t,x) \in [0,T]\times\mathbb{R}^k$, $u^{i,k}(t,x)$ converges w.r.t k to $u^i(t,x)$. Moreover by (2.3.8), u^i is of polynomial growth since v is so and finally by (2.3.7), $Y^{i,t,x}_s = u^i(s, X^{t,x}_s)$, $ds \times d\mathbb{P}$ on $[t,T] \times \mathbb{R}^k$.

We now deal with the second point. Assume that $f_i(t,x,0,0)$ and $h_i(x)$ are bounded. Then the solution $\bar{Y}$ is bounded. This is obtained by a change of probability, applying Girsanov's theorem and by multiplying both hand-sides of the equation by $e^{-m\bar{C}s}$, conditionning and taking into account of the inequality $\bar{Y} \geq 0$. Therefore the deterministic function v is a also bounded. Consequently, $u^{i,k}$ are uniformly bounded and so are u^i, $i \in \mathcal{I}$. □

Remark 2.3.4 *At this point we do not know whether the functions u^i, $i \in \mathcal{I}$, are continuous or not. However we will show later that they can be chosen continuous.* □

2.4 The main result:

In this section, we study the existence and uniqueness in viscosity sense of the solution of the system of m partial differential equations with interconnected obstacles (2.2.9). The candidate to be the solution are the functions

$(u^1, \dots, u^m)$ defined in Proposition 2.3.3 by which we represent $(Y^i)_{i\in\mathcal{I}}$. So, firstly we are going to show that those functions u^i, $i \in \mathcal{I}$, can be chosen continuous.

Proposition 2.4.1 *Assume that (H1)-(H3) hold. Then we can choose the functions u^i, $i \in \mathcal{I}$, defined in Proposition 2.3.3, continuous in (t,x) and of polynomial growth.*

Proof: It will be given in two steps. In the first one we are going to suppose moreover that h_i and $f_i(t,x,0,0)$, $i \in \mathcal{I}$, are bounded. Later on we deal with the general case, i.e., without assuming the boundedness of those latter functions.

Step 1: Suppose that for any $i \in \mathcal{I}$, h_i and $f_i(t,x,0,0)$ are bounded.
Recall the continuous functions $u^{i,k}$, $i \in \mathcal{I}$ and $k \geq 0$, defined in (2.3.7). By (2.3.9) they verify: $\forall k \geq 1$, $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$|u^{i,k}(t,x) - u^{i,p}(t,x)|^2 \leq \frac{2C}{\alpha}\mathbb{E}[\int_t^T e^{\alpha(r-t)} \sum_{j=1,m} |u^{j,k-1}(r, X_r^{t,x}) - u^{j,p-1}(r, X_r^{t,x})|^2 dr], \tag{2.4.1}$$

where, as pointed out in (2.3.4), C is the uniform Lipschitz constant of $f_i's$ w.r.t $(\vec{y}, z)$ and $\alpha \geq C$.

On the other hand we know, by Proposition 2.3.3-ii), that $u^{i,k}$ are uniformly bounded for any $i \in \mathcal{I}$ and $k \geq 0$. Now let us take $\alpha = C$ and let η be a constant such that $2C^{-1}m(e^{C\eta} - 1) = \frac{3}{4}$ and finally let us set

$$\|u^{i,k} - u^{i,p}\|_{\infty,\eta} := \sup_{(t,x)\in[T-\eta,T]\times\mathbb{R}^k} |u^{i,k}(t,x) - u^{i,p}(t,x)|.$$

Going back to (2.3.9) and taking the summations over all i, we deduce that for any $k, p \geq 1$,

$$\sum_{i=1,m} \|u^{i,k} - u^{i,p}\|^2_{\infty,\eta} \leq \underbrace{2mC^{-1}(e^{C\eta} - 1)}_{=\frac{3}{4}} \sum_{i=1,m} \|u^{i,k-1} - u^{i,p-1}\|^2_{\infty,\eta}$$
$$\leq \frac{3}{4} \sum_{i=1,m} \|u^{i,k-1} - u^{i,p-1}\|^2_{\infty,\eta},$$

which means that the sequence $((u^{i,k})_{i\in\mathcal{I}})_{k\geq 0}$ is uniformly convergent in $[T-\eta, T]\times\mathbb{R}^k$. Thus, their limits, i.e., the functions $(u^i)_{i\in\mathcal{I}}$ are also continuous on the set $[T-\eta, T]\times\mathbb{R}^k$.

Next let $t\in[T-2\eta, T-\eta]$, then once more by (2.4.1) we have:

$$
\begin{aligned}
&|u^{i,k}(t,x)-u^{i,p}(t,x)|^2\\
&\qquad\leq 2C\alpha^{-1}\mathbb{E}[\int_t^{T-\eta} e^{\alpha(r-t)}\sum_{j=1,m}|u^{j,k-1}(r,X_r^{t,x})-u^{j,p-1}(r,X_r^{t,x})|^2dr]\\
&\qquad\quad+2C\alpha^{-1}\mathbb{E}[\int_{T-\eta}^{T} e^{\alpha(r-t)}\sum_{j=1,m}|u^{j,k-1}(r,X_r^{t,x})-u^{j,p-1}(r,X_r^{t,x})|^2dr].
\end{aligned}
\tag{2.4.2}
$$

Then, if we choose $t=T-2\eta$ and set

$$
\|u^{i,k}-u^{i,p}\|_{\infty,2\eta}:=\sup_{(t,x)\in[T-2\eta,T-\eta]\times\mathbb{R}^k}|u^{i,k}(t,x)-u^{i,p}(t,x)|,
$$

we obtain:

$$
\begin{aligned}
&\sum_{i=1,m}\|u^{i,k}-u^{i,p}\|^2_{\infty,2\eta}\\
&\leq 2mC^{-1}\Big((e^{C\eta}-1)\|u^{i,k-1}-u^{i,p-1}\|^2_{\infty,2\eta}+(e^{2C\eta}-e^{C\eta})\sum_{i=1,m}\|u^{i,k-1}-u^{i,p-1}\|^2_{\infty,\eta}\Big)\\
&\leq\frac{3}{4}\sum_{i=1,m}\|u^{i,k-1}-u^{i,p-1}\|^2_{\infty,2\eta}+2mC^{-1}(e^{2C\eta}-e^{C\eta})\sum_{i=1,m}\|u^{i,k-1}-u^{i,p-1}\|^2_{\infty,\eta}.
\end{aligned}
$$

It implies that

$$
\limsup_{k,p\to\infty}\sum_{i=1,m}\|u^{i,k}-u^{i,p}\|^2_{\infty,2\eta}\leq\frac{3}{4}\limsup_{k,p\to\infty}\sum_{i=1,m}\|u^{i,k-1}-u^{i,p-1}\|^2_{\infty,2\eta}
$$

since $\limsup_{k,p\to\infty}\sum_{i=1,m}\|u^{i,k-1}-u^{i,p-1}\|^2_{\infty,\eta}=0$. Therefore

$$
\limsup_{k,p\to\infty}\sum_{i=1,m}\|u^{i,k}-u^{i,p}\|^2_{\infty,2\eta}=0.
$$

Consequently the sequence $((u^{i,k})_{i\in\mathcal{I}})_{k\geq 0}$ is uniformly convergent in $[T-2\eta, T-\eta]\times\mathbb{R}^k$. Thus, their limits, the functions $(u^i)_{i\in\mathcal{I}}$ are also continuous in $[T-2\eta, T-\eta]\times\mathbb{R}^k$, which implies that $(u^i)_{i\in\mathcal{I}}$ are continuous in $[T-2\eta, T]\times\mathbb{R}^k$. Continuing now this reasoning as many times as necessary on $[T-3\eta, T-2\eta]$,

$[T-4\eta, T-3\eta]$ etc. we obtain the continuity of $(u^i)_{i\in\mathcal{I}}$ in $[0,T]\times\mathbb{R}^k$, since η is fixed.

Step 2 : We now deal with the general case. Firstly by (H1)-iii), (H2) and (H3), there exist two constants C and $p\in\mathbb{N}$ such $f_i(t,x,0,...,0)$, $h_i(x)$ and $g_{ij}(t,x)$ are of polynomial growth, i.e., for any $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$|f_i(t,x,0,...,0)|+|h_i(x)|+|g_{ij}(t,x)|\leq C(1+|x|^p). \tag{2.4.3}$$

To proceed for $s\in[t,T]$ let us define,

$$\overline{Y}^i_s := Y^i_s\varphi(X^{t,x}_s),$$

where for $x\in\mathbb{R}$, $\varphi(x):=\frac{1}{(1+|x|^2)^p}$ (p is the same constant as in (2.4.3)). Then by the integration-by-parts formula we have:

$$\begin{aligned}
d\overline{Y}^i_s &= \varphi(X^{t,x}_s)dY^i_s + Y^i_s d\varphi(X^{t,x}_s) + d\langle Y^i, \varphi(X^{t,x})\rangle_s\\
&= \varphi(X^{t,x}_s)\{-f_i(s,X^{t,x}_s,(Y^k_s)_{k=1,m},Z^i_s)ds - dK^i_s + Z^i_s dB_s\}\\
&\quad + Y^i_s\{\mathcal{L}\varphi(X^{t,x}_s)ds + D_x\varphi(X^{t,x}_s)\sigma(s,X^{t,x}_s)dB_s\} + Z^i_s D_x\varphi(X^{t,x}_s)\sigma(s,X^{t,x}_s)ds\\
&= \{-\varphi(X^{t,x}_s)f_i(s,X^{t,x}_s,(Y^k_s)_{k=1,m},Z^i_s) + \mathcal{L}\varphi(X^{t,x}_s)Y^i_s + D_x\varphi(X^{t,x}_s)\sigma(s,X^{t,x}_s)Z^i_s\}ds\\
&\quad - \varphi(X^{t,x}_s)dK^i_s + \{Z^i_s\varphi(X^{t,x}_s) + Y^i_s D_x\varphi(X^{t,x}_s)\sigma(s,X^{t,x}_s)\}dB_s,
\end{aligned}$$

where $\mathcal{L}\varphi$ is given in (2.2.10). Next let us set, for $s\in[t,T]$,

$$d\overline{K}^i_s := \varphi(X^{t,x}_s)dK^i_s \text{ and } \overline{Z}^i_s := Z^i_s\varphi(X^{t,x}_s) + Y^i_s D_x\varphi(X^{t,x}_s)\sigma(s,X^{t,x}_s).$$

Then $((\overline{Y}^i,\overline{Z}^i,\overline{K}^i))_{i\in\mathcal{I}}$ satisfies: $\forall s\in[t,T]$,

$$\begin{cases}
\overline{Y}^i_s = \overline{h}_i(X^{t,x}_T) + \int_s^T \overline{f}_i(r,X^{t,x}_r,(\overline{Y}^k_r)_{k=1,...,m},\overline{Z}^i_r)dr + \overline{K}^i_T - \overline{K}^i_s - \int_s^T \overline{Z}^i_r dB_r,\\
\overline{Y}^i_s \geqslant \max\limits_{j\in\mathcal{I}^{-i}}(\overline{Y}^j_s - \overline{g}_{ij}(s,X^{t,x}_s)),\\
\int_0^T(\overline{Y}^i_s - \max\limits_{j\in\mathcal{I}^{-i}}(\overline{Y}^j_s - \overline{g}_{ij}(s,X^{t,x}_s)))d\overline{K}^i_s = 0,
\end{cases} \tag{2.4.4}$$

where for any $i,j\in\mathcal{I}$,

$$\overline{h}_i(X^{t,x}_T) := h_i(X^{t,x}_T)\varphi(X^{t,x}_T),\ \overline{g}_{ij}(s,X^{t,x}_s) := g_{ij}(s,X^{t,x}_s)\varphi(X^{t,x}_s),$$

and

$$\begin{aligned}\overline{f}_i(s,x,\vec{y},z) &:= \varphi(x)f_i(s,x,\varphi^{-1}(x)\vec{y},\varphi^{-1}(x)z - D_x\varphi(x)\sigma(s,x)\varphi^{-1}(x)y^i)) \\ &- \mathcal{L}\varphi(x)\varphi^{-1}(x)y^i - D_x\varphi(x)\sigma(s,x)\varphi^{-1}(x)\{z - D_x\varphi(x)\sigma(s,x)\varphi^{-1}(x)y^i\}; \\ &(\varphi^{-1}(x) = (1+|x|^2)^p).\end{aligned}$$

Here let us notice that the functions $\overline{f}_i(t,x,\vec{0},0)$, $\overline{g}_{ij}$ and $\overline{h}_i$ are bounded. Then by the result of the first step, there exists bounded continuous functions $(\bar{u}^i)_{i\in\mathcal{I}}$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, and $s\in[t,T]$, $\bar{Y}^i_s = \bar{u}^i(s,X^{t,x}_s)$, $\forall i\in\mathcal{I}$. Thus for any $(t,x)\in[0,T]\times\mathbb{R}^k$, and $s\in[t,T]$, $Y^i_s = \varphi^{-1}(X^{t,x}_s)\bar{u}^i(s,X^{t,x}_s)$, $\forall i\in\mathcal{I}$. Then it is enough to take $u^i(t,x) := \varphi^{-1}(x)\bar{u}^i(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}^k$ and $i\in\mathcal{I}$, which are continuous functions and of polynomial growth. □

We are now ready to give the main result of this paper. Let $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$ be the unique solution of (2.3.1) and let $(u^i)_{i\in\mathcal{I}}$ be the continuous functions with polynomial growth such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$ and $i\in\mathcal{I}$,

$$\mathbb{P}-a.s.,\ \forall\, s\in[t,T],\ Y^{i,t,x}_s = u^i(s,X^{t,x}_s).$$

We then have:

Theorem 2.4.2 *The function $(u^i)_{i\in\mathcal{I}}$ is a solution in viscosity sense of system (2.2.9). Moreover it is unique in the class of continuous functions of polynomial growth.*

Proof: First let us show that $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of system (2.2.9).

Recall that $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$ is a solution of the system of reflected BSDEs with interconnected obstacles (2.3.1) and for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $i\in\mathcal{I}$ and $s\in[t,T]$, $Y^{i,t,x}_s = u^i(s,X^{t,x}_s)$. Then $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$ verify: for any $s\in[t,T]$ and $i\in\mathcal{I}$,

$$\begin{cases} Y^{i,t,x}_s = h_i(X^{t,x}_T) + \int_s^T f_i(r,X^{t,x}_r,(u^k(r,X^{t,x}_r))_{k\in\mathcal{I}},Z^{i,t,x}_r)dr + K^{i,t,x}_T - K^{i,t,x}_s \\ \qquad\qquad - \int_s^T Z^{i,t,x}_r dB_r, \\ Y^{i,t,x}_s \geqslant \max\limits_{j\in\mathcal{I}^{-i}}(u^j(s,X^{t,x}_s) - g_{ij}(s,X^{t,x}_s)), \\ \int_0^T (Y^{i,t,x}_r - \max\limits_{j\in\mathcal{I}^{-i}}(u^j(r,X^{t,x}_r) - g_{ij}(r,X^{t,x}_r)))dK^{i,t,x}_r = 0. \end{cases} \tag{2.4.5}$$

But system (2.4.5) is decoupled and using a result by El-Karoui et al. (Theorem 8.5 in [28]) one obtains that, for any i_0, u^{i_0} is a solution in viscosity sense of the following PDE with obstacle:

$$\begin{cases} \min\{u^{i_0}(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); \\ \qquad -\partial_t u^{i_0}(t,x) - \mathcal{L}u^{i_0}(t,x) - f_{i_0}(t,x,(u^k(t,x))_{k=1,,m},(\sigma^\top D_x u^{i_0})(t,x))\} = 0; \\ \\ u^{i_0}(T,x) = h_i(x). \end{cases} \tag{2.4.6}$$

As i_0 is arbitrary in $\mathcal{I}$, then the functions $(u^i)_{i\in\mathcal{I}}$ is a solution in viscosity sense of (2.2.9).

Next let us show that $(u^i)_{i\in\mathcal{I}}$ is the unique solution in the class of continuous functions with polynomial growth. It is based on the uniqueness of the solution of the system of reflected BSDEs with interconnected obstacles (2.3.1).

So suppose that there exists another continuous with polynomial growth solution $(\tilde{u}^i)_{i=1,\dots,m}$ of (2.2.9), i.e., for any $i \in \mathcal{I}$,

$$\begin{cases} \min\{\tilde{u}^i(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}(\tilde{u}^j(t,x) - g_{ij}(t,x)); \\ \qquad -\partial_t \tilde{u}^i(t,x) - \mathcal{L}\tilde{u}^i(t,x) - f_i(t,x,(\tilde{u}^k(t,x))_{k=1,\dots,m},(\sigma^\top D_x \tilde{u}^i)(t,x))\} = 0\ ; \\ \tilde{u}^i(T,x) = h_i(x). \end{cases} \tag{2.4.7}$$

Let $(\tilde{Y}^i)_{i\in\mathcal{I}} \in \mathcal{H}^{2,m}$ be such that for any $i \in \mathcal{I}$ and $s \in [t,T]$,

$$\tilde{Y}_s^{i,t,x} = \tilde{u}^i(s, X_s^{t,x}).$$

Next let us define $(\overline{Y}^{i,t,x})_{i\in\mathcal{I}}$ as follows:

$$(\overline{Y}^{i,t,x})_{i\in\mathcal{I}} = \Theta((\tilde{Y}_s^{i,t,x})_{i\in\mathcal{I}}), \tag{2.4.8}$$

that is to say, $(\overline{Y}^{i,t,x}, \overline{Z}^{i,t,x}, \overline{K}^{i,t,x})_{i\in\mathcal{I}}$ is the solution of the following system of

reflected BSDEs with oblique reflection: $\forall s \in [t,T]$, $\forall i \in \mathcal{I}$

$$\begin{cases} \overline{Y}^{i,t,x} \in \mathcal{S}^2,\ \overline{Z}^{i,t,x} \in \mathcal{H}^{2,d} \text{ and } \overline{K}^{i,t,x} \in \mathcal{A}^2; \\ \\ \overline{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, (\tilde{u}^k(s, X_s^{t,x})_{k=1,m}, \overline{Z}_r^{i,t,x})dr + \overline{K}_T^{i,t,x} - \overline{K}_s^{i,t,x} \\ \qquad\qquad - \int_s^T \overline{Z}_r^{i,t,x} dB_r, \\ \overline{Y}_s^{i,t,x} \geqslant \max_{j \in \mathcal{I}^{-i}} (\overline{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_t^T (\overline{Y}_s^{i,t,x} - \max_{j \in \mathcal{I}^{-i}} (\overline{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))d\overline{K}_s^{i,t,x} = 0. \end{cases} \tag{2.4.9}$$

As the deterministic functions $(\tilde{u}^i)_{i=1,...,m}$ are continuous and of polynomial growth, then by using a result by Hamadène-Morlais ([38], Theorem 1), one can infer the existence of deterministic continuous functions with polynomial growth $(v^i)_{i=1,...,m}$ such that: $\forall i \in \mathcal{I}$ and $s \in [t,T]$,

$$\overline{Y}_s^{i,t,x} = v^i(s, X_s^{t,x}).$$

Moreover, $(v^i)_{i=1,...,m}$ is the unique viscosity solution (in the class of functions with polynomial growth) of the following system of PDEs with interconnected obstacles : $\forall i = 1, ..., m$

$$\begin{cases} \min\{v^i(t,x) - \max_{j \in \mathcal{I}^{-i}} (v^j(t,x) - g_{ij}(t,x)); \\ \qquad -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) - f_i(t,x,(\tilde{u}^k(t,x))_{k=1,,...,m}, (\sigma^\top D_x v^i)(t,x))\} = 0\ ; \\ \\ v^i(T,x) = h_i(x). \end{cases} \tag{2.4.10}$$

Let us notice that in system (2.4.10), in the arguments of f_i we have $\tilde{u}^k$ and not v^k. On the other hand the functions $(t,x) \mapsto f_i(t,x,(\tilde{u}^k(t,x))_{k=1,,...,m}, z)$, $i \in \mathcal{I}$, are continuous uniformly w.r.t z, i.e., they satisfy (H1-(i)). This property is needed in order to use the results of [38]).

Now as the functions $(\tilde{u}^i)_{i=1,...,m}$ solve system (2.4.10), hence by uniqueness of the solution of this system (2.4.10) (see [38], Thm. 1, pp.175), one deduces that

$$v^i = \tilde{u}^i \ \text{ and then } \tilde{Y}^{i,t,x} = \overline{Y}^{i,t,x},\ \forall i \in \mathcal{I}.$$

Therefore $(\tilde{Y}_s^{i,t,x})_{i\in\mathcal{I}}$ verify

$$(\tilde{Y}^{i,t,x})_{i\in\mathcal{I}} = \Theta((\tilde{Y}_s^{i,t,x})_{i\in\mathcal{I}}).$$

But $(Y^i)_{i\in\mathcal{I}}$ is the unique fixed point of Θ in $(\mathcal{H}^{2,m}, \|.\|_{\alpha_0})$ then we have that for any $s \in [t,T]$ and $i \in \mathcal{I}$, $\tilde{Y}_s^{i,t,x} = Y_s^{i,t,x}$. Henceforth, in taking $s = t$, we obtain that for any $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, $\tilde{u}^i(t,x) = u^i(t,x)$. Thus $(u^i)_{i=1,\dots,m}$ is the unique solution of system (2.4.10) in the class of continuous functions with polynomial growth. □

Chapter 3

Viscosity solution of system of integral-partial differential equations with interconnected obstacles of non-local type without Monotonicity Conditions

The content of this chapter is from an article in collaboration with Saïd Hamadène and Mohamed Mnif [37], published in Journal of Dynamics and Differential Equations.

3.1 Introduction

Let us consider the following system of integral-partial differential equations (IPDEs for short) with interconnected obstacles with non-local terms:
$\forall i \in \mathcal{I} := \{1, ..., m\}$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) - \mathcal{K}u^i(t,x) \\ \quad -\bar{f}_i(t,x,(u^k(t,x))_{k=1,m}, (\sigma^\top D_x u^i)(t,x), B_i u^i(t,x))\} = 0, \ (t,x) \in [0,T]\times\mathbb{R}^k; \\ u^i(T,x) = h_i(x), \end{cases} \tag{3.1.1}$$

where $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$ for any $i \in \mathcal{I}$ and the operators $\mathcal{L}$, $\mathcal{K}$ and B_i are defined as follows: For any $i \in \mathcal{I}$,

$$\begin{aligned}
&\mathcal{L}u^i(t,x) := b(t,x)^\top D_x u^i(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x)D^2_{xx}u^i(t,x)],\\
&\mathcal{K}u^i(t,x) := \int_E (u^i(t,x+\beta(x,e)) - u^i(t,x) - \beta(x,e)^\top D_x u^i(t,x))\lambda(de) \quad \text{and}\\
&B_i u^i(t,x) := \int_E \gamma_i(x,e)(u^i(t,x+\beta(x,e)) - u^i(t,x))\lambda(de).
\end{aligned} \tag{3.1.2}$$

In the above, $D_x u^i$ and $D^2_{xx}u^i$ are the gradient and Hessian matrix of u^i with respect to its second variable x, respectively; $(.)^\top$ is the transpose and $\lambda(.)$ is a finite Lévy measure on $E := \mathbb{R}^l - \{0\}$.

We note that, due to the presence of $B_i u^i$ and Ku^i in equation (3.1.1), such an IPDE is called of non-local type. The non-local setting has been studied by several authors (see e.g. [3], [5], [34], [39], [45], [55]). Actually, in [45], Hamadène-Zhao have shown that, if for any $i \in \mathcal{I}$,

(i) $\gamma_i \geqslant 0$;

(ii) $q \in \mathbb{R} \mapsto \bar{f}_i(t,x,(y_k)_{k=1,m},z,q)$ is non-decreasing, when the other components (t,x,y,z) are fixed;

then, there exist functions $(u^i)_{i\in\mathcal{I}}$ unique continuous viscosity solution of system (3.1.1) in the class of functions with polynomial growth. Conditions (i)-(ii), which will be referred as the monotonicity conditions, are needed in order to have the comparison result and to treat the operator $B_i u^i$ which is not well-defined for an arbitrary u^i. The above monotonicity conditions are classically assumed in the literature of viscosity solution for equation with a non local term. Therefore, without assuming the conditions neither on γ_i nor on $\bar{f}_i$, $i = 1, ..., m$, the problem of existence and uniqueness of the viscosity solution of system (3.1.1) remains open. To deal with this problem is the main objective of this paper.

A special case of this type of system of IPDEs with interconnected obstacles occurs in the context of optimal switching control problems when the dynamics of the state variables are described by a jump diffusion process $(X^{t,x}_s)_{s\leq T}$

solving the following stochastic differential equation:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s + \int_E \beta(X_{s^-}^{t,x}, e)\tilde{\mu}(ds, de), \ \ s \in [t, T]; \\ X_s^{t,x} = x \in \mathbb{R}^k, \ \ s \leq t, \end{cases} \tag{3.1.3}$$

where $B := (B_s)_{s \leq T}$ is a d-dimensional Brownian motion, μ an independent Poisson random measure with compensator $ds\lambda(de)$ and $\tilde{\mu}(ds, de) := \mu(ds, de) - ds\lambda(de)$ its compensated random measure.

In this setting, if for any $i \in \mathcal{I}$, $\bar{f}_i$ does not depend on $(u^k)_{k=1,m}$, $D_x u^i$ and $B_i u^i$ (see e.g. [44]), the IPDEs (3.1.1) reduce to the Hamilton-Jacobi-Bellman system associated with the switching control problem whose value function is defined by: $\forall i \in \mathcal{I}$ and $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$u^i(t,x) = \sup_{\delta := (\theta_k, \alpha_k)_{k \geq 0}} \mathbb{E}\Big[\int_t^T \bar{f}^\delta(s, X_s^{t,x})ds \quad - \sum_{k \geq 1} g_{\alpha_{k-1}\alpha_k}(\theta_k, X_{\theta_k}^{t,x}) \mathbf{1}_{\{\theta_k < T\}} + h^\delta(X_T^{t,x}) \Big],$$

where :

(a) $\delta := (\theta_k, \alpha_k)_{k \geq 0}$ is a strategy of switching in which $(\theta_k)_{k \geq 0}$ is an increasing sequence of stopping times and $(\alpha_k)_{k \geq 0}$ is a sequence of random variables with values in $\{1, ..., m\}$ ($\theta_0 = t$ and $\alpha_0 = i$);

(b) $\bar{f}^\delta(s, X_s^{t,x})$ is the instantaneous payoff when the strategy δ is implemented on the system under switching, and $h^\delta(X_T^{t,x})$ is the terminal payoff ;

(c) g_{ij} is the switching cost function when moving from mode i to mode j ($i, j \in \mathcal{I}$, $i \neq j$).

The main tool to tackle system (3.1.1) is to deal with the following system of reflected backward stochastic differential equations (RBSDEs for short) with jumps and interconnected obstacles: $\forall i \in \mathcal{I}$ and $s \in [t, T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k \in \mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad + K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,t,x} \geqslant \max_{j \in \mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})); \\ \int_t^T (Y_s^{i,t,x} - \max_{j \in \mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,t,x} = 0. \end{cases} \tag{3.1.4}$$

Note that, without the jump process, the system of RBSDEs with oblique reflection has been investigated in several papers including ([13, 43, 38, 46], etc.). With the presence of the jump process, Hamadène-Zhao in [45], have proved, under conditions (i)-(ii) on γ_i and $\bar{f}_i$, $i = 1, ..., m$, the existence and uniqueness of the solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ of RBSDEs (3.1.4). Moreover, they have made the link between this RBSDEs and the IPDEs (3.1.1) through the Feynman-Kac representation, i.e., for any $(t, x) \in [0, T] \times \mathbb{R}^k$ and $i \in \mathcal{I}$,

$$u^i(t, x) = Y_t^{i,t,x} \text{ and } Y_s^{i,t,x} = u^i(s, X_s^{t,x}), \forall s \in [t, T]. \tag{3.1.5}$$

Therefore, in the first part of this paper, the main issue is to deal with RBSDEs (3.1.4) without assuming the two points (i)-(ii) mentioned above. Actually we show that when the measure $\lambda(.)$ is finite, the system of RBSDEs (3.1.4) has a solution which is unique among the Markovian solutions, that is to say, which have the representation (3.1.5). Our method relies mainly on the characterization of the jump part of the RBSDEs (3.1.4) by means of the functions $(u^i)_{i=1,m}$ defined in (3.1.5) and the jump-diffusion process $X^{t,x}$. In the second part, we deal with the problem of existence and uniqueness in viscosity sense of the solution of system (3.1.1). We show that the functions $(u^i)_{i=1,m}$ defined in (3.1.5), through the unique solution of (3.1.4), is the unique solution of system (3.1.1).

The paper is organized as follows. In Section 2, we provide all the necessary notations and assumptions concerning the study of IPDEs (3.1.1) and related RBSDEs with jumps as well. In Section 3, we study the existence of a solution for system of RBSDEs with jumps (3.1.4) and Feynman-Kac representation (3.1.5). Actually we introduce an approximating scheme (system (3.3.2) below) which we show that it converges to the solution of system (3.1.4) when the functions h_i and $\bar{f}_i(t, x, \vec{0}, 0, 0)$, $i \in \mathcal{I}$, are bounded. On the other hand, the Feynman-Kac representation (3.1.5) holds true. Later on, by a weighting technique, we get rid of those latter boundedness conditions. Finally we show that the Markovian solution of (3.1.4) is unique. At the end of the paper, in Section 4, we prove that the functions $(u^i)_{i=1,m}$ are the unique viscosity so-

lution of (3.1.1) in the class of continuous functions with polynomial growth. In the Appendix, we give another definition of the viscosity solution of system (3.1.1) which is inspired by the work by Hamadène-Zhao in [45]. □

3.2 Preliminaries and notations

Let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ be a stochastic basis such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, $\mathcal{F}_{t^+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, and we suppose that the filtration is generated by the two following mutually independent processes :

(i) a d-dimensional standard Brownian motion $B := (B_t)_{0\leq t\leq T}$ and

(ii) a Poisson random measure μ on $\mathbb{R}^+ \times E$, where $E := \mathbb{R}^l - \{0\}$ is equipped with its Borel field $\mathcal{B}(E), (l \geqslant 1$ fixed). Let $\nu(dt, de) = dt\lambda(de)$ be its compensated process, i.e., $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t\leq T}$ is a martingale for every $A \in \mathcal{B}(E)$ satisfying $\lambda(A) < \infty$. Throughout this paper the measure $\lambda(.)$ is assumed to be finite on $(E, \mathcal{B}(E))$. An example when $l = 1$ is $\lambda(de) = (|e|^{-\theta}1_{\{|e|\leq 1\}} + |e|^{-\rho}1_{\{|e|\geq 1\}})de$ with $\theta \in (0,1)$ and $\rho > 1$.

Let us now introduce the following spaces:

a) $\mathcal{P}$ (resp. $\mathbf{P}$) is the σ-algebra of $\mathbb{F}$-progressively measurable (resp. $\mathbb{F}$-predictable) sets on $\Omega \times [0,T]$;

b) $\mathcal{L}^2(\lambda)$ is the space of Borel measurable functions $(\varphi(e))_{e\in E}$ from E into $\mathbb{R}$ such that $\int_E |\varphi(e)|^2\lambda(de) < \infty$;

c) $\mathcal{S}^2$ is the space of RCLL (right continuous with left limits) $\mathcal{P}$-measurable and $\mathbb{R}$-valued processes $Y := (Y_s)_{s\leq T}$ such that $\mathbb{E}\big[\sup_{0\leq t\leq T} |Y_s|^2\big] < \infty$;

d) $\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of continuous non-decreasing processes $K := (K_t)_{t\leq T}$ such that $K_0 = 0$;

e) $\mathcal{H}^{2,d}$ is the space of $\mathcal{P}$-measurable and $\mathbb{R}^d$-valued processes $Z := (Z_s)_{s\leq T}$ such that $\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty$;

f) $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ is the space of $\mathbf{P}$-measurable and $\mathcal{L}^2(\lambda)$-valued processes $U := (U_s)_{s\leq T}$ such that
$\mathbb{E}\left[\int_0^T \int_E |U_s(e)|^2 \lambda(de)ds\right] < \infty$.

For a RCLL process $(\theta_s)_{s\leq T}$, we define for any $s \in (0,T]$, $\theta_{s^-} = \lim_{r\nearrow s} \theta_r$ and $\Delta_s\theta = \theta_s - \theta_{s^-}$ is the jump size of θ at s.

Now, for any $(t,x) \in [0,T]\times\mathbb{R}^k$, let $(X_s^{t,x})_{s\leq T}$ be the stochastic process solution of the following stochastic differential equation (SDE for short) of diffusion-jump type:

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dB_s + \int_E \beta(X_{s^-}^{t,x},e)\tilde{\mu}(ds,de), \quad s\in[t,T] \\ X_s^{t,x} = x \in \mathbb{R}^k, \quad 0\leq s\leq t \end{cases} \tag{3.2.1}$$

where $b : [0,T]\times\mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : [0,T]\times\mathbb{R}^k \to \mathbb{R}^{k\times d}$ are two continuous functions in (t,x) and Lipschitz w.r.t x, i.e., there exists a positive constant C such that

$$|b(t,x)-b(t,x')| + |\sigma(t,x)-\sigma(t,x')| \leq C|x-x'|, \ \ \forall (t,x,x') \in [0,T]\times\mathbb{R}^{k+k}. \tag{3.2.2}$$

Note that the continuity of b, σ and (3.2.2) imply the existence of a constant C such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|), \ \ \forall (t,x) \in [0,T]\times\mathbb{R}^k. \tag{3.2.3}$$

Next, let $\beta : \mathbb{R}^k \times E \to \mathbb{R}^k$ be a measurable function such that for some real constant c,

$$\begin{aligned} &|\beta(x,e)| \leq c(1\wedge|e|) \quad \text{and} \\ &|\beta(x,e)-\beta(x',e)| \leq c|x-x'|(1\wedge|e|), \quad \forall e\in E \text{ and } x,x'\in\mathbb{R}^k \end{aligned} \tag{3.2.4}$$

Conditions (3.2.2), (3.2.3) and (3.2.4) ensure, for any $(t,x) \in [0,T]\times\mathbb{R}^k$, the existence and uniqueness of a solution of equation (3.2.1) (see [32] for more details). Moreover, it satisfies the following estimate: $\forall p \geqslant 1$,

$$\mathbb{E}[\sup_{s\leq T} |X_s^{t,x}|^p] \leqslant C(1+|x|^p). \tag{3.2.5}$$

Next, let us introduce the following deterministic functions $(\bar{f}_i)_{i\in\mathcal{I}}$, $(h_i)_{i\in\mathcal{I}}$ and $(g_{ij})_{i,j\in\mathcal{I}}$ defined as follows : for any $i,j\in\mathcal{I}$,

a) $\bar{f}_i : (t,x,\vec{y},z,q)\in[0,T]\times\mathbb{R}^{k+m+d+1}\longmapsto\bar{f}_i(t,x,\vec{y},z,q)\in\mathbb{R}$ $(\vec{y}:=(y^1,...,y^m))$;

b) $g_{ij} : (t,x)\in[0,T]\times\mathbb{R}^k\longmapsto g_{ij}(t,x)\in\mathbb{R}$;

c) $h_i : x\in\mathbb{R}^k\longmapsto h_i(x)\in\mathbb{R}$.

Additionally we assume that they satisfy:

(H1) For any $i\in\{1,...,m\}$,

(i) The function $(t,x)\mapsto\bar{f}_i(t,x,\vec{y},z,q)$ is continuous, uniformly w.r.t. the variables $(\vec{y},z,q)$.

(ii) The function $\bar{f}_i$ is Lipschitz continuous w.r.t. the variables $(\vec{y},z,q)$ uniformly in (t,x), i.e., there exists a positive constant C_i such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $(\vec{y},z,q)$ and $(\vec{y}_1,z_1,q_1)$ elements of $\mathbb{R}^{m+d+1}$:

$$|\bar{f}_i(t,x,\vec{y},z,q)-\bar{f}_i(t,x,\vec{y}_1,z_1,q_1)|\leq C_i(|\vec{y}-\vec{y}_1|+|z-z_1|+|q-q_1|). \tag{3.2.6}$$

(iii) The mapping $(t,x)\mapsto\bar{f}_i(t,x,0,0,0)$ has polynomial growth in x, i.e., there exist two constants $C>0$ and $p\geqslant 1$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$|\bar{f}_i(t,x,0,0,0)|\leq C(1+|x|^p). \tag{3.2.7}$$

(iv) For any $i\in\mathcal{I}$ and $j\in\mathcal{I}^{-i}$,
the mapping $y^j\mapsto\bar{f}_i(t,x,y^1,...,y^{j-1},y^j,y^{j+1},...,y^m,z,q)$ is non-decreasing whenever the components $(t,x,y^1,...,y^{j-1},y^{j+1},...,y^m,z,q)$ are fixed.

(v) Let $\gamma_i : \mathbb{R}^k \times E \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(E)$-measurable functions such that for some constant $C > 0$:

$$|\gamma_i(x,e)| \leq C(1 \wedge |e|), \quad \forall (x,e) \in \mathbb{R}^k \times E. \tag{3.2.8}$$

Finally let us define the function $(f_i)_{i=1,...,m}$ on $[0,T] \times \mathbb{R}^{k+m+d} \times \mathcal{L}^2(\lambda)$, as follows:

$$f_i(t,x,\vec{y},z,v) := \bar{f}_i(t,x,\vec{y},z,\textstyle\int_E v(e)\gamma_i(x,e)\lambda(de)). \tag{3.2.9}$$

Note that since $\bar{f}_i$ is uniformly Lipschitz in $(\vec{y},z,q)$ and γ_i verifies (3.2.8) then the function f_i enjoy the two following properties:

(a) f_i is Lipschitz continuous w.r.t. the variables $(\vec{y},z,v)$ uniformly in (t,x),

(b) The mapping $(t,x) \mapsto f_i(t,x,\vec{0},0,0) = \bar{f}_i(t,x,\vec{0},0,0)$ is continuous with polynomial growth.

(H2) $\forall i,j \in \{1,...,m\}$, $g_{ii} = 0$ and for $i \neq j$, $g_{ij}(t,x)$ is non-negative, jointly continuous in (t,x) with polynomial growth and satisfies the following non free loop property :

For any $(t,x) \in [0,T] \times \mathbb{R}^k$, for any sequence of indices $i_1,...,i_k$ such that $i_1 = i_k$ and $card\{i_1,...,i_k\} = k-1$ $(k \geq 3)$ we have

$$g_{i_1 i_2}(t,x) + g_{i_2 i_3}(t,x) + ... + g_{i_{k-1} i_1}(t,x) > 0. \tag{3.2.10}$$

(H3) For $i \in \{1,...,m\}$, the function h_i, which stands for the terminal condition, is continuous with polynomial growth and satisfies the following consistency condition:

$$\forall x \in \mathbb{R}^k, \ h_i(x) \geqslant \max_{j \in \mathcal{I}^{-i}} (h_j(x) - g_{ij}(T,x)). \tag{3.2.11}$$

(H4)-(i) $\forall i \in \mathcal{I}$, $\gamma_i \geqslant 0$;

(H4)-(ii) The mapping $q \in \mathbb{R} \longmapsto \bar{f}_i(t,x,\vec{y},z,q)$ is non-decreasing when the other components $(t,x,\vec{y},z)$ are fixed. □

Remark 3.2.1 *The condition (H1)-i) is needed, e.g. in [39] or [45] in order to apply Ishii's Lemma to show comparison in the systems considered in those papers and then to deduce uniqueness and continuity of the viscosity solution. However instead of requiring* $(H1)-i)$ *it is enough to require other sufficient conditions which make comparison of sub. and super-solutions hold. If* $\bar{f}_i$, $i \in \mathcal{I}$, *do not depend on* q *it is enough to require the following conditions:*

(a) For any $i \in \mathcal{I}$, $\bar{f}_i$ *is jointly continuous in* $(t, x, \vec{y}, z)$;
(b) For any $R > 0$, *there exists a continuous function* m_R *from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$ *such that* $m_R(0) = 0$ *and for any* $|x| \leq R$, $|x'| \leq R$ *and* $|\vec{y}| \leq R$ *we have,*

$$|\bar{f}_i(t, x, \vec{y}, z) - \bar{f}_i(t, x', \vec{y}, z)| \leq m_R(|x - x'|(1 + |z|)). \tag{3.2.12}$$

One can see e.g. the paper by El-Karoui et al. [28] on this latter condition. In the case when $(\bar{f}_i)_{i\in\mathcal{I}}$ *depend on* q, *similar results exist (one can see e.g. [3] for more details).*

The main objective of this paper is to study the following system of integral-partial differential equations (IPDEs) with interconnected obstacles: for any $i \in \mathcal{I} := \{1, ..., m\}$,

$$\begin{cases} \min\{u^i(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t u^i(t,x) - \mathcal{L}u^i(t,x) - \mathcal{K}u^i(t,x) \\ \qquad -\bar{f}_i(t,x,(u^k(t,x))_{k=1,m}, (\sigma^T D_x u^i)(t,x), B_i u^i(t,x))\} = 0; \\ u^i(T,x) = h_i(x), \end{cases} \tag{3.2.13}$$

where $\mathcal{L}$ is the second-order local operator

$$\mathcal{L}\varphi(t,x) := b(t,x)^\top D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t,x)D^2_{xx}\varphi(t,x)]; \tag{3.2.14}$$

and the two non-local operators $\mathcal{K}$ and B_i, $i \in \mathcal{I}$, are defined as follows

$$\begin{aligned} &K\varphi(t,x) := \textstyle\int_E(\varphi(t,x+\beta(x,e)) - \varphi(t,x) - \beta(x,e)^\top D_x\varphi(t,x))\lambda(de) \quad \text{and} \\ &B_i\varphi(t,x) := \textstyle\int_E \gamma_i(x,e)(\varphi(t,x+\beta(x,e)) - \varphi(t,x))\lambda(de). \end{aligned} \tag{3.2.15}$$

for any $\mathbb{R}$-valued function $\varphi(t,x)$ such that $D_x\varphi(t,x)$ and $D^2_{xx}\varphi(t,x)$ are defined.

3.3 Systems of Reflected BSDEs with Jumps with Oblique Reflection

The system of IPDEs (3.2.13) is deeply related to the following system of reflected BSDEs with jumps with interconnected obstacles (or oblique reflection) associated with $((\bar{f}_i)_{i\in\mathcal{I}}, (g_{ij})_{i,j\in\mathcal{I}}, (h_i)_{i\in\mathcal{I}})$:
$\forall i = 1, ..., m$ and $s \in [0, T]$,

$$\begin{cases} Y^{i,t,x} \in \mathcal{S}^2, Z^{i,t,x} \in \mathcal{H}^{2,d}, V^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } K^{i,t,x} \in \mathcal{A}^2; \\ Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,t,x} = 0. \end{cases} \tag{3.3.1}$$

This system of reflected BSDEs with jumps with interconnected obstacles (3.3.1) has been considered by Hamadène and Zhao in [45] where issues of existence and uniqueness of the solution, and the relationship between the solution of (3.3.1) and the one of system (3.2.13), are considered. Actually in [45], it is shown:

Theorem 3.3.1 *(see [45]).*
Assume that the deterministic functions $(\bar{f}_i)_{i\in\mathcal{I}}, (g_{ij})_{i,j\in\mathcal{I}}, (h_i)_{i\in\mathcal{I}}$ and $(\gamma)_{i\in\mathcal{I}}$ verify Assumptions (H1)-(H3) and (H4) holds. Then, we have:

i) The system (3.3.1) has a unique solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$.

ii) There exists a deterministic continuous functions $(u^i)_{i\in\mathcal{I}}$ of polynomial growth, defined on $[0, T] \times \mathbb{R}^k$, such that:

$$\forall s \in [t, T],\ Y_s^{i,t,x} = u^i(s, X_s^{t,x}).$$

In our setting, we also consider the system (3.3.1) but without assuming Assumption (H4). We then have the following first result (as an intermediary step).

Proposition 3.3.2 *Assume that:*

(i) The functions $(\bar{f}_i)_{i\in\mathcal{I}}, (g_{ij})_{i,j\in\mathcal{I}}$ *and* $(h_i)_{i\in\mathcal{I}}$ *verify Assumptions (H1)-(H3).*

(ii) There exist a constant $\bar{C}$ *such that, for any* $i \in \mathcal{I}$,

$$|h_i(x)| + |\bar{f}_i(t, x, \vec{0}, 0, 0)| \leq \bar{C}.$$

Then the system (3.3.1) has a solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. *Moreover there exist bounded continuous functions* $(u^i)_{i\in\mathcal{I}}$ *such that for any* $i \in \mathcal{I}$, $(t, x) \in [0, T] \times \mathbb{R}^k$,

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ \forall s \in [t, T].$$

Proof: The proof is divided into four steps.

Step 1: The iterative construction

For any $n \geq 0$, let $(Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{i\in\mathcal{I}}$ be the sequence of processes defined recursively as follows:

$$(Y^{i,0}, Z^{i,0}, V^{i,0}, K^{i,0}) = (0, 0, 0, 0) \quad \text{for all } i \in \mathcal{I}, \quad \text{for } n \geq 1 \text{ and } s \leq T,$$

$$\begin{cases} Y^{i,n} \in \mathcal{S}^2, Z^{i,n} \in \mathcal{H}^{2,d}, V^{i,n} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } K^{i,n} \in \mathcal{A}^2; \\ Y_s^{i,n} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, \int_E V_r^{i,n-1}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,n} - K_s^{i,n} - \int_s^T Z_r^{i,n} dB_r - \int_s^T \int_E V_r^{i,n}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,n} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T (Y_s^{i,n} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n} - g_{ij}(s, X_s^{t,x})))dK_s^{i,n} = 0. \end{cases} \tag{3.3.2}$$

First we notice that by Theorem (3.3.1), the solution of this system (3.3.2) exists and is unique. More precisely, for any $i \in \mathcal{I}$, the generators $\bar{f}_i$ do not depend on $V^{i,n}$, noting that $V^{i,n-1}$ is already given. The functions $(h_i)_{i\in\mathcal{I}}$ and $(g_{ij})_{i,j\in\mathcal{I}}$ satisfy the Assumptions (H2)-(H3) as well. Next, since the setting is Markovian and using an induction argument on n, there exist deterministic

continuous functions of polynomial growth $u^{i,n} : [0,T] \times \mathbb{R}^k \to \mathbb{R}$, such that for any $s \in [t,T]$:

$$\begin{aligned} &\text{(a)}\ \ Y_s^{i,n} := u^{i,n}(s, X_s^{t,x}) \text{ and} \\ &\text{(b)}\ \ V_s^{i,n}(e) := u^{i,n}(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^{i,n}(s, X_{s^-}^{t,x}), \\ &\qquad\qquad ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [t,T] \times \Omega \times E. \end{aligned} \tag{3.3.3}$$

Indeed, for $n = 0$, the representations (a), (b) are valid with $u^{i,0} = 0$, $i \in \mathcal{I}$. Assume now that they are satisfied for some $n-1$, with $n \geq 1$. Then $(Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})$ verifies: for any $s \in [t,T]$ and $i \in \mathcal{I}$,

$$\begin{cases} Y_s^{i,n} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, \int_E \{u^{i,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) \\ \qquad\qquad -u^{i,n-1}(r, X_{r^-}^{t,x})\}\gamma_i(X_r^{t,x}, e)\lambda(de))dr + K_T^{i,n} - K_s^{i,n} \\ \qquad\qquad - \int_s^T Z_r^{i,n} dB_r - \int_s^T \int_E V_r^{i,n}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,n} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n} - g_{ij}(s, X_s^{t,x})); \\ \int_t^T (Y_s^{i,n} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n} - g_{ij}(s, X_s^{t,x})))dK_s^{i,n} = 0. \end{cases}$$

Hence, by Proposition 4.2 in [45], we deduce the existence of $u^{i,n}$ which is continuous and of polynomial growth. Finally as the measure λ is finite, i.e., $\lambda(E) < \infty$, then we have the following relationship between the process $(V^{i,n})_{i\in\mathcal{I}}$ and the deterministic functions $(u^{i,n})_{i\in\mathcal{I}}$ (see [39], Proposition 3.3):

$$V_s^{i,n}(e) = u^{i,n}(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^{i,n}(s, X_{s^-}^{t,x}),\ ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [t,T] \times \Omega \times E. \tag{3.3.4}$$

Thus, the two representations (a) and (b) hold true for any $n \geq 0$.

Remark 3.3.3 *For $s \in [0,t]$, $X_s^{t,x} = x$ and $Y_t^{i,n} = u^{i,n}(t,x)$, therefore in considering the declination of system (3.3.2) on the time interval $[0,t]$, we can easily show by induction that $Z_s^{i,n} 1_{[s\leq t]} = 0$, $ds \otimes d\mathbb{P} - a.e$ and $V_s^{i,n}(e) 1_{[s\leq t]} = 0$, $ds \otimes d\mathbb{P} \otimes d\lambda - a.e.$ since the data are continuous and deterministic.*

Step 2: Switching representation

In this step, we represent $Y^{i,n}$ as the value of an optimal switching problem. Indeed, let $\delta := (\theta_k, \alpha_k)_{k\geq 0}$ be an admissible strategy of switching, i.e., $(\theta_k)_{k\geq 0}$

is an increasing sequence of stopping times with values in $[0,T]$ such that $\mathbb{P}[\theta_k < T, \forall k \geq 0] = 0$ and $\forall k \geq 0$, α_k is a random variable $\mathcal{F}_{\theta_k}$-measurable with values in $\mathcal{I}$.

Next, with the admissible strategy $\delta := (\theta_k, \alpha_k)_{k\geq 0}$ is associated a switching cost process $(A_s^\delta)_{s\leq T}$ defined by:

$$A_s^\delta := \sum_{k\geq 1} g_{\alpha_{k-1}\alpha_k}(\theta_k, X_{\theta_k}^{t,x})\mathbf{1}_{\{\theta_k \leq s\}} \text{ for } s < T, \text{ and } A_T^\delta = \lim_{s\to T} A_s^\delta. \tag{3.3.5}$$

The process $(A_s^\delta)_{s\leq T}$ is non-decreasing and RCLL. Now, for $s \leq T$, let us set $\eta_s := \alpha_0 \mathbf{1}_{\{\theta_0\}}(s) + \sum_{k\geq 1} \alpha_k \mathbf{1}_{[\theta_k \leq s < \theta_{k+1})}$ which stands for the indicator of the system at time s. The process η is in bijection with the strategy δ. Finally, for any fixed $s \leq T$ and $i \in \mathcal{I}$, let us denote by $\mathcal{A}_s^i$ the following set of admissible strategies:

$$\begin{aligned}\mathcal{A}_s^i := \{&\delta := (\theta_k, \alpha_k)_{k\geq 0} \text{ admissible strategy such that } \theta_0 = s, \alpha_0 = i \\ &\text{and } \mathbb{E}[(A_T^\delta)^2] < \infty\}.\end{aligned}$$

Now, let $\delta := (\theta_k, \alpha_k)_{k\geq 0} \in \mathcal{A}_s^i$ and let us define the triplet of adapted processes $(P_s^{n,\delta}, N_s^{n,\delta}, Q_s^{n,\delta})_{s\leq T}$ as follows: $\forall s \leq T$,

$$\begin{cases} P^{n,\delta} \text{ is RCLL and } \mathbb{E}[\sup_{s\leq T} |P_s^{n,\delta}|^2] < \infty\,; N^{n,\delta} \in \mathcal{H}^{2,d} \text{ and } Q^{n,\delta} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)); \\ P_s^{n,\delta} = h^\delta(X_T^{t,x}) - A_T^\delta + A_s^\delta - \int_s^T N_r^{n,\delta} dB_r - \int_s^T \int_E Q_r^{n,\delta}(e)\tilde{\mu}(dr,de) \\ + \int_s^T \bar{f}^\delta(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, N_r^{n,\delta}, \\ \underbrace{\int_E \{u^{\delta,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^{\delta,n-1}(r, X_{r^-}^{t,x})\} 1_{\{r\geq t\}} \gamma^\delta(X_r^{t,x}, e)\lambda(de))}_{\Sigma_r^{\delta,n-1}} dr; \end{cases} \tag{3.3.6}$$

where

$$\begin{aligned} &h^\delta(x) := \sum_{k\geq 0} h_{\alpha_k}(x)\mathbf{1}_{[\theta_k \leq T < \theta_{k+1})} \text{ and} \\ &\bar{f}^\delta(s, x, (Y_s^{k,n})_{k\in\mathcal{I}}, z, \Sigma_s^{\delta,n-1}) := \sum_{k\geq 0} \bar{f}_{\alpha_k}(s, x, (Y_s^{k,n})_{k\in\mathcal{I}}, z, \Sigma_r^{\alpha_k,n-1})\mathbf{1}_{[\theta_k \leq s < \theta_{k+1})}. \end{aligned} \tag{3.3.7}$$

Those series contain only a finite many terms as δ is admissible and then $\mathbb{P}[\theta_n < T, \forall n \geq 0] = 0$. Note that, in (3.3.6), the generators $\bar{f}^\delta$ does not depend

neither on $P^{n,\delta}$ nor on $Q^{n,\delta} \in \mathcal{H}^2(\mathcal{L}^2(\lambda))$. Now, by a change of variables, the existence of $(P^{n,\delta} - A^\delta, N^{n,\delta}, Q^{n,\delta})$ stems from the standard existence result of solutions of BSDEs with jumps by Tang-Li [76] since its generator $z \mapsto \bar{f}^\delta(s, X_s^{t,x}, (Y_s^{k,n})_{k\in\mathcal{I}}, z, \Sigma_s^{\delta,n-1})$ is Lipschitz w.r.t z and A_T^δ is square integrable.

Next, let us consider the following system of RBSDEs: $\forall i \in \mathcal{I}$ and $s \leq T$

$$\begin{cases} \underline{Y}^{i,n} \in \mathcal{S}^2, \underline{Z}^{i,n} \in \mathcal{H}^{2,d}, \underline{V}^{i,n} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } \underline{K}^{i,n} \in \mathcal{A}^2; \\ \underline{Y}_s^{i,n} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, \underline{Z}_r^{i,n}, \int_E V_r^{i,n-1}(e)\gamma^i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad + \underline{K}_T^{i,n} - \underline{K}_s^{i,n} - \int_s^T \underline{Z}_r^{i,n} dB_r - \int_s^T \int_E \underline{V}_r^{i,n}(e)\tilde{\mu}(dr, de); \\ \underline{Y}_s^{i,n} \geqslant \max_{j\in\mathcal{I}^{-i}}(\underline{Y}_s^{j,n} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T (\underline{Y}_s^{i,n} - \max_{j\in\mathcal{I}^{-i}}(\underline{Y}_s^{j,n} - g_{ij}(s, X_s^{t,x})))d\underline{K}_s^{i,n} = 0 \end{cases} \tag{3.3.8}$$

whose solution exists and is unique by Theorem 3.3.1. Therefore, we have the following representation of $\underline{Y}^{i,n}$ (see e.g. [44] for more details on this representation):

$$\underline{Y}_s^{i,n} = \text{esssup}_{\delta\in\mathcal{A}_s^i}(P_s^{n,\delta} - A_s^\delta).$$

But $(Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{i\in\mathcal{I}}$ is also solution of (3.3.8), then by uniqueness one deduces that

$$\forall s \leq T,\ \underline{Y}_s^{i,n} = Y_s^{i,n} = \text{esssup}_{\delta\in\mathcal{A}_s^i}(P_s^{n,\delta} - A_s^\delta) = (P_s^{n,\delta^*} - A_s^{\delta^*}), \tag{3.3.9}$$

for some $\delta^* \in \mathcal{A}_s^i$, which means that δ^* is an optimal strategy of the switching control problem.

Step 3: Convergence of $(\mathbf{u^{i,n}})_{\mathbf{n\geq 0}}$

We now adapt the argument already used in [13, 39, 44] to justify a convergence result for the sequence $((u^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$. For this, let us set: $\forall i \in \mathcal{I}$ and $n,\ p \geq 1$

$$\begin{aligned} F_i^{n,p}(s, X_s^{t,x}, \omega, z) := \bar{f}_i(s, X_s^{t,x}, (Y_s^{k,n})_{k\in\mathcal{I}}, z, \int_E V_s^{i,n-1}(e)\gamma^i(X_s^{t,x}, e)\lambda(de)) \\ \vee \bar{f}_i(s, X_s^{t,x}, (Y_s^{k,p})_{k\in\mathcal{I}}, z, \int_E V_s^{i,p-1}(e)\gamma^i(X_s^{t,x}, e)\lambda(de)). \end{aligned}$$

Next, let us consider the solution, denoted by $(\hat{Y}^{i,n,p}, \hat{Z}^{i,n,p}, \hat{V}^{i,n,p}, \hat{K}^{i,n,p})_{i\in\mathcal{I}}$, of the obliquely reflected BSDEs with jumps associated with $((F_i^{n,p})_{i\in\mathcal{I}}, (g_{ij})_{i,j\in\mathcal{I}}$,

$(h_i)_{i\in\mathcal{I}}$), which exists and is unique according to Theorem 3.3.1. Moreover, as in (3.3.9), we have: $\forall s \leq T$,

$$\hat{Y}_s^{i,n,p} = \text{esssup}_{\delta\in\mathcal{A}_s^i}(\hat{P}_s^{n,p,\delta} - A_s^{\delta}) = (\hat{P}_s^{n,p,\tilde{\delta}^*} - A_s^{\tilde{\delta}^*}), \tag{3.3.10}$$

where $(\hat{P}^{n,p,\delta}, \hat{N}^{n,p,\delta}, \hat{Q}^{n,p,\delta})$ is the solution of the BSDE (3.3.6) with generator $F^{\delta,n,p}(...)$. Then by the comparison result (see Proposition 4.2 in [45]), between the solutions $Y^{i,n}$ and $\hat{Y}^{i,n,p}$, and $Y^{i,p}$ and $\hat{Y}^{i,n,p}$ (this possible since the generators of the systems do not depend on the jump parts), one deduces that: $\forall i \in \mathcal{I}$ and $s \leq T$,

$$Y_s^{i,n} \leq \hat{Y}_s^{i,n,p} \quad \text{and} \quad Y_s^{i,p} \leq \hat{Y}_s^{i,n,p}.$$

This combined with (3.3.9) and (3.3.10), lead to:

$$P_s^{n,\tilde{\delta}^*} - A_s^{\tilde{\delta}^*} \leq Y_s^{i,n} \leq \hat{P}_s^{n,p,\tilde{\delta}^*} - A_s^{\tilde{\delta}^*} \text{ and } P_s^{p,\tilde{\delta}^*} - A_s^{\tilde{\delta}^*} \leq Y_s^{i,p} \leq \hat{P}_s^{n,p,\tilde{\delta}^*} - A_s^{\tilde{\delta}^*},$$

which implies

$$\forall s \leq T,\ |Y_s^{i,n} - Y_s^{i,p}|^2 \leq 2\{|\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{n,\tilde{\delta}^*}|^2 + |\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{p,\tilde{\delta}^*}|^2\}. \tag{3.3.11}$$

Since both terms on the right-hand side of (3.3.11) are treated similarly, we focus only on the first one. Actually applying Itô's formula with $e^{\alpha s}|\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{n,\tilde{\delta}^*}|^2$ ($\alpha > 0$), yields: $\forall s \leq T$,

$$\begin{aligned}
&e^{\alpha s}|\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{n,\tilde{\delta}^*}|^2 + \int_s^T e^{\alpha r}|\hat{N}_r^{n,p,\tilde{\delta}^*} - N_r^{n,\tilde{\delta}^*}|^2 dr \\
&\quad + \sum_{s<r\leq T} e^{\alpha r}\Delta_r(\hat{P}^{n,p,\tilde{\delta}^*} - P^{n,\tilde{\delta}^*})^2 \\
&= -\alpha\int_s^T e^{\alpha r}|\hat{P}_r^{n,p,\tilde{\delta}^*} - P_r^{n,\tilde{\delta}^*}|^2 dr + 2\int_s^T e^{\alpha r}(\hat{P}_r^{n,p,\tilde{\delta}^*} - P_r^{n,\tilde{\delta}^*})\big\{F^{\tilde{\delta}^*,n,p}(r, X_r^{t,x}, \hat{N}_r^{n,p,\tilde{\delta}^*}) \\
&\quad - \bar{f}^{\tilde{\delta}^*}(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, N_r^{n,\tilde{\delta}^*}, \textstyle\int_E V_r^{\tilde{\delta}^*,n-1}(e)\gamma^{\tilde{\delta}^*}(X_r^{t,x}, e)\lambda(de))\big\}dr \\
&\quad - 2\int_s^T e^{\alpha r}(\hat{P}_r^{n,p,\tilde{\delta}^*} - P_r^{n,\tilde{\delta}^*})(\hat{N}_r^{n,p,\tilde{\delta}^*} - N_r^{n,\tilde{\delta}^*})dB_r \\
&\quad - 2\int_s^T \int_E e^{\alpha r}(\hat{P}_{r-}^{n,p,\tilde{\delta}^*} - P_{r-}^{n,\tilde{\delta}^*})(\hat{Q}_r^{n,p,\tilde{\delta}^*}(e) - Q_r^{n,\tilde{\delta}^*}(e))\tilde{\mu}(de, dr).
\end{aligned} \tag{3.3.12}$$

Observe that the inequality $|x \vee y - x| \leq |x - y|$, $\forall x, y \in \mathbb{R}$, combined with the

Lipschitz property of $f^{\tilde{\delta}^*}$ leads to:

$$
\begin{aligned}
&|F^{\tilde{\delta}^*,n,p}(r, X_r^{t,x}, \hat{N}_r^{n,p,\tilde{\delta}^*}) \\
&\quad - \bar{f}^{\tilde{\delta}^*}(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, N_r^{n,\tilde{\delta}^*}, \textstyle\int_E V_r^{\tilde{\delta}^*,n-1}(e)\gamma^{\tilde{\delta}^*}(X_r^{t,x}, e)\lambda(de))| \\
&\leq C\big\{|(Y_r^{k,n})_{k\in\mathcal{I}} - (Y_r^{k,p})_{k\in\mathcal{I}}| + |\hat{N}_r^{n,p,\tilde{\delta}^*} - N_r^{n,\tilde{\delta}^*}| \\
&\quad + |\textstyle\int_E\{V_r^{\tilde{\delta}^*,n-1}(e) - V_r^{\tilde{\delta}^*,p-1}(e)\}\gamma^{\tilde{\delta}^*}(X_r^{t,x}, e)\lambda(de))|\big\}.
\end{aligned}
$$

Going back to (3.3.12), taking expectation and using the inequality $2|ab| \leq \varepsilon|a|^2 + \frac{1}{\varepsilon}|b|^2 (\varepsilon > 0)$, we obtain:

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha s}|\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{n,\tilde{\delta}^*}|^2 + \textstyle\int_s^T e^{\alpha r}|\hat{N}_r^{n,p,\tilde{\delta}^*} - N_r^{n,\tilde{\delta}^*}|^2 dr \\
&\quad + \textstyle\sum_{s<r\leq T} e^{\alpha r}\Delta_r(\hat{P}^{n,p,\tilde{\delta}^*} - P^{n,\tilde{\delta}^*})^2\Big] \\
&\leq (-\alpha + 3\varepsilon)\mathbb{E}\Big[\textstyle\int_s^T e^{\alpha r}|\hat{P}_r^{n,p,\tilde{\delta}^*} - P_r^{n,\tilde{\delta}^*}|^2 dr\Big] \\
&\quad + \frac{C^2}{\varepsilon}\Big\{\mathbb{E}\Big[\textstyle\int_s^T e^{\alpha r}|(Y_r^{k,n})_{k\in\mathcal{I}} - (Y_r^{k,p})_{k\in\mathcal{I}}|^2 dr\Big] + \mathbb{E}\Big[\textstyle\int_s^T e^{\alpha r}|\hat{N}_r^{n,p,\tilde{\delta}^*} - N_r^{n,\tilde{\delta}^*}|^2 dr\Big] \\
&\quad + \mathbb{E}\Big[\int_s^T e^{\alpha r}\big(\int_E |\{V_r^{\tilde{\delta}^*,n-1}(e) - V_r^{\tilde{\delta}^*,p-1}(e)\}\gamma^{\tilde{\delta}^*}(X_r^{t,x}, e)|\lambda(de)\big)^2 dr\Big]\Big\}.
\end{aligned}
$$

If we choose $\alpha = \alpha_0 = 3\varepsilon$ and $\varepsilon > C^2$, we get : $\forall s \leq T$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha_0 s}|\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{n,\tilde{\delta}^*}|^2\Big] \leq \frac{3C^2}{\alpha_0}\Big\{\mathbb{E}\Big[\textstyle\int_s^T e^{\alpha_0 r}|(Y_r^{k,n})_{k\in\mathcal{I}} - (Y_r^{k,p})_{k\in\mathcal{I}}|^2 dr\Big] \\
&\quad + \mathbb{E}\Big[\int_s^T e^{\alpha_0 r}\big(\int_E \textstyle\sum_{k=1,m} |\{V_r^{k,n-1}(e) - V_r^{k,p-1}(e)\}\gamma^k(X_r^{t,x}, e)|\lambda(de)\big)^2 dr\Big]\Big\}.
\end{aligned}
$$

The same reasoning leads to the same estimate for $e^{\alpha s}|\hat{P}_s^{n,p,\tilde{\delta}^*} - P_s^{p,\tilde{\delta}^*}|^2$. Therefore, we deduce from (3.3.11) that:

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha_0 s}|Y_s^{i,n} - Y_s^{i,p}|^2\Big] \leq \frac{6C^2}{\alpha_0}\Big\{\mathbb{E}\Big[\textstyle\int_s^T e^{\alpha_0 r}|(Y_r^{k,n})_{k\in\mathcal{I}} - (Y_r^{k,p})_{k\in\mathcal{I}}|^2 dr\Big] \\
&\quad + \mathbb{E}\Big[\int_s^T e^{\alpha_0 r}\big(\int_E \textstyle\sum_{k=1,m} |\{V_r^{k,n-1}(e) - V_r^{k,p-1}(e)\}\gamma^k(X_r^{t,x}, e)|\lambda(de)\big)^2 dr\Big]\Big\}.
\end{aligned} \tag{3.3.13}
$$

Then, by summing over $i \in \mathcal{I}$, there exists a constant κ such that: $\forall s \leq T$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha_0 s}|(Y_s^{k,n})_{k\in\mathcal{I}} - (Y_s^{i,p})_{k\in\mathcal{I}}|^2\Big] \leq \kappa\Big\{\mathbb{E}\Big[\textstyle\int_s^T e^{\alpha_0 r}|(Y_r^{k,n})_{k\in\mathcal{I}} - (Y_r^{k,p})_{k\in\mathcal{I}}|^2 dr\Big] \\
&\quad + \mathbb{E}\Big[\int_s^T e^{\alpha_0 r}\big(\int_E \textstyle\sum_{k=1,m} |\{V_r^{k,n-1}(e) - V_r^{k,p-1}(e)\}\gamma^k(X_r^{t,x}, e)|\lambda(de)\big)^2 dr\Big]\Big\}.
\end{aligned}
$$

Finally by using Gronwall's inequality one can find a constant κ_1 such that: $\forall s \leq T$,

$$\mathbb{E}\Big[e^{\alpha_0 s}|(Y_s^{k,n})_{k\in\mathcal{I}} - (Y_s^{i,p})_{k\in\mathcal{I}}|^2\Big]$$
$$\leq \kappa_1 \mathbb{E}\Big[\int_s^T e^{\alpha_0 r}\big(\int_E \sum_{k=1,m} |\{V_r^{k,n-1}(e) - V_r^{k,p-1}(e)\}\gamma^k(X_r^{t,x},e)|\lambda(de)\big)^2 dr\Big].$$

Taking $s = t$ and considering (3.3.3)-((a),(b)), we obtain : for any $i \in \mathcal{I}$,

$$|u^{i,n}(t,x) - u^{i,p}(t,x)|^2$$
$$\leq \kappa_1 \mathbb{E}\Big[\int_t^T e^{\alpha_0(r-t)}\big(\int_E \sum_{k=1,m} |\{u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,n-1}(r, X_{r^-}^{t,x})$$
$$- (u^{k,p-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,p-1}(r, X_{r^-}^{t,x}))\}\gamma^k(X_r^{t,x},e)|\lambda(de)\big)^2 dr\Big].$$

Next, using Cauchy-Schwarz inequality, (3.2.8) and the inequality $|a+b|^2 \leq 2(|a|^2 + |b|^2)$, we get:

$$|u^{i,n}(t,x) - u^{i,p}(t,x)|^2$$
$$\leq \kappa_1 \mathbb{E}\big[\int_t^T e^{\alpha_0(r-t)}\big(\int_E \{\sum_{k=1,m} \big|\gamma^k(X_r^{t,x},e)\big|^2\}\lambda(de)\big)$$
$$\times \big(\int_E \{\sum_{k=1,m} \big|u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,n-1}(r, X_{r^-}^{t,x})$$
$$- u^{k,p-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) + u^{k,p-1}(r, X_{r^-}^{t,x})\big|^2\}\lambda(de)\big)dr\big]$$
$$\leq C\mathbb{E}\big[\int_t^T e^{\alpha_0(r-t)}\big(\int_E \{\sum_{k=1,m} \big|u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,n-1}(r, X_{r^-}^{t,x})$$
$$- u^{k,p-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) + u^{k,p-1}(r, X_{r^-}^{t,x})\big|^2\}\lambda(de)\big)dr\big]$$
$$\leq 2C\mathbb{E}\big[\int_t^T e^{\alpha_0(r-t)} \int_E \sum_{k=1,m} \{\big|(u^{k,n-1} - u^{k,p-1})(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e))\big|^2$$
$$+ \big|(u^{k,n-1} - u^{k,p-1})(r, X_{r^-}^{t,x})\big|^2\}\lambda(de)dr\big], \tag{3.3.14}$$

for some constant C (which may change from line to line).

Now, in order to take the supremum on the inequality (3.3.14), we need to show the boundedness of $(u^{i,n})_{i\in\mathcal{I}}$. For this, let $(\bar{Y}, \bar{Z})$ be the solution of the following standard BSDE: for any $s \leq T$,

$$\begin{cases} \bar{Y} \in \mathcal{S}^2,\ \bar{Z} \in \mathcal{H}^{2,d}; \\ \bar{Y}_s = \bar{C} + \int_s^T \{\bar{C} + m\, C_f^y\, \bar{Y}_r + C_f^z\, |\bar{Z}_r| + 2\theta \bar{Y}_r\} dr - \int_s^T \bar{Z}_r dB_r; \end{cases}$$

where C_f^y , C_f^z and C_f^v are the maximum of the Lipschitz constants of the $f_i's$ w.r.t. $\vec{y}$, z and v respectively, and

$$\theta = C_f^v C_\gamma \int_E (1 \wedge |e|)\lambda(de).$$

Note that the solution of this BSDE exists and is unique by Pardoux and Peng's result [62]. Then, there exists a constant $\underline{C}$ such that $|\bar{Y}| \leq \underline{C}$. Finally, noting that $\bar{Y}$ is deterministic and $\bar{Z} = 0$.

Now, recall that $((Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ verify:

$$(Y^{i,0}, Z^{i,0}, V^{i,0}, K^{i,0}) = (0,0,0,0) \text{ and for } n \geq 1,$$

$$\begin{cases} Y_s^{i,n} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, \int_E V_r^{i,n-1}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,n} - K_s^{i,n} - \int_s^T Z_r^{i,n} dB_r - \int_s^T \int_E V_r^{i,n}(e)\tilde{\mu}(dr, de), \quad s \leq T; \\ Y_s^{i,n} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n} - g_{ij}(s, X_s^{t,x})), \quad s \leq T; \\ \int_0^T (Y_s^{i,n} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n} - g_{ij}(s, X_s^{t,x})))dK_s^{i,n} = 0, \end{cases} \tag{3.3.15}$$

with, $\forall s \in [t,T]$, $Y_s^{i,n} = u^{i,n}(s, X_s^{t,x})$. Then, by an induction argument on n, it follows that: $\forall n \geq 1$ and $i \in \mathcal{I}$,

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k, \quad |u^{i,n}(t,x)| \leq \bar{Y}_t. \tag{3.3.16}$$

Indeed, for $n = 1$, we have:

$$\begin{cases} Y_s^{i,1} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,1})_{k\in\mathcal{I}}, Z_r^{i,1}, 0)dr + K_T^{i,1} - K_s^{i,1} \\ \qquad\qquad - \int_s^T Z_r^{i,1} dB_r - \int_s^T \int_E V_r^{i,1}(e)\tilde{\mu}(dr, de), \quad s \leq T; \\ Y_s^{i,1} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,1} - g_{ij}(s, X_s^{t,x})), \quad s \leq T; \\ \int_0^T (Y_s^{i,1} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,1} - g_{ij}(s, X_s^{t,x})))dK_s^{i,1} = 0. \end{cases} \tag{3.3.17}$$

Next, let us set, for $i \in \mathcal{I}$,

$$\underline{Y}^i = \bar{Y}, \quad \underline{Z}^i = \bar{Z}, \quad \underline{V}^i = 0, \quad \text{and} \quad \underline{K}^i = 0.$$

Therefore, $(\underline{Y}^i, \underline{Z}^i, \underline{V}^i, \underline{K}^i)_{i\in\mathcal{I}}$ is a solution of the following system: $\forall i \in \mathcal{I}$ and $s \leq T$,

$$\begin{cases} \underline{Y}^i_s = \bar{C} + \int_s^T \{\bar{C} + m\, C^y_f\, \underline{Y}^i_r + +2\theta\underline{C} + C^z_f\, |\underline{Z}^i_r|\}dr \\ \qquad\qquad +\underline{K}^i_T - \underline{K}^i_s - \int_s^T \underline{Z}^i_r dB_r - \int_s^T \int_E \underline{V}^i_r(e)\tilde{\mu}(dr,de); \\ \underline{Y}^i_s \geqslant \max_{j\in\mathcal{I}^{-i}}(\underline{Y}^j_s - g_{ij}(s, X^{t,x}_s)); \\ \int_0^T (\underline{Y}^i_s - \max_{j\in\mathcal{I}^{-i}}(\underline{Y}^j_s - g_{ij}(s, X^{t,x}_s)))d\underline{K}^i_s = 0. \end{cases}$$

On the other hand, let $\vec{\Gamma} := (\Gamma^i)_{i=1,\dots,m} \in \mathcal{H}^{2,m}$ and let us consider the following mapping:

$$\begin{aligned} \Theta : \mathcal{H}^{2,m} &\to \mathcal{H}^{2,m} \\ \vec{\Gamma} \quad &\mapsto \Theta(\vec{\Gamma}) := (Y^{\Gamma,i})_{i=1,\dots,m} \end{aligned} \tag{3.3.18}$$

where $(Y^{\Gamma,i})_{i\in\mathcal{I}}$ verifies: $\forall i \in \mathcal{I}$ and $s \leq T$,

$$\begin{cases} Y^{\Gamma,i}_s = h_i(X^{t,x}_T) + \int_s^T \bar{f}_i(r, X^{t,x}_r, \vec{\Gamma}_r, Z^{\Gamma,i}_r, 0)dr + K^{\Gamma,i}_T - K^{\Gamma,i}_s \\ \qquad\qquad - \int_s^T Z^{\Gamma,i}_r dB_r - \int_s^T \int_E V^{\Gamma,i}_r(e)\tilde{\mu}(dr,de); \\ Y^{\Gamma,i}_s \geqslant \max_{j\in\mathcal{I}^{-i}}(Y^{\Gamma,j}_s - g_{ij}(s, X^{t,x}_s)); \\ \int_0^T (Y^{\Gamma,i}_s - \max_{j\in\mathcal{I}^{-i}}(Y^{\Gamma,j}_s - g_{ij}(s, X^{t,x}_s)))dK^{\Gamma,i}_s = 0. \end{cases} \tag{3.3.19}$$

As Θ is a contraction in $\mathcal{H}^{2,m}$ equipped with an appropriate equivalent norm (see Proposition 3.3 in [13]), then it has a unique fixed point $(Y^{i,1})_{i\in\mathcal{I}}$ which, combined with the associated processes $(Z^{i,1}, V^{i,1}, K^{i,1})_{i\in\mathcal{I}}$, makes that $(Y^{i,1}, Z^{i,1}, V^{i,1}, K^{i,1})_{i\in\mathcal{I}}$ is the unique solution of system (3.3.17).

Now, let us consider the following sequence of processes $((Y^i_k, Z^i_k, V^i_k, K^i_k)_{i\in\mathcal{I}})_{k\geq 1}$:

$$Y^i_0 = 0, \text{ for all } i \in \mathcal{I} \text{ and for } k \geq 1, \ (Y^i_k)_{i\in\mathcal{I}} = \Theta((Y^i_{k-1})_{i\in\mathcal{I}}),$$

where Θ is the mapping defined in (3.3.18) and Z^i_k, V^i_k, K^i_k are associated with Y^i_k, $i \in \mathcal{I}$, through equation (3.3.19). Therefore, as Θ is a contraction, the sequence $((Y^i_k)_{i\in\mathcal{I}})_{k\geq 0}$ converges to $(Y^{i,1})_{i\in\mathcal{I}}$ in $\mathcal{H}^{2,m}$. On the other hand by

an induction argument on k and by using the comparison result, we have that:

$$\forall k \geq 0,\, \forall i \in \mathcal{I}, \quad -\bar{Y} \leq Y_k^i \leq \bar{Y}. \tag{3.3.20}$$

In fact, for $k = 0$, this obviously holds since $\bar{Y} \geq 0$. Next suppose that (3.3.20) holds for some $k-1$ with $k \geq 1$, i.e.

$$\forall i \in \mathcal{I}, -\bar{Y} \leq Y_{k-1}^i \leq \bar{Y}.$$

Then, by a linearization procedure of $\bar{f}_i$, which is possible since it is Lipschitz w.r.t $(\vec{y}, z)$, and using the induction hypothesis, we obtain: for any $i \in \mathcal{I}$,

$$\begin{aligned}\bar{f}_i(s, X_s^{t,x}, (Y_{k-1}^a(s))_{a\in\mathcal{I}}, z, 0) &\leq \bar{C} + C_f^y \sum_{a=1,m} |Y_{k-1}^a(s)| + C_f^z\,|z| \\ &\leq \bar{C} + m\,C_f^y\,\bar{Y}_s + C_f^z\,|z|,\end{aligned}$$

and

$$\bar{f}_i(s, X_s^{t,x}, (Y_{k-1}^a(s))_{a\in\mathcal{I}}, z, 0) \geq -(\bar{C} + m\,C_f^y\,\bar{Y}_s + C_f^z\,|z|).$$

Finally, by the comparison result (see Proposition 4.2 in [45]) (this is possible since the generators of the systems do not depend on the jump parts), one deduces that: $\forall i \in \mathcal{I}$,

$$-\bar{Y} = -\underline{Y}^i \leq Y_k^i \leq \underline{Y}^i = \bar{Y}.$$

Taking the limit w.r.t. k, we get: $\forall i \in \mathcal{I}$,

$$-\bar{Y} \leq Y^{i,1} \leq \bar{Y}.$$

But, for any $s \in [t,T]$ $Y_s^{i,1} = u^{i,1}(s, X_s^{t,x})$. Then, by taking $s = t$ in the previous inequalities, we obtain:

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k, \quad |u^{i,1}(t,x)| \leq \bar{Y}_t,$$

which implies that the inequality (3.3.16) is true for $n = 1$. Now, suppose that, its holds for some $n-1$ with $n \geq 1$, i.e.,

$$\forall i \in \mathcal{I},\, \forall (t,x) \in [0,T] \times \mathbb{R}^k, \quad |u^{i,n-1}(t,x)| \leq \bar{Y}_t. \tag{3.3.21}$$

Next, we are going to prove that, for any $i \in \mathcal{I}$, $(t,x) \in [0,T]\times\mathbb{R}^k$, $|u^{i,n}(t,x)| \leq \bar{Y}_t$.

Recall that $((Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ the solution of (3.3.15) and let us introduce the following mapping:

$$\begin{aligned}\Theta^{v,n-1}:\ \mathcal{H}^{2,m} &\longrightarrow \mathcal{H}^{2,m}\\ \vec{\rho} := (\rho^i)_{i\in\mathcal{I}} &\longmapsto \Theta^{v,n-1}(\vec{\rho}) := (Y^{i,n,\rho})_{i\in\mathcal{I}}\end{aligned} \tag{3.3.22}$$

where $(Y^{i,n,\rho})_{i\in\mathcal{I}}$ verifies: $\forall s \leq T$,

$$\begin{cases} Y_s^{i,n,\rho} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (\rho_r^k)_{k\in\mathcal{I}}, Z_r^{i,n,\rho}, \int_E V_r^{i,n-1}(e)\gamma_i(X_r^{t,x},e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,n,\rho} - K_s^{i,n,\rho} - \int_s^T Z_r^{i,n,\rho}dB_r - \int_s^T\int_E V_r^{i,n,\rho}(e)\tilde{\mu}(dr,de); \\ Y_s^{i,n,\rho} \geq \max\limits_{j\in\mathcal{I}^{-i}}(Y_s^{j,n,\rho} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T (Y_s^{i,n,\rho} - \max\limits_{j\in\mathcal{I}^{-i}}(Y_s^{j,n,\rho} - g_{ij}(s,X_s^{t,x})))dK_s^{i,n,\rho} = 0. \end{cases} \tag{3.3.23}$$

Note that $(Y^{i,n})_{i\in\mathcal{I}}$ verifies $(Y^{i,n})_{i\in\mathcal{I}} = \Theta^{v,n-1}((Y^{i,n})_{i\in\mathcal{I}})$ and it is the unique fixed point of $\Theta^{v,n-1}$ in $\mathcal{H}^{2,m}$ equipped with an appropriate equivalent norm (see Proposition 3.3 in [13]). Next, let us consider the following sequence of processes $((Y^{i,n,l}, Z^{i,n,l}, V^{i,n,l}, K^{i,n,l})_{i\in\mathcal{I}})_{l\geq 0}$:

$$Y^{i,n,0} = 0, \ \text{ for all } i \in \mathcal{I} \text{ and for } l \geq 1,\ (Y^{i,n,l})_{i\in\mathcal{I}} = \Theta^{v,n-1}((Y^{i,n,l-1})_{i\in\mathcal{I}}),$$

where $(Y^{i,n,l})_{i\in\mathcal{I}}$ verifies: $\forall s \leq T$,

$$\begin{cases} Y_s^{i,n,l} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,n,l-1})_{k\in\mathcal{I}}, Z_r^{i,n,l}, \int_E V_r^{i,n-1}(e)\gamma_i(X_r^{t,x},e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,n,l} - K_s^{i,n,l} - \int_s^T Z_r^{i,n,l}dB_r - \int_s^T\int_E V_r^{i,n,l}(e)\tilde{\mu}(dr,de); \\ Y_s^{i,n,l} \geqslant \max\limits_{j\in\mathcal{I}^{-i}}(Y_s^{j,n,l} - g_{ij}(s, X_s^{t,x})); \\ \int_0^T (Y_s^{i,n,l} - \max\limits_{j\in\mathcal{I}^{-i}}(Y_s^{j,n,l} - g_{ij}(s, X_s^{t,x})))dK_s^{i,n,l} = 0. \end{cases}$$

Similarly as above, since $\Theta^{v,n-1}$ is a contraction, then the sequence $((Y^{i,n,l})_{i\in\mathcal{I}})_{l\geq 0}$ converges to $(Y^{i,n})_{i\in\mathcal{I}}$, as $l \to \infty$, in $\mathcal{H}^{2,m}$. Next, by an induction argument on l and by using the comparison result, we have that:

$$\forall i \in \mathcal{I},\ -\bar{Y} \leq Y^{i,n,l} \leq \bar{Y}. \tag{3.3.24}$$

Actually for $l = 0$, the property holds true and if we assume that it is satisfied for some $l-1$ and by using the induction hypotheses, we deduce: $\forall i \in \mathcal{I}$, $\forall s \in [t,T]$,

$$\begin{aligned}
&|\bar{f}_i(s, X_s^{t,x}, (Y_s^{k,n,l-1})_{k\in\mathcal{I}}, z, \textstyle\int_E V_s^{i,n-1}(e)\gamma_i(X_s^{t,x}, e)\lambda(de))| \\
&= |\bar{f}_i(s, X_s^{t,x}, (Y_s^{k,n,l-1})_{k\in\mathcal{I}}, z, \textstyle\int_E \{u^{i,n-1}(s, X_{s-}^{t,x} + \beta(X_{s-}^{t,x}, e)) \\
&\qquad - u^{i,n-1}(s, X_{s-}^{t,x})\}\gamma_i(X_s^{t,x}, e)\lambda(de))| \\
&\leq \bar{C} + C_f^y \sum_{i=1,m} |Y_s^{i,n,l-1}| + C_f^z |z| + 2\theta \bar{Y}_s \\
&\leq \bar{C} + m\, C_f^y\, \bar{Y}_s + C_f^z |z| + 2\theta \bar{Y}_s.
\end{aligned}$$

Now by using comparison (Proposition 4.2 in [45]) we deduce that

$$\forall i \in \mathcal{I},\ -\bar{Y} = -\underline{Y}^i \leq Y^{i,n,l} \leq \underline{Y}^i = \bar{Y}$$

and in taking the limit w.r.t $l \to \infty$ we obtain:

$$\forall i \in \mathcal{I},\ -\bar{Y} \leq Y^{i,n} \leq \bar{Y}.$$

Finally, as $Y_s^{i,n} = u^{i,n}(s, X_s^{t,x})$, $\forall s \in [t,T]$, then

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k,\ |u^{i,n}(t,x)| \leq \bar{Y}_t \leq \underline{C},$$

which implies that $(u^{i,n})_{i\in\mathcal{I}}$, $n \geq 0$, are uniformly bounded. The proof of the claim is now completed.

Next recall the inequality (3.3.14). Let us choose η a constant such that $\frac{4}{\alpha_0} C m \lambda(E)(e^{\alpha_0 \eta} - 1) = \frac{3}{4}$. Note that η does not depend on the terminal conditions $(h_i)_{i\in\mathcal{I}}$. Finally let us set

$$\|u^{i,n} - u^{i,p}\|_{\infty,\eta} := \sup_{(t,x)\in[T-\eta,T]\times\mathbb{R}^k} |u^{i,n}(t,x) - u^{i,p}(t,x)|.$$

From (3.3.14), after summation over i, we obtain for any $n, p \geq 1$,

$$\begin{aligned}
&\sum_{i=1,m} \|u^{i,n} - u^{i,p}\|^2_{\infty,\eta} \\
&\quad \leq \underbrace{\frac{4}{\alpha_0} Cm\lambda(E)(e^{\alpha_0\eta} - 1)}_{=\frac{3}{4}} \sup_{(t,x)\in[T-\eta,T]\times\mathbb{R}^k} \sum_{i=1,m} |u^{i,n-1}(t,x) - u^{i,p-1}(t,x)|^2 \\
&\quad = \frac{3}{4} \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,\eta}
\end{aligned}$$

which means that the sequence $((u^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ is uniformly convergent in $[T-\eta, T]\times\mathbb{R}^k$. Next, let $t \in [T-2\eta, T-\eta]$, then once more by (3.3.14), we have:

$$\begin{aligned}
&|u^{i,n}(t,x) - u^{i,p}(t,x)|^2 \\
&\quad \leq 2C\mathbb{E}\Big[\int_{T-2\eta}^{T-\eta} e^{\alpha_0(r-t)} \int_E \sum_{k=1,m} \{\big|(u^{k,n-1} - u^{k,p-1})(r, X^{t,x}_{r^-} + \beta(X^{t,x}_{r^-}, e))\big|^2 \\
&\qquad + \big|(u^{k,n-1} - u^{k,p-1})(r, X^{t,x}_{r^-})\big|^2\}\lambda(de)dr\Big] \\
&\qquad + 2C\mathbb{E}\Big[\int_{T-\eta}^{T} e^{\alpha_0(r-t)} \int_E \sum_{k=1,m} \{\big|(u^{k,n-1} - u^{k,p-1})(r, X^{t,x}_{r^-} + \beta(X^{t,x}_{r^-}, e))\big|^2 \\
&\qquad + \big|(u^{k,n-1} - u^{k,p-1})(r, X^{t,x}_{r^-})\big|^2\}\lambda(de)dr\Big].
\end{aligned} \tag{3.3.25}$$

Then, if we set

$$\|u^{i,n} - u^{i,p}\|_{\infty,2\eta} := \sup_{(t,x)\in[T-2\eta,T-\eta]\times\mathbb{R}^k} |u^{i,n}(t,x) - u^{i,p}(t,x)|,$$

we have:

$$\begin{aligned}
\sum_{i=1,m} \|u^{i,n} - u^{i,p}\|^2_{\infty,2\eta} &\leq \frac{4}{\alpha_0} Cm\lambda(E)\Big((e^{\alpha_0\eta} - 1) \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,2\eta} \\
&\qquad + (e^{2\alpha_0\eta} - e^{\alpha_0\eta}) \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,\eta}\Big) \\
&\leq \frac{3}{4} \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,2\eta} \\
&\qquad + \frac{4}{\alpha_0} Cm\lambda(E)(e^{2\alpha_0\eta} - e^{\alpha_0\eta}) \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,\eta}.
\end{aligned}$$

As $\limsup_{n,p\to\infty} \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,\eta} = 0$, we obtain:

$$\limsup_{n,p\to\infty} \sum_{i=1,m} \|u^{i,n} - u^{i,p}\|^2_{\infty,2\eta} \leq \frac{3}{4} \limsup_{n,p\to\infty} \sum_{i=1,m} \|u^{i,n-1} - u^{i,p-1}\|^2_{\infty,2\eta}.$$

Therefore

$$\limsup_{n,p\to\infty} \sum_{i=1,m} \|u^{i,n} - u^{i,p}\|^2_{\infty,2\eta} = 0.$$

Thus, the sequence $((u^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ is uniformly convergent in $[T-2\eta, T-\eta]\times\mathbb{R}^k$. Continuing now this reasoning as many times as necessary on $[T-3\eta, T-2\eta]$, $[T-4\eta, T-3\eta]$ etc. we obtain the uniform convergence of $((u^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ in $[0,T]\times\mathbb{R}^k$. So for $i\in\mathcal{I}$ and $(t,x)\in[0,T]\times\mathbb{R}^k$, let us set $u^i(t,x) = \lim_{n\to\infty} u^{i,n}(t,x)$, $i\in\mathcal{I}$. Note that $(u^i)_{i\in\mathcal{I}}$ are continuous and bounded functions on $[0,T]\times\mathbb{R}^k$.

Step 4: Convergence of $(\mathbf{Y^{i,n}}, \mathbf{Z^{i,n}}, \mathbf{V^{i,n}}, \mathbf{K^{i,n}})_{\mathbf{n\geq 0}}$

We are now ready to study the convergence of the sequences $(Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{n\geq 0}$.

Convergence of $(Y^{i,n})_{n\geq 0}$ *on* $[t,T]$: For any $i\in\mathcal{I}$ and $s\in[t,T]$ let us set $Y^i_s = u^i(s, X^{t,x}_s)$. Next let $n\geq 1$, then:

$$\begin{aligned}\mathbb{E}\big[\sup_{t\leq s\leq T} |Y^{i,n}_s - Y^{i,t,x}_s|^2\big] &= \mathbb{E}\big[\sup_{t\leq s\leq T} |u^{i,n}(s, X^{t,x}_s) - u^i(s, X^{t,x}_s))|^2\big] \\ &\leq \|u^{i,n} - u^i\|_\infty := \sup_{(t,x)\in[0,T]\times\mathbb{R}^k} |u^{i,n}(t,x) - u^i(t,x)|\end{aligned} \tag{3.3.26}$$

As the right hand-side converges to 0 as $n\to\infty$, then $(Y^{i,n})_{n\geq 0}$ converges to $(Y^i_s)_{s\in[t,T]}$ in $\mathcal{S}^2_{[t,T]}$ which is $\mathcal{S}^2$ reduced to $[t,T]$. The same is valid for $\mathcal{A}^2_{[0,t]}$ which is $\mathcal{A}^2$ reduced to $[0,t]$ (it is introduced in (3.3.27) below).

Convergence of $(Y^{i,n})_{n\geq 0}$ *on* $[0,t]$: By Remark 3.3.3, on the time interval $[0,t]$ the sequences
$(Y^{i,n}, Z^{i,n}, V^{i,n}, K^{i,n})_{n\geq 0}$ verify:

$$(Y^{i,0}, Z^{i,0}, V^{i,0}, K^{i,0}) = (0,0,0,0) \quad \text{for all } i\in\mathcal{I}, \quad \text{for } n\geq 1 \text{ and } s\leq t,$$

$$\begin{cases} Y^{i,n} \in \mathcal{S}^2_{[0,t]} \text{ and } K^{i,n} \in \mathcal{A}^2_{[0,t]}; \\ Y^{i,n}_s = u^{i,n}(t,x) + \int_s^t \bar{f}_i(r,x,(Y^{k,n}_r)_{k\in\mathcal{I}},0,0)dr + K^{i,n}_t - K^{i,n}_s, \ s \le t; \\ Y^{i,n}_s \geqslant \max_{j\in\mathcal{I}^{-i}}(Y^{j,n}_s - g_{ij}(s,x)), \ s \le t; \\ \int_0^t (Y^{i,n}_s - \max_{j\in\mathcal{I}^{-i}}(Y^{j,n}_s - g_{ij}(s,x)))dK^{i,n}_s = 0. \end{cases} \tag{3.3.27}$$

But $(Y^{i,n}_s)_{s\le t}$ is deterministic, continuous and still have the representation property (3.3.9) in connection with the switching problem in $[0,t]$. Next in considering $(\check{P}^{n,p,\delta}, \check{N}^{n,p,\delta})$ the solution on $[0,t]$ of the BSDE (3.3.6) with generator $\check{F}^{\delta,n,p}(r,x) := \bar{f}_i(r,x,(Y^{k,n}_r)_{k\in\mathcal{I}},0,0) \vee \bar{f}_i(r,x,(Y^{k,p}_r)_{k\in\mathcal{I}},0,0)$ and terminal value $h^{n,p}(t,x) := u^{i,n}(t,x) \vee u^{i,p}(t,x)$ and arguing as in Step 3, we deduce a similar inequality as in (3.3.13) that reads: $\forall s \in [0,t]$,

$$\mathbb{E}\Big[e^{\alpha_0 s}|Y^{i,n}_s - Y^{i,p}_s|^2\Big]$$
$$\le 2\sum_{k=1,m} e^{\alpha_0 t}|u^{k,n}(t,x) - u^{k,p}(t,x)|^2 + \frac{6C^2}{\alpha_0}\mathbb{E}\Big[\int_s^t e^{\alpha_0 r}|(Y^{k,n}_r)_{k\in\mathcal{I}} - (Y^{k,p}_r)_{k\in\mathcal{I}}|^2 dr\Big].$$

Now as we know that for any $i \in \mathcal{I}$, the sequence $(u^{i,n}(t,x))_{n\ge 0}$ converges to $u^i(t,x)$ then it is enough to mimic the arguments of Step 3 to obtain that $(Y^{i,n}_s)_{s\le t}$ converges uniformly on $[0,t]$ to some continuous deterministic function (and then bounded) $(Y^i_s)_{s\le t}$.

As a consequence, for any $i \in \mathcal{I}$, the sequence $(Y^{i,n})_{n\ge 0}$ converges in $\mathcal{S}^2$ to some process Y^i, which is moreover bounded since the functions $(u^i)_{i\in\mathcal{I}}$ are bounded.

Next the measure λ is finite, then by Remark 3.3.3, the characterization (3.3.4) on $[t,T]$ of the sequence $(V^{i,n})_{n\ge 0}$ by means of the function $(u^{i,n})_{n\ge 0}$ and the uniform convergence of $(u^{i,n})_{n\ge 0}$ we deduce that the sequence $(V^{i,n})_{n\ge 0}$ converges in $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ to some process $V^{i,t,x}$ which belongs also to $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ and which has the following representation:

$$V^i_s(e) := \{u^i(s, X^{t,x}_{s^-} + \beta(X^{t,x}_{s^-}, e)) - u^i(s, X^{t,x}_{s^-})\}1_{\{s\ge t\}},$$
$$ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [0,T]\times\Omega\times E. \tag{3.3.28}$$

This representation imply that $V^{i,t,x}$ are uniformly bounded.

We now focus on the convergence of the components $(Z^{i,n}, K^{i,n})_{n\geq 0}$. For this, we first establish a priori estimates, uniform on n of the sequences $(Z^{i,n}, K^{i,n})_{n\geq 0}$. Applying Itô's formula to $|Y_s^{i,n}|^2$, we have: $\forall s \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\big[|Y_s^{i,n}|^2\big] + \mathbb{E}\big[\int_s^T |Z_r^{i,n}|^2 dr\big] + \mathbb{E}\big[\int_s^T \int_E |V_r^{i,n}(e)|^2 \lambda(de)dr\big] \\
&\qquad = \mathbb{E}\big[|h_i(X_T^{t,x}|^2\big] + 2\mathbb{E}\big[\int_s^T Y_r^{i,n} f_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, V_r^{i,n-1})dr\big] \\
&\qquad\quad + 2\mathbb{E}\big[\int_s^T Y_r^{i,n} dK_r^{i,n}\big].
\end{aligned}$$

Then by a linearization procedure of f_i, which is possible since it is Lipschitz w.r.t $(\vec{y}, z, q)$ and using the inequality $2ab \leq \frac{1}{\epsilon}a^2 + \epsilon b^2$ for any constant $\epsilon > 0$, we have:

$$\begin{aligned}
&\mathbb{E}\big[\textstyle\int_0^T |Z_r^{i,n}|^2 dr\big] \\
&\quad \leq \mathbb{E}\big[|h_i(X_T^{t,x}|^2\big] + 2\mathbb{E}\big[\textstyle\int_0^T |Y_r^{i,n}|\{|f_i(r, X_r^{t,x}, 0, 0, 0)| + \sum_{l=1,m} a_r^{i,l,n}|Y_r^{l,n}| + b_r^{i,n}|Z_r^i| \\
&\qquad + c_r^{i,n}\textstyle\int_E |V_r^{i,n-1}(e)\gamma(X_r^{t,x}, e)|\lambda(de)\}dr\big] + \frac{1}{\epsilon}\mathbb{E}\big[\sup_{s\leq T}|Y_s^{i,n}|^2\big] + \epsilon\mathbb{E}\big[\big(K_T^{i,n}\big)^2\big],
\end{aligned}$$

where $a^{i,l,n} \in \mathbb{R}$, $b^{i,l,n} \in \mathbb{R}^d$ are $\mathcal{P}$-measurable non-negative bounded processes while $c^{i,n} \in \mathbb{R}$ is non-negative bounded and $\mathbf{P}$-measurable process. Using again the inequality $2ab \leq \frac{1}{\nu}a^2 + \nu b^2$ for $\nu > 0$, yields

$$\begin{aligned}
&\mathbb{E}\big[\textstyle\int_0^T |Z_r^{i,n}|^2 dr\big] \\
&\quad \leq \mathbb{E}\big[|h_i(X_T^{t,x})|^2\big] + \frac{1}{\nu}\mathbb{E}\big[\textstyle\int_0^T |Y_r^{i,n}|^2 dr\big] + \nu\mathbb{E}\big[\textstyle\int_0^T \{|f_i(r, X_r^{t,x}, 0, 0, 0)| \\
&\qquad + \textstyle\sum_{l=1,m} a_r^{i,l,n}|Y_r^{l,n}| + b_r^{i,n}|Z_r^i| + c_r^{i,n}\textstyle\int_E |V_r^{i,n-1}(e)\gamma(X_r^{t,x}, e)|\lambda(de)\}^2 dr\big] \\
&\qquad + \frac{1}{\epsilon}\mathbb{E}\big[\sup_{s\leq T}|Y_s^{i,n}|^2\big] + \epsilon\mathbb{E}\big[\big(K_T^{i,n}\big)^2\big].
\end{aligned}$$

From the boundedness of $f_i(t, x, 0, 0, 0)$ and $h_i(x)$, the inequality $|a+b+c+d|^2 \leq 4\{|a|^2 + |b|^2 + |c|^2 + |d|^2\}$, $\forall a, b, c, d \in \mathbb{R}$ and finally the Cauchy-Schwarz

one, we have:

$$\begin{aligned}
&\mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr\big] \\
&\quad \leq \bar{C}^2 + 4\nu\bar{C}^2 T + \tfrac{1}{\nu}\mathbb{E}\big[\int_0^T |Y_r^{i,n}|^2 dr\big] + 4\nu C_3 \mathbb{E}\big[\int_0^T \textstyle\sum_{l=1,m} |Y_r^{l,n}|^2 dr\big] \\
&\qquad + 4\nu C_3 \mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr\big] + 4\nu C_3 \mathbb{E}\big[\int_0^T \int_E |V_r^{i,n-1}(e)|^2 \lambda(de) dr\big] \\
&\qquad + \tfrac{1}{\epsilon}\mathbb{E}\big[\sup_{s\leq T} |Y_s^{i,n}|^2\big] + \epsilon \mathbb{E}\big[\big(K_T^{i,n}\big)^2\big],
\end{aligned}$$

for a suitable positive constants C_1, C_2 and C_3. Choose now ν such that $4\nu C_3 < 1$, and taking the sum over all $i \in \mathcal{I}$, we obtain:

$$\begin{aligned}
&\textstyle\sum_{i=1,m} \mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr\big] \\
&\quad \leq C\Big(1 + \textstyle\sum_{i=1,m} \mathbb{E}\big[\sup_{s\leq T} |Y_s^{i,n}|^2\big] + \sum_{i=1,m} \mathbb{E}\big[\int_0^T \int_E |V_r^{i,n-1}(e)|^2 \lambda(de) dr\big]\Big) \\
&\qquad + \epsilon \textstyle\sum_{i=1,m} \mathbb{E}\big[\big(K_T^{i,n}\big)^2\big],
\end{aligned}$$

where $C = C(T, m, \nu, \epsilon) > 0$ is an appropriate constant independent of n. Through the convergence of $(Y^{i,n})_n$ in $\mathcal{S}^2$, we have $\sup_{n\geq 0} \mathbb{E}\big[\sup_{s\leq T} |Y_s^{i,n}|^2\big] \leq C$, and then taking into consideration the convergence of $(V^{i,n})_n$ in $\mathcal{H}^2(\mathcal{L}^2(\lambda))$, we finally obtain

$$\textstyle\sum_{i=1,m} \mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr\big] \leq C + \epsilon \sum_{i=1,m} \mathbb{E}\big[\big(K_T^{i,n}\big)^2\big]. \tag{3.3.29}$$

Now, from the relation

$$\begin{aligned}
K_T^{i,n} = Y_0^{i,n} - h_i(X_T^{t,x}) - \textstyle\int_0^T f_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, V_r^{i,n-1}) dr \\
+ \textstyle\int_0^T Z_r^{i,n} dB_r + \int_0^T \int_E V_r^{i,n}(e) \tilde{\mu}(dr, de),
\end{aligned} \tag{3.3.30}$$

and once again, by the linearization procedure of the Lipschitz function f_i and the boundedness of $f_i(t, x, 0, 0, 0)$ and $h_i(x)$, there exist some positive constant C' such that

$$\begin{aligned}
\textstyle\sum_{i=1,m} \mathbb{E}\big[\big(K_T^{i,n}\big)^2\big] &\leq C'\Big(1 + \textstyle\sum_{i=1,m} \mathbb{E}\big[\sup_{s\leq T} |Y_s^{i,n}|^2\big] + \sum_{i=1,m} \mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr\big] \\
&\qquad + \textstyle\sum_{i=1,m} \mathbb{E}\big[\int_0^T \int_E |V_r^{i,n-1}(e)|^2 \lambda(de) dr\big]\Big) \\
&\leq C'\Big(1 + \textstyle\sum_{i=1,m} \mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr\big]\Big).
\end{aligned}$$

Combining this last estimate with (3.3.29) and choosing ϵ small enough since it is arbitrary, then we obtain a constant $\bar{C}$ such that

$$\textstyle\sum_{i=1,m} \mathbb{E}\big[\int_0^T |Z_r^{i,n}|^2 dr + \big(K_T^{i,n}\big)^2\big] \leq \bar{C}. \tag{3.3.31}$$

Now, for any $n, p \geq 1$, it follows from Itô's formula that

$$\begin{aligned}
&\mathbb{E}\big[\int_0^T |Z_r^{i,n} - Z_r^{i,p}|^2 dr\big] \\
&\quad \leq 2\mathbb{E}\big[\int_0^T \big(Y_r^{i,n} - Y_r^{i,p}\big)\big(f_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, V_r^{i,n-1}) \\
&\qquad - f_i(r, X_r^{t,x}, (Y_r^{k,p})_{k\in\mathcal{I}}, Z_r^{i,p}, V_r^{i,p-1})\big) dr\big] \\
&+ \qquad 2\mathbb{E}\big[\int_0^T \big(Y_r^{i,n} - Y_r^{i,p}\big)\big(dK_r^{i,n}(r) - dK_r^{i,p}\big)\big].
\end{aligned}$$

By Cauchy-Schwarz inequality and using the inequality $2ab \leq \frac{1}{\eta}a^2 + \eta b^2$ for $\eta > 0$, we have

$$\begin{aligned}
&\mathbb{E}\big[\int_0^T |Z_r^{i,n} - Z_r^{i,p}|^2 dr\big] \leq 2\sqrt{\mathbb{E}\big[\sup_{s\leq T} |Y_r^{i,n} - Y_r^{i,p}|^2\big]} \times \\
&\sqrt{\mathbb{E}\big[\int_0^T \big|f_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, V_r^{i,n-1}) - f_i(r, X_r^{t,x}, (Y_r^{k,p})_{k\in\mathcal{I}}, Z_r^{i,p}, V_r^{i,p-1})\big|^2 dr\big]} \\
&\quad + \frac{1}{\eta}\,\mathbb{E}\big[\sup_{s\leq T} |Y_s^{i,n} - Y_s^{i,p}|^2\big] + \eta\,\mathbb{E}\big[\big(K_T^{i,n} + K_T^{i,p}\big)^2\big].
\end{aligned}$$

But there exists a constant $C \geq 0$ (independent of n and p) such that, for all $n, p \geq 1$,

$$\mathbb{E}\big[\int_0^T \big|f_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, V_r^{i,n-1}) - f_i(r, X_r^{t,x}, (Y_r^{k,p})_{k\in\mathcal{I}}, Z_r^{i,p}, V_r^{i,p-1})\big|^2 dr\big] \leq C. \tag{3.3.32}$$

Then taking the limit w.r.t n, p in the previous inequality and taking into account of (3.3.31) and the convergence of $Y^{i,n}$ in $\mathcal{S}^2$, we deduce that:

$$\limsup_{n,p\to\infty} \mathbb{E}\big[\int_0^T |Z_r^{i,n} - Z_r^{i,p}|^2 dr\big] \leq \bar{C}\eta.$$

As η is arbitrary then $(Z^{i,n})_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}^{2,d}$. Therefore there exists a process $Z^{i,t,x}$ which belongs to $\mathcal{H}^{2,d}$ such that $(Z^{i,n})_{n\geq 0}$ converges to $Z^{i,t,x}$ in $\mathcal{H}^{2,d}$. Finally, since for $s \leq T$,

$$\begin{aligned}
K_s^{i,n} = Y_0^{i,n} - Y_s^{i,n} - \int_0^s f_i(r, X_r^{t,x}, (Y_r^{k,n})_{k\in\mathcal{I}}, Z_r^{i,n}, V_r^{i,n-1}) dr \\
+ \int_0^s Z_r^{i,n} dB_r + \int_0^s \int_E V_r^{i,n}(e)\tilde{\mu}(dr, de),
\end{aligned}$$

then, we have also $\mathbb{E}\big[\sup_{s\leq T} |K_s^{i,n} - K_s^{i,p}|^2\big] \to 0$ as $n, p \to \infty$. Thus, there exists a process $(K_s^{i,t,x})_{s\leq T}$ which belongs to $\mathcal{A}^2$ such that $\mathbb{E}\big[\sup_{s\leq T} |K_s^{i,n} -$

$K_s^{i,t,x}|^2 dr\big] \to 0$ as $n \to \infty$. Moreover we have: $\forall s \in [0,T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad +K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})). \end{cases} \tag{3.3.33}$$

Finally, let us show that the third condition in (3.3.1) is satisfied by $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. Actually

$$\begin{aligned} \int_0^T (Y_s^{i,t,x} &- \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,t,x} \\ &= \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))(dK_s^{i,t,x} - dK_s^{i,n}) \\ &+ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,n}. \end{aligned} \tag{3.3.34}$$

Let ω be fixed. It follows from the uniform convergence of $(Y^{i,n})_n$ to $(Y^{i,t,x})_{i\in\mathcal{I}}$ that, for any $\epsilon \geq 0$, there exist $N_\epsilon(\omega) \in \mathbb{N}$, such that for any $n \geq N_\epsilon(\omega)$ and $s \leq T$,

$$\begin{aligned} Y_s^{i,t,x}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x}(\omega) - g_{ij}(s, X_s^{t,x}(\omega))) \\ \leq Y_s^{i,n}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n}(\omega) - g_{ij}(s, X_s^{t,x}(\omega))) + \epsilon. \end{aligned}$$

Therefore, for $n \geq N_\epsilon(\omega)$ we have

$$\int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,n} \leq \epsilon K_T^{i,n}(\omega). \tag{3.3.35}$$

On the other hand, the function

$$\begin{aligned} Y^{i,t,x}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y^{j,t,x}(\omega) - g_{ij}(., X_.^{t,x}(\omega))) : \\ s \in [0,T] \longmapsto Y_s^{i,t,x}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x}(\omega) - g_{ij}(s, X_s^{t,x}(\omega))) \end{aligned}$$

is RCLL and then bounded, then there exists a sequence of step functions $(f^m(\omega,.))_{m\geq 1}$ which converges uniformly on $[0,T]$ to $Y^{i,t,x}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y^{j,t,x}(\omega) - g_{ij}(., X_.^{t,x}(\omega)))$, i.e., there exist $m_\epsilon(\omega) \geq 0$ such that for $m \geq m_\epsilon(\omega)$, we have

$$\forall s \leq T, \ \left|Y_s^{i,t,x}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x}(\omega) - g_{ij}(s, X_s^{t,x}(\omega))) - f^m(\omega, s)\right| < \epsilon.$$

It follows that

$$\begin{aligned}
&\int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))(dK_s^{i,t,x} - dK_s^{i,n}) \\
&= \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})) - f^m(\omega, s))(dK_s^{i,t,x} - dK_s^{i,n}) \\
&\quad + \int_0^T f^m(\omega, s)(dK_s^{i,t,x} - dK_s^{i,n}) \\
&\leq \int_0^T f^m(\omega, s)(dK_s^{i,t,x} - dK_s^{i,n}) + \epsilon(K_T^{i,t,x}(\omega) + K_T^{i,n}(\omega)).
\end{aligned}$$

But the right-hand side converges to $2\epsilon K_T^{i,t,x}(\omega)$, as $n \to \infty$, since $f^m(\omega, .)$ is a step function and then $\int_0^T f^m(\omega, s)(dK_s^{i,t,x} - dK_s^{i,n}) \to 0$ as $n \to \infty$. Therefore, we have

$$\limsup_{n\to\infty} \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))(dK_s^{i,t,x} - dK_s^{i,n}) \leq 2\epsilon K_T^{i,t,x}. \tag{3.3.36}$$

Finally, from (3.3.34), (3.3.35) and (3.3.36) we deduce that

$$\int_0^T (Y_s^{i,t,x}(\omega) - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x}(\omega) - g_{ij}(s, X_s^{t,x})(\omega)))dK_s^{i,t,x}(\omega) \leq 3\epsilon K_T^{i,t,x}(\omega).$$

As ϵ is arbitrary and $Y_s^{i,t,x} \geq \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x}))$, then

$$\int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,t,x} = 0,$$

which completes the proof. □

Now, we study the system (3.3.1) in the general case i.e., without assuming the boundedness of the functions $f_i(t, x, \vec{0}, 0, 0)$ and $h_i(x)$. We need an extra assumption that the functions $\bar{f}_i$, $i \in \mathcal{I}$, should satisfy since we are going to introduce a transformed system in order to fall in the previous framework. This transformation induces some perturbations of the Assumptions (H1) and specifically (H1)-(i). To remedy to this situation we are led to weaken this latter assumption. For completeness, let us precise once more the assumptions that the data should satisfy and which are:

(H5) For any $i \in \{1, ..., m\}$,

(a) For any Φ, a bounded continuous function from $[0, T] \times \mathbb{R}^k$ to $\mathbb{R}$, the function $(t, x, \vec{y}, z) \mapsto \bar{f}_i(t, x, \vec{y}, z, \Phi(t, x))$ is continuous and verifies the condition (3.2.12).

(b) The function $\bar{f}_i$ verifies (H1)-ii), iii), iv).

(c) The function γ_i verifies (H1)-v).

Note that the assumption (H5)-(a) is satisfied if for any $i \in \mathcal{I}$, $\bar{f}_i(t,x,\vec{y},z,q) = \Psi_i^1(t,x)+\Psi_i^2(\vec{y},z,q)$ where, in their respective spaces, Ψ_i^1 is continuous and Ψ_i^2 is Lipschitz. It is also satisfied if $\bar{f}_i$, $i \in \mathcal{I}$, are Lipschitz in $(x,\vec{y},z,q)$ uniformly w.r.t t.

The following is the main result of this section.

Theorem 3.3.4 *Assume that the functions $(\bar{f}_i)_{i\in\mathcal{I}}$ and $(\gamma_i)_{i\in\mathcal{I}}$ verify Assumption (H5) and, $(g_{ij})_{i,j\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ verify Assumptions (H2) and (H3). Then the system (3.3.1) has a solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. Moreover there exists continuous functions $(u^i)_{i\in\mathcal{I}}$ of polynomial growth such that for any $i \in \mathcal{I}$, $(t,x) \in [0,T]\times\mathbb{R}^k$,*

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ \forall s \in [t,T].$$

Proof: First we are going to transform the system (3.3.1) in such a way to fall in the same framework as the one of Proposition 3.3.2. So let ϕ be a function defines as follows (p is the same or greater than the exponents which are involved in (H1)-iii) and (H3)):

$$\phi(x) := \frac{1}{(1+|x|^2)^p},\ x \in \mathbb{R}^k, \tag{3.3.37}$$

and for $s \in [0,T]$ let us define,

$$\overline{Y}_s^{i,t,x} := Y_s^{i,t,x}\phi(X_s^{t,x}). \tag{3.3.38}$$

Then, by Itô's formula we have: $\forall s \in [0,T]$,

$$\begin{aligned}\phi(X_s^{t,x}) =& \phi(X_0^{t,x}) + \textstyle\int_0^s D_x\phi(X_{r^-}^{t,x})dX_r^{t,x} + \frac{1}{2}\int_0^s \mathrm{Tr}(D_{xx}^2\phi(X_{r^-}^{t,x})\sigma\sigma^\top(r,X_r^{t,x}))dr \\ &+ \textstyle\sum_{0<r\le s}\{\phi(X_r^{t,x}) - \phi(X_{r^-}^{t,x}) - D_x\phi(X_{r^-}^{t,x})\Delta_r X^{t,x}\}.\end{aligned}$$

Since $X^{t,x}$ satisfies the SDE (3.2.1), then for $s \in [0,T]$,

$$\begin{aligned}
&\sum_{0<r\leq s}\{\phi(X_r^{t,x}) - \phi(X_{r^-}^{t,x}) - D_x\phi(X_{r^-}^{t,x})\Delta_r X^{t,x}\}\\
&= \textstyle\sum_{0<r\leq s}\{\phi(X_{r^-}^{t,x} + \Delta_r X^{t,x}) - \phi(X_{r^-}^{t,x}) - D_x\phi(X_{r^-}^{t,x})\Delta_r X^{t,x}\}\\
&= \textstyle\int_0^s\int_E\{\phi(X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - \phi(X_{r^-}^{t,x}) - D_x\phi(X_{r^-}^{t,x})\beta(X_{r^-}^{t,x},e)\}\mu(dr,de)\\
&= \textstyle\int_0^s\int_E\{\phi(X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - \phi(X_{r^-}^{t,x}) - D_x\phi(X_{r^-}^{t,x})\beta(X_{r^-}^{t,x},e)\}\tilde{\mu}(dr,de)\\
&\quad + \textstyle\int_0^s\int_E\{\phi(X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - \phi(X_{r^-}^{t,x}) - D_x\phi(X_{r^-}^{t,x})\beta(X_{r^-}^{t,x},e)\}\lambda(de)ds.
\end{aligned}$$

Next, going back to (3.3.38) and using the Integration by Parts formula we obtain: $\forall s \in [0,T]$,

$$d\overline{Y}_s^{i,t,x} = Y_{s^-}^{i,t,x}d\phi(X_s^{t,x}) + \phi(X_{s^-}^{t,x})dY_s^{i,t,x} + d[Y^{i,t,x},\phi(X^{t,x})]_s,$$

where

$$[Y^{i,t,x},\phi(X^{t,x})]_s = \langle Y^{i,t,x},\phi(X^{t,x})\rangle_s^c + \sum_{0<r\leq s}\Delta_r Y^{i,t,x}\Delta_r\phi(X^{t,x}).$$

But

$$d\langle Y^{i,t,x},\phi(X^{t,x})\rangle_s^c = Z_s^{i,t,x}D_x\phi(X_{s^-}^{t,x})\sigma(s,X_s^{t,x})ds;$$

and

$$\begin{aligned}
\sum_{0<r\leq s}\Delta_r Y^{i,t,x}\Delta_r\phi(X^{t,x}) &= \sum_{0<r\leq s}\Delta_r Y^{i,t,x}\{\phi(X_r^{t,x}) - \phi(X_{r^-}^{t,x})\}\\
&= \sum_{0<r\leq s}\Delta_r Y^{i,t,x}\{\phi(X_{r^-}^{t,x} + \Delta_r X^{t,x}) - \phi(X_{r^-}^{t,x})\}\\
&= \textstyle\int_0^s\int_E V_r^{i,t,x}(e)\{\phi(X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - \phi(X_{r^-}^{t,x})\}\mu(dr,de)\\
&= \textstyle\int_0^s\int_E V_r^{i,t,x}(e)\{\phi(X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - \phi(X_{r^-}^{t,x})\}\tilde{\mu}(dr,de)\\
&\quad + \textstyle\int_0^s\int_E V_r^{i,t,x}(e)\{\phi(X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - \phi(X_{r^-}^{t,x})\}\lambda(de)dr.
\end{aligned}$$

Then it follows that: $\forall s \in [0,T]$

$$
\begin{aligned}
d\overline{Y}_s^{i,t,x} = \Big\{ & -\phi(X_{s^-}^{t,x})\bar{f}_i(s, X_s^{t,x}, (Y_s^{k,t,x})_{k\in\mathcal{I}}, Z_s^{i,t,x}, \textstyle\int_E V_s^{i,t,x}(e)\gamma_i(X_s^{t,x}, e)\lambda(de)) \\
& + Y_{s^-}^{t,x}\big(D_x\phi(X_{s^-}^{t,x})b(s, X_s^{t,x}) + \frac{1}{2}\mathrm{Tr}(D_{xx}^2\phi(X_{s^-}^{t,x})\sigma\sigma^\top(s, X_s^{t,x}))\big) \\
& + Y_{s^-}^{t,x}\textstyle\int_E \big(\phi(X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \phi(X_{s^-}^{t,x}) - D_x\phi(X_{s^-}^{t,x})\beta(X_{s^-}^{t,x}, e)\big)\lambda(de) \\
& + Z_s^{i,t,x}D_x\phi(X_{s^-}^{t,x})\sigma(s, X_s^{t,x}) + \textstyle\int_E V_s^{i,t,x}(e)\{\phi(X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) \\
& - \phi(X_{s^-}^{t,x})\}\lambda(de)\Big\}ds - \phi(X_{s^-}^{t,x})dK_s^{i,t,x} + \Big\{\phi(X_{s^-}^{t,x})Z_s^{i,t,x} \\
& + Y_{s^-}^{i,t,x}D_x\phi(X_{s^-}^{t,x})\sigma(s, X_s^{t,x})\Big\}dB_s + \textstyle\int_E \Big\{Y_{s^-}^{i,t,x}\{\phi(X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) \\
& - \phi(X_{s^-}^{t,x})\} + V_s^{i,t,x}(e)\phi(X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e))\Big\}\tilde{\mu}(ds, de).
\end{aligned}
$$

Next let us set, for $s \in [0,T]$,

$$
\begin{aligned}
& d\overline{K}_s^{i,t,x} := \phi(X_{s-}^{t,x})dK_s^{i,t,x} \text{ and } \overline{K}_0^{i,t,x} = 0, \\
& \overline{Z}_s^{i,t,x} := \phi(X_{s^-}^{t,x})Z_s^{i,t,x} + Y_{s^-}^{i,t,x}D_x\phi(X_{s^-}^{t,x})\sigma(s, X_s^{t,x}) \quad \text{and} \\
& \overline{V}_s^{i,t,x}(e) := Y_{s^-}^{i,t,x}\{\phi(X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \phi(X_{s^-}^{t,x})\} + V_s^{i,t,x}(e)\phi(X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)).
\end{aligned}
$$

Then $((\overline{Y}^{i,t,x}, \overline{Z}^{i,t,x}, \overline{V}^{i,t,x}, \overline{K}^{i,t,x}))_{i\in\mathcal{I}}$ verifies: $\forall s \in [0,T]$,

$$
\begin{cases}
\overline{Y}_s^{i,t,x} = \check{h}_i(X_T^{t,x}) + \int_s^T \check{F}_i(r, X_r^{t,x}, (\overline{Y}_r^{k,t,x})_{k\in\mathcal{I}}, \overline{Z}_r^{i,t,x}, \int_E \overline{V}_r^{i,t,x}\gamma_i(X_r^{t,x}, e))dr \\
\qquad\qquad + \overline{K}_T^{i,t,x} - \overline{K}_s^{i,t,x} - \int_s^T \overline{Z}_r^{i,t,x}dB_r - \int_s^T\int_E \overline{V}_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\
\overline{Y}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(\overline{Y}_s^{j,t,x} - \check{g}_{ij}(s, X_s^{t,x})) \\
\int_0^T (\overline{Y}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(\overline{Y}_s^{j,t,x} - \check{g}_{ij}(s, X_s^{t,x})))d\overline{K}_s^{i,t,x} = 0,
\end{cases}
\tag{3.3.39}
$$

where for any $i, j \in \mathcal{I}$,

$$
\check{h}_i(X_T^{t,x}) := \phi(X_T^{t,x})h_i(X_T^{t,x}),\ \check{g}_{ij}(s, X_s^{t,x}) := \phi(X_s^{t,x})g_{ij}(s, X_s^{t,x}),
$$

and

$$\begin{aligned}\check{F}_i(s,x,\vec{y},z,v) := \phi(x) f_i\Big(&s,x,\phi(x)^{-1}\vec{y},\phi(x)^{-1}z - y^i\phi(x)^{-2}D_x\phi(x)\sigma(s,x),\\
&\textstyle\int_E \gamma^i(x,e)\phi(x+\beta(x,e))^{-1}v(e)\lambda(de) - y^i\int_E \gamma^i(x,e)\phi(x+\beta(x,e))^{-1}\\
&\times \phi(x)^{-1}(\phi(x+\beta(x,e))-\phi(x))\lambda(de)\Big) - y^i\phi(x)^{-1}\Big\{b(s,x)D_x\phi(x)\\
&+\frac{1}{2}\mathrm{Tr}(D^2_{xx}\phi(x)\sigma\sigma^\top(s,x)) + \textstyle\int_E(\phi(x+\beta(x,e))-\phi(x)-D_x\phi(x)\beta(x,e))\lambda(de)\\
&+\textstyle\int_E \lambda(de)(\phi(x+\beta(x,e))-\phi(x))^2\phi(x+\beta(x,e))^{-1}\Big\}\\
&-z\phi(x)^{-1}D_x\phi(x)^\top\sigma(s,x) - y^i\phi(x)^{-2}D_x\phi(x)^\top\sigma(s,x)\sigma(s,x)^\top D_x\phi(x)\\
&-\textstyle\int_E(\phi(x+\beta(x,e))-\phi(x))\phi(x+\beta(x,e))^{-1}v(e)\lambda(de).\end{aligned}$$

Here, let us notice that the functions $(\check{g}_{ij})_{i,j\in\mathcal{I}}$ and $(\check{h}_i)_{i\in\mathcal{I}}$ verify Assumptions (H2)-(H3) while $(\check{F}_i)_{i\in\mathcal{I}}$ satisfy (H1)-ii), iii), iv). The functions $(\check{F}_i)_{i\in\mathcal{I}}$ do not satisfy (H1)-i) but they satisfy the condition (3.2.12). However the following scheme $(\check{Y}^{i,n,t,x},\check{Z}^{i,n,t,x},\check{V}^{i,n,t,x},\check{K}^{i,n,t,x})_{i\in\mathcal{I}}$, $n\geq 1$, is well-posed (by Theorem 3.3.1): $\check{V}^{i,0,t,x}=0$ and for $n\geq 1$ (we omit the dependence on t,x as there is no confusion) and $s\in[0,T]$,

$$\begin{cases}\check{Y}^{i,n}\in\mathcal{S}^2,\check{Z}^{i,n}\in\mathcal{H}^{2,d},\check{V}^{i,n}\in\mathcal{H}^2(\mathcal{L}^2(\lambda)),\text{ and }\check{K}^{i,n}\in\mathcal{A}^2;\\
\check{Y}^{i,n}_s=\check{h}_i(X^{t,x}_T)+\int_s^T\check{F}_i(r,X^{t,x}_r,(\check{Y}^{k,n}_r)_{k\in\mathcal{I}},\check{Z}^{i,n}_r,\check{V}^{i,n-1}_r)dr\\
\qquad\qquad +\check{K}^{i,n}_T-\check{K}^{i,n}_s-\int_s^T\check{Z}^{i,n}_r dB_r-\int_s^T\int_E\check{V}^{i,n}_r(e)\tilde{\mu}(dr,de);\\
\check{Y}^{i,n}_s\geqslant\max_{j\in\mathcal{I}^{-i}}(\check{Y}^{j,n}_s-\check{g}_{ij}(s,X^{t,x}_s));\\
\int_0^T(\check{Y}^{i,n}_s-\max_{j\in\mathcal{I}^{-i}}(\check{Y}^{j,n}_s-\check{g}_{ij}(s,X^{t,x}_s)))d\check{K}^{i,n}_s=0.\end{cases}\tag{3.3.40}$$

Next we need to have a representation for $\check{Y}^{i,n}$ and $\check{V}^{i,n}$ similar to (3.3.3), i.e., there exist deterministic continuous bounded functions $(\check{u}^i)_{i\in\mathcal{I}}$ such that for

any $(t,x) \in [0,T] \times \mathbb{R}^k$ and any $s \in [t,T]$:

$$\begin{aligned} &\text{(a)}\ \ \check{Y}_s^{i,n} := \check{u}^{i,n}(s, X_s^{t,x}) \text{ and} \\ &\text{(b)}\ \ \check{V}_s^{i,n}(e) := \check{u}^{i,n}(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \check{u}^{i,n}(s, X_{s^-}^{t,x}), \\ &\qquad\qquad ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [t,T] \times \Omega \times E. \end{aligned} \tag{3.3.41}$$

But this can be shown by induction. For $n = 1$ the property holds true since $\check{V}_s^{i,0} = 0$ and then
$(\check{Y}^{i,1,t,x}, \check{Z}^{i,1,t,x}, \check{V}^{i,1,t,x}, \check{K}^{i,1,t,x})_{i\in\mathcal{I}}$ verify: $\forall s \leq T$,

$$\begin{cases} \check{Y}^{i,1} \in \mathcal{S}^2, \check{Z}^{i,1} \in \mathcal{H}^{2,d}, \check{V}^{i,1} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } \check{K}^{i,1} \in \mathcal{A}^2; \\ \check{Y}_s^{i,1} = \check{h}_i(X_T^{t,x}) + \int_s^T \check{F}_i(r, X_r^{t,x}, (\check{Y}_r^{k,1})_{k\in\mathcal{I}}, \check{Z}_r^{i,1}, 0)dr \\ \qquad\qquad + \check{K}_T^{i,1} - \check{K}_s^{i,1} - \int_s^T \check{Z}_r^{i,1} dB_r - \int_s^T \int_E \check{V}_r^{i,1}(e)\tilde{\mu}(dr,de); \\ \check{Y}_s^{i,1} \geqslant \max_{j\in\mathcal{I}^{-i}} (\check{Y}_s^{j,1} - \check{g}_{ij}(s, X_s^{t,x})); \\ \int_0^T (\check{Y}_s^{i,1} - \max_{j\in\mathcal{I}^{-i}} (\check{Y}_s^{j,1} - \check{g}_{ij}(s, X_s^{t,x})))d\check{K}_s^{i,1} = 0. \end{cases} \tag{3.3.42}$$

Since we are in the Markovian framework and the functions $(\check{F}_i(s,x,\vec{y},z,0))_{i\in\mathcal{I}}$ verify (H1),ii), iii), iv) and (3.2.12), and $\check{F}_i(t,x,\vec{0},0,0)$ and $\check{h}_i$, $i \in \mathcal{I}$, are bounded, then there exist deterministic continuous bounded functions $(\check{u}^{i,1})_{i\in\mathcal{I}}$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and $s \in [t,T]$, $\check{Y}_s^{i,1} := \check{u}^{i,1}(s, X_s^{t,x})$, $i \in \mathcal{I}$. Next by continuity and since the Levy measure $\lambda(.)$ is bounded, we have

$$\begin{aligned} \check{V}_s^{i,1}(e) := 1_{\{s\geq t\}}(\check{u}^{i,1}(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \check{u}^{i,1}(s, X_{s^-}^{t,x}), \\ ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [0,T] \times \Omega \times E. \end{aligned}$$

Therefore the property holds for $n = 1$. Next if it is satisfied for some n, by the same reasoning and by using Assumption (H5), we deduce that the property holds for $n+1$ and then the property holds for any $n \geq 1$. The proof of this claim is deeply related to the use of Assumption (H5)-i) and the result of the first part.

To proceed it is enough to follow the same steps as in Steps 3 and 4 in the proof of Proposition 3.3.2 to show that:

i) Let $i \in \mathcal{I}$ be fixed. The sequence $(\check{u}^{i,n})_{n\geq 0}$ converges uniformly on $[0,T]\times\mathbb{R}$ to some bounded continuous function $\check{u}^i$. The representation (3.3.41) allows to show that the sequence $(\check{Y}^{i,n})_{n\geq 0}$ converges to some process $\bar{Y}^i$ in $\mathcal{S}^2_{[t,T]}$. Next as in Step 4, we have also the convergence of $(\check{Y}^{i,n})_{n\geq 0}$ in $\mathcal{S}^2_{[0,t]}$ to $\bar{Y}^i$ a deterministic continuous bounded function. Therefore, the sequence $(\check{Y}^{i,n})_{n\geq 0}$ converges in $\mathcal{S}^2$ to some process $\bar{Y}^i$. On the other hand we have also the convergence of $(\check{V}^{i,n})_{n\geq 0}$ in $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ to $\bar{V}^i(t,e) :=:= 1_{\{s\geq t\}}(\check{u}^i(s, X^{t,x}_{s^-} + \beta(X^{t,x}_{s^-}, e)) - \check{u}^i(s, X^{t,x}_{s^-}),\ ds\otimes d\mathbb{P}\otimes d\lambda$ on $[0,T]\times\Omega\times E$ and the convergence of $(\check{Z}^{i,n})_{n\geq 0}$ (resp. $(\check{K}^{i,n})_{n\geq 0}$) in $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ (resp. $\mathcal{S}^2$) to a process $\bar{Z}^i$ (resp. $\bar{K}^i$), $i\in\mathcal{I}$;

ii) $((\overline{Y}^{i,t,x}, \overline{Z}^{i,t,x}, \overline{V}^{i,t,x}, \overline{K}^{i,t,x}))_{i\in\mathcal{I}}$ is a solution of the system associated with $((\check{F}_i)_{i\in\mathcal{I}}, (\check{h}_i)_{i\in\mathcal{I}}, \check{g}_{ij})_{i,j\in\mathcal{I}})$.

To proceed for $s\in[0,T]$, let us set:

$$
\begin{aligned}
Y^{i,t,x}_s &:= (\phi(X^{t,x}_s))^{-1}\overline{Y}^{i,t,x}_s,\\
dK^{i,t,x}_s &:= (\phi(X^{t,x}_{s-}))^{-1}\overline{Y}^{i,t,x}_s \text{ and } K^{i,t,x}_0 = 0,\\
Z^{i,t,x}_s &:= (\phi(X^{t,x}_s))^{-1}\big\{\overline{Z}^{i,t,x}_s - ((\phi(X^{t,x}_s))^{-1}\overline{Y}^{i,t,x}_s D_x\phi(x)\sigma(s,x)\big\},\\
V^{i,t,x}_s(e) &:= (\phi(X^{t,x}_{s^-} + \beta(X^{t,x}_{s^-}, e))^{-1}\big\{\overline{V}^{i,t,x}_s - \phi(X^{t,x}_{s^-})^{-1}\overline{Y}^{i,t,x}_s\big(\phi(X^{t,x}_{s^-} + \beta(X^{t,x}_{s^-}, e))\\
&\qquad - \phi(X^{t,x}_{s^-})\big)\big\}.
\end{aligned}
$$

Then $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ is a solution of system (3.3.1). Moreover in setting $u^i(t,x) := (\phi(x))^{-1}\overline{u}^i(t,x)$, $(t,x)\in[0,T]\times\mathbb{R}^k$ and $i\in\mathcal{I}$ we obtain that for any $s\in[t,T]$, $Y^{i,t,x}_s = u^i(s, X^{t,x}_s)$ for any $i\in\mathcal{I}$ and u^i is of polynomial growth as $\bar{u}^i$ is bounded. □

As a by-product of the Proposition 3.2 and Theorem 3.3 we have the following

Corollary 3.3.5 *For any $i\in\mathcal{I}$ and $(t,x)\in[0,T]\times\mathbb{R}^k$,*

$$
V^{i,t,x}_s(e) = 1_{\{s\geq t\}}\{u^i(s, X^{t,x}_{s^-} + \beta(X^{t,x}_{s^-}, e)) - u^i(s, X^{t,x}_{s^-})\},
$$
$$
ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E.
$$

Now, we provide the uniqueness of the Markovian solution of the system of reflected BSDEs (3.3.1).

Proposition 3.3.6 *Let $(\tilde{u}^i)_{i\in\mathcal{I}}$ be a deterministic continuous functions of polynomial growth such that*

$$\forall s \in [t,T],\ Y_s^{i,t,x} = \tilde{u}^i(s, X_s^{t,x}). \tag{3.3.43}$$

Then, for any $i \in \mathcal{I}$, $\tilde{u}^i = u^i$.

Proof: In order to show that the Markovian solution of the system of reflected BSDEs is unique (3.3.1), we suppose that there exist other continuous with polynomial growth functions $(\tilde{u}^i)_{i\in\mathcal{I}}$ such that:

$$\forall s \in [t,T],\ \tilde{Y}_s^{i,t,x} = \tilde{u}^i(s, X_s^{t,x}),$$

where $(\tilde{Y}^{i,t,x})_{i\in\mathcal{I}}$ is the first component of the solution of the following system of RBSDEs with jumps with interconnected obstacles: for any $i \in \mathcal{I}$ and $s \in [t,T]$,

$$\begin{cases} \tilde{Y}^{i,t,x} \in \mathcal{S}^2, \tilde{Z}^{i,t,x} \in \mathcal{H}^{2,d}, \tilde{V}^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } \tilde{K}^{i,t,x} \in \mathcal{A}^2; \\ \tilde{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (\tilde{Y}_r^{k,t,x})_{k\in\mathcal{I}}, \tilde{Z}_r^{i,t,x}, \int_E \gamma_i(X_r^{t,x}, e)\tilde{V}_r^{i,t,x}(e)\lambda(de))dr \\ \qquad\qquad +\tilde{K}_T^{i,t,x} - \tilde{K}_s^{i,t,x} - \int_s^T \tilde{Z}_r^{i,t,x} dB_r - \int_s^T \int_E \tilde{V}_r^{i,t,x}(e)\tilde{\mu}(dr,de), \\ \tilde{Y}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}} (\tilde{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_t^T (\tilde{Y}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}} (\tilde{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))d\tilde{K}_s^{i,t,x} = 0. \end{cases} \tag{3.3.44}$$

On the other hand, as for any $i \in \{1, ...m\}$, $\tilde{u}^i$ is continuous function of polynomial growth and since the Lévy measure $\lambda(.)$ is finite, one has

$$\tilde{V}_s^{i,t,x}(e) = \tilde{u}^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \tilde{u}^i(s, X_{s^-}^{t,x}),\ ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [t,T]\times\Omega\times E.$$

Now, let $s \in [t,T]$ and an admissible strategy $\delta \in \mathcal{A}_s^i$. Let $(P_r^\delta, N_r^\delta, Q_r^\delta)_{r\in[s,T]}$ be the triplet of processes associated with δ and which solves the following

BSDE: $\forall r \in [s,T]$

$$P_r^\delta = h^\delta(X_T^{t,x}) + \int_r^T f^\delta(\tau, X_\tau^{t,x}, N_\tau^\delta)d\tau - \int_r^T N_\tau^\delta dB_\tau - \int_r^T \int_E Q_\tau^\delta(e)\tilde{\mu}(d\tau, de) - A_T^\delta + A_r^\delta,$$

where, when $\delta_\tau = i$, $f^\delta(\tau, X_\tau^{t,x}, z)$ is equal to

$$\bar{f}_i(\tau, X_\tau^{t,x}, (\tilde{u}^k(\tau, X_\tau^{t,x})_{k\in\mathcal{I}}, z, \int_E \gamma_i(X_\tau^{t,x}, e)\{\tilde{u}^i(\tau, X_{\tau-}^{t,x} + \beta(X_{\tau-}^{t,x}, e)) - \tilde{u}^i(\tau, X_{\tau-}^{t,x})\}\lambda(de)).$$

Therefore, we have the following representation of $\tilde{Y}^i$:

$$\tilde{Y}_s^i = \operatorname{esssup}_{\delta\in\mathcal{A}_s^i}(P_s^\delta - A_s^\delta).$$

Next, the same procedure as the one which leads to inequality (3.3.14) can be used here to deduce that for any $i \in \mathcal{I}$,

$$|u^i(t,x) - \tilde{u}^i(t,x)|^2 \le 2C\mathbb{E}\big[\int_t^T e^{\alpha_0(r-t)} \int_E \sum_{k=1,m} \big\{|(u^i - \tilde{u}^i)(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e))|^2 + |(u^i - \tilde{u}^i)(r, X_{r^-}^{t,x})|^2\big\}\lambda(de)dr\big].$$

We now consider two cases.

Case 1: The functions u^i and $\tilde{u}^i$, $i \in \mathcal{I}$, are bounded.

Let η be the constant give in Step 3 and which does not depend on the terminal condition $(h_i)_{i\in\mathcal{I}}$ and verifies $\frac{4}{\alpha_0} m\lambda(E)(e^{\alpha_0\eta} - 1) = \frac{3}{4}$. Then, we deduce from (3.3.14), that for any $i \in \mathcal{I}$,

$$\|u^i - \tilde{u}^i\|_{\infty,\eta}^2 \le \frac{3}{4}\|u^i - \tilde{u}^i\|_{\infty,\eta}^2$$

which implies that, for any $i \in \mathcal{I}$, $u^i = \tilde{u}^i$ on $[T-\eta, T]$. Consequently, for any $s \in [T-\eta, T]$ and $i \in \mathcal{I}$, $Y_s^{i,t,x} = \tilde{Y}_s^{i,t,x}$. Next, on $[T-2\eta, T-\eta]$, we have

$$\|u^i - \tilde{u}^i\|_{\infty,2\eta}^2 \le \frac{3}{4}\|u^i - \tilde{u}^i\|_{\infty,2\eta}^2 + \frac{4}{\alpha_0} m\lambda(E)(e^{2\alpha_0\eta} - e^{\alpha_0\eta})\|u^i - \tilde{u}^i\|_{\infty,\eta}^2.$$

Since $u^i = \tilde{u}^i$ on $[T-\eta, T]$, we then obtain:

$$\|u^i - \tilde{u}^i\|_{\infty,2\eta}^2 \le \frac{3}{4}\|u^i - \tilde{u}^i\|_{\infty,2\eta}^2.$$

Consequently, for any $i \in \mathcal{I}$, $u^i = \tilde{u}^i$ on $[T-2\eta, T-\eta]$. Thus, for any $s \in [T-2\eta, T-\eta]$ and $i \in \mathcal{I}$, $Y_s^{i,t,x} = \tilde{Y}_s^{i,t,x}$. Repeating now this procedure

on $[T-3\eta, T-2\eta]$, $[T-4\eta, T-3\eta]$ etc., we obtain, for any $i \in \mathcal{I}$, $u^i = \tilde{u}^i$. Thus, for any $s \in [t,T]$ and $i \in \mathcal{I}$, $Y_s^{i,t,x} = \tilde{Y}_s^{i,t,x}$. Henceforth, $(Y^{i,t,x})_{i\in\mathcal{I}}$ is the unique Markovian solution to the system of BSDEs (3.3.1).

Case 2 : We now deal with the general case, i.e., without assuming the boundedness of the functions u^i and $\tilde{u}^i$, $i \in \mathcal{I}$, but only polynomial growth.

Let us define, for $s \in [t,T]$,

$$\overline{Y}_s^{i,t,x} := Y_s^{i,t,x}\phi(X_s^{t,x}) \quad \text{and} \quad \underline{Y}_s^{i,t,x} := \tilde{Y}_s^{i,t,x}\phi(X_s^{t,x}),$$

where ϕ is the function defined by (3.3.37). Therefore $(\overline{Y}^{i,t,x}, \overline{Z}^{i,t,x}, \overline{K}^{i,t,x}, \overline{U}^{i,t,x})_{i\in\mathcal{I}}$ and
$(\underline{Y}^{i,t,x}, \underline{Z}^{i,t,x}, \underline{K}^{i,t,x}, \underline{U}^{i,t,x})_{i\in\mathcal{I}}$ are solutions of the system (3.3.44) associated with $(\check{F}_i)_{i\in\mathcal{I}}, (\check{g}_{ij})_{i,j\in\mathcal{I}}$ and $(\check{h}_i)_{i\in\mathcal{I}}$. But for any $i \in \mathcal{I}$, $\overline{Y}^{i,t,x}$ and $\underline{Y}^{i,t,x}$ have representations through deterministic continuous bounded functions ϕu^i and $\phi\tilde{u}^i$, respectively. Therefore by using the result of Step 1 we deduce that $\phi u^i = \phi\tilde{u}^i$ for any $i \in \mathcal{I}$ and then $\overline{Y}_s^{i,t,x} = \underline{Y}_s^{i,t,x}$ for any $s \in [t,T]$ and $i \in \mathcal{I}$, which implies that $Y_s^{i,t,x} = \tilde{Y}_s^{i,t,x}$, for any $s \in [t,T]$ and $i \in \mathcal{I}$. Thus the Markovian solution is unique. □

3.4 The main result : Existence and uniqueness of the solution for the system of IPDEs with interconnected obstacles

We now turn to the study of the existence and uniqueness in viscosity sense of the solution of the system of integral-partial differential equations with interconnected obstacles (3.2.13). Before doing so, we first precise the meaning of the definition of the viscosity solution of this system. It is not exactly the same as in [45] (see also Definition (3.4.4) in the Appendix).

Definition 3.4.1 *We say that a family of deterministic continuous functions* $\vec{u} := (u^i)_{i\in\mathcal{I}}$ *is a viscosity supersolution (resp. subsolution) of* (3.2.13) *if:*

$\forall i \in \{1, ..., m\}$,

a) $u^i(T,x) \geq (resp.\ \leq)\ h_i(x),\ \forall x \in \mathbb{R}^k$;

b) if $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ *is such that* $(t,x) \in [0,T) \times \mathbb{R}^k$ *a global minimum (resp. maximum) point of* $u^i - \phi$

then

$$\min\Big\{u^i(t,x) - \max_{j \in \mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{K}\phi(t,x) - \bar{f}_i(t,x,(u^k(t,x))_{k=1,...,m}, (\sigma^\top D_x\phi)(t,x), \mathcal{B}_i u^i(t,x))\Big\} \geq\ (resp.\ \leq)\ 0.$$

We say that $\vec{u} := (u^i)_{i\in\mathcal{I}}$ *is a viscosity solution of* (3.2.13) *if it is both a supersolution and subsolution of* (3.2.13).

Remark 3.4.2 *In our definition, the last argument of* $\bar{f}_i$ *is* $\mathcal{B}_i u^i(t,x)$ *instead of* $\mathcal{B}_i\phi(t,x)$*, where* ϕ *is the test function. Indeed,* $\mathcal{B}_i u^i(t,x)$ *is well-posed since* u^i *has a polynomial growth,* β *is bounded and the measure* $\lambda(.)$ *is finite.*

We are now able to state the main result of this paper.
Let $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ be the solution of (3.3.1) and let $(u^i)_{i\in\mathcal{I}}$ be the continuous functions with polynomial growth such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$, $i \in \mathcal{I}$ and $s \in [t,T]$,

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}).$$

We then have:

Theorem 3.4.3 *Assume that the functions* $(\bar{f}_i)_{i\in\mathcal{I}}$ *and* $(\gamma_i)_{i\in\mathcal{I}}$ *verify Assumption (H5) and,* $(g_{ij})_{i,j\in\mathcal{I}}$ *and* $(h_i)_{i\in\mathcal{I}}$ *verify Assumptions (H2) and (H3). Then the functions* $(u^i)_{i\in\mathcal{I}}$ *is the unique viscosity solution of the system* (3.2.13)*, according to Definition* (3.4.1)*, in the class of continuous functions with polynomial growth.*

Proof: We first show that $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of system (3.2.13).

So let us consider the following system of reflected BSDEs:

$$\begin{cases} \underline{\mathrm{Y}}^{i,t,x} \in \mathcal{S}^2, \underline{\mathrm{Z}}^{i,t,x} \in \mathcal{H}^{2,d}, \underline{\mathrm{V}}^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } \underline{\mathrm{K}}^{i,t,x} \in \mathcal{A}^2; \\ \underline{\mathrm{Y}}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (\underline{\mathrm{Y}}_r^{k,t,x})_{k\in\mathcal{I}}, \underline{\mathrm{Z}}_r^{i,t,x}, \int_E \gamma(X_r^{t,x}, e)\times \\ \qquad \{u^i(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^i(r, X_{r^-}^{t,x})\}\lambda(de))dr + \underline{\mathrm{K}}_T^{i,t,x} - \underline{\mathrm{K}}_s^{i,t,x} \\ \qquad - \int_s^T \underline{\mathrm{Z}}_r^{i,t,x} dB_r - \int_s^T \int_E \underline{\mathrm{V}}_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ \underline{\mathrm{Y}}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}} (\underline{\mathrm{Y}}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})) \\ \int_t^T (\underline{\mathrm{Y}}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}} (\underline{\mathrm{Y}}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))d\underline{\mathrm{K}}_s^{i,t,x} = 0. \end{cases} \tag{3.4.1}$$

As the deterministic functions $(u^i)_{i\in\mathcal{I}}$ are continuous and of polynomial growth, $\beta(x,e)$ and $\gamma_i(x,e)$ verify respectively (3.2.4) and (3.2.8) and finally by Theorem 3.3.1, the solution of this system exists and is unique. Since the functions $(h_i)_{i\in\mathcal{I}}$, $(g_{ij})_{i,j\in\mathcal{I}}$ and

$$(t,x,y,z) \longmapsto \bar{f}_i(t,x,y,z, \int_E \gamma(x,e)\{u^i(t, x+\beta(x,e)) - u^i(t,x)\}\lambda(de))$$

satisfy the Assumptions (H1)-(H3) and (H4). Moreover, again by Theorem 3.3.1, there exist deterministic continuous functions of polynomial growth $(\underline{\mathrm{u}}^i)_{i\in\mathcal{I}}$, such that: $i \in \mathcal{I}$ and $s \in [t,T]$,

$$\underline{\mathrm{Y}}_s^{i,t,x} = \underline{\mathrm{u}}^i(s, X_s^{t,x}).$$

Finally, using a result by Hamadène-Zhao [45], we deduce that $(\underline{\mathrm{u}}^i)_{i\in\mathcal{I}}$ is a solution in viscosity sense of the following system of IPDE with interconnected obstacle:

$$\begin{cases} \min\{\underline{\mathrm{u}}^i(t,x) - \max_{j\in\mathcal{I}^{-i}} (\underline{\mathrm{u}}^j(t,x) - g_{ij}(t,x)); -\partial_t \underline{\mathrm{u}}^i(t,x) - \mathcal{L}\underline{\mathrm{u}}^i(t,x) - \mathcal{K}\underline{\mathrm{u}}^i(t,x) \\ \quad -\bar{f}_i(t,x,(\underline{\mathrm{u}}^k(t,x))_{k=1,,\ldots,m}, (\sigma^\top D_x \underline{\mathrm{u}}^i)(t,x), \mathcal{B}_i u^i(t,x))\} = 0; \\ \underline{\mathrm{u}}^i(T,x) = h_i(x), \end{cases} \tag{3.4.2}$$

Let us notice that, in this system (3.4.2), the last component of $\bar{f}_i$ is $\mathcal{B}_i u^i(t,x)$ and not $\mathcal{B}_i \underline{\mathrm{u}}^i(t,x)$. Next, recall that $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ solves the system of reflected BSDEs with jumps with interconnected obstacles (3.3.1).

Therefore, we know, by Corollary 3.3.5, that for any $(t,x) \in [0,T] \times \mathbb{R}^k$, $i \in \mathcal{I}$ and $s \in [t,T]$,

$$V_s^{i,t,x}(e) = u^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x}),\ ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [t,T] \times \Omega \times E.$$

Then $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ verify: for any $s \in [t,T]$ and $i \in \mathcal{I}$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E \gamma(X_r^{t,x}, e) \times \\ \qquad \{u^i(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^i(r, X_{r^-}^{t,x})\}\lambda(de))dr + K_T^{i,t,x} - K_s^{i,t,x} \\ \qquad - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{t,xj} - g_{ij}(s, X_s^{t,x})), \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))dK_s^{i,t,x} = 0. \end{cases} \tag{3.4.3}$$

Therefore, by uniqueness of the Markovian solution of the system of reflected BSDEs (3.4.1), we deduce that for any $s \in [t,T]$ and $i \in \mathcal{I}$, $\underline{Y}_s^{i,t,x} = Y_s^{i,t,x}$. Then, for any $i \in \mathcal{I}$, $\underline{u}^i = u^i$. Consequently, $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of (3.2.13) in the sense of Definition 3.4.1.

Now, let us show that $(u^i)_{i\in\mathcal{I}}$ is the unique solution in the class of continuous functions of polynomial growth. It is based on the uniqueness of the Markovian solution of the system of reflected BSDEs (3.3.1).

So let $(\bar{u}^i)_{i\in i\in\mathcal{I}}$ be another continuous with polynomial growth solution of (3.2.13) in the sense of Definition 3.4.1, i.e., for any $i \in \mathcal{I}$,

$$\begin{cases} \min\{\bar{u}^i(t,x) - \max_{j\in\mathcal{I}^{-i}}(\bar{u}^j(t,x) - g_{ij}(t,x)); -\partial_t \bar{u}^i(t,x) - \mathcal{L}\bar{u}^i(t,x) - \mathcal{K}\bar{u}^i(t,x) \\ \quad -\bar{f}_i(t,x,(\bar{u}^k(t,x))_{k=1,,\dots,m}, (\sigma^\top D_x \bar{u}^i)(t,x), \mathcal{B}_i\bar{u}^i(t,x))\} = 0; \\ \bar{u}^i(T,x) = h_i(x). \end{cases} \tag{3.4.4}$$

Next, let us consider the following system of reflected BSDEs:

$$\begin{cases} \bar{Y}^{i,t,x} \in \mathcal{S}^2, \bar{Z}^{i,t,x} \in \mathcal{H}^{2,d}, \bar{V}^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } \bar{K}^{i,t,x} \in \mathcal{A}^2; \\ \bar{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (\bar{Y}_r^{k,t,x})_{k\in\mathcal{I}}, \bar{Z}_r^{i,t,x}, \int_E \gamma(X_r^{t,x}, e)\times \\ \qquad \{\bar{u}^i(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - \bar{u}^i(r, X_{r^-}^{t,x})\}\lambda(de))dr + \bar{K}_T^{i,t,x} - \bar{K}_s^{i,t,x} \\ \qquad - \int_s^T \bar{Z}_r^{i,t,x} dB_r - \int_s^T \int_E \bar{V}_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ \bar{Y}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(\bar{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_t^T (\bar{Y}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(\bar{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))d\bar{K}_s^{i,t,x} = 0. \end{cases} \tag{3.4.5}$$

As for the reflected BSDEs (3.4.1), the solution of the system (3.4.5) exists and is unique since the deterministic functions $(\bar{u}^i)_{i\in i\in\mathcal{I}}$ are continuous and of polynomial growth. Moreover, there exists a deterministic continuous functions of polynomial growth $(v^i)_{i\in\mathcal{I}}$, such that:

$$\forall s \in [t, T],\ \bar{Y}_s^{i,t,x} = v^i(s, X_s^{t,x}).$$

and

$$\bar{V}_s^{i,t,x}(e) = v^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - v^i(s, X_{s^-}^{t,x}),\ ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [t, T] \times \Omega \times E. \tag{3.4.6}$$

Then, by using a result by Hamadène-Zhao [45], $(v^i)_{i\in\mathcal{I}}$ is the unique viscosity solution, in the class of continuous functions with polynomial growth, of the following system:

$$\begin{cases} \min\{v^i(t,x) - \max_{j\in\mathcal{I}^{-i}}(v^j(t,x) - g_{ij}(t,x)); -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) - \mathcal{K}v^i(t,x) \\ \quad -\bar{f}_i(t, x, (v^k(t,x))_{k=1,,\ldots,m}, (\sigma^\top D_x v^i)(t,x), \mathcal{B}_i\bar{u}^i(t,x))\} = 0; \\ v^i(T,x) = h_i(x), \end{cases} \tag{3.4.7}$$

Now, as the functions $(\bar{u}^i)_{i\in i\in\mathcal{I}}$ solves system (3.4.7), hence by uniqueness of the solution of this system (3.4.7) (see [45], Proposition 4.2), for any $i \in \mathcal{I}$ one deduces that $\bar{u}^i = v^i$. Next, by the characterization of the jumps (3.4.6), for

any $i \in \mathcal{I}$, it holds:

$$\bar{V}_s^{i,t,x}(e) = \bar{u}^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \bar{u}^i(s, X_{s^-}^{t,x}),\ ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [t,T] \times \Omega \times E. \tag{3.4.8}$$

Going back now to (3.4.5) and replace the quantity $\bar{u}^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - \bar{u}^i(s, X_{s^-}^{t,x})$ with $\bar{V}_s^{i,t,x}(e)$, yields: for any $i \in \mathcal{I}$ and $s \in [t,T]$,

$$\begin{cases} \bar{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (\bar{Y}_r^{k,t,x})_{k\in\mathcal{I}}, \bar{Z}_r^{i,t,x}, \int_E \gamma(X_r^{t,x}, e)\bar{V}_r^{i,t,x}(e)\lambda(de))dr \\ \qquad\qquad + \bar{K}_T^{i,t,x} - \bar{K}_s^{i,t,x} - \int_s^T \bar{Z}_r^{i,t,x} dB_r - \int_s^T \int_E \bar{V}_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ \bar{Y}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(\bar{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})), \\ \int_t^T (\bar{Y}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(\bar{Y}_s^{j,t,x} - g_{ij}(s, X_s^{t,x})))d\bar{K}_s^{i,t,x} = 0. \end{cases} \tag{3.4.9}$$

But $(Y^{i,t,x}, Z^{i,t,x}, K^{i,t,x}, V^{i,t,x})_{i\in\mathcal{I}}$ is a solution of system (3.4.9) and $Y^{i,t,x}$ is Markovian. Then, by the uniqueness result of Proposition (3.3.6), one deduces that

$$\forall i \in \mathcal{I},\ \bar{Y}_s^{i,t,x} = Y_s^{i,t,x},\ \forall s \in [t,T].$$

Hence, for any $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, $Y_t^{i,t,x} = \bar{Y}_t^{i,t,x} = u^i(t,x) = \bar{u}^i(t,x) = v^i(t,x)$ which means that the solution of (3.2.13), in the sense of Definition (3.4.1), is unique in the class of continuous functions with polynomial growth. □

Appendix

In the paper by Hamadène and Zhao [45], the definition of the viscosity solution of the system (3.2.13), is given as follows.

Definition 3.4.4 *Let $\vec{u} := (u^i)_{i\in\mathcal{I}}$ be a function of $C([0,T] \times \mathbb{R}^k; \mathbb{R}^m)$.*
(i) We say that $\vec{u}$ is a viscosity supersolution (resp. subsolution) of (3.2.13) if: $\forall i \in \{1, ..., m\}$,

a) $u^i(T,x) \geq (resp.\ \leq)\ h_i(x),\ \forall x \in \mathbb{R}^k$;

b) if $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k)$ *is such that* $(t,x) \in [0,T) \times \mathbb{R}^k$ *a global minimum (resp. maximum) point of* $u^i - \phi$,

then

$$\min\Big\{u^i(t,x) - \max_{j\in\mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{K}\phi(t,x) \\ - \bar{f}_i(t,x,(u^k(t,x))_{k=1,\dots,m}, (\sigma^\top D_x\phi)(t,x), \mathcal{B}_i\phi(t,x))\Big\} \geq \ (resp.\ \leq)\ 0.$$

(ii) We say that $\vec{u} := (u^i)_{i\in\mathcal{I}}$ *is a viscosity solution of* (3.2.13) *if it is both a supersolution and subsolution of* (3.2.13).

Chapter 4

Systems of Markovian Obliquely Reflected BSDEs with Jumps and without Monotonicity Conditions and related systems of integral-partial differential equations of non-local type: The case of Infinite Lévy measure

The content of this chapter is from a work in progress, in collaboration with Said Hamadène and Mohamed Mnif.

4.1 Introduction

In this paper we are concerned with systems of Markovian reflected backward stochastic differential equations (RBSDEs for short) with interconnected obstacles when the noise is driven by a Brownian motion $B := (B_s)_{s\leq T}$ and an independent Poisson random measure μ. A solution for such a system is quadruple of adapted stochastic processes $(Y^{i,t,x}, Z^{i,t,x}, K^{i,t,x}, V^{i,t,x})_{i=1,\dots,m}$

such that: $\forall i \in \mathcal{I} := \{1, ..., m\}$ and $s \in [0, T]$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad + K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)); \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0, \end{cases} \tag{4.1.1}$$

where $(t, x) \in [0, T] \times \mathbb{R}^k$, $ds\lambda(de)$ is the compensator of μ and $\tilde{\mu}(ds, de) := \mu(ds, de) - ds\lambda(de)$ its compensated random measure, and finally $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$. The process $X^{t,x}$ is the solution of the following standard differential equation:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s + \int_E \beta(s, X_{s^-}^{t,x}, e)\tilde{\mu}(ds, de), \quad s \in [t, T]; \\ X_s^{t,x} = x \in \mathbb{R}^k, \quad 0 \leq s \leq t. \end{cases} \tag{4.1.2}$$

This system of reflected BSDEs (4.1.1) is termed of Marovian type since randomness stems from the process $X^{t,x}$ which is a Markovian. On the other hand, it is deeply related to the optimal stochastic switching problem (see e.g. [13, 38, 39, 43, 45], for more details).

With the system (4.1.1), is associated the following system of integral-partial differential equations (IPDEs for short) with interconnected obstacles: $\forall i \in \mathcal{I}$,

$$\begin{cases} \min\{u^i(t, x) - \max_{j\in\mathcal{I}^{-i}}(u^j(t, x) - g_{ij}(t)); -\partial_t u^i(t, x) - \mathcal{L}u^i(t, x) - \mathcal{K}u^i(t, x) \\ \quad -\bar{f}_i(t, x, (u^k(t, x))_{k=1,m}, (\sigma^T D_x u^i)(t, x), B_i u^i(t, x))\} = 0, \ (t, x) \in [0, T] \times \mathbb{R}^k; \\ u^i(T, x) = h_i(x), \end{cases} \tag{4.1.3}$$

where the operators $\mathcal{L}$, $\mathcal{K}$ and B_i are defined as follows:

$$\begin{aligned} &\mathcal{L}u^i(t, x) := b(t, x)^\top D_x u^i(t, x) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(t, x)D_{xx}^2 u^i(t, x)]; \\ &\mathcal{K}u^i(t, x) := \int_E (u^i(t, x + \beta(x, e)) - u^i(t, x) - \beta(x, e)^\top D_x u^i(t, x))\lambda(de) \quad \text{and} \\ &B_i u^i(t, x) := \int_E \gamma_i(x, e)(u^i(t, x + \beta(x, e)) - u^i(t, x))\lambda(de). \end{aligned} \tag{4.1.4}$$

The system of reflected BSDEs (4.1.1) has been considered in several works including [45] and [37]. In [45], Hamadène and Zhao have shown that if, mainly, the two following monotonicity conditions: For any $i \in \mathcal{I}$,

(a) $\gamma_i \geqslant 0$;

(b) The function $q \in \mathbb{R} \mapsto \bar{f}_i(t, x, (y_k)_{k=1,m}, z, q)$ is non-decreasing, when the components (t, x, y, z) are fixed;

are satisfied, then system (4.1.1) has a unique solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. Moreover the Feynman-Kac representation of the processes $(Y^{i,t,x})_{i\in\mathcal{I}}$ holds true, i.e., there exist deterministic continuous functions $(u^i)_{i\in\mathcal{I}}$ defined on $[0,T] \times \mathbb{R}^k$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and $i \in \mathcal{I}$,

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}),\ \forall s \in [t,T] \text{ and then } u^i(t,x) = Y_t^{i,t,x}. \tag{4.1.5}$$

Finally, it is proved that the functions $(u^i)_{i\in\mathcal{I}}$ are the unique continuous viscosity solution of IPDEs system (4.1.3) in the class of functions with polynomial growth. Later in [37], the authors also considered both systems (4.1.1) and (4.1.3) as well, but without assuming the above monotonicity conditions (a) and (b). It is proved that if the Lévy measure $\lambda(.)$, associated with the Poisson random measure μ is finite, i.e. $\lambda(E) < \infty$, then system (4.1.1) has a unique Markovian solution, i.e. for which the Feynman-Kac representation (4.1.5) holds. Moreover those functions $(u^i)_{i\in\mathcal{I}}$ of (4.1.5) are the unique solution in viscosity sense of system (4.1.3). In [37], a property which plays an important role is the representation of the process $(V^{i,t,x})_{i\in\mathcal{I}}$ via the continuous functions $(u^i)_{i\in\mathcal{I}}$ and the process $X^{t,x}$ and which reads:

$$V_s^{i,t,x}(e) = 1_{\{s\geq t\}}\{u^i(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x})\},$$
$$ds \otimes d\mathbb{P} \otimes d\lambda \text{ on } [0,T] \times \Omega \times E. \tag{4.1.6}$$

Therefore the main objective of this paper is to deal with system (4.1.1) in the case when $\lambda(.)$ is not finite, i.e., $\lambda(E) = \infty$. Actually, in this work, we show that if $\lambda(.)$ is infinite and integrates the function $(1 \wedge |e|)_{e\in E}$, in combination with other regularity properties on the data $(\bar{f}_i)_{i\in\mathcal{I}}$, $(h_i)_{i\in\mathcal{I}}$ and $(g_{ij})_{i,j\in\mathcal{I}}$,

then the system (4.1.1) has a Markovian solution, moreover the Feynman-Kac representation (4.1.5) holds true. Finally we show that those functions $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of (4.1.3). The relation (4.1.6) which binds the processes $(V^{i,t,x})_{i\in\mathcal{I}}$, the functions $(u^i)_{i\in\mathcal{I}}$ and the process $X^{t,x}$ is also valid. Once more it plays an important role in the proof of our result. Our method relies first on the truncation of the measure $\lambda(.)$ in such a way to fall in the framework of a finite Lévy measure. Then to take the limit to obtain a solution for system (4.1.1) in the case when the generators do not depend on the jump parts and, on the other hand and more importantly, the characterization (4.1.6) of the jump components $(V^{i,t,x})_{i\in\mathcal{I}}$. To state this result we need to assume that the functions $g_{ij}, i, j \in \mathcal{I}$ do not depend on x. Then we deal with the case where the generators depend on the jump components by the introduction of a recursive scheme which we show that is convergent and its limit provides a solution for system (4.1.1). Finally we show that the functions $(u^i)_{i\in\mathcal{I}}$ of the Feynman-Kac representation of the processes $(Y^{i,t,x})_{i\in\mathcal{I}}$ is a viscosity solution of system (4.1.3).

The paper is organized as follows. In Section 2, we provide notations and assumptions needed in the study of the obliquely RBSDEs with jumps system (4.1.1) and the related IPDEs system (4.1.3) as well. Section 3 is mainly devoted to prove the relation (4.1.6) in the case when $(\bar{f}_i)_{i\in\mathcal{I}}$ do not depend on the jump part $(V^{i,t,x})_{i\in\mathcal{I}}$. For that we first truncate the Lévy measure in order to fall in the case of a finite Lévy measure since in this latter framework the relation (4.1.6) is rather easy to obtain. Then we take the limit and we recover (4.1.6) in the general case. Next, we study the existence of solution for system of RBSDEs (4.1.1), Feynman-Kac representation (4.1.5) and the representation (4.1.6). Finally, in Section 4, we prove that the functions $(u^i)_{i=1,m}$ are a viscosity solution of (4.1.3). □

4.2 Framework and state of the art

Let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ be a stochastic basis such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, $\mathcal{F}_{t^+} = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, and we suppose that the filtration is generated by the two following mutually independent processes :

(i) a d-dimensional standard Brownian motion $B := (B_t)_{0 \leq t \leq T}$ and

(ii) a Poisson random measure μ on $\mathbb{R}^+ \times E$, where $E := \mathbb{R}^l - \{0\}$ is equipped with its Borel field $\mathcal{B}(E), (l \geqslant 1$ fixed). Let $\nu(dt, de) = dt\lambda(de)$ be its compensated process such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t \leq T}$ is a martingale for every $A \in \mathcal{B}(E)$ satisfying $\lambda(A) < \infty$. We also assume that $\lambda(.)$ is a σ-finite measure on $(E, \mathcal{B}(E))$, integrates the function $(1 \wedge |e|)_{e \in E}$ and $\lambda(E) = \infty$. If $l = 1$, $\lambda(e) = |e|^{-\frac{3}{2}} de$ is an example of such a Lèvy measure.

Now, let us introduce the following spaces:

a) $\mathcal{P}$ (resp. $\mathbf{P}$) is the σ-algebra of $\mathbb{F}$-progressively measurable (resp. $\mathbb{F}$-predictable) sets on $\Omega \times [0,T]$;

b) $\mathcal{L}^2(\lambda)$ is the space of Borel measurable functions $(\varphi(e))_{e \in E}$ from E into $\mathbb{R}$ such that
$\int_E |\varphi(e)|^2 \lambda(de) < \infty$;

c) $\mathcal{S}^2$ is the space of RCLL (right continuous with left limits), $\mathcal{P}$-measurable and $\mathbb{R}$-valued processes
$Y := (Y_s)_{s \leq T}$ such that $\mathbb{E}\big[\sup_{0 \leq t \leq T} |Y_s|^2\big] < \infty$;

d) $\mathcal{A}^2$ is the subspace of $\mathcal{S}^2$ of continuous non-decreasing processes $K := (K_t)_{t \leq T}$ such that $K_0 = 0$;

e) $\mathcal{H}^{2,d}$ is the space of $\mathcal{P}$-measurable and $\mathbb{R}^d$-valued processes $Z := (Z_s)_{s \leq T}$ such that $\mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty$;

f) $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ is the space of $\mathbf{P}$-measurable and $\mathcal{L}^2(\lambda)$-valued processes $V := (V_s)_{s\leq T}$ such that
$\mathbb{E}\left[\int_0^T \int_E |V_s(e)|^2 \lambda(de)ds\right] < \infty$.

For an RCLL process $(\theta_s)_{s\leq T}$, we define, for any $s \in (0,T]$, $\theta_{s^-} := \lim_{r\nearrow s} \theta_r$ and $\Delta_s\theta := \theta_s - \theta_{s^-}$ is the jump size of θ at s.

Now, for any $(t,x) \in [0,T]\times\mathbb{R}^k$, let $(X_s^{t,x})_{s\leq T}$ be the stochastic process solution of the following
stochastic differential equation (SDE for short) of diffusion-jump type:

$$\begin{cases} dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dB_s + \int_E \beta(s,X_{s^-}^{t,x},e)\tilde{\mu}(ds,de), \quad s\in[t,T] \\ X_s^{t,x} = x \in \mathbb{R}^k, \quad 0\leq s\leq t \end{cases} \tag{4.2.1}$$

where $b : [0,T]\times\mathbb{R}^k \to \mathbb{R}^k$ and $\sigma : [0,T]\times\mathbb{R}^k \to \mathbb{R}^{k\times d}$ are two continuous functions in (t,x) and Lipschitz w.r.t x, i.e., there exists a positive constant C such that

$$|b(t,x)-b(t,x')| + |\sigma(t,x)-\sigma(t,x')| \leq C|x-x'|, \quad \forall (t,x,x') \in [0,T]\times\mathbb{R}^{k+k}. \tag{4.2.2}$$

Note that the continuity of b, σ and (4.2.2) imply the existence of a constant C such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|), \quad \forall (t,x) \in [0,T]\times\mathbb{R}^k. \tag{4.2.3}$$

Next, let $\beta : \mathbb{R}^k \times E \to \mathbb{R}^k$ be a $\mathcal{B}(\mathbb{R}^k)\otimes\mathcal{B}(E)$-measurable function such that for some real constant c, $\forall e \in E$ and $x, x' \in \mathbb{R}^k$,

$$|\beta(x,e)| \leq c(1\wedge|e|) \quad \text{and} \quad |\beta(x,e)-\beta(x',e)| \leq c|x-x'|(1\wedge|e|). \tag{4.2.4}$$

Conditions (4.2.2), (4.2.3) and (4.2.4) ensure, for any $(t,x) \in [0,T]\times\mathbb{R}^k$, the existence and uniqueness of a solution of equation (4.2.1) (see [32] for more details). Moreover, it satisfies: For any $p \geq 2$ and $x, x' \in \mathbb{R}^k$,

$$\mathbb{E}[\sup_{s\leq T}|X_s^{t,x}-x|^p] \leqslant C(1+|x|^p) \text{ and } \mathbb{E}[\sup_{s\leq T}|X_s^{t,x}-X_s^{t,x'}-(x-x')|^p] \leqslant C|x-x'|^p. \tag{4.2.5}$$

Next, let us introduce the following deterministic functions $(\bar{f}_i)_{i\in\mathcal{I}}$, $(h_i)_{i\in\mathcal{I}}$ and $(g_{ij})_{i,j\in\mathcal{I}}$ defined as follows : for any $i,j\in\mathcal{I}$,

a) $\bar{f}_i : (t,x,\vec{y},z,q)\in[0,T]\times\mathbb{R}^{k+m+d+1}\longmapsto\bar{f}_i(t,x,\vec{y},z,q)\in\mathbb{R}$ $(\vec{y}:=(y^1,...,y^m))$;

b) $g_{ij} : t\in[0,T]\longmapsto g_{ij}(t)\in\mathbb{R}^+$;

c) $h_i : x\in\mathbb{R}^k\longmapsto h_i(x)\in\mathbb{R}$.

Additionally we assume that they satisfy:

(H1) For any $i\in\{1,...,m\}$,

(i) The function $\bar{f}_i$ is continuous in (t,x) uniformly w.r.t. the variables $(\vec{y},z,q)$.

(ii) The mapping $(t,x)\mapsto\bar{f}_i(t,x,\vec{0},0,0)$ has polynomial growth in x, i.e., there exist two constants $C>0$ and $p\geqslant 1$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$|\bar{f}_i(t,x,\vec{0},0,0)|\leq C(1+|x|^p). \tag{4.2.6}$$

(iii) The function $\bar{f}_i$ is Lipschitz continuous w.r.t. the variables $(\vec{y},z,q)$ uniformly in (t,x), i.e., there exists a positive constant C_i such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $(\vec{y},z,q)$ and $(\vec{y}_1,z_1,q_1)$ elements of $\mathbb{R}^{m+d+1}$:

$$|\bar{f}_i(t,x,\vec{y},z,q)-\bar{f}_i(t,x,\vec{y}_1,z_1,q_1)|\leq C_i(|\vec{y}-\vec{y}_1|+|z-z_1|+|q-q_1|). \tag{4.2.7}$$

(iv) For any $i\in\mathcal{I}$ and $j\in\mathcal{I}^{-i}$, the mapping $y^j\mapsto\bar{f}_i(t,x,y^1,...,y^{j-1},y^j,y^{j+1},...,y^m,z,q)$ is non-decreasing whenever the other components $(t,x,y^1,...,y^{j-1},y^{j+1},...,y^m,z,q)$ are fixed.

(v) For any $i\in\{1,...,m\}$, let $\gamma_i:\mathbb{R}^k\times E\to\mathbb{R}$ be a $\mathcal{B}(\mathbb{R}^k)\otimes\mathcal{B}(E)$-measurable functions such that for some constant $C>0$, and for all $e\in E$,

$$(a)\,|\gamma_i(x,e)|\leq c(1\wedge|e|),\quad\forall x\in\mathbb{R}^k,\quad\text{and}$$

$$(b)\,|\gamma_i(x,e)-\gamma_i(x',e)|\leq c(1\wedge|e|)|x-x'|,\quad\forall x,x'\in\mathbb{R}^k.$$

Finally let us define the function $(f_i)_{i=1,...,m}$ on $[0,T] \times \mathbb{R}^{k+m+d} \times \mathcal{L}^2(\lambda)$, as follows:

$$f_i(t,x,\vec{y},z,v) := \bar{f}_i(t,x,\vec{y},z,\textstyle\int_E v(e)\gamma_i(x,e)\lambda(de)). \tag{4.2.8}$$

Note that since $\bar{f}_i$ is uniformly Lipschitz in $(\vec{y},z,q)$ and γ_i verifies (3.2.8) then the function f_i enjoy the two following properties:

(a) f_i is Lipschitz continuous w.r.t. the variables $(\vec{y},z,v)$ uniformly in (t,x).

(b) The mapping $(t,x) \mapsto f_i(t,x,0,0,0) = \bar{f}_i(t,x,0,0,0)$ is continuous with polynomial growth.

(H2) $\forall i,j \in \{1,...,m\}$, $g_{ii} = 0$ and for $i \neq j$, $g_{ij}(t)$ is non-negative, continuous in t and satisfies the following non free loop property :

For any $t \in [0,T]$, for any sequence of indices $i_1,...,i_k$ such that $i_1 = i_k$ and $card\{i_1,...,i_k\} = k-1$ $(k \geq 3)$ we have

$$g_{i_1 i_2}(t) + g_{i_2 i_3}(t) + ... + g_{i_{k-1} i_1}(t) > 0. \tag{4.2.9}$$

(H3) For $i \in \{1,...,m\}$, the function h_i, which stands for the terminal condition (the function of terminal payoff), is continuous with polynomial growth and satisfies the following consistency condition:

$$h_i(x) \geqslant \max_{j \in \mathcal{I}^{-i}} (h_j(x) - g_{ij}(T)), \quad \forall x \in \mathbb{R}^k. \tag{4.2.10}$$

(H4)-(i) $\forall i \in \mathcal{I}$, $\gamma_i \geqslant 0$;

(H4)-(ii) The mapping $q \in \mathbb{R} \longmapsto \bar{f}_i(t,x,\vec{y},z,q)$ is non-decreasing when the other components $(t,x,\vec{y},z)$ are fixed.

(H5) The measure $\lambda(.)$ is finite, i.e. $\lambda(E) < \infty$.

(H6) The functions h_i and $f_i(t,x,0,0,0)$, $i \in \mathcal{I}$, are bounded, i.e., there exists a constant $\overline{C}$ such that

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k \text{ and } i \in \mathcal{I},\ |h_i(x)| + |f_i(t,x,0,0,0)| \leq \overline{C}.$$

Remark 4.2.1 *The condition (H1)-i) is needed, e.g. in [39] or [45] in order to apply Ishii's Lemma to show comparison in the systems considered in those papers and then to deduce uniqueness and continuity of the viscosity solution. However instead of requiring* $(H1)-i)$ *it is enough to require other sufficient conditions which make comparison of sub. and super-solutions hold. If* $\bar{f}_i$, $i \in \mathcal{I}$, *do not depend on* q *it is enough to require the following conditions:*

i) For any $i \in \mathcal{I}$, $\bar{f}_i$ *is jointly continuous in* $(t, x, \vec{y}, z)$;
ii) For any $R > 0$, *there exists a continuous function* $m_R(.)$ *from* $\mathbb{R}^+$ *to* $\mathbb{R}^+$ *such that* $m_R(0) = 0$ *and for any* $|x| \leq R$, $|x'| \leq R$ *and* $|\vec{y}| \leq R$ *we have,*

$$|\bar{f}_i(t, x, \vec{y}, z) - \bar{f}_i(t, x', \vec{y}, z)| \leq m_R(|x - x'|(1 + |z|)). \tag{4.2.11}$$

One can see e.g. the paper by El-Karoui et al. [28] on this latter condition. In the case when $(\bar{f}_i)_{i\in\mathcal{I}}$ *depend on* q, *similar results exist (one can see e.g. [3] for more details).* □

Now, let us consider the following system of obliquely RBSDEs with jumps: $\forall i \in \mathcal{I}$ and $s \in [0, T]$,

$$\begin{cases} Y^{i,t,x} \in \mathcal{S}^2, Z^{i,t,x} \in \mathcal{H}^{2,d}, V^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } K^{i,t,x} \in \mathcal{A}^2; \\ Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad + K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)), \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0. \end{cases} \tag{4.2.12}$$

In [37], we have shown that this system (4.2.12) has a solution, without assuming the monotonicity condition (i.e. (H4)(i)-(ii)) on the generators $(\bar{f}_i)_{i\in\mathcal{I}}$ and when the measure $\lambda(.)$ is finite (i.e. (H5) is satisfied). Moreover, we have provided a representation of $(Y^{i,t,x}, V^{i,t,x})_{i\in\mathcal{I}}$ as deterministic functions of $(s, X_s^{t,x})$. Actually, we have proved:

Proposition 4.2.2 *(see [37], Proposition 3.2).*
Assume that Assumptions (H1),(i)-(v), (H2), (H3), (H5) and (H6) hold. Then, the system (4.2.12) has a solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. *Moreover, there exist deterministic bounded continuous functions* $(u^i)_{i\in\mathcal{I}}$, *defined on* $[0,T]\times\mathbb{R}^k$, *such that:*

$$\forall s\in[t,T],\ Y_s^{i,t,x}=u^i(s,X_s^{t,x}), \tag{4.2.13}$$

and

$$V_s^{i,t,x}(e)=1_{\{s\geq t\}}(u^i(s,X_{s^-}^{t,x}+\beta(X_{s^-}^{t,x},e))-u^i(s,X_{s^-}^{t,x})),$$
$$ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E. \tag{4.2.14}$$

Remark 4.2.3 *Assumption (H5) is not only used to obtain the characterization (4.2.14) of the jump part of the RBSDEs system (4.2.12) but also to show the existence of the solution* $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ *itself. Additionally we do not know whether or not the solution* $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ *of the RBSDEs system (4.2.12) is unique, however the Markovian solution which has the representation (4.2.13) is unique (see Proposition 3.6 in [37]).*

4.3 Systems of Obliquely RBSDEs with Jumps: The Infinite Activity Case

In this section, we also consider the system (4.2.12) without assuming Assumption (H5) on boundedness of $\lambda(.)$, as we did in [37]. Relaxing the last hypothesis, we establish a new existence result of a solution for the system (4.2.12). For this, we consider firstly the following system of RBSDEs with

jumps and interconnected obstacles: $\forall i \in \mathcal{I}$ and $s \in [0,T]$,

$$\begin{cases} Y^{i,t,x} \in \mathcal{S}^2, Z^{i,t,x} \in \mathcal{H}^{2,d}, V^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } K^{i,t,x} \in \mathcal{A}^2; \\ Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \widehat{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x})dr + K_T^{i,t,x} - K_s^{i,t,x} \\ \qquad\qquad - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr,de); \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)); \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0. \end{cases} \tag{4.3.1}$$

Note that in this system (4.3.1), the generators $(\widehat{f}_i)_{i\in\mathcal{I}}$ do not depend on the jump component and, on the other hand, we assume that they satisfy the following assumptions:

(H7) For any $i \in \mathcal{I}$,

(a) For any $i \in \mathcal{I}$, the function $\widehat{f}_i$, $i \in \mathcal{I}$, verifies: There exists a continuous concave function Φ_i, from $\mathbb{R}^k$ into $\mathbb{R}$, such that $\Phi_i(0) = 0$ and

$$\forall x, x', \vec{y}, \text{ and } z, \ |\widehat{f}_i(t,x,\vec{y},z) - \widehat{f}_i(t,x',\vec{y},z)| \leq \Phi_i(|x-x'|). \tag{4.3.2}$$

(b) The functions h_i are uniformly continuous.

Remark 4.3.1 *Condition (H7)-(a) is satisfied if for any $i \in \mathcal{I}$, $\widehat{f}_i$ is Lipschitz w.r.t $(x,\vec{y},z)$ uniformly in t. It is also satisfied if for any $i \in \mathcal{I}$, $\widehat{f}_i(t,x,\vec{y},z) = \widehat{f}_i^{(1)}(t,x) + \widehat{f}_i^{(2)}(\vec{y},z)$ where $f_i^{(1)}$ is uniformly continuous and $f_i^{(2)}$ is Lipschitz.*

We then have the following first result.

Theorem 4.3.2 *Assume that the functions $(g_{ij})_{i,j\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$ verify Assumptions (H2) and (H3) respectively and for any $i \in \mathcal{I}$ the function:*

(i) $\widehat{f}_i$ verify (H1), iii)-iv) and (H7)-(a).

(ii) h_i verifies (H7)-(b).

(iii) h_i and $\widehat{f}_i(t,x,0,0)$ are bounded, i.e., they satisfy (H6).

(iv) γ_i verifies (H1)-(v).

Then, we have:

a) The system (4.3.1) has a unique solution $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$.

b) There exist deterministic continuous bounded functions $(u^i)_{i\in\mathcal{I}}$, *defined on* $[0,T]\times\mathbb{R}^k$, *such that:*

$$\forall s\in[t,T],\ Y_s^{i,t,x}=u^i(s,X_s^{t,x}).$$

c) For any $i\in\mathcal{I}$,

$$V_s^{i,t,x}(e)=1_{\{s\geq t\}}(u^i(s,X_{s^-}^{t,x}+\beta(X_{s^-}^{t,x},e))-u^i(s,X_{s^-}^{t,x})),$$
$$ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E. \tag{4.3.3}$$

Proof: The proof is rather long and divided into 5 steps.

Step 1:

First let us notice that by the result of Hamadène and Zhao in [45], the system (4.3.1) has a unique solution
$(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$. On the other hand there exist deterministic continuous functions $(u^i)_{i\in\mathcal{I}}$, defined on $[0,T]\times\mathbb{R}^k$ and of polynomial growth such that:

$$\forall s\in[t,T],\ Y_s^{i,t,x}=u^i(s,X_s^{t,x}).$$

Therefore it remains to show that: (i) the functions $(u^i)_{i\in\mathcal{I}}$ are bounded; (ii) the property (4.3.3) holds true.

So let $(\bar{Y},\bar{Z})$ be the solution of the following standard BSDE: for any $s\leq T$,

$$\begin{cases}\bar{Y}\in\mathcal{S}^2,\ \bar{Z}\in\mathcal{H}^{2,d};\\ \bar{Y}_s=\bar{C}+\int_s^T\{\bar{C}+m\,C_{\hat{f}}^y\,\bar{Y}_r+C_{\hat{f}}^z\,|\bar{Z}_r|\}dr-\int_s^T\bar{Z}_r dB_r;\end{cases}$$

where $C_{\hat{f}}^y$, $C_{\hat{f}}^z$ are the maximum of the Lipschitz constants of the functions $\widehat{f}_i$, $i\in\mathcal{I}$, w.r.t. $\vec{y}$ and z respectively. Then by comparison, which is valid in our case since $(\widehat{f}_i)_{i\in\mathcal{I}}$, do not depend the jump component, we have:

$$|Y_s^{i,t,x}|\leq\bar{Y}_s,\ \forall s\in[t,T].$$

But $\bar{Z}=0$, $ds\otimes d\mathbb{P}$-a.e and $\bar{Y}$ is deterministic and continuous. Finally it is enough to take $s=t$ in the previous equality to conclude that $(u^i)_{i\in\mathcal{I}}$ are bounded.

Step 2: Truncation of the measure $\lambda(.)$

For any $n \geq 1$, let us introduce a new Poisson random measure μ_n and its associated compensator ν_n as follows :

$$\mu_n(ds,de) = \mathbf{1}_{\{|e|\geq \frac{1}{n}\}}\mu(ds,de) \text{ and } \nu_n(ds,de) = \lambda_n(de)ds := \mathbf{1}_{\{|e|\geq \frac{1}{n}\}}\lambda(de)ds.$$

The random measure $\tilde{\mu}_n(ds,de) := (\mu_n - \nu_n)(ds,de)$, is the compensated one associated with μ_n. The main point to notice is that

$$\lambda_n(E) = \int_E \lambda_n(de) < \infty.$$

Next, let us introduce the process ${}^nX^{t,x}$ solution of the following SDE of jump-diffusion type:

$$\begin{cases} d{}^nX_s^{t,x} = b(s,{}^nX_s^{t,x})ds + \sigma(s,{}^nX_s^{t,x})dB_s + \int_E \beta({}^nX_{s^-}^{t,x},e)\tilde{\mu}_n(ds,de), \quad s\in[t,T] \\ {}^nX_s^{t,x} = x \in \mathbb{R}^k, \quad 0 \leq s \leq t. \end{cases} \tag{4.3.4}$$

Note that thanks to the assumptions on b, σ and β, the solution of (4.3.4) exists and is unique. Moreover, it satisfies the same estimates as in (4.2.5). Indeed, the measure $\lambda_n(.)$ is just a truncation at the origin of the measure $\lambda(.)$ which integrates $(1\wedge|e|)_{e\in E}$ and then $(1\wedge|e|^2)_{e\in E}$. In the following lemma we collect some results of the process ${}^nX^{t,x}$.

Lemma 4.3.3 *The process* ${}^nX^{t,x}$ *satisfies the following properties:*

(i) For any $p \geq 2$*, there exists a constant* C *such that*

$$\mathbb{E}\Big[\sup_{s\leq T}|{}^nX_s^{t,x}|^p\Big] \leq C(1+|x|^p). \tag{4.3.5}$$

(ii) For any $p \geq 1$ *and* $m \geq n \geq 1$*,*

$$\mathbb{E}\Big[\sup_{s\leq T}|{}^nX_s^{t,x} - {}^mX_s^{t,x}|^{2p}\Big] \leq C\Big\{\int_{\{\frac{1}{m}\leq|e|\leq\frac{1}{n}\}}(1\wedge|e|^2)\lambda(de)\Big\}^{\frac{p}{2}}. \tag{4.3.6}$$

Proof: We prove only (4.3.6) as (4.3.5) is classical. Let $p \geq 1$ and $m \geq n$. For any $s \in [0,T]$, we have:

$${}^nX_s^{t,x} - {}^mX_s^{t,x} = \int_0^s (b(r,{}^nX_r^{t,x}) - b(r,{}^mX_r^{t,x}))dr + \int_0^s (\sigma(r,{}^nX_r^{t,x}) - \sigma(r,{}^mX_r^{t,x}))dB_r$$
$$\int_0^s\int_E (\beta(e,{}^nX_{r_-}^{t,x})\mathbf{1}_{\{|e|\geq\frac{1}{n}\}} - \beta(e,{}^mX_{r_-}^{t,x}))\tilde{\mu}_m(dr,de)$$

Since $|a+b+c|^{2p} \leq 3^{2(p-1)}(|a|^{2p}+|b|^{2p}+|c|^{2p})$, for $p \geq 1$ and for any a, b $c \in \mathbb{R}$, then by the Cauchy-Schwarz and the Burkholder-Davis-Gundy inequalities we have: $\forall \eta \in [0,T]$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq s\leq \eta}\big|{}^nX_s^{t,x}-{}^mX_s^{t,x}\big|^{2p}\Big]\\
&\leq 3^{2(p-1)}\mathbb{E}\Big[\sup_{0\leq s\leq \eta}\Big|\int_0^s b(r,{}^nX_r^{t,x})-b(r,{}^mX_r^{t,x})dr\Big|^{2p}\\
&\quad+\sup_{0\leq s\leq \eta}\Big|\int_0^s \sigma(r,{}^nX_r^{t,x})-\sigma(r,{}^mX_r^{t,x})dB_r\Big|^{2p}\\
&\quad+\sup_{0\leq s\leq \eta}\Big|\int_0^s\int_E(\beta(e,{}^nX_{r_-}^{t,x})\mathbf{1}_{\{|e|\geq\frac{1}{n}\}}-\beta(e,{}^mX_{r_-}^{t,x}))\tilde{\mu}_m(dr,de)\Big|^{2p}\Big]\\
&\leq C_p\Big\{\mathbb{E}\Big[\int_0^\eta\sup_{0\leq\tau\leq r}\big|b(\tau,{}^nX_\tau^{t,x})-b(\tau,{}^mX_\tau^{t,x})\big|^{2p}d\tau\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_0^\eta\sup_{0\leq\tau\leq r}\big|\sigma(\tau,{}^nX_\tau^{t,x})-\sigma(\tau,{}^mX_\tau^{t,x})\big|^2d\tau\Big)^p\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_0^\eta\int_E\big|\beta(e,{}^nX_{\tau_-}^{t,x})-\beta(e,{}^mX_{\tau_-}^{t,x}))\big|^2\lambda_n(de)d\tau\Big)^p\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_0^\eta\int_{\{|e|<\frac{1}{n}\}}\big|\beta(e,{}^mX_{\tau_-}^{t,x})\big|^2\lambda_m(de)d\tau\Big)^p\Big]\Big\}.
\end{aligned}
$$

As b is Lipschitz w.r.t x, then we have (for some constant c_p which may depend on p and change from line to line): $\forall 0 \leq \eta \leq T$,

$$
\mathbb{E}\Big[\int_0^\eta\sup_{0\leq\tau\leq r}\big|b(\tau,{}^nX_\tau^{t,x})-b(\tau,{}^mX_\tau^{t,x})\big|^{2p}d\tau\Big]\leq c_p\mathbb{E}\Big[\int_0^\eta\sup_{0\leq\tau\leq r}\big|{}^nX_\tau^{t,x}-{}^mX_\tau^{t,x}\big|^{2p}d\tau\Big].
$$

Besides using Jensen'inequality and the fact that σ is Lipschitz w.r.t x, we get: $\forall 0 \leq \eta \leq T$,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^\eta\sup_{0\leq\tau\leq r}\big|\sigma(\tau,{}^nX_\tau^{t,x})-\sigma(\tau,{}^mX_\tau^{t,x})\big|^2d\tau\Big)^p\Big]\\
&\leq c_p\mathbb{E}\Big[\int_0^\eta\sup_{0\leq\tau\leq r}\big|\sigma(\tau,{}^nX_\tau^{t,x})-\sigma(\tau,{}^mX_\tau^{t,x})\big|^{2p}d\tau\Big]\\
&\leq c_p\mathbb{E}\Big[\int_0^\eta\sup_{0\leq\tau\leq r}\big|{}^nX_\tau^{t,x}-{}^mX_\tau^{t,x}\big|^{2p}d\tau\Big].
\end{aligned}
$$

Finally as β verifies (4.2.4), then

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^\eta\int_E \big|\beta(e,{}^nX^{t,x}_{\tau_-})-\beta(e,{}^mX^{t,x}_{\tau_-})\big|^2\lambda_n(de)d\tau\Big)^p\Big]\\
&\quad\le c_p\mathbb{E}\Big[\Big(\int_0^\eta \sup_{0\le\tau\le r}\big|{}^nX^{t,x}_\tau-{}^mX^{t,x}_\tau\big|^2 d\tau\int_E(1\wedge|e|)^2\lambda_n(de)d\tau\Big)^p\Big]\\
&\quad\le c_p\mathbb{E}\Big[\int_0^\eta d\tau\sup_{0\le\tau\le r}\big|{}^nX^{t,x}_\tau-{}^mX^{t,x}_\tau\big|^{2p}\Big].
\end{aligned}$$

The last inequality stems from the fact that $\lambda_n(.)$ is $\lambda(.)$ truncated and this latter integrates $(1\wedge|e|^2)_{e\in E}$. Plug now those three last inequalities in the previous one to obtain: $\forall\, 0\le\eta\le T$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le\eta}\big|{}^nX^{t,x}_s-{}^mX^{t,x}_s\big|^{2p}\Big]\\
&\quad\le C_p\mathbb{E}\Big[\int_0^\eta\sup_{0\le\tau\le r}\big|{}^nX^{t,x}_\tau-{}^mX^{t,x}_\tau\big|^{2p}d\tau+\Big(\int_{\{\frac1m\le|e|\le\frac1n\}}(1\wedge|e|^2)\lambda(de)\Big)^p\Big].
\end{aligned}$$

Finally, using Gronwall's inequality we obtain: For any $\eta\in[0,T]$,

$$\mathbb{E}\Big[\sup_{0\le s\le\eta}\big|{}^nX^{t,x}_s-{}^mX^{t,x}_s\big|^{2p}\Big]\le C\Big(\int_{\{\frac1m\le|e|\le\frac1n\}}(1\wedge|e|^2)\lambda(de)\Big)^p. \tag{4.3.7}$$

Then taking $\eta=T$, we obtain the desired result.

Now, let us consider the following obliquely RBSDEs with jumps (similar as RBSDEs (4.3.1)):
for any $i\in\mathcal{I}$ and $s\in[0,T]$,

$$\begin{cases}
{}^nY^{i,t,x}\in\mathcal{S}^2,\,{}^nZ^{i,t,x}\in\mathcal{H}^{2,d},\,{}^nV^{i,t,x}\in\mathcal{H}^2(\mathcal{L}^2(\lambda_n)),\text{ and }{}^nK^{i,t,x}\in\mathcal{A}^2;\\
{}^nY^{i,t,x}_s=h_i({}^nX^{t,x}_T)+\int_s^T\widehat{f}_i(r,{}^nX^{t,x}_r,({}^nY^{k,t,x}_r)_{k\in\mathcal{I}},{}^nZ^{i,t,x}_r)dr\\
\qquad\qquad+{}^nK^{i,t,x}_T-{}^nK^{i,t,x}_s-\int_s^T{}^nZ^{i,t,x}_r dB_r-\int_s^T\int_E{}^nV^{i,t,x}_r(e)\tilde{\mu}_n(dr,de);\\
{}^nY^{i,t,x}_s\geqslant\max_{j\in\mathcal{I}^{-i}}({}^nY^{j,t,x}_s-g_{ij}(s));\\
\int_0^T({}^nY^{i,t,x}_s-\max_{j\in\mathcal{I}^{-i}}({}^nY^{j,t,x}_s-g_{ij}(s)))d{}^nK^{i,t,x}_s=0.
\end{cases} \tag{4.3.8}$$

First, note that this latter RBSDEs is related to the filtration $\mathbb{F}^{\mu_n}:=(\mathcal{F}^{\mu_n}_t)_{t\le T}$ generated by the Brownian motion B and the independent random measure μ_n. The filtration $\mathbb{F}^{\mu_n}$ is defined in the same way as $\mathbb{F}$ in the beginning of this

section, except that one replaces μ with μ_n. Next, under (H2), (H3) and (i)-(v), thanks to Proposition 3.2 in [37], there exists a unique markovian solution $({}^nY^{i,t,x}, {}^nZ^{i,t,x}, {}^nV^{i,t,x}, {}^nK^{i,t,x})$ which solves (4.3.8). The following characterization of ${}^nY^{i,t,x}$ as a Snell envelope holds true: $\forall s \leq T$,

$$ {}^nY_s^{i,t,x} = \text{esssup}_{\tau \geq s} \mathbb{E}\Big[\int_s^\tau \widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x}) dr + h_i({}^nX_\tau^{t,x}) \mathbf{1}_{\{\tau=T\}} + \max_{j\in\mathcal{I}^{-i}} ({}^nY_\tau^{j,t,x} - g_{ij}(\tau)) \mathbf{1}_{\{\tau<T\}} \big| \mathcal{F}_s \Big]. \tag{4.3.9} $$

Next there exist unique deterministic continuous functions $({}^nu^i)_{i\in\mathcal{I}}$, defined on $[0,T]\times\mathbb{R}^k$ and of polynomial growth such that:

$$ \forall s \in [t,T],\ {}^nY_s^{i,t,x} := {}^nu^i(s, {}^nX_s^{t,x}), \quad \mathbb{P}-a.s. \tag{4.3.10} $$

and

$$ {}^nV_s^{i,t,x}(e) = 1_{\{s\geq t\}}({}^nu^i(s, {}^nX_{s^-}^{t,x} + \beta({}^nX_{s^-}^{t,x}, e)) - {}^nu^i(s, {}^nX_{s^-}^{t,x})), \quad ds\otimes d\mathbb{P}\otimes d\lambda_n \text{ on } [0,T]\times\Omega\times E. \tag{4.3.11} $$

Finally note that, as in Step 1, the functions $u^{i,n}$, $i\in\mathcal{I}$, are bounded uniformly w.r.t n.

Step 3: Switching representation of ${}^nY^{i,t,x}$

In this step, we aim at representing ${}^nY^{i,t,x}$ as the value of an optimal switching problem. Indeed, let $\mathbb{F}^{\mu_n} := (\mathcal{F}_t^{\mu_n})_{t\leq T}$ be the filtration generated by the Brownian motion B and the independent random measure μ_n. Next let $\delta := (\theta_k, \alpha_k)_{k\geq 0}$ be an admissible strategy of switching, i.e., $(\theta_k)_{k\geq 0}$ is an increasing sequence of $\mathbb{F}^{\mu_n}$-stopping times with values in $[0,T]$ such that $\mathbb{P}[\theta_k < T, \forall k \geq 0] = 0$ and for any $k\geq 0$, α_k is a random variable $\mathcal{F}_{\theta_k}^{\mu_n}$-measurable with values in $\mathcal{I}$. Next, with the admissible strategy $\delta := (\theta_k, \alpha_k)_{k\geq 0}$ is associated a switching cost process $(A_s^\delta)_{s\leq T}$ defined by:

$$ A_s^\delta := \sum_{k\geq 1} g_{\alpha_{k-1}\alpha_k}(\theta_k) \mathbf{1}_{\{\theta_k\leq s\}} \text{ for } s<T \text{ and } A_T^\delta = \lim_{s\to T} A_s^\delta. \tag{4.3.12} $$

The process $(A_s^\delta)_{s\leq T}$ is RCLL and non-decreasing. Now, for $s\leq T$, let us set $\eta_s := \alpha_0 \mathbf{1}_{\{\theta_0\}}(s) + \sum_{k\geq 1} \alpha_k \mathbf{1}_{[\theta_k\leq s<\theta_{k+1})}$ which stands for the mode indicator of

the system at time s. The process $(\eta_s)_{s\leq T}$ is in bijection with the strategy δ. Finally, for any fixed $s \leq T$ and $i \in \mathcal{I}$, let us denote by $\mathcal{A}_s^{i,\mu_n}$ the following set of admissible strategies:

$$\mathcal{A}_s^{i,\mu_n} := \{\delta := (\theta_k, \alpha_k)_{k\geq 0} \text{ admissible strategy such that } \theta_0 = s, \alpha_0 = i \\ \text{and } \mathbb{E}[(A_T^\delta)^2] < \infty\}.$$

Now, let $\delta := (\theta_k, \alpha_k)_{k\geq 0} \in \mathcal{A}_t^{i,\mu_n}$ and let us define the triplet of $\mathbb{F}^{\mu_n}$-progressively measurable processes $({}^nP_s^{\delta,t,x}, {}^nN_s^{\delta,t,x}, {}^nQ_s^{\delta,t,x})_{s\leq T}$ as follows:

$$\begin{cases} {}^nP^{\delta,t,x} \text{ is RCLL and } \mathbb{E}[\sup_{s\leq T} |{}^nP_s^{\delta,t,x}|^2] < \infty\,; \\ \mathbb{E}[\int_0^T |{}^nN_s^{\delta,t,x}|^2 ds] + \mathbb{E}[\int_0^T \int_E |{}^nQ_s^{\delta,t,x}(e)|^2 \lambda_n(de)ds] < \infty; \\ {}^nP_s^{\delta,t,x} = h^\delta({}^nX_T^{t,x}) + \int_s^T \widehat{f}^\delta(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nN_r^{\delta,t,x})dr - \int_s^T {}^nN_r^{\delta,t,x} dB_r \\ \qquad\qquad - \int_s^T \int_E {}^nQ_r^{\delta,t,x}(e)\tilde{\mu}_n(dr,de) - A_T^\delta + A_s^\delta,\ s \leq T, \end{cases} \tag{4.3.13}$$

where

$$h^\delta(x) := \sum_{k\geq 0} h_{\alpha_k}(x)\mathbf{1}_{[\theta_k\leq T<\theta_{k+1}]} \quad \text{and}$$
$$\widehat{f}^\delta(s,x,({}^nY_s^{k,t,x})_{k\in\mathcal{I}},z) := \sum_{k\geq 0} \widehat{f}_{\alpha_k}(s,x,({}^nY_s^{k,t,x})_{k\in\mathcal{I}},z)\mathbf{1}_{[\theta_k\leq s<\theta_{k+1}]}. \tag{4.3.14}$$

Those series contain only a finite many terms as δ is admissible and then $\mathbb{P}[\theta_n < T, \forall n \geq 0] = 0$. Note that, in (4.3.13), the generators $\widehat{f}^\delta$ doe not depend neither on ${}^nP^{\delta,t,x}$ nor on ${}^nQ^{\delta,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda_n))$. However it depends on $({}^nY_r^{k,t,x})_{k\in\mathcal{I}}$ which is already defined. Next, by a change of variable, the existence of
$({}^nP^{\delta,t,x} - A^\delta, {}^nN^{\delta,t,x}, {}^nQ^{\delta,t,x})$ stems from the standard existence result of solutions of BSDEs with jumps by Tang-Li [76] since its generator
$\widehat{f}^\delta(s, {}^nX_s^{t,x}, ({}^nY_s^{k,t,x})_{k\in\mathcal{I}}, z)$ is Lipschitz w.r.t z and A_T^δ is square integrable. On the other hand let $({}^n\underline{Y}^{i,t,x}, {}^n\underline{Z}^{i,t,x}, {}^n\underline{V}^{i,t,x}, {}^n\underline{K}^{i,t,x})_{i\in\mathcal{I}}$ be the solution of the

following system: $\forall i = 1, ..., m$ and $s \in [0, T]$,

$$\begin{cases} {}^n\underline{Y}^{i,t,x} \in \mathcal{S}^2, {}^n\underline{Z}^{i,t,x} \in \mathcal{H}^{2,d}, {}^n\underline{V}^{i,t,x} \in \mathcal{H}^2(\mathcal{L}^2(\lambda_n)), \text{ and } {}^n\underline{K}^{i,t,x} \in \mathcal{A}^2; \\ {}^n\underline{Y}^{i,t,x}_s = h_i({}^nX^{t,x}_T) + \int_s^T \widehat{f}_i(r, {}^nX^{t,x}_r, ({}^nY^{k,t,x}_r)_{k\in\mathcal{I}}, {}^n\underline{Z}^{i,t,x}_r)dr + {}^n\underline{K}^{i,t,x}_T - {}^n\underline{K}^{i,t,x}_s \\ \qquad\qquad - \int_s^T {}^n\underline{Z}^{i,t,x}_r dB_r - \int_s^T \int_E {}^n\underline{V}^{i,t,x}_r(e)\tilde{\mu}_n(dr, de); \\ {}^n\underline{Y}^{i,t,x}_s \geqslant \max_{j\in\mathcal{I}^{-i}}({}^n\underline{Y}^{j,t,x}_s - g_{ij}(s)); \\ \int_0^T ({}^n\underline{Y}^{i,t,x}_s - \max_{j\in\mathcal{I}^{-i}}({}^n\underline{Y}^{j,t,x}_s - g_{ij}(s)))d{}^n\underline{K}^{i,t,x}_s = 0, \end{cases} \tag{4.3.15}$$

whose solution exists and is unique. Therefore the link between the solution of system (4.3.15) and optimal switching problems implies that (one can see [44] for more details) :

$${}^n\underline{Y}^{i,t,x}_s = \text{esssup}_{\delta\in\mathcal{A}^{i,\mu_n}_s}({}^nP^{\delta,t,x}_s - A^\delta_s). \tag{4.3.16}$$

But the processes $({}^nY^{i,t,x}, {}^nZ^{i,t,x}, {}^nV^{i,t,x}, {}^nK^{i,t,x})_{i\in\mathcal{I}}$ solution of (4.3.8) is also a solution of (4.3.15), therefore by uniqueness of the solution of (4.3.15) one deduces that

$${}^n\underline{Y}^{j,t,x}_s = {}^nY^{i,t,x}_s = \underset{\delta\in\mathcal{A}^{i,\mu_n}_s}{\text{esssup}}\,({}^nP^{\delta,t,x}_s - A^\delta_s) = ({}^nP^{\delta^*,t,x}_s - A^{\delta^*}_s). \tag{4.3.17}$$

for some $\delta^* \in \mathcal{A}^{i,\mu_n}_s$. It means that δ^* is an optimal strategy of this switching control problem.

Step 4: Uniform convergence of $({}^n\mathbf{u}^\mathbf{i})_{\mathbf{n}\geq\mathbf{1}}$, $\mathbf{i} \in \mathcal{I}$.

We are going to show that for any $i \in \mathcal{I}$, $({}^nu^i)_{n\geq1}$ is a Cauchy sequence for the uniform norm.

So let $m \geq n \geq 1$. Then we obviously have $\mathbb{F}^{\mu_n} \subset \mathbb{F}^{\mu_m}$. Next for any $i \in \mathcal{I}$ and $s \in [t, T]$ let us set:

$$F^{n,m}_i(s, \omega, z) := \widehat{f}_i(s, {}^nX^{t,x}_s, ({}^nY^{k,t,x}_s)_{k\in\mathcal{I}}, z) \vee \widehat{f}_i(s, {}^mX^{t,x}_s, ({}^mY^{k,t,x}_s)_{k\in\mathcal{I}}, z) \text{ and}$$

$$H^{n,m}_i := h^i({}^nX^{t,x}_T) \vee h^i({}^mX^{t,x}_T).$$

Now let us consider the solution, denoted by $({}^{n,m}Y^{i,t,x}, {}^{n,m}Z^{i,t,x}, {}^{n,m}V^{i,t,x}, {}^{n,m}K^{i,t,x})_{i\in\mathcal{I}}$, of the obliquely reflected BSDEs with jumps associated with

$((F_i^{n,m})_{i\in\mathcal{I}}, (g_{ij})_{i,j\in\mathcal{I}}, (H_i^{n,m})_{i\in\mathcal{I}})$ which exists and is unique (see Proposition 4.3.1) since

$$\forall i \in \mathcal{I}, H_i^{n,m} \geq \max_{j\neq i}(H_j^{n,m} - g_{ij}(T)).$$

Note that this condition is satisfied since g_{ij}, $i,j \in \mathcal{I}$, do not depend on x. Next, let $s \in [t,T]$ and $\delta := (\theta_k, \alpha_k)_{k\geq 0}$ a strategy of $\mathcal{A}_s^{i,\mu_m}$. Let us define the triplet of $\mathbb{F}^{\mu_m}$-progressively measurable processes $({}^{n,m}P_r^{\delta,t,x}, {}^{n,m}N_r^{\delta,t,x}, {}^{n,m}Q_r^{\delta,t,x})_{r\leq T}$ as follows:

$$\begin{cases} {}^{n,m}P^{\delta,t,x} \text{ is RCLL and } \mathbb{E}[\sup_{r\leq T}|{}^{n,m}P_r^{\delta,t,x}|^2] < \infty; \\ \mathbb{E}[\int_0^T |{}^{n,m}N_r^{\delta,t,x}|^2 dr] + \mathbb{E}[\int_0^T \int_E |{}^{n,m}Q_r^{\delta,t,x}(e)|^2 \lambda_m(de)dr] < \infty; \\ {}^{n,m}P_\tau^{\delta,t,x} = H_\delta^{n,m} + \int_\tau^T F_\delta^{n,m}(r, {}^{n,m}N_r^{\delta,t,x})dr - \int_\tau^T {}^{n,m}N_r^{\delta,t,x} dB_r \\ \qquad\qquad - \int_\tau^T \int_E {}^{n,m}Q_r^{\delta,t,x}(e)\tilde{\mu}_m(dr,de) \\ \quad -A_T^\delta + A_\tau^\delta, \ \tau \leq T, \end{cases} \tag{4.3.18}$$

where, as in (4.3.14),

$$H_\delta^{n,m} := \sum_{k\geq 0} H_{\alpha_k}^{n,m} \mathbf{1}_{[\theta_k \leq T < \theta_{k+1}]} \text{ and } F_\delta^{n,m}(r,z) = \sum_{k\geq 0} F_{\alpha_k}^{n,m}(r,z)\mathbf{1}_{[\theta_k \leq r < \theta_{k+1}]}. \tag{4.3.19}$$

Next as in (4.3.16), we have:

$${}^{n,m}Y_s^{i,t,x} = \text{esssup}_{\delta\in\mathcal{A}_s^{i,\mu_m}}({}^{n,m}P_s^{\delta,t,x} - A_s^{\delta,t,x}) = ({}^{n,m}P_s^{\bar{\delta},t,x} - A_s^{\bar{\delta},t,x}), \tag{4.3.20}$$

where $\bar{\delta}$ belongs to $\mathcal{A}_s^{i,\mu_m}$ and depends on n,m which we omit as there is no possible confusion ($\bar{\delta}$ is the optimal strategy for the underlying switching problem). Next recall that $({}^nY^{i,t,x}, {}^nZ^{i,t,x}, {}^nV^{i,t,x}, {}^nK^{i,t,x})_{i\in\mathcal{I}}$ is a solution of the following system of obliquely RBSDEs with jumps: $\forall i \in \mathcal{I}$ and $s \in [0,T]$,

$$\begin{cases} {}^nY_s^{i,t,x} = h_i({}^nX_T^{t,x}) + \int_s^T \widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x})dr \\ \qquad\qquad + {}^nK_T^{i,t,x} - {}^nK_s^{i,t,x} - \int_s^T {}^nZ_r^{i,t,x} dB_r - \int_s^T \int_E {}^nV_r^{i,t,x}(e)\tilde{\mu}_n(dr,de); \\ {}^nY_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}({}^nY_s^{j,t,x} - g_{ij}(s)); \\ \int_0^T ({}^nY_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}({}^nY_s^{j,t,x} - g_{ij}(s)))d{}^nK_s^{i,t,x} = 0. \end{cases} \tag{4.3.21}$$

But $\tilde{\mu}_n = \tilde{\mu}_m 1_{\{|e|\geq\frac{1}{n}\}}$ and $\mathbb{F}^{\mu_n} \subset \mathbb{F}^{\mu_m}$, then $({}^nY^{i,t,x}, {}^nZ^{i,t,x}, {}^nV^{i,t,x}1_{\{|e|\geq\frac{1}{n}\}}, {}^nK^{i,t,x})_{i\in\mathcal{I}}$ is a solution of the following system of obliquely RBSDEs with jumps: $\forall i = 1, ..., m$ and $s \in [0, T]$,

$$\begin{cases} {}^nY_s^{i,t,x} = h_i({}^nX_T^{t,x}) + \int_s^T \widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x})dr \\ \qquad\qquad + {}^nK_T^{i,t,x} - {}^nK_s^{i,t,x} - \int_s^T {}^nZ_r^{i,t,x}dB_r - \int_s^T\int_E {}^nV_r^{i,t,x}(e)1_{\{|e|\geq\frac{1}{n}\}}\tilde{\mu}_m(dr,de); \\ {}^nY_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}({}^nY_s^{j,t,x} - g_{ij}(s)); \\ \int_0^T({}^nY_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}({}^nY_s^{j,t,x} - g_{ij}(s)))d{}^nK_s^{i,t,x} = 0. \end{cases} \tag{4.3.22}$$

Therefore we have also:

$${}^nY_s^{i,t,x} = \text{esssup}_{\delta\in\mathcal{A}_s^{i,\mu_m}}({}^n\bar{P}_s^{\delta,t,x} - A_s^{\delta,t,x}), \tag{4.3.23}$$

where $({}^n\bar{P}^{\delta,t,x}, {}^n\bar{N}^{\delta,t,x}, {}^n\bar{Q}^{\delta,t,x})$ is the solution of the BSDE similar to that one satisfied by $({}^nP^{\delta,t,x}, {}^nN^{\delta,t,x}, {}^nQ^{\delta,t,x})$ in (3.3.6) but in the stochastic basis $(\mathbb{F}^{\mu_m}, \mu_m)$ since the strategy δ belongs to $\mathcal{A}^{i,\mu_m}$. Next by the comparison result (see Proposition 4.2 in [45]), between the solutions ${}^nY^{i,t,x}$ and ${}^{n,m}Y^{i,t,x}$, and ${}^mY^{i,t,x}$ and ${}^{n,m}Y^{i,t,x}$ (this is possible since: (i) the generators of the systems do not depend on the jump parts; (ii) the corresponding generators and terminal values are comparable), one deduces that: For any $i \in \mathcal{I}$,

$${}^nY_s^{i,t,x} \leq {}^{n,m}Y_s^{i,t,x} \quad \text{and} \quad {}^mY_s^{i,t,x} \leq {}^{n,m}Y_s^{i,t,x}.$$

This combined with (4.3.17) and (4.3.23), leads to:

$${}^n\bar{P}_s^{\bar{\delta},t,x} - A_s^{\bar{\delta},t,x} \leq {}^nY_s^{i,t,x} \leq {}^{n,m}P_s^{\bar{\delta}} - A_s^{\bar{\delta},t,x} \quad \text{and} \quad {}^mP_s^{\bar{\delta}} - A_s^{\bar{\delta},t,x} \leq {}^mY_s^{i,t,x} \leq {}^{n,m}P_s^{\bar{\delta}} - A_s^{\bar{\delta},t,x},$$

which implies:

$$|{}^nY_s^{i,t,x} - {}^mY_s^{i,t,x}| \leq |{}^{n,m}P_s^{\bar{\delta}} - {}^n\bar{P}_s^{\bar{\delta}}| + |{}^{n,m}P_s^{\bar{\delta}} - {}^mP_s^{\bar{\delta}}|. \tag{4.3.24}$$

Since both terms on the right-hand side of (4.3.24) are treated similarly, we focus only on the first one. Applying Itô's formula with $e^{\alpha\tau}|{}^{n,m}P_\tau^{\bar{\delta},t,x} - {}^n\bar{P}_\tau^{\bar{\delta},t,x}|^2$

and $\alpha \in \mathbb{R}$, yields: $\forall \tau \leq T$,

$$
\begin{aligned}
&e^{\alpha\tau}|^{n,m}P_\tau^{\bar\delta} - {}^n\bar P_\tau^{\bar\delta}|^2 + \int_\tau^T e^{\alpha r}|^{n,m}N_r^{\bar\delta} - {}^n\bar N_r^{\bar\delta}|^2 dr + \textstyle\sum_{\tau<r\leq T} e^{\alpha r}\Delta_r({}^{n,m}P^{\bar\delta} - {}^n\bar P^{\bar\delta})^2 \\
&= e^{\alpha T}|h^{\bar\delta}({}^mX_T^{t,x}) \vee h^{\bar\delta}({}^nX_T^{t,x}) - h^{\bar\delta}({}^nX_T^{t,x})|^2 - \alpha\int_\tau^T e^{\alpha r}|^{n,m}P_r^{\bar\delta} - {}^n\bar P_r^{\bar\delta}|^2 dr \\
&\quad + 2\int_\tau^T e^{\alpha r}({}^{n,m}P_r^{\bar\delta} - {}^n\bar P_r^{\bar\delta}) \times \Big\{ \widehat{f}^{\bar\delta}(r, {}^mX_r^{t,x}, ({}^mY_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar\delta}) \\
&\quad \vee \widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar\delta}) - \widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^n\bar N_r^{\bar\delta}) \Big\} dr \\
&\quad - 2\int_\tau^T e^{\alpha r}({}^{n,m}P_r^{\bar\delta} - {}^n\bar P_r^{\bar\delta})({}^{n,m}N_r^{\bar\delta} - {}^n\bar N_r^{\bar\delta}) dB_r \\
&\quad - 2\int_\tau^T\int_E e^{\alpha r}({}^{n,m}P_{r-}^{\bar\delta} - {}^n\bar P_{r-}^{\bar\delta})({}^{n,m}Q_r^{\bar\delta}(e) - {}^n\bar Q_r^{\bar\delta}(e))\tilde\mu_m(de, dr).
\end{aligned}
$$

Next, taking expectation and using the inequality $|x\vee y - y| \leq |x-y|$ $(x, y \in \mathbb{R})$ to obtain: $\forall \tau \in [0, T]$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha\tau}|^{n,m}P_\tau^{\bar\delta} - {}^n\bar P_\tau^{\bar\delta}|^2 + \int_\tau^T e^{\alpha r}|^{n,m}N_r^{\bar\delta} - {}^n\bar N_r^{\bar\delta}|^2 dr + \textstyle\sum_{\tau<r\leq T} e^{\alpha r}\Delta_r({}^{n,m}P^{\bar\delta} - {}^n\bar P^{\bar\delta})^2\Big] \\
&\leq \mathbb{E}\Big[e^{\alpha T}|h^{\bar\delta}({}^mX_T^{t,x}) - h^{\bar\delta}({}^nX_T^{t,x})|^2 - \alpha\int_\tau^T e^{\alpha r}|^{n,m}P_r^{\bar\delta} - {}^n\bar P_r^{\bar\delta}|^2 dr \\
&\quad + 2\int_\tau^T e^{\alpha r}|^{n,m}P_r^{\bar\delta} - {}^n\bar P_r^{\bar\delta}| \times \Big\{ |\widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar\delta}) \\
&\quad - \widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^n\bar N_r^{\bar\delta})| \\
&\quad + |\underbrace{\widehat{f}^{\bar\delta}(r, {}^mX_r^{t,x}, ({}^mY_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar\delta}) - \widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar\delta})}_{\Sigma^{n,m,\bar\delta}(r)}| \Big\} dr\Big].
\end{aligned}
\tag{4.3.25}
$$

Now as $(\vec y, z) \mapsto \widehat{f}^{\bar\delta}(t, x, \vec y, z)$ is Lipschitz uniformly w.r.t. (t, x) then (for some constant κ_1 which may change from line to line):

$$
|\widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar\delta}) - \widehat{f}^{\bar\delta}(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^n\bar N_r^{\bar\delta})| \leq \kappa_1|^{n,m}N_r^{\bar\delta} - {}^n\bar N_r^{\bar\delta}|.
$$

On the other hand, by (4.3.2), we have:

$$
\begin{aligned}
|\Sigma^{n,m,\bar{\delta}}(r)| &\leq \kappa_1 |({}^m Y_r^{k,t,x})_{k\in\mathcal{I}} - ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}| \\
&\quad + |\widehat{f}^{\bar{\delta}}(r, {}^m X_r^{t,x}, ({}^m Y_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}N_r^{\bar{\delta}}) - \widehat{f}^{\bar{\delta}}(r, {}^n X_r^{t,x}, ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}, {}^{n,m}\bar{N}_r^{\bar{\delta}})| \\
&\leq \kappa_1 |({}^m Y_r^{k,t,x})_{k\in\mathcal{I}} - ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}| + \sum_{\ell=1,m} \Phi_\ell(|{}^m X_r^{t,x} - {}^n X_r^{t,x}|).
\end{aligned}
\tag{4.3.26}
$$

Therefore we get from (4.3.25) and (4.3.26) : $\forall \tau \leq T$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha\tau}|{}^{n,m}P_\tau^{\bar{\delta}} - {}^n\bar{P}_\tau^{\bar{\delta}}|^2 + \int_\tau^T e^{\alpha r}|{}^{n,m}N_r^{\bar{\delta}} - {}^n\bar{N}_r^{\bar{\delta}}|^2 dr + \textstyle\sum_{\tau<r\leq T} e^{\alpha r}\Delta_r({}^{n,m}P^{\bar{\delta}} - {}^n\bar{P}^{\bar{\delta}})^2\Big] \\
&\leq e^{\alpha T}\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|^2] + (-\alpha + 2 + 4\kappa_1^2)\mathbb{E}\Big[\int_\tau^T e^{\alpha r}|{}^{n,m}P_r^{\bar{\delta}} - {}^n\bar{P}_r^{\bar{\delta}}|^2 dr\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha r}|({}^m Y_r^{k,t,x})_{k\in\mathcal{I}} - ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}|^2 dr\Big] + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha r}|{}^{n,m}N_r^{\bar{\delta}} - {}^n\bar{N}_r^{\bar{\delta}}|^2 dr\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha r}|\sum_{\ell=1,m}\Phi_\ell(|{}^m X_r^{t,x} - {}^n X_r^{t,x}|)|^2 dr\Big].
\end{aligned}
$$

Take now $\alpha = \alpha_0 = 2 + 4\kappa_1^2$ to deduce that for any $\tau \leq T$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha_0\tau}|{}^{n,m}P_\tau^{\bar{\delta}} - {}^n\bar{P}_\tau^{\bar{\delta}}|^2\Big] \\
&\leq e^{\alpha_0 T}\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|^2] + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha_0 r}|({}^m Y_r^{k,t,x})_{k\in\mathcal{I}} - ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}|^2 dr\Big] + \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha_0 r}|\sum_{\ell=1,m}\Phi_\ell(|{}^m X_r^{t,x} - {}^n X_r^{t,x}|)|^2 dr\Big].
\end{aligned}
$$

In the same way as presley we have also for any $\tau \leq T$,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{\alpha_0\tau}|{}^{n,m}P_\tau^{\bar{\delta}} - {}^m\bar{P}_\tau^{\bar{\delta}}|^2\Big] \\
&\leq e^{\alpha_0 T}\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|^2] + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha_0 r}|({}^m Y_r^{k,t,x})_{k\in\mathcal{I}} - ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}|^2 dr\Big] + \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\int_\tau^T e^{\alpha_0 r}|\sum_{\ell=1,m}\Phi_\ell(|{}^m X_r^{t,x} - {}^n X_r^{t,x}|)|^2 dr\Big].
\end{aligned}
\tag{4.3.27}
$$

Next take $\tau = s$ and using the inequality (4.3.24) to deduce that

$$\begin{aligned}
&\mathbb{E}\Big[e^{\alpha_0 s}|{}^m Y_s^{i,t,x} - {}^m Y_s^{i,t,x}|^2\Big] \\
&\leq 4e^{\alpha_0 T}\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|^2] + 2\mathbb{E}\Big[\int_s^T e^{\alpha_0 r}|({}^m Y_r^{k,t,x})_{k\in\mathcal{I}} - ({}^n Y_r^{k,t,x})_{k\in\mathcal{I}}|^2 dr\Big] \\
&\quad + 2\underbrace{\mathbb{E}\Big[\int_s^T e^{\alpha_0 r}|\sum_{\ell=1,m}\Phi_\ell(|{}^m X_r^{t,x} - {}^n X_r^{t,x}|)|^2 dr\Big]}_{\Sigma^{n,m}(t,x)(s)}.
\end{aligned} \tag{4.3.28}$$

Next let us deal with $\mathbb{E}\Big[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|^2\Big]$. Indeed

$$|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})| \leq \sum_{i=1,m}|h_i({}^m X_T^{t,x}) - h_i({}^n X_T^{t,x})|.$$

As h_i is uniformly continuous we denote by ϖ_i its modulus of continuity and then

$$|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})| \leq \sum_{i=1,m}\varpi_i(|{}^m X_T^{t,x} - {}^n X_T^{t,x}|).$$

Next by concavity of the modulus of continuity, Jensen's inequality and the estimate (4.3.6) it holds

$$\begin{aligned}
\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|] &\leq \sum_{i=1,m}\varpi_i(\mathbb{E}[|{}^m X_T^{t,x} - {}^n X_T^{t,x}|]) \\
&\leq \sum_{i=1,m}\varpi_i((C\int_{\{\frac{1}{m}\leq|e|\leq\frac{1}{n}\}}(1\wedge|e|^2)\lambda(de))^{\frac{1}{2}}),
\end{aligned}$$

and then

$$\lim_{n,m\to\infty}\underbrace{\sup_{(t,x)\in[0,T]\times\mathbb{R}^k}\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|]}_{\Gamma_h^{n,m}} = 0.$$

But as $h^{\bar{\delta}}(.)$ is bounded then for any $p \geq 1$

$$\lim_{n,m\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^k}\mathbb{E}[|h^{\bar{\delta}}({}^m X_T^{t,x}) - h^{\bar{\delta}}({}^n X_T^{t,x})|^p] = 0.$$

Finally let us deal with the term $\Sigma^{n,m}(t,x)$ of inequality (4.3.28). Using the

Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}\Sigma^{n,m}(t,x)(s) &= \int_s^T e^{\alpha_0 r}\mathbb{E}\Big[\Big(\sum_{\ell=1,m}\Phi_\ell(|{}^mX_r^{t,x}-{}^nX_r^{t,x}|)\Big)^2\Big]dr\\ &\leq \int_s^T e^{\alpha_0 r}\Big\{\mathbb{E}\Big[\sum_{\ell=1,m}\Phi_\ell(|{}^mX_r^{t,x}-{}^nX_r^{t,x}|)\Big]\Big\}^{\frac{1}{2}}\times\\ &\qquad\Big\{\mathbb{E}\Big[\Big(\sum_{\ell=1,m}\Phi_\ell(|{}^mX_r^{t,x}-{}^nX_r^{t,x}|)\Big)^3\Big]\Big\}^{\frac{1}{2}}dr\\ &\leq \int_s^T e^{\alpha_0 r}\Big\{\Big[\sum_{\ell=1,m}\Phi_\ell(\mathbb{E}[\sup_{r\leq T}|{}^mX_r^{t,x}-{}^nX_r^{t,x}|])\Big]\Big\}^{\frac{1}{2}}\times\\ &\qquad\Big\{\mathbb{E}\Big[\Big(\sum_{\ell=1,m}\Phi_\ell(|{}^mX_r^{t,x}-{}^nX_r^{t,x}|)\Big)^3\Big]\Big\}^{\frac{1}{2}}dr\end{aligned}$$

where, for the last inequality, we have made use of Jensen's and Cauchy-Schwarz's ones. Now it is enough to take into account of (4.3.6) and the fact that the modulus of continuity is of linear growth to deduce that

$$\limsup_{n,m\to\infty}\underbrace{\sup_{(t,x)\in[0,T]\times\mathbb{R}^k}\sup_{s\in[t,T]}\Sigma^{n,m}(t,x)(s)}_{\Gamma_f^{n,m}}\to 0 \text{ as } n,m\to\infty.$$

Now go back to (4.3.28), take the sum over $i\in\mathcal{I}$ and make use of the Gronwall inequality to deduce that: $\forall s\in[t,T]$,

$$\mathbb{E}\Big[e^{\alpha_0 s}\sum_{i=1,m}|{}^nY_s^{i,t,x}-{}^mY_s^{i,t,x}|^2\Big]\leq C(4me^{\alpha_0 T}\Gamma_h^{n,m}+2m\Gamma_f^{n,m})$$

Finally take $s=t$ and the supremum over $(t,x)\in[0,T]\times\mathbb{R}^k$ in the previous inequality to deduce that for any $i\in\mathcal{I}$, the sequence $(u^{i,n})_{n\geq 0}$ is uniformly convergent in $[0,T]\times\mathbb{R}^k$. Thus its limit $\check{u}^i$ is a continuous and bounded function on $[0,T]\times\mathbb{R}^k$.

Step 5: Convergence of the auxiliary processes

First, let us prove that $(({}^nY_s^{i,t,x})_{s\in[t,T]})_{n\geq 1}$ is a Cauchy sequence in $\mathcal{S}^2_{[t,T]}$ which

is $\mathcal{S}^2$ reduced to $[t,T]$. For $n,m \geq 1$ we have:

$$\begin{aligned}\mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^{n}Y_s^{i,t,x}-{}^{m}Y_s^{i,t,x}\big|^2\big] &= \mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^{n}u^i(s,{}^{n}X_s^{t,x})-{}^{m}u^i(s,{}^{m}X_s^{t,x})\big|^2\big]\\ &\leq 2\mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^{n}u^i(s,{}^{n}X_s^{t,x})-{}^{m}u^i(s,{}^{n}X_s^{t,x})\big|^2\big]\\ &\quad + 2\mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^{m}u^i(s,{}^{n}X_s^{t,x})-{}^{m}u^i(s,{}^{m}X_s^{t,x})\big|^2\big].\end{aligned} \tag{4.3.29}$$

Besides

$$\begin{aligned}&\mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^{n}u^i(s,{}^{n}X_s^{t,x})-{}^{m}u^i(s,{}^{n}X_s^{t,x})\big|^2\big] \\ &\quad\leq ||{}^{n}u^i-{}^{m}u^i||_\infty^2 := \sup_{(t,x)\in[0,T]\times\mathbb{R}^k}|{}^{n}u^i(t,x)-{}^{m}u^i(t,x)|^2\to 0 \text{ as } n,m\to\infty.\end{aligned} \tag{4.3.30}$$

As for the second term, we use the inequality $|a+b+c|^2 \leq 3\{|a|^2+|b|^2+|c|^2|\}$ to obtain :

$$\begin{aligned}&\mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^{m}u^i(s,{}^{n}X_s^{t,x})-{}^{m}u^i(s,{}^{m}X_s^{t,x})\big|^2\big]\\ &\quad\leq 3\mathbb{E}\Big[\sup_{t\leq s\leq T}\big|{}^{m}u^i(s,{}^{n}X_s^{t,x})-\check{u}^i(s,{}^{n}X_s^{t,x})\big|^2+\sup_{t\leq s\leq T}\big|\check{u}^i(s,{}^{n}X_s^{t,x})-\check{u}^i(s,{}^{m}X_s^{t,x})\big|^2\\ &\qquad+\sup_{t\leq s\leq T}\big|\check{u}^i(s,{}^{m}X_s^{t,x})-{}^{m}u^i(s,{}^{m}X_s^{t,x})\big|^2\Big].\end{aligned}$$

By the uniform convergence of $(u^{i,n})_{n\geq 0}$ to $\check{u}^i$, the first term and the third one converge to 0 as $m\to\infty$. Next let us focus on the second one. So let $\rho>0$ and let us denote by $\omega_\rho^i(.)$ the concave modulus of continuity of $\check{u}^i$ on $[0,T]\times\bar{B}(0,\rho)$. Then by the boundedness of $\check{u}^i$, we have,

$$\begin{aligned}&\mathbb{E}\big[\sup_{t\leq s\leq T}\big|\check{u}^i(s,{}^{m}X_s^{t,x})-\check{u}^i(s,{}^{n}X_s^{t,x})\big|^2\big]\\ &\leq C\Big\{\mathbb{E}\big[\mathbf{1}_{\{\sup_{s\leq T}\{|{}^{m}X_s^{t,x}|+|{}^{n}X_s^{t,x}|\}\leq\rho\}}\sup_{t\leq s\leq T}\big\{\big|\check{u}^i(s,{}^{m}X_s^{t,x})-\check{u}^i(s,{}^{n}X_s^{t,x})\big|\big\}\big]\\ &\qquad+\mathbb{E}\big[\mathbf{1}_{\{\sup_{s\leq T}\{|{}^{m}X_s^{t,x}|+|{}^{n}X_s^{t,x}|\}>\rho\}}\sup_{t\leq s\leq T}\big\{\big|\check{u}^i(s,{}^{m}X_s^{t,x})-\check{u}^i(s,{}^{n}X_s^{t,x})\big|\big\}\big]\Big\}\\ &\leq C\Big\{\mathbb{E}\big[\sup_{t\leq s\leq T}\big\{\big|\check{u}^i(s,{}^{m}X_s^{t,x})-\check{u}^i(s,{}^{n}X_s^{t,x})\big|\big\}\mathbf{1}_{\{\sup_{s\leq T}\{|{}^{m}X_s^{t,x}|+|{}^{n}X_s^{t,x}|\}>\rho\}}\big\}\big]\\ &\qquad+\mathbb{E}\big[\sup_{t\leq s\leq T}\big\{\omega_\rho^i(|{}^{m}X_s^{t,x}-{}^{n}X_s^{t,x}|)\big\}\mathbf{1}_{\{\sup_{s\leq T}\{|{}^{m}X_s^{t,x}|+|{}^{n}X_s^{t,x}|\}\leq\rho\}}\big]\Big\}.\end{aligned} \tag{4.3.31}$$

But Markov's inequality implies that the first term is dominated by $C_x\rho^{-1}$, where C_x is an appropriate constant which may depend on x, since $\check{u}^i$ is bounded and $\mathbb{E}[\sup_{s\leq T}\{|{}^mX_s^{t,x}|+|{}^nX_s^{t,x}|\}]\leq C(1+|x|)$. On the other hand, the second one verifies

$$\begin{aligned}
&\mathbb{E}\big[\sup_{t\leq s\leq T}\{\omega_\rho^i(|{}^mX_s^{t,x}-{}^nX_s^{t,x}|)\}\mathbf{1}_{\{\sup_{s\leq T}\{|{}^mX_s^{t,x}|+|{}^nX_s^{t,x}|\}\leq\rho\}}\big]\\
&\leq\mathbb{E}\big[\{\omega_\rho^i(\sup_{t\leq s\leq T}|{}^mX_s^{t,x}-{}^nX_s^{t,x}|)\}\mathbf{1}_{\{\sup_{s\leq T}\{|{}^mX_s^{t,x}|+|{}^nX_s^{t,x}|\}\leq\rho\}}\big]\\
&=\mathbb{E}\big[\omega_\rho^i\Big(\sup_{t\leq s\leq T}|{}^mX_s^{t,x}-{}^nX_s^{t,x}|\times\mathbf{1}_{\{\sup_{s\leq T}\{|{}^mX_s^{t,x}|+|{}^nX_s^{t,x}|\}\leq\rho\}}\Big)\big]\\
&\leq\omega_\rho^i\Big(\mathbb{E}\big[\sup_{t\leq s\leq T}|{}^mX_s^{t,x}-{}^nX_s^{t,x}|\times\mathbf{1}_{\{\sup_{s\leq T}\{|{}^mX_s^{t,x}|+|{}^nX_s^{t,x}|\}\leq\rho\}}\big]\Big).
\end{aligned}$$

The last inequality stems from concavity of $\omega_\rho^i(.)$ and Jensen's inequality. Finally using (4.3.6) we deduce that

$$\limsup_{n,m\to\infty}\mathbb{E}\big[\sup_{t\leq s\leq T}\{\omega_\rho^i(|{}^mX_s^{t,x}-{}^nX_s^{t,x}|)\}\mathbf{1}_{\{\sup_{s\leq T}\{|{}^mX_s^{t,x}|+|{}^nX_s^{t,x}|\}\leq\rho\}}\big]=0.$$

Now as ρ is arbitrary then going back to (4.3.31), take the superior limit w.r.t m,n then the limit w.r.t ρ to deduce that

$$\limsup_{n,m\to\infty}\mathbb{E}\big[\sup_{t\leq s\leq T}\big|\check{u}^i(s,{}^mX_s^{t,x})-\check{u}^i(s,{}^nX_s^{t,x})\big|^2\big]=0.$$

Thus we have

$$\limsup_{n,m\to\infty}\mathbb{E}\big[\sup_{t\leq s\leq T}\big|{}^nY_s^{i,t,x}-{}^mY_s^{i,t,x}\big|^2\big]=0$$

which means that for any $i\in\mathcal{I}$, the sequence $({}^nY^{i,t,x})_{n\geq0}$ is convergent in $\mathcal{S}^2_{[t,T]}$ and whose limit is $(\check{Y}_s^{i,t,x}=\check{u}^i(s,X_s^{t,x}))_{s\in[t,T]}$ after substracting a subsequence of $({}^nX^{t,x})_{n\geq0}$ which converges pointwise. The proof of the claim is now terminated.

Now, we focus on the convergence of $({}^nZ^{i,t,x})_{n\geq0}$. For this, we first establish a priori estimates, uniform on n of the processes $({}^nZ^{i,t,x},{}^nV^{i,t,x},{}^nK^{i,t,x})$.

Applying Itô's formula to $|{}^nY_s^{i,t,x}|^2$, we obtain: $\forall s \in [t,T]$ and $i \in \mathcal{I}$,

$$\begin{aligned}
&\mathbb{E}\big[|{}^nY_s^{i,t,x}|^2\big] + \mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr\big] + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr\big] \\
&\quad \leq \mathbb{E}\big[|h_i({}^nX_T^{t,x})|^2\big] + 2\mathbb{E}\big[\int_s^T {}^nY_r^{i,t,x} \widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x})dr\big] \\
&\qquad + 2\mathbb{E}\big[\int_s^T {}^nY_r^{i,t,x} d{}^nK_r^{i,t,x}\big].
\end{aligned}$$

Then by a linearization procedure of $\widehat{f}_i$, which is possible since it is Lipschitz w.r.t $(\vec{y}, z)$ and using the inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$, for any $\epsilon > 0$ and $a, b \in \mathbb{R}$, we have:

$$\begin{aligned}
&\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr\big] + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr\big] \\
&\quad \leq \mathbb{E}\big[|h_i({}^nX_T^{t,x})|^2\big] + 2\mathbb{E}\big[\int_s^T |{}^nY_r^{i,t,x}| \times \Big\{|\widehat{f}_i(r, {}^nX_r^{t,x}, 0, 0)| + \sum_{l=1,m} a_r^{i,l,n}|{}^nY_r^{l,t,x}| \\
&\qquad + b_r^{i,n}|{}^nZ_r^{i,t,x}|\Big\}dr\big] + \frac{1}{\epsilon}\mathbb{E}\big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x}|^2\big] + \epsilon\mathbb{E}\big[\big({}^nK_T^{i,t,x} - {}^nK_s^{i,t,x}\big)^2\big],
\end{aligned}$$

where $a^{i,l,n} \in \mathbb{R}$, $b^{i,n} \in \mathbb{R}^d$ are $\mathcal{P}$-measurable non-negative bounded processes. Using again the inequality $2ab \leq \nu a^2 + \frac{1}{\nu}b^2$, for $\nu > 0$, we also get:

$$\begin{aligned}
&\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr\big] + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr\big] \\
&\quad \leq \mathbb{E}\big[|h_i({}^nX_T^{t,x})|^2\big] + \frac{1}{\nu}\mathbb{E}\big[\int_s^T |{}^nY_r^{i,t,x}|^2 dr\big] + \nu\mathbb{E}\big[\int_s^T \big\{|\widehat{f}_i(r, {}^nX_r^{t,x}, 0, 0, 0)| \\
&\qquad + \sum_{l=1,m} a_r^{i,l,n}|{}^nY_r^{l,t,x}| + b_r^{i,n}|{}^nZ_r^{i,t,x}|\big\}^2 dr\big] \\
&\qquad + \frac{1}{\epsilon}\mathbb{E}\big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x}|^2\big] + \epsilon\mathbb{E}\big[\big({}^nK_T^{i,t,x} - {}^nK_s^{i,t,x}\big)^2\big].
\end{aligned}$$

From the boundedness of $\widehat{f}_i(t,x,0,0)$ and $h_i(x)$, the inequality $|a+b+c|^2 \leq 3\{|a|^2 + |b|^2 + |c|^2\}$ $\forall a, b, c \in \mathbb{R}$ and finally the Cauchy-Schwarz one, we have:

$$\begin{aligned}
&\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,n}|^2 dr\big] + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr\big] \\
&\quad \leq \bar{C}^2 + 3\nu\bar{C}^2(T-s) + \frac{1}{\nu}\mathbb{E}\big[\int_s^T |{}^nY_r^{i,t,x}|^2 dr\big] + 3\nu C\mathbb{E}\big[\int_s^T \sum_{l=1,m} |{}^nY_r^{l,t,x}|^2 dr\big] \\
&\qquad + 3\nu C\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr\big] + \frac{1}{\epsilon}\mathbb{E}\big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x}|^2\big] + \epsilon\mathbb{E}\big[\big({}^nK_T^{i,t,x} - {}^nK_s^{i,t,x}\big)^2\big],
\end{aligned}$$

for a suitable positive constant C. Choose now ν such that $3\nu C < 1$ and taking the summation over all $i \in \mathcal{I}$, we obtain:

$$\sum_{i=1,m} \Big(\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr \big] + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr \big] \Big)$$
$$\leq C\Big(1 + \sum_{i=1,m} \mathbb{E}\big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x}|^2 \big]\Big) + \epsilon \sum_{i=1,m} \mathbb{E}\big[\big({}^nK_T^{i,t,x} - {}^nK_s^{i,t,x}\big)^2 \big],$$

where $C = C(T, m, \nu, \epsilon) > 0$ is an appropriate constant independent of n. Through the convergence of $({}^nY^{i,t,x})_n$ in $\mathcal{S}^2_{[t,T]}$, we have $\mathbb{E}\big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x}|^2\big] \leq C$ and then we get:

$$\sum_{i=1,m} \Big(\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr \big] + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr \big] \Big)$$
$$\leq C + \epsilon \sum_{i=1,m} \mathbb{E}\big[\big({}^nK_T^{i,n} - {}^nK_s^{i,t,x}\big)^2 \big].$$

Now, from the equality:

$$\begin{aligned} {}^nK_T^{i,t,x} - {}^nK_s^{i,t,x} = {}^nY_s^{i,t,x} - h_i({}^nX_T^{t,x}) - \int_s^T \widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x})dr \\ + \int_s^T {}^nZ_r^{i,t,x} dB_r + \int_s^T \int_E {}^nV_r^{i,t,x}(e)\tilde{\mu}_n(dr, de), \end{aligned} \tag{4.3.32}$$

and, once again, by a linearization procedure of the Lipschitz function $\widehat{f}_i$, the boundedness of $\widehat{f}_i(t, x, \vec{0}, 0)$ and $h_i(x)$ and finally the use of the Burkholder-Davis-Gundy inequality, there exists a positive constant C' such that

$$\sum_{i=1,m} \mathbb{E}\big[\big({}^nK_T^{i,t,x} - {}^nK_s^{i,t,x}\big)^2 \big]$$
$$\leq C'\Big(1 + \sum_{i=1,m} \mathbb{E}\big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x}|^2 \big] + \sum_{i=1,m} \mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr \big]$$
$$+ \sum_{i=1,m} \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr \big]\Big)$$
$$\leq C'\Big(1 + \sum_{i=1,m} \mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr \big] + \sum_{i=1,m} \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr \big]\Big).$$

Combining this last estimate with (4.3.32) and choosing ϵ small enough since

it is arbitrary, we obtain a constant $\bar{C}$ independent of n such that

$$\sum_{i=1,m} \mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x}|^2 dr + \big({}^nK_T^{i,t,x} - {}^nK_s^{i,t,x}\big)^2 + \int_s^T \int_E |{}^nV_r^{i,t,x}(e)|^2 \lambda_n(de)dr\big] \leq \bar{C}. \tag{4.3.33}$$

Now, for any $n, m \geq 1$, it follows from Itô's formula that:

$$\begin{aligned}
&\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x} - {}^mZ_r^{i,t,x}|^2 dr\big] \\
&\quad + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)\mathbf{1}_{\{|e|\geq \frac{1}{n}\}} - {}^mV_r^{i,t,x}(e)\mathbf{1}_{\{|e|\geq \frac{1}{m}\}}|^2 \lambda(de)dr\big] \\
&\leq \mathbb{E}\Big[|h_i({}^nX_T^{t,x}) - h_i({}^mX_T^{t,x})|^2\Big] + 2\mathbb{E}\big[\int_s^T \big({}^nY_r^{i,t,x} - {}^mY_r^{i,t,x}\big)\times \\
&\quad \Big\{\widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x}) - \widehat{f}_i(r, {}^mX_r^{t,x}, ({}^mY_r^{k,t,x})_{k\in\mathcal{I}}, {}^mZ_r^{i,t,x})\Big\}dr\big] \\
&\quad + 2\mathbb{E}\big[\int_s^T \big({}^nY_r^{i,t,x} - {}^mY_r^{i,t,x}\big)\big(d{}^nK_r^{i,t,x} - d{}^mK_r^{i,t,x}\big)\big].
\end{aligned}$$

By Cauchy-Schwarz inequality and by using the inequality $2ab \leq \frac{1}{\eta}a^2 + \eta b^2$ for $\eta > 0$, we obtain:

$$\begin{aligned}
&\mathbb{E}\big[\int_s^T |{}^nZ_r^{i,t,x} - {}^mZ_r^{i,t,x}|^2 dr\big] \\
&\quad + \mathbb{E}\big[\int_s^T \int_E |{}^nV_r^{i,t,x}(e)\mathbf{1}_{\{|e|\geq \frac{1}{n}\}} - {}^mV_r^{i,t,x}(e)\mathbf{1}_{\{|e|\geq \frac{1}{m}\}}|^2 \lambda(de)dr\big] \\
&\leq \mathbb{E}\Big[|h_i({}^nX_T^{t,x}) - h_i({}^mX_T^{t,x})|^2\Big] + 2\sqrt{\mathbb{E}\Big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x} - {}^mY_s^{i,t,x}|^2\Big]}\times \\
&\quad \Big\{\mathbb{E}\Big[\int_s^T |\widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x}) - \widehat{f}_i(r, {}^mX_r^{t,x}, ({}^mY_r^{k,t,x})_{k\in\mathcal{I}}, {}^mZ_r^{i,t,x})|^2 dr\Big]\Big\}^{\frac{1}{2}} \\
&\quad + \frac{1}{\eta}\mathbb{E}\Big[\sup_{t\leq s\leq T} |{}^nY_s^{i,t,x} - {}^mY_s^{i,t,x}|^2\Big] + \eta\mathbb{E}\Big[({}^nK_T^{i,t,x} + {}^mK_T^{i,t,x})^2\Big].
\end{aligned} \tag{4.3.34}$$

But there exists a constant $C \geq 0$ (independent of n and m) such that, for all $n, m \geq 1$,

$$\mathbb{E}\big[\int_s^T \big|f_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x}) - f_i(r, {}^mX_r^{t,x}, ({}^mY_r^{k,t,x})_{k\in\mathcal{I}}, {}^mZ_r^{i,t,x})\big|^2 dr\big] \leq C. \tag{4.3.35}$$

Besides, as h_i is uniformly continuous, we denote by ω_i its modulus of continuity and then we obtain:

$$\mathbb{E}\big[|h_i({}^nX_T^{t,x}) - h_i({}^mX_T^{t,x})|\big] \leq \mathbb{E}\big[\omega_i(|{}^nX_T^{t,x} - {}^mX_T^{t,x}|)\big].$$

From concavity of $\omega_i(.)$, Jensen'inequality and (4.3.6), we get:

$$\begin{aligned}\mathbb{E}\big[|h_i({}^nX_T^{t,x}) - h_i({}^mX_T^{t,x})|\big] &\leq \omega_i\big(\mathbb{E}\big[|{}^nX_T^{t,x} - {}^mX_T^{t,x}|\big]\big)\\ &\leq \omega_i\Big(\sqrt{C\int_{\{\frac{1}{m}\leq|e|\leq\frac{1}{n}\}}(1\wedge|e|^2)\lambda(de)}\Big).\end{aligned}$$

Since h_i is bounded then,

$$\lim_{n,m\to\infty}\sup_{(t,x)\in[0,T]\times\mathbb{R}^k}\mathbb{E}[|h_i({}^nX_T^{t,x}) - h_i({}^mX_T^{t,x})|^2] = 0. \tag{4.3.36}$$

Then taking the limit w.r.t n, m in (4.3.34) and taking into account of (4.3.33) and the convergence of ${}^nY^{i,t,x}$ in $\mathcal{S}^2_{[t,T]}$, we deduce that: we deduce that:

$$\begin{aligned}\limsup_{n,m\to\infty}\Big\{&\mathbb{E}\big[\int_t^T|{}^nZ_r^{i,t,x} - {}^mZ_r^{i,t,x}|^2dr\big]\\ &+\mathbb{E}\big[\int_t^T\int_E|{}^nV_r^{i,t,x}(e)\mathbf{1}_{\{|e|\geq\frac{1}{n}\}} - {}^mV_r^{i,t,x}(e)\mathbf{1}_{\{|e|\geq\frac{1}{m}\}}|^2\lambda(de)dr\big]\Big\} \leq \eta\, C.\end{aligned} \tag{4.3.37}$$

As η is arbitrary then $({}^nZ^{i,t,x})_n$ and $({}^nV^{i,t,x})_n$ are Cauchy sequences in the complete spaces $\mathcal{H}^{2,d}$ and $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ respectively. Then there exist processes $\check{Z}^{i,t,x}$ and $\check{V}^{i,t,x}$, respectively $\mathcal{P}$-measurable and $\mathbf{P}$-measurable such that the sequences $({}^nZ^{i,t,x})_n$ and $({}^nV^{i,t,x})_n$ converge respectively toward $\check{Z}^{i,t,x}$ and $\check{V}^{i,t,x}$ in $\mathcal{H}^{2,d}$ and $\mathcal{H}^2(\mathcal{L}^2(\lambda))$.

Now, going back to (4.3.9) and taking the limit w.r.t. n, we obtain: for any $i\in\mathcal{I}$ and $s\in[t,T]$,

$$\begin{aligned}\check{Y}_s^{i,t,x} = \text{esssup}_{\tau\geq s}\mathbb{E}\Big[&\int_s^\tau\widehat{f}_i(r,X_r^{t,x},(\check{Y}_r^{k,t,x})_{k\in\mathcal{I}},\check{Z}_r^{i,t,x})dr + h_i(X_\tau^{t,x})\mathbf{1}_{\{\tau=T\}}\\ &+\max_{j\in\mathcal{I}^{-i}}(Y_\tau^{j,t,x} - g_{ij}(\tau))\mathbf{1}_{\{\tau<T\}}\Big|\mathcal{F}_s\Big].\end{aligned}$$

Therefore, there exist processes $(\tilde{Z}^{i,t,x},\tilde{V}^{i,t,x},\tilde{K}^{i,t,x})_{i\in\mathcal{I}}\in\mathcal{H}^{2,d}\times\mathcal{H}^2(\mathcal{L}^2(\lambda))\times\mathcal{A}^2$ such that: for any $i\in\mathcal{I}$ and $s\in[t,T]$,

$$\begin{cases}\check{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T\widehat{f}_i(r,X_r^{t,x},(\check{Y}_r^{k,t,x})_{k\in\mathcal{I}},\check{Z}_r^{i,t,x})dr + \tilde{K}_T^{i,t,x} - \tilde{K}_s^{i,t,x}\\ \qquad\qquad - \int_s^T\tilde{Z}_r^{i,t,x}dB_r - \int_s^T\int_E\tilde{V}_r^{i,t,x}(e)\tilde{\mu}(dr,de);\\ \check{Y}_s^{i,t,x}\geqslant\max_{j\in\mathcal{I}^{-i}}(\check{Y}_s^{j,t,x} - g_{ij}(s));\\ \int_0^T(\check{Y}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(\check{Y}_s^{j,t,x} - g_{ij}(s)))d\tilde{K}_s^{i,t,x} = 0.\end{cases} \tag{4.3.38}$$

Now, applying Itô's formula with $|{}^nY_s^{i,t,x} - \check{Y}_s^{i,t,x}|^2$, yields: $\forall s \in [t,T]$,

$$\begin{aligned}
&\mathbb{E}\big[|{}^nY_s^{i,t,x} - \check{Y}_s^{i,t,x}|^2 + \int_s^T |{}^nZ_r^{i,t,x} - \tilde{Z}_r^{i,t,x}|^2 dr \\
&\quad + \int_s^T \int_E |{}^nV_r^{i,t,x}\mathbf{1}_{\{|e|\geq \frac{1}{n}\}} - \tilde{V}_r^{i,t,x}|^2 \lambda(de)dr\big] \\
&\leq \mathbb{E}\big[|h_i({}^nX_T^{t,x}) - h_i(X_T^{t,x})|^2\big] + 2\mathbb{E}\big[\int_s^T ({}^nY_r^{i,t,x} - \check{Y}_r^{i,t,x})\times \\
&\qquad \{\widehat{f}_i(r, {}^nX_r^{t,x}, ({}^nY_r^{k,t,x})_{k\in\mathcal{I}}, {}^nZ_r^{i,t,x}) - \widehat{f}_i(r, X_r^{t,x}, (\check{Y}_r^{k,t,x})_{k\in\mathcal{I}}, \check{Z}_r^{i,t,x})\}dr\big] \\
&\quad + 2\mathbb{E}\big[\int_s^T ({}^nY_r^{i,t,x} - \check{Y}_r^{i,t,x})d({}^nK_r^{i,t,x} - \tilde{K}_r^{i,t,x})\big].
\end{aligned}$$

Next, the same procedure as the one which leads to (4.3.37) can be used here to deduce that:

$$\mathbb{E}\big[\int_t^T |{}^nZ_r^{i,t,x} - \tilde{Z}_r^{i,t,x}|^2 dr + \int_t^T \int_E |{}^nV_r^{i,t,x}\mathbf{1}_{\{|e|\geq \frac{1}{n}\}} - \tilde{V}_r^{i,t,x}|^2 \lambda(de)dr\big] \to 0 \text{ as } n \to \infty.$$

This implies that the sequences $({}^nZ^{i,t,x})_n$ and $({}^nV^{i,t,x})_n$ converge toward $\tilde{Z}^{i,t,x}$ and $\tilde{V}^{i,t,x}$ in $\mathcal{H}^{2,d}$ and $\mathcal{H}^2(\mathcal{L}^2(\lambda))$ respectively. Therefore, by uniqueness of limits we deduce that for any $s \in [t,T]$ and $i \in \mathcal{I}$,

$$\tilde{Z}_s^{i,t,x} = \check{Z}_s^{i,t,x} \text{ and } \tilde{V}_s^{i,t,x} = \check{V}_s^{i,t,x}.$$

Then $(\check{Y}^{i,t,x}, \check{Z}^{i,t,x}, \tilde{V}^{i,t,x}, \tilde{K}^{i,t,x})_{i\in\mathcal{I}}$ verify: for any $s \in [t,T]$ and $i \in \mathcal{I}$,

$$\begin{cases}
\check{Y}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \widehat{f}_i(r, X_r^{t,x}, (\check{Y}_r^{k,t,x})_{k\in\mathcal{I}}, \check{Z}_r^{i,t,x})dr + \tilde{K}_T^{i,t,x} - \tilde{K}_s^{i,t,x} \\
\qquad\qquad - \int_s^T \check{Z}_r^{i,t,x} dB_r - \int_s^T \int_E \tilde{V}_r^{i,t,x}(e)\tilde{\mu}(dr,de); \\
\check{Y}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(\check{Y}_s^{j,t,x} - g_{ij}(s)); \\
\int_0^T (\check{Y}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(\check{Y}_s^{j,t,x} - g_{ij}(s)))d\tilde{K}_s^{i,t,x} = 0.
\end{cases}$$

It means that the quadruples of processes $((\check{Y}^{i,t,x}, \check{Z}^{i,t,x}, \tilde{V}^{i,t,x}, \tilde{K}^{i,t,x}))_{i\in\mathcal{I}}$ is a solution of system (4.3.1) in $[t,T]$. But the solution of this latter is unique, therefore we have:

$$(\check{Y}^{i,t,x}, \check{Z}^{i,t,x}, \tilde{V}^{i,t,x}, \tilde{K}^{i,t,x})_{i\in\mathcal{I}} = (Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}.$$

Then for any $i \in \mathcal{I}$, $\check{u}^i = u^i$.

Now, let us consider a subsequence which be denote by $\{n_k\}$ such that

$$|{}^{n_k}V_s^{i,t,x}\mathbf{1}_{\{|e|\geq \frac{1}{n_k}\}} - V_s^{i,t,x}|^2 \longrightarrow_{n_k} 0,\ ds\otimes\lambda\otimes\mathbb{P} - a.e. \text{ on } [t,T]\times E\times\Omega.$$

Besides, the process ${}^nV^{i,t,x}$ has a representation in terms of ${}^nu^i$ (see (4.3.11)) and $({}^nu^i)_{n\geq 0}$ converges uniformly to u^i. Then, if $(x_n)_n$ is a sequence of $\mathbb{R}^k$ which converges to x, then ${}^nu^i(t,x_n)$ converges to $u^i(t,x)$. Next, let us consider a subsequence which we denote by $\{n_{k,l}\}$ such that

$$|{}^{n_{k,l}}X^{t,x}_{s-} - X^{t,x}_{s-}|^2 \leq \sup_{s\leq T}|{}^{n_{k,l}}X^{t,x}_s - X^{t,x}_s|^2 \longrightarrow_{n_{k,l}} 0,\ \mathbb{P}-a.s.$$

As the mapping $x \mapsto \beta(x,e)$ is Lipschitz then the sequence

$$\begin{aligned}&({}^{n_{k,l}}V^{i,t,x}_s(e)\mathbf{1}_{\{|e|\geq\frac{1}{n_k}\}})_{n_k\geq 1}\\ &= (({}^{n_{k,l}}u^i(s,{}^{n_{k,l}}X^{t,x}_{s^-}+\beta({}^{n_{k,l}}X^{t,x}_{s^-},e)) - {}^{n_{k,l}}u^i(s,{}^{n_{k,l}}X^{t,x}_{s^-}))\mathbf{1}_{\{|e|\geq\frac{1}{n_{k,l}}\}})_{n_{k,l}\geq 1} \longrightarrow_{n_k}\\ &\quad u^i(s,X^{t,x}_{s^-}+\beta(X^{t,x}_{s^-},e)) - u^i(s,X^{t,x}_{s^-}),\ ds\otimes d\mathbb{P}\otimes d\lambda - a.e.\text{ on } [t,T]\times\Omega\times E.\end{aligned}$$

Therefore, we deduce that:

$$V^{i,t,x}_s(e) = u^i(s,X^{t,x}_{s^-}+\beta(X^{t,x}_{s^-},e)) - u^i(s,X^{t,x}_{s^-}),\ ds\otimes d\mathbb{P}\otimes d\lambda - a.e.\text{ on } [t,T]\times\Omega\times E, \tag{4.3.39}$$

which is the desired result. □

We are now in position to show existence of a solution for system (4.2.12) in the case when $\lambda(.)$ is not finite and integrates $(1\wedge|e|)_{e\in E}$. For that, we need to assume additionally the following hypotheses on the functions $(\bar{f}_i)_{i\in\mathcal{I}}$ and $(h_i)_{i\in\mathcal{I}}$.

(H8) For any $i\in\{1,...,m\}$,

(a) For any Φ, a bounded continuous function from $[0,T]\times\mathbb{R}^k$ to $\mathbb{R}$, the function $\bar{f}_i(t,x,\vec{y},z,\Phi(t,x))$ is continuous in $(t,x,\vec{y},z)$. Moreover there exists a continuous concave function Ψ_i, from $\mathbb{R}^k$ into $\mathbb{R}$, such that $\Psi_i(0)=0$ and

$$|\bar{f}_i(t,x,\vec{y},z,\Phi(t,x)) - \bar{f}_i(t,x',\vec{y},z,\Phi(t,x'))| \leq \Psi_i(|x-x'|). \tag{4.3.40}$$

(b) The function h_i is uniformly continuous.

Remark 4.3.4 *Note that in Assumption (H8)-(b), the function Ψ_i can depend on the function Φ. This assumption is satisfied if, e.g., for any $i\in\mathcal{I}$, $\bar{f}_i(t,x,\vec{y},z,q) = \bar{g}_i(t,x,\vec{y},z,q\varphi(x))$ where $\bar{g}_i(t,x,\vec{y},z,\zeta)$ is Lipschitz in $(x,\vec{y},z,\zeta)$ uniformly in t and $\varphi(x)$ is a continuous function such that $\lim_{|x|\to\infty}\varphi(x)=0$.*

Theorem 4.3.5 *Assume that the functions:*

(i) $(\bar{f}_i)_{i\in\mathcal{I}}$ *verify (H1)-iii),iv) and (H8)-(a).*

(ii) $(h_i)_{i\in\mathcal{I}}$ *verify (H2) and (H8)-(b).*

(iii) $(g_{ij})_{i,j\in\mathcal{I}}$ *verify (H3).*

(iv) $(\bar{f}_i(t,x,\vec{0},0,0))_{i\in\mathcal{I}}$ *and* $(h_i)_{i\in\mathcal{I}}$ *are bounded.*

(v) $(\gamma_i)_{i\in\mathcal{I}}$ *verify (H1)-(v).*

Then the system (4.2.12) has a solution $(Y^{i,t,x},Z^{i,t,x},V^{i,t,x},K^{i,t,x})_{i\in\mathcal{I}}$. *Moreover there exist bounded continuous functions* $(u^i)_{i\in\mathcal{I}}$ *such that for any* $i\in\mathcal{I}$, $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$Y_s^{i,t,x}=u^i(s,X_s^{t,x}),\ \forall s\in[t,T].$$

Proof: The proof follows the same steps as in the proof of Proposition 3.2 in [37], except that in our framework we should take into account of the non-boundedness of $\lambda(.)$. This difficulty is tackled in using the fact that for any $i\in\mathcal{I}$, $|\gamma_i(x,e)|\leq c(1\wedge|e|)$ and $\int_E(1\wedge|e|)\lambda(de)<\infty$ as well.

Step 1: The iterative construction

Let us consider $((Y^{i,n;t,x},Z^{i,n;t,x},V^{i,n;t,x},K^{i,n;t,x})_{i\in\mathcal{I}})_{n\geq0}$ the sequence of processes defined recursively as follows:

$(Y^{i,0;t,x},Z^{i,0;t,x},V^{i,0;t,x},K^{i,0;t,x})=(0,0,0,0)$ for all $i\in\mathcal{I}$ and for $n\geq1$ and $s\leq T$,

$$\begin{cases}
Y^{i,n;t,x}\in\mathcal{S}^2,Z^{i,n;t,x}\in\mathcal{H}^{2,d},V^{i,n;t,x}\in\mathcal{H}^2(\mathcal{L}^2(\lambda)),\text{ and }K^{i,n;t,x}\in\mathcal{A}^2;\\
Y_s^{i,n;t,x}=h_i(X_T^{t,x})+\int_s^T\bar{f}_i(r,X_r^{t,x},(Y_r^{k,n;t,x})_{k\in\mathcal{I}},Z_r^{i,n;t,x},\int_E V_r^{i,n-1;t,x}(e)\times\\
\qquad\gamma_i(X_r^{t,x},e)\lambda(de))dr+K_T^{i,n;t,x}-K_s^{i,n;t,x}-\int_s^T Z_r^{i,n;t,x}dB_r\\
\qquad-\int_s^T\int_E V_r^{i,n;t,x}(e)\tilde{\mu}(dr,de);\\
Y_s^{i,n;t,x}\geqslant\max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n;t,x}-g_{ij}(s));\\
\int_0^T(Y_s^{i,n;t,x}-\max_{j\in\mathcal{I}^{-i}}(Y_s^{j,n;t,x}-g_{ij}(s)))dK_s^{i,n;t,x}=0.
\end{cases}\tag{4.3.41}$$

Thanks to the result in [45], the system admits a unique solution. Indeed, the generators $\check{F}_i(\dots)$, $i \in \mathcal{I}$, of the system do not depend on $V^{i,n;t,x}$, noting that $V^{i,n-1;t,x}$ is already determined. Next, by an induction argument on n, there exist a deterministic continuous bounded functions $(u^{i,n})_{i\in\mathcal{I}}$ such that for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and any $s \in [t,T]$:

(a) $Y_s^{i,n;t,x} := u^{i,n}(s, X_s^{t,x})$ and

(b) $V_s^{i,n;t,x}(e) = u^{i,n}(s, X_{s^-}^{t,x} + \beta(X_{s^-}^{t,x}, e)) - u^{i,n}(s, X_{s^-}^{t,x})$, $ds \otimes d\mathbb{P} \otimes d\lambda$ on $[t,T] \times \Omega \times E$.

(c) There exists a constant C such that for any $n \geq 0$ and $i \in \mathcal{I}$,

$$|u^{i,n}(t,x)| \leq C, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

Indeed, for $n = 0$, the property holds true with $u^{i,0} = 0$, $i \in \mathcal{I}$. Suppose that it is satisfied for some n. Then $(Y^{i,n+1;t,x}, Z^{i,n+1;t,x}, V^{i,n+1;t,x}, K^{i,n+1;t,x})$ verifies (we omit the dependence on t, x as there is no confusion): $\forall s \in [t,T]$ and $i \in \mathcal{I}$,

$$\begin{cases} Y_s^{i,n+1} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,n+1})_{k\in\mathcal{I}}, Z_r^{i,n+1}, \int_E \{u^{i,n}(r, X_{r^-}^{t,x} + \\ \qquad \beta(X_{r^-}^{t,x}, e)) - u^{i,n}(r, X_{r^-}^{t,x})\}\gamma_i(X_r^{t,x}, e)\lambda(de))dr + K_T^{i,n+1} - K_s^{i,n+1} \\ \qquad - \int_s^T Z_r^{i,n+1} dB_r - \int_s^T \int_E V_r^{i,n+1}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,n+1} \geqslant \max_{j\in\mathcal{I}^{-i}} (Y_s^{j,n+1} - g_{ij}(s)); \\ \int_t^T (Y_s^{i,n+1} - \max_{j\in\mathcal{I}^{-i}} (Y_s^{j,n+1} - g_{ij}(s)))dK_s^{i,n+1} = 0. \end{cases} \tag{4.3.42}$$

The generators of the system of reflected BSDEs (4.3.42) are given by: For any $i \in \mathcal{I}$,

$$\check{F}_i(t,x,\vec{y},z) = \bar{f}_i(t,x,\vec{y},z, \int_E \{u^{i,n}(t, x+\beta(x,e)) - u^{i,n}(t,x)\}\gamma_i(x,e)\lambda(de))$$

since the process $X^{t,x}$ is RCLL and the functions $u^{i,n}, \beta^i$ and γ_i are continuous w.r.t. x. On the other hand, note that the function $(t,x) \mapsto \int_E \{u^{i,n}(t, x + \beta(x,e)) - u^{i,n}(t,x)\}\gamma_i(x,e)\lambda(de))$ is continuous and bounded by the induction

hypothesis.

Therefore:

i) As for any $i \in \mathcal{I}$, $\check{F}_i(t,x,\vec{y},z)$ is continuous in $(t,x,\vec{y},z)$ and by Assumption (H8)-(a), it satisfies the condition (4.2.11) and assumptions (H1),(ii)-(iv), then by Proposition 4.2 in [45], there exist deterministic continuous functions of polynomial growth $(u^{i,n+1})_{i\in\mathcal{I}}$ such that for any $i \in \mathcal{I}$ and $s \in [t,T]$

$$Y_s^{i,n+1;t,x} = u^{i,n+1}(s, X_s^{t,x}).$$

ii) Let $(\bar{Y},\bar{Z})$ be the solution of the following standard BSDE: for any $s \leq T$,

$$\begin{cases} \bar{Y} \in \mathcal{S}^2,\ \bar{Z} \in \mathcal{H}^{2,d}; \\ \bar{Y}_s = \bar{C} + \int_s^T \left\{\bar{C} + m\, C_f^y\, \bar{Y}_r + C_f^z\, |\bar{Z}_r| + 2\theta \bar{Y}_r\right\} dr - \int_s^T \bar{Z}_r dB_r; \end{cases}$$

where: (a) C_f^y , C_f^z and C_f^q are the maximum of the Lipschitz constants of the functions $\bar{f}_i(t,x,\vec{y},z,q)$ w.r.t. $\vec{y}$, z and q respectively; (b) $\bar{C}$ is the constant of boundedness which appears in (H6); (c)

$$\theta = C_f^q\, C_\gamma \int_E (1 \wedge |e|)\lambda(de).$$

Therefore if for any $i \in \mathcal{I}$ and $s \in [t,T]$, $|u^{i,n}(s, X_s^{t,x})| \leq \bar{Y}_s$, then for any $i \in \mathcal{I}$ and $s \in [t,T]$, $|u^{i,n+1}(s, X_s^{t,x})| \leq \bar{Y}_s$ (one can see [36], pp.12 for more details). As the process $\bar{Y}$ is deterministic continuous, then it is bounded. Finally the proof of the claim of point c) is proved by induction since it is satisfied for $n = 0$.

iii) The representation of the processes $V^{i,n+1;t,x}$ stems from Proposition 4.3.2 since in using the assumption (H8) and the induction hypothesis for n, the functions $(\check{F}_i)_{i\in\mathcal{I}}$, $(h_i)_{i\in\mathcal{I}}$ and $(g_{ij})_{i,j\in\mathcal{I}}$ fulfill the requirements of Theorem (4.3.2).

It follows that points (a), (b) and (c) above are valid for $n+1$ and then they hold true for any $n \geq 0$.

Remark 4.3.6 *For $s \in [0,t]$, $X_s^{t,x} = x$ and $Y_t^{i,n;t,x} = u^{i,n}(t,x)$, therefore in considering the declination of system (4.3.41) on the time interval $[0,t]$, we can*

easily show by induction that $Z_s^{i,n}1_{[s\leq t]} = 0,\ ds\otimes d\mathbb{P}-a.e$ *and* $V_s^{i,n}(e)1_{[s\leq t]} = 0,\ ds\otimes d\mathbb{P}\otimes d\lambda - a.e.$ *since the data are continuous and deterministic.*

Step 2: Convergence of $(\mathbf{u^{i,n}})_\mathbf{n}$

Following similar steps as in the proof of Proposition 3.2 in [37] (pp. 10), we obtain for some $\alpha_0 > 0$: $\forall n, q \geq 1, s \in [t, T]$,

$$\mathbb{E}\Big[e^{\alpha_0 s}|(Y_s^{k,n;t,x})_{k\in\mathcal{I}} - (Y_s^{i,q;t,x})_{k\in\mathcal{I}}|^2\Big]$$
$$\leq C\mathbb{E}\Big[\int_s^T e^{\alpha_0 r}\big(\int_E \sum_{k=1,m}|\{V_r^{k,n-1;t,x}(e) - V_r^{k,q-1;t,x}(e)\}\gamma^k(X_r^{t,x},e)|\lambda(de)\big)^2 dr\Big].$$

Taking $s = t$ and considering (4.3)-(a)-(b), we obtain : $\forall i \in \mathcal{I}$,

$$|u^{i,n}(t,x) - u^{i,q}(t,x)|^2$$
$$\leq \kappa_1\mathbb{E}\Big[\int_t^T e^{\alpha_0(r-t)}\big(\int_E \sum_{k=1,m}|\{u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,n-1}(r, X_{r^-}^{t,x})$$
$$- (u^{k,q-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,q-1}(r, X_{r^-}^{t,x}))\}\gamma^k(X_r^{t,x},e)|\lambda(de)\big)^2 dr\Big]. \tag{4.3.43}$$

Now, let $\eta > 0$ and $t \in [T-\eta, T]$. Then by (4.3.43), we get:

$$|u^{i,n}(t,x) - u^{i,q}(t,x)|^2$$
$$\leq \kappa_1\mathbb{E}\Big[\int_{T-\eta}^T e^{\alpha_0(r-t)}\big(\int_E \sum_{k=1,m}|\{u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,n-1}(r, X_{r^-}^{t,x})$$
$$- (u^{k,q-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x},e)) - u^{k,q-1}(r, X_{r^-}^{t,x}))\}\gamma^k(X_r^{t,x},e)|\lambda(de)\big)^2 dr\Big].$$

Then if we set

$$\|u^{i,n} - u^{i,q}\|_{\infty,\eta} := \sup_{(t,x)\in[T-\eta,T]\times\mathbb{R}^k}|u^{i,n}(t,x) - u^{i,q}(t,x)|,$$

and since $u^{i,n}$ is uniformly bounded, we have: For any $t \in [T-\eta, T]$ and $i \in \mathcal{I}$,

$$
\begin{aligned}
&|u^{i,n}(t,x) - u^{i,q}(t,x)|^2 \\
&\quad \leq \kappa_2 \mathbb{E}\Big[\int_{T-\eta}^{T} e^{\alpha_0(r-t)} \big(\int_E \sum_{k=1,m} |\{u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^{k,n-1}(r, X_{r^-}^{t,x}) \\
&\qquad - (u^{k,q-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^{k,q-1}(r, X_{r^-}^{t,x}))\}(1 \wedge |e|)\lambda(de)\big)^2 dr \Big] \\
&\quad \leq \kappa_2 \int_{T-\eta}^{T} dr e^{\alpha_0(r-t)} \{2 \sum_{k=1,m} \|u^{k,n} - u^{,q}\|_{\infty,\eta}\}^2 \big(\int_E (1 \wedge |e|)\lambda(de)\big)^2 \\
&\quad \leq \kappa_2 \sum_{k=1,m} \|u^{k,n} - u^{,q}\|_{\infty,\eta}^2 \int_{T-\eta}^{T} dr e^{\alpha_0(r-t)}
\end{aligned}
$$

where κ_2 is a constant which may change from line to line and since $\lambda(.)$ integrates $(1 \wedge |e|)_{e \in E}$. Now, let η be a constant such that $\frac{\kappa_2}{\alpha_0} m(e^{\alpha_0 \eta} - 1) = \frac{3}{4}$. Note that η does not depend on the terminal conditions $(h_i)_{i \in \mathcal{I}}$. Therefore we deduce from the last inequality, in taking the supremum over (t,x), that for any $n, q \geq 1$,

$$
\sum_{i=1,m} \|u^{i,n} - u^{i,q}\|_{\infty,\eta}^2 \leq \frac{3}{4} \sum_{i=1,m} \|u^{i,n-1} - u^{i,q-1}\|_{\infty,\eta}^2.
$$

It means that the sequence $((u^{i,n})_{i \in \mathcal{I}})_{n \geq 0}$ is uniformly convergent in $[T-\eta, T] \times \mathbb{R}^k$. So for $(t,x) \in [T-\eta, T] \times \mathbb{R}^k$, let us set $u^i(t,x) = \lim_{n \to \infty} u^{i,n}(t,x)$, $i \in \mathcal{I}$. Note that $(u^i)_{i \in \mathcal{I}}$ are continuous bounded functions on $[T-\eta, T] \times \mathbb{R}^k$.

Next let $t \in [T-2\eta, T-\eta]$. From (4.3.43) we have:

$$
\begin{aligned}
&|u^{i,n}(t,x) - u^{i,q}(t,x)|^2 \\
&\quad \leq \kappa_1 \mathbb{E}\Big[\int_t^{T-\eta} e^{\alpha_0(r-t)} \big(\int_E \textstyle\sum_{k=1,m} |\{u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) \\
&\qquad - u^{k,n-1}(r, X_{r^-}^{t,x}) - (u^{k,q-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) \\
&\qquad - u^{k,q-1}(r, X_{r^-}^{t,x}))\}\gamma^k(X_r^{t,x}, e)|\lambda(de)\big)^2 dr \Big] \\
&\qquad + \kappa_1 \mathbb{E}\Big[\int_{T-\eta}^{T} e^{\alpha_0(r-t)} \big(\int_E \textstyle\sum_{k=1,m} |\{u^{k,n-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) \\
&\qquad - u^{k,n-1}(r, X_{r^-}^{t,x}) - (u^{k,q-1}(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) \\
&\qquad - u^{k,q-1}(r, X_{r^-}^{t,x}))\}\gamma^k(X_r^{t,x}, e)|\lambda(de)\big)^2 dr \Big].
\end{aligned}
$$

But the second term converges to 0 uniformly w.r.t $(t,x) \in [0,T] \times \mathbb{R}^k$ (by the result of Step 1). Now in arguing as previously and since η verifies $\frac{\kappa_2}{\alpha_0} m(e^{\alpha_0 \eta} - 1) = \frac{3}{4}$ we deduce that for any $i \in \mathcal{I}$,

$$\limsup_{n,q\to\infty} \|u^{i,n} - u^{i,q}\|_{\infty,[T-2\eta,T-\eta]} := \limsup_{n,q\to\infty} \sup_{(t,x)\in[T-\eta,T]\times\mathbb{R}^k} |u^{i,n}(t,x) - u^{i,q}(t,x)| = 0.$$

Then once more the sequence $((u^{i,n})_{i\in\mathcal{I}})_{n\geq 0}$ is uniformly convergent in $[T-2\eta, T-\eta] \times \mathbb{R}^k$. So for $(t,x) \in [T-2\eta, T-\eta] \times \mathbb{R}^k$, let us set $u^i(t,x) = \lim_{n\to\infty} u^{i,n}(t,x)$, $i \in \mathcal{I}$. Note that $(u^i)_{i\in\mathcal{I}}$ are continuous bounded functions on $[T-2\eta, T] \times \mathbb{R}^k$ by concatenation. Now by applying repeatedly the same reasoning on each time interval $[T-(j+1)\eta, T-j\eta]$ of fixed length η and pasting the solutions, we obtain the uniform convergence of $((u^{i,n})_{i\in\mathcal{I}})_n$ in $[0,T] \times \mathbb{R}^k$. So for $i \in \mathcal{I}$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, let us set $u^i(t,x) = \lim_{n\to\infty} u^{i,n}(t,x)$, $i \in \mathcal{I}$. Note that $(u^i)_{i\in\mathcal{I}}$ are continuous bounded functions on $[0,T] \times \mathbb{R}^k$.

Step 3: Convergence of $(\mathbf{Y^{i,n;t,x}}, \mathbf{Z^{i,n;t,x}}, \mathbf{V^{i,n;t,x}}, \mathbf{K^{i,n;t,x}})_\mathbf{n}$

In this step, we use the same techniques as in [37] (pp.17) to deduce the existence of a processes
$(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})$ which belong to $\mathcal{S}^2 \times \mathcal{H}^{2,d} \times \mathcal{H}^2(\mathcal{L}^2(\lambda)) \times \mathcal{A}^2$ such that

$$\mathbb{E}\Big[\sup_{s\leq T} |Y_s^{i,n:t,x} - Y_s^{i,t,x}|^2 + \int_0^T |Z_s^{i,n:t,x} - Z_s^{i,t,x}|^2 ds$$
$$+ \int_0^T \int_E |V_s^{i,n:t,x}(e) - V_s^{i,t,x}(e)|^2 \lambda(ds)de + \sup_{s\leq T} |K_s^{i,n:t,x} - K_s^{i,t,x}|^2\Big] \to 0 \ \text{ as } n \to \infty.$$

Moreover,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E V_r^{i,t,x}(e)\gamma_i(X_r^{t,x}, e)\lambda(de))dr \\ \qquad\qquad + K_T^{i,t,x} - K_s^{i,t,x} - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de); \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}} (Y_s^{j,t,x} - g_{ij}(s)). \end{cases} \tag{4.3.44}$$

Similarly, as in [37] (pp. 20) we also have

$$\int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}} (Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0.$$

Therefore $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ fulfils the third condition in (4.2.12), which means that its is a solution of (4.2.12). The proof is completed. □

As a by-product of Theorem 4.3.5, we obtain the following result:

Corollary 4.3.7 *For any $i \in \mathcal{I}$ and $(t,x) \in [0,T]\times\mathbb{R}^k$,*

$$V_s^{i,t,x}(e) = 1_{\{s\geq t\}}u^i(s, X_{s^-}^{t,x}+\beta(X_{s^-}^{t,x}, e)) - u^i(s, X_{s^-}^{t,x}),$$
$$ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [0,T]\times\Omega\times E. \quad (4.3.45)$$

4.4 The second main result: existence of the solution for system of IPDEs

In this section, we study the existence of the viscosity solution of the IPDEs system (4.1.3). The candidate to be the solution are the functions $(u^i)_{i\in\mathcal{I}}$ defined in Theorem 4.3.5 by which we represent $(Y^{i,t,x})_{i\in\mathcal{I}}$. So, firstly we recall the notion of viscosity solution we deal with. This definition has been already introduced in [37].

Definition 4.4.1 *A family of deterministic continuous functions $\vec{u} := (u^i)_{i\in\mathcal{I}}$ is a viscosity supersolution (resp. subsolution) of (4.1.3) if: $\forall i \in \mathcal{I}$,*

a) $u^i(T,x) \geq$ (resp. $\leq$) $h_i(x)$, $\forall x \in \mathbb{R}^k$;

b) if $\phi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ is such that $(t,x) \in [0,T)\times\mathbb{R}^k$ a global minimum (resp. maximum) point of $u^i - \phi$

then

$$\min\Big\{u^i(t,x) - \max_{j\in\mathcal{I}^{-i}}(u^j(t,x) - g_{ij}(t,x)); -\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{K}\phi(t,x)$$
$$- \bar{f}_i(t,x,(u^k(t,x))_{k=1,\dots,m}, (\sigma^\top D_x\phi)(t,x), \mathcal{B}_i u^i(t,x))\Big\} \geq \text{ (resp. } \leq) \ 0.$$

We say that $\vec{u} := (u^i)_{i\in\mathcal{I}}$ is a viscosity solution of (4.1.3) if it is both a supersolution and subsolution of (4.1.3).

Remark 4.4.2 *In our definition, we have $\mathcal{B}_i u^i(t,x)$ instead of $\mathcal{B}_i\phi(t,x)$ in the argument of $\bar{f}_i$, where ϕ is the test function. Indeed, $\mathcal{B}_i u^i(t,x)$ is well defined since u^i and β are bounded.*

We are now ready to state the second main result of this paper.
Let $(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ be the solution of (4.2.12) and let $(u^i)_{i\in\mathcal{I}}$ be the bounded continuous functions such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $i\in\mathcal{I}$ and $s\in[t,T]$,

$$Y_s^{i,t,x} = u^i(s, X_s^{t,x}).$$

We then have:

Theorem 4.4.3 *Assume that the functions:*

(i) $(\bar{f}_i)_{i\in\mathcal{I}}$ *verify (H1)-iii),iv) and (H8)-(a).*

(ii) $(h_i)_{i\in\mathcal{I}}$ *verify (H2) and (H8)-(c).*

(iii) $(g_{ij})_{i,j\in\mathcal{I}}$ *verify (H3).*

(iv) $(\bar{f}_i(t,x,\vec{0},0,0))_{i\in\mathcal{I}}$ *and* $(h_i)_{i\in\mathcal{I}}$ *are bounded.*

(v) $(\gamma_i)_{i\in\mathcal{I}}$ *verify (H1)-(v).*

Then the functions $(u^i)_{i\in\mathcal{I}}$ *is a viscosity solution of the system (4.1.3), according to Definition 4.4.1.*

Proof:
Let us consider the following system of reflected BSDEs with jumps and interconnected obstacles: $\forall s\in[t,T]$ and $i\in\mathcal{I}$,

$$\begin{cases} \underline{\mathrm{Y}}^{i,t,x}\in\mathcal{S}^2, \underline{\mathrm{Z}}^{i,t,x}\in\mathcal{H}^{2,d}, \underline{\mathrm{V}}^{i,t,x}\in\mathcal{H}^2(\mathcal{L}^2(\lambda)), \text{ and } \underline{\mathrm{K}}^{i,t,x}\in\mathcal{A}^2; \\ \underline{\mathrm{Y}}_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (\underline{\mathrm{Y}}_r^{k,t,x})_{k\in\mathcal{I}}, \underline{\mathrm{Z}}_r^{i,t,x}, \int_E \gamma_i(X_r^{t,x}, e)\times \\ \qquad \{u^i(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^i(r, X_{r^-}^{t,x})\}\lambda(de))dr + \underline{\mathrm{K}}_T^{i,t,x} - \underline{\mathrm{K}}_s^{i,t,x} \\ \quad - \int_s^T \underline{\mathrm{Z}}_r^{i,t,x} dB_r - \int_s^T\int_E \underline{\mathrm{V}}_r^{i,t,x}(e)\tilde{\mu}(dr,de), \\ \underline{\mathrm{Y}}_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(\underline{\mathrm{Y}}_s^{j,t,x} - g_{ij}(s)) \text{ and } \int_t^T(\underline{\mathrm{Y}}_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(\underline{\mathrm{Y}}_s^{j,t,x} - g_{ij}(s)))d\underline{\mathrm{K}}_s^{i,t,x} = 0. \end{cases} \tag{4.4.1}$$

Thanks to the result in [45], the system admits a unique solution $(\underline{\mathrm{Y}}^{i,t,x}, \underline{\mathrm{Z}}^{i,t,x}, \underline{\mathrm{V}}^{i,t,x}, \underline{\mathrm{K}}^{i,t,x})_{i\in\mathcal{I}}$. In fact, the generators of the system (4.4.1) do not depend on $\underline{\mathrm{V}}^{i,t,x}$.

Next by Theorem 4.3.2 there exist deterministic bounded continuous functions $(\underline{\mathrm{u}}^i)_{i\in\mathcal{I}}$, such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\forall s\in[t,T],\quad \underline{\mathrm{Y}}_s^{i,t,x}=\underline{\mathrm{u}}^i(s,X_s^{t,x}).$$

In fact, for any $i\in\mathcal{I}$ the function

$$(t,x,\vec{y},z)\longmapsto \bar{f}_i(t,x,\vec{y},z,\textstyle\int_E\gamma(x,e)\{u^i(t,x+\beta(x,e))-u^i(t,x)\}\lambda(de))$$

is continuous and by Assumption (H8)-(a), it satisfies the assumptions (H7)-(a) and (H1),iii)-iv).
Finally, using a result by Hamadène-Zhao [45], we deduce that $(\underline{\mathrm{u}}^i)_{i\in\mathcal{I}}$ is a viscosity solution of the following system: For any $(t,x)\in[0,T]\times\mathbb{R}^k$,

$$\begin{cases}\min\{\underline{\mathrm{u}}^i(t,x)-\max\limits_{j\in\mathcal{I}^{-i}}(\underline{\mathrm{u}}^j(t,x)-g_{ij}(t));-\partial_t\underline{\mathrm{u}}^i(t,x)-\mathcal{L}\underline{\mathrm{u}}^i(t,x)-\mathcal{K}\underline{\mathrm{u}}^i(t,x)\\ \qquad -\bar{f}_i(t,x,(\underline{\mathrm{u}}^k(t,x))_{k=1,,\dots,m},(\sigma^\top D_x\underline{\mathrm{u}}^i)(t,x),\mathcal{B}_iu^i(t,x))\}=0;\\ \underline{\mathrm{u}}^i(T,x)=h_i(x),\end{cases}\tag{4.4.2}$$

Let us notice that, in this system (4.4.2), the last component of $\bar{f}_i$ is $\mathcal{B}_iu^i(t,x)$ and not $\mathcal{B}_i\underline{\mathrm{u}}^i(t,x)$. Now, recall that $(Y^{i,t,x},Z^{i,t,x},V^{i,t,x},K^{i,t,x})_{i\in\mathcal{I}}$ solves the system (4.2.12) and by Corollary 4.3.7, we know that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $i\in\mathcal{I}$ and $s\in[t,T]$,

$$V_s^{i,t,x}(e)=u^i(s,X_{s^-}^{t,x}+\beta(X_{s^-}^{t,x},e))-u^i(s,X_{s^-}^{t,x}),\ ds\otimes d\mathbb{P}\otimes d\lambda \text{ on } [t,T]\times\Omega\times E.$$

Plug this relation in the second term of the right-hand side of the second equality of (4.2.12), we obtain that

$(Y^{i,t,x}, Z^{i,t,x}, V^{i,t,x}, K^{i,t,x})_{i\in\mathcal{I}}$ verify: for any $s \in [t,T]$ and $i \in \mathcal{I}$,

$$\begin{cases} Y_s^{i,t,x} = h_i(X_T^{t,x}) + \int_s^T \bar{f}_i(r, X_r^{t,x}, (Y_r^{k,t,x})_{k\in\mathcal{I}}, Z_r^{i,t,x}, \int_E \gamma(X_r^{t,x}, e)\times \\ \qquad \{u^i(r, X_{r^-}^{t,x} + \beta(X_{r^-}^{t,x}, e)) - u^i(r, X_{r^-}^{t,x})\}\lambda(de))dr + K_T^{i,t,x} - K_s^{i,t,x} \\ \qquad - \int_s^T Z_r^{i,t,x} dB_r - \int_s^T \int_E V_r^{i,t,x}(e)\tilde{\mu}(dr, de), \\ Y_s^{i,t,x} \geqslant \max_{j\in\mathcal{I}^{-i}}(Y_s^{t,xj} - g_{ij}(s)), \\ \int_0^T (Y_s^{i,t,x} - \max_{j\in\mathcal{I}^{-i}}(Y_s^{j,t,x} - g_{ij}(s)))dK_s^{i,t,x} = 0. \end{cases} \tag{4.4.3}$$

Then, by uniqueness of the solution of the system of reflected BSDEs (4.4.1), we deduce that any $s \in [t,T]$ and $i \in \mathcal{I}$, $\underline{Y}_s^{i,t,x} = Y_s^{i,t,x}$. Then, for any $i \in \mathcal{I}$, $\underline{u}^i = u^i$. Consequently, $(u^i)_{i\in\mathcal{I}}$ is a viscosity solution of (4.1.3) in the sense of Definition 4.4.1. □

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Titre : Systèmes d'EDP et d'EDSR réfléchies à obstacles interconnectés et leurs liens avec les problèmes de switching optimal

Mots clés : Equations aux dérivées partielles, Equations différentielles stochastique rétrogrades, Obstacles interconnectés, Switching optimal, Solution de viscosité, Processus de Lèvy.

Résumé : La thèse est composée de trois parties. dans la première partie, nous nous intéressons à l'étude du système d'équations aux dérivées partielles (EDP) avec obstacles interconnectés. Nous montrons l'existence d'une unique solution de viscosité qui est continue et à croissance polynomiale, sans la condition de monotonie sur le générateur. Des cas particuliers de ce système d'EDP ne sont rien d'autres que les systèmes d'Hamilton Jacobi Bellman associés à un problème de switching optimal. Dans la seconde partie, nous considérons aussi un système d'EDP avec obstacles interconnectés mais associé à un processus de diffusion avec sauts. En retirant la condition de monotonie sur le générateur par rapport à la composante de saut, nous montrons que si la mesure de Lèvy associée au processus de poisson est finie, le système d'EDP admet une unique solution de viscosité. L'outil principal pour étudier le système en question est de considérer le système d'EDSRs réfléchies avec sauts et à obstacles interconnectés pour lequel on montre l'existence d'une solution markovienne unique. Dans la dernière partie, nous traitons le cas où la mesure n'est pas finie. Nous donnons des conditions nécessaires et suffisantes assurant l'existence d'une solution du système d'EDSRs réfléchies avec sauts et à obstacles interconnectés. Ainsi nous en déduisons une représentation de Feynman-Kac ce qui donne une solution du système d'EDP associé.

Title : Systems of PDEs and reflected BSDEs with interconnected obstacles and related optimal switching problems

Keywords: Partial differential equations, Backward stochastic differential equations, Interconnected obstacles, Optimal switching, Viscosity solution, Lèvy process.

Abstract: This thesis is divided into three parts. In the first part, we study a system of partial differential equations (PDEs) with interconnected obstacles for which we establish a new existence and uniqueness result of continuous viscosity solution. The novelty is that we relax the so-called monotonicity condition on the generator. In particular cases, this system of PDEs is nothing else but the Hamilton Jacobi Bellman one associated with the stochastic optimal switching problem. In the second part, we study a system of second order integral-partial differential equations (IPDEs) with interconnected obstacles with non-local terms, related to an optimal switching problem with jump-diffusion model. By getting rid of the monotonicity condition on the generators with respect to the jump component, we construct a continuous viscosity solution which is unique in the class of functions with polynomial growth. In our study, the main tool is the associated system of reflected backward stochastic differential equations (RBSDEs) with jumps and interconnected obstacles for which we show the existence of a unique Markovian solution. At last, we prove existence of a solution to a system of Markovian RBSDEs with jumps and interconnected obstacles. Our motivation is that we deal with the case when the Lévy measure is not finite. In fact, under appropriate assumptions and using a truncation of the Lévy measure, we show that the system of RBSDEs has a Markovian solution. Hence the Feynman-Kac representation holds true and provides a solution for the associated system of IPDEs with interconnected obstacles.