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# Gamma-bounded $C_0$ -semigroups and power gamma-bounded operators: characterizations and functional calculi

Loris Arnold

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# **$C_0$ -semigroupes gamma-bornés et opérateurs à puissances gamma-bornées : caractérisations et calculs fonctionnels**

Thèse de doctorat de  
l'Université Bourgogne Franche-Comté

École doctorale CARNOT-PASTEUR

Soutenue à Besançon le 4 mars 2021  
en vue de l'obtention du grade de

Docteur de Mathématiques de l'Université Bourgogne Franche-Comté

par

**Loris ARNOLD**

*Composition du jury :*

<i>Président du jury :</i>	M. Gilles LANCIEN	Université de Franche-Comté
<i>Rapporteurs :</i>	M. Markus HAASE	Université de Kiel
	M. Catalin BADEA	Université des Sciences et Technologies de Lille I
<i>Examineurs :</i>	Mme. Sylvie MONNIAUX	Université Aix-Marseille
	M. El Maati OUHABAZ	Université de Bordeaux
<i>Directeur de thèse :</i>	M. Christian LE MERDY	Université de Franche-Comté

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# $\gamma$ -bounded $C_0$ -semigroups and power $\gamma$ -bounded operators: characterizations and functional calculi

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# Introduction générale

Le principal objectif de cette thèse est d'étudier les  $C_0$ -semigroupes bornés et les opérateurs à puissances bornées sur un espace de Banach (parfois spécifiquement un espace de Hilbert) et les différents calculs fonctionnels que l'on peut construire à partir de ces objets. À la fin des années 1990, Gomilko ([16]) ainsi que Shi et Feng ([53]), démontrent indépendamment qu'un  $C_0$ -semigroupe borné  $(T_t)_{t \geq 0}$  sur un espace de Hilbert est caractérisé par la condition suivante : il existe  $C > 0$  tel que pour tout  $\alpha < 0$  on a

$$-\alpha \int_{\mathbb{R}} \|R(\alpha + it, A)x\|^2 \leq C \|x\|^2, \quad x \in H \quad (0.1)$$

et

$$-\alpha \int_{\mathbb{R}} \|R(\alpha + it, A)^*x\|^2 \leq C \|x\|^2, \quad x \in H, \quad (0.2)$$

où  $-A$  est le générateur de  $(T_t)_{t \geq 0}$ . Ce résultat est maintenant connu sous le nom de Théorème de Gomilko Shi-Feng. Ce dernier est un outil très puissant qui est utilisé pour démontrer beaucoup de résultats liés aux  $C_0$ -semigroupes bornés sur un espace de Hilbert. On peut par exemple voir le fameux Théorème de Gerhart-Prüss ([1, Théorème 5.2.1.]) comme un simple corollaire du Théorème de Gomilko Shi-Feng. La question qu'on peut se poser est alors "qu'advient-il dans les espaces de Banach généraux ?". Les inégalités (0.1) et (0.2) sont peu appropriées aux espace de Banach. On introduit la condition suivante, adaptée aux espaces de Banach généraux, que j'appelle condition de Gomilko Shi-Feng (GFS) : il existe  $C > 0$  tel que

$$\int_{\mathbb{R}} |\langle R(\alpha + it, A)^2x, y \rangle| dt \leq \frac{C}{-\alpha} \|x\| \|y\|, \quad x \in X, y \in X^* \text{ et } \alpha < 0. \quad (0.3)$$

Dans les espaces de Hilbert la condition de Gomilko Shi-Feng est aussi équivalente à la génération d'un  $C_0$ -semigroupe borné. En revanche on peut trouver des contres-exemples dans des espaces non Hilbertiens, par exemple il est connu (voir [16]) que les semigroupes de shift dans  $L^p(\mathbb{R})$  pour  $1 < p \neq 2 < \infty$  ne vérifient pas la condition (0.3) alors que ces semigroupes sont bornés.

La condition (0.3) est largement étudiée dans l'article [9] de Batty, Haase et Mubeen. Ils y introduisent un nouveau type de calcul fonctionnel borné, le calcul fonctionnel dérivé borné. Prenons  $A$  un opérateur sur un espace de Banach  $X$  qui est densément défini, dont le spectre est inclus dans le demi-plan  $\overline{\mathbb{C}}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$  et qui vérifie que sa resolvante  $R(\lambda, A)$  est bornée sur tout demi-plan de la forme  $\{z \in \mathbb{C} : \operatorname{Re}(z) \leq \alpha\}$  avec  $\alpha < 0$ . On appellera un tel  $A$  un opérateur de type demi-plan. À partir d'un tel opérateur on peut construire un calcul fonctionnel sur un sous-espace particulier de  $H^\infty(\overline{\mathbb{C}}_+)$ , l'ensemble des fonctions holomorphes et bornées sur  $\overline{\mathbb{C}}_+$ , qui est noté  $\mathcal{E}$  dans [9]. C'est-à-dire que l'on peut construire, en partie grâce au Théorème de Cauchy, un morphisme d'algèbre  $\Phi : \mathcal{E} \mapsto B(X)$  à partir de  $A$ , qui vérifie notamment que si  $f \in \mathcal{E}$  et  $\lambda < 0$  alors  $\Phi(f(z)(\lambda - z)) = \Phi(f)R(A, \lambda)$ . On notera aussi  $f(A) := \Phi(f)$ . Grâce à



une technique que l'on appelle régularisation on peut réussir à donner une définition de  $\Phi(f)$  (qui n'est pas forcément un opérateur borné) pour  $f \in H^\infty(R_\alpha)$  et pour tout  $\alpha < 0$  où  $R_\alpha := \{z \in \mathbb{C} : \operatorname{Re}(z) > \alpha\}$ . Maintenant on dit qu'un opérateur  $A$  de type demi-plan a un calcul fonctionnel  $H^\infty(\mathbb{C}_+)$ -borné, lorsque qu'il existe une constante  $C > 0$  tel que pour tout  $\alpha < 0$  et  $f \in H^\infty(R_\alpha)$  on a

$$f(A) \in B(X) \quad \text{et} \quad \|f(A)\| \leq C \|f\|_{H^\infty(\mathbb{C}_+)}.$$

Cette condition, à l'instar de la notion d'opérateur polynomialement borné pour les opérateurs bornés, n'a pas de bonne caractérisation. On ne connaît actuellement que deux classes d'exemples d'opérateurs admettant un calcul fonctionnel  $H^\infty(\mathbb{C}_+)$ -borné, les opérateurs semblables à des générateurs de  $C_0$ -semigroupes de contractions sur les espaces de Hilbert, et les opérateurs sectoriels qui possèdent un calcul fonctionnel sectoriel borné d'angle strictement inférieur à  $\frac{\pi}{2}$ . Le calcul fonctionnel dérivé borné est une condition plus faible que le calcul fonctionnel  $H^\infty$ -borné. On dit qu'un opérateur  $A$  de type demi-plan a un calcul fonctionnel dérivé borné s'il existe une constante  $C > 0$  tel que pour  $\alpha < 0$  et tout  $f \in H^\infty(R_\alpha)$  on a

$$f'(A) \in B(X) \quad \text{et} \quad \|f'(A)\| \leq \frac{C}{-\alpha} \|f\|_{H^\infty(R_\alpha)}. \quad (0.4)$$

Pour tout  $m \in \mathbb{N}^*$ , on peut de la même manière définir un calcul fonctionnel  $m$ -dérivé borné où (0.4) est remplacé par

$$f^{(m)}(A) \in B(X) \quad \text{et} \quad \|f^{(m)}(A)\| \leq \frac{C}{(-\alpha)^m} \|f\|_{H^\infty(R_\alpha)}.$$

La cohérence de ces définitions viennent des inégalités de Cauchy pour les dérivés. Un des résultats principaux de l'article de Batty, Haase et Mubeen est de montrer que le calcul  $m$ -dérivé borné ne dépend pas de  $m \in \mathbb{N}^*$  et est équivalent à la condition de Gomilko Shi-Feng (0.3).

On retrouve par ailleurs la condition (0.3) dans les deux articles récents de Batty, Gomilko et Tomilov [7] et [8] dans lesquels elle joue un rôle central. Notons

$$\mathcal{B}(\mathbb{C}_+) := \left\{ f \in \operatorname{Hol}(\mathbb{C}_+) : \int_0^\infty \sup_{y \in \mathbb{R}} |f'(x + iy)| dx < \infty \text{ et } \lim_{x \rightarrow +\infty} f(x) = 0 \right\},$$

muni de la norme  $\|f\|_{\mathcal{B}} = \int_0^\infty \sup_{y \in \mathbb{R}} |f'(x + iy)| dx$  qui en fait une algèbre de Banach. Lorsque  $A$  vérifie la condition de Gomilko Shi-Feng (0.3) alors il est possible de montrer que l'application  $\Psi_A : \mathcal{B}(\mathbb{C}_+) \mapsto B(X, X^{**})$  définie par

$$\langle \Psi_A(f)x, x^* \rangle := \frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \langle R(-\alpha + i\beta, A)^2 x, x^* \rangle F'(\alpha + i\beta) d\beta d\alpha. + \langle f(\infty)x, x^* \rangle$$

est bien définie et bornée. Un résultat majeur de Batty, Gomilko et Tomilov est que pour tout  $f \in \mathcal{B}(\mathbb{C}_+)$ ,  $\Psi_A(f)$  est en fait un élément de  $B(X)$  (voir [7, Lemme 4.3.]), et donc que  $\Psi_A$  est un morphisme d'algèbres de  $\mathcal{B}(\mathbb{C}_+)$  dans  $B(X)$  borné. On note alors  $f(A) := \Psi_A(f)$  et on dit alors que  $A$  possède un calcul de Besov borné. Réciproquement si  $A$  possède un calcul de Besov borné, c'est-à-dire qu'il existe un morphisme d'algèbres  $\Psi_A$  comme ci-avant qui en plus vérifie que pour tout  $\lambda \in \mathbb{C}_+$ ,  $\Psi_A((\lambda + z)^{-1}) = -R(-\lambda, A)$  alors  $A$  vérifie la condition de Gomilko Shi-Feng (0.3).

Dans ce mémoire, on abordera différentes problématiques liées aux résultats de [7, 8, 9, 16, 53]. La première est d'obtenir un analogue du théorème de Gomilko Shi-Feng dans les espaces de Banach. Pour répondre à cette question on utilise les notions de  $\gamma$ -bornitude et d'opérateurs  $\gamma$ -radonifiants (voir le livre de Hytönen, Van Neerven, Veraar et Weis [26, chapitre 8 et 9] pour plus de détails). Ces notions jouent un rôle majeur dans différents aspects du calcul fonctionnel, notamment grâce aux travaux de Kalton et Weis [29, 28] qui ont très largement inspiré et contribué à l'essor du sujet. Soient  $X$  et  $Y$  deux espaces de Banach. On rappelle qu'un ensemble d'opérateurs  $\mathcal{T} \subset B(X, Y)$  est  $\gamma$ -borné s'il existe  $C > 0$  tel que pour tout  $N \in \mathbb{N}^*$ ,  $x_1, \dots, x_n \in X$  et  $T_1, \dots, T_N \in \mathcal{T}$  on a :

$$\int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s) T_k x_k \right\|_Y ds \leq C \int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s) x_k \right\|_X ds,$$

où  $(\gamma_k)_{k \geq 1}$  est une suite de Gaussiennes indépendantes sur un espace de probabilité fixé. On note que la  $\gamma$ -bornitude implique l'uniforme bornitude et que la réciproque est vraie sur les espaces de Hilbert. Plusieurs fois dans ce mémoire on cherchera à étendre aux espaces de Banach différentes caractérisations des  $C_0$ -semigroupes sur des espaces de Hilbert. Pour réussir à faire cela il est nécessaire de remplacer l'hypothèse d'uniforme bornitude de l'ensemble  $\{T_t : t \geq 0\}$  par l'hypothèse de  $\gamma$ -bornitude. Il se trouve que remplacer l'uniforme bornitude par la  $\gamma$ -bornitude dans ce genre de problème est souvent fructueux. Sachant cela on comprend l'importance de l'étude des  $C_0$ -semigroupes  $(T_t)_{t \geq 0}$   $\gamma$ -bornés c'est-à-dire tel que l'ensemble  $\{T_t : t \geq 0\}$  est  $\gamma$ -borné. Une caractérisation des  $C_0$ -semigroupes  $\gamma$ -bornés dans les espaces de Banach analogue au Théorème de Gomilko Shi-Feng sera obtenue dans le chapitre 2. On considérera aussi un calcul  $m$ -dérivé  $\gamma$ -borné et on verra qu'il est indépendant de  $m \in \mathbb{N}^*$  et que ce calcul caractérise les  $C_0$ -semigroupes  $\gamma$ -bornés.

Le lien étroit entre les  $C_0$ -semigroupes bornés et les opérateurs à puissances bornées amène naturellement à la question suivante : peut-on obtenir une caractérisation analogue des opérateurs bornés  $T \in B(X)$  vérifiant que l'ensemble  $\{T^n : n \in \mathbb{N}\}$  est  $\gamma$ -borné ? Cette question est étudiée dans le chapitre 3 dans lequel on obtiendra une caractérisation de ces opérateurs similaire au théorème de Gomilko Shi-Feng.

La question sous-jacente aux thèmes abordés ci-dessus est la suivante : lorsque  $f \in H^\infty(R_\alpha)$  pour  $\alpha < 0$  comment peut-on estimer la norme de l'opérateur  $f(A)$  et quelle est la meilleure estimation que l'on peut obtenir ? Dans [7, 8], Batty Gomilko et Tomilov donnent une bonne estimation dans un cadre général, en effet les résultats présentés ci-dessus nous disent que lorsque  $f \in \mathcal{B}(\mathbb{C}_+)$  et  $A$  a la condition de Gomilko Shi-Feng alors on a

$$\|f(A)\| \leq C \|f\|_{\mathcal{B}}.$$

On note qu'une estimation similaire était déjà obtenue par Haase [21] dans le cas Hilbertien. Dans le chapitre 4 on se propose d'améliorer cette estimation lorsque  $-A$  est le générateur d'un  $C_0$ -semigroupe borné sur un espace de Hilbert. Pour cela on introduira une algèbre qui est inspirée d'un article de Peller [47] et qui se trouve être un analogue analytique des algèbres de Figa-Talamanca-Herz.

## Contenu de la thèse

Cette thèse se compose de quatre chapitres rédigés en anglais. Le premier chapitre est un article [4] intitulé : « New counterexamples on Ritt operators, sectorial operators and

$R$ -boundedness » qui est un travail joint avec Christian Le Merdy, publié dans la revue Bulletin of the Australian Mathematical Society. Le deuxième chapitre reprend mon article [3] «  $\gamma$ -boundedness of  $C_0$ -semigroups and their  $H^\infty$ -functional calculi » publié dans la revue Studia Mathematica tandis que le chapitre 3 reprend mon article [2] « Derivative bounded functional calculus of power bounded operators on Banach spaces » accepté à la publication dans la revue Acta Szeged. Le chapitre 4 s'intitule « Functional calculus for a bounded  $C_0$ -semigroup on Hilbert space » et présente un travail joint [5] avec Christian Le Merdy. Enfin on conclut avec un appendice qui complète le chapitre 2.

### Chapitre 1 : Nouveaux contre-exemples d'opérateurs de Ritt et sectoriels qui ne sont pas $\gamma$ -bornés

La  $\gamma$ -bornitude joue un rôle majeur dans l'étude des opérateurs de Ritt et des opérateurs sectoriels (pour plus d'informations voir [39]). On dira qu'un opérateur  $A$  sectoriel de type  $\omega$  est  $\gamma$ -sectoriel de type  $\omega$  si pour tout  $\omega < \theta < \pi$  l'ensemble

$$\{\lambda R(\lambda, A) : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}\} \quad (0.5)$$

est  $\gamma$ -borné avec  $\Sigma_\theta = \{z \in \mathbb{C} : |\arg(z)| < \theta\}$ . D'après [58] un opérateur est  $\gamma$ -sectoriel de type  $< \frac{\pi}{2}$  si et seulement si les ensembles

$$\{e^{-tA} : t \geq 0\} \quad \text{et} \quad \{tAe^{-tA} : t \geq 0\} \quad (0.6)$$

sont  $\gamma$ -bornés. L'existence d'opérateurs sectoriels qui ne sont pas  $\gamma$ -sectoriels est un problème très important relié au problème de régularité maximal- $L^p$  résolu par Kalton et Lancien dans leur article [30]. Dans l'article [30], les auteurs construisent des opérateurs sectoriels qui ne sont pas  $\gamma$ -sectoriels. Fackler lui, dans son article plus récent [14] simplifie la construction de [30] et construit des exemples qui sont des multiplicateurs relativement à des bases de Schauder. Les exemples de Fackler vérifient que l'ensemble de (0.5) n'est pas  $\gamma$ -borné. Néanmoins on ne sait pas si l'un des deux ou les deux ensembles de (0.6) ne sont pas  $\gamma$ -bornés.

Dans ce chapitre, après quelques rappels sur les multiplicateurs relativement à des bases de Schauder, on construira un exemple « simple » d'opérateur sectoriel qui est un multiplicateur relativement à une base de Schauder de tel sorte que l'ensemble  $\{e^{-tA} : t \geq 0\}$  n'est pas  $\gamma$ -borné. On construira aussi un opérateur de Ritt  $T \in B(X)$  tel que l'ensemble  $\{T^n : n \geq 0\}$  n'est pas  $\gamma$ -borné.

On utilisera les résultats de ce chapitre pour un exemple présenté dans la dernière section du chapitre 2.

### Chapitre 2 : $\gamma$ -bornitude des $C_0$ -semigroupes et leurs calculs $H^\infty$ -fonctionnels

Dans ce chapitre on étudie les  $C_0$ -semigroupes  $\gamma$ -bornés, c'est-à-dire les  $C_0$ -semigroupes  $(T_t)_{t \geq 0}$  tel que  $\{T_t : t \geq 0\}$  est  $\gamma$ -borné. Jusqu'à maintenant ceux-ci n'ont été étudiés principalement qu'à travers les opérateurs  $\gamma$ -sectoriels qui sont à fortiori générateurs (négatifs) de  $C_0$ -semigroupes  $\gamma$ -bornés. Pour faire cette étude on commence par donner des résultats analogues à ceux de Batty, Haase et Mubeen [9] dans le cas faiblement  $\gamma$ -bornés. Pour ce faire on introduit une version  $\gamma$ -bornée de la condition de Gomilko Shi-Feng (0.3) qui devient alors : il existe  $C > 0$  tel que pour tout  $N \in \mathbb{N}^*$ , tout  $\alpha_1, \dots, \alpha_N < 0$ , et pour tout  $x_1, \dots, x_N \in X$  et  $y_1, \dots, y_N \in X^*$ , on a

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} |\langle (-\alpha_k)R(\alpha_k + it, A)^2 x_k, y_k \rangle| dt \\ & \leq C \left( \int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s)x_k \right\|_X ds \right) \left( \int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s)y_k \right\|_{X^*} ds \right). \end{aligned} \quad (0.7)$$

Un premier résultat important est que si  $X$  est  $K$ -convexe alors, pour tout  $m \in \mathbb{N}^*$ , cette condition est équivalente au fait que  $A$  a un calcul fonctionnel  $m$ -dérivé  $\gamma$ -borné, c'est-à-dire que  $A$  a un calcul fonctionnel  $m$ -dérivé borné et l'ensemble

$$\bigcup_{\alpha < 0} \{(-\alpha)^m f^{(m)}(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1\}$$

est  $\gamma$ -borné. Il en viendra une extension aux espaces  $K$ -convexe du théorème de Gomilko Shi-Feng dans les espaces de Hilbert. En particulier dans les espaces  $L^p$  pour  $1 < p \neq 2 < \infty$  (on notera  $q$  le conjugué de  $p$ ) ce résultat prendra la forme suivante :

**Théorème.** *Soit  $X = L^p$  avec  $1 < p \neq 2 < \infty$ . Supposons que  $A$  soit de type demi-plan sur  $X$ . Alors les assertions suivantes sont équivalentes*

(i)  $-A$  engendre un  $C_0$ -semigroupe  $(T_t)_{t \geq 0}$   $\gamma$ -borné;

(ii) il existe  $C > 0$  tel que pour tout  $N \in \mathbb{N}^*$ , pour tout  $x_1, \dots, x_N \in L^p$  et  $y_1, \dots, y_N \in L^q$ , et pour tout  $\alpha_1, \dots, \alpha_N < 0$ , on a

$$\left\| \left( \sum_{k=1}^N \int_{\mathbb{R}} (-\alpha_k) |R(\alpha_k + it, A)x_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| \left( \sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

et

$$\left\| \left( \sum_{k=1}^N \int_{\mathbb{R}} (-\alpha_k) |R(\alpha_k + it, A^*)y_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \left\| \left( \sum_{k=1}^N |y_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q}.$$

Grâce à cela on est capable de démontrer une version  $\gamma$ -bornée du théorème de Gearhart-Prüss et dont un corollaire direct et remarquable est le suivant :

**Théorème.** *Soit  $(T_t)_{t \geq 0}$  un  $C_0$ -semigroupe borné sur un espace  $K$ -convexe. S'il existe  $\delta > 0$  tel que  $\{e^{-\delta t}T_t : t \geq 0\}$  est  $\gamma$ -borné, alors  $\{e^{-\delta t}T_t : t \geq 0\}$  est  $\gamma$ -borné pour tout  $\delta > 0$ .*

De plus utilisant les résultats du chapitre 1, on pourra préciser ce théorème en donnant un exemple de semigroupe  $(T_t)_{t \geq 0}$  vérifiant que pour tout  $\delta > 0$ ,  $\{e^{-\delta t}T_t : t \geq 0\}$  est  $\gamma$ -borné alors que  $\{T_t : t \geq 0\}$  n'est pas  $\gamma$ -borné.

*On notera en appendice qu'en utilisant les  $\gamma$ -espaces introduits dans [29] on pourra étendre nos résultats au cas où  $X$  est un espace de Banach sans aucune hypothèse de  $K$ -convexité.*

Ce chapitre s'organise de la façon suivante. Dans la section 2 on donnera des résultats préliminaires utiles pour la suite. Notamment, on rappellera comment se construit le

calcul fonctionnel des opérateurs de type demi-plan et on donnera les résultats clés liés à la  $\gamma$ -bornitude et aux opérateurs  $\gamma$ -radonifiants. La section 3 est dédiée à montrer que la condition (0.7) est équivalente au calcul fonctionnel  $m$ -dérivé  $\gamma$ -borné. Dans la section 4 on énoncera notre version du théorème de Gomilko-Shi Feng dans les espaces de Banach ce qui nous donnera une caractérisation des  $C_0$ -semigroupes  $\gamma$ -bornés, puis dans la section 5 on utilisera cette caractérisation pour montrer la version  $\gamma$ -bornée du théorème de Gearhart-Prüss. Enfin dans la section 6 on résumera les résultats obtenus et on donnera des contre-exemples.

### Chapitre 3 : Calcul fonctionnel dérivé borné des opérateurs à puissances bornées sur un espace de Banach

Le but de ce chapitre est de donner les analogues discrets des résultats de Batty, Haase et Mubeen [9] et des résultats du chapitre 2. Une version discrète du théorème de Gomilko Shi-Feng est donnée par Gomilko et Cojuhari [17]. Ils introduisent notamment ce que l'on appellera la condition discrète de Gomilko Shi-Feng : pour  $T \in B(X)$  avec  $\sigma(T) \subset \overline{\mathbb{D}}$ , il existe  $C > 0$  tel que

$$\int_0^{2\pi} |\langle R(re^{it}, T)^2 x, x^* \rangle| dt \leq \frac{C}{(r+1)(r-1)} \|x\| \|x^*\|, \quad r > 1, x \in X, x^* \in X^*. \quad (0.8)$$

Une première observation simple est qu'un opérateur vérifiant (0.8) est à puissances bornées, c'est-à-dire  $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$ . On fera un parallèle avec un calcul fonctionnel dérivé borné. On dira qu'un opérateur  $T \in B(X)$  vérifiant  $\sigma(T) \subset \overline{\mathbb{D}}$  a un calcul fonctionnel  $m$ -dérivé borné discret lorsqu'il existe  $C > 0$  tel que

$$\|\phi^{(m)}(T)\| \leq \frac{C}{(r-1)^m} \|\phi\|_{H^\infty(r\mathbb{D})} \quad \text{pour tout } \phi \in H^\infty(r\mathbb{D}) \text{ et } r > 1.$$

On retrouvera la même conclusion que dans [9], c'est-à-dire que pour  $m \in \mathbb{N}^*$ ,  $T$  a la condition de Gomilko Shi-Feng discrète (0.8) si et seulement s'il a un calcul fonctionnel  $m$ -dérivé borné discret. Par ailleurs on retrouvera le résultat connu dans les espaces de Hilbert (voir [17]) que  $T$  est à puissances bornées si et seulement s'il a la condition de Gomilko Shi-Feng discrète.

On étudiera aussi le calcul fonctionnel de Besov sous la condition de Gomilko Shi-Feng discrète (en s'inspirant des résultats de Batty, Gomilko et Tomilov [7, 8]). On dit que  $f : \mathbb{D} \rightarrow \mathbb{C}$  est dans  $\mathcal{B}(\mathbb{D})$  si  $f$  est holomorphe sur le disque unité  $\mathbb{D}$  et vérifie

$$\int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du < \infty.$$

On munit  $B(\mathbb{D})$  de la norme  $\|\cdot\|$  définie par

$$\forall f \in \mathcal{B}(\mathbb{D}), \quad \|f\|_{\mathcal{B}(\mathbb{D})} = \int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du + \sup_{z \in \mathbb{D}} |f(z)|.$$

Peller montre dans [47] que si  $T$  est à puissances bornées sur un espace de Hilbert, alors il existe  $C > 0$  tel que pour tout polynôme  $P$  on a

$$\|P(T)\| \leq C \|P\|_{\mathcal{B}(\mathbb{D})}.$$

On va réussir à étendre ce résultat à tout opérateur  $T$  vérifiant la condition de Gomilko Shi-Feng (0.8) et on verra que l'on peut donner une forme explicite de  $f(T)$  quand  $f \in \mathcal{B}(\mathbb{D})$ . En effet on aura, pour  $x \in X$  et  $x^* \in X^*$ ,

$$\langle f(T)x, x^* \rangle = \frac{1}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} f' \left( \frac{e^{it}}{r} \right) \langle R^2(re^{it}, T)x, x^* \rangle dt dr + \langle f(0)x, x^* \rangle.$$

La deuxième partie de ce chapitre se focalisera sur les analogues discrets du chapitre 2. En particulier on étudiera les opérateurs  $T$  à puissances  $\gamma$ -bornées, c'est-à-dire tels que l'ensemble  $\{T^n : n \in \mathbb{N}^*\}$  est  $\gamma$ -borné. On verra que ces opérateurs vérifient la condition de Gomilko Shi-Feng discrète (0.8). On obtiendra une caractérisation des opérateurs à puissances  $\gamma$ -bornées lorsque l'espace sous-jacent est un espace de Banach général. Dans  $L^p$  le théorème s'énoncera comme suit :

**Théorème.** *Soit  $X = L^p$  avec  $1 < p \neq 2 < \infty$ . Soit  $T$  un opérateur borné sur  $X$ . Les assertions suivantes sont équivalentes:*

- (i) *L'ensemble  $\{T^n : n \in \mathbb{N}\}$  est  $\gamma$ -borné;*
- (ii) *Le spectre  $\sigma(T)$  est inclus dans  $\overline{\mathbb{D}}$  et il existe une constante  $C > 0$  tel que pour tout  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in L^p$  et  $y_1, \dots, y_N \in L^q$*

$$\left\| \left( \sum_{k=1}^N \int_0^{2\pi} (r_k^2 - 1) |R(r_k e^{it}, T)x_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| \left( \sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

et

$$\left\| \left( \sum_{k=1}^N \int_0^{2\pi} (r_k^2 - 1) |R(r_k e^{it}, T^*)y_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \left\| \left( \sum_{k=1}^N |y_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q}.$$

Ce chapitre s'organise de la façon suivante. Dans la section 2 on montrera que la condition de Gomilko Shi-Feng discrète est équivalente au calcul fonctionnel  $m$ -dérivé discret tandis que la section 3 sera dédiée au calcul de Besov. Enfin dans la section 4 on étudiera les opérateurs à puissances  $\gamma$ -bornées, on énoncera notre caractérisation des opérateurs à puissances  $\gamma$ -bornées dans un espace de Banach. De plus on montrera qu'un opérateur à puissances  $\gamma$ -bornées possède un calcul de Peller (voir [47, section 3]), et ce calcul est de plus  $\gamma$ -borné.

## Chapitre 4 : Calcul fonctionnel pour les $C_0$ -semigroupes bornés sur un espace de Hilbert

On se focalise pour l'essentiel sur les générateurs (négatifs) de  $C_0$ -semigroupes bornés sur un espace de Hilbert. Soit  $H$  un espace de Hilbert et  $-A$  le générateur d'un  $C_0$ -semigroupe  $(T_t)_{t \geq 0}$ . Pour  $b \in L^1(\mathbb{R}_+)$ , on définit  $L_b$  la transformée de Laplace par

$$L_b : \mathbb{C}_+ \rightarrow \mathbb{C}, \quad z \mapsto \int_0^\infty b(t)e^{-tz} dt,$$

et on peut lui associer l'opérateur  $L_b(A) \in B(H)$  défini par

$$L_b(A) = \int_0^\infty b(t)T_t dt.$$

L'application  $b \mapsto L_b(A)$  est appelée calcul de Hille-Phillips et on a évidemment

$$\|L_b(A)\| \leq C \|b\|_1, \quad b \in L^1(\mathbb{R}_+).$$

où  $C = \sup_{t \geq 0} \|T_t\|$ . Lorsque  $(T_t)_{t \geq 0}$  est semblable à un  $C_0$ -semigroupe de contractions alors il existe  $C > 0$  tel que

$$\|L_b(A)\| \leq C \|\hat{b}\|_\infty,$$

où  $\hat{b}$  désigne la transformée de Fourier de  $b$ . Il se trouve que tous les générateurs négatifs de  $C_0$ -semigroupes bornés sur un espace de Hilbert ne satisfont pas une telle estimation. Mais [21, Corollaire 5.5] donne que pour tout générateur négatif  $A$  de  $C_0$ -semigroupe borné sur un espace de Hilbert il existe  $C > 0$  tel que

$$\|L_b(A)\| \leq C \|b\|_{\mathcal{B}}. \quad (0.9)$$

La motivation de ce chapitre est de trouver une meilleure estimation de  $\|L_b(A)\|$ .

Pour ce faire on a besoin d'introduire une algèbre de Banach inspirée des algèbres de Figa-Talamanca-Herz (voir [12]). On note  $\mathcal{A}_0(\mathbb{C}_+)$  (respectivement  $\mathcal{A}(\mathbb{C}_+)$ ) l'ensemble des fonctions  $F : \overline{\mathbb{C}_+} \rightarrow \mathbb{C}$  telles qu'il existe deux suites de fonctions  $(f_k)_{k \in \mathbb{N}}$  de  $C_0(\mathbb{R})$  (respectivement de  $BUC(\mathbb{R})$ ) et  $(h_k)_{k \in \mathbb{N}}$  de  $H^1(\mathbb{R})$  satisfaisant

$$\sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1 < \infty \quad (0.10)$$

et

$$\forall z \in \overline{\mathbb{C}_+}, \quad F(z) = \sum_{k=1}^{\infty} (f_k \star h_k)(iz). \quad (0.11)$$

Une première remarque est que si  $F$  est un élément de  $\mathcal{A}_0(\mathbb{C}_+)$  (respectivement de  $\mathcal{A}(\mathbb{C}_+)$ ) alors  $F|_{\mathbb{C}_+} \in H^\infty(\mathbb{C}_+)$  et  $x \mapsto F(ix)$  est dans  $C_0(\mathbb{R})$  (respectivement dans  $BUC(\mathbb{R})$ ). On s'emploiera aussi à démontrer que  $\mathcal{A}_0(\mathbb{C}_+)$  et  $\mathcal{A}(\mathbb{C}_+)$  sont des algèbres de Banach et on montrera que  $\mathcal{B}(\mathbb{C}_+)$  est continûment inclus dans  $\mathcal{A}(\mathbb{C})$  et que  $\mathcal{B}(\mathbb{C}_+)$  et  $\mathcal{A}(\mathbb{C}_+)$  sont différents.

Maintenant le résultat principal de ce chapitre est le suivant :

**Théorème.** *Il existe un unique morphisme borné  $\rho_A : \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$  tel que*

$$\rho_A(L_b) = \int_0^\infty b(t) T_t dt \quad (0.12)$$

pour tout  $b \in L^1(\mathbb{R}_+)$ .

L'intérêt de ce résultat est que comme  $\mathcal{A}(\mathbb{C}_+)$  et  $\mathcal{B}(\mathbb{C}_+)$  sont différents et que  $\mathcal{B}(\mathbb{C}_+)$  est continûment inclus dans  $\mathcal{A}(\mathbb{C})$  alors ce résultat améliore l'estimation (0.9).

Afin de montrer que  $\mathcal{B}(\mathbb{C}_+)$  et  $\mathcal{A}(\mathbb{C}_+)$  sont différents on aura besoin de déterminer le dual de  $\mathcal{A}_0(\mathbb{C}_+)$ . On dira que  $m \in L^\infty$  est un multiplicateur de Fourier borné sur  $H^1(\mathbb{R})$  si l'application

$$\begin{aligned} T_m & : H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \\ h & \mapsto \mathcal{F}^{-1}(m\hat{h}) \end{aligned}$$

est bornée par rapport à la norme de  $H^1(\mathbb{R})$  et on notera  $\mathcal{M}(H^1(\mathbb{R}))$  l'ensemble des multiplicateurs de Fourier bornés sur  $H^1(\mathbb{R})$ . On montrera le résultat important suivant, où  $\tau_s$  est défini par  $(\tau_s f)(t) = f(t - s)$ :

**Théorème.** *Soit  $T \in B(H^1(\mathbb{R}))$ . Si  $T$  commute avec les translations, c'est-à-dire pour tout  $s \in \mathbb{R}$ ,  $T\tau_s = \tau_s T$ , alors il existe  $m : \mathbb{R}_+^* \rightarrow \mathbb{C}$  bornée et continue telle que  $T = T_m$ . De plus dans ce cas on a*

$$\|m\|_\infty \leq \|T\|.$$

L'espace  $B(H^1(\mathbb{R}))$  étant naturellement un dual, nous noterons  $\mathcal{PM} \subset B(H^1(\mathbb{R}))$  la fermeture pour la topologie faible\* de l'ensemble  $\{h \mapsto \mu \star h : \mu \in M(\mathbb{R})\}$ . Cet ensemble  $\mathcal{PM}$  est un sous-espace de  $\mathcal{M}(H^1(\mathbb{R}))$  et on montre que

$$\mathcal{PM}^* = \mathcal{A}_0(\mathbb{C}_+).$$

On conclura ce chapitre en s'intéressant une nouvelle fois aux cas des  $C_0$ -semigroupes  $\gamma$ -bornés, et on montrera que dans ce cas on peut de la même manière construire un  $\mathcal{A}(\mathbb{C}_+)$ -calcul fonctionnel borné qui se trouve en fait être  $\gamma$ -borné. Pour être plus précis on obtiendra le résultat :

**Théorème.** *Soient  $X$  un espace de Banach et  $(T_t)_{t \geq 0}$  un  $C_0$ -semigroupe  $\gamma$ -borné sur  $X$ . Alors il existe un unique morphisme borné  $\rho_A : \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$  tel que  $\rho_A$  vérifie (4.4.8) et l'ensemble*

$$\{\rho_A(f) : f \in \mathcal{A}(\mathbb{C}_+), \|f\|_{\mathcal{A}} \leq 1\}$$

*est  $\gamma$ -borné.*

Le chapitre se construit comme suit. Dans la section 2 on étudie les multiplicateurs de Fourier sur l'espace de Hardy  $H^1(\mathbb{R})$ . Dans la section 3 on introduit nos nouvelles algèbres  $\mathcal{A}_0(\mathbb{C}_+)$  et  $\mathcal{A}_0(\mathbb{C}_+)$  et on en donne des propriétés utiles avant d'en déterminer le dual de  $\mathcal{A}_0(\mathbb{C}_+)$ . Dans la section 4 on construit notre calcul fonctionnel et on démontre le résultat principal. Dans la section 5 on montre que  $\mathcal{B}(\mathbb{C}_+)$  et  $\mathcal{A}(\mathbb{C}_+)$  sont différents. Enfin la section 6 est consacrée à étendre notre principal résultat au cas des  $C_0$ -semigroupes  $\gamma$ -bornés sur un espace de Banach  $X$ .





# General introduction

The main goal of this thesis is to study bounded  $C_0$ -semigroups and power-bounded operators on Banach spaces (sometimes specifically on Hilbert spaces). In the late of 1990s, Gomilko [16] and, independently, Shi and Feng [53], show that a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on Hilbert space is characterized by the following condition : there exists  $C > 0$  such that for each  $\alpha < 0$  we have

$$-\alpha \int_{\mathbb{R}} \|R(\alpha + it, A)x\|^2 \leq C \|x\|^2, \quad x \in H \quad (0.13)$$

and

$$-\alpha \int_{\mathbb{R}} \|R(\alpha + it, A)^*x\|^2 \leq C \|x\|^2, \quad x \in H, \quad (0.14)$$

where  $-A$  is the generator of  $(T_t)_{t \geq 0}$ . This result is now known as the Gomilko Shi-Feng Theorem. The latter is a powerful tool used for the proof of many results related to bounded  $C_0$ -semigroups on Hilbert spaces. For example, it is possible to prove the famous Gearhart-Prüss Theorem ([1, Theorem 5.2.1.]) as an application of Gomilko Shi-Feng Theorem. The question we can ask is : “What happens on general Banach spaces ?”. Inequalities (0.13) and (0.14) are not appropriate to Banach space setting. We then use the following condition, which I call Gomilko Shi-Feng condition (GFS) : there exists  $C > 0$  such that

$$\int_{\mathbb{R}} |\langle R(\alpha + it, A)^2x, y \rangle| dt \leq \frac{C}{-\alpha} \|x\| \|y\|, \quad x \in X, y \in X^* \text{ and } \alpha < 0. \quad (0.15)$$

On Hilbert space, the Gomilko Shi-Feng condition is also equivalent to the generation of a bounded  $C_0$ -semigroup. On the other hand, we can find counter-examples on non Hilbert spaces, for instance the shift semigroup on  $L^p(\mathbb{R})$  for  $1 < p \neq 2 < \infty$  does not satisfy the condition (0.15) while these semigroups are bounded.

The condition (0.15) is studied in the article [9] of Batty, Haase and Mubeen. They introduce a new bounded functional calculus, the derivative bounded functional calculus. Let  $A$  be an operator on Banach space  $X$  which is densely defined, whose spectrum is included in the half-plane  $\overline{\mathbb{C}}_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$  and which satisfies that its resolvent  $R(\lambda, A)$  is bounded on each half-plane  $\{z \in \mathbb{C} : \operatorname{Re}(z) \leq \alpha\}$  with  $\alpha < 0$ . We will call such an  $A$  a half-plane type operator. For such operator one can construct a functional calculus on a particular subspace of  $H^\infty(\mathbb{C}_+)$ , the set of all holomorphic functions bounded on  $\mathbb{C}_+$ , which is denoted by  $\mathcal{E}$  in [9]. This means that we can construct from  $A$ , thanks to the Cauchy Theorem, an algebra homomorphism  $\Phi : \mathcal{E} \mapsto B(X)$  which satisfies  $\Phi(f(z)(\lambda - z)) = \Phi(f)R(A, \lambda)$  for each  $f \in \mathcal{E}$  and each  $\alpha < 0$ . We will write  $f(A) := \phi(f)$ . Thanks to regularization it is possible to give a definition of  $\Phi(f)$  (which can be an unbounded operator) when  $f \in H^\infty(R_\alpha)$  with  $\alpha < 0$  where  $R_\alpha := \{z \in \mathbb{C} : \operatorname{Re}(z) > \alpha\}$ . Now we say that a half-plane type operator admits a

bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus if there exists  $C > 0$  such that for each  $\alpha < 0$  and  $f \in H^\infty(R_\alpha)$  we have

$$f(A) \in B(X) \quad \text{and} \quad \|f(A)\| \leq C \|f\|_{H^\infty(\mathbb{C}_+)}.$$

The bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus, like the notion of polynomially bounded operator, has no good characterisation. At present only two classes of operators with a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus are known: negative generators of  $C_0$ -semigroup similar to a contractive semigroup on Hilbert space and sectorial operators with a bounded  $H^\infty$ -functional calculus of angle  $< \frac{\pi}{2}$ . The derivative bounded functional calculus is weaker than  $H^\infty(\mathbb{C}_+)$ -bounded functional calculus. We say that an operator  $A$  of half-plane type admits a derivative bounded functional calculus if there exists  $C > 0$  such that for each  $\alpha < 0$  and each  $f \in H^\infty(R_\alpha)$  we have

$$f'(A) \in B(X) \quad \text{and} \quad \|f'(A)\| \leq \frac{C}{-\alpha} \|f\|_{H^\infty(R_\alpha)}. \quad (0.16)$$

For each  $m \in \mathbb{N}$ , one can also define a  $m$ -derivative functional calculus where (0.16) is replaced by

$$f^{(m)}(A) \in B(X) \quad \text{and} \quad \|f^{(m)}(A)\| \leq \frac{C}{(-\alpha)^m} \|f\|_{H^\infty(R_\alpha)}.$$

These definitions are meaningful thanks to Cauchy's inequalities. One of the main results of Batty, Haase and Mubeen [9] is that the  $m$ -derivative bounded functional calculus does not depend on  $m \in \mathbb{N}$  and is equivalent to the Gomilko Shi-Feng condition (0.15).

Besides this, the condition (0.15) plays also a major role in the two recent articles of Batty, Gomilko and Tomilov [7] and [8]. Let  $\mathcal{B}(\mathbb{C}_+)$  be a Besov algebra defined by

$$\mathcal{B}(\mathbb{C}_+) := \left\{ f \in \text{Hol}(\mathbb{C}_+) : \int_0^\infty \sup_{y \in \mathbb{R}} |f'(x + iy)| dx < \infty \text{ and } \lim_{x \rightarrow +\infty} f(x) = 0 \right\},$$

equipped with the norm  $\|f\|_{\mathcal{B}} = \int_0^\infty \sup_{y \in \mathbb{R}} |f'(x + iy)| dx$ . When  $A$  satisfies the Gomilko Shi-Feng condition (0.15) then it is possible to show that the map  $\Psi_A : \mathcal{B}(\mathbb{C}_+) \mapsto B(X, X^{**})$  defined by

$$\langle \Psi_A(f)x, x^* \rangle := \frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \langle R(-\alpha + i\beta, A)^2 x, x^* \rangle F'(\alpha + i\beta) d\beta d\alpha. + \langle f(\infty)x, x^* \rangle$$

is well-defined and bounded. According to [7] it turns out for each  $f \in \mathcal{B}(\mathbb{C}_+)$ ,  $\Psi_A(f)$  is in fact an element of  $B(X)$  (see [7, Lemma 4.3.]). Therefore  $\Psi_A$  is a bounded homomorphism from  $\mathcal{B}(\mathbb{C}_+)$  into  $B(X)$ . We write  $f(A) := \Psi_A(f)$  and we say that  $A$  admits a bounded Besov calculus. Conversely if  $A$  admits a bounded Besov calculus, that is there exists a bounded homomorphism from  $\mathcal{B}(\mathbb{C}_+)$  into  $B(X)$  such that for each  $\lambda \in \mathbb{C}_+$ ,  $\Psi_A((\lambda + z)^{-1}) = -R(-\lambda, A)$ , then  $A$  satisfies the Gomilko Shi-Feng condition (0.15).

In this thesis, we will investigate various issues related to the results of [7, 8, 9, 16, 53]. The first one is to obtain an analogue of the Gomilko Shi-Feng Theorem on Banach space. To achieve this we need the notion of  $\gamma$ -boundedness and  $\gamma$ -radonifying operators (see the book of Hytönen, Van Neerven, Veraar and Weis [26, chapitre 8 and 9] for more details). These notions play an important role in various aspects of functional calculus, especially thanks the work of Kalton and Weis [29, 28].

Let  $X$  and  $Y$  be two Banach spaces. We recall that a set of operators  $\mathcal{T} \subset B(X, Y)$  is  $\gamma$ -bounded if there exists  $C > 0$  such that for each  $N \in \mathbb{N}$ ,  $x_1, \dots, x_n \in X$  and  $T_1, \dots, T_N \in \mathcal{T}$  we have

$$\int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s) T_k x_k \right\|_Y ds \leq C \int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s) x_k \right\|_X ds,$$

where  $(\gamma_k)_{k \geq 1}$  is a Gaussian sequence on a fixed probability space. Note that  $\gamma$ -boundedness implies uniform boundedness and that the converse is true on Hilbert spaces. An important part of our work is to generalize to the Banach setting some characterizations of bounded  $C_0$ -semigroups which are stated in Hilbert spaces. A fruitful way to do this is to replace the assumption of uniform boundedness (of the set  $\{T_t : t \geq 0\}$ ) by the assumption of  $\gamma$ -boundedness. Consequently it is important to study  $\gamma$ -bounded  $C_0$ -semigroup, that is,  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  such that  $\{T_t : t \geq 0\}$  is  $\gamma$ -bounded. A characterisation of  $\gamma$ -bounded  $C_0$ -semigroups will be obtained in chapter 2. This characterisation is an analogue of the Gomilko Shi-Feng Theorem on Banach spaces. We also investigate in the notion of  $m$ -derivative  $\gamma$ -bounded functional calculus and we will see that this is independent of  $m \in \mathbb{N}$  and characterises  $\gamma$ -bounded  $C_0$ -semigroups.

The link between bounded  $C_0$ -semigroups and power-bounded operators leads to the following question : can we obtain analogous characterisation of operators  $T \in B(X)$  which satisfy  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded ? This question is studied in chapter 3 in which we will obtain a characterisation of these operators similar to the Gomilko Shi-Feng Theorem.

An important question related to the above topics is the following: when  $f \in H^\infty(R_\alpha)$  for  $\alpha < 0$  how can we estimate the norm of  $f(A)$  and obtain the sharp estimate ? In [7, 8], Batty Gomilko and Tomilov obtain an estimate in the Banach setting. Indeed according above, if  $f \in \mathcal{B}(\mathbb{C}_+)$  and  $A$  satisfies the Gomilko Shi-Feng condition (0.15) then

$$\|f(A)\| \leq C \|f\|_{\mathcal{B}}.$$

In [21], Haase had already obtained a similar estimate on Hilbert space. The goal of chapter 4 is to improve this estimate when  $-A$  is the generator of a  $C_0$ -semigroup on a Hilbert space. To this end we will introduce an algebra inspired by an article of Peller [47]. It turns out that this algebra is an analytic analogue of Figa-Talamanca-Herz algebras.

## Content of the thesis

This thesis consists of four chapters. The first chapter is an article [4] called : “New counterexamples on Ritt operators, sectorial operators and R-boundedness” which is a joint work with Christian Le Merdy, published in Bulletin of the Australian Mathematical Society. The second chapter is my article [3] “ $\gamma$ -boundedness of  $C_0$ -semigroups and their  $H^\infty$ -functional calculi” published in Studia Mathematica while the third chapter is my article [2] “Derivative bounded functional calculus of power bounded operators on Banach spaces” accepted in Acta Szeged. The last chapter is a joint work [5] with Christian Le Merdy called “Functional calculus for a bounded  $C_0$ -semigroup on Hilbert space”. I conclude with an appendix which complements Chapter 2

## Chapter 1 : New counterexamples on Ritt operators, sectorial operators and $R$ -boundedness

$\gamma$ -boundedness plays a key role in the study of Ritt operators and sectorial operators (see [39]). A sectorial operator  $A$  of type  $\omega$  is  $\gamma$ -sectorial of type  $\omega$  if for each  $\omega < \theta < \pi$  the set

$$\{\lambda R(\lambda, A) : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}\} \quad (0.17)$$

is  $\gamma$ -bounded with  $\Sigma_\theta = \{z \in \mathbb{C} : |\arg(z)| < \theta\}$ . According to [58] an operator is  $\gamma$ -sectorial of type  $< \frac{\pi}{2}$  if and only if the sets

$$\{e^{-tA} : t \geq 0\} \quad \text{and} \quad \{tAe^{-tA} : t \geq 0\} \quad (0.18)$$

are  $\gamma$ -bounded. The existence of sectorial operators which are not  $\gamma$ -sectorial is an important issue related to the  $L^p$ -maximal regularity problem solved by Kalton and Lancien in [30]. In [30], the authors construct sectorial operators which are not  $\gamma$ -sectorial. Fackler, in his more recent article [14], provides a simpler construction using multipliers with respect to Schauder bases. The examples of Fackler satisfy that the set (0.17) is not  $\gamma$ -bounded. Nevertheless we do not know if one of the both or the both sets of (0.18) are not  $\gamma$ -bounded.

In this chapter, after some background about multipliers with respect to Schauder bases we will construct a “simple” sectorial operator of this type such that the set  $\{e^{-tA} : t \geq 0\}$  is not  $\gamma$ -bounded. Furthermore we will construct a Ritt operator  $T$  such that the set  $\{T^n : n \in \mathbb{N}\}$  is not  $\gamma$ -bounded.

We will use these results to provide an example in the last section of chapter 2.

## Chapter 2 : $\gamma$ -boundedness of $C_0$ -semigroups and their $H^\infty$ -functional calculi

In this chapter we study  $\gamma$ -bounded  $C_0$ -semigroups, that is  $C_0$ -semigroups  $(T_t)_{t \geq 0}$  such that  $\{T_t : t \geq 0\}$  is  $\gamma$ -bounded. Until now, the latter have been only studied through  $\gamma$ -sectorial operators which are (negative) generators of  $\gamma$ -bounded  $C_0$ -semigroups (see chapter 1). To that end we give analogous results of [9] in the setting of weak  $\gamma$ -boundedness. We will introduce a  $\gamma$ -version of Gomilko Shi-Feng condition : there exists  $C > 0$  such that for  $N \in \mathbb{N}^*$ ,  $\alpha_1, \dots, \alpha_N < 0$ , and for each  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in X^*$ , we have

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} | \langle (-\alpha_k) R(\alpha_k + it, A)^2 x_k, y_k \rangle | dt \\ & \leq C \left( \int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s) x_k \right\|_X ds \right) \left( \int_{\Omega} \left\| \sum_{k=1}^N \gamma_k(s) y_k \right\|_{X^*} ds \right). \end{aligned} \quad (0.19)$$

A first important result is that if  $X$  is  $K$ -convex then, for each  $m \in \mathbb{N}$  the above condition is equivalent to the fact that  $A$  admits a  $m$ -derivative  $\gamma$ -bounded functional calculus, that is  $A$  admits a  $m$ -derivative bounded functional calculus and the set

$$\bigcup_{\alpha < 0} \{(-\alpha)^m f^{(m)}(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1\}$$

is  $\gamma$ -bounded. We will deduce an extension of the Gomilko Shi-Feng Theorem in  $K$ -convex Banach spaces. In particular in  $L^p$ -space for  $1 < p \neq 2 < \infty$  (we will denote by  $q$  the conjugate of  $p$ ) the theorem is stated as follows :

**Theorem.** *Let  $X = L^p$  with  $1 < p \neq 2 < \infty$ . Assume that  $A$  is of half-plane type on  $X$ . Then the following assertions are equivalent*

- (i)  $-A$  is the generator of a  $\gamma$ -bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$ ;
- (ii) there exists  $C > 0$  such that for each  $N \in \mathbb{N}$ ,  $x_1, \dots, x_N \in L^p$ ,  $y_1, \dots, y_N \in L^q$ , and each  $\alpha_1, \dots, \alpha_N < 0$ , we have

$$\left\| \left( \sum_{k=1}^N \int_{\mathbb{R}} (-\alpha_k) |R(\alpha_k + it, A)x_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| \left( \sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

and

$$\left\| \left( \sum_{k=1}^N \int_{\mathbb{R}} (-\alpha_k) |R(\alpha_k + it, A^*)y_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \left\| \left( \sum_{k=1}^N |y_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q}.$$

Thanks to this result we are able to prove a  $\gamma$ -bounded version of the Gearhart-Prüss Theorem whose a direct corollary is the following :

**Theorem.** *Let  $(T_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $K$ -convex space. If there exists  $\delta > 0$  such that  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded, then  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded for all  $\delta > 0$ .*

Furthermore, using results of chapter 1, we will be able to make this result more precise this result by giving an example of  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  such that for each  $\delta > 0$ ,  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded while  $\{T_t : t \geq 0\}$  is not  $\gamma$ -bounded.

*In appendix we will see, using  $\gamma$ -spaces introduced in [29], that the assumption of  $K$ -convexity is not necessary.*

This chapter is constructed as follows. In section 2 we will give preliminaries. In particular we will recall how the holomorphic functional calculus of half-plane type operators is obtained and we will give key results about  $\gamma$ -boundedness and  $\gamma$ -radonifying operators. The section 3 is dedicated to show that condition (0.19) is equivalent to  $m$ -derivative  $\gamma$ -bounded on a  $K$ -convex space. In section 4 we will state our version of Gomilko Shi-Feng Theorem and in section 5 we will use this Theorem to show the  $\gamma$ -version of Gearhart-Prüss Theorem. Finally in section 6 we will sup up our results and we will give some counter-examples.

### Chapter 3 : Derivative bounded functional calculus of power bounded operators on Banach spaces

The aim of this chapter is to give discrete analogues of results of Batty, Haase and Mubeen [9] and of results of chapter 2. A discrete version of Gomilko Shi-Feng is given in [17]. They introduce which we will call discrete Gomilko Shi-Feng condition : for  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ , there exists  $C > 0$  such that

$$\int_0^{2\pi} |\langle R(re^{it}, T)^2 x, x^* \rangle| dt \leq \frac{C}{(r+1)(r-1)} \|x\| \|x^*\|, \quad r > 1, x \in X, x^* \in X^*. \quad (0.20)$$

A first simple observation is that an operator satisfying (0.20) is power-bounded, that is  $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$ . We will draw a parallel with derivative bounded functional calculus. We

will say that an operator  $T \in B(X)$  satisfying  $\sigma(T) \subset \overline{\mathbb{D}}$  admits a discrete  $m$ -derivative bounded functional calculus if there exists  $C > 0$  such that

$$\left\| \phi^{(m)}(T) \right\| \leq \frac{C}{(r-1)^m} \|\phi\|_{H^\infty(r\mathbb{D})} \quad \text{for each } \phi \in H^\infty(r\mathbb{D}) \text{ and } r > 1.$$

We will establish a result similar to [9], that is for each  $m \in \mathbb{N}^*$ ,  $T$  satisfies the discrete Gomilko Shi-Feng condition (0.20) if and only if  $T$  admits a discrete  $m$ -derivative functional calculus. Moreover, we will recover the known result (see [17]) that on Hilbert space,  $T$  is power-bounded if and only if  $T$  has the discrete Gomilko Shi-Feng condition (0.20).

We will also study a Besov calculus under the discrete Gomilko Shi-Feng condition (0.20) (we will be inspired by the results of Batty, Gomilko and Tomilov [7, 8]). We say that  $f : \mathbb{D} \rightarrow \mathbb{C}$  is in  $\mathcal{B}(\mathbb{D})$  if  $f$  is holomorphic on  $\mathbb{D}$  and satisfies

$$\int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du < \infty.$$

We equip  $B(\mathbb{D})$  with the norm  $\|\cdot\|$  defined by

$$\forall f \in \mathcal{B}(\mathbb{D}), \quad \|f\|_{\mathcal{B}(\mathbb{D})} = \int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du + \sup_{z \in \mathbb{D}} |f(z)|.$$

Peller shows in [47] that if  $T$  is power bounded on a Hilbert space, then there exists  $C > 0$  such that for each polynomial  $P$  we have

$$\|P(T)\| \leq C \|P\|_{\mathcal{B}(\mathbb{D})}.$$

We will be able to extend this result to each operator  $T$  satisfying the discrete Gomilko Shi-Feng condition (0.20). Furthermore we will show that for each  $f \in \mathcal{B}(\mathbb{D})$  and each  $x \in X$  and  $x^* \in X^*$  we have

$$\langle f(T)x, x^* \rangle = \frac{1}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} f'\left(\frac{e^{it}}{r}\right) \langle R^2(re^{it}, T)x, x^* \rangle dt dr + \langle f(0)x, x^* \rangle.$$

The second part of this chapter will focus on discrete analogues of chapter 2. In particular we will study power  $\gamma$ -bounded operators  $T$ , that is, operators  $T$  such that the set  $\{T^n : n \in \mathbb{N}^*\}$  is  $\gamma$ -bounded. We will show that these operators satisfy the discrete Gomilko Shi-Feng condition (0.20). We will obtain a characterisation of power  $\gamma$ -bounded operators on Banach space. On  $L^p$ -spaces, this characterisation is stated as follows:

**Theorem.** *Let  $X = L^p$  with  $1 < p \neq 2 < \infty$ . Let  $T$  be a bounded operator on  $X$ . The following assertions are equivalent:*

- (i) *The set  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded;*
- (ii) *The spectrum  $\sigma(T)$  is included in  $\overline{\mathbb{D}}$  and there exists  $C > 0$  such that for each  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in L^p$  and  $y_1, \dots, y_N \in L^q$*

$$\left\| \left( \sum_{k=1}^N \int_0^{2\pi} (r_k^2 - 1) |R(r_k e^{it}, T)x_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^p} \leq C \left\| \left( \sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

and

$$\left\| \left( \sum_{k=1}^N \int_0^{2\pi} (r_k^2 - 1) |R(r_k e^{it}, T^*)y_k|^2 dt \right)^{\frac{1}{2}} \right\|_{L^q} \leq C \left\| \left( \sum_{k=1}^N |y_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q}.$$

This chapter is written as follows. In section 2 we will show that the Gomilko Shi-Feng condition is equivalent to discrete  $m$ -derivative bounded functional calculus while section 3 is dedicated to Besov calculus. Finally in section 4 we will study the power  $\gamma$ -bounded operators on a Banach space. Furthermore we will show that power  $\gamma$ -bounded operators admit a Peller calculus (see [47, section 3]) which is  $\gamma$ -bounded.

**Chapter 4 : Functional calculus for a bounded  $C_0$ -semigroup on Hilbert space**

We mainly focus on (negative) generators of bounded  $C_0$ -semigroup on a Hilbert space. Let  $H$  be a Hilbert space and  $-A$  be the generator of a  $C_0$ -semigroup on  $H$ . For  $b \in L^1(\mathbb{R}_+)$ , its Laplace transform  $L_b$  is defined by

$$L_b : \mathbb{C}_+ \rightarrow \mathbb{C}, \quad z \mapsto \int_0^\infty b(t)e^{-tz} dt,$$

and we can associate the operator  $L_b(A) \in B(H)$  defined by

$$L_b(A) = \int_0^\infty b(t)T_t dt.$$

This is the Hille-Phillips calculus. Obviously we have

$$\|L_b(A)\| \leq C \|b\|_1, \quad b \in L^1(\mathbb{R}_+),$$

where  $C = \sup_{t \geq 0} \|T_t\|$ . If  $(T_t)_{t \geq 0}$  is similar to a contractive semigroup then there exists  $C > 0$  such that

$$\|L_b(A)\| \leq C \|\hat{b}\|_\infty,$$

where  $\hat{b}$  is the Fourier transform of  $b$ . It turns out there exists negative generators of bounded  $C_0$ -semigroup on Hilbert space which do not satisfy such and estimate. But [21, Corollaire 5.5] shows that for all negative generators  $A$  of bounded  $C_0$ -semigroups on Hilbert spaces, there exists  $C > 0$  such that

$$\|L_b(A)\| \leq C \|b\|_{\mathcal{B}}. \tag{0.21}$$

The motivation of this chapter is the search of a sharp estimate for  $\|L_b(A)\|$ .

For this purpose we introduce Banach algebras which are analytic Figa-Talamanca - Herz algebras (see [12]). We denote by  $\mathcal{A}_0(\mathbb{C}_+)$  (respectively  $\mathcal{A}(\mathbb{C}_+)$ ) the set of all functions  $F : \overline{\mathbb{C}_+} \rightarrow \mathbb{C}$  such that there exists two sequences  $(f_k)_{k \in \mathbb{N}}$  of  $C_0(\mathbb{R})$  (respectively  $BUC(\mathbb{R})$ ) and  $(h_k)_{k \in \mathbb{N}}$  in  $H^1(\mathbb{R})$  satisfying

$$\sum_{k=1}^\infty \|f_k\|_\infty \|h_k\|_1 < \infty \tag{0.22}$$

and

$$\forall z \in \overline{\mathbb{C}_+}, \quad F(z) = \sum_{k=1}^\infty \widehat{f_k \star h_k}(iz). \tag{0.23}$$

A first observation is that if  $F$  is an element of  $\mathcal{A}_0(\mathbb{C}_+)$  (respectively  $\mathcal{A}(\mathbb{C}_+)$ ) then  $F|_{\mathbb{C}_+} \in H^\infty(\mathbb{C}_+)$  and  $x \mapsto F(ix)$  is in  $C_0(\mathbb{R})$  (respectively  $BUC(\mathbb{R})$ ). We will show that  $\mathcal{A}(\mathbb{C}_+)$  and  $\mathcal{A}_0(\mathbb{C}_+)$  are Banach algebras. Further  $\mathcal{B}(\mathbb{C}_+)$  is continuously embedded in  $\mathcal{A}(\mathbb{C}_+)$  and different from  $\mathcal{A}(\mathbb{C}_+)$ . The main result of this chapter is the following:



**Theorem.** *There exists a unique bounded algebras homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$  such that*

$$\rho_A(L_b) = \int_0^\infty b(t)T_t dt \quad (0.24)$$

for each  $b \in L^1(\mathbb{R}_+)$ .

Since  $\mathcal{A}(\mathbb{C}_+)$  and  $\mathcal{B}(\mathbb{C}_+)$  are different and  $\mathcal{B}(\mathbb{C}_+)$  is continuously embedded in  $\mathcal{A}(\mathbb{C}_+)$ , this result improves (0.21).

In order to prove that  $\mathcal{B}(\mathbb{C}_+)$  and  $\mathcal{A}(\mathbb{C}_+)$  are different we will need to determine the dual of  $\mathcal{A}_0(\mathbb{C}_+)$ . Let  $m \in L^\infty(\mathbb{R}_+)$ . For  $h \in H^2(\mathbb{R})$ , the map  $m\hat{h}$  is in  $L^2(\mathbb{R}_+)$  and then its fourier inverse  $\mathcal{F}^{-1}(m\hat{h})$  is in  $H^2(\mathbb{R})$ . If

$$\begin{aligned} T_m &: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \\ h &\mapsto \mathcal{F}^{-1}(m\hat{h}) \end{aligned}$$

is bounded with respect to  $H^1(\mathbb{R})$ -norm then  $T_m$  will be called a bounded Fourier multiplier on  $H^1(\mathbb{R})$  and we denote by  $\mathcal{M}(H^1(\mathbb{R}))$  the linear space of all bounded Fourier multipliers. We will state the following result which characterizes bounded Fourier multiplier on  $H^1(\mathbb{R})$ . In the next statement,  $\tau_s$  is defined by  $(\tau_s f)(t) = f(t - s)$ :

**Theorem.** *Let  $T \in B(H^1(\mathbb{R}))$ . If  $T$  commutes with translations, that is, for each  $s \in \mathbb{R}$ ,  $T\tau_s = \tau_s T$ , then there exists a bounded and continuous function  $m: \mathbb{R}_+ \rightarrow \mathbb{C}$  such that  $T = T_m$ . Furthermore in this case one has*

$$\|m\|_\infty \leq \|T\|.$$

Now denote by  $\mathcal{PM}$  the closure of  $\{h \mapsto \mu \star h\} \subset B(H^1(\mathbb{R}))$  with respect to the  $w^*$ -topology (We can naturally regard  $B(H^1(\mathbb{R}))$  as a dual space). We are able to show that  $\mathcal{PM} \subset \mathcal{M}(H^1(\mathbb{R}))$  and

$$\mathcal{A}_0(\mathbb{C}_+)^* = \mathcal{PM}.$$

We will conclude this chapter in studying again the case of  $\gamma$ -bounded  $C_0$ -semigroups and we will show that we can similarly defined a  $\mathcal{A}(\mathbb{C}_+)$ -functional calculus for them. Further this functional calculus turns out to be  $\gamma$ -bounded. More precisely we have the following result:

**Theorem.** *Let  $X$  be a Banach space and  $(T_t)_{t \geq 0}$  a  $\gamma$ -bounded  $C_0$ -semigroup on  $X$ . Then there exists a unique bounded homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$  such that  $\rho_A$  satisfies (0.24) and the set*

$$\{\rho_A(f) : f \in \mathcal{A}(\mathbb{C}_+), \|f\|_{\mathcal{A}} \leq 1\}$$

is  $\gamma$ -bounded.

This chapter is constructed as following. In section 2 we study Fourier multipliers on Hardy space  $H^1(\mathbb{R})$ . In section 3 we introduce our new algebras  $\mathcal{A}_0(\mathbb{C}_+)$  and  $\mathcal{A}(\mathbb{C}_+)$  and we give their main properties. Next we determine the dual of  $\mathcal{A}_0(\mathbb{C}_+)$ . In section 4 we construct our functional calculus and we state our main result. In section 5 we show that  $\mathcal{B}(\mathbb{C}_+)$  and  $\mathcal{A}(\mathbb{C}_+)$  are different. Finally section 6 is dedicaced to extend our main result to the Banach setting.

# Chapter 1

## New counterexamples on Ritt operators, sectorial operators and $R$ -boundedness

### 1.1. Introduction

$R$ -boundedness plays a prominent role in the study of sectorial operators and Ritt operators. Namely the notions of  $R$ -sectorial operators and  $R$ -Ritt operators have been instrumental in the development of  $H^\infty$ -functional calculus, square function estimates and applications to maximal regularity and to many other aspects of the harmonic analysis of semigroups (in either the continuous or the discrete case).

The existence of sectorial operators which are not  $R$ -sectorial was discovered by Kalton and Lancien in their paper solving the  $L^p$ -maximal regularity problem [30]. The existence of Ritt operators which are not  $R$ -Ritt was established a bit later by Portal [52]. More recently, Fackler [14] extended the work of Kalton-Lancien in various directions. In contrast with [30], which focused on existence results, [14] supplied explicit constructions of sectorial operators which are not  $R$ -sectorial. Further it is easy to derive from the latter paper explicit constructions of Ritt operators which are not  $R$ -Ritt. In [14, 30, 52], sectorial operators which are not  $R$ -sectorial (resp. Ritt operators which are not  $R$ -Ritt) are defined as multipliers with respect to Schauder decompositions having various "bad" properties. In particular, these Schauder decompositions cannot be  $R$ -Schauder (see Lemma 1.2.2).

The aim of this note is two-fold. First we show that given *any* Schauder decomposition  $\mathbb{D}$  which is not  $R$ -Schauder, one can define a sectorial operator  $A$  which is a multiplier with respect to  $\mathbb{D}$  and which is not  $R$ -sectorial (resp. a Ritt operator  $T$  which is a multiplier with respect to  $\mathbb{D}$  and which is not  $R$ -Ritt). Second we strengthen these negative results in both cases by showing that  $A$  can be chosen such that  $\{e^{-tA} : t \geq 0\}$  is not  $R$ -bounded and  $A$  is bounded, whereas  $T$  is taken such that  $\{T^n : n \geq 0\}$  is not  $R$ -bounded. (See Remark 1.3.4 for more comments.)

In addition to the above mentioned papers, we refer the reader to [10, 32, 39, 58] for relevant information on  $R$ -sectorial and  $R$ -Ritt operators. We also mention [33] which contains examples of Ritt operators which are not  $R$ -Ritt. They are of a different nature to those in [52].

## 1.2. Preliminaries

In this section we introduce the relevant definitions and constructions to be used in this paper, as long as preliminary facts. Throughout we let  $X$  be a complex Banach space and we let  $B(X)$  denote the Banach algebra of all bounded operators on  $X$ . We let  $I_X$  denote the identity operator on  $X$ .

Let  $(\varepsilon_j)_{j \geq 1}$  be an independent sequence of Rademacher variables on some probability space  $(\Omega, d\mathbb{P})$ . Given any  $x_1, \dots, x_k$  in  $X$ , we set

$$\left\| \sum_{j=1}^k \varepsilon_j \otimes x_j \right\|_{R,X} = \int_{\Omega} \left\| \sum_{j=1}^k \varepsilon_j(u) x_j \right\|_X d\mathbb{P}(u).$$

Then we say that a subset  $F \subset B(X)$  is  $R$ -bounded provided that there exists a constant  $K \geq 0$  such that for any  $k \geq 1$ , for any  $T_1, \dots, T_k$  in  $F$  and for any  $x_1, \dots, x_k$  in  $X$ ,

$$\left\| \sum_{j=1}^k \varepsilon_j \otimes T_j(x_j) \right\|_{R,X} \leq K \left\| \sum_{j=1}^k \varepsilon_j \otimes x_j \right\|_{R,X}.$$

We refer the reader to e.g. [26, Chap. 8] for basic information on  $R$ -boundedness.

For any  $\omega \in (0, \pi)$ , we let  $\Sigma_{\omega} = \{\lambda \in \mathbb{C}^* : |\text{Arg}(\lambda)| < \omega\}$ . Let  $A$  be a densely defined closed operator  $A: D(A) \rightarrow X$ , with domain  $D(A) \subset X$ . Let  $\sigma(A)$  denote the spectrum of  $A$  and let  $R(\lambda, A) = (\lambda I_X - A)^{-1}$  denote the resolvent operator for  $\lambda \notin \sigma(A)$ . We say that  $A$  is sectorial of type  $\omega \in (0, \pi)$  if  $\sigma(A) \subset \overline{\Sigma_{\omega}}$  and for any  $\theta \in (\omega, \pi)$ , the set

$$\{\lambda R(\lambda, A) : \lambda \in \mathbb{C} \setminus \overline{\Sigma_{\theta}}\} \quad (1.2.1)$$

is bounded. We further say that  $A$  is sectorial of type 0 if it is sectorial of type  $\omega$  for any  $\omega \in (0, \pi)$ .

Note that if  $A$  is sectorial of type  $\omega$  and  $A$  is invertible, then  $A^{-1} \in B(X)$  is sectorial of type  $\omega$  as well. This readily follows from the fact that for any  $\lambda \notin \overline{\Sigma_{\omega}}$ , we have  $\lambda^{-1} \notin \overline{\Sigma_{\omega}}$  and

$$\lambda R(\lambda, A^{-1}) = I_X - \lambda^{-1} R(\lambda^{-1}, A). \quad (1.2.2)$$

We recall that  $A$  is sectorial of type  $< \frac{\pi}{2}$  if and only if  $-A$  generates a bounded analytic semigroup. In this case, the latter is denoted by  $(e^{-tA})_{t \geq 0}$ .

Next we say that  $A$  is  $R$ -sectorial of  $R$ -type  $\omega \in (0, \pi)$  if  $A$  is sectorial of type  $\omega$  and for any  $\theta \in (\omega, \pi)$ , the set (1.2.1) is  $R$ -bounded.

The following lemma is a straightforward consequence of (1.2.2).

**Lemma 1.2.1.** *Let  $A$  be  $R$ -sectorial of  $R$ -type  $\omega$  and assume that  $A$  is invertible. Then  $A^{-1}$  is also  $R$ -sectorial of  $R$ -type  $\omega$ .*

Let  $A$  be a sectorial operator of type  $< \frac{\pi}{2}$ . We recall that by [58],  $A$  is  $R$ -sectorial of  $R$ -type  $< \frac{\pi}{2}$  if and only if the two sets

$$\{e^{-tA} : t \geq 0\} \quad \text{and} \quad \{tAe^{-tA} : t \geq 0\} \quad (1.2.3)$$

are  $R$ -bounded.

Let  $T \in B(X)$ . We say that  $T$  is a Ritt operator if the two sets

$$\{T^n : n \geq 0\} \quad \text{and} \quad \{nT^n(I_X - T) : n \geq 1\} \quad (1.2.4)$$

are bounded. We further say that  $T$  is  $R$ -Ritt if these two sets are  $R$ -bounded.

These notions are closely related to sectoriality. Indeed let  $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$  be the open unit disc. Then  $T$  is a Ritt operator if and only if  $\sigma(T) \subset \mathbb{D} \cup \{1\}$  and  $I_X - T$  is sectorial of type  $< \frac{\pi}{2}$ . Further in this case,  $T$  is  $R$ -Ritt if and only if  $I_X - T$  is  $R$ -sectorial of  $R$ -type  $< \frac{\pi}{2}$ . We refer the reader to [10, 39] and the references therein for these results and various informations on Ritt operators and their applications.

We recall from [42, Section 1.g] that a Schauder decomposition on  $X$  is a sequence  $\mathbb{D} = \{X_n : n \geq 1\}$  of closed subspaces of  $X$  such that for any  $x \in X$ , there exists a unique sequence  $(x_n)_{n \geq 1}$  of  $X$  such that  $x_n \in X_n$  for any  $n \geq 1$  and  $x = \sum_{n=1}^{\infty} x_n$ . For any  $n \geq 1$ , we let  $p_n \in B(X)$  be the projection defined for  $x$  as above by  $p_n(x) = x_n$ . For any integer  $N \geq 1$ , consider their sum  $P_N = \sum_{n=1}^N p_n$ . This is a projection and the set

$$\{P_N : N \geq 1\} \tag{1.2.5}$$

is bounded.

We say that  $\mathbb{D}$  is an  $R$ -Schauder decomposition if this set is actually  $R$ -bounded. Then a Schauder basis is called  $R$ -Schauder if its associated Schauder decomposition is  $R$ -Schauder.

Let  $c = (c_n)_{n \geq 1}$  be a sequence of complex numbers. Assume that the sum  $\sum_{n=1}^{\infty} |c_n - c_{n+1}|$  is finite (in which case we say that the sequence has a bounded variation). Then  $c$  has a limit. Let  $\ell_c$  denote this limit and set

$$\text{var}(c) = |\ell_c| + \sum_{n=1}^{\infty} |c_n - c_{n+1}|.$$

For any  $x \in X$ , the series  $\sum_n c_n p_n(x)$  converges. This follows from an Abel transformation argument, using the boundedness of  $\{P_N : N \geq 1\}$ . Let  $M_c : X \rightarrow X$  be defined by  $M_c(x) = \sum_{n=1}^{\infty} c_n p_n(x)$ , then we actually have

$$M_c(x) = \ell_c x + \sum_{N=1}^{\infty} (c_N - c_{N+1}) P_N(x), \quad x \in X. \tag{1.2.6}$$

This implies that

$$\|M_c\| \leq \text{var}(c) \sup_{N \geq 1} \|P_N\|. \tag{1.2.7}$$

Let  $(a_n)_{n \geq 1}$  be a nondecreasing sequence of  $(0, \infty)$ . Then we may define an operator  $A : D(A) \rightarrow X$  as follows. We let  $D(A)$  be the space of all  $x \in X$  such that the series  $\sum_n a_n p_n(x)$  converges and for any  $x \in D(A)$ , we set

$$A(x) = \sum_{n=1}^{\infty} a_n p_n(x). \tag{1.2.8}$$

Such operators were first introduced in [6, 56]. It is well-known that

$$\sigma(A) = \overline{\{a_n : n \geq 1\}} \tag{1.2.9}$$

and that  $A$  is a sectorial operator of type 0 (see [30, 34]). More precisely, for any  $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ ,  $R(\lambda, A)$  is the operator  $M_{c(\lambda)}$  associated with the sequence  $c(\lambda) = ((\lambda - a_n)^{-1})_{n \geq 1}$  and for any  $\theta \in (0, \pi)$ , we have

$$K_\theta = \sup\{|\lambda| \text{var}(c(\lambda)) : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}\} < \infty, \tag{1.2.10}$$

see e.g. [34, Section 2]. This estimate and (1.2.7) show that  $A$  is sectorial of type 0.

We note for further use that by (1.2.9), the above operator  $A$  is invertible.

In the sequel, any sectorial operator  $A$  of this form will be called a  $\mathbb{D}$ -multiplier.

Likewise let  $c = (c_n)_{n \geq 1}$  be a nondecreasing sequence of  $(0, 1)$ . Then  $c$  has a bounded variation, which allows the definition of  $T = M_c \in B(X)$  given by

$$T(x) = \sum_{n=1}^{\infty} c_n p_n(x), \quad x \in X. \quad (1.2.11)$$

It turns out that  $T$  is a Ritt operator on  $X$ . Indeed, let  $A$  be the sectorial operator (1.2.8) associated with the sequence  $(a_n)_{n \geq 1}$  defined by  $a_n = (1 - c_n)^{-1}$ . Then  $I_X - T = A^{-1}$  is sectorial of type 0 and  $\sigma(T) \subset [0, 1]$ , which ensures that  $T$  is a Ritt operator.

In the sequel, any Ritt operator  $T$  of this form will be called a  $\mathbb{D}$ -multiplier.

The following is well-known to specialists.

**Lemma 1.2.2.** *Let  $\mathbb{D} = \{X_n : n \geq 1\}$  be an  $R$ -Schauder decomposition on  $X$ .*

(a) *Any sectorial operator  $A$  on  $X$  which is a  $\mathbb{D}$ -multiplier is  $R$ -sectorial of  $R$ -type 0.*

(b) *Any Ritt operator  $T \in B(X)$  which is a  $\mathbb{D}$ -multiplier is  $R$ -Ritt.*

*Proof.* Let  $F = \{P_N : N \geq 1\}$ . Let  $A$  be given by (1.2.8) and let  $\theta \in (0, \pi)$ . We may assume that  $\lim_n a_n = \infty$ . It follows from the above discussion that for any  $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ ,

$$[\lambda R(\lambda, A)](x) = \sum_{N=1}^{\infty} \lambda (c(\lambda)_N - c(\lambda)_{N+1}) P_N(x), \quad x \in X,$$

with  $c(\lambda) = ((\lambda - a_n)^{-1})_{n \geq 1}$ . This implies that

$$\{\lambda R(\lambda, A) : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\theta}\} \subset K_\theta \cdot \overline{\text{aco}}^{so}(F),$$

where  $K_\theta$  is given by (1.2.10) and  $\overline{\text{aco}}^{so}(F)$  stands for the the closure of the absolute convex hull of  $F$  in the strong operator topology of  $B(X)$ . Since  $F$  is  $R$ -bounded,  $\overline{\text{aco}}^{so}(F)$  is  $R$ -bounded as well, see e.g. [26, Subsection 8.1.e]. Then the set (1.2.1) is  $R$ -bounded, which shows (a).

Let  $T$  be given by (1.2.11). It follows from the above discussion that  $T = I_X - A^{-1}$  for some sectorial operator  $A$  on  $X$  which is a  $\mathbb{D}$ -multiplier. By (a) and Lemma 1.2.1,  $A^{-1}$  is  $R$ -sectorial of type  $< \frac{\pi}{2}$ . This entails that  $T$  is  $R$ -Ritt.  $\square$

### 1.3. Main result

Our main result is the following.

**Theorem 1.3.1.** *Let  $\mathbb{D}$  be a Schauder decomposition on  $X$  and assume that  $\mathbb{D}$  is not  $R$ -Schauder.*

(a) *There exists a sectorial operator  $A$  on  $X$  which is a  $\mathbb{D}$ -multiplier, such that the set*

$$\{e^{-tA^{-1}} : t \geq 0\}$$

*is not  $R$ -bounded.*

(b) There exists a Ritt operator  $T \in B(X)$  which is a  $\mathbb{D}$ -multiplier, such that the set

$$\{T^n : n \geq 0\}$$

is not  $R$ -bounded.

*Proof.* We introduce  $Q_N = I_X - P_N$  for any  $N \geq 1$ . The idea of the proof is to construct  $A$  (resp.  $T$ ) such that each  $Q_N$  is close to  $e^{-tA^{-1}}$  for some  $t \geq 0$  (resp. to  $T^n$  for some  $n \geq 0$ ).

Let  $c = (c_n)_{n \geq 1}$  be a complex sequence with a bounded variation and let  $N \geq 1$  be a fixed integer. For any  $x \in X$ ,  $Q_N(x) = \sum_{n=N+1}^{\infty} p_n(x)$  hence

$$(M_c - Q_N)(x) = \sum_{n=1}^N c_n p_n(x) + \sum_{n=N+1}^{\infty} (c_n - 1)p_n(x).$$

On the one hand, we have

$$\sum_{n=1}^N c_n p_n(x) = c_N P_N(x) + \sum_{n=1}^{N-1} (c_n - c_{n+1}) P_n(x).$$

On the other hand,

$$\begin{aligned} \sum_{n=N+1}^{\infty} (c_n - 1)p_n(x) &= \sum_{n=N+1}^{\infty} (c_n - 1)(Q_{n-1}(x) - Q_n(x)) \\ &= (c_{N+1} - 1)Q_N(x) + \sum_{n=N+1}^{\infty} (c_{n+1} - c_n)Q_n(x). \end{aligned}$$

Let  $K = \sup_{N \geq 1} \|P_N\|$ . It follows from these identities that

$$\|M_c - Q_N\| \leq (1 + K) \left( \sum_{n=1}^{N-1} |c_{n+1} - c_n| + |c_N| + |1 - c_{N+1}| + \sum_{n=N+1}^{\infty} |c_{n+1} - c_n| \right).$$

Let  $(a_n)_{n \geq 1}$  be a nondecreasing sequence of  $(0, \infty)$ , with  $\lim_n a_n = \infty$ , and let  $A$  be the associated sectorial operator defined by (1.2.8). Let  $t > 0$  and apply the above with

$$c_n = e^{-ta_n^{-1}}, \quad n \geq 1.$$

Then  $c_n \in (0, 1)$  for any  $n \geq 1$ , the sequence  $(c_n)_{n \geq 1}$  is nondecreasing (hence has a bounded variation) and  $M_c = e^{-tA^{-1}}$ . Further  $\lim_n c_n = 1$ . Consequently we have

$$|c_N| = c_N = e^{-ta_N^{-1}}, \quad |1 - c_{N+1}| = 1 - c_{N+1} = 1 - e^{-ta_{N+1}^{-1}},$$

$$\sum_{n=1}^{N-1} |c_{n+1} - c_n| = c_N - c_1 \leq c_N = e^{-ta_N^{-1}},$$

$$\sum_{n=N+1}^{\infty} |c_{n+1} - c_n| = 1 - c_{N+1} = 1 - e^{-ta_{N+1}^{-1}},$$

and hence

$$\|e^{-tA^{-1}} - Q_N\| \leq 2(1 + K)(e^{-ta_N^{-1}} + (1 - e^{-ta_{N+1}^{-1}})). \quad (1.3.1)$$

We apply the above with

$$a_n = (n!)^3, \quad n \geq 1.$$

Next we consider the sequence  $(t_N)_{N \geq 1}$  of positive integers given by

$$t_N = N(N!)^3, \quad N \geq 1.$$

Then by (1.3.1), we have

$$\|e^{-t_N A^{-1}} - Q_N\| \leq 2(1+K)(e^{-N} + (1 - e^{-\frac{N}{(N+1)^3}})) \leq 2(1+K)(e^{-N} + N^{-2})$$

for any  $N \geq 1$ . This estimate implies that

$$\sum_{N=1}^{\infty} \|e^{-t_N A^{-1}} - Q_N\| < \infty.$$

Let  $C$  be the above sum. Then for any  $x_1, \dots, x_k$  in  $X$ , we have

$$\begin{aligned} \left\| \sum_{N=1}^k \varepsilon_N \otimes (e^{-t_N A^{-1}} - Q_N)(x_N) \right\|_{R,X} &\leq \sum_{N=1}^k \|e^{-t_N A^{-1}} - Q_N\| \|x_N\| \\ &\leq C \sup\{\|x_N\| : 1 \leq N \leq k\} \\ &\leq C \left\| \sum_{N=1}^k \varepsilon_N \otimes x_N \right\|_{R,X}. \end{aligned}$$

Hence

$$\left\| \sum_{N=1}^k \varepsilon_N \otimes Q_N(x_N) \right\|_{R,X} \leq \left\| \sum_{N=1}^k \varepsilon_N \otimes e^{-t_N A^{-1}}(x_N) \right\|_{R,X} + C \left\| \sum_{N=1}^k \varepsilon_N \otimes x_N \right\|_{R,X}.$$

By assumption, the set  $\{P_N : N \geq 1\}$  is not  $R$ -bounded, hence  $\{Q_N : N \geq 1\}$  is not  $R$ -bounded. The above estimate therefore shows that the set  $\{e^{-t A^{-1}} : t \geq 0\}$  cannot be  $R$ -bounded. This proves (a).

To prove (b), we consider  $T = e^{-A^{-1}}$ . Then  $T$  is the Ritt operator defined by (1.2.11) for the sequence  $c_n = e^{-a_n^{-1}}$ . For any  $N \geq 1$ ,  $t_N$  is an integer and  $T^{t_N} = e^{-t_N A^{-1}}$ . Hence the above argument shows that  $\{T^n : n \geq 1\}$  is not  $R$ -bounded, which proves (b).  $\square$

Theorem 1.3.1 provides a converse to Lemma 1.2.2, as follows.

**Corollary 1.3.2.** *Let  $\mathbb{D}$  be a Schauder decomposition on  $X$ . Then  $\mathbb{D}$  is  $R$ -Schauder if and only if any sectorial operator on  $X$  which is a  $\mathbb{D}$ -multiplier is  $R$ -sectorial, if and only if any Ritt operator on  $X$  which is a  $\mathbb{D}$ -multiplier is  $R$ -Ritt.*

*Proof.* Let  $\mathbb{D}$  be a Schauder decomposition on  $X$  which is not  $R$ -Schauder. Let  $A$  be verifying (a) in Theorem 1.3.1, and let  $B = A^2$ . Then  $B$  is a sectorial operator on  $X$  which is a  $\mathbb{D}$ -multiplier. Assume that  $B$  is  $R$ -sectorial, with some  $R$ -type  $\omega \in (0, \pi)$ . By Lemma 1.2.1, its inverse  $B^{-1}$  is  $R$ -sectorial of  $R$ -type  $\omega$  as well. Hence by [27, Proposition 3.4],  $A^{-1} = (B^{-1})^{\frac{1}{2}}$  is  $R$ -sectorial of  $R$ -type  $\frac{\omega}{2} < \frac{\pi}{2}$ . This implies (see (1.2.3)) that the set  $\{e^{-t A^{-1}} : t \geq 0\}$  is  $R$ -bounded, a contradiction. Hence  $B$  is not  $R$ -sectorial.

Combining the above fact with Theorem 1.3.1 (b) and Lemma 1.2.2, we deduce both ‘if and only if’ results.  $\square$

It follows from [14, 30] that if  $X$  has an unconditional basis and  $X$  is not isomorphic to a Hilbert space, then  $X$  has a Schauder basis which is not  $R$ -Schauder. The above theorem therefore applies to all these spaces.

Further the arguments in [30, Theorem 3.7 & Corollary 3.8] show that we actually have the following.

**Corollary 1.3.3.** *Let  $X$  be isomorphic to a separable Banach lattice and assume that  $X$  is not isomorphic to a Hilbert space.*

- (a) *There exists a bounded sectorial operator  $A$  of type 0 on  $X$  such that  $\{e^{-tA} : t \geq 0\}$  is not  $R$ -bounded.*
- (b) *There exists a Ritt operator  $T \in B(X)$  such that the set  $\{T^n : n \geq 0\}$  is not  $R$ -bounded.*

**Remark 1.3.4.** This final remark compares the above corollary with existing results. Let  $X$  be isomorphic to a separable Banach lattice without being isomorphic to a Hilbert space.

(1) It follows from [52] that there exists a Ritt operator  $T \in B(X)$  such that  $T$  is not  $R$ -Ritt. Recall that by definition,  $T$  is not  $R$ -Ritt if and only if one of the two sets in (1.2.4) is not  $R$ -bounded. Part (b) of Corollary 1.3.3 strengthens [52] by providing a Ritt operator  $T$  on  $X$  for which we know that the first of the two sets in (1.2.4) is not  $R$ -bounded. This is an important step in the understanding of the class of power bounded operators  $T$  such that  $\{T^n : n \geq 0\}$  is  $R$ -bounded. This class will be investigated in chapter 3. We refer to [38] for the study of invertible operators  $T \in B(X)$  such that  $\{T^n : n \in \mathbb{Z}\}$  is  $R$ -bounded.

(2) The existence of a sectorial operators  $A$  of type 0 on  $X$  such that  $\{e^{-tA} : t \geq 0\}$  is not  $R$ -bounded follows from [14]. Part (a) of Corollary 1.3.3 shows that this can be achieved with a bounded  $A$ . Note that the existence of bounded  $A$  such that either  $\{e^{-tA} : t \geq 0\}$  or  $\{tAe^{-tA} : t \geq 0\}$  is not  $R$ -bounded follows from [52]. We refer to chapter 2 for various results on bounded  $C_0$ -semigroups  $(T_t)_{t \geq 0}$  on Banach space such that the set  $\{T_t : t \geq 0\}$  is/is not  $R$ -bounded.





# Chapter 2

## $\gamma$ -boundedness of $C_0$ -semigroups and their $H^\infty$ -functional calculi

### 2.1. Introduction

The  $H^\infty$ -functional calculus for a sectorial operator and a strip-type operator have played an important role in the spectral theory and evolution equations [19]. The  $H^\infty$ -functional calculus for a half-plane type operator is a recent tool studied in [9]. As for sectorial and strip-type operators, it is natural to construct a holomorphic functional calculus for a half-plane type operator  $A$  via the Dunford formula

$$f(A) = \frac{1}{2\pi i} \int_{\partial R} f(\lambda)R(\lambda, A)d\lambda.$$

Here,  $R$  is a half-plane and  $f$  is a bounded analytic function on  $R$  with good properties which ensure that  $f(A)$  is bounded. This construction allows us to define a notion of bounded  $H^\infty$ -functional calculus for a half-plane type operator. Contrary to the bounded  $H^\infty$ -functional calculus for a sectorial or a strip-type operator, the bounded  $H^\infty$ -functional calculus for a half-plane type operator has no characterization with simple estimates, even on Hilbert space. However, a weaker notion of bounded  $H^\infty$ -functional calculus, called  $m$ -bounded functional calculus (Definition 2.2.7) turns out to be equivalent to a condition studied independently by Gomilko [16] and Shi and Feng [53] (Definition 2.2.5), called the GFS condition in this paper. Furthermore, they show that this condition is sufficient for the generation of bounded  $C_0$ -semigroups. Therefore, the set (containing all sectorial operators of type  $< \frac{\pi}{2}$ ) of all half-plane type operators which have GFS is included (equal when  $X$  is Hilbert) in the set of all negative generators of bounded  $C_0$ -semigroups.

The aim of this paper is to study the generation of a  $\gamma$ -bounded  $C_0$ -semigroup on a Banach space  $X$ , that is, a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on a Banach space  $X$  such that the set  $\{T(t) : t \geq 0\}$  is  $\gamma$ -bounded. A first step is to consider a stronger condition than the GFS condition (however equivalent on Hilbert spaces), that we call  $\gamma$ -Gomilko-Shi-Feng (Definition 2.3.4) and abbreviate as the  $\gamma$ -GFS condition. It turns out that the  $\gamma$ -GFS condition is equivalent to a notion of  $\gamma$ -bounded  $m$ -bounded functional calculus. Furthermore the latter condition is sufficient for the generation of weak  $\gamma$ -bounded  $C_0$ -semigroups, and hence for the generation of  $\gamma$ -bounded  $C_0$ -semigroups when the underlying space is  $K$ -convex. Therefore when  $X$  is  $K$ -convex, the set (containing all  $\gamma$ -sectorial operators of  $\gamma$ -type  $< \frac{\pi}{2}$ ) of all operators which satisfy the  $\gamma$ -GFS condition is equal to the set of all negative generators of  $\gamma$ -bounded  $C_0$ -semigroups. This last statement is contained in our main result (Theorem 2.4.1), which gives equivalence of the generation of  $\gamma$ -bounded

$C_0$ -semigroup with the  $\gamma$ - $m$ -bounded functional calculus, as well as with some estimates of the resolvent of the negative generator and its adjoint. From this theorem, we deduce, when  $X$  is  $K$ -convex space, a  $\gamma$ -bounded version of the Gearhart-Prüss Theorem (Corollary 2.5.3) and the following result: if  $(T_t)_{t \geq 0}$  is a bounded  $C_0$ -semigroup on a  $K$ -convex space and if the set  $\{e^{-\delta t} T_t : t \geq 0\}$  is  $\gamma$ -bounded for one  $\delta > 0$ , then it is  $\gamma$ -bounded for each  $\delta > 0$ . Moreover it is possible to find an example of bounded  $C_0$ -semigroup such that  $\{e^{-\delta t} T_t : t \geq 0\}$  is  $\gamma$ -bounded for all  $\delta > 0$  but not for  $\delta = 0$ .

Now we describe the structure of the paper. Section 2 only contains preliminary results. We recall important results of [9], then we collect some results about  $\gamma$ -boundedness and weak  $\gamma$ -boundedness and generalized square functions. In Section 3, we discuss the  $\gamma$ -GFS condition and  $\gamma$ - $m$ -bounded  $H^\infty$ -functional calculus. We give our main result in Section 4, namely a version of the Gomilko-Shi-Feng Theorem on  $K$ -convex spaces. Section 5 is devoted to some consequences of the results of Section 4, in particular we state a version of the Gearhart-Prüss Theorem in  $K$ -convex spaces. Finally, Section 6 is dedicated to an overview of the implications between the different notions of  $H^\infty$ -bounded functional calculus for half-plane type operators and the generation of bounded and  $\gamma$ -bounded  $C_0$ -semigroups.

## 2.2. Background and preliminary results

For any Banach spaces  $X, Y$ , we let  $\mathcal{L}(X, Y)$  denote the space of all bounded linear operators from  $X$  into  $Y$ . If  $Y = X$ , we write  $\mathcal{L}(X)$  instead of  $\mathcal{L}(X, X)$ . If  $A$  is a closed operator on  $X$ , we denote by  $Dom(A)$ ,  $\rho(A)$  and  $\sigma(A)$  the domain, the resolvent set and the spectrum of  $A$ , respectively. When  $\lambda \in \rho(A)$ , we let  $R(\lambda, A) = (\lambda I - A)^{-1}$  denote the corresponding resolvent operator.

### Half-plane type operators

In this subsection we review the definitions of half-plane type operators and their functional calculi, following [9].

We fix a real number  $\omega$ . For any  $\alpha \leq \omega$ , we consider the right open half-plane

$$R_\alpha = \{z \in \mathbb{C} : Re(z) > \alpha\}.$$

**Definition 2.2.1.** Let  $A$  be a closed and densely defined operator on  $X$ . We will say that  $A$  is of *half-plane type*  $\omega$  if  $\sigma(A) \subset \overline{R_\omega}$  and

$$\forall \alpha < \omega, \quad \sup\{\|R(z, A)\| : Re(z) \leq \alpha\} < \infty.$$

We will say that a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  is of type  $\omega \in \mathbb{R}$  if there exists a constant  $C > 0$  such that  $\|T_t\| \leq Ce^{-\omega t}$  for all  $t \geq 0$ . Note that such  $\omega$  always exists [46, Theorem 2.2 section 1.2]. It follows from the Laplace formula (see e.g. [46, Formula (7.1) section 1.7]) that if  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  of type  $\omega$ , then  $A$  is an operator of half-plane type  $\omega$  (in fact  $A$  is even strong half-plane in the sense of [9, Definition 2.1]).

Throughout the rest of this subsection, we let  $A$  be an operator of half-plane type  $\omega$ .

For any  $\alpha \leq \omega$ , let  $H^\infty(R_\alpha)$  be the space of all bounded analytic functions  $f : R_\alpha \rightarrow \mathbb{C}$ , equipped with the norm  $\|f\|_{H^\infty(R_\alpha)} := \sup_{z \in R_\alpha} |f(z)|$ . Then  $H^\infty(R_\alpha)$  is a Banach algebra.

Next we consider the auxiliary space

$$\mathcal{E}(R_\alpha) := \{f \in H^\infty(R_\alpha) : \exists s > 0, f(z) = O(|z|^{-(1+s)}) \text{ as } |z| \rightarrow \infty\}.$$

Whenever  $\alpha < \delta < \omega$  and  $f \in \mathcal{E}(R_\alpha)$ , the integral

$$f(A) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\delta + it)R(\delta + it, A)dt \quad (2.2.1)$$

is absolutely convergent in  $\mathcal{L}(X)$ . Further its value is independent of  $\delta \in (\alpha, \omega)$ . This is due to Cauchy's theorem for vector-valued holomorphic functions.

If  $f \in H^\infty(R_\alpha)$ , we can define a closed, densely defined, operator  $f(A)$  by regularisation as follows (see [9] and [19] for more details). Let  $\mu < \alpha$  and set  $e(z) := (\mu - z)^{-2}$ . Then  $e \in \mathcal{E}(R_\alpha)$ ,  $ef \in \mathcal{E}(R_\alpha)$  and  $e(A) = R(\mu, A)^2$  is injective. Then  $f(A)$  is defined by

$$f(A) = e(A)^{-1}(ef)(A),$$

with  $Dom(f(A))$  equal to the space of all  $x \in X$  such that  $[(ef)(A)](x)$  belongs to the range of  $e(A)$  ( $= Dom(A^2)$ ). It turns out that this definition does not depend on the choice of  $\mu$ .

In particular, for any  $t \geq 0$ , the function  $z \mapsto e^{-tz}$  belongs to  $H^\infty(R_\alpha)$ , hence the above construction provides an operator

$$e^{-tA} := (e^{-tz})(A).$$

The following proposition (see [9, Proposition 2.5]) gives an expected link between the generator of a  $C_0$ -semigroup and its exponential in the previous sense.

**Proposition 2.2.2.** *The operator  $-A$  is the generator of a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  if and only if  $e^{-tA}$  is a bounded operator for all  $t \in [0, 1]$  and  $\sup_{t \in [0, 1]} \|e^{-tA}\| < \infty$ . In this case, we have  $T_t = e^{-tA}$  for all  $t \geq 0$ .*

**Definition 2.2.3.** We say that  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$  if there exists  $C > 0$  such that for any  $\alpha < \omega$  and for any  $f \in H^\infty(R_\alpha)$ ,  $f(A)$  belongs to  $\mathcal{L}(X)$  and

$$\|f(A)\| \leq C \|f\|_{H^\infty(R_\omega)}. \quad (2.2.2)$$

*Remark 2.2.4.* The reasoning at the beginning of [9, Section 5] and the so-called convergence lemma (see [9, Theorem 3.1]) show that to prove that an operator  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$ , it suffices to prove an estimate (2.2.2) for any  $f \in \mathcal{E}(R_\alpha)$  and any  $\alpha < \omega$ .

As a direct consequence of Proposition 2.2.2, we see that if  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$ , then  $-A$  generates a  $C_0$ -semigroup of type  $\omega$ . The converse does not hold true, even on Hilbert space (see Section 6).

The following condition, called Gomilko-Shi-Feng condition (GFS), is important to connect  $C_0$ -semigroups and functional calculi.

**Definition 2.2.5.** Let  $m \geq 1$  be an integer. We say that  $A$  satisfies  $(GFS)_{m, \omega}$  if there exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}} |\langle R(\alpha + it, A)^{m+1}x, y \rangle| dt \leq \frac{C}{(\omega - \alpha)^m} \|x\| \|y\|$$

for any  $\alpha < \omega$ , any  $x \in X$  and any  $y \in X^*$ .

Gomilko [16] and Shi-Feng [53] have shown the following two results: if  $A$  has  $(GFS)_{1,0}$ , then  $-A$  generates a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$ ; conversely if  $X$  is a Hilbert space, the negative generator of a bounded  $C_0$ -semigroup satisfies  $(GFS)_{1,0}$ . This is now known as the Gomilko-Shi-Feng Theorem.

*Remark 2.2.6.* It is easy to check that a sectorial operator of type  $< \frac{\pi}{2}$  has  $(GFS)_{1,0}$ . We refer e.g. to [19] for information about sectorial operators. We recall that  $A$  is a sectorial operator of type  $< \frac{\pi}{2}$  if and only if  $-A$  generates a bounded analytic semigroup and that in this case, there exists a constant  $C > 0$  such that

$$\forall \lambda \in \mathbb{C} \setminus \overline{R_0}, \quad \|\lambda R(\lambda, A)\| \leq C.$$

This implies that for any  $\alpha < 0$  and any  $t \in \mathbb{R}$ ,  $\|R(\alpha + it, B)\| \leq \frac{C}{|\alpha + it|}$ . Hence for any  $\alpha < 0$  and arbitrary  $x \in X$  and  $y \in X^*$ , we have

$$\begin{aligned} (-\alpha) \int_{\mathbb{R}} |\langle R(\alpha + it, A)^2 x, y \rangle| dt &\leq (-\alpha) \int_{\mathbb{R}} \|R(\alpha + it, A)\|^2 dt \|x\| \|y\| \\ &\leq \int_{\mathbb{R}} \frac{-C\alpha}{(-\alpha)^2 + t^2} dt \|x\| \|y\| \\ &= C\pi \|x\| \|y\|, \end{aligned}$$

which proves the result.

It is noticed in [9, Lemma 5.4] that for any  $\alpha < \beta < \omega$  and any  $f \in H^\infty(R_\alpha)$ , the  $m$ -th derivative  $f^{(m)}$  of  $f$  belongs to  $H^\infty(R_\beta)$  for any integer  $m \geq 1$ . Thus it makes sense to define  $f^{(m)}(A)$ .

**Definition 2.2.7.** Let  $m \geq 1$  be an integer. We say that  $A$  has a  $m$ -bounded functional calculus of type  $\omega$  if there exists a constant  $C > 0$  such that for each  $\alpha < \omega$  and each  $f \in H^\infty(R_\alpha)$ ,

$$f^{(m)}(A) \in \mathcal{L}(X) \text{ and } \|f^{(m)}(A)\| \leq \frac{C}{(\omega - \alpha)^m} \|f\|_{H^\infty(R_\alpha)}.$$

In [9], authors use the terminology *strong  $m$ -bounded functional calculus*. For convenience, the word “strong” is omitted here.

The following remarkable results are proved in [9, Theorem 6.4]:  $A$  has  $(GFS)_{m,\omega}$  if and only if  $A$  has a  $m$ -bounded functional calculus of type  $\omega$ , if and only if  $A$  has a 1-bounded functional calculus of type  $\omega$ . (In particular,  $(GFS)_{m,\omega}$  does not depend on  $m$ .) Further if  $A$  has a  $m$ -bounded functional calculus of type  $\omega$ , then  $-A$  generates a  $C_0$ -semigroup of type  $\omega$ . It is further shown in [9, Theorem 7.1] that if  $X$  is a Hilbert space, then conversely,  $A$  has a  $m$ -bounded functional calculus of type  $\omega$  if  $-A$  generates a  $C_0$ -semigroup of type  $\omega$ . This converse is wrong in general, see Section 6 for more on this.

*Remark 2.2.8.* Let us consider the following conditions, for  $\omega \in \mathbb{R}$  :

$$(\omega - \alpha) \int_{\mathbb{R}} \|R(\alpha + it, A)x\|^2 \leq C \|x\|^2, \quad (\alpha < \omega, x \in X) \quad (2.2.3)$$

$$(\omega - \alpha) \int_{\mathbb{R}} \|R(\alpha + it, A^*)x^*\|^2 \leq C \|x^*\|^2, \quad (\alpha < \omega, x^* \in X^*). \quad (2.2.4)$$

Thanks to Cauchy-Schwarz inequality, it is easy to show that (2.2.3) and (2.2.4) together imply  $A$  has  $(GFS)_{m,\omega}$ . So we can see these two conditions as a strong GFS condition. In a Hilbert space, due to our previous observations,  $(GFS)_{m,\omega}$  is equivalent to (2.2.3) and (2.2.4).

The following is implicit in [9].

**Proposition 2.2.9.** *If  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$ , then  $A$  has a  $m$ -bounded functional calculus of type  $\omega$ , for any  $m \geq 1$ .*

*Proof.* According to [9, Lemma 5.4], there is a constant  $K > 0$ , such that for any  $\alpha < \omega$  and any  $f \in H^\infty(R_\alpha)$ , we have  $\|f^{(m)}\|_{H^\infty(R_\omega)} \leq \frac{K}{(\omega - \alpha)^m} \|f\|_{H^\infty(R_\alpha)}$ .

Assume that  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$ . Then for any  $f$  as above,  $f^{(m)}(A) \in \mathcal{L}(X)$  and we have

$$\|f^{(m)}(A)\| \leq C \|f^{(m)}\|_{H^\infty(R_\omega)} \leq \frac{KC}{(\omega - \alpha)^m} \|f\|_{H^\infty(R_\alpha)}.$$

This proves that  $A$  has a  $m$ -bounded functional calculus of type  $\omega$ .  $\square$

### The Sun Dual of a $C_0$ -Semigroup

In this subsection we collect a few facts from [45] which are useful when dealing with non reflexive Banach spaces.

Let  $X$  be a Banach space and let  $Z \subset X^*$  be a closed subspace. We say that  $Z$  is norming if there exists a constant  $c > 0$  such that for any  $x \in X$ ,

$$c \|x\| \leq \sup\{|\langle x, y \rangle| : y \in Z, \|y\| \leq 1\}.$$

Let  $(T_t)_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$ , with generator  $-A$ . It may happen that the dual semigroup  $(T_t^*)_{t \geq 0}$  is not strongly continuous on  $X^*$ . We denote by  $X^\odot$  (pronounced  $X$ -sun) the set

$$X^\odot := \{x \in X^*, \quad \|T_t^* x - x\| \xrightarrow[t \rightarrow 0]{} 0\}. \quad (2.2.5)$$

This set trivially satisfies  $T_t^*(X^\odot) \subset X^\odot$  for every  $t \geq 0$ . Moreover  $X^\odot$  is a closed and weak\*-dense subspace of  $X^*$ . Indeed we have  $X^\odot = \overline{D(A^*)}$ .

We let  $T_t^\odot$  denote the restriction of  $T_t^*$  to  $X^\odot$ . Then by definition,  $(T_t^\odot)_{t \geq 0}$  is a  $C_0$ -semigroup on  $X^\odot$ . It is called the *sun dual* of  $(T_t)_{t \geq 0}$ . Let  $A^\odot$  be its negative generator. Then  $A^\odot$  is the part of  $A^*$  in  $X^\odot$  (see [19, page 6] for definition).

We will use the following two results.

**Theorem 2.2.10.** *Let  $(T_t)_{t \geq 0}$  be a  $C_0$ -semigroup on  $X$ , with generator  $-A$ .*

(1) *We have  $\rho(A) = \rho(A^*) = \rho(A^\odot)$  and  $R(\lambda, A)^* y = R(\lambda, A^*) y = R(\lambda, A^\odot) y$  for all  $\lambda \in \rho(A)$  and for any  $y \in X^\odot$ .*

(2) *The space  $X^\odot$  is norming.*

About (2), we note that more precisely, if we define

$$\|x\|' := \sup\{|\langle x, x^\odot \rangle|, x^\odot \in X^\odot, \|x^\odot\| \leq 1\},$$

and if we let  $M := \overline{\lim}_{t \rightarrow 0} \|T(t)\|$ , then we have

$$\|x\|' \leq \|x\| \leq M \|x\|', \quad x \in X.$$

### $\gamma$ -boundedness on Banach spaces.

In recent years,  $\gamma$ -boundedness and  $\mathcal{R}$ -boundedness played an important role in the operator valued harmonic analysis, multiplier theory and functional calculi (see [26] for more details). Throughout  $X, Y$  denote arbitrary Banach spaces and we let  $(\gamma_n)_{n \geq 1}$  be a sequence of independent complex valued standard Gaussian variables on some probability space  $\Sigma$ . We denote by  $G(X)$  the closure of

$$\left\{ \sum_{k=1}^n \gamma_k \otimes x_k : x_k \in X, n \in \mathbb{N} \right\}$$

in  $L^2(\Sigma, X)$ . For  $x_1, \dots, x_n \in X$ , we let

$$\left\| \sum_{k=1}^n \gamma_k \otimes x_k \right\|_{G(X)} := \left( \int_{\Sigma} \left\| \sum_{k=1}^n \gamma_k(\lambda) x_k \right\|^2 d\lambda \right)^{\frac{1}{2}}$$

denote the induced norm.

**Definition 2.2.11.** Let  $\mathcal{T} \subset \mathcal{L}(X, Y)$  be a set of operators. We say that  $\mathcal{T}$  is  $\gamma$ -bounded if there exists a constant  $C \geq 0$  such that for all finite sequences  $(T_n)_{n=1}^N \subset \mathcal{T}$  and  $(x_n)_{n=1}^N \subset X$ , the following inequality holds:

$$\left\| \sum_{n=1}^N \gamma_n \otimes T_n x_n \right\|_{G(Y)} \leq C \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)}. \quad (2.2.6)$$

The least admissible constant in the above inequality is called the  $\gamma$ -bound of  $\mathcal{T}$  and we denote this quantity by  $\gamma(\mathcal{T})$ . If  $\mathcal{T}$  fails to be  $\gamma$ -bounded, we set  $\gamma(\mathcal{T}) = \infty$ .

Replacing the sequence  $(\gamma_k)_{k \geq 1}$  by a sequence  $(\epsilon_k)_{k \geq 1}$  of independent Rademacher variables in the above definition, we obtain the definition of  $\mathcal{R}$ -boundedness. It is well known that any  $\mathcal{R}$ -bounded set is  $\gamma$ -bounded, and that these notions are equivalent when  $X$  has finite cotype (see [26, theorem 8.6.4] for a more general result). Furthermore if  $X$  has cotype 2 and  $Y$  has type 2 (especially when  $X = Y$  is a Hilbert space) then  $\mathcal{R}$ -boundedness,  $\gamma$ -boundedness and uniform boundedness are equivalent.

Assume that  $X$  and  $Y$  are Banach lattices with finite cotype. By the Khintchine-Maurey inequality [26, Theorem 7.2.13], there exist  $c, C > 0$  such that for every  $x_1, \dots, x_n \in X$ ,

$$c \left\| \left( \sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}} \right\|_X \leq \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)} \leq C \left\| \left( \sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}} \right\|_X,$$

and  $Y$  satisfies a similar property. Hence a set  $\mathcal{T} \subset \mathcal{L}(X, Y)$  is  $\gamma$ -bounded if and only if there exists  $C \geq 0$  such that for all finite sequences  $(T_n)_{n=1}^N \subset \mathcal{T}$  and  $(x_n)_{n=1}^N \subset X$ ,

$$\left\| \left( \sum_{n=1}^N |T_n x_n|^2 \right)^{\frac{1}{2}} \right\|_Y \leq C \left\| \left( \sum_{n=1}^N |x_n|^2 \right)^{\frac{1}{2}} \right\|_X. \quad (2.2.7)$$

We recall for further use that  $\gamma$ -boundedness is stable under the strong operator topology.

**Proposition 2.2.12.** *Let  $\mathcal{T} \subset \mathcal{L}(X, Y)$  be a  $\gamma$ -bounded set. Then the closure  $\overline{\mathcal{T}}^{so}$  of  $\mathcal{T}$  in the strong operator topology is  $\gamma$ -bounded with  $\gamma(\overline{\mathcal{T}}^{so}) = \gamma(\mathcal{T})$ .*

We will need the following lemma, for which we refer to [26, Theorem 8.5.4].

**Lemma 2.2.13** ( $L^\infty$ -integral means). *Let  $(\Omega, \mu)$  be a measure space and let  $F : \Omega \rightarrow \mathcal{L}(X, Y)$  be an operator-valued function. Assume that  $F(\cdot)x$  belongs to  $L^1(\Omega; Y)$  for any  $x \in X$  and that there exists a constant  $K > 0$  such that*

$$\|F(\cdot)x\|_{L^1(\Omega; X)} \leq K \|x\|, \quad x \in X.$$

Then, for any  $\phi \in L^\infty(\Omega)$ , we can define a bounded operator  $T_\phi^F \in \mathcal{L}(X, Y)$  by

$$T_\phi^F x = \int_\Omega \phi(s)F(s)x d\mu(s), \quad x \in X,$$

and for any  $C > 0$ , the set

$$\mathcal{T}_\infty^F := \{T_\phi^F : \phi \in L^\infty(\Omega), \|\phi\|_{L^\infty(\Omega)} \leq C\}$$

is  $\gamma$ -bounded.

The adjoint set of a  $\gamma$ -bounded set may not be  $\gamma$ -bounded ([26, example 8.4.2]). Following [28], we introduce weaker notions to circumvent this difficulty.

**Definition 2.2.14.**

- 1) Let  $\mathcal{T} \subset \mathcal{L}(X, Y)$  be a set of operators. We say that  $\mathcal{T}$  is weak  $\gamma$ -bounded ( $W\gamma$ -bounded in short) if there exists  $C$  such that for all finite sequences  $(T_n)_{n=1}^N \subset \mathcal{T}$ ,  $(x_n)_{n=1}^N \subset X$  and  $(y_n^*)_{n=1}^N \subset Y^*$ , the following inequality holds:

$$\sum_{n=1}^N |\langle T_n x_n, y_n^* \rangle| \leq C \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)} \left\| \sum_{n=1}^N \gamma_n \otimes y_n^* \right\|_{G(Y^*)}. \quad (2.2.8)$$

- 2) Let  $\mathcal{T} \subset \mathcal{L}(X, Y^*)$  be a set of operators. We say that  $\mathcal{T}$  is weak\*  $\gamma$ -bounded ( $W^*\gamma$ -bounded in short) if there exists  $C$  such that for all finite sequences  $(T_n)_{n=1}^N \subset \mathcal{T}$ ,  $(x_n)_{n=1}^N \subset X$  and  $(y_n)_{n=1}^N \subset Y$ , the following inequality holds:

$$\sum_{n=1}^N |\langle T_n x_n, y_n \rangle| \leq C \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)} \left\| \sum_{n=1}^N \gamma_n \otimes y_n \right\|_{G(Y)}. \quad (2.2.9)$$

If  $\mathcal{T}$  is  $W\gamma$ -bounded (respectively  $W^*\gamma$ -bounded) then the adjoint set  $\mathcal{T}^*$  is  $W\gamma$ -bounded (respectively  $W^*\gamma$ -bounded) (see [25, Lemma 2.4]). It is clear that  $\gamma$ -boundedness implies  $W\gamma$ -boundedness, however the converse is false in general. Indeed take a  $\gamma$ -bounded set  $\mathcal{T}$  such that  $\mathcal{T}^*$  is not  $\gamma$ -bounded, then as  $\mathcal{T}$  is  $W\gamma$ -bounded,  $\mathcal{T}^*$  is also  $W\gamma$ -bounded but not  $\gamma$ -bounded.

**Lemma 2.2.15** (weak  $L^1$ -integral means). *Let  $(\Omega, \mu)$  be a measure space, let  $\mathcal{T} \subset \mathcal{L}(X, Y^*)$  be a  $W^*\gamma$ -bounded set and let  $G : \Omega \rightarrow \mathcal{L}(X, Y^*)$  be an operator-valued function such that  $G$  takes values in  $\mathcal{T}$  and for all  $x \in X$  and  $y \in Y$ , the scalar function  $s \mapsto \langle G(s)x, y \rangle$  is measurable.*

Then for any  $\phi \in L^1(\Omega)$ , one can define a bounded operator  $T_G^\phi \in \mathcal{L}(X, Y^*)$  by

$$\langle T_G^\phi x, y \rangle := \int_\Omega \phi(s) \langle G(s)x, y \rangle d\mu(s), \quad \text{for } x \in X \text{ and } y \in Y,$$

and for any  $C > 0$ , the set

$$\left\{ T_G^\phi : \phi \in L^1(\Omega), \|\phi\|_{L^1(\Omega)} \leq C \right\} \subset \mathcal{L}(X, Y^*)$$

is  $W^*\gamma$ -bounded.



*Proof.* Let  $\sigma$  be the topology on  $\mathcal{L}(X, Y^*)$  generated by the family of seminorms  $\rho_{x,y}$  defined by  $\rho_{x,y}(T) = |\langle T(x), y \rangle|$ , for any  $x \in X$  and  $y \in Y$ . The topology  $\sigma$  is called *weak\* operator topology on  $\mathcal{L}(X, Y^*)$* . It is clear that if  $\mathcal{T} \subset \mathcal{L}(X, Y^*)$  is  $W^*\gamma$ -bounded, then its closure  $\overline{\mathcal{T}}^\sigma$  is also  $W^*\gamma$ -bounded. Moreover if  $\mathcal{T} \subset \mathcal{L}(X, Y^*)$  is  $W^*\gamma$ -bounded then the absolute convex hull of  $\mathcal{T}$  is  $W^*\gamma$ -bounded as well.

With these two facts in hand, one can obtain the result by mimicking the proof of [26, Theorem 8.5.2]. Details are left to the reader.  $\square$

We will need the notion of  $K$ -convexity, for which we refer to [43] or [26]. We recall that a Banach space  $X$  is  $K$ -convex if and only if there exists a constant  $K > 0$  such that for all  $x_1, \dots, x_N \in X$  the following inequality holds:

$$\left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)} \leq K \sup \left\{ \left| \sum_{n=1}^N \langle x_n, y_n \rangle \right| : y_1, \dots, y_N \in X^*, \left\| \sum_{n=1}^N \gamma_n \otimes y_n \right\|_{G(X^*)} \leq 1 \right\}. \quad (2.2.10)$$

It turns out that  $X$  is  $K$ -convex if and only if  $X^*$  is  $K$ -convex. If this is the case, then according to [26, Corollary 7.4.6], there exists a constant  $K' > 0$  such that for all  $y_1, \dots, y_N \in X^*$ ,

$$\left\| \sum_{n=1}^N \gamma_n \otimes y_n \right\|_{G(X^*)} \leq K' \sup \left\{ \left| \sum_{n=1}^N \langle x_n, y_n \rangle \right| : x_1, \dots, x_N \in X, \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)} \leq 1 \right\}. \quad (2.2.11)$$

We recall that all UMD spaces are  $K$ -convex. In particular,  $L^p$ -spaces are  $K$ -convex for any  $1 < p < \infty$ . Further any closed subspace of a  $K$ -convex space is  $K$ -convex. We also recall that any  $K$ -convex space has a finite cotype. In particular, a  $K$ -convex Banach space cannot contain  $c_0$ . A fundamental result on  $K$ -convexity is Pisier's Theorem [26, Theorem 7.4.23] which asserts that  $X$  is  $K$ -convex if and only if  $X$  has non-trivial type.

We note that there exist non reflexive  $K$ -convex Banach spaces. It readily follows from (2.2.10) that if  $Y$  is a  $K$ -convex space, then a set  $\mathcal{T} \subset \mathcal{L}(X, Y)$  is  $W\gamma$ -bounded if and only if it is  $\gamma$ -bounded. Likewise using (2.2.11), we obtain that if  $Y$  is  $K$ -convex, then a set  $\mathcal{T} \subset \mathcal{L}(X, Y^*)$  is  $W^*\gamma$ -bounded if and only if it is  $\gamma$ -bounded.

We now turn to the definition of  $\gamma$ -spaces, which play a fundamental role in this paper. Let  $H$  be a Hilbert space. A linear operator  $T : H \rightarrow X$  is called  $\gamma$ -summing if

$$\|T\|_\gamma := \sup \left\| \sum_{n=1}^N \gamma_n \otimes Th_n \right\|_{G(X)} < \infty,$$

where the supremum is taken over all finite orthonormal systems  $\{h_1, \dots, h_n\}$  in  $H$ . We let  $\gamma_\infty(H; X)$  denote the space of all  $\gamma$ -summing operators and we endow it with the norm  $\|\cdot\|_\gamma$ . Then  $\gamma_\infty(H; X)$  is a Banach space. Clearly any finite rank (bounded) operator is a  $\gamma$ -summing operator. We let  $\gamma(H; X)$  be the closure in  $\gamma_\infty(H; X)$  of the space of finite rank operators from  $H$  into  $X$ . The spaces  $\gamma_\infty(H; X)$  and  $\gamma(H; X)$  do not coincide in general [26, Example 9.1.21] but when  $X$  does not contain a copy of  $c_0$  (in particular when  $X$  is  $K$ -convex) then these spaces coincide.

Let  $(S, \mu)$  be a measure space. We say that a function  $f : S \rightarrow X$  is weakly  $L^2$  if for each  $x^* \in X^*$ , the function  $s \mapsto \langle f(s), x^* \rangle$  is measurable and belongs to  $L^2(S)$ . If  $f : S \rightarrow X$  is measurable and weakly  $L^2$ , one can define an operator  $\mathbb{I}_f : L^2(S) \rightarrow X$ , given by

$$\mathbb{I}_f(g) := \int_S g(s)f(s)d\mu, \quad g \in L^2(S),$$

where this integral is defined in the Pettis sense.

We let  $\gamma(S; X)$  be the space of all measurable and weakly  $L^2$  functions  $f : S \rightarrow X$  such that  $\mathbb{I}_f$  belongs to  $\gamma(L^2(S); X)$ . We endow it with  $\|f\|_{\gamma(S; X)} := \|\mathbb{I}_f\|_{\gamma(L^2(S); X)}$ . A remarkable fact is the density of simple functions in the set  $\gamma(S; X)$  [26, Proposition 9.2.5].

Now we collect some important results, which will be useful in the following sections. We start with the so-called Multiplier Theorem [26, Theorem 9.5.1], a high ranking result involving  $\gamma$ -boundedness. We state it under the assumption that  $X$  does not contain  $c_0$ . Thus the following statement applies to  $K$ -convex spaces.

**Theorem 2.2.16** ( $\gamma$ -Multiplier theorem). *Let  $X$  be a Banach space not containing  $c_0$ . Let  $M : S \rightarrow \mathcal{L}(X)$  be a strongly measurable function and assume that its range  $\mathcal{M} := \{M(s) : s \in S\}$  is  $\gamma$ -bounded. Then for every function  $\psi : S \rightarrow X$  in  $\gamma(S; X)$ , the function  $M\psi : S \rightarrow X$  belongs to  $\gamma(S; X)$ , and we have*

$$\|M\psi\|_{\gamma(S; X)} \leq \gamma(\mathcal{M}) \|\psi\|_{\gamma(S; X)}.$$

The next result is an inequality of Hölder type [26, Theorem 9.2.14 (1)].

**Theorem 2.2.17** ( $\gamma$ -Hölder inequality). *If  $f : S \rightarrow X$  and  $g : S \rightarrow X^*$  belongs to  $\gamma(S; X)$  and  $\gamma(S; X^*)$ , respectively, then  $\langle f, g \rangle$  belongs to  $L^1(S)$  and we have*

$$\|\langle f, g \rangle\|_{L^1(S)} \leq \|f\|_{\gamma(S; X)} \|g\|_{\gamma(S; X^*)}.$$

Now we give an extension result, for which we refer to [26, Theorem 9.6.1]. We identify the algebraic tensor product  $H^* \otimes X$  with the space of finite rank bounded operator operators from  $H$  into  $X$  in the usual way, that is, we set  $(h^* \otimes x)(h) = h^*(h)x$  for any  $h \in H$ ,  $h^* \in H^*$  and  $x \in X$ .

**Theorem 2.2.18.** [*Extension theorem*] *Let  $H_1$  and  $H_2$  be Hilbert spaces. For any bounded operator  $U : H_1^* \rightarrow H_2^*$ , the mapping*

$$U \otimes I_X : H_1^* \otimes X \longrightarrow H_2^* \otimes X,$$

*taking  $h^* \otimes x$  to  $U(h^*) \otimes x$  for any  $h^* \in H_1^*$  and  $x \in X$ , has a unique extension to a bounded linear operator  $\tilde{U} : \gamma(H_1; X) \rightarrow \gamma(H_2; X)$  of the same norm. Furthermore for all  $T \in \gamma(H_1; X)$ ,*

$$\tilde{U}(T) = T \circ {}^tU, \tag{2.2.12}$$

*where  ${}^tU$  denotes the Banach space adjoint of  $U$ .*

To conclude this part, we apply the above principles to the Fourier-Plancherel transform  $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ . We identify the dual of  $L^2(\mathbb{R})$  with  $L^2(\mathbb{R})$  via the usual duality map provided by integration on  $\mathbb{R}$ .

**Lemma 2.2.19.** For any  $f \in L^1(\mathbb{R}; X)$ , let  $\hat{f}$  be its Fourier transform defined by

$$\hat{f}(t) = \int_{\mathbb{R}} e^{-its} f(s) ds.$$

Let  $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  be the Fourier-Plancherel transform (which coincides with  $f \mapsto \hat{f}$  on  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ ). Let  $\tilde{\mathcal{F}} : \gamma(L^2(\mathbb{R}); X) \rightarrow \gamma(L^2(\mathbb{R}); X)$  be its extension provided by Theorem 2.2.18. If  $f \in \gamma(\mathbb{R}; X) \cap L^1(\mathbb{R}; X)$ , then we have

$$\hat{f} \in \gamma(\mathbb{R}; X) \quad \text{and} \quad \tilde{\mathcal{F}}(\mathbb{I}_f) = \mathbb{I}_{\hat{f}}$$

and further,

$$\|\hat{f}\|_{\gamma(\mathbb{R}; X)} = \sqrt{2\pi} \|f\|_{\gamma(\mathbb{R}; X)}.$$

*Proof.* Obviously  $\hat{f}$  is measurable, and as  $f$  is weakly  $L^2$ ,  $\hat{f}$  is also weakly  $L^2$ . Indeed for  $x \in X^*$  one has by Fourier-Plancherel theorem :

$$\begin{aligned} \|\langle \hat{f}, x^* \rangle\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |\langle \int_{\mathbb{R}} f(s) e^{-ist} ds, x^* \rangle|^2 dt = \int_{\mathbb{R}} |\int_{\mathbb{R}} e^{-its} \langle f(s), x^* \rangle ds|^2 dt \\ &= \|\mathcal{F}(\langle f, x^* \rangle)\|_{L^2(\mathbb{R})}^2 = 2\pi \|\langle f, x^* \rangle\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

It follows that  $\mathbb{I}_{\hat{f}}$  is well defined and bounded. Now let  $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ . Then by Fubini's theorem, using equality  ${}^t\mathcal{F} = \mathcal{F}$  and (2.2.12):

$$\begin{aligned} \tilde{\mathcal{F}}(\mathbb{I}_f)(g) &= \mathbb{I}_f \circ \mathcal{F}(g) = \int_{\mathbb{R}} \mathcal{F}(g)(t) f(t) dt = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} g(s) e^{-ist} ds \right) f(t) dt \\ &= \int_{\mathbb{R}} g(s) \left( \int_{\mathbb{R}} f(t) e^{-its} dt \right) ds = \mathbb{I}_{\hat{f}}(g). \end{aligned}$$

By density and since  $\tilde{\mathcal{F}}(\mathbb{I}_f)$  and  $\mathbb{I}_{\hat{f}}$  are bounded, the equality  $\tilde{\mathcal{F}}(\mathbb{I}_f) = \mathbb{I}_{\hat{f}}$  follows. Hence  $\hat{f} \in \gamma(\mathbb{R}; X)$ .

Finally since  $(\sqrt{2\pi})^{-1}\mathcal{F}$  is an isometry, the extension principle yields the equalities

$$\|\hat{f}\|_{\gamma(\mathbb{R}; X)} = \|\mathbb{I}_{\hat{f}}\|_{\gamma(L^2(\mathbb{R}); X)} = \|\tilde{\mathcal{F}}(\mathbb{I}_f)\|_{\gamma(L^2(\mathbb{R}); X)} = \sqrt{2\pi} \|\mathbb{I}_f\|_{\gamma(L^2(\mathbb{R}); X)},$$

and then

$$\|\hat{f}\|_{\gamma(\mathbb{R}; X)} = \sqrt{2\pi} \|f\|_{\gamma(\mathbb{R}; X)}.$$

□

## 2.3. $\gamma$ -m-Bounded functional calculus

We will say that a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on Banach space is of  $\gamma$ -type  $\omega$  (resp. of  $W\gamma$ -type  $\omega$ ) if the set  $\{e^{\omega t} T_t : t \geq 0\}$  is  $\gamma$ -bounded (resp.  $W\gamma$ -bounded). If no such  $\omega$  exists, we will say that  $(T_t)_{t \geq 0}$  has no  $\gamma$ -type (resp. no  $W\gamma$ -type).

*Example 2.3.1.* It is easy to exhibit  $C_0$ -semigroups with no  $\gamma$ -type. Let  $1 \leq p < \infty$  and let  $(S_t)_{t \geq 0}$  be the right translation  $C_0$ -semigroup on  $L^p(\mathbb{R})$ , defined by  $S_t(f) = f(\cdot - t)$  for any  $t \geq 0$  and any  $f \in L^p(\mathbb{R})$ . Then, for  $p \neq 2$ ,  $(S_t)_{t \geq 0}$  has no  $\gamma$ -type. This follows from

the well-known fact that  $\{S_t, t \in [0, 1]\}$  is not  $\gamma$ -bounded. Indeed, assume that  $p \in [1, 2)$  and for  $n \in \mathbb{N}$  and  $i \in \{1, \dots, n\}$ , let  $t_i^n = \frac{i-1}{n}$  and  $f_i^n = \chi_{[0,1/n]}$ . Then we have

$$\left\| \left( \sum_{i=1}^n |S_{t_i^n} f_i^n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} = \|\chi_{[0,1]}\|_{L^p(\mathbb{R})} = 1.$$

whereas

$$\left\| \left( \sum_{i=1}^n |f_i^n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} = \left\| n^{1/2} \chi_{[0,1/n]} \right\|_{L^p(\mathbb{R})} = n^{1/2-1/p}.$$

Hence the inequality (2.2.7) cannot be true. The proof in the case  $p \in (2, \infty)$  is similar.

The fact that the set  $\{S_t : t \in [0, 1]\}$  is not  $\gamma$ -bounded immediately implies that for any  $\omega$ , the set  $\{e^{\omega t} S_t : t \in [0, 1]\}$  is not  $\gamma$ -bounded, and hence  $\{e^{\omega t} S_t : t \geq 0\}$  cannot be  $\gamma$ -bounded.

Throughout this section, we let  $X$  be a Banach space. Then we let  $A$  be a half-plane type operator of type  $\omega$  on  $X$ .

The condition that  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$  can be rephrased by saying that the set

$$\bigcup_{\alpha < \omega} \{f(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\omega)} \leq 1\}$$

is uniformly bounded. This formulation and Definition 2.2.7 motivate the following definitions.

**Definition 2.3.2.**

- 1) We say that  $A$  has a  $\gamma$ -bounded (resp.  $W\gamma$ -bounded)  $H^\infty$ -functional calculus of type  $\omega$  if  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$  and the set

$$\bigcup_{\alpha < \omega} \{f(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\omega)} \leq 1\}$$

is  $\gamma$ -bounded (resp.  $W\gamma$ -bounded).

- 2) Let  $m \geq 1$  be an integer. We say that  $A$  has a  $\gamma$ - $m$ -bounded (resp. a  $W\gamma$ - $m$ -bounded) functional calculus of type  $\omega$  if  $A$  has a  $m$ -bounded functional calculus of type  $\omega$  and the set

$$\bigcup_{\alpha < \omega} \{(\omega - \alpha)^m f^{(m)}(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1\} \tag{2.3.1}$$

is  $\gamma$ -bounded (resp.  $W\gamma$ -bounded).

*Remark 2.3.3.*

- 1) To prove that the set

$$\bigcup_{\alpha < \omega} \{(\omega - \alpha)^m f^{(m)}(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1\}$$

it is enough to prove that

$$\bigcup_{\alpha < \omega} \bigcup_{\delta < \alpha} \{(\omega - \alpha)^m f^{(m)}(A) : f \in \mathcal{E}(R_\delta), \|f\|_{H^\infty(R_\alpha)} \leq 1\}$$

This follows from the convergence lemma [9, Theorem 3.1], the argument in the proof of [9, Theorem 5.6 (a)], and Proposition 2.2.12. Details are left to the reader.

- 2) Likewise to prove that  $A$  has a  $\gamma$ -bounded  $H^\infty$ -functional calculus of type  $\omega$ , it is enough to prove that the set

$$\bigcup_{\alpha < \omega} \{f(A) : f \in \mathcal{E}(R_\alpha), \|f\|_{H^\infty(R_\omega)} \leq 1\}$$

is  $\gamma$ -bounded.

- 3) If  $A$  has a  $\gamma$ -bounded  $H^\infty$ -functional calculus of type  $\omega$ , then it has a  $\gamma$ - $m$ -bounded functional calculus of type  $\omega$ . Indeed consider the set

$$\Delta := \bigcup_{\beta < \omega} \{g(A) : g \in H^\infty(R_\beta), \|g\|_{H^\infty(R_\omega)} \leq 1\},$$

and assume that  $\Delta$  is  $\gamma$ -bounded. Let  $\alpha < \omega$  and let  $f \in H^\infty(R_\alpha)$  with  $\|f\|_{H^\infty(R_\alpha)} \leq 1$ . According to [9, Lemma 5.4],  $f^{(m)} \in H^\infty(R_\beta)$  for any  $\beta \in (\alpha, \omega)$  and

$$\left\| \frac{(\omega - \alpha)^m}{m!} f^{(m)} \right\|_{H^\infty(R_\omega)} \leq 1.$$

Consequently,

$$\bigcup_{\alpha < \omega} \left\{ \frac{(\omega - \alpha)^m}{m!} f^{(m)}(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1 \right\} \subset \Delta.$$

Hence the above set is  $\gamma$ -bounded, which shows that  $A$  has a  $\gamma$ - $m$ -bounded functional calculus of type  $\omega$ .

The previous three statements hold as well with  $W\gamma$ -boundedness replacing  $\gamma$ -boundedness.

Recall the condition  $(GFS)_{m,\omega}$  from Definition 2.2.5. We introduce the following stronger form.

**Definition 2.3.4.** Let  $m \geq 1$  be an integer. We say that  $A$  has property  $(W\gamma\text{-}GFS)_{m,\omega}$  if there exists a constant  $C > 0$  such that for any  $N \in \mathbb{N}$ , for any  $\alpha_1, \dots, \alpha_N < \omega$ , and for any  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in X^*$ , we have

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} | \langle (\omega - \alpha_k)^m R(\alpha_k + it, A)^{m+1} x_k, y_k \rangle | dt \\ & \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}. \end{aligned}$$

Clearly,  $(W\gamma\text{-}GFS)_{m,\omega}$  implies  $(GFS)_{m,\omega}$ . Further if  $X$  is a Hilbert space, then  $(GFS)_{m,\omega}$  and  $(W\gamma\text{-}GFS)_{m,\omega}$  are equivalent.

In the sequel we let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  denote the open unit disc. The following statement is straightforward.

**Lemma 2.3.5.** Let  $m \geq 1$  be an integer. Assume that  $A$  has property  $(GFS)_{m,\omega}$ . For any measurable function  $\epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}}$  and for any  $\alpha < \omega$ , let

$$\int_{\mathbb{R}} \epsilon(t) R(\alpha + it, A)^{m+1} dt \in \mathcal{L}(X, X^{**})$$

denote the operator defined, for  $x \in X$  and  $y \in X^*$  by

$$\left\langle \left( \int_{\mathbb{R}} \epsilon(t) R(\alpha + it, A)^{m+1} dt \right) x, y \right\rangle = \int_{\mathbb{R}} \langle \epsilon(t) R(\alpha + it, A)^{m+1} x, y \rangle dt.$$

Then, the operator  $A$  has property  $(W\gamma\text{-GFS})_{m,\omega}$  if and only if the set

$$\left\{ (\omega - \alpha)^m \int_{\mathbb{R}} \epsilon(t) R(\alpha + it, A)^{m+1} dt : \epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}} \text{ measurable, } \alpha < \omega \right\} \quad (2.3.2)$$

is  $W^*\gamma$ -bounded.

*Remark 2.3.6.* Arguing as in Remark 2.2.6, one shows that if  $A$  is  $\gamma$ -sectorial of  $\gamma$ -type  $< \frac{\pi}{2}$ , then  $A$  has property  $(W\gamma\text{-GFS})_{1,0}$ . (We refer e.g. to [26] for information on  $\gamma$ -sectorial operators.)

Indeed assume that  $A$  is  $\gamma$ -sectorial of  $\gamma$ -type  $< \frac{\pi}{2}$ . Then the set

$$\left\{ (\alpha + it) R(\alpha + it, A) : \alpha < 0, t \in \mathbb{R} \right\}$$

is  $\gamma$ -bounded. Next for any measurable  $\epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}}$ , we can write

$$(-\alpha) \int_{\mathbb{R}} \epsilon(t) R(\alpha + it, A)^2 dt = \int_{\mathbb{R}} (-\alpha)(\alpha + it)^{-2} \epsilon(t) ((\alpha + it) R(\alpha + it, A))^2 dt.$$

Since  $\|t \mapsto (-\alpha)(\alpha + it)^{-2} \epsilon(t)\|_{L^1} \leq \pi$ , Lemma 2.2.13 ensures that the set

$$\left\{ (-\alpha) \int_{\mathbb{R}} \epsilon(t) R(\alpha + it, A)^2 dt : \epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}} \text{ measurable, } \alpha < 0 \right\}$$

is  $W^*\gamma$ -bounded. Hence the result follows from the above Lemma 2.3.5.

**Proposition 2.3.7.** *Let  $m \geq 1$  be an integer and assume that  $A$  has property  $(W\gamma\text{-GFS})_{m,\omega}$ . Then for any integer  $1 \leq p \leq m$ ,  $A$  has property  $(W\gamma\text{-GFS})_{p,\omega}$ .*

*Proof.* We proceed by induction, showing that if  $m \geq 2$ , then  $(W\gamma\text{-GFS})_{m,\omega}$  implies  $(W\gamma\text{-GFS})_{m-1,\omega}$ .

Suppose that  $A$  has property  $(W\gamma\text{-GFS})_{m,\omega}$ , with  $m \geq 2$ . Let  $\epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}}$  be a measurable function and let  $\alpha < \omega$ . Applying [9, Proposition 6.3. (a)], we have

$$R(\alpha + it, A)^m = -m \int_{-\infty}^{\alpha} R(u + it, A)^{m+1} du$$

for any  $t \in \mathbb{R}$ . Hence for any  $x \in X$ , for any  $y \in X^*$  and for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} \langle \epsilon(t) R(\alpha + it, A)^m x, y \rangle &= - \int_{-\infty}^{\alpha} m \langle \epsilon(t) R(u + it, A)^{m+1} x, y \rangle du \\ &= - \int_{-\infty}^0 m \langle \epsilon(t) R(\alpha + u + it, A)^{m+1} x, y \rangle du. \end{aligned}$$

We now integrate over  $t$ . Property  $(GFS)_{m,\omega}$  ensures that we can apply Fubini's theorem in the following computation:

$$\begin{aligned} &(\omega - \alpha)^{m-1} \int_{\mathbb{R}} \langle \epsilon(t) R(\alpha + it, A)^m x, y \rangle dt \\ &= - \int_{\mathbb{R}} m (\omega - \alpha)^{m-1} \int_{-\infty}^0 \langle \epsilon(t) R(\alpha + u + it, A)^{m+1} x, y \rangle dt du \\ &= - \int_{-\infty}^0 m (\omega - \alpha)^{m-1} \int_{\mathbb{R}} \langle \epsilon(t) R(\alpha + u + it, A)^{m+1} x, y \rangle dt du \\ &= - \int_{-\infty}^0 \frac{m (\omega - \alpha)^{m-1}}{(\omega - \alpha - u)^m} \int_{\mathbb{R}} \langle (\omega - \alpha - u)^m \epsilon(t) R(\alpha + u + it, A)^{m+1} x, y \rangle dt du. \end{aligned}$$

Let  $\mathcal{T}$  be the set (2.3.2). By assumption,  $\mathcal{T}$  is  $W^*\gamma$ -bounded hence by Lemma 2.2.15, the set

$$\Gamma := \left\{ \int_{-\infty}^0 \phi(u) \int_{\mathbb{R}} (\omega - \alpha - u)^m \epsilon(t) R(\alpha + u + it, A)^{m+1} dt du : \right. \\ \left. \epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}} \text{ measurable, } \alpha < \omega, \phi \in L^1((-\infty, 0)), \|\phi\|_{L^1} \leq 2 \right\}$$

is  $W^*\gamma$ -bounded. Since

$$\left\| m \frac{(\omega - \alpha)^{m-1}}{(\omega - \alpha - \cdot)^m} \right\|_{L^1((-\infty, 0))} = \frac{m}{m-1} \leq 2,$$

the above calculation shows that the set

$$\left\{ (\omega - \alpha)^{m-1} \int_{\mathbb{R}} \epsilon(t) R(\alpha + it, A)^m dt : \epsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}} \text{ measurable, } \alpha < \omega \right\}$$

is included in  $\Gamma$ , hence is  $W^*\gamma$ -bounded. Hence, by Lemma 2.3.5, the operator  $A$  has property  $(W\gamma\text{-GFS})_{m-1, \omega}$ .  $\square$

We recall that  $(GFS)_{m, \omega}$  is equivalent to  $m$ -bounded functional calculus of type  $\omega$ . The following theorem provides a similar statement in the context of  $W\gamma$ -boundedness.

**Theorem 2.3.8.** *The following assertions are equivalent for  $m \geq 1$ :*

- (i) *A has  $(W\gamma\text{-GFS})_{m, \omega}$ ;*
- (ii) *A has a  $W\gamma$ -1-bounded functional calculus of type  $\omega$ ;*
- (iii) *A has a  $W\gamma$ - $m$ -bounded functional calculus of type  $\omega$ .*

*Additionally, if A satisfies these conditions, then  $-A$  generates a  $C_0$ -semigroup of  $W\gamma$ -type  $\omega$ .*

*Proof.*

(i)  $\Rightarrow$  (ii): First, by Proposition 3.4.14,  $A$  has  $(W\gamma\text{-GFS})_{1, \omega}$ .

Let  $\alpha_1, \dots, \alpha_N < \omega$ ,  $\delta_1 < \alpha_1, \dots, \delta_N < \alpha_N$ , and  $f_1 \in \mathcal{E}(R_{\delta_1}), \dots, f_N \in \mathcal{E}(R_{\delta_N})$  with  $\|f_k\|_{H^\infty(R_{\alpha_k})} \leq 1$ . Let  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in X^*$ . It follows from the proof of [9, Theorem 5.6 (a)] that for any  $k = 1, \dots, N$ ,

$$f'_k(A) = \frac{-1}{2\pi} \int_{\mathbb{R}} f(\alpha_k + it) R(\alpha_k + it, A)^2 dt.$$

Consequently,

$$\begin{aligned} & \sum_{k=1}^N |\langle (\omega - \alpha_k) f'_k(A) x_k, y_k \rangle| \\ &= \sum_{k=1}^N \left| \left\langle \frac{-1}{2\pi} \left( \int_{\mathbb{R}} (\omega - \alpha_k) f_k(\alpha_k + it) R(\alpha_k + it, A)^2 dt \right) x_k, y_k \right\rangle \right| \\ &\leq \frac{1}{2\pi} \sum_{k=1}^N \int_{\mathbb{R}} (\omega - \alpha_k) |f_k(\alpha_k + it)| |\langle R(\alpha_k + it, A)^2 x_k, y_k \rangle| dt \\ &\leq \frac{1}{2\pi} \sum_{k=1}^N \|f_k\|_{H^\infty(R_{\alpha_k})} \int_{\mathbb{R}} |\langle (\omega - \alpha_k) R(\alpha_k + it, A)^2 x_k, y_k \rangle| dt. \end{aligned}$$

Since  $\|f_k\|_{H^\infty(R_{\alpha_k})} \leq 1$  and  $A$  has  $(W\gamma\text{-}GFS)_{1,\omega}$ , this yields an estimate

$$\sum_{k=1}^N |\langle (\omega - \alpha_k) f'_k(A) x_k, y_k \rangle| \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}.$$

According to Remark 2.3.3 (1), this implies that  $A$  has a  $W\gamma$ -1-bounded functional calculus of type  $\omega$ .

(ii)  $\Rightarrow$  (iii): It follows from the assumption that the set

$$\Delta := \bigcup_{\alpha < \beta < \omega} \{(\omega - \beta) g^{(m)}(A) : g \in H^\infty(R_\alpha), \|g^{(m-1)}\|_{H^\infty(R_\beta)} \leq 1\}$$

is  $W\gamma$ -bounded. For any  $\alpha < \beta < \omega$  and  $f \in H^\infty(R_\alpha)$  with  $\|f\|_{H^\infty(R_\alpha)} \leq 1$ , we have

$$\left\| \frac{(\beta - \alpha)^{m-1}}{(m-1)!} f^{(m-1)} \right\|_{H^\infty(R_\beta)} \leq 1$$

by [9, Lemma 5.4]. Hence

$$\bigcup_{\alpha < \beta < \omega} \left\{ \frac{(\omega - \beta)(\beta - \alpha)^{m-1}}{(m-1)!} f^{(m)}(A), f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1 \right\} \subset \Delta.$$

Taking  $\beta = \frac{\alpha + \omega}{2}$  in the above set, we obtain

$$\bigcup_{\alpha < \omega} \left\{ \frac{(\omega - \alpha)^m}{2^m (m-1)!} f^{(m)}(A) : f \in H^\infty(R_\alpha), \|f\|_{H^\infty(R_\alpha)} \leq 1 \right\} \subset \Delta.$$

Hence the above is  $W\gamma$ -bounded. Thus  $A$  has a  $W\gamma$ - $m$ -bounded functional calculus of type  $\omega$ .

(iii)  $\Rightarrow$  (i): Let  $\alpha_1, \dots, \alpha_N < \omega$ ,  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in X^*$ . Arguing as in the proof of [9, Theorem 5.6 (b)], we consider  $\beta_k \in (\alpha_k, \omega)$  for any  $k = 1, \dots, N$  and introduce measurable functions  $\epsilon_1, \dots, \epsilon_N : \mathbb{R} \rightarrow \overline{\mathbb{D}}$  such that

$$|\langle R(\alpha_k + it, A)^{m+2} x_k, y_k \rangle| = \langle R(\alpha_k + it, A)^{m+2} x_k, y_k \rangle \epsilon_k(t)$$

for all  $k = 1, \dots, N$  and all  $t \in \mathbb{R}$ . Next for any  $R > 0$  and any  $Re(z) > \beta_k$ , we set

$$\phi_{k,R}(z) := \frac{\beta_k - \alpha_k}{\pi} \int_{-R}^R \frac{\epsilon_k(t)}{(\alpha_k + it - z)^2} dt$$

Since  $R$  is finite, it is easy to show that  $\phi_{k,R}(z) = O(|z|^{-2})$  as  $|z| \rightarrow \infty$ , and hence  $\phi_{k,R} \in \mathcal{E}(R_\alpha)$ . Furthermore, it follows from the proof of [9, Theorem 5.6 (b)] that  $\|\phi_{k,R}\|_{H^\infty(R_{\beta_k})} \leq 1$  and

$$\phi_k^{(m)}(A) = \int_{-R}^R \frac{\beta_k - \alpha_k}{\pi} (m+1)! \epsilon_k(t) R(\alpha_k + it, A)^{m+2} dt.$$

It therefore follows from (iii) that we have an estimate

$$\begin{aligned} & \sum_{k=1}^N \int_{-R}^R \left| \left\langle \frac{(\beta_k - \alpha_k)(\omega - \beta_k)^m}{\pi} (m+1)! R(\alpha_k + it, A)^{m+2} x_k, y_k \right\rangle \right| dt \\ &= \sum_{k=1}^N \left| \langle (\omega - \beta_k)^m \phi_{k,R}^{(m)}(A) x_k, y_k \rangle \right| \\ &\leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}. \end{aligned}$$



Passing to the limit when  $R \rightarrow \infty$ , one obtains

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} \left| \left\langle \frac{(\beta_k - \alpha_k)(\omega - \beta_k)^m}{\pi} (m+1)! R(\alpha_k + it, A)^{m+2} x_k, y_k \right\rangle \right| dt \\ & \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}. \end{aligned}$$

Now we choose  $\beta_k = \frac{\omega + \alpha_k}{2}$  in the above estimate. We obtain the following inequality

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} \left| \left\langle (\omega - \alpha_k)^{m+1} R(\alpha_k + it, A)^{m+2} x_k, y_k \right\rangle \right| dt \\ & \leq \frac{2^{m+1} C \pi}{(m+1)!} \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}. \end{aligned}$$

This shows that  $A$  has  $(W\gamma\text{-GFS})_{m+1, \omega}$ . Then by Proposition 3.4.14,  $A$  has  $(W\gamma\text{-GFS})_{m, \omega}$ .

Finally, assume that (ii) holds true. In particular  $A$  has a  $m$ -1-bounded functional calculus of type  $\omega$  hence by Theorem [9, Theorem 6.4],  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  of type  $\omega$ .

For any  $t \geq 0$  and  $\alpha < \omega$ , let  $\phi_{\alpha, t}(z) = e^{-tz} e^{t\alpha}$  for  $z \in R_\alpha$ . Then  $\|\phi_{\alpha, t}\|_{H^\infty(R_\alpha)} = 1$ . Hence by (ii), the set

$$\bigcup_{\alpha < \omega} \{(\omega - \alpha)\phi'_{\alpha, t}(A) : t \geq 0\} \text{ is } W\gamma\text{-bounded.}$$

Therefore the set

$$\bigcup_{\alpha < \omega} \{(\omega - \alpha)(-t)e^{-tA} e^{t\alpha} : t > 0\} \text{ is } W\gamma\text{-bounded.}$$

We noticed in Proposition 2.2.2 that  $e^{-tA} = T_t$  for any  $t \geq 0$ . Hence taking  $\alpha = \omega - \frac{1}{t}$  for any  $t > 0$ , we deduce that the set

$$\{e^{\omega t} T_t : t \geq 0\} \text{ is } W\gamma\text{-bounded.}$$

Hence  $(T_t)_{t \geq 0}$  is a  $C_0$ -semigroup of  $W\gamma$ -type  $\omega$ . □

*Remark 2.3.9.* The above proof shows as well that if  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $\omega$ , then  $-A$  generates a  $C_0$ -semigroup of  $\gamma$ -type  $\omega$ .

In the  $K$ -convex case, Theorem 2.3.8 can be strengthened as follows.

**Theorem 2.3.10.** *Assume that  $X$  is  $K$ -convex and let  $m \geq 1$  be an integer. The following assertions are equivalent:*

- (i) *There exist a norming subspace  $Z \subset X^*$  and a constant  $C > 0$  such that for any  $N \in \mathbb{N}$ , for any  $\alpha_1, \dots, \alpha_N < \omega$ , for any  $x_1, \dots, x_N \in X$  and for any  $y_1, \dots, y_N \in Z$ ,*

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} \left| \left\langle (\omega - \alpha_k)^m R(\alpha_k + it, A)^{m+1} x_k, y_k \right\rangle \right| dt \\ & \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}. \end{aligned}$$

(ii)  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $\omega$ ;

(iii)  $A$  has a  $\gamma$ - $m$ -bounded functional calculus of type  $\omega$ .

*Proof.* The proofs of (ii)  $\Leftrightarrow$  (iii) and (iii)  $\Rightarrow$  (i) are obvious by the equivalence of  $W\gamma$ -boundedness and  $\gamma$ -boundedness on a  $K$ -convex space, and Theorem 2.3.8.

Now assume (i). By [26, Corollary 7.4.6], there exists  $M > 0$  such that for all  $N \in \mathbb{N}$  and  $x_1, \dots, x_N \in X$ , we have

$$\left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \leq M \sup \left\{ \left| \sum_{k=1}^N \langle x_k, y_k \rangle \right|, y_k \in Z, \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)} \leq 1 \right\}. \quad (2.3.3)$$

The argument in the proof of Theorem 2.3.8 shows that for all  $\alpha_1, \dots, \alpha_N < \omega$ ,  $\delta_1 < \alpha_1, \dots, \delta_N < \alpha_N$ ,  $f_1 \in \mathcal{E}(R_{\delta_1}), \dots, f_N \in \mathcal{E}(R_{\delta_N})$  with  $\|f_k\|_{H^\infty(R_{\alpha_k})} \leq 1$ ,  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in Z$ , we have an estimate

$$\sum_{k=1}^N |(\omega - \alpha_k) f'_k(A) x_k, y_k| \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}.$$

Applying (2.3.3), this implies

$$\left\| \sum_{k=1}^N \gamma_n \otimes (\omega - \alpha_k) f'_k(A) x_k \right\|_{G(X)} \leq MC \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}.$$

Hence  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $\omega$ .  $\square$

## 2.4. A Shi-Feng-Gomilko Theorem on $K$ -convex spaces

Let  $(\Omega, \mu)$  be a measure space, let  $N \in \mathbb{N}$  and let  $(e_1, \dots, e_N)$  denote the canonical basis of  $l_N^2$ . Let  $l_N^2 \otimes L^2(\Omega)$  denote the Hilbert space tensor product of  $l_N^2$  and  $L^2(\Omega)$ . With the notation  $\mathbb{N}_N = \{1, \dots, N\}$ , we have a unitary isomorphism

$$l_N^2 \otimes L^2(\Omega) = L^2(\Omega \times \mathbb{N}_N).$$

Let  $X$  be a Banach space. For any bounded operator  $u : l_N^2 \otimes L^2(\Omega) \rightarrow X$ , let  $u_k : L^2(\Omega) \rightarrow X$  be defined by  $u_k(f) = u(e_k \otimes f)$ , for any  $k = 1, \dots, N$ . Then the mapping  $u \mapsto \sum_k e_k \otimes u_k$  induces an algebraic isomorphism

$$\mathcal{L}(l_N^2 \otimes L^2(\Omega), X) \simeq l_N^2 \otimes \mathcal{L}(L^2(\Omega), X).$$

It is easy to check that  $u : l_N^2 \otimes L^2(\Omega) \rightarrow X$  belongs to  $\gamma(L^2(\Omega \times \mathbb{N}_N); X)$  if and only if  $u_k$  belongs to  $\gamma(L^2(\Omega); X)$  for any  $k = 1, \dots, N$ . This leads to an algebraic isomorphism

$$\gamma(L^2(\Omega \times \mathbb{N}_N); X) \simeq l_N^2 \otimes \gamma(L^2(\Omega); X).$$

Likewise, a function  $f : \Omega \times \mathbb{N}_N \rightarrow X$  belongs to  $\gamma(\Omega \times \mathbb{N}_N; X)$  if and only if  $f(\cdot, k)$  belongs to  $\gamma(\Omega; X)$  for any  $k = 1, \dots, N$ .

Recall the sun dual  $X^\odot$  from Subsection 2.2. We now state and prove the main result of this article.

**Theorem 2.4.1.** *Let  $X$  be a  $K$ -convex Banach space and let  $m \geq 1$  be an integer. Let  $A$  be an operator of half-plane type  $\omega$  on  $X$ . The following assertions are equivalent:*

- (i)  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  of  $\gamma$ -type  $\omega$ ;
- (ii) *There exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$ , for all  $x_1, \dots, x_N \in X$ , for all  $y_1, \dots, y_N \in \overline{\text{dom}(A^*)} = X^\odot$ , and for all  $\alpha_1, \dots, \alpha_N < \omega$ , the functions  $(t, k) \mapsto \sqrt{\omega - \alpha_k} R(\alpha_k + it, A)x_k$  and  $(t, k) \mapsto \sqrt{\omega - \alpha_k} R(\alpha_k + it, A^*)y_k$  belong to  $\gamma(\mathbb{R} \times \mathbb{N}_N; X)$  and  $\gamma(\mathbb{R} \times \mathbb{N}_N; X^*)$ , respectively, and satisfy*

$$\left\| (t, k) \mapsto \sqrt{\omega - \alpha_k} R(\alpha_k + it, A)x_k \right\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; X)} \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \quad (2.4.1)$$

and

$$\left\| (t, k) \mapsto \sqrt{\omega - \alpha_k} R(\alpha_k + it, A^*)y_k \right\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; X^*)} \leq C \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}; \quad (2.4.2)$$

- (iii)  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $\omega$ ;
- (iv)  $A$  has a  $\gamma$ - $m$ -bounded functional calculus of type  $\omega$ .

*Proof.* Without loss of generality, one may assume  $\omega = 0$ .

The equivalence (iii)  $\Leftrightarrow$  (iv) and the implication (iv)  $\Rightarrow$  (i) follow from Theorems 2.3.8 and 2.3.10.

(i)  $\Rightarrow$  (ii): Since  $\mathcal{T} := \{T_t : t \geq 0\}$  is  $\gamma$ -bounded, the strongly measurable function  $M : \mathbb{R}_+ \times \mathbb{N}_N \rightarrow \mathcal{L}(X)$  defined by  $M(t, k) := T_t$  has  $\gamma$ -bounded range  $\mathcal{T}$ . Then by Theorem 2.2.16, for each  $\psi \in \gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)$ , the function  $M\psi$  belongs to  $\gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)$ , with

$$\|M\psi\|_{\gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)} \leq \gamma(\mathcal{T}) \|\psi\|_{\gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)}. \quad (2.4.3)$$

Consider  $\alpha_1, \dots, \alpha_N < 0$  and  $x_1, \dots, x_N \in X$ , and define

$$\psi(t, k) = \sqrt{-\alpha_k} e^{\alpha_k t} x_k, \quad t \in \mathbb{R}_+, \quad k = 1, \dots, N.$$

Then  $\psi \in \gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)$  and

$$\|\psi\|_{\gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)} = \frac{1}{\sqrt{2}} \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}. \quad (2.4.4)$$

Indeed,  $\psi = \sum_{k=1}^N h_k \otimes x_k$  where  $h_k : \mathbb{R}^+ \times \mathbb{N}_N \rightarrow \mathbb{R}$  is defined by

$$h_k : (t, j) \mapsto \begin{cases} \sqrt{-\alpha_k} e^{\alpha_k t} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

Further  $h_1, \dots, h_N$  are pairwise orthogonal with

$$\|h_k\|_{L^2(\mathbb{R}_+ \times \mathbb{N}_N)} = \left\| \sqrt{-\alpha_k} e^{\alpha_k t} \right\|_{L^2(\mathbb{R}_+)} \frac{1}{\sqrt{2}}.$$

Hence (2.4.4) follows from [26, Example 9.2.4].

Recall that for each  $k \in \{1, \dots, N\}$ , we have

$$R(\alpha_k + it, A)x_k = - \int_0^\infty e^{its} e^{\alpha_k s} T_s x_k ds.$$

Applying Lemma 2.2.19, we deduce that  $(t, k) \mapsto R(\alpha_k + it, A)x_k$  belongs to  $\gamma(\mathbb{R}_+ \times \mathbb{N}_N; X)$ , with

$$\begin{aligned} & \|(t, k) \mapsto \sqrt{-\alpha_k} R(\alpha_k + it, A)x_k\|_{\gamma(\mathbb{R} \times \mathbb{N}_N, X)} \\ &= \sqrt{2\pi} \|(t, k) \mapsto \sqrt{-\alpha_k} e^{\alpha_k t} T_t x_k\|_{\gamma(\mathbb{R} \times \mathbb{N}_N, X)}. \end{aligned} \quad (2.4.5)$$

Combining (2.4.3), (2.4.4) and (2.4.5), we actually obtain

$$\|(t, k) \mapsto \sqrt{-\alpha_k} R(\alpha_k + it, A)x_k\|_{\gamma(\mathbb{R} \times \mathbb{N}_N, X)} \leq \sqrt{\pi} \gamma(\mathcal{T}) \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)},$$

which proves (2.4.1).

Finally, since  $X$  is  $K$ -convex, the set  $\{T_t^* : t \geq 0\}$  is  $\gamma$ -bounded. Further  $(T_t^\odot)_{t \geq 0}$  is a  $C_0$ -semigroup on the sun dual  $X^\odot$ , with generator equal to  $-A^\odot$ . Then the above computations together with Theorem 2.2.10 lead to (2.4.2).

(ii)  $\Rightarrow$  (iii): Let  $\alpha_1, \dots, \alpha_N < 0$ ,  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in X^\odot$ . Applying Theorem 3.4.3, one obtains

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} |\langle -\alpha_k R(\alpha_k + it, A)^2 x_k, y_k \rangle| dt \\ &= \sum_{k=1}^N \int_{\mathbb{R}} |\langle \sqrt{-\alpha_k} R(\alpha_k + it, A)x_k, \sqrt{-\alpha_k} R(\alpha_k + it, A)^* y_k \rangle| dt \\ &= \|(t, k) \mapsto \langle \sqrt{-\alpha_k} R(\alpha_k + it, A)x_k, \sqrt{-\alpha_k} R(\alpha_k + it, A)^* y_k \rangle\|_{L^1(\mathbb{R} \times \mathbb{N}_N)} \\ &\leq \|(t, k) \mapsto \sqrt{-\alpha_k} R(\alpha_k + it, A)x_k\|_{\gamma(\mathbb{R} \times \mathbb{N}_N, X)} \\ &\quad \times \|(t, k) \mapsto \sqrt{-\alpha_k} R(\alpha_k + it, A)^* y_k\|_{\gamma(\mathbb{R} \times \mathbb{N}_N, X^*)} \\ &\leq C^2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}. \end{aligned}$$

Since  $X^\odot$  is norming in  $X$ , it follows from the above estimate and Theorem 2.3.10 that  $A$  has a  $\gamma$ -1-bounded calculus of type 0.  $\square$

Conditions (2.4.1) and (2.4.2) are the  $\gamma$ -version of (2.2.3) and (2.2.4). Moreover, it is obvious that (ii) implies  $(W\gamma\text{-}GFS)_{m,\omega}$  and these two conditions are equivalent when  $X$  is a  $K$ -convex space.

*Remark 2.4.2.* Let  $(S, \mu)$  be a measure space and let  $E(S)$  be a  $K$ -convex Banach function space over  $(S, \mu)$  (see [26, appendix F] for definition). Then  $E(S)$  has finite cotype hence according to [26, Proposition 9.3.8], there exist  $c > 0$  and  $C > 0$  such that for each  $f \in \gamma(\mathbb{R} \times \mathbb{N}_N; E(S))$ ,

$$c \|f\|_{E(S; L^2(\mathbb{R} \times \mathbb{N}_N))} \leq \|f\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; E(S))} \leq C \|f\|_{E(S; L^2(\mathbb{R} \times \mathbb{N}_N))}. \quad (2.4.6)$$

Furthermore, the following equality holds,

$$\|f\|_{E(S; L^2(\mathbb{R} \times \mathbb{N}_N))} = \left\| \left( \int_{\mathbb{R}} \sum_{k=1}^N |f(\cdot, k)|^2 \right)^{\frac{1}{2}} \right\|_{E(S)}.$$

The space  $E(S)^*$  satisfies similar properties. Hence using (3.4.11) and the Khintchine-Maurey inequality [26, Theorem 7.2.13], the condition (ii) of Theorem 2.4.1 can be replaced by:

(ii)' There exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$ , for all  $x_1, \dots, x_N \in X$ , for all  $y_1, \dots, y_N \in \overline{\text{dom}(A^*)} = X^\odot$ , and for all  $\alpha_1, \dots, \alpha_N < \omega$ ,

$$\left\| \left( \sum_{k=1}^N \int_{\mathbb{R}} (\omega - \alpha_k) |R(\alpha_k + it, A)x_k|^2 dt \right)^{\frac{1}{2}} \right\|_{E(S)} \leq C \left\| \left( \sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}} \right\|_{E(S)}$$

and

$$\left\| \left( \sum_{k=1}^N \int_{\mathbb{R}} (\omega - \alpha_k) |R(\alpha_k + it, A^*)y_k|^2 dt \right)^{\frac{1}{2}} \right\|_{E(S)^*} \leq C \left\| \left( \sum_{k=1}^N |y_k|^2 \right)^{\frac{1}{2}} \right\|_{E(S)^*}.$$

Thus  $-A$  generates a  $C_0$ -semigroup of  $\gamma$ -type  $\omega$  on  $E(S)$  if and only if (ii)' holds true.

Of course the above applies when  $E(S) = L^p(S)$  for some  $1 < p < \infty$ .

*Remark 2.4.3.* In [22, Theorem 6.4], Haase and Rozendaal state that if  $-A$  generates a  $\gamma$ -bounded  $C_0$ -semigroup on a Banach space  $X$ , then  $A$  has a  $m$ -bounded functional calculus of type 0, for any  $m \geq 1$ . If  $X$  is  $K$ -convex, this is a formal consequence of Theorem 2.4.1 and in this case, the latter is a strengthening of the Haase-Rozendaal theorem.

For general  $X$ , a proof of [22, Theorem 6.4] can be derived from the arguments in the proof of Theorem 2.4.1. Indeed assume that  $-A$  generates a  $\gamma$ -bounded  $C_0$ -semigroup, consider  $\alpha < 0$  and let  $x \in X$  and  $y \in X^\odot$ . The proof of Theorem 2.4.1 shows that  $t \mapsto R(\alpha + it, A)x$  belongs to  $\gamma(\mathbb{R}; X)$  and that

$$\sqrt{-\alpha} \|t \mapsto R(\alpha + it, A)x\|_{\gamma(\mathbb{R}, X)} \leq C \|x\| \quad (2.4.7)$$

for some constant  $C > 0$  not depending either on  $\alpha$  or  $x$ . Then let  $\gamma'(\mathbb{R}; X^*)$  be the space introduced in [29, Section 5]. Using [29, Remark 5.12, (S2)] instead of Theorem 2.2.16, one obtains in a similar manner that

$$\sqrt{-\alpha} \|t \mapsto R(\alpha + it, A)^*y\|_{\gamma'(\mathbb{R}, X^*)} \leq C \|y\|. \quad (2.4.8)$$

Finally, using [29, Remark 5.12, (S1)] instead of Theorem 3.4.3, one deduces from (2.4.7) and (2.4.8) that

$$(-\alpha) \int_{\mathbb{R}} |R(\alpha + it, A)^2 x, y| dt \leq C^2 \|x\| \|y\|.$$

Since the sun dual  $X^\odot$  is  $w^*$ -dense in  $X^*$ , this shows that  $A$  has a 1-bounded functional calculus of type 0 (and hence a  $m$ -bounded functional calculus of type 0 for any  $m \geq 1$ ).

## 2.5. A Gearhart-Prüss Theorem on $K$ -convex spaces

Let  $A$  be a half-plane type operator on some Banach space  $X$ . Its abscissa of uniform boundedness  $s_0(A)$  is defined by

$$s_0(A) := \sup \left\{ \alpha \in \mathbb{R} : \sigma(A) \subset R_\alpha \text{ and } \sup_{\text{Re}(z) \leq \alpha} \|R(z, A)\| < \infty \right\}.$$

If  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$ , then the exponential growth bound  $\omega(A)$  is defined as the supremum of all  $\omega \in \mathbb{R}$  such that  $(T_t)_{t \geq 0}$  is of type  $\omega$ , that is,

$$\omega(A) := \sup \left\{ \omega \in \mathbb{R} : \text{there exists } M_\omega > 0 \text{ such that} \right. \\ \left. \|T_t\| \leq M_\omega e^{-\omega t} \text{ for all } t \geq 0 \right\}.$$

We introduce  $\gamma$ -bounded analogues of these notions, as follows. First we set

$$s_0^\gamma(A) := \sup \left\{ \alpha \in \mathbb{R} : \sigma(A) \subset R_\alpha \text{ and} \right. \\ \left. \text{the set } \{R(z, A) : \operatorname{Re}(z) \leq \alpha\} \text{ is } \gamma\text{-bounded} \right\},$$

with the convention that  $s_0^\gamma(A) = -\infty$  if no set  $\{R(z, A) : \operatorname{Re}(z) \leq \alpha\}$  is  $\gamma$ -bounded. Second, if  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$ , and if the latter admits a  $\gamma$ -type, then we set

$$\omega^\gamma(A) := \sup \left\{ \omega \in \mathbb{R} : \text{the set } \{e^{\omega t} T_t : t \geq 0\} \text{ is } \gamma\text{-bounded} \right\}.$$

By convention we set  $\omega^\gamma(A) = -\infty$  if  $(T_t)_{t \geq 0}$  has no  $\gamma$ -type. See Example 2.3.1 for simple examples of such semigroups.

When  $X$  is a Hilbert space, the Gearhart-Prüss Theorem [1, Theorem 5.2.1] asserts that  $\omega(A) = s_0(A)$ . The main purpose of this section is to give an analogous equality  $\omega^\gamma(A) = s_0^\gamma(A)$  on  $K$ -convex Banach spaces.

It is obvious that  $\omega^\gamma(A) \leq \omega(A)$  and  $s_0^\gamma(A) \leq s_0(A)$ . The next inequality is more significant.

**Lemma 2.5.1.** *Assume that  $-A$  generates a  $C_0$ -semigroup. Then  $\omega(A) \leq s_0^\gamma(A)$ .*

*Proof.* Let  $\omega < \omega_0 < \omega(A)$ . By assumption, there exists  $M > 0$  such that  $\|T_s\| \leq M e^{-\omega_0 s}$  for any  $s \geq 0$ . Writing  $e^{\omega s} T_s = e^{(\omega - \omega_0)s} e^{\omega_0 s} T_s$ , we obtain that  $s \mapsto e^{\omega s} T_s x$  belongs to  $L^1((0, \infty), X)$  for any  $x \in X$ , with

$$\|s \mapsto e^{\omega s} T_s x\|_{L^1((0, \infty), X)} \leq \frac{M}{\omega - \omega_0} \|x\|. \tag{2.5.1}$$

For any  $\alpha \leq \omega$  and any  $t \in \mathbb{R}$ , we have

$$R(\alpha + it, A)x = - \int_0^\infty e^{its} e^{\alpha s} T(s)x ds \\ = - \int_0^\infty e^{its} e^{(\alpha - \omega)s} e^{\omega s} T_s x ds.$$

Since  $|e^{its} e^{(\alpha - \omega)s}| \leq 1$  for any  $s > 0$ , we derive that the set

$$\{R(\alpha + it, A) : t \in \mathbb{R}, \alpha \leq \omega\} \tag{2.5.2}$$

is included in the set

$$\left\{ \int_0^\infty \psi(s) e^{\omega s} T_s ds, : \psi \in L^\infty((0, \infty)), \|\psi\|_\infty \leq 1 \right\}.$$

By Lemma 2.2.13 and (2.5.1), the above set is  $\gamma$ -bounded. Therefore the set (2.5.2) is  $\gamma$ -bounded. Hence  $\omega < s_0^\gamma(A)$ . Passing to the supremum, this yields  $\omega(A) \leq s_0^\gamma(A)$ .  $\square$

Summarizing, we have

$$\omega^\gamma(A) \leq \omega(A) \leq s_0^\gamma(A) \leq s_0(A) \tag{2.5.3}$$

whenever  $-A$  generates a  $C_0$ -semigroup.

**Theorem 2.5.2.** *Let  $X$  be a  $K$ -convex Banach space. Let  $-A$  be the generator of a  $C_0$ -semigroup of  $\gamma$ -type  $\omega$  on  $X$ . Then  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $s$  for each  $s < s_0^\gamma(A)$ .*

*Proof.* We fix some  $s < s_0^\gamma(A)$ . If  $s \leq \omega$ , then  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $s$  by Theorem 2.4.1. Thus we may now assume that  $\omega < s$ .

Let  $N \in \mathbb{N}$ , let  $x_1, \dots, x_N \in X$  and let  $y_1, \dots, y_N \in X^\odot$ . According to Theorem 2.4.1, an estimate (2.4.1) is satisfied for any  $\alpha_1, \dots, \alpha_N < \omega$ . Consider  $s_1, \dots, s_N < s$  and let  $\alpha_1, \dots, \alpha_N < \omega$  be chosen such that

$$\omega - \alpha_k = s - s_k, \quad k \in \mathbb{N}_N. \quad (2.5.4)$$

By the resolvent identity, we have

$$R(s_k + it, A)x_k = (I + (\alpha_k - s_k)R(s_k + it, A))R(\alpha_k + it, A)x_k$$

for any  $k \in \mathbb{N}_N$  and any  $t \in \mathbb{R}$ . According to (2.5.4), this implies that

$$\sqrt{s - s_k}R(s_k + it, A)x_k = (I + (\omega - s)R(s_k + it, A))\sqrt{\omega - \alpha_k}R(\alpha_k + it, A)x_k.$$

Now define  $M_{s_1, \dots, s_n} : \mathbb{R} \times \mathbb{N}_N \rightarrow \mathcal{L}(X)$  by

$$M_{s_1, \dots, s_n}(t, k) := I + (\omega - s)R(s_k + it, A), \quad t \in \mathbb{R}, \quad k \in \mathbb{N}_N.$$

The range of  $M_{s_1, \dots, s_n}$  is included in the set

$$\{I + (\omega - s)R(\alpha + it, A), \quad \alpha \leq s\},$$

which is independent of  $s_1, \dots, s_k$ . The latter set is  $\gamma$ -bounded, by the definition of  $s_0^\gamma(A)$ . Let  $K > 0$  denote its  $\gamma$ -bounded constant. Applying Theorem 2.2.16 and (2.4.1), we obtain that

$$\begin{aligned} & \left\| (t, k) \mapsto \sqrt{s - s_k}R(s_k + it, A)x_k \right\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; X)} \\ & \leq K \left\| (t, k) \mapsto \sqrt{\omega - \alpha_k}R(\alpha_k + it, A)x_k \right\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; X)} \\ & \leq KC \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}. \end{aligned}$$

Since  $X$  is  $K$ -convex, the set  $\{I + (\omega - s)R(\alpha + it, A)^*, \quad \alpha \leq s\}$  is  $\gamma$ -bounded as well. Hence using (2.4.2) we obtain a similar estimate

$$\left\| (t, k) \mapsto \sqrt{s - s_k}R(s_k + it, A^*)y_k \right\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; X^*)} \leq K^*C \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G(X^*)}.$$

Now applying the implication “(ii)  $\Rightarrow$  (iii)” of Theorem 2.4.1, we obtain the desired result.  $\square$

**Corollary 2.5.3.** *Let  $X$  be a  $K$ -convex Banach space and let  $-A$  be the generator of a  $C_0$ -semigroup on  $X$ . If  $\omega^\gamma(A) > -\infty$ , then we have*

$$\omega^\gamma(A) = \omega(A) = s_0^\gamma(A).$$

*Proof.* If  $\omega^\gamma(A) > -\infty$ , then by Theorem 2.5.2,  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $s$  for any  $s < s_0^\gamma(A)$ . According to Theorem 2.4.1, this implies that  $(T_t)_{t \geq 0}$  is of  $\gamma$ -type  $s$  for any  $s < s_0^\gamma(A)$ . Thus  $\omega^\gamma(A) \geq s_0^\gamma(A)$ . Combining with (2.5.3), we obtain the result.  $\square$

*Example 2.5.4.* Let  $(\Omega, \mu)$  be a measure space and let  $1 < p < \infty$ . According to [1, Theorem 5.3.6], if  $-A$  is the generator of a *positive*  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on  $L^p(\Omega)$ , then  $s_0(A) = \omega(A)$ . If in addition  $\omega^\gamma(A) > -\infty$  then the equalities  $\omega^\gamma(A) = \omega(A) = s_0^\gamma(A) = s_0(A)$  hold by Corollary 2.5.3.

The equality  $\omega^\gamma(A) = \omega(A)$  in Corollary 2.5.3 implies the following statement.

**Corollary 2.5.5.** *Let  $(T_t)_{t \geq 0}$  is a bounded  $C_0$ -semigroup on some  $K$ -convex Banach space. If there exists  $\delta > 0$  such that  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded, then  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded for any  $\delta > 0$ .*

*Remark 2.5.6.* Let  $(T_t)_{t \geq 0}$  is a bounded  $C_0$ -semigroup on  $X$ , with generator  $-A$ . If  $A$  is bounded (equivalently, if  $(T_t)_{t \geq 0}$  is uniformly continuous), then the property considered in the above statement is true, that is,  $\{e^{-\delta t}T_t, t \geq 0\}$  is  $\gamma$ -bounded for any  $\delta > 0$ .

Indeed, consider  $\delta > 0$ . Since  $A$  is bounded and  $\sigma(A) \subset \overline{R_0}$ , there exists an open disk  $D$  such that  $\sigma(A + \delta) \subset D \subset R_0$ . Let  $\partial D$  be the boundary of  $D$  oriented counterclockwise. Then by the Dunford-Riesz calculus, we have

$$e^{-\delta t}T_t = \frac{1}{2\pi i} \int_{\partial D} e^{-t\lambda} R(\lambda, A + \delta) d\lambda$$

for any  $t \geq 0$ . Then a straightforward application of Lemma 2.2.13 shows that  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded.

We conclude this section with an observation and two questions. First we state a result that we recently obtained (with C. Le Merdy).

**Theorem 2.5.7.** *(Corollary 1.3.3) Let  $X$  be isomorphic to a separable Banach lattice with finite cotype such that  $X$  is not isomorphic to a Hilbert space. Then there exists  $A \in \mathcal{L}(X)$  such that  $\{e^{-tA} : t \geq 0\}$  is bounded but not  $\gamma$ -bounded.*

Combining this theorem with Remark 2.5.6, we obtain that Corollary 2.5.5 is sharp in the class of uniformly continuous semigroups. Namely on any  $K$ -convex separable Banach lattice not isomorphic to a Hilbert space (on  $L^p$  for  $1 < p \neq 2 < \infty$ , say) we obtain a uniformly continuous semigroup  $(T_t)_{t \geq 0}$  such that  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded for any  $\delta > 0$ ,  $\{T_t : t \geq 0\}$  is bounded but  $\{T_t : t \geq 0\}$  is not  $\gamma$ -bounded.

The role of  $K$ -convexity in Corollary 2.5.5 is rather unclear. This leads to the following question:

*Question 2.5.8.* Can the assumption in Corollary 2.5.5 that  $X$  is a  $K$ -convex space be dropped?

We recall (2.5.3) and the existence of  $-A$  generating a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  such that  $\omega(A) < s_0(A)$ . So we ask

*Question 2.5.9.* Does there exist an operator  $A$  such that  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$ , satisfying  $\omega(A) < s_0^\gamma(A)$  ?



## 2.6. An overview

Let  $\omega \in \mathbb{R}$  and let  $A$  be a half-plane type operator on some Banach space  $X$ . Either in [9] or in the present paper, the following six properties are considered:

- (i)  $A$  has a bounded  $H^\infty$ -functional calculus of type  $\omega$ .
- (ii)  $A$  has a 1-bounded functional calculus of type  $\omega$ .
- (iii)  $-A$  generates a  $C_0$ -semigroup of type  $\omega$ .
- (i) $_\gamma$   $A$  has a  $\gamma$ -bounded  $H^\infty$ -functional calculus of type  $\omega$ .
- (ii) $_\gamma$   $A$  has a  $\gamma$ -1-bounded functional calculus of type  $\omega$ .
- (iii) $_\gamma$   $-A$  generates a  $C_0$ -semigroup of  $\gamma$ -type  $\omega$ .

The aim of this last section is to give an overview of the relations between these properties, at least on  $K$ -convex spaces. This will require the analysis of a specific example, see Proposition 2.6.1 below. In the above list, we have deliberately omitted the  $m$ -bounded and  $\gamma$ - $m$ -bounded functional calculi.

It follows from Proposition 2.2.9 and [9, Theorem 6.4] that

$$(i) \Rightarrow (ii) \Rightarrow (iii).$$

Likewise it follows from Remark 2.3.3 (3) and Remark 2.3.9 that

$$(i)_\gamma \Rightarrow (ii)_\gamma \Rightarrow (iii)_\gamma.$$

The implication “ $(ii) \Rightarrow (i)$ ” is wrong. Indeed it follows from either [6] or [44] that on any infinite dimensional Hilbert space  $H$ , there exists a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on  $H$  whose negative generator  $A$  does not have a bounded  $H^\infty$ -functional calculus of type 0. Thus with  $\omega = 0$ ,  $A$  satisfies (iii) and does not satisfy (i). Moreover (ii) and (iii) are equivalent on Hilbert space, by [9, Theorem 7.1]. This proves the result.

Since  $\gamma$ -boundedness and uniform boundedness are equivalent on Hilbert space, the above also shows that the implication “ $(ii)_\gamma \Rightarrow (i)_\gamma$ ” is wrong.

The implication “ $(iii) \Rightarrow (ii)$ ” is wrong. Indeed let  $1 < p \neq 2 < +\infty$  and let  $(T_t)_{t \in \mathbb{R}}$  be the right translation group on  $L^p(\mathbb{R})$ , which is a bounded  $C_0$ -group. Let  $-A$  denote its generator. It follows from Gomilko’s paper [16] that either  $A$  or  $-A$  does not have a 1-bounded functional calculus of type 0.

We have shown in Theorem 2.4.1 that if  $X$  is  $K$ -convex, then the implication “ $(iii)_\gamma \Rightarrow (ii)_\gamma$ ” holds true. We do not know whether “ $(iii)_\gamma \Rightarrow (ii)_\gamma$ ” holds true on any Banach space.

The implication “ $(iii)_\gamma \Rightarrow (ii)$ ” holds true, by [22, Theorem 6.4] (see Remark 2.4.3 for more on this.)

We noticed above that  $(iii)$  does not imply  $(i)$  on Hilbert space. Consequently, the implication “ $(iii)_\gamma \Rightarrow (i)$ ” is wrong.

The only remaining question is whether  $(i)$  implies  $(iii)_\gamma$ . We are going to show that this is wrong on sufficiently bad spaces, see Example 2.6.2 below.

For this purpose we introduce a class of  $C_0$ -(semi)groups of independent interest. Recall the Gaussian space  $G(X)$  from Subsection 2.3. We will use the so-called ‘contraction principle’ [26, Theorem 6.1.13], which says that for any  $x_1, \dots, x_n \in X$  and any

$\alpha_1, \dots, \alpha_n \in \mathbb{C}$ , we have

$$\left\| \sum_{k=1}^n \alpha_k \gamma_k \otimes x_k \right\|_{G(X)} \leq \sup_k |\alpha_k| \left\| \sum_{k=1}^n \gamma_k \otimes x_k \right\|_{G(X)}. \quad (2.6.1)$$

We recall that  $X$  has property  $(\alpha)$  (see [26, section 7.5] for more details) if there exists a constant  $C \geq 1$  such that for any finite family  $(x_{ij})$  in  $X$  and any finite family  $(t_{ij})$  in  $\mathbb{C}$ , we have

$$\left\| \sum_{i,j} \gamma_i \otimes \gamma_j \otimes t_{ij} x_{ij} \right\|_{G(G(X))} \leq C \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \gamma_i \otimes \gamma_j \otimes x_{ij} \right\|_{G(G(X))}. \quad (2.6.2)$$

Banach spaces with property  $(\alpha)$  have a finite cotype, thus  $R$ -boundedness and  $\gamma$ -boundedness are equivalent on such spaces. We recall that the class of all Banach spaces with property  $(\alpha)$  is stable under taking subspaces and that all Banach lattices with a finite cotype have property  $(\alpha)$ . In particular, for any  $1 \leq p < \infty$ ,  $L^p$ -spaces and their subspaces have property  $(\alpha)$ .

Let  $(\xi_k)_{k \geq 1}$  be a sequence of distinct points of  $\mathbb{R}$ . For any finite Gaussian sum  $\sum_{k=1}^n \gamma_k \otimes x_k$ , with  $x_1, \dots, x_n \in X$ , we let

$$T_t \left( \sum_{k=1}^n \gamma_k \otimes x_k \right) := \sum_{k=1}^n e^{-it\xi_k} \gamma_k \otimes x_k, \quad t \in \mathbb{R}. \quad (2.6.3)$$

Then we have

$$\left\| T_t \left( \sum_{k=1}^n \gamma_k \otimes x_k \right) \right\| \leq \left\| \sum_{k=1}^n \gamma_k \otimes x_k \right\|,$$

by (2.6.1). Since the finite Gaussian sums are dense in  $G(X)$ , each  $T_t$  extends to a bounded linear operator on  $G(X)$  (still denoted by  $T_t$ ), with  $\|T_t\| = 1$ . Furthermore  $(T_t)_{t \in \mathbb{R}}$  is a  $C_0$ -group. Indeed it is plain that for any finite Gaussian sum  $z = \sum_{k=1}^n \gamma_k \otimes x_k$ ,  $T_t(z) \rightarrow z$  when  $t \rightarrow 0$ . Then the strong continuity of  $(T_t)_{t \in \mathbb{R}}$  follows from the uniform boundedness of  $(T_t)_{t \in \mathbb{R}}$  and the density of the set of all finite Gaussian sums in  $G(X)$ .

**Proposition 2.6.1.** *Let  $-A$  denote the generator of the  $C_0$ -group  $(T_t)_{t \in \mathbb{R}}$  defined by (2.6.3).*

- (1)  *$A$  has a bounded  $H^\infty$ -functional calculus of type 0.*
- (2) *The  $C_0$ -group  $(T_t)_{t \in \mathbb{R}}$  is  $\gamma$ -bounded if and only if  $X$  has property  $(\alpha)$ .*

*Proof.* For any  $b \in L^1(\mathbb{R})$ , we let  $\int_{\mathbb{R}} b(t) T_t dt \in \mathcal{L}(G(X))$  denote the operator defined by

$$\left( \int_{\mathbb{R}} b(t) T_t dt \right) (z) = \int_0^\infty b(t) T_t(z) dt, \quad z \in G(X).$$

If  $b \in L^1(\mathbb{R}_+)$ , we let  $\mathcal{L}b$  denote the Laplace transform of  $b$ , that is,

$$\mathcal{L}b(z) = \int_0^\infty e^{-zt} b(t) dt, \quad z \in \overline{R_0}.$$

Obviously  $\mathcal{L}b$  is continuous and bounded on  $\overline{R_0}$  and its restriction to  $R_0$  belongs to  $H^\infty(R_0)$ .

For any  $b \in L^1(\mathbb{R})$  and any  $x_1, \dots, x_n \in X$ , we have

$$\left( \int_{\mathbb{R}} b(t) T_t dt \right) \left( \sum_{k=1}^n \gamma_k \otimes x_k \right) = \int_{\mathbb{R}} b(t) \sum_{k=1}^n e^{-i\xi_k t} \gamma_k \otimes x_k dt = \sum_{k=1}^n \widehat{b}(\xi_k) \gamma_k \otimes x_k. \quad (2.6.4)$$

If  $b \in L^1(\mathbb{R}_+)$ , this implies, using (2.6.1), that

$$\left\| \int_0^\infty b(t) T_t dt \right\|_{\mathcal{L}(G(X))} \leq \| \mathcal{L}b \|_{H^\infty(R_0)}. \quad (2.6.5)$$

Now let  $\alpha < 0$  and let  $f \in \mathcal{E}(R_\alpha)$ . According to [20, Lemma 5.1], there is a (necessarily unique)  $b \in L^1(\mathbb{R}_+)$  such that  $f = \mathcal{L}b$  on  $R_0$  and

$$f(A) = \int_0^\infty b(t) T_t dt.$$

The estimate (2.6.5) therefore implies that

$$\| f(A) \| \leq \| \mathcal{L}b \|_{H^\infty(R_0)} = \| f \|_{H^\infty(R_0)}.$$

This shows (1).

We now turn to the proof of (2). First assume that  $X$  has property  $(\alpha)$ . Let  $(t_j)_j$  be a finite family of real numbers and for any  $j$ , let  $z_j = \sum_k \gamma_j \otimes x_{jk}$  be a finite Gaussian sum. We have

$$\sum_j \gamma_j \otimes T_{t_j}(z_j) = \sum_{i,j} e^{-it_j \xi_k} \gamma_j \otimes \gamma_k \otimes x_{jk}.$$

Applying (2.6.2) we deduce that

$$\begin{aligned} \left\| \sum_j \gamma_j \otimes T_{t_j}(z_j) \right\|_{G(G(X))} &\leq C \left\| \sum_{i,j} \gamma_j \otimes \gamma_k \otimes x_{jk} \right\|_{G(G(X))} \\ &= C \left\| \sum_j \gamma_j \otimes z_j \right\|_{G(G(X))}. \end{aligned}$$

Since the set of all finite Gaussian sums is dense in  $G(X)$ , this shows that  $(T_t)_{t \in \mathbb{R}}$  is  $\gamma$ -bounded.

Assume on the contrary that  $X$  does not have property  $(\alpha)$ . By (2.6.1), there exists a (necessarily unique) contractive, non degenerate, homomorphism

$$w : C_0(\mathbb{R}) \rightarrow \mathcal{L}(G(X))$$

such that

$$w(f) \left( \sum_{k=1}^n \gamma_k \otimes x_k \right) = \sum_{k=1}^n f(\xi_k) \gamma_k \otimes x_k$$

for any  $n \geq 1$  and any  $x_1, \dots, x_n \in X$ . We claim that the set

$$S = \{ w(f) : f \in C_0(\mathbb{R}), \|f\|_\infty \leq 1 \}$$

is not  $\gamma$ -bounded. Indeed let  $(t_{ik})$  and  $(x_{ik})$  be finite families in  $\mathbb{C}$  and  $X$ , respectively, and assume that  $|t_{ik}| \leq 1$  for any  $i, k$ . There exist  $f_i \in C_0(\mathbb{R})$  such that  $f_i(\xi_k) = t_{ik}$  and  $\|f_i\|_\infty \leq 1$  for any  $i, k$ . Then

$$\sum_{i,k} \gamma_i \otimes \gamma_k \otimes t_{ik} x_{ik} = \sum_i \gamma_i \otimes w(f_i) \left( \sum_k \gamma_k \otimes x_{ik} \right).$$

If  $S$  were  $\gamma$ -bounded, this would imply that the norm of the left hand side is dominated by the norm of  $\sum_{i,k} \gamma_i \otimes \gamma_k \otimes x_{ik}$ , which would imply property  $(\alpha)$ .

Note that the homomorphism  $w$  ‘extends’  $(T_t)_{t \in \mathbb{R}}$  in the sense of [38, Definition 2.4]. Indeed, according to (2.6.4), we have

$$\int_{\mathbb{R}} b(t) T_t dt = w(\hat{b})$$

for any  $b \in L^1(\mathbb{R})$ . Therefore if  $(T_t)_{t \in \mathbb{R}}$  were  $\gamma$ -bounded, then according to [38, Theorem 4.4], the above set  $S$  would be  $\gamma$ -bounded. We just noticed that this does not hold true. Hence,  $(T_t)_{t \in \mathbb{R}}$  is not  $\gamma$ -bounded.  $\square$

*Example 2.6.2.* Proposition 2.6.1 provides an example of an operator which satisfies (i) without satisfying  $(iii)_\gamma$ , for  $\omega = 0$ . Indeed, assume that  $X$  does not have property  $(\alpha)$  and let  $(T_t)_{t \in \mathbb{R}}$  (with generator  $-A$ ) be given by the above proposition. Changing  $T_t$  into  $T_{-t}$ , part (1) of Proposition 2.6.1 shows that  $A$  and  $-A$  have a bounded  $H^\infty$ -bounded functional calculus of type 0. However by part (2) of Proposition 2.6.1, either  $(T_t)_{t \geq 0}$  or  $(T_{-t})_{t \geq 0}$  is not  $\gamma$ -bounded.

We do not know if (i) implies  $(iii)_\gamma$  on non Hilbertian Banach spaces with property  $(\alpha)$ . In particular, we do not know if (i) implies  $(iii)_\gamma$  on  $L^p$ -spaces, for  $1 < p \neq 2 < \infty$ .



# Chapter 3

## Derivative bounded functional calculus of power bounded operators on Banach spaces

### 3.1. Introduction

Let  $X$  be a Banach space, a bounded operator  $T : X \rightarrow X$  is called power-bounded when  $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$ . In this case the spectrum of  $T$  is contained in the closed unit ball  $\overline{\mathbb{D}}$ . We say that  $T$  is polynomially bounded when it satisfies an estimate

$$\|P(T)\| \leq C \sup\{|P(z)| : z \in \mathbb{D}\}, \quad P \text{ polynomial.} \quad (3.1.1)$$

When  $X = H$  is an Hilbert space and  $T$  is a contraction, that is  $\|T\| \leq 1$  (which is a power-bounded operator) it is well-known that  $T$  satisfies (3.1.1) with  $C = 1$ . This is the so-called von Neumann's inequality. However even on Hilbert space we cannot expect that any power-bounded operator satisfies (3.1.1) (see [40]). A natural question is whether one can obtain similar estimates, for power-bounded operators, replacing the norm uniform in the right-hand side of (3.1.1) by an another function norm. An answer is given by Peller in [47] for power-bounded operators on Hilbert space. If  $T$  is such operator, then it satisfies

$$\|P(T)\| \leq C \|P\|_{\mathcal{B}(\mathbb{D})}, \quad P \text{ polynomial.} \quad (3.1.2)$$

where  $\|\cdot\|_{\mathcal{B}(\mathbb{D})}$  is an appropriate Besov norm (see section 3).

In the first part of this paper we deal with operators which satisfy the discrete Gomilko Shi-Feng condition: an operator  $T : X \rightarrow X$  satisfies this condition if the spectrum of  $T$  is included in  $\overline{\mathbb{D}}$  and

$$\int_0^{2\pi} |\langle R(re^{it}, T)^2 x, x^* \rangle| dt \leq \frac{C}{(r^2 - 1)} \|x\| \|x^*\|, \quad r > 1, x \in X, x^* \in X^*. \quad (3.1.3)$$

This condition was first introduced in [17]. The continuous case had been introduced before in [16] and [53] and studied extensively in [9], [7] and chapter 2. We will show that this condition implies power boundedness and that the converse is true when  $X = H$  is an Hilbert space. We will be able to show (see section 3) that a bounded operator  $T$  on a general Banach space  $X$  satisfies an estimate (3.1.2) if and only if  $T$  satisfies the discrete Gomilko Shi-Feng condition (3.1.3). Furthermore one of the main results of this paper is

to show that the discrete Gomilko Shi-Feng condition is equivalent to the boundedness of the set

$$\left\{ (r-1)\phi'(T) : r > 1 \text{ and } \phi \in H^\infty(r\mathbb{D}) \text{ with } \sup\{|\phi(z)| : z \in r\mathbb{D}\} \leq 1 \right\}. \quad (3.1.4)$$

This is weaker than (3.1.1) in the sense that if an operator  $T : X \rightarrow X$  satisfies (3.1.1) then (3.1.4) is bounded. The converse is false even on Hilbert space. The continuous analogue of the equivalence of (3.1.3) and (3.1.4) is in [9].

The boundedness of (3.1.4) is a way to characterize power bounded operators on Hilbert space or operators with discrete Gomilko Shi-Feng condition on general Banach space. However (3.1.4) is not a usual bounded functional calculus because we have an estimate of  $\|\phi'(T)\|$  and not  $\|\phi(T)\|$ .

The second part of this article (section 4) is devoted to ‘ $\gamma$ -versions’ of the previous results. By a ‘ $\gamma$ -version’, we mean replacing operator norm boundedness by the so-called stronger notion of  $\gamma$ -boundedness (see [26] and references therein). This section starts with some basics about  $\gamma$ -boundedness,  $\gamma$ -operators and  $\gamma'$ -operators (see [29]). We will introduce a  $\gamma$ -analogue of the discrete Gomilko Shi-Feng condition for a bounded operator  $T : X \rightarrow X$  with spectrum included in  $\overline{\mathbb{D}}$  as follows : there exists  $C > 0$  such that for any  $N \in \mathbb{N}$ , for any  $r_1, \dots, r_N > 1$ , and for any  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ , we have

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} | \langle (r_k + 1)(r_k - 1)R(r_k e^{it}, T)^2 x_k, x_k^* \rangle | dt \\ & \leq C \| (x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)} \| (x_k^*)_{k \in \mathbb{N}_N} \|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned} \quad (3.1.5)$$

We will show that  $T$  satisfies this condition if and only if the set in (3.1.4) is  $\gamma$ -bounded. Moreover, we will show that  $T$  is power  $\gamma$ -bounded, that is the set  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded, if and only if  $T$  satisfies (3.1.5). It is important to notice that these results are stated without any assumption on  $X$ . We will conclude this paper with a  $\gamma$ -version of [47, Corollary 3.7].

Finally we give some notation to be used along this paper. We write  $\mathbb{N} = \{1, 2, \dots\}$  for the natural numbers and for  $N \in \mathbb{N}$  we write  $\mathbb{N}_N = \{1, 2, \dots, N\}$  the first  $N$  natural numbers. For any Banach spaces  $X$  and  $Y$ , we denote by  $B(X, Y)$  the algebra of all bounded operators from  $X$  into  $Y$  equipped with the operator norm, and we set  $B(X) := B(X, X)$ . For  $T \in B(X)$  we denote by  $\sigma(T)$  the spectrum of  $T$ . For  $\lambda \in \mathbb{C} \setminus \sigma(T)$ , we put  $R(\lambda, T) = (\lambda I_X - T)^{-1}$  the resolvent operator. We let  $\mathbb{D}$  and  $\overline{\mathbb{D}}$  respectively the open and closed unit disk. Also the open disk of radius  $r$  centered at 0 and will be denoted by  $r\mathbb{D}$  while the closed disk of radius  $r$  centered at 0 and will be denoted by  $r\overline{\mathbb{D}}$ . Also. For any  $r \in \mathbb{R}$  we denote by  $H^\infty(r\mathbb{D})$  the space of all bounded analytic functions  $\phi : r\mathbb{D} \rightarrow \mathbb{C}$ . This is a Banach space for the norm

$$\|\phi\|_{H^\infty(r\mathbb{D})} := \sup\{|\phi(z)| : z \in r\mathbb{D}\}.$$

### 3.2. Discrete Gomilko Shi-Feng condition and derivative functional calculus.

#### Discrete Gomilko Shi-Feng condition

Let  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . For  $r > 1$ ,  $\phi \in H^\infty(r\mathbb{D})$  and  $m \in \mathbb{N}$ , we denote by  $\phi^{(m)}(T)$  the operator obtained with Riesz-Dunford calculus. Since  $\phi \in H^\infty(r\mathbb{D})$  then for each  $1 < \rho < r$ ,  $\phi^{(m)} \in H^\infty(\rho\mathbb{D})$  and therefore the Riesz-Dunford calculus can be applied to the function  $\phi^{(m)}$  and the operator  $T$ . We obtain

$$\phi^{(m)}(T) = \frac{1}{2\pi} \int_0^{2\pi} \rho e^{it} \phi^{(m)}(\rho e^{it}) R(\rho e^{it}, T) dt \in B(X), \quad 1 < \rho < r. \quad (3.2.1)$$

**Lemma 3.2.1.** *Let  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$  and let  $m \in \mathbb{N}$ . Then for each  $r > 1$  and  $\phi \in H^\infty(r\mathbb{D})$ , one has*

$$\phi^{(m)}(T) = \frac{m!}{2\pi} \int_0^{2\pi} \rho e^{it} \phi(\rho e^{it}) R(\rho e^{it}, T)^{m+1} dt, \quad 1 < \rho < r. \quad (3.2.2)$$

*Proof.* Let  $r > 1$ ,  $\phi \in H^\infty(r\mathbb{D})$  and  $1 < \rho < r$ . For each  $n \in \mathbb{N}$ ,

$$\frac{d}{dt} R(\rho e^{it}, T)^n = -in\rho e^{it} R(\rho e^{it}, T)^{n+1}.$$

Using (3.2.1) it follows by  $m$  integrations by parts,

$$\begin{aligned} \phi^{(m)}(T) &= \frac{1}{2\pi} \int_0^{2\pi} \rho e^{it} \phi^{(m-1)}(\rho e^{it}) R(\rho e^{it}, T)^2 dt \\ &= \dots = \frac{m!}{2\pi} \int_0^{2\pi} \rho e^{it} \phi(\rho e^{it}) R(\rho e^{it}, T)^{m+1} dt. \end{aligned}$$

□

We now investigate the discrete Gomilko Shi-Feng condition which appears in [17] in the case  $m = 1$ . The case  $m > 1$  is inspired by [9, Proposition 6.3].

**Definition 3.2.2.** Let  $m \geq 1$  be an integer. We say that  $T \in B(X)$  has the property  $(GFS)_m$  if  $\sigma(T) \subset \overline{\mathbb{D}}$  and there exists  $C > 0$  such that

$$\int_0^{2\pi} |\langle R(re^{it}, T)^{m+1} x, x^* \rangle| dt \leq \frac{C}{(r+1)(r-1)^m} \|x\| \|x^*\|, \quad r > 1, x \in X, x^* \in X^*. \quad (3.2.3)$$

**Proposition 3.2.3.** *Let  $T$  with  $(GFS)_m$ . Then  $T$  is power bounded.*

*Proof.* Let  $\phi(z) = \frac{z^{n+m}}{(n+1)\dots(n+m)}$ , then  $\phi \in H^\infty(r\mathbb{D})$  for any  $r > 1$  with

$$\|\phi\|_{H^\infty(r\mathbb{D})} = \frac{r^{n+m}}{(n+1)\dots(n+m)}$$

and furthermore  $\phi^{(m)}(z) = z^n$ . By (3.2.2), one has for  $r > 1$ ,

$$T^n = \frac{m!}{2\pi} \int_0^{2\pi} r e^{it} \phi(r e^{it}) R(r e^{it}, T)^{m+1} dt.$$



It follows, for  $x \in X$  and  $x^* \in X^*$ ,

$$\begin{aligned} |\langle T^n x, x^* \rangle| &\leq \frac{rm!}{2\pi} \|\phi\|_{H^\infty(r\mathbb{D})} \int_0^{2\pi} |\langle R(re^{it}, T)^{m+1} x, x^* \rangle| dt \\ &\leq \frac{Cr^{n+m+1}m!}{2\pi(n+1)\dots(n+m)(r+1)(r-1)^m} \|x\| \|x^*\|. \end{aligned} \quad (3.2.4)$$

Therefore,

$$\|T^n\| \leq \frac{m!Cr^{n+m+1}}{2\pi(n+1)\dots(n+m)(r-1)^m}.$$

Letting  $r = 1 + \frac{1}{n}$  one obtains

$$\frac{m!Cr^{n+m+1}}{2\pi(n+1)\dots(n+m)(r-1)^m} = \frac{m!Cn^m(1 + \frac{1}{n})^n(1 + \frac{1}{n})^{m+1}}{2\pi(n+1)\dots(n+m)}.$$

But for each  $n \in \mathbb{N}$  one has

$$\frac{n^m}{(n+1)\dots(n+m)} \leq 1 \quad \text{and} \quad \left(1 + \frac{1}{n}\right)^{m+1} \leq 2^{m+1}$$

and it is well-known that

$$\sup_{n \in \mathbb{N}} \left(1 + \frac{1}{n}\right)^n = e.$$

Finally, for each  $n \in \mathbb{N}$ ,

$$\|T^n\| \leq \frac{2^m m! C e}{\pi}.$$

□

*Remark 3.2.4.* We recall that  $T \in B(X)$  is a Ritt operator if  $\sigma(T) \subset \overline{\mathbb{D}}$  and there exists  $C \geq 0$  such that

$$\forall \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}, \quad \|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|}.$$

Ritt operators play a prominent role in the theory of functional calculus. Indeed they have a specific  $H^\infty$  functional calculus (see [39] and references therein).

It is easy to check that a Ritt operator has  $(GFS)_1$ . Indeed let  $T$  be a Ritt operator and  $r > 1$ , let  $x \in X$  and  $x^* \in X^*$ , one has

$$\begin{aligned} \int_0^{2\pi} |\langle R(re^{it}, T)^2 x, x^* \rangle| dt &\leq \int_0^{2\pi} \frac{C}{|re^{it} - 1|^2} \|x\| \|x^*\| dt \\ &= \int_0^{2\pi} \frac{C}{r^2 + 1 - 2r\cos(t)} \|x\| \|x^*\| dt. \end{aligned}$$

Now by the residue method (see for example [11, p.99])

$$\int_0^{2\pi} \frac{1}{r^2 + 1 - 2r\cos(t)} dt = \frac{2\pi}{(r+1)(r-1)}, \quad (3.2.5)$$

therefore one has

$$\int_0^{2\pi} |\langle R(re^{it}, T)^2 x, x^* \rangle| dt \leq \frac{2\pi C}{(r+1)(r-1)} \|x\| \|x^*\|.$$

We will need the following stability property. We skip the easy proof.

**Proposition 3.2.5.** *Let  $S \in B(X)$  which has  $(GFS)_m$ . If  $T$  is similar to  $S$ , that is there exists  $U \in B(X)$  invertible such that  $T = USU^{-1}$ , then  $T$  has  $(GFS)_m$ .*

We will see that  $T$  has  $(GFS)_m$  if and only if  $T$  has  $(GFS)_1$ . Let us begin by one implication. For the reverse implication we will use a derivative functional calculus that we introduce in the next sub-section.

**Proposition 3.2.6.** *Let  $m \geq 1$  an integer and  $T \in B(X)$ . If  $T$  has  $(GFS)_m$  then  $T$  has  $(GFS)_k$  for each  $1 \leq k \leq m$ .*

*Proof.* Let  $r > 1$  and assume (3.2.3). We show by downward induction that for  $1 \leq k \leq m$ ,

$$\int_0^{2\pi} |\langle R(re^{it}, T)^{k+1} x, x^* \rangle| dt \leq \frac{Cm}{k(r+1)(r-1)^k} \|x\| \|x^*\|, \quad x \in X, x^* \in X^*. \quad (3.2.6)$$

Let  $x \in X, x^* \in X^*$ . Suppose that (3.2.6) is satisfied for  $k$  with  $1 < k+1 \leq m$ . Using the fact that  $\frac{d}{dr} R(re^{it}, T)^n = -ne^{it} R(re^{it}, T)^{n+1}$  for each  $n \in \mathbb{N}$  and Fubini's Theorem, one has

$$\begin{aligned} \int_0^{2\pi} \langle R(re^{it}, T)^k x, x^* \rangle dt &= \int_0^{2\pi} \int_r^{+\infty} \langle ke^{it} R(ue^{it}, T)^{k+1} x, x^* \rangle dudt \\ &= k \int_r^{+\infty} \int_0^{2\pi} e^{it} \langle R(ue^{it}, T)^{k+1} x, x^* \rangle dt du. \end{aligned} \quad (3.2.7)$$

It follows using (3.2.6),

$$\begin{aligned} \int_0^{2\pi} |\langle R(re^{it}, T)^k x, x^* \rangle| dt &\leq k \int_r^{+\infty} \int_0^{2\pi} |\langle R(ue^{it}, T)^{k+1} x, x^* \rangle| dt du \\ &\leq k \int_r^{+\infty} \frac{Cm}{k(u+1)(u-1)^k} \|x\| \|x^*\| du \\ &\leq \int_r^{+\infty} \frac{Cm}{(r+1)(u-1)^k} \|x\| \|x^*\| du \\ &= \frac{Cm}{(k-1)(r+1)(r-1)^{k-1}} \|x\| \|x^*\|. \end{aligned}$$

□

### $m$ -derivative bounded functional calculus

We begin by giving a basic result which is a straightforward consequence of Cauchy inequalities.

**Lemma 3.2.7.** *Let  $r > 0$  and  $\phi \in H^\infty(r\mathbb{D})$ . Then for each  $\rho < r$ ,  $\phi^{(m)} \in H^\infty(\rho\mathbb{D})$  with*

$$\left\| \phi^{(m)} \right\|_{H^\infty(\rho\mathbb{D})} \leq \frac{m!}{(r-\rho)^m} \|\phi\|_{H^\infty(r\mathbb{D})}. \quad (3.2.8)$$

This Lemma justifies the following definition.

**Definition 3.2.8.** Let  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$  and  $m \in \mathbb{N}$ . Then  $T$  is said to have  $m$ -derivative bounded functional calculus if there exists  $C > 0$  such that

$$\left\| \phi^{(m)}(T) \right\| \leq \frac{C}{(r-1)^m} \|\phi\|_{H^\infty(r\mathbb{D})} \quad \text{for each } \phi \in H^\infty(r\mathbb{D}) \text{ and } r > 1. \quad (3.2.9)$$

*Remark 3.2.9.* It is easy to check that (3.2.9) is equivalent to:

$$\left\| \phi^{(m)}(rT) \right\| \leq \frac{C}{(1-r)^m} \|\phi\|_{H^\infty(\mathbb{D})} \quad \text{for each } \phi \in H^\infty(\mathbb{D}) \quad \text{and } 0 < r < 1.$$

**Theorem 3.2.10.** *Let  $T \in B(X)$  with  $\sigma(T) \subset \bar{\mathbb{D}}$ . The following assertions are equivalent for  $m \in \mathbb{N}$ ,*

- (i)  $T$  has  $(GFS)_m$ ;
- (ii)  $T$  has 1-derivative bounded functional calculus;
- (iii)  $T$  has  $m$ -derivative bounded functional calculus.

*Proof.* (i)  $\Rightarrow$  (ii): According to Proposition 3.2.6  $T$  has  $(GFS)_1$ . Let  $r > \rho > 1$  and  $\phi \in H^\infty(r\mathbb{D})$ . Then using (3.2.2),

$$\begin{aligned} \|\phi'(T)\| &= \frac{1}{2\pi} \sup_{\|x\|=1} \sup_{\|x^*\|=1} \left| \int_0^{2\pi} \langle \rho e^{it} \phi(\rho e^{it}) R(\rho e^{it}, T)^2 x, x^* \rangle dt \right| \\ &\leq \frac{\rho \|\phi\|_{H^\infty(\rho\mathbb{D})}}{2\pi} \sup_{\|x\|=1} \sup_{\|x^*\|=1} \int_0^{2\pi} |\langle R(\rho e^{it}, T)^2 x, x^* \rangle| dt \\ &\leq \frac{C\rho}{2\pi(\rho-1)(\rho+1)} \|\phi\|_{H^\infty(\rho\mathbb{D})} \leq \frac{C}{2\pi(\rho-1)} \|\phi\|_{H^\infty(r\mathbb{D})}. \end{aligned}$$

The second to last inequality comes from (3.2.3) with  $m = 1$ . Letting  $\rho \rightarrow r$  yields

$$\|\phi'(T)\| \leq \frac{C}{2\pi(r-1)} \|\phi\|_{H^\infty(r\mathbb{D})}.$$

(ii)  $\Rightarrow$  (iii): Let  $1 < \rho < r$  and  $\phi \in H^\infty(r\mathbb{D})$ . According to Lemma 3.2.7  $\phi^{(m-1)} \in H^\infty(\rho\mathbb{D})$  so applying the 1-derivative bounded functional calculus to  $\phi^{(m-1)}$  and again Lemma 3.2.7 one has

$$\begin{aligned} \left\| \phi^{(m)}(T) \right\| &= \left\| (\phi^{(m-1)})'(T) \right\| \leq \frac{C}{\rho-1} \left\| \phi^{(m-1)} \right\|_{H^\infty(\rho\mathbb{D})} \\ &\leq \frac{(m-1)!C}{(\rho-1)(r-\rho)^{m-1}} \|\phi\|_{H^\infty(r\mathbb{D})}. \end{aligned}$$

Taking  $\rho = \frac{r+1}{2}$  yields

$$\left\| \phi^{(m)}(T) \right\| \leq \frac{2^m(m-1)!C}{(r-1)^m} \|\phi\|_{H^\infty(r\mathbb{D})}.$$

(iii)  $\Rightarrow$  (i): Let  $r > 1$ ,  $1 < \rho < r$ ,  $x \in X$  and  $x^* \in X^*$ . There exists a measurable function  $\varepsilon : [0, 2\pi] \rightarrow \{z \in \mathbb{C}, |z| = 1\}$  such that for each  $t \in [0, 2\pi]$

$$|\langle R(re^{it}, T)^{m+2} x, x^* \rangle| = \varepsilon(t) \langle R(re^{it}, T)^{m+2} x, x^* \rangle.$$

Let define  $\phi(z) := \frac{1}{(m+1)!} \int_0^{2\pi} \frac{\varepsilon(t)}{(re^{it} - z)^2} dt$  for  $z \in \rho\mathbb{D}$ . Then  $\phi \in H^\infty(\rho\mathbb{D})$ , with

$$\|\phi\|_{H^\infty(\rho\mathbb{D})} \leq \frac{2\pi}{(m+1)!(r+\rho)(r-\rho)}. \quad (3.2.10)$$

Indeed for each  $\theta \in [0, 2\pi)$ , using (3.2.5),

$$\begin{aligned} \int_0^{2\pi} \frac{1}{|re^{it} - \rho e^{i\theta}|^2} dt &= \frac{1}{\rho^2} \int_0^{2\pi} \frac{1}{\left|\frac{r}{\rho}e^{it} - 1\right|^2} dt \\ &= \frac{2\pi}{\rho^2\left(\frac{r}{\rho} + 1\right)\left(\frac{r}{\rho} - 1\right)} \\ &= \frac{2\pi}{(r + \rho)(r - \rho)}. \end{aligned}$$

Moreover

$$\phi^{(m)}(z) = \int_0^{2\pi} \frac{\varepsilon(t)}{(re^{it} - z)^{m+2}} dt, \quad z \in \rho\mathbb{D}.$$

By Fubini's Theorem one has

$$\phi^{(m)}(T) = \int_0^{2\pi} \varepsilon(t) R(re^{it}, T)^{m+2} dt,$$

and finally using (3.2.9) and (3.2.10),

$$\begin{aligned} \int_0^{2\pi} |\langle R(re^{it}, T)^{m+2} x, x^* \rangle| dt &= \int_0^{2\pi} \varepsilon(t) |\langle R(re^{it}, T)^{m+2} x, x^* \rangle| dt \\ &= |\langle \phi^{(m)}(T)x, x^* \rangle| \leq \|\phi^{(m)}(T)\| \|x\| \|x^*\| \\ &\leq \frac{C}{(\rho - 1)^m} \|\phi\|_{H^\infty(\rho\mathbb{D})} \|x\| \|x^*\| \\ &\leq \frac{2\pi C}{(m+1)!(r+\rho)(\rho-1)^m(r-\rho)} \|x\| \|x^*\| \\ &\leq \frac{2\pi C}{(m+1)!(r+1)(\rho-1)^m(r-\rho)} \|x\| \|x^*\|. \end{aligned}$$

Taking  $\rho = \frac{r+1}{2}$  one obtains

$$\int_0^{2\pi} |\langle R(re^{it}, T)^{m+2} x, x^* \rangle| \leq \frac{2^{m+2}\pi C}{(m+1)!(r+1)(r-1)^{m+1}} \|x\| \|x^*\|.$$

Thus  $T$  has  $(GFS)_{m+1}$  and hence by Proposition 3.2.6,  $T$  has  $(GFS)_m$ .  $\square$

**Corollary 3.2.11.** *The condition  $(GFS)_m$  does not depend on  $m \in \mathbb{N}$ . In other words, if  $T \in B(X)$  satisfies condition  $(GFS)_m$  for one  $m \in \mathbb{N}$  then it has  $(GFS)_k$  for all  $k \in \mathbb{N}$ .*

**Definition 3.2.12.** Let  $T \in B(X)$ . If  $T$  satisfies one of the three conditions of Theorem 3.2.10 then we will say that  $T$  is a Gomilko Shi-Feng operator or GFS operator.

When  $X$  is an Hilbert space one obtains the following characterisation.

**Corollary 3.2.13.** *Let  $H$  be a Hilbert space and let  $T \in B(H)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . The following assertions are equivalent,*

(i)  $T$  is power-bounded;

(ii) there is a constant  $C > 0$  such that for all  $r > 1$ ,

$$\begin{aligned} (r^2 - 1) \int_0^{2\pi} \|R(re^{it}, T)x\|^2 dt &\leq C \|x\|^2 \quad (x \in H) \\ (r^2 - 1) \int_0^{2\pi} \|R(re^{it}, T)^*y\|^2 dt &\leq C \|y\|^2 \quad (y \in H); \end{aligned}$$

(iii)  $T$  is a GFS operator.

*Proof.* The implication (iii)  $\implies$  (i) is given by Proposition 3.2.3.

(i)  $\implies$  (ii): Let  $r > 1$ , one has

$$\int_0^{2\pi} \|R(re^{it}, T)x\|^2 dt = \int_0^{2\pi} \left\| \sum_{n=0}^{\infty} \frac{T^n x}{(re^{it})^{n+1}} \right\|^2 dt.$$

The Fourier-Plancherel Theorem gives

$$\int_0^{2\pi} \left\| \sum_{n=0}^{\infty} \frac{T^n x}{(re^{it})^{n+1}} \right\|^2 dt = 2\pi \sum_{n=0}^{\infty} \left( \frac{\|T^n x\|}{r^{n+1}} \right)^2.$$

Then taking  $M = \sup_n \|T^n\|$ , we have

$$2\pi \sum_{n=0}^{\infty} \left( \frac{\|T^n\|}{r^{n+1}} \right)^2 \leq 2\pi M^2 \sum_{n=0}^{\infty} \frac{1}{r^{2(n+1)}} = \frac{2\pi M^2}{r^2 - 1},$$

whence the first inequality in (ii). Since  $T^*$  is also power bounded, the second inequality in (ii) holds as well.

(ii)  $\implies$  (iii): by Cauchy-Schwarz inequality one has

$$\begin{aligned} \int_0^{2\pi} |\langle R(re^{it}, T)x, x^* \rangle| dt &= \int_0^{2\pi} |\langle R(re^{it}, T)x, R(re^{it}, T)^* x^* \rangle| dt \\ &\leq \left( \int_0^{2\pi} \|R(re^{it}, T)x\|^2 dt \right)^{\frac{1}{2}} \left( \int_0^{2\pi} \|R(re^{it}, T)^* x^*\|^2 dt \right)^{\frac{1}{2}} \\ &\leq \frac{C}{r^2 - 1} \|x\| \|x^*\|. \end{aligned}$$

It follows that  $T$  has  $(GFS)_1$ , and so  $T$  is a GFS operator.  $\square$

*Remarks 3.2.14.*

1. We say that  $T \in B(X)$  is polynomially bounded when there exists  $C > 0$  such that for each polynomial  $P$  one has

$$\|P(T)\| \leq C \|P\|_{H^\infty(\mathbb{D})}.$$

It is easy to show that if  $T$  is polynomially bounded, then  $T$  is a GFS operator. Indeed let  $P$  be a polynomial and  $r > 1$ . Then according to Lemma 3.2.7 and the assumption that  $T$  is polynomially bounded:

$$\|P^{(m)}(T)\| \leq C \|P^{(m)}\|_{H^\infty(\mathbb{D})} \leq \frac{Cm!}{(r-1)^m} \|P^{(m)}\|_{H^\infty(r\mathbb{D})}.$$

The converse implication is false. In fact, on any infinite-dimensional Hilbert space there exists a power bounded operator which is not polynomially bounded (see [40]). But, according to Corollary 3.2.13, in a Hilbert space an operator is power bounded if and only if it is a GFS operator.

2. When  $X$  is not an Hilbert space, the implication (i)  $\implies$  (iii) in Corollary (3.2.13) is false. Indeed according to [17, Theorem 2.2], if  $X$  is a reflexive Banach space and  $T \in B(X)$  with  $\sigma(T) \subset \mathbb{T}$  then  $T$  is a scalar type spectral operator if and only if  $T$  and  $T^{-1}$  have  $(GFS)_1$ . Let  $U$  denote the shift operator defined on  $l_p(\mathbb{Z})$  ( $1 < p < \infty$ ) by

$$U((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}. \tag{3.2.11}$$

It is well-known that for  $1 < p \neq 2 < \infty$ ,  $U$  is not a scalar type spectral operator. It follows, that either  $U$  or  $U^{-1}$  does not have  $(GFS)_1$ . Further  $U$  and  $U^{-1}$  are similar. Hence by Proposition 3.2.5,  $U$  does not have  $(GFS)_1$ . Thus  $U$  is a power bounded operator which is not a GFS operator.

### 3.3. Polynomial Besov calculus on general Banach spaces

We denote by  $\mathcal{B}(\mathbb{D})$  the Banach algebra of all holomorphic functions  $f$  on  $\mathbb{D}$  such that

$$\int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du < \infty, \tag{3.3.1}$$

endowed with the norm

$$\|f\|_{\mathcal{B}} = \int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du + \|f\|_{H^\infty(\mathbb{D})}.$$

We remark that by a change of variables  $u = \frac{1}{r}$  one has

$$\int_0^1 \sup_{t \in [0, 2\pi)} |f'(ue^{it})| du = \int_1^\infty \frac{1}{r^2} \sup_{t \in [0, 2\pi)} \left| f' \left( \frac{e^{it}}{r} \right) \right| dr. \tag{3.3.2}$$

Let  $H$  be a Hilbert space and let  $T \in B(H)$  be a power-bounded operator. According to [47, Theorem 3.8.] there exists  $C > 0$  such that for each polynomial  $P$  one has

$$\|P(T)\| \leq C \|P\|_{\mathcal{B}}. \tag{3.3.3}$$

Now since the set of polynomials is dense in  $\mathcal{B}(\mathbb{D})$  the bounded algebra homomorphism  $P \rightarrow P(T)$  extends to a bounded algebra homomorphism on  $\mathcal{B}(\mathbb{D})$ . We note that in the context of  $C_0$ -semigroups, there is also a notion of Besov functional calculus for a negative generator of a  $C_0$ -semigroup (see [59], [21], [57], [7] and [8]). According to [7] and [8] the semigroup version of the GFS condition is equivalent to having the Besov functional calculus.

We now consider the issue of obtaining similar results in discrete case that is obtaining estimation (3.3.3) on an arbitrary Banach space  $X$ .

**Proposition 3.3.1.** *Let  $T \in B(X)$  be a GFS operator. Then there exists  $C > 0$  such that for each polynomial  $P$ ,*

$$\|P(T)\| \leq C \|P\|_{\mathcal{B}}.$$

*Proof.* Let  $n \in \mathbb{N}$  and consider  $P_n(z) = z^n$  Applying (3.2.2) with  $m = 1$  and  $\phi(z) = \frac{z^{n+1}}{n+1}$ , one has the following representation,

$$T^n = \frac{1}{2\pi(n+1)} \int_0^{2\pi} r^{n+2} e^{i(n+2)t} R(re^{it}, T)^2 dt,$$

for any  $r > 1$ . We deduce that for any  $x \in X, x^* \in X^*$ ,

$$\begin{aligned}
\langle T^n x, x^* \rangle &= 2n(n+1) \int_1^\infty \frac{r^2 - 1}{r^{2n+3}} dr \langle T^n x, x^* \rangle \\
&= 2n(n+1) \int_1^\infty \frac{r^2 - 1}{r^{2n+3}} \frac{1}{2\pi(n+1)} \int_0^{2\pi} r^{n+2} e^{i(n+2)t} \langle R(re^{it}, T)^2 x, x^* \rangle dt dr \\
&= \frac{n}{\pi} \int_1^\infty \frac{r^2 - 1}{r^{n+1}} \int_0^{2\pi} e^{i(n+2)t} \langle R(re^{it}, T)^2 x, x^* \rangle dt dr \\
&= \frac{n}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} \frac{e^{i(n-1)t}}{r^{n-1}} \langle R(re^{it}, T)^2 x, x^* \rangle dt dr \\
&= \frac{1}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} P'_n \left( \frac{e^{it}}{r} \right) \langle R(re^{it}, T)^2 x, x^* \rangle dt dr, \tag{3.3.4}
\end{aligned}$$

where, in the first equality, we have used the following identity

$$\int_1^\infty \frac{r^2 - 1}{r^{2n+3}} dr = \frac{1}{2n(n+1)}.$$

Now let  $P$  be a polynomial. Using the linearity of derivative and integration, and previous calculations one obtains

$$\langle P(T)x, x^* \rangle = \frac{1}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} P' \left( \frac{e^{it}}{r} \right) \langle R(re^{it}, T)^2 x, x^* \rangle dt dr + P(0) \langle x, x^* \rangle. \tag{3.3.5}$$

Since  $T$  has  $(GFS)_1$  there exists  $C > 0$  such that

$$(r^2 - 1) \int_0^{2\pi} |\langle R(re^{it}, T)^2 x, x^* \rangle| dt \leq C \|x\| \|x^*\|, \quad r > 1, x \in X, x^* \in X^*.$$

Therefore combining this with (3.3.2),

$$|\langle P(T)x, x^* \rangle| \leq C' \left( \int_1^\infty \frac{1}{r^2} \sup_{t \in [0, 2\pi]} |P' \left( \frac{e^{it}}{r} \right)| dr + |P(0)| \right) \|x\| \|x^*\| \leq C' \|P\|_{\mathcal{B}} \|x\| \|x^*\|, \tag{3.3.6}$$

which proves the result.  $\square$

The above proof leads to the following result:

**Theorem 3.3.2.** *Let  $T \in B(X)$  a GFS operator. Then the mapping  $P \rightarrow P(T)$  on the set of polynomials can be extended uniquely to a bounded homomorphism from the Besov algebra  $\mathcal{B}(\mathbb{D})$  to the algebra of bounded operators  $B(X)$ . When  $f \in \mathcal{B}(\mathbb{D})$  let  $f(T)$  denote the operator obtained by this extension.*

*Then for each  $f \in \mathcal{B}(\mathbb{D})$  one has the following representation: for  $x \in X$  and  $x^* \in X^*$ :*

$$\langle f(T)x, x^* \rangle = \frac{1}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} f' \left( \frac{e^{it}}{r} \right) \langle R(re^{it}, T)^2 x, x^* \rangle dt dr + \langle f(0)x, x^* \rangle. \tag{3.3.7}$$

*Proof.* The first assertion is straightforward since the set of polynomials is dense in the Besov algebra  $\mathcal{B}(\mathbb{D})$ . Let  $f \in \mathcal{B}(\mathbb{D})$  and denote by  $f_{\mathcal{B}}(T)$  the operator defined for  $x \in X$  and  $x^* \in X^*$  by

$$\langle f_{\mathcal{B}}(T)x, x^* \rangle = \frac{1}{\pi} \int_1^\infty (r^2 - 1) \int_0^{2\pi} \frac{e^{3it}}{r^2} f' \left( \frac{e^{it}}{r} \right) \langle R(re^{it}, T)^2 x, x^* \rangle dt dr + \langle f(0)x, x^* \rangle.$$

The calculation at the end of the proof of Proposition 3.3.1 shows that this is a well-defined element of  $B(X, X^{**})$ . Now let  $(P_n)_{n \in \mathbb{N}}$  be a sequence of polynomials such that

$$\|f - P_n\|_{\mathcal{B}} \xrightarrow{n \rightarrow \infty} 0.$$

By definition of  $f(T)$ , one has for  $x \in X$  and  $x^* \in X^*$

$$\langle P_n(T)x, x^* \rangle \xrightarrow{n \rightarrow \infty} \langle f(T)x, x^* \rangle.$$

Furthermore by the same arguments leading to the estimate (3.3.6) one has for  $x \in X$  and  $x^* \in X^*$ ,

$$|\langle (f_{\mathcal{B}}(T) - P_n(T))x, x^* \rangle| \leq C \|f - P_n\|_{\mathcal{B}} \|x\| \|x^*\|.$$

We conclude that  $f(T) = f_{\mathcal{B}}(T)$  and therefore we have shown equality (3.3.7).  $\square$

In general when we extend a functional calculus to a larger class, we do not know what the operators obtained on this larger class look like. The above Theorem has more value because we do not only extend polynomial calculus to the Besov algebra but also we are able to give a good representation of operators obtained by extension.

The next result is a converse to Proposition 3.3.1.

**Proposition 3.3.3.** *Let  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . If there exists  $C > 0$  such that for each polynomial  $P$*

$$\|P(T)\| \leq C \|P\|_{\mathcal{B}},$$

*is a GFS operator.*

*Proof.* Let  $r > 1$  and  $\phi \in H^\infty(r\mathbb{D})$ . Then  $\phi' \in \mathcal{B}(\mathbb{D})$ , indeed by (3.2.8)

$$\begin{aligned} \int_0^1 \sup_{t \in [0, 2\pi)} |\phi''(se^{it})| ds &= \int_0^1 \|\phi''\|_{H^\infty(s\mathbb{D})} ds \\ &\leq \int_0^1 \frac{2}{(r-s)^2} \|\phi\|_{H^\infty(r\mathbb{D})} ds \\ &= \frac{2}{r(r-1)} \|\phi\|_{H^\infty(r\mathbb{D})}. \end{aligned}$$

It follows

$$\|\phi'(T)\| \leq C \|\phi'\|_{\mathcal{B}} \leq \frac{4C}{(r^2-1)} \|\phi\|_{H^\infty(r\mathbb{D})}.$$

Therefore  $T$  has 1-derivative bounded functional calculus, hence  $T$  is a GFS operator.  $\square$

*Remark 3.3.4.* It is known (see [59, Lemma 2.3.7] for details) that there exists  $C > 0$  such that for each polynomial of degree  $N$  one has

$$\|P\|_{\mathcal{B}} \leq C \log(N+2) \|P\|_{H^\infty(\mathbb{D})}.$$

Therefore if  $T$  has  $(GFS)_1$  then there exists  $C > 0$  such that for each polynomial of degree  $N$ ,

$$\|P(T)\| \leq C \log(N+2) \|P\|_{H^\infty(\mathbb{D})}.$$



### 3.4. Generalizations involving $\gamma$ -boundedness

This main goal of this section is to obtain an extension of Corollary 3.2.13 to Banach spaces. We have noticed that it is not possible to replace the Hilbert space  $H$  by a Banach space  $X$  without additional assumptions. We need assumptions on the operator  $T$ . We prove an analogue of Corollary 3.2.13 if  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded. We start with some background on the so-called  $\gamma$ -spaces introduced by Kalton and Weis in [29].

Let  $X, Y$  be Banach spaces and we let  $(\gamma_n)_{n \geq 1}$  be a sequence of independent complex valued standard Gaussian variables on some probability space  $\Sigma$ . We denote by  $G(X)$  the closure of

$$\left\{ \sum_{k=1}^n \gamma_k \otimes x_k : x_k \in X, n \in \mathbb{N} \right\}$$

in  $L^2(\Sigma, X)$ . For  $x_1, \dots, x_n \in X$ , we let

$$\left\| \sum_{k=1}^n \gamma_k \otimes x_k \right\|_{G(X)} := \left( \int_{\Sigma} \left\| \sum_{k=1}^n \gamma_k(\lambda) x_k \right\|^2 d\lambda \right)^{\frac{1}{2}}$$

denote the induced norm.

**Definition 3.4.1.** Let  $\mathcal{T} \subset B(X, Y)$  be a set of operators. We say that  $\mathcal{T}$  is  $\gamma$ -bounded if there exists a constant  $C \geq 0$  such that for all finite sequences  $(T_n)_{n=1}^N \subset \mathcal{T}$  and  $(x_n)_{n=1}^N \subset X$ , the following inequality holds:

$$\left\| \sum_{n=1}^N \gamma_n \otimes T_n x_n \right\|_{G(Y)} \leq C \left\| \sum_{n=1}^N \gamma_n \otimes x_n \right\|_{G(X)}. \quad (3.4.1)$$

The least admissible constant in the above inequality is called the  $\gamma$ -bound of  $\mathcal{T}$  and we denote this quantity by  $\gamma(\mathcal{T})$ . If  $\mathcal{T}$  fails to be  $\gamma$ -bounded, we set  $\gamma(\mathcal{T}) = \infty$ .

The notion of  $\gamma$ -boundedness is stronger than uniform boundedness, indeed using the definition of  $\gamma$ -boundedness with  $N = 1$  it is easy to see that  $\gamma$ -boundedness implies uniform boundedness. We will use also two important facts which we sum up in the following proposition.

**Proposition 3.4.2.** *Let  $\mathcal{T} \subset B(X, Y)$  be a  $\gamma$ -bounded set. Then*

1. *the closure  $\overline{\mathcal{T}}^{so}$  of  $\mathcal{T}$  in the strong operator topology is  $\gamma$ -bounded with  $\gamma(\overline{\mathcal{T}}^{so}) = \gamma(\mathcal{T})$ .*
2. *The absolute convex hull of  $\mathcal{T}$ ,  $\text{absconv}(\mathcal{T})$  is  $\gamma$ -bounded with  $\gamma(\text{absconv}(\mathcal{T})) = \gamma(\mathcal{T})$ .*

We now turn to the definition of  $\gamma$ -spaces.

Let  $H$  be a Hilbert space. A linear operator  $T : H \rightarrow X$  is called  $\gamma$ -summing if

$$\|T\|_{\gamma} := \sup \left\| \sum_{n=1}^N \gamma_n \otimes T h_n \right\|_{G(X)} < \infty,$$

where the supremum is taken over all finite orthonormal systems  $\{h_1, \dots, h_n\}$  in  $H$ . We let  $\gamma_{\infty}(H; X)$  denote the space of all  $\gamma$ -summing operators and we endow it with the norm

$\|\cdot\|_\gamma$ . Then  $\gamma_\infty(H; X)$  is a Banach space. Clearly any finite rank (bounded) operator is a  $\gamma$ -summing operator. We let  $\gamma(H; X)$  be the closure in  $\gamma_\infty(H; X)$  of the space of finite rank operators from  $H$  into  $X$ .

We let  $\gamma'_\infty(H; X)$  be the space of bounded operators  $T : H \mapsto X$  such that

$$\|T\|_{\gamma'(H;X)} = \sup\{\text{trace}(T^* \circ S) \mid S : H \mapsto X^*, \|S\|_{\gamma(H;X)} \leq 1, \dim S(H) < \infty\}$$

and we denote by  $\gamma'(H; X)$  the closure of the finite dimensional operators in  $\gamma'_\infty(H; X)$ . See [29, section 5] for details about spaces  $\gamma'(H; X)$  and  $\gamma'_\infty(H; X)$ . When  $X$  is  $K$ -convex (see [26, Section 7.4.] for details about  $K$ -convexity) then one has

$$\gamma'(H; X^*) = \gamma(H; X^*).$$

Let  $(S, \mu)$  be a measure space. We say that a function  $f : S \rightarrow X$  is weakly  $L^2$  if for each  $x^* \in X^*$ , the function  $s \mapsto \langle f(s), x^* \rangle$  is measurable and belongs to  $L^2(S)$ . If  $f : S \rightarrow X$  is measurable and weakly  $L^2$ , one can define an operator  $\mathbb{I}_f : L^2(S) \rightarrow X$ , given by

$$\mathbb{I}_f(g) := \int_S g(s)f(s)d\mu, \quad g \in L^2(S),$$

where this integral is defined in the Pettis sense. We let  $\gamma(S; X)$  (resp.  $\gamma'(S; X)$ ) be the space of all measurable and weakly  $L^2$  functions  $f : S \rightarrow X$  such that  $\mathbb{I}_f$  belongs to  $\gamma(L^2(S); X)$  (resp.  $\gamma'(L^2(S); X)$ ). We endow it with  $\|f\|_{\gamma(S;X)} := \|\mathbb{I}_f\|_{\gamma(L^2(S);X)}$  (resp.  $\|f\|_{\gamma'(S;X)} := \|\mathbb{I}_f\|_{\gamma'(L^2(S);X)}$ ).

The next result is an inequality of Hölder type [29, Corollary 5.5.].

**Theorem 3.4.3** ( $\gamma$ -Hölder inequality). *If  $f : S \rightarrow X$  and  $g : S \rightarrow X^*$  belong to  $\gamma(S; X)$  and  $\gamma'(S; X^*)$ , respectively, then  $\langle f, g \rangle$  belongs to  $L^1(S)$  and we have*

$$\|\langle f, g \rangle\|_{L^1(S)} \leq \|f\|_{\gamma(S;X)} \|g\|_{\gamma'(S;X^*)}.$$

*Remark 3.4.4.* Let  $N \in \mathbb{N}$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . By construction,

$$\left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} = \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}. \tag{3.4.2}$$

As a particular case of Theorem 3.4.3 one has

$$\left| \sum_{k=1}^N \langle x_k, x_k^* \rangle \right| \leq \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}.$$

Further this inequality is optimal, namely

$$\|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} = \sup \left\{ \left| \sum_{k=1}^N \langle x_k, x_k^* \rangle \right| : (x_k^*)_{k \in \mathbb{N}_N} \subset X^*, \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)} \leq 1 \right\} \tag{3.4.3}$$

and

$$\|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)} = \sup \left\{ \left| \sum_{k=1}^N \langle z_k, x_k^* \rangle \right| : (x_k)_{k \in \mathbb{N}_N} \subset X, \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \leq 1 \right\}. \tag{3.4.4}$$

Furthermore for each sequence  $(x_k^*)_{k \in \mathbb{N}} \subset X^*$ , one has

$$\|(x_k^*)_{k \in \mathbb{N}}\|_{\gamma'(\mathbb{N}; X^*)} = \lim_{N \rightarrow +\infty} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}. \tag{3.4.5}$$

We finally state an extension result ([26, Theorem 9.6.1.] and [29, Remark 5.4.]).

**Lemma 3.4.5.** *Let  $H$  and  $K$  be Hilbert spaces and  $U \in B(H^*, K^*)$ . Then  $U \otimes I_X$  extends uniquely to a bounded linear operator  $\tilde{U} \in B(\gamma(H; X), \gamma(K; X))$  of the same norm. Moreover  $U \otimes I_{X^*}$  extends uniquely to a bounded linear operator  $\hat{U} \in B(\gamma'(H; X^*), \gamma'(K; X^*))$  of the same norm.*

### Power $\gamma$ -bounded operators

**Definition 3.4.6.** Let  $T \in B(X)$  be power bounded. We say that  $T$  is power  $\gamma$ -bounded if the set  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded. When  $T$  is power  $\gamma$ -bounded we will denote by  $C_\gamma$  the  $\gamma$ -bound  $\gamma(\{T^n : n \in \mathbb{N} \cup \{0\}\})$ .

*Example 3.4.7.* The shift operator  $U$  defined by (3.2.11) is obviously power bounded but for  $1 < p \neq 2 < \infty$ ,  $U$  is not power  $\gamma$ -bounded. Assume that  $1 < p < 2$  and let  $(\delta_0)_{n \in \mathbb{Z}}$  the sequence defined by  $\delta_0(0) = 1$  and  $\delta_0(k) = 0$  for each  $k \neq 0$ . Then one has

$$\left\| \left( \sum_{i=1}^n |U^{(i-1)} \delta_0|^2 \right)^{\frac{1}{2}} \right\|_{l_p} = n^{\frac{1}{p}}$$

whereas

$$\left\| \left( \sum_{i=1}^n |\delta_0|^2 \right)^{\frac{1}{2}} \right\|_{l_p} = n^{\frac{1}{2}}.$$

The proof in the case  $2 < p < \infty$  is similar.

*Example 3.4.8.* A Ritt operator  $T$  is called  $\gamma$ -Ritt if the set

$$\{(\lambda - 1)R(\lambda, T) : \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}\}$$

is  $\gamma$ -bounded. It turns out  $T$  is  $\gamma$ -Ritt if and only if the two sets

$$\{T^n : n \in \mathbb{N}\} \quad \text{and} \quad \{n(I - T)T^n : n \in \mathbb{N}\}$$

are  $\gamma$ -bounded. It follows that any  $\gamma$ -Ritt operator is power  $\gamma$ -bounded. In chapter 1 we construct a Ritt operator such that the set  $\{T^n : n \in \mathbb{N}\}$  is not  $\gamma$ -bounded (in particular  $T$  is not  $\gamma$ -Ritt). An interesting problem which is still open is to construct a Ritt operator which is not  $\gamma$ -Ritt but which is power  $\gamma$ -bounded.

*Example 3.4.9.* Let  $(\Omega, \mu)$  be a measure space,  $m : \Omega \rightarrow \overline{\mathbb{D}}$  an element of  $L^\infty(\Omega)$  and  $1 \leq p < \infty$ . We define the bounded multiplier  $T_m \in B(L^p(\Omega))$  by

$$(T_m f)(s) := m(s)f(s), \quad s \in \Omega.$$

For each  $n \in \mathbb{N}$  one has

$$(T_m)^n = T_{m^n}.$$

Since for each  $n \in \mathbb{N}$ ,  $\|m^n\|_\infty \leq 1$ , it follows (see [26, Example 8.1.9]) that  $T_m$  is power  $\gamma$ -bounded.

Polynomial boundedness does not imply power  $\gamma$ -boundedness. Indeed according to [38, Proposition 6.6] there exist a Banach space  $X$  and an invertible operator  $T \in B(X)$  such that  $T$  is not power  $\gamma$ -bounded but there exists a bounded unital homomorphism  $\omega : C(\mathbb{T}) \rightarrow B(X)$  such that  $\omega(z \mapsto z) = T$ . Obviously that operator  $T$  is polynomially bounded. In the following Theorem we use the property  $(\alpha)$ , we refer the reader to [26, section 7.5] for more details.

**Theorem 3.4.10.** *Suppose  $X$  has property  $(\alpha)$ . Let  $T \in B(X)$  be invertible and let  $m \in \mathbb{N}$ . Then  $T$  and  $T^{-1}$  are GFS operators if and only if  $\{T^n, n \in \mathbb{Z}\}$  is  $\gamma$ -bounded.*

*Proof.* Since  $X$  has property  $(\alpha)$  then according to [38, Proposition 6.3. (2)], the set  $\{T^n, n \in \mathbb{Z}\}$  is  $\gamma$ -bounded if and only if there is a bounded unital homomorphism  $\omega : C(\mathbb{T}) \rightarrow B(X)$  such that  $\omega(z \mapsto z) = T$ . Furthermore, according to the proof of [17, Theorem 2.2.], there is a bounded unital homomorphism  $\omega : C(\mathbb{T}) \rightarrow B(X)$  such that  $\omega(z \mapsto z) = T$  if and only if  $T$  and  $T^{-1}$  have  $(GFS)_1$ . The result is now straightforward.  $\square$

### $\gamma$ -discrete Gomilko Shi-Feng condition

**Definition 3.4.11.** Let  $m \geq 1$  be an integer and  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . We say that  $T$  has property  $(\gamma\text{-GFS})_m$  if there exists a constant  $C > 0$  such that for any  $N \in \mathbb{N}$ , for any  $r_1, \dots, r_N > 1$ , and for any  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ , we have

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} | \langle (r_k + 1)(r_k - 1)^m R(r_k e^{it}, T)^{m+1} x_k, x_k^* \rangle | dt \\ & \leq C \| (x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)} \| (x_k^*)_{k \in \mathbb{N}_N} \|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned} \quad (3.4.6)$$

Obviously if  $T$  has  $(\gamma\text{-GFS})_m$  then  $T$  has  $(GFS)_m$ . Moreover on Hilbert spaces, properties  $(\gamma\text{-GFS})_m$  and  $(GFS)_m$  are equivalent.

**Proposition 3.4.12.** *If  $T$  has  $(\gamma\text{-GFS})_m$  then  $T$  is power  $\gamma$ -bounded, that is, the set  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded.*

*Proof.* The proof is similar to the proof of Proposition 3.2.3. Let  $N \in \mathbb{N}$ ,  $n_1, \dots, n_N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . For  $x_1^*, \dots, x_N^* \in X^*$  such that  $\| (x_k^*)_{k \in \mathbb{N}_N} \|_{\gamma'(\mathbb{N}_N; X^*)} \leq 1$  the proof of (3.2.4) and the assumption yield

$$\begin{aligned} & \sum_{k=1}^N \frac{2\pi(n_k + 1) \dots (n_k + m)(r_k + 1)(r_k - 1)^m}{r_k^{n_k + m + 1} m!} | \langle T^{n_k} x_k, x_k^* \rangle | \\ & \leq \sum_{k=1}^N \int_0^{2\pi} (r_k + 1)(r_k - 1)^m | \langle R(r_k e^{it}, T)^{m+1} x_k, x_k^* \rangle | dt \\ & \leq C \| (x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)} \| (x_k^*)_{k \in \mathbb{N}_N} \|_{\gamma'(\mathbb{N}_N; X^*)} \\ & \leq C \| (x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)}. \end{aligned}$$

Taking  $r_k = 1 + \frac{1}{n_k}$  one obtains

$$\sum_{k=1}^N | \langle T^{n_k} x_k, x_k^* \rangle | \leq \frac{2^m m! C e}{\pi} \| (x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)}.$$

Taking the supremum over  $x_1^*, \dots, x_N^*$ , such that  $\| (x_k^*)_{k \in \mathbb{N}_N} \|_{\gamma'(\mathbb{N}_N; X^*)} \leq 1$ , one obtains according to (3.4.3)

$$\| (T^{n_k} x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)} \leq \frac{2^m m! C e}{\pi} \| (x_k)_{k \in \mathbb{N}_N} \|_{\gamma(\mathbb{N}_N; X)}.$$

Hence the set  $\{T^n : n \in \mathbb{N}\}$  is  $\gamma$ -bounded.  $\square$

**Lemma 3.4.13.** *Let  $m \in \mathbb{N}$  and  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . Then  $T$  has property  $(\gamma\text{-GFS})_m$  if and only if the set*

$$\left\{ (r+1)(r-1)^m \int_0^{2\pi} \varepsilon(t) R(re^{it}, T)^{m+1} dt \mid \varepsilon : [0, 2\pi) \rightarrow \overline{\mathbb{D}} \text{ measurable, } r > 1 \right\} \quad (3.4.7)$$

is  $\gamma$ -bounded.

*Proof.* Let  $T \in B(X)$  with  $(\gamma\text{-GFS})_m$ . Let  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in X$  and  $\varepsilon_k : [0, 2\pi) \rightarrow \overline{\mathbb{D}}$  measurable. For  $x_1^*, \dots, x_N^* \in X^*$  with  $\|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)} \leq 1$  one has, using (3.4.6),

$$\begin{aligned} & \left| \sum_{k=1}^N \left\langle \left( (r_k+1)(r_k-1)^m \int_0^{2\pi} \varepsilon_k(t) R(r_k e^{it}, T)^{m+1} dt \right) x_k, x_k^* \right\rangle \right| \\ & \leq C \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}. \end{aligned}$$

Now take the supremum over  $x_1^*, \dots, x_N^* \in X^*$  one obtains by (3.4.3)

$$\left\| \left( (r_k+1)(r_k-1)^m \int_0^{2\pi} \varepsilon_k(t) R(r_k e^{it}, T)^{m+1} dt x_k \right)_{k \in \mathbb{N}_N} \right\|_{\gamma(\mathbb{N}_N; X)} \leq C \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}.$$

This means that the set in (3.4.7) is  $\gamma$ -bounded.

Suppose that the set in (3.4.7) is  $\gamma$ -bounded. Let  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . For  $k \in \mathbb{N}_N$ , let  $\varepsilon_k : [0, 2\pi) \rightarrow \overline{\mathbb{D}}$  measurable such that

$$\begin{aligned} & \int_{\mathbb{R}} |\langle (r_k+1)(r_k-1)^m R(r_k e^{it}, T)^{m+1} x_k, x_k^* \rangle| dt \\ & = \left\langle \left( (r_k+1)(r_k-1)^m \int_{\mathbb{R}} \varepsilon_k(t) R(r_k e^{it}, T)^{m+1} dt \right) x_k, x_k^* \right\rangle. \end{aligned}$$

Then by Theorem 3.4.3 and the assumption,

$$\begin{aligned} & \left\langle \left( (r_k+1)(r_k-1)^m \int_{\mathbb{R}} \varepsilon_k(t) R(r_k e^{it}, T)^{m+1} dt \right) x_k, x_k^* \right\rangle \\ & \leq \left\| \left( \sum_{k=1}^N (r_k+1)(r_k-1)^m \int_0^{2\pi} \varepsilon_k(t) R(r_k e^{it}, T)^{m+1} dt x_k \right)_{k \in \mathbb{N}_N} \right\|_{\gamma(\mathbb{N}_N; X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)} \\ & \leq C \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned}$$

□

**Proposition 3.4.14.** *Let  $T \in B(X)$  with  $(\text{GFS})_m$ . If  $T$  has  $(\gamma\text{-GFS})_m$  then  $T$  has  $(\gamma\text{-GFS})_p$  for  $1 \leq p \leq m$ .*

*Proof.* We proceed by induction, showing that if  $m \geq 2$ , then  $(\gamma\text{-GFS})_m$  implies  $(\gamma\text{-GFS})_{m-1}$ .

Suppose that  $T$  has property  $(\gamma\text{-GFS})_m$ , with  $m \geq 2$ . Let  $\varepsilon : [0, 2\pi) \rightarrow \overline{\mathbb{D}}$  be a measurable function and let  $r > 1$ . Then by the argument showing (3.2.7) and a change of variable one has

$$\begin{aligned} & (r+1)(r-1)^{m-1} \int_0^{2\pi} \varepsilon(t) R(re^{it}, T)^m dt \\ & = r(m-1) \int_1^{+\infty} (r+1)(r-1)^{m-1} \int_0^{2\pi} \varepsilon(t) R(rue^{it}, T)^{m+1} dt du \\ & = r(m-1) \int_1^{+\infty} \frac{(r+1)(r-1)^{m-1}}{(ru+1)(ru-1)^m} \int_0^{2\pi} (ru+1)(ru-1)^m \varepsilon(t) R(rue^{it}, T)^{m+1} dt du. \end{aligned}$$

Let  $\mathcal{T}$  be the set (3.4.7). By assumption and Lemma 3.4.13,  $\mathcal{T}$  is  $\gamma$ -bounded hence by [26, Theorem 8.5.2.], the set

$$\Gamma := \left\{ \int_1^\infty \phi(u) \int_0^{2\pi} (ru+1)(ru-1)^m \varepsilon(t) R(rue^{it}, T)^{m+1} dt du \mid \right. \\ \left. \varepsilon : [0, 2\pi) \rightarrow \overline{\mathbb{D}} \text{ measurable, } r > 1, \phi \in L^1((1, \infty)), \|\phi\|_{L^1} \leq 1 \right\}$$

is  $\gamma$ -bounded. Since

$$\left\| r(m-1) \frac{(r+1)(r-1)^{m-1}}{(r \cdot +1)(r \cdot -1)^m} \right\|_{L^1((1, \infty))} \leq 1,$$

the above calculation shows that the set

$$\left\{ (r+1)(r-1)^{m-1} \int_0^{2\pi} \varepsilon(t) R(re^{it}, T)^m dt : \varepsilon : \mathbb{R} \rightarrow \overline{\mathbb{D}} \text{ measurable, } r > 1 \right\}$$

is included in  $\Gamma$ , hence is  $\gamma$ -bounded. Hence, by Lemma 3.4.13, the operator  $T$  has property  $(\gamma\text{-GFS})_{m-1}$ .  $\square$

### $m$ -derivative $\gamma$ -bounded functional calculus

**Definition 3.4.15.** Let  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . Then  $T$  is said to have  $m$ -derivative  $\gamma$ -bounded functional calculus if the set

$$\left\{ (r-1)^m \phi^{(m)}(T) : r > 1, \phi \in H^\infty(r\mathbb{D}), \|\phi\|_{H^\infty(r\mathbb{D})} \leq 1 \right\}$$

is  $\gamma$ -bounded.

We give now the  $\gamma$ -bounded version of Theorem 3.2.10.

**Theorem 3.4.16.** Let  $T \in B(X)$  with  $\sigma(T) \subset \overline{\mathbb{D}}$ . The following assertions are equivalent for  $m \in \mathbb{N}$ ,

- (i)  $T$  has  $(\gamma\text{-GFS})_m$ ;
- (ii)  $T$  has 1-derivative  $\gamma$ -bounded functional calculus;
- (iii)  $T$  has  $m$ -derivative  $\gamma$ -bounded functional calculus.

*Proof.* (i)  $\Rightarrow$  (ii): First, by Proposition 3.4.14,  $T$  has  $(\gamma\text{-GFS})_1$ . Let  $N \in \mathbb{N}$  and  $r_1, \dots, r_N > 1, \phi_1 \in H^\infty(r_1\mathbb{D}), \dots, \phi_N \in H^\infty(r_N\mathbb{D})$  with  $\|\phi_k\|_{H^\infty(r_k\mathbb{D})} \leq 1$ . Let  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . Using (3.2.2) and arguing as in the proof of (i)  $\Rightarrow$  (ii) in Theorem 3.2.10, one has

$$\sum_{k=1}^N |\langle (r_k - 1) \phi_k'(T) x_k, x_k^* \rangle| \leq \frac{1}{2\pi} \sum_{k=1}^N \|\phi_k\|_{H^\infty(r_k\mathbb{D})} \int_0^{2\pi} |\langle (r_k^2 - 1) R(r_k e^{it}, T)^2 x_k, x_k^* \rangle| dt.$$

Suppose  $\|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)} \leq 1$ . Since  $\|\phi_k\|_{H^\infty(r_k\mathbb{D})} \leq 1$  and  $T$  has  $(\gamma\text{-GFS})_1$ , one obtains

$$\sum_{k=1}^N |\langle (r_k - 1) \phi_k'(T) x_k, x_k^* \rangle| \leq \frac{C}{2\pi} \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}.$$

Taking the supremum over  $x_1^*, \dots, x_N^*$  this implies, thanks to (3.4.3), that  $T$  has 1-derivative  $\gamma$ -bounded functional calculus.

(ii)  $\Rightarrow$  (iii): It follows from the assumption that the set

$$\Delta := \left\{ (\rho - 1)\phi^{(m)}(T) : 1 < \rho < r, \phi \in H^\infty(r\mathbb{D}), \left\| \phi^{(m-1)} \right\|_{H^\infty(\rho\mathbb{D})} \leq 1 \right\}$$

is  $\gamma$ -bounded. For any  $1 < \rho < r$  and  $\phi \in H^\infty(r\mathbb{D})$  with  $\|\phi\|_{H^\infty(r\mathbb{D})} \leq 1$ , by Lemma 3.2.7 one has

$$\left\| \frac{(r - \rho)^{m-1}}{(m-1)!} \phi^{(m-1)} \right\|_{H^\infty(\rho\mathbb{D})} \leq 1.$$

Hence

$$\left\{ \frac{(\rho - 1)(r - \rho)^{m-1}}{(m-1)!} \phi^{(m)}(T) : 1 < \rho < r, \phi \in H^\infty(r\mathbb{D}), \|\phi\|_{H^\infty(r\mathbb{D})} \leq 1 \right\} \subset \Delta.$$

Taking  $\rho = \frac{r+1}{2}$  in the above set, we obtain

$$\left\{ \frac{(r-1)^m}{2^m(m-1)!} \phi^{(m)}(T) : r > 1, \phi \in H^\infty(r\mathbb{D}), \|\phi\|_{H^\infty(r\mathbb{D})} \leq 1 \right\} \subset \Delta.$$

Hence the above set is  $\gamma$ -bounded. Thus  $T$  has  $m$ -derivative  $\gamma$ -bounded functional calculus.

(iii)  $\Rightarrow$  (i): Let  $r_1, \dots, r_N > 1$ ,  $1 < \rho_1 < r_1, \dots, 1 < \rho_N < r_N$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . Let us introduce measurable functions  $\varepsilon_1, \dots, \varepsilon_N : [0, 2\pi) \rightarrow \overline{\mathbb{D}}$  such that

$$|\langle R((r_k)e^{it}, T)^{m+2}x_k, x_k^* \rangle| = \varepsilon_k(t) \langle R((r_k)e^{it}, T)^{m+2}x_k, x_k^* \rangle.$$

for all  $k = 1, \dots, N$  and all  $t \in [0, 2\pi)$ . Next we set

$$\phi_k(z) := \frac{(r_k + \rho_k)(r_k - \rho_k)}{2\pi} \int_0^{2\pi} \frac{\varepsilon_k(t)}{(r_k e^{it} - z)^2} dt, \quad z \in r_k\mathbb{D}.$$

The computations in the proof of Theorem 3.2.10 show that  $\|\phi_k\|_{H^\infty(\rho_k\mathbb{D})} \leq 1$  and

$$\phi_k^{(m)}(T) = \frac{(m+1)!(r_k + \rho_k)(r_k - \rho_k)}{2\pi} \int_0^{2\pi} \varepsilon_k(t) R(r_k e^{it}, T)^{m+2} dt.$$

Therefore it follows from (iii) that we have the estimate

$$\begin{aligned} & \sum_{k=1}^N \frac{(m+1)!(\rho_k - 1)^m (r_k - \rho_k)(r_k + 1)}{2\pi} \int_0^{2\pi} |\langle R(r_k e^{it}, T)^{m+2}x_k, x_k^* \rangle| dt \\ & \leq \sum_{k=1}^N \frac{(m+1)!(\rho_k - 1)^m (r_k - \rho_k)(r_k + \rho_k)}{2\pi} \int_0^{2\pi} \varepsilon_k(t) \langle R(r_k e^{it}, T)^{m+2}x_k, x_k^* \rangle dt \\ & = \sum_{k=1}^N |\langle (\rho_k - 1)^m \phi_k^{(m)}(T)x_k, x_k^* \rangle| dt \\ & \leq C \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned}$$

The last inequality comes from Theorem 3.4.3. Now we choose  $\rho_k = \frac{r_k+1}{2}$  in the above estimate. We obtain the following inequality

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} |\langle (r_k + 1)(r_k - 1)^{m+1} R(r_k e^{it}, T)^{m+2}x_k, x_k^* \rangle| dt \\ & \leq \frac{2^{m+2}\pi C}{(m+1)!} \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned}$$

This shows that  $T$  has  $(\gamma\text{-GFS})_{m+1}$ . Then by Proposition 3.4.14,  $T$  has  $(\gamma\text{-GFS})_m$ .  $\square$

**Definition 3.4.17.** Let  $T \in B(X)$ . If  $T$  satisfies one of the three conditions of Theorem 3.4.16 then we will say that  $T$  is a  $\gamma$ -GFS operator.

**Characterization of power  $\gamma$ -bounded on Banach space  $X$** 

In the following, the space  $L^2((0, 2\pi))$  will be equipped with the norm

$$\|f\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(s)|^2 ds, \quad f \in L^2((0, 2\pi)).$$

Therefore, the Fourier-Parseval operator

$$\begin{aligned} \mathcal{F} : L^2([0, 2\pi)) &\rightarrow l_{\mathbb{Z}}^2 \\ f &\mapsto (c_n(f))_{n \in \mathbb{Z}}, \end{aligned}$$

is an isometry. Here  $c_n(f)$  is the  $n$ -th Fourier coefficient defined by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

We give a characterization of power  $\gamma$ -bounded operators.

**Theorem 3.4.18.** *Let  $X$  be a Banach space. Following assertions are equivalent :*

- (i) *The operator  $T$  is power  $\gamma$ -bounded;*
- (ii) *The spectrum set  $\sigma(T)$  is included in  $\overline{\mathbb{D}}$  and there exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$  the functions  $(t, k) \mapsto \sqrt{r_k^2 - 1} R(r_k e^{it}, T)x_k$  and  $(t, k) \mapsto \sqrt{r_k^2 - 1} R(r_k e^{it}, T^*)x_k^*$  are in  $\gamma([0, 2\pi) \times \mathbb{N}_N; X)$  and  $\gamma'([0, 2\pi) \times \mathbb{N}_N; X^*)$  respectively, and satisfy*

$$\left\| (t, k) \mapsto \sqrt{r_k^2 - 1} R(r_k e^{it}, T)x_k \right\|_{\gamma([0, 2\pi) \times \mathbb{N}_N; X)} \leq C \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \quad (3.4.8)$$

and

$$\left\| (t, k) \mapsto \sqrt{r_k^2 - 1} R(r_k e^{it}, T^*)x_k^* \right\|_{\gamma'([0, 2\pi) \times \mathbb{N}_N; X^*)} \leq \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}; \quad (3.4.9)$$

- (iii)  *$T$  is a  $\gamma$ -GFS operator.*

*Proof.* (iii)  $\implies$  (i) is Proposition 3.4.12.

(i)  $\implies$  (ii) : Recall that for each  $k \in \mathbb{N}_N$  one has

$$R(r_k e^{it}, T)x_k = \sum_{n=0}^{\infty} \frac{T^n x_k}{(r_k e^{it})^{n+1}} = \sum_{n=0}^{\infty} \frac{T^n x_k}{r_k^{n+1}} e^{-i(n+1)t}.$$

We apply Lemma 3.4.5 with  $H = l_N^2 \otimes L^2((0, 2\pi))$ ,  $K = l_N^2 \otimes l_{\mathbb{Z}}^2$  and  $U : H \rightarrow K$  defined by

$$U\left(\sum_{k=1}^N e_k \otimes f_k\right) = \sum_{k=1}^N e_k \otimes (c_n(f_k))_{n \in \mathbb{Z}}, \quad f_1, \dots, f_N \in L^2((0, 2\pi)).$$

We obtain that  $(n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^n x_k}{r_k^{n+1}}$  belongs to  $\gamma(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X)$  and

$$\left\| (t, k) \mapsto \sqrt{r_k^2 - 1} R(r_k e^{it}, T)x_k \right\|_{\gamma([0, 2\pi) \times \mathbb{N}_N; X)} = \left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^n x_k}{r_k^{n+1}} \right\|_{\gamma(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X)}.$$



But, one has by (3.4.2)

$$\left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^n x_k}{r_k^{n+1}} \right\|_{\gamma(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X)} = \left\| \sum_{k=1}^N \sqrt{r_k^2 - 1} \sum_{n=0}^{\infty} \frac{1}{r_k^{n+1}} \gamma_{k,n} \otimes T^n x_k \right\|_{G(X)},$$

where  $(\gamma_{k,n})_{k \in \mathbb{N}_N, n \in \mathbb{N} \cup \{0\}}$  is a family of independant complex valued standard Gaussian variables. Using the  $\gamma$ -boundedness of  $\{T^n : n \in \mathbb{N}\}$  it follows

$$\left\| \sum_{k=1}^N \sqrt{r_k^2 - 1} \sum_{n=0}^{\infty} \frac{1}{r_k^{n+1}} \gamma_{k,n} \otimes T^n x_k \right\|_{G(X)} \leq C_\gamma \left\| \sum_{k=1}^N \sqrt{r_k^2 - 1} \sum_{n=0}^{\infty} \frac{1}{r_k^{n+1}} \gamma_{k,n} \otimes x_k \right\|_{G(X)}.$$

Furthermore  $\left\| \left( \sqrt{r_k^2 - 1} \frac{1}{r_k^{n+1}} \right)_{n \in \mathbb{N} \cup \{0\}} \right\|_{l^2} = 1$  for each  $k \in \mathbb{N}_N$  therefore

$$\left( \sum_{n=0}^{\infty} \sqrt{r_k^2 - 1} \frac{1}{r_k^{n+1}} \gamma_{k,n} \right)_{k \in \mathbb{N}_N}$$

is a sequence of independent standard Gaussian variables, which implies

$$\left\| \sum_{k=1}^N \sqrt{r_k^2 - 1} \sum_{n=0}^{\infty} \frac{1}{r_k^{n+1}} \gamma_{k,n} \otimes x_k \right\|_{G(X)} = \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}.$$

Hence one obtains (3.4.8).

Likewise, using Lemma 3.4.5 we obtain  $(n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^{*n} x_k}{r_k^{n+1}}$  belongs to  $\gamma'(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X^*)$  and

$$\left\| (t, k) \mapsto \sqrt{r_k^2 - 1} R(r_k e^{it}, T^*) x_k^* \right\|_{\gamma'([0, 2\pi] \times \mathbb{N}_N; X^*)} = \left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \right\|_{\gamma'(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X^*)}.$$

Furthermore, one has by (3.4.5)

$$\left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \right\|_{\gamma'(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X^*)} = \lim_{M \rightarrow \infty} \left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \right\|_{\gamma'(\mathbb{N}_M \cup \{0\} \times \mathbb{N}_N; X^*)}.$$

But, by (3.4.4)

$$\begin{aligned} & \left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \right\|_{\gamma'(\mathbb{N}_M \cup \{0\} \times \mathbb{N}_N; X^*)} \\ &= \sup \left\{ \left| \sum_{n=1}^M \sum_{k=1}^N \langle x_{n,k}, \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \rangle \right| : \|(x_{n,k})_{n \in \mathbb{N}_M \cup \{0\}, k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_M \cup \{0\} \times \mathbb{N}_N, X)} \leq 1 \right\}, \end{aligned} \quad (3.4.10)$$

and for  $M \in \mathbb{N}$ , using Theorem 3.4.3,

$$\begin{aligned} \left| \sum_{n=1}^M \sum_{k=1}^N \langle x_{n,k}, \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \rangle \right| &= \left| \sum_{n=1}^M \sum_{k=1}^N \langle \sqrt{r_k^2 - 1} \frac{T^n x_{n,k}}{r_k^{n+1}}, x_k^* \rangle \right| \\ &\leq \left\| \sqrt{r_k^2 - 1} \frac{T^n x_{n,k}}{r_k^{n+1}} \right\|_{\gamma(\mathbb{N}_M \cup \{0\} \times \mathbb{N}_N, X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned}$$

By previous computations which leads to (3.4.8), for each  $M \in \mathbb{N}$  one has

$$\left\| \sqrt{r_k^2 - 1} \frac{T^n x_{n,k}}{r_k^{n+1}} \right\|_{\gamma(\mathbb{N}_M \cup \{0\} \times \mathbb{N}_N; X)} \leq C_\gamma \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}.$$

This implies

$$\left\| (n, k) \mapsto \sqrt{r_k^2 - 1} \frac{T^{*n} x_k^*}{r_k^{n+1}} \right\|_{\gamma'(\mathbb{N}_M \cup \{0\} \times \mathbb{N}_N; X^*)} \leq C_\gamma \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}.$$

Hence one obtains (3.4.9).

(ii)  $\implies$  (iii) : Let  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . One has

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} |\langle (r_k^2 - 1)R(r_k e^{it}, T)^2 x_k, x_k^* \rangle| dt \\ &= \sum_{k=1}^N \int_{\mathbb{R}} |\langle (r_k^2 - 1)R(r_k e^{it}, T)x_k, R(r_k e^{it}, T)^* x_k^* \rangle| dt \\ &= \left\| (t, k) \mapsto \langle \sqrt{r_k^2 - 1}R(r_k e^{it}, T)x_k, \sqrt{r_k^2 - 1}R(r_k e^{it}, T)^* x_k^* \rangle \right\|_{L^1((0, 2\pi) \times \mathbb{N}_N)}. \end{aligned}$$

By Theorem 3.4.3 and assumptions (3.4.8) and (3.4.9),

$$\begin{aligned} & \left\| (t, k) \mapsto \langle \sqrt{r_k^2 - 1}R(r_k e^{it}, T)x_k, \sqrt{r_k^2 - 1}R(r_k e^{it}, T)^* x_k^* \rangle \right\|_{L^1((0, 2\pi) \times \mathbb{N}_N)} \\ & \leq \left\| \sqrt{r_k^2 - 1}R(r_k e^{it}, T)x_k \right\|_{\gamma((0, 2\pi) \times \mathbb{N}_N; X)} \left\| \sqrt{r_k^2 - 1}R(r_k e^{it}, T)^* x_k^* \right\|_{\gamma'((0, 2\pi) \times \mathbb{N}_N; X^*)} \\ & \leq C^2 \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)} \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}. \end{aligned}$$

Hence  $T$  has  $(\gamma\text{-GFS})_1$  and therefore it is a  $\gamma$ -GFS operator.  $\square$

*Remark 3.4.19.* When  $X$  is  $K$ -convex (3.4.9) can be replaced by

$$\left\| (t, k) \mapsto \sqrt{r_k^2 - 1}R(r_k e^{it}, T^*)x_k^* \right\|_{\gamma([0, 2\pi] \times \mathbb{N}_N; X^*)} \leq \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X^*)}.$$

Moreover let  $(S, \mu)$  be a measure space and let  $E(S)$  be a  $K$ -convex Banach function space over  $(S, \mu)$  (see [26, appendix F] for definition). Then  $E(S)$  has finite cotype hence according to [26, Proposition 9.3.8], there exist  $c > 0$  and  $C > 0$  such that for each  $f \in \gamma([0, 2\pi] \times \mathbb{N}_N; E(S))$ ,

$$c \|f\|_{E(S; L^2([0, 2\pi] \times \mathbb{N}_N))} \leq \|f\|_{\gamma([0, 2\pi] \times \mathbb{N}_N; E(S))} \leq C \|f\|_{E(S; L^2([0, 2\pi] \times \mathbb{N}_N))}. \quad (3.4.11)$$

Furthermore, the following equality holds,

$$\|f\|_{E(S; L^2([0, 2\pi] \times \mathbb{N}_N))} = \left\| \left( \int_0^{2\pi} \sum_{k=1}^N |f(\cdot, k)|^2 \right)^{\frac{1}{2}} \right\|_{E(S)}.$$

The space  $E(S)^*$  satisfies similar properties. Hence using (3.4.11) and the Khintchine-Maurey inequality [26, Theorem 7.2.13], (ii) in Theorem 3.4.18 can be replaced by:

(ii)' The spectrum set  $\sigma(T)$  is included in  $\overline{\mathbb{D}}$  and there exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$ ,  $r_1, \dots, r_N > 1$ ,  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$

$$\left\| \left( \sum_{k=1}^N \int_0^{2\pi} (r_k^2 - 1) |R(r_k e^{it}, T)x_k|^2 dt \right)^{\frac{1}{2}} \right\|_{E(S)} \leq C \left\| \left( \sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}} \right\|_{E(S)}$$

and

$$\left\| \left( \sum_{k=1}^N \int_0^{2\pi} (r_k^2 - 1) |R(r_k e^{it}, T^*)x_k^*|^2 dt \right)^{\frac{1}{2}} \right\|_{E(S)^*} \leq C \left\| \left( \sum_{k=1}^N |x_k^*|^2 \right)^{\frac{1}{2}} \right\|_{E(S)^*}.$$

Thus  $T \in B(E(S))$  is power  $\gamma$ -bounded if and only if (ii)' holds true.

In particular the above applies when  $E(S) = L^p(S)$  for some  $1 < p < \infty$ .

### Peller calculus

We denote by  $H^p$  the classical Hardy space over the unit disk. Let  $\mathcal{A}(\mathbb{D})$  be the set of all functions  $F : \overline{\mathbb{D}} \rightarrow \mathbb{C}$  such that there exist two sequences  $(f_k)_{k \in \mathbb{N}} \subset C(\mathbb{T})$  and  $(h_k)_{k \in \mathbb{N}} \subset H^1$  satisfying

$$\sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 < \infty \quad (3.4.12)$$

and

$$\forall z \in \overline{\mathbb{D}}, \quad F(z) = \sum_{k=1}^{\infty} f_k \star h_k(z). \quad (3.4.13)$$

We endow  $\mathcal{A}(\mathbb{D})$  with the norm

$$\|F\|_{\mathcal{A}} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\}$$

where the infimum runs over all sequences  $(f_k)_{k \in \mathbb{N}} \subset C(\mathbb{T})$  and  $(h_k)_{k \in \mathbb{N}} \subset H^1$  satisfying (4.3.1) and (3.4.13). It is known (see [47, Lemma 3.6.]) that with this norm  $\mathcal{A}(\mathbb{D})$  is a Banach algebra for pointwise multiplication.

Let  $H$  be a Hilbert space and let  $T \in B(H)$  be a power-bounded operator. According to [47, Theorem 3.5.] or [50, Proposition 4.11] there exists  $C > 0$  such that for each polynomial  $P$  one has

$$\|P(T)\| \leq C \|P\|_{\mathcal{A}}.$$

Now since the set of polynomials is dense in  $\mathcal{A}(\mathbb{D})$  the bounded algebra homomorphism  $P \rightarrow P(T)$  extends to a bounded algebra homomorphism from  $\mathcal{A}(\mathbb{D})$  into  $B(H)$ . Our aim is now to give a  $\gamma$ -version of this result.

**Theorem 3.4.20.** *Let  $X$  be a Banach space. Let  $T \in B(X)$  be a power  $\gamma$ -bounded operator. Then the set*

$$\{P(T) : P \text{ is a polynomial with } \|P\|_{\mathcal{A}} \leq 1\}$$

*is  $\gamma$ -bounded.*

*Proof.* We adapt an argument from [50, Proposition 4.16.]. First we show that the set

$$\Gamma = \{f \star (uv)(T) : f \in C(\mathbb{T}), u, v \text{ are polynomials}, \|f\|_\infty \|u\|_2 \|v\|_2 \leq 1\}$$

is  $\gamma$ -bounded. Let  $N \in \mathbb{N}$ ,  $f_1, \dots, f_N \in C(\mathbb{T})$  and let  $u_1, \dots, u_N, v_1, \dots, v_N$  be polynomials with  $\|f_k\|_\infty \|u_k\|_2 \|v_k\|_2 \leq 1$  for each  $k \in \mathbb{N}_N$ , and let  $x_1, \dots, x_N \in X$ ,  $x_1^*, \dots, x_N^* \in X^*$ . One has, for each  $k \in \mathbb{N}_N$

$$\langle (f_k \star (u_k v_k))(T)x_k, x_k^* \rangle = \frac{1}{2\pi} \int_0^{2\pi} f_k(e^{is}) \langle u_k(e^{-is}T)x_k, v_k(e^{-is}T)^* x_k^* \rangle ds.$$

It follows, using Theorem 3.4.3, that

$$\begin{aligned} \sum_{k=1}^N |\langle (f_k \star (u_k v_k))(T)x_k, x_k^* \rangle| &\leq \frac{1}{2\pi} \sum_{k=1}^N \int_0^{2\pi} \frac{|f_k(e^{is})|}{\|f_k\|_\infty \|u_k\|_2 \|v_k\|_2} |\langle u_k(e^{-is}T)x_k, v_k(e^{-is}T)^* x_k^* \rangle| ds \\ &\leq \sum_{k=1}^N \int_0^{2\pi} \left| \left\langle \frac{u_k(e^{-is}T)x_k}{\|u_k\|_2}, \frac{v_k(e^{-is}T)^* x_k^*}{\|v_k\|_2} \right\rangle \right| ds \\ &= \left\| (s, k) \mapsto \left\langle \frac{u_k(e^{-is}T)x_k}{\|u_k\|_2}, \frac{v_k(e^{-is}T)^* x_k^*}{\|v_k\|_2} \right\rangle \right\|_{L^1((0,2\pi) \times \mathbb{N}_N)} \\ &\leq \frac{1}{2\pi} \left\| (s, k) \mapsto \frac{u_k(e^{-is}T)x_k}{\|u_k\|_2} \right\|_{\gamma((0,2\pi) \times \mathbb{N}_N; X)} \left\| (s, k) \mapsto \frac{v_k(e^{-is}T)^* x_k^*}{\|v_k\|_2} \right\|_{\gamma'((0,2\pi) \times \mathbb{N}_N; X^*)}. \end{aligned}$$

By Lemma 3.4.5 and the  $\gamma$ -boundedness of  $\{T^n : n \in \mathbb{N}\}$  one obtains

$$\begin{aligned} \left\| (s, k) \mapsto \frac{u_k(e^{-is}T)x_k}{\|u_k\|_2} \right\|_{\gamma((0,2\pi) \times \mathbb{N}_N; X)} &= \left\| (n, k) \mapsto c_n(u_k) T^n \left( \frac{x_k}{\|u_k\|_2} \right) \right\|_{\gamma(\mathbb{N} \cup \{0\} \times \mathbb{N}_N; X)} \\ &= \left\| \sum_{n=0}^{\infty} \sum_{k=1}^N \gamma_{n,k} \otimes T^n \left( \frac{c_n(u_k)x_k}{\|u_k\|_2} \right) \right\|_{G(X)} \\ &\leq C_\gamma \left\| \sum_{n=0}^{\infty} \sum_{k=1}^N \gamma_{n,k} \otimes \frac{c_n(u_k)x_k}{\|u_k\|_2} \right\|_{G(X)}. \end{aligned}$$

For each  $k \in \mathbb{N}_N$ ,  $\frac{\|(c_n(u_k))_{n \in \mathbb{N} \cup \{0\}}\|_{l^2}}{\|u_k\|_2} = 1$ , therefore

$$\left\| \sum_{n=0}^{\infty} \sum_{k=1}^N \gamma_{n,k} \otimes \frac{c_n(u_k)x_k}{\|u_k\|_2} \right\|_{G(X)} \leq \|(x_k)_{k \in \mathbb{N}_N}\|_{\gamma(\mathbb{N}_N; X)}.$$

Then arguing as in the proof of Theorem 3.4.18,

$$\left\| (s, k) \mapsto \frac{v_k(e^{-is}T)^* x_k^*}{\|v_k\|_2} \right\|_{\gamma'((0,2\pi) \times \mathbb{N}_N; X^*)} \leq C_\gamma \|(x_k^*)_{k \in \mathbb{N}_N}\|_{\gamma'(\mathbb{N}_N; X^*)}.$$

This shows that  $\Gamma$  is  $\gamma$ -bounded.

Now if  $P = \sum_{k \in \mathbb{N}} f_k \star h_k$  is a polynomial with  $f_k \in C(\mathbb{T})$ ,  $h_k \in H^1$  and  $\sum_{k \in \mathbb{N}} \|f_k\|_\infty \|h_k\|_1 \leq 1$  then using the facts that for each  $k \in \mathbb{N}$ ,  $h_k$  can be written as a product  $h_k = u_k v_k$  with  $u_k, v_k \in H^2$  and  $\|h_k\|_1 = \|u_k\|_2 \|v_k\|_2$  and that the set of polynomials is dense in  $H^2$  one obtains that  $P \in \overline{\text{absconv}(\Gamma)}^{\|\cdot\|}$ . Finally one has the following inclusion

$$\{P(T) : P \text{ is a polynomial with } \|P\|_{\mathcal{A}} \leq 1\} \subset \overline{\text{absconv}(\Gamma)}^{\|\cdot\|},$$

and the latter set is  $\gamma$ -bounded thanks to Proposition 3.4.2 whence the desired result.  $\square$

*Remark 3.4.21.* Since the set  $\{P(T) : P \text{ is a polynomial with } \|P\|_{\mathcal{A}} \leq 1\}$  is  $\gamma$ -bounded it is uniformly bounded. Therefore, since polynomials are dense in  $\mathcal{A}(\mathbb{D})$ , the homomorphism

$$u : P \mapsto P(T)$$

extends to a bounded algebra homomorphism from  $\mathcal{A}(\mathbb{D})$  into  $B(X)$ . Theorem 3.4.20 implies that this homomorphism is  $\gamma$ -bounded, that is  $\{u(f) : f \in \mathcal{A}(\mathbb{D}) : \|f\|_{\mathcal{A}} \leq 1\}$  is  $\gamma$ -bounded.

# Chapter 4

## Functional calculus for a bounded $C_0$ -semigroup on Hilbert space

### 4.1. Introduction

Let  $H$  be a Hilbert space and let  $-A$  be the infinitesimal generator of a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on  $H$ . To any  $b \in L^1(\mathbb{R}_+)$ , one may associate the operator  $\Gamma(A, b) \in B(H)$  defined by

$$[\Gamma(A, b)](x) = \int_0^\infty b(t)T_t(x) dt, \quad x \in H.$$

The mapping  $b \mapsto \Gamma(A, b)$  is the so-called Hille-Phillips functional calculus ([23], see also [19, Section 3.3]) and we obviously have

$$\|\Gamma(A, b)\| \leq C\|b\|_1, \quad b \in L^1(\mathbb{R}_+),$$

where  $C = \sup_{t \geq 0} \|T_t\|$ . This holds true as well for any bounded  $C_0$ -semigroup on Banach space. However we focus here on semigroups acting on Hilbert space.

If  $(T_t)_{t \geq 0}$  is a contractive semigroup (i.e.  $\|T_t\| \leq 1$  for all  $t \geq 0$ ) on  $H$ , then we have the much stronger estimate  $\|\Gamma(A, b)\| \leq \|\widehat{b}\|_\infty$  for any  $b \in L^1(\mathbb{R}_+)$ , where  $\widehat{b}$  denotes the Fourier transform of  $b$ . This is a semigroup version of von Neumann's inequality, see [19, Section 7.1.3] for a proof. Hence more generally, if  $(T_t)_{t \geq 0}$  is similar to a contractive semigroup, then there exists a constant  $C \geq 1$  such that

$$\|\Gamma(A, b)\| \leq C\|\widehat{b}\|_\infty, \quad b \in L^1(\mathbb{R}_+). \quad (4.1.1)$$

However not all negative generators of bounded  $C_0$ -semigroups satisfy such an estimate. Indeed if  $A$  is sectorial of type  $< \frac{\pi}{2}$ , it follows from [19, Section 3.3] that  $A$  satisfies an estimate of the form (4.1.1) exactly when  $A$  has a bounded  $H^\infty$ -functional calculus, see Subsection 4.4 for more on this.

The motivation for this paper is the search of sharp estimates of  $\|\Gamma(A, b)\|$ , and of the norms of other functions of  $A$ , valid for all negative generators of bounded  $C_0$ -semigroups. A major breakthrough was achieved by Haase [21, Corollary 5.5] who proved an estimate

$$\|\Gamma(A, b)\| \leq C\|L_b\|_{\mathcal{B}},$$

where  $\|\cdot\|_{\mathcal{B}}$  denotes the norm with respect to a suitable Besov algebra  $\mathcal{B}(\mathbb{C}_+)$  of analytic functions, and

$$L_b: \mathbb{C}_+ = \{\operatorname{Re}(\cdot) > 0\} \rightarrow \mathbb{C}, \quad L_b(z) = \int_0^\infty b(t)e^{-tz} dt,$$

is the Laplace transform of  $b$ . It is worth mentioning the related works by White [59] and Vitse [57]. More recently, Batty-Gomilko-Tomilov [7] (see also [8]) extended Haase's result by providing an explicit construction of a bounded functional calculus  $\mathcal{B}(\mathbb{C}_+) \rightarrow B(H)$  associated with  $A$ , extending the Hille-Phillips functional calculus.

In this paper we introduce the space  $\mathcal{A}(\mathbb{R}) \subset H^\infty(\mathbb{R})$  defined by

$$\mathcal{A}(\mathbb{R}) = \left\{ F = \sum_{k=1}^{\infty} f_k \star h_k : (f_k)_{k \in \mathbb{N}} \subset BUC(\mathbb{R}), (h_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R}), \sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1 < \infty \right\},$$

equipped with the norm  $\|F\|_{\mathcal{A}} = \inf\{\sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1\}$ , where the infimum runs over all sequences  $(f_k)_{k \in \mathbb{N}} \subset BUC(\mathbb{R})$  and  $(h_k)_{k \in \mathbb{N}} \subset H^1(\mathbb{R})$  such that  $\sum_{k=1}^{\infty} \|f_k\|_\infty \|h_k\|_1 < \infty$  and  $F = \sum_{k=1}^{\infty} f_k \star h_k$ . The definition of this space is inspired by Peller's paper [47], where a discrete analogue of  $\mathcal{A}(\mathbb{R})$  was introduced to study functions of power bounded operators on Hilbert space. Also,  $\mathcal{A}(\mathbb{R})$  can be regarded as an analytic version of the the Figa-Talamanca-Herz algebras  $A_p(\mathbb{R})$ ,  $1 < p < \infty$ , for which we refer e.g. to [12, Chapter 3].

We prove in Section 4.3 that  $\mathcal{A}(\mathbb{R})$  is indeed a Banach algebra for pointwise multiplication. Next in Section 4.4 we introduce the natural half-plane version  $\mathcal{A}(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+)$  of  $\mathcal{A}(\mathbb{R})$  and we show (Corollary 4.4.7) that whenever  $A$  is the negative generator of a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on Hilbert space, there is a unique bounded homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$ , such that

$$\rho_A(L_b) = \Gamma(A, b), \quad b \in L^1(\mathbb{R}_+). \quad (4.1.2)$$

In particular we show that

$$\|\Gamma(A, b)\| \leq \left( \sup_{t \geq 0} \|T_t\| \right)^2 \|L_b\|_{\mathcal{A}}, \quad b \in L^1(\mathbb{R}_+).$$

This improves Haase's estimate mentioned above. Our work also improves [7, Theorem 4.4] in the Hilbert space case. Indeed we show in Section 4.5 that the Besov algebra considered in [19, 7] is included in  $\mathcal{A}(\mathbb{C}_+)$ , with an estimate  $\|\cdot\|_{\mathcal{A}} \lesssim \|\cdot\|_{\mathcal{B}}$ , and we also show that the converse is wrong.

In general, our main result (Corollary 4.4.7) does hold true on non Hilbertian Banach spaces. In Section 4.6, following ideas from chapter 2, chapter 3 and [38], we give a Banach space version of Corollary 4.4.7, using the notion of  $\gamma$ -boundedness. Namely we show that if  $A$  is the negative generator of a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  on a Banach space  $X$ , then the set  $\{T_t : t \geq 0\} \subset B(X)$  is  $\gamma$ -bounded if and only if there exists a  $\gamma$ -bounded homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$  satisfying (4.1.2). This should be regarded as a semigroup version of [38, Theorem 4.4], where a characterization of  $\gamma$ -bounded continuous representations of amenable groups was established.

Our results make crucial use of Fourier multipliers on the Hardy space  $H^1(\mathbb{R})$ . Section 4.2 is devoted to this topic. In particular we establish the following result of independent interest: if a bounded operator  $T: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  commutes with translations, then there exists a bounded continuous function  $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$  such that  $\|m\|_\infty \leq \|T\|$  and for any  $h \in H^1(\mathbb{R})$ ,  $\widehat{T(h)} = m\widehat{h}$ .

## Notation and convention.

We will use the following open half-planes of  $\mathbb{C}$ ,

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}, \quad P_+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}, \quad P_- := \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}.$$

Also for any real  $\alpha \in \mathbb{R}$ , we set

$$\mathcal{H}_\alpha = \{z \in \mathbb{C} : \operatorname{Re}(z) > \alpha\}.$$

In particular,  $\mathcal{H}_0 = \mathbb{C}_+$ .

For any  $s \in \mathbb{R}$ , we let  $\tau_s: L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \rightarrow L^1(\mathbb{R}) + L^\infty(\mathbb{R})$  denote the translation operator defined by

$$\tau_s f(t) = f(t - s), \quad t \in \mathbb{R},$$

for any  $f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ .

The Fourier transform of any  $f \in L^1(\mathbb{R})$  is defined by

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(t) e^{-itu} dt, \quad u \in \mathbb{R}.$$

Sometimes we write  $\mathcal{F}(f)$  instead of  $\widehat{f}$ . We will also let  $\mathcal{F}(f)$  or  $\widehat{f}$  denote the Fourier transform of any  $f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ . Wherever it makes sense, we will use  $\mathcal{F}^{-1}$  to denote the inverse Fourier transform.

We will use several times the following elementary result (which follows from Fubini's Theorem and the Fourier inversion Theorem).

**Lemma 4.1.1.** *Let  $f_1, f_2 \in L^1(\mathbb{R})$  such that either  $\widehat{f}_1$  or  $\widehat{f}_2$  belongs to  $L^1(\mathbb{R})$ . Then*

$$\int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}_1(u) \widehat{f}_2(-u) du.$$

The norm on  $L^p(\mathbb{R})$  will be denoted by  $\|\cdot\|_p$ . We let  $C_0(\mathbb{R})$  (resp.  $BUC(\mathbb{R})$ , resp.  $C_b(\mathbb{R})$ ) denote the Banach algebra of continuous functions on  $\mathbb{R}$  which vanish at infinity (resp. of bounded and uniformly continuous functions on  $\mathbb{R}$ , resp. of bounded continuous functions on  $\mathbb{R}$ ), equipped with the sup-norm  $\|\cdot\|_\infty$ . We set

$$C_{00}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \widehat{f} \in L^1(\mathbb{R})\}, \quad (4.1.3)$$

this is a dense subspace of  $C_0(\mathbb{R})$ . Further we let  $\mathcal{S}(\mathbb{R})$  denote the Schwartz space on  $\mathbb{R}$  and we let  $M(\mathbb{R})$  denote the Banach algebra of all bounded Borel measures on  $\mathbb{R}$ .

We will use the identification  $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$  (Riesz's Theorem) provided by the duality pairing

$$\langle \mu, f \rangle = \int_{\mathbb{R}} f(-t) d\mu(t), \quad \mu \in M(\mathbb{R}), f \in C_0(\mathbb{R}). \quad (4.1.4)$$

For any non empty open set  $\mathcal{O} \subset \mathbb{C}$ , we let  $H^\infty(\mathcal{O})$  denote the Banach algebra of all bounded analytic functions on  $\mathcal{O}$ , equipped with the sup-norm  $\|\cdot\|_\infty$ .

Let  $X, Y$  be (complex) Banach spaces. We let  $B(X, Y)$  denote the Banach space of all bounded operators  $X \rightarrow Y$ . We simply write  $B(X)$  instead of  $B(X, X)$ , when  $Y = X$ . We let  $I_X$  denote the identity operator on  $X$ .

The domain of an operator  $A$  on some Banach space  $X$  is denoted by  $\operatorname{Dom}(A)$ . Its kernel and range are denoted by  $\operatorname{Ker}(A)$  and  $\operatorname{Ran}(A)$ , respectively. If  $z \in \mathbb{C}$  belongs to the resolvent set of  $A$ , we let  $R(z, A) = (zI_X - A)^{-1}$  denote the corresponding resolvent operator.



## 4.2. Fourier multipliers on $H^1(\mathbb{R})$

We denote by  $H^1(\mathbb{R})$  the classical Hardy space, defined as the closed subspace of  $L^1(\mathbb{R})$  of all functions  $h$  such that  $\widehat{h}(u) = 0$  for any  $u \leq 0$ . For any  $1 < p < \infty$ , we denote by  $H^p(\mathbb{R})$  the closure of  $H^1(\mathbb{R}) \cap L^p(\mathbb{R})$  in  $L^p(\mathbb{R})$ . Also we let  $H^\infty(\mathbb{R})$  denote the  $w^*$ -closure of  $H^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  in  $L^\infty(\mathbb{R})$ . We recall (see e.g [15], [24] or [31]) that for any  $1 \leq p \leq \infty$ ,  $H^p(\mathbb{R})$  coincides with the subspace of all functions  $f \in L^p(\mathbb{R})$  whose Poisson integral  $\mathcal{P}[f]: P_+ \rightarrow \mathbb{C}$  is analytic.

It is well-known that  $H^2(\mathbb{R})$  is the subspace of all functions in  $L^2(\mathbb{R})$  whose Fourier transform vanishes almost everywhere on  $\mathbb{R}_-$ . This can be expressed by the identification

$$\mathcal{F}(H^2(\mathbb{R})) = L^2(\mathbb{R}_+). \quad (4.2.1)$$

Let  $m \in L^\infty(\mathbb{R}_+)$ . Using (4.2.1), we may associate

$$\begin{aligned} T_m &: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R}) \\ h &\mapsto \mathcal{F}^{-1}(m\widehat{h}), \end{aligned}$$

and we have  $\|T_m\| = \|m\|_\infty$ . The function  $m$  is called the symbol of  $T_m$ .

Let  $1 \leq p < \infty$ . Assume that  $T_m$  is bounded with respect to the  $H^p(\mathbb{R})$ -norm, that is, there exists a constant  $C > 0$  such that

$$\|\mathcal{F}^{-1}(m\widehat{h})\|_p \leq C \|h\|_p, \quad h \in H^p(\mathbb{R}) \cap H^2(\mathbb{R}). \quad (4.2.2)$$

Then since  $H^p(\mathbb{R}) \cap H^2(\mathbb{R})$  is dense in  $H^p(\mathbb{R})$ ,  $T_m$  uniquely extends to a bounded operator on  $H^p(\mathbb{R})$  whose norm is the least possible constant  $C$  satisfying (4.2.2). In this case we keep the same notation  $T_m: H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$  for this extension. Operators of this form are called bounded Fourier multipliers on  $H^p(\mathbb{R})$ . They form a subspace of  $B(H^p(\mathbb{R}))$ , that we denote by  $\mathcal{M}(H^p(\mathbb{R}))$ . It is plain that  $\mathcal{M}(H^2(\mathbb{R})) \simeq L^\infty(\mathbb{R}_+)$  isometrically. In the sequel we will be mostly interested by  $\mathcal{M}(H^1(\mathbb{R}))$ .

The above definitions parallel the classical definitions of bounded Fourier multipliers on  $L^p(\mathbb{R})$ , that we will use without any further reference.

*Example 4.2.1.* Let  $s \in \mathbb{R}$ . For all  $h \in L^1(\mathbb{R})$ , one has  $\widehat{\tau_s h}(u) = e^{-isu}\widehat{h}(u)$  for any  $u \in \mathbb{R}$ . Hence  $\tau_s$  maps  $H^1(\mathbb{R})$  into itself. Further  $\tau_s$  is a bounded Fourier multiplier on  $H^1(\mathbb{R})$ , with symbol  $m(u) = e^{-isu}$ .

In the sequel we say that a bounded operator  $T: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  commutes with translations if  $T\tau_s = \tau_s T$  for each  $s \in \mathbb{R}$ . Classical properties of the Fourier transform easily imply that any bounded Fourier multiplier on  $H^1(\mathbb{R})$  commutes with translations. The next result implies that the converse is true. This is the  $H^1(\mathbb{R})$ -version of the well-known characterization of  $L^p(\mathbb{R})$ -Fourier multipliers as the operators  $L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  commuting with translations (see e.g. [54, Theorem 3.16]). The  $H^1(\mathbb{R})$ -case requires different arguments.

**Theorem 4.2.2.** *Let  $T \in B(H^1(\mathbb{R}))$  and assume that  $T$  commutes with translations. Then there exists a bounded continuous function  $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$  such that  $\widehat{Th} = m\widehat{h}$  for any  $h \in H^1(\mathbb{R})$  (and hence  $T = T_m$ ). In this case, we have*

$$\|m\|_\infty \leq \|T\|. \quad (4.2.3)$$

*Proof.* We use Bochner spaces and Bochner integrals, for which we refer to [13]. Let  $T \in B(H^1(\mathbb{R}))$  and assume that  $T$  commutes with translations.

Let  $h, g \in H^1(\mathbb{R})$ . The identification  $L^1(\mathbb{R}^2) = L^1(\mathbb{R}; L^1(\mathbb{R}))$  and the fact that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(t-s)||g(s)| dt ds = \|h\|_1 \|g\|_1 < \infty$$

imply that  $s \mapsto g(s)\tau_s h$  in an almost everywhere defined function belonging to the Bochner space  $L^1(\mathbb{R}; L^1(\mathbb{R}))$ . Since  $\tau_s h$  belongs to  $H^1(\mathbb{R})$  for any  $s \in \mathbb{R}$ , the latter is actually an element of  $L^1(\mathbb{R}; H^1(\mathbb{R}))$ . Further its integral (which is an element of  $H^1(\mathbb{R})$ ) is equal to the convolution of  $h$  and  $g$ , that is,

$$h \star g = \int_{-\infty}^{\infty} \tau_s h g(s) ds. \quad (4.2.4)$$

It follows, using the assumption, that

$$\begin{aligned} T(h \star g) &= \int_{-\infty}^{\infty} T(\tau_s h)g(s) ds \\ &= \int_{-\infty}^{\infty} \tau_s (Th)g(s) ds \\ &= Th \star g. \end{aligned}$$

(The last equality comes from (4.2.4) replacing  $h$  by  $Th$ .) Likewise  $T(h \star g) = h \star Tg$ , whence  $Th \star g = h \star Tg$ . Applying the Fourier transform to the latter equality, one obtains

$$\widehat{Th} \cdot \widehat{g} = \widehat{h} \cdot \widehat{Tg}. \quad (4.2.5)$$

Now let  $f \in \mathcal{S}(\mathbb{R})$  satisfying  $f = 0$  on  $\mathbb{R}_-$  and  $f > 0$  on  $\mathbb{R}_+^*$ . The function  $g = \mathcal{F}^{-1}(f)$  belongs to  $H^1(\mathbb{R})$  and we may define  $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$  by

$$m(u) = \frac{\widehat{Tg}(u)}{\widehat{g}(u)}, \quad u > 0. \quad (4.2.6)$$

Obviously  $m$  is continuous. Furthermore it follows from (4.2.5) that for any  $h \in H^1(\mathbb{R})$ , we have  $\widehat{Th} = m\widehat{h}$  on  $\mathbb{R}_+^*$ . It therefore suffices to show that  $m$  is bounded and that (4.2.3) holds true.

We adapt an argument from [37]. Let us first prove the inequality

$$|m(1)| \leq \|T\|. \quad (4.2.7)$$

To obtain this, we let  $\gamma_1, \gamma_2 \in C_{00}(\mathbb{R})$  (see (4.1.3)) such that  $\|\gamma_1\|_1 = 1$ ,  $\text{Supp}(\widehat{\gamma}_1) \subset [-1, 1]$  and  $\|\gamma_2\|_{\infty} = 1$ . For each  $0 < \varepsilon \leq \frac{1}{2}$ , we define  $h_{\varepsilon}, g_{\varepsilon}: \mathbb{R} \rightarrow \mathbb{C}$  by

$$h_{\varepsilon}(t) = \varepsilon e^{it} \gamma_1(\varepsilon t) \quad \text{and} \quad g_{\varepsilon}(t) = e^{-it} \gamma_2(\varepsilon t), \quad t \in \mathbb{R}.$$

Then,  $h_{\varepsilon} \in H^1(\mathbb{R})$ ,  $g_{\varepsilon} \in C_{00}(\mathbb{R})$ ,  $\|h_{\varepsilon}\|_1 = 1$ ,  $\|g_{\varepsilon}\|_{\infty} = 1$  and we both have

$$\widehat{h_{\varepsilon}}(u) = \widehat{\gamma}_1\left(\frac{u-1}{\varepsilon}\right) \quad \text{and} \quad \widehat{g_{\varepsilon}}(u) = \frac{1}{\varepsilon} \widehat{\gamma}_2\left(\frac{u+1}{\varepsilon}\right)$$

for any  $u \in \mathbb{R}$ . Moreover using Lemma 4.1.1, we have

$$\int_{-\infty}^{\infty} [T(h_{\varepsilon})](t)g_{\varepsilon}(t) dt = \frac{1}{2\pi} \int_0^{\infty} \widehat{T(h_{\varepsilon})}(u)\widehat{g_{\varepsilon}}(-u) du.$$

We infer that

$$\begin{aligned} \int_{-\infty}^{\infty} [T(h_\varepsilon)](t)g_\varepsilon(t) dt &= \frac{1}{2\pi} \int_0^\infty m(u)\widehat{h}_\varepsilon(u)\widehat{g}_\varepsilon(-u)du \\ &= \frac{1}{2\pi} \int_{1-\varepsilon}^{1+\varepsilon} \frac{m(u)}{\varepsilon} \widehat{\gamma}_1\left(\frac{u-1}{\varepsilon}\right) \widehat{\gamma}_2\left(\frac{-u+1}{\varepsilon}\right) du. \end{aligned}$$

Therefore,

$$\frac{1}{\varepsilon} \left| \int_{1-\varepsilon}^{1+\varepsilon} m(u) \widehat{\gamma}_1\left(\frac{u-1}{\varepsilon}\right) \widehat{\gamma}_2\left(\frac{-u+1}{\varepsilon}\right) du \right| \leq 2\pi \|T\|.$$

By the change of variable  $u \mapsto \frac{u-1}{\varepsilon}$ , this reads

$$\left| \int_{-1}^1 m(1+\varepsilon u) \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du \right| \leq 2\pi \|T\|.$$

Since  $m$  is continuous on  $\mathbb{R}_+^*$  and  $\varepsilon \in (0, \frac{1}{2})$ , the function  $u \mapsto m(1+\varepsilon u)$  is bounded on  $(-1, 1)$ , independently of  $\varepsilon$ . Moreover  $\widehat{\gamma}_1$  is bounded on  $(-1, 1)$  and  $\widehat{\gamma}_2$  is integrable. Hence by Lebesgue's dominated convergence Theorem,

$$\int_{-1}^1 m(1+\varepsilon u) \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du \xrightarrow{\varepsilon \rightarrow 0} m(1) \int_{-1}^1 \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du.$$

Since  $\int_0^\infty \widehat{\gamma}_1(u) \widehat{\gamma}_2(-u) du = 2\pi \int_{-\infty}^\infty \gamma_1(t) \gamma_2(t) dt$ , by Lemma 4.1.1, we deduce the inequality

$$|m(1)| \left| \int_{-\infty}^\infty \gamma_1(t) \gamma_2(t) dt \right| \leq \|T\|.$$

It turns out that the supremum of  $\left\{ \left| \int_{-\infty}^\infty \gamma_1(t) \gamma_2(t) dt \right| \right\}$ , for  $\gamma_1$  and  $\gamma_2$  defined as above, is equal to 1. Hence (4.2.7) holds true.

Now consider an arbitrary  $a > 0$ . Define  $r_a: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  by

$$[r_a h](t) = \frac{1}{a} h\left(\frac{t}{a}\right), \quad h \in H^1(\mathbb{R}), t \in \mathbb{R}.$$

Then  $r_a$  is an isometry satisfying  $\widehat{r_a h}(u) = \widehat{h}(au)$  for any  $h \in H^1(\mathbb{R})$  and  $u \in \mathbb{R}$ . Set

$$T_{(a)} = r_a T r_{\frac{1}{a}}: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}).$$

Then  $\tau_s T_{(a)} = T_{(a)} \tau_s$  for any  $s \in \mathbb{R}$  and the function  $\mathbb{R}_+^* \rightarrow \mathbb{C}$  associated with  $T_{(a)}$  by the formula (4.2.6) is  $u \mapsto m(au)$ . Further  $\|T_{(a)}\| = \|T\|$ . Hence applying (4.2.7) to  $T_{(a)}$  we deduce that

$$|m(a)| \leq \|T\|.$$

This proves the boundness of  $m$  and (4.2.3).  $\square$

*Remark 4.2.3.* Let  $1 < p < \infty$ .

(1) Let  $S: L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  be a bounded Fourier multiplier. Then  $S$  maps  $H^p(\mathbb{R})$  into itself and the restriction  $S|_{H^p}: H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$  is a bounded Fourier multiplier.

Let  $Q: L^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$  be the Riesz projection and let  $J: H^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  be the canonical embedding. Then conversely, for any bounded Fourier multiplier  $T: H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$ ,  $S = JTQ$  is a bounded Fourier multiplier on  $L^p(\mathbb{R})$ , whose restriction to  $H^p(\mathbb{R})$  coincides with  $T$ . Thus  $\mathcal{M}(H^p(\mathbb{R}))$  can be simply regarded as a subspace of  $\mathcal{M}(L^p(\mathbb{R}))$ , the space of bounded Fourier multipliers on  $L^p(\mathbb{R})$ .

It is also easy to check that a bounded operator  $H^p(\mathbb{R}) \rightarrow H^p(\mathbb{R})$  belongs to  $\mathcal{M}(H^p(\mathbb{R}))$  if and only if it commutes with translations, using the similar result on  $L^p(\mathbb{R})$ .

(2) Let  $p' = \frac{p}{p-1}$  be the conjugate number of  $p$ . Using  $Q$  again, we see that given any  $m \in L^\infty(\mathbb{R}_+)$ , the operator  $T_m: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$  extends to a bounded Fourier multiplier on  $H^p(\mathbb{R})$  if and only if it extends to a bounded Fourier multiplier on  $H^{p'}(\mathbb{R})$ . Thus,

$$\mathcal{M}(H^p(\mathbb{R})) \simeq \mathcal{M}(H^{p'}(\mathbb{R})) \quad (4.2.8)$$

isomorphically.

Recall that the bounded Fourier multipliers on  $L^1(\mathbb{R})$  are the operators of the form  $h \mapsto \mu \star h$ , with  $\mu \in M(\mathbb{R})$ , and that the norm of the latter operator is equal to  $\|\mu\|_{M(\mathbb{R})}$  (see e.g. [54, Theorem 3.19]).

For any  $\mu \in M(\mathbb{R})$ , let  $R_\mu: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  be the restriction of  $h \mapsto \mu \star h$  to  $H^1(\mathbb{R})$ . This is a bounded Fourier multiplier whose symbol is equal to the restriction of  $\hat{\mu}$  to  $\mathbb{R}_+$ . We set

$$\mathcal{R} = \{R_\mu : \mu \in M(\mathbb{R})\} \subset \mathcal{M}(H^1(\mathbb{R})). \quad (4.2.9)$$

In contrast with the result in Remark 4.2.3 (1), we will see in Proposition 4.2.4 below that  $\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R}))$ .

Following [49], we say that an operator  $T \in B(H^1(\mathbb{R}))$  is regular if there exists a constant  $C \geq 0$  such that

$$\left\| \sup_k |T(h_k)| \right\|_1 \leq C \left\| \sup_k |h_k| \right\|_1$$

for any finite family  $(h_k)_k$  of  $H^1(\mathbb{R})$ .

For any Banach space  $X$ , let  $H^1(\mathbb{R}; X)$  denote the closure of  $H^1(\mathbb{R}) \otimes X$  in the Bochner space  $L^1(\mathbb{R}; X)$ . As is well-known (see [51, Chapter 1]),  $T \in B(H^1(\mathbb{R}))$  is regular if and only if  $T \otimes I_X: H^1(\mathbb{R}) \otimes X \rightarrow H^1(\mathbb{R}) \otimes X$  extends to a bounded operator from  $H^1(\mathbb{R}; X)$  into itself, for any Banach space  $X$ .

#### Proposition 4.2.4.

(1) Let  $T \in \mathcal{M}(H^1(\mathbb{R}))$ . Then  $T \in \mathcal{R}$  if and only if  $T$  is regular.

(2)  $\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R}))$ .

*Proof.* Let  $\mu \in M(\mathbb{R})$ . For any  $F = \sum_j h_j \otimes x_j$  (finite sum) in  $H^1(\mathbb{R}) \otimes X$ , we have  $(R_\mu \otimes I_X)(F) = \mu \star F$ . Hence arguing as in the scalar case, we find that  $\|(R_\mu \otimes I_X)(F)\|_1 \leq \|\mu\|_{M(\mathbb{R})} \|F\|_1$ . This shows that any element of  $\mathcal{R}$  is regular.

Conversely, let  $T \in \mathcal{M}(H^1(\mathbb{R}))$  and assume that  $T$  is regular. By [41] (see also [49, Theorem 3]),  $T$  admits a bounded extension  $S: L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ . Using an entirely classical averaging argument, we are going to show that  $T$  actually admits a bounded extension on  $L^1(\mathbb{R})$  which commutes with translations.

Let us regard  $L^1(\mathbb{R}) \subset M(\mathbb{R})$  in the usual way. Owing to the fact that  $\mathbb{R}$  is amenable, let  $\nu: L^\infty(\mathbb{R}) \rightarrow \mathbb{C}$  be a state such that  $\nu \circ \tau_s = \nu$  for any  $s \in \mathbb{R}$ . Define  $d: L^1(\mathbb{R}) \times C_0(\mathbb{R}) \rightarrow \mathbb{C}$  by

$$d(h, f) = \nu \left( s \mapsto \langle \tau_{-s} S \tau_s(h), f \rangle \right), \quad h \in L^1(\mathbb{R}), \quad f \in C_0(\mathbb{R}),$$

where the duality pairing is given by (4.1.4). Then  $d$  is a bounded bilinear map which satisfies

$$d(\tau_s h, f) = d(h, \tau_s f), \quad h \in L^1(\mathbb{R}), \quad f \in C_0(\mathbb{R}), \quad s \in \mathbb{R}. \quad (4.2.10)$$

Let  $S_\nu: L^1(\mathbb{R}) \rightarrow M(\mathbb{R})$  be induced by  $d$ , that is,  $\langle S_\nu(h), f \rangle = d(h, f)$  for any  $h \in L^1(\mathbb{R})$  and any  $f \in C_0(\mathbb{R})$ . Let  $q: M(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  be the projection which takes any element of  $M(\mathbb{R})$  to (the density of) its absolutely continuous part. Then  $\tau_s q = q \tau_s$  for any  $s \in \mathbb{R}$ . Hence (4.2.10) implies that  $T_\nu := q S_\nu: L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$  satisfies

$$\tau_s T_\nu = T_\nu \tau_s, \quad s \in \mathbb{R}. \quad (4.2.11)$$

Since  $T$  commutes with translations,  $\langle \tau_{-s} S \tau_s(h), f \rangle = \langle S(h), f \rangle$  for any  $h \in L^1(\mathbb{R})$ ,  $f \in C_0(\mathbb{R})$  and  $s \in \mathbb{R}$ . Further  $\nu(1) = 1$ . Consequently,  $T_\nu h = Th$  for any  $h \in H^1(\mathbb{R})$ , that is,  $T_\nu$  is an extension of  $T$ .

By (4.2.11), there exists  $\mu \in M(\mathbb{R})$  such that  $T_\nu = \mu \star \cdot$ . This implies that  $T = R_\mu$ , which shows (1).

Let us now prove (2). Fix  $s \in \mathbb{R}^*$  and define  $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$  by  $m(u) = u^{is}$ . By [37, Lemma 4.1 and Theorem 1.2],  $m$  is the symbol of a bounded Fourier multiplier  $T_m: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  and if  $X$  is a Banach space without the so-called AUMD property, then  $T_m \otimes I_X$  does not extend to a bounded operator on  $H^1(\mathbb{R}; X)$ . Using (1), this implies that  $T_m \notin \mathcal{R}$ .  $\square$

The following lemma will play a crucial role.

**Lemma 4.2.5.** *For any  $1 \leq p < \infty$ , we have  $\mathcal{M}(H^1(\mathbb{R})) \subset \mathcal{M}(H^p(\mathbb{R}))$ .*

*Proof.* By definition we have  $\mathcal{M}(H^1(\mathbb{R})) \subset \mathcal{M}(H^2(\mathbb{R}))$ . By (4.2.8) we may assume that  $p \in (1, 2)$ . Let  $\theta = 2(1 - \frac{1}{p})$ . Then in the complex interpolation method, we have

$$[H^1(\mathbb{R}), H^2(\mathbb{R})]_\theta \simeq H^p(\mathbb{R}),$$

by [48, Theorem 4.3]. The result follows at once.  $\square$

### 4.3. Algebras $\mathcal{A}_0$ and $\mathcal{A}$

We introduce and study new algebras of functions which will be used in Section 4.4 to establish a functional calculus for negative generators of bounded  $C_0$ -semigroups on Hilbert space. The next definitions are inspired by [47], see also the "Notes and Remarks on Chapter 6" in [50].

#### Definitions and properties

**Definition 4.3.1.** We let  $\mathcal{A}_0(\mathbb{R})$  (resp.  $\mathcal{A}(\mathbb{R})$ ) be the set of all functions  $F: \mathbb{R} \rightarrow \mathbb{C}$  such that there exist two sequences  $(f_k)_{k \in \mathbb{N}}$  in  $C_0(\mathbb{R})$  (resp.  $BUC(\mathbb{R})$ ) and  $(h_k)_{k \in \mathbb{N}}$  in  $H^1(\mathbb{R})$  satisfying

$$\sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 < \infty \quad (4.3.1)$$

and

$$\forall s \in \mathbb{R}, \quad F(s) = \sum_{k=1}^{\infty} f_k \star h_k(s). \quad (4.3.2)$$

Note that for any  $f \in C_0(\mathbb{R})$  (resp.  $BUC(\mathbb{R})$ ) and any  $h \in H^1(\mathbb{R})$ ,  $f \star h$  belongs to  $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$  (resp.  $BUC(\mathbb{R}) \cap H^\infty(\mathbb{R})$ ). Further for any  $(f_k)_{k \in \mathbb{N}}$  and  $(h_k)_{k \in \mathbb{N}}$  as in Definition 4.3.1, we have  $\|f_k \star h_k\|_{\infty} \leq \|f_k\|_{\infty} \|h_k\|_1$ , and hence  $\sum_{k=1}^{\infty} \|f_k \star h_k\|_{\infty} < \infty$ , by (4.3.1). This insures the convergence of the series in (4.3.2) and implies that

$$\mathcal{A}_0(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{A}(\mathbb{R}) \subset BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}).$$

**Definition 4.3.2.** For all  $F \in \mathcal{A}_0(\mathbb{R})$  (resp.  $F \in \mathcal{A}(\mathbb{R})$ ), we set

$$\|F\|_{\mathcal{A}_0} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\} \quad (\text{resp. } \|F\|_{\mathcal{A}} = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 \right\}),$$

where the infimum runs over all sequences  $(f_k)_{k \in \mathbb{N}}$  in  $C_0(\mathbb{R})$  (resp.  $BUC(\mathbb{R})$ ) and  $(h_k)_{k \in \mathbb{N}}$  in  $H^1(\mathbb{R})$  satisfying (4.3.1) and (4.3.2).

It is clear that

$$\|F\|_{\infty} \leq \|F\|_{\mathcal{A}}, \quad F \in \mathcal{A}(\mathbb{R}).$$

To show that  $\|\cdot\|_{\mathcal{A}_0}$  and  $\|\cdot\|_{\mathcal{A}}$  are complete norms, we make a connection with projective tensor products, which will be useful along this section.

If  $X$  and  $Y$  are any Banach spaces, the projective norm of  $\zeta \in X \otimes Y$  is defined by

$$\|\zeta\|_{\wedge} = \inf \left\{ \sum_k \|x_k\| \|y_k\| \right\},$$

where the infimum runs over all finite families  $(x_k)_k$  in  $X$  and  $(y_k)_k$  in  $Y$  satisfying  $\zeta = \sum_k x_k \otimes y_k$ . The completion of  $(X \otimes Y, \|\cdot\|_{\wedge})$ , denoted by  $X \widehat{\otimes} Y$ , is called the projective tensor product of  $X$  and  $Y$ .

Let  $Z$  be a third Banach space. To any  $\ell \in B_2(X \times Y, Z)$ , the space of bounded bilinear maps from  $X \times Y$  into  $Z$ , one can associate a linear map  $\overset{\circ}{\ell}: X \otimes Y \rightarrow Z$  by the formula

$$\overset{\circ}{\ell}(x \otimes y) = \ell(x, y), \quad x \in X, y \in Y.$$

Then  $\overset{\circ}{\ell}$  extends to a bounded operator (still denoted by)  $\overset{\circ}{\ell}: X \widehat{\otimes} Y \rightarrow Z$ , with  $\|\overset{\circ}{\ell}\| = \|\ell\|$ . Further the mapping  $\ell \mapsto \overset{\circ}{\ell}$  yields an isometric identification

$$B_2(X \times Y, Z) \simeq B(X \widehat{\otimes} Y, Z). \quad (4.3.3)$$

Let us apply the above property in the case when  $Z = \mathbb{C}$ . Using the standard identification  $B_2(X \times Y, \mathbb{C}) = B(X, Y^*)$ , we obtain an isometric identification

$$(X \widehat{\otimes} Y)^* \simeq B(X, Y^*). \quad (4.3.4)$$

We refer to [13, Chapter 8, Theorem 1 & Corollary 2] for these classical facts.

Consider the bilinear map  $\sigma: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$  defined by

$$\sigma(f, h) = f \star h, \quad f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}). \quad (4.3.5)$$

Applying (4.3.3), let

$$\overset{\circ}{\sigma}: C_0(\mathbb{R}) \widehat{\otimes} H^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$$

be associated with  $\sigma$ . Then  $\mathcal{A}_0(\mathbb{R}) = \text{Ran}(\overset{\circ}{\sigma})$ . Through the resulting algebraic isomorphism between  $\mathcal{A}_0(\mathbb{R})$  and  $(C_0(\mathbb{R}) \widehat{\otimes} H^1(\mathbb{R})) / \text{Ker}(\overset{\circ}{\sigma})$ ,  $\|\cdot\|_{\mathcal{A}_0}$  corresponds to the quotient norm on the latter space. Thus  $(\mathcal{A}_0(\mathbb{R}), \|\cdot\|_{\mathcal{A}_0})$  is a Banach space and  $\overset{\circ}{\sigma}$  induces an isometric identification

$$\mathcal{A}_0(\mathbb{R}) \simeq \frac{C_0(\mathbb{R}) \widehat{\otimes} H^1(\mathbb{R})}{\text{Ker}(\overset{\circ}{\sigma})}. \quad (4.3.6)$$

Similarly,  $\|\cdot\|_{\mathcal{A}}$  is a norm on  $\mathcal{A}(\mathbb{R})$  and  $(\mathcal{A}(\mathbb{R}), \|\cdot\|_{\mathcal{A}})$  is a Banach space.

*Remark 4.3.3.* It is clear from Definition 4.3.2 that  $\mathcal{A}_0(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$  and that for any  $F \in \mathcal{A}_0(\mathbb{R})$ , we have

$$\|F\|_{\mathcal{A}} \leq \|F\|_{\mathcal{A}_0}. \quad (4.3.7)$$

### $\mathcal{A}$ and $\mathcal{A}_0$ are Banach algebras

**Proposition 4.3.4.** *The spaces  $\mathcal{A}_0(\mathbb{R})$  and  $\mathcal{A}(\mathbb{R})$  are Banach algebras for the pointwise multiplication. Furthermore,  $\mathcal{A}_0(\mathbb{R})$  is an ideal of  $\mathcal{A}(\mathbb{R})$  and for any  $F \in \mathcal{A}(\mathbb{R})$  and  $G \in \mathcal{A}_0(\mathbb{R})$ , we have*

$$\|FG\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}} \|G\|_{\mathcal{A}_0}. \quad (4.3.8)$$

*Proof.* Let  $f_1, f_2 \in C_0(\mathbb{R})$  and  $h_1, h_2 \in H^1(\mathbb{R})$ . We note that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h_1(t)| |h_2(t+s)| dt ds = \|h_1\|_1 \|h_2\|_1. \quad (4.3.9)$$

We define, for  $s \in \mathbb{R}$ ,  $\varphi_s: \mathbb{R} \rightarrow \mathbb{C}$  and  $\psi_s: \mathbb{R} \rightarrow \mathbb{C}$  by

$$\varphi_s(t) = f_1(t)f_2(t-s) \quad \text{and} \quad \psi_s(t) = h_1(t)h_2(t+s).$$

Since  $f_2$  is uniformly continuous, the function  $s \mapsto \varphi_s$  is continuous from  $\mathbb{R}$  into  $C_0(\mathbb{R})$ . Thus  $s \mapsto \varphi_s$  belongs to the Bochner space  $L^\infty(\mathbb{R}; C_0(\mathbb{R}))$ . Using (4.3.9) and arguing as at the beginning of the proof of Theorem 4.2.2, we see that  $s \mapsto \psi_s$  belongs to  $L^1(\mathbb{R}; H^1(\mathbb{R}))$ . It follows that  $s \mapsto \varphi_s \star \psi_s$  is defined almost everywhere and belongs to  $L^1(\mathbb{R}; \mathcal{A}_0(\mathbb{R}))$ . Moreover,

$$\int_{-\infty}^{\infty} \|\varphi_s \star \psi_s\|_{\mathcal{A}_0} ds \leq \|f_1\|_\infty \|f_2\|_\infty \|h_1\|_1 \|h_2\|_1. \quad (4.3.10)$$

Indeed, using (4.3.9) and Fubini's Theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} \|\varphi_s \star \psi_s\|_{\mathcal{A}_0} ds &\leq \int_{-\infty}^{\infty} \|\varphi_s\|_\infty \|\psi_s\|_1 ds \\ &\leq \|f_1\|_\infty \|f_2\|_\infty \int_{-\infty}^{\infty} \|\psi_s\|_1 ds, \end{aligned}$$

which is equal to the right-hand side of (4.3.10).

The integral of  $s \mapsto \varphi_s \star \psi_s$  is an element of  $\mathcal{A}_0(\mathbb{R})$ . We claim that we actually have

$$\int_{-\infty}^{\infty} \varphi_s \star \psi_s ds = (f_1 \star h_1)(f_2 \star h_2). \quad (4.3.11)$$

Indeed, using again (4.3.9) and Fubini's Theorem, we have for all  $u \in \mathbb{R}$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_s \star \psi_s(u) ds &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(t)f_2(t-s)h_1(u-t)h_2(u-t+s) dt ds \\ &= \int_{-\infty}^{\infty} f_1(t)h_1(u-t) \int_{-\infty}^{\infty} f_2(t-s)h_2(u-(t-s)) ds dt \\ &= (f_1 \star h_1)(u)(f_2 \star h_2)(u). \end{aligned}$$

Combining (4.3.11) and (4.3.10), we obtain that  $(f_1 \star h_1)(f_2 \star h_2) \in \mathcal{A}_0(\mathbb{R})$ , with

$$\|(f_1 \star h_1)(f_2 \star h_2)\|_{\mathcal{A}_0} \leq \|f_1\|_\infty \|f_2\|_\infty \|h_1\|_1 \|h_2\|_1.$$

Now let  $F, G \in \mathcal{A}_0(\mathbb{R})$  and let  $\varepsilon > 0$ . Consider sequences  $(f_k^1)_{k \in \mathbb{N}}$ ,  $(f_k^2)_{k \in \mathbb{N}}$  in  $C_0(\mathbb{R})$  and  $(h_k^1)_{k \in \mathbb{N}}$ ,  $(h_k^2)_{k \in \mathbb{N}}$  in  $H^1(\mathbb{R})$  such that

$$F = \sum_{k=1}^{\infty} f_k^1 \star h_k^1 \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k^1\|_\infty \|h_k^1\|_1 \leq \|F\|_{\mathcal{A}_0} + \varepsilon,$$

as well as

$$G = \sum_{k=1}^{\infty} f_k^2 \star h_k^2 \quad \text{and} \quad \sum_{k=1}^{\infty} \|f_k^2\|_{\infty} \|h_k^2\|_1 \leq \|G\|_{\mathcal{A}_0} + \varepsilon.$$

Then, using summation in  $C_0(\mathbb{R})$ , we have

$$FG = \sum_{k,l=1}^{\infty} (f_k^1 \star h_k^1)(f_l^2 \star h_l^2).$$

Further  $(f_k^1 \star h_k^1)(f_l^2 \star h_l^2) \in \mathcal{A}_0(\mathbb{R})$  for any  $k, l \geq 1$ , and

$$\begin{aligned} \sum_{k,l=1}^{\infty} \|(f_k^1 \star h_k^1)(f_l^2 \star h_l^2)\|_{\mathcal{A}_0} &\leq \sum_{k,l=1}^{\infty} \|f_k^1\|_{\infty} \|f_l^2\|_{\infty} \|h_k^1\|_1 \|h_l^2\|_1 \\ &= \left( \sum_{k=1}^{\infty} \|f_k^1\|_{\infty} \|h_k^1\|_1 \right) \left( \sum_{l=1}^{\infty} \|f_l^2\|_{\infty} \|h_l^2\|_1 \right) \\ &\leq (\|F\|_{\mathcal{A}_0} + \varepsilon) \leq (\|G\|_{\mathcal{A}_0} + \varepsilon). \end{aligned}$$

Since  $\mathcal{A}_0(\mathbb{R})$  is complete, this shows that  $FG \in \mathcal{A}_0(\mathbb{R})$ . Since  $\varepsilon > 0$  is arbitrary, we obtain that  $\|FG\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}_0} \|G\|_{\mathcal{A}_0}$ . This shows that  $\mathcal{A}_0(\mathbb{R})$  is a Banach algebra.

Analogously,  $\mathcal{A}(\mathbb{R})$  is a Banach algebra. Moreover if  $f_1 \in BUC(\mathbb{R})$  and  $f_2 \in C_0(\mathbb{R})$ , then for each  $s \in \mathbb{R}$ ,  $\varphi_s: t \mapsto f_1(t)f_2(t-s)$  belongs to  $C_0(\mathbb{R})$ . Hence the computations above show that  $\mathcal{A}_0(\mathbb{R})$  is an ideal of  $\mathcal{A}(\mathbb{R})$ , as well as (4.3.8).  $\square$

## Duality results and consequences

The main aim of this subsection is to identify  $\mathcal{A}_0(\mathbb{R})^*$  with a subspace of  $\mathcal{M}(H^1(\mathbb{R}))$ , the space of bounded Fourier multipliers on  $H^1(\mathbb{R})$ . This requires the use of duality tools.

Recall the identification  $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$  provided by (4.1.4) and regard  $H^1(\mathbb{R}) \subset L^1(\mathbb{R}) \subset M(\mathbb{R})$  in the usual way. By [15, Chapter II, Theorem 3.8], we have

$$(C_0(\mathbb{R}) \cap H^{\infty}(\mathbb{R}))^{\perp} = H^1(\mathbb{R}). \quad (4.3.12)$$

Set  $Z_0 := C_0(\mathbb{R}) \cap H^{\infty}(\mathbb{R})$  for convenience, then the above result yields an isometric identification

$$\left( \frac{C_0(\mathbb{R})}{Z_0} \right)^* \simeq H^1(\mathbb{R}). \quad (4.3.13)$$

In the sequel, we let  $\dot{f} \in \frac{C_0(\mathbb{R})}{Z_0}$  denote the class of any  $f \in C_0(\mathbb{R})$ .

We note that for any  $f \in C_0(\mathbb{R})$ ,  $h \in H^1(\mathbb{R})$  and  $s \in \mathbb{R}$ , we have

$$(f \star h)(s) = \langle \tau_{-s}h, f \rangle. \quad (4.3.14)$$

Thus  $f \star h = 0$  for any  $f \in Z_0$  and  $h \in H^1(\mathbb{R})$ . The bilinear map  $\sigma: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$  defined by (4.3.5) therefore induces a bilinear map  $\delta: \frac{C_0(\mathbb{R})}{Z_0} \times H^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$  given by

$$\delta(\dot{f}, h) = f \star h, \quad f \in C_0(\mathbb{R}), \quad h \in H^1(\mathbb{R}).$$

Let

$$\mathring{\delta}: \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R}) \longrightarrow C_0(\mathbb{R}) \quad (4.3.15)$$



be the bounded map induced by  $\delta$ . Then the argument leading to (4.3.6) shows as well that  $\mathcal{A}_0(\mathbb{R})$  is equal to the range of  $\overset{\circ}{\delta}$  and that

$$\mathcal{A}_0(\mathbb{R}) \simeq \left( \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R}) \right) / \text{Ker}(\overset{\circ}{\delta}). \quad (4.3.16)$$

Since  $H^1(\mathbb{R})$  is a dual space, by (4.3.13),  $B(H^1(\mathbb{R}))$  is a dual space as well. Indeed applying (4.3.4), we have an isometric identification

$$B(H^1(\mathbb{R})) \simeq \left( \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R}) \right)^*. \quad (4.3.17)$$

If we unravel the identifications leading to (4.3.17), we obtain that the latter is given by

$$\langle T, \dot{f} \otimes h \rangle = \int_{-\infty}^{\infty} (Th)(t) f(-t) dt, \quad T \in B(H^1(\mathbb{R})), f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}). \quad (4.3.18)$$

**Lemma 4.3.5.** *The space  $\mathcal{M}(H^1(\mathbb{R}))$  is  $w^*$ -closed in  $B(H^1(\mathbb{R}))$ .*

*Proof.* According to Theorem 4.2.2, an operator  $T \in B(H^1(\mathbb{R}))$  belongs to  $\mathcal{M}(H^1(\mathbb{R}))$  if and only if  $\tau_s T = T \tau_s$  for any  $s \in \mathbb{R}$ . Hence it suffices to show that the maps  $T \mapsto \tau_s T$  and  $T \mapsto T \tau_s$  are  $w^*$ -continuous on  $B(H^1(\mathbb{R}))$ , for any  $s \in \mathbb{R}$ .

Fix such an  $s$  and let  $(T_i)_i$  be a bounded net of  $B(H^1(\mathbb{R}))$  converging to some  $T \in B(H^1(\mathbb{R}))$  in the  $w^*$ -topology. It readily follows from (4.3.18) that for any  $f \in C_0(\mathbb{R})$  and  $h \in H^1(\mathbb{R})$ , we have

$$\langle \tau_s T, \dot{f} \otimes h \rangle = \langle T, \overset{\frown}{\tau_s f} \otimes h \rangle \quad \text{and} \quad \langle \tau_s T_i, \dot{f} \otimes h \rangle = \langle T_i, \overset{\frown}{\tau_s f} \otimes h \rangle,$$

for all  $i$ . This implies that  $\langle \tau_s T_i, \dot{f} \otimes h \rangle \rightarrow \langle \tau_s T, \dot{f} \otimes h \rangle$  when  $i \rightarrow \infty$ . By linearity, this implies that for any  $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R})$ , we have  $\langle \tau_s T_i, \Phi \rangle \rightarrow \langle \tau_s T, \Phi \rangle$  when  $i \rightarrow \infty$ . Since  $(T_i)_i$  is bounded, this implies that  $\tau_s T_i \rightarrow \tau_s T$  in the  $w^*$ -topology. This shows that  $T \mapsto \tau_s T$  is  $w^*$ -continuous. The proof that  $T \mapsto T \tau_s$  is  $w^*$ -continuous is similar.  $\square$

We introduce

$$\mathcal{PM} = \overline{\text{Span}}^{w^*} \{ \tau_s : s \in \mathbb{R} \} \subset B(H^1(\mathbb{R})).$$

A direct consequence of Lemma 4.3.5 is that

$$\mathcal{PM} \subset \mathcal{M}(H^1(\mathbb{R})). \quad (4.3.19)$$

**Lemma 4.3.6.** *Recall the mapping  $\overset{\circ}{\delta}$  from (4.3.15). Then  $[\text{Ker}(\overset{\circ}{\delta})]^\perp = \mathcal{PM}$ .*

*Proof.* Let  $s \in \mathbb{R}$  and let  $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R})$ . Choose two sequences  $(f_k)_{k \in \mathbb{N}}$  in  $C_0(\mathbb{R})$  and  $(h_k)_{k \in \mathbb{N}}$  in  $H^1(\mathbb{R})$  such that

$$\sum_{k=1}^{\infty} \|f_k\|_{\infty} \|h_k\|_1 < \infty \quad \text{and} \quad \Phi = \sum_{k=1}^{\infty} \dot{f}_k \otimes h_k.$$

Then by (4.3.14),

$$[\overset{\circ}{\delta}(\Phi)](s) = \sum_{k=1}^{\infty} (f_k \star h_k)(s) = \sum_{k=1}^{\infty} \langle \tau_{-s} h_k, \dot{f}_k \rangle = \langle \tau_{-s}, \Phi \rangle.$$

This shows that

$$\text{Span} \{ \tau_s : s \in \mathbb{R} \}_\perp = \text{Ker}(\overset{\circ}{\delta}).$$

The result follows at once.  $\square$

By standard duality and (4.3.16), the dual space  $\mathcal{A}_0(\mathbb{R})^*$  may be identified with  $[\text{Ker}(\overset{\circ}{\delta})]^\perp$ . Applying Lemma 4.3.6 and (4.3.18), we therefore obtain the following.

**Theorem 4.3.7.**

(1) For any  $T \in \mathcal{PM}$ , there exists a unique  $\eta_T \in \mathcal{A}_0(\mathbb{R})^*$  such that

$$\langle \eta_T, f \star h \rangle = \int_{-\infty}^{\infty} (Th)(t) f(-t) dt$$

for any  $f \in C_0(\mathbb{R})$  and any  $h \in H^1(\mathbb{R})$ .

(2) The mapping  $T \mapsto \eta_T$  induces a  $w^*$ -homeomorphic and isometric identification

$$\mathcal{A}_0(\mathbb{R})^* \simeq \mathcal{PM}.$$

*Remark 4.3.8.*

(1) Recall  $\mathcal{R}$  from (4.2.9). It turns out that

$$\mathcal{PM} = \overline{\mathcal{R}}^{w^*}.$$

Indeed for any  $s \in \mathbb{R}$ ,  $\tau_s: H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R})$  is equal to the convolution by the Dirac mass at  $s$ , which yields  $\subset$ . To show the converse inclusion, we observe that for any  $\mu \in M(\mathbb{R})$  and any  $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \widehat{\otimes} H^1(\mathbb{R})$ , we have

$$\langle \mu, \overset{\circ}{\delta}(\Phi) \rangle = \langle R_\mu, \Phi \rangle. \quad (4.3.20)$$

Here we use the identification  $M(\mathbb{R}) \simeq C_0(\mathbb{R})^*$  given by (4.1.4) on the left-hand side and we use (4.3.17) on the right-hand side. To prove this identity, let  $f \in C_0(\mathbb{R})$  and  $h \in H^1(\mathbb{R})$ . Then

$$\begin{aligned} \langle \mu, \overset{\circ}{\delta}(f \otimes h) \rangle &= \langle \mu, f \star h \rangle \\ &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} f(-s) h(s-t) ds d\mu(t) \\ &= \langle \mu \star h, f \rangle. \end{aligned}$$

This shows (4.3.20) when  $\Phi = f \otimes h$ . By linearity, this implies (4.3.20) for  $\Phi \in \frac{C_0(\mathbb{R})}{Z_0} \otimes H^1(\mathbb{R})$ . Then by density, we deduce (4.3.20) for all  $\Phi$ .

It clearly follows from (4.3.20) that  $\mathcal{R} \subset [\text{Ker}(\overset{\circ}{\delta})]^\perp$ . By Lemma 4.3.6, this yields  $\supset$ .

(2) The Banach algebra  $\mathcal{A}_0(\mathbb{R})$  can be naturally regarded as an analytic version of the Figa-Talamanca-Herz algebras  $A_p(\mathbb{R})$ ,  $1 < p < \infty$  (see e.g. [12, Chapter 3]). A remarkable property of these algebras is that the dual of  $A_p(\mathbb{R})$  can be identified with the space  $\mathcal{M}(L^p(\mathbb{R}))$  of bounded Fourier multipliers on  $L^p(\mathbb{R})$ . By analogy, one may expect that the dual of  $\mathcal{A}_0(\mathbb{R})$  identifies with  $\mathcal{M}(H^1(\mathbb{R}))$ . However at the time of this writing we do not know if the inclusion (4.3.19) is an equality.

We now give a few consequences of the above duality results.

**Proposition 4.3.9.** For any  $b$  is  $L^1(\mathbb{R}_+)$ , the function  $\widehat{b}(-\cdot): \mathbb{R} \rightarrow \mathbb{C}$  belongs to  $\mathcal{A}_0(\mathbb{R})$  and

$$\|\widehat{b}(-\cdot)\|_{\mathcal{A}_0} \leq \|b\|_1.$$

Moreover the mapping  $L^1(\mathbb{R}_+) \rightarrow \mathcal{A}_0(\mathbb{R})$  taking  $b$  to  $\widehat{b}(-\cdot)$  is a Banach algebra homomorphism with dense range.

*Proof.* We will use the space  $C_{00}(\mathbb{R})$  defined by (4.1.3). Let  $C_{00} \star H^1 \subset \mathcal{A}_0(\mathbb{R})$  be the linear span of the functions  $f \star h$ , for  $f \in C_{00}(\mathbb{R})$  and  $h \in H^1(\mathbb{R})$ . By definition of  $C_{00}(\mathbb{R})$ , the Fourier transform maps  $C_{00} \star H^1$  into  $L^1(\mathbb{R}_+)$ . Then we consider

$$\mathcal{C}_{0,1} = \{\widehat{F} : F \in C_{00} \star H^1\} \subset L^1(\mathbb{R}_+).$$

Let  $b \in C^\infty(\mathbb{R}_+^*)$  with compact support. Let  $c \in C^\infty(\mathbb{R}_+^*)$  with compact support such that  $c \equiv 1$  on the support of  $b$ , so that  $b = bc$ . Then  $\mathcal{F}^{-1}(b) \in C_{00}(\mathbb{R})$ ,  $\mathcal{F}^{-1}(c) \in H^1(\mathbb{R})$  and the Fourier transform of  $\mathcal{F}^{-1}(b) \star \mathcal{F}^{-1}(c)$  is equal to  $b$ . Thus  $b \in \mathcal{C}_{0,1}$ . Consequently,  $\mathcal{C}_{0,1}$  is dense in  $L^1(\mathbb{R}_+)$ .

Let  $b \in \mathcal{C}_{0,1}$  and let  $F = \widehat{b}(-\cdot)$ , so that

$$\widehat{F} = (2\pi)b.$$

Take finite families  $(f_k)_k$  in  $C_{00}(\mathbb{R})$  and  $(h_k)_k$  in  $H^1(\mathbb{R})$  such that  $F = \sum_k f_k \star h_k$ . Pick  $\eta \in \mathcal{A}_0(\mathbb{R})^*$  such that  $\|\eta\| = 1$  and  $\|F\|_{\mathcal{A}_0} = \langle \eta, F \rangle$ . By Theorem 4.3.7, there exists  $T \in \mathcal{M}(H^1(\mathbb{R}))$  such that  $\|T\|_{B(H^1)} = 1$  and for any  $k$ ,

$$\langle \eta, f_k \star h_k \rangle = \int_{-\infty}^{\infty} (Th_k)(u) f_k(-u) du.$$

By Theorem 4.2.2, the symbol  $m$  of the multiplier  $T$  satisfies  $\|m\|_\infty \leq 1$ . Moreover by Lemma 4.1.1, we have

$$\begin{aligned} \int_{-\infty}^{\infty} (Th_k)(u) f_k(-u) du &= \frac{1}{2\pi} \int_0^\infty \widehat{Th_k}(t) \widehat{f_k}(t) dt \\ &= \frac{1}{2\pi} \int_0^\infty m(t) \widehat{h_k}(t) \widehat{f_k}(t) dt, \end{aligned}$$

for all  $k$ . Summing over  $k$ , we deduce that

$$\langle \eta, F \rangle = \frac{1}{2\pi} \sum_k \int_0^\infty m(t) \widehat{h_k}(t) \widehat{f_k}(t) dt = \frac{1}{2\pi} \int_0^\infty m(t) \widehat{F}(t) dt,$$

and hence

$$\langle \eta, F \rangle = \int_0^\infty m(t) b(t) dt.$$

We deduce that

$$\|F\|_{\mathcal{A}_0} \leq \|b\|_1.$$

Since  $\mathcal{C}_{0,1}$  is dense in  $L^1(\mathbb{R}_+)$  and  $\mathcal{A}_0(\mathbb{R})$  is complete, this estimate implies that  $\widehat{b}(-\cdot) : \mathbb{R} \rightarrow \mathbb{C}$  belongs to  $\mathcal{A}_0(\mathbb{R})$  for any  $b \in L^1(\mathbb{R}_+)$ , with  $\|\widehat{b}(-\cdot)\|_{\mathcal{A}_0} \leq \|b\|_1$ .

It is plain that  $b \mapsto \widehat{b}(-\cdot)$  is a Banach algebra homomorphism. Its range contains  $C_{00} \star H^1$  and the latter is dense in  $\mathcal{A}_0(\mathbb{R})$ , because  $C_{00}(\mathbb{R})$  is dense in  $C_0(\mathbb{R})$ .  $\square$

*Remark 4.3.10.* Let  $\eta \in \mathcal{A}_0(\mathbb{R})^*$ , let  $T \in \mathcal{M}(H^1(\mathbb{R}))$  be the multiplier associated with  $\eta$ , according to Theorem 4.3.7, and let  $m \in C_b(\mathbb{R}_+^*)$  be the symbol of  $T$ . Then it follows from the previous result and its proof that for any  $b \in L^1(\mathbb{R}_+)$ ,

$$\langle \eta, \widehat{b}(-\cdot) \rangle = \int_0^\infty m(t) b(t) dt.$$

*Remark 4.3.11.* For any  $\lambda \in P_-$ , we let

$$b_\lambda(t) = ie^{-i\lambda t}, \quad t > 0.$$

Then  $b_\lambda \in L^1(\mathbb{R}_+)$  and we have

$$\widehat{b}_\lambda(-u) = \frac{1}{\lambda - u}, \quad u \in \mathbb{R}.$$

Applying Proposition 4.3.9, we obtain that  $(\lambda - \cdot)^{-1}$  belongs to  $\mathcal{A}_0(\mathbb{R})$  for any  $\lambda \in P_-$ . Since  $\mathcal{A}_0(\mathbb{R})$  is a Banach algebra, this implies that any rational function  $F: \mathbb{R} \rightarrow \mathbb{C}$  with degree  $\deg(F) \leq -1$  and poles in  $P_-$  belongs to  $\mathcal{A}_0(\mathbb{R})$ .

We can now strengthen Remark 4.3.3 as follows.

**Proposition 4.3.12.** *For any  $F \in \mathcal{A}_0(\mathbb{R})$ , we have*

$$\|F\|_{\mathcal{A}_0} = \|F\|_{\mathcal{A}}.$$

*Proof.* We will use Proposition 4.3.9 again. For any  $N \in \mathbb{N}$ , let  $G_N: \mathbb{R} \rightarrow \mathbb{C}$  be defined by

$$G_N(u) = \frac{N}{N - iu}, \quad u \in \mathbb{R}.$$

Then  $G_N = \widehat{Ne^{-N\cdot}}(-\cdot)$  belongs to  $\mathcal{A}_0(\mathbb{R})$ . We note that the sequence  $(Ne^{-N\cdot})_{N \in \mathbb{N}}$  is a contractive approximate unit of  $L^1(\mathbb{R}_+)$ . This implies that  $(G_N)_{N \in \mathbb{N}}$  is a contractive approximate unit of  $\mathcal{A}_0(\mathbb{R})$ .

Let  $F \in \mathcal{A}_0(\mathbb{R})$ . By Proposition 4.3.4, we have

$$\|FG_N\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}} \|G_N\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}}. \quad (4.3.21)$$

Moreover  $FG_N \rightarrow F$  in  $\mathcal{A}_0(\mathbb{R})$  hence  $\|FG_N\|_{\mathcal{A}_0} \rightarrow \|F\|_{\mathcal{A}_0}$  when  $N \rightarrow \infty$ . We deduce that

$$\|F\|_{\mathcal{A}_0} \leq \|F\|_{\mathcal{A}}.$$

Combining with (4.3.7), we obtain  $\|F\|_{\mathcal{A}_0} = \|F\|_{\mathcal{A}}$ . □

*Remark 4.3.13.* According to [15, Chapter II, Theorem 3.8],  $H^1(\mathbb{R}) \subset M(\mathbb{R})$  is the orthogonal space of the functions  $u \mapsto \frac{1}{\lambda - u}$ , for  $\lambda$  running in  $P_-$ . Hence we deduce from Remark 4.3.11 that  $\mathcal{A}_0(\mathbb{R})^\perp$  contains  $H^1(\mathbb{R})$ . Combining with (4.3.12) we deduce that we actually have  $\mathcal{A}_0(\mathbb{R})^\perp = H^1(\mathbb{R}) = (C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}))^\perp$ . Consequently,  $\mathcal{A}_0(\mathbb{R})$  is dense in  $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ .

Using Proposition 4.3.9, or repeating the above argument, we also obtain that the space  $\{\widehat{b}(-\cdot) : b \in L^1(\mathbb{R}_+)\}$  is dense in  $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ .

## 4.4. Functional calculus on $\mathcal{A}_0$ and $\mathcal{A}$

The goal of this section is to construct a bounded functional calculus for the negative generator of a bounded  $C_0$ -semigroup on Hilbert space, defined on a suitable half-plane version of  $\mathcal{A}(\mathbb{R})$ .

This half-plane version  $\mathcal{A}(\mathbb{C}_+)$  (and its companion  $\mathcal{A}_0(\mathbb{C}_+)$ ) are defined as follows. Let  $F \in H^\infty(\mathbb{R})$ . We may consider its Poisson integral  $\mathcal{P}[F]: P_+ \rightarrow \mathbb{C}$  and the latter is a bounded holomorphic function (see e.g. [15, Sect. I.3]). Then we define

$$\widetilde{F}: \mathbb{C}_+ \longrightarrow \mathbb{C}$$

by setting

$$\tilde{F}(z) = \mathcal{P}[F](iz), \quad z \in \mathbb{C}_+.$$

We set

$$\mathcal{A}_0(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{A}_0(\mathbb{R})\} \quad \text{and} \quad \mathcal{A}(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{A}(\mathbb{R})\}.$$

We equip these spaces with the norms induced by  $\mathcal{A}_0(\mathbb{R})$  and  $\mathcal{A}(\mathbb{R})$ , respectively. That is, we set  $\|\tilde{F}\|_{\mathcal{A}_0} = \|F\|_{\mathcal{A}_0}$  (resp.  $\|\tilde{F}\|_{\mathcal{A}} = \|F\|_{\mathcal{A}}$ ) for any  $F \in \mathcal{A}_0(\mathbb{R})$  (resp.  $F \in \mathcal{A}(\mathbb{R})$ ). By construction, we have

$$\mathcal{A}_0(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad \mathcal{A}(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+). \quad (4.4.1)$$

It clearly follows from Section 4.3 that  $\mathcal{A}(\mathbb{C}_+)$  is a Banach algebra for pointwise multiplication, that the second inclusion in (4.4.1) is contractive, that  $\mathcal{A}_0(\mathbb{C}_+)$  is an ideal of  $\mathcal{A}(\mathbb{C}_+)$  and that the first inclusion in (4.4.1) is an isometry.

Let  $b \in L^1(\mathbb{R}_+)$  and consider  $F = \hat{b}(-\cdot) : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $\tilde{F}$  coincides with the Laplace transform  $L_b : \mathbb{C}_+ \rightarrow \mathbb{C}$  of  $b$  defined by

$$L_b(z) = \int_0^\infty e^{-tz} b(t) dt, \quad z \in \mathbb{C}_+. \quad (4.4.2)$$

As a consequence of Proposition 4.3.9, we therefore obtain the following.

**Lemma 4.4.1.** *For any  $b \in L^1(\mathbb{R}_+)$ ,  $L_b$  belongs to  $\mathcal{A}_0(\mathbb{C}_+)$  and  $\|L_b\|_{\mathcal{A}_0} \leq \|b\|_1$ . Moreover the mapping  $b \mapsto L_b$  is a Banach algebra homomorphism from  $L^1(\mathbb{R}_+)$  into  $\mathcal{A}_0(\mathbb{C}_+)$ , and the space  $\{L_b : b \in L^1(\mathbb{R}_+)\}$  is dense in  $\mathcal{A}_0(\mathbb{C}_+)$ .*

### Half-plane holomorphic functional calculus

We need some background on the half-plane holomorphic functional calculus introduced by Batty-Haase-Mubeen in [9], to which we refer for details.

Let  $X$  be an arbitrary Banach space. Let  $\omega \in \mathbb{R}$  and let  $A$  be a closed and densely defined operator on  $X$  such that the spectrum of  $A$  is included in the closed half-plane  $\overline{\mathcal{H}_\omega}$  and

$$\forall \alpha < \omega, \quad \sup\{\|R(z, A)\| : \operatorname{Re}(z) \leq \alpha\} < \infty. \quad (4.4.3)$$

Consider the auxiliary algebra

$$\mathcal{E}(\mathcal{H}_\alpha) := \{\varphi \in H^\infty(\mathcal{H}_\alpha) : \exists s > 0, |\varphi(z)| = O(|z|^{-(1+s)}) \text{ as } |z| \rightarrow \infty\},$$

for any  $\alpha < \omega$ . For any  $\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$  and for any  $\beta \in (\alpha, \omega)$ , the assumption (4.4.3) insures that the integral

$$\varphi(A) := \frac{-1}{2\pi} \int_{-\infty}^\infty \varphi(\beta + is) R(\beta + is, A) ds \quad (4.4.4)$$

is absolutely convergent in  $B(X)$ . Further its value is independent of the choice of  $\beta$  (this is due to Cauchy's Theorem for vector-valued holomorphic functions) and the mapping  $\varphi \mapsto \varphi(A)$  in an algebra homomorphism from  $\mathcal{E}(\mathcal{H}_\alpha)$  into  $B(X)$ . This definition is compatible with the usual rational functional calculus; indeed for any  $\mu \in \mathbb{C}$  with  $\operatorname{Re}(\mu) < \alpha$  and any integer  $m \geq 2$ , the function

$$e_{\mu, m} : z \mapsto (\mu - z)^{-m},$$

which belongs to  $\mathcal{E}(\mathcal{H}_\alpha)$ , satisfies  $e_{\mu, m}(A) = R(\mu, A)^m$ .

Let  $\varphi \in H^\infty(\mathcal{H}_\alpha)$ . We can define a closed, densely defined, operator  $\varphi(A)$  by regularisation, as follows (see [9] and [19] for more on such constructions). Take  $\mu \in \mathbb{C}$  with  $\operatorname{Re}(\mu) < \alpha$ , and set  $e = e_{\mu,2}$ . Then  $e\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$  and  $e(A) = R(\mu, A)^2$  is injective. Then  $\varphi(A)$  is defined by

$$\varphi(A) = e(A)^{-1}(e\varphi)(A), \quad (4.4.5)$$

with domain  $\operatorname{Dom}(\varphi(A))$  equal to the space of all  $x \in X$  such that  $[(e\varphi)(A)](x)$  belongs to the range of  $e(A)$  ( $= \operatorname{Dom}(A^2)$ ). It turns out that this definition does not depend on the choice of  $\mu$ .

The half-plane holomorphic functional calculus  $\varphi \mapsto \varphi(A)$  satisfies the following ‘‘Convergence Lemma’’, provided by [9, Theorem 3.1].

**Lemma 4.4.2.** *Assume that  $A$  satisfies (4.4.3) for some  $\omega \in \mathbb{R}$  and let  $\alpha < \omega$ . Let  $(\varphi_i)_i$  be a net of  $H^\infty(\mathcal{H}_\alpha)$  such that  $\varphi_i(A) \in B(X)$  for any  $i$  and let  $\varphi \in H^\infty(\mathcal{H}_\alpha)$  such that  $\varphi_i \rightarrow \varphi$  pointwise on  $R_\alpha$ , when  $i \rightarrow \infty$ . If*

$$\sup_i \|\varphi_i\|_{H^\infty(R_\alpha)} < \infty \quad \text{and} \quad \sup_i \|\varphi_i(A)\|_{B(X)} < \infty,$$

then  $\varphi(A) \in B(X)$  and for any  $x \in X$ ,  $[\varphi_i(A)](x) \rightarrow [\varphi(A)](x)$  when  $i \rightarrow \infty$ .

Let  $(T_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  and let  $-A$  denote its infinitesimal generator. For any  $b \in L^1(\mathbb{R}_+)$ , we set

$$\Gamma(A, b) := \int_0^\infty b(t) T_t dt,$$

where the latter integral is defined in the strong sense. The mapping  $b \mapsto \Gamma(A, b)$  is the so-called Hille-Phillips functional calculus. We refer to [19, Section 3.3] for information and background. We recall that this functional calculus is a Banach algebra homomorphism

$$L^1(\mathbb{R}_+) \longrightarrow B(X).$$

We now provide a compatibility result between the half-plane holomorphic functional calculus and the Hille-Phillips functional calculus. Note that any  $A$  as above satisfies (4.4.3) for  $\omega = 0$ . Thus for any  $\varepsilon > 0$ , the operator  $A + \varepsilon$  satisfies (4.4.3) for  $\omega = \varepsilon$ . For any  $b \in L^1(\mathbb{R}_+)$ , this allows to define  $L_b(A + \varepsilon)$ , where  $L_b$  is the Laplace transform defined by (4.4.2).

**Lemma 4.4.3.** *Let  $b \in L^1(\mathbb{R}_+)$  and let  $-A$  be the generator of a bounded  $C_0$ -semigroup on  $X$ . Then for any  $\varepsilon > 0$ , we have*

$$L_b(A + \varepsilon) = \Gamma(A + \varepsilon, b).$$

*Proof.* Let  $\varepsilon > 0$ , let  $\beta \in (0, \varepsilon)$ , and let  $b \in L^1(\mathbb{R}_+)$ .

First suppose that  $L_b \in \mathcal{E}(\mathcal{H}_0)$ . Then by the Laplace formula,

$$L_b(A + \varepsilon) = \frac{1}{2\pi} \int_{-\infty}^\infty L_b(\beta + is) \left( \int_0^\infty e^{ist} e^{\beta t} e^{-\varepsilon t} T_t dt \right) ds.$$

Since  $L_b \in \mathcal{E}(\mathcal{H}_0)$ , the function  $s \mapsto L_b(\beta + is)$  belongs to  $L^1(\mathbb{R})$ . Hence by Fubini’s Theorem,

$$L_b(A + \varepsilon) = \frac{1}{2\pi} \int_0^\infty e^{\beta t} \left( \int_{-\infty}^\infty L_b(\beta + is) e^{ist} ds \right) e^{-\varepsilon t} T_t dt.$$

By the Fourier inversion Theorem, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} L_b(\delta + is)e^{ist} ds = e^{-\beta t}b(t)$$

for each  $t \in \mathbb{R}$ . We deduce that

$$L_b(A + \varepsilon) = \int_0^{\infty} b(t)e^{-\varepsilon t}T_t dt = \Gamma(A + \varepsilon, b).$$

For the general case, let us consider  $e \in \mathcal{E}(\mathcal{H}_0)$  defined by  $e(z) = (1 + z)^{-2}$ . We note that  $e$  is the Laplace transform of the function  $c \in L^1(\mathbb{R}_+)$  defined by  $c(t) = te^{-t}$ . The product  $eL_b$ , which belongs to  $\mathcal{E}(\mathcal{H}_0)$ , is therefore the Laplace transform of  $b \star c$ . Hence by the first part of this proof,

$$(eL_b)(A + \varepsilon) = \Gamma(A + \varepsilon, b \star c).$$

The multiplicativity of the Hille-Phillips functional calculus yields  $\Gamma(A + \varepsilon, b \star c) = \Gamma(A + \varepsilon, c)\Gamma(A + \varepsilon, b)$ . Further  $e(A + \varepsilon) = \Gamma(A + \varepsilon, c)$  by the Laplace formula. Thus we have

$$e(A + \varepsilon)\Gamma(A + \varepsilon, b) = (eL_b)(A + \varepsilon).$$

Applying (4.4.5), we obtain that  $L_b(A + \varepsilon) = \Gamma(A + \varepsilon, b)$  as wanted.  $\square$

### Functional calculus on $\mathcal{A}_0(\mathbb{C}_+)$

Throughout this subsection, we fix a Hilbert space  $H$ , we let  $(T_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $H$  and we let  $-A$  denote its infinitesimal generator. We set

$$C := \sup\{\|T_t\| : t \geq 0\}.$$

For any  $f \in C_{00}(\mathbb{R})$  and any  $h \in H^1(\mathbb{R})$ , the function

$$b = (2\pi)^{-1}\widehat{f\widehat{h}}$$

belongs to  $L^1(\mathbb{R}_+)$  and we have  $\widehat{b(-\cdot)} = f \star h$ . Consequently,

$$(f \star h)^\sim = L_b.$$

Further we have the following key estimate, which is inspired by [50, Proposition 4.16].

**Lemma 4.4.4.** *For any  $f \in C_{00}(\mathbb{R})$  and any  $h \in H^1(\mathbb{R})$ ,*

$$\|\Gamma(A, (2\pi)^{-1}\widehat{f\widehat{h}})\| \leq C^2 \|f\|_\infty \|h\|_1. \quad (4.4.6)$$

*Proof.* We fix  $f \in C_{00}(\mathbb{R})$ . Let  $w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$  and let  $h = wv$ . By definition,

$$\Gamma(A, (2\pi)^{-1}\widehat{f\widehat{h}}) = \frac{1}{2\pi} \int_0^{\infty} \widehat{f}(t)\widehat{wv}(t)T_t dt.$$

By assumption,  $\widehat{w}$  and  $\widehat{v}$  belong to  $L^1(\mathbb{R}_+)$  hence  $\widehat{wv} = (2\pi)^{-1}\widehat{w} \star \widehat{v}$  belongs to  $L^1(\mathbb{R}_+)$ . Further  $f$  belongs to  $L^1(\mathbb{R})$ . We can therefore apply Fubini's Theorem and we obtain that

$$\frac{1}{2\pi} \int_0^{\infty} \widehat{f}(t)\widehat{wv}(t)T_t dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \left( \int_0^{\infty} \widehat{wv}(t)e^{-its}T_t dt \right) ds.$$

Note that for any  $s \in \mathbb{R}$ ,

$$\int_0^\infty \widehat{wv}(t)e^{-its}T_t dt = \Gamma(A + is, \widehat{wv}).$$

According to the multiplicativity of the Hille-Phillips functional calculus, this implies that

$$\int_0^\infty \widehat{wv}(t)e^{-its}T_t dt = \frac{1}{2\pi} \left( \int_0^\infty \widehat{w}(r)e^{-irs}T_r dr \right) \left( \int_0^\infty \widehat{v}(t)e^{-its}T_t dt \right).$$

Let  $W, V: \mathbb{R} \rightarrow B(H)$  be defined by

$$W(s) = \int_0^\infty \widehat{w}(r)e^{-irs}T_r dr \quad \text{and} \quad V(s) = \int_0^\infty \widehat{v}(t)e^{-its}T_t dt, \quad s \in \mathbb{R}.$$

It follows from above that for any  $x, x^* \in H$ , we have

$$\langle \Gamma(A, (2\pi)^{-1}\widehat{f\widehat{h}})x, x^* \rangle = \frac{1}{4\pi^2} \int_{-\infty}^\infty f(s) \langle W(s)x, V(s)^*x^* \rangle ds. \quad (4.4.7)$$

Applying the Cauchy-Schwarz inequality, we deduce

$$|\langle \Gamma(A, (2\pi)^{-1}\widehat{f\widehat{h}})x, x^* \rangle| \leq \frac{1}{4\pi^2} \|f\|_\infty \left( \int_{-\infty}^\infty \|W(s)x\|^2 ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^\infty \|V(s)^*x^*\|^2 ds \right)^{\frac{1}{2}}.$$

According to the Fourier-Plancherel equality on  $L^2(\mathbb{R}; H)$ , we have

$$\int_{-\infty}^\infty \|W(s)x\|^2 ds = 2\pi \int_{-\infty}^\infty |\widehat{w}(r)|^2 \|T_r x\|^2 dr.$$

This implies

$$\begin{aligned} \int_{-\infty}^\infty \|W(s)x\|^2 ds &\leq 2\pi C^2 \int_{-\infty}^\infty |\widehat{w}(r)|^2 \|x\|^2 dr \\ &= 4\pi^2 C^2 \|w\|_2^2 \|x\|^2. \end{aligned}$$

Similarly, we have

$$\int_{-\infty}^\infty \|V(s)^*x^*\|^2 ds \leq 4\pi^2 C^2 \|v\|_2^2 \|x^*\|^2.$$

Hence,

$$|\langle \Gamma(A, (2\pi)^{-1}\widehat{f\widehat{h}})x, x^* \rangle| \leq C^2 \|f\|_\infty \|w\|_2 \|v\|_2 \|x\| \|x^*\|.$$

Since this is true for any  $x, x^*$ , we have proved that

$$\|\Gamma(A, (2\pi)^{-1}\widehat{f\widehat{h}})\| \leq C^2 \|f\|_\infty \|w\|_2 \|v\|_2.$$

Now let  $h$  be an arbitrary element of  $H^1(\mathbb{R})$ . As is well-known (see e.g. [15, Exercise 1, p.84]), there exist  $w, v \in H^2(\mathbb{R})$  such that  $h = wv$  and  $\|w\|_2^2 = \|v\|_2^2 = \|h\|_1$ .

Since  $\mathcal{F}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$ , it follows from (4.2.1) that  $\mathcal{F}(H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})) = L^2(\mathbb{R}_+) \cap \mathcal{S}(\mathbb{R})$ . Since  $L^2(\mathbb{R}_+) \cap \mathcal{S}(\mathbb{R})$  is dense in  $L^2(\mathbb{R}_+)$ , we readily deduce that  $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$  is dense in  $H^2(\mathbb{R})$ .

Thus, there exist sequences  $(w_k)_{k \in \mathbb{N}}, (v_k)_{k \in \mathbb{N}}$  of  $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$  such that  $w_k \rightarrow w$  and  $v_k \rightarrow v$  in  $H^2(\mathbb{R})$ , when  $k \rightarrow \infty$ . This implies that

$$\|w_k\|_2 \|v_k\|_2 \longrightarrow \|h\|_1,$$



when  $k \rightarrow \infty$ . Moreover,  $w_k v_k \rightarrow wv = h$  in  $H^1(\mathbb{R})$ , when  $k \rightarrow \infty$ . Consequently,

$$\|\Gamma(A, (2\pi)^{-1} \widehat{f w_k v_k}) - \Gamma(A, (2\pi)^{-1} \widehat{f h})\| \rightarrow 0$$

when  $k \rightarrow \infty$ . Indeed,

$$\begin{aligned} \|\Gamma(A, (2\pi)^{-1} \widehat{f w_k v_k}) - \Gamma(A, (2\pi)^{-1} \widehat{f h})\| &= \left\| \int_0^\infty \widehat{f}(t) \overline{(w_k v_k - wv)(t)} T_t dt \right\| \\ &\leq C \|\widehat{f}\|_1 \|(w_k v_k - wv)\|_\infty \\ &\leq C \|\widehat{f}\|_1 \|w_k v_k - wv\|_1. \end{aligned}$$

For all  $k \in \mathbb{N}$ , we have

$$\|\Gamma(A, (2\pi)^{-1} \widehat{f w_k v_k})\| \leq C^2 \|f\|_\infty \|u_k\|_2 \|v_k\|_2,$$

by the first part of the proof. Passing to the limit, we obtain (4.4.6).  $\square$

We now arrive at the main result of this subsection.

**Theorem 4.4.5.** *There exists a unique bounded homomorphism  $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(H)$  such that*

$$\rho_{0,A}(L_b) = \int_0^\infty b(t) T_t dt \quad (4.4.8)$$

for any  $b \in L^1(\mathbb{R}_+)$ . Moreover  $\|\rho_{0,A}\| \leq C^2$ .

*Proof.* By Lemma 4.4.4 and the density of  $C_{00}(\mathbb{R})$  in  $C_0(\mathbb{R})$ , there exists a unique bounded bilinear map

$$u_A: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \longrightarrow B(H)$$

such that  $u_A(f, h) = \Gamma(A, (2\pi)^{-1} \widehat{f h})$  for each  $(f, h) \in C_{00}(\mathbb{R}) \times H^1(\mathbb{R})$ . Moreover  $\|u_A\| \leq C^2$ .

For each  $\varepsilon > 0$ ,  $(A + \varepsilon)$  is the negative generator of the semigroup  $(e^{-\varepsilon t} T_t)_{t \geq 0}$ . Therefore, in the same manner as above, one can define  $u_{A+\varepsilon}: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow B(H)$  and we have the uniform estimate

$$\forall \varepsilon > 0, \quad \|u_{A+\varepsilon}: C_0(\mathbb{R}) \times H^1(\mathbb{R}) \longrightarrow B(H)\| \leq C^2. \quad (4.4.9)$$

We claim that for each  $\varepsilon > 0$ , we have

$$u_{A+\varepsilon}(f, h) = (f \star h)^\sim(A + \varepsilon), \quad f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}). \quad (4.4.10)$$

(We recall that the operator on the right-hand side is defined by the half-plane holomorphic functional calculus. In particular the above formula shows that  $(f \star h)^\sim(A + \varepsilon)$  is bounded.) Recall that if  $f \in C_{00}(\mathbb{R})$ , then  $b = (2\pi)^{-1} \widehat{f h} \in L^1(\mathbb{R}_+)$  and  $(f \star h)^\sim = L_b$ . Hence (4.4.10) is given by Lemma 4.4.3 in this case. In the general case, let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of  $C_{00}(\mathbb{R})$  such that  $f_n \rightarrow f$  in  $C_0(\mathbb{R})$ , when  $n \rightarrow \infty$ . Then  $u_{A+\varepsilon}(f_n, h) \rightarrow u_{A+\varepsilon}(f, h)$ , hence  $(f_n \star h)^\sim(A + \varepsilon) \rightarrow u_{A+\varepsilon}(f, h)$ . Moreover  $(f_n \star h)^\sim \rightarrow (f \star h)^\sim$  in  $H^\infty(\mathbb{C}_+)$ . Therefore by the Convergence Lemma 4.4.2,  $(f \star h)^\sim(A + \varepsilon)$  is bounded and (4.4.10) holds true.

Next we show that in  $B(H)$ , we have

$$u_{A+\varepsilon}(f, h) \xrightarrow{\varepsilon \rightarrow 0} u_A(f, h), \quad f \in C_0(\mathbb{R}), h \in H^1(\mathbb{R}). \quad (4.4.11)$$

In the case when  $f \in C_{00}(\mathbb{R})$ ,

$$u_{A+\varepsilon}(f, h) = \frac{1}{2\pi} \int_0^\infty \widehat{f}(t)\widehat{h}(t)e^{-\varepsilon t}T_t dt$$

for any  $\varepsilon \geq 0$ . Hence

$$\|u_A(f, h) - u_{A+\varepsilon}(f, h)\| \leq \frac{C}{2\pi} \int_0^\infty |\widehat{f}(t)\widehat{h}(t)|(1 - e^{-\varepsilon t}) dt.$$

This integral goes to 0 when  $\varepsilon \rightarrow 0$ , by Lebesgue's dominated convergence Theorem. This yields the result in this case. The general case follows from the density of  $C_{00}(\mathbb{R})$  in  $C_0(\mathbb{R})$  and the uniform estimate (4.4.9).

We now construct  $\rho_{0,A}$ . Let  $F \in \mathcal{A}_0(\mathbb{R})$  and consider two sequences  $(f_k)_{k \in \mathbb{N}}$  of  $C_0(\mathbb{R})$  and  $(h_k)_{k \in \mathbb{N}}$  of  $H^1(\mathbb{R})$  satisfying (4.3.1) and (4.3.2). We let

$$F_N = \sum_{k=1}^N f_k \star h_k, \quad N \geq 1.$$

For any fixed  $\varepsilon > 0$ , it follows from (4.4.10) that for any  $N \geq 1$ ,

$$\widetilde{F}_N(A + \varepsilon) = \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k).$$

We both have that  $\widetilde{F}_N \rightarrow \widetilde{F}$  in  $H^\infty(\mathbb{C}_+)$  and that  $\sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) \rightarrow \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k)$  in  $B(H)$ . Appealing again to Lemma 4.4.2, we deduce that  $\widetilde{F}(A + \varepsilon) \in B(H)$  and that

$$\widetilde{F}(A + \varepsilon) = \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k). \quad (4.4.12)$$

We observe that

$$\sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) \xrightarrow{\varepsilon \rightarrow 0} \sum_{k=1}^\infty u_A(f_k, h_k) \quad (4.4.13)$$

in  $B(H)$ . To check this, let  $a > 0$  and choose  $N \geq 1$  such that  $\sum_{k=N+1}^\infty \|f_k\|_\infty \|h_k\|_1 \leq a$ . We have

$$\begin{aligned} \left\| \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^\infty u_A(f_k, h_k) \right\| &\leq \left\| \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| \\ &\quad + \sum_{k=N+1}^\infty \|u_{A+\varepsilon}(f_k, h_k)\| + \sum_{k=N+1}^\infty \|u_A(f_k, h_k)\|. \end{aligned}$$

Using the uniform estimate (4.4.9), this implies that

$$\left\| \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^\infty u_A(f_k, h_k) \right\| \leq \left\| \sum_{k=1}^N u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^N u_A(f_k, h_k) \right\| + 2Ca.$$

Applying (4.4.11), we deduce that

$$\left\| \sum_{k=1}^\infty u_{A+\varepsilon}(f_k, h_k) - \sum_{k=1}^\infty u_A(f_k, h_k) \right\| \leq 3Ca$$

for  $\varepsilon > 0$  small enough, which shows the result.

Combining (4.4.12) and (4.4.13) we obtain that  $\tilde{F}(A + \varepsilon)$  has a limit in  $B(H)$ , when  $\varepsilon \rightarrow 0$ . We set

$$\rho_{0,A}(\tilde{F}) := \lim_{\varepsilon \rightarrow 0} \tilde{F}(A + \varepsilon).$$

It is plain that  $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(H)$  is a linear map. It follows from the construction that

$$\|\rho_{0,A}(\tilde{F})\|_{\mathcal{A}_0} \leq C^2 \|\tilde{F}\|_{\mathcal{A}_0}$$

for any  $F \in \mathcal{A}_0(\mathbb{R})$ , hence  $\rho_{0,A}$  is bounded with  $\|\rho_{0,A}\| \leq C^2$ .

Let  $b \in L^1(\mathbb{R}_+)$ . By the compatibility Lemma 4.4.3, we have

$$L_b(A + \varepsilon) = \int_0^\infty \widehat{b}(t) e^{-\varepsilon t} T_t dt$$

for any  $\varepsilon > 0$ . Passing to the limit and using Lebesgue's dominated convergence Theorem, we obtain (4.4.8).

It follows from the density of  $\{L_b : b \in L^1(\mathbb{R}_+)\}$  in  $\mathcal{A}_0(\mathbb{C}_+)$ , given by Lemma 4.4.1, that  $\rho_{0,A}$  is unique. Moreover the multiplicativity of the Hille-Phillips functional calculus insures that  $\rho_{0,A}$  is a Banach algebra homomorphism.  $\square$

*Remark 4.4.6.* Let  $F \in \mathcal{A}_0(\mathbb{R})$  and let  $(f_k)_{k \in \mathbb{N}}$  and  $(h_k)_{k \in \mathbb{N}}$  be sequences of  $C_0(\mathbb{R})$  and  $H^1(\mathbb{R})$ , respectively, satisfying (4.3.1) and (4.3.2). It follows from the proof of Theorem 4.4.5 that

$$\rho_{0,A}(\tilde{F}) = \sum_{k=1}^{\infty} u_A(f_k, h_k). \quad (4.4.14)$$

This equality shows that the right-hand side of (4.4.14) does not depend on the choice of  $(f_k)_{k \in \mathbb{N}}$  and  $(h_k)_{k \in \mathbb{N}}$ . The reason why we did not take (4.4.14) as a definition of  $\rho_{0,A}$  is precisely that we did not know a priori that  $\sum_{k=1}^{\infty} u_A(f_k, h_k)$  was independent of the representation of  $F$ .

### Functional calculus on $\mathcal{A}(\mathbb{C}_+)$

We keep the notation from the previous subsection. We can extend Theorem 4.4.5 as follows.

**Corollary 4.4.7.** *There exists a unique bounded homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$  extending  $\rho_{0,A}$ . Moreover  $\|\rho_A\| \leq C^2$ .*

*Proof.* We follow an idea from [7], using regularization. Consider the sequence  $(G_N)_{N \in \mathbb{N}}$  defined in the proof of Proposition 4.3.12. Then

$$\widetilde{G}_N(z) = \frac{N}{N+z}, \quad z \in \mathbb{C}_+, N \geq 1.$$

For any  $\varphi \in \mathcal{A}(\mathbb{C}_+)$ , we let  $S_\varphi$  be the operator defined by

$$S_\varphi = (1 + A)\rho_{0,A}(\varphi \widetilde{G}_1),$$

with domain  $\text{Dom}(S_\varphi) = \{x \in H : [\rho_{0,A}(\varphi \widetilde{G}_1)](x) \in \text{Dom}(A)\}$ . In this definition, we use the fact that  $\varphi \widetilde{G}_1$  belongs to  $\mathcal{A}_0(\mathbb{C}_+)$ , which follows from Proposition 4.3.4. It is clear that  $S_\varphi$  is closed. Further  $\text{Dom}(A) \subset \text{Dom}(S_\varphi)$ , hence  $S_\varphi$  is densely defined. More precisely, if  $x \in \text{Dom}(A)$ , then  $x = \rho_{A,0}(\widetilde{G}_1)(1 + A)(x)$  hence

$$[\rho_{0,A}(\varphi \widetilde{G}_1)](x) = \rho_{0,A}(\varphi \widetilde{G}_1^2)(1 + A)(x) = (1 + A)^{-1} \rho_{0,A}(\varphi \widetilde{G}_1)(1 + A)(x)$$

belongs to  $\text{Dom}(A)$  and we have

$$S_\varphi(x) = \rho_{0,A}(\varphi\widetilde{G}_1)(1+A)(x). \quad (4.4.15)$$

Since  $\rho_{0,A}$  is multiplicative, we have  $\rho_{0,A}(\varphi\widetilde{G}_N\widetilde{G}_1) = \rho_{0,A}(\varphi\widetilde{G}_N)(1+A)^{-1}$  for any  $N \geq 1$ . Moreover as noticed in the proof of Proposition 4.3.12,  $(G_N)_{N \in \mathbb{N}}$  is an approximate unit of  $\mathcal{A}_0(\mathbb{R})$ , hence  $\varphi\widetilde{G}_N\widetilde{G}_1 \rightarrow \varphi\widetilde{G}_1$  in  $\mathcal{A}_0(\mathbb{C}_+)$ , when  $N \rightarrow \infty$ . We deduce, using (4.4.15), that for any  $x \in \text{Dom}(A)$ ,

$$S_\varphi(x) = \lim_N \rho_{0,A}(\varphi\widetilde{G}_N\widetilde{G}_1)(1+A)(x) = \lim_N \rho_{0,A}(\varphi\widetilde{G}_N)(x).$$

For any  $N \geq 1$ ,

$$\|\rho_{0,A}(\varphi\widetilde{G}_N)\| \leq C^2 \|\varphi\widetilde{G}_N\|_{\mathcal{A}_0} \leq C^2 \|\varphi\|_{\mathcal{A}},$$

by (4.3.8). Consequently,  $\|S_\varphi(x)\| \leq C^2 \|\varphi\|_{\mathcal{A}} \|x\|$  for any  $x \in \text{Dom}(A)$ . This shows that  $\text{Dom}(S_\varphi) = H$  and  $S_\varphi \in B(H)$ .

We now define  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$  by  $\rho_A(\varphi) = S_\varphi$ . It is clear from above that  $\rho_A$  is linear and bounded, with  $\|\rho_A\| \leq C^2$ . It extends  $\rho_{0,A}$  because if  $F \in \mathcal{A}_0(\mathbb{C}_+)$ , then we have  $\rho_{0,A}(\varphi\widetilde{G}_1) = \rho_{0,A}(\widetilde{G}_1)\rho_{0,A}(\varphi) = (1+A)^{-1}\rho_{0,A}(\varphi)$ , hence  $S_\varphi = \rho_{0,A}(\varphi)$ . Using similar arguments, one easily obtains that  $\rho_A$  is multiplicative, as well as the uniqueness property.  $\square$

### Operators with a bounded $H^\infty(\mathbb{C}_+)$ -functional calculus

The goal of this subsection is to explain the connections between our main results (Theorem 4.4.5, Corollary 4.4.7) and  $H^\infty$ -functional calculus.

We will assume that the reader is familiar with sectorial operators and their  $H^\infty$ -functional calculus, for which we refer to [19] or [26, Chapter 10]. Using usual notation, for any  $\theta \in (0, \pi)$  we let  $\Sigma_\theta = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta\}$  and

$$H_0^\infty(\Sigma_\theta) = \{\varphi \in H^\infty(\Sigma_\theta) : \exists s > 0, |\varphi(z)| \lesssim \min\{|z|^s, |z|^{-s}\} \text{ on } \Sigma_\theta\}.$$

Let  $(T_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on some Banach space  $X$ , with generator  $-A$ . Recall that  $A$  is a sectorial operator of type  $\frac{\pi}{2}$ .

The following lemma is probably known to specialists, we include a proof for the sake of completeness. In part (i), the operator  $\varphi(A)$  is defined by (4.4.4) whereas in part (ii), the operator  $\varphi(A)$  is defined by [19, (2.5)]. It is worth noting that if  $\varphi \in \mathcal{E}(\mathcal{H}_\alpha) \cap H_0^\infty(\Sigma_\theta)$ , then these two definitions coincide.

**Lemma 4.4.8.** *The following assertions are equivalent.*

(i) *There exists a constant  $C > 0$  such that for any  $\alpha < 0$  and for any  $\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$ ,*

$$\|\varphi(A)\| \leq C \|\varphi\|_{H^\infty(\mathbb{C}_+)}. \quad (4.4.16)$$

(ii) *There exists a constant  $C > 0$  such that for any  $\theta \in (\frac{\pi}{2}, \pi)$  and for any  $\varphi \in H_0^\infty(\Sigma_\theta)$ ,*

$$\|\varphi(A)\| \leq C \|\varphi\|_{H^\infty(\mathbb{C}_+)}. \quad (4.4.16)$$

(iii) *There exists a constant  $C > 0$  such that for any  $b \in L^1(\mathbb{R}_+)$ ,*

$$\left\| \int_0^\infty b(t)T_t dt \right\| \leq C \|\widehat{b}\|_\infty. \quad (4.4.17)$$

*Proof.* Assume (i). By the approximation argument at the beginning of [9, Section 5], (4.4.16) holds as well for any  $\varphi \in H^\infty(\mathcal{H}_\alpha)$ . Let  $b \in L^1(\mathbb{R}_+)$ . The function  $L_b(\cdot + \varepsilon)$  belongs to  $H^\infty(\mathcal{H}_{-\varepsilon})$  for any  $\varepsilon > 0$ , hence we have

$$\|L_b(A + \varepsilon)\| \leq C \|L_b(\cdot + \varepsilon)\|_{H^\infty(\mathbb{C}_+)} \leq C \|L_b\|_{H^\infty(\mathbb{C}_+)} = C \|\widehat{b}\|_\infty.$$

Applying Lemma 4.4.3 and letting  $\varepsilon \rightarrow 0$ , we obtain (4.4.17), which proves (iii).

The fact that (iii) implies (ii) follows from [19, Lemma 3.3.1 & Proposition 3.3.2], see also [36, Lemma 2.12].

Assume (ii) and let us prove (i). For any  $\varepsilon \in (0, 1)$ , consider the rational function  $q_\varepsilon$  defined by

$$q_\varepsilon(z) = \frac{\varepsilon + z}{1 + \varepsilon z}, \quad z \neq \frac{-1}{\varepsilon}.$$

We may and do assume that  $\alpha \in (-1, 0)$  when proving (i). Fix some  $\varepsilon \in (0, 1)$ . It is easy to check (left to the reader) that  $q_\varepsilon$  maps  $\mathcal{H}_\alpha$  into itself. Moreover there exists  $\theta \in (\frac{\pi}{2}, \pi)$  such that  $q_\varepsilon$  maps  $\Sigma_\theta$  into  $\mathbb{C}_+$ .

Let  $\varphi \in \mathcal{E}(\mathcal{H}_\alpha)$ , then

$$\varphi_\varepsilon := \varphi \circ q_\varepsilon : \mathcal{H}_\alpha \cup \Sigma_\theta \longrightarrow \mathbb{C}$$

is a well-defined bounded holomorphic function. Moreover we have

$$\|\varphi_\varepsilon\|_{H^\infty(\mathcal{H}_\alpha)} \leq \|\varphi\|_{H^\infty(\mathcal{H}_\alpha)}. \tag{4.4.18}$$

By [19, Lemma 2.2.3],  $\varphi_\varepsilon$  belongs to  $H_0^\infty(\Sigma_\theta) \oplus \text{Span}\{1, (1 + \cdot)^{-1}\}$ . Further the definition of  $\varphi_\varepsilon(A)$  provided by the functional calculus of sectorial operators coincides with the definition of  $\varphi_\varepsilon(A)$  provided by the half-plane functional calculus. Hence for some constant  $C' > 0$  not depending on  $\varepsilon$ , we have

$$\|\varphi_\varepsilon(A)\| \leq C' \|\varphi_\varepsilon\|_{H^\infty(\mathbb{C}_+)} \leq C' \|\varphi\|_{H^\infty(\mathbb{C}_+)},$$

by (ii). Since  $\varphi_\varepsilon \rightarrow \varphi$  pointwise on  $\mathcal{H}_\alpha$ , it now follows from (4.4.18) and the Convergence Lemma 4.4.2 that  $\|\varphi(A)\| \leq C' \|\varphi\|_{H^\infty(\mathbb{C}_+)}$ , which proves (i).  $\square$

We say that  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus if one of (equivalently, all of) the properties of Lemma 4.4.8 hold true. If  $A$  is sectorial of type  $< \frac{\pi}{2}$ , the latter is equivalent to  $A$  having a bounded  $H^\infty$ -functional calculus of angle  $\frac{\pi}{2}$  in the usual sense. The main feature of the "bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus" property considered here is that it may apply to the case when the sectorial type of  $A$  is not  $< \frac{\pi}{2}$ .

We now come back to the specific case when  $X = H$  is a Hilbert space. Here are a few known facts in this setting:

- (f1) If  $(T_t)_{t \geq 0}$  is a contractive semigroup (that is,  $\|T_t\| \leq 1$  for all  $t \geq 0$ ), then  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus. See [19, Section 7.1.3] for a proof and more on this theme.
- (f2) We say that  $(T_t)_{t \geq 0}$  is similar to a contractive semigroup if there exists an invertible operator  $S \in B(H)$  such that  $(ST_tS^{-1})_{t \geq 0}$  is a contractive semigroup. A straightforward application of the previous result is that in this case,  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus.
- (f3) If  $A$  is sectorial of type  $< \frac{\pi}{2}$ , then  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus (if and) only if  $(T_t)_{t \geq 0}$  is similar to a contractive semigroup. This goes back to [35, Section 4].

- (f4) There exist sectorial operators of type  $< \frac{\pi}{2}$  which do not admit a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus, by [44, 6] (see also [19, Section 7.3.4]).
- (f5) There exists a bounded  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  such that  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus but  $(T_t)_{t \geq 0}$  is not similar to a contractive semigroup. This follows from [35, Proposition 4.8] and its proof.

We now establish analogues of Theorem 4.4.5 and Corollary 4.4.7 in the case when  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus. Just as we did at the beginning of Section 4.4, we set

$$\mathcal{C}_0(\mathbb{C}_+) = \{\tilde{F} : F \in C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})\} \quad \text{and} \quad \mathcal{C}(\mathbb{C}_+) = \{\tilde{F} : F \in C_b(\mathbb{R}) \cap H^\infty(\mathbb{R})\}.$$

Since  $\{\widehat{b}(-\cdot) : b \in L^1(\mathbb{R}_+)\}$  is dense in  $C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$ , by Remark 4.3.13, the following is straightforward.

**Proposition 4.4.9.** *Assume that  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus. Then there exists a unique bounded homomorphism  $\nu_{0,A} : \mathcal{C}_0(\mathbb{C}_+) \rightarrow B(H)$  such that*

$$\nu_{0,A}(L_b) = \int_0^\infty b(t)T_t dt \tag{4.4.19}$$

for any  $b \in L^1(\mathbb{R}_+)$ .

Now arguing as in the proof of Corollary 4.4.7, we deduce the following.

**Corollary 4.4.10.** *Assume that  $A$  admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus. Then there exists a unique bounded homomorphism  $\nu_A : \mathcal{C}(\mathbb{C}_+) \rightarrow B(H)$  such that (4.4.19) holds true for any  $b \in L^1(\mathbb{R}_+)$ .*

Of course when the above corollary applies,  $\nu_A$  is an extension of the mapping  $\rho_A$  from Corollary 4.4.7. Thus our main results (Theorem 4.4.5, Corollary 4.4.7) should be regarded as a way to obtain a “good” functional calculus for negative generators of bounded  $C_0$ -semigroups which do not admit a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus.

*Remark 4.4.11.* We can easily deduce from the above results a new proof of Proposition 4.2.4 (2), as follows. Recall the identifications  $C_0(\mathbb{R})^* = M(\mathbb{R})$  and  $\mathcal{A}_0(\mathbb{R})^* \subset \mathcal{M}(H^1(\mathbb{R}))$  given by (4.1.4) and Subsection 4.3. The identity (4.3.20) shows that through these identifications, the mapping  $R : M(\mathbb{R}) \rightarrow \mathcal{M}(H^1(\mathbb{R}))$  taking any  $\mu \in M(\mathbb{R})$  to  $R_\mu$  is the adjoint of the canonical embedding  $j : \mathcal{A}_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ . Assume, by contradiction, that  $\mathcal{R} = \mathcal{M}(H^1(\mathbb{R}))$ . Then  $R$  is onto, hence  $j$  induces an isomorphism from  $\mathcal{A}_0(\mathbb{R})$  onto its range. This implies that  $\mathcal{A}_0(\mathbb{R}) = C_0(\mathbb{R}) \cap H^\infty(\mathbb{R})$  and that  $\|\cdot\|_{\mathcal{A}_0}$  is equivalent to  $\|\cdot\|_\infty$  on  $\mathcal{A}_0(\mathbb{R})$ . By Theorem 4.4.5, this implies that the negative generator of any bounded  $C_0$ -semigroup on Hilbert space admits a bounded  $H^\infty(\mathbb{C}_+)$ -functional calculus. This is false, as noted in (f4), hence we have  $\mathcal{R} \neq \mathcal{M}(H^1(\mathbb{R}))$ .

## 4.5. Comparison with the Besov functional calculus

In this section we compare the functional calculus constructed in Section 4.4 (Theorem 4.4.5 and Corollary 4.4.7) with the Besov functional calculus from [21, Subsection 5.5] and [7]. We start with some background on the analytic homogeneous Besov space used in the latter paper. We refer to [7] for further details.

Let  $\psi \in \mathcal{S}(\mathbb{R})$  such that  $\text{Supp}(\psi) \subset [\frac{1}{2}, 2]$ ,  $\psi(t) \geq 0$  for all  $t \in \mathbb{R}$ , and  $\psi(t) + \psi(\frac{t}{2}) = 1$  for all  $t \in [1, 2]$ . For any  $k \in \mathbb{Z}$ , we let  $\psi_k \in \mathcal{S}(\mathbb{R})$  be defined by  $\psi_k(t) = \psi(2^{-k}t)$ ,  $t \in \mathbb{R}$ . A key property of the sequence  $(\psi_k)_{k \in \mathbb{Z}}$  is that for any  $k_0 \in \mathbb{Z}$ , we have

$$\forall t \in [2^{k_0}, 2^{k_0+1}) : \quad \psi_{k_0}(t) + \psi_{k_0+1}(t) = 1 \quad \text{and} \quad \psi_k(t) = 0 \text{ if } k \notin \{k_0, k_0 + 1\}. \quad (4.5.1)$$

Next define  $\phi_k = \mathcal{F}^{-1}(\psi_k)$ . It is plain that for any  $k \in \mathbb{Z}$ ,

$$\phi_k \in H^1(\mathbb{R}) \quad \text{and} \quad \|\phi_k\|_1 = \|\phi_0\|_1. \quad (4.5.2)$$

It follows that for any  $F \in BUC(\mathbb{R})$  and any  $k \in \mathbb{Z}$ ,  $F \star \phi_k$  belongs to  $BUC(\mathbb{R}) \cap H^\infty(\mathbb{R})$ . We define a Besov space  $\mathcal{B}(\mathbb{R})$  by

$$\mathcal{B}(\mathbb{R}) = \left\{ F \in BUC(\mathbb{R}) : \sum_{k \in \mathbb{Z}} \|F \star \phi_k\|_\infty < \infty \text{ and } F = \sum_{k \in \mathbb{Z}} F \star \phi_k \right\}.$$

This is a Banach space for the norm

$$\|F\|_{\mathcal{B}} = \sum_{k \in \mathbb{Z}} \|F \star \phi_k\|_\infty.$$

Next we set  $\mathcal{B}_0(\mathbb{R}) = \mathcal{B}(\mathbb{R}) \cap C_0(\mathbb{R})$ , equipped with the norm of  $\mathcal{B}(\mathbb{R})$ . Then  $\mathcal{B}_0(\mathbb{R})$  is a closed subspace of  $\mathcal{B}(\mathbb{R})$  and we clearly have

$$\mathcal{B}_0(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{B}(\mathbb{R}) \subset BUC(\mathbb{R}) \cap H^\infty(\mathbb{R}).$$

We warn the reader that our notation  $\mathcal{B}/\mathcal{B}_0$  differs from the one introduced in [7]. We make this choice to match with the notation in Section 4.3.

We wish to underline that the above definitions of  $\mathcal{B}(\mathbb{R})$  and  $\mathcal{B}_0(\mathbb{R})$  do not depend on the choice of the function  $\psi$ . More precisely if  $\psi^{(1)}, \psi^{(2)}$  are two functions as above and if we let  $\mathcal{B}^{\psi^{(1)}}(\mathbb{R})$  and  $\mathcal{B}^{\psi^{(2)}}(\mathbb{R})$  denote the associated spaces, then  $\mathcal{B}^{\psi^{(1)}}(\mathbb{R})$  and  $\mathcal{B}^{\psi^{(2)}}(\mathbb{R})$  coincide as vector spaces and the norms  $\|\cdot\|_{\mathcal{B}^{\psi^{(1)}}}$  and  $\|\cdot\|_{\mathcal{B}^{\psi^{(2)}}}$  are equivalent. We refer to [7] and the references therein for these properties.

Similarly to the beginning of Section 4.4, we introduce half-plane versions of  $\mathcal{B}(\mathbb{R})$  and  $\mathcal{B}_0(\mathbb{R})$ , by setting

$$\mathcal{B}_0(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{B}_0(\mathbb{R})\} \quad \text{and} \quad \mathcal{B}(\mathbb{C}_+) = \{\tilde{F} : F \in \mathcal{B}(\mathbb{R})\}.$$

In [7], Batty-Gomilko-Tomilov provided an equivalent definition of  $\mathcal{B}(\mathbb{C}_+)$ , from which it follows that

$$\{L_b : b \in L^1(\mathbb{R}_+)\} \subset \mathcal{B}_0(\mathbb{C}_+), \quad (4.5.3)$$

see [7, Subsection 2.4 & Proposition 6.2].

Moreover they established the following remarkable functional calculus result.

**Theorem 4.5.1.** ([7, Theorem 4.4], [8, Theorem 6.1]) *Let  $X$  be a Banach space, let  $(T_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup and let  $-A$  denote its generator. The following are equivalent.*

(i) *There exists a constant  $K > 0$  such that*

$$\int_{-\infty}^{\infty} |\langle R(\beta + it, A)^2(x), x^* \rangle| dt \leq \frac{-K}{\beta} \|x\| \|x^*\|$$

*for any  $\beta < 0$ , any  $x \in X$  and any  $x^* \in X^*$ .*

(ii) *There exists a bounded homomorphism  $\gamma_A: \mathcal{B}(\mathbb{C}_+) \rightarrow B(X)$  such that*

$$\gamma_A(L_b) = \int_0^\infty b(t)T_t dt, \quad b \in L^1(\mathbb{R}_+).$$

*In this case,  $\gamma_A$  is unique.*

Condition (i) in Theorem 4.5.1 goes back at least to [16] and [53]. In fact, condition (i) can be defined for any closed and densely defined operator  $A$  satisfying (4.4.3) for  $\omega = 0$ . Then it follows from [16, 53] that (i) actually implies that  $-A$  generates a bounded  $C_0$ -semigroup on  $X$ . (See also [9, Theorem 6.4]). Conversely, if  $X = H$  is a Hilbert space, it is proved in [16, 53] that if  $-A$  generates a bounded  $C_0$ -semigroup, then  $A$  satisfies (i). (The assumption that  $X = H$  is a Hilbert space is crucial here, see the beginning of Section 4.6 for more on this.)

Thus if  $(T_t)_{t \geq 0}$  is a bounded  $C_0$ -semigroup with generator  $-A$  on Hilbert space, then the property (ii) in Theorem 4.5.1 holds true. It is therefore natural to compare Corollary 4.4.7 with that property. This is the aim of the rest of this section.

**Proposition 4.5.2.** *We both have*

$$\mathcal{B}(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad \mathcal{B}_0(\mathbb{C}_+) \subset \mathcal{A}_0(\mathbb{C}_+).$$

*Moreover there exists a constant  $K > 0$  such that  $\|\varphi\|_{\mathcal{A}} \leq K\|\varphi\|_{\mathcal{B}}$  for any  $\varphi \in \mathcal{B}(\mathbb{C}_+)$ .*

*Proof.* It follows from (4.5.1) that

$$\psi_k = \psi_k(\psi_{k-1} + \psi_k + \psi_{k+1}), \quad k \in \mathbb{Z}. \tag{4.5.4}$$

Consequently,

$$\phi_k = \phi_k \star (\phi_{k-1} + \phi_k + \phi_{k+1}), \quad k \in \mathbb{Z}.$$

Let  $F \in \mathcal{B}(\mathbb{R})$ . Applying the above identity, we have

$$F = \sum_{k \in \mathbb{Z}} F \star \phi_k \star (\phi_{k-1} + \phi_k + \phi_{k+1}).$$

Appealing to (4.5.2), we observe that  $F \star \phi_k \in BUC(\mathbb{R})$  and  $\phi_{k-1} + \phi_k + \phi_{k+1} \in H^1(\mathbb{R})$  for each  $k \in \mathbb{Z}$ , and that

$$\sum_{k \in \mathbb{Z}} \|F \star \phi_k\|_\infty \|\phi_{k-1} + \phi_k + \phi_{k+1}\|_1 \leq 3\|\phi_0\|_1 \|F\|_{\mathcal{B}}.$$

This shows that  $F \in \mathcal{A}(\mathbb{R})$ , with

$$\|F\|_{\mathcal{A}} \leq 3\|\phi_0\|_1 \|F\|_{\mathcal{B}}.$$

This yields  $\mathcal{B}(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+)$ . The above argument also shows that  $\mathcal{B}_0(\mathbb{C}_+) \subset \mathcal{A}_0(\mathbb{C}_+)$ .  $\square$

Let  $H$  be a Hilbert space and let  $A$  be the negative generator of a bounded  $C_0$ -semigroup on  $H$ . We already noticed that  $A$  satisfies property (ii) in Theorem 4.5.1. According to Proposition 4.5.2, the functional calculus  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(H)$  from Corollary 4.4.7 extends the functional calculus  $\gamma_A: \mathcal{B}(\mathbb{C}_+) \rightarrow B(H)$ .

It turns out that the extension from  $\gamma_A$  to  $\rho_A$  is an actual improvement, because of the following result.



**Theorem 4.5.3.** *We both have*

$$\mathcal{B}(\mathbb{C}_+) \neq \mathcal{A}(\mathbb{C}_+) \quad \text{and} \quad \mathcal{B}_0(\mathbb{C}_+) \neq \mathcal{A}_0(\mathbb{C}_+).$$

We need some preparation before coming to the proof. We use an idea from [55, Paragraph 2.6.4]. First for the definition of the Besov space  $\mathcal{B}(\mathbb{R})$ , we make the additional assumption that  $\psi(t) = 1$  for any  $t \in [\frac{3}{4}, 1]$ . This is allowed by the aforementioned fact that the definition of  $\mathcal{B}(\mathbb{R})$  does not depend on  $\psi$ . This implies that  $\text{Supp}(\psi) \subset [\frac{1}{2}, \frac{3}{2}]$ . Second we fix a non-zero function  $f_0 \in \mathcal{S}(\mathbb{R})$  such that  $\text{Supp}(f_0) \subset [\frac{3}{4}, 1]$ . Next for any integer  $n \geq 0$ , we set  $N_n = 2^n - 1$  and  $f_n = \tau_{N_n} f = f(\cdot - N_n)$ . By construction,  $\text{Supp}(\psi_k) \subset [2^{k-1}, \frac{3}{2}2^k]$  for all  $k \in \mathbb{Z}$  and  $\text{Supp}(f_n) \subset [2^n - \frac{1}{4}, 2^n]$  for all  $n \geq 0$ . We derive that

$$\forall k \geq 0 : \quad f_k \psi_k = f_k \quad \text{and} \quad f_n \psi_k = 0 \text{ if } n \neq k, \quad (4.5.5)$$

as well as

$$\forall n, n' \geq 0 : \quad f_n f_{n'} = 0 \text{ if } n \neq n'. \quad (4.5.6)$$

**Lemma 4.5.4.** *There exists a bounded continuous function  $m: \mathbb{R}_+^* \rightarrow \mathbb{C}$  such that*

$$\sup_{k \in \mathbb{Z}} \|\widehat{m\psi_k}\|_1 < \infty \quad (4.5.7)$$

and the mapping  $T_m: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$  does not belong to  $\mathcal{M}(H^1(\mathbb{R}))$ .

*Proof.* Using the definitions preceding the lemma, we set

$$m(t) = \sum_{n=0}^{\infty} e^{iN_n t} f_n(t), \quad t > 0.$$

At most one term is non zero in this sum, hence this is well-defined and  $m \in C_b(\mathbb{R}_+^*)$ . Let  $k \geq 0$ . According to (4.5.5), we have  $m\psi_k = e^{iN_k \cdot} f_k$  hence

$$\|\widehat{m\psi_k}\|_1 = \|\widehat{f_k}(\cdot - N_k)\|_1 = \|\widehat{f_k}\|_1 = \|\widehat{f_0}\|_1.$$

Since  $m\psi_k = 0$  if  $k < 0$ , this shows (4.5.7).

Define

$$g_N = \mathcal{F}^{-1} \left( \sum_{n=0}^N e^{-iN_n \cdot} f_n \right)$$

for all  $N \geq 0$ . Then  $g_N \in \mathcal{S}(\mathbb{R}) \cap H^1(\mathbb{R})$  hence  $g_N \in H^p(\mathbb{R})$  for any  $1 \leq p \leq \infty$ . Let us estimate its  $L^p$ -norm. On the one hand, we have

$$\|g_N\|_1 \leq \sum_{n=0}^N \|\mathcal{F}^{-1}(e^{-iN_n \cdot} f_n)\|_1 = \sum_{n=0}^N \|\mathcal{F}^{-1}(f_n)(\cdot - N_n)\|_1 = \sum_{n=0}^N \|\mathcal{F}^{-1}(f_n)\|_1,$$

hence

$$\|g_N\|_1 \leq (N + 1) \|\mathcal{F}^{-1}(f_0)\|_1.$$

On the other hand, for any  $t \in \mathbb{R}$ , we have

$$g_N(t) = \sum_{n=0}^N \mathcal{F}^{-1}(f_n)(t - N_n),$$

hence

$$|g_N(t)| \leq \sum_{n=0}^N |g_0(t - N_n)|.$$

Since  $g_0 = \mathcal{F}^{-1}(f_0) \in \mathcal{S}(\mathbb{R})$  we infer that

$$\sup_{N \geq 0} \|g_N\|_\infty < \infty.$$

For any  $1 < p < \infty$ , we have  $\|g_N\|_p \leq \|g_N\|_1^{\frac{1}{p}} \|g_N\|_\infty^{1-\frac{1}{p}}$ , hence the above estimates imply the existence of a constant  $K > 0$  such that

$$\|g_N\|_p \leq K N^{\frac{1}{p}}, \quad N \geq 1. \quad (4.5.8)$$

By (4.5.6), we have

$$m\widehat{g}_N = \left( \sum_{n=0}^{\infty} e^{iN_n \cdot} f_n \right) \left( \sum_{n=0}^N e^{-iN_n \cdot} f_n \right) = \sum_{n=0}^N f_n^2.$$

For any  $n \geq 0$ ,  $\text{Supp}(f_n^2) \subset [2^{n-1}, 2^n]$  hence by [18, Theorem 5.1.5.], we have an estimate

$$\|\mathcal{F}^{-1}(m\widehat{g}_N)\|_p \approx \left\| \left( \sum_{n=0}^N |\mathcal{F}^{-1}(f_n^2)|^2 \right)^{\frac{1}{2}} \right\|_p.$$

Further,  $f_n^2 = f_0^2(\cdot - N_n)$  hence  $|\mathcal{F}^{-1}(f_n^2)| = |\mathcal{F}^{-1}(f_0^2)|$  for any  $n \geq 0$ . Consequently,

$$\left( \sum_{n=0}^N |\mathcal{F}^{-1}(f_n^2)|^2 \right)^{\frac{1}{2}} = (N+1)^{\frac{1}{2}} |\mathcal{F}^{-1}(f_0^2)|.$$

Thus we have

$$\|\mathcal{F}^{-1}(m\widehat{g}_N)\|_p \approx N^{\frac{1}{2}}.$$

Comparing with (4.5.8) we deduce that if  $2 < p < \infty$ , then  $T_m: H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$  is not a bounded Fourier multiplier on  $H^p(\mathbb{R})$ . By Lemma 4.2.5, we deduce that  $T_m \notin \mathcal{M}(H^1(\mathbb{R}))$ .  $\square$

*Proof of Theorem 4.5.3.* If  $\mathcal{A}(\mathbb{C}_+)$  were equal to  $\mathcal{B}(\mathbb{C}_+)$ , we would have  $\mathcal{A}_0(\mathbb{C}_+) = \mathcal{B}_0(\mathbb{C}_+)$  which in turn is equivalent to  $\mathcal{A}_0(\mathbb{R}) = \mathcal{B}_0(\mathbb{R})$ . So it suffices to show that this equality fails. Let us assume, by contradiction, that  $\mathcal{A}_0(\mathbb{R}) = \mathcal{B}_0(\mathbb{R})$ .

Let  $m$  be given by Lemma 4.5.4. Let  $b \in L^1(\mathbb{R}_+)$ . For any  $k \in \mathbb{Z}$ , we have

$$\psi_k b = \frac{1}{2\pi} \mathcal{F}(\phi_k \star \widehat{b}(-\cdot)).$$

Hence using (4.5.1), (4.5.4) and Lemma 4.1.1, we have

$$\begin{aligned} \int_{-\infty}^{\infty} m(t)b(t) dt &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} m(t)\psi_k(t)b(t) dt \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} (\psi_{k-1}(t) + \psi_k(t) + \psi_{k+1}(t))m(t)\psi_k(t)b(t) dt \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} [\mathcal{F}((\psi_{k-1} + \psi_k + \psi_{k+1})m)](u) [\mathcal{F}(\psi_k b)](-u) du \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} [\mathcal{F}((\psi_{k-1} + \psi_k + \psi_{k+1})m)](u) [\phi_k \star \widehat{b}(-\cdot)](u) du \end{aligned}$$

Therefore,

$$\left| \int_{-\infty}^{\infty} m(t)b(t) dt \right| \leq \sum_{k \in \mathbb{Z}} \|\mathcal{F}((\psi_{k-1} + \psi_k + \psi_{k+1})m)\|_1 \|\phi_k \star \widehat{b}(-\cdot)\|_{\infty}.$$

Applying (4.5.7), we deduce the existence of a constant  $K > 0$  such that

$$\left| \int_{-\infty}^{\infty} m(t)b(t) dt \right| \leq K \sum_{k \in \mathbb{Z}} \|\phi_k \star \widehat{b}(-\cdot)\|_{\infty} = K \|\widehat{b}(-\cdot)\|_{\mathcal{B}}.$$

Therefore there exists  $\eta \in \mathcal{B}_0(\mathbb{R})^*$  such that

$$\langle \eta, \widehat{b}(-\cdot) \rangle = \int_{-\infty}^{\infty} m(t)b(t) dt, \quad b \in L^1(\mathbb{R}_+).$$

By assumption,  $\eta \in \mathcal{A}_0(\mathbb{R})^*$ . Applying Theorem 4.3.7, let  $T \in \mathcal{M}(H^1(\mathbb{R}))$  be associated to  $\eta$  and let  $m_0 \in C_b(\mathbb{R}_+^*)$  be the symbol of  $T$ . Then by Remark 4.3.10, we have

$$\langle \eta, \widehat{b}(-\cdot) \rangle = \int_{-\infty}^{\infty} m_0(t)b(t) dt, \quad b \in L^1(\mathbb{R}_+).$$

We deduce that  $m_0 = m$ , and this contradicts the fact that  $T_m \notin \mathcal{M}(H^1(\mathbb{R}))$ . □

*Remark 4.5.5.* Let  $\mathcal{D} \subset H^1(\mathbb{R})$  be the space of all  $h \in H^1(\mathbb{R})$  such that  $\text{Supp}(\widehat{h})$  is a compact subset of  $\mathbb{R}_+^*$ . It is well-known that  $\mathcal{D}$  is dense in  $H^1(\mathbb{R})$ . To check this, take any  $h \in H^1(\mathbb{R})$  and recall that there exist  $v, w \in H^2(\mathbb{R})$  such that  $h = vw$ . Let  $(d_n)_{n \in \mathbb{N}}$  and  $(c_n)_{n \in \mathbb{N}}$  be sequences of  $C_b(\mathbb{R}_+^*)$  with compact supports such that  $d_n \rightarrow \widehat{w}$  and  $c_n \rightarrow \widehat{v}$  in  $L^2(\mathbb{R}_+)$ . Then  $\mathcal{F}^{-1}(d_n) \rightarrow w$  and  $\mathcal{F}^{-1}(c_n) \rightarrow v$  in  $H^2(\mathbb{R})$ , hence  $\mathcal{F}^{-1}(d_n)\mathcal{F}^{-1}(c_n) \rightarrow h$  in  $H^1(\mathbb{R})$ . Now it is easy to see that  $\mathcal{F}^{-1}(d_n)\mathcal{F}^{-1}(c_n)$  belongs to  $\mathcal{D}$  for any  $n \in \mathbb{N}$ .

Let  $BUC \star \mathcal{D} \subset \mathcal{A}(\mathbb{R})$  be the linear span of the functions  $f \star h$ , for  $f \in BUC(\mathbb{R})$  and  $h \in \mathcal{D}$ . It follows from above that this is a dense subspace of  $\mathcal{A}(\mathbb{R})$ .

Let  $\mathcal{G} \subset H^\infty(\mathbb{R})$  be the space of all  $F \in H^\infty(\mathbb{R})$  such that  $\text{Supp}(\widehat{F})$  is a compact subset of  $\mathbb{R}_+^*$ . Then we have

$$BUC \star \mathcal{D} \subset \mathcal{G} \subset \mathcal{B}(\mathbb{R}).$$

The first inequality is obvious and the second one is given by [7, Proposition 6.2 & Lemma 2.4]. Note that [7, Proposition 2.9] also asserts that  $\mathcal{G}$  is dense in  $\mathcal{B}(\mathbb{R})$ .

It follows that  $\mathcal{B}(\mathbb{R})$  is dense in  $\mathcal{A}(\mathbb{R})$ , or equivalently that  $\mathcal{B}(\mathbb{C}_+)$  is dense in  $\mathcal{A}(\mathbb{C}_+)$ .

Also  $\mathcal{B}_0(\mathbb{C}_+)$  is dense in  $\mathcal{A}_0(\mathbb{C}_+)$ , by (4.5.3).

*Remark 4.5.6.*

(1) Let  $A$  be as in Subsections 4.4 and 4.4. Let  $\mu \in M(\mathbb{R})$ , with  $\mu(\{0\}) = 0$ . According to [7, Subsection 2.2 & Proposition 6.2], its Laplace transform  $L_\mu: \mathbb{C}_+ \rightarrow \mathbb{C}$  belongs to  $\mathcal{B}(\mathbb{C}_+)$ . Hence  $L_\mu$  belongs to  $\mathcal{A}(\mathbb{C}_+)$ , by Proposition 4.5.2. The argument in the proof of Corollary 4.4.7 shows that  $\rho_A(L_\mu)$  is the strong limit of  $\rho_{0,A}(L_\mu \widetilde{G}_N)$ , when  $N \rightarrow \infty$ . Define  $c_N(t) = Ne^{-Nt}$  for any  $t > 0$  and recall that  $\widetilde{G}_N = L_{c_N}$ . Then  $L_\mu \widetilde{G}_N = L_{\mu \star c_N}$  for any  $N \geq 1$ . Further  $\mu \star c_N \rightarrow \mu$  narrowly, when  $N \rightarrow \infty$ . It therefore follows from (4.4.8) that

$$[\rho_A(L_\mu)](x) = \int_{\mathbb{R}} T_t(x) d\mu(t), \quad x \in H.$$

(2) It follows from [7, Subsection 2.2] that  $\lim_{y \rightarrow \infty} \varphi(y) = 0$  for any  $\varphi \in \mathcal{B}(\mathbb{C}_+)$ . We noticed in Remark 4.5.5 that  $\mathcal{B}(\mathbb{C}_+)$  is dense in  $\mathcal{A}(\mathbb{C}_+)$ . Since  $\|\cdot\|_{H^\infty(\mathbb{C}_+)} \leq \|\cdot\|_{\mathcal{A}(\mathbb{C}_+)}$ , this implies that any element of  $\mathcal{A}(\mathbb{C}_+)$  is the uniform limit of a sequence of  $\mathcal{B}(\mathbb{C}_+)$ . Consequently,  $\lim_{y \rightarrow \infty} \varphi(y) = 0$  for any  $\varphi \in \mathcal{A}(\mathbb{C}_+)$ .

Thus the algebra  $\mathcal{A}(\mathbb{C}_+)$  (equivalently, the algebra  $\mathcal{A}(\mathbb{R})$ ) does not contain any non zero constant function and hence is not unital.

### 4.6. $\gamma$ -Bounded semigroups on Banach spaces

In general, Theorem 4.4.5 and Corollary 4.4.7 do not hold true if  $H$  is replaced by an arbitrary Banach space. Indeed as a consequence of [16, Proposition], the translation semigroup  $(T_t)_{t \geq 0}$  on  $L^p(\mathbb{R})$ , for  $1 \leq p \neq 2 < \infty$ , does not satisfy condition (i) in Theorem 4.5.1. Hence by the latter theorem and Proposition 4.5.2, the mapping

$$L_b \mapsto \int_0^\infty b(t)T_t dt, \quad b \in L^1(\mathbb{R}_+),$$

is not bounded with respect to the  $\mathcal{A}_0(\mathbb{C}_+)$ -norm.

In this section we will however establish Banach space versions of Theorem 4.4.5 and Corollary 4.4.7 on Banach spaces, involving  $\gamma$ -boundedness. We start with some background and basic facts on this topic and refer to [26, Chapter 9] for details and more information.

Let  $X$  be a Banach space. Let  $(\gamma_n)_{n \geq 1}$  be a sequence of independent complex valued standard Gaussian variables on some probability space  $\Sigma$  and let  $G_0 \subset L^2(\Sigma)$  be the linear span of the  $\gamma_n$ . We denote by  $G(X)$  the closure of

$$G_0 \otimes X = \left\{ \sum_{k=1}^N \gamma_k \otimes x_k : x_k \in X, N \in \mathbb{N} \right\}$$

in the Bochner space  $L^2(\Sigma; X)$ , equipped with the induced norm. Next we let  $G'(X^*)$  denote the closure of  $G_0 \otimes X^*$  in the dual space  $G(X)^*$ .

A bounded set  $\mathcal{T} \subset B(X)$  is called  $\gamma$ -bounded if there exists a constant  $C \geq 0$  such that for all finite sequences  $(S_k)_{k=1}^N \subset \mathcal{T}$  and  $(x_k)_{k=1}^N \subset X$ , we have:

$$\left\| \sum_{k=1}^N \gamma_k \otimes S_k(x_k) \right\|_{G(X)} \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}. \tag{4.6.1}$$

The least admissible constant  $C$  in the above inequality is called the  $\gamma$ -bound of  $\mathcal{T}$  and is denoted by  $\gamma(\mathcal{T})$ .

Let  $Z$  be any Banach space and let  $\text{Ball}(Z)$  denote its closed unit ball. A bounded operator  $\rho: Z \rightarrow B(X)$  is called  $\gamma$ -bounded if the set  $\rho(\text{Ball}(Z)) \subset B(X)$  is  $\gamma$ -bounded. In this case we set  $\gamma(\rho) = \gamma(\rho(\text{Ball}(Z)))$ .

We now turn to the definition of  $\gamma$ -spaces, which goes back to the paper [29] (which began to circulate 20 years ago). Let  $H$  be a Hilbert space. A bounded operator  $T: H \rightarrow X$  is called  $\gamma$ -summing if

$$\|T\|_\gamma := \sup \left\{ \left\| \sum_{k=1}^N \gamma_k \otimes T(e_k) \right\|_{G(X)} \right\} < \infty,$$

where the supremum is taken over all finite orthonormal systems  $(e_k)_{k=1}^N$  in  $H$ . We let  $\gamma_\infty(H; X)$  denote the space of all  $\gamma$ -summing operators and we endow it with the norm  $\|\cdot\|_\gamma$ . Then  $\gamma_\infty(H; X)$  is a Banach space. Any finite rank bounded operator is  $\gamma$ -summing and we let  $\gamma(H; X)$  denote their closure in  $\gamma_\infty(H; X)$ . In the sequel, finite rank bounded operator are represented by the algebraic tensor product  $H^* \otimes X$  in the usual way.

Following [29, Section 5], we let  $\gamma'_+(H^*; X^*)$  be the space of all bounded operators  $S: H^* \rightarrow X^*$  such that

$$\|S\|_{\gamma'} := \sup \{ |\text{tr}(T^*S)| \mid T: H \rightarrow X, \text{rk}(T) < \infty, \|T\|_\gamma \leq 1 \} < \infty.$$

Then  $\|\cdot\|_{\gamma'}$  is a norm on  $\gamma'_+(H^*; X^*)$  and according to [29, Proposition 5.1], we have

$$\gamma'_+(H^*; X^*) = \gamma(H; X)^* \quad (4.6.2)$$

isometrically, through the duality pairing

$$(S, T) \mapsto \text{tr}(T^*S), \quad T \in \gamma(H; X), \quad S \in \gamma'_+(H^*; X^*).$$

We will focus on the case when  $H$  is an  $L^2$ -space. Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space. We identify  $L^2(\Omega)^*$  and  $L^2(\Omega)$  in the usual way. A function  $\xi: \Omega \rightarrow X$  is called weakly- $L^2$  if for each  $x^* \in X^*$ , the function  $\langle x^*, \xi(\cdot) \rangle$  belongs to  $L^2(\Omega)$ . Then the operator  $x^* \mapsto \langle x^*, \xi(\cdot) \rangle$  from  $X^*$  into  $L^2(\Omega)$  is bounded. If  $\xi$  is both measurable and weakly- $L^2$ , then its adjoint takes values in  $X$  and we let  $\mathbb{I}_\xi: L^2(\Omega) \rightarrow X$  denote the resulting operator. More explicitly,

$$\langle x^*, \mathbb{I}_\xi(g) \rangle = \int_{\Omega} g(t) \langle x^*, \xi(t) \rangle d\mu(t), \quad g \in L^2(\Omega), \quad x^* \in X^*.$$

We let  $\gamma(\Omega; X)$  be the space of all measurable and weakly- $L^2$  functions  $\xi: \Omega \rightarrow X$  such that  $\mathbb{I}_\xi$  belongs to  $\gamma(L^2(\Omega); X)$ , and we write  $\|\xi\|_{\gamma} = \|\mathbb{I}_\xi\|_{\gamma}$  for any such function.

Likewise a function  $\zeta: \Omega \rightarrow X^*$  is called weakly\*- $L^2$  if for each  $x \in X$ , the function  $\langle \zeta(\cdot), x \rangle$  belongs to  $L^2(\Omega)$ . In this case, the operator  $x \mapsto \langle \zeta(\cdot), x \rangle$  from  $X$  into  $L^2(\Omega)$  is bounded and we let  $\mathbb{I}_\zeta: L^2(\Omega) \rightarrow X^*$  denote its adjoint. We let  $\gamma'_+(\Omega; X^*)$  be the space of all weakly\*- $L^2$  functions  $\zeta: \Omega \rightarrow X^*$  such that  $\mathbb{I}_\zeta$  belongs to  $\gamma'_+(L^2(\Omega); X)$ , and we write  $\|\zeta\|_{\gamma'} = \|\mathbb{I}_\zeta\|_{\gamma'}$  for any such function.

Note that our space  $\gamma'_+(\Omega; X^*)$  is a priori bigger than the one from [29, Definition 4.5], where only measurable functions  $\Omega \rightarrow X^*$  are considered.

**Lemma 4.6.1.** *For any  $\xi \in \gamma(\Omega; X)$  and any  $\zeta \in \gamma'_+(\Omega; X^*)$ , the function  $t \mapsto \langle \zeta(t), \xi(t) \rangle$  belongs to  $L^1(\Omega)$  and in the duality (4.6.2), we have*

$$\langle \mathbb{I}_\zeta, \mathbb{I}_\xi \rangle = \int_{\Omega} \langle \zeta(t), \xi(t) \rangle d\mu(t).$$

Moreover

$$\int_{\Omega} |\langle \zeta(t), \xi(t) \rangle| d\mu(t) \leq \|\xi\|_{\gamma} \|\zeta\|_{\gamma'}.$$

If we consider measurable functions  $\zeta: \Omega \rightarrow X^*$  only, the above statement is provided by [29, Corollary 5.5]. The fact that it holds as well in the more general setting of the present paper follows from the proof of [26, Theorem 9.2.14].

The main result of this section is the following.

**Theorem 4.6.2.** *Let  $(T_t)_{t \geq 0}$  be a bounded  $C_0$ -semigroup on  $X$  and let  $A$  be its negative generator. The following assertions are equivalent.*

- (i) *The semigroup  $(T_t)_{t \geq 0}$  is  $\gamma$ -bounded, that is, the set  $\mathcal{T}_A = \{T_t : t \geq 0\}$  is  $\gamma$ -bounded;*
- (ii) *There exists a  $\gamma$ -bounded homomorphism  $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(X)$  such that (4.4.8) holds true for any  $b \in L^1(\mathbb{R}_+)$ .*

*In this case,  $\rho_{0,A}$  is unique and  $\gamma(\mathcal{T}_A) \leq \gamma(\rho_{0,A}) \leq \gamma(\mathcal{T}_A)^2$ .*

*Further there exists a unique bounded homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$  extending  $\rho_{0,A}$ , this homomorphism is  $\gamma$ -bounded and  $\gamma(\rho_A) = \gamma(\rho_{0,A})$ .*

A thorough look at the proofs of Theorem 4.4.5 and Corollary 4.4.7 reveals that in Subsections 4.4 and 4.4, the Hilbertian structure was used only in Lemma 4.4.4. So without any surprise the main point in proving Theorem 4.6.2 is the following  $\gamma$ -bounded version of Lemma 4.4.4.

**Lemma 4.6.3.** *Let  $(T_t)_{t \geq 0}$  be a  $\gamma$ -bounded  $C_0$ -semigroup on  $X$  and let  $A$  be its negative generator. Let  $C = \gamma(\mathcal{T}_A)$ . Then the set*

$$\left\{ \Gamma(A, (2\pi)^{-1} \widehat{f w v}) : f \in C_{00}(\mathbb{R}), w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}), \{\|f\|_\infty, \|w\|_2, \|v\|_2\} \leq 1 \right\} \tag{4.6.3}$$

is  $\gamma$ -bounded, with  $\gamma$ -bound  $\leq C^2$ .

*Proof.* Let  $N \in \mathbb{N}$  and let  $f_1, \dots, f_N \in C_{00}(\mathbb{R}), w_1, \dots, w_N, v_1, \dots, v_N \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$  such that  $\|f_k\|_\infty \leq 1, \|w_k\|_2 \leq 1$  and  $\|v_k\|_2 \leq 1$  for any  $k = 1, \dots, N$ . We set

$$S_k = \Gamma(A, (2\pi)^{-1} \widehat{f_k w_k v_k}), \quad k = 1, \dots, N.$$

Let  $x_1, \dots, x_N \in X$  and  $x_1^*, \dots, x_N^* \in X^*$ . Following the notation in the proof of Lemma 4.4.4, we define, for any  $k = 1, \dots, N$ , two strongly continuous functions  $W_k, V_k: \mathbb{R} \rightarrow B(X)$  by

$$W_k(s) = \int_0^\infty \widehat{w_k}(r) e^{-irs} T_r dr \quad \text{and} \quad V_k(s) = \int_0^\infty \widehat{v_k}(t) e^{-its} T_t dt, \quad s \in \mathbb{R}.$$

According to (4.4.7),

$$\sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle = \frac{1}{4\pi^2} \sum_{k=1}^N \int_{-\infty}^\infty f_k(s) \langle W_k(s)x_k, V_k(s)^* x_k^* \rangle ds,$$

hence

$$\left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq \frac{1}{4\pi^2} \sum_{k=1}^N \int_{-\infty}^\infty |\langle W_k(s)x_k, V_k(s)^* x_k^* \rangle| ds.$$

We let  $\mathbb{N}_N = \{1, \dots, N\}$  for convenience. We will use  $\gamma$ -spaces on either  $\mathbb{R}$  or  $\mathbb{N}_N \times \mathbb{R}$ . For any  $k = 1, \dots, N$ , the function

$$\alpha_k := W_k(\cdot)x_k: \mathbb{R} \longrightarrow X$$

is measurable and weakly- $L^2$ . Likewise,

$$\beta_k := V_k(\cdot)^* x_k^*: \mathbb{R} \longrightarrow X^*$$

is weakly\*- $L^2$ . If we are able to show that  $\alpha_k \in \gamma(\mathbb{R}; X)$  and  $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$  for any  $k = 1, \dots, N$ , then Lemma 4.6.1 insures that

$$\left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq \frac{1}{4\pi^2} \left\| (k, s) \mapsto W_k(s)x_k \right\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} \left\| (k, s) \mapsto V_k^*(s)x_k^* \right\|_{\gamma'(\mathbb{N}_N \times \mathbb{R}; X^*)}. \tag{4.6.4}$$

Our aim is now to check that  $\alpha_k \in \gamma(\mathbb{R}; X)$  and  $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$  for any  $k$  and to estimate the right-hand side of (4.6.4).

By assumption,  $\mathcal{T}_A = \{T_t : t \geq 0\}$  is  $\gamma$ -bounded. According to the Multiplier Theorem stated as [22, Theorem 6.1], there exists a bounded operator

$$M: \gamma(L^2(\mathbb{R}); X) \longrightarrow \gamma(L^2(\mathbb{R}); X)$$

with norm  $\leq C = \gamma(\mathcal{T}_A)$ , mapping  $\gamma(\mathbb{R}; X)$  into itself, and such that for any  $\xi \in \gamma(\mathbb{R}; X)$ ,  $[M(\xi)](t) = T_t(\xi(t))$  if  $t \geq 0$ , and  $[M(\xi)](t) = 0$  if  $t < 0$ . Further by the Extension Theorem stated as [26, Theorem 9.6.1],  $\mathcal{F} \otimes I_X: L^2(\mathbb{R}) \otimes X \rightarrow L^2(\mathbb{R}) \otimes X$  admits a (necessarily unique) bounded extension

$$\Psi: \gamma(L^2(\mathbb{R}); X) \longrightarrow \gamma(L^2(\mathbb{R}); X),$$

with norm  $\leq \sqrt{2\pi}$ . According to Lemma 2.2.19,  $\mathbb{I}_{\alpha_k} = (\Psi \circ M)(\widehat{w}_k \otimes x_k)$  for any  $k = 1, \dots, N$ . This shows that  $\alpha_k \in \gamma(\mathbb{R}; X)$ . Let  $(e_k)_{k=1}^N$  be the canonical basis of  $\ell_N^2$ . It follows from above that

$$\begin{aligned} \|(k, s) \mapsto W_k(s)x_k\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} &= \left\| \sum_{k=1}^N e_k \otimes (\Psi \circ M)(\widehat{w}_k \otimes x_k) \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)} \\ &\leq \sqrt{2\pi} C \left\| \sum_{k=1}^N e_k \otimes \widehat{w}_k \otimes x_k \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)}. \end{aligned}$$

The finite sequence  $(e_k \otimes \widehat{w}_k)_{k=1}^N$  is an orthogonal family of  $L^2(\mathbb{N}_N \times \mathbb{R})$ . Consequently,

$$\begin{aligned} \left\| \sum_{k=1}^N e_k \otimes \widehat{w}_k \otimes x_k \right\|_{\gamma(L^2(\mathbb{N}_N \times \mathbb{R}); X)} &= \left\| \sum_{k=1}^N \|\widehat{w}_k\|_2 \gamma_k \otimes x_k \right\|_{G(X)} \\ &\leq \max_k \|\widehat{w}_k\|_2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}. \end{aligned}$$

Since  $\|\widehat{w}_k\|_2 = \sqrt{2\pi} \|w_k\|_2 \leq \sqrt{2\pi}$  for any  $k = 1, \dots, N$ , we finally obtain that

$$\|(k, s) \mapsto W_k(s)x_k\|_{\gamma(\mathbb{N}_N \times \mathbb{R}; X)} \leq 2\pi C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}.$$

We now analyse the  $\beta_k$ . Fix  $k$  and consider  $g \in L^2(\mathbb{R})$  and  $x \in X$ . Using Lemma 4.6.1, we have

$$\begin{aligned} \langle \mathbb{I}_{\beta_k}(g), x \rangle &= \int_{-\infty}^{\infty} g(s) \langle x_k^*, V_k(s)x \rangle ds \\ &= \int_{-\infty}^{\infty} g(s) \mathcal{F}(\widehat{v}_k \langle x_k^*, T(\cdot)x \rangle)(s) ds \\ &= \int_0^{\infty} \widehat{g}(t) \widehat{v}_k(t) \langle x_k^*, T_t(x) \rangle dt \\ &= \langle \widehat{v}_k \otimes x_k^*, M(\widehat{g} \otimes x) \rangle \\ &= \langle \widehat{v}_k \otimes x_k^*, (M \circ \Psi)(g \otimes x) \rangle \\ &= \langle (\Psi^* \circ M^*)(\widehat{v}_k \otimes x_k^*), g \otimes x \rangle. \end{aligned}$$

This shows that  $\beta_k \in \gamma'_+(\mathbb{R}; X^*)$ , with  $\mathbb{I}_{\beta_k} = (\Psi^* \circ M^*)(\widehat{v}_k \otimes x_k^*)$ . Now arguing as in the  $W_k(\cdot)x_k$  case, we obtain that

$$\|(k, s) \mapsto V_k^*(s)x_k^*\|_{\gamma'(\mathbb{N}_N \times \mathbb{R}; X^*)} \leq 2\pi C \left\| \sum_{k=1}^N \gamma_k \otimes x_k^* \right\|_{G'(X)}.$$

We now implement these estimates in (4.6.4) to obtain that

$$\left| \sum_{k=1}^N \langle S_k(x_k), x_k^* \rangle \right| \leq C^2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes x_k^* \right\|_{G'(X)}.$$

By the very definition of  $G'(X)$ , this means that

$$\left\| \sum_{k=1}^N \gamma_k \otimes S_k(x_k) \right\|_{G(X)} \leq C^2 \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)},$$

which completes the proof. □

*Proof of Theorem 4.6.2.* Assume (i). By Lemma 4.6.3, any element in the set (4.6.3) has norm  $\leq C$ . Hence the proof of Theorem 4.4.5 shows the existence of a unique bounded homomorphism  $\rho_{0,A}: \mathcal{A}_0(\mathbb{C}_+) \rightarrow B(X)$  such that (4.4.8) holds true for any  $b \in L^1(\mathbb{R}_+)$ .

To prove  $\gamma$ -boundedness of  $\rho_{0,A}$ , we introduce the set

$$\mathcal{L} = \{(f \star wv)^\sim : f \in C_{00}(\mathbb{R}), w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R}), \{\|f\|_\infty, \|w\|_2, \|v\|_2\} \leq 1\} \subset \mathcal{A}_0(\mathbb{C}_+).$$

Recall (see the proof of Lemma 4.4.4) that any  $h \in H^1(\mathbb{R})$  can be written as a product  $h = wv$ , with  $w, v \in H^2(\mathbb{R})$  and  $\|w\|_2^2 = \|v\|_2^2 = \|h\|_1$ , and that  $H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$  is dense in  $H^2(\mathbb{R})$ . Going back to Definition 4.3.2, we derive that

$$\text{Ball}(\mathcal{A}_0(\mathbb{C}_+)) = \overline{\text{Conv}}\{\mathcal{L}\}.$$

This implies that

$$\rho_{0,A}(\text{Ball}(\mathcal{A}_0(\mathbb{C}_+))) \subset \overline{\text{Conv}}\{\rho_{0,A}(\mathcal{L})\}.$$

Since  $\rho_{0,A}((f \star wv)^\sim) = \Gamma(A, (2\pi)^{-1} \widehat{f \star wv})$  for any  $f \in C_{00}(\mathbb{R})$  and any  $w, v \in H^2(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ , Lemma 4.6.3 says that  $\rho_{0,A}(\mathcal{L})$  is  $\gamma$ -bounded, with  $\gamma$ -bound  $\leq C^2$ . Owing to the fact that  $\gamma$ -boundedness and  $\gamma$ -bounds are preserved by convex hulls (see e.g. [26, Proposition 8.1.21]) and uniform limits, we infer that  $\rho_{0,A}$  is  $\gamma$ -bounded, with  $\gamma(\rho_{0,A}) \leq C^2$ . This proves (ii).

Conversely assume (ii). The proof of Corollary 4.4.7 shows the existence of a unique bounded homomorphism  $\rho_A: \mathcal{A}(\mathbb{C}_+) \rightarrow B(X)$  extending  $\rho_{0,A}$  as well as the fact that  $\rho_A(\text{Ball}(\mathcal{A}(\mathbb{C}_+)))$  belongs to the strong closure of  $\rho_{0,A}(\text{Ball}(\mathcal{A}_0(\mathbb{C}_+)))$ . Since  $\gamma$ -boundedness and  $\gamma$ -bounds are preserved by strong limits, we obtain that  $\rho_A$  is  $\gamma$ -bounded, with  $\gamma(\rho_A) = \gamma(\rho_{0,A})$ .

Finally the argument in Remark 4.5.6 (1) shows that for any  $t > 0$ ,

$$T_t \in \rho_A(\text{Ball}(\mathcal{A}(\mathbb{C}_+))).$$

This implies (i), with  $\gamma(\mathcal{T}_A) \leq \gamma(\rho_A)$ . □





# Appendices



# Appendix A

## $\gamma$ -boundedness of $C_0$ -semigroups on non $K$ -convex spaces

After Chapter 2 was published as a paper in *Studia Mathematica*, I realised that some of its results could be slightly improved. Namely most results in Chapter 2 are stated under a  $K$ -convexity assumption which can be lifted, up to some simple modifications. Since Chapter 2 was already published, I decided not to modify it and to explain the modifications in this appendix.

We will use the spaces  $G'(X^*)$  and  $\gamma'_+(\Omega)$  introduced in [29] (see also Chapter 4 section 6). In this context, we introduce the condition  $(\gamma\text{-}GFS)_{m,\omega}$  which is defined as the condition  $(W\gamma\text{-}GFS)_{m,\omega}$  except the norm  $\|\cdot\|_{G(X^*)}$  is replaced by  $\|\cdot\|_{G'(X)}$  (see definition 2.3.4).

**Definition A.1.** Let  $m \geq 1$  be an integer. We say that  $A$  has property  $(\gamma\text{-}GFS)_{m,\omega}$  if there exists a constant  $C > 0$  such that for any  $N \in \mathbb{N}$ , for any  $\alpha_1, \dots, \alpha_N < \omega$ , and for any  $x_1, \dots, x_N \in X$  and  $y_1, \dots, y_N \in X^*$ , we have

$$\begin{aligned} & \sum_{k=1}^N \int_{\mathbb{R}} | \langle (\omega - \alpha_k)^m R(\alpha_k + it, A)^{m+1} x_k, y_k \rangle | dt \\ & \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G'(X^*)}. \end{aligned}$$

Now using Lemma 4.6.1, the equality

$$\left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)} = \sup \left\{ \left| \sum_{k=1}^N \langle x_k, y_k \rangle \right| : \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G'(X^*)} \leq 1 \right\}$$

and the proofs of Theorem 2.3.8 and Theorem 2.4.1 one obtains the following result (which sums up an improvement of Theorem 2.3.8 and Theorem 2.4.1):

**Theorem A.2.** *Let  $X$  be a Banach space and let  $m \geq 1$  be an integer. Let  $A$  be an operator of half-plane type  $\omega$  on  $X$ . The following assertions are equivalent:*

- (i)  $-A$  generates a  $C_0$ -semigroup  $(T_t)_{t \geq 0}$  of  $\gamma$ -type  $\omega$ ;
- (ii) There exists a constant  $C > 0$  such that for all  $N \in \mathbb{N}$ , for all  $x_1, \dots, x_N \in X$ , for all  $y_1, \dots, y_N \in X^*$ , and for all  $\alpha_1, \dots, \alpha_N < \omega$ , the functions  $(t, k) \mapsto \sqrt{\omega - \alpha_k} R(\alpha_k +$

$it, A)x_k$  and  $(t, k) \mapsto \sqrt{\omega - \alpha_k}R(\alpha_k + it, A^*)y_k$  belong to  $\gamma(\mathbb{R} \times \mathbb{N}_N; X)$  and  $\gamma'_+(\mathbb{R} \times \mathbb{N}_N; X^*)$ , respectively, and satisfy

$$\|(t, k) \mapsto \sqrt{\omega - \alpha_k}R(\alpha_k + it, A)x_k\|_{\gamma(\mathbb{R} \times \mathbb{N}_N; X)} \leq C \left\| \sum_{k=1}^N \gamma_k \otimes x_k \right\|_{G(X)}$$

and

$$\|(t, k) \mapsto \sqrt{\omega - \alpha_k}R(\alpha_k + it, A^*)y_k\|_{\gamma'_+(\mathbb{R} \times \mathbb{N}_N; X^*)} \leq C \left\| \sum_{k=1}^N \gamma_k \otimes y_k \right\|_{G'(X^*)};$$

(iii)  $A$  has  $(\gamma\text{-GFS})_{m, \omega}$ ;

(iv)  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $\omega$ ;

(v)  $A$  has a  $\gamma$ - $m$ -bounded functional calculus of type  $\omega$ .

A thorough look at the proof of 2.5.2 convinces us that we have the following result (which is an improvement of Theorem 2.5.2) :

**Theorem A.3.** *Let  $X$  be a Banach space. Let  $-A$  be the generator of a  $C_0$ -semigroup of  $\gamma$ -type  $\omega$  on  $X$ . Then  $A$  has a  $\gamma$ -1-bounded functional calculus of type  $s$  for each  $s < s_0^\gamma(A)$ .*

In sight of the proof of 2.5.3, it is easy to obtain the following improvement of Corollary 2.5.5:

**Corollary A.4.** *Let  $(T_t)_{t \geq 0}$  is a bounded  $C_0$ -semigroup on some Banach space. If there exists  $\delta > 0$  such that  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded, then  $\{e^{-\delta t}T_t : t \geq 0\}$  is  $\gamma$ -bounded for any  $\delta > 0$ .*

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## $C_0$ -semigroupes $\gamma$ -bornés et opérateurs à puissance $\gamma$ -bornées, caractérisations et calculs fonctionnels

Dans un premier temps, on montre l'existence d'opérateurs sectoriels  $A$  bornés de type 0 (respectivement d'opérateurs de Ritt  $T$ ) tels que l'ensemble  $\{e^{-tA} : t \geq 0\}$  n'est pas  $\gamma$ -borné (respectivement l'ensemble  $\{T^n : n \in \mathbb{N}\}$  n'est pas  $\gamma$ -borné).

Dans le second chapitre nous étudions les  $C_0$ -semigroupes  $\gamma$ -bornés sur un espace de Banach. Nous généralisons le Théorème de Gomilko Shi-Feng aux espaces de Banach ce qui nous donne une caractérisation des  $C_0$ -semigroupes  $\gamma$ -bornés. De plus nous étudions le calcul dérivé introduit par Batty Haase et Mubeen dans ce contexte.

Le chapitre suivant est consacré à l'étude des opérateurs qui satisfont une condition appelée dans le mémoire condition de Gomilko Shi Feng. Nous montrons que cette condition est équivalente à différents calculs fonctionnels bornés. Nous étudions aussi les opérateurs à puissances  $\gamma$ -bornées que nous caractérisons par un résultat similaire au cas des  $C_0$ -semigroupes  $\gamma$ -bornés.

Dans le dernier chapitre on s'intéresse au  $C_0$ -semigroupes sur un espace de Hilbert. Notre but est de construire un calcul fonctionnel borné sur une nouvelle algèbre  $\mathcal{A}(\mathbb{C}_+)$  inspirée des algèbres de Figa-Talamanca-Herz. Nous verrons que ce calcul fonctionnel améliore les résultats qui existent déjà sur le sujet. Nous obtenons aussi des résultats sur l'espace des multiplicateurs de Fourier bornés sur l'espace de Hardy  $H^1(\mathbb{R})$  qui sont utiles pour l'étude de l'algèbre  $\mathcal{A}(\mathbb{C}_+)$ .

**Mots-clés :**  $C_0$ -semigroupes; opérateurs à puissances bornés;  $\gamma$ -bornitude; multiplicateur de Fourier; espaces de Besov; calcul fonctionnel.

**Classification AMS :** 47A99, 46B15, 47A60, 47D06, 46B28, 42B35, 30H05.

## $\gamma$ -bounded $C_0$ -semigroups and power $\gamma$ -bounded operators: characterizations and functional calculi

First and foremost we show that there exist bounded sectorial operators  $A$  of type 0 (respectively Ritt operators  $T$ ) such that the set  $\{e^{-tA} : t \geq 0\}$  is not  $\gamma$ -bounded (respectively the set  $\{T^n : n \in \mathbb{N}\}$  is not  $\gamma$ -bounded).

In the second chapter, we study  $\gamma$ -bounded  $C_0$ -semigroups on Banach spaces. We will be able to generalize Gomilko Shi-Feng Theorem in Banach settings. This generalization gives us a characterization of  $\gamma$ -bounded  $C_0$ -semigroups. Further, in this context, we study the derivative bounded functional calculus introduced by Batty Haase and Mubeen.

The next chapter is dedicated to operators which satisfy a condition called discrete Gomilko Shi-Feng condition. We show that this condition is equivalent to various bounded functional calculi. We also study power  $\gamma$ -bounded operators and we characterize them in a similar way as for  $\gamma$ -bounded  $C_0$ -semigroups.

In the final chapter, we focus on  $C_0$ -semigroups on Hilbert space. Our goal is to construct a bounded functional calculus on a new algebra  $\mathcal{A}(\mathbb{C}_+)$  inspired by Figa-Talamanca-Herz algebras. We show that this bounded functional calculus improves existing results. We also get results about bounded Fourier multipliers on the Hardy space  $H^1(\mathbb{R})$  which are useful for the study of  $\mathcal{A}(\mathbb{C}_+)$ .

**Key words :**  $C_0$ -semigroups; power bounded operators; Fourier multipliers; Besov spaces; functional calculus.

**AMS Classification :** 47A99, 46B15, 47A60, 47D06, 46B28, 42B35, 30H05.