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Contributions to the analysis of communications systems in the finite blocklength regime

Dadja Toussaint Anade Akpo

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Abstract

IN this thesis, a real-valued function that approximates the cumulative distribution function (CDF) of a finite sum of real-valued independent random vectors is presented. The approximation error is upper bounded and thus, as a byproduct, an upper bound and a lower bound on the CDF are obtained. It is observed that in the case of lattice and absolutely continuous random variables, the proposed approximation is identical to the saddlepoint approximation of the CDF. This result is used to approximate decoding error probability (DEP) bounds for the point to point and Multiple Access channels (MAC). For the point to point channel, this result has pointed out the impertinence of the normal approximation, especially for small values of the DEP. For the MAC, the introduction of the notion of the individual (per user) DEP has revealed the almost non-interference between transmitters in the regime of small values of DEP and blocklength that is not captured by the system DEP.

Résumé

DANS cette thèse, une fonction qui approxime la fonction de répartition d'une somme de vecteurs aléatoires indépendants et identiquement distribués est présentée. L'erreur d'approximation est majorée, et par conséquent, une borne supérieure et une borne inférieure sur la fonction de répartition sont obtenues. Pour des vecteurs aléatoires absolument continus ou lattices, l'approximation proposée est identique à l'approximation du point de selle de la fonction de répartition. Ce résultat est utilisé pour approcher les bornes de probabilité d'erreur de décodage pour les canaux point à point et à accès multiple. Sur le canal point à point, cette approche a permis de constater l'insuffisance de l'approximation normale, particulièrement pour des probabilité d'erreur de décodage de faibles valeurs. Concernant les canaux à accès multiple, la considération de la notion d'erreur individuel a révélé le comportement presque non interférant des transmetteurs pour des petites valeurs de la probabilité d'erreur de décodage et de la longueur des paquets.

Publications

Journals

- [1] “*Fundamental Limits of Non Orthogonal Multiple Access (NOMA) for the Massive Gaussian Broadcast Channel in Finite Block-Length*”. Jean-Marie Gorce, Philippe MARY, Dadja Anade, and Kélif, Jean-Marc. MDPI Special Issue: Massive and Reliable Sensor Communications with LPWANs Technologies, vol. 21 no. 3, Jan., 2021.
- [2] “*An Upper Bound on the Error Induced by Saddlepoint Approximations - Applications to Information Theory*”. Dadja Anade, Jean-Marie Gorce, Philippe MARY, and Samir M. Perlaza. MDPI Entropy, vol. 22 no. 6, pp. 690, Jun., 2020.

Conferences

- [1] “*Saddlepoint Approximation of Cumulative Distribution Functions of Sums of Random Vectors*”. Dadja Anade, Jean-Marie Gorce, Philippe MARY, and Samir M. Perlaza. IEEE Information Symposium on Information Theory (ISIT), Melbourne, Victoria, Australia, Jul., 2021.
- [2] “*On the saddlepoint approximation of the dependence testing bound in memoryless channels*”. Dadja Anade, Jean-Marie Gorce, Philippe MARY, and Samir M. Perlaza. IEEE International Conference on Communications (ICC), Dublin, Ireland, Jun., 2020.

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- [1] “*Saddlepoint Approximations of Cumulative Distribution Functions of Sums of Random Vectors*”. Dadja Anade, Jean-Marie Gorce, Philippe MARY, and Samir M. Perlaza. Technical Report, INRIA, No. 9388, Lyon, France, Feb., 2021.
- [2] “*An upper bound on the error induced by saddlepoint approximations - Applications to information theory*”. Dadja Anade, Jean-Marie Gorce, Philippe MARY, and Samir M. Perlaza. Technical Report, INRIA, No. 9329, Lyon, France, Feb., 2020.

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- [1] “*Nouvelle borne atteignable de la probabilité d'erreur pour des transmissions en paquets courts*”. Dadja Toussaint Anade Akpo, Jean-Marie Gorce, and Philippe MARY. GRETSI 2019 - XXVIIème Colloque francophonede traitement du signal et des images, Lille, Aug., 2019.

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Acronyms

A-IDEP	Average Individual Decoding Error Probability
ASαS	Additive Symmetric α -Stable
A-SDEP	Average System Decoding Error Probability
AWGN	Additive White Gaussian Noise
BSC	Binary Symmetric Channel
CDF	Cumulative Distribution Function
CGF	Cumulant Generating Function
DEP	Decoding Error Probability
DMC	Discrete Memoryless Channel
DT	Dependence Testing
IDEP	Individual Decoding Error Probability
IID	Independent and Identically Distributed
IoT	Internet of Things
MAC	Multiple Access Channels
MC	Meta Converse
NOMA	Non-Orthogonal Multiple Access
OMA	Orthogonal Multiple Access
PDF	Probability Density Function
PMF	Probability Mass Function
SαS	Symmetric α -Stable
SDEP	System Decoding Error Probability
RCU	Random Coding Union
URLLC	Ultra Reliable Low Latency Communication

Notation

Sets are represented by calligraphic letters, e.g, \mathcal{X} . Sets, whose elements are sets, are represented by script letters, e.g, \mathcal{X} . The empty set is denoted by \emptyset . The sets of all integer numbers, all real numbers, and all complex numbers are respectively denoted by \mathbb{N} , \mathbb{R} , and \mathbb{C} . In particular, $0 \notin \mathbb{N}$. The Borel sigma field on \mathbb{R}^k , with $k \in \mathbb{N}$, is denoted by $\mathcal{B}(\mathbb{R}^k)$. The Lebesgue measure on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ is denoted by ν_k . Given a discrete set \mathcal{K} , the biggest sigma field, i.e., the set of all its subsets, is denoted by $2^{\mathcal{K}}$. The Euclidian norm in \mathbb{R}^k is denoted by $\|\cdot\|$. Given a set $\mathcal{A} \subseteq \mathbb{R}^k$, the closure of the set \mathcal{A} , denoted by $\text{clo}\mathcal{A}$, is defined by $\text{clo}\mathcal{A} \triangleq \{\mathbf{x} \in \mathbb{R}^k : \forall r > 0, \exists \mathbf{y} \in \mathcal{A}, \|\mathbf{x} - \mathbf{y}\| < r\}$. A diagonal matrix whose diagonal is the vector $\mathbf{x} \in \mathbb{R}^k$ is denoted by $\text{diag}(\mathbf{x})$.

Random variables and their realizations are respectively represented by uppercase and lowercase letters, e.g, X and x . Random vectors and their realizations are respectively represented by uppercase and lowercase boldface letters, e.g, \mathbf{X} and \mathbf{x} . Random matrices and their realizations are respectively represented by uppercase and lowercase boldface underline letters, e.g, $\underline{\mathbf{X}}$ and $\underline{\mathbf{x}}$.

The probability measure induced by a random variable/vectors/matrix is denoted by the letter P indexed by the random variable/vector/matrix, e.g, P_X , $P_{\mathbf{X}}$, and $P_{\underline{\mathbf{X}}}$. When an additional measure is needed for the same random variable, the letter Q is used in the place of P to differentiate between measures and avoid confusion. The conditional probability measure induced by a random variable Y given another random vector X is denoted by $P_{Y|X}$. The same notation for conditional probability measures holds for any combination of random variable, vector and matrix. The set of all probability measures, whose support is subset of a given set, is represented by Δ (*< given set >*), e.g, $\Delta(\mathcal{S})$ represents the set of all probability measures whose supports are subsets of \mathcal{S} .

The logarithm function \ln is assumed to be in base e and is denoted by \ln .



Introduction

SMART and autonomous are now common adjectives; for instance, smartphone, smart/autonomous car, smart home, and smart environment to cite a few. These qualifications designate the ability of an object to exchange information with its surroundings and to consequently act with respect to a given objective. The Internet of Things (IoT) and Internet of Everything (IoE) are seen as first steps towards smart environments thanks to the importance of a communication interface to support information exchange between each object. Even though the acronym IoT appeared many more than a decade ago, many challenges remain before the IoT becomes a reality.

The advent of the IoT challenges key assumptions present in traditional communication systems. The first assumption is the number of users, which are dramatically large to allow massive numbers of objects to communicate. Companies like Cisco and SigFox forecast many billions of new objects to be connected [1]. The multiplication of users leads to a scarcity of resources and questions how the latter must be allocated. For example, a key question is whether the existing orthogonal multiple access (OMA) protocols are sufficient, or if non-orthogonal multiple access (NOMA) protocols are required?

The second assumption is the number of different types of users in the network. Heterogeneity among users may lead to a loss in quality of service (QoS). For example, an autonomous car may have a lower latency requirement than a temperature sensor that must send its measurement message each hour. A telemedicine system may have higher reliability requirement than a video surveillance system. Due to the expected simultaneous access of the channel by a large number of users, which may exhibit some heterogeneity, the decoding error probability (DEP) may need refinement as it does not capture the requirement of individual objects. For instance, must the IoT minimize the system error associated with all users or to ensure some success probability to each user separately?

Third, the simultaneous access of the channel by different users may impact the distribution of the interference. Indeed, the interference produced by a set of sensors is observed in [1] to exhibit impulsivity, which does not align with Gaussian assumption.

The failure of standard hypotheses in the IoT setting leads to the question of whether traditional protocols are sufficient. To determine whether new protocols are required, a key

approach is to characterize theoretical limits of IoT communication.

The theoretical study of communication systems relies on information theory, which was started by Shannon in [2]. The tools developed in information theory for the analysis of communication systems can be divided in three groups: (a) the channel (Shannon) capacity (Asymptotic regime), (b) general bounds (Non-asymptotic regime), and (c) second order achievability (Almost non-symptotic regime).

The channel capacity is defined as the maximum information rate at which it is possible to transmit with an arbitrary low DEP when communication is allowed to take infinite time. As consequence, the channel capacity does not allow to study the latency constraint due to an infinite time transmission consideration.

General bounds are the proposition of upper and lower bounds on the DEP that are developed with no particular assumption on the latency or reliability constraints. Such bounds are the answer of information theorists [3–6] to overcome the limitation of the channel capacity framework. Despite the generality of the proposed bounds, their evaluation complexity makes them useless without an approximation. Indeed, as it will be figured out in Chapter 2, such bounds are functions of unknown cumulative distribution functions (CDFs) of random vectors. The blind computation of such CDFs is impossible due to the high dimension of the random vectors.

The second order achievability has been a middle point between the channel capacity and general bounds. The second order achievability results from the approximation of general bounds, which is a consequence of the normal (Gaussian) approximation of CDFs in these bounds. These approximations are obtained under memoryless and stationary assumptions that allow to work with sums of independent and identically distributed (IID) random vectors and then to apply the normal approximation. As a consequence, the second order achievability suffers from the limitation of the normal approximation that fails on the CDF tail and on a sum with few IID random vectors. The CDF tail corresponds to the regime of low values of DEP that is equivalent to a high reliability requirement. Few IID random vectors correspond to the regime of small channel uses that is equivalent to the low latency regime. Hence, the second order achievability is not suitable to analyze ultra-reliable low latency communication (URLLC), which is a critical component for autonomous vehicle networks and for telemedicine systems.

The previous paragraphs have shown limitations of the tools developed by information theorists for understanding URLLC systems, which are part of the IoT. The objective of this thesis is to reduce the limitations due to the normal approximation by introducing a new family: Exponentially tilted Gaussian approximations of CDFs (Chapter 3 and Chapter 4). This family is obtained from a change of measure that introduced a parametrized random vector, which is also a sum of IID random vectors, followed by its normal approximation. Thus, the normal approximation is applied to the intermediate random vector in the place of the initial summation of IID random vectors. Due to this parametrization and the proposition of an upper bound on the induced approximation error, the choice of intermediate random vector can be optimized. Such an optimization seems to overcome the limitations of the normal approximation on the CDF tail but also for sums with few IID random vectors. The application of exponentially tilted Gaussian approximations to the general bounds (Chapter 5) leads to easy computable bounds. The contributions of this thesis are:

- A proposition of the exponentially tilted Gaussian approximations of CDFs of sums of independent random vectors and a characterization of the induced error by providing

an explicit upper bound. Gaussian and saddlepoint approximations of CDFs are shown to be particular cases of exponentially tilted Gaussian approximations. The non-lattice discrete random vectors were covered by the exponentially tilted Gaussian approximation compared to the saddlepoint approximation that is constrained to lattice random vectors. Chapter 3 and Chapter 4 focus on the exponentially tilted Gaussian approximations of CDFs of sums of independent random variable and vectors, respectively.

- The exponentially tilted Gaussian approximation is used to provide easy computable bounds on DEPs point-to-point channels and multiple access channels (MAC) and are presented Chapter 5.
- A formalization of the notion of individual (per user) DEP for MAC presented in Chapter 2.

The sequel of the manuscript is organized as follows: Chapter 2 reviews general bounds on DEPs for point-to-point channels and MAC. The notion of individual error is also formalized in Chapter 2 with the proposition of upper bounds on DEPs. Chapter 3 and Chapter 4 present the exponentially tilted Gaussian approximations of CDFs of sums of independent random variables and vectors, respectively. Chapter 5 is an application chapter based on the theoretical materials presented in the previous chapters and it provides easy computable bounds on DEPs. Chapter 6 concludes this work and discusses points of improvement of our contributions.

— 2 —

System Models and the State of the Art

THIS chapter introduces the point to point channel and the MAC and reviews the main results on the DEP, achievability and converse bounds, that have been proposed in the literature. A contribution on the improvement of the dependence testing (DT) bound for the point to point channel is additionally presented. This chapter also formulates an alternative approach to tackle the fundamental bounds of the MAC by considering the individual DEP (IDEP). Usually, the DEP in the MAC is considered as what we call the system DEP (SDEP), i.e. an error occurs if at least one user is wrongly decoded. Unfortunately, this definition is not pertinent for applications with many transmitting users. The consideration of the IDEP, i.e. the DEP of each transmitter in the network, is far more insightful to design systems with high connectivity. It is worth mentioning that this vision has been completely eluded from the information theory community and we have been among the first people to get interest about it in the ANR ARBURST project, independently followed by the work of Kowshik and Polyanskiy [7].

The chapter is organized as follows. Section 2.1 focuses on the point to point channel and presents the main results on the achievable DEP (RCU, DT bound) and the converse (meta converse (MC) bound). Section 2.2 deals with the MAC system model and presents the multi-user RCU and DT bounds for the achievable DEP (system and individual) as well as the multi-user version of the meta converse bound.

2.1. Point-to-Point Channels

Starting from the system model formalization of the point-to-point channel in Sub-section 2.1.1, lower and upper bounds on the DEP are reviewed and shown to be related of the CDFs of information density random variables in Sub-section 2.1.2. The evaluation difficulty of these bounds due to unknown CDFs are discussed, which leads to Subsection 2.1.3 on the approximations of the DEP, mainly the normal approximation. Finally, Sub-section 2.1.4

discusses the complexity of the bound evaluation and the limitation of the state of the art (normal) approximations that leads to the development of exponentially tilted Gaussian approximations in Chapter 3.

2.1.1. System Model

Consider a point to point communication in which a transmitter aims at sending information to one receiver through a noisy channel. Such a channel can be modelled by a random transformation

$$(\mathcal{X}^n, \mathcal{Y}^n, P_{\mathbf{Y}|\mathbf{X}}), \quad (2.1)$$

where $n \in \mathbb{N}$ is the number of channel uses, \mathcal{X} and \mathcal{Y} are respectively the channel input set and output set. Given the channel inputs $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top \in \mathcal{X}^n$, the output is a random vector $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)^\top$ that induces the probability measure

$$P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}} \quad (2.2)$$

on the measurable space $(\mathcal{Y}^n, \mathcal{B}(\mathcal{Y}^n))$.

The objective of the communication is to transmit a message index i , which is a realization of a random variable W that is uniformly distributed over the set

$$\mathcal{W} \triangleq \{1, 2, \dots, M\}, \quad (2.3)$$

with $M \in \mathbb{N}$, $M > 1$. To achieve this objective, the transmitter uses an (n, M) -code.

Definition 1 ((n, M) -code) *Given a tuple $(M, n) \in \mathbb{N}^2$, an (n, M) -code for the random transformation in (2.1) is a system*

$$\left\{ \left(\mathbf{u}(1), \mathcal{D}(1) \right), \left(\mathbf{u}(2), \mathcal{D}(2) \right), \dots, \left(\mathbf{u}(M), \mathcal{D}(M) \right) \right\}, \quad (2.4)$$

where for all $(j, \ell) \in \mathcal{W}^2$, with $j \neq \ell$:

$$\mathbf{u}(j) = (u_1(j), u_2(j), \dots, u_n(j))^\top \in \mathcal{X}^n, \quad (2.5a)$$

$$\mathcal{D}(j) \cap \mathcal{D}(\ell) = \emptyset, \text{ and } \bigcup_{j \in \mathcal{W}} \mathcal{D}(j) \subseteq \mathcal{Y}^n. \quad (2.5b)$$

Remark 1 *Given an (n, M) -code $\mathcal{C}_{n,M} \triangleq \{(\mathbf{u}(1), \mathcal{D}(1)), \dots, (\mathbf{u}(M), \mathcal{D}(M))\}$, the codebook is the set of codewords $\mathbf{u}(1), \dots, \mathbf{u}(M)$.*

Given an (n, M) -code $\mathcal{C}_{n,M} \triangleq \{(\mathbf{u}(1), \mathcal{D}(1)), \dots, (\mathbf{u}(M), \mathcal{D}(M))\}$, to transmit the message index $i \in \mathcal{W}$, the transmitter uses the codeword $\mathbf{u}(i)$. At channel use t , for all $t \in \{1, 2, \dots, n\}$, the transmitter inputs the symbol $u_t(i)$ into the channel. Assume that, at the end of channel use t , the receiver observes the output y_t . After n channel uses, the receiver observes the vector $\mathbf{y} = (y_1, y_2, \dots, y_n)^\top$. The receiver determines that the symbol j was transmitted if $\mathbf{y} \in \mathcal{D}(j)$, with $j \in \mathcal{W}$.

A decoding error occurs when the codeword $\mathbf{u}(i)$, with $i \in \mathcal{W}$, is transmitted and the received vector \mathbf{y} satisfies:

$$\mathbf{y} \notin \mathcal{D}(i). \quad (2.6)$$

The DEP associated to the transmission of the message index i can be computed as

$$\mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}(i)\}}], \quad (2.7)$$

where $P_{\mathbf{Y}|\mathbf{X}}$ is the random transformation in (2.1).

The average DEP associated to the (n, M) -code $\mathcal{C}_{n, M}$, denoted by $\lambda(\mathcal{C}_{n, M})$, can be computed as

$$\lambda(\mathcal{C}_{n, M}) \triangleq \frac{1}{M} \sum_{i=1}^M \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}(i)\}}], \quad (2.8)$$

where $P_{\mathbf{Y}|\mathbf{X}}$ is the random transformation in (2.1).

Given $(n, M) \in \mathbb{N}^2$, the minimum average DEP associated to the random transformation in (2.1) is defined hereunder.

Definition 2 *Given a pair $(n, M) \in \mathbb{N}^2$, the minimum average DEP for the random transformation in (2.1), denoted by $\lambda^*(n, M)$, is defined as:*

$$\lambda^*(n, M) = \inf_{\mathcal{C}_{n, M} \in \mathcal{C}_{n, M}} \lambda(\mathcal{C}_{n, M}), \quad (2.9)$$

where $\lambda(\mathcal{C}_{n, M})$ is defined in (2.8) and $\mathcal{C}_{n, M}$ is the set of all (n, M) -codes.

The minimum average DEP (2.9) is a fundamental limit of the random transformation in (2.1). Indeed, it is possible to construct an (n, M) -code $\mathcal{C}_{n, M}$ for the random transformation in (2.1) with average DEP $\lambda(\mathcal{C}_{n, M}) > \lambda^*(n, M)$. However, the construction of an (n, M) -code for the random transformation in (2.1) with average DEP $\lambda(\mathcal{C}_{n, M}) < \lambda^*(n, M)$ is impossible.

The objective of many works in information theory [4–6, 8] consists in determining this fundamental limit of the point to point channel represented by the random transformation in (2.1).

The minimum average DEP, $\lambda^*(n, M)$ in (2.9), is an increasing function of M for a fixed value of n . Such a claim is based on the observation that for all (n, M) -code achieving $\lambda^*(n, M)$, an (n, M') -code, with $M' < M$, can be constructed using the following steps. $(M - M')$ codewords with the highest DEPs are removed and their decoding sets are merged with the decoding set of the codeword with the lowest DEP. Denote this new code by $\mathcal{C}_{n, M'}$, this construction leads to an average DEP $\lambda(\mathcal{C}_{n, M'})$ that satisfies $\lambda(\mathcal{C}_{n, M'}) \leq \lambda^*(n, M)$. This implies $\lambda^*(n, M') \leq \lambda^*(n, M)$.

The monotonicity of $\lambda^*(n, M)$ with respect to M allows to observe that the maximum M for which $\lambda^*(n, M) \leq \epsilon$, with $\epsilon \in [0, 1]$, is also a fundamental limit of the point to pint channel. In order to study the latter fundamental limit, it is helpful to consider the definition of an (n, M, ϵ) -code.

Definition 3 *(n, M, ϵ) -code* Given a tuple $(M, n, \epsilon) \in \mathbb{N}^2 \times [0, 1]$, an (n, M, ϵ) -code for

the random transformation in (2.1) is an (n, M) -code $\mathcal{C}_{n,M}$ whose average DEP satisfies

$$\lambda(\mathcal{C}_{n,M}) \leq \epsilon. \quad (2.10)$$

Using the Definition 3, it can be shown that $\lambda^*(n, M)$ in (2.9) satisfies

$$\lambda^*(n, M) = \inf \{ \epsilon \in [0, 1] : \exists (n, M, \epsilon)\text{-code} \}. \quad (2.11)$$

An (n, M, ϵ) -code is said to transmit at an information rate

$$R = \frac{\log_2(M)}{n} \quad (2.12)$$

bits per channel use.

For a fixed value of n , studying R or M is equivalent. Operationally, R is more interesting than M since it is directly connected to the information rate. Hence, the study of the maximum of R instead of M will be considered in the sequel. To go further in this study, consider the following definition.

Definition 4 (Capacity region) *Given a blocklength n and an average DEP ϵ , the capacity region of the random transformation in (2.1), denoted by $\mathcal{A}^*(n, \epsilon)$, is defined as:*

$$\mathcal{A}^*(n, \epsilon) \triangleq \{ R \in \mathbb{R} : \exists (n, 2^{nR}, \epsilon)\text{-code} \}. \quad (2.13)$$

The supremum of $\mathcal{A}^*(n, \epsilon)$ is called the (ϵ, n) -capacity and is denoted by $R^*(n, \epsilon)$:

$$R^*(n, \epsilon) = \sup \{ R \in \mathbb{R} : \exists (n, 2^{nR}, \epsilon)\text{-code} \}. \quad (2.14)$$

The determination of $R^*(n, \epsilon)$ relies on the knowledge of $\lambda^*(n, M)$ in (2.9), which requires to solve a difficult problem (2.9). Indeed, the evaluation of the average DEP $\lambda(\mathcal{C}_{n,M})$ in (2.8) relies on the evaluation of all expectations $\mathbb{E}_{P_{Y|X=u(i)}} [\mathbb{1}_{\{Y \notin \mathcal{D}(i)\}}]$, with $i \in \mathcal{W}$, that are generally difficult to calculate. For instance in [3], an additive white Gaussian noise (AWGN) channel, where input codewords belong to the power shell and decoding sets $\mathcal{D}(i)$ are n -dimensional pyramids, with $i \in \mathcal{W}$, is considered. Therein, the evaluation of $\mathbb{E}_{P_{Y|X=u(i)}} [\mathbb{1}_{\{Y \notin \mathcal{D}(i)\}}]$ turns out to be difficult. As result, $\mathcal{D}(i)$, with $i \in \mathcal{W}$, are approximated by n -dimensional cones in order to facilitate the evaluation. Furthermore, solving (2.9) may require to evaluate $\lambda(\mathcal{C}_{n,M})$ for all (n, M) -codes $\mathcal{C}_{n,M}$. For example with $M = 5$, $n = 10$, and the size $|\mathcal{X}| = 2$ of the input alphabet \mathcal{X} , there exists a least $\binom{|\mathcal{X}|^n}{M} = 9.2912 \cdot 10^{12}$, which is a big number.

The difficult evaluation (2.9) calls for the proposition of upper and lower bounds on $\lambda^*(n, M)$ in (2.9). These bounds allow to characterise the (ϵ, n) -capacity.

The next section summarizes some of the existing upper and lower bounds on $\lambda^*(n, M)$ in (2.9).

Remark 2 *Throughout this thesis, an upper bound on the minimum average DEP will be referred to as an achievability bound and a lower bound as a converse bound.*

2.1.2. Bounds on the Decoding Error Probability

The study of achievability bounds and converse bounds dates back to the works in [2, 3, 5, 8–10] and was recently renewed in [6]. Starting from the definition of the minimum average DEP

in (2.9), any upper bound on the average DEP of any (n, M) -code is an achievability bound. Thus, an achievability bound is derived by assuming an (n, M) -code and deriving an upper bound on its average DEP. However, a converse is a lower bound of all average DEPs of all (n, M) -codes and its derivation must consider all possible (n, M) -codes, which is often difficult.

Achievability Bounds

Evaluating the average DEP of a given (n, M) -code, i.e. $\lambda(\mathcal{C}_{n,M})$ in (2.8), is generally difficult due to the interplay between the decoding sets $\mathcal{D}(i)$, for all $i \in \mathcal{W}$, which form a partition of \mathcal{Y}^n . This interdependence between the decoding sets may lead to heterogeneity of methods to evaluate each $\mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}(i)\}}]$ in (2.8), with $i \in \mathcal{W}$. To avoid using a different method for each decoding set, a randomization has been introduced in [2, 3] to bring some homogeneity. The randomization consists in studying a set of (n, M) -codes instead of a particular (n, M) -code. The main interest is to study the mean of the average DEPs across this set. Indeed, the mean is greater than the minimum. Thus, the mean is an achievability bound. The main idea consists in creating a symmetry between codewords through a rotation among them. From an (n, M) -code to another, the same codeword can be associated to a different message index. The same holds for the decoding sets. Following this process, a symmetry is created between decoding sets associated to different message indices that breaks the heterogeneity among them. Such a process is known as random coding and makes it easier to deal with the average of $\lambda(\mathcal{C}_{n,M})$ in (2.8) by only concentrating on the average of $\mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}(i)\}}]$ with respect to a probability measure $P_{\mathbf{X}}$ on the codeword set \mathcal{X}^n . As a result of the induced symmetry, the average of $\mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}(i)\}}]$ becomes a constant function of i .

Depending on the method used to construct the decoding sets, the random coding leads to different DEPs. In this thesis, two methods that summarize the majority of the decoding set construction are presented: (a) the maximum likelihood and (b) a threshold decoding.

A decoder (decoding sets) is said to rely on the maximum likelihood if for a given signal $\mathbf{y} \in \mathcal{Y}^n$, the estimated message index \hat{m} satisfies:

$$\hat{m} \in \left\{ i \in \mathcal{W} : P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{u}(i)) = \max_{m \in \mathcal{W}} P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{u}(m)) \right\}. \quad (2.15)$$

A decoder is said to be based on a threshold decoding if for a given signal $\mathbf{y} \in \mathcal{Y}^n$, the estimated message index \hat{m} satisfies:

$$\hat{m} \in \{ i \in \mathcal{W} : P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{u}(i)) > t(\mathbf{y}, \mathbf{u}(i), i) \}, \quad (2.16)$$

where the function $t : \mathcal{Y}^n \times \mathcal{X}^n \times \mathcal{W} \rightarrow \mathbb{R}_+$ is a threshold function.

Note that decoding sets based on the maximum likelihood lead to a lower average DEP than those based on a threshold decoding. However, the average DEP based on the maximum likelihood is more complex than the average DEP based on a threshold decoding.

These two methods of decoding set construction respectively lead to random coding union (RCU) bound [6, Theorem 16] for the maximum likelihood and dependence testing (DT) bound for a threshold decoding [6, Theorem 18].

To ease the presentation of these bounds, consider the following notation. For all $\mathbf{x} \in \mathcal{X}^n$, $\mathbf{y} \in \mathcal{Y}^n$ and for all probability measures $Q_{\mathbf{Y}} \in \Delta(\mathcal{Y}^n)$, let the function $\tilde{l} : \mathcal{X}^n \times \mathcal{Y}^n \times \Delta(\mathcal{Y}^n) \rightarrow \mathbb{R}$ be

$$\tilde{l}(\mathbf{x}; \mathbf{y} | Q_{\mathbf{Y}}) \triangleq \ln \left(\frac{dP_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}}{dQ_{\mathbf{Y}}}(\mathbf{y}) \right), \quad (2.17)$$

where

(i) for all $\mathbf{y} \in \text{supp}Q_{\mathbf{Y}}$, the function $\frac{dP_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}}{dQ_{\mathbf{Y}}} : \mathcal{Y}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is equal to the Radon–Nikodym derivative of the measure $\underline{P}_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}$ with respect to the probability measures $Q_{\mathbf{Y}}$, with $\underline{P}_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}$ defined as:

$$\underline{P}_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}(\mathcal{S}) = P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}(\mathcal{S}) \text{ for all } \mathcal{S} \subseteq \text{supp}Q_{\mathbf{Y}}, \text{ and} \quad (2.18a)$$

$$\underline{P}_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}(\{\mathbf{y} \in \mathbb{R} : \mathbf{y} \notin \text{supp}Q_{\mathbf{Y}}\})=0. \quad (2.18b)$$

(ii) for all $\mathbf{y} \in \{\mathbf{t} \in \text{supp}P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}} : \mathbf{t} \notin \text{supp}Q_{\mathbf{Y}}\}$,

$$\frac{dP_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}}{dQ_{\mathbf{Y}}}(\mathbf{y}) = \infty. \quad (2.18c)$$

Note that the definition of the $\tilde{l}(\mathbf{x}; \mathbf{y}|Q_{\mathbf{Y}})$ in (2.17) appears also in [11] as “the density of the modified mutual information” due to the change of measure from induced output probability measure $P_{\mathbf{Y}}$ to a chosen probability measure $Q_{\mathbf{Y}}$.

For all $(n, \gamma) \in \mathbb{N} \times \mathbb{R}$ and for all probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$ and $Q_{\mathbf{Y}} \in \Delta(\mathcal{Y}^n)$, let the function $T : \mathbb{N} \times \Delta(\mathcal{X}^n) \times \Delta(\mathcal{Y}^n) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be

$$T(n, P_{\mathbf{X}}, Q_{\mathbf{Y}}, \gamma) \triangleq \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbb{1}_{\{\tilde{l}(\mathbf{X}; \mathbf{Y}|Q_{\mathbf{Y}}) \leq \ln(\gamma)\}} \right] + \gamma \mathbb{E}_{P_{\mathbf{X}}Q_{\mathbf{Y}}} \left[\mathbb{1}_{\{\tilde{l}(\mathbf{X}; \mathbf{Y}|Q_{\mathbf{Y}}) > \ln(\gamma)\}} \right]. \quad (2.19)$$

Using these notations, the following lemmas introduce the RCU bound and the DT bound, respectively.

Lemma 1 (RCU bound [6, Theorem 16]) *Given a pair $(n, M) \in \mathbb{N}^2$, the following holds for all probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$, with respect to the random transformation in (2.1):*

$$\lambda^*(n, M) \leq \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\min \left\{ 1, (M-1) \mathbb{E}_{P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{\tilde{l}(\mathbf{X}; \mathbf{Y}|P_{\mathbf{Y}}) \leq \tilde{l}(\bar{\mathbf{X}}; \mathbf{Y}|P_{\mathbf{Y}})\}} \right] \right\} \right], \quad (2.20)$$

where the function \tilde{l} is defined in (2.17); the probability measure $P_{\mathbf{Y}}$ is the marginal of $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}$; and the random variable $\bar{\mathbf{X}}$ is identically distributed as \mathbf{X} but independent of \mathbf{X} and \mathbf{Y} ;

For given constraints on the codewords, e.g. power, amplitude constrains, the upper bound in (2.20) must be considered with respect to the input probability measures $P_{\mathbf{X}}$ that satisfy these constraints instead of all $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$.

Note that the right hand side (RHS) in (2.20) can be minimized with respect to the input probability measures $P_{\mathbf{X}}$ under the constraints on the codewords to obtain a tighter upper bound.

Comments 1 *Unfortunately, the bound in Lemma 1 is difficult to evaluate. The evaluation of the RCU bound can be facilitated if the cumulative distribution function (CDF) of the random variables $\tilde{l}(\mathbf{x}; \mathbf{y}|P_{\mathbf{Y}}) - \tilde{l}(\bar{\mathbf{X}}; \mathbf{y}|P_{\mathbf{Y}})$ for all $\mathbf{x} \in \mathcal{X}^n$ and $\mathbf{y} \in \mathcal{Y}^n$ is known. In this particular case, the evaluation of the RHS in (2.20) reduces to the expectation of a function of this CDF with respect to $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}$ that is still difficult to compute. Generally, this CDF is not known, and the evaluation of RHS in (2.20) requires the evaluation of $3 \times n$ -dimensional integral:*

$$\begin{aligned} & \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\min \left\{ 1, (M-1) \mathbb{E}_{P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{\tilde{l}(\mathbf{X}; \mathbf{Y}|P_{\mathbf{Y}}) \leq \tilde{l}(\bar{\mathbf{X}}; \mathbf{Y}|P_{\mathbf{Y}})\}} \right] \right\} \right] \\ &= \int_{\mathcal{X}^n} \int_{\mathcal{Y}^n} \min \left\{ 1, (M-1) \int_{\mathcal{X}^n} \mathbb{1}_{\{\tilde{l}(\mathbf{x}; \mathbf{y}|P_{\mathbf{Y}}) \leq \tilde{l}(\bar{\mathbf{x}}; \mathbf{y}|P_{\mathbf{Y}})\}} P_{\bar{\mathbf{X}}}(\mathrm{d}\bar{\mathbf{x}}) \right\} P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}(\mathrm{d}\mathbf{y}) P_{\mathbf{X}}(\mathrm{d}\mathbf{x}). \quad (2.21) \end{aligned}$$

This integration is difficult due to the number of the dimension ($3n$). Hence, the proposition of another bound on the average DEP based on a threshold decoding.

Lemma 2 (DT bound [6, Theorem 18]) *Given a pair $(n, M) \in \mathbb{N}^2$, the following holds for all probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$, with respect to the random transformation in (2.1):*

$$\lambda^*(n, M) \leq T\left(n, P_{\mathbf{X}}, P_{\mathbf{Y}}, \frac{M-1}{2}\right), \quad (2.22)$$

where the function T is defined in (2.19) and $P_{\mathbf{Y}}$ is the marginal of the joint probability measure $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}$.

For given constraints on the codewords, the upper bound in (2.22) must be considered with respect to the input probability measures $P_{\mathbf{X}}$ that satisfy these constraints. The upper bound in (2.22) can be minimized with respect to the input probability measures $P_{\mathbf{X}}$ that satisfy the constraints on the codewords.

The bound in Lemma 2 suggests to use a constant threshold for decoding. However, as some codewords are discarded along the sequential decoding process, the error probability also evolves and a tighter bound can be obtained by considering an adaptive threshold according to the number of codewords that remains to test. The following lemma summarises this result [12].

Lemma 3 *Given a pair $(n, M) \in \mathbb{N}^2$, the following holds for all probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$, with respect to the random transformation in (2.1):*

$$\lambda^*(n, M) \leq \frac{1}{M} \sum_{i=1}^M T(n, P_{\mathbf{X}}, P_{\mathbf{Y}}, M-i), \quad (2.23)$$

where the function T is defined in (2.19) and $P_{\mathbf{Y}}$ is the marginal of the joint probability measure $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}$.

Proof: The proof is presented in Appendix A. ■

Note that the improvement of the bound (2.23) with respect to the bound in (2.22) is a consequence of the concavity of the function $T(n, P_{\mathbf{X}}, P_{\mathbf{Y}}, \gamma)$ in γ .

Comments 2 *The evaluation of the function T in (2.19) is facilitated if the CDFs of the random variable $\tilde{i}(\mathbf{X}; \mathbf{Y}|Q_{\mathbf{Y}})$ under the probability measures $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}$ and $P_{\mathbf{X}}Q_{\mathbf{Y}}$ are known. In this particular case, the evaluations of the first and second expectations in RHS of (2.19) respectively reduce to the evaluations of the CDF and the complementary CDF of the random variable $\tilde{i}(\mathbf{X}; \mathbf{Y}|Q_{\mathbf{Y}})$ under the probability measures $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}$ and $P_{\mathbf{X}}Q_{\mathbf{Y}}$. Generally, these CDFs are not known and the evaluation of the function T in (2.19) requires $2 \times n$ -dimensional integral:*

$$\mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbb{1}_{\{\tilde{i}(\mathbf{X}; \mathbf{Y}|Q_{\mathbf{Y}}) \leq \ln(\gamma)\}} \right] = \int_{\mathcal{X}^n} \int_{\mathcal{Y}^n} \mathbb{1}_{\{\tilde{i}(x; y|Q_{\mathbf{Y}}) \leq \ln(\gamma)\}} P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}(d\mathbf{y}) P_{\mathbf{X}}(d\mathbf{x}) \quad \text{and} \quad (2.24)$$

$$\mathbb{E}_{P_{\mathbf{X}}Q_{\mathbf{Y}}} \left[\mathbb{1}_{\{\tilde{i}(\mathbf{X}; \mathbf{Y}|Q_{\mathbf{Y}}) > \ln(\gamma)\}} \right] = \int_{\mathcal{X}^n} \int_{\mathcal{Y}^n} \mathbb{1}_{\{\tilde{i}(x; y|Q_{\mathbf{Y}}) > \ln(\gamma)\}} P_{\mathbf{Y}}(d\mathbf{y}) P_{\mathbf{X}}(d\mathbf{x}). \quad (2.25)$$

This integration is difficult due to the number of the dimension ($2n$) but remains less complex than the integration in (2.20).

Converse Bound

The lower bound on $\lambda^*(n, M)$ presented in this thesis is based on the Neyman-Pearson hypothesis testing [10, 13]. The following lemma presents this bound known as the meta converse (MC) bound [6, Theorem 27]. To unify the presentation with respect to the bound appearing in Lemma 2, the MC bound is presented using the form in [14, 15, Lemma 1].

Lemma 4 (MC Bound [14, Lemma 1]) *Given a pair $(n, M) \in \mathbb{N}^2$, the following holds for all probability measure $Q_{\mathbf{Y}}$ on the measurable space $(\mathcal{Y}^n, \mathcal{B}(\mathcal{Y}^n))$, with respect to the random transformation in (2.1):*

$$\lambda^*(n, M) \geq \inf_{P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)} \max_{\gamma \geq 0} \left(T(n, P_{\mathbf{X}}, Q_{\mathbf{Y}}, \gamma) - \frac{\gamma}{M} \right), \quad (2.26)$$

where the function T is defined in (2.19).

The infimum in (2.26) is taken with respect to the input probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$. If additional constraints hold on the codewords, the space of the probability measures is reduced to reflect this constraints. Note that the lower bound in (2.26) can be maximized with respect to output probability measures $Q_{\mathbf{Y}} \in \Delta(\mathcal{Y}^n)$.

Remark 3 *The lower and upper bounds proposed by Claude Shannon in [3] are not discussed here. Their derivations are based on the euclidean distance that is strongly related to Gaussian distributions. This unique property of the AWGN channels does not allow to generalize this technique to other channels.*

Comments 3 *The evaluation of the reviewed upper and lower bounds on the minimum average DEP are greatly facilitated if the CDFs of $\tilde{I}(\mathbf{X}; \mathbf{Y} | Q_{\mathbf{Y}})$ under different probability measures are known. Unfortunately, this is not generally the case. Hence, the approximation of the CDF of $\tilde{I}(\mathbf{X}; \mathbf{Y} | Q_{\mathbf{Y}})$ is pursued.*

2.1.3. Approximations to the Decoding Error Probability

This section focuses on approximations to the DEP under the memoryless and stationary assumptions on the channel.

The channel is said to be memoryless if for all input signal $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathcal{X}^n$ and for all boxes $\mathcal{B} \in \mathcal{B}(\mathcal{Y}^n)$, i.e.,

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \dots \times \mathcal{B}_n, \quad (2.27)$$

where, for all $t \in \{1, 2, \dots, n\}$, \mathcal{B}_t is an interval of \mathcal{Y} , it holds that

$$P_{\mathbf{Y}|\mathbf{X}}(\mathcal{B}|\mathbf{x}) = \prod_{t=1}^n P_{Y_t|X_t}(\mathcal{B}_t|x_t), \quad (2.28)$$

such that for all $t \in \{1, 2, \dots, n\}$, $x_t \in \mathcal{X}$, and $P_{Y_t|X_t=x_t} \in \Delta(\mathcal{Y})$ are given. This implies that at time t the probability measure induced by channel output random variable Y_t is totally defined by the input signal x_t .

The channel is said to be stationary if for all $t \in \{1, 2, \dots, n\}$ the conditional probability measure $P_{Y_t|X_t}$ satisfies for all $x \in \mathcal{X}$ and for all $\mathcal{S} \subseteq \mathcal{Y}$, it holds that

$$P_{Y_t|X_t}(\mathcal{S}|x) = P_{Y|X}(\mathcal{S}|x), \quad (2.29)$$

where $P_{Y|X=x} \in \Delta(\mathcal{Y})$.

Stationary and memoryless assumptions allow to expand the bound in Lemma 1 to the following theorem known as the Gallager's error exponent [5, Theorem 8].

Theorem 5 (Random code error exponent [5, Theorem 8]) *Given a pair $(n, M) \in \mathbb{N}^2$, for all input probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$, for all product probability measures $Q_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$ such that $P_{\mathbf{X}}$ is absolutely continuous with respect to $Q_{\mathbf{X}}$, the following holds with respect to the random transformation in (2.1) subject to (2.28) and (2.29):*

$$\lambda^*(n, M) \leq \max_{\rho \in (0, 1]} (M - 1)^\rho (\alpha(P_{\mathbf{X}}, Q_{\mathbf{X}}))^{1+\rho} \left(\mathbb{E}_{Q_{\mathbf{Y}}} \left[\mathbb{E}_{Q_{\mathbf{X}}} \left[\exp \left(\frac{\tilde{I}(\mathbf{X}; \mathbf{Y} | Q_{\mathbf{Y}})}{1 + \rho} \right) \right] \right] \right)^n \quad (2.30)$$

where, for all $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{X}^n$,

$$\alpha(P_{\mathbf{X}}, Q_{\mathbf{X}}) \triangleq \sup_{\mathbf{x} \in \mathcal{X}^n} \frac{dP_{\mathbf{X}}}{dQ_{\mathbf{X}}}(\mathbf{x}), \quad (2.31)$$

$$Q_{\mathbf{X}}(\mathbf{x}) = \prod_{t=1}^n Q_X(x_t), \quad (2.32)$$

$Q_{\mathbf{Y}}$ is the marginal of $Q_{\mathbf{X}}P_{Y|X}$, and the function \tilde{I} is defined in (2.17).

Note that the function α in (2.31) does not appear in [5, Theorem 8]. Indeed, [5] relies on the maximum constraint on the input codeword power. However with few algebra steps, the function α is obtained. The notation in (2.31) is preferred in the aim to harmonize between product input probability measures, i.e $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$ satisfying for all $\mathbf{x} \in \mathcal{X}^n$:

$$P_{\mathbf{X}}(\mathbf{x}) = \prod_{t=1}^n P_X(x_t), \quad (2.33)$$

and non product input probability measures.

Note that the upper bound in (2.30) can be minimized with respect to probability measures $P_{\mathbf{X}}$ and $Q_{\mathbf{X}}$. A first step of this minimization consists in finding for each $P_{\mathbf{X}}$ a probability measure $Q_{\mathbf{X}}$ that minimizes the function α in (2.31).

The main limitation of the bound in (2.30) as pointed out in [5] is that it is loose for low information rate. A solution is proposed for discrete memoryless channels (DMC). However, the generalization of the proposed solution to other channels is not trivial.

Normal Approximation

The normal approximation of the function T in (2.19) is useful to approximate the bound in Lemma 2, and Lemma 4. In that case, the random variable $\tilde{I}(\mathbf{X}; \mathbf{Y} | Q_{\mathbf{Y}})$ in (2.19) is assumed to follow a Gaussian distribution. More specifically, for all $P_{\mathbf{X}} \in \Delta(\mathcal{X})$, let

$$\mu(P_{\mathbf{X}}, Q_{\mathbf{Y}}) \triangleq \mathbb{E}_{P_{\mathbf{X}}P_{Y|X}} [\tilde{I}(X; Y | Q_{\mathbf{Y}})], \quad (2.34)$$

$$\sigma(P_X, Q_Y) \triangleq \mathbb{E}_{P_X P_{Y|X}} \left[(\tilde{\iota}(X; Y|Q_Y) - \mu(P_X, Q_Y))^2 \right], \text{ and} \quad (2.35)$$

$$\xi(P_X, Q_Y) \triangleq c_1 \left(\frac{\mathbb{E}_{P_X P_{Y|X}} \left[|\tilde{\iota}(X; Y|Q_Y) - \mu(P_X, Q_Y)|^3 \right]}{\sigma(P_X)^{\frac{3}{2}}} + c_2 \right), \quad (2.36)$$

with $c_1 = 0.33554$ and $c_2 = 0.415$, be functions of the input probability measure P_X . In particular, $\mu(P_X)$ and $\sigma(P_X)$ are respectively the first moment and the second central moment of the random variables $\iota(X_1; Y_1)$, $\iota(X_2; Y_2) \dots \iota(X_n; Y_n)$. Using this notation, consider the functions $\underline{N} : \mathbb{N} \times \mathbb{R}_+ \times \Delta(\mathcal{X}) \rightarrow \mathbb{R}_+$ and $\bar{N} : \mathbb{N} \times \mathbb{R}_+ \times \Delta(\mathcal{X}) \rightarrow \mathbb{R}_+$ such that for all $(n, \gamma) \in \mathbb{N} \times \mathbb{R}_+$ and for all $P_X \in \Delta(\mathcal{X})$,

$$\underline{N}(n, \gamma, P_X, Q_Y) = \max \left\{ 0, \hat{N}(n, \gamma, P_X, Q_Y) - \frac{\xi(P_X, Q_Y)}{\sqrt{n}} \right\}, \text{ and} \quad (2.37)$$

$$\bar{N}(n, \gamma, P_X) = \min \left\{ 1, \hat{N}(n, \gamma, P_X, Q_Y) + \frac{5 \xi(P_X, Q_Y)}{\sqrt{n}} + \frac{2 \ln(2)}{\sigma(P_X)^{\frac{1}{2}} \sqrt{2n\pi}} \right\}, \quad (2.38)$$

where

$$\hat{N}(n, M, P_X, Q_Y) \triangleq Q \left(\frac{n\mu(P_X, Q_Y) - \ln(\gamma)}{\sqrt{n\sigma(P_X, Q_Y)}} \right) \quad (2.39)$$

is referred as the normal approximation of $T(n, P_X, Q_Y, \gamma)$. Using this notation, the following theorem introduces lower and upper bounds on the function T in (2.19).

Theorem 6 ([16]) *Given a pair $(n, M) \in \mathbb{N}^2$, for all input product probability measures $P_X \in \Delta(\mathcal{X}^n)$, the following holds with respect to the random transformation in (2.1) subject to (2.28),*

$$\underline{N}(n, \gamma, P_X, Q_Y) \leq T(n, P_X, Q_Y, \gamma) \leq \bar{N}(n, \gamma, P_X, Q_Y), \quad (2.40)$$

where the functions T , \underline{N} and \bar{N} are defined in (2.19), (2.37) and (2.38), respectively.

Note that Theorem 6 does not directly appeared in the literature. Indeed, Theorem 6 are used as an intermediate result by focusing on the upper bound to develop second order achievabilities or the lower bound for converse bound.

Note that the upper bound on the function T in (2.40) leads to an upper bound on the DEP through the Lemma 2. In order to derive a lower bound on the DEP, the function

$$C(n, \gamma, P_X, Q_Y, M) = T(n, P_X, Q_Y, \gamma) - \frac{\gamma}{M}, \quad (2.41)$$

which appears in the MC bound (Lemma 4), must be bounded. To do so, consider the functions

$$\underline{N}_c(n, M, P_X, Q_Y, \gamma) = \underline{N}(n, \gamma, P_X, Q_Y) - \frac{\gamma}{M}, \quad (2.42)$$

$$\bar{N}_c(n, M, P_X, Q_Y, \gamma) = \bar{N}(n, \gamma, P_X, Q_Y) - \frac{\gamma}{M}, \text{ and} \quad (2.43)$$

$$\hat{N}_c(n, M, P_X, Q_Y, \gamma) \triangleq \hat{N}(n, M, P_X, Q_Y) - \frac{\gamma}{M}. \quad (2.44)$$

Then, it holds that

$$\underline{N}_c(n, M, P_X, Q_Y, \gamma) \leq C(n, \gamma, P_X, Q_Y, M) \leq \bar{N}_c(n, M, P_X, Q_Y, \gamma), \quad (2.45)$$

and the term $\hat{N}_c(n, M, P_X, Q_Y, \gamma)$ is referred as the normal approximation to the term $C(n, \gamma, P_X, Q_Y, M)$. Finally, the lower bound on the function C after an optimal choice of parameter γ and P_X leads to an lower bound on the DEP.

The normal approximation is less accurate for small values of the DEP and n . Indeed, the normal approximation of CDF is less precise on its tail and also for sums with a small number of IID random variables. This is captured by a small decay factor $\frac{1}{\sqrt{n}}$ of the normal approximation error provided by the Berry-Esseen theorem, which will be presented in Chapter 3. The probability evaluated on the tail of the CDF may be several order of magnitude lower than the approximation error especially for small values of n due to a small decay factor $\frac{1}{\sqrt{n}}$. This can be seen by terms containing a factor $\frac{1}{\sqrt{n}}$ in the lower bound in (2.37) and the upper bound in (2.38) and being independent of M . As consequence, for small values of M , which correspond to the tail of the CDF of $\tilde{i}(\mathbf{X}; \mathbf{Y}|Q_Y)$, such terms dominate the bounds, which makes the approximations useless. However, for large values of n , such terms in $\frac{1}{\sqrt{n}}$ can be neglected, which validate the asymptotic expansion obtained from the normal.

Saddlepoint Approximation

The limitation of the normal approximation and random code error exponent leads the authors of [14, 15, 17–20] to be the first to consider the saddlepoint approximation of the DEP bounds. The saddlepoint approximation of the function T in (2.19) is introduced in [15, Equation (18)] in which the function α_β represents that maximum of the function T with respect to variable γ and $\beta = \frac{1}{M}$. To introduce this approximation, denote by $K_{l,P} : \mathbb{R} \rightarrow \mathbb{R}$ the cumulant generating function of the random variable $\tilde{i}(X; Y|Q_Y)$. That is, for all $\theta \in \mathbb{R}$,

$$K_{l,P}(\theta) = \ln \left(\mathbb{E}_{P_X P_Y|X} [\exp(\theta \tilde{i}(X; Y|Q_Y))] \right). \quad (2.46)$$

The first and second derivatives of $K_{l,P}$ denoted respectively by $K_{l,P}^{(1)}$ and $K_{l,P}^{(2)}$. Using this notation consider the function $\hat{T} : \mathbb{N} \times \mathbb{R} \times \Delta(\mathcal{X}) \times \Delta(\mathcal{Y}) \rightarrow \mathbb{R}_+$:

$$\begin{aligned} \hat{T}(n, \gamma, P_X, Q_Y) = & \mathbb{1}_{\{\theta > 0\}} + \gamma \mathbb{1}_{\{\theta \leq -1\}} + \exp(nK_{l,P}(\theta) - \theta \ln(\gamma)) \left(\right. \\ & (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp\left(\frac{1}{2}\theta^2 nK_{l,P}^{(2)}(\theta)\right) Q\left(\sqrt{nK_{l,P}^{(2)}(\theta)}|\theta\right) \\ & \left. + (-1)^{\mathbb{1}_{\{\theta \leq -1\}}} \exp\left(\frac{1}{2}(\theta + 1)^2 nK_{l,P}^{(2)}(\theta)\right) Q\left(\sqrt{nK_{l,P}^{(2)}(\theta)}|\theta + 1\right) \right), \end{aligned} \quad (2.47)$$

in which the parameter θ is the solution in t to

$$nK_{l,P}^{(1)}(t) = \ln(\gamma). \quad (2.48)$$

The function \hat{T} is referred as the saddlepoint approximation of the function T . The saddlepoint

approximation is observed in [15] to be an accurate approximation even for small values of n . However, the approximation error is neglected. Without the characterization of the approximation error, the approximation accuracy cannot be validated without a heavy evaluation of bounds on the DEP. The lack of the approximation error characterization limits the utility of the approximation. In our work, that will be discussed later, we provide the characterization of the approximation error, which allows us to derive the easy computable and avoid a heavy evaluation of bounds on the DEP.

The second constraint from the work in [14, 15, 19, 20], in order to derive the saddlepoint approximation, the mutual information random variable $\tilde{I}(\mathbf{X}; \mathbf{Y}|Q_Y)$ must satisfy a lot of conditions. That is, the mutual information random variable $\tilde{I}(\mathbf{X}; \mathbf{Y}|Q_Y)$ must absolutely continuous or lattice with some assumptions on its cumulant generation function that are discussed in [21]. In our work to be presented in Chapter 3, we only required the existence of the cumulant generation function.

Asymptotic Expansion

The asymptotic expansion refers to the approximation of upper and lower bounds on $\lambda^*(n, M)$ in (2.9) or (n, ϵ) -capacity $R^*(n, \epsilon)$ in (2.14) for large values of n .

Following the seminal work of Claude Shannon in [2], it has been proven that:

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} R^*(n, \epsilon) = \max_{P_X \in \Delta(\mathcal{X})} \mathbb{E}_{P_X P_Y|X} [\tilde{I}(X; Y|P_Y)], \quad (2.49)$$

with \tilde{I} defined in (2.17).

The expression $\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} R^*(n, \epsilon)$ is known as the channel capacity and is denoted by C .

Following the work in [10], it was recently shown in [6, Theorem 49 and Theorem 54] and [22, Theorem 1] that for DMC and additive white Gaussian noise (AWGN) memoryless channels the function $R^*(n, \epsilon)$ in (2.14) can be approximated for large values of n by:

$$R^*(n, \epsilon) = C - \sqrt{\frac{V}{n}} Q^{-1}(\epsilon) + O\left(\frac{\ln(n)}{n}\right), \quad (2.50)$$

where C , V are respectively the channel capacity and channel dispersion, and Q is the complementary CDF of the standard Gaussian random variable with zero-mean and unit-variance:

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{+\infty} \exp\left(-\frac{x^2}{2}\right) dx. \quad (2.51)$$

Explicitly, the channel dispersion V is defined as:

$$V = \min_{P_X \in \Delta^*(\mathcal{X})} \mathbb{E}_{P_X P_Y|X} \left[|\tilde{I}(X; Y|P_Y) - C|^2 \right], \quad (2.52)$$

where $\Delta^*(\mathcal{X})$ is the set of input probability measures archiving channel capacity C in (2.49).

2.1.4. Discussion

Starting from the DEP formalization, upper and lower bounds on the minimum average $\lambda^*(n, M)$ in (2.9) (Lemma 1, Lemma 2, Lemma 3, and Lemma 4) are reviewed. The evaluation

complex of these bounds calls for their approximations, the normal approximation under memoryless and stationary assumptions. Despite the fact that the normal approximation leads to correct second order achievability derivation, the normal approximation fails on small values of DEP, especially for small values of blocklength n . Indeed, such a limitation is due to the poor performance of the normal approximation of the CDF of a sum of IID random variables on the CDF tail. Hence, the need to find an alternative approximation of CDFs of sums of IID random variables that is the main focus of Chapter 3.

2.2. Multiple Access Channels

MAC refers to the multiple-user communication scenario in the uplink, i.e. from the devices to the base station. In classical information theory framework, the DEP is the SDEP, i.e. there is an error if at least one message is wrongly decoded. An alternative notion of the DEP, i.e. the IDEP, where DEPs are considered with respect to each transmitter is introduced.

Starting from the system model formalization of MAC, lower and upper bounds on the SDEP are reviewed followed by the proposition of upper bounds on the IDEP. The evaluation difficulty of these bounds are discussed that leads to the next sub-section on the approximations of the DEPs. Finally, the last sub-section discusses the limitation of the state of the art approximations, which open the subject of Chapter 4.

2.2.1. System Model

Consider a communication system in which K transmitters aim at simultaneously sending information to one receiver through a noisy channel, with $K \in \mathbb{N}$. Such a channel can be modelled by a random transformation

$$(\mathcal{X}_1^n \times \mathcal{X}_2^n \times \dots \times \mathcal{X}_K^n, \mathcal{Y}^n, P_{\mathbf{Y}|\mathbf{X}_1\mathbf{X}_2\dots\mathbf{X}_K}), \quad (2.53)$$

where $n \in \mathbb{N}$ is the number of channel uses. For all $k \in \{1, 2, \dots, K\}$, given the inputs $\mathbf{x}_k \in \mathcal{X}_k^n$, the channel output is a random vector $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)^\top$ that induces the probability measure

$$P_{\mathbf{Y}|\mathbf{X}_1=\mathbf{x}_1, \mathbf{X}_2=\mathbf{x}_2, \dots, \mathbf{X}_K=\mathbf{x}_K}. \quad (2.54)$$

on the measurable space $(\mathcal{Y}^n, \mathcal{B}(\mathcal{Y}^n))$.

For all $k \in \{1, 2, \dots, K\}$, the message index sent by Transmitter k to Receiver is a realization of a random variable W_k that is uniformly distributed over the set

$$\mathcal{W}_k \triangleq \{1, 2, \dots, M_k\}, \quad (2.55)$$

with $M_k \in \mathbb{N}$. To send the message indices within n channel uses, the transmitters use an (n, \mathbf{M}) -code.

Definition 5 ((n, \mathbf{M}) -code) *Given $\mathbf{M} = (M_1, M_2, \dots, M_K) \in \mathbb{N}^K$ and a blocklength $n \in \mathbb{N}$, an (n, \mathbf{M}) -code for the random transformation in (2.53) is a system*

$$\{(\underline{\mathbf{u}}(\mathbf{i}), \mathcal{D}(\mathbf{i})) : \mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K\}, \quad (2.56)$$

where for all $\mathbf{i} = (i_1, i_2, \dots, i_K)^T \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K$, $\mathbf{j} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K$, with $\mathbf{i} \neq \mathbf{j}$:

$$\mathbf{u}(\mathbf{i}) \triangleq (\mathbf{u}_1(i_1), \mathbf{u}_2(i_2), \dots, \mathbf{u}_K(i_K)), \quad (2.57a)$$

$$\mathcal{D}(\mathbf{i}) \cap \mathcal{D}(\mathbf{j}) = \emptyset, \quad \text{and} \quad \bigcup_{\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K} \mathcal{D}(\mathbf{i}) \subseteq \mathcal{Y}^n, \quad (2.57b)$$

with for all $k \in \{1, 2, \dots, K\}$

$$\mathbf{u}_k(i_k) \triangleq (u_{k,1}(i_k), u_{k,2}(i_k), \dots, u_{k,n}(i_k))^T \in \mathcal{X}_k^n. \quad (2.57c)$$

Given an (n, \mathbf{M}) -code $\mathcal{C}_{n, \mathbf{M}} \triangleq \{(\mathbf{u}(\mathbf{i}), \mathcal{D}(\mathbf{i})) : \mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K\}$, the transmitter k uses the codeword $\mathbf{u}_k(i)$ to transmit the message index $i \in \mathcal{W}_k$. At channel use t , with $t \in \{1, 2, \dots, n\}$, the transmitter k inputs the symbol $u_{k,t}(i)$ to the channel. After n channel uses, the receiver observes an output vector $\mathbf{y} \in \mathcal{Y}^n$ and determines that the indices $\mathbf{i} = (i_1, i_2, \dots, i_K)$ were transmitted if

$$\mathbf{y} \in \mathcal{D}(\mathbf{i}). \quad (2.58)$$

The decoding error occurs when message indices \mathbf{i} were transmitted and the received vector \mathbf{y} satisfies:

$$\mathbf{y} \notin \mathcal{D}(\mathbf{i}). \quad (2.59)$$

The definition of the decoding error in (2.59) is restrictive. It may be useful to consider other definitions like the decoding error occurrence with respect to a given transmitter or a set of transmitters, especially for a large number of users. For instance, considering that decoding error occurs if at least a certain number (percentage) of transmitters get their message indices wrongly decoded may be more relevant for a telecommunication operator, especially for a large number of users.

In this thesis, we limit ourselves to two kinds of decoding error: (a) System decoding error that occurs if at least one message index is wrongly decoded; (b) Individual decoding error associated to a given transmitter that occurs if a message index of the associated transmitter is wrongly decoded.

To ease the notation, for all $K \in \mathbb{N}$, the sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_K$ of same dimension is called a matrix and is denoted by $\underline{\mathbf{x}}$. As consequences, for all $k \in \{1, 2, \dots, K\}$ and for all $\mathbf{x}_k \in \mathcal{X}_k^n$, the terms $P_{\mathbf{Y}|\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_K}$ and $P_{\mathbf{Y}|\mathbf{X}_1=\mathbf{x}_1, \mathbf{X}_2=\mathbf{x}_2, \dots, \mathbf{X}_K=\mathbf{x}_K}$ are respectively denoted by $P_{\mathbf{Y}|\underline{\mathbf{X}}}$ and $P_{\mathbf{Y}|\underline{\mathbf{X}}=\underline{\mathbf{x}}}$ for the rest of this manuscript.

System Decoding Error Probability

The system decoding error probability (SDEP) associated to the transmission of message indices $\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K$ is given by

$$\mathbb{E}_{P_{\mathbf{Y}|\underline{\mathbf{X}}=\underline{\mathbf{u}}(\mathbf{i})}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}(\mathbf{i})\}}], \quad (2.60)$$

where $P_{\mathbf{Y}|\underline{\mathbf{X}}} = P_{\mathbf{Y}|\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_K}$ is random transformation in (2.53).

Let an (n, \mathbf{M}) -code be denoted by $\mathcal{C}_{n, \mathbf{M}}$. The average SDEP (A-SDEP) associate to an

(n, \mathbf{M}) -code, denoted by $\lambda(\mathcal{C}_{n, \mathbf{M}})$, is

$$\lambda(\mathcal{C}_{n, \mathbf{M}}) \triangleq \frac{1}{M_1 M_2 \dots M_K} \sum_{\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K} \mathbb{E}_{P_{Y|\mathbf{X}=\mathbf{u}(\mathbf{i})}} [\mathbb{1}_{\{Y \neq D(\mathbf{i})\}}]. \quad (2.61)$$

Given a fixed pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the minimum average DEP associated to the random transformation in (2.53) is defined hereunder.

Definition 6 *Given a tuple $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the minimum average A-SDEP for the random transformation in (2.53), denoted by $\lambda^*(n, \mathbf{M})$, is given by*

$$\lambda^*(n, \mathbf{M}) = \inf_{\mathcal{C}_{n, \mathbf{M}} \in \mathcal{C}(n, \mathbf{M})} \lambda(\mathcal{C}_{n, \mathbf{M}}), \quad (2.62)$$

where $\lambda(\mathcal{C}_{n, \mathbf{M}})$ is defined in (2.61) and $\mathcal{C}(n, \mathbf{M})$ is the set of all (n, \mathbf{M}) -codes.

Note that given $\mathbf{M}_1 = (M_{1,1}, M_{1,2}, \dots, M_{1,K}) \in \mathbb{N}^K$ and $\mathbf{M}_2 = (M_{2,1}, M_{2,2}, \dots, M_{2,K}) \in \mathbb{N}^K$ such that for all $k \in \{1, 2, \dots, K\}$, $M_{1,k} \leq M_{2,k}$, it holds that $\lambda^*(n, \mathbf{M}_1) \leq \lambda^*(n, \mathbf{M}_2)$.

Definition 7 ($(n, \mathbf{M}, \epsilon)$ -code) *Let $\epsilon \in [0, 1)$ be fixed and consider an (n, \mathbf{M}) -code defined by the system in (2.56) such that $\lambda(\mathcal{C}_{n, \mathbf{M}}) \leq \epsilon$. Then, such a code is said to be an $(n, \mathbf{M}, \epsilon)$ -code with A-SDEP.*

When ϵ is chosen accordingly with the reliability constraints, an $(n, \mathbf{M}, \epsilon)$ -code is said to transmit at an information rate vector

$$\mathbf{R} = (R_1, R_2, \dots, R_K), \quad (2.63)$$

with for all $i \in [K]$,

$$R_i = \frac{\log_2(M_i)}{n} \quad (2.64)$$

bits per channel use.

Definition 8 (A-SDEP capacity region) *The A-SDEP capacity region, denoted by $\mathcal{A}^*(n, \epsilon)$, is defined as:*

$$\mathcal{A}^*(n, \epsilon) = \{\mathbf{R} \in \mathbb{R} : \exists (n, 2^{n\mathbf{R}}, \epsilon)\text{-code}\}, \quad (2.65)$$

where

$$2^{n\mathbf{R}} = (2^{nR_1}, 2^{nR_2}, \dots, 2^{nR_K}). \quad (2.66)$$

Any subset of $\mathcal{A}^*(n, \epsilon)$ is called an A-SDEP achievable region.

Individual Decoding Error Probabilities

The individual decoding error probability (IDEP) associated to Transmitter k , with $k \in \{1, 2, \dots, K\}$, and the transmission of message indices $\mathbf{i} = (i_1, i_2, \dots, i_K)^\top$, with $\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K$, is given by

$$\mathbb{E}_{P_{\mathbf{Y}|\underline{\mathbf{X}}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}_k(i_k)\}}], \quad (2.67)$$

where

$$\mathcal{D}_k(i_k) = \bigcup_{\substack{j=(j_1, j_2, \dots, j_K) \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \\ j_k = i_k}} \mathcal{D}(j), \quad (2.68)$$

and $P_{\mathbf{Y}|\underline{\mathbf{X}}} = P_{\mathbf{Y}|\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_K}$ is the random transformation in (2.53).

The average IDEP (A-IDEP) associated to Transmitter k , denoted by $\lambda_k(\mathcal{C}_{n, \mathbf{M}})$, is

$$\lambda_k(\mathcal{C}_{n, \mathbf{M}}) \triangleq \frac{1}{M_1 M_2 \dots M_K} \sum_{i \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K} \mathbb{E}_{P_{\mathbf{Y}|\underline{\mathbf{X}}=\mathbf{u}(i)}} [\mathbb{1}_{\{\mathbf{Y} \notin \mathcal{D}_k(i_k)\}}]. \quad (2.69)$$

The tuple $(\lambda_1(\mathcal{C}_{n, \mathbf{M}}), \lambda_2(\mathcal{C}_{n, \mathbf{M}}), \dots, \lambda_K(\mathcal{C}_{n, \mathbf{M}}))$ forms the A-IDEP vector associated to (n, \mathbf{M}) -code and is denoted by $\boldsymbol{\lambda}(\mathcal{C}_{n, \mathbf{M}})$.

Definition 9 ($(n, \mathbf{M}, \boldsymbol{\epsilon})$ -code) *Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_K) \in [0, 1]^K$ be fixed and consider an (n, \mathbf{M}) -code defined by the system in (2.56) subject to (2.68) such that, for all $k \in \{1, 2, \dots, K\}$, $\lambda_k(\mathcal{C}_{n, \mathbf{M}}) \leq \epsilon_k$. Then, such a code is said to be an $(n, \mathbf{M}, \boldsymbol{\epsilon})$ -code with A-IDEP.*

Definition 10 *Given $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the minimum A-IDEPs for the random transformation in (2.53) associated to Transmitter k , with $k \in \{1, 2, \dots, K\}$, denoted by $\lambda_k^*(n, \mathbf{M})$ is given by*

$$\lambda_k^*(n, \mathbf{M}) = \inf \{ \epsilon_k \in [0, 1] : \exists (n, \mathbf{M}, \boldsymbol{\epsilon})\text{-code with A-IDEP} \}. \quad (2.70)$$

When $\boldsymbol{\epsilon}$ is chosen accordingly with respect to the reliability constraints, an $(n, \mathbf{M}, \boldsymbol{\epsilon})$ -code is said to transmit at the information rate vector \mathbf{R} defined (2.63).

Definition 11 (IDEP capacity Region) *The A-IDEP capacity region, denoted by $\mathcal{E}^*(n, \boldsymbol{\epsilon})$, is defined as:*

$$\mathcal{E}^*(n, \boldsymbol{\epsilon}) = \{ \mathbf{R} \in \mathbb{R} : \exists (n, 2^{n\mathbf{R}}, \boldsymbol{\epsilon})\text{-code} \}, \quad (2.71)$$

where $2^{n\mathbf{R}}$ is defined in (2.66).

Any subset of $\mathcal{E}^*(n, \boldsymbol{\epsilon})$ is called an A-IDEP achievable region.

2.2.2. Bounds on the System Decoding Error Probability

This section presents previous works on upper and lower bounds on the minimum average SDEP, i.e., $\lambda^*(n, \mathbf{M})$ in (2.62) proposed in [11, 23–25].

To ease the presentation of bounds, consider the following notation. Given a $K \in \mathbb{N}$, let the collection $\mathcal{S}(K)$ be defined by

$$\mathcal{S}(K) \triangleq \{ \mathcal{S} \subseteq \{1, 2, \dots, K\} : \mathcal{S} \neq \{1, 2, \dots, K\} \}. \quad (2.72)$$

Given an integer $K \in \mathbb{N}$ and a set $\mathcal{S} \in \mathcal{S}(K)$ and $\mathcal{S} \neq \emptyset$, let the matrix $\underline{\mathbf{X}}_{\mathcal{S}}$ be a sub-matrix of the matrix $\underline{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_K)$ such that

$$\underline{\mathbf{X}}_{\mathcal{S}} = (\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \dots, \mathbf{X}_{i_s}), \quad (2.73)$$

with

$$i_1 < i_2 < \dots < i_s, \text{ and} \quad (2.74a)$$

$$\{i_1, i_2, \dots, i_s\} = \mathcal{S}. \quad (2.74b)$$

Given the probability measures $P_{\underline{X}} \in \Delta(\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K)$, $Q_{\underline{X}} \in \Delta(\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K)$, and $Q_{\mathbf{Y}|\underline{X}} \in \Delta(\mathcal{Y}^n)$, for all sets $\mathcal{S} \in \mathcal{S}(K)$, let the random variable $\tilde{i}(\underline{X}; \mathbf{Y}|Q_{\mathbf{Y}|\underline{X}}, \mathcal{S})$ be defined by

$$\tilde{i}(\underline{X}; \mathbf{Y}|Q_{\mathbf{Y}|\underline{X}}, \mathcal{S}) \triangleq \ln \left(\frac{dP_{\underline{X}}P_{\mathbf{Y}|\underline{X}}}{dQ_{\underline{X}}Q_{\mathbf{Y}|\underline{X}_S}}(\underline{X}, \mathbf{Y}) \right), \quad (2.75)$$

where the function $\frac{dP_{\underline{X}}P_{\mathbf{Y}|\underline{X}}}{dQ_{\underline{X}}Q_{\mathbf{Y}|\underline{X}_S}} : \mathcal{X}_1^n \times \mathcal{X}_2^n \times \dots \times \mathcal{X}_K^n \times \mathcal{Y}^n \rightarrow \mathbb{R}$ is the Radon-Nikodym derivatives of the joint probability measures $P_{\underline{X}}P_{\mathbf{Y}|\underline{X}}$ with respect to $Q_{\underline{X}}Q_{\mathbf{Y}|\underline{X}_S} = Q_{\mathbf{Y}|\underline{X}_S}$; and the probability measure $Q_{\underline{X}_S}Q_{\mathbf{Y}|\underline{X}_S}$ is the marginal of $Q_{\underline{X}}Q_{\mathbf{Y}|\underline{X}}$, with $Q_{\mathbf{Y}|\underline{X}_S} = Q_{\mathbf{Y}}$ for $\mathcal{S} = \emptyset$.

Remark 4 For the rest of this manuscript, the probability of union of events will be represented by the expectation of the maximum over their indicator functions. Indeed, let \mathcal{E}_1 and \mathcal{E}_2 be two events. Then,

$$\mathbb{P}[\mathcal{E}_1 \cup \mathcal{E}_2] = \mathbb{E}[\max\{\mathbf{1}_{\{\mathcal{E}_1\}}, \mathbf{1}_{\{\mathcal{E}_2\}}\}]. \quad (2.76)$$

This choice allows us to explicitly index the expectation operator \mathbb{E} by the distribution used to evaluate the probability. Finally, the expectation operator \mathbb{E} allows us to simultaneously manipulate discrete and continuous measures.

Achievability

The following lemmas present the bounds of [16, 25] that are the extensions of RCU and DT bounds to MAC.

Lemma 7 (RCU bound [25, (14)]) Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the following holds for all $k \in \{1, 2, \dots, K\}$, $P_{X_k} \in \Delta(\mathcal{X}_k^n)$, with respect to the random transformation in (2.53):

$$\lambda^*(n, \mathbf{M}) \leq \mathbb{E}_{P_{\underline{X}}P_{\mathbf{Y}|\underline{X}}} \left[\min \left(1, \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}_{\mathcal{S}^c}}} \left[\mathbf{1}_{\{\tilde{i}(\underline{X}; \mathbf{Y}|P_{\mathbf{Y}|\underline{X}}, \mathcal{S}) \geq \tilde{i}(\underline{X}; \mathbf{Y}|P_{\mathbf{Y}|\underline{X}}, \mathcal{S})\}} \right] \right) \right], \quad (2.77)$$

where $\mathcal{S}(K)$ is defined in (2.72); the random matrix $\underline{\bar{X}}$ is such that its sub-random matrix $\underline{\bar{X}}_{\mathcal{S}} = \underline{X}_{\mathcal{S}}$ and $\underline{\bar{X}}_{\mathcal{S}^c}$ is independent from $\underline{X}_{\mathcal{S}^c}$ but identically distributed, i.e., $(P_{\underline{\bar{X}}_{\mathcal{S}^c}} = P_{\underline{X}_{\mathcal{S}^c}})$; the function \tilde{i} is defined in (2.75); for all $\underline{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_K) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \dots \times \mathcal{X}_K^n$:

$$P_{\underline{X}}(\underline{x}) = \prod_{k=1}^K P_{X_k}(\mathbf{x}_k); \quad (2.78)$$

and the set \mathcal{S}^c is the complementary of \mathcal{S} in $\{1, 2, \dots, K\}$.

Note that as in the case of the point to point channel, the RHS of (2.77) is not generally easy to evaluate. It involves CDFs of random variables that are function of $K \times n$ -dimensional random variables. Such CDFs are generally not known.

Lemma 8 (DT bound [16, (5.133)]) *Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the following holds for all $k \in \{1, 2, \dots, K\}$, $P_{\mathbf{X}_k} \in \Delta(\mathcal{X}_k^n)$, with respect to the random transformation in (2.53):*

$$\lambda^*(n, \mathbf{M}) \leq \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\max_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1} \left\{ \tilde{i}(\mathbf{X}; \mathbf{Y} | P_{\mathbf{Y}|\mathbf{X}, \mathcal{S}}) \leq \ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right] + \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}, \mathcal{S}}} \left[\mathbb{1} \left\{ \tilde{i}(\mathbf{X}; \mathbf{Y} | P_{\mathbf{Y}|\mathbf{X}, \mathcal{S}}) > \ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right], \quad (2.79)$$

where the function \tilde{i} , the sets $\mathcal{S}(K)$, the probability measure $P_{\mathbf{X}}$ are respectively defined in (2.75), (2.72), and (2.78).

Even though, the expectation terms in the RHS of (2.79) can be expressed as functions of CDFs of random variables that are function of $K \times n$ -dimensional random variables, the bound in (2.79) is generally difficult because such CDFs are generally unknown.

Converse

The following lemma is an extension of the results of Verdù-Han in [26] to MAC.

Lemma 9 (Verdù-Han [16]) *Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the following holds for all probability measures $Q_{\mathbf{Y}|\mathbf{X}}$ with respect to the random transformation in (2.53):*

$$\lambda^*(n, \mathbf{M}) \geq \inf \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\max_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1} \left\{ \tilde{i}(\mathbf{X}; \mathbf{Y} | Q_{\mathbf{Y}|\mathbf{X}, \mathcal{S}}) \leq \ln(\gamma_{\mathcal{S}}) \right\} \right] - \left(\sum_{\mathcal{S} \in \mathcal{S}(K)} \frac{\gamma_{\mathcal{S}}}{\prod_{s \in \mathcal{S}^c} M_s} \right), \quad (2.80)$$

where the function \tilde{i} , the sets $\mathcal{S}(K)$ are respectively defined in (2.75), (2.72) and the infimum is with respect to all input distributions $P_{\mathbf{X}}$ satisfying (2.78).

The following lemma is an extension of the MC bound to MAC.

Lemma 10 (MC bound [24]) *Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, for all set $\mathcal{S} \in \mathcal{S}(K)$ and for all $Q_{\mathbf{Y}|\mathbf{X}} \in \Delta(\mathcal{Y}^n \times \mathcal{X}_1^n \times \dots \times \mathcal{X}_K^n)$, with respect to the random transformation in (2.53), it holds that*

$$\lambda^*(n, \mathbf{M}) \geq \inf_{P_{\mathbf{X}} \in \Delta(\mathcal{X}_1^n \times \mathcal{X}_2^n \times \dots \times \mathcal{X}_K^n)} \max_{\gamma \geq 0} \left(T_{\mathcal{S}}(P_{\mathbf{X}}, Q_{\mathbf{Y}|\mathbf{X}}, \gamma) - \frac{\gamma}{\prod_{s \in \mathcal{S}^c} M_s} \right), \quad (2.81)$$

where,

$$T_S(P_{\underline{X}}, Q_{Y|\underline{X}}, \gamma) = \mathbb{E}_{P_{\underline{X}}P_{Y|\underline{X}}} \left[\mathbb{1}_{\{\tilde{i}(\underline{X}; Y|Q_{Y|\underline{X}}, S) \leq \ln(\gamma)\}} \right] + \gamma \mathbb{E}_{P_{\underline{X}}Q_{Y|\underline{X}}S} \left[\mathbb{1}_{\{\tilde{i}(\underline{X}; Y|Q_{Y|\underline{X}}, S) > \ln(\gamma)\}} \right], \quad (2.82)$$

the function \tilde{i} is defined in (2.75) and the infimum is with respect to all input distributions $P_{\underline{X}}$ satisfying (2.78).

2.2.3. Bounds on the Individual Decoding Error Probability

For $k \in \{1, 2, \dots, K\}$, let the set $\mathcal{P}(k, K)$ be defined by

$$\mathcal{P}(k, K) = \{S \subseteq [K] : k \notin S\}. \quad (2.83)$$

Achievability The following lemmas extends the RCU and DT bounds to MAC.

Lemma 11 Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, the following holds for all $P_{\mathbf{X}_k} \in \Delta(\mathcal{X}_k^n)$, with $k \in \{1, 2, \dots, K\}$, with respect to the random transformation in (2.53):

$$\lambda_k^*(n, \mathbf{M}) \leq \mathbb{E}_{P_{\underline{X}}P_{Y|\underline{X}}} \left[\min \left(1, \sum_{S \in \mathcal{P}(k, K)} \left(\prod_{s \in S^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}_{S^c}}} \left[\mathbb{1}_{\{\tilde{i}(\underline{X}; Y|P_{Y|\underline{X}}, S) \geq \tilde{i}(\underline{X}; Y|P_{Y|\underline{X}}, S)\}} \right] \right) \right], \quad (2.84)$$

where the random matrix \underline{X} is such that its sub-random matrix $\underline{X}_S = \underline{X}_S$ and \underline{X}_{S^c} is independent from \underline{X}_{S^c} but identically distributed, i.e., $(P_{\underline{X}_{S^c}} = P_{\underline{X}_{S^c}})$; the function \tilde{i} , the set $\mathcal{P}(k, K)$, the probability measure $P_{\underline{X}}$ are respectively defined in (2.75), (2.83), and (2.78).

Proof: The proof borrows the same lines in the proof of Lemma 7, which relies on the maximum likelihood. Assume that the conditional probability measure $P_{Y|\underline{X}}$ is absolutely continuous with respect to measure ν and denote the density (Radon Nikodym derivative) of $P_{Y|\underline{X}}$ with respect to ν by $f_{Y|\underline{X}}$. Then, for a given transmitted message index vector $\mathbf{i} = (i_1, i_2, \dots, i_K) \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K$ and the observed vector \mathbf{y} at the output of the channel, the message index i_k of the transmitter k is successively decoded if the following condition

$$\sum_{\substack{\hat{\mathbf{i}} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \\ m_k = i_k}} f_{Y|\underline{X}}(\mathbf{y}|\underline{\mathbf{u}}(\mathbf{m})) > \max \left\{ \sum_{\substack{\hat{\mathbf{m}} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \\ \hat{m}_k = j_k}} f_{Y|\underline{X}}(\mathbf{y}|\underline{\mathbf{u}}(\hat{\mathbf{m}})) : \right. \\ \left. j_k \in \mathcal{W}_K, \text{ and } j_k \neq i_k \right\} \quad (2.85)$$

is satisfied. The summation in (2.85) is due to the computation of the conditional marginal density with respect to the input of the transmitter k under uniform distribution assumption of message index vectors. The uniform distribution assumption leads to simplification of (2.85) by factorizing with the message index vector probability. Due to the difficulty of working with a such summation, the following suboptimal maximum likelihood is considered. That is, for a given transmitted message indices vector $\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K$ and the observed vector \mathbf{y}

at the output of the channel, the message index i_k of the transmitter k is successively decoded if the following condition

$$f_{\mathbf{Y}|\underline{\mathbf{X}}}(\mathbf{y}|\underline{\mathbf{u}}(i)) > \max \left\{ f_{\mathbf{Y}|\underline{\mathbf{X}}}(\mathbf{y}|\underline{\mathbf{u}}(\hat{\mathbf{m}})) : \hat{\mathbf{m}} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K \right. \\ \left. \hat{m}_k \in \mathcal{W}_K, \text{ and } \hat{m}_k \neq i_k \right\} \quad (2.86)$$

is satisfied. Such a condition is equivalent to for all sets $\mathcal{S} \in \mathcal{P}(k, K)$, with $\mathcal{P}(k, K)$ in (2.83),

$$\tilde{t}(\underline{\mathbf{u}}(i); \mathbf{y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S}) > \max \left\{ \tilde{t}(\underline{\mathbf{u}}(j); \mathbf{y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S}) : \mathbf{j} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \right. \\ \left. \mathbf{j}_{\mathcal{S}} = \mathbf{i}_{\mathcal{S}}, \text{ and } \forall \ell \in \mathcal{S}^c, j_\ell \neq i_\ell \right\}. \quad (2.87)$$

Thus, the A-IDEP $\lambda_k(\mathcal{C}_{n, \mathbf{M}})$ associated to the suboptimal decoder in (2.87) is given by

$$\lambda_k(\mathcal{C}_{n, \mathbf{M}}) = \frac{1}{\prod_{t=1}^K M_t} \sum_{\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K} \mathbb{E}_{P_{\mathbf{Y}|\underline{\mathbf{X}}=\underline{\mathbf{u}}(i)}} \left[\max \left\{ \mathbb{1}_{\{\tilde{t}(\underline{\mathbf{u}}(i); \mathbf{Y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S}) \leq \tilde{t}(\underline{\mathbf{u}}(j); \mathbf{Y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S})\}} : \right. \right. \\ \left. \left. \mathcal{S} \in \mathcal{P}(k, K), \mathbf{j} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \mathbf{j}_{\mathcal{S}} = \mathbf{i}_{\mathcal{S}}, \text{ and } \forall \ell \in \mathcal{S}^c, j_\ell \neq i_\ell \right\} \right] \quad (2.88)$$

$$\leq \sum_{\mathcal{S} \in \mathcal{P}(k, K)} \frac{1}{\prod_{t=1}^K M_t} \sum_{\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K} \mathbb{E}_{P_{\mathbf{Y}|\underline{\mathbf{X}}=\underline{\mathbf{u}}(i)}} \left[\max \left\{ \mathbb{1}_{\{\tilde{t}(\underline{\mathbf{u}}(i); \mathbf{Y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S}) \leq \tilde{t}(\underline{\mathbf{u}}(j); \mathbf{Y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S})\}} : \right. \right. \\ \left. \left. \mathbf{j} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \mathbf{j}_{\mathcal{S}} = \mathbf{i}_{\mathcal{S}}, \text{ and } \forall \ell \in \mathcal{S}^c, j_\ell \neq i_\ell \right\} \right]. \quad (2.89)$$

$$\leq \sum_{\mathcal{S} \in \mathcal{P}(k, K)} \frac{1}{\prod_{t=1}^K M_t} \sum_{\mathbf{i} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K} \min \left\{ 1, \right. \\ \left. \sum_{\substack{\mathbf{j} \in \mathcal{W}_1 \times \mathcal{W}_2 \times \dots \times \mathcal{W}_K, \\ \mathbf{j}_{\mathcal{S}} = \mathbf{i}_{\mathcal{S}}, \text{ and } \forall \ell \in \mathcal{S}^c, j_\ell \neq i_\ell}} \mathbb{E}_{P_{\mathbf{Y}|\underline{\mathbf{X}}=\underline{\mathbf{u}}(i)}} \left[\mathbb{1}_{\{\tilde{t}(\underline{\mathbf{u}}(i); \mathbf{Y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S}) \leq \tilde{t}(\underline{\mathbf{u}}(j); \mathbf{Y}|P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{S})\}} \right] \right\}. \quad (2.90)$$

Finally, the random coding argument leads to (2.84).

Similarly to (2.84), the RHS of (2.84) is not generally easy to evaluate. Hence, the consideration of the following bound. ■

Lemma 12 *Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, $P_{\mathbf{X}_k} \in \Delta(\mathcal{X}_k^n)$, with $k \in \{1, 2, \dots, K\}$, with respect*

to the random transformation in (2.53):

$$\lambda_k^*(n, \mathbf{M}) \leq \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\max_{\mathcal{S} \in \mathcal{P}(k, K)} \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right]^+ + \sum_{\mathcal{S} \in \mathcal{P}(k, K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) > \ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right] \quad (2.91)$$

with the function \tilde{i} , the set $\mathcal{P}(k, K)$, and the probability measure $P_{\underline{X}}$ are respectively defined in (2.75), (2.83), and (2.78).

Proof: The proof of Lemma 12 follows from Lemma 12. From (2.77), for all $\alpha = (\alpha_{S_1}, \alpha_{S_2}, \dots, \alpha_{S_\ell}) \in \mathbb{R}^\ell$, with $\ell = |\mathcal{P}(k, K)|$, it holds that

$$\lambda_k^*(n, \mathbf{M}) \leq \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\min \left(1, \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}, \mathcal{S}^c}} \left[\mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \geq i(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \right\} \right] \right) \right. \\ \left. 1 - \max \left\{ \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}} \right\} : \mathcal{S} \in \mathcal{S}(K) \right\} + \max \left\{ \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}} \right\} : \mathcal{S} \in \mathcal{S}(K) \right\} \right] \quad (2.92)$$

$$\leq \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\max \left\{ \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}} \right\} : \mathcal{S} \in \mathcal{S}(K) \right\} \right] + \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\prod_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) > \alpha_{\mathcal{S}} \right\} \right. \\ \left. \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}, \mathcal{S}^c}} \left[\mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \geq i(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \right\} \right] \right] \quad (2.93)$$

$$\leq \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\max \left\{ \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}} \right\} : \mathcal{S} \in \mathcal{S}(K) \right\} \right] + \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}} P_{\underline{X}, \mathcal{S}^c}} \left[\mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) > \alpha_{\mathcal{S}} \right\} \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \geq i(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \right\} \right] \quad (2.94)$$

$$\leq \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\max \left\{ \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}} \right\} : \mathcal{S} \in \mathcal{S}(K) \right\} \right] + \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}} P_{\underline{X}, \mathcal{S}^c}} \left[\mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) > \alpha_{\mathcal{S}} \right\} \right] \quad (2.95)$$

$$\leq \mathbb{E}_{P_{\underline{X}} P_{\underline{Y}|\underline{X}}} \left[\max \left\{ \mathbb{1} \left\{ \tilde{i}(\underline{X}; \underline{Y} | P_{\underline{Y}\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}} \right\} : \mathcal{S} \in \mathcal{S}(K) \right\} \right] +$$

$$\sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{X}} P_{Y|\underline{X}, \mathcal{S}}} \left[\mathbb{1}_{\{i(\underline{X}; Y | P_{Y|\underline{X}, \mathcal{S}}) > \alpha_{\mathcal{S}}\}} \right]. \quad (2.96)$$

Note that (2.93) follows from splitting the min function in (2.92) and the fact that

$$1 - \max \left\{ \mathbb{1}_{\{i(\underline{X}; Y | P_{Y|\underline{X}, \mathcal{S}}) \leq \alpha_{\mathcal{S}}\}} : \mathcal{S} \in \mathcal{S}(K) \right\} = \prod_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1}_{\{i(\underline{X}; Y | P_{Y|\underline{X}, \mathcal{S}}) > \alpha_{\mathcal{S}}\}}. \quad (2.97)$$

Finally, choosing $\alpha_{\mathcal{S}} = \ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right)$ concludes the proof. \blacksquare

Similarly to the RHS of (2.79), the evaluation of (2.79) remains generally difficult.

2.2.4. Approximations to the Decoding Error probability

The evaluation of the presented bounds calls for their approximations. Such approximations need further assumptions like memoryless and stationary hypotheses on the channel.

The channel is said to be memoryless if for all $k \in \{1, 2, \dots, K\}$, $\mathbf{x}_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})^\top \in \mathcal{X}_k^n$ and for all boxes $\mathcal{B} \in \mathcal{B}(\mathcal{Y}^n)$, *i.e.*,

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \dots \times \mathcal{B}_n, \quad (2.98)$$

where for all $t \in \{1, 2, \dots, n\}$, \mathcal{B}_t is an interval of \mathcal{Y} , for all $k \in \{1, 2, \dots, K\}$, it holds that

$$P_{Y|X_1 X_2 \dots X_K}(\mathcal{B} | \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_K) = \prod_{t=1}^n P_{Y_t | X_{1,t} X_{2,t} \dots X_{K,t}}(\mathcal{B}_t | x_{1,t}, x_{2,t}, \dots, x_{K,t}), \quad (2.99)$$

where, for all $(x_1, x_2, \dots, x_K) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K$, $P_{Y|X_1=x_1, X_2=x_2, \dots, X_K=x_K} \in \Delta(\mathcal{Y})$.

The channel is said to be stationary when for all $t \in \{1, 2, \dots, n\}$ the parameter of the conditional distribution $P_{Y_t | X_{1,t} X_{2,t} \dots X_{K,t}}$ is insensible to the value of t :

$$P_{Y_t | X_{1,t} X_{2,t} \dots X_{K,t}} = P_{Y | X_1 X_2 \dots X_K} \quad (2.100)$$

Based on memoryless and stationary assumptions, suitable expansions of the A-SDEP are provided in [16, 23, 25, 27].

Theorem 13 (Random code error exponent [27]) *Given a tuple $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, for all $k \in \{1, 2, \dots, K\}$, for all input product probability measures $P_{X_k} \in \Delta(\mathcal{X}_k)$, for all $\mathcal{S} \in \mathcal{S}(K)$, $\rho_{\mathcal{S}} \in [0, 1]$, the following holds with respect to the random transformation in (2.53) subject to (2.99):*

$$\lambda^*(n, \mathbf{M}) \leq \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right)^{\rho_{\mathcal{S}}} E(P_{\underline{X}}, \mathcal{S}), \quad (2.101)$$

where,

$$E(P_{\mathbf{X}}, \mathcal{S}) = \left(\mathbb{E}_{P_{Y|P_{Y|X_S}}} \left[\mathbb{E}_{P_{\bar{X}_{S^c}}} \left[\exp \left(\frac{\tilde{t}(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_K; Y | P_{Y|X_1 X_2 \dots X_K}, \mathcal{S})}{1 + \rho_{\mathcal{S}}} \right) \right]^{1+\rho_{\mathcal{S}}} \right] \right)^n, \quad (2.102)$$

with for all $k \in \{1, 2, \dots, K\}$, $\bar{X}_k = X_k$ if $k \notin \mathcal{S}$ and \bar{X}_k and X_k are independent but identically distributed otherwise.

A similar bound hold for A-IDEP by replacing the summation set in (2.101) by the set $\mathcal{P}(k, K)$ defined in (2.83).

Normal Approximation

The normal approximation consists in assuming that the random variables $\tilde{t}(\mathbf{X}; \mathbf{Y} | Q_{Y\mathbf{X}}, \mathcal{S})$, with $Q_{Y\mathbf{X}} = P_{Y\mathbf{X}}$, involved in the previous bounds are Gaussian random variables. More precisely, for all $\mathcal{C} \subseteq \mathcal{S}(k)$, denote by

$$\tilde{\mathbf{t}}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{C}) \triangleq \begin{pmatrix} \tilde{t}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{S}_1) \\ \tilde{t}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{S}_2) \\ \vdots \\ \tilde{t}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{S}_\ell) \end{pmatrix}, \quad (2.103)$$

such that

$$\{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\ell\} = \mathcal{C}, \quad (2.104)$$

$$\mathbf{X} = (X_1, X_2, \dots, X_K)^\top \quad (2.105)$$

and the tuple (\mathbf{X}, Y) induces the probability measure $P_{\mathbf{X}}P_{Y|\mathbf{X}}$. Consider also, for all $P_{\mathbf{X}}P_{Y|\mathbf{X}}$, for all $Q_{Y|\mathbf{X}}$ and for all $\mathcal{C} \subseteq \mathcal{S}(k)$:

$$\begin{aligned} \boldsymbol{\mu}_\ell(P_{\mathbf{X}}, Q_{Y\mathbf{X}}, \mathcal{C}) &= \mathbb{E}_{P_{\mathbf{X}}P_{Y|\mathbf{X}}} [\tilde{\mathbf{t}}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{C})], \text{ and} \\ \mathbf{v}_\ell(P_{\mathbf{X}}, Q_{Y\mathbf{X}}, \mathcal{C}) &= \mathbb{E}_{P_{\mathbf{X}}P_{Y|\mathbf{X}}} \left[(\tilde{\mathbf{t}}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{C}) - \boldsymbol{\mu}_\ell(P_{\mathbf{X}}, Q_{Y\mathbf{X}}, \mathcal{C})) \right. \\ &\quad \left. \cdot (\tilde{\mathbf{t}}(\mathbf{X}; Y | Q_{Y\mathbf{X}}, \mathcal{C}) - \boldsymbol{\mu}_\ell(P_{\mathbf{X}}, Q_{Y\mathbf{X}}, \mathcal{C}))^\top \right]. \end{aligned} \quad (2.106)$$

That $\boldsymbol{\mu}_\ell$ and \mathbf{v}_ℓ are respectively the mean vector and the covariance matrix of the random vector in (2.103). Using this notation, consider the function $\hat{G} : \mathbb{N}^{1+K} \times \Delta(\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K) \times \mathcal{B}(\mathcal{S}(K)) \rightarrow \mathbb{R}$ such that for all $(n, \mathbf{K}) \in \mathbb{N}^{1+K}$, for all $P_{X_1 X_2 \dots X_K} \in \Delta(\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_K)$ and for all $\mathcal{C} \in \mathcal{B}(\mathcal{S}(K))$, with $\mathcal{B}(\mathcal{S}(K))$ the collection all subsets of $\mathcal{S}(K)$:

$$\hat{G}(n, \mathbf{M}, P_{\mathbf{X}}, \mathcal{C}) = 1 - \Phi(\boldsymbol{\gamma}(\mathcal{C}) + \boldsymbol{\mu}_\ell(P_{\mathbf{X}}, P_{Y|\mathbf{X}}, \mathcal{C}); \mathbf{v}_\ell(P_{\mathbf{X}}, P_{Y|\mathbf{X}}, \mathcal{C})), \quad (2.107)$$

where

$$\gamma(\mathcal{C}) \triangleq \begin{pmatrix} -\ln \left(\prod_{s \in \mathcal{S}_1^c} (M_s - 1) \right) \\ -\ln \left(\prod_{s \in \mathcal{S}_2^c} (M_s - 1) \right) \\ \vdots \\ -\ln \left(\prod_{s \in \mathcal{S}_\ell^c} (M_s - 1) \right) \end{pmatrix}, \quad (2.108)$$

with $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\ell$ satisfying (2.104); and $\Phi(\cdot; \underline{\mathbf{v}})$ is the CDF of a Gaussian vector with mean vector zero and covariance matrix $\underline{\mathbf{v}}$.

For $\mathcal{C} = \mathcal{S}(K)$, $\hat{G}(n, \mathbf{M}, P_{\mathbf{X}}, \mathcal{C})$ is referred as the normal approximation of the DT bound (Lemma 8). The error induced by this normal approximation is upper bounded by $\frac{C}{\sqrt{n}}$, where C is a constant that does not depend on the blocklength n . This constant term implies that for some values of \mathbf{M} and n , the upper bound on the error is at several order of magnitude greater than the normal approximation. Under this particular case, the normal approximation can lead to false conclusion. Indeed, small values of \mathbf{M} correspond to the evaluation of the CDF on tail where the normal approximation is known to be less precise for small values of n .

Asymptotic Expansion

In order to introduce asymptotic expansion, for all positive matrices $\underline{\mathbf{v}} \in \mathbb{R}^{3 \times 3}$ and $\epsilon \in [0, 1]$, let the following set $\mathcal{P}(\underline{\mathbf{v}}, \epsilon)$ be defined as:

$$\mathcal{P}(\underline{\mathbf{v}}, \epsilon) = \left\{ \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3 : \mathbb{E}_{P_{\mathbf{Z}}} \left[\prod_{t=1}^3 \mathbb{1}_{\{Z_t \leq z_t\}} \right] \geq 1 - \epsilon \right\} \quad (2.109)$$

where $\mathbf{Z} = (Z_1, Z_2, Z_3)$ is a zero mean Gaussian vector with covariance matrix $\underline{\mathbf{v}}$.

For all input probability measures $P_{X_1} \in \Delta(\mathcal{X}_1)$, $P_{X_2} \in \Delta(\mathcal{X}_2)$, and $Q_{X_1 X_2 Y} \in \Delta(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y})$, let the following set $\mathcal{R}(P_{X_1}, P_{X_2}, Q_{X_1 X_2 Y})$ be defined as

$$\begin{aligned} & \mathcal{R}(P_{X_1}, P_{X_2}, Q_{X_1 X_2 Y}) \\ &= \mu_t(P_{X_1 X_2}, Q_{Y X_1 X_2}, \mathcal{S}(2)) - \mathcal{P}(\underline{\mathbf{v}}_t(P_{X_1 X_2}, Q_{Y X_1 X_2}, \mathcal{S}(2)), \epsilon) + O\left(\frac{\ln(n)}{n}\right) \mathbf{1}, \end{aligned} \quad (2.110)$$

where

$$\mathbf{1} = (1, 1, 1)^\top. \quad (2.111)$$

In [23, Theorem 4] it is proven that for large values of n , the set $\mathcal{R}(P_{X_1}, P_{X_2}, Q_{X_1 X_2 Y})$ in (2.110) is an achievable region for two transmitter discrete memoryless MAC (DM), where P_{X_1} and P_{X_2} are chosen to be product probability measures.

In [25, Theorem 2], it is proven that for large values of n , the set $\mathcal{R}(P_{X_1}, P_{X_2}, Q_{X_1 X_2 Y})$ in (2.110) is an achievable region for two transmitter Gaussian MAC under equal power constraints on the codewords, where P_{X_1}, P_{X_2} are uniform on power shell and Q_{X_1}, Q_{X_2} are product Gaussian distributions.

2.2.5. Discussion

The main advantage of the asymptotic expansion in (2.110) [23, Theorem 4] [25, Theorem 2] compared to the bounds of Lemma 7, Lemma 8 and Lemma 10 is the simplicity of the computation. Indeed, the evaluation of the bounds in Lemma 7, Lemma 8 and Lemma 10 involves the complex computation of unknown CDFs of random vector of $(2K + 1) \times n$ and $(K + 1) \times n$ dimension. However, the asymptotic expansion in (2.110) is limited to large values of n compared to the bounds of Lemma 7 and Lemma 8 that hold for all values of n . Indeed, the asymptotic expansion in (2.110) is the result of the normal approximation. Hence, the asymptotic expansion in (2.110) suffers from the low convergence rate of $O\left(\frac{1}{\sqrt{n}}\right)$ of the normal approximation of CDFs of sums of IID random vectors. This characteristic justifies the search for another approximation for such CDFs, which is the topic of Chapter 4.

2.3. Conclusion

The system models of the point to point channel and MAC have been presented, followed by the literature of the state of the art bounds on the DEPs for both channels. A slight improvement of the DT bound of the point to point channel is also proposed. The notion of the IDEP, which separately associates the DEP to each transmitter compared to the tradition conception where the DEP is indistinguishably associated to all transmitters, is introduced in MAC. The evaluation complexity of these bounds, which involved unknown CDFs of random vectors/variables, is discussed and leads to the survey of their approximations: the normal approximation of such CDFs. Unfortunately, the limitation of the normal approximation on the CDF tail makes it impertinent to analyse the performance of the emerging URLLC. Hence, the search of other approximation of CDFs, which is the main object of Chapter 3 and Chapter 4.

— 3 —

Approximation of the CDF of Sums of Independent Random Variables

A new approximation of the CDF of sums of IID random variables is proposed. Such approximation, referred as exponentially tilted Gaussian approximation, relies on a change of the measure (exponentially tilted measure) and a normal approximation. The exponentially tilted measure introduces a parametrized (θ) sum of IID random variables whose measure is approximated by a Gaussian measure. An upper bound on the approximation error is proposed that depends on the parameter θ . Our approximation covers the Gaussian approximation (for $\theta = 0$) and the saddlepoint approximation for $\theta = \theta^*$. This work has been published in [28].

3.1. Preliminaries

Let Y_1, Y_2, \dots, Y_n , with n an integer and $2 \leq n < \infty$, be real-valued random variables. For all $i \in \{1, 2, \dots, n\}$, the random variable Y_i induces the probability measure P_Y on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Denote by F_Y the CDF associated with P_Y . That is, for all $(a, b) \in \mathbb{R}^2$, with $a < b$,

$$P_Y([a, b]) = F_Y(b) - F_Y(a). \quad (3.1)$$

When the Radon-Nikodym derivative of P_Y with respect to the Lebesgue measure exists, it is denoted by f_Y . Note that f_Y is the probability density function (PDF) of the random

variable Y_i , with $i \in \{1, 2, \dots, n\}$. Let also

$$X_n = \sum_{t=1}^n Y_t \quad (3.2)$$

be a random variable that induces the probability measure P_{X_n} on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Denote by F_{X_n} the CDF and if it exists, denote by f_{X_n} the PDF associated with P_{X_n} . The objective is to provide a positive function that approximates F_{X_n} and an upper bound on the resulting approximation error. In the following, a positive function $g : \mathbb{R} \rightarrow \mathbb{R}_+$ is said to approximate F_{X_n} with an *approximation error* that is upper bounded by a function $\epsilon : \mathbb{R} \rightarrow \mathbb{R}_+$, if, for all $x \in \mathbb{R}$,

$$|F_{X_n}(x) - g(x)| \leq \epsilon(x). \quad (3.3)$$

The case in which Y_1, Y_2, \dots, Y_n in (3.2) are stable random variables with F_Y analytically expressible is trivial. This is essentially because the sum X_n induces the same probability measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as a random variable $a_n Y + b_n$, where $(a_n, b_n) \in \mathbb{R}^2$ and Y is a random variable whose CDF is F_Y . Examples of this case are random variables following the Gaussian, Cauchy, or Levy distributions [29].

In general, the problem of calculating the CDF of X_n boils down to calculating $n - 1$ convolutions. More specifically, it holds that

$$f_{X_n}(x) = \int_{-\infty}^{\infty} f_{X_{n-1}}(x-t) f_Y(t) dt, \quad (3.4)$$

where $f_{X_1} = f_Y$. Even for discrete random variables and small values of n , the integral in (3.4) often requires excessive computation resources [21].

When the PDF of the random variable X_n cannot be conveniently obtained but only the r first moments are known, with $r \in \mathbb{N}$, an approximation of the PDF can be obtained by using an Edgeworth expansion. Nonetheless, the resulting relative error in the large deviation regime makes these approximations inaccurate [30].

When the cumulant generating function (CGF) associated with F_Y , denoted by $K_Y : \mathbb{R} \rightarrow \mathbb{R}$, is known, the PDF f_{X_n} can be obtained via the Laplace inversion lemma [21]. That is, given two reals $\alpha_- < 0$ and $\alpha_+ > 0$, if K_Y is analytic for all $z \in \{a + b\sqrt{-1} \in \mathbb{C} : (a, b) \in \mathbb{R}^2 \text{ and } \alpha_- \leq a \leq \alpha_+\} \subset \mathbb{C}$, then,

$$f_{X_n}(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma-\infty\sqrt{-1}}^{\gamma+\infty\sqrt{-1}} \exp(nK_Y(z) - zx) dz, \quad (3.5)$$

with $\gamma \in (\alpha_-, \alpha_+)$. Note that the domain of K_Y in (3.5) has been extended to the complex plane and thus it is often referred to as the complex CGF. With an abuse of notation, both the CGF and the complex CGF are identically denoted.

In the case in which n is sufficiently large, an approximation to the Bromwich integral in (3.5) can be obtained by choosing the contour to include the unique saddlepoint of the integrand as suggested in [31]. The intuition behind this lies on the following observations:

- (i) the saddlepoint, denoted by z_0 , is unique, real and $z_0 \in (\alpha_-, \alpha_+)$;
- (ii) within a neighborhood around the saddlepoint of the form $|z - z_0| < \epsilon$, with $z \in \mathbb{C}$ and $\epsilon > 0$ sufficiently small, $\text{Im}[nK_Y(z) - zx] = 0$ and $\text{Re}[nK_Y(z) - zx]$ can be assumed

constant; and

(iii) outside such neighborhood, the integrand is negligible.

From (i), it follows that the derivative of $nK_Y(t) - tx$ with respect to t , with $t \in \mathbb{R}$, is equal to zero when it is evaluated at the saddlepoint z_0 . More specifically, for all $t \in \mathbb{R}$,

$$\frac{d}{dt}K_Y(t) = \mathbb{E}_{P_Y} [Y \exp(tY - K_Y(t))], \quad (3.6)$$

and thus

$$\mathbb{E}_{P_Y} [Y \exp(z_0 Y - K_Y(z_0))] = \frac{x}{n}, \quad (3.7)$$

which shows the dependence of z_0 on both x and n .

A Taylor series expansion of the exponent $nK_Y(z) - zx$ in the neighborhood of z_0 , leads to the following asymptotic expansion in powers of $\frac{1}{n}$ of the Bromwich integral in (3.5):

$$f_{X_n}(x) = \hat{f}_{X_n}(x) \left(1 + \frac{1}{n} \left(\frac{1}{8} \frac{K_Y^{(4)}(z_0)}{(K_Y^{(2)}(z_0))^2} - \frac{5}{24} \frac{(K_Y^{(3)}(z_0))^2}{(K_Y^{(2)}(z_0))^3} \right) + O\left(\frac{1}{n^2}\right) \right), \quad (3.8)$$

where $\hat{f}_{X_n} : \mathbb{R} \rightarrow \mathbb{R}_+$ is

$$\hat{f}_{X_n}(x) = \sqrt{\frac{1}{2\pi n K_Y^{(2)}(z_0)}} \exp(nK_Y(z_0) - z_0 x), \quad (3.9)$$

and for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$, the notation $K_Y^{(k)}(t)$ represents the k -th real derivative of the CGF K_Y evaluated at t . The first two derivatives $K_Y^{(1)}$ and $K_Y^{(2)}$ play a central role, and thus it is worth providing explicit expressions. That is,

$$K_Y^{(1)}(t) \triangleq \mathbb{E}_{P_Y} [Y \exp(tY - K_Y(t))], \quad \text{and} \quad (3.10)$$

$$K_Y^{(2)}(t) \triangleq \mathbb{E}_{P_Y} \left[\left| Y - K_Y^{(1)}(t) \right|^2 \exp(tY - K_Y(t)) \right]. \quad (3.11)$$

The function \hat{f}_{X_n} in (3.9) is referred to as the *saddlepoint approximation* of the PDF f_{X_n} and was first introduced in [31]. Nonetheless, \hat{f}_{X_n} is not necessarily a PDF as often its integral on \mathbb{R} is not equal to one. A particular exception is observed only in three cases [32]. First, when f_Y is the PDF of a Gaussian random variable, the saddlepoint approximation \hat{f}_{X_n} is identical to f_{X_n} , for all $n > 0$. Second and third, when f_Y is the PDF associated with a Gamma distribution and an inverse normal distribution, respectively, the saddlepoint approximation \hat{f}_{X_n} is exact up to a normalization constant for all $n > 0$.

An approximation to the CDF F_{X_n} can be obtained by integrating the PDF in (3.5), cf., [33–35]. In particular, the result reported in [33] leads to an asymptotic expansion of the CDF of X_n , for all $x \in \mathbb{R}$, of the form:

$$F_{X_n}(x) = \hat{F}_{X_n}(x) + O\left(\frac{1}{\sqrt{n}} \exp(nK_Y(z_0) - xz_0)\right), \quad (3.12)$$

where the function $\hat{F}_{X_n} : \mathbb{R} \rightarrow \mathbb{R}$ is the *saddlepoint approximation* of F_{X_n} . That is, for all

$x \in \mathbb{R}$,

$$\hat{F}_{X_n}(x) = \mathbb{1}_{\{z_0 > 0\}} + (-1)^{\mathbb{1}_{\{z_0 > 0\}}} \exp\left(nK_Y(z_0) - z_0x + \frac{1}{2}z_0^2 nK_Y^{(2)}(z_0)\right) Q\left(|z_0| \sqrt{nK_Y^{(2)}(z_0)}\right), \quad (3.13)$$

where the function $Q : \mathbb{R} \rightarrow [0, 1]$ is the complementary CDF of a Gaussian random variable with zero mean and unit variance. That is, for all $t \in \mathbb{R}$,

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp\left(-\frac{x^2}{2}\right) dx. \quad (3.14)$$

Finally, from the central limit theorem [30], for large values of n and for all $x \in \mathbb{R}$, a reasonable approximation to $F_{X_n}(x)$ is $1 - Q(x)$. In the following, this approximation is referred to as the *Gaussian approximation* of F_{X_n} .

The main contribution of this work is an upper bound on the error induced by the saddlepoint approximation \hat{F}_{X_n} in (3.13) (Theorem 17 in Section 3.3). This result builds upon two observations. The first observation is that the CDF F_{X_n} can be written for all $x \in \mathbb{R}$ in the form (see Appendix B),

$$\begin{aligned} F_{X_n}(x) &= \mathbb{1}_{\{z_0 \leq 0\}} \mathbb{E}_{P_{S_n}} \left[\exp(nK_Y(z_0) - z_0 S_n) \mathbb{1}_{\{S_n \leq x\}} \right] + \mathbb{1}_{\{z_0 > 0\}} \left(1 - \mathbb{E}_{P_{S_n}} \left[\exp(nK_Y(z_0) - z_0 S_n) \mathbb{1}_{\{S_n > x\}} \right] \right), \end{aligned} \quad (3.15)$$

where the random variable

$$S_n = \sum_{t=1}^n Y_t^{(z_0)} \quad (3.16)$$

induces the probability measure P_{S_n} on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and the random variables $Y_1^{(z_0)}, Y_2^{(z_0)}, \dots, Y_n^{(z_0)}$ are independent that induce probability measure $P_{Y^{(z_0)}}$ on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The probability measure $P_{Y^{(z_0)}}$ is an exponentially tilted measure [36] with respect to the probability measure P_Y at the saddlepoint z_0 . More specifically, the Radon–Nikodym derivative of the probability measure $P_{Y^{(z_0)}}$ with respect to the probability measure P_Y satisfies for all $y \in \text{supp} P_Y$,

$$\frac{dP_{Y^{(z_0)}}}{dP_Y}(y) = \exp\left(-\left(K_Y(z_0) - z_0 y\right)\right). \quad (3.17)$$

The second observation is that the saddlepoint approximation \hat{F}_{X_n} in (3.13) can be written for all $x \in \mathbb{R}$ in the form (see Appendix B),

$$\begin{aligned} \hat{F}_{X_n}(x) &= \mathbb{1}_{\{z_0 \leq 0\}} \mathbb{E}_{P_{Z_n}} \left[\exp(nK_Y(z_0) - z_0 Z_n) \mathbb{1}_{\{Z_n \leq x\}} \right] + \mathbb{1}_{\{z_0 > 0\}} \left(1 - \mathbb{E}_{P_{Z_n}} \left[\exp(nK_Y(z_0) - z_0 Z_n) \mathbb{1}_{\{Z_n > x\}} \right] \right), \end{aligned} \quad (3.18)$$

where Z_n is a Gaussian random variable with mean x , variance $nK_Y^{(2)}(z_0)$, and induces the probability measure P_{Z_n} on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Note that the means of the random variables S_n in (3.15) and Z_n in (3.18) are equal to $nK_Y^{(1)}(z_0)$, whereas their variances

are equal to $nK_Y^{(2)}(z_0)$. Note also that, from (3.7), it holds that $x = nK_Y^{(1)}(z_0)$.

Using these observations, it holds that the absolute difference between F_{X_n} in (3.15) and \hat{F}_{X_n} in (3.18) satisfies for all $x \in \mathbb{R}$,

$$\begin{aligned} & \left| F_{X_n}(x) - \hat{F}_{X_n}(x) \right| \\ &= \mathbf{1}_{\{z_0 \leq 0\}} \left| \mathbb{E}_{P_{S_n}} \left[\exp(nK_Y(z_0) - z_0 S_n) \mathbf{1}_{\{S_n \leq x\}} \right] - \mathbb{E}_{P_{Z_n}} \left[\exp(nK_Y(z_0) - z_0 Z_n) \mathbf{1}_{\{Z_n \leq x\}} \right] \right| \\ &+ \mathbf{1}_{\{z_0 > 0\}} \left| \mathbb{E}_{P_{S_n}} \left[\exp(nK_Y(z_0) - z_0 S_n) \mathbf{1}_{\{S_n > x\}} \right] - \mathbb{E}_{P_{Z_n}} \left[\exp(nK_Y(z_0) - z_0 Z_n) \mathbf{1}_{\{Z_n > x\}} \right] \right| \end{aligned} \quad (3.19)$$

A step forward (Lemma 35 in Appendix B) is to note that, when x is such that $z_0 \leq 0$, then,

$$\begin{aligned} & \left| \mathbb{E}_{P_{S_n}} \left[\exp(nK_Y(z_0) - z_0 S_n) \mathbf{1}_{\{S_n \leq x\}} \right] - \mathbb{E}_{P_{Z_n}} \left[\exp(nK_Y(z_0) - z_0 Z_n) \mathbf{1}_{\{Z_n \leq x\}} \right] \right| \\ & \leq \exp(nK_Y(z_0) - z_0 x) \min \left\{ 1, 2 \sup_{a \in \mathbb{R}} |F_{S_n}(a) - F_{Z_n}(a)| \right\}, \end{aligned} \quad (3.20)$$

and when x is such that $z_0 > 0$, it holds that

$$\begin{aligned} & \left| \mathbb{E}_{P_{S_n}} \left[\exp(nK_Y(z_0) - z_0 S_n) \mathbf{1}_{\{S_n > x\}} \right] - \mathbb{E}_{P_{Z_n}} \left[\exp(nK_Y(z_0) - z_0 Z_n) \mathbf{1}_{\{Z_n > x\}} \right] \right| \\ & \leq \exp(nK_Y(z_0) - z_0 x) \min \left\{ 1, 2 \sup_{a \in \mathbb{R}} |F_{S_n}(a) - F_{Z_n}(a)| \right\}, \end{aligned} \quad (3.21)$$

where F_{S_n} and F_{Z_n} are the CDFs of the random variables S_n and Z_n , respectively. The final result is obtained by observing that $\sup_{a \in \mathbb{R}} |F_{S_n}(a) - F_{Z_n}(a)|$ can be upper bounded using the Berry–Esseen Theorem (Theorem 14 in Section 3.2). This is essentially due to the fact that the random variable S_n is the sum of n independent random variables, i.e., (3.16), and Z_n is a Gaussian random variable, and both S_n and Z_n possess identical means and variances. Thus, the main result (Theorem 17 in Section 3.3) is that, for all $x \in \mathbb{R}$,

$$\left| F_{X_n}(x) - \hat{F}_{X_n}(x) \right| \leq \frac{2\xi_Y(z_0)}{\sqrt{n}} \exp(nK_Y(z_0) - z_0 x), \quad (3.22)$$

where

$$\xi_Y(z_0) = c_1 \left(\frac{\mathbb{E}_{P_Y} \left[\left| Y - K_Y^{(1)}(z_0) \right|^3 \exp(z_0 Y - K_Y(z_0)) \right]}{\left(K_Y^{(2)}(z_0) \right)^{3/2}} + c_2 \right), \quad (3.23)$$

with

$$c_1 \triangleq 0.33554, \text{ and} \quad (3.24a)$$

$$c_2 \triangleq 0.415. \quad (3.24b)$$

Finally, note that (3.22) holds for any finite value of n and admits the asymptotic scaling law with respect to n suggested in (3.12).

In the following sections, upper bounds on the absolute error of approximating F_{X_n} by the *Gaussian approximation* and the *saddlepoint approximation* are presented.

3.2. Gaussian Approximation

Given a random variable Y , let the function $\xi_Y : \mathbb{R} \rightarrow \mathbb{R}$ be for all $t \in \mathbb{R}$:

$$\xi_Y(t) \triangleq c_1 \left(\frac{\mathbb{E}_{P_Y} \left[\left| Y - K_Y^{(1)}(t) \right|^3 \exp(tY - K_Y(t)) \right]}{\left(K_Y^{(2)}(t) \right)^{3/2}} + c_2 \right), \quad (3.25)$$

where c_1 and c_2 are defined in (3.24).

The following theorem, known as the Berry–Esseen theorem [30], introduces an upper bound on the approximation error induced by the Gaussian approximation.

Theorem 14 (Berry–Esseen [37]) *Let Y_1, Y_2, \dots, Y_n be IID random variables, such that each of them induces the probability measure P_Y on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let also Z_n be a Gaussian random variable with mean $n K_Y^{(1)}(0)$, variance $n K_Y^{(2)}(0)$, and CDF denoted by F_{Z_n} . Then, the CDF of the random variable $X_n = Y_1 + Y_2 + \dots + Y_n$, denoted by F_{X_n} , satisfies*

$$\sup_{a \in \mathbb{R}} |F_{X_n}(a) - F_{Z_n}(a)| \leq \min \left\{ 1, \frac{\xi_Y(0)}{\sqrt{n}} \right\}, \quad (3.26)$$

where the functions $K_Y^{(1)}$, $K_Y^{(2)}$ and ξ_Y are defined in (3.10), (3.11), and (3.25).

An immediate result from Theorem 14 gives the following upper and lower bounds on $F_{X_n}(a)$, for all $a \in \mathbb{R}$,

$$F_{X_n}(a) \leq F_{Z_n}(a) + \min \left\{ 1, \frac{\xi_Y(0)}{\sqrt{n}} \right\} \triangleq \bar{\Sigma}(a, n), \quad \text{and} \quad (3.27)$$

$$F_{X_n}(a) \geq F_{Z_n}(a) - \min \left\{ 1, \frac{\xi_Y(0)}{\sqrt{n}} \right\} \triangleq \underline{\Sigma}(a, n). \quad (3.28)$$

The main drawback of Theorem 14 is that the upper bound on the approximation error does not depend on the exact value of a . More importantly, for some values of a and n , the upper bound on the approximation error might be particularly big, which leads to irrelevant results.

3.3. Exponentially Tilted Gaussian Approximation

The following theorem introduces an upper bound on the approximation error induced by approximating the CDF F_{X_n} of X_n in (3.2) by the function $\eta_Y : \mathbb{R}^2 \times \mathbb{N} \rightarrow \mathbb{R}$ defined such that for all $(\theta, a, n) \in \mathbb{R}^2 \times \mathbb{N}$,

$$\begin{aligned} & \eta_Y(\theta, a, n) \\ & \triangleq \mathbb{1}_{\{\theta > 0\}} + (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp \left(\frac{1}{2} n \theta^2 K_Y^{(2)}(\theta) + n K_Y(\theta) - n \theta K_Y^{(1)}(\theta) \right) \\ & Q \left((-1)^{\mathbb{1}_{\{\theta \leq 0\}}} \frac{a + n \theta K_Y^{(2)}(\theta) - n K_Y^{(1)}(\theta)}{\sqrt{n K_Y^{(2)}(\theta)}} \right), \end{aligned} \quad (3.29)$$

where the function $Q : \mathbb{R} \rightarrow [0, 1]$ is the complementary CDF of the standard Gaussian distribution defined in (3.14). Note that $\eta_Y(\theta, n, a)$ is identical to $\hat{F}_{X_n}(a)$, when θ is chosen to satisfy the saddlepoint $K_Y^{(1)}(\theta) = \frac{a}{n}$. Note also that $\eta_Y(0, n, a)$ is the CDF of a Gaussian random variable with mean $nK_Y^{(1)}(0)$ and variance $nK_Y^{(2)}(0)$, which are the mean and the variance of X_n in (3.2), respectively.

Theorem 15 *Let Y_1, Y_2, \dots, Y_n be IID random variables, such that each of them induces the probability measure P_Y on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and denote by K_Y the corresponding CGF. Let also F_{X_n} be the CDF of the random variable $X_n = Y_1 + Y_2 + \dots + Y_n$. Hence, for all $a \in \mathbb{R}$ and for all $\theta \in \Theta_Y$, it holds that*

$$|F_{X_n}(a) - \eta_Y(\theta, a, n)| \leq \exp(nK_Y(\theta) - \theta a) \min \left\{ 1, \frac{2 \xi_Y(\theta)}{\sqrt{n}} \right\}, \quad (3.30)$$

where

$$\Theta_Y \triangleq \{t \in \mathbb{R} : K_Y(t) < \infty\}; \quad (3.31)$$

and the functions ξ_Y and η_Y are defined in (3.25) and (3.29), respectively.

Proof: The proof of Theorem 15 is presented in Appendix B. ■

This result leads to the following upper and lower bounds on $F_{X_n}(a)$, for all $a \in \mathbb{R}$,

$$F_{X_n}(a) \leq \eta_Y(\theta, a, n) + \exp(nK_Y(\theta) - \theta a) \min \left\{ 1, \frac{2 \xi_Y(\theta)}{\sqrt{n}} \right\}, \quad \text{and} \quad (3.32)$$

$$F_{X_n}(a) \geq \eta_Y(\theta, a, n) - \exp(nK_Y(\theta) - \theta a) \min \left\{ 1, \frac{2 \xi_Y(\theta)}{\sqrt{n}} \right\}, \quad (3.33)$$

with $\theta \in \Theta_Y$.

The advantages of approximating F_{X_n} by using Theorem 15 instead of Theorem 14 are twofold. First, both the approximation η_Y and the corresponding approximation error depend on the exact value of a . In particular, the approximation can be optimized for each value of a via the parameter θ . Second, the parameter θ in (3.30) can be optimized to improve either the upper bound in (3.32) or the lower bound in (3.33) for some $a \in \mathbb{R}$. Nonetheless, such optimizations are not necessarily simple.

An alternative to the optimization on θ in (3.32) and (3.33) is to choose θ such that it minimizes $nK_Y(\theta) - \theta a$. We wrote this solution θ^* . In addition, θ^* corresponds to the saddlepoint of Z_0 defined in (3.7). This follows the intuition that, for some values of a and n , the term $\exp(nK_Y(\theta) - \theta a)$ is the one that influences the most the value of the right-hand side of (3.30). To build upon this idea, consider the following lemma.

Lemma 16 *Consider a random variable Y that induces the probability measure P_Y on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and denote by K_Y the corresponding CGF. Given $n \in \mathbb{N}$, let the function $h : \mathcal{D}_n \rightarrow \mathbb{R}$, with*

$$\mathcal{D}_n \triangleq \left\{ t \in \mathbb{R} : \frac{t}{n} \in \text{int}\mathcal{C}_Y \right\}, \quad (3.34)$$

and $\text{int}C_Y$ denoting the interior of the convex hull of $\text{supp } P_{X_n}$, be such that for all $a \in \mathcal{D}_n$,

$$h(a) = \inf_{\theta \in \Theta_Y} nK_Y(\theta) - \theta a, \quad (3.35)$$

where Θ_Y is defined in (3.31). Then, the function h is concave and for all $a \in \mathbb{R}$,

$$h(a) \leq h(n\mathbb{E}_{P_Y}[Y]) = 0. \quad (3.36)$$

Furthermore,

$$h(a) = nK_Y(\theta^*) - \theta^* a, \quad (3.37)$$

where θ^* is the unique solution in θ to

$$nK_Y^{(1)}(\theta) = a, \quad (3.38)$$

with $K_Y^{(1)}$ is defined in (3.10).

Proof: The proof of Lemma 16 is presented in Appendix C. ■

Given $(a, n) \in \mathbb{R} \times \mathbb{N}$, the value of $h(a)$ in (3.35) is the argument that minimizes the exponential term in (3.30). An interesting observation from Lemma 16 is that the maximum of h is zero, and it is reached when $a = n\mathbb{E}_{P_Y}[Y] = \mathbb{E}_{P_{X_n}}[X_n]$. In this case, $\theta^* = 0$, and thus, from (3.32) and (3.33), it holds that

$$\begin{aligned} F_{X_n}(a) &\leq \eta_Y(0, a, n) + \min \left\{ 1, \frac{2 \xi_Y(0)}{\sqrt{n}} \right\} \\ &= F_{Z_n}(a) + \min \left\{ 1, \frac{2 \xi_Y(0)}{\sqrt{n}} \right\}, \text{ and} \end{aligned} \quad (3.39)$$

$$\begin{aligned} F_{X_n}(a) &\geq \eta_Y(0, a, n) - \min \left\{ 1, \frac{2 \xi_Y(0)}{\sqrt{n}} \right\} \\ &= F_{Z_n}(a) - \min \left\{ 1, \frac{2 \xi_Y(0)}{\sqrt{n}} \right\}, \end{aligned} \quad (3.40)$$

where F_{Z_n} is the CDF defined in Theorem 14. Hence, the upper bound in (3.39) and the lower bound in (3.40) obtained from Theorem 15 are worse than those in (3.27) and (3.28) obtained from Theorem 14. In a nutshell, for values of a around the vicinity of $n\mathbb{E}_{P_Y}[Y] = \mathbb{E}_{P_{X_n}}[X_n]$, it is more interesting to use Theorem 14 instead of Theorem 15.

Alternatively, given that h is non-positive and concave, when $|a - n\mathbb{E}_{P_Y}[Y]| = |a - \mathbb{E}_{P_{X_n}}[X_n]| > \gamma$, with γ sufficiently large, it follows that

$$\exp(nK_Y(\theta^*) - \theta^* a) < \min \left\{ 1, \frac{\xi_Y(0)}{\sqrt{n}} \right\}, \quad (3.41)$$

with θ^* defined in (3.38). Hence, in this case, the right-hand side of (3.30) is always smaller than the right-hand side of (3.26). That is, for such values of a and n , the upper and lower bounds in (3.32) and (3.33) are better than those in (3.27) and (3.28), respectively. The following theorem leverages this observation.

Theorem 17 Let Y_1, Y_2, \dots, Y_n be IID random variables, such that each of them induces the probability measure P_Y on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and denote by K_Y the corresponding CGF. Let also F_{X_n} be the CDF of the random variable $X_n = Y_1 + Y_2 + \dots + Y_n$. Hence, for all $a \in \text{int } \mathcal{C}_{X_n}$, with $\text{int } \mathcal{C}_{X_n}$ the interior of the convex hull of $\text{supp } P_{X_n}$, it holds that

$$\left| F_{X_n}(a) - \hat{F}_{X_n}(a) \right| \leq \exp(nK_Y(\theta^*) - \theta^* a) \min \left\{ 1, \frac{2\xi_Y(\theta^*)}{\sqrt{n}} \right\}, \quad (3.42)$$

where θ^* is defined in (3.38), and the functions \hat{F}_{X_n} and ξ_Y are defined in (3.13), and (3.25), respectively.

Proof: The proof of Theorem 17 is presented in Appendix D. ■

An immediate result from Theorem 17 gives the following upper and lower bounds on $F_X(a)$, for all $a \in \mathbb{R}$,

$$F_{X_n}(a) \leq \hat{F}_{X_n}(a) + \exp(nK_Y(\theta^*) - \theta^* a) \min \left\{ 1, \frac{2\xi_Y(\theta^*)}{\sqrt{n}} \right\} \triangleq \bar{\Omega}(a, n), \quad \text{and} \quad (3.43)$$

$$F_{X_n}(a) \geq \hat{F}_{X_n}(a) - \exp(nK_Y(\theta^*) - \theta^* a) \min \left\{ 1, \frac{2\xi_Y(\theta^*)}{\sqrt{n}} \right\} \triangleq \underline{\Omega}(a, n). \quad (3.44)$$

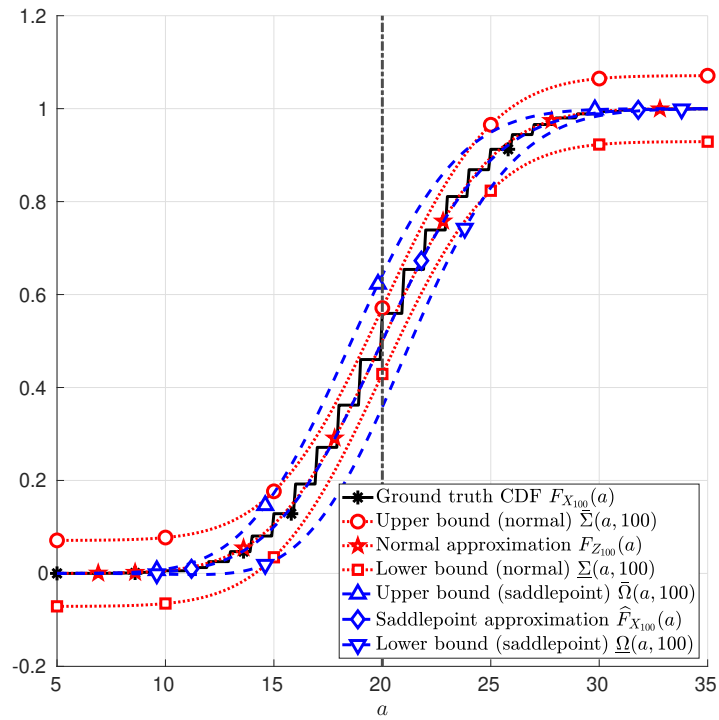
The following section presents two examples that highlight the observations mentioned above.

3.4. Examples

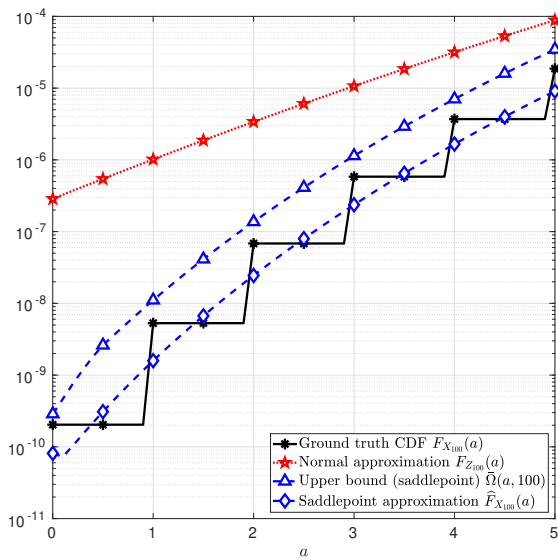
Example 1 (Discrete random variable) Let the random variables Y_1, Y_2, \dots, Y_n in (3.2) be IID Bernoulli random variables with parameter $p = 0.2$ and $n = 100$. In this case, $\mathbb{E}_{P_{X_n}}[X_n] = n\mathbb{E}_{P_Y}[Y] = 20$. Figure 3.1 depicts the CDF $F_{X_{100}}$ of X_{100} in (3.2); the normal approximation $F_{Z_{100}}$ in (3.26); and the saddlepoint approximation $\hat{F}_{X_{100}}$ in (3.13). Therein, it is also depicted the upper and lower bounds due to the normal approximation $\bar{\Sigma}$ in (3.27) and $\underline{\Sigma}$ in (3.28), respectively; and the upper and lower bounds due to the saddlepoint approximation $\bar{\Omega}$ in (3.43) and $\underline{\Omega}$ in (3.44), respectively. These functions are plotted as a function of a , with $a \in [0, 60]$.

Example 2 (Continuous random variable) Let the random variables Y_1, Y_2, \dots, Y_n in (3.2) be IID chi-squared random variables with parameter $k = 1$ and $n = 50$. In this case, $\mathbb{E}_{P_{X_n}}[X_n] = n\mathbb{E}_{P_Y}[Y] = 50$. Figure 3.2 depicts the CDF $F_{X_{50}}$ of X_{50} in (3.2); the normal approximation $F_{Z_{50}}$ in (3.26); and the saddlepoint approximation $\hat{F}_{X_{50}}$ in (3.13). Therein, it is also depicted the upper and lower bounds due to the normal approximation $\bar{\Sigma}$ in (3.27) and $\underline{\Sigma}$ in (3.28), respectively; and the upper and lower bounds due to the saddlepoint approximation $\bar{\Omega}$ in (3.43) and $\underline{\Omega}$ in (3.44), respectively. These functions are plotted as a function of a , with $a \in [0, 100]$.

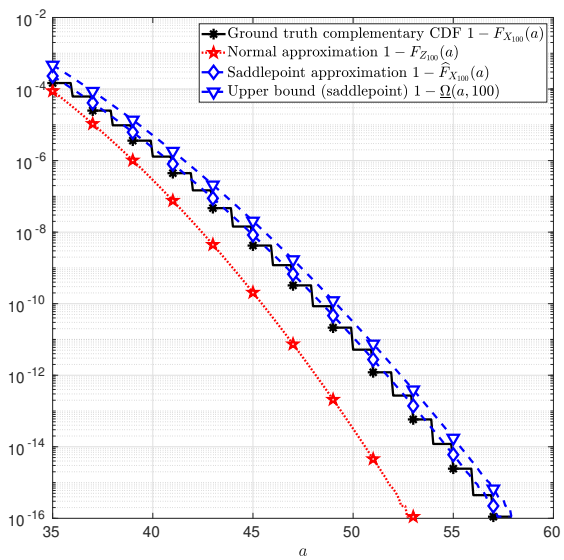
3. Approximation of the CDF of Sums of Independent Random Variables



(a) $a \in [5, 35]$

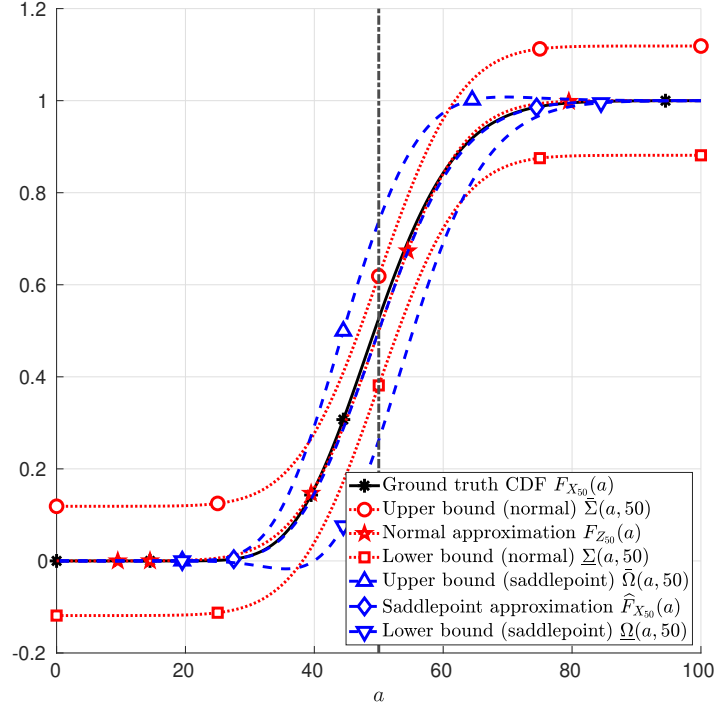


(b) $a \in [0, 5]$

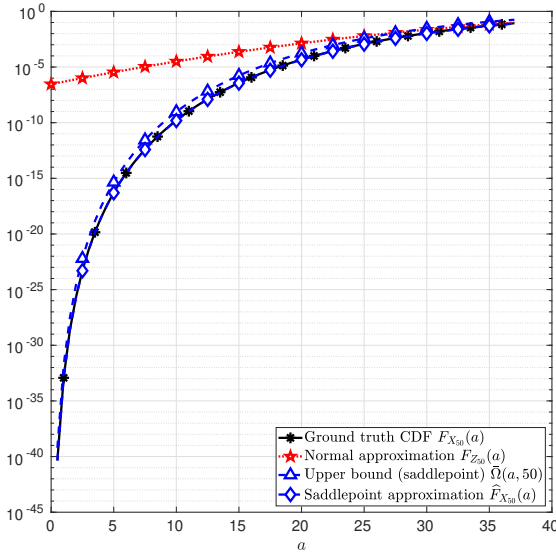


(c) $a \in [35, 60]$

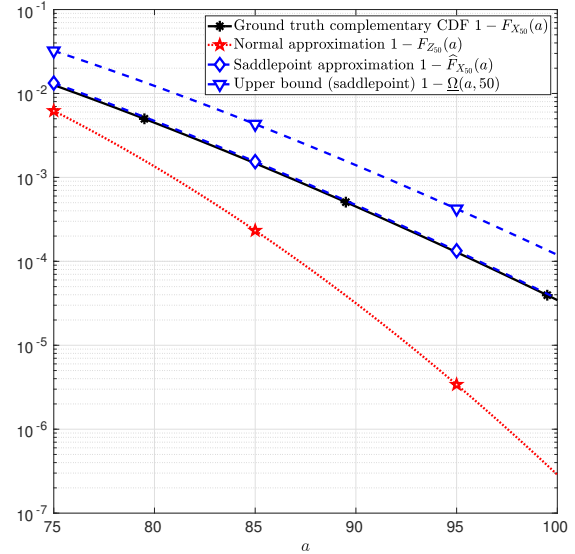
Figure 3.1.: Sum of the independent Bernoulli random variables Y_1, Y_2, \dots, Y_n , with $n = 100$, and $p = 0.2$.



(a) $a \in [0, 100]$



(b) $a \in [0, 40]$



(c) $a \in [75, 100]$

Figure 3.2.: Sum of the independent chi-square random variables Y_1, Y_2, \dots, Y_n , with $n = 50$, and $k = 1$.

Note that the CGF K_Y of a Bernoulli random variable with parameter $p = 0.2$ is given by:

$$K_Y(\theta) = \ln(0.2 \exp(\theta) + 0.8). \tag{3.45}$$

Hence,

$$K_Y^{(1)}(\theta) = \frac{p \exp(\theta)}{(1-p) \exp(\theta) + 0.8} = \frac{0.2 \exp(\theta)}{0.2 \exp(\theta) + 0.8}, \quad (3.46)$$

$$\theta^* = \ln\left(\frac{(1-p)a}{p(n-a)}\right) = \ln\left(\frac{4a}{(n-a)}\right), \quad (3.47)$$

and

$$K_Y^{(2)}(\theta) = \frac{0.2 \exp(\theta)}{0.2 \exp(\theta) + 0.8} - \left(\frac{0.2 \exp(\theta)}{0.2 \exp(\theta) + 0.8}\right)^2. \quad (3.48)$$

Note that the CGF K_Y of a chi-squared random variable with parameter $k = 1$ is given by:

$$K_Y(\theta) = -\frac{k}{2} \ln(1 - 2\theta) \text{ for } \theta < 0.5. \quad (3.49)$$

Consequently

$$K_Y^{(1)}(\theta) = \frac{k}{1 - 2\theta} = \frac{1}{1 - 2\theta}, \quad (3.50)$$

$$\theta^* = 0.5 - \frac{kn}{2a} = 0.5 - \frac{n}{2a}, \quad (3.51)$$

and

$$K_Y^{(2)}(\theta) = \frac{-2k}{(1 - 2\theta)^2}. \quad (3.52)$$

3.5. Conclusion

In this chapter, exponentially tilted Gaussian approximations of the CDF of a sum of IID random variables is introduced (Theorem 15). The Gaussian and saddlepoint approximations are special cases. Thank to this result, an upper bound on the saddlepoint approximation error is given. This upper bound confirms the precision of the saddlepoint approximation on the tail of the CDF. Finally, unlike the Gaussian approximation that fails on the CDF tail, the exponentially tilted Gaussian approximations when properly optimized overcome these limitation both on the tail and for sums with a small IID random variables.

However, The bound on the approximation error presented in Theorem 15 are looser that the bound provided by Berry-Esseen Theorem in the vicinity of the mean of the sum in (3.2). Indeed, the bound on the approximation error presented in Theorem 15 uses a triangle inequality in the proof of Lemma 35, which is loose. This is essentially the reason why Theorem 15 is not reduced to the Berry-Esseen Theorem when the parameter θ is equal to zero. An interesting extension of this work is to tighten the inequality in Lemma 35 such that the Berry-Esseen Theorem can be obtained as a special case of Theorem 15, i.e., when $\theta = 0$. If such improvement on Theorem 15 is possible, Theorem 17 will be strongly improved and it would be more precise everywhere and in particular in the vicinity of the mean of the sum in (3.2).

— 4 —

Approximation of the CDF of Sums of Independent Random Vectors

The objective of this chapter is to extend the results of Chapter 3 to sums of independent random vectors. The approach follows the same arguments (exponentially tilted measure and Gaussian approximations). This work has been partly published in [38].

4.1. Introduction

Let n be a finite integer, with $n > 1$, and let $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ be independent random vectors such that each of them induces the probability measure $P_{\mathbf{Y}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, with $k \in \mathbb{N}$. Denote by $K_{\mathbf{Y}} : \mathbb{R}^k \rightarrow \mathbb{R}$ the CGF of each of these random variables. That is, for all $\mathbf{t} \in \mathbb{R}^k$,

$$K_{\mathbf{Y}}(\mathbf{t}) = \ln \left(\mathbb{E}_{P_{\mathbf{Y}}} \left[\exp \left(\mathbf{t}^{\top} \mathbf{Y} \right) \right] \right). \quad (4.1)$$

The gradient of the CGF $K_{\mathbf{Y}}$ is a function denoted by $K_{\mathbf{Y}}^{(1)} : \mathbb{R}^k \rightarrow \mathbb{R}^k$. More specifically, for all $\mathbf{t} \in \mathbb{R}^k$,

$$K_{\mathbf{Y}}^{(1)}(\mathbf{t}) = \mathbb{E}_{P_{\mathbf{Y}}} \left[\mathbf{Y} \exp \left(\mathbf{t}^{\top} \mathbf{Y} - K_{\mathbf{Y}}(\mathbf{t}) \right) \right]. \quad (4.2)$$

The Hessian of the CGF $K_{\mathbf{Y}}$ is a function denoted by $K_{\mathbf{Y}}^{(2)} : \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k}$. That is, for all $\mathbf{t} \in \mathbb{R}^k$,

$$K_{\mathbf{Y}}^{(2)}(\mathbf{t}) = \mathbb{E}_{P_{\mathbf{Y}}} \left[\left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\mathbf{t}) \right) \left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\mathbf{t}) \right)^{\top} \exp \left(\mathbf{t}^{\top} \mathbf{Y} - K_{\mathbf{Y}}(\mathbf{t}) \right) \right]. \quad (4.3)$$

Note that $K_{\mathbf{Y}}^{(2)}(\mathbf{0})$ is the covariance matrix of the random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$. In the following, $K_{\mathbf{Y}}^{(2)}(\mathbf{0})$ is assumed to be positive definite (instead of positive semidefinite).

Let also

$$\mathbf{X}_n \triangleq \sum_{t=1}^n \mathbf{Y}_t \quad (4.4)$$

be a random vector that induces the probability measure $P_{\mathbf{X}_n}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, with CDF denoted by $F_{\mathbf{X}_n}$.

Often, the calculation of the CDF $F_{\mathbf{X}_n}$ requires elaborated numerical methods. From this perspective, approximations to the CDF $F_{\mathbf{X}_n}$, e.g., Gaussian approximations and saddlepoint approximations [21, 39–41], are rather popular in the realm of applied mathematics. In the particular case of information theory, Gaussian and saddlepoint approximations play central roles in the approximation of the fundamental limits of data transmission, c.f., [14, 15, 18–20, 42–44].

When for all $i \in \{1, 2, \dots, n\}$ the random vector \mathbf{Y}_i in (4.4) is absolutely continuous and its corresponding CGF $K_{\mathbf{Y}}$ is such that the set

$$\mathcal{C}_{\mathbf{Y}} \triangleq \left\{ \boldsymbol{\theta} \in \mathbb{R}^k : K_{\mathbf{Y}}(\boldsymbol{\theta}) < \infty \right\} \cap]-\infty, 0[^k \quad (4.5)$$

is not empty, the CDF $F_{\mathbf{X}_n}$ can be written as a complex integral [39]. In particular, for all $\mathbf{x} \in \mathbb{R}^k$,

$$F_{\mathbf{X}_n}(\mathbf{x}) = \int_{\mathbf{c}-i\mathbf{e}}^{\mathbf{c}+i\mathbf{e}} \frac{\exp(nK_{\mathbf{Y}}(\boldsymbol{\tau}) - \boldsymbol{\tau}^T \mathbf{x})}{(2\pi i)^k \prod_{t=1}^k \tau_t} d\boldsymbol{\tau}, \quad (4.6)$$

where i is the imaginary unit; $\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_k)$; the constant \mathbf{c} is arbitrarily chosen to satisfy $\mathbf{c} \in \mathcal{C}_{\mathbf{Y}}$; and the vector $\mathbf{e} = (e_1, e_2, \dots, e_k)$ is such that for all $t \in \{1, 2, \dots, k\}$, $e_t = +\infty$.

The complex integral in (4.6) results from the multivariate Laplace inverse transform [45], and can be approximated with high precision, as shown hereunder. Denote by \mathcal{D} the following set

$$\mathcal{D} \triangleq \left\{ \mathbf{u} \in \mathbb{R}^k : \exists \mathbf{t} \in]-\infty, 0[^k, nK_{\mathbf{Y}}^{(1)}(\mathbf{t}) = \mathbf{u} \right\}, \quad (4.7)$$

and denote by $\boldsymbol{\tau}_0 \in \mathbb{R}^k$ the unique solution in $\boldsymbol{\tau}$ to

$$K_{\mathbf{Y}}^{(1)}(\boldsymbol{\tau}) = \frac{1}{n} \mathbf{x}. \quad (4.8)$$

For all $\mathbf{x} \in \mathcal{D}$, a Taylor series expansion of $nK_{\mathbf{Y}}(\boldsymbol{\tau}) - \boldsymbol{\tau}^T \mathbf{x}$ in the neighborhood of $\boldsymbol{\tau}_0$, leads to the following asymptotic expansion of the integral in (4.6):

$$F_{\mathbf{X}_n}(\mathbf{x}) = \hat{F}_{\mathbf{X}_n}(\mathbf{x}) + O\left(\frac{\exp(nK_{\mathbf{Y}}(\boldsymbol{\tau}_0) - \boldsymbol{\tau}_0^T \mathbf{x})}{\sqrt{n}}\right), \quad (4.9)$$

where the function $\hat{F}_{\mathbf{X}_n} : \mathcal{D} \rightarrow \mathbb{R}$ is

$$\hat{F}_{\mathbf{X}_n}(\mathbf{x}) = \exp\left(nK_{\mathbf{Y}}(\boldsymbol{\tau}_0) - \boldsymbol{\tau}_0^T \mathbf{x} + \frac{n\boldsymbol{\tau}_0^T K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0}{2}\right) F_{\mathbf{G}_n^{(\boldsymbol{\tau}_0)}}\left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0\right), \quad (4.10)$$

and the function $F_{\mathbf{G}_n(\boldsymbol{\tau}_0)} : \mathbb{R}^k \rightarrow [0, 1]$ is the CDF of a Gaussian random vector with mean vector $(0, 0, \dots, 0)$ and covariance matrix $nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0)$.

The vector $\boldsymbol{\tau}_0$ and the function $\hat{F}_{\mathbf{X}_n}$ in (4.10) are respectively referred to as the *saddlepoint* and the *saddlepoint approximation* of the CDF $F_{\mathbf{X}_n}$. In [39], it is shown that the approximation $\hat{F}_{\mathbf{X}_n}$ in (4.10) also holds for the case in which for all $i \in \{1, 2, \dots, n\}$ the vector \mathbf{Y}_i in (4.4) is a lattice random vector. Moreover, when for all $i \in \{1, 2, \dots, n\}$ the random vector \mathbf{Y}_i in (4.4) is a Gaussian random vector, then *saddlepoint approximation* is exact. That is, $\hat{F}_{\mathbf{X}_n}$ and $F_{\mathbf{X}_n}$ are identical.

The main drawback of saddlepoint approximations, despite their well known precision, c.f., [21] and [41], is that the approximation error lacks of a tight upper bound. This is the main motivation of this chapter, whose main contributions are:

- (a) A real-valued function that approximates the CDF $F_{\mathbf{X}_n}$ is presented. This approximation turns out to be identical to the saddlepoint approximation $\hat{F}_{\mathbf{X}_n}$ in (4.9) when $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ are either absolutely continuous or lattices random vectors; and
- (b) an upper bound on the error induced by the proposed approximation is also presented. The asymptotic behaviour with n of the proposed upper bound is consistent with the one suggested in (4.9).
- (c) An extension of (a) and (b) for CDFs of sums of independent random vectors but not necessarily identically distributed.

This chapter is structured as follows. Section 4.2 introduces the Gaussian approximation to $F_{\mathbf{X}_n}$, which is used as a benchmark. Section 4.3 and Section 4.4 introduce the main results of this report. Mainly, an approximation to the measure $P_{\mathbf{X}_n}$ and an approximation to the CDF $F_{\mathbf{X}_n}$ are presented in Section 4.3. Section 4.4 generalizes the approximation of the measure $P_{\mathbf{X}_n}$ when \mathbf{X}_n is a sum of independent random vectors but not necessarily identically distributed. Section 4.5 presents an example and numerical results. Section 4.6 concludes this work with some final remarks and a discussion on the main results.

4.2. Gaussian Approximation

Let $\boldsymbol{\mu}_{\mathbf{X}_n} \in \mathbb{R}^k$ and $\underline{\mathbf{v}}_{\mathbf{X}_n} \in \mathbb{R}^{k \times k}$ be the mean vector and covariance matrix of the random vector \mathbf{X}_n in (4.4). The Gaussian approximation of the measure $P_{\mathbf{X}_n}$ induced by \mathbf{X}_n is the probability measure induced by a Gaussian vector with mean vector $\boldsymbol{\mu}_{\mathbf{X}_n}$ and covariance matrix $\underline{\mathbf{v}}_{\mathbf{X}_n}$. The following theorem, known as the multivariate Berry–Esseen theorem [46], introduces an upper bound on the approximation error.

Theorem 18 ([46, Theorem 1.1]) *Assume that the measure $P_{\mathbf{Y}}$ induced by each of the random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ in (4.4) satisfies,*

$$\mathbb{E}_{P_{\mathbf{Y}}}[\mathbf{Y}] = (0, 0, \dots, 0), \text{ and} \tag{4.11}$$

$$\mathbb{E}_{P_{\mathbf{Y}}}[\mathbf{Y}\mathbf{Y}^T] = \frac{1}{n} \text{diag}(1, 1, \dots, 1). \tag{4.12}$$

Let $P_{\mathbf{Z}_n}$ be the probability measure induced on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ by a Gaussian random vector \mathbf{Z}_n with mean vector $(0, 0, \dots, 0)$ and covariance matrix $\text{diag}(1, 1, \dots, 1)$. Then,

$$\sup_{\mathcal{A} \in \mathcal{C}_k} |P_{\mathbf{X}_n}(\mathcal{A}) - P_{\mathbf{Z}_n}(\mathcal{A})| \leq \min \left(1, c(k) n \mathbb{E}_{P_{\mathbf{Y}}} [\|\mathbf{Y}\|^3] \right), \tag{4.13}$$

where \mathcal{C}_k is the collection of all convex sets in $\mathcal{B}(\mathbb{R}^k)$; and the function $c : \mathbb{N} \rightarrow \mathbb{R}$ satisfies for all $k \in \mathbb{N}$,

$$c(k) = 42k^{\frac{1}{4}} + 16. \quad (4.14)$$

The measure P_{Z_n} in (4.13) is often referred to as the Gaussian approximation of P_{X_n} . Similarly, F_{Z_n} , the CDF of the measure P_{Z_n} , is referred to as the Gaussian approximation of F_{X_n} . Theorem 18 leads to the following inequalities for all $\mathbf{x} \in \mathbb{R}^k$,

$$\underline{\Sigma}(n, \mathbf{x}) \leq F_{X_n}(\mathbf{x}) \leq \bar{\Sigma}(n, \mathbf{x}), \quad (4.15)$$

where,

$$\bar{\Sigma}(n, \mathbf{x}) \triangleq F_{Z_n}(\mathbf{x}) + \min\left(1, c(k) n \mathbb{E}_{P_Y} \left[\|\mathbf{Y}\|^3 \right]\right), \quad \text{and} \quad (4.16a)$$

$$\underline{\Sigma}(n, \mathbf{x}) \triangleq F_{Z_n}(\mathbf{x}) - \min\left(1, c(k) n \mathbb{E}_{P_Y} \left[\|\mathbf{Y}\|^3 \right]\right). \quad (4.16b)$$

That is, the functions $\underline{\Sigma}(n, \cdot)$ and $\bar{\Sigma}(n, \cdot)$ are respectively a lower and an upper bound on the CDF F_{X_n} .

4.3. Exponentially Tilted Gaussian Approximation

This section introduces two central results. First, given a convex set \mathcal{A} in $\mathcal{B}(\mathbb{R}^k)$, the probability $P_{X_n}(\mathcal{A})$, with P_{X_n} the probability measure induced by the random vector \mathbf{X}_n in (4.4), is approximated by a function that is a measure but not necessary a probability measure. This function, which is parametrized by a vector in \mathbb{R}^k that can be arbitrarily chosen to locally minimize the approximation error, is often referred to as the exponentially tilted Gaussian approximation of P_{X_n} . Second, using the first result, the CDF of \mathbf{X}_n is approximated by a function that is not necessarily a CDF. Additionally, an upper bound on the approximation error induced by both functions is provided. As a by product, an upper bound and a lower bound are provided for both the measure P_{X_n} and the CDF F_{X_n} .

4.3.1. Approximation of the Measure

Given $\boldsymbol{\theta} \in \Theta_Y$, with

$$\Theta_Y \triangleq \{\mathbf{t} \in \mathbb{R}^k : K_Y(\mathbf{t}) < \infty\}, \quad (4.17)$$

let $\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}$ be independent random vectors such that each of them induces the probability measure $P_{Y^{(\boldsymbol{\theta})}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ that satisfies for all $\mathbf{y} \in \mathbb{R}^k$,

$$\frac{dP_{Y^{(\boldsymbol{\theta})}}}{dP_Y}(\mathbf{y}) = \exp\left(\boldsymbol{\theta}^\top \mathbf{y} - K_Y(\boldsymbol{\theta})\right). \quad (4.18)$$

That is, the probability measure $P_{Y^{(\boldsymbol{\theta})}}$ is an exponentially tilted measure with respect to P_Y . Denote by $P_{Y_1^{(\boldsymbol{\theta})} Y_2^{(\boldsymbol{\theta})} \dots Y_n^{(\boldsymbol{\theta})}}$ and $P_{Y_1 Y_2 \dots Y_n}$ the joint probability measures respectively induced by the independent random vectors $\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}$ and $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ in (4.4) on the measurable space $(\mathbb{R}^{k \times n}, \mathcal{B}(\mathbb{R}^{k \times n}))$. Then, for all $j \in \{1, 2, \dots, n\}$ and for all $\mathbf{y}_j \in \mathbb{R}^k$, it

holds that

$$\frac{dP_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}{dP_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n}}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_2 \dots \mathbf{y}_n) = \prod_{j=1}^n \frac{dP_{\mathbf{Y}^{(\theta)}}}{dP_{\mathbf{Y}}}(\mathbf{y}_j) \quad (4.19)$$

$$= \exp\left(\sum_{j=1}^n \left(\boldsymbol{\theta}^\top \mathbf{y}_j - K_{\mathbf{Y}}(\boldsymbol{\theta})\right)\right). \quad (4.20)$$

Using this notation, for all $\mathcal{A} \subseteq \mathbb{R}^k$ and for all $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$, with $\Theta_{\mathbf{Y}}$ defined in (4.17), it holds that

$$P_{\mathbf{X}_n}(\mathcal{A}) = \mathbb{E}_{P_{\mathbf{X}_n}}[\mathbb{1}_{\{\mathbf{X}_n \in \mathcal{A}\}}] \quad (4.21a)$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n}}\left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j \in \mathcal{A}\}}\right] \quad (4.21b)$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}\left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\theta)} \in \mathcal{A}\}} \frac{dP_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n}}{dP_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}\left(\mathbf{Y}_1^{(\theta)}, \mathbf{Y}_2^{(\theta)}, \dots, \mathbf{Y}_n^{(\theta)}\right)\right] \quad (4.21c)$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}\left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\theta)} \in \mathcal{A}\}} \left(\frac{dP_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}{dP_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n}}\left(\mathbf{Y}_1^{(\theta)}, \mathbf{Y}_2^{(\theta)}, \dots, \mathbf{Y}_n^{(\theta)}\right)\right)^{-1}\right] \quad (4.21d)$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}\left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\theta)} \in \mathcal{A}\}} \exp\left(\sum_{j=1}^n \left(K_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{Y}_j^{(\theta)}\right)\right)\right] \quad (4.21e)$$

$$= \exp(nK_{\mathbf{Y}}(\boldsymbol{\theta})) \mathbb{E}_{P_{\mathbf{Y}_1^{(\theta)}\mathbf{Y}_2^{(\theta)}\dots\mathbf{Y}_n^{(\theta)}}}\left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\theta)} \in \mathcal{A}\}} \exp\left(-\boldsymbol{\theta}^\top \sum_{j=1}^n \mathbf{Y}_j^{(\theta)}\right)\right]. \quad (4.21f)$$

To ease the notation, consider the random vector

$$\mathbf{S}_n^{(\boldsymbol{\theta})} = \sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})}, \quad (4.22)$$

which induces the probability measure $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Hence, plugging (4.22) in (4.21f) yields,

$$P_{\mathbf{X}_n}(\mathcal{A}) = \exp(nK_{\mathbf{Y}}(\boldsymbol{\theta})) \mathbb{E}_{P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}}\left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\{\mathbf{S}_n^{(\boldsymbol{\theta})} \in \mathcal{A}\}}\right]. \quad (4.23)$$

The equality in (4.23) is central as it expresses the probability $P_{\mathbf{X}_n}(\mathcal{A})$ in terms of another measure $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$, which is the sum of n independent random vectors. From this observation, it follows that an approximation on $P_{\mathbf{X}_n}(\mathcal{A})$ can be obtained by arbitrarily replacing $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$ by its Gaussian approximation, i.e., a probability measure $P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}$ induced on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ by a Gaussian random vector $\mathbf{Z}_n^{(\boldsymbol{\theta})}$ with the same mean vector and covariance matrix as the random vector $\mathbf{S}_n^{(\boldsymbol{\theta})}$ in (4.22). Denote by $\boldsymbol{\mu}_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \in \mathbb{R}^k$ and $\boldsymbol{\nu}_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \in \mathbb{R}^{k \times k}$ the mean vector and the covariance matrix of the random vector $\mathbf{Z}_n^{(\boldsymbol{\theta})}$, respectively. Hence,

$$\boldsymbol{\mu}_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \triangleq \mathbb{E}_{P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}}\left[\mathbf{S}_n^{(\boldsymbol{\theta})}\right] \quad (4.24a)$$

$$= n\mathbb{E}_{P_{\mathbf{Y}(\boldsymbol{\theta})}} \left[\mathbf{Y}^{(\boldsymbol{\theta})} \right] \quad (4.24b)$$

$$= n\mathbb{E}_{P_{\mathbf{Y}}} \left[\mathbf{Y} \exp \left(\boldsymbol{\theta}^\top \mathbf{Y} - K_{\mathbf{Y}}(\boldsymbol{\theta}) \right) \right] \quad (4.24c)$$

$$= nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}), \quad (4.24d)$$

where $K_{\mathbf{Y}}^{(1)}$ is the gradient vector of the CGF $K_{\mathbf{Y}}$ defined in (4.2). Alternatively,

$$\underline{\mathbf{v}}_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \triangleq \mathbb{E}_{P_{S_n^{(\boldsymbol{\theta})}}} \left[\left(S_n^{(\boldsymbol{\theta})} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \left(S_n^{(\boldsymbol{\theta})} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (4.25a)$$

$$= n\mathbb{E}_{P_{\mathbf{Y}(\boldsymbol{\theta})}} \left[\left(\mathbf{Y}^{(\boldsymbol{\theta})} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y}^{(\boldsymbol{\theta})} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (4.25b)$$

$$= n\mathbb{E}_{P_{\mathbf{Y}}} \left[\left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \exp \left(\boldsymbol{\theta}^\top \mathbf{Y} - K_{\mathbf{Y}}(\boldsymbol{\theta}) \right) \right] \quad (4.25c)$$

$$= nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}), \quad (4.25d)$$

where $K_{\mathbf{Y}}^{(2)}$ is the Hessian matrix of the CGF $K_{\mathbf{Y}}$ defined in (4.3). The equality (4.25b) is a consequence of the random vector $S_n^{(\boldsymbol{\theta})}$ in (4.22) being a sum of independent random vectors. Hence, the central idea for providing an approximation to $P_{\mathbf{X}_n}$ is to approximate the RHS of (4.23) by the function $\eta_{\mathbf{Y}} : \mathbb{R}^k \times \mathcal{B}(\mathbb{R}^k) \times \mathbb{N} \rightarrow \mathbb{R}$, which is such that

$$\eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n) \triangleq \exp(nK_{\mathbf{Y}}(\boldsymbol{\theta})) \mathbb{E}_{P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\boldsymbol{\theta})} \right) \mathbb{1}_{\{\mathbf{Z}_n^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \right]. \quad (4.26)$$

Note that $\eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$ can also be expressed as follows:

$$\eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n) = \exp(nK_{\mathbf{Y}}(\boldsymbol{\theta})) \int_{\mathbb{R}^k} \exp(-\boldsymbol{\theta}^\top \mathbf{z}) \mathbb{1}_{\{\mathbf{z} \in \mathcal{A}\}} dP_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}(\mathbf{z}) \quad (4.27a)$$

$$= \int_{\mathcal{A}} \frac{\exp(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{z})}{\sqrt{(2\pi)^k \det(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}))}} \exp \left(-\frac{\left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)}{2} \right) d\nu_k(\mathbf{z}) \quad (4.27b)$$

$$= \frac{\exp(nK_{\mathbf{Y}}(\boldsymbol{\theta}))}{\sqrt{(2\pi)^k \det(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}))}} \int_{\mathcal{A}} \exp \left(-\frac{\left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)}{2} \right) \exp \left(-\frac{2\boldsymbol{\theta}^\top \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \mathbf{z}}{2} \right) d\nu_k(\mathbf{z}) \quad (4.27c)$$

$$= \frac{\exp \left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^\top nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)}{\sqrt{(2\pi)^k \det(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}))}}$$

$$\int_{\mathcal{A}} \exp \left(- \frac{\left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) + \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{\top} \boldsymbol{\theta} \right)^{\top} \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) + \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{\top} \boldsymbol{\theta} \right)}{2} \right) d\nu_k(\mathbf{z}) \quad (4.27d)$$

$$= \exp \left(n \left(K_{\mathbf{Y}}(\boldsymbol{\theta}) \frac{\boldsymbol{\theta}^{\top} K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^{\top} K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \right) \frac{1}{\sqrt{(2\pi)^k \det \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)}}$$

$$\int_{\mathcal{A}} \exp \left(- \frac{\left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) + \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{\top} \boldsymbol{\theta} \right)^{\top} \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \left(\mathbf{z} - nK_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) + \left(nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \right)^{\top} \boldsymbol{\theta} \right)}{2} \right) d\nu_k(\mathbf{z}) \quad (4.27e)$$

$$= \exp \left(n \left(K_{\mathbf{Y}}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^{\top} K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^{\top} K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \right) P_{\mathbf{H}_n^{(\boldsymbol{\theta})}}(\mathcal{A}) \quad (4.27f)$$

where the probability measure $P_{\mathbf{H}_n^{(\boldsymbol{\theta})}}$ is the probability measure induced by a Gaussian random vector $\mathbf{H}_n^{(\boldsymbol{\theta})}$ with mean vector $n \left(K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) - K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta} \right)$ and covariance matrix $nK_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta})$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

Hence, the approximation of the probability $P_{\mathbf{X}_n}(\mathcal{A})$ by $\eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$ follows from the assumption that the probability measure $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$ can be approximated by the probability measure $P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}$. Given that $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$ is an exponentially tilted measure with respect to $P_{\mathbf{X}_n}$ and $P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}$ is the Gaussian approximation of $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$, the function $\eta_{\mathbf{Y}}$ in (4.26) is referred to as the *exponentially tilted Gaussian approximation of $P_{\mathbf{X}_n}$* .

The rest of this section follows by upper bounding the error induced by replacing $P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}$ by its Gaussian approximation $P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}$ in (4.23). That is, establishing an upper-bound on

$$|P_{\mathbf{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)|, \quad (4.28)$$

which is the purpose of the following lemma.

Lemma 19 *Given $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_k) \in \Theta_{\mathbf{Y}}$, with $\Theta_{\mathbf{Y}}$ in (4.17), and a convex set $\mathcal{A} \in \mathcal{B}(\mathbb{R}^k)$, it holds that*

$$|P_{\mathbf{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)| \leq \exp \left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^{\top} \mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \right) \Delta \left(P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}, P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \right), \quad (4.29)$$

where

$$\Delta \left(P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}, P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \right) \triangleq \sup_{\mathcal{B} \in \mathcal{C}_k} \left| P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}(\mathcal{B}) - P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}(\mathcal{B}) \right|; \quad (4.30)$$

the collection \mathcal{C}_k contains all convex sets in $\mathcal{B}(\mathbb{R}^k)$; and the vector $\mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) = (a_1(\mathcal{A}, \boldsymbol{\theta}), \dots, a_k(\mathcal{A}, \boldsymbol{\theta}))$,

$a_2(\mathcal{A}, \boldsymbol{\theta}), \dots, a_k(\mathcal{A}, \boldsymbol{\theta})$ is such that for all $i \in \{1, 2, \dots, k\}$,

$$a_i(\mathcal{A}, \boldsymbol{\theta}) \triangleq \begin{cases} 0 & \text{if } \theta_i = 0 \\ \inf_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i & \text{if } \theta_i > 0 \\ \sup_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i & \text{if } \theta_i < 0. \end{cases} \quad (4.31)$$

Proof: The proof of Lemma 19 is presented in Appendix E. ■

Note that the term $\Delta \left(P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}, P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \right)$ in (4.30) can be upper bounded by using Theorem 18. For doing so, consider the function $\xi_{\mathbf{Y}} : \mathbb{R}^k \rightarrow \mathbb{R}$, such that for all $\mathbf{t} \in \mathbb{R}^k$,

$$\xi_{\mathbf{Y}}(\mathbf{t}) \triangleq \mathbb{E}_{P_{\mathbf{Y}}} \left[\left(\left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\mathbf{t}) \right)^\top \left(K_{\mathbf{Y}}^{(2)}(\mathbf{t}) \right)^{-1} \left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\mathbf{t}) \right) \right)^{3/2} \exp(\mathbf{t}^\top \mathbf{Y} - K_{\mathbf{Y}}(\mathbf{t})) \right]. \quad (4.32)$$

Using this notation, the following theorem introduces an upper bound on the error induced by the approximation of the probability $P_{\mathbf{X}_n}(\mathcal{A})$ by $\eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)$, where $\mathcal{A} \subseteq \mathbb{R}^k$ is a convex Borel measurable set and $\boldsymbol{\theta} \in \mathbb{R}^k$ is a fixed parameter.

Theorem 20 For all $\mathcal{A} \in \mathcal{C}_k$, with \mathcal{C}_k the collection of all convex sets in $\mathcal{B}(\mathbb{R}^k)$, and for all $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_k) \in \Theta_{\mathbf{Y}}$, with $\Theta_{\mathbf{Y}}$ in (4.17), it holds that

$$|P_{\mathbf{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)| \leq \exp \left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \right) \min \left(1, \frac{c(k)\xi_{\mathbf{Y}}(\boldsymbol{\theta})}{\sqrt{n}} \right), \quad (4.33)$$

where the functions c and $\eta_{\mathbf{Y}}$ are respectively defined in (4.14) and (4.27f); the vector $\mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \triangleq (a_1(\mathcal{A}, \boldsymbol{\theta}), a_2(\mathcal{A}, \boldsymbol{\theta}), \dots, a_k(\mathcal{A}, \boldsymbol{\theta}))$ is defined in (4.31); and the function $\xi_{\mathbf{Y}}$ is defined in (4.32).

Proof: The proof of Theorem 20 is presented in Appendix F. ■

4.3.2. Approximation of the CDF

The CDF $F_{\mathbf{X}_n}$ can be written in the form of the probability of a convex set in $\mathcal{B}(\mathbb{R}^k)$. That is, for all $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$, let the set $\mathcal{A}_{\mathbf{x}}$ be such that

$$\mathcal{A}_{\mathbf{x}} = \left\{ (t_1, t_2, \dots, t_k) \in \mathbb{R}^k : \forall i \in \{1, 2, \dots, k\}, t_i \leq x_i \right\}. \quad (4.34)$$

Then, for all $\mathbf{x} \in \mathbb{R}^k$, it holds that

$$F_{\mathbf{X}_n}(\mathbf{x}) = P_{\mathbf{X}_n}(\mathcal{A}_{\mathbf{x}}). \quad (4.35)$$

This observation allows to use Theorem 20 to approximate the CDF $F_{\mathbf{X}_n}$ of the random vector \mathbf{X}_n in (4.4). Explicitly, for all $\mathbf{x} \in \mathbb{R}^k$ and for all $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$, with $\Theta_{\mathbf{Y}}$ in (4.17), it holds that

$$|F_{\mathbf{X}_n}(\mathbf{x}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}_{\mathbf{x}}, n)| \leq \exp \left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}_{\mathbf{x}}, \boldsymbol{\theta}) \right) \min \left(1, \frac{c(k)\xi_{\mathbf{Y}}(\boldsymbol{\theta})}{\sqrt{n}} \right). \quad (4.36)$$

The approximation of the CDF $F_{\mathbf{X}_n}$ in (4.36) can be enhanced by choosing the parameter $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$ that minimizes the right-hand side (RHS) of (4.36). From this standpoint, the

parameter $\boldsymbol{\theta}$ must be searched within a subset of Θ_Y in which

$$\boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}_x, \boldsymbol{\theta}) = \boldsymbol{\theta}^\top \mathbf{x} < +\infty. \quad (4.37)$$

More specifically, given \mathbf{A}_x in (4.34), it follows from (4.31) that the minimization must be restricted to the set

$$\Theta_Y^- \triangleq \{(t_1, t_2, \dots, t_k) \in \Theta_Y, \forall i \in \{1, 2, \dots, k\}, t_i \leq 0\}. \quad (4.38)$$

An arbitrary choice of $\boldsymbol{\theta}$ is the one that minimizes the exponential term $\exp(nK_Y(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}_x, \boldsymbol{\theta}))$, which depends on \mathbf{x} . Denote such a choice by $\boldsymbol{\theta}(\mathbf{x})$, which is defined in terms of the following quantity:

$$\boldsymbol{\tau}(\mathbf{x}) \triangleq \arg \min_{t \in \text{clo}\Theta_Y^-} (nK_Y(t) - t^\top \mathbf{x}), \quad (4.39)$$

where $\text{clo}\Theta_Y^-$ is the closure of Θ_Y^- . The uniqueness of $\boldsymbol{\tau}(\mathbf{x})$ in (4.39), for a given \mathbf{x} , follows from the fact that the set Θ_Y^- in (4.38) is convex and the function $nK_Y(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}_x, \boldsymbol{\theta})$ is strictly convex with respect to $\boldsymbol{\theta}$. More specifically, the difference between a strictly convex function, i.e., $nK_Y(\boldsymbol{\theta})$ and a linear function, i.e., $\boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}_x, \boldsymbol{\theta})$ is strictly convex. The former is strictly convex due to the fact that the covariance matrix $K_Y^{(2)}(\mathbf{0})$ is a positive definite matrix, c.f., [21, Section 1.2] and [47, Theorem 7.1].

Given $\boldsymbol{\tau}(\mathbf{x})$ in (4.39), the choice of the vector $\boldsymbol{\theta}$ to reduce the RHS of (4.36) is

$$\boldsymbol{\theta}(\mathbf{x}) = \begin{cases} \boldsymbol{\tau}(\mathbf{x}) & \text{if } \boldsymbol{\tau}(\mathbf{x}) \in \Theta_Y^- \\ \boldsymbol{\tau}(\mathbf{x}) + \boldsymbol{\epsilon} & \text{otherwise,} \end{cases} \quad (4.40)$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^k$ is chosen such that two conditions are simultaneously met: First, $\|\boldsymbol{\epsilon}\| < r$, with $r > 0$ arbitrary small; and second, $\boldsymbol{\theta}(\mathbf{x}) \in \Theta_Y^-$.

The following lemma, presents some of the properties of $\boldsymbol{\theta}(\mathbf{x})$.

Lemma 21 For all $\mathbf{x} \in \mathbb{R}^k$, $\boldsymbol{\theta}(\mathbf{x})$ in (4.40) satisfies

$$(\mathbf{x} - \boldsymbol{\mu}_{X_n})^\top \boldsymbol{\theta}(\mathbf{x}) \geq 0, \quad (4.41)$$

and

$$\boldsymbol{\theta}(\boldsymbol{\mu}_{X_n}) = \mathbf{0}, \quad (4.42)$$

where,

$$\boldsymbol{\mu}_{X_n} = (\mu_{X_{n,1}}, \mu_{X_{n,2}}, \dots, \mu_{X_{n,k}})^\top \quad (4.43)$$

is the mean of the random vector \mathbf{X}_n in (4.4).

Proof: The proof of Lemma 21 is presented in Appendix G. ■

Let the set \mathcal{E}_{X_n} be defined by

$$\mathcal{E}_{X_n} \triangleq \{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k : \forall i \in \{1, 2, \dots, k\}, x_i > \mu_{X_{n,i}}\}, \quad (4.44)$$

where for all $i \in \{1, 2, \dots, k\}$, $\mu_{X_{n,i}}$ is defined in (4.43). From (4.44), it holds that for all $\mathbf{x} \in \mathcal{E}_{X_n}$, the vector $\mathbf{x} - \boldsymbol{\mu}_{X_n}$ is such that all components are strictly positive. Similarly,

from (4.38), it follows that $\boldsymbol{\theta}(\mathbf{x})$ is a vector whose components are all nonpositive. Hence, from (4.41), it follows that,

$$\boldsymbol{\theta}(\mathbf{x}) = \mathbf{0}, \quad (4.45)$$

which leads to the Gaussian approximation of the CDF $F_{\mathbf{X}_n}$ at the point \mathbf{x} . Hence, for all $\mathbf{x} \in \mathcal{E}_{\mathbf{X}_n}$ the choice of $\boldsymbol{\theta}$ in (4.40) can still be improved. In this case, the objective is to focus on $1 - F_{\mathbf{X}_n}(\mathbf{x})$ and write it as a sum of probability measures of convex sets with respect to $P_{\mathbf{X}_n}$. The following lemma provides such a result.

Lemma 22 For all $\mathbf{x} = (x_1, x_2, \dots, x_k)^T \in \mathbb{R}^k$, with $k \in \mathbb{N}$, it holds that

$$1 - F_{\mathbf{X}_n}(\mathbf{x}) = \sum_{i=1}^k P_{\mathbf{X}_n}(\mathcal{B}(\mathbf{x}, i)), \quad (4.46)$$

where the set $\mathcal{B}(\mathbf{x}, i)$, with $i \in \{1, 2, \dots, k\}$, is,

$$\mathcal{B}(\mathbf{x}, i) = \left\{ \mathbf{t} = (t_1, t_2, \dots, t_k) \in \mathbb{R}^k : \forall j \in \{1, 2, \dots, k\}, t_j \leq x_j \text{ if } j < i, \text{ and } t_j > x_j \text{ if } j = i \right\}. \quad (4.47)$$

Proof: The proof of Lemma 22 is presented in Appendix H ■

For all $i \in \{1, 2, \dots, k\}$, the probability $P_{\mathbf{X}_n}(\mathcal{B}(\mathbf{x}, i))$ in (4.46) can be approximated by using Theorem 20. More specifically, for all $i \in \{1, 2, \dots, k\}$ and for all $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$,

$$|P_{\mathbf{X}_n}(\mathcal{B}(\mathbf{x}, i)) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{B}(\mathbf{x}, i), n)| \leq \exp\left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^T \mathbf{a}(\mathcal{B}(\mathbf{x}, i), \boldsymbol{\theta})\right) \min\left(1, \frac{c^{(k)} \xi_{\mathbf{Y}}(\boldsymbol{\theta})}{\sqrt{n}}\right). \quad (4.48)$$

Similar to the previous discussion on the minimization of the RHS of (4.36), the minimization of the RHS of (4.48) is focused only on the term $\exp\left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^T \mathbf{a}(\mathcal{B}(\mathbf{x}, i), \boldsymbol{\theta})\right)$, and thus, the choice of $\boldsymbol{\theta}$ must be constrained to a subset of $\Theta_{\mathbf{Y}}$ in which $\boldsymbol{\theta}^T \mathbf{a}(\mathcal{B}(\mathbf{x}, i), \boldsymbol{\theta})$ is finite. That is, for all $\mathbf{x} \in \mathbb{R}^k$, the choice of $\boldsymbol{\theta}$ must satisfy that

$$\boldsymbol{\theta}^T \mathbf{a}(\mathcal{B}(\mathbf{x}, i), \boldsymbol{\theta}) = \boldsymbol{\theta}^T \mathbf{x} < +\infty. \quad (4.49)$$

More specifically, given a set $\mathcal{B}(\mathbf{x}, i)$, it follows from (4.31), that the the choice of $\boldsymbol{\theta}$ must be restricted to the set

$$\Theta_{\mathbf{Y}}^i \triangleq \{(\theta_1, \theta_2, \dots, \theta_k) \in \Theta_{\mathbf{Y}} : \forall j \in \{1, 2, \dots, k\}, \theta_j \leq 0 \text{ if } j < i, \theta_j \geq 0 \text{ if } j = i, \text{ and } \theta_j = 0 \text{ otherwise}\}. \quad (4.50)$$

Denote such a choice by $\boldsymbol{\theta}_i(\mathbf{x})$, which is defined in terms of the following quantity:

$$\boldsymbol{\tau}_i(\mathbf{x}) = \arg \min_{\mathbf{t} \in \text{clo} \Theta_{\mathbf{Y}}^i} \left(nK_{\mathbf{Y}}(\mathbf{t}) - \mathbf{t}^T \mathbf{x} \right) \quad (4.51)$$

with $\text{clo} \Theta_{\mathbf{Y}}^i$ the closure of $\Theta_{\mathbf{Y}}^i$. The uniqueness of $\boldsymbol{\tau}_i(\mathbf{x})$ in (4.51), for a given \mathbf{x} , follows from the fact that the set $\Theta_{\mathbf{Y}}^i$ in (4.50) is convex and the function $nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^T \mathbf{a}(\mathcal{B}(\mathbf{x}, i), \boldsymbol{\theta})$ is strictly convex with respect to $\boldsymbol{\theta}$, as discussed above.

Given $\boldsymbol{\tau}(\mathbf{x})$ in (4.51), the choice of the vector $\boldsymbol{\theta}$ to reduce the RHS of (4.48) is

$$\boldsymbol{\theta}_i(\mathbf{x}) = \begin{cases} \boldsymbol{\tau}_i(\mathbf{x}) & \text{if } \boldsymbol{\tau}_i(\mathbf{x}) \in \Theta_Y^i \\ \boldsymbol{\tau}_i(\mathbf{x}) + \boldsymbol{\epsilon} & \text{otherwise,} \end{cases} \quad (4.52)$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^k$ is chosen such that two conditions are simultaneously met: First, $\|\boldsymbol{\epsilon}\| < r$, with $r > 0$ arbitrary small; and second, $\boldsymbol{\theta}_i(\mathbf{x}) \in \Theta_Y^i$.

Finally, for all $i \in \{1, 2, \dots, k\}$, it holds from (4.48) that the probability $P_{\mathbf{X}_n}(\mathcal{B}(\mathbf{x}, i))$ can be approximated by $\eta_Y(\boldsymbol{\theta}_i, \mathcal{B}(\mathbf{x}, i), n)$. Using these approximations in (4.46), the approximation error can be upper bounded as follows:

$$\begin{aligned} & \left| 1 - F_{\mathbf{X}_n}(\mathbf{x}) - \sum_{i=1}^k \eta_Y(\boldsymbol{\theta}_i(\mathbf{x}), \mathcal{B}(\mathbf{x}, i), n) \right| \\ &= \left| \sum_{i=1}^k P_{\mathbf{X}_n}(\mathcal{B}(\mathbf{x}, i)) - \sum_{i=1}^k \eta_Y(\boldsymbol{\theta}_i(\mathbf{x}), \mathcal{B}(\mathbf{x}, i), n) \right| \end{aligned} \quad (4.53)$$

$$\leq \sum_{i=1}^k |P_{\mathbf{X}_n}(\mathcal{B}(\mathbf{x}, i)) - \eta_Y(\boldsymbol{\theta}_i(\mathbf{x}), \mathcal{B}(\mathbf{x}, i), n)| \quad (4.54)$$

$$\leq \sum_{i=1}^k \exp\left(nK_Y(\boldsymbol{\theta}_i(\mathbf{x})) - \boldsymbol{\theta}_i^\top(\mathbf{x}) \mathbf{x}\right) \min\left(1, \frac{c(k) \xi_Y(\boldsymbol{\theta}_i(\mathbf{x}))}{\sqrt{n}}\right), \quad (4.55)$$

where the inequality in (4.54) follows from the triangular inequality; and the inequality in (4.55) follows from (4.48).

In order to ease the notation, let the functions $\zeta_Y : \mathbb{N} \times \mathbb{R}^k \rightarrow \mathbb{R}$ and $\delta_Y : \mathbb{N} \times \mathbb{R}^k \rightarrow \mathbb{R}$ be such that for all $(n, \mathbf{x}) \in \mathbb{N} \times \mathbb{R}^k$,

$$\zeta_Y(n, \mathbf{x}) \triangleq \begin{cases} \eta_Y(\boldsymbol{\theta}(\mathbf{x}), \mathcal{A}_x, n) & \text{if } \mathbf{x} \notin \mathcal{E}_{\mathbf{X}_n} \\ 1 - \sum_{i=1}^k \eta_Y(\boldsymbol{\theta}_i(\mathbf{x}), \mathcal{B}(\mathbf{x}, i), n) & \text{if } \mathbf{x} \in \mathcal{E}_{\mathbf{X}_n}, \end{cases} \quad (4.56)$$

and

$$\delta_Y(n, \mathbf{x}) \triangleq \begin{cases} \exp\left(nK_Y(\boldsymbol{\theta}(\mathbf{x})) - \boldsymbol{\theta}(\mathbf{x})^\top \mathbf{x}\right) \min\left(1, \frac{c(k) \xi_Y(\boldsymbol{\theta}(\mathbf{x}))}{\sqrt{n}}\right) & \text{if } \mathbf{x} \notin \mathcal{E}_{\mathbf{X}_n} \\ \sum_{i=1}^k \exp\left(nK_Y(\boldsymbol{\theta}_i(\mathbf{x})) - \boldsymbol{\theta}_i^\top(\mathbf{x}) \mathbf{x}\right) \min\left(1, \frac{c(k) \xi_Y(\boldsymbol{\theta}_i(\mathbf{x}))}{\sqrt{n}}\right) & \text{if } \mathbf{x} \in \mathcal{E}_{\mathbf{X}_n}. \end{cases} \quad (4.57)$$

Using this notation, the following theorem summarizes the discussion above.

Theorem 23 For all $\mathbf{x} \in \mathbb{R}^k$, it holds that

$$|F_{\mathbf{X}_n}(\mathbf{x}) - \zeta_Y(n, \mathbf{x})| \leq \delta_Y(n, \mathbf{x}), \quad (4.58)$$

where the functions ζ_Y and δ_Y are respectively defined in (4.56) and (4.57).

An immediate result from Theorem 23 is the following upper and lower bounds on $F_{\mathbf{X}_n}(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{R}^k$,

$$\underline{\Omega}(n, \mathbf{x}) \leq F_{\mathbf{X}_n}(\mathbf{x}) \leq \bar{\Omega}(n, \mathbf{x}), \quad (4.59)$$

where,

$$\bar{\Omega}(n, \mathbf{x}) \triangleq \zeta_{\mathbf{Y}}(n, \mathbf{x}) + \delta_{\mathbf{Y}}(n, \mathbf{x}), \text{ and} \quad (4.60)$$

$$\underline{\Omega}(n, \mathbf{x}) \triangleq \zeta_{\mathbf{Y}}(n, \mathbf{x}) - \delta_{\mathbf{Y}}(n, \mathbf{x}). \quad (4.61)$$

4.3.3. Connexion with the Saddlepoint Approximation

The following claim underlines the relation between the saddlepoint approximation $\hat{F}_{\mathbf{X}_n}$ in (4.10) and the function $\zeta_{\mathbf{Y}}$ in (4.56).

Claim 24 For all $\mathbf{x} \in \mathcal{D}$, with \mathcal{D} in (4.7), it holds that

$$\zeta_{\mathbf{Y}}(n, \mathbf{x}) = \hat{F}_{\mathbf{X}_n}(\mathbf{x}), \quad (4.62)$$

where the function $\hat{F}_{\mathbf{X}_n}$ and $\zeta_{\mathbf{Y}}$ are respectively defined in (4.10) and (4.56).

Proof: From (4.7), note that for all $\mathbf{x} \in \mathcal{D}$ the solution in \mathbf{t} to

$$nK_{\mathbf{Y}}^{(1)}(\mathbf{t}) = \mathbf{x} \quad (4.63)$$

denoted by $\boldsymbol{\tau}_0$ exists and the components of $\boldsymbol{\tau}_0$ are all strictly negative. Thus, $\boldsymbol{\tau}_0 \in \Theta_{\mathbf{Y}}^-$, with $\Theta_{\mathbf{Y}}^-$ in (4.38).

Note that the vector $\boldsymbol{\tau}_0$ is also a solution to (4.39) for all $\mathbf{x} \in \mathcal{D}$. This follows from the fact that the CGF $K_{\mathbf{Y}}$ is strictly convex and $K_{\mathbf{Y}}^{(1)}$ is the gradient vector of the CGF $K_{\mathbf{Y}}$. Thus, the vector $\boldsymbol{\theta}(\mathbf{x})$ in (4.40) satisfies

$$\boldsymbol{\theta}(\mathbf{x}) = \boldsymbol{\tau}_0. \quad (4.64)$$

Then, for all $\mathbf{x} \in \mathcal{D}$, from (4.64), all the components of $\boldsymbol{\theta}(\mathbf{x})$ are strictly negative and thus, $\mathbf{x} \notin \mathcal{E}_{\mathbf{X}_n}$. Then, plugging (4.64) in (4.56) yields

$$\zeta_{\mathbf{Y}}(n, \mathbf{x}) = \eta_{\mathbf{Y}}(\boldsymbol{\tau}_0, \mathcal{A}_{\mathbf{x}}, n), \quad (4.65)$$

$$= \exp \left(n \left(K_{\mathbf{Y}}(\boldsymbol{\tau}_0) - \boldsymbol{\tau}_0^{\top} K_{\mathbf{Y}}^{(1)}(\boldsymbol{\tau}_0) + \frac{\boldsymbol{\tau}_0^{\top} K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0}{2} \right) \right) P_{\mathbf{H}_n^{(\boldsymbol{\tau}_0)}}(\mathcal{A}_{\mathbf{x}}) \quad (4.66)$$

$$= \exp \left(n K_{\mathbf{Y}}(\boldsymbol{\tau}_0) - n \boldsymbol{\tau}_0^{\top} K_{\mathbf{Y}}^{(1)}(\boldsymbol{\tau}_0) + \frac{n \boldsymbol{\tau}_0^{\top} K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0}{2} \right) P_{\mathbf{H}_n^{(\boldsymbol{\tau}_0)}}(\mathcal{A}_{\mathbf{x}}) \quad (4.67)$$

$$= \exp \left(n K_{\mathbf{Y}}(\boldsymbol{\tau}_0) - \boldsymbol{\tau}_0^{\top} \mathbf{x} + \frac{n \boldsymbol{\tau}_0^{\top} K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0}{2} \right) P_{\mathbf{H}_n^{(\boldsymbol{\tau}_0)}}(\mathcal{A}_{\mathbf{x}}) \quad (4.68)$$

$$= \exp \left(n K_{\mathbf{Y}}(\boldsymbol{\tau}_0) - \boldsymbol{\tau}_0^{\top} \mathbf{x} + \frac{n \boldsymbol{\tau}_0^{\top} K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0}{2} \right) F_{\mathbf{H}_n^{(\boldsymbol{\tau}_0)}}(\mathbf{x}), \quad (4.69)$$

where $P_{\mathbf{H}_n^{(\boldsymbol{\tau}_0)}}$ is the probability measure induced by a Gaussian random vector $\mathbf{H}_n^{(\boldsymbol{\tau}_0)}$ with mean vector $n \left(K_{\mathbf{Y}}^{(1)}(\boldsymbol{\tau}_0) - K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0) \boldsymbol{\tau}_0 \right)$ and covariance matrix $n K_{\mathbf{Y}}^{(2)}(\boldsymbol{\tau}_0)$ on the measurable space

$(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$; and $F_{\mathbf{H}_n^{(\tau_0)}}$ is the CDF of the random vector $\mathbf{H}_n^{(\tau_0)}$. The equality in (4.66) follows from (4.27f). The equality in (4.68) follows from τ_0 being the solution in \mathbf{t} to $nK_{\mathbf{Y}}^{(1)}(\mathbf{t}) = \mathbf{x}$. Let the random vector $\mathbf{Z}_n^{(\tau_0)}$ be such that

$$\mathbf{Z}_n^{(\tau_0)} = \mathbf{H}_n^{(\tau_0)} - n \left(K_{\mathbf{Y}}^{(1)}(\tau_0) - K_{\mathbf{Y}}^{(2)}(\tau_0)\tau_0 \right), \quad (4.70)$$

which induces the probability measure $P_{\mathbf{Z}_n^{(\tau_0)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and the corresponding CDF is denoted by $F_{\mathbf{Z}_n^{(\tau_0)}}$. Then, the mean vector and the covariance matrix of $\mathbf{Z}_n^{(\tau_0)}$ are respectively $(0, 0, \dots, 0)$ and $nK_{\mathbf{Y}}^{(2)}(\tau_0)$. Thus, for all $\mathbf{x} \in \mathbb{R}^k$, it holds that

$$F_{\mathbf{H}_n^{(\tau_0)}}(\mathbf{x}) = F_{\mathbf{Z}_n^{(\tau_0)}} \left(\mathbf{x} - n \left(K_{\mathbf{Y}}^{(1)}(\tau_0) - K_{\mathbf{Y}}^{(2)}(\tau_0)\tau_0 \right) \right) \quad (4.71)$$

$$= F_{\mathbf{Z}_n^{(\tau_0)}} \left(nK_{\mathbf{Y}}^{(2)}(\tau_0)\tau_0 \right), \quad (4.72)$$

where the equality in (4.72) follows from τ_0 being the solution in \mathbf{t} to $nK_{\mathbf{Y}}^{(1)}(\mathbf{t}) = \mathbf{x}$. Plugging (4.72) in (4.69) yields

$$\zeta_{\mathbf{Y}}(n, \mathbf{x}) = \exp \left(nK_{\mathbf{Y}}(\tau_0) - \tau_0^\top \mathbf{x} + \frac{n\tau_0^\top K_{\mathbf{Y}}^{(2)}(\tau_0)\tau_0}{2} \right) F_{\mathbf{Z}_n^{(\tau_0)}} \left(nK_{\mathbf{Y}}^{(2)}(\tau_0)\tau_0 \right) \quad (4.73)$$

$$= \hat{F}_{\mathbf{X}_n}(\mathbf{x}), \quad (4.74)$$

where the equality in (4.74) follows from (4.10). This concludes the proof. \blacksquare

4.4. Sums of Independent Random Vectors

This section generalizes the results of Section 4.3 to CDFs of sums of independent random vectors but not necessarily identically distributed. That is, the independent random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ in (4.4) respectively induce on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ the probability measures $P_{\mathbf{Y}_1}, P_{\mathbf{Y}_2}, \dots, P_{\mathbf{Y}_n}$ that are not necessarily identical. More specifically, Lemma 19, Theorem 20, and Theorem 23 are generalised when the random vector \mathbf{X}_n in (4.4) is a sum of independent random vectors.

First, given a convex set \mathcal{A} in $\mathcal{B}(\mathbb{R}^k)$, the probability $P_{\mathbf{X}_n}(\mathcal{A})$, with $P_{\mathbf{X}_n}$ the probability measure induced the random vector \mathbf{X}_n in (4.4), is approximated by a function that is a measure but not necessary a probability measure (Lemma 26). This function is parametrized by a vector in \mathbb{R}^k that can be chosen to minimize the approximation error. Second, using the previous result (Lemma 26), the CDF of \mathbf{X}_n is approximated by a function that is not necessarily a CDF but inherits the parametrization by a vector in \mathbb{R}^k that can be chosen to minimize the approximation error (Theorem 23).

Denote by $\eta_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n} : \mathbb{R}^k \times \mathcal{B}(\mathbb{R}^k) \rightarrow \mathbb{R}$ the function such that for all $\boldsymbol{\theta} \in \mathbb{R}^k$

$$\eta_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A}) \triangleq \exp \left(\sum_{j=1}^n \left(K_{\mathbf{Y}_j}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta})\boldsymbol{\theta}}{2} \right) \right) P_{\mathbf{W}_n^{(\boldsymbol{\theta})}}(\mathcal{A}), \quad (4.75)$$

where for all $j \in \{1, 2, \dots, n\}$, the functions $K_{\mathbf{Y}_j}^{(1)} : \mathbb{R}^k \rightarrow \mathbb{R}^k$ and $K_{\mathbf{Y}_j}^{(2)} : \mathbb{R}^k \rightarrow \mathbb{R}^{k \times k}$ are respectively the gradient vector and the Hessian matrix of the CGF $K_{\mathbf{Y}_j}$ of the random

vector \mathbf{Y}_j in (4.4); the probability measure $P_{\mathbf{W}_n^{(\theta)}}$ is induced by a Gaussian random vector $\mathbf{W}_n^{(\theta)}$ with mean vector $\sum_{j=1}^n \left(K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) - K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta} \right)$ and covariance matrix $\sum_{j=1}^n K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta})$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

The objective is to show that $P_{\mathbf{X}_n}(\mathcal{A})$ can be approximated by $\eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A})$ for some $\boldsymbol{\theta} \in \mathbb{R}^k$. The intuition behind this becomes clear in the proof of Lemma 25 and in the following lines only a discussion is presented.

Note that the probability $P_{\mathbf{X}_n}(\mathcal{A})$ can be written as follows

$$P_{\mathbf{X}_n}(\mathcal{A}) = \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \mathbb{E}_{P_{\mathbf{S}^{(n, \boldsymbol{\theta})}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{S}^{(n, \boldsymbol{\theta})} \right) \mathbb{1}_{\{\mathbf{S}^{(n, \boldsymbol{\theta})} \in \mathcal{A}\}} \right], \quad (4.76)$$

where

$$\mathbf{S}^{(n, \boldsymbol{\theta})} = \sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})}, \quad (4.77)$$

and for all $j \in \{1, 2, \dots, n\}$, the random vector $\mathbf{Y}_j^{(\boldsymbol{\theta})}$ induces the probability measure $P_{\mathbf{Y}_j^{(\boldsymbol{\theta})}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ that satisfies for all $\mathbf{y} \in \mathbb{R}^k$,

$$\frac{dP_{\mathbf{Y}_j^{(\boldsymbol{\theta})}}}{dP_{\mathbf{Y}_j}}(\mathbf{y}) = \exp \left(\boldsymbol{\theta}^\top \mathbf{y} - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right). \quad (4.78)$$

That is, the probability measure $P_{\mathbf{Y}_j^{(\boldsymbol{\theta})}}$ is an exponentially tilted measure with respect to the probability measure $P_{\mathbf{Y}_j}$. The equality in (4.76) is obtained by a change of measure with which the expectation $\mathbb{E}_{P_{\mathbf{X}_n}}[\mathbb{1}_{\{\mathbf{X}_n \in \mathcal{A}\}}] = P_{\mathbf{X}_n}(\mathcal{A})$ is calculated with respect to the probability measure $P_{\mathbf{S}^{(n, \boldsymbol{\theta})}}$ induced by the random variable $\mathbf{S}^{(n, \boldsymbol{\theta})}$ in (4.77).

Note also that $\eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A})$ can be written as follows

$$\eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A}) = \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \mathbb{E}_{P_{\mathbf{Z}^{(n, \boldsymbol{\theta})}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}^{(n, \boldsymbol{\theta})} \right) \mathbb{1}_{\{\mathbf{Z}^{(n, \boldsymbol{\theta})} \in \mathcal{A}\}} \right], \quad (4.79)$$

where the probability measure $P_{\mathbf{Z}^{(n, \boldsymbol{\theta})}}$ is induced on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ by the Gaussian random vector $\mathbf{Z}^{(n, \boldsymbol{\theta})}$ with the same mean vector and covariance matrix as the random vector $\mathbf{S}^{(n, \boldsymbol{\theta})}$ in (4.77).

From this perspective, the approximation of the probability $P_{\mathbf{X}_n}(\mathcal{A})$ by $\eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A})$, follows from the arbitrary assumption that the probability measure $P_{\mathbf{S}^{(n, \boldsymbol{\theta})}}$ is sufficiently close to the probability measure $P_{\mathbf{Z}^{(n, \boldsymbol{\theta})}}$.

The idea behind the calculation of an upper bound on the error induced by such approximation consists in upper bounding the following absolute difference

$$|P_{\mathbf{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A})|, \quad (4.80)$$

which is the aim of the following lemma.

Lemma 25 *Given $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_k) \in \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$, with*

$$\Theta_{Y_1 Y_2 \dots Y_n} \triangleq \{t \in \mathbb{R}^k : \forall j \in \{1, 2, \dots, n\}, K_{Y_j}(t) < \infty\}, \quad (4.81)$$

and a convex set $\mathcal{A} \in \mathcal{B}(\mathbb{R}^k)$, it holds that

$$|P_{X_n}(\mathcal{A}) - \eta_{Y_1 Y_2 \dots Y_n}(\theta, \mathcal{A})| \leq \exp\left(\sum_{j=1}^n K_{Y_j}(\theta) - \theta^\top \mathbf{a}(\mathcal{A}, \theta)\right) \Delta(P_{\mathbf{S}^{(n, \theta)}}, P_{\mathbf{Z}^{(n, \theta)}}), \quad (4.82)$$

where

$$\Delta(P_{\mathbf{S}^{(n, \theta)}}, P_{\mathbf{Z}^{(n, \theta)}}) \triangleq \sup_{\mathcal{B} \in \mathcal{C}_k} |P_{\mathbf{S}^{(n, \theta)}}(\mathcal{B}) - P_{\mathbf{Z}^{(n, \theta)}}(\mathcal{B})|, \quad (4.83)$$

and \mathcal{C}_k is the collection of all convex sets in $\mathcal{B}(\mathbb{R}^k)$; and the vector $\mathbf{a}(\mathcal{A}, \theta) = (a_1(\mathcal{A}, \theta), a_2(\mathcal{A}, \theta), \dots, a_k(\mathcal{A}, \theta))$ is defined in (4.31).

Proof: The proof of Lemma 25 is presented in Appendix E. ■

Note that the Lemma 25 is a generalization of Lemma 19 when the the random vector \mathbf{X}_n in (4.4) is a sum of independent random vectors but not necessarily identically distributed.

Note that the term $\Delta(P_{\mathbf{S}^{(n, \theta)}}, P_{\mathbf{Z}^{(n, \theta)}})$ in (4.83) can be upper bounded by leveraging the observation that the random vector $\mathbf{S}^{(n, \theta)}$ is the sum of n independent random vectors and $\mathbf{Z}^{(n, \theta)}$ is a Gaussian random vector with the same mean vector and covariance matrix as $\mathbf{S}^{(n, \theta)}$. This allows the use of the Multivariate Berry-Esseen Theorem [46, Theorem 1.1] for upper bounding $\Delta(P_{\mathbf{S}^{(n, \theta)}}, P_{\mathbf{Z}^{(n, \theta)}})$ in (4.83). For doing so, consider the functions $\xi_{Y_1 Y_2 \dots Y_n}^j : \mathbb{R}^k \rightarrow \mathbb{R}$, with $j \in \{1, 2, \dots, n\}$, such that for all $t \in \mathbb{R}^k$,

$$\xi_{Y_1 Y_2 \dots Y_n}^j(t) \triangleq \mathbb{E}_{P_{Y_j}} \left[\left(\left(\mathbf{Y}_j - K_{Y_j}^{(1)}(t) \right)^\top \left(\sum_{s=1}^n K_{Y_s}^{(2)}(t) \right)^{-1} \left(\mathbf{Y}_j - K_{Y_j}^{(1)}(t) \right) \right)^{3/2} \exp\left(t^\top \mathbf{Y}_j - K_{Y_j}(t)\right) \right]. \quad (4.84)$$

Using this notation, the following lemma introduces an upper bound on the error induced by the approximation of the probability $P_{X_n}(\mathcal{A})$ by $\eta_{Y_1 Y_2 \dots Y_n}(\theta, \mathcal{A})$, where $\mathcal{A} \subseteq \mathbb{R}^k$ is a convex Borel measurable set and $\theta \in \mathbb{R}^k$ is a vector parameter.

Lemma 26 *Given a pair $(k, n) \in \mathbb{N}^2$, let $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ be independent random vectors that respectively induce the probability measures $P_{Y_1}, P_{Y_2}, \dots, P_{Y_n}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, with CGFs $K_{Y_1}, K_{Y_2}, \dots, K_{Y_n}$ on \mathbb{R}^k . Denoted by P_{X_n} the probability measure induced by the random vector $\mathbf{X}_n = \mathbf{Y}_1 + \mathbf{Y}_2 + \dots + \mathbf{Y}_n$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Then, for all $\mathcal{A} \in \mathcal{C}_k$, with \mathcal{C}_k the collection of all convex sets in $\mathcal{B}(\mathbb{R}^k)$, and for all $\theta = (\theta_1, \theta_2, \dots, \theta_k) \in \Theta_{Y_1 Y_2 \dots Y_n}$, with $\Theta_{Y_1 Y_2 \dots Y_n}$ in (4.17), it holds that*

$$|P_{X_n}(\mathcal{A}) - \eta_{Y_1 Y_2 \dots Y_n}(\theta, \mathcal{A})| \leq \exp\left(\sum_{j=1}^n K_{Y_j}(\theta) - \theta^\top \mathbf{a}(\mathcal{A}, \theta)\right) \min\left(1, c(k) \sum_{j=1}^n \xi_{Y_1 Y_2 \dots Y_n}^j(\theta)\right), \quad (4.85)$$

where the functions $c : \mathbb{N} \rightarrow \mathbb{R}$ and $\eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$ are respectively defined in (4.14) and (4.75); the vector $\mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \triangleq (a_1(\mathcal{A}, \boldsymbol{\theta}), a_2(\mathcal{A}, \boldsymbol{\theta}), \dots, a_k(\mathcal{A}, \boldsymbol{\theta}))$ is defined in (4.31); and for all $j \in \{1, 2, \dots, n\}$, the function $\xi_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}^j$ is defined in (4.84).

Proof: The proof of Lemma 26 is presented in Appendix I. ■

Note that the Lemma 26 is a generalization of Lemma 20 when the the random vector \mathbf{X}_n in (4.4) is a sum of independent random vectors but not necessarily identically distributed.

4.5. Examples

Consider the case in which the independent random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ in (4.4), with n fixed, are such that for all $i \in \{1, 2, \dots, n\}$,

$$\mathbf{Y}_i \triangleq \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad (4.86)$$

where $\rho \in [0, 1)$ is the Pearson correlation coefficient between the components of \mathbf{Y}_i ; and both B_1 and B_2 are independent Bernoulli random variables with parameter $p = 0.25$. The mean of \mathbf{X}_n in (4.4) is $\boldsymbol{\mu}_{\mathbf{X}_n} = np \left(1, \rho + \sqrt{1 - \rho^2}\right)^\top$.

Given a vector $\mathbf{x} \in \mathbb{R}^2$, Figure 4.1 depicts the set \mathcal{A}_x in (4.34) (blue rectangle); the set $\mathcal{B}_x(\{1\}) \setminus \mathcal{B}_x(\{1, 2\})$ (grey rectangle), the set $\mathcal{B}_x(\{2\}) \setminus \mathcal{B}_x(\{1, 2\})$ (yellow rectangle); and the set $\mathcal{B}_x(\{1, 2\})$ (red rectangle) in (4.47). Four cases are distinguished with respect to the given vector $\mathbf{x} \in \mathbb{R}^2$ and the mean vector $\boldsymbol{\mu}_{\mathbf{X}_n}$ in (4.43). In Sub-figure 4.1a, the mean vector $\boldsymbol{\mu}_{\mathbf{X}_n}$ belongs to the set $\mathcal{B}_x(\{1, 2\})$. In Sub-figure 4.1b, the mean vector $\boldsymbol{\mu}_{\mathbf{X}_n}$ belongs to the set $\mathcal{B}_x(\{2\})$. In Sub-figure 4.1c, the mean vector $\boldsymbol{\mu}_{\mathbf{X}_n}$ belongs to the set $\mathcal{B}_x(\{1\})$. In these three cases, the approximation of the CDF $F_{\mathbf{X}_n}$ is done using the set \mathcal{A}_x , i.e., using the equality in (4.35). In Sub-figure 4.1d, the mean vector $\boldsymbol{\mu}_{\mathbf{X}_n}$ belongs to the set \mathcal{A}_x . In this case, the approximation of the CDF $F_{\mathbf{X}_n}$ is done using the sets $\mathcal{B}_x(\{1\})$, $\mathcal{B}_x(\{2\})$, and $\mathcal{B}_x(\{1, 2\})$. That is, using the equality in (4.46).

Figures 4.2-4.4 depict the CDF $F_{\mathbf{X}_n}$ of \mathbf{X}_n in (4.4); the Gaussian approximation $F_{\mathbf{Z}_n}$ in (4.16); the saddlepoint approximation $\zeta_{\mathbf{Y}}$ in (4.56); and the saddlepoint upper and lower bounds $\bar{\Omega}$ in (4.60) and $\underline{\Omega}$ in (4.61); through the line $a\mathbf{d} + \boldsymbol{\mu}_{\mathbf{X}_n}$. The plots on the left and the center in Figures 4.2-4.4 are respectively for fixed vectors $\mathbf{d} = (1, 1)^\top$ and $\mathbf{d} = (1, -1)^\top$, as a function of a . The plots on the right in Figures 4.2-4.4 are in function of ρ for a fixed point in the line $a\mathbf{d} + \boldsymbol{\mu}_{\mathbf{X}_n}$, with $a \in \{-6, -12, -24\}$ and $\mathbf{d} = (1, 1)^\top$, i.e., the tail of the distribution in the direction of the vector $\mathbf{d} = (1, 1)^\top$. Note that Gaussian and saddlepoint approximations are particularly precise near to the mean $\boldsymbol{\mu}_{\mathbf{X}_n}$. That is, when $a = 0$. Nonetheless, away from the mean, i.e., $a < -4$ when $n = 25$, or $a < -10$ when $n \in \{50, 100\}$, the Gaussian approximation induces a large approximation error. Note that this is in sharp contrast with the saddlepoint approximation.

For the value of $n = 50$, Figure 4.3, the lower bound $\underline{\Omega}$ is negative, except when $a > 5$. Alternatively, the Gaussian upper and lower bounds $\bar{\Sigma}$ in (4.16a) and $\underline{\Sigma}$ in (4.16b) are trivial. That is, the lower bound is negative and the upper bound is bigger than one, which highlights the lack of formal mathematical arguments to evaluate the Gaussian approximation. For instance, note that when $a < -10$, the Gaussian approximation is bigger than the upper

bound due to the saddle point approximation. In particular, note that Figure 4.3 (Right) highlights the fact that the same observation holds for all values of ρ .

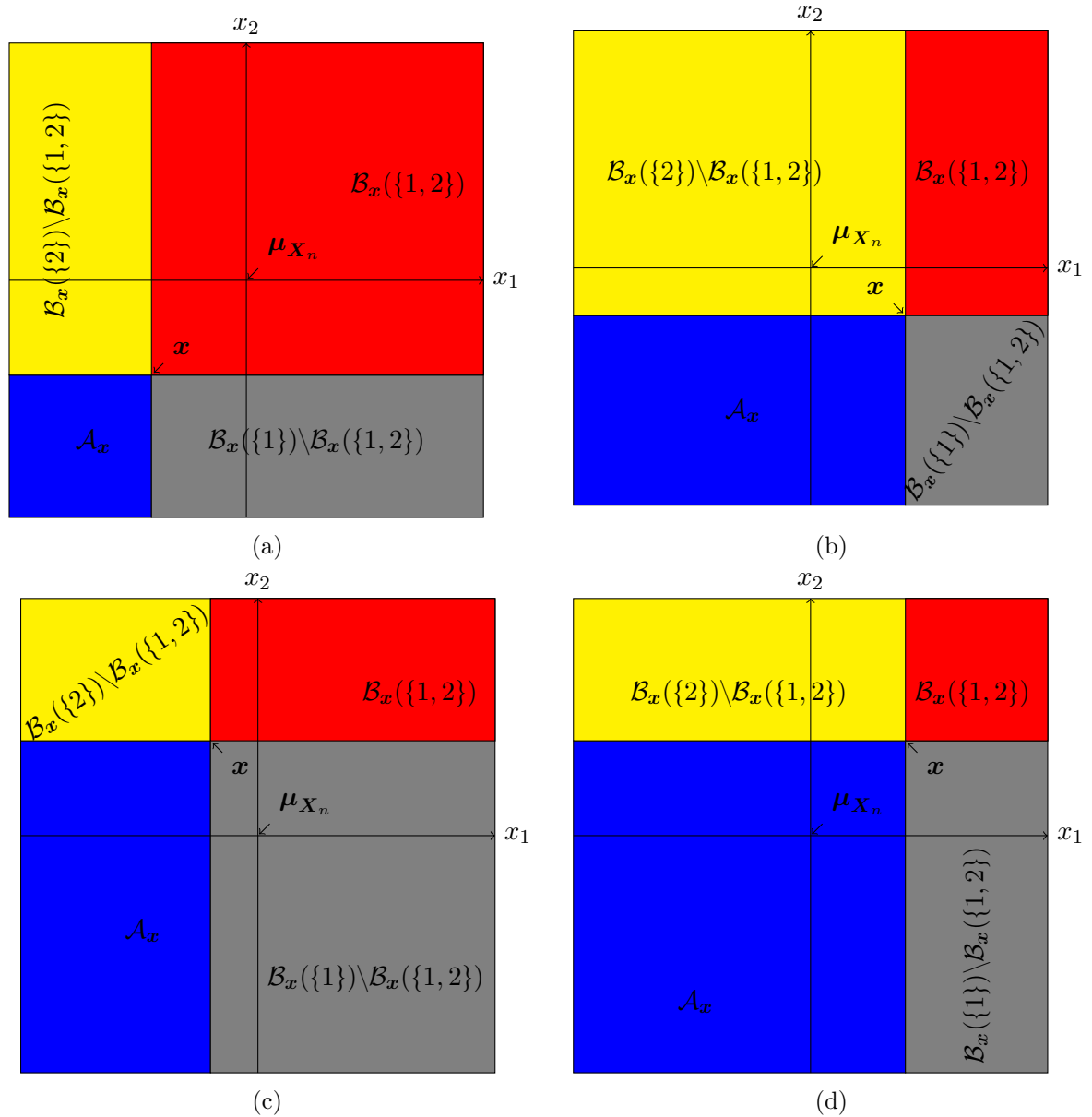


Figure 4.1.: Example of the set \mathcal{A}_x in (4.34) (blue rectangle); the set $\mathcal{B}_x(\{1\}) \setminus \mathcal{B}_x(\{1,2\})$ (grey rectangle), the set $\mathcal{B}_x(\{2\}) \setminus \mathcal{B}_x(\{1,2\})$ (yellow rectangle); and the set $\mathcal{B}_x(\{1,2\})$ (red rectangle) in (4.47).

4. Approximation of the CDF of Sums of Independent Random Vectors

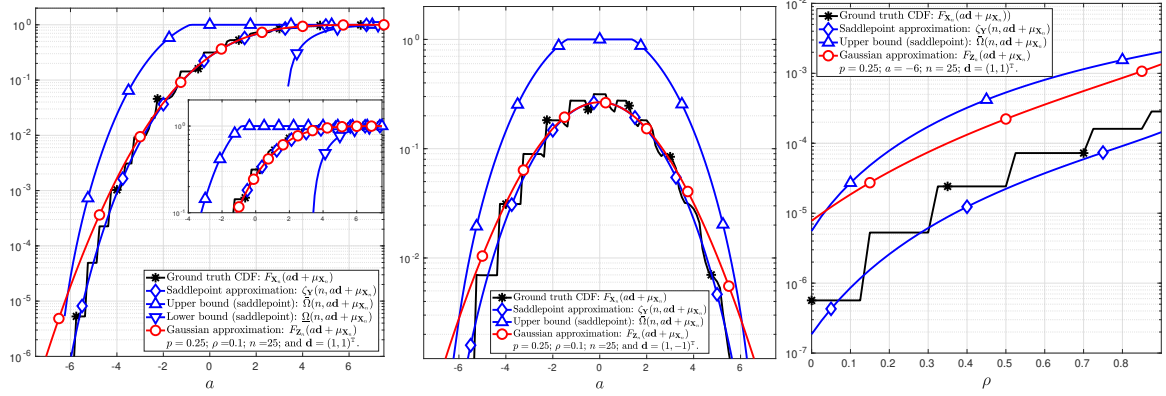


Figure 4.2.: Sum of the independent random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$, with $n = 25$, such that for all $i \in \{1, 2, \dots, n\}$, \mathbf{Y}_i satisfies (4.86).

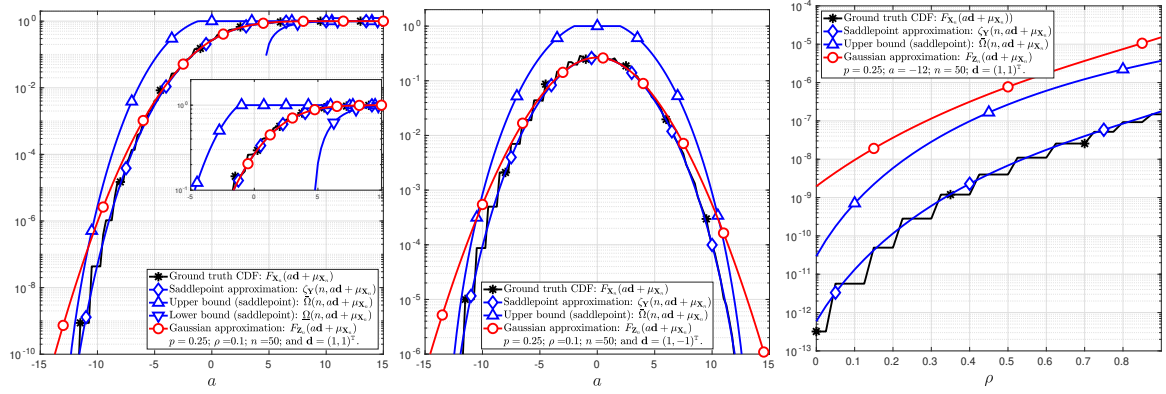


Figure 4.3.: Sum of the independent random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$, with $n = 50$, such that for all $i \in \{1, 2, \dots, n\}$, \mathbf{Y}_i satisfies (4.86).

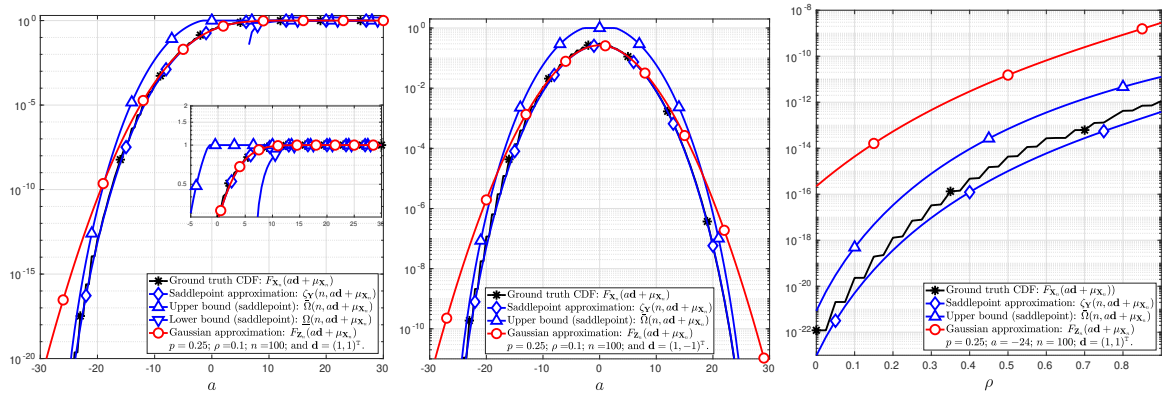


Figure 4.4.: Sum of the independent random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$, with $n = 100$, such that for all $i \in \{1, 2, \dots, n\}$, \mathbf{Y}_i satisfies (4.86).

4.6. Final Remarks and Discussion

A final remark is the fact that for all $\mathbf{x} \in \mathcal{D}$, the saddlepoint approximation $\hat{F}_{\mathbf{X}_n}(\mathbf{x})$ in (4.10) is identical to $\zeta_{\mathbf{Y}}(n, \mathbf{x})$ in (4.56). That is, the saddlepoint approximation $\hat{F}_{\mathbf{X}_n}$ can be obtained from Theorem 23 in the special case in which the vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ in (4.4) are either lattice or absolutely continuous random vectors. Additionally, it is worth to highlight that Theorem 23 holds for all random variables whose CGF exists. Under this condition, the Multivariate Berry-Essen theorem in [46, Theorem 1.1], Theorem 18, is a special case of Theorem 20 for the choice $\boldsymbol{\theta} = \mathbf{0}$.

The advantages of approximating the probability of a convex set in $\mathcal{B}(\mathbb{R}^k)$ by using Theorem 20 instead of Theorem 18 are twofold. First, the proposed upper bound on the approximation error depends on the set to be approximated. Second, both the approximation and the upper bound on the approximation error are parametrized by $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$, with $\Theta_{\mathbf{Y}}$ in (4.17). Thus, the vector $\boldsymbol{\theta}$ in (4.33) can be tuned to minimize the upper bound on the approximation error. Nonetheless, such optimization is not trivial. In this work, a non necessarily optimal choice has been made for obtaining an approximation of the CDF $F_{\mathbf{X}_n}$ in Theorem 23. That being said, the possibility to obtain tighter upper bounds on the approximation error on the measure $P_{\mathbf{X}_n}$ from Theorem 20; and on the approximation error on the CDF $F_{\mathbf{X}_n}$ from Theorem 23 is not negligible.

In the single dimension case, i.e., $k = 1$, Theorem 20 leads to the same approximation on the measure $P_{\mathbf{X}_n}$ in [28, Theorem 2]. Nonetheless, the upper bound provided in [28, Theorem 2] on the approximation error is better than the one provided by Theorem 20.

— 5 —

Memoryless Channels

THIS chapter introduces lower and upper bounds on the DEPs of point-to-point channels and multiple access memoryless channels (MAC) that are easier to compute than the previous lower and upper bounds. These bounds are the consequence of the application of the new upper bound on the saddlepoint approximation error in chapter 3 and 4 to the CDFs appearing in lower and upper bounds on the DEPs in Chapter 2. Memoryless point-to-point channels and multiple access channels with memoryless inputs are respectively considered in Section 5.1 and Section 5.2. The main results of these sections are lower and upper bounds on the DT bounds (Lemma 2, Lemma 8, Lemma 12), and the RCU bounds (Lemma 1, Lemma 7, Lemma 11). Section 5.1 also provides lower and upper bounds on the MC bounds (Lemma 4) introduced in Chapter 2.

5.1. Point-to-Point Channels with Memoryless Inputs

Memoryless inputs refer to the assumption that the probability measure induced by input channel vector on the corresponding measurable space can be written as a product of probability measures induced by each of the coordinates of the channel input vector. More specifically, the chapter focuses on the case in which each coordinate of the channel input vector induces the same probability measure, *i.e.*, the input is stationary. In other words, the inputs to the channel are assumed to be independent and identically distributed. That is, given a probability measure $P_{\mathbf{X}}$ on the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ for point-to-point channels, a probability measure P_X on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is assumed to exist such that for all boxes $\mathcal{B} \in \mathcal{B}(\mathbb{R}^n)$, *i.e.*,

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \dots \times \mathcal{B}_n, \quad (5.1)$$

where for all $t \in \{1, 2, \dots, n\}$, \mathcal{B}_t is an interval of \mathbb{R} , for all $k \in \{1, 2, \dots, K\}$, it holds that

$$P_{\mathbf{X}}(\mathcal{B}) = \prod_{t=1}^n P_X(\mathcal{B}_t). \quad (5.2)$$

The memoryless assumptions on the input and the channel result in the random variable

$\tilde{I}(\mathbf{X}; \mathbf{Y} | P_Y)$ in (2.20) that is a sum of IID random variables. Additionally, choosing Q_Y as a product of probability measures results in the random variable $\tilde{I}(\mathbf{X}; \mathbf{Y} | Q_Y)$ in (2.19) that is a sum of IID random variables. That is, assuming the existence of a probability measure Q_Y on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that for any set \mathcal{B} in (5.1),

$$Q_Y(\mathcal{B}) = \prod_{t=1}^n Q_Y(\mathcal{B}_t), \quad (5.3)$$

under the assumption in (5.2),

$$\tilde{I}(\mathbf{X}; \mathbf{Y} | Q_Y) = \sum_{t=1}^n \tilde{I}(X_t; Y_t | Q_Y), \quad (5.4)$$

where the function \tilde{I} is defined in (2.17) and $\tilde{I}(X_1; Y_1 | Q_Y)$, $\tilde{I}(X_2; Y_2 | Q_Y)$, \dots , $\tilde{I}(X_n; Y_n | Q_Y)$ are IID random variables.

Under the assumptions in (5.2) and (5.3), the DT bound (Lemma 2) and the MC (Lemma 4) bound through the function T in (2.19) become functions of CDFs of sums of IID random variables. As consequence, their approximations result directly from the approximations to these CDFs of sums of IID random variables.

In contrast to the DT bound and the MC bound, the RCU bound (Lemma 1) is not a direct function of CDFs of random variables. However, as it will be shown in Section 5.1.2, starting from the RCU bound (Lemma 1), a bound that is a direct function of CDFs of random variables is derived.

5.1.1. Approximations to the Dependence Testing and Meta Converse Bounds

The approximations to the DT bound and the MC bound essentially consist in the approximation to the function T in (2.19). The function T in (2.19) is a function of CDFs of random variables, which are sums of IID random variables under the assumptions in (5.2) and (5.3). Hence, the usage of Theorem 17 to approximate the function T in (2.19). To introduce this approximation, consider the following definitions.

The CGF $K_{l,P} : \mathbb{R} \rightarrow \mathbb{R}$ of the random variable $\tilde{I}(X; Y | Q_Y)$ is defined for all $\theta \in \mathbb{R}$ by

$$K_{l,P}(\theta) = \ln \left(\mathbb{E}_{P_X P_{Y|X}} \left[\exp(\theta \tilde{I}(X; Y | Q_Y)) \right] \right). \quad (5.5)$$

The first and second derivatives of $K_{l,P}$ denoted respectively by $K_{l,P}^{(1)}$ and $K_{l,P}^{(2)}$ are given by

$$K_{l,P}^{(1)}(\theta) = \mathbb{E}_{P_X P_{Y|X}} \left[\tilde{I}(X; Y | Q_Y) \exp(\theta \tilde{I}(X; Y | Q_Y) - K_{l,P}(\theta)) \right], \text{ and} \quad (5.6)$$

$$K_{l,P}^{(2)}(\theta) = \mathbb{E}_{P_X P_{Y|X}} \left[\left(\tilde{I}(X; Y | Q_Y) - K_{l,P}^{(1)}(\theta) \right)^2 \exp(\theta \tilde{I}(X; Y | Q_Y) - K_{l,P}(\theta)) \right]. \quad (5.7)$$

Let also the function $\xi_{l,P} : \mathbb{R} \rightarrow \mathbb{R}$ be, for all $\theta \in \mathbb{R}$,

$$\xi_{l,P}(\theta) \triangleq c_1 \left(\frac{\mathbb{E}_{P_X P_{Y|X}} \left[\left| \tilde{I}(X; Y | Q_Y) - K_{l,P}^{(1)}(\theta) \right|^3 \exp(\theta \tilde{I}(X; Y | Q_Y) - K_{l,P}(\theta)) \right]}{\left(K_{l,P}^{(2)}(\theta) \right)^{3/2}} + c_2 \right), \quad (5.8)$$

where the constants c_1 and c_2 are defined in (3.24).

Using this notation consider the functions $\hat{T}_1 : \mathbb{N} \times \mathbb{R} \times \Delta(\mathcal{X}) \times \Delta(\mathcal{Y}) \rightarrow \mathbb{R}_+$, $\hat{T}_2 : \mathbb{N} \times \mathbb{R} \times \Delta(\mathcal{X}) \times \Delta(\mathcal{Y}) \rightarrow \mathbb{R}_+$, and $E : \mathbb{N} \times \mathbb{R} \times \Delta(\mathcal{X}) \times \Delta(\mathcal{Y}) \rightarrow \mathbb{R}_+$:

$$\hat{T}_1(n, \gamma, P_X, Q_Y) = \mathbf{1}_{\{\theta > 0\}} + (-1)^{\mathbf{1}_{\{\theta > 0\}}} \exp\left(nK_{L,P}(\theta) - \theta \ln(\gamma) + \frac{1}{2}\theta^2 nK_{L,P}^{(2)}(\theta)\right) Q\left(\sqrt{nK_{L,P}^{(2)}(\theta)}|\theta|\right), \quad (5.9)$$

$$\hat{T}_2(n, \gamma, P_X, Q_Y) = \exp\left(nK_{L,P}(\theta) - (\theta + 1) \ln(\gamma) + \frac{1}{2}(\theta + 1)^2 nK_{L,P}^{(2)}(\theta)\right) Q\left(\sqrt{nK_{L,P}^{(2)}(\theta)}|\theta + 1|\right) (-1)^{\mathbf{1}_{\{\theta \leq -1\}}} + \mathbf{1}_{\{\theta \leq -1\}}, \quad \text{and} \quad (5.10)$$

$$E(n, \gamma, P_X, Q_Y) = \frac{2\xi_{L,P}(\theta)}{\sqrt{n}} \exp(nK_{L,P}(\theta) - \theta \ln(\gamma)), \quad (5.11)$$

where the parameter θ is the solution in t to

$$nK_{L,P}^{(1)}(t) = \ln(\gamma). \quad (5.12)$$

Note that \hat{T}_1 and \hat{T}_2 are the saddlepoint approximations to the CDF and the complementary CDF of the random variable $\tilde{t}(\mathbf{X}; \mathbf{Y}|Q_Y)$ when (\mathbf{X}, \mathbf{Y}) follows the probability measures $P_X P_{Y|X}$ and the $P_X Q_Y$, respectively. Consider also the following functions:

$$\underline{T}_1(n, \gamma, P_X, Q_Y) = \max\left\{0, \hat{T}_1(n, \gamma, P_X, Q_Y) - E(n, \gamma, P_X, Q_Y)\right\}, \quad (5.13)$$

$$\underline{T}_2(n, \gamma, P_X, Q_Y) = \max\left\{0, \hat{T}_2(n, \gamma, P_X, Q_Y) - \frac{E(n, \gamma, P_X, Q_Y)}{\gamma}\right\}, \quad (5.14)$$

$$\underline{T}(n, \gamma, P_X, Q_Y) = \underline{T}_1(n, \gamma, P_X, Q_Y) + \gamma \underline{T}_2(n, 2\gamma, \theta, P_X, Q_Y), \quad (5.15)$$

$$\hat{T}(n, \gamma, P_X, Q_Y) = \hat{T}_1(n, \gamma, P_X, Q_Y) + \gamma \hat{T}_2(n, \gamma, P_X, Q_Y), \quad \text{and} \quad (5.16)$$

$$\bar{T}(n, \gamma, P_X, Q_Y) = \min\left(1, \hat{T}(n, \gamma, P_X, Q_Y) + 2E(n, \gamma, P_X, Q_Y)\right). \quad (5.17)$$

The function $\hat{T}(n, \gamma, P_X, Q_Y)$ in (5.16) is referred to as the *saddlepoint approximation* to $T(n, P_X, Q_Y, \gamma)$ in (2.19), which is indeed a language abuse. The function $\hat{T}(n, \gamma, P_X, Q_Y)$ in (5.16) was first derived in [14] for continuous channels only in contrast to arbitrary channels in this work. The following Lemma introduces a new lower bound and a new upper bound on the function T in (2.19).

Lemma 27 *Given a pair $(n, \gamma) \in \mathbb{N} \times \mathbb{R}$, for all probability measures $P_X \in \Delta(\mathcal{X})$ subject to (5.2) and $Q_Y \in \Delta(\mathcal{X})$ subject to (5.3) such that $P_{Y|X}$ is absolutely continuous with respect to Q_Y , it holds that*

$$\underline{T}(n, \gamma, P_X, Q_Y) \leq T(n, P_X, Q_Y, \gamma) \leq \bar{T}(n, \gamma, P_X, Q_Y) \quad (5.18)$$

where the functions T , \underline{T} and \bar{T} are respectively defined in (2.19), (5.15) and (5.17).

Proof: The proof of Lemma 27 is presented in Appendix J. ■

Lower and upper bounds on the DT bound are obtained from Lemma 27, by replacing γ with $\ln\left(\frac{M-1}{2}\right)$ and Q_Y with the channel output probability measure P_Y induced by the channel input P_X . To provide lower and upper bounds on the MC bound, an optimal probability

measure $P_{\mathbf{X}}^*$ and an optimal constant γ^* in (2.26) must be determined. Then, lower and upper bounds on the MC are obtained by replacing γ with γ^* and P_X with $P_{\mathbf{X}}^*$. The following corollary of Lemma 27 establishes these lower and upper bounds.

Corollary 28 *Given a pair $(n, \gamma) \in \mathbb{N} \times \mathbb{R}$, denote by $P_{\mathbf{X}}^* \in \Delta(\mathcal{X})$ subject to (5.2) that minimizes the RHS of (2.26). Denote by γ^* the parameter that maximizes the RHS of (2.26) under $P_{\mathbf{X}}^*$. Then, for all $Q_{\mathbf{Y}} \in \Delta(\mathcal{Y})$ subject to (5.3) such that $P_{Y|X}$ is absolutely continuous with respect to $Q_{\mathbf{Y}}$, the following lower and upper hold on the MC bound $C(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*, M)$:*

$$\underline{C}(n, \gamma^*, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, M) \leq C(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*, M) \leq \bar{C}(n, \gamma^*, P_X, Q_{\mathbf{Y}}, M), \quad (5.19)$$

where

$$\underline{C}(n, \gamma^*, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, M) = \underline{T}(n, \gamma^*, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}) - \frac{\gamma^*}{M}, \quad (5.20)$$

$$C(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*, M) = T(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*) - \frac{\gamma^*}{M}, \quad (5.21)$$

$$\bar{C}(n, \gamma^*, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, M) = \bar{T}(n, \gamma^*, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}) - \frac{\gamma^*}{M}, \quad (5.22)$$

with the functions T , \underline{T} , \bar{T} defined in (2.19), (5.15) and (5.17), respectively. Furthermore, the MC bound $C(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*, M)$ is approximated by

$$\hat{C}(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*, M) = \hat{T}(n, P_{\mathbf{X}}^*, Q_{\mathbf{Y}}, \gamma^*) - \frac{\gamma^*}{M}, \quad (5.23)$$

which is referred as its saddlepoint approximation, with \hat{T} defined in (5.16).

5.1.2. Approximation of Random Coding Union Bound

The RCU bound in Lemma 1 is not directly expressible in term of CDFs of random variables. The following Lemma provides a bound derived from Lemma 1 that is a function of CDFs of random vectors.

Lemma 29 *Given a pair $(n, M) \in \mathbb{N}^2$, for all probability measures $P_{\mathbf{X}} \in \Delta(\mathcal{X}^n)$, and for all $\alpha \in \mathbb{R}$, the following holds with respect to the random transformation in (2.1):*

$$\lambda^*(n, M) \leq F_{\tilde{\mathbf{I}}, n}(\alpha) + (M - 1)F_{\tilde{\mathbf{I}}_2, n} \left((0, -\alpha)^T \right) \triangleq R(n, P_{\mathbf{X}}, \alpha, M) \quad (5.24)$$

where the function $F_{\tilde{\mathbf{I}}, n}$ is the CDF of the random variable $\tilde{\mathbf{I}}(\mathbf{X}; \mathbf{Y} | P_{\mathbf{Y}})$; the function $F_{\tilde{\mathbf{I}}_2, n}$ is the CDF of the vector

$$\tilde{\mathbf{I}}(\mathbf{X}, \bar{\mathbf{X}}; \mathbf{Y}) \triangleq \begin{pmatrix} \tilde{\mathbf{I}}(\mathbf{X}; \mathbf{Y} | P_{\mathbf{Y}}) - \tilde{\mathbf{I}}(\bar{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}}) \\ -\tilde{\mathbf{I}}(\mathbf{X}; \mathbf{Y} | P_{\mathbf{Y}}) \end{pmatrix}, \quad (5.25)$$

with the function $\tilde{\mathbf{I}}$ defined in (2.17) and the tuple $(\mathbf{X}, \bar{\mathbf{X}}, \mathbf{Y})$ inducing the probability measure $P_{\mathbf{X}}P_{\mathbf{Y}|X}P_{\bar{\mathbf{X}}}$ on the measurable space $(\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{X}^n, \mathcal{B}(\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{X}^n))$ with $P_{\mathbf{X}} = P_{\bar{\mathbf{X}}}$.

Proof: Starting from (2.20), for all $\alpha \in \mathbb{R}$, it holds that

$$\lambda^*(n, M)$$

$$\begin{aligned} &\leq \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\alpha\}} \min \left\{ 1, (M-1)\mathbb{E}_{P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\tilde{I}(\bar{\mathbf{X}};\mathbf{Y}|P_{\mathbf{Y}})\}} \right] \right\} \right. \\ &\quad \left. + \mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})>\alpha\}} \min \left\{ 1, (M-1)\mathbb{E}_{P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\tilde{I}(\bar{\mathbf{X}};\mathbf{Y}|P_{\mathbf{Y}})\}} \right] \right\} \right] \end{aligned} \quad (5.26)$$

$$\begin{aligned} &\leq \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\alpha\}} + (M-1)\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})>\alpha\}} \mathbb{E}_{P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\tilde{I}(\bar{\mathbf{X}};\mathbf{Y}|P_{\mathbf{Y}})\}} \right] \right] \\ &\leq \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\alpha\}} \right] + (M-1)\mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\geq\alpha\}} \mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\tilde{I}(\bar{\mathbf{X}};\mathbf{Y}|P_{\mathbf{Y}})\}} \right] \end{aligned} \quad (5.27)$$

$$= \mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq\alpha\}} \right] + (M-1)\mathbb{E}_{P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}P_{\bar{\mathbf{X}}}} \left[\mathbb{1}_{\{-\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})\leq-\alpha\}} \mathbb{1}_{\{\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})-\tilde{I}(\bar{\mathbf{X}};\mathbf{Y}|P_{\mathbf{Y}})\leq 0\}} \right] \quad (5.28)$$

$$= F_{\tilde{I},n}(\alpha) + (M-1)F_{\tilde{I}_2,n} \left((0, -\alpha)^\top \right), \quad (5.29)$$

which completes the proof. \blacksquare

Under the assumption in (5.2), the random variables $\tilde{I}(\mathbf{X};\mathbf{Y}|P_{\mathbf{Y}})$ and $\tilde{I}(\bar{\mathbf{X}};\mathbf{Y}|P_{\mathbf{Y}})$ in (5.28) are sums of IID random variables. Thus, the random vector in (5.25) is a sum of IID random vectors. That is,

$$\tilde{\mathbf{I}}(\mathbf{X}, \bar{\mathbf{X}}; \mathbf{Y}) \triangleq \sum_{t=1}^n \tilde{\mathbf{I}}(X_t, \bar{X}_t; Y_t), \quad (5.30)$$

where

$$\tilde{\mathbf{I}}(X_t, \bar{X}_t; Y_t) \triangleq \begin{pmatrix} \tilde{I}(X_t; Y_t|P_{\mathbf{Y}}) - \tilde{I}(\bar{X}_t; Y_t|P_{\mathbf{Y}}) \\ -\tilde{I}(X_t; Y_t|P_{\mathbf{Y}}) \end{pmatrix}, \quad (5.31)$$

with $(X_t, \bar{X}_t; Y_t)$ inducing the probability measure $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}}P_{\bar{\mathbf{X}}}$ on the measurable space $(\mathcal{X} \times \mathcal{Y} \times \mathcal{X}, \mathcal{B}(\mathcal{X} \times \mathcal{Y} \times \mathcal{X}))$ with $P_{\mathbf{X}} = P_{\bar{\mathbf{X}}}$. From this fact, the CDFs $F_{\tilde{I},n}$ and $F_{\tilde{I}_2,n}$ are the CDFs of sums of IID random vectors, which can be approximated using Theorem 17 and Theorem 23.

From Section 5.1.1, note that, for all $\alpha \in \mathbb{R}_+$, $F_{\tilde{I},n}(\alpha)$ is approximated by $\hat{T}_1(n, \alpha, P_{\mathbf{X}}, P_{\mathbf{Y}})$, and it holds that

$$\underline{T}_1(n, \alpha, P_{\mathbf{X}}, Q_{\mathbf{Y}}) \leq F_{\tilde{I},n}(\alpha) \leq \bar{T}_1(n, \alpha, P_{\mathbf{X}}, Q_{\mathbf{Y}}), \quad (5.32)$$

with

$$\bar{T}_1(n, \alpha, P_{\mathbf{X}}, Q_{\mathbf{Y}}) = \min \left(1, \hat{T}_1(n, \alpha, P_{\mathbf{X}}, Q_{\mathbf{Y}}) + E(n, \alpha, P_{\mathbf{X}}, Q_{\mathbf{Y}}) \right). \quad (5.33)$$

To introduce the saddlepoint approximation to the CDF $F_{\tilde{I}_2,n}$ of the random vector in (5.30), consider the following notation. Denote by $\tilde{\mathbf{t}}_2$ the random vector in (5.31) and $P_{\tilde{\mathbf{t}}_2}$ the probability induced by the random vector $\tilde{\mathbf{t}}_2$. Denote by $K_{\tilde{\mathbf{t}}_2}^{(1)}$ and $K_{\tilde{\mathbf{t}}_2}^{(2)}$ the gradient vector and Hessian matrix of the CGF $K_{\tilde{\mathbf{t}}_2}$ of the random vector $\tilde{\mathbf{t}}_2$.

Consider the functions $\eta_{\tilde{\mathbf{t}}_2} : \mathbb{R}^2 \times \mathcal{B}(\mathbb{R}^2) \times \mathbb{N} \rightarrow \mathbb{R}$ and $\zeta_{\tilde{\mathbf{t}}_2} : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for all $\boldsymbol{\theta} \in \mathbb{R}^2$,

for all $\mathcal{A} \in \mathcal{B}(\mathbb{R}^2)$, and for all $n \in \mathbb{N}$,

$$\eta_{\tilde{t}_2}(\boldsymbol{\theta}, \mathcal{A}, n) = \exp\left(n\left(K_{\tilde{t}_2}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top K_{\tilde{t}_2}^{(1)}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\tilde{t}_2}^{(2)}(\boldsymbol{\theta})\boldsymbol{\theta}}{2}\right)\right) P_{I_n^{(\boldsymbol{\theta})}}(\mathcal{A}), \quad \text{and} \quad (5.34)$$

$$\xi_{\tilde{t}_2}(\boldsymbol{\theta}) = \mathbb{E}_{P_{\tilde{t}_2}}\left[\left[\left(\tilde{t}_2 - K_{\tilde{t}_2}^{(1)}(\boldsymbol{\theta})\right)^\top \left(K_{\tilde{t}_2}^{(2)}(\boldsymbol{\theta})\right)^{-1} \left(\tilde{t}_2 - K_{\tilde{t}_2}^{(1)}(\boldsymbol{\theta})\right)\right]^{3/2} \exp\left(\boldsymbol{\theta}^\top \tilde{t}_2 - K_{\tilde{t}_2}(\boldsymbol{\theta})\right)\right], \quad (5.35)$$

where the probability measure $P_{I_n^{(\boldsymbol{\theta})}}$ is the probability measure induced by a Gaussian random vector with mean vector $n\left(K_{\tilde{t}_2}^{(1)}(\boldsymbol{\theta}) - K_{\tilde{t}_2}^{(2)}(\boldsymbol{\theta})\boldsymbol{\theta}\right)$ and covariance matrix $nK_{\tilde{t}_2}^{(2)}(\boldsymbol{\theta})$ on the measurable space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Denote by $\boldsymbol{\mu}_{\tilde{t}_2, n} = (\mu_{\tilde{t}_2, 1}, \mu_{\tilde{t}_2, 2})$ the mean vector of the random vector in (5.30). Let the set $\mathcal{E}_{\tilde{t}_2, n}$ be

$$\mathcal{E}_{\tilde{t}_2, n} = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : \forall i \in \{1, 2\}, x_i > \mu_{\tilde{t}_2, i}\}. \quad (5.36)$$

Using these notations, let the functions $\zeta_{\tilde{t}_2} : \mathbb{N} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\delta_{\tilde{t}_2} : \mathbb{N} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be for all $(n, \boldsymbol{x}) \in \mathbb{N} \times \mathbb{R}^2$,

$$\zeta_{\tilde{t}_2}(n, \boldsymbol{x}) \triangleq \begin{cases} \eta_{\tilde{t}_2}(\boldsymbol{\theta}_0(\boldsymbol{x}), \mathcal{A}_{\boldsymbol{x}}, n) & \text{if } \boldsymbol{x} \notin \mathcal{E}_{\tilde{t}_2, n} \\ 1 - \sum_{i=1}^2 \eta_{\tilde{t}_2}(\boldsymbol{\theta}_i, \mathcal{B}(\boldsymbol{x}, i), n) & \text{otherwise,} \end{cases} \quad (5.37)$$

$$\delta_{\tilde{t}_2}(n, \boldsymbol{x}) \triangleq \begin{cases} \exp\left(nK_{\tilde{t}_2}(\boldsymbol{\theta}_0(\boldsymbol{x})) - \boldsymbol{\theta}_0^\top(\boldsymbol{x})\boldsymbol{x}\right) \min\left(1, \frac{c(k)\xi_{\tilde{t}_2}(\boldsymbol{\theta}_0(\boldsymbol{x}))}{\sqrt{n}}\right) & \text{if } \boldsymbol{x} \notin \mathcal{E}_{\tilde{t}_2, n} \\ \sum_{i=1}^2 \exp\left(nK_{\tilde{t}_2}(\boldsymbol{\theta}_i(\boldsymbol{x})) - \boldsymbol{\theta}_i^\top(\boldsymbol{x})\boldsymbol{x}\right) \min\left(1, \frac{c(k)\xi_{\tilde{t}_2}(\boldsymbol{\theta}_i(\boldsymbol{x}))}{\sqrt{n}}\right) & \text{otherwise,} \end{cases} \quad (5.38)$$

with the function c in (4.14), the sets $\mathcal{A}_{\boldsymbol{x}}$ and $\mathcal{B}(\boldsymbol{x}, i)$ in (4.34) and (4.47), respectively, such that: (a) for all $i \in \{0, 1, 2\}$, the vector $\boldsymbol{\theta}(\boldsymbol{x})$ satisfies

$$\boldsymbol{\theta}_i(\boldsymbol{x}) = \begin{cases} \boldsymbol{\tau}_i(\boldsymbol{x}) & \text{if } \boldsymbol{\tau}_i(\boldsymbol{x}) \in \Theta_{\tilde{t}_2}^i \\ \boldsymbol{\tau}_i(\boldsymbol{x}) + \boldsymbol{\epsilon} & \text{otherwise,} \end{cases} \quad (5.39)$$

where

$$\boldsymbol{\tau}_i(\boldsymbol{x}) \in \arg \min_{\boldsymbol{t} \in \Theta_{\tilde{t}_2}^i} \left(nK_{\tilde{t}_2}(\boldsymbol{t}) - \boldsymbol{t}^\top \boldsymbol{x}\right), \quad (5.40)$$

with

$$\begin{aligned} \Theta_{\tilde{t}_2}^0 &= \{\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2 : K_{\tilde{t}_2}(\boldsymbol{t}) < +\infty, \text{ and } \forall i \in \{1, 2\}, t_i \leq 0\}, \text{ and } \forall i \in \{1, 2\}, \\ \Theta_{\tilde{t}_2}^i &= \{\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2 : K_{\tilde{t}_2}(\boldsymbol{t}) < +\infty, \text{ and } \forall j \in \{1, 2\}, t_j \leq 0 \text{ if } j < i, t_j \geq 0 \text{ if } j = i, \end{aligned} \quad (5.41)$$

$$\text{and } t_j = 0 \text{ otherwise} \}; \quad (5.42)$$

and $\epsilon \in \mathbb{R}^2$ is chosen such that two conditions are simultaneously met: First, $\|\epsilon\| < r$, with $r > 0$ arbitrary small; and second, $\theta_i(\mathbf{x}) \in \Theta_{i_2}^i$.

Using these notations, the saddlepoint approximation of the CDF $F_{\tilde{t}_2, n}$ of the random vector in (5.30) is given by the following corollary of Theorem 23.

Corollary 30 *For all $\mathbf{v} \in \mathbb{R}^2$, it holds that*

$$|F_{\tilde{t}_2, n}(\mathbf{v}) - \zeta_{\tilde{t}_2}(n, \mathbf{v})| \leq \delta_{\tilde{t}_2}(n, \mathbf{v}), \quad (5.43)$$

where the functions $\zeta_{\tilde{t}_2}$ and $\delta_{\tilde{t}_2}$ are defined in (5.37) and in (5.38), respectively.

Finally, the combination of (5.32) and (5.43) leads to lower and upper bounds on the function R in (5.24). To introduce these lower and upper bounds, consider the following definition. For all $(M, n) \in \mathbb{N}^2$, for all $\alpha \in \mathbb{R}$, and for all probability measures P_X , let the functions $\underline{R}(n, P_X, \alpha, M)$, $\hat{R}(n, P_X, \alpha, M)$, and $\bar{R}(n, P_X, \alpha, M)$ be defined by

$$\underline{R}(n, P_X, \alpha, M) \triangleq \underline{T}_1(n, \alpha, P_X, P_Y) + (M - 1) \max \left\{ 0, \zeta_{\tilde{t}_2}(n, (0, -\alpha)^\top) - \delta_{\tilde{t}_2}(n, (0, -\alpha)^\top) \right\}, \quad (5.44)$$

$$\hat{R}(n, P_X, \alpha, M) \triangleq \hat{T}_1(n, \alpha, P_X, P_Y) + (M - 1) \zeta_{\tilde{t}_2}(n, (0, -\alpha)^\top), \text{ and} \quad (5.45)$$

$$\bar{R}(n, P_X, \alpha, M) \triangleq \bar{T}_1(n, \alpha, P_X, P_Y) + (M - 1) \min \left\{ 1, \zeta_{\tilde{t}_2}(n, (0, -\alpha)^\top) + \delta_{\tilde{t}_2}(n, (0, -\alpha)^\top) \right\}. \quad (5.46)$$

Using these definitions, given a couple $(n, M) \in \mathbb{N}^2$, for all probability measures P_X subject to (5.2), and for all $\alpha \in \mathbb{R}$, the term $R(n, P_X, \alpha, M)$ is approximated by $\hat{R}(n, P_X, \alpha, M)$ and satisfies

$$\underline{R}(n, P_X, \alpha, M) \leq R(n, P_X, \alpha, M) \leq \bar{R}(n, P_X, \alpha, M). \quad (5.47)$$

5.1.3. Numerical Analysis

The normal and the saddlepoint approximations to the DT and MC bounds as well as their corresponding upper and lower bounds presented in Section 5.1.1 are studied in the cases of the BSC, the AWGN channel, and the S α S channel. The latter is defined by the random transformation in (2.1) subject to (2.28) and for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$:

$$P_{Y|X}(y|x) = P_Z(y - x), \quad (5.48)$$

where P_Z is a probability distribution satisfying for all $t \in \mathbb{R}$,

$$\mathbb{E}_{P_Z} [\exp(itZ)] = \exp(-|\sigma t|^\alpha), \quad (5.49)$$

with $i = \sqrt{-1}$. The reals $\alpha \in (0, 2]$ and $\sigma \in \mathbb{R}_+$ in (5.49) are parameters of the S α S channel.

In the following figures, Figures 5.1–5.3, the channel inputs are discrete $\mathcal{X} = \{-1, 1\}$, and P_X is the uniform distribution. For the results relative to the MC bound, Q_Y is chosen to be equal to the distribution P_Y , i.e., the marginal of $P_X P_{Y|X}$. The parameter γ^* is chosen to maximize

the function $C(n, 2^{nR}, P_X, P_Y, \gamma^*)$ in (5.21). The plots in Figures 5.1a–5.3a illustrate the function T in (2.19) as well as the bounds in (5.18). Figures 5.1b–5.3b illustrate the function C in (5.21) and the bounds in (5.19). The normal approximations, i.e., $\hat{N}(n, 2^{nR}, P_X, P_Y)$ in (2.39) and $\hat{N}_c(n, 2^{nR}, P_X, P_Y, \gamma^*)$ in (2.44), to the DT and MC bounds, respectively, are plotted in black diamonds. The upper bounds, i.e., $\bar{N}(n, 2^{nR}, P_X, P_Y)$ in (2.38) and $\bar{N}_c(n, 2^{nR}, P_X, P_Y, \gamma^*)$ in (2.43), are plotted in blue squares. The lower bounds of the DT and MC bounds, i.e., $\underline{N}(n, M, P_X, P_Y)$ in (2.37) and $\underline{N}_c(n, 2^{nR}, P_X, P_Y, \gamma^*)$ in (2.42), are non-positive in these cases, and thus do not appear in the figures. The saddlepoint approximations to the DT and MC bounds, i.e., $\hat{T}(n, 2^{nR}, P_X, P_Y)$ in (5.16) and $\hat{C}(n, \gamma^*, P_X, P_Y, 2^{nR})$ in (5.23), respectively, are plotted in black stars. The upper bounds, i.e., $\bar{T}(n, 2^{nR}, P_X, P_Y)$ in (5.17) and $\bar{C}(n, \gamma^*, P_X, P_Y, 2^{nR})$ in (5.20), are plotted in blue upward-pointing triangles. The lower bounds, i.e., $\underline{T}(n, 2^{nR}, P_X, P_Y)$ in (5.15) and $\underline{C}(n, \gamma^*, P_X, P_Y, 2^{nR})$ in (5.22), are plotted in red downward-pointing triangles.

Figure 5.1 illustrates the case of a BSC with cross-over probability $\delta = 0.11$. The information rates are chosen to be $R = 0.32$ and $R = 0.42$ bits per channel use in Figure 5.1a,b, respectively. The functions T and C can be calculated exactly and thus they are plotted in magenta asterisks in Figure 5.1a,b, respectively. In these figures, it can be observed that the saddlepoint approximations to the DT and MC bounds, i.e., \hat{T} and \hat{C} , respectively, overlap with the functions T and C . These observations are in line with those reported in [14]. Therein, the saddlepoint approximations to the RCU bound and the MC bound are both shown to be precise approximations. Alternatively, the normal approximations to the DT and MC bounds, i.e., \hat{N} and \hat{N}_c , do not overlap with T and C respectively.

In Figure 5.1, it can be observed that the new bounds on the DT and MC provided in (5.18) and (5.19), respectively, are tighter than those of the normal approximation. Indeed, the upper-bounds \bar{N} and \bar{N}_c on the DT and MC bounds derived from the normal approximations \hat{N} and \hat{N}_c , are several order of magnitude above T and C , respectively. This observation remains valid for AWGN channels in Figure 5.2 and S α S channels in Figure 5.3, respectively. Note that, in Figure 5.1a, for $n > 1000$, the normal approximation \hat{N} is below the lower bound \underline{T} showing that approximating T by \hat{N} is too optimistic. These results show that the use of the Berry–Esseen Theorem to approximate the DT and MC bounds may lead to erroneous conclusions due to the uncontrolled error made on the approximation.

Figures 5.2 and 5.3 illustrate the cases of a real-valued AWGN channel and a S α S channel, respectively. The signal-to-noise ratio (SNR) is $\text{SNR} = 1$ for the AWGN channel. The information rate is $R = 0.425$ bits per channel use for the AWGN channel and $R = 0.38$ bits per channel use for the S α S channel with $(\alpha, \sigma) = (1.4, 0.6)$. In both cases, the functions T in (2.19) and C in (5.21) can not be computed explicitly and hence does not appear in Figures 5.2 and 5.3. In addition, the lower bounds $\underline{N}(n, M, P_X, P_Y)$ and $\underline{N}_c(n, 2^{nR}, P_X, P_Y, \gamma^*)$ obtained from the normal approximation are non-positive in these cases, and thus, do not appear on these figures.

In Figure 5.2, note that the saddlepoint approximations, \hat{T} and \hat{C} , are well bounded by (5.18) and (5.19) for a large range of blocklengths. Alternatively, the lower bounds \underline{N} and \underline{N}_c based on the normal approximation do not even exist in that case.

In Figure 5.3, note that the upper bounds \bar{T} and \bar{C} on the DT and MC respectively are relatively tight compared to those in AWGN channel case. This characteristic is of a particular importance in a channel such as S α S channel, where the DT and MC bounds remain computable only by Monte Carlo simulations.

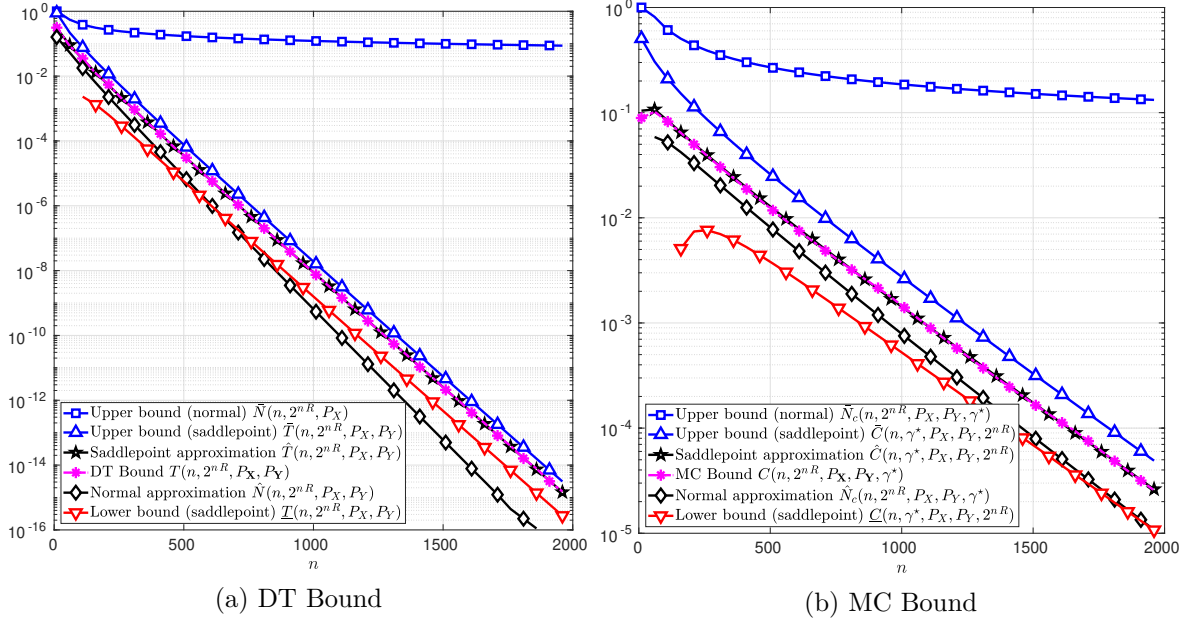


Figure 5.1.: Normal and saddlepoint approximations to the functions T (Figure 5.1a) in (2.19) and C (Figure 5.1b) in (5.21) as functions of the blocklength n for the case of a BSC with cross-over probability $\delta = 0.11$. The information rate is $R = 0.32$ and $R = 0.42$ bits per channel use for Figure 5.1a,b, respectively. The channel input distribution P_X is chosen to be the uniform distribution, the output distribution P_Y is the induced channel output distribution, and the parameter γ^* is chosen to maximize C in (5.21).

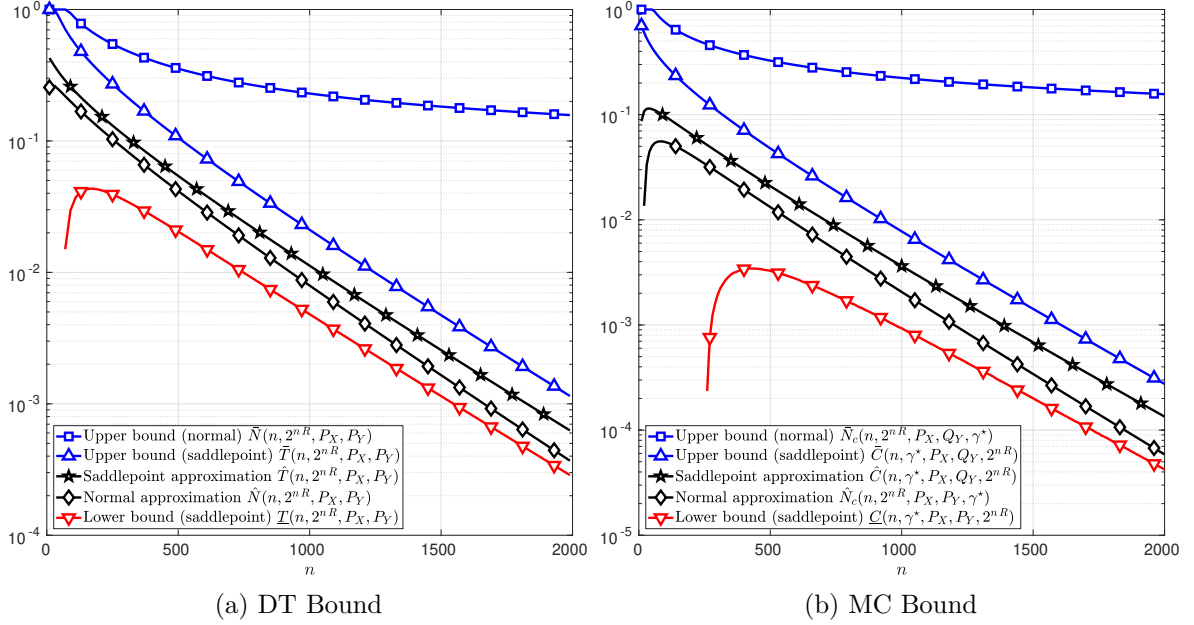


Figure 5.2.: Normal and saddlepoint approximations to the functions T (Figure 5.2a) in (2.19) and C (Figure 5.2b) in (5.21) as functions of the blocklength n for the case of a real-valued AWGN channel with discrete channel inputs, $\mathcal{X} = \{-1, 1\}$, signal to noise ratio $\text{SNR} = 1$, and information rate $R = 0.425$ bits per channel use. The channel input distribution P_X is chosen to be the uniform distribution, the output distribution P_Y is the induced channel output distribution, and the parameter γ^* is chosen to maximize C in (5.21).

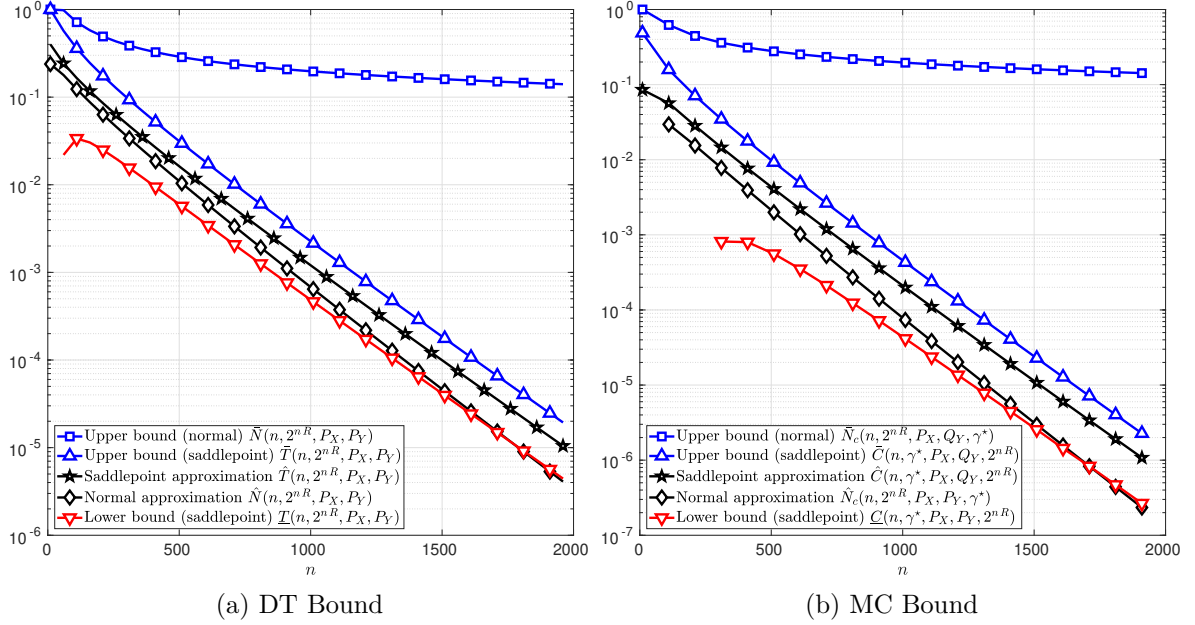


Figure 5.3.: Normal and saddlepoint approximation to the functions T (Figure 5.3a) in (2.19) and C (Figure 5.3b) in (5.21) as functions of the blocklength n for the case of a real-valued symmetric α -stable noise channel with discrete channel inputs $\mathcal{X} = \{-1, 1\}$, shape parameter $\alpha = 1.4$, dispersion parameter $\sigma = 0.6$, and information rate $R = 0.38$ bits per channel use. The channel input distribution P_X is chosen to be the uniform distribution, the output distribution P_Y is the induced channel output distribution, and the parameter γ^* is chosen to maximize C in (5.21).

5.2. Multiple Access Channels With Memoryless Inputs

Given probability measures $P_{X_1}, P_{X_2}, \dots, P_{X_K}$ on the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ for multiple access channels with K transmitters, probability measures $P_{X_1}, P_{X_2}, \dots, P_{X_K}$ on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are assumed to exist such that for all boxes \mathcal{B} in (5.1), for all $k \in \{1, 2, \dots, K\}$, it holds that

$$P_{X_k}(\mathcal{B}) = \prod_{t=1}^n P_{X_k}(\mathcal{B}_t). \quad (5.50)$$

In other words, the inputs to the channel at each transmitter are independent and identically distributed.

For memoryless multiple access channels, the memoryless assumption on the channel input results in the random variable $\tilde{I}(\underline{\mathbf{X}}; \mathbf{Y} | P_{Y, \underline{\mathbf{X}}}, \mathcal{S})$ in (2.75) being a sum of IID random variables. That is, for all input probability measures $P_{X_1}, P_{X_2}, \dots, P_{X_K}$ satisfying (5.50),

$$\tilde{I}(\underline{\mathbf{X}}; \mathbf{Y} | P_{Y, \underline{\mathbf{X}}}, \mathcal{S}) = \sum_{t=1}^n \tilde{I}(\mathbf{X}_t; Y_t | P_{Y, \mathbf{X}}, \mathcal{S}), \quad (5.51)$$

such that for all $t \in \{1, 2, \dots, n\}$,

$$\tilde{t}(\mathbf{X}_t; Y_t | P_{Y\mathbf{X}}, \mathcal{S}) \triangleq \ln \left(\frac{dP_{\mathbf{X}} P_{Y|\mathbf{X}}}{dP_{\mathbf{X}} P_{Y|\mathbf{X}_S}}(\mathbf{X}_t, Y_t) \right), \quad (5.52)$$

where $P_{Y|\mathbf{X}_S}$ is the marginal of $P_{Y|\mathbf{X}}$, with $P_{Y|\mathbf{X}_S} = P_{Y|\mathbf{X}}$ for $\mathcal{S} = \emptyset$; \mathbf{X}_S is a sub-random vector of $\mathbf{X} = (X_1, X_2, \dots, X_K)^\top$; and \mathbf{X}_t is the t -th row of the random matrix $\underline{\mathbf{X}}$. For all $t \in \{1, 2, \dots, n\}$, the random variables $\tilde{t}(\mathbf{X}_t; Y_t | P_{Y\mathbf{X}}, \mathcal{S})$ are IID.

Under the assumption in (5.50), the DT bounds (Lemma 8 and Lemma 12) becomes a function of CDFs of sums of IID random vectors. Their approximations result from the approximations to these CDFs of sums of IID random vectors.

In contrast to the DT bounds, the RCU bounds (Lemma 7 and Lemma 11) are not functions of CDFs of random vectors. However, as it will be shown in Section 5.2.2, starting from the RCU bounds (Lemma 7, Lemma 11), bounds that are functions of CDFs of random variables are derived.

5.2.1. Approximations of the Dependence Testing Bounds

The DT bounds in Lemma 8 and Lemma 12 are compactly represented by a single function. More specifically, for all $K \in \mathbb{N}$, for all $\mathbf{M} \in \mathbb{N}^K$, $n \in \mathbb{N}$, for all $\mathcal{C} \subseteq \mathcal{S}(K)$, with $\mathcal{S}(K)$ in (2.72), and for all probability measures $P_{\underline{\mathbf{X}}}$, let the function $T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C})$ be defined by

$$T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \triangleq \mathbb{E}_{P_{\underline{\mathbf{X}}} P_{Y|\underline{\mathbf{X}}}} \left[\max_{S \in \mathcal{C}} \mathbb{1} \left\{ \tilde{t}(\underline{\mathbf{X}}; Y | P_{Y\underline{\mathbf{X}}}, \mathcal{S}) \leq \ln \left(\prod_{s \in S^c} (M_s - 1) \right) \right\} \right] + \sum_{S \in \mathcal{C}} \left(\prod_{s \in S^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{\mathbf{X}}} P_{Y|\underline{\mathbf{X}}}} \left[\mathbb{1} \left\{ \tilde{t}(\underline{\mathbf{X}}; Y | P_{Y\underline{\mathbf{X}}}, \mathcal{S}) > \ln \left(\prod_{s \in S^c} (M_s - 1) \right) \right\} \right] \quad (5.53)$$

Then, the DT bounds in Lemma 8 and Lemma 12 are identical to $T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C})$ in (5.53) with $\mathcal{C} = \mathcal{S}(K)$ and $\mathcal{C} = \mathcal{P}(k, K)$, respectively. The value of $\mathcal{C} = \mathcal{S}(K)$ stands for the SDEP, whereas the value of $\mathcal{C} = \mathcal{P}(k, K)$ stands for the IDEP.

Note that the function T in (5.53) is a function of CDFs of random vectors. The following lemma states this fact.

Lemma 31 *For all $(K, n) \in \mathbb{N}^2$, for all $\mathbf{M} \in \mathbb{N}^K$, for all $\mathcal{C} \subseteq \mathcal{S}(K)$, with $\mathcal{S}(K)$ in (2.72), and for all probability measures $P_{\underline{\mathbf{X}}}$, it holds that*

$$T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \triangleq 1 - F_{\tilde{t}, \mathcal{C}, [K], n}(\gamma(\mathcal{C})) + \sum_{S \in \mathcal{C}} \left(\prod_{s \in S^c} (M_s - 1) \right) F_{\tilde{t}, \{S\}, S, n}(\gamma(\{S\})), \quad (5.54)$$

where for all $\mathcal{B} \subseteq \mathcal{S}(K)$

$$\gamma(\mathcal{B}) \triangleq \begin{pmatrix} -\ln \left(\prod_{s \in \mathcal{S}_1^c} (M_s - 1) \right) \\ -\ln \left(\prod_{s \in \mathcal{S}_2^c} (M_s - 1) \right) \\ \vdots \\ -\ln \left(\prod_{s \in \mathcal{S}_\ell^c} (M_s - 1) \right) \end{pmatrix}, \quad (5.55)$$

with

$$\{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\ell\} = \mathcal{B}, \quad (5.56)$$

and for all $\mathcal{A} \in \mathcal{S}(K)$, the notation $F_{\tilde{t}, \mathcal{B}, \mathcal{A}, n}$ denotes the CDF of the random vector

$$\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{B}) \triangleq \begin{pmatrix} -\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}_1) \\ -\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}_2) \\ \vdots \\ -\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}_\ell) \end{pmatrix}, \quad (5.57)$$

such that the couple $(\underline{\mathbf{X}}, \mathbf{Y})$ induces the probability measure $P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}$, with $P_{\mathbf{Y}|\underline{\mathbf{X}}_{\mathcal{A}}} = P_{\mathbf{Y}}$ for $\mathcal{A} = \emptyset$.

Proof: From (5.53), it holds that

$$\begin{aligned} T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) &= 1 - \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\prod_{\mathcal{S} \in \mathcal{C}} \mathbb{1} \left\{ \tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) > \ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right] + \\ &\quad \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\mathbb{1} \left\{ -\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) < -\ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right] \end{aligned} \quad (5.58)$$

$$\begin{aligned} &= 1 - \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\prod_{\mathcal{S} \in \mathcal{C}} \mathbb{1} \left\{ -\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) < -\ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \right\} \right] + \\ &\quad \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) F_{\tilde{t}, \{\mathcal{S}\}, \mathcal{S}, n}(\gamma(\{\mathcal{S}\})) \end{aligned} \quad (5.59)$$

$$= 1 - F_{\tilde{t}, \mathcal{C}, [K], n}(\gamma(\mathcal{C})) + \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) F_{\tilde{t}, \{\mathcal{S}\}, \mathcal{S}, n}(\gamma(\{\mathcal{S}\})), \quad (5.60)$$

which completes the proof. Note that with a little abuse of notation, the strict inequality is used in the definition of the CDF. \blacksquare

Note that by assumption in (5.50), for all $\mathcal{S} \in \mathcal{S}(K)$, the random variable $\tilde{t}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S})$ is a sum of IID random variables. Thus, the random vector in (5.57) is a sum of IID random

vectors. That is,

$$\tilde{\mathbf{t}}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{C}) = \sum_{t=1}^n \tilde{\mathbf{t}}(\mathbf{X}_t; Y_t | P_{\mathbf{Y}\mathbf{X}}, \mathcal{C}), \quad (5.61)$$

where

$$\tilde{\mathbf{t}}(\mathbf{X}; Y | P_{\mathbf{Y}\mathbf{X}}, \mathcal{C}) \triangleq \begin{pmatrix} \tilde{\mathbf{t}}(\mathbf{X}; Y | P_{\mathbf{Y}\mathbf{X}}, \mathcal{S}_1) \\ \tilde{\mathbf{t}}(\mathbf{X}; Y | P_{\mathbf{Y}\mathbf{X}}, \mathcal{S}_2) \\ \vdots \\ \tilde{\mathbf{t}}(\mathbf{X}; Y | P_{\mathbf{Y}\mathbf{X}}, \mathcal{S}_l) \end{pmatrix}, \quad (5.62)$$

such that the sets $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_l$ satisfy (5.56).

Hence, for all $\mathcal{C} \subseteq \mathcal{S}(K)$, and for all $\mathcal{S} \subseteq \mathcal{S}(K)$, with $\mathcal{S}(K)$ in (2.72), the CDF $F_{\tilde{\mathbf{t}}, \mathcal{C}, \mathcal{S}, n}$ is the CDF of a sum of IID random vectors. This observation allows to apply Theorem 23 for approximating the function T in (5.53).

To introduce the saddlepoint approximation to the function T in (5.53), consider the following notation. Denote by $\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}$ the random vector $\tilde{\mathbf{t}}(\mathbf{X}; Y | P_{\mathbf{Y}\underline{\mathbf{X}}}, \mathcal{C})$ in (5.62) with (\mathbf{X}, Y) inducing the probability measure $P_{\mathbf{X}}P_{\mathbf{Y}|\mathbf{X}\mathcal{S}}$. Denote by $P_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}$ the probability induced by the random vector $\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}$. Denote by $K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(1)}$ and $K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(2)}$ the gradient vector and Hessian matrix of the CGF $K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}$ of the random vector $\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}$.

Consider the functions $\eta_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}} : \mathbb{R}^\ell \times \mathcal{B}(\mathbb{R}^\ell) \times \mathbb{N} \rightarrow \mathbb{R}$ and $\zeta_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}} : \mathbb{R}^\ell \rightarrow \mathbb{R}$ such that for all $\boldsymbol{\theta} \in \mathbb{R}^\ell$, for all $\mathcal{A} \in \mathcal{B}(\mathbb{R}^\ell)$, and for all $n \in \mathbb{N}$,

$$\begin{aligned} & \eta_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}(\boldsymbol{\theta}, \mathcal{A}, n) \\ &= \exp \left(n \left(K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(1)}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} \right) \right) P_{I_n^{(\boldsymbol{\theta})}}(\mathcal{A}), \text{ and} \end{aligned} \quad (5.63)$$

$$\begin{aligned} & \xi_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}(\boldsymbol{\theta}) \\ &= \mathbb{E}_{P_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}} \left[\left(\left(\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}} - K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(1)}(\boldsymbol{\theta}) \right)^\top \left(K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \left(\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}} - K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(1)}(\boldsymbol{\theta}) \right) \right)^{3/2} \exp(\boldsymbol{\theta}^\top \tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}} - K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}(\boldsymbol{\theta})) \right], \end{aligned} \quad (5.64)$$

where the probability measure $P_{I_n^{(\boldsymbol{\theta})}}$ is the probability measure induced by a Gaussian random vector with mean vector $n \left(K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(1)}(\boldsymbol{\theta}) - K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta} \right)$ and covariance matrix $n K_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}}^{(2)}(\boldsymbol{\theta})$ on the measurable space $(\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell))$. Denote by $\boldsymbol{\mu}_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n}} = (\mu_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n, 1}}, \mu_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n, 2}}, \dots, \mu_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n, \ell}})$ the mean vector of the random vector in (5.57). Let the set $\mathcal{E}_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n}}$ be

$$\mathcal{E}_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n}} = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_\ell) \in \mathbb{R}^\ell : \forall i \in \{1, 2, \dots, \ell\}, x_i > \mu_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}, n, i}} \right\}. \quad (5.65)$$

Using these notations, let the functions $\zeta_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}} : \mathbb{N} \times \mathbb{R}^\ell \rightarrow \mathbb{R}$ and $\delta_{\tilde{\mathbf{t}}_{\mathcal{C}, \mathcal{S}}} : \mathbb{N} \times \mathbb{R}^\ell \rightarrow \mathbb{R}$ be such

that for all $(n, \mathbf{x}) \in \mathbb{N} \times \mathbb{R}^\ell$,

$$\zeta_{\bar{\mathcal{L}},S}(n, \mathbf{x}) \triangleq \begin{cases} \eta_{\bar{\mathcal{L}},S}(\boldsymbol{\theta}_0(\mathbf{x}), \mathcal{A}_{\mathbf{x}}, n) & \text{if } \mathbf{x} \notin \mathcal{E}_{\bar{\mathcal{L}},n} \\ 1 - \sum_{i=1}^{\ell} \eta_{\bar{\mathcal{L}},S}(\boldsymbol{\theta}_i, \mathcal{B}(\mathbf{x}, i), n) & \text{otherwise,} \end{cases} \quad (5.66)$$

$$\delta_{\bar{\mathcal{L}},S}(n, \mathbf{x}) \triangleq \begin{cases} \exp(nK_{\bar{\mathcal{L}},S}(\boldsymbol{\theta}_0(\mathbf{x})) - \boldsymbol{\theta}_0^\top(\mathbf{x}) \mathbf{x}) \min\left(1, \frac{c^{(k)} \xi_{\bar{\mathcal{L}},S}(\boldsymbol{\theta}_0(\mathbf{x}))}{\sqrt{n}}\right) & \text{if } \mathbf{x} \notin \mathcal{E}_{\bar{\mathcal{L}},n} \\ \sum_{i=1}^{\ell} \exp(nK_{\bar{\mathcal{L}},S}(\boldsymbol{\theta}_i(\mathbf{x})) - \boldsymbol{\theta}_i^\top(\mathbf{x}) \mathbf{x}) \min\left(1, \frac{c^{(k)} \xi_{\bar{\mathcal{L}},S}(\boldsymbol{\theta}_i(\mathbf{x}))}{\sqrt{n}}\right) & \text{otherwise,} \end{cases} \quad (5.67)$$

with the function c in (4.14); the sets $\mathcal{A}_{\mathbf{x}}$ and $\mathcal{B}(\mathbf{x}, i)$ respectively defined in (4.34) and (4.47); such that for all $i \in \{0, 1, 2, \dots, \ell\}$, the vector $\boldsymbol{\theta}_i(\mathbf{x})$ satisfies

$$\boldsymbol{\theta}_i(\mathbf{x}) = \begin{cases} \boldsymbol{\tau}_i(\mathbf{x}) & \text{if } \boldsymbol{\tau}_i(\mathbf{x}) \in \Theta_{\bar{\mathcal{L}},S}^i \\ \boldsymbol{\tau}_i(\mathbf{x}) + \boldsymbol{\epsilon} & \text{otherwise,} \end{cases} \quad (5.68)$$

where

$$\boldsymbol{\tau}_i(\mathbf{x}) \in \arg \min_{\mathbf{t} \in \Theta_{\bar{\mathcal{L}},S}^i} (nK_{\bar{\mathcal{L}},S}(\mathbf{t}) - \mathbf{t}^\top \mathbf{x}); \quad (5.69)$$

$\boldsymbol{\epsilon} \in \mathbb{R}^k$ is chosen such that two conditions are simultaneously met: First, $\|\boldsymbol{\epsilon}\| < r$, with $r > 0$ arbitrary small; and second, $\boldsymbol{\theta}_i(\mathbf{x}) \in \Theta_{\bar{\mathcal{L}},S}^i$; and

$$\Theta_{\bar{\mathcal{L}},S}^0 = \left\{ \mathbf{t} = (t_1, t_2, \dots, t_\ell) \in \mathbb{R}^\ell : K_{\bar{\mathcal{L}},S}(\mathbf{t}) < +\infty, \text{ and } \forall i \in \{1, 2, \dots, \ell\}, t_i \leq 0 \right\}; \quad (5.70)$$

and for all $i \in \{1, 2, \dots, \ell\}$,

$$\Theta_{\bar{\mathcal{L}},S}^i = \left\{ \mathbf{t} = (t_1, t_2, \dots, t_\ell) \in \mathbb{R}^\ell : K_{\bar{\mathcal{L}},S}(\mathbf{t}) < +\infty, \text{ and } \forall j \in \{1, 2, \dots, \ell\}, t_j \leq 0 \text{ if } j < i, \right. \\ \left. t_j \geq 0 \text{ if } j = i, \text{ and } t_j = 0 \text{ otherwise} \right\}. \quad (5.71)$$

Using these notations, the following corollary of Theorem 23 gives the saddlepoint approximation to the CDF $F_{\bar{\mathcal{L}},S,n}$ of the random vector in (5.57).

Corollary 32 *For all $\mathbf{v} \in \mathbb{R}^\ell$, it holds that*

$$|F_{\bar{\mathcal{L}},S,n}(\mathbf{v}) - \zeta_{\bar{\mathcal{L}},S}(n, \mathbf{v})| \leq \delta_{\bar{\mathcal{L}},S}(n, \mathbf{v}), \quad (5.72)$$

where the functions $\zeta_{\bar{\mathcal{L}},S}$ and $\delta_{\bar{\mathcal{L}},S}$ are defined in (5.66) and in (5.67), respectively.

From Corollary 32, an approximation, lower and upper bounds are derived for the function T in (5.53). For doing so, consider the following definitions. For all $K \in \mathbb{N}$, for all $\mathbf{M} \in \mathbb{N}^K$, $n \in \mathbb{N}$, for all $\mathcal{C} \subseteq \mathcal{S}(K)$, with $\mathcal{S}(K)$ in (2.72), and for all probability measures $P_{\mathbf{X}}$, let the functions $\underline{T}(n, \mathbf{M}, P_{\mathbf{X}}, \mathcal{C})$, $\bar{T}(n, \mathbf{M}, P_{\mathbf{X}}, \mathcal{C})$, $\bar{T}_c(n, \mathbf{M}, P_{\mathbf{X}}, \mathcal{C})$ and $\hat{T}(n, \mathbf{M}, P_{\mathbf{X}}, \mathcal{C})$ be defined

by

$$\begin{aligned} \underline{T}(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) &\triangleq \max \left(0, 1 - \zeta_{\bar{\mathcal{I}}_{\mathcal{C}}, [K]}(n, \gamma(\mathcal{C})) - \delta_{\bar{\mathcal{I}}_{\mathcal{C}}, [K]}(n, \gamma(\mathcal{C})) \right) + \\ &\quad \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \max \left(0, \zeta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})) - \delta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})) \right), \end{aligned} \quad (5.73)$$

$$\begin{aligned} \bar{T}(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) &\triangleq \min \left(1, 1 - \zeta_{\bar{\mathcal{I}}_{\mathcal{C}}, [K]}(n, \gamma(\mathcal{C})) + \delta_{\bar{\mathcal{I}}_{\mathcal{C}}, [K]}(n, \gamma(\mathcal{C})) \right) + \\ &\quad \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \min \left(1, \zeta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})) + \delta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})) \right), \end{aligned} \quad (5.74)$$

$$\begin{aligned} \bar{T}_c(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) &\triangleq \sum_{\mathcal{S} \in \mathcal{C}} \min \left(1, 1 - \zeta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, [K]}(n, \gamma(\{\mathcal{S}\})) + \delta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, [K]}(n, \gamma(\{\mathcal{S}\})) \right) + \\ &\quad \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \min \left(1, \zeta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})) + \delta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})) \right), \end{aligned} \quad (5.75)$$

$$\hat{T}(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \triangleq 1 - \zeta_{\bar{\mathcal{I}}_{\mathcal{C}}, [K]}(n, \gamma(\mathcal{C})) + \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \zeta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}}(n, \gamma(\{\mathcal{S}\})). \quad (5.76)$$

with γ in (5.55).

Then, the function \hat{T} in (5.76) is referred as the saddlepoint approximation to the function T in (5.53) and given a tuple $(n, \mathbf{M}) \in \mathbb{N}^{1+K}$, for all $k \in \{1, 2, \dots, K\}$, and for all probability measures $P_{\mathbf{X}_k} \in \Delta(\mathcal{X}_K)$, it holds that

$$\underline{T}(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \leq T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \leq \bar{T}(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \quad \text{and} \quad (5.77)$$

$$T(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}) \leq \bar{T}_c(n, \mathbf{M}, P_{\underline{\mathbf{X}}}, \mathcal{C}). \quad (5.78)$$

5.2.2. Approximation of Random Coding Union Bound

The RCU bounds in Lemma 7 and Lemma 11 are not directly expressible in term of CDFs of random variables. The following Lemma provides bounds derived from Lemma 7 and Lemma 11 that are functions of CDFs of random vectors.

Lemma 33 *Given a pair $(n, \mathbf{M}) \in \mathbb{N}^{K+1}$, for all $P_{\underline{\mathbf{X}}} \in \Delta(\mathcal{X}_1^n \times \mathcal{X}_2^n \times \dots \times \mathcal{X}_K^n)$, for all $k \in \{1, 2, \dots, K\}$, for all $\boldsymbol{\alpha} \in \mathbb{R}^\ell$, with $\ell = |\mathcal{S}(K)|$, and for all $\boldsymbol{\alpha}_k \in \mathbb{R}^{\ell_k}$, with $\ell_k = |\mathcal{P}(k, K)|$, the following holds with respect to the random transformation in (2.53):*

$$\lambda^*(n, M) \leq R(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\alpha}, \mathbf{M}, \mathcal{S}(K)), \quad (5.79)$$

$$\lambda_k^*(n, M) \leq R(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\alpha}_k, \mathbf{M}, \mathcal{P}(k, K)), \quad (5.80)$$

such that for all $\mathcal{C} \subseteq \mathcal{S}(K)$, and for all $\boldsymbol{\beta} \in \mathbb{R}^m$, with $m = |\mathcal{C}|$

$$R(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C}) \triangleq 1 - F_{\bar{\mathcal{I}}_{\mathcal{C}}, \mathcal{S}, n}(\boldsymbol{\beta}) + \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) F_{\mathcal{I}_{\mathcal{S}}, n} \left((0, -\boldsymbol{\beta}_{\mathcal{S}})^T \right), \quad (5.81)$$

where the CDF $F_{\bar{\mathcal{I}}_{\mathcal{C}}, \mathcal{S}, n}$ is defined according to Lemma 31; the function $F_{\mathcal{I}_{\mathcal{S}}, n}$ is the CDF of

the random vector

$$\tilde{i}(\underline{\mathbf{X}}, \bar{\underline{\mathbf{X}}}; \mathbf{Y}, \mathcal{S}) \triangleq \begin{pmatrix} \tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) - \tilde{i}(\bar{\underline{\mathbf{X}}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \\ -\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \end{pmatrix}, \quad (5.82)$$

with $\bar{\underline{\mathbf{X}}}_{\mathcal{S}} = \underline{\mathbf{X}}_{\mathcal{S}}$, $(\underline{\mathbf{X}}, \bar{\underline{\mathbf{X}}}, \mathbf{Y})$ inducing the probability measure $P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}P_{\bar{\underline{\mathbf{X}}}_{\mathcal{S}^c}}$ on the measurable space $(\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{X}^n, \mathcal{B}(\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{X}^n))$, $P_{\bar{\underline{\mathbf{X}}}_{\mathcal{S}^c}} = P_{\underline{\mathbf{X}}_{\mathcal{S}^c}}$ and $\underline{\mathcal{X}}^n = \mathcal{X}_1^n \times \mathcal{X}_2^n \times \dots \times \mathcal{X}_K^n$; and for all $\mathcal{S} \in \mathcal{C}$, $\beta_{\mathcal{S}}$ are the coordinates of the vector β .

Proof: The proof is made for $\lambda^*(n, \mathbf{M})$ and the proof for $\lambda_k^*(n, \mathbf{M})$ follows the similar steps. From (2.77), for all $\alpha = (\alpha_{\mathcal{S}_1}, \alpha_{\mathcal{S}_2}, \dots, \alpha_{\mathcal{S}_\ell}) \in \mathbb{R}^\ell$, with $\ell = |\mathcal{S}(K)|$, it holds that

$$\begin{aligned} & \lambda^*(n, \mathbf{M}) \\ & \leq \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\left(1 - \prod_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1}_{\{\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) > \alpha_{\mathcal{S}}\}} \right) \right. \\ & \quad \left. \min \left(1, \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\bar{\underline{\mathbf{X}}}_{\mathcal{S}^c}}} \left[\mathbb{1}_{\{\tilde{i}(\bar{\underline{\mathbf{X}}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \geq \tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S})\}} \right] \right) \right] \\ & \quad + \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\prod_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1}_{\{\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \geq \alpha_{\mathcal{S}}\}} \right] \\ & \quad \left. \min \left(1, \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\bar{\underline{\mathbf{X}}}_{\mathcal{S}^c}}} \left[\mathbb{1}_{\{\tilde{i}(\bar{\underline{\mathbf{X}}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \geq \tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S})\}} \right] \right) \right] \end{aligned} \quad (5.83)$$

$$\begin{aligned} & \leq 1 - \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\prod_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1}_{\{-\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) < -\alpha_{\mathcal{S}}\}} \right] + \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}} \left[\prod_{\mathcal{S} \in \mathcal{S}(K)} \mathbb{1}_{\{-\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \leq -\alpha_{\mathcal{S}}\}} \right] \\ & \quad \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\bar{\underline{\mathbf{X}}}_{\mathcal{S}^c}}} \left[\mathbb{1}_{\{\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) - \tilde{i}(\bar{\underline{\mathbf{X}}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \leq 0\}} \right] \end{aligned} \quad (5.84)$$

$$\begin{aligned} & \leq 1 - F_{\tilde{i}, \mathcal{S}(K), \mathcal{S}, n}(\alpha) + \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \mathbb{E}_{P_{\underline{\mathbf{X}}}P_{\mathbf{Y}|\underline{\mathbf{X}}}P_{\bar{\underline{\mathbf{X}}}_{\mathcal{S}^c}}} \left[\mathbb{1}_{\{-\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \leq -\alpha_{\mathcal{S}}\}} \right] \\ & \quad \mathbb{1}_{\{\tilde{i}(\underline{\mathbf{X}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) - \tilde{i}(\bar{\underline{\mathbf{X}}}; \mathbf{Y} | P_{\mathbf{Y}|\underline{\mathbf{X}}}, \mathcal{S}) \leq 0\}} \end{aligned} \quad (5.85)$$

$$= 1 - F_{\tilde{i}, \mathcal{S}(K), \mathcal{S}, n}(\alpha) + \sum_{\mathcal{S} \in \mathcal{S}(K)} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) F_{i, \mathcal{S}, n} \left((0, -\alpha_{\mathcal{S}})^{\top} \right), \quad (5.86)$$

$$= R(n, P_{\underline{\mathbf{X}}}, \alpha, \mathbf{M}, \mathcal{S}(K)), \quad (5.87)$$

which concludes the proof on $\lambda^*(n, \mathbf{M})$. Similar steps induce to

$$\lambda_k^*(n, \mathbf{M}) \leq R(n, P_{\underline{\mathbf{X}}}, \alpha_k, \mathbf{M}, \mathcal{P}(k, K)). \quad (5.88)$$

■

The random vector in (5.82), as a consequence of the assumption in (5.50), is a sum of IID

random vectors. That is,

$$\tilde{\mathbf{z}}(\underline{\mathbf{X}}, \bar{\mathbf{X}}; \mathbf{Y}, \mathcal{S}) = \sum_{t=1}^n \tilde{\mathbf{z}}(\mathbf{X}_t, \bar{\mathbf{X}}_t; Y_t, \mathcal{S}), \quad (5.89)$$

where

$$\begin{aligned} & \tilde{\mathbf{z}}(\mathbf{X}_t, \bar{\mathbf{X}}_t; Y_t, \mathcal{S}) \\ \triangleq & \begin{pmatrix} \tilde{\mathbf{z}}(\mathbf{X}_t; Y_t | P_{Y\mathbf{X}}, \mathcal{S}) - \tilde{\mathbf{z}}(\bar{\mathbf{X}}_t; Y_t | P_{Y\mathbf{X}}, \mathcal{S}) \\ -\tilde{\mathbf{z}}(\mathbf{X}_t; Y | P_{Y\mathbf{X}}, \mathcal{S}) \end{pmatrix}, \end{aligned} \quad (5.90)$$

such that the random sub-vector $\bar{\mathbf{X}}_{t,\mathcal{S}}, \mathbf{X}_{t,\mathcal{S}}$ of respectively $\bar{\mathbf{X}}_t$ and \mathbf{X}_t satisfy $\bar{\mathbf{X}}_{t,\mathcal{S}} = \mathbf{X}_{t,\mathcal{S}}$ and the tuple $(\mathbf{X}_t, Y, \bar{\mathbf{X}}_t)$ induces the probability measure $P_{\mathbf{X}}P_{Y|\mathbf{X}} P_{\bar{\mathbf{X}}_{\mathcal{S}^c}},$ with $P_{\bar{\mathbf{X}}_{\mathcal{S}^c}} = P_{\mathbf{X}_{\mathcal{S}^c}}.$ This observation allows to exploit Theorem 23 to approximate the CDF $F_{i,\mathcal{S},n}.$

To introduce the saddlepoint approximation to the CDF $F_{i,\mathcal{S},n}$ of the random vector in (5.89), consider the following notation. Denote by $\tilde{\mathbf{z}}_{\mathcal{S}}$ the random vector in (5.90) and $P_{\tilde{\mathbf{z}}_{\mathcal{S}}}$ the probability induced by the random vector $\tilde{\mathbf{z}}_{\mathcal{S}}.$ Denote by $K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(1)}$ and $K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(2)}$ the gradient vector and Hessian matrix of the CGF $K_{\tilde{\mathbf{z}}_{\mathcal{S}}}$ of the random vector $\tilde{\mathbf{z}}_{\mathcal{S}},$ respectively.

Consider the functions $\eta_{\tilde{\mathbf{z}}_{\mathcal{S}}} : \mathbb{R}^2 \times \mathcal{B}(\mathbb{R}^2) \times \mathbb{N} \rightarrow \mathbb{R}$ and $\zeta_{\tilde{\mathbf{z}}_{\mathcal{S}}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for all $\boldsymbol{\theta} \in \mathbb{R}^2,$ for all $\mathcal{A} \in \mathcal{B}(\mathbb{R}^2),$ and for all $n \in \mathbb{N},$

$$\eta_{\tilde{\mathbf{z}}_{\mathcal{S}}}(\boldsymbol{\theta}, \mathcal{A}, n) = \exp\left(n\left(K_{\tilde{\mathbf{z}}_{\mathcal{S}}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(1)}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(2)}(\boldsymbol{\theta})\boldsymbol{\theta}}{2}\right)\right) P_{I_n^{(\boldsymbol{\theta})}}(\mathcal{A}), \text{ and} \quad (5.91)$$

$$\zeta_{\tilde{\mathbf{z}}_{\mathcal{S}}}(\boldsymbol{\theta}) = \mathbb{E}_{P_{\tilde{\mathbf{z}}_{\mathcal{S}}}}\left[\left(\left(\tilde{\mathbf{z}}_{\mathcal{S}} - K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(1)}(\boldsymbol{\theta})\right)^\top \left(K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(2)}(\boldsymbol{\theta})\right)^{-1} \left(\tilde{\mathbf{z}}_{\mathcal{S}} - K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(1)}(\boldsymbol{\theta})\right)\right)^{3/2} \exp\left(\boldsymbol{\theta}^\top \tilde{\mathbf{z}}_{\mathcal{S}} - K_{\tilde{\mathbf{z}}_{\mathcal{S}}}(\boldsymbol{\theta})\right)\right], \quad (5.92)$$

where the probability measure $P_{I_n^{(\boldsymbol{\theta})}}$ is the probability measure induced by a Gaussian random vector with mean vector $n\left(K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(1)}(\boldsymbol{\theta}) - K_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(2)}(\boldsymbol{\theta})\boldsymbol{\theta}\right)$ and covariance matrix $nK_{\tilde{\mathbf{z}}_{\mathcal{S}}}^{(2)}(\boldsymbol{\theta})$ on the measurable space $\left(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)\right).$ Denote by $\boldsymbol{\mu}_{\tilde{\mathbf{z}}_{\mathcal{S},n}} = (\mu_{\tilde{\mathbf{z}}_{\mathcal{S},1}}, \mu_{\tilde{\mathbf{z}}_{\mathcal{S},2}})$ the mean vector of the random vector in (5.89). Let the set $\mathcal{E}_{\tilde{\mathbf{z}}_{\mathcal{S},n}}$ be

$$\mathcal{E}_{\tilde{\mathbf{z}}_{\mathcal{S},n}} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : \forall i \in \{1, 2\}, x_i > \mu_{\tilde{\mathbf{z}}_{\mathcal{S},i}}\}. \quad (5.93)$$

Using these notations, let the functions $\zeta_{\tilde{\mathbf{z}}_{\mathcal{S}}} : \mathbb{N} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\delta_{\tilde{\mathbf{z}}_{\mathcal{S}}} : \mathbb{N} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be such that for all $(n, \mathbf{x}) \in \mathbb{N} \times \mathbb{R}^2,$

$$\zeta_{\tilde{\mathbf{z}}_{\mathcal{S}}}(n, \mathbf{x}) \triangleq \begin{cases} \eta_{\tilde{\mathbf{z}}_{\mathcal{S}}}(\boldsymbol{\theta}_0(\mathbf{x}), \mathcal{A}_{\mathbf{x}}, n) & \text{if } \mathbf{x} \notin \mathcal{E}_{\tilde{\mathbf{z}}_{\mathcal{S},n}} \\ 1 - \sum_{i=1}^2 \eta_{\tilde{\mathbf{z}}_{\mathcal{S}}}(\boldsymbol{\theta}_i, \mathcal{B}(\mathbf{x}, i), n) & \text{otherwise,} \end{cases} \quad (5.94)$$

$$\delta_{\bar{\mathcal{I}}_S}(n, \mathbf{x}) \triangleq \begin{cases} \exp(nK_{\bar{\mathcal{I}}_S}(\boldsymbol{\theta}_0(\mathbf{x})) - \boldsymbol{\theta}_0^\top(\mathbf{x}) \mathbf{x}) \min\left(1, \frac{c(k) \xi_{\bar{\mathcal{I}}_S}(\boldsymbol{\theta}_0(\mathbf{x}))}{\sqrt{n}}\right) & \text{if } \mathbf{x} \notin \mathcal{E}_{\bar{\mathcal{I}}_S, n} \\ \sum_{i=1}^2 \exp(nK_{\bar{\mathcal{I}}_S}(\boldsymbol{\theta}_i(\mathbf{x})) - \boldsymbol{\theta}_i^\top(\mathbf{x}) \mathbf{x}) \min\left(1, \frac{c(k) \xi_{\bar{\mathcal{I}}_S}(\boldsymbol{\theta}_i(\mathbf{x}))}{\sqrt{n}}\right) & \text{otherwise,} \end{cases} \quad (5.95)$$

where for all $i \in \{0, 1, 2\}$ the vector $\boldsymbol{\theta}_i(\mathbf{x})$ satisfies

$$\boldsymbol{\theta}_i(\mathbf{x}) = \begin{cases} \boldsymbol{\tau}_i(\mathbf{x}) & \text{if } \boldsymbol{\tau}_i(\mathbf{x}) \in \Theta_{\bar{\mathcal{I}}_S}^i \\ \boldsymbol{\tau}_i(\mathbf{x}) + \boldsymbol{\epsilon} & \text{otherwise,} \end{cases} \quad (5.96)$$

where

$$\boldsymbol{\tau}_i(\mathbf{x}) \in \arg \min_{\mathbf{t} \in \Theta_{\bar{\mathcal{I}}_S}^i} (nK_{\bar{\mathcal{I}}_S}(\mathbf{t}) - \mathbf{t}^\top \mathbf{x}) \quad (5.97)$$

and $\boldsymbol{\epsilon} \in \mathbb{R}^2$ is chosen such that two conditions are simultaneously met: First, $\|\boldsymbol{\epsilon}\| < r$, with $r > 0$ arbitrary small; and second, $\boldsymbol{\theta}_i(\mathbf{x}) \in \Theta_{\bar{\mathcal{I}}_S}^i$; and with

$$\Theta_{\bar{\mathcal{I}}_S}^0 = \{\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2 : K_{\bar{\mathcal{I}}_S} < +\infty, \text{ and } \forall i \in \{1, 2\}, t_i \leq 0\}, \text{ and } \forall i \in \{1, 2\} \quad (5.98)$$

$$\Theta_{\bar{\mathcal{I}}_S}^i = \left\{ \mathbf{t} = (t_1, t_2) \in \mathbb{R}^2 : K_{\bar{\mathcal{I}}_S} < +\infty, \text{ and } \forall i \in \{1, 2\}, t_j \leq 0 \text{ if } j < i, \right. \\ \left. t_j \geq 0 \text{ if } j = i, \text{ and } t_j = 0 \text{ otherwise} \right\}; \quad (5.99)$$

the function c is respectively defined in (4.14); the sets $\mathcal{A}_{\mathbf{x}}$ and $\mathcal{B}(\mathbf{x}, i)$ are respectively defined in (4.34) and (4.47).

Then, the saddlepoint approximation to the CDF $F_{\bar{\mathcal{I}}_S, n}$ of the random vector in (5.89) is given by the following corollary of Theorem 23.

Corollary 34 *For all $\mathbf{v} \in \mathbb{R}^2$, it holds that*

$$|F_{\bar{\mathcal{I}}_S, n}(\mathbf{v}) - \zeta_{\bar{\mathcal{I}}_S}(n, \mathbf{v})| \leq \delta_{\bar{\mathcal{I}}_S}(n, \mathbf{v}), \quad (5.100)$$

where the functions $\zeta_{\bar{\mathcal{I}}_S}$ and $\delta_{\bar{\mathcal{I}}_S}$ are defined in (5.94) and in (5.95), respectively.

Finally, the approximation, lower and upper bounds for the function R in (5.81) are obtained from Corollary 32 and Corollary 34. To do so, consider the following definitions.

For all $K \in \mathbb{N}$, for all $\mathbf{M} \in \mathbb{N}^K$, for all $n \in \mathbb{N}$, for all $\mathcal{C} \subseteq \mathcal{S}(K)$, and for all $\boldsymbol{\beta} \in \mathbb{R}^m$, with $m = |\mathcal{C}|$, and for all probability measures $P_{\underline{\mathbf{X}}}$, let the functions $\underline{R}(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C})$, $\hat{R}(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C})$, $\bar{R}(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C})$, and $\bar{R}_c(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C})$ be defined by

$$\underline{R}(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C}) = \max(0, 1 - \zeta_{\bar{\mathcal{I}}_S}(n, \boldsymbol{\beta}) - \delta_{\bar{\mathcal{I}}_S}(n, \boldsymbol{\beta})) + \\ \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \max\left(0, \zeta_{\bar{\mathcal{I}}_S}\left(n, (0, -\boldsymbol{\beta}_S)^\top\right) - \delta_{\bar{\mathcal{I}}_S}\left(n, (0, -\boldsymbol{\beta}_S)^\top\right)\right) \quad (5.101)$$

$$\hat{R}(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C}) = 1 - \zeta_{\bar{\mathcal{I}}_S}(n, \boldsymbol{\beta}) + \sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \zeta_{\bar{\mathcal{I}}_S}\left(n, (0, -\boldsymbol{\beta}_S)^\top\right), \quad (5.102)$$

$$\bar{R}(n, P_{\underline{\mathbf{X}}}, \boldsymbol{\beta}, \mathbf{M}, \mathcal{C}) = \min(1, 1 - \zeta_{\bar{\mathcal{I}}_S}(n, \boldsymbol{\beta}) + \delta_{\bar{\mathcal{I}}_S}(n, \boldsymbol{\beta})) +$$

$$\sum_{\mathcal{S} \in \mathcal{C}} \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \min \left(1, \zeta_{\bar{\mathcal{I}}_{\mathcal{S}}} \left(n, (0, -\beta_{\mathcal{S}})^{\top} \right) + \delta_{\bar{\mathcal{I}}_{\mathcal{S}}} \left(n, (0, -\beta_{\mathcal{S}})^{\top} \right) \right) \quad (5.103)$$

$$\begin{aligned} \bar{R}_c(n, P_{\underline{\mathbf{X}}}, \beta, \mathbf{M}, \mathcal{C}) = & \sum_{\mathcal{S} \in \mathcal{C}} \min \left(1, 1 - \zeta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}} \left(n, \beta_{\mathcal{S}} \right) + \delta_{\bar{\mathcal{I}}_{\{\mathcal{S}\}, \mathcal{S}}} \left(n, \beta_{\mathcal{S}} \right) \right) + \\ & \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) \min \left(1, \zeta_{\bar{\mathcal{I}}_{\mathcal{S}}} \left(n, (0, -\beta_{\mathcal{S}})^{\top} \right) + \delta_{\bar{\mathcal{I}}_{\mathcal{S}}} \left(n, (0, -\beta_{\mathcal{S}})^{\top} \right) \right). \end{aligned} \quad (5.104)$$

Using these definitions, it holds that

$$\underline{R}(n, P_{\underline{\mathbf{X}}}, \beta, \mathbf{M}, \mathcal{C}) \leq R(n, P_{\underline{\mathbf{X}}}, \beta, \mathbf{M}, \mathcal{C}) \leq \bar{R}(n, P_{\underline{\mathbf{X}}}, \beta, \mathbf{M}, \mathcal{C}) \quad (5.105)$$

$$R(n, P_{\underline{\mathbf{X}}}, \beta, \mathbf{M}, \mathcal{C}) \leq \bar{R}_c(n, P_{\underline{\mathbf{X}}}, \beta, \mathbf{M}, \mathcal{C}). \quad (5.106)$$

The function \hat{R} is referred as the saddlepoint approximation to the function R in (5.81).

5.2.3. Numerical Analysis

This section firstly compared the saddlepoint approximations to the DT bounds and the RCU bounds as well as their corresponding upper and lower bounds presented from Section 5.2 to the normal approximation in the case the memoryless Gaussian MACs. Secondly, the best bound is used to study the achievable region of two transmitters memoryless Gaussian MAC under A-SDEP, average and maximum IDEP.

In the following figures, Figures 5.4–5.7, the channel inputs at transmitter k , with $k \in \{1, 2, \dots, K\}$, are discrete $\mathcal{X} = \{-A_k, A_k\}$, with $A_k \in \mathbb{R}$ and P_{X_k} is the uniform distribution. For the results relative to the RCU bounds, for all collection $\mathcal{C} \subseteq \mathcal{S}(K)$, the parameter $\beta = (\beta_{\mathcal{S}_1}, \beta_{\mathcal{S}_2}, \dots, \beta_{\mathcal{S}_\ell})$ is chosen such that for all $\mathcal{S} \in \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\ell\} = \mathcal{C}$,

$$\beta_{\mathcal{S}} = -\ln \left(\prod_{s \in \mathcal{S}^c} (M_s - 1) \right) + \ln(n). \quad (5.107)$$

Bound Comparison

The plots in Figures 5.4 and 5.5 illustrate the functions T in (5.53) and R in (5.81) as well as the bounds in (5.77) (5.78), (5.105) and (5.106), with $\mathcal{C} = \mathcal{S}(K)$. The choice of $\mathcal{C} = \mathcal{S}(K)$, with $\mathcal{S}(K)$ in (2.72), is equivalent to study of the bounds on the SDEP $\lambda^*(n, \mathbf{M})$ in (2.62).

The normal approximation, i.e., \hat{G} in (2.107) to the DT bound is plotted in $(-\bullet-)$. The upper and lower bound on the normal approximation provided by Berry-Theorem Theorem are respectively greater than one and negative. For this reason, they are not plotted. The saddlepoint approximations to the DT and RCU bounds, i.e., \hat{T} in (5.76) and \hat{R} in (5.102), respectively, are plotted in $(-\blacksquare-)$ and $(-\blacklozenge-)$. The upper bounds, i.e., \bar{T} in (5.77), \bar{T}_c , in (5.78), \bar{R} in (5.105), and \bar{R}_c in (5.106), are respectively plotted in $(-\triangle-)$, $(-\blacktriangle-)$, $(-\diamond-)$, and $(-\blacklozenge-)$. The lower bounds, i.e., \underline{T} in (5.77) and \underline{R} in (5.105), are non-positive in these cases, and thus do not appear in the figures.

Figures 5.4 and (5.5) respectively illustrate the case of Gaussian MAC with two transmitters identical power levels and information rates on one hand, and asymmetric power levels and

information rates. For both Figure 5.4 and Figure 5.5, the approximation \hat{R} to the RCU bound R in(5.81) is the lowest. The upper \bar{R} on the RCU bound R in(5.81) is also the lowest upper bound on the SDEP. The missing of lower bounds to not allow to conclude on the validity of the normal approximation as in the case of the point to point channels. However, it is important to note that upper bounds obtained from the upper bound on the saddlepoint approximation error are close to the approximation and exhibit the same behaviour as illustrate by a constant gap for the DT bound on both figures and RCU bound starting from $n = 300$. Such behaviour enhance the confidence in the saddlepoint approximation. Additionally, the upper bounds can be used alone as firms bounds on SDEP, which removes the confusion brought by the approximation. For instance, in Figure 5.4, for $n > 320$, the upper bound \bar{R}_c allows to state that the normal approximation G is achievable but for $n < 320$, nothing can be said about the achievability of the normal approximation G .

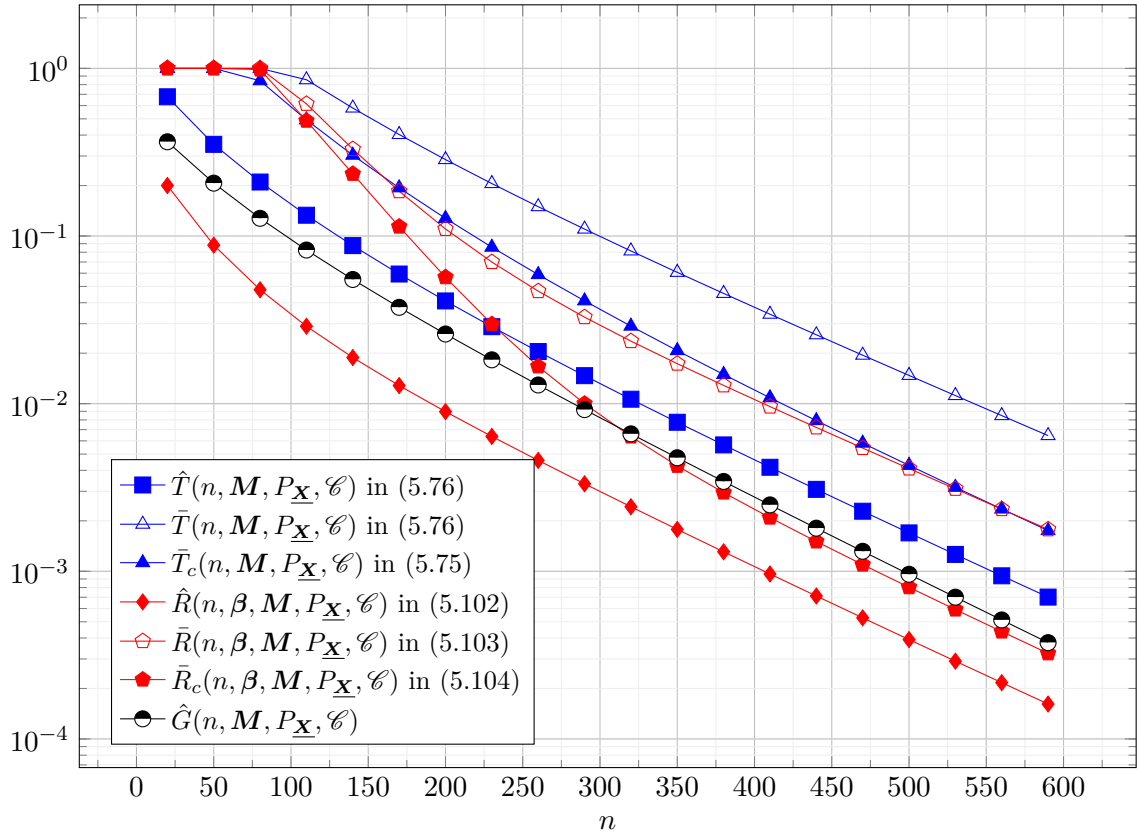


Figure 5.4.: Normal and saddlepoint approximation to the functions T in (5.53) and R in (5.81) as functions of the blocklength n for the case of two transmitter Gaussian MAC with discrete channel inputs $\mathcal{X}_1 = \mathcal{X}_2 = \{-1, 1\}$, noise variance $\sigma^2 = 1$, and information rates $R_1 = R_2 = 0.32$ bits per channel use.

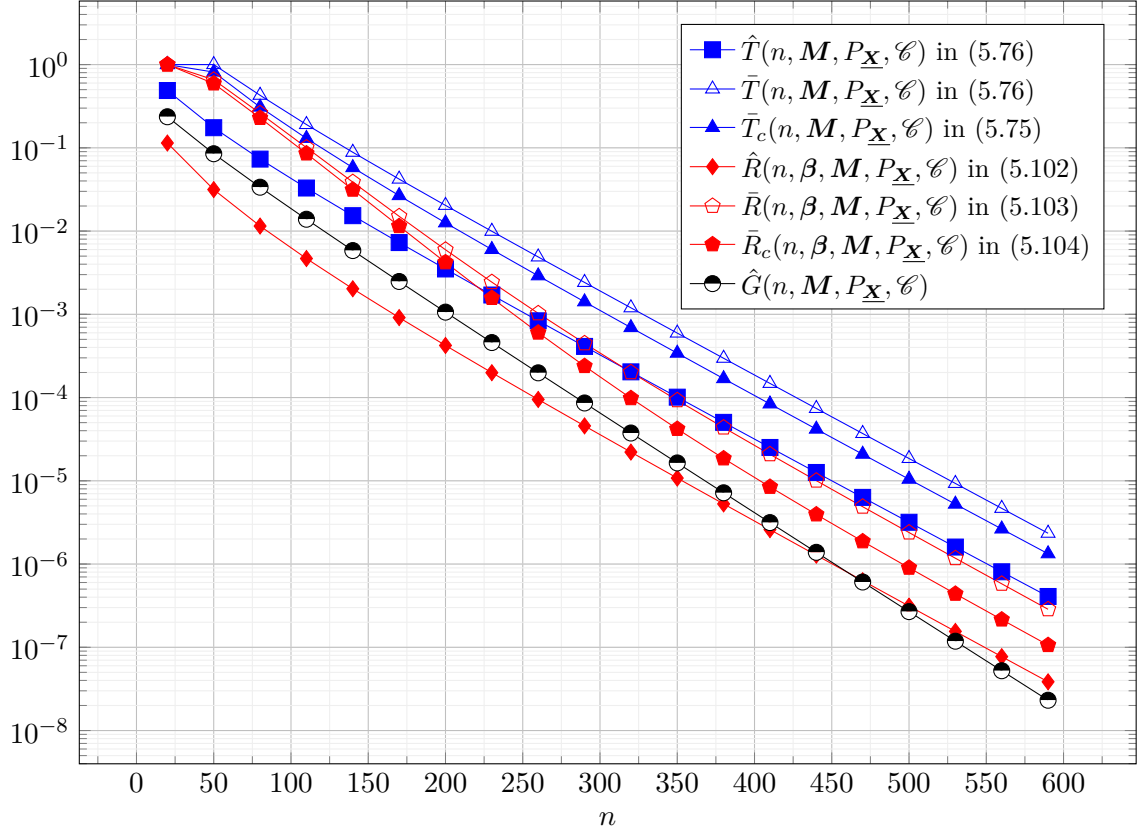


Figure 5.5.: Normal and saddlepoint approximation to the functions T in (5.53) and R in (5.81) as functions of the blocklength n for the case of a two transmitter Gaussian MAC with discrete channel inputs $\mathcal{X}_1 = \{-1, 1\}$, $\mathcal{X}_2 = \{-\sqrt{1.5}, \sqrt{1.5}\}$, noise variance $\sigma^2 = 1$, and information rates $R_1 = 0.3$ and $R_2 = 0.35$ bits per channel use.

Achievable Region

The plots in figures 5.6–5.7 illustrate the achievable regions (red curves) and their approximations (blue curves) based on the upper bound \bar{R}_c in (5.104) and the approximation \hat{R} of the RCU bound for two transmitters Gaussian MAC. Different achievable regions and their approximations are plotted based on the SDEP $\lambda^*(n, \mathbf{M})$ in (2.62), *i.e.*, the curves in $(-\triangle-)$ and $(-\square-)$; the maximum and average IDEPs $\lambda_k^*(n, \mathbf{M})$ in (2.62), *i.e.*, the curves in $(-\blacktriangle-)$ and $(-\blacksquare-)$ for the maximum; and $(-\circ-)$ and $(-\ominus-)$ for the average. Figure 5.6 focuses on two transmitters Gaussian MAC with symmetric power levels, whereas Figure 5.7 targets asymmetric power levels.

The average IDEP and A-IDEP achievable regions have the shape that is different than the maximum IDEP achievable region. Such a similar shape implies a possible equivalence between the average IDEP and A-IDEP. The almost square shape of the achievable region induced by the maximum IDEP seems to demonstrate that the two transmitters do not interfere with each other as was observed in the work of [7]. This observation is missed by the A-SDEP achievable region. It is also interesting to observe that for the special case of the symmetric information rates and power levels, an equivalence can be found between the maximum and average IDEPs as captured by the overlapping point in Figure 5.6. As expected, for identical constraint on

this tree considerations of the DEP, the maximum IDEP achievable region is included in the A-SDEP achievable region, which is itself included in the average IDEP achievable region.

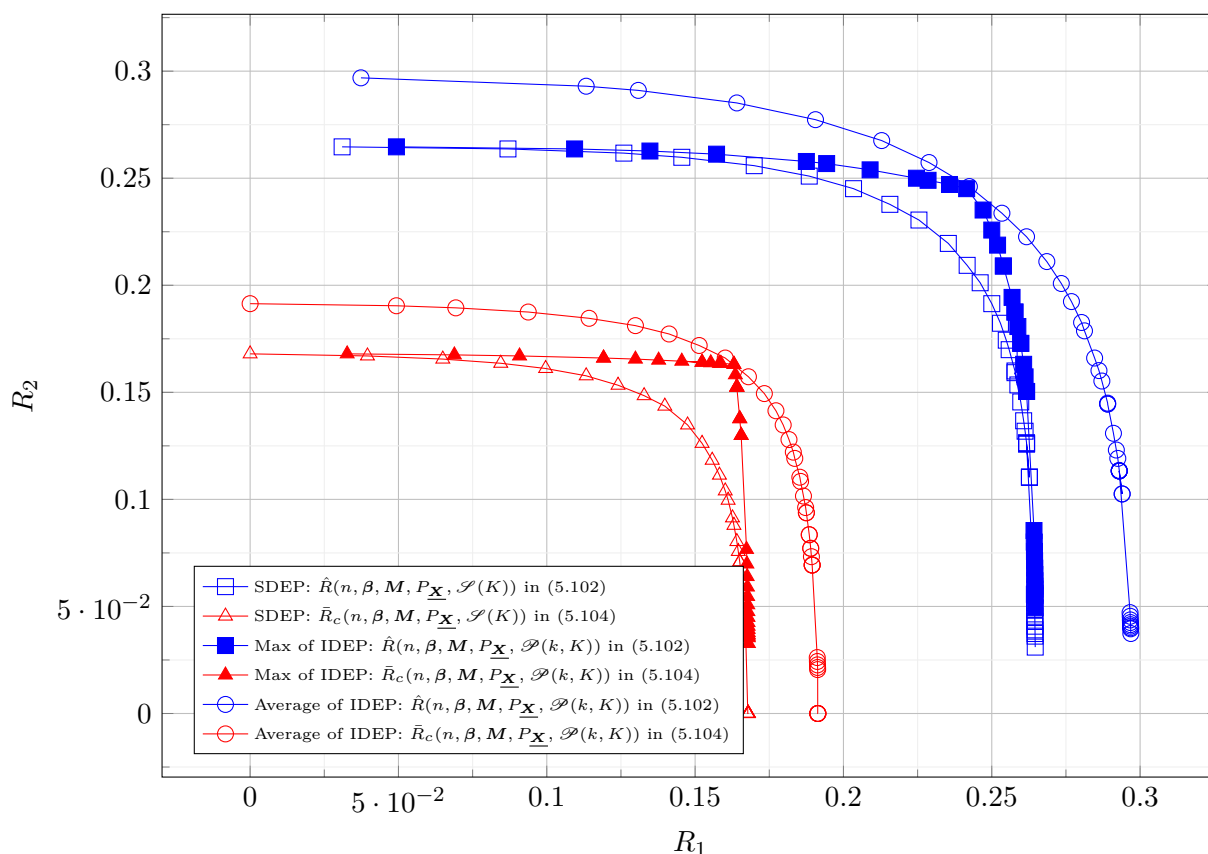


Figure 5.6.: Achievable regions and their approximations for the case of a two transmitters Gaussian MAC with discrete channel inputs $\mathcal{X}_1 = \mathcal{X}_2 = \{-1, 1\}$, noise variance $\sigma^2 = 1$, blocklength $n = 50$, and DEP $\epsilon = 10^{-2}$.

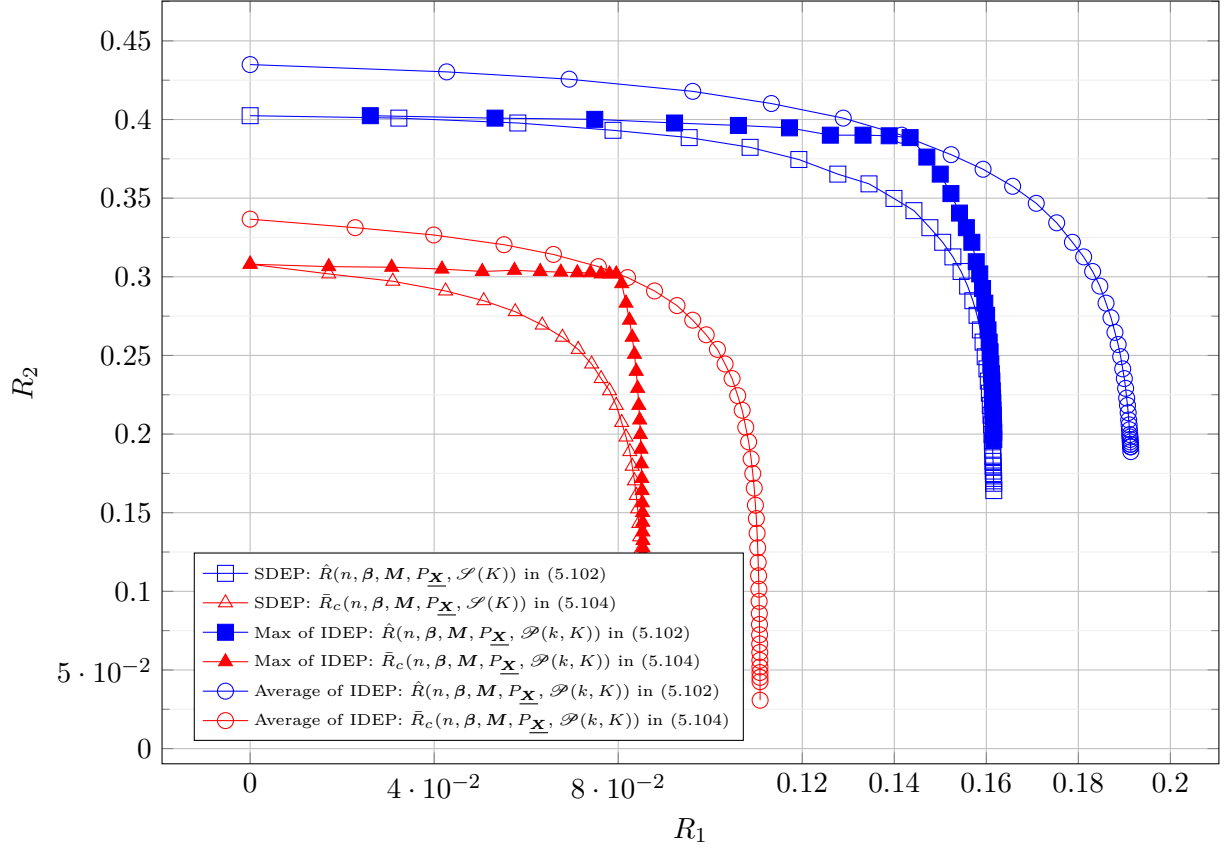


Figure 5.7.: Achievable regions and their approximations for the case of a two transmitters Gaussian MAC with discrete channel inputs $\mathcal{X}_1 = \{-1, 1\}$, $\mathcal{X}_2 = \{-\sqrt{2}, \sqrt{2}\}$, noise variance $\sigma^2 = 1$, blocklength $n = 50$ and DEP $\epsilon = 10^{-3}$.

5.3. Conclusion

In this chapter, the saddlepoint approximations to the DEP bounds of Chapter 2 have been derived. For the point to point channel, the bounds induced by the saddlepoint approximation have shown the limitation of the normal approximation, especial for small values of the DEP. These bounds are very close to the approximation with the advantage of being firm bounds. For the MAC, the saddlepoint approximation of the RCU bound and related upper bound are observed to outperform others. Unlike the normal approximation, the upper bound induced by the saddlepoint approximation is close to the approximation, which boots the confidence on the approximation. Thank to the firm upper bound induced by the RCU bound approximation, achievable regions for two transmitters Gaussian MAC have been studied based on different DEPs. It is has been observed that the consideration of the IDEP seems to show the no-interference nature between the two transmitters, which was completely eluded by the sole consideration of the A-SDEP.

6

Conclusion

In this thesis, the problem of the characterization of the fundamental limits of communication at low values of DEP and finite blocklength has been tackled. The fundamental bounds are of particular importance for emerging URLLC systems. Chapter 2 has discussed the evaluation difficulty of such fundamental bounds on DEPs for point to point channels and MAC, which leads information theorists to consider the second order achievabilities. The strong foundation of the second order achievabilities on the normal approximation of CDFs of sums of IID random vectors makes them irrelevant to the study of small values of the DEP and the small blocklengths.

In this thesis, by observing that the evaluation of the CDFs contained in the bounds on the DEP constitutes the main bottleneck, new methods (Chapter 3 and Chapter 4) to approach these CDFs are proposed, under the stationary and memoryless assumptions where the involved random variables are sums of IID random variables. The method, which is referred as the *exponentially tilted Gaussian approximation*, relies on a changed of measure that introduced a parametrized random vector, which is also a sum of independent random vectors, followed by its Gaussian approximation. The resulting family of approximations, which contain saddlepoint and Gaussian approximations, when properly optimized has shown to overcome the limitation of the normal approximation. Thank to these results, easy computable bounds on DEPs have been provided in Chapter 5. For point to point, such new bounds have pointed out the limitation of the normal approximation for small values of the DEP. The analyse of achievable regions of MAC under different types of the DEP has shown the interest of the notion of the IDEP. Indeed, the almost rectangular shape of the maximum IDEP achievable region seems to infer the non-interference nature between transmitters as observed in [7]. This observation was completely missing by the sole consideration of the SDEP.

Despite the importance of this thesis contribution, the concentration on the approximation of CDFs of sums of independent random variables or vectors limits the focus to the memoryless channels with also the memoryless assumption on the input. Dropping the independence assumption in Theorem 15 and Lemma 25 must be useful to the study of any channel DEP bounds. The principal challenge is to overcome the CGF of arbitrary sums that is no more factorizable and may lead to a computation bottleneck.

The negative lower bound for the CDFs of random vectors does not allow to study converse bounds for MACs as in the case of the point to point channels. However, the close gap between

the upper bound and the approximation encourages to investigate the approximation error upper bounds in Theorem 15 and Lemma 20. The factor two in Theorem 15 that is obtained by the triangular inequality in the proof of Lemma 35 can be improved to one by finding an alternative to the triangular inequality. If such an improvement is made, the study of converse bound for the point to point channel can be extended to arbitrary small values of the blocklength. Theorem 17 can be enhanced by improving the factor $c(k)$ in the Berry-Esseen Theorem or finding an alternative to the final step involving the Berry-Esseen Theorem. The choice of parameters θ in Theorem 15 and θ in Lemma 20, which respectively leads to main results Theorem 17 and Theorem 23, can be optimized by minimizing the upper bound instead of the exponential factor as carried out in this thesis due the difficulty reason.

The almost rectangle shape of the the maximum IDEP achievable region must be investigated by studying the converse region. If the almost rectangle shape still holds, the NOMA can replace OMA protocols with almost no-interference, which is a considerable change in nowadays resource allocation.

Part I.

APPENDICES



Proof of Lemma 3

Assume that an (n, M) -code $\mathcal{C}_{n,M}$ defined by the system in (2.4) is used for the transmission. Given an observation \mathbf{y} , the receiver successively performs the following tests starting from $i = 1$ up to $i = M$, with $i \in \mathcal{W}$:

$$\tilde{t}(\mathbf{u}(i), \mathbf{y}) > \gamma(i), \quad (\text{A.1})$$

where the function \tilde{t} is defined in (2.17) and $\gamma : \mathcal{W} \rightarrow \mathbb{R}$. Then, the receiver chooses the lowest message index i satisfying the condition in (A.1), if it exists, as being the transmitted message index, or declares an error if no message index satisfies (A.1).

Note that the decoding scheme based on (A.1) is a particular instance of the threshold decoding defined in (2.16).

Considering the transmission of message index i , two events lead to a decoding error: (a) message index i does not satisfies the condition in (A.1); (b) there exists a message index j , with $j < i$, that satisfies the condition (A.1). The DEP associated to message index i , denoted by ϵ_i , is given by:

$$\epsilon_i = \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} \left[\max \left\{ \mathbb{1}_{\{\tilde{t}(\mathbf{u}(i), \mathbf{Y}) \leq \gamma(i)\}}, \max_{j \in \{1, 2, \dots, i-1\}} \mathbb{1}_{\{\tilde{t}(\mathbf{u}(j), \mathbf{Y}) > \gamma(j)\}} \right\} \right] \quad (\text{A.2a})$$

$$\leq \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} \left[\mathbb{1}_{\{\tilde{t}(\mathbf{u}(i), \mathbf{Y}) \leq \gamma(i)\}} + \sum_{j \in \{1, 2, \dots, i-1\}} \mathbb{1}_{\{\tilde{t}(\mathbf{u}(j), \mathbf{Y}) > \gamma(j)\}} \right]. \quad (\text{A.2b})$$

Note that the right hand side term in (A.2a) follows from ϵ_i being the probability of union of events that can be written in the form of the expectation of the maximum of their indicator functions. The upper bound in (A.2b) results from the observation that the sum of positive elements (in this case indicator functions) is greater than their maximum. Then, the average DEP $\lambda(\mathcal{C}_{n,M})$ associated to this (n, M) -code $\mathcal{C}_{n,M}$ is given by:

$$\lambda(\mathcal{C}_{n,M}) = \frac{1}{M} \sum_{i=1}^M \epsilon_i \quad (\text{A.3a})$$

$$\leq \frac{1}{M} \sum_{i=1}^M \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{u}(i), \mathbf{Y}) \leq \gamma(i)\}} + \sum_{j \in \{1, 2, \dots, i-1\}} \mathbf{1}_{\{\tilde{t}(\mathbf{u}(j), \mathbf{Y}) > \gamma(j)\}} \right] \quad (\text{A.3b})$$

$$= \frac{1}{M} \sum_{i=1}^M \left(\mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{u}(i), \mathbf{Y}) \leq \gamma(i)\}} \right] + \sum_{j \in \{1, 2, \dots, i-1\}} \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{u}(j), \mathbf{Y}) > \gamma(j)\}} \right] \right) \quad (\text{A.3c})$$

$$= \frac{1}{M} \sum_{i=1}^M \left(\mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(i)}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{u}(i), \mathbf{Y}) \leq \gamma(i)\}} \right] + \sum_{j \in \{i+1, i+2, \dots, M\}} \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{u}(j)}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{u}(i), \mathbf{Y}) > \gamma(i)\}} \right] \right). \quad (\text{A.3d})$$

The right hand side term in (A.3c) follows from the expectation being a linear operator. By regrouping the terms containing $\tilde{t}(\mathbf{u}(i), \mathbf{Y})$ for all $i \in \{1, 2, \dots, M\}$ in (A.3c), the right term in (A.3d) is obtained. Taking the expectation of the bound in (A.3d) with respect to an input probability measure $P_{\mathbf{X}}$ induced by random coding and using the definition of $\lambda^*(n, M)$ in (2.9), the following holds:

$$\lambda^*(n, M) \leq \frac{1}{M} \sum_{i=1}^M \left(\int_{\mathcal{X}^n} \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{x}, \mathbf{Y}) \leq \gamma(i)\}} \right] P_{\mathbf{X}}(d\mathbf{x}) + \sum_{j \in \{i+1, i+2, \dots, M\}} \int_{\mathcal{X}^n} \int_{\mathcal{X}^n} \mathbb{E}_{P_{\mathbf{Y}|\mathbf{X}=\mathbf{x}_1}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{x}, \mathbf{Y}) > \gamma(i)\}} \right] P_{\mathbf{X}}(d\mathbf{x}_1) P_{\mathbf{X}}(d\mathbf{x}) \right) \quad (\text{A.4a})$$

$$= \frac{1}{M} \sum_{i=1}^M \left(\mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}} \right] + \sum_{j \in \{i+1, i+2, \dots, M\}} \int_{\mathcal{X}^n} \mathbb{E}_{P_{\mathbf{Y}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{x}, \mathbf{Y}) > \gamma(i)\}} \right] P_{\mathbf{X}}(d\mathbf{x}) \right) \quad (\text{A.4b})$$

$$= \frac{1}{M} \sum_{i=1}^M \left(\mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}} \right] + \sum_{j \in \{i+1, i+2, \dots, M\}} \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) > \gamma(i)\}} \right] \right) \quad (\text{A.4c})$$

$$= \frac{1}{M} \sum_{i=1}^M \left(\mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}} \right] + (M - i) \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) > \gamma(i)\}} \right] \right). \quad (\text{A.4d})$$

Finally, choosing $\gamma(i) = \ln(M - i)$ in (A.4d) and using the definition of the function T in (2.19) leads to

$$\lambda^*(n, M) \leq \frac{1}{M} \sum_{i=1}^M T(n, P_{\mathbf{X}}, P_{\mathbf{Y}}, M - i), \quad (\text{A.5})$$

which concludes the proof.

Note that $\left(\mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}} \right] + (M - i) \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) > \gamma(i)\}} \right] \right)$ in (A.4d) can be written in the form:

$$\begin{aligned} & \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}} \right] + (M - i) \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) > \gamma(i)\}} \right] \\ &= (M - i + 1) \left(\frac{1}{M - i + 1} \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}} \right] + \frac{M - i}{M - i + 1} \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}}} \left[\mathbf{1}_{\{\tilde{t}(\mathbf{X}, \mathbf{Y}) > \gamma(i)\}} \right] \right), \end{aligned} \quad (\text{A.6})$$

where by choosing $\gamma(i) = \ln(M - i)$,

$$\left(\frac{1}{M - i + 1} \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}} [\mathbb{1}_{\{\tilde{i}(\mathbf{X}, \mathbf{Y}) \leq \gamma(i)\}}] + \frac{M - i}{M - i + 1} \mathbb{E}_{P_{\mathbf{X}} P_{\mathbf{Y}}} [\mathbb{1}_{\{\tilde{i}(\mathbf{X}, \mathbf{Y}) > \gamma(i)\}}] \right), \quad (\text{A.7})$$

which is the minimum Bayesian error probability of a binary hypothesis test with: (a) the null hypothesis having the probability $\frac{M-i}{M-i+1}$ and inducing the probability measure $P_{\mathbf{X}} P_{\mathbf{Y}}$; and (b) the alternative hypothesis having the probability $\frac{1}{M-i+1}$ and inducing the probability measure $P_{\mathbf{X}} P_{\mathbf{Y}|\mathbf{X}}$. Thus, the choice of $\gamma(i) = \ln\left(\frac{M-1}{2}\right)$ that leads to the bound in Lemma 2 can be improved with $\gamma(i) = \ln(M - i)$ instead.

— B —

Proof of Theorem 15

The proof of Theorem 15 relies on the notion of exponentially tilted distributions. The proof is based on three steps

- Writing the probability $P_{X_n}(\mathcal{A})$ with respect to exponentially tilted measure of P_{X_n} in (B.4).
- Approximating the exponentially tilted measure of P_{X_n} by a Gaussian distribution in (B.5).
- The distance between $P_{X_n}(\mathcal{A})$ and the approximation is bounded by using the Kolmogorov–Smirnov distance ((B.6)) in (B.7) as detailed in Appendix K. The Kolmogorov–Smirnov distance is in turn bounded Berry-Esseen Theorem.

Let φ_Y be the moment generating function of the distribution P_Y . Given $\theta \in \Theta_Y$, let $Y_1^{(\theta)}, Y_2^{(\theta)}, \dots, Y_n^{(\theta)}$ be random variables whose joint probability distribution, denoted by $P_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}$, satisfies for all $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,

$$\frac{dP_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}{dP_{Y_1Y_2\dots Y_n}}(y_1, y_2, \dots, y_n) = \frac{\exp\left(\theta \sum_{j=1}^n y_j\right)}{(\varphi_Y(\theta))^n}. \quad (\text{B.1})$$

That is, the distribution $P_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}$ is an exponentially tilted distribution with respect to $P_{Y_1Y_2\dots Y_n}$. Using this notation, for all $\mathcal{A} \subseteq \mathbb{R}$ and for all $\theta \in \Theta_Y$,

$$P_{X_n}(\mathcal{A}) = \mathbb{E}_{P_{X_n}}[\mathbb{1}_{\{X_n \in \mathcal{A}\}}] \quad (\text{B.2a})$$

$$= \mathbb{E}_{P_{Y_1Y_2\dots Y_n}}\left[\mathbb{1}_{\{\sum_{j=1}^n Y_j \in \mathcal{A}\}}\right] \quad (\text{B.2b})$$

$$= \mathbb{E}_{P_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}\left[\frac{dP_{Y_1Y_2\dots Y_n}}{dP_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}(Y_1^{(\theta)}, Y_2^{(\theta)}, \dots, Y_n^{(\theta)})\mathbb{1}_{\{\sum_{j=1}^n Y_j^{(\theta)} \in \mathcal{A}\}}\right] \quad (\text{B.2c})$$

$$= \mathbb{E}_{P_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}\left[\left(\frac{dP_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}{dP_{Y_1Y_2\dots Y_n}}(Y_1^{(\theta)}, Y_2^{(\theta)}, \dots, Y_n^{(\theta)})\right)^{-1}\mathbb{1}_{\{\sum_{j=1}^n Y_j^{(\theta)} \in \mathcal{A}\}}\right] \quad (\text{B.2d})$$

$$= \mathbb{E}_{P_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}\left[\left(\frac{\exp\left(\theta \sum_{j=1}^n Y_j^{(\theta)}\right)}{(\varphi_Y(\theta))^n}\right)^{-1}\mathbb{1}_{\{\sum_{j=1}^n Y_j^{(\theta)} \in \mathcal{A}\}}\right] \quad (\text{B.2e})$$

$$= (\varphi_Y(\theta))^n \mathbb{E}_{P_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}}\left[\exp\left(-\theta \sum_{j=1}^n Y_j^{(\theta)}\right)\mathbb{1}_{\{\sum_{j=1}^n Y_j^{(\theta)} \in \mathcal{A}\}}\right] \quad (\text{B.2f})$$

For the ease of the notation, consider the random variable

$$S_{n,\theta} = \sum_{j=1}^n Y_j^{(\theta)}, \quad (\text{B.3})$$

whose probability distribution is denoted by $P_{S_{n,\theta}}$. Hence, plugging (B.3) in (B.2f) yields,

$$P_{X_n}(\mathcal{A}) = (\varphi_Y(\theta))^n \mathbb{E}_{P_{S_{n,\theta}}}\left[\exp(-\theta S_{n,\theta})\mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}}\right]. \quad (\text{B.4})$$

The proof continues by upper bounding the following absolute difference

$$\left|P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}}\left[\exp(-\theta Z_{n,\theta})\mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}}\right]\right|, \quad (\text{B.5})$$

where $Z_{n,\theta}$ is a Gaussian random variable with the same mean and variance as $S_{n,\theta}$, and probability distribution denoted by $P_{Z_{n,\theta}}$. The relevance of the absolute difference in (B.5) is that it is equal to the error of calculating $P_{X_n}(\mathcal{A})$ under the assumption that the resulting random variable S_n follows a Gaussian distribution. The following lemma provides an upper bound on the absolute difference in (B.5) in terms of the Kolmogorov–Smirnov distance between the distributions $P_{S_{n,\theta}}$ and $P_{Z_{n,\theta}}$, denoted by

$$\Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) \triangleq \sup_{x \in \mathbb{R}} |F_{S_{n,\theta}}(x) - F_{Z_{n,\theta}}(x)|, \quad (\text{B.6})$$

where $F_{S_{n,\theta}}$ and $F_{Z_{n,\theta}}$ are the CDFs of the random variables $S_{n,\theta}$ and $Z_{n,\theta}$, respectively.

Lemma 35 *Given $\theta \in \Theta_Y$ and $a \in \mathbb{R}$, consider the following conditions:*

- (i) $\theta \leq 0$ and $\mathcal{A} = (-\infty, a]$, and
- (ii) $\theta > 0$ and $\mathcal{A} = (a, \infty)$.

If at least one of the above conditions is satisfied, then the absolute difference in (B.5) satisfies

$$\left| P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right| \leq \frac{(\varphi_Y(\theta))^n}{\exp(\theta a)} \min \{1, 2 \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}})\} \quad (\text{B.7})$$

Proof: The proof of Lemma 35 is presented in Appendix K. ■

The proof continues by providing an upper bound on $\Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}})$ in (B.7) leveraging the observation that $S_{n,\theta}$ is the sum of n independent and identically distributed random variables. This follows immediately from the assumptions of Theorem 15, nonetheless, for the sake of completeness, the following lemma provides a proof of this statement.

Lemma 36 *For all $\theta \in \Theta_Y$, $Y_1^{(\theta)}, Y_2^{(\theta)}, \dots, Y_n^{(\theta)}$ are mutually independent and identically distributed random variables with probability distribution $P_{Y^{(\theta)}}$. Moreover, $P_{Y^{(\theta)}}$ is an exponential tilted distribution with respect to P_Y . That is, $P_{Y^{(\theta)}}$ satisfies for all $y \in \mathbb{R}$,*

$$\frac{dP_{Y^{(\theta)}}}{dP_Y}(y) = \frac{\exp(\theta y)}{\varphi_Y(\theta)}. \quad (\text{B.8})$$

Proof: The proof of Lemma 36 is presented in Appendix L. ■

Lemma 36 paves the way for obtaining an upper bound on $\Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}})$ in (B.7) via the Berry–Esseen Theorem (Theorem 14). Let μ_θ , V_θ , and T_θ be the mean, the variance, and the third absolute central moment of the random variable $Y^{(\theta)}$, whose probability distribution is $P_{Y^{(\theta)}}$ in (B.8). More specifically:

$$\mu_\theta = \mathbb{E}_{P_{Y^{(\theta)}}}[Y^{(\theta)}] = \mathbb{E}_{P_Y} \left[\frac{Y \exp(\theta Y)}{\varphi_Y(\theta)} \right], \quad (\text{B.9})$$

$$V_\theta = \mathbb{E}_{P_{Y^{(\theta)}}}[(Y^{(\theta)} - \mu_\theta)^2] = \mathbb{E}_{P_Y} \left[\frac{(Y - \mu_\theta)^2 \exp(\theta Y)}{\varphi_Y(\theta)} \right], \text{ and} \quad (\text{B.10})$$

$$T_\theta = \mathbb{E}_{P_{Y^{(\theta)}}}[|Y^{(\theta)} - \mu_\theta|^3] = \mathbb{E}_{P_Y} \left[\frac{|Y - \mu_\theta|^3 \exp(\theta Y)}{\varphi_Y(\theta)} \right]. \quad (\text{B.11})$$

Let also ξ_θ be

$$\xi_\theta = c_1 \left(\frac{T_\theta}{V_\theta^{3/2}} + c_2 \right), \quad (\text{B.12})$$

with c_1 and c_2 defined in (3.24).

From Theorem 14, it follows that $\Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}})$ in (B.7) satisfies:

$$\Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) \leq \min \left\{ 1, \frac{\xi_\theta}{\sqrt{n}} \right\} \leq \frac{\xi_\theta}{\sqrt{n}}. \quad (\text{B.13})$$

Plugging (B.13) in (B.7) yields

$$\left| P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}} [\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}}] \right| \leq \frac{(\varphi_Y(\theta))^n}{\exp(\theta a)} \min \left\{ 1, 2 \frac{\xi_\theta}{\sqrt{n}} \right\}, \quad (\text{B.14})$$

under the assumption that at least one of the conditions of Lemma 35 is met.

The proof ends by obtaining a closed-form expression of the term $\mathbb{E}_{P_{Z_{n,\theta}}} [\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}}]$ in (B.14) under the assumption that at least one of the conditions of Lemma 35 is met. First, assuming that condition (i) in Lemma 35 holds, it follows that:

$$\begin{aligned} & \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \\ &= \int_{-\infty}^a \exp(-\theta z) \frac{1}{\sqrt{2\pi n V_\theta}} \exp\left(-\frac{(z - n\mu_\theta)^2}{2nV_\theta}\right) dz \end{aligned} \quad (\text{B.15a})$$

$$= \int_{-\infty}^a \frac{1}{\sqrt{2\pi n V_\theta}} \exp\left(-\frac{z^2 - 2z n\mu_\theta + n^2\mu_\theta^2 + 2n\theta V_\theta z}{2nV_\theta}\right) dz \quad (\text{B.15b})$$

$$= \int_{-\infty}^a \frac{1}{\sqrt{2\pi n V_\theta}} \exp\left(-\frac{(z - n\mu_\theta + n\theta V_\theta)^2 - n^2\theta^2 V_\theta^2 + 2n\mu_\theta n\theta V_\theta}{2nV_\theta}\right) dz \quad (\text{B.15c})$$

$$= \exp\left(-\theta n\mu_\theta + \frac{1}{2}nV_\theta\theta^2\right) \int_{-\infty}^a \frac{1}{\sqrt{2\pi n V_\theta}} \exp\left(-\frac{(z - n\mu_\theta + n\theta V_\theta)^2}{2nV_\theta}\right) dz \quad (\text{B.15d})$$

$$= \exp\left(-\theta n\mu_\theta + \frac{1}{2}nV_\theta\theta^2\right) \int_{-\infty}^{\frac{a - n\mu_\theta + n\theta V_\theta}{\sqrt{nV_\theta}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \quad (\text{B.15e})$$

$$= \exp\left(-\theta n\mu_\theta + \frac{1}{2}nV_\theta\theta^2\right) Q\left(-\frac{a - n\mu_\theta + n\theta V_\theta}{\sqrt{nV_\theta}}\right). \quad (\text{B.15f})$$

Second, assuming that condition (ii) in Lemma 35 holds, it follows that:

$$\mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] = \int_a^\infty \exp(-\theta z) \frac{1}{\sqrt{2\pi n V_\theta}} \exp\left(-\frac{(z - n\mu_\theta)^2}{2nV_\theta}\right) dz \quad (\text{B.16a})$$

$$= \exp\left(-\theta n\mu_\theta + \frac{1}{2}nV_\theta\theta^2\right) \int_{\frac{a - n\mu_\theta + n\theta V_\theta}{\sqrt{nV_\theta}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \quad (\text{B.16b})$$

$$= \exp\left(-\theta n\mu_\theta + \frac{1}{2}nV_\theta\theta^2\right) Q\left(\frac{a - n\mu_\theta + n\theta V_\theta}{\sqrt{nV_\theta}}\right), \quad (\text{B.16c})$$

where Q in (B.15f) and (B.16c) is the complementary CDF of the standard Gaussian distribution defined in (3.14).

The expressions in (B.15f) and (B.16c) can be jointly written as follows:

$$\mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] = \exp\left(-\theta n\mu_\theta + \frac{1}{2}nV_\theta\theta^2\right) Q\left((-1)^{\mathbb{1}_{\{\theta \leq 0\}}} \frac{a - n\mu_\theta + n\theta V_\theta}{\sqrt{nV_\theta}}\right) \quad (\text{B.17})$$

under the assumption that at least one of the conditions (i) or (ii) in Lemma 35 holds.

Finally, under the same assumption, plugging (B.17) in (B.14) yields

$$\left| P_{X_n}(\mathcal{A}) - \exp\left(n \ln(\varphi_Y(\theta)) - n\theta\mu_\theta + \frac{1}{2}n\theta^2 V_\theta\right) Q\left((-1)^{\mathbb{1}_{\{\theta \leq 0\}}} \frac{a + n\theta V_\theta - n\mu_\theta}{\sqrt{nV_\theta}}\right) \right|$$

$$\leq \exp(n \ln(\varphi_Y(\theta)) - \theta a) \min \left\{ 1, \frac{2 \xi_\theta}{\sqrt{n}} \right\}. \quad (\text{B.18})$$

Under condition (i) in Lemma 35, the inequality in (B.18) can be written as follows:

$$\begin{aligned} & \left| F_{X_n}(a) - \exp \left(n \ln(\varphi_Y(\theta)) - n\theta\mu_\theta + \frac{1}{2}n\theta^2V_\theta \right) \cdot Q \left((-1)^{\mathbb{1}_{\{\theta \leq 0\}}} \frac{a + n\theta V_\theta - n\mu_\theta}{\sqrt{nV_\theta}} \right) \right| \\ & \leq \exp(n \ln(\varphi_Y(\theta)) - \theta a) \min \left\{ 1, \frac{2 \xi_\theta}{\sqrt{n}} \right\}. \end{aligned} \quad (\text{B.19})$$

Alternatively, under condition (ii) in Lemma 35, it follows from (B.18) that

$$\begin{aligned} & \left| 1 - F_{X_n}(a) - \exp \left(n \ln(\varphi_Y(\theta)) - n\theta\mu_\theta + \frac{1}{2}n\theta^2V_\theta \right) \cdot Q \left((-1)^{\mathbb{1}_{\{\theta \leq 0\}}} \frac{a + n\theta V_\theta - n\mu_\theta}{\sqrt{nV_\theta}} \right) \right| \\ & \leq \exp(n \ln(\varphi_Y(\theta)) - \theta a) \min \left\{ 1, \frac{2 \xi_\theta}{\sqrt{n}} \right\}, \end{aligned} \quad (\text{B.20})$$

Then, jointly writing (B.19) and (B.20), it follows that, for all $a \in \mathbb{R}$ and for all $\theta \in \Theta_Y$,

$$\begin{aligned} & \left| F_{X_n}(a) - \mathbb{1}_{\{\theta > 0\}} - (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp \left(n \ln(\varphi_Y(\theta)) - n\theta\mu_\theta + \frac{1}{2}n\theta^2V_\theta \right) Q \left((-1)^{\mathbb{1}_{\{\theta \leq 0\}}} \frac{a + n\theta V_\theta - n\mu_\theta}{\sqrt{nV_\theta}} \right) \right| \\ & \leq \exp(n \ln(\varphi_Y(\theta)) - \theta a) \min \left\{ 1, \frac{2 \xi_\theta}{\sqrt{n}} \right\}, \end{aligned} \quad (\text{B.21})$$

which can also be written as

$$|F_{X_n}(a) - \eta_Y(\theta, a, n)| \leq \exp(nK_Y(\theta) - \theta a) \min \left\{ 1, \frac{2 \xi_Y(\theta)}{\sqrt{n}} \right\}. \quad (\text{B.22})$$

This completes the proof.



Proof of Lemma 16

LET $g : \mathbb{R}^2 \times \mathbb{N} \rightarrow \mathbb{R}$ be for all $(\theta, a, n) \in \mathbb{R}^2 \times \mathbb{N}$,

$$g(\theta, a, n) = nK_Y(\theta) - \theta a = n \ln(\varphi_Y(\theta)) - \theta a. \quad (\text{C.1})$$

First, note that for all $\theta \in \Theta_Y$ and for all $n \in \mathbb{N}$, the function g is a concave function of a . Hence, from the definition of the function h in (3.35), h is concave.

Second, note that $0 \in \Theta_Y$ given that $\varphi_Y(0) = 1 < \infty$. Hence, from (3.35), it holds that, for all $a \in \mathbb{R}$,

$$h(a) \leq nK_Y(0) = n \ln(\varphi_Y(0)) = n \ln(1) = 0. \quad (\text{C.2a})$$

This shows that the function h in (3.35) is not positive.

Third, the next step of the proof consists of proving the equality in (3.37). For doing so, let $\theta^* : \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$ be for all $(a, n) \in \mathbb{R} \times \mathbb{N}$,

$$\theta^*(a, n) = \arg \inf_{\theta \in \Theta_Y} g(\theta, a, n). \quad (\text{C.3})$$

Note that the function g is a convex in θ . This follows by verifying that its second derivative with respect to θ is positive. That is,

$$\frac{d}{d\theta} g(\theta, a, n) = \frac{n}{\varphi_Y(\theta)} \frac{d}{d\theta} \varphi_Y(\theta) - a, \text{ and} \quad (\text{C.4a})$$

$$\frac{d^2}{d\theta^2} g(\theta, a, n) = \frac{n}{(\varphi_Y(\theta))^2} \left(\varphi_Y(\theta) \frac{d^2}{d\theta^2} \varphi_Y(\theta) - \left(\frac{d}{d\theta} \varphi_Y(\theta) \right)^2 \right) \quad (\text{C.4b})$$

$$= n \left(\frac{1}{\varphi_Y(\theta)} \frac{d^2}{d\theta^2} \varphi_Y(\theta) - \left(\frac{1}{\varphi_Y(\theta)} \frac{d}{d\theta} \varphi_Y(\theta) \right)^2 \right) \quad (\text{C.4c})$$

$$= n \left(\frac{1}{\varphi_Y(\theta)} \frac{d^2}{d\theta^2} \mathbb{E}_{P_Y}[\exp(\theta Y)] - \left(\frac{1}{\varphi_Y(\theta)} \frac{d}{d\theta} \mathbb{E}_{P_Y}[\exp(\theta Y)] \right)^2 \right) \quad (\text{C.4d})$$

$$= n \left(\frac{\mathbb{E}_{P_Y}[Y^2 \exp(\theta Y)]}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} - \left(\frac{\mathbb{E}_{P_Y}[Y \exp(\theta Y)]}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right)^2 \right) \quad (\text{C.4e})$$

$$=n \left(\mathbb{E}_{P_Y} \left[\frac{Y^2 \exp(\theta Y)}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right] - \left(\mathbb{E}_{P_Y} \left[\frac{Y \exp(\theta Y)}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right] \right)^2 \right) \quad (\text{C.4f})$$

$$=n \left(\mathbb{E}_{P_Y} \left[\frac{Y^2 \exp(\theta Y)}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right] - 2 \mathbb{E}_{P_Y} \left[\frac{Y \exp(\theta Y)}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right] K_Y^{(1)}(\theta) + \left(K_Y^{(1)}(\theta) \right)^2 \right)$$

$$=n \mathbb{E}_{P_Y} \left[\frac{\left(Y - K_Y^{(1)}(\theta) \right)^2 \exp(\theta Y)}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right] > 0. \quad (\text{C.4g})$$

Hence, if the first derivative of g with respect to θ (see (C.4a)) admits a zero in Θ_Y , then $\theta^*(a, n)$ is the unique solution in θ to the following equality:

$$\frac{d}{d\theta} g(\theta, a, n) = \frac{n}{\varphi_Y(\theta)} \frac{d}{d\theta} \varphi_Y(\theta) - a = 0. \quad (\text{C.5})$$

Equation (C.5) in θ can be rewritten as follows:

$$\frac{a}{n} = \frac{1}{\varphi_Y(\theta)} \frac{d}{d\theta} \varphi_Y(\theta) \quad (\text{C.6a})$$

$$= \frac{1}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \frac{d}{d\theta} \mathbb{E}_{P_Y}[\exp(\theta Y)] \quad (\text{C.6b})$$

$$= \frac{1}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \mathbb{E}_{P_Y}[Y \exp(\theta Y)] \quad (\text{C.6c})$$

$$= \mathbb{E}_{P_Y} \left[\frac{Y \exp(\theta Y)}{\mathbb{E}_{P_Y}[\exp(\theta Y)]} \right] \quad (\text{C.6d})$$

$$= K_Y^{(1)}(\theta). \quad (\text{C.6e})$$

From (C.6d), it follows that $\frac{a}{n}$ is the mean of a random variable that follows an exponentially tilted distribution with respect to P_Y . Thus, there exists a solution in θ for (C.6d) if and only if $\frac{a}{n} \in \text{int}\mathcal{C}_Y$ —hence the equality in (3.37).

Finally, from (C.6d), $a = n \mathbb{E}_{P_Y}[Y]$ implies that $\theta^*(a, n) = 0$. Hence, $h(n \mathbb{E}_{P_Y}[Y]) = 0$ from (3.37). This completes the proof for $h(n \mathbb{E}_{P_Y}[Y]) = 0$.

— D —

Proof of Theorem 17

FROM Lemma 16, it holds that given $(a, n) \in \mathbb{R} \times \mathbb{N}$ such that $\frac{a}{n} \in \text{int}\mathcal{C}_Y$,

$$nK_Y^{(1)}(\theta^*) = a. \quad (\text{D.1})$$

Then, plugging (D.1) in the expression of $\eta_Y(\theta^*, a, n)$, with function η_Y defined in (3.29), the following holds:

$$\begin{aligned} & \eta_Y(\theta^*, a, n) \\ &= \mathbb{1}_{\{\theta^* > 0\}} + (-1)^{\mathbb{1}_{\{\theta^* > 0\}}} \exp\left(\frac{1}{2}n(\theta^*)^2 K_Y^{(2)}(\theta) + nK_Y(\theta^*) - \theta^* a\right) Q\left((-1)^{\mathbb{1}_{\{\theta^* \leq 0\}}} \frac{a + n\theta^* K_Y^{(2)}(\theta^*) - a}{\sqrt{nK_Y^{(2)}(\theta^*)}}\right) \end{aligned} \quad (\text{D.2a})$$

$$= \mathbb{1}_{\{\theta^* > 0\}} + (-1)^{\mathbb{1}_{\{\theta^* > 0\}}} \exp\left(\frac{1}{2}n(\theta^*)^2 K_Y^{(2)}(\theta) + nK_Y(\theta^*) - \theta^* a\right) Q\left((-1)^{\mathbb{1}_{\{\theta^* \leq 0\}}} \theta^* \sqrt{nK_Y^{(2)}(\theta^*)}\right) \quad (\text{D.2b})$$

$$= \mathbb{1}_{\{\theta^* > 0\}} + (-1)^{\mathbb{1}_{\{\theta^* > 0\}}} \exp\left(\frac{1}{2}n(\theta^*)^2 K_Y^{(2)}(\theta) + nK_Y(\theta^*) - \theta^* a\right) Q\left(|\theta^*| \sqrt{nK_Y^{(2)}(\theta^*)}\right) \quad (\text{D.2c})$$

$$= \hat{F}_{X_n}(a), \quad (\text{D.2d})$$

where equality in (D.2d) follows (3.13). Finally, plugging (D.2d) in (3.30) yields

$$\left|F_{X_n}(a) - \hat{F}_{X_n}(a)\right| \leq \exp(nK_Y(\theta^*) - \theta^* a) \min\left\{1, \frac{2 \xi_Y(\theta^*)}{\sqrt{n}}\right\}. \quad (\text{D.3})$$

This completes the proof by observing that $\frac{a}{n} \in \text{int}\mathcal{C}_Y$ is equivalent to $a \in \text{int}\mathcal{C}_{X_n}$.



Proof of Lemma 19

First, the proof of Lemma 20 is given under more general conditions where the random vectors $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ in (4.4) are independent and not necessarily identically distributed. That is, for $i \in \{1, 2, \dots, k\}$, the random \mathbf{Y}_i induces the probability measures $P_{\mathbf{Y}_i}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and the corresponding CGF is denoted by $K_{\mathbf{Y}_i}$. Then, the second part of proof concentrated on the identically distributed random vectors by relying on the results of first part of the proof.

E.1. Independent Random vectors

The proof relies on the notion of exponentially tilted measures. Given $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$, with

$$\Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n} = \left\{ \mathbf{t} \in \mathbb{R}^k : \forall i \in \{1, 2, \dots, k\}, K_{\mathbf{Y}_i}(\mathbf{t}) < +\infty \right\}, \quad (\text{E.1})$$

let $\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}$ be independent random vectors that respectively induce the probability measures $P_{\mathbf{Y}_1^{(\boldsymbol{\theta})}}, P_{\mathbf{Y}_2^{(\boldsymbol{\theta})}}, \dots, P_{\mathbf{Y}_n^{(\boldsymbol{\theta})}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, such that for all $j \in \{1, 2, \dots, n\}$ and for all $\mathbf{y} \in \mathbb{R}^k$,

$$\frac{dP_{\mathbf{Y}_j^{(\boldsymbol{\theta})}}}{dP_{\mathbf{Y}_j}}(\mathbf{y}) = \exp\left(\boldsymbol{\theta}^\top \mathbf{y} - K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right). \quad (\text{E.2})$$

That is, for all $j \in \{1, 2, \dots, n\}$, the probability measure $P_{\mathbf{Y}_j^{(\boldsymbol{\theta})}}$ is an exponentially tilted measure with respect to $P_{\mathbf{Y}_j}$. Denote by $P_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}$ and $P_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$ the joint probability measures respectively induced by the independent random vectors $\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}$ and $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_n$ on the measurable space $(\mathbb{R}^{k \times n}, \mathcal{B}(\mathbb{R}^{k \times n}))$. Then, for all $j \in \{1, 2, \dots, n\}$ and for all $\mathbf{y}_j \in \mathbb{R}^k$, it holds that

$$\frac{dP_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}}{dP_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_2 \dots \mathbf{y}_n) = \prod_{j=1}^n \frac{dP_{\mathbf{Y}_j^{(\boldsymbol{\theta})}}}{dP_{\mathbf{Y}_j}}(\mathbf{y}_j) \quad (\text{E.3})$$

$$= \exp\left(\sum_{j=1}^n \left(\boldsymbol{\theta}^\top \mathbf{y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right)\right). \quad (\text{E.4})$$

Using this notation, for all $\mathcal{A} \subseteq \mathbb{R}^k$ and for all $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$, with $\Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$ defined in (E.1),

$$P_{\mathbf{X}_n}(\mathcal{A}) = \mathbb{E}_{P_{\mathbf{X}_n}}[\mathbb{1}_{\{\mathbf{X}_n \in \mathcal{A}\}}] \quad (\text{E.5a})$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}} \left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j \in \mathcal{A}\}} \right] \quad (\text{E.5b})$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}} \left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \frac{dP_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}}{dP_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}}(\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}) \right] \quad (\text{E.5c})$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}} \left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \left(\frac{dP_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}}{dP_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}}(\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}) \right)^{-1} \right] \quad (\text{E.5d})$$

$$= \mathbb{E}_{P_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}} \left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \exp\left(\sum_{j=1}^n \left(K_{\mathbf{Y}_j}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{Y}_j^{(\boldsymbol{\theta})}\right)\right) \right] \quad (\text{E.5e})$$

$$= \exp\left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\mathbf{Y}_1^{(\boldsymbol{\theta})} \mathbf{Y}_2^{(\boldsymbol{\theta})} \dots \mathbf{Y}_n^{(\boldsymbol{\theta})}}} \left[\mathbb{1}_{\{\sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \exp\left(-\boldsymbol{\theta}^\top \sum_{j=1}^n \mathbf{Y}_j^{(\boldsymbol{\theta})}\right) \right] \quad (\text{E.5f})$$

$$= \exp\left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\boldsymbol{\theta})}\right) \mathbb{1}_{\{\mathbf{S}_n^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \right], \quad (\text{E.5g})$$

the random vector $\mathbf{S}_n^{(\theta)}$ is defined in (4.77) under the condition that for all $j \in \{1, 2, \dots, k\}$, $\mathbf{Y}_j^{(\theta)}$ satisfies (E.2). This proves the observation made in (4.76). The proof continues by upper bounding the following absolute difference

$$\left| P_{\mathbf{X}_n}(\mathcal{A}) - \exp\left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right] \right|, \quad (\text{E.6})$$

where the random vector $\mathbf{Z}_n^{(\theta)}$ is defined in (4.26).

Plugging (E.5g) in (E.6) yields

$$\begin{aligned} & \left| P_{\mathbf{X}_n}(\mathcal{A}) - \exp\left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right) \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right] \right| \\ &= \exp\left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta})\right) \left| \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{A}\}} \right] - \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right] \right|. \end{aligned} \quad (\text{E.7})$$

The following lines focuses on obtaining explicit expressions for the terms

$$\mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{A}\}} \right], \quad \text{and} \quad (\text{E.8})$$

$$\mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right], \quad (\text{E.9})$$

in (E.7).

E.1.1. Explicit Expression of (E.8)

From (E.8), the following holds

$$\mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\exp\left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\theta)}\right) \mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{A}\}} \right] = \int_{\mathbb{R}^k} \exp\left(-\boldsymbol{\theta}^\top \mathbf{x}\right) \mathbb{1}_{\{\mathbf{x} \in \mathcal{A}\}} dP_{\mathbf{S}_n^{(\theta)}}(\mathbf{x}). \quad (\text{E.10})$$

The next step consists in writing the function $\exp\left(-\boldsymbol{\theta}^\top \mathbf{x}\right)$ in the right hand-side of (E.10) as a Lebesgue integral. For doing so, consider the set $\mathcal{K} = \{1, 2, \dots, k\}$ and let the set-valued functions $\mathcal{I}^- : \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n} \rightarrow 2^{\mathcal{K}}$, $\mathcal{I} : \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n} \rightarrow 2^{\mathcal{K}}$, and $\mathcal{I}^+ : \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n} \rightarrow 2^{\mathcal{K}}$ be respectively defined such that for all $\mathbf{u} = (u_1, u_2, \dots, u_k) \in \Theta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}$

$$\mathcal{I}^-(\mathbf{u}) = \{i \in \{1, 2, \dots, k\} : u_i < 0\}, \quad (\text{E.11})$$

$$\mathcal{I}(\mathbf{u}) = \{i \in \{1, 2, \dots, k\} : u_i = 0\}, \quad \text{and} \quad (\text{E.12})$$

$$\mathcal{I}^+(\mathbf{u}) = \{i \in \{1, 2, \dots, k\} : u_i > 0\}. \quad (\text{E.13})$$

Then, for all $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$, the following holds

$$\begin{aligned} & \exp\left(-\boldsymbol{\theta}^\top \mathbf{x}\right) \\ &= \exp\left(-\theta_1 x_1 - \theta_2 x_2 - \dots - \theta_k x_k\right) \end{aligned} \quad (\text{E.14})$$

$$= \exp(-\theta_1 x_1) \exp(-\theta_2 x_2) \dots \exp(-\theta_k x_k) \quad (\text{E.15})$$

$$= \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \exp(-\theta_i x_i) \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \exp(-\theta_j x_j) \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \exp(-\theta_s x_s) \right) \quad (\text{E.16})$$

$$= \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \int_{-\infty}^{x_i} -\theta_i \exp(-\theta_i t_i) dt_i \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \int_0^1 dt_j \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \int_{x_s}^{\infty} \theta_s \exp(-\theta_s t_s) dt_s \right) \quad (\text{E.17})$$

$$= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \int_{-\infty}^{x_i} \theta_i \exp(-\theta_i t_i) dt_i \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \int_0^1 dt_j \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \int_{x_s}^{\infty} \theta_s \exp(-\theta_s t_s) dt_s \right). \quad (\text{E.18})$$

To ease the notation, for all $u \in \mathbb{R}$ and for all $a \in \mathbb{R}$, let the set $\mathcal{B}_{u,a}$ be:

$$\mathcal{B}_{u,a} \triangleq \begin{cases} [0, 1] & \text{if } u = 0 \\ (-\infty, a] & \text{if } u < 0 \\ [a, \infty) & \text{if } u > 0. \end{cases} \quad (\text{E.19})$$

Then, using the notation in (E.19), the equality in (E.18) can be written as follow

$$\begin{aligned} & \exp(-\boldsymbol{\theta}^\top \mathbf{x}) \\ &= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \int_{\mathcal{B}_{\theta_i, x_i}} \theta_i \exp(-\theta_i t_i) dt_i \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \int_{\mathcal{B}_{\theta_j, x_j}} dt_j \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \int_{\mathcal{B}_{\theta_s, x_s}} \theta_s \exp(-\theta_s t_s) dt_s \right) \end{aligned} \quad (\text{E.20})$$

$$= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{B}_{\theta_1, x_1}} \int_{\mathcal{B}_{\theta_2, x_2}} \dots \int_{\mathcal{B}_{\theta_k, x_k}} \exp(-\boldsymbol{\theta}^\top \mathbf{t}) dt_k \dots dt_2 dt_1, \quad (\text{E.21})$$

where the equality in (E.21) follows from the linearity of the integration operator. To ease the notation, consider the set defined as

$$\mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}} = \mathcal{B}_{\theta_1, x_1} \times \mathcal{B}_{\theta_2, x_2} \times \dots \times \mathcal{B}_{\theta_k, x_k}, \quad (\text{E.22})$$

where for all $i \in \{1, 2, \dots, k\}$, the set $\mathcal{B}_{\theta_i, x_i}$ is defined in (E.19). Then, plugging (E.22) in (E.18) yields

$$\begin{aligned} & \exp(-\boldsymbol{\theta}^\top \mathbf{x}) \\ &= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}} \exp(-\boldsymbol{\theta}^\top \mathbf{t}) d\nu_k(\mathbf{t}) \end{aligned} \quad (\text{E.23})$$

$$= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \exp(-\boldsymbol{\theta}^\top \mathbf{t}) \mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} d\nu_k(\mathbf{t}), \quad (\text{E.24})$$

where ν_k is the Lebesgue measure on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

Then, plugging (E.24) in (E.10) yields

$$\begin{aligned} & \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\theta)} \right) \mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{A}\}} \right] \\ &= \int_{\mathbb{R}^k} (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} d\nu_k(\mathbf{t}) \mathbb{1}_{\{\mathbf{x} \in \mathcal{A}\}} dP_{\mathbf{S}_n^{(\theta)}}(\mathbf{x}) \quad (\text{E.25}) \end{aligned}$$

$$= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} \mathbb{1}_{\{\mathbf{x} \in \mathcal{A}\}} d\nu_k(\mathbf{t}) dP_{\mathbf{S}_n^{(\theta)}}(\mathbf{x}) \quad (\text{E.26})$$

$$= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^{k \times 2}} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} \mathbb{1}_{\{\mathbf{x} \in \mathcal{A}\}} d\nu_k \cdot P_{\mathbf{S}_n^{(\theta)}}(\mathbf{t}, \mathbf{x}), \quad (\text{E.27})$$

where the Lebesgue integral in (E.27) is with respect to the product measure $\nu_k \cdot P_{\mathbf{S}_n^{(\theta)}}$ on the measurable space $(\mathbb{R}^{k \times 2}, \mathcal{B}(\mathbb{R}^{k \times 2}))$. Note that the integral in (E.27) is absolutely integrable. Thus, using Fubini's Theorem [48], the right hand-side of (E.27) can be written as follows

$$\begin{aligned} & \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\theta)} \right) \mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{A}\}} \right] \\ &= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} \mathbb{1}_{\{\mathbf{x} \in \mathcal{A}\}} dP_{\mathbf{S}_n^{(\theta)}}(\mathbf{x}) d\nu_k(\mathbf{t}). \quad (\text{E.28}) \end{aligned}$$

From (E.22), the indicator $\mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}}$ in (E.28) can be written as follows

$$\mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} = \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0, 1]\}} \right) \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \mathbb{1}_{\{t_i \leq x_i\}} \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \mathbb{1}_{\{t_s \geq x_s\}} \right). \quad (\text{E.29})$$

To ease the notation, let the set

$$\bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} = \bar{\mathcal{B}}_{\theta_1, t_1} \times \bar{\mathcal{B}}_{\theta_2, t_2} \times \dots \times \bar{\mathcal{B}}_{\theta_k, t_k}, \quad (\text{E.30})$$

where, for all $i \in \{1, 2, \dots, k\}$, the set $\bar{\mathcal{B}}_{\theta_i, t_i}$ is defined by:

$$\bar{\mathcal{B}}_{\theta_i, t_i} = \begin{cases} \mathbb{R} & \text{if } \theta_i = 0 \\ (-\infty, t_i] & \text{if } \theta_i > 0 \\ [t_i, \infty) & \text{if } \theta_i < 0. \end{cases} \quad (\text{E.31})$$

Then, plugging (E.30) in (E.29) yields

$$\mathbb{1}_{\{\mathbf{t} \in \mathcal{B}_{\boldsymbol{\theta}, \mathbf{x}}\}} = \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0, 1]\}} \right) \mathbb{1}_{\{\mathbf{x} \in \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}\}}. \quad (\text{E.32})$$

Hence, plugging (E.32) in (E.28) yields

$$\begin{aligned} & \mathbb{E}_{P_{S_n^{(\theta)}}} \left[\exp \left(-\boldsymbol{\theta}^\top S_n^{(\theta)} \right) \mathbb{1}_{\{S_n^{(\theta)} \in \mathcal{A}\}} \right] \\ &= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0,1]\}} \right) \\ & \mathbb{1}_{\{x \in \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}\}} \mathbb{1}_{\{x \in \mathcal{A}\}} dP_{S_n^{(\theta)}}(\mathbf{x}) d\nu_k(\mathbf{t}) \end{aligned} \quad (\text{E.33})$$

$$\begin{aligned} &= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0,1]\}} \right) \\ & \int_{\mathbb{R}^k} \mathbb{1}_{\{x \in \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}\}} \mathbb{1}_{\{x \in \mathcal{A}\}} dP_{S_n^{(\theta)}}(\mathbf{x}) d\nu_k(\mathbf{t}) \end{aligned} \quad (\text{E.34})$$

$$\begin{aligned} &= (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \left(\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0,1]\}} \right) P_{S_n^{(\theta)}}(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}) d\nu_k(\mathbf{t}) \end{aligned} \quad (\text{E.35})$$

Note that the support of the integrand in (E.35) is a proper set of \mathbb{R}^k . Hence, the objective of the following lines is to characterize a proper subset of \mathbb{R}^k , denoted by $\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})$, that contains the support of the integrand in (E.35). The integrand is different from zero if $P_{S_n^{(\theta)}}(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}})$ and $\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0,1]\}}$ are simultaneously strictly positive. On the first hand, given a vector $\mathbf{t} = (t_1, t_2, \dots, t_k) \in \mathbb{R}^k$, the product $\prod_{j \in \mathcal{I}(\boldsymbol{\theta})} \mathbb{1}_{\{t_j \in [0,1]\}}$ is strictly positive if and only if for all $i \in \{1, 2, \dots, k\}$, it holds that

$$\begin{cases} t_i \in [0, 1] & \text{if } \theta_i = 0 \\ t_i \in \mathbb{R} & \text{if } \theta_i \neq 0. \end{cases} \quad (\text{E.36})$$

On the other hand, given $\mathbf{t} = (t_1, t_2, \dots, t_k) \in \mathbb{R}^k$, a necessary condition for $P_{S_n^{(\theta)}}(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}})$ to be strictly positive is that the set $\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}$ is not empty. Now a necessary condition for the non-emptiness of $\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}$ is that the set $\bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}$ contains at least one element $\mathbf{e} = (e_1, e_2, \dots, e_k)$ such that for all $i \in \{1, 2, \dots, k\}$,

$$\inf_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i \leq e_i \leq \sup_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i, \quad (\text{E.37})$$

which imposes some conditions the given vector \mathbf{t} . More specifically, from the definition of the set $\bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}}$ in (E.30), the vector $\mathbf{t} = (t_1, t_2, \dots, t_k)$ must satisfy for all $i \in \{1, 2, \dots, k\}$,

$$\begin{cases} t_i \geq \inf_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i & \text{if } \theta_i > 0 \\ t_i \leq \sup_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i & \text{if } \theta_i < 0. \end{cases} \quad (\text{E.38})$$

Hence, from (E.36) and (E.38) the set $\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})$ can be defined as follows

$$\mathcal{D}(\mathcal{A}, \boldsymbol{\theta}) \triangleq \left\{ (t_1, t_2, \dots, t_k) \in \mathbb{R}^k : \forall i \in \{1, 2, \dots, k\}, t_i \in [0, 1] \text{ if } \theta_i = 0, \right. \\ \left. t_i \geq \inf_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i \text{ if } \theta_i > 0, \text{ and } t_i \leq \sup_{(b_1, b_2, \dots, b_k) \in \mathcal{A}} b_i \text{ if } \theta_i < 0 \right\}. \quad (\text{E.39})$$

Then, the equality in (E.35) can be written as follows

$$\mathbb{E}_{P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{S}_n^{(\boldsymbol{\theta})} \right) \mathbb{1}_{\{\mathbf{S}_n^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \right] \\ = (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) P_{\mathbf{S}_n^{(\boldsymbol{\theta})}} \left(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \right) d\nu_k(\mathbf{t}). \quad (\text{E.40})$$

E.1.2. Explicit Expression of (E.9)

Following similar steps as in Subsection E.1.1, the following holds with the random vector $\mathbf{Z}_n^{(\boldsymbol{\theta})}$:

$$\mathbb{E}_{P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\boldsymbol{\theta})} \right) \mathbb{1}_{\{\mathbf{Z}_n^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \right] \\ = (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \left(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \right) d\nu_k(\mathbf{t}). \quad (\text{E.41})$$

E.1.3. Upper Bound on (E.7)

The proof ends by plugging (E.40) and (E.41) in the right hand-side of (E.7). This yields,

$$\left| P_{\mathbf{X}_n}(\mathcal{A}) - \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \mathbb{E}_{P_{\mathbf{Z}_n, \boldsymbol{\theta}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\boldsymbol{\theta})} \right) \mathbb{1}_{\{\mathbf{Z}_n^{(\boldsymbol{\theta})} \in \mathcal{A}\}} \right] \right| \\ = \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \left| (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) P_{\mathbf{S}_n^{(\boldsymbol{\theta})}} \left(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \right) d\nu_k(\mathbf{t}) \right. \\ \left. - (-1)^{|\mathcal{I}^-(\boldsymbol{\theta})|} \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \left(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \right) d\nu_k(\mathbf{t}) \right| \quad (\text{E.42})$$

$$= \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \left| \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp \left(-\boldsymbol{\theta}^\top \mathbf{t} \right) \left(P_{\mathbf{S}_n^{(\boldsymbol{\theta})}} \left(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \right) - P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}} \left(\mathcal{A} \cap \bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \right) \right) d\nu_k(\mathbf{t}) \right|. \quad (\text{E.43})$$

The set $\bar{\mathcal{B}}_{\boldsymbol{\theta}, \mathbf{t}} \cap \mathcal{A}$ in (E.43) is convex Borel measurable, given that \mathcal{A} is convex Borel measurable

from the assumptions of the lemma. From (4.30), it holds that

$$\left| P_{\mathbf{S}_n^{(\theta)}}(\mathcal{A} \cap \bar{\mathcal{B}}_{\theta, t}) - P_{\mathbf{Z}_n^{(\theta)}}(\mathcal{A} \cap \bar{\mathcal{B}}_{\theta, t}) \right| \leq \Delta \left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}} \right). \quad (\text{E.44})$$

Then, plugging (E.44) in (E.43) yields

$$\begin{aligned} & \left| P_{\mathbf{X}_n}(\mathcal{A}) - \exp \left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta}) \right) \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)} \right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right] \right| \\ & \leq \exp \left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta}) \right) \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} -\theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \left| \exp(-\boldsymbol{\theta}^\top \mathbf{t}) \right| \Delta \left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}} \right) \nu_k(d\mathbf{t}) \end{aligned} \quad (\text{E.45})$$

$$= \exp \left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta}) \right) \Delta \left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}} \right) \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} -\theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right) \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp(-\boldsymbol{\theta}^\top \mathbf{t}) \nu_k(d\mathbf{t}). \quad (\text{E.46})$$

The expression $\int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp(-\boldsymbol{\theta}^\top \mathbf{t}) d\nu_k(\mathbf{t})$ in (E.46) using the notation in (E.19) and (E.22) can be written in the form

$$\begin{aligned} & \int_{\mathcal{D}(\mathcal{A}, \boldsymbol{\theta})} \exp(-\boldsymbol{\theta}^\top \mathbf{t}) d\nu_k(\mathbf{t}) \\ & = \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \int_{-\infty}^{a_i(\mathcal{A}, \boldsymbol{\theta})} \exp(-\theta_i t_i) dt_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \int_{a_s(\mathcal{A}, \boldsymbol{\theta})}^{\infty} \exp(-\theta_s t_s) dt_s \right) \end{aligned} \quad (\text{E.47})$$

$$= \left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} \frac{\exp(-\theta_i a_i(\mathcal{A}, \boldsymbol{\theta}))}{\theta_i} \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \frac{\exp(-\theta_s a_s(\mathcal{A}, \boldsymbol{\theta}))}{\theta_s} \right) \quad (\text{E.48})$$

$$= \frac{\exp(-\boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta}))}{\left(\prod_{i \in \mathcal{I}^-(\boldsymbol{\theta})} -\theta_i \right) \left(\prod_{s \in \mathcal{I}^+(\boldsymbol{\theta})} \theta_s \right)}, \quad (\text{E.49})$$

where the vector $\mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) = (a_1(\mathcal{A}, \boldsymbol{\theta}), a_2(\mathcal{A}, \boldsymbol{\theta}), \dots, a_k(\mathcal{A}, \boldsymbol{\theta}))$ is defined in (4.31). Hence, plugging (E.49) in (E.46) yields

$$\begin{aligned} & \left| P_{\mathbf{X}_n}(\mathcal{A}) - \exp \left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta}) \right) \mathbb{E}_{P_{\mathbf{Z}_n, \boldsymbol{\theta}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)} \right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right] \right| \\ & \leq \exp \left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \right) \Delta \left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}} \right), \end{aligned} \quad (\text{E.50})$$

which provides an upper bound.

The last part of the proof consists in proving the equality between the RHS of (4.26) and the RHS of (4.26). For doing so, the expectation term in (E.50) is expanded in the following subsection.

E.1.4. Expansion of the Expectation Term in (E.50)

Denote by $\boldsymbol{\mu}_{\mathbf{Z}_n^{(\theta)}} \in \mathbb{R}^k$ and $\mathbf{v}_{\mathbf{Z}_n^{(\theta)}} \in \mathbb{R}^{k \times k}$ the mean vector and the covariance matrix of the random vector $\mathbf{Z}_n^{(\theta)}$ in (4.26), respectively. More specifically,

$$\boldsymbol{\mu}_{\mathbf{Z}_n^{(\theta)}} = \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\mathbf{Z}_n^{(\theta)} \right] \quad (\text{E.51a})$$

$$= \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\mathbf{S}_n^{(\theta)} \right] \quad (\text{E.51b})$$

$$= \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\mathbf{Y}_j^{(\theta)} \right] \quad (\text{E.51c})$$

$$= \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j}} \left[\mathbf{Y}_j \exp \left(\boldsymbol{\theta}^\top \mathbf{Y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \right] \quad (\text{E.51d})$$

$$= \sum_{j=1}^n K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}), \quad (\text{E.51e})$$

$$(\text{E.51f})$$

where for all $j \in \{1, 2, \dots, n\}$, $K_{\mathbf{Y}_j}^{(1)}$ is the gradient vector of the CGF $K_{\mathbf{Y}_j}$ of the random vector \mathbf{Y}_j ; and

$$\mathbf{v}_{\mathbf{Z}_n^{(\theta)}} = \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\left(\mathbf{Z}_n^{(\theta)} - \sum_{j=1}^n K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Z}_n^{(\theta)} - \sum_{j=1}^n K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (\text{E.52a})$$

$$= \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\left(\mathbf{S}_n^{(\theta)} - \sum_{j=1}^n K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{S}_n^{(\theta)} - \sum_{j=1}^n K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (\text{E.52b})$$

$$= \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\mathbf{Y}_j^{(\theta)} - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y}_j^{(\theta)} - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (\text{E.52c})$$

$$= \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j}} \left[\left(\mathbf{Y}_j - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y}_j - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \exp \left(\boldsymbol{\theta}^\top \mathbf{Y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \right] \quad (\text{E.52d})$$

$$= \sum_{j=1}^n K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta}), \quad (\text{E.52e})$$

where $K_{\mathbf{Y}_j}^{(2)}$ is the Hessian matrix of the CGF $K_{\mathbf{Y}_j}$. Note that the equality (E.52c) is a consequence of the random vector $\mathbf{S}_n^{(\theta)}$ in (4.77) being a sum of independent random vectors.

Denote by ν_k the Lebesgue measure on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, then,

$$\begin{aligned} & \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \mathbb{E}_{P_{\mathbf{Z}_n^{(\theta)}}} \left[\exp \left(-\boldsymbol{\theta}^\top \mathbf{Z}_n^{(\theta)} \right) \mathbb{1}_{\{\mathbf{Z}_n^{(\theta)} \in \mathcal{A}\}} \right] \\ &= \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \int_{\mathbb{R}^k} \exp \left(-\boldsymbol{\theta}^\top \mathbf{z} \right) \mathbb{1}_{\{\mathbf{z} \in \mathcal{A}\}} dP_{\mathbf{Z}_n^{(\theta)}}(\mathbf{z}) \end{aligned} \quad (\text{E.53a})$$

$$= \exp\left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta})\right) \int_{\mathcal{A}} \frac{\exp(-\boldsymbol{\theta}^\top \mathbf{z})}{\sqrt{(2\pi)^k \det(\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}} \exp\left(-\frac{(\mathbf{z} - \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})})}^\top (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}^{-1} (\mathbf{z} - \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})})})}{2}\right) d\nu_k(\mathbf{z}) \quad (\text{E.53b})$$

$$= \frac{\exp\left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta})\right)}{\sqrt{(2\pi)^k \det(\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}} \int_{\mathcal{A}} \exp\left(-\frac{(\mathbf{z} - \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})})}^\top (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}^{-1} (\mathbf{z} - \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})})})}{2}\right) \exp\left(-\frac{2\boldsymbol{\theta}^\top (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})} (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}^{-1} \mathbf{z})}{2}\right) d\nu_k(\mathbf{z}) \quad (\text{E.53c})$$

$$= \exp\left(\left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta})\right) + \frac{\boldsymbol{\theta}^\top \bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})}} \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^\top \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})}}\right) \frac{1}{\sqrt{(2\pi)^k \det(\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}} \int_{\mathcal{A}} \exp\left(-\frac{(\mathbf{z} - \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})}} + (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}^\top \boldsymbol{\theta})^\top (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}^{-1} (\mathbf{z} - \boldsymbol{\mu}_{Z_n^{(\boldsymbol{\theta})}} + (\bar{\mathbf{v}}_{Z_n^{(\boldsymbol{\theta})})}^\top \boldsymbol{\theta}))}{2}\right) d\nu_k(\mathbf{z}) \quad (\text{E.53d})$$

$$= \exp\left(\sum_{j=1}^n \left(K_{Y_j}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{Y_j}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^\top K_{Y_j}^{(1)}(\boldsymbol{\theta})\right)\right) P_{\mathbf{W}_n^{(\boldsymbol{\theta})}}(\mathcal{A}), \quad (\text{E.53e})$$

where $P_{\mathbf{W}_n^{(\boldsymbol{\theta})}}$ is the probability measure induced on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ by a Gaussian random vector $\mathbf{W}_n^{(\boldsymbol{\theta})}$ with mean vector $\sum_{j=1}^n (K_{Y_j}^{(1)}(\boldsymbol{\theta}) - K_{Y_j}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta})$ and covariance matrix $\sum_{j=1}^n K_{Y_j}^{(2)}(\boldsymbol{\theta})$. The equality in (E.53e) follows by plugging (E.51e) and (E.52e) in (E.53d).

Finally, plugging (E.53e) in (E.50) yields

$$\left| P_{\mathbf{X}_n}(\mathcal{A}) - \exp\left(\sum_{j=1}^n \left(K_{Y_j}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{Y_j}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^\top K_{Y_j}^{(1)}(\boldsymbol{\theta})\right)\right) P_{\mathbf{W}_n^{(\boldsymbol{\theta})}}(\mathcal{A}) \right| \leq \exp\left(\sum_{j=1}^n K_{Y_j}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta})\right) \Delta(P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}, P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}), \quad (\text{E.54})$$

which completes the proof for the case of independent random vectors.

E.2. Independent and Identically Distributed Random Vectors

In this section, the approximation in (E.54) is simplified for IID random vectors. For identically distributed random vectors, $\Theta_{\mathbf{Y}}$ in (??) $\Theta_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n}$ in (E.1) satisfy

$$\Theta_{\mathbf{Y}} = \Theta_{\mathbf{Y}_1\mathbf{Y}_2\dots\mathbf{Y}_n} \quad (\text{E.55})$$

and for all $j \in \{1, 2, \dots, n\}$, for all $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$

$$K_{\mathbf{Y}_j}(\boldsymbol{\theta}) = K_{\mathbf{Y}}(\boldsymbol{\theta}), \quad (\text{E.56})$$

$$K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) = K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}), \quad (\text{E.57})$$

$$K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta}) = K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}), \quad (\text{E.58})$$

which yields

$$\begin{aligned} & \exp\left(\sum_{j=1}^n \left(K_{\mathbf{Y}_j}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2} - \boldsymbol{\theta}^\top K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta})\right)\right) P_{\mathbf{W}_n^{(\boldsymbol{\theta})}}(\mathcal{A}) \\ &= \exp\left(n \left(K_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2}\right)\right) P_{\mathbf{W}^{(n,\boldsymbol{\theta})}}(\mathcal{A}) \end{aligned} \quad (\text{E.59})$$

$$= \exp\left(n \left(K_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) + \frac{\boldsymbol{\theta}^\top K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}) \boldsymbol{\theta}}{2}\right)\right) P_{\mathbf{H}^{(n,\boldsymbol{\theta})}}(\mathcal{A}) \quad (\text{E.60})$$

$$= \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n), \quad (\text{E.61})$$

where $\mathbf{H}_n^{(\boldsymbol{\theta})}$ is the Gaussian random vector in (4.26) and the function $\eta_{\mathbf{Y}}$ is defined in (4.26). Finally, plugging (E.56) and (E.61) in (E.54) yields

$$|P_{\mathbf{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)| \leq \exp\left(n K_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta})\right) \Delta\left(P_{\mathbf{S}_n^{(\boldsymbol{\theta})}}, P_{\mathbf{Z}_n^{(\boldsymbol{\theta})}}\right), \quad (\text{E.62})$$

which completes the proof.



Proof of Lemma 20

From Lemma 19, it holds that

$$|P_{\mathcal{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)| \leq \exp\left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta})\right) \Delta\left(P_{\mathcal{S}_n^{(\boldsymbol{\theta})}}, P_{\mathcal{Z}_n^{(\boldsymbol{\theta})}}\right). \quad (\text{F.1})$$

Hence, the objective is to provide an upper bound on $\Delta\left(P_{\mathcal{S}_n^{(\boldsymbol{\theta})}}, P_{\mathcal{Z}_n^{(\boldsymbol{\theta})}}\right)$. An upper bound would be obtained immediately from Theorem 18, except for the fact that the vectors $\mathbf{Y}_1^{(\boldsymbol{\theta})}, \mathbf{Y}_2^{(\boldsymbol{\theta})}, \dots, \mathbf{Y}_n^{(\boldsymbol{\theta})}$ in (4.22) do not have means $(0, 0, \dots, 0)$ and variances $\frac{1}{n} \text{diag}(1, 1, \dots, 1)$, as required by Theorem 18. Denote by $\boldsymbol{\mu}_{\mathbf{Y}^{(\boldsymbol{\theta})}} \in \mathbb{R}^k$ and $\mathbf{v}_{\mathbf{Y}^{(\boldsymbol{\theta})}} \in \mathbb{R}^{k \times k}$, respectively, the mean vector and the covariance matrix of these random vectors, for some $\boldsymbol{\theta} \in \Theta_{\mathbf{Y}}$, with $\Theta_{\mathbf{Y}}$ in (4.17). Then, the following holds,

$$\boldsymbol{\mu}_{\mathbf{Y}^{(\boldsymbol{\theta})}} \triangleq \mathbb{E}_{P_{\mathbf{Y}^{(\boldsymbol{\theta})}}}[\mathbf{Y}^{(\boldsymbol{\theta})}] \quad (\text{F.2a})$$

$$= \mathbb{E}_{P_{\mathbf{Y}}} \left[\mathbf{Y} \exp\left(\boldsymbol{\theta}^\top \mathbf{Y} - K_{\mathbf{Y}}(\boldsymbol{\theta})\right) \right] \quad (\text{F.2b})$$

$$= K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}), \quad (\text{F.2c})$$

where $K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta})$ is the gradient vector of the CGF $K_{\mathbf{Y}}$ defined in (4.2). Alternatively,

$$\mathbf{v}_{\mathbf{Y}^{(\boldsymbol{\theta})}} \triangleq \mathbb{E}_{P_{\mathbf{Y}^{(\boldsymbol{\theta})}}} \left[\left(\mathbf{Y}^{(\boldsymbol{\theta})} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y}^{(\boldsymbol{\theta})} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (\text{F.2d})$$

$$= \mathbb{E}_{P_{\mathbf{Y}}} \left[\left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}) \right)^\top \exp\left(\boldsymbol{\theta}^\top \mathbf{Y} - K_{\mathbf{Y}}(\boldsymbol{\theta})\right) \right] \quad (\text{F.2e})$$

$$= K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}), \quad (\text{F.2f})$$

where $K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta})$ is the Hessian matrix of the CGF $K_{\mathbf{Y}}$ defined in (4.3). Let the Cholesky decomposition of the matrix $\mathbf{v}_{\mathbf{Y}^{(\boldsymbol{\theta})}}$ be

$$\mathbf{v}_{\mathbf{Y}^{(\boldsymbol{\theta})}} = \mathbf{l}_{\mathbf{Y}^{(\boldsymbol{\theta})}} \mathbf{l}_{\mathbf{Y}^{(\boldsymbol{\theta})}}^\top, \quad (\text{F.3})$$

where $\mathbf{L}_{\mathbf{Y}^{(\theta)}}$ is a real lower triangular matrix. Note that the matrix $\mathbf{v}_{\mathbf{Y}^{(\theta)}}$ is nonsingular. This follows from the assumption that the covariance matrix $K_{\mathbf{Y}^{(2)}}(\mathbf{0})$ is positive definite, which implies that the CGF $K_{\mathbf{Y}}$ is strictly convex and thus, its Hessian matrix $K_{\mathbf{Y}^{(2)}}$ is positive definite.

Let the random vector $\mathbf{R}_n^{(\theta)}$ be such that

$$\mathbf{R}_n^{(\theta)} \triangleq \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{S}_n^{(\theta)} - n\boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \quad (\text{F.4})$$

which induces the probability measure $P_{\mathbf{R}_n^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Plugging (4.22) in (F.4) yields:

$$\mathbf{R}_n^{(\theta)} = \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \sum_{j=1}^n \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \quad (\text{F.5})$$

$$= \sum_{j=1}^n \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \quad (\text{F.6})$$

$$= \sum_{j=1}^n \mathbf{U}_j^{(\theta)}, \quad (\text{F.7})$$

where

$$\mathbf{U}_j^{(\theta)} \triangleq \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \quad (\text{F.8})$$

is a random vector that induces the probability measure $P_{\mathbf{U}^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Thus, the random vector $\mathbf{R}_n^{(\theta)}$ in (F.4) is the sum of n IID random vectors $\mathbf{U}_1^{(\theta)}, \mathbf{U}_2^{(\theta)}, \dots, \mathbf{U}_n^{(\theta)}$ such that each of them induces the probability measure $P_{\mathbf{U}^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, which satisfies,

$$\mathbb{E}_{P_{\mathbf{U}^{(\theta)}}} \left[\mathbf{U}^{(\theta)} \right] = \frac{1}{\sqrt{n}} \mathbb{E}_{P_{\mathbf{Y}^{(\theta)}}} \left[\mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{Y}^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \right] \quad (\text{F.9})$$

$$= \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \mathbb{E}_{P_{\mathbf{Y}^{(\theta)}}} \left[\left(\mathbf{Y}^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \right] \quad (\text{F.10})$$

$$= 0, \quad (\text{F.11})$$

and

$$\mathbb{E}_{P_{\mathbf{U}^{(\theta)}}} \left[\mathbf{U}^{(\theta)} \mathbf{U}^{(\theta)\top} \right] = \frac{1}{n} \mathbb{E}_{P_{\mathbf{Y}^{(\theta)}}} \left[\left(\mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{Y}^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \right) \left(\mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{Y}^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \right)^\top \right] \quad (\text{F.12})$$

$$= \frac{1}{n} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \mathbb{E}_{P_{\mathbf{Y}^{(\theta)}}} \left[\left(\mathbf{Y}^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right) \left(\mathbf{Y}^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right)^\top \right] \left(\mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \right)^\top \quad (\text{F.13})$$

$$= \frac{1}{n} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \mathbf{v}_{\mathbf{Y}^{(\theta)}} \left(\mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \right)^\top \quad (\text{F.14})$$

$$= \frac{1}{n} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \mathbf{L}_{\mathbf{Y}^{(\theta)}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^\top \left(\mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \right)^\top \quad (\text{F.15})$$

$$= \frac{1}{n} \text{diag}(1, 1, \dots, 1). \quad (\text{F.16})$$

Similarly, let the random vector $\mathbf{W}_n^{(\theta)}$ be such that

$$\mathbf{W}_n^{(\theta)} \triangleq \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} \left(\mathbf{Z}_n^{(\theta)} - n\boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}} \right), \quad (\text{F.17})$$

which induce the probability measure $P_{\mathbf{W}_n^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. The mean vector and the covariance matrix of the random vector $\mathbf{Z}_n^{(\theta)}$ are identical to those of the random vector $\mathbf{S}_n^{(\theta)}$. See for instance (4.24) and (4.25). Then, from (F.4) and (F.17), it holds that the mean vector and the covariance matrix of the random vector $\mathbf{W}_n^{(\theta)}$ are identical to those of the random vector $\mathbf{R}_n^{(\theta)}$.

The rest of the proof follows by noticing that for all $\mathcal{B} \in \mathcal{B}(\mathbb{R}^k)$, the set $\mathcal{S}(\mathcal{B})$ defined by

$$\mathcal{S}(\mathcal{B}) \triangleq \left\{ \mathbf{y} \in \mathbb{R}^k : \exists \mathbf{x} \in \mathcal{B}, \mathbf{y} = \frac{1}{\sqrt{n}} \mathbf{L}_{\mathbf{Y}^{(\theta)}}^{-1} (\mathbf{x} - n\boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}}) \right\} \quad (\text{F.18})$$

$$= \left\{ \mathbf{y} \in \mathbb{R}^k : (\sqrt{n} \mathbf{L}_{\mathbf{Y}^{(\theta)}} \mathbf{y} + n\boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}}) \in \mathcal{B} \right\}. \quad (\text{F.19})$$

allows writing that

$$P_{\mathbf{S}_n^{(\theta)}}(\mathcal{B}) = \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{B}\}} \right] \quad (\text{F.20})$$

$$= \mathbb{E}_{P_{\mathbf{R}_n^{(\theta)}}} \left[\mathbb{1}_{\{(\sqrt{n} \mathbf{L}_{\mathbf{Y}^{(\theta)}} \mathbf{R}_n^{(\theta)} + n\boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}}) \in \mathcal{B}\}} \right] \quad (\text{F.21})$$

$$= P_{\mathbf{R}_n^{(\theta)}} \left(\left\{ \mathbf{y} \in \mathbb{R}^k : (\sqrt{n} \mathbf{L}_{\mathbf{Y}^{(\theta)}} \mathbf{y} + n\boldsymbol{\mu}_{\mathbf{Y}^{(\theta)}}) \in \mathcal{B} \right\} \right) \quad (\text{F.22})$$

$$= P_{\mathbf{R}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})). \quad (\text{F.23})$$

Similarly, from (F.17) and (F.18), it holds that

$$P_{\mathbf{Z}_n^{(\theta)}}(\mathcal{B}) = P_{\mathbf{W}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})). \quad (\text{F.24})$$

This implies that,

$$\Delta(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}}) = \sup_{\mathcal{B} \in \mathcal{C}_k} \left| P_{\mathbf{R}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})) - P_{\mathbf{W}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})) \right|, \quad (\text{F.25a})$$

$$\leq \sup_{\mathcal{S} \in \mathcal{C}_k} \left| P_{\mathbf{R}_n^{(\theta)}}(\mathcal{S}) - P_{\mathbf{W}_n^{(\theta)}}(\mathcal{S}) \right| \quad (\text{F.25b})$$

$$= \Delta(P_{\mathbf{R}_n^{(\theta)}}, P_{\mathbf{W}_n^{(\theta)}}), \quad (\text{F.25c})$$

where the inequality in (F.25b) is a consequence of the fact that the collection \mathcal{C}_k of all convex sets in $\mathcal{B}(\mathbb{R}^k)$ is stable under linear transformations. Then, from the multivariate Berry-Essen Theorem (Theorem 18), it holds that

$$\begin{aligned} & \Delta(P_{\mathbf{R}_n^{(\theta)}}, P_{\mathbf{W}_n^{(\theta)}}) \\ & \leq \min \left(1, c(k) n \mathbb{E}_{P_{\mathbf{U}^{(\theta)}}} \left[\|\mathbf{U}^{(\theta)}\|^3 \right] \right) \end{aligned} \quad (\text{F.26a})$$

$$= \min \left(1, c(k) n \mathbb{E}_{P_{\mathbf{U}^{(\theta)}}} \left[\left(\mathbf{U}^{(\theta)\top} \mathbf{U}^{(\theta)} \right)^{\frac{3}{2}} \right] \right) \quad (\text{F.26b})$$

$$= \min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\mathbf{Y}(\theta)}} \left[\left(\left((\mathbf{l}_{\mathbf{Y}(\theta)})^{-1} (\mathbf{Y}(\theta) - \boldsymbol{\mu}_{\mathbf{Y}(\theta)}) \right)^\top (\mathbf{l}_{\mathbf{Y}(\theta)})^{-1} (\mathbf{Y}(\theta) - \boldsymbol{\mu}_{\mathbf{Y}(\theta)}) \right)^{\frac{3}{2}} \right] \right) \quad (\text{F.26c})$$

$$= \min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\mathbf{Y}(\theta)}} \left[\left((\mathbf{Y}(\theta) - \boldsymbol{\mu}_{\mathbf{Y}(\theta)})^\top (\mathbf{l}_{\mathbf{Y}(\theta)})^{-1} (\mathbf{Y}(\theta) - \boldsymbol{\mu}_{\mathbf{Y}(\theta)}) \right)^{\frac{3}{2}} \right] \right) \quad (\text{F.26d})$$

$$= \min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\mathbf{Y}(\theta)}} \left[\left((\mathbf{Y}(\theta) - \boldsymbol{\mu}_{\mathbf{Y}(\theta)})^\top (\mathbf{v}_{\mathbf{Y}(\theta)})^{-1} (\mathbf{Y}(\theta) - \boldsymbol{\mu}_{\mathbf{Y}(\theta)}) \right)^{\frac{3}{2}} \right] \right) \quad (\text{F.26e})$$

$$= \min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\mathbf{Y}}} \left[\left((\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}(\theta)})^\top (\mathbf{v}_{\mathbf{Y}(\theta)})^{-1} (\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}(\theta)}) \right)^{3/2} \exp(\boldsymbol{\theta}^\top \mathbf{Y} - K_{\mathbf{Y}}(\boldsymbol{\theta})) \right] \right) \quad (\text{F.26f})$$

where c is the function defined in (4.14).

Finally, plugging (F.2c) and (F.2f) in (F.26f) yields

$$\begin{aligned} & \Delta \left(P_{\mathbf{R}_n^{(\theta)}}, P_{\mathbf{W}_n^{(\theta)}} \right) \\ & \leq \min \left(1, \frac{c(k)}{\sqrt{n}} \mathbb{E}_{P_{\mathbf{Y}}} \left[\left((\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta}))^\top (K_{\mathbf{Y}}^{(2)}(\boldsymbol{\theta}))^{-1} (\mathbf{Y} - K_{\mathbf{Y}}^{(1)}(\boldsymbol{\theta})) \right)^{3/2} \exp(\boldsymbol{\theta}^\top \mathbf{Y} - K_{\mathbf{Y}}(\boldsymbol{\theta})) \right] \right) \end{aligned} \quad (\text{F.27a})$$

$$= \min \left(1, \frac{c(k)\xi_{\mathbf{Y}}(\boldsymbol{\theta})}{\sqrt{n}} \right), \quad (\text{F.27b})$$

where the function $\xi_{\mathbf{Y}}$ is defined in (4.32). Plugging (F.27b) in (F.25c) yields

$$\Delta \left(P_{\mathcal{S}_n^{(\theta)}}, P_{\mathcal{Z}_n^{(\theta)}} \right) \leq \min \left(1, \frac{c(k)\xi_{\mathbf{Y}}(\boldsymbol{\theta})}{\sqrt{n}} \right). \quad (\text{F.28})$$

Finally, plugging (F.28) in (F.1) yields

$$|P_{\mathcal{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}}(\boldsymbol{\theta}, \mathcal{A}, n)| \leq \exp \left(nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \right) \min \left(1, \frac{c(k)\xi_{\mathbf{Y}}(\boldsymbol{\theta})}{\sqrt{n}} \right), \quad (\text{F.29})$$

which completes the proof.



Proof of Lemma 21

Note that for all $\mathbf{x} \in \mathbb{R}^k$, it holds from (4.40) and from the fact that $\mathbf{0} \in \Theta_{\mathbf{Y}}^-$, that

$$nK_{\mathbf{Y}}(\boldsymbol{\theta}(\mathbf{x})) - \boldsymbol{\theta}^\top(\mathbf{x}) \mathbf{x} \leq (nK_{\mathbf{Y}}(\mathbf{0}) - \mathbf{0}^\top \mathbf{x}) = 0. \quad (\text{G.1})$$

Moreover, for all $\mathbf{x} \in \mathbb{R}^k$ and for all $\boldsymbol{\theta} \in \text{clo}\Theta_{\mathbf{Y}}^-$, it holds that

$$nK_{\mathbf{Y}}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{x} = n \log \left(\mathbb{E}_{P_{\mathbf{Y}}} \left[\exp \left(\boldsymbol{\theta}^\top \mathbf{Y} \right) \right] \right) - \boldsymbol{\theta}^\top \mathbf{x} \quad (\text{G.2})$$

$$\geq n \mathbb{E}_{P_{\mathbf{Y}}} \left[\log \left(\exp \left(\boldsymbol{\theta}^\top \mathbf{Y} \right) \right) \right] - \boldsymbol{\theta}^\top \mathbf{x} \quad (\text{G.3})$$

$$= n \mathbb{E}_{P_{\mathbf{Y}}} \left[\boldsymbol{\theta}^\top \mathbf{Y} \right] - \boldsymbol{\theta}^\top \mathbf{x} \quad (\text{G.4})$$

$$= \boldsymbol{\theta}^\top n \mathbb{E}_{P_{\mathbf{Y}}} [\mathbf{Y}] - \boldsymbol{\theta}^\top \mathbf{x} \quad (\text{G.5})$$

$$= \boldsymbol{\theta}^\top (n \mathbb{E}_{P_{\mathbf{Y}}} [\mathbf{Y}] - \mathbf{x}) \quad (\text{G.6})$$

$$= \boldsymbol{\theta}^\top (\mathbb{E}_{P_{\mathbf{X}_n}} [\mathbf{X}_n] - \mathbf{x}) \quad (\text{G.7})$$

$$= \boldsymbol{\theta}^\top (\boldsymbol{\mu}_{\mathbf{X}_n} - \mathbf{x}), \quad (\text{G.8})$$

where the inequality in (G.3) follows from Jensen's inequality [49, Section 2.6]; the equality in (G.7) follows from (4.4); and the equality in (G.8) follows from (4.43).

From (G.1) and (G.8), it holds that for all $\mathbf{x} \in \mathbb{R}^k$, $\boldsymbol{\theta}(\mathbf{x})$ in (4.40) satisfies

$$\boldsymbol{\theta}^\top(\mathbf{x}) (\boldsymbol{\mu}_{\mathbf{X}_n} - \mathbf{x}) \leq 0, \quad (\text{G.9})$$

which implies that $(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}_n})^\top \boldsymbol{\theta}(\mathbf{x}) \geq 0$ and proves the inequality in (4.41).

From (G.1) and (G.8), it holds that for $\mathbf{x} = \boldsymbol{\mu}_{\mathbf{X}_n}$,

$$nK_{\mathbf{Y}}(\boldsymbol{\tau}(\boldsymbol{\mu}_{\mathbf{X}_n})) - \boldsymbol{\tau}^\top(\boldsymbol{\mu}_{\mathbf{X}_n}) \boldsymbol{\mu}_{\mathbf{X}_n} = (nK_{\mathbf{Y}}(\mathbf{0}) - \mathbf{0}^\top \boldsymbol{\mu}_{\mathbf{X}_n}) = 0. \quad (\text{G.10})$$

Thus, the uniqueness of $\boldsymbol{\tau}(\boldsymbol{\mu}_{\mathbf{X}_n})$ implies from (G.10) that

$$\boldsymbol{\tau}(\boldsymbol{\mu}_{\mathbf{X}_n}) = \mathbf{0}. \quad (\text{G.11})$$

Finally, note that $\mathbf{0} \in \Theta_{\mathbf{Y}}^-$ and thus, from (4.40), it holds that

$$\theta(\boldsymbol{\mu}_{\mathbf{X}_n}) = \boldsymbol{\tau}(\boldsymbol{\mu}_{\mathbf{X}_n}) = \mathbf{0}, \quad (\text{G.12})$$

which concludes the proof.



Proof of Lemma 22

For all $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$, it holds that

$$1 - F_{\mathbf{X}_n}(\mathbf{x}) = 1 - \mathbb{E}_{\mathbf{X}_n} \left[\prod_{t=1}^k \mathbb{1}_{\{X_{n,t} \leq x_t\}} \right] \quad (\text{H.1})$$

$$= \mathbb{E}_{\mathbf{X}_n} \left[1 - \prod_{t=1}^k \mathbb{1}_{\{X_{n,t} \leq x_t\}} \right] \quad (\text{H.2})$$

$$= \mathbb{E}_{\mathbf{X}_n} \left[\max \{ \mathbb{1}_{\{X_{n,t} > x_t\}} : t \in \{1, 2, \dots, k\} \} \right]. \quad (\text{H.3})$$

The proof continues by using a property of the max function provided by the following lemma.

Lemma 37 For all $k \in \mathbb{N}$, and for all $(a_1, a_2, \dots, a_k) \in \{0, 1\}^k$, it holds that

$$\max\{a_1, a_2, \dots, a_k\} = \sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j, \quad (\text{H.4})$$

where the collection $\mathcal{S}(k)$ is defined in (??).

Proof: The proof is made by recurrence. For $k = 1$, the results is trivial. For $k = 2$, it holds that

$$\max\{a_1, a_2\} = a_1 + a_2 - a_1 a_2. \quad (\text{H.5})$$

Assume that for all $k \in \mathbb{N}$ and for all $(a_1, a_2, \dots, a_k) \in \{0, 1\}^k$, it holds that

$$\max\{a_1, a_2, \dots, a_k\} = \sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j, \quad (\text{H.6})$$

where the collection $\mathcal{S}(k)$ is defined in (??). Let a_{k+1} a real be such that $a_{k+1} \in \{0, 1\}$. Then, it holds that

$$\max\{a_1, a_2, \dots, a_k, a_{k+1}\} = \max\{a_{k+1}, \max\{a_1, a_2, \dots, a_k\}\}. \quad (\text{H.7})$$

Note that $\max\{a_1, a_2, \dots, a_k\} \in \{0, 1\}$. Then, plugging (H.5) in (H.7) yields

$$\max\{a_1, a_2, \dots, a_k, a_{k+1}\} = a_{k+1} + \max\{a_1, a_2, \dots, a_k\} - a_{k+1} \max\{a_1, a_2, \dots, a_k\}. \quad (\text{H.8})$$

Then, plugging (H.6) in (H.8) yields

$$\begin{aligned} & \max\{a_1, a_2, \dots, a_k, a_{k+1}\} \\ &= a_{k+1} + \left(\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j \right) - a_{k+1} \left(\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j \right) \end{aligned} \quad (\text{H.9})$$

$$= a_{k+1} + \left(\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j \right) + \left(\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{2+|\mathcal{J}|} a_{k+1} \prod_{j \in \mathcal{J}} a_j \right) \quad (\text{H.10})$$

$$= a_{k+1} + \left(\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j \right) + \left(\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{2+|\mathcal{J}|} \prod_{j \in \mathcal{J} \cup \{k+1\}} a_j \right). \quad (\text{H.11})$$

Note for all $k \in \mathbb{N}$, using the definition in (??), it holds that

$$\mathcal{S}(k+1) = \{\{k+1\}\} \cup \mathcal{S}(k) \cup \{\{k+1\} \cup \mathcal{J} : \mathcal{J} \in \mathcal{S}(k)\}. \quad (\text{H.12})$$

Then, plugging (H.12) in (H.11) yields

$$\max\{a_1, a_2, \dots, a_k, a_{k+1}\} = \sum_{\mathcal{J} \in \mathcal{S}(k+1)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} a_j, \quad (\text{H.13})$$

which concludes the proof by recurrence. ■

Using Lemma 37 in (H.3), it follows that

$$1 - F_{\mathbf{X}_n}(\mathbf{x}) = \mathbb{E}_{\mathbf{X}_n} \left[\sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \prod_{j \in \mathcal{J}} \mathbb{1}_{\{X_{n,j} > x_j\}} \right] \quad (\text{H.14})$$

$$= \sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} \mathbb{E}_{\mathbf{X}_n} \left[\prod_{j \in \mathcal{J}} \mathbb{1}_{\{X_{n,j} > x_j\}} \right] \quad (\text{H.15})$$

$$= \sum_{\mathcal{J} \in \mathcal{S}(k)} (-1)^{1+|\mathcal{J}|} P_{\mathbf{X}_n}(\mathcal{B}_{\mathbf{x}}(\mathcal{J})), \quad (\text{H.16})$$

where the set $\mathcal{B}_{\mathbf{x}}(\mathcal{J})$ is defined in (4.47). The equality in (H.15) follows from the linearity of the expectation. This concludes the proof.



Proof of Lemma 26

The proof of Lemma 20 follows from Lemma 19 by providing an upper bound on $\Delta \left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}} \right)$ in (4.30) leveraging on the fact that the random vector $\mathbf{S}_n^{(\theta)}$ in (4.77) is the sum of independent random vectors $\mathbf{Y}_1^{(\theta)}, \mathbf{Y}_2^{(\theta)}, \dots, \mathbf{Y}_n^{(\theta)}$ in (4.77).

For all $j \in \{1, 2, \dots, n\}$, denote by $\boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \in \mathbb{R}^k$ and $\mathbf{v}_{\mathbf{Y}_j^{(\theta)}} \in \mathbb{R}^{k \times k}$, respectively, the mean vector and the covariance matrix of the random vector $\mathbf{Y}_j^{(\theta)}$ that induces the probability measure $P_{\mathbf{Y}_j^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ described in (4.78). That is,

$$\boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \triangleq \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} [\mathbf{Y}_j^{(\theta)}] \quad (\text{I.1a})$$

$$= \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\mathbf{Y}_j \exp \left(\boldsymbol{\theta}^\top \mathbf{Y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \right] \quad (\text{I.1b})$$

$$= K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}), \quad (\text{I.1c})$$

$$(\text{I.1d})$$

where $K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta})$ is the gradient vector of the CGF $K_{\mathbf{Y}_j}$; and

$$\mathbf{v}_{\mathbf{Y}_j^{(\theta)}} \triangleq \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\mathbf{Y}_j^{(\theta)} - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y}_j^{(\theta)} - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \right] \quad (\text{I.1e})$$

$$= \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\mathbf{Y}_j - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \left(\mathbf{Y}_j - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \exp \left(\boldsymbol{\theta}^\top \mathbf{Y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \right] \quad (\text{I.1f})$$

$$= K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta}), \quad (\text{I.1g})$$

where $K_{\mathbf{Y}_j}^{(2)}(\boldsymbol{\theta})$ is the Hessian matrix of the CGF $K_{\mathbf{Y}_j}$. Note that by positive-definite assumption on the covariance matrix of $P_{\mathbf{Y}}$, $K_{\mathbf{Y}}$ is strictly convex [47, Theorem 7.1]. Thus, the Hessian matrix $K_{\mathbf{Y}}^{(2)}$ is positive-definite.

Denote by $\boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \in \mathbb{R}^k$ and $\mathbf{v}_{\mathbf{S}_n^{(\theta)}} \in \mathbb{R}^{k \times k}$, respectively, the mean vector and the covariance

matrix of the random vector $\mathbf{S}_n^{(\theta)}$ in (4.77). That is,

$$\boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} = \sum_{j=1}^n \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \quad (\text{I.2a})$$

$$= \sum_{j=1}^n K_{\mathbf{Y}_j^{(\theta)}}^{(1)}(\boldsymbol{\theta}), \text{ and} \quad (\text{I.2b})$$

$$\underline{\mathbf{v}}_{\mathbf{S}_n^{(\theta)}} = \sum_{j=1}^n \underline{\mathbf{v}}_{\mathbf{Y}_j^{(\theta)}} \quad (\text{I.2c})$$

$$= \sum_{j=1}^n K_{\mathbf{Y}_j^{(\theta)}}^{(2)}(\boldsymbol{\theta}). \quad (\text{I.2d})$$

Let the Cholesky decomposition of the covariance matrix $\bar{\mathbf{v}}_{\mathbf{S}_n^{(\theta)}}$ of the random vector $\mathbf{S}_n^{(\theta)}$ in (4.77) be

$$\bar{\mathbf{v}}_{\mathbf{S}_n^{(\theta)}} = \underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}} \underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}}^{\top}, \quad (\text{I.3})$$

where $\underline{\mathbf{L}}_{\mathbf{Y}^{(\theta)}}$ is a real lower triangular matrix. Let the random vectors $\mathbf{R}_n^{(\theta)}$ and $\mathbf{W}_n^{(\theta)}$ be such that

$$\mathbf{R}_n^{(\theta)} \triangleq \underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{S}_n^{(\theta)} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right), \text{ and} \quad (\text{I.4})$$

$$\mathbf{W}_n^{(\theta)} \triangleq \underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Z}_n^{(\theta)} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right), \quad (\text{I.5})$$

which respectively induce the probability measures $P_{\mathbf{R}_n^{(\theta)}}$ and $P_{\mathbf{W}_n^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

For all set $\mathcal{B} \in \mathcal{B}(\mathbb{R}^k)$, let the set $\mathcal{S}(\mathcal{B})$ be defined by

$$\mathcal{S}(\mathcal{B}) \triangleq \left\{ \mathbf{y} \in \mathbb{R}^k : \exists \mathbf{x} \in \mathcal{B}, \mathbf{y} = \underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \right\} \quad (\text{I.6})$$

$$= \left\{ \mathbf{y} \in \mathbb{R}^k : \left(\underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}} \mathbf{y} + \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \in \mathcal{B} \right\}. \quad (\text{I.7})$$

Then, from (I.4) and (I.6), it holds that

$$P_{\mathbf{S}_n^{(\theta)}}(\mathcal{B}) = \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\mathbb{1}_{\{\mathbf{S}_n^{(\theta)} \in \mathcal{B}\}} \right] \quad (\text{I.8})$$

$$= \mathbb{E}_{P_{\mathbf{R}_n^{(\theta)}}} \left[\mathbb{1}_{\left\{ \left(\underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}} \mathbf{R}_n^{(\theta)} + \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \in \mathcal{B} \right\}} \right] \quad (\text{I.9})$$

$$= P_{\mathbf{R}_n^{(\theta)}} \left(\left\{ \mathbf{y} \in \mathbb{R}^k : \left(\underline{\mathbf{L}}_{\mathbf{S}_n^{(\theta)}} \mathbf{y} + \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \in \mathcal{B} \right\} \right) \quad (\text{I.10})$$

$$= P_{\mathbf{R}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})). \quad (\text{I.11})$$

Similarly, from (I.5) and (I.6), it holds that

$$P_{\mathbf{Z}_n^{(\theta)}}(\mathcal{B}) = P_{\mathbf{W}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})). \quad (\text{I.12})$$

Then, using (4.30)

$$\Delta\left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}}\right) = \sup_{\mathcal{B} \in \mathcal{C}_k} \left| P_{\mathbf{R}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})) - P_{\mathbf{W}_n^{(\theta)}}(\mathcal{S}(\mathcal{B})) \right|, \quad (\text{I.13a})$$

$$\leq \sup_{\mathcal{S} \in \mathcal{C}_k} \left| P_{\mathbf{R}_n^{(\theta)}}(\mathcal{S}) - P_{\mathbf{W}_n^{(\theta)}}(\mathcal{S}) \right| \quad (\text{I.13b})$$

$$= \Delta\left(P_{\mathbf{R}_n^{(\theta)}}, P_{\mathbf{W}_n^{(\theta)}}\right), \quad (\text{I.13c})$$

where the inequality in (I.13b) is a consequence of the collection \mathcal{C}_k of all convex sets in $\mathcal{B}(\mathbb{R}^k)$ being stable under linear transformation.

Plugging (4.77) and (I.2a) in (I.4) yields:

$$\mathbf{R}_n^{(\theta)} = \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \sum_{j=1}^n \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \quad (\text{I.14a})$$

$$= \sum_{j=1}^n \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \quad (\text{I.14b})$$

$$= \sum_{j=1}^n \mathbf{U}_j^{(\theta)}, \quad (\text{I.14c})$$

where

$$\mathbf{U}_j^{(\theta)} \triangleq \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \quad (\text{I.14d})$$

induces the probability measure $P_{\mathbf{U}_j^{(\theta)}}$ on the measurable space $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. Thus, $\mathbf{R}_n^{(\theta)}$ in (I.4) is the sum of independent random vectors $\mathbf{U}_1^{(\theta)}, \mathbf{U}_2^{(\theta)}, \dots, \mathbf{U}_n^{(\theta)}$ such that for all $j \in \{1, 2, \dots, n\}$,

$$\mathbb{E}_{P_{\mathbf{U}_j^{(\theta)}}} \left[\mathbf{U}_j^{(\theta)} \right] = \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \right] \quad (\text{I.14e})$$

$$= \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \right] \quad (\text{I.14f})$$

$$= 0, \text{ and} \quad (\text{I.14g})$$

$$\sum_{j=1}^n \mathbb{E}_{P_{\mathbf{U}_j^{(\theta)}}} \left[\mathbf{U}_j^{(\theta)} \mathbf{U}_j^{(\theta)\top} \right] = \mathbb{E}_{P_{\mathbf{R}_n^{(\theta)}}} \left[\mathbf{R}_n^{(\theta)} \mathbf{R}_n^{(\theta)\top} \right] \quad (\text{I.14h})$$

$$= \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{S}_n^{(\theta)} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \left(\mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{S}_n^{(\theta)} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \right)^\top \right] \quad (\text{I.14i})$$

$$= \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \mathbb{E}_{P_{\mathbf{S}_n^{(\theta)}}} \left[\left(\mathbf{S}_n^{(\theta)} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right) \left(\mathbf{S}_n^{(\theta)} - \boldsymbol{\mu}_{\mathbf{S}_n^{(\theta)}} \right)^\top \right] \left(\mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \right)^\top \quad (\text{I.14j})$$

$$= \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \bar{\boldsymbol{\nu}}_{\mathbf{S}_n^{(\theta)}} \left(\mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \right)^\top \quad (\text{I.14k})$$

$$= \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \mathbf{l}_{\mathbf{S}_n^{(\theta)}} \mathbf{l}_{\mathbf{S}_n^{(\theta)}}^\top \left(\mathbf{l}_{\mathbf{S}_n^{(\theta)}}^{-1} \right)^\top \quad (\text{I.14l})$$

$$= \text{diag}(1, 1, \dots, 1). \quad (\text{I.14m})$$

Note that by assumption of Lemma 19, the mean vector and the covariance matrix of the random vector $\mathbf{Z}_n^{(\theta)}$ are identical to the the mean vector and the covariance matrix of the random vector $\mathbf{S}_n^{(\theta)}$. Then, from (I.4) and (I.5), the mean vector and the covariance matrix of the random vector $\mathbf{W}_n^{(\theta)}$ are identical to the mean vector and the covariance matrix of the random vector $\mathbf{R}_n^{(\theta)}$. Hence, from (I.14c), (I.14g) and (I.14m), the random vector $\mathbf{W}_n^{(\theta)}$ in (I.5) is a Gaussian random vector with mean vector $(0, 0, \dots, 0)$ and covariance matrix $\text{diag}(1, 1, \dots, 1)$. Then, from the multivariate Berry-Essen Theorem ([46, Theorem 1.1]), it holds that

$$\Delta \left(P_{\mathbf{R}_n^{(\theta)}}, P_{\mathbf{W}_n^{(\theta)}} \right) \leq \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{U}_j^{(\theta)}}} \left[\|\mathbf{U}_j^{(\theta)}\|^3 \right] \right) \quad (\text{I.15a})$$

$$= \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{U}_j^{(\theta)}}} \left[\left(\mathbf{U}_j^{(\theta)\top} \mathbf{U}_j^{(\theta)} \right)^{\frac{3}{2}} \right] \right) \quad (\text{I.15b})$$

$$= \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\left(\mathbf{L}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \right)^\top \mathbf{L}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \right)^{\frac{3}{2}} \right] \right) \quad (\text{I.15c})$$

$$= \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right)^\top \left(\mathbf{L}_{\mathbf{S}_n^{(\theta)}}^{-1} \right)^\top \mathbf{L}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \right)^{\frac{3}{2}} \right] \right) \quad (\text{I.15d})$$

$$= \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j^{(\theta)}}} \left[\left(\left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right)^\top \mathbf{v}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j^{(\theta)} - \boldsymbol{\mu}_{\mathbf{Y}_j^{(\theta)}} \right) \right)^{\frac{3}{2}} \right] \right) \quad (\text{I.15e})$$

$$= \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j}} \left[\left(\left(\mathbf{Y}_j - \boldsymbol{\mu}_{\mathbf{Y}_j} \right)^\top \mathbf{v}_{\mathbf{S}_n^{(\theta)}}^{-1} \left(\mathbf{Y}_j - \boldsymbol{\mu}_{\mathbf{Y}_j} \right) \right)^{3/2} \exp \left(\boldsymbol{\theta}^\top \mathbf{Y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \right] \right), \quad (\text{I.15f})$$

where c is the function defined in (4.14).

Plugging (I.1c) and (I.1g) in (I.15f) yields

$$\Delta \left(P_{\mathbf{R}_n^{(\theta)}}, P_{\mathbf{W}_n^{(\theta)}} \right) \leq \min \left(1, c(k) \sum_{j=1}^n \mathbb{E}_{P_{\mathbf{Y}_j}} \left[\left(\left(\mathbf{Y}_j - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right)^\top \left(\sum_{s=1}^n K_{\mathbf{Y}_s}^{(2)}(\boldsymbol{\theta}) \right)^{-1} \left(\mathbf{Y}_j - K_{\mathbf{Y}_j}^{(1)}(\boldsymbol{\theta}) \right) \right)^{3/2} \exp \left(\boldsymbol{\theta}^\top \mathbf{Y}_j - K_{\mathbf{Y}_j}(\boldsymbol{\theta}) \right) \right] \right) \quad (\text{I.15g})$$

$$= \min \left(1, c(k) \sum_{j=1}^n \xi_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}^j(\boldsymbol{\theta}) \right), \quad (\text{I.15h})$$

where the function $\xi_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}^j$ is defined in (4.84). Plugging (I.15h) in (I.13b), it holds that

$$\Delta \left(P_{\mathbf{S}_n^{(\theta)}}, P_{\mathbf{Z}_n^{(\theta)}} \right) \leq \min \left(1, c(k) \sum_{j=1}^n \xi_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}^j (\boldsymbol{\theta}) \right). \quad (\text{I.16})$$

Finally, plugging (I.16) in (4.82) yields

$$\begin{aligned} & |P_{\mathbf{X}_n}(\mathcal{A}) - \eta_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}(\boldsymbol{\theta}, \mathcal{A})| \\ & \leq \exp \left(\sum_{j=1}^n K_{\mathbf{Y}_j}(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbf{a}(\mathcal{A}, \boldsymbol{\theta}) \right) \min \left(1, c(k) \sum_{j=1}^n \xi_{\mathbf{Y}_1 \mathbf{Y}_2 \dots \mathbf{Y}_n}^j (\boldsymbol{\theta}) \right), \end{aligned} \quad (\text{I.17})$$

which completes the proof.



Proof of Lemma 27

Note that the function T in (2.19) can be written in the form

$$T(n, P_X, Q_Y, \gamma) = F_{W_n}(\ln(\gamma)) + \gamma(1 - F_{V_n}(\ln(\gamma))), \quad (\text{J.1})$$

where F_{W_n} and F_{V_n} are respectively the CDFs of W_n and V_n that are sums i.i.d. random variables, i.e,

$$W_n = \sum_{t=1}^n \tilde{t}(X_t; Y_t | Q_Y), \quad (\text{J.2})$$

$$V_n = \sum_{t=1}^n \tilde{t}(X_t; \bar{Y}_t | Q_Y), \quad (\text{J.3})$$

such that the couple (X_t, Y_t) induces the probability measure $P_X P_{Y|X}$ whereas (X_t, \bar{Y}_t) induces the probability measure $P_X Q_Y$. This observation allows to use the theorem 17 to bound the function T in (2.19). That is, the following bounds hold on the CDFs F_{W_n} and F_{V_n} in (J.1):

$$\left| F_{W_n}(\ln(\gamma)) - \hat{F}_{W_n}(\ln(\gamma)) \right| \leq \exp(nK_{j,P}(\theta) - \theta \ln(\gamma)) \min\left(1, \frac{2\xi_{j,P}(\theta)}{\sqrt{n}}\right) \quad (\text{J.4})$$

$$\leq E(n, \gamma, P_X, Q_Y), \text{ and} \quad (\text{J.5})$$

$$\left| F_{V_n}(\ln(\gamma)) - \hat{F}_{V_n}(\ln(\gamma)) \right| \leq \exp(nK_{j,G}(s) - s \ln(\gamma)) \min\left(1, \frac{2\xi_{j,G}(s)}{\sqrt{n}}\right), \quad (\text{J.6})$$

where for all $x \in \mathbb{R}$

$$\hat{F}_{W_n}(x) = \mathbb{1}_{\{\theta > 0\}} + (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp\left(nK_{j,P}(\theta) - \theta x + \frac{1}{2}\theta^2 nK_{j,P}^{(2)}(\theta)\right) Q\left(|\theta| \sqrt{nK_{j,P}^{(2)}(\theta)}\right), \quad (\text{J.7})$$

$$= \hat{T}_1(n, \exp(x), \theta, P_X, Q_Y) \quad (\text{J.8})$$

$$\hat{F}_{V_n}(x) = \mathbb{1}_{\{s > 0\}} + (-1)^{\mathbb{1}_{\{s > 0\}}} \exp\left(nK_{j,G}(s) - sx + \frac{1}{2}s^2 nK_{j,G}^{(2)}(s)\right) Q\left(|s| \sqrt{nK_{j,G}^{(2)}(s)}\right), \quad (\text{J.9})$$

$$\xi_{J,P}(\theta) = c_1 \left(\frac{\mathbb{E}_{P_X P_{Y|X}} \left[\left| \tilde{l}(X; Y|Q_Y) - K_{j,P}^{(1)}(\theta) \right|^3 \exp(\theta \tilde{l}(X; Y|Q_Y)) - K_{j,G}(\theta) \right]}{\left(K_{j,P}^{(2)}(\theta) \right)^{3/2}} + c_2 \right), \quad (\text{J.10})$$

$$\xi_{J,G}(s) = c_1 \left(\frac{\mathbb{E}_{P_X Q_Y} \left[\left| \tilde{l}(X; Y|Q_Y) - K_{j,G}^{(1)}(s) \right|^3 \exp(s \tilde{l}(X; Y|Q_Y)) - K_{j,G}(s) \right]}{\left(K_{j,G}^{(2)}(s) \right)^{3/2}} + c_2 \right), \quad (\text{J.11})$$

and

$$nK_{j,P}^{(1)}(\theta) = \ln(\gamma) = nK_{j,G}^{(1)}(s), \quad (\text{J.12})$$

with the constants c_1 and c_2 in (3.24), $K_{j,P}$ and $K_{j,G}$ the CGFs of the random variables $\tilde{l}(X; Y|Q_Y)$ and $\tilde{l}(X; \bar{Y}|Q_Y)$, respectively.

More specifically, for all $t \in \mathbb{R}$

$$K_{j,G}(t) = \ln \left(\mathbb{E}_{P_X Q_Y} \left[\exp(t \tilde{l}(X; Y|Q_Y)) \right] \right), \quad \text{and} \quad (\text{J.13})$$

$$K_{j,P}(t) = \ln \left(\mathbb{E}_{P_X P_{Y|X}} \left[\exp(t \tilde{l}(X; Y|Q_Y)) \right] \right) \quad (\text{J.14})$$

$$= \ln \left(\mathbb{E}_{P_X Q_Y} \left[\frac{dP_X P_{Y|X}}{dP_X Q_Y}(X, Y) \exp(t \tilde{l}(X; Y|Q_Y)) \right] \right) \quad (\text{J.15})$$

$$= \ln \left(\mathbb{E}_{P_X Q_Y} \left[\exp((t+1) \tilde{l}(X; Y|Q_Y)) \right] \right) \quad (\text{J.16})$$

$$= K_{j,G}(t+1) \quad (\text{J.17})$$

where the expression in (J.15) follows from change of a measure from $P_X P_{Y|X}$ to $P_X Q_Y$ due to the absolute continuity of the former with respect to the latter; the expression in (J.16) follows from the definition of \tilde{l} in (2.17); and the expression in (J.17) follows from (J.13). Combining the observation in (J.17) to the equality in (J.12) leads to

$$\theta + 1 = s. \quad (\text{J.18})$$

Plugging (J.17) and (J.18) in (J.10) yields

$$\begin{aligned} & \xi_{J,P}(\theta) \\ &= \xi_{J,P}(s-1) \end{aligned} \quad (\text{J.19})$$

$$= c_1 \left(\frac{\mathbb{E}_{P_X P_{Y|X}} \left[\left| \tilde{l}(X; Y|Q_Y) - K_{j,P}^{(1)}(s-1) \right|^3 \exp((s-1) \tilde{l}(X; Y|Q_Y)) - K_{j,P}(s-1) \right]}{\left(K_{j,P}^{(2)}(s-1) \right)^{3/2}} + c_2 \right) \quad (\text{J.20})$$

$$= c_1 \left(\frac{\mathbb{E}_{P_X P_{Y|X}} \left[\left| \tilde{l}(X; Y|Q_Y) - K_{j,G}^{(1)}(s) \right|^3 \exp((s-1) \tilde{l}(X; Y|Q_Y)) - K_{j,G}(s) \right]}{\left(K_{j,G}^{(2)}(s) \right)^{3/2}} + c_2 \right) \quad (\text{J.21})$$

$$=c_1 \left(\frac{\mathbb{E}_{P_X Q_Y} \left[\frac{dP_X P_Y | X}{dP_X Q_Y}(U) \left| \tilde{l}(X; Y | Q_Y) - K_{j,G}^{(1)}(s) \right|^3 \exp((s-1)\tilde{l}(X; Y | Q_Y)) - K_{j,G}(s) \right]}{\left(K_{j,G}^{(2)}(s) \right)^{3/2}} + c_2 \right) \quad (\text{J.22})$$

$$=c_1 \left(\frac{\mathbb{E}_{P_X Q_Y} \left[\left| \tilde{l}(X; Y | Q_Y) - K_{j,G}^{(1)}(s) \right|^3 \exp(s\tilde{l}(X; Y | Q_Y)) - K_{j,G}(s) \right]}{\left(K_{j,G}^{(2)}(s) \right)^{3/2}} + c_2 \right) \quad (\text{J.23})$$

$$= \xi_{J,G}(s), \quad (\text{J.24})$$

where the expression in (J.24) follows from (J.11).

Plugging (J.17) and (J.18) in (J.9) yields

$$\begin{aligned} \hat{F}_{V_n}(x) &= \mathbb{1}_{\{\theta > -1\}} + (-1)^{\mathbb{1}_{\{\theta > -1\}}} Q \left(|\theta + 1| \sqrt{n K_{j,G}^{(2)}(\theta + 1)} \right) \\ &\quad \exp \left(n K_{j,G}(\theta + 1) - (\theta + 1)x + \frac{1}{2}(\theta + 1)^2 n K_{j,G}^{(2)}(\theta + 1) \right) \end{aligned} \quad (\text{J.25})$$

$$\begin{aligned} &= \mathbb{1}_{\{\theta > -1\}} + (-1)^{\mathbb{1}_{\{\theta > -1\}}} Q \left(|\theta + 1| \sqrt{n K_{j,P}^{(2)}(\theta)} \right) \\ &\quad \exp \left(n K_{j,G}(\theta + 1) - (\theta + 1)x + \frac{1}{2}(\theta + 1)^2 n K_{j,P}^{(2)}(\theta) \right) \end{aligned} \quad (\text{J.26})$$

$$= 1 - \hat{T}_2(n, \exp(x), P_X, Q_Y), \quad (\text{J.27})$$

where the expression in (J.27) follows from (5.10). Plugging (J.17), (J.18), (J.24) and (J.27) in (J.6) yields

$$\left| F_{V_n}(\ln(\gamma)) - 1 + \hat{T}_2(n, \gamma, \theta, P_X, Q_Y) \right| \leq \exp(n K_{j,P}(\theta) - (\theta + 1)\ln(\gamma)) \min \left(1, \frac{2\xi_{j,P}(\theta)}{\sqrt{n}} \right) \quad (\text{J.28})$$

$$\leq \exp(n K_{j,P}(\theta) - (\theta + 1)\ln(\gamma)) \frac{2\xi_{j,P}(\theta)}{\sqrt{n}} \quad (\text{J.29})$$

$$= \frac{E(n, \gamma, P_X, Q_Y)}{\gamma} \quad (\text{J.30})$$

which can be written in the form

$$1 - F_{V_n}(\ln(\gamma)) \leq \hat{T}_2(n, \gamma, P_X, Q_Y) + \frac{E(n, \gamma, P_X, Q_Y)}{\gamma}, \quad \text{and} \quad (\text{J.31})$$

$$1 - F_{V_n}(\ln(\gamma)) \geq \hat{T}_2(n, \gamma, P_X, Q_Y) - \frac{E(n, \gamma, P_X, Q_Y)}{\gamma}. \quad (\text{J.32})$$

The combination of the positivity of the function $1 - F_{V_n}(\ln(\gamma))$ with (J.32) yields

$$1 - F_{V_n}(\ln(\gamma)) \geq \max \left(0, \hat{T}_2(n, \gamma, \theta, P_X, Q_Y) - \frac{E(n, \gamma, P_X, Q_Y)}{\gamma} \right) \quad (\text{J.33})$$

$$= \underline{T}_2(n, \gamma, P_X, Q_Y), \quad (\text{J.34})$$

where the expression in (J.34) follows from (5.14).

From (J.5) and (J.8), it holds that

$$F_{W_n}(\ln(\gamma)) \leq \hat{T}_1(n, \gamma, P_X, Q_Y) + E(n, \gamma, P_X, Q_Y), \text{ and} \quad (\text{J.35})$$

$$F_{W_n}(\ln(\gamma)) \geq \hat{T}_1(n, \gamma, \theta, P_X, Q_Y) - E(n, \gamma, P_X, Q_Y). \quad (\text{J.36})$$

The combination of the positivity of the function $1 - F_{W_n}(\ln(\gamma))$ with (J.36) yields

$$F_{W_n}(\ln(\gamma)) \geq \max\left(0, \hat{T}_1(n, \gamma, P_X, Q_Y) - E(n, \gamma, P_X, Q_Y)\right) \quad (\text{J.37})$$

$$= \underline{T}_1(n, \gamma, P_X, Q_Y), \quad (\text{J.38})$$

where the expression in (J.38) follows from (5.13).

Plugging (J.34) and (J.38) in (J.1) yields

$$T(n, P_X, Q_Y, \gamma) \geq \underline{T}_1(n, \gamma, P_X, Q_Y) + \gamma \underline{T}_2(n, \gamma, P_X, Q_Y) \quad (\text{J.39})$$

$$= \underline{T}(n, \gamma, P_X, Q_Y), \quad (\text{J.40})$$

which complete the first part of the proof.

Plugging (J.31) and (J.35) in (J.1) yields

$$T(n, P_X, Q_Y, \gamma) \leq \hat{T}_1(n, \gamma, P_X, Q_Y) + \gamma \hat{T}_2(n, \gamma, P_X, Q_Y) + 2E(n, \gamma, P_X, Q_Y) \quad (\text{J.41})$$

$$= \hat{T}(n, \gamma, P_X, Q_Y) + 2E(n, \gamma, P_X, Q_Y). \quad (\text{J.42})$$

Note that $T(n, P_X, Q_Y, \gamma) \leq 1$. Hence,

$$T(n, P_X, Q_Y, \gamma) \leq \min\left(1, \hat{T}(n, \gamma, P_X, Q_Y) + 2E(n, \gamma, P_X, Q_Y)\right) \quad (\text{J.43})$$

$$= \bar{T}(n, \gamma, P_X, Q_Y), \quad (\text{J.44})$$

which complete the second part of the proof.



Proof of Lemma 35

The proof is derived in two steps:

- The terms in (K.1) are expressed as a function of the CDFs of $S_{n,\theta}$ and $Z_{n,\theta}$. The derivation is done for discrete and absolute continuous random variables and leads to a unique formulation in (K.8).
- The difference is bounded by using the Kolmogorov–Smirnov distance in (K.12).

The left-hand side of (B.7) satisfies

$$\begin{aligned} & \left| P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right| \\ &= (\varphi_Y(\theta))^n \left| \mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right] - \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right|. \end{aligned} \quad (\text{K.1})$$

The focus is on obtaining explicit expressions for the terms $\mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right]$ and $\mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right]$ in (K.1). First, consider the case in which the random variable $S_{n,\theta}$ is absolutely continuous and denote its probability density function by $f_{S_{n,\theta}}$ and its CDF by $F_{S_{n,\theta}}$. Then,

$$\mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right] = \int_{\mathcal{A}} \exp(-\theta x) f_{S_{n,\theta}}(x) dx. \quad (\text{K.2})$$

Using integration by parts in (K.2), under the assumption (i) or (ii) in Lemma 35, the following holds:

$$\mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right] = (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp(-\theta a) F_{S_{n,\theta}}(a) + \int_{\mathcal{A}} \theta \exp(-\theta x) F_{S_{n,\theta}}(x) dx \quad (\text{K.3})$$

Second, consider the case in which the random variable $S_{n,\theta}$ is discrete and denote its probability mass function by $p_{S_{n,\theta}}$ and its CDF by $F_{S_{n,\theta}}$. Let the support of $S_{n,\theta}$ be $\{s_0, s_1, \dots, s_\ell\} \subset \mathbb{R}$, with $\ell \in \mathbb{N}$. Assume that condition (i) in Lemma 35 is satisfied. Then,

$$\mathcal{A} \cap \{s_0, s_1, \dots, s_\ell\} = \{s_0, s_1, \dots, s_u\}, \quad (\text{K.4})$$

with $u \leq \ell$, and

$$\begin{aligned} & \mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right] \\ &= \sum_{k=0}^u \exp(-\theta s_k) p_{S_{n,\theta}}(s_k) \end{aligned} \quad (\text{K.5a})$$

$$= F_{S_{n,\theta}}(s_0) \exp(-\theta s_0) + \sum_{k=1}^u (F_{S_{n,\theta}}(s_k) - F_{S_{n,\theta}}(s_{k-1})) \exp(-\theta s_k) \quad (\text{K.5b})$$

$$= \sum_{k=0}^u F_{S_{n,\theta}}(s_k) \exp(-\theta s_k) - \sum_{k=1}^u F_{S_{n,\theta}}(s_{k-1}) \exp(-\theta s_k) \quad (\text{K.5c})$$

$$= \sum_{k=0}^u F_{S_{n,\theta}}(s_k) \exp(-\theta s_k) - \sum_{k=0}^{u-1} F_{S_{n,\theta}}(s_k) \exp(-\theta s_{k+1}) \quad (\text{K.5d})$$

$$= F_{S_{n,\theta}}(s_u) \exp(-\theta s_u) - \sum_{k=0}^{u-1} F_{S_{n,\theta}}(s_k) (\exp(-\theta s_{k+1}) - \exp(-\theta s_k)) \quad (\text{K.5e})$$

$$= F_{S_{n,\theta}}(s_u) \exp(-\theta s_u) - \sum_{k=0}^{u-1} \int_{s_k}^{s_{k+1}} -\theta \exp(-\theta t) F_{S_{n,\theta}}(s_k) dt \quad (\text{K.5f})$$

$$= F_{S_{n,\theta}}(s_u) \exp(-\theta s_u) + \int_{s_0}^{s_u} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.5g})$$

$$= F_{S_{n,\theta}}(a) \exp(-\theta a) - F_{S_{n,\theta}}(a) \exp(-\theta a) + F_{S_{n,\theta}}(s_u) \exp(-\theta s_u) + \int_{s_0}^{s_u} F_{S_{n,\theta}}(t) \theta \exp(-\theta t) dt \quad (\text{K.5h})$$

$$= F_{S_{n,\theta}}(a) \exp(-\theta a) - F_{S_{n,\theta}}(s_u) \exp(-\theta a) + F_{S_{n,\theta}}(s_u) \exp(-\theta s_u) + \int_{s_0}^{s_u} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.5i})$$

$$= F_{S_{n,\theta}}(a) \exp(-\theta a) - F_{S_{n,\theta}}(s_u) (\exp(-\theta a) - \exp(-\theta s_u)) + \int_{s_0}^{s_u} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.5j})$$

$$= F_{S_{n,\theta}}(a) \exp(-\theta a) + \int_{s_u}^a \theta \exp(-\theta t) F_{S_{n,\theta}}(s_u) dt + \int_{s_0}^{s_u} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.5k})$$

$$= \exp(-\theta a) F_{S_{n,\theta}}(a) + \int_{s_0}^a \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.5l})$$

$$= \exp(-\theta a) F_{S_{n,\theta}}(a) + \int_{-\infty}^a \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt, \quad (\text{K.5m})$$

which is an expression of the same form as the one in (K.3). Alternatively, assume that condition (ii) in Lemma 35 holds. Then,

$$\mathcal{A} \cap \{s_0, s_1, \dots, s_l\} = \{s_u, s_{u+1}, \dots, s_l\}, \quad (\text{K.6})$$

with $u \leq \ell$, and

$$\begin{aligned} & \mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right] \\ &= \sum_{k=u}^l \exp(-\theta s_k) p_{S_{n,\theta}}(s_k) \end{aligned} \quad (\text{K.7a})$$

$$= (F_{S_{n,\theta}}(s_u) - F_{S_{n,\theta}}(a)) \exp(-\theta s_u) + \sum_{k=u+1}^l (F_{S_{n,\theta}}(s_k) - F_{S_{n,\theta}}(s_{k-1})) \exp(-\theta s_k) \quad (\text{K.7b})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta s_u) + \sum_{k=u}^l F_{S_{n,\theta}}(s_k) \exp(-\theta s_k) - \sum_{k=u+1}^l F_{S_{n,\theta}}(s_{k-1}) \exp(-\theta s_k) \quad (\text{K.7c})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta s_u) + \sum_{k=u}^l F_{S_{n,\theta}}(s_k) \exp(-\theta s_k) - \sum_{k=u}^{l-1} F_{S_{n,\theta}}(s_k) \exp(-\theta s_{k+1}) \quad (\text{K.7d})$$

$$= F_{S_{n,\theta}}(s_l) \exp(-\theta s_l) - F_{S_{n,\theta}}(a) \exp(-\theta s_u) - \sum_{k=u}^{l-1} F_{S_{n,\theta}}(s_k) (\exp(-\theta s_{k+1}) - \exp(-\theta s_k)) \quad (\text{K.7e})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta s_u) - \int_{s_l}^{\infty} -\theta \exp(-\theta s_t) F_{S_{n,\theta}}(s_l) dt - \sum_{k=u}^{l-1} \int_{s_k}^{s_{k+1}} -\theta \exp(-\theta t) F_{S_{n,\theta}}(s_k) dt \quad (\text{K.7f})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta s_u) + \int_{s_u}^{\infty} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.7g})$$

$$= F_{S_{n,\theta}}(a) \exp(-\theta a) - F_{S_{n,\theta}}(a) \exp(-\theta a) - F_{S_{n,\theta}}(a) \exp(-\theta s_u) + \int_{s_u}^{\infty} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.7h})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta a) - F_{S_{n,\theta}}(a) (\exp(-\theta s_u) - \exp(-\theta a)) + \int_{s_u}^{\infty} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.7i})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta a) - \int_a^{s_u} -\theta \exp(-\theta t) F_{S_{n,\theta}}(a) dt + \int_{s_u}^{\infty} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt \quad (\text{K.7j})$$

$$= -F_{S_{n,\theta}}(a) \exp(-\theta a) + \int_a^{\infty} \theta \exp(-\theta t) F_{S_{n,\theta}}(t) dt, \quad (\text{K.7k})$$

which is an expression of the same form as those in (K.3) and (K.5m).

Note that, under the assumption that at least one of the conditions in Lemma 35 holds, the expressions in (K.3), (K.5m), and (K.7k) can be jointly written as follows:

$$\mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right] = (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp(-\theta a) F_{S_{n,\theta}}(a) + \int_{\mathcal{A}} \theta \exp(-\theta x) F_{S_{n,\theta}}(x) dx \quad (\text{K.8})$$

The expression in (K.8) does not involve particular assumptions on the random variable $S_{n,\theta}$ other than being discrete or absolutely continuous. Hence, the same expression holds with respect to the random variable $Z_{n,\theta}$ in (K.1). More specifically,

$$\mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] = (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp(-\theta a) F_{Z_{n,\theta}}(a) + \int_{\mathcal{A}} \theta \exp(-\theta x) F_{Z_{n,\theta}}(x) dx \quad (\text{K.9})$$

where $F_{Z_{n,\theta}}$ is the CDF of the random variable $Z_{n,\theta}$.

The proof ends by plugging (K.8) and (K.9) into the right-hand side of (K.1). This yields

$$\begin{aligned} & \left| P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right| \\ &= (\varphi_Y(\theta))^n \left| (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp(-\theta a) F_{S_{n,\theta}}(a) + \int_{\mathcal{A}} \theta \exp(-\theta x) F_{S_{n,\theta}}(x) dx \right. \\ & \quad \left. - (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp(-\theta a) F_{Z_{n,\theta}}(a) - \int_{\mathcal{A}} \theta \exp(-\theta x) F_{Z_{n,\theta}}(x) dx \right| \end{aligned} \quad (\text{K.10a})$$

$$= (\varphi_Y(\theta))^n \left| (-1)^{\mathbb{1}_{\{\theta > 0\}}} \exp(-a) (F_{S_{n,\theta}}(a) - F_{Z_{n,\theta}}(a)) + \int_{\mathcal{A}} \theta \exp(-\theta x) (F_{S_{n,\theta}}(x) - F_{Z_{n,\theta}}(x)) dx \right| \quad (\text{K.10b})$$

$$\leq (\varphi_Y(\theta))^n \left(|\exp(-\theta a) (F_{S_{n,\theta}}(a) - F_{Z_{n,\theta}}(a))| + \left| \int_{\mathcal{A}} \theta \exp(-\theta x) (F_{S_{n,\theta}}(x) - F_{Z_{n,\theta}}(x)) dx \right| \right) \quad (\text{K.10c})$$

$$\leq (\varphi_Y(\theta))^n \left(\exp(-\theta a) \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) + \int_{\mathcal{A}} |\theta \exp(-\theta x)| \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) dx \right) \quad (\text{K.10d})$$

$$= (\varphi_Y(\theta))^n \left(\exp(-\theta a) \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) + \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) \left| \int_{\mathcal{A}} \theta \exp(-\theta x) dx \right| \right) \quad (\text{K.10e})$$

$$= (\varphi_Y(\theta))^n \left(\exp(-\theta a) \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) + \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}) \exp(-\theta a) \right) \quad (\text{K.10f})$$

$$= 2 \frac{(\varphi_Y(\theta))^n}{\exp(\theta a)} \Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}). \quad (\text{K.10g})$$

Finally, under the assumption that at least one of the conditions in Lemma 35 holds, then

$$\left| P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right| \leq (\varphi_Y(\theta))^n \max \left(\mathbb{E}_{P_{S_{n,\theta}}} \left[\exp(-\theta S_{n,\theta}) \mathbb{1}_{\{S_{n,\theta} \in \mathcal{A}\}} \right], \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right) \quad (\text{K.11a})$$

$$\leq (\varphi_Y(\theta))^n \exp(-\theta a) = \frac{(\varphi_Y(\theta))^n}{\exp(\theta a)}. \quad (\text{K.11b})$$

Under the same assumption, the expressions in (K.10g) and (K.11b) can be jointly written as follows:

$$\left| P_{X_n}(\mathcal{A}) - (\varphi_Y(\theta))^n \mathbb{E}_{P_{Z_{n,\theta}}} \left[\exp(-\theta Z_{n,\theta}) \mathbb{1}_{\{Z_{n,\theta} \in \mathcal{A}\}} \right] \right| \leq \frac{(\varphi_Y(\theta))^n}{\exp(\theta a)} \min \{ 2\Delta(P_{S_{n,\theta}}, P_{Z_{n,\theta}}), 1 \} \quad (\text{K.12})$$

This concludes the proof of Lemma 35.



Proof of Lemma 36

IN the case in which Y is discrete ($p_Y, p_{Y^{(\theta)}}, p_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}$ denote probability mass functions) or absolutely continuous random variables ($p_Y, p_{Y^{(\theta)}}, p_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}$ denote probability density functions), the following holds for all $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,

$$\frac{dP_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}(y_1, y_2, \dots, y_n)}{dP_{Y_1Y_2\dots Y_n}}(y_1, y_2, \dots, y_n) = \frac{p_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}(y_1, y_2, \dots, y_n)}{\prod_{j=1}^n p_Y(y_j)}, \quad (\text{L.1})$$

and for all $y \in \mathbb{R}$,

$$\frac{dP_{Y^{(\theta)}}(y)}{dP_Y}(y) = \frac{p_{Y^{(\theta)}}(y)}{p_Y(y)}. \quad (\text{L.2})$$

Equating the right-hand side of both (B.1) and (L.1), it yields for all $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

$$p_{Y_1^{(\theta)}Y_2^{(\theta)}\dots Y_n^{(\theta)}}(y_1, y_2, \dots, y_n) = \prod_{j=1}^n \frac{\exp(\theta y_j)}{\varphi_Y(\theta)} p_Y(y_j). \quad (\text{L.3})$$

Hence, $Y_1^{(\theta)}, Y_2^{(\theta)}, \dots, Y_n^{(\theta)}$ are mutually independent and identically distributed. Moreover, for all $y \in \mathbb{R}$,

$$p_{Y^{(\theta)}}(y) = \frac{\exp(\theta y)}{\varphi_Y(\theta)} p_Y(y). \quad (\text{L.4})$$

Finally, plugging (L.4) in (L.2) yields, for all $y \in \mathbb{R}$,

$$\frac{dP_{Y^{(\theta)}}(y)}{dP_Y}(y) = \frac{\exp(\theta y)}{\varphi_Y(\theta)}, \quad (\text{L.5})$$

which completes the proof.



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FOLIO ADMINISTRATIF

THESE DE L'UNIVERSITE DE LYON OPEREE AU SEIN DE L'INSA LYON

NOM : ANADE
(avec précision du nom de jeune fille, le cas échéant)

DATE de SOUTENANCE : 07/10/2021

Prénoms : Dadja

TITRE : Contribution à l'analyse des systèmes de communication en régime paquets courts.

NATURE : Doctorat

Numéro d'ordre : 2021LYSEI065

Ecole doctorale : Électronique, Électrotechnique et Automatique de l'université de Lyon

Spécialité : Traitement du signal et des images

RESUME : Dans cette thèse, une fonction qui approxime la fonction de répartition d'une somme de vecteurs aléatoires indépendants et identiquement distribués est présentée. L'erreur d'approximation est majorée, et par conséquent, une borne supérieure et une borne inférieure sur la fonction de répartition sont obtenues. Pour des vecteurs aléatoires absolument continus ou discrets réguliers ("lattice"), l'approximation proposée est identique à l'approximation du point de selle de la fonction de répartition. Ce résultat est utilisé pour approcher les bornes de probabilité d'erreur de décodage pour les canaux point à point et à accès multiple. Sur le canal point à point, cette approche a permis de constater l'insuffisance de l'approximation normale, particulièrement pour des probabilités d'erreur de décodage de faibles valeurs. Concernant les canaux à accès multiple, la considération de la notion d'erreur individuel a révélé le comportement presque non interférant des transmetteurs pour des petites valeurs de la probabilité d'erreur de décodage et de la longueur des paquets.

MOTS-CLÉS : Fonction de répartition, somme de vecteurs indépendant, canal point à point, canal multiple access

Laboratoire (s) de recherche : CITI

Directeur de thèse: Prof. GORCE Jean-Marie

Président de jury : Prof. POOR H. Vincent

Composition du jury : BACCELLI François, GORCE Jean-Marie, MARY Philippe, PERLAZA M. Samir, POOR H.Vincent, ROUMY Aline, WIGGER Michèle.

